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TIME - FRACTIONAL DIFFUSION EQUATION AND ITS APPLICATIONS IN PHYSICS

Tesi di laurea in Fisica

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Abstract

In physics, process involving the phenomena of diffusion and wave propagation have great relevance; these physical process are governed, from a mathematical point of view, by differential equations of order 1 and 2 in time. By introducing a fractional derivatives of order α in time, with $0 < \alpha < 1$ or $1 \leq \alpha \leq 2$, we lead to process in mathematical physics which we may refer to as fractional phenomena; this is not merely a phenomenological procedure providing an additional fit parameter.

The aim of this thesis is to provide a description of such phenomena adopting a mathematical approach to the fractional calculus.

The use of Fourier-Laplace transform in the analysis of the problem leads to certain special functions, scilicet transcendental functions of the Wright type, nowadays known as $M-$Wright functions.

We will distinguish slow-diffusion processes $(0 < \alpha < 1)$ from intermediate processes $(1 \leq \alpha \leq 2)$, and we point out the attention to the applications of fractional calculus in certain problems of physical interest, such as the Neuronal Cable Theory.

Sommario

In fisica, i processi che coinvolgono i fenomeni della diffusione e della propagazione hanno una grande rilevanza; questi processi fisici sono matematicamente governati da equazioni differenziali di ordine 1 e 2 nel tempo.
Con l'introduzione di una derivata frazionaria di ordine α nel tempo, con $0 < \alpha < 1$ o $1 \leq \alpha \leq 2$, otteniamo dei processi della fisica matematica ai quali possiamo riferirci con il termine di fenomeni frazionari; questa non è semplicemente una procedura fenomenologica che ci fornisce un parametro di fit aggiuntivo.
Lo scopo di questa tesi è di fornire una descrizione di questi fenomeni adottando un approccio matematico volto al calcolo frazionario.
L'utilizzo delle trasformate di Fourier e Laplace nell'analisi del problema ci porterà a considerare alcune funzioni speciali, ovvero funzioni trascendenti del tipo Wright, note ai giorni d'oggi come funzioni $M-$Wright.
Distingueremo i processi di diffusione lenta ($0 < \alpha < 1$) dai processi intermedi ($1 \leq \alpha \leq 2$), ed evidenzieremo le applicazioni del calcolo frazionario in alcuni problemi di interesse fisico, come la Neuronal Cable Theory.

Introduction

The link between mathematics and physics should not surprise the reader; We can think of mathematics as the only way to describe the physics, and not just as a useful tool.
This doesn't means that this two disciplines cannot develop independently, but important milestones are reached through and intense interplay of this two camps, as pointed in [4].
The study of differential equations, differential geometry and operator theory, motivated respectively by physical topics such as Newtonian mechanics, general relativity and quantum mechanics, is a clear example of this. On the other hand, some of the most fundamental physics has been expressed in abstract mathematical formulations, such as mechanics in symplectic geometry.
Largely, the laws of physics are written in the form of differential equations. The presence of a single independent variable, as seen in point particle mechanics, leads to ordinary differential equations (ODEs). Other areas of physics studies extended objects, and partial derivatives join the differential equations, which are hence called partial differential equations (PDEs).
Some of the most frequently recurring and most famous PDEs in mathematical physics are the *Poisson's equation*, and its vacuum version known as *Laplace's equation*. Togheter with them, we have to mention the *heat equation*, the *D'Alambert equation*, the *Schröedinger equation*, and his relativistic generalization for a free particle, known as *Klein-Gordon equation*.
All these equations just mentioned have partial derivatives with respect to

time.
With the introduction of a fractional derivative, we obtain partial differential equations of fractional order.
When talking about time-fractional diffusion processes, we mean certain diffusion-like phenomena governed by master equations containing fractional derivatives in time, that are linear integro partial differential equations, and whose fundamental solutions can be interpreted as a probability density function (pdf) in space evolving in time.
For one of the most elementary diffusion process, the Brownian motion, the master equation is the standard linear diffusion equation, whose fundamental solution is the Gaussian density with a spatial variance growing linearly in time. We are so talking about normal diffusion, and we reserve the term anomalous diffusion when the variance grows differently.

In recent years sizable interest has been shown in *Fractional Calculus*; most authors will cite a particular date as the birthday of Fractional Calculus; In a letter dated September 30th, 1695 L'Hopital wrote to Leibniz asking him about a particular notation he had used in his publications for the nth-derivative of the linear function $f(x) = x$, $\frac{d^n x}{dx^n}$. L'Hopital's posed the question to Leibniz, what would the result be if $n = \frac{1}{2}$, and Leibniz's response: "An apparent paradox, from which one day useful consequences will be drawn" (see [5]).
Fractional calculus surely has a long history, dating back to famous matematician such as Euler, Laplace, Fourier, Liouville and Riemann; The first applications in physics were dealt with by Abel and Heaviside.
Such interest has been stimulated by the applications that this calculus finds in different areas of physics and engineering, possibly including fractal phenomena.
In more recent years, a fractional equation was introduced to describe diffusion in special types of porous media that exhibit a fractal geometry, to provides an opportunity to extend the use of the Bloch equation to describe

a wider range of experimental situations in Nuclear Magnetic Resonance (NMR) (see [7]), was showed that the fractional diffusion-wave equation governs a propagation of the mechanical diffusive waves in viscoelastic media that exhibit a simple power-law creep, and that can provide a concise model for the description of the dynamic events that occur in biological tissues.

The **outline** of this thesis is the following: In *Chapter 1* we will give some physical and mathematical understanding about pure diffusion ($\alpha = 1$) phenomena. In *Chapter 2* we will provide the reader with the essential notions and notations concerning the integral transforms and some special functions which are necessary in the rest of the thesis; moreover we introduce in the complex plane $\mathbb{C}$ the series and integral representations of the general Wright function denoted by $W_{\lambda,\mu}(z)$ and of the two related auxiliary functions $F_\nu(z)$, $M_\nu(z)$, considering then our auxiliary functions in real domain and pointing out their main properties, involving their integrals and their asymptotic representations. In *Chapter 3* we devote our attention to the fractional calculus, giving some basics definitions, and providing the fundamental solutions of the basic *Cauchy* and *Signalling* for the *time-fractional diffusion equation*, introducing a generalized derivative defined in the framework of the so-called *Riemann-Liouville fractional calculus.* Finally, in *Chapter 4* we consider some applications of the time-fractional diffusion equation, dealing with the *Time Fractional Neuronal Cable Equation.*

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Chapter 1

The Linear Diffusion Equation

The simplest linear evolution equations in Mathematical Physics are the classical diffusion and wave equations. Denoting by $\boldsymbol{x} := (x, y, z),\ t$ the space and time variables and by $u = u(\boldsymbol{x}, t)$ the field variable, these equations read:

$$\frac{\partial u}{\partial t} = D\nabla^2 u \tag{1.1}$$

for the *Diffusion Equation*, and

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u \tag{1.2}$$

for the *Wave Equation*.
D and c are both positive constants, and in above equations ∇ denotes the *Nabla Operator*, so that ∇^2 is the *Laplacian*.
Deffering the analysis of wave phenomena, now we consider the *Diffusion Equation* which is known to describe a number of physical models, such as the conduction of heat in a solid or the spread of solute particles in a solvent.

1.1 Physical insights

We want to have a physical understanding of how (1.1) arises, considering a simple model of heat conduction in a solid: in this case $u(\boldsymbol{x}, t)$ represents the temperature in a solid at position $\boldsymbol{x}$ and time t.

The heat can be transmitted in three fundamental ways: *radiation*, *conduction* and *advection* (or convection). In the simplest cases, the *radiation* is described by the laws of the black body; *Conduction* consists in the thermal agitation spread as a result of interactions between molecules: conduction is not associated then at transport of matter. In the case that a body is at rest, the heat can propagate only by conduction or by radiation; if the body is in motion, we can face with the *advection* of heat, that consists in the transport of the heat stored in the moving material: macroscopic portions of the material carries heat stored in it to new regions of space.

1.1.1 Fourier conduction law

If we think to a wall of width h, whose surfaces are maintained at temperatures T_1 and $T_2 < T_1$ by two thermostats, we can wait for the system reaches a stationary configuration and estimate the heat Q absorbed by the colder thermostat.
Calling A the surfaces area, ΔT the difference in temperature between the two surfaces, and t the time passed, it is experimentally shown that:

$$Q = kAt\frac{\Delta T}{h} \tag{1.3}$$

known as *Fourier Law*, which in differential form reads:

$$\mathbf{q} = -k\nabla T \tag{1.4}$$

The constant of proportionality k is a property of the material and is defined as thermal conductivity, and then $\mathbf{q}$, defined as heat flux, is the vector whose module provides the heat that flows through the unitary isotherm surface area in the unit of time, with the normal direction to the isotherm surface and with direction opposite to that in which the temperature increases.

1.1.2 Heat transmission law

As described in [1], lets think to a portion dV of material with specific heat c_p, temperature T, density ρ and thermal conductivity k; this portion is in motion with velocity $\mathbf{v}$. Furnishing to a body of mass $dM = \rho dV$ a certain amount of heat at costant pressure, in a reversible way, we will raise its entalpy of ΔH, and its temperature will rise of a quantity ΔT, basing on the law:

$$\Delta H = \rho c_p \Delta T dV$$

So, for an object of finished dimension, its increment of entalpy for unit of time is:

$$\dot{H} = \int_V \rho c_p \frac{dT}{dt}\, dV = \int_V \rho c_p \left(\frac{\partial T}{\partial t} + \mathbf{v} \nabla T \right) dV$$

If some heat is absorbed through the walls, or if the heat it is produced within the body, we then have an increase in temperature.
Mathematically, the two contributions will be respectively written as:

$$-\oint_{\partial V} \mathbf{q} \cdot \mathbf{n}\, dS$$

and

$$\int_V \rho \mathcal{Q}\, dV$$

∂V is the border of V, and $\mathcal{Q}$ is the specific rate of heat production, expressed in W/kg.

By the balance of these 3 terms, we get:

$$\int_V \rho c_p \frac{dT}{dt}\, dV = -\oint_{\partial V} \mathbf{q} \cdot \mathbf{n}\, dS + \int_V \rho \mathcal{Q}\, dV$$

and using the divergence theorem, we obtain:

$$\int_V \left(\rho c_p \frac{dT}{dt} + \nabla \cdot \mathbf{q} - \rho \mathcal{Q} \right) dV = 0$$

In the last steps we have to consider an arbitrary domain and the Fourier Law (1.4), getting the *heat transmission law*:

$$\rho c_p \left(\frac{\partial T}{\partial t} + \mathbf{v} \nabla T \right) = \rho \mathcal{Q} + \nabla \cdot (k \nabla T) \tag{1.5}$$

Eq. (1.1) is so obteined in absence of work, heat sources and heat sinks, and when the material is homogeneous, so that k does not depend on spatial coordinates.
The *diffusivity* D turns out to be given by:

$$D = \frac{k}{\rho c}$$

It's dimensions are $L^2 T^{-1}$, i.e. those of the kinematic viscosity of a fluid. When the heat conduction can be considered as a one-dimensional phenomenon, e.g. when the solid is a thin rod extended along the x-axis, what we wrote previously can be reduced to:

$$q(x,t) = -k u_x(x,t), \tag{1.6}$$

$$\rho c \frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{1.7}$$

In these Chapter we consider the one-dimensional diffusion equation,

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} \tag{1.8}$$

1.2 Mathematical insights: boundary value problems

In this section we refer to [13]. Like for any partial differential equation occurring in mathematical physics, we must specify some boundary conditions (i.e. the values attained by the field variable and/or by certain its derivatives on the boundary of the space-time domain) in order to guarantee the existence, the uniqueness and the determination of a solution of physical interest to the problem. There are many possible *boundary conditions* for the diffusion equation (1.8); we will consider the most simple boundary conditions, and for an easier visuaization, we could retain the terminology of heat transfer.
As far as the space-time domain is concerning, we presume that t varies in the semi-infinite interval $0 \leq t \leq \infty$, while the variable x may range in an interval which may be bounded or unbounded at one or both sides.
We specify the *initial temperature* of the conductor by a function of x, say $f(x)$, and the *temperature at the ends points* by two functions of t, say $g(t)$, $h(t)$; these *boundary conditions* are understood as limits as (x, t) approaches the respective boundary along a line orthogonal to it.
We write the boundary conditions in mathematical terms as follows

$$\boxed{\lim_{x \to 0^+} u(x,t) := u(x,0^+) = f(x), \quad a < x < b} \tag{1.9}$$

$$\boxed{\begin{array}{ll} \lim_{x \to a^+} u(x,t) := u(a^+,t) = g(t), & \\ \lim_{x \to b^-} u(x,t) := u(b^-,t) = h(t), & t > 0 \end{array}} \tag{1.10}$$

If the medium is bounded at both sides or unbounded at one side, it may be convenient to refer to the intervals $0 < x < L$ or $0 < x < +\infty$, respectively.
Because of the linearity of the diffusion equation, the above boundary value

problem (BVP) [(1.1)+ (1.9-10)] can be formatted as the superposition of three distinct BVP.

Denoting by $\mathcal{D}$ the differential operator of diffusion, i.e.

$$\mathcal{D} := \frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2} \tag{1.11}$$

and writing

$$u(x,t) = u_1(x,t) + u_2(x,t) + u_3(x,t) \tag{1.12}$$

we require

$$\begin{aligned}
&(a)\ \mathcal{D}u_1(x,t) = 0, u_1(x,0^+) = f(x),\ u_1(a^+,t) = 0,\ u_1(b^-,t) = 0;\\
&(b)\ \mathcal{D}u_2(x,t) = 0,\ u_2(x,0^+) = 0,\ u_2(a^+,t) = g(t),\ u_2(b^-,t) = 0;\\
&(c)\ \mathcal{D}u_3(x,t) = 0, u_3(x,0^+) = 0,\ \ u_3(a^+,t) = 0,\ u_3(b^-,t) = h(t);
\end{aligned} \tag{1.13}$$

The functions f, g, h, refered to as *data functions* are requested to satisfy some regularity conditions; we intend to privilege the application of trasforms methods based on the *space Fourier transforms* and the *time Laplace transforms* to find the solutions in the space-time domain. Therefore the following requirements for the *data functions* are sufficient: the space function $f(x)$ must admit the Fourier transform (if their support is finite, the Fourier series expansion), whereas the time functions $g(t)$, $h(t)$ must admit the Laplace transform.

Having in mind the application of the Laplace transform in the time variable, we have implicity assumed, for $t < 0$, the medium to be quiescent (i.e. at a constant equiibrium temperature); we require, without loosing in generality, that $u(x,t) \equiv u(x,0^-) \equiv 0$ for $a < x < b,\ \ t < 0$. In our approach so, any function of t is assumed to be *causal*, where with *causal function* we mean a function $\phi(t)$ vanishin for $t < 0$.

Sometimes it may be convenient to point out this fact writing

$$\phi_+ := \phi(t)H(t), \qquad where \quad H(t) := \begin{cases} 1 & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t < 0 \end{cases} \tag{1.14}$$

denotes the *Heaviside* or *unit-step function*.

Among the causal functions, a relevant role is played by the following function, referred to as *Gel'fand - Shilov function*,

$$\boxed{\Phi_\lambda(t) := \frac{t_+^{\lambda-1}}{\Gamma(\lambda)}, \quad \lambda \in \mathbb{C}}, \tag{1.15}$$

which is proved to be an *entire* function of λ [see Gel'fand-Shilov], and where the suffix $+$ just denotes the causality property of vanishing for $t < 0$. Furthermore, for $\lambda = 0, -1, -2, ...$ it is proved to reduce to the delta distribution and its derivatives (in the distribution sense), so providing an interesting alternative representation of these distributions. To point out the causality condition, we agree to write

$$\boxed{\delta_+^{(n)}(t) := \frac{t_+^{-(n-1)}}{\Gamma(-n)}, \quad n = 0, 1, 2, ...}, \tag{1.16}$$

and denote δ_+ the *causal delta generalized function.*

1.2.1 Cauchy and Signalling problems

Two basic problems for the diffusion equation are the *Cauchy problem* and the Signalling problem.
In the *Cauchy problem* the medium, supposed unlimited ($-\infty < x < \infty$) is subjected, for $t = 0$, to a known disturbance, provided by a function $f(x)$ thus we have

$$\begin{cases} u(x, 0^+) = f(x), & -\infty < x < +\infty; \\ u(\pm\infty, t) = 0, & t > 0 \end{cases} \tag{1.17}$$

Because the boundary values are specified along the boundary $t = 0$, usually are called *initial values*, and so this problem can be considered a pure *initial-value problem* (IVP).
In the *Signalling problem* the medium, supposed semi-infinite ($0 \leq x \leq +\infty$) and initially undisturbed is subjected, at $x = 0$ (the accessible end) and for $t > 0$, to a known disturbance, provided by a causal function $g(t)$; the conditions read

$$\begin{cases} u(x, 0^+) = 0, & 0 < x < +\infty; \\ u(0^+, t) = g(t), \quad u(+\infty, t) = 0, & t > 0 \end{cases} \tag{1.18}$$

This problem is also referred to as the *initial boundary value problem (IBVP)* in the quadrant $\{x, t\} > 0$.
For each problem the solutions turns out to be expressed by a proper convolution between the source function and a characteristic function (the so-called *Green function* or *fundamental solution* of the problem).

$$\textit{Cauchy problem}: \boxed{u(x,t) = \int_{-\infty}^{+\infty} \mathcal{G}_c(\xi,t) f(x-\xi)\, d\xi = \mathcal{G}_c(x,t) * f(x)}, \tag{1.19}$$

where $*$ denotes the (bilateral) space convolution. The function $\mathcal{G}_c(x, t)$ referred to as the *Green function* for the *Cauchy problem*, turns out to be

$$\boxed{\mathcal{G}_c(x,t) = \frac{1}{2\sqrt{\pi D}} t^{-1/2} e^{-x^2/(4Dt)}} \tag{1.20}$$

It represents the solutions for $f(x) = \delta(x)$, where $\delta(x)$ denotes the *Dirac* or *delta generalized function.*

$$\textit{Signalling problem}: \boxed{u(x,t) = \int_0^t \mathcal{G}_s(x,\tau) g(t-\tau)\, d\tau = \mathcal{G}_s(x,t) * g(t)}, \tag{1.21}$$

where now $*$ denotes the (unilateral) time convolution. (For causal functions the unlimited interval of the integral simply reduces to [0,t]).
The function $\mathcal{G}_s(x, t)$ referred to as the *Green function* for the *Signalling problem*, turns out to be

$$\boxed{\mathcal{G}_s(x,t) = \frac{x}{2\sqrt{\pi D}} t^{-3/2} e^{-x^2/(4Dt)}} \tag{1.22}$$

It represents the solutions for $g(t) = \delta_+(t)$.
The Green functions (1.20) and (1.22) are also referred to as the *fundamental*

solutions for the respective problems because of their relevance to construct the appropriate solution.

We can introduce in Eqs (1.21)-(1.22) the new variable of integration $\eta = \eta(\xi)$,

$$\eta = \frac{\xi}{2\sqrt{Dt}}, \tag{1.23}$$

and obtain an alternative representation for the general solution of the *Cauchy problem*,

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\eta^2} f(x - 2\sqrt{Dt}\eta) d\eta. \tag{1.24}$$

We can introduce in Eqs (1.19)-(1.20) the new variable of integration $\sigma = \sigma(\tau)$,

$$\sigma = \frac{x}{2\sqrt{D\tau}}, \tag{1.25}$$

and obtain an alternative representation for the general solution of the *Signalling problem*,

$$u(x,t) = \frac{2}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{Dt}}}^{+\infty} e^{-\sigma^2} g\left(t - \frac{x^2}{4D\sigma^2}\right) d\sigma. \tag{1.26}$$

A particular case of (1.26) is obtained when $g(t) = H(t)$; in this case the solutions turns out to be

$$\boxed{u(x,t) := \mathcal{H}_s(x,t) = erfc\left(\frac{x}{2\sqrt{Dt}}\right)} \tag{1.27}$$

where erfc denotes the *complementary error function.*

This solution is related to the corresponding *Green function* by the relation

$$\boxed{\mathcal{H}_s(x,t) = \int_0^t \mathcal{G}_s(x,\tau) d\tau} \tag{1.28}$$

From Eqs (1.20), (1.22) we note that the following relevant property is valid for $\{x,t\} > 0$,

$$\boxed{x\mathcal{G}_c(x,t) = t\mathcal{G}_s(x,t) = F(z)}, \tag{1.29}$$

where

$$z = \frac{x}{\sqrt{Dt}}, \quad F(z) = \frac{z}{2} M(z), \quad M(z) = \frac{1}{\sqrt{\pi}} e^{-z^2/4} \tag{1.30}$$

The first equality in (1.29) can be referred as *reciprocity relation* between the two Green functions. In Eqs (1.29)-(1.30) z represents the *similarity variable* and $F(z)$, $M(z)$ are referred to as the *auxiliary functions* in that through them the Green functions can easily be derived.

We recall that the *error function* erf(x) is usually defined as

$$erf(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi, \quad x \in \mathbb{R} \tag{1.31}$$

We note: $erf(-x) = -erf(x)$ with erf($\pm\infty$) = ± 1. Its Taylor series (around $x = 0$) reads

$$erf(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1} = \frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{n=0}^{\infty} \frac{2^n}{(2n+1)!!} x^{2n+1} \tag{1.32}$$

Its derivative is erf'(x)= $2e^{-x^2}/\sqrt{\pi}$. The *complementary error function* erfc(x) is

$$erfc(x) := 1 - erf(x) := \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-\xi^2} d\xi, \quad x \in \mathbb{R} \tag{1.33}$$

We point out the asymptotic representation: $erfc(x) \sim e^{-x^2}/(\sqrt{\pi}x)$ as $x \to \infty$, that provides a good approximation already from $x \geq 1,5$.
The next Sections are devoted to find the solutions to the previous problems by using methods based on integral transforms.

1.2.2 The Green function for the Cauchy problem via Fourier Transform

Is straightforward the use of the Fourier transform (FT) with respect to x for the *Cauchy problem*.

We adopt the following notation ($\kappa \in \mathbb{R}$)

$$\boxed{\hat{u}(\kappa,t) := \int_{-\infty}^{+\infty} e^{\pm i\kappa x} u(x,t)\,dx \div u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\pm i\kappa x}\hat{u}(\kappa,t)\,d\kappa} \tag{1.34}$$

Applying FT to [(1.8)+(1.17)] we get the first-order differential equation in t,

$$\frac{d\hat{u}}{dt} + D\kappa^2\hat{u} = 0, \quad t > 0, \tag{1.35}$$

with the initial condition

$$\hat{u}(\kappa, 0^+) = \hat{f}(\kappa). \tag{1.36}$$

The solution of this initial value problem is

$$\hat{u}(\kappa,t) = \hat{f}(\kappa)e^{-Dt\kappa^2}. \tag{1.37}$$

Introducing

$$\boxed{\hat{\mathcal{G}}_c(\kappa,t) := e^{-Dt\kappa^2}}, \tag{1.38}$$

the transform solution reads

$$\hat{u}(\kappa,t) = \hat{f}(\kappa)\hat{\mathcal{G}}_c(\kappa,t). \tag{1.39}$$

Apllying the convolution theorem for Fourier transforms, the solution in the space-time domain turns out to be

$$\boxed{u(x,t) = \int_{-\infty}^{+\infty} \mathcal{G}_c(\xi,t) f(x-\xi)\,d\xi = \mathcal{G}_c(x,t) * f(x)}, \tag{1.40}$$

where $\mathcal{G}_c(x,t)$ is the *Green function* for the *Cauchy problem.* For the inversion of $\hat{\mathcal{G}}_c(\kappa,t)$ we recall the Fourier transform pair

$$\boxed{e^{-a\kappa^2} \div \frac{1}{2\sqrt{\pi a}} e^{-x^2/(4a)} \qquad (a > 0)}, \tag{1.41}$$

which yelds to $(a = Dt)$

$$\boxed{\mathcal{G}_c(x,t) = \frac{1}{2\sqrt{\pi D t}} e^{-x^2/(4Dt)}} \tag{1.42}$$

1.2.3 The Green function for the Cauchy problem via Laplace Transform

The *Cauchy Problem* can also be treated by the Laplace Transform (LT) technique with respect to t, though the corresponding analysis turns out to be more cumbersome with respect to that via Fourier transform; by the way, it is easier to be generalized to treat the *Cauchy problem* for the *time fractional diffusion-wave equation* afterwards.
Applying (LT) to [(1.8)+(1.17)], we get the following inhomogeneous, second-order differential equation in x,

$$s\tilde{u} - D\frac{d^2\tilde{u}}{dx^2} = f(x), \quad -\infty < x < +\infty, \tag{1.43}$$

with the boundary conditions

$$\tilde{u}(\pm\infty, s) = 0. \tag{1.44}$$

Now, in order to obtain the *fundamental solution* $\mathcal{G}_c(x,t)$ of the Cauchy problem, we have to put $f(x) = \delta(x)$, i.e. to solve the following (singular) differential equation

$$s\tilde{\mathcal{G}}_c - D\frac{d^2\tilde{G}_c}{dx^2} = \delta(x). \tag{1.45}$$

Therefore, $\tilde{\mathcal{G}}_c$ is in the form

$$\tilde{\mathcal{G}}_c(x,s) = \begin{cases} c_1(s)e^{-(x/\sqrt{D})s^{1/2}} + c_2(s)e^{+(x/\sqrt{D})s^{1/2}}, & if\ x > 0; \\ c_3(s)e^{-(x/\sqrt{D})s^{1/2}} + c_4(s)e^{+(x/\sqrt{D})s^{1/2}}, & if\ x < 0. \end{cases} \tag{1.46}$$

Cleary, we must set $c_2(s) = c_3(s) = 0$ to ensure that the solution vanishes as $|x| \to \infty$. We recognize from (1.45) that in $x = 0$ the function $\tilde{\mathcal{G}}_c(x,s)$ is continuous, but not its first derivative: we write

$$\tilde{\mathcal{G}}_c(0^+,s) - \tilde{\mathcal{G}}_c(0^-,s) = 0, \tag{1.47}$$

and

$$\frac{d}{dx}\tilde{\mathcal{G}}_c(0^+,s) - \frac{d}{dx}\tilde{\mathcal{G}}_c(0^-,s) = -\frac{1}{D} \tag{1.48}$$

Therefore, using (1.46-48) we obtain

$$c_1(s) = c_4(s) = \frac{1}{2\sqrt{D}s^{1/2}}, \tag{1.49}$$

and

$$\boxed{\tilde{\mathcal{G}}_c(x,s) = \frac{1}{2\sqrt{D}s^{1/2}} e^{-(|x|/\sqrt{D})s^{1/2}}}. \tag{1.50}$$

For the inversion of $\tilde{\mathcal{G}}_c(x,s)$ we recall the Laplace transform pair

$$\boxed{\frac{e^{-as^{1/2}}}{s^{1/2}} \div \frac{1}{\sqrt{\pi}} t^{-1/2} e^{-a^2/(4t)} := \chi(a,t) \quad (a \geq 0)}, \tag{1.51}$$

which yelds ($a = |x|/\sqrt{D}$)

$$\boxed{\mathcal{G}_c(x,t) = \frac{1}{2\sqrt{\pi D t}} e^{-x^2/(4Dt)}} \tag{1.52}$$

1.2.4 The Green function for the Signalling problem via Laplace transform

The *Signalling problem* is conveniently treated by the Laplace transform (LT) technique with respect to t.
We adopt the following notation ($s \in \mathbb{C}$)

$$\boxed{\tilde{u}(x,s) := \int_0^{+\infty} e^{-st} u(x,t)\,dt \div u(x,t) = \frac{1}{2\pi i}\int_{Br} e^{st}\tilde{u}(x,s)\,ds}\,. \qquad (1.53)$$

Applying LT to [(1.8)+(1.18)] we get the second-order differential equation in x,

$$\frac{d^2\tilde{u}}{dx^2}(x,s) - \frac{s}{D}\tilde{u}(x,s) = 0, \qquad x > 0, \qquad (1.54)$$

with the boundary conditions

$$\tilde{u}(0^+,s) = \tilde{g}(s), \qquad \tilde{u}(+\infty,s) = 0. \qquad (1.55)$$

Solving (1.54) we find $\tilde{u}(x,s) = c_1(s)e^{-(x/\sqrt{D})s^{1/2}} + c_2(s)e^{+(x/\sqrt{D})s^{1/2}}$; then, choosing $c_1(s) = \tilde{f}(s)$ and $c_2(s) = 0$ to satisfy the boundary conditions (1.55), we get

$$\tilde{u}(x,s) = \tilde{g}(s)e^{-(x/\sqrt{D})s^{1/2}} \qquad (1.56)$$

Introducing

$$\boxed{\tilde{\mathcal{G}}_s(x,s) := e^{-(x/\sqrt{D})s^{1/2}}} \qquad (1.57)$$

the transform solution (1.56) reads

$$\tilde{u}(x,s) = \tilde{g}(s)\tilde{\mathcal{G}}_s(x,s) \qquad (1.58)$$

Applying the convolution theorem for Laplace transforms, the solution in the space-time domain turns out to be

$$\boxed{u(x,t) = \int_0^t \mathcal{G}_s(x,\tau) g(t-\tau) d\tau = \mathcal{G}_s(x,t) * g(t)}, \tag{1.59}$$

where $\mathcal{G}_s(x,t)$ is the *Green function* for the *Signalling problem.* For the inversion of $\tilde{\mathcal{G}}(x,s)$ we recall the Laplace transform pair

$$\boxed{e^{-as^{1/2}} \div \frac{a}{2\sqrt{\pi}} t^{-3/2} e^{-a^2/(4t)} := \psi(a,t) \qquad (a>0)}, \tag{1.60}$$

which yelds $(a = x/\sqrt{D})$

$$\boxed{\mathcal{G}_s(x,t) = \frac{x}{2\sqrt{\pi D t^3}} e^{-x^2/(4Dt)}} \tag{1.61}$$

We can consider the particular case $g(t) = H(t)$ that provides the *step response* $\mathcal{H}_s(x,t) = \int_0^t \mathcal{G}_s(x,\tau) d\tau$ according to eq. (1.28); In fact, from Eqs (1.57)-(1.58), we have:

$$\tilde{\mathcal{H}}_s(x,s) = \frac{\tilde{\mathcal{G}}_s(x,s)}{s} \tag{1.62}$$

For the inversion of $\tilde{\mathcal{H}}_s(x,s)$ we recall the transform pair

$$\boxed{\frac{e^{-as^{1/2}}}{s} \div erfc\left(\frac{x}{2\sqrt{t}}\right) := \phi(a,t) \qquad (a>0)}, \tag{1.63}$$

which yields $(a = x/\sqrt{D})$

$$\boxed{\mathcal{H}_s(x,t) = erfc\left(\frac{x}{2\sqrt{Dt}}\right)}. \tag{1.64}$$

in agreement with (1.27).

An interesting signalling problem is concerning the medium, supposed semi-infinite $(0 \le x \le +\infty)$ and initially undisturbed, when it is subjected at $x = 0$ (the accessible end) and for $t > 0$ to a known gradient (*heat flux* for the heat conduction model) provided by a causal function

$$q(t) = -K \frac{\partial u}{\partial x}\bigg|_{x=0} . \tag{1.65}$$

We refer to it as to the *flux-signalling problem.* In this case the conditions read

$$\begin{cases} u(x, 0^+) = 0, & 0 < x < +\infty; \\ \frac{\partial u}{\partial x}(0^+, t) = -q(t)/K, \quad u(+\infty, t) = 0, & t > 0 \end{cases} \tag{1.66}$$

Applying LT to [(1.8)+(1.66)] we get the second-order differential equation in x,

$$\frac{d^2 \tilde{u}}{dx^2}(x, s) - \frac{s}{D} \tilde{u}(x, s) = 0, \quad x > 0, \tag{1.67}$$

with the (new) boundary conditions

$$\frac{d\tilde{u}}{dx}(0^+, s) = -\tilde{q}(s)/K, \quad \tilde{u}(+\infty, s) = 0. \tag{1.68}$$

Solving (1.55) we find

$$\tilde{u}(x, s) = c_1(s) e^{-(x/\sqrt{D}) s^{1/2}} + c_2(s) e^{+(x/\sqrt{D}) s^{1/2}}; \tag{1.69}$$

and choosing

$$c_1(s) = \frac{\sqrt{D}}{K} \frac{\tilde{q}(s)}{s^{1/2}}, \quad c_2(s) = 0, \tag{1.70}$$

to satisfy the boundary conditions (1.68), we get

$$\tilde{u}(x, s) = \frac{\sqrt{D}}{K} \tilde{q}(s) \frac{e^{-(x/\sqrt{D}) s^{1/2}}}{s^{1/2}} \tag{1.71}$$

Introducing

$$\boxed{\tilde{G}_q(x,s) := \frac{e^{(-x/\sqrt{D})s^{1/2}}}{s^{1/2}}}, \tag{1.72}$$

the transform solution (1.71) reads

$$\tilde{u}(x,s) = \frac{\sqrt{D}}{K}\tilde{q}(s)\tilde{\mathcal{G}}_q(x,s). \tag{1.73}$$

Then, applying the convolution theorem for Laplace transforms, the solution in the space-time domain turns out to be

$$\boxed{u(x,t) = \frac{\sqrt{D}}{K}\int_0^t \mathcal{G}_q(x,\tau)q(t-\tau)d\tau = \frac{\sqrt{D}}{K}\int_0^t \mathcal{G}_q(x,t) * q(t)} \tag{1.74}$$

where $\mathcal{G}_q(x,t)$ is the *Green function* for the *Heat-Flow Signalling problem.* For the inversion of $\tilde{\mathcal{G}}_q(x,s)$ we recall the Laplace transform pair

$$\boxed{\frac{e^{-as^{1/2}}}{s^{1/2}} \div \frac{1}{\sqrt{\pi}}t^{-1/2}e^{-a^2/(4t)} := \chi(a,t) \quad (a>0)}, \tag{1.75}$$

which yelds ($a = x/\sqrt{D}$)

$$\boxed{\mathcal{G}_q(x,t) = \frac{1}{\sqrt{\pi Dt}}e^{-x^2/(4Dt)}}. \tag{1.76}$$

We point out the relevance in diffusion problems of the three functions $\phi(a,t)$, $\psi(a,t)$ and $\xi(a,t)$. We note that the relations among the three functions turn out to be easily derived by working in the Laplace transform domain, and so we call these functions the *three sisters.*

1.2.5 The three sisters functions

The *three sisters* are three functions that are defined through their Laplace Transforms (see [11]).

$$\phi(a,t) \div \frac{e^{-a\sqrt{s}}}{s} = \tilde{\phi}(a,s) \tag{1.77}$$

$$\psi(a,t) \div e^{-a\sqrt{s}} = \tilde{\psi}(a,s) \tag{1.78}$$

$$\chi(a,t) \div \frac{e^{-a\sqrt{s}}}{\sqrt{s}} = \tilde{\chi}(a,s) \tag{1.79}$$

where $a, t \in \mathbb{R}^+$ and $\Re[s] > 0$.
We can express each of them as a function of the 2 other *three sisters* (table 1).

	$\tilde{\phi}$	$\tilde{\psi}$	$\tilde{\chi}$
$\tilde{\phi}$	$\frac{e^{-a\sqrt{s}}}{s}$	$\frac{\tilde{\psi}}{s}$	$-\frac{1}{s}\frac{\partial\tilde{\chi}}{\partial a}$
$\tilde{\psi}$	$s\tilde{\phi}$	$e^{-a\sqrt{s}}$	$-\frac{\partial\tilde{\chi}}{\partial a}$
$\tilde{\chi}$	$-\frac{\partial\tilde{\phi}}{\partial a}$	$-\frac{2}{a}\frac{\partial\tilde{\psi}}{\partial s}$	$\frac{e^{-a\sqrt{s}}}{\sqrt{s}}$

Table 1: Relations between the *three sisters* in the Laplace domain.

The three sisters in the t domain may be all directly calculated, by making use of the Bromwich formula, with which we obtain:

$$\tilde{\phi}(a,s) \div \phi(a,t) = 1 - \frac{1}{\pi}\int_0^\infty e^{-rt} sin(a\sqrt{r})\frac{dr}{r} \tag{1.80}$$

$$\tilde{\psi}(a,s) \div \psi(a,t) = \frac{1}{\pi}\int_0^\infty e^{-rt} sin(a\sqrt{r})dr \tag{1.81}$$

$$\tilde{\chi}(a,s) \div \chi(a,t) = \frac{1}{\pi}\int_0^\infty e^{-rt} cos(a\sqrt{r})\frac{dr}{\sqrt{r}} \tag{1.82}$$

Then, through the substitution $\rho = \sqrt{r}$, we may arrive at the Gaussian integral, and consequently at the required explicit expressions of the *three sisters*:

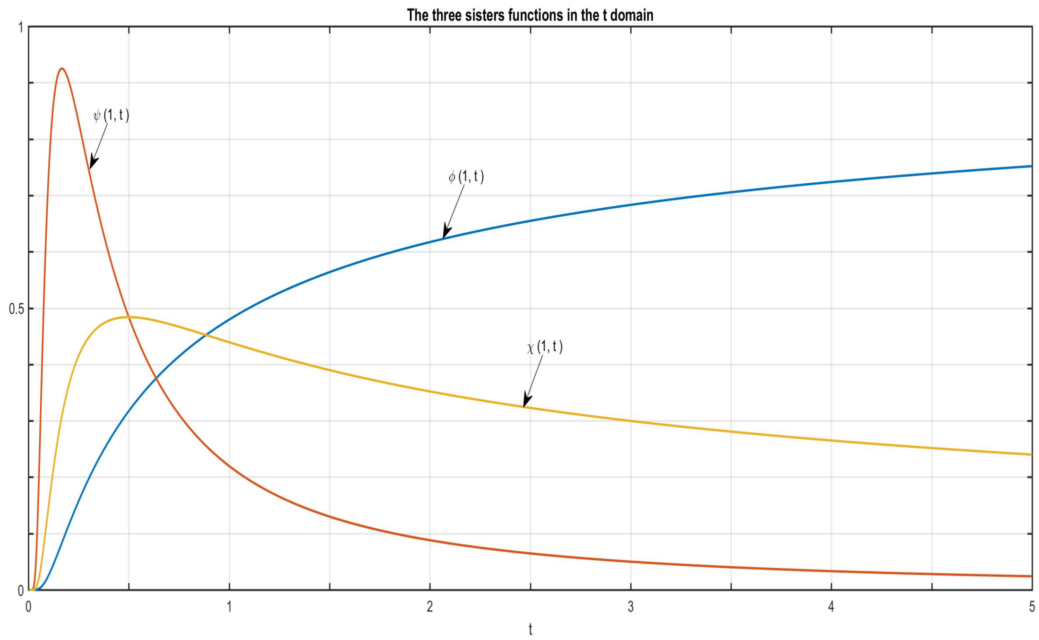


Figure 1.1: The three sisters functions $\phi(a,t)$, $\psi(a,t)$ and $\chi(a,t)$ (with $a = 1$) in the t domain.

$$\phi(a,t) = erfc\left(\frac{a}{2\sqrt{t}}\right) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{a/(2\sqrt{t})} e^{-u^2} du \tag{1.83}$$

$$\psi(a,t) = \frac{a}{2\sqrt{\pi}} t^{-3/2} e^{-a^2/(4t)} \tag{1.84}$$

$$\chi(a,t) = \frac{1}{\sqrt{\pi}} t^{-1/2} e^{-a^2/(4t)}, \tag{1.85}$$

Alternatively we can compute the three sisters in the t domain by using the relations between the functions in the Laplace domain listed in table 1, but in this case one of the three sisters in the t domain must be already known.

To derive $\phi(a,t)$ we may refer to [2].

In fig.1.1 the three sisters, with $a = 1$, are plotted.

1.3 MATLAB plots

Here are shown the plots of $\mathcal{G}_c(x,t)$ either versus x, at fixed t, or versus t, at fixed x.

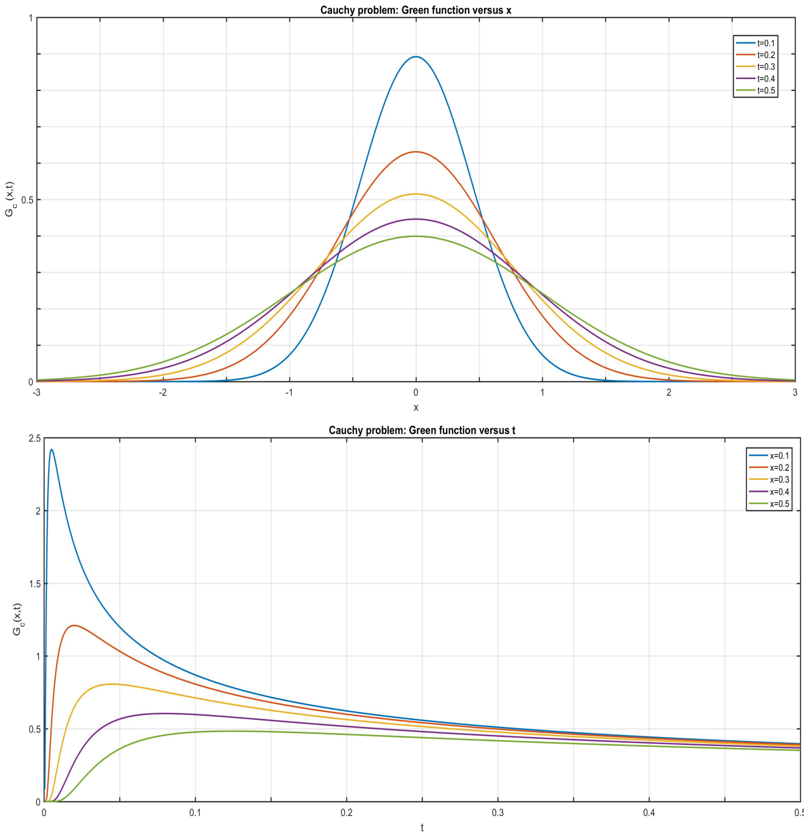


Figure 1.2: The Green function for the Cauchy problem versus x and versus t.

Here are shown the plots of $\mathcal{G}_s(x,t)$ either versus x, at fixed t, or versus t, at fixed x.

Here are shown the plots of $\mathcal{H}_s$, which is referred to as the *step response* for the Signalling problem (either versus x, at fixed t, or versus t, at fixed x), and the plots of the *error function* erf(x), of its first derivative erf'(x), of the *complementary error function* erfc(x), and of its asymptotic representation erfc(x)$\sim e^{-x^2}/(\sqrt{\pi}x)$.

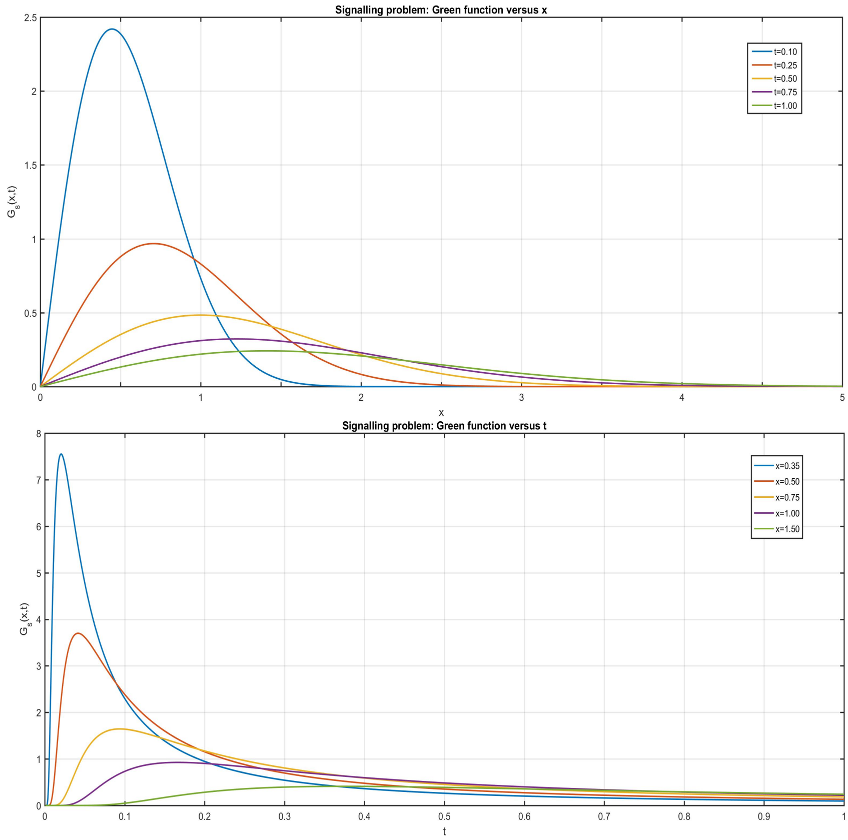


Figure 1.3: The Green function for the Signalling problem versus x and versus t.

Finally, we plot the Green function $\mathcal{G}_q$ for the *Heat-Flux Signalling problem.*

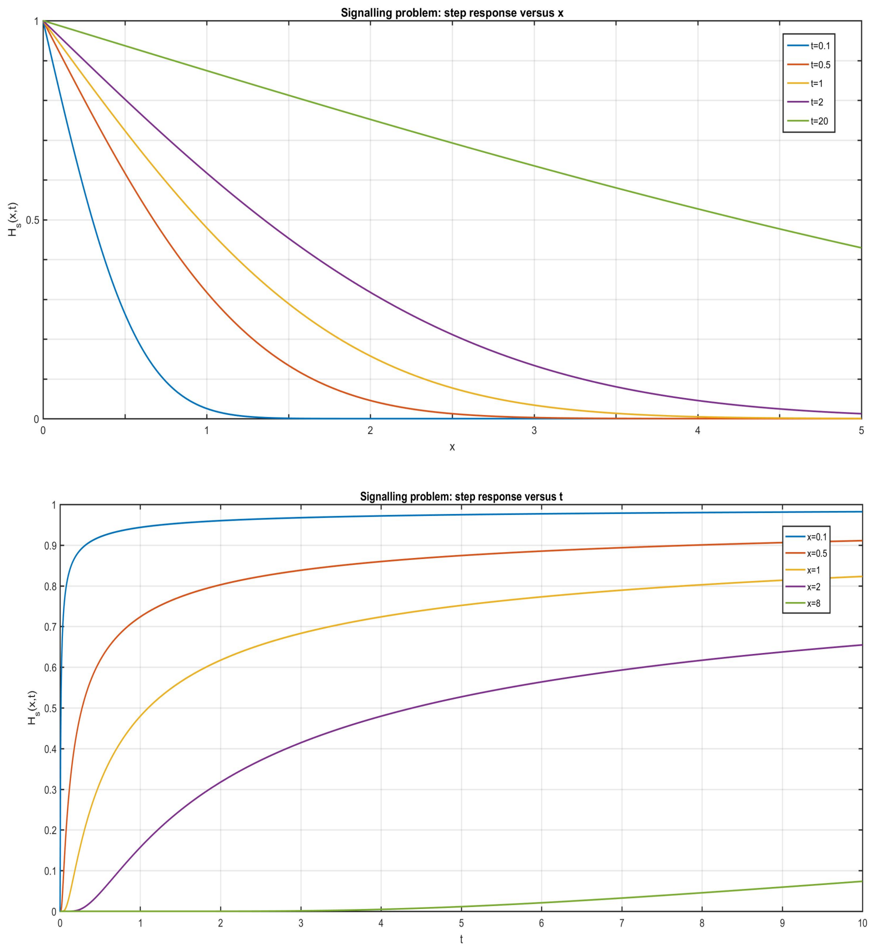


Figure 1.4: The step response for the Signalling problem versus x and versus t.

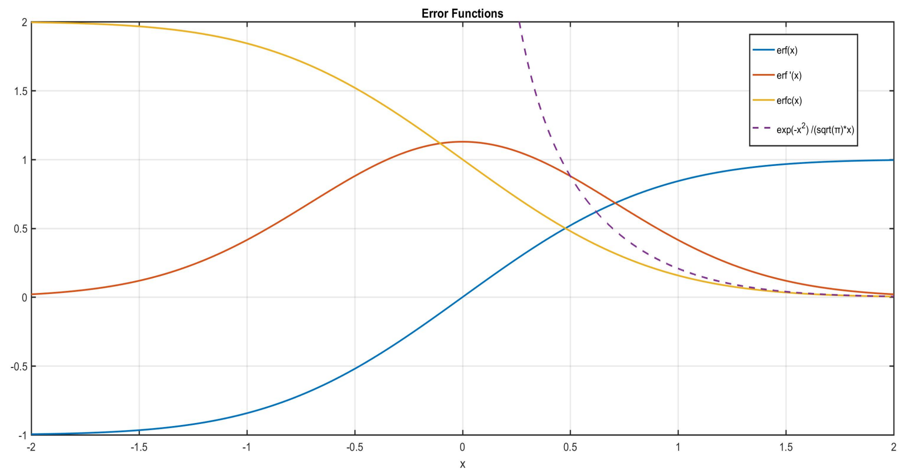
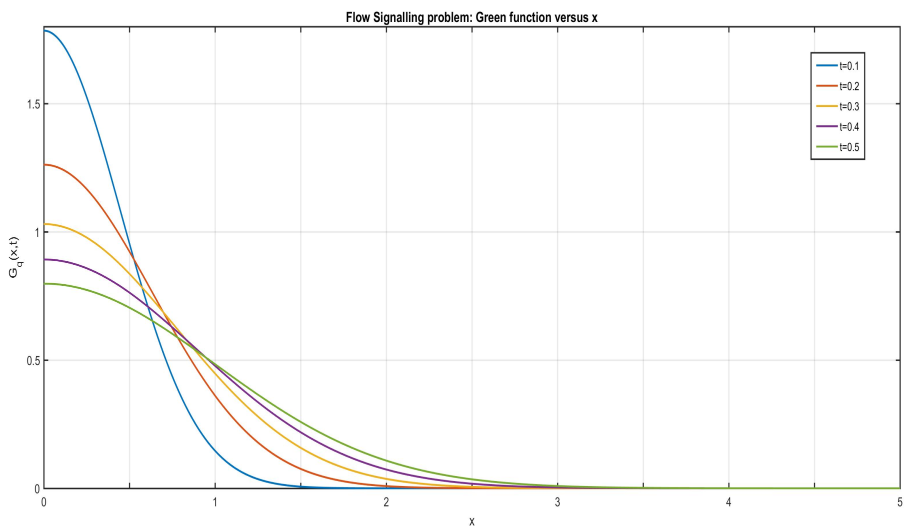


Figure 1.5: Plots of erf(x), erf'(x) and erfc(x) in the interval $-2 \le x \le +2$



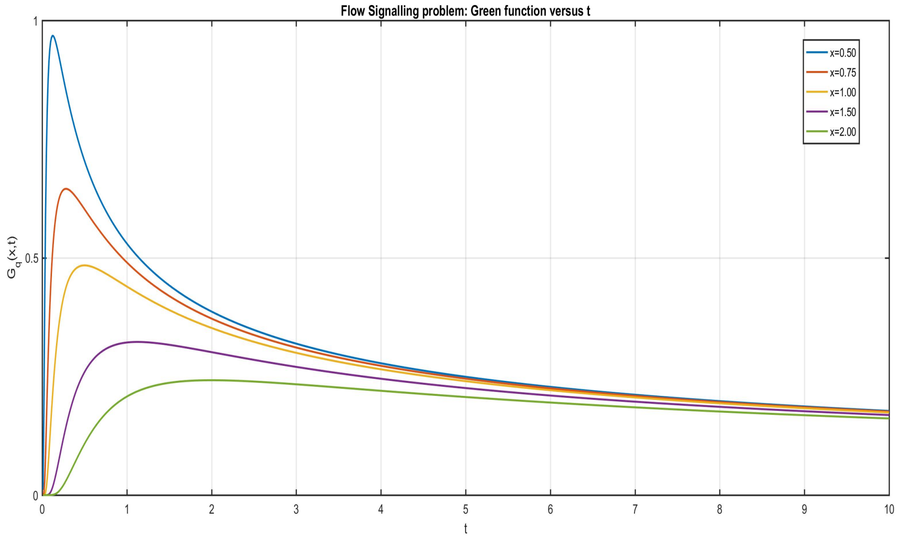


Figure 1.6: The step response for the Signalling problem versus x and versus t.

1.4 The Linear Diffusion Equation with a shift

We want to proceed with the study of the diffusion equation, and so now we consider the following equation, where the diffusion coeffient is $D = 1$, and which contains an additional term $-u(x,t)$, that result in a shift in the solution:

$$\boxed{\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} - u(x,t) = 0} \tag{1.86}$$

The mathematical treatment that is developed here, will be reconsidered from a physical point of view in the next sections.

1.4.1 Signalling problem via Laplace transform

Also in this case the *Signalling problem* is conveniently treated by the Laplace transform (LT) technique with respect to t.

Although a more in-depth analysis about Laplace trasform will be carried out in the appendix, here we remember that the LT of the derivative of a function f is

$$\mathcal{L}\left[\frac{df}{dt}\right] = s\tilde{f}(s) - f(0^+) \tag{1.87}$$

and so, applying Laplace transform to $u(x,t)$ with respect to the variable t, we obtain:

$$\frac{d^2\tilde{u}(x,s)}{dx^2} - s\tilde{u}(x,s) + u(x,0^+) - \tilde{u}(x,s) = 0 \tag{1.88}$$

From which:

$$\frac{d^2\tilde{u}(x,s)}{dx^2} - (s+1)\tilde{u}(x,s) + u(x,0^+) = 0 \tag{1.89}$$

What we have said before about partial differential equation is still valid, and so we must specify some boundary conditions.
As before, we have a IBVP problem in the quadrant $\{x,t\} > 0$, in which a semi-infinite and initially undisturbed medium is subjected at $x = 0$ and for $t > 0$ to a known disturbance, provided by a causal function $g(t)$; the conditions read as in (1.18).

We can then write:

$$\frac{d^2\tilde{u}(x,s)}{dx^2} - (s+1)\tilde{u}(x,s) = 0 \tag{1.90}$$

The general solution of this homogeneous equation is:

$$\tilde{u}(x,s) = A(s)e^{-x\sqrt{s+1}} + B(s)e^{x\sqrt{s+1}} \tag{1.91}$$

where A(s) and B(s) are constant or functions of s, and must be determined from prescrived initial and boundary conditions. By our assumption of a semi-infinite medium, we can expect $u(x,t)$ to decay to 0 as x increases toward infinity, as stated in (1.18).
This condition is satisfied when $B(s) = 0$, and thus we obtain:

$$\tilde{u}(x,s) = A(s)e^{-x\sqrt{s+1}} \tag{1.92}$$

In order to evaluate $A(s)$ we must specify the initial stimulation condition, that we consider as a step input $g(t) = u_0 H(t)$, follows that

$$\tilde{u}(x,s) = \frac{u_0}{s}e^{-x\sqrt{s+1}} \tag{1.93}$$

These first steps suggest that the formal solution of the problem is equivalent to the classical diffusion one; this clarifies the fact that the addition of the term $-u(x,t)$ just result in a shift, which transforms s into $s+1$.
The shift property of the Laplace Transform tell us that

$$\mathcal{L}^{-1}[\tilde{f}(s+a)] = e^{-at}f(t) \tag{1.94}$$

We can then use the Laplace rules to invert (1.81), obtaining

$$\begin{aligned} u(x,t) &= \int_0^t \frac{x}{2\sqrt{\pi\tau^3}}e^{-(\frac{x^2}{4\tau}+\tau)}u_0 H(t-\tau)d\tau \\ &= \int_0^t \frac{u_0 x}{2\sqrt{\pi\tau^3}}e^{-(\frac{x^2}{4\tau}+\tau)}d\tau \end{aligned} \tag{1.95}$$

These expression can be written in a more similar form to that of the diffusion in the so called step-response case, by means of the complementary error function.

$$u(x,t) = \frac{u_0}{2}e^{x}erfc\left(\frac{x}{2\sqrt{t}}+\sqrt{t}\right)+\frac{u_0}{2}e^{-x}erfc\left(\frac{x}{2\sqrt{t}}-\sqrt{t}\right) \tag{1.96}$$

Let's note that the difference between this case and the step-response diffusion is given by the multiplication term $e^{\pm x}$ and the additive term $\pm\sqrt{t}$. Some graphics will be included later, during the physical analysis related to this problem.

Now we treat a *signalling problem* concerning with a semi-infinite medium ($0 \leq x < +\infty$) initially undisturbed when its subjected, at $x = 0$ and for $t > 0$, to a known gradiente provided by a causal function

$$q(t) = -k\frac{\partial u}{\partial x}\bigg|_{x=0}. \tag{1.97}$$

This is the so called *flux-signalling problem*, and the conditions read

$$\begin{cases} u(x, 0^+) = 0, & 0 < x < +\infty; \\ \frac{\partial u}{\partial x}(0^+, t) = -q(t)/k, \quad u(+\infty, t) = 0, & t > 0. \end{cases} \tag{1.98}$$

Applying LT to [(1.8)+(1.98)] we get the usual second-order differential equation in x,

$$\frac{d^2\tilde{u}}{dx^2}(x, s) - s\tilde{u}(x, s) - \tilde{u}(x, s) = 0, \quad x > 0 \tag{1.99}$$

and so

$$\frac{d^2\tilde{u}}{dx^2}(x, s) - (s + 1)\tilde{u}(x, s) = 0, \quad x > 0 \tag{1.100}$$

The new boundary conditions read

$$\frac{d\tilde{u}}{dx}(0^+, s) = -\tilde{q}(s)/k, \quad \tilde{u}(+\infty, s) = 0. \tag{1.101}$$

Solving (1.100) we find

$$\tilde{u}(x, s) = c_1(s)e^{-x\sqrt{s+1}} + c_2(s)e^{-x\sqrt{s+1}} \tag{1.102}$$

Choosing

$$c_1(s) = \frac{1}{k}\frac{\tilde{q}(s)}{\sqrt{s+1}}, \quad c_2(s) = 0 \tag{1.103}$$

to satisfy the boundary conditions (1.101) we get

$$\tilde{u}(x,s) = \frac{1}{k}\tilde{q}(s)\frac{e^{-x\sqrt{s+1}}}{\sqrt{s+1}}. \tag{1.104}$$

Introducing

$$\boxed{\tilde{\mathcal{G}}_q(x,s) := \frac{e^{-x\sqrt{s+1}}}{\sqrt{s+1}},} \tag{1.105}$$

The transform solution (1.104) reads

$$\tilde{u}(x,s) = \frac{1}{k}\tilde{q}(s)\tilde{\mathcal{G}}_q(x,s). \tag{1.106}$$

Then, applying the convolution theorem for Laplace Transform, the solution in the space-time domain turns out to be

$$\boxed{u(x,t) = \frac{1}{k}\int_0^t \tilde{\mathcal{G}}_q(x,\tau)q(t-\tau)d\tau = \frac{1}{k}\mathcal{G}_q(x,t) * q(t)}, \tag{1.107}$$

where $\mathcal{G}_q(x,t)$ is the *Green function* for the *Heat-Flow Signalling problem.* For the inversion of $\mathcal{G}_q(x,\tau)$ we recall the Laplace transform

$$\boxed{\frac{e^{-x\sqrt{s+1}}}{\sqrt{s+1}} \div \frac{1}{\sqrt{\pi}}t^{-1/2}e^{-\left(\frac{x^2}{4t}+t\right)}} \tag{1.108}$$

and so, finally, we obtain the expression of the *Green function*

$$\boxed{\mathcal{G}_q(x,t) = \frac{1}{\sqrt{\pi t}}e^{\left(\frac{x^2}{4t}+t\right)}} \tag{1.109}$$

1.4.2 Cauchy problem via Laplace transform

Dealing with the Cauchy problem, we take into account an unlimited medium, with the boundary conditions $u(\pm\infty,t) = 0$ and initial condition $u(x,0) = f(x)$.

The general solution of the Cauchy problem is related to the Green function $\mathcal{G}_c(x,t)$ through the following relation

$$u(x,t) = \int_{-\infty}^{+\infty} \mathcal{G}_c(\xi,t) f(x-\xi) d\xi. \tag{1.110}$$

$\mathcal{G}_c(x,t)$ is obtained via Laplace Transform:

$$(s+1)\tilde{\mathcal{G}}_c(x,s) - \frac{\partial^2 \tilde{\mathcal{G}}_c}{\partial x^2} = \delta(x), \tag{1.111}$$

and boundary conditions imposes:

$$\tilde{\mathcal{G}}_c(x,s) = \begin{cases} c_1(s) e^{-x\sqrt{s+1}}, & if\ x > 0 \\ c_2(s) e^{+x\sqrt{s+1}}, & if\ x < 0 \end{cases} \tag{1.112}$$

Imposing $\tilde{\mathcal{G}}(0^-,s) = \tilde{\mathcal{G}}(0^+,s)$, leads to $c_1(s) = c_2(s)$. Integrating Eq. (1.86) over x from 0^- to 0^+ we have:

$$\frac{\partial \tilde{\mathcal{G}}_c(0^+,s)}{\partial x} - \frac{\partial \tilde{\mathcal{G}}_c(0^-,s)}{\partial x} = -1 \tag{1.113}$$

and the coefficients result:

$$c_1(s) = c_2(s) = \frac{1}{2\sqrt{s+1}} \tag{1.114}$$

The resulting LT of the Green function reads:

$$\tilde{\mathcal{G}}_c(x,s) = \frac{1}{2\sqrt{s+1}} e^{-x\sqrt{s+1}} \tag{1.115}$$

The inversion can be easily performed, and reads

$$\mathcal{G}_c(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-\left(\frac{x^2}{4t}+t\right)} \tag{1.116}$$

1.5 Final remarks

We point out that a number of evolution equations can be reduced by appropriate transformations (change of independent and dependent variables) to the simple linear diffusion equation (1.8).
Examples of physical relevance are the (linear) *Fokker-Planck-Kolmogorov equation*

$$\boxed{\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + u}, \tag{1.117}$$

found in non-equilibrium statistical mechanics and the (quasi-linear) *Burgers equation*

$$\boxed{\frac{\partial u}{\partial t} + \lambda u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}}, \quad \lambda, \nu > 0, \tag{1.118}$$

which is found in non linear acoustics. The last equation was just introduced by Burgers as a simple model of turbulence.
The transformation required for the *Fokker-Planck-Kolmogorov equation* (1.74) is

$$x \longrightarrow \xi = xe^t, \quad t \longrightarrow \tau = \frac{e^{2t} - 1}{2}, \quad u \longrightarrow \Phi = ue^{-t} \tag{1.119}$$

which leads to

$$\frac{\partial \Phi}{\partial \tau} = \frac{\partial^2 \Phi}{\partial \xi^2}. \tag{1.120}$$

The transformation required for the *Burgers* equation 81.75) is the celebrated *Cole-Hopf transformation*

$$u(x,t) = -\frac{2\nu}{\lambda}\frac{1}{\Phi}\frac{\partial \Phi}{\partial x} = -\frac{2\nu}{\lambda}\frac{\partial}{\partial x} ln\Phi, \quad \Phi = \Phi(x,t) > 0 \tag{1.121}$$

which leads to

$$\frac{\partial \Phi}{\partial t} = \nu\frac{\partial^2 \Phi}{\partial x^2}. \tag{1.122}$$

The *Cole-Hopf transformation* can be better understood in the following two steps

$$u = -\frac{\partial V}{\partial x} \tag{1.123}$$

$$V = \frac{2\nu}{\lambda} ln\Phi \tag{1.124}$$

The equation satisfied by V = V(x,t), known as the *Potential Burgers* equation, turns out to be

$$\boxed{\frac{\partial V}{\partial t} - \frac{\lambda}{2}\left(\frac{\partial V}{\partial x}\right)^2 = \nu \frac{\partial^2 V}{\partial x^2}}. \tag{1.125}$$

Chapter 2

Special Functions

This chapter gathers some of the most fundamentals *special funtions* that are used in previous and subsequent chapters.
Actually it's very difficult to talk about physics without recourse to mathematics, and so it was inevitable to resort to some concepts; here we want to take those concepts, together with others, and give a more accurate description of them. By the way, for the aim of this work we will not deal with a complete treatment.

2.1 The Eulerian Functions

In this section we want to introduce some *special functions*, called in this way because of their importance in mathematical analysis, physics and so on.

2.1.1 The Gamma Function

The *Gamma function* is the most widely used of all the special functions, and this is why we discuss first about it.
This function, denoted by $\Gamma(z)$, can be defined by *Euler's Integral Representation* (see [15])

$$\boxed{\Gamma(z) = \int_0^\infty e^{-u} u^{z-1} du, \quad Re(z) > 0}. \tag{2.1}$$

This representation is the most common for $\Gamma(z)$, even if its valid only in the right half-plane of $\mathbb{C}$, and it's referred to as the *Euler integral of the second kind*; later the analytic continuation to the left half-plane will be considered to obtain its *Domain of Analyticity*

$$\boxed{D_\Gamma = \mathbb{C} - \{0, -1, -2, ...\}}. \tag{2.2}$$

Using integration by parts, we have that at least for $Re(z) > 0$, $\Gamma(z)$ satisfies the simple *Difference Equation*

$$\boxed{\Gamma(z+1) = z\Gamma(z)}, \tag{2.3}$$

which can be iterated to yield

$$\boxed{\Gamma(z+n) = z(z+1)...(z+n-1)\Gamma(z), \quad n \in \mathbb{N}}. \tag{2.4}$$

The recurrence formulas (2.3-4) can be extended to any $z \in D_\Gamma$. In particular, being $\Gamma(1) = 1$ we get, for non negative *integer values*

$$\boxed{\Gamma(n+1) = n! \quad n = 0, 1, 2, ..} \tag{2.5}$$

As a consequence, $\Gamma(z)$ can be used to define the *Complex Factorial Function*

$$\boxed{z! := \Gamma(z+1)}. \tag{2.6}$$

By the substitution $u = v^2$ in (2.1) we get the *Gaussian Integral Representation*

$$\boxed{\Gamma(z) = 2\int_0^\infty e^{-v^2} v^{2z-1} dv, \quad Re(z) > 0} \tag{2.7}$$

which can be used to obtain $\Gamma(z)$ when z assumes positive *semi-integer values*, as follows

$$\Gamma\left(\frac{1}{2}\right)= \int_{-\infty}^{+\infty} e^{-v^2} dv = \sqrt{\pi} \approx 1.77245 \tag{2.8}$$

$$\boxed{\Gamma\left(n+\frac{1}{2}\right)= \int_{-\infty}^{+\infty} e^{-v^2} v^{2n} dv = \Gamma\left(\frac{1}{2}\right)\frac{(2n-1)!!}{2^n} = \sqrt{\pi}\frac{(2n)!}{2^{2n}n!}, n \in \mathbb{N}} \tag{2.9}$$

The formula (2.4) can be used to obtain the *Domain of Analiticity* D_Γ by means of the so-called *Analytical Continuation by the Recurrence Formula*

$$\boxed{\Gamma(z) = \frac{\Gamma(z+n)}{(z+n-1)(z+n-2)...(z+1)z}}. \tag{2.10}$$

Since n can be arbitrarily large, we deduce that $\Gamma(z)$ is analytic in the entire complex plane, except the points $z_n = -n \ \ (n = 0, 1, 2..)$, which turns out to be simple poles with residues $R_n = (-1)^n/n!$. The point at the infinity, being accumulation point of poles, is an essential non isolated singularity. Thus $\Gamma(z)$ is a trascendental *meromorphic* function.

The integration by parts in the basic representation (2.1) provides the *Analytical Continuation by the Cauchy-Saalschütz Representation*

$$\boxed{\Gamma(z) = \int_0^\infty u^{z-1}\left[e^{-u} - 1 + u - \frac{u^2}{2!} + ... + (-1)^{n+1}\frac{u^n}{n!}\right]du} \tag{2.11}$$

which holds for any integer $n \geq 0$ with $-(n+1) < Re(z) < -n$.
The representation can be understood by iterating the first step. In fact, for $-1 < Re(z) < 0$:

$$\int_0^\infty u^{z-1}[e^{-u} - 1]du = \frac{1}{z}\int_0^\infty u^z e^{-u} du = \frac{1}{z}\Gamma(z+1) = \Gamma(z).$$

Finally, another instructive manner to obtain the *Domain of Analyticity* is to use the so-called *Analytical Continuation by the Mixed Representation*

$$\boxed{\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)} + \int_1^{\infty} e^{-u} u^{z-1} du \quad z \in D_\Gamma.} \tag{2.12}$$

This representation can be obtained splitting the integral in (2.1) into 2 integrals, the former over the interval $0 \le u \le 1$ which is then developed in series, the latter over the interval $1 < u \le \infty$, which, being uniformly convergent inside $\mathbb{C}$, provides an entire function. The terms of the series (uniformly convergent inside D_Γ) provide the principal parts of $\Gamma(z)$ at the corresponding poles $z_n = -n$.

The *Reflection or Complementary Formula*

$$\boxed{\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}}. \tag{2.13}$$

is a formula which shows the relationship between the Γ function and the trigonometric function, and it is of great importance together with the recurrence formula.
It can be proven in several manners; the simplest proof consists in proving (2.13) for $0 < Re(z) < 1$ and extend the result by analytic continuation to $\mathbb{C}$ except the points $0, \pm 1, \pm 2, \ldots.$
The reflection formula shows that $\Gamma(z)$ has no zeros. In fact, the zeros cannot be in $z = 0, \pm 1, \pm 2, \ldots$ and, if $\Gamma(z)$ vanished for a not integer z, because of (2.13) this zero be a pole of $\Gamma(1-z)$, that cannot be true.

Gauss proved the following *Multiplication Formula*

$$\boxed{\Gamma(nz) = (2\pi)^{(1-n)/2} n^{nz-1/2} \prod_{k=0}^{n-1} \Gamma\big(z + \frac{k}{n}\big), \quad n = 2, 3, \ldots} \tag{2.14}$$

which reduces, for $n = 2$, to *Legendre's Duplication Formula*

$$\boxed{\Gamma(2z) = \frac{1}{\sqrt{2\pi}} 2^{2z-1/2}\Gamma(z)\Gamma\big(z+\frac{1}{2}\big)} \tag{2.15}$$

and, for $n = 3$ to the *Triplication Formula*

$$\boxed{\Gamma(3z) = \frac{1}{2\pi} 3^{3z-1/2}\Gamma(z)\Gamma\big(z+\frac{1}{3}\big)\Gamma\big(z+\frac{2}{3}\big)} \tag{2.16}$$

Pochhammer's symbols $(z)_n$ are defined for any non negative integer n as

$$\boxed{(z)_n := z(z+1)(z+2)...(z+n-1) = \frac{\Gamma(z+n)}{\Gamma(z)}, \quad n \in \mathbb{N}} \tag{2.17}$$

with $(z)_0 = 1$.
In particular, for $z = 1/2$, we obtain from (2.9)

$$\left(\frac{1}{2}\right)_n := \frac{\Gamma(n+1/2)}{\Gamma(1/2)} = \frac{(2n-1)!!}{2^n}$$

Here, we take the liberty of extending the above notation to negative integers, defining

$$\boxed{(z)_{-n} := z(z-1)(z-2)...(z-n+1) = \frac{\Gamma(z+1)}{\Gamma(z-n+1)}, \quad n \in \mathbb{N}} \tag{2.18}$$

One can have an idea of the graph of the Gamma function on the real axis using the formulas

$$\Gamma(x+1) = x\Gamma(x), \quad \Gamma(x-1) = \frac{\Gamma(x)}{x-1}$$

to be iterated starting from the interval $0 < x \leq 1$, where $\Gamma(x) \to +\infty$ as $x \to 0^+$, and $\Gamma(1) = 1$. For $x > 0$ Euler's integral representation (2.1) yelds $\Gamma(x) > 0$ and $\Gamma''(x) > 0$ since

$$\Gamma(x) = \int_0^\infty e^{-u} u^{x-1} du, \qquad \Gamma''(x) = \int_0^\infty e^{-u} u^{x-1} (\log u)^2 du$$

As a consequence, on the positive real axis $\Gamma(x)$ turns out to be positive and convex, so that it is either a monotonic decreasing function, or it first decreases and then increases exhibiting a minimum value. Since $\Gamma(1) = \Gamma(2) = 1$, we must have a minimum at some $x_0, 1 < x_0 < 2$. It turns out to be $x_0 = 1.4616...$ where $\Gamma(x_0) = 0.8856...$; hence x_0 is quite close to the point x = 1.5 where Γ attains the value $\sqrt{\pi}/2 = 0.8862....$

On the negative real axis $\Gamma(x)$ exhibits vertical asymptotes at $x = -n$ ($n = 0, 1, 2, ...$); it turns out to be positive for $-2 < x < -1$, $-4 < x < -3, ...,$ and negative for $-1 < x < 0$, $-3 < x < -2, ...$

Plots of $\Gamma(x)$ (continuous line) and $1/\Gamma(x)$ (dashed line) for $-4 \leq x \leq 4$, and for $0 \leq x \leq 3$ are shown

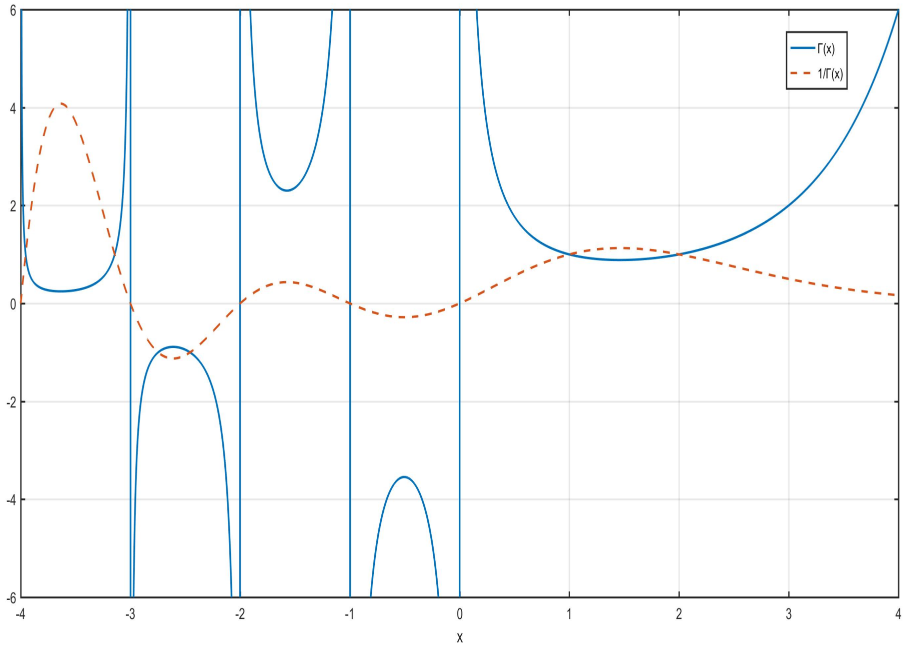


Figure 2.1: Plots of $\Gamma(x)$ and $1/\Gamma(x)$ for $-4 \leq x \leq 4$

Hankel, in 1864, provided a complex integral representation of the function $1/\Gamma(z)$ valid for unrestricted z; it reads

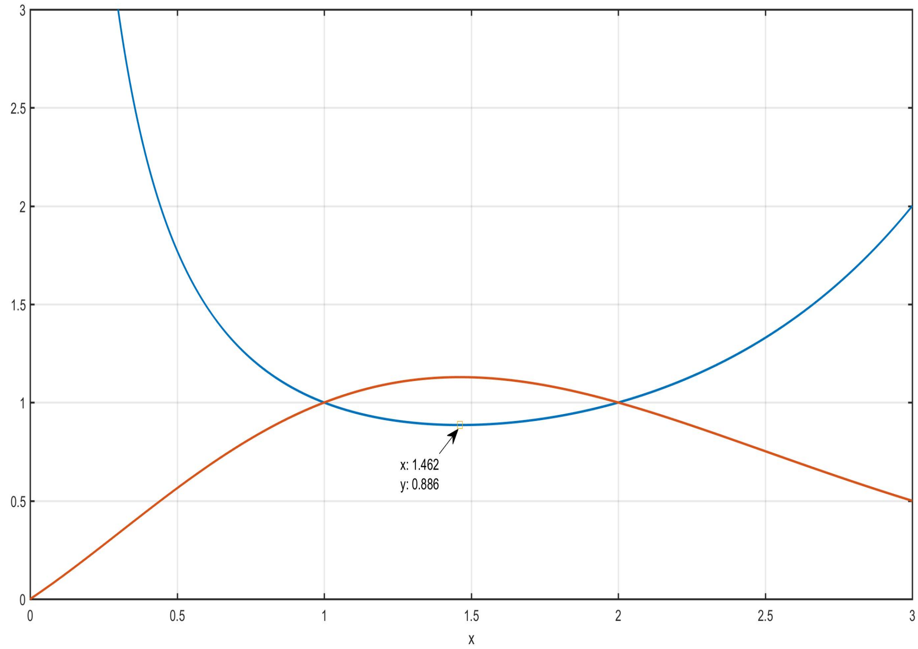


Figure 2.2: Plots of $\Gamma(x)$ and $1/\Gamma(x)$ for $0 \leq x \leq 3$

$$\boxed{\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{Ha_-} \frac{e^t}{t^z}dt, \quad z \in \mathbb{C}} \tag{2.19}$$

where Ha_- denotes the Hankel path defined as a contour that begins at $t = -\infty - ia$ $(a > 0)$, encircles the branch cut that lies along the negative real axis, and ends up at $t = -\infty + ib$ $(b > 0)$; the branch cut is present when z is not integer because t^{-z} is a multivalued function.
When z is an integer, the contour can be taken to be simply a circle around the origin, described in the counterclockwise direction.
An alternative representation is obtained assuming the branch cut along the positive real axis; in this case we get

$$\boxed{\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{Ha_+} \frac{e^{-t}}{(-t)^z}dt, \quad z \in \mathbb{C}} \tag{2.20}$$

where Ha_+ denotes the Hankel path defined as a contour that begins at $t = +\infty + ib$ $(b > 0)$, encircles the branch cut that lies along the positive real axis, and ends up at $t = +\infty - ia$ $(a > 0)$.

The advantage of Hankel's representations over Euler's integral representation is that they converge for all complex z and not just for $Re(z) > 0$. As a consequence $1/\Gamma(z)$ is a transcendental entire function (of maximum exponential type); the point at infinity is an essential non isolated singularity, which is an accumulation point of zeros ($z_n = -n$, $n = 0, 1, ...$). Since $1/\Gamma(z)$ is entire, $\Gamma(z)$ does not vanish in $\mathbb{C}$.
Furthermore, using the *reflection formula*, we can get the integral representations of $\Gamma(z)$ itself in terms of the Hankel paths (referred to as *Hankel's Integral Representations for* $\Gamma(z)$, which turns out to be valid in the whole Domain of Analyticity D_Γ. These representations, which provide the required analytical continuation of $\Gamma(z)$, are using the path Ha_-

$$\boxed{\Gamma(z) = \frac{1}{2i \sin \pi z} \int_{Ha_-} e^t t^{z-1} dt, \quad z \in D_\Gamma;} \tag{2.21}$$

using the path Ha_+

$$\boxed{\Gamma(z) = -\frac{1}{2i \sin \pi z} \int_{Ha_+} e^{-t} (-t)^{z-1} dt, \quad z \in D_\Gamma.} \tag{2.22}$$

Finally, we treat the asymptotic expression of $\Gamma(z)$, usually referred to as *Stirling's Formula*, originally given for $n!$.

$$\boxed{\Gamma(z) \simeq \sqrt{2\pi} e^{-z} z^{z-1/2} \left[1 + \frac{1}{12z} + \frac{1}{288z^2} + ... \right]; \quad z \to \infty, \ |\arg z| < \pi}. \tag{2.23}$$

The accuracy of the formula is surprisingly very good on the positive real axis also for moderate values of $z = x > 0$.

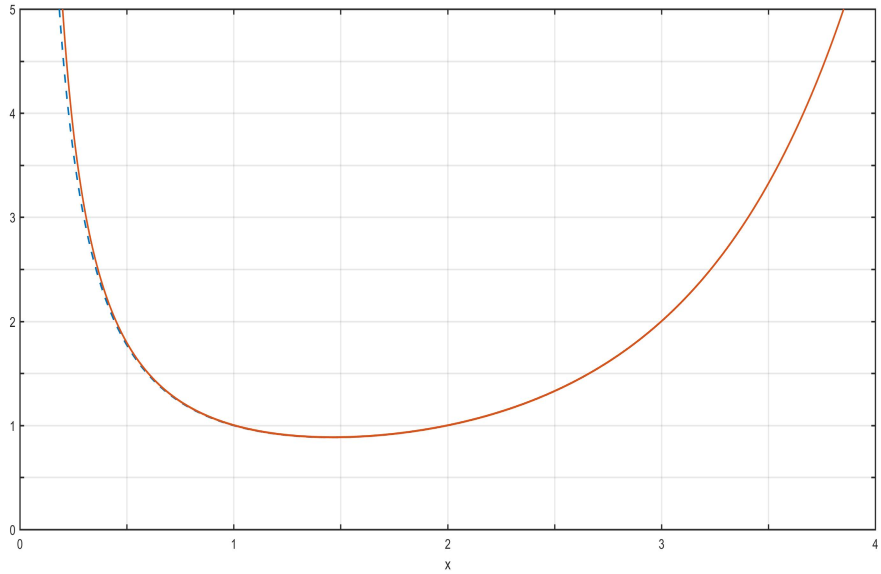


Figure 2.3: The $\Gamma(x)$ (continuous line) compared with its second order Stirling aproximation (dashed line).

2.1.2 The Beta Function

The *beta function*, also called *Euler integral of the first kind*, is a special function defined by the *Euler's Integral Representation* as

$$\boxed{B(p,q) = \int_0^1 u^{p-1}(1-u)^{q-1}du, \quad Re(p) > 0, \quad Re(q) > 0}. \tag{2.24}$$

The Beta function is therefore a complex function of two complex variables whose analyticity properties can be deduced later, as soon as the relation with the Gamma function will be established.

Let's first note the property of *simmetry*, for which

$$\boxed{B(p,q) = B(q,p)} \tag{2.25}$$

that is a simple consequence of the definition (2.24).

We can have a *Trigonometric Integral Representation*

$$\boxed{B(p,q) = 2\int_0^{\pi/2} (\cos\theta)^{2p-1}(\sin\theta)^{2q-1}d\theta, \quad Re(p) > 0, \quad Re(q) > 0}. \tag{2.26}$$

This representation follows from (2.1) by setting $u = (\cos\theta)^2$.

A relation of fundamental importance is the one that link the Beta Function with the Gamma Function, which reads

$$\boxed{B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}}. \tag{2.27}$$

Furthermore, it allows us to obtain the analytical continuation of the Beta function. The proof of (2.27) can be easily obtained by writing the product $\Gamma(p)\Gamma(q)$ as a double integral which is to be evaluated introducing polar coordinates. In this respect we must use the Gaussian representation (2.7) for the Gamma function and the trigonometric representation (2.26) for the Beta function.
In fact,

$$\begin{aligned}\Gamma(p)\Gamma(q) &= 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}u^{2p-1}v^{2q-1}dudv \\ &= 4\int_0^\infty e^{-\rho^2}\rho^{2(p+q)-1}d\rho\int_0^{\pi/2}(\cos\theta)^{2p-1}(\sin\theta)^{2q-1}d\theta \\ &= \Gamma(p+q)B(p,q).\end{aligned}$$

The Beta function plays a fundamental role in the Laplace convolution of power functions.
We recall that the Laplace convolution is the convolution between causal functions (i.e. vanishing for $t < 0$),

$$f(t) * g(t) = \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau = \int_0^t f(\tau)g(t-tau)d\tau$$

The convolution satisfies both the commutative and associative properties in that

$$f(t) * g(t) = g(t) * f(t), \quad f(t) * [g(t) * h(t)] = [f(t) * g(t)] * h(t).$$

It is straightforward to show by setting in (2.49) $u = \tau/t$, the *Convolution Representation*

$$\boxed{t^{p-1} * t^{q-1} = \int_0^t \tau^{p-1}(t-\tau)^{q-1} d\tau = t^{p+q-1} B(p,q)}. \tag{2.28}$$

Remembering the definition of the *causal Gel'fand-Shilov function* (1.15), we can write the previous result in the following interesting form

$$\boxed{\Phi_p(t) * \Phi_q(t) = \Phi_{p+q}(t)}. \tag{2.29}$$

that is a *convolution between Gel'fand-Shilov Function*; in fact, dividing by $\Gamma(p)\Gamma(q)$ the L.H.S of (2.28), and using (2.27), we just obtain (2.29).

2.2 Mittag-Leffler Functions $E_\alpha(z), E_{\alpha,\beta}(z)$

The *Mittag-Leffler function* is a generalization of the exponential function that plays an important role in fractional calculus.
The Mittag-Leffler function $E_\alpha(z)$ with $\alpha > 0$ is defined by its power series, which converges in the whole complex plane (see [12] , [17]),

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0, \quad z \in \mathbb{C}. \tag{2.30}$$

It turns out that $E_\alpha(z)$ is an *entire function* of order $\rho = 1/\alpha$ and type 1. This property is still valid but with $\rho = 1/Re(\alpha)$, if $\alpha \in \mathbb{C}$ with *positive real part*, as formerly noted by Mittag-Leffler himself.

The Mittag-Leffler function reduces to familiar functions for specific values of its parameters.
When $\alpha = 1$ we obtain the exponential function; in fact

$$E_1(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{2.31}$$

Other particular cases of from which elementary functions are recovered, are

$$E_2(+z^2) = \cosh z, \qquad E_2(-z^2) = \cos z, \qquad z \in \mathbb{C} \tag{2.32}$$

$$E_{1/2}(\pm z^{1/2}) = e^z[1 + erf(\pm z^{1/2})] = e^z erfc(\mp z^{1/2}), \quad z \in \mathbb{C}, \tag{2.33}$$

where erf /erfc) denotes the (complementary) error function defined as

$$erf(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2} du, \quad erfc(z) := 1 - erf(z), \quad z \in \mathbb{C}. \tag{2.34}$$

By $z^{1/2}$ we mean the principal value of the square root of z in the complex plane cut along the negative real semi-axis. With this choice $\pm z^{1/2}$ turns out to be positive/negative for $z \in \mathbb{R}^+$. A straightforward generalization of the Mittag- Leffler function, originally due to Agarwal in 1953 based on a note by Humbert, is obtained by replacing the additive constant 1 in the argument of the Gamma function in (2.30) by an arbitrary complex parameter β. For the generalized Mittag-Leffler function we agree to use the notation

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha > 0,\ \beta \in \mathbb{C}, \quad z \in \mathbb{C}. \tag{2.35}$$

Particular cases are

$$E_{1,2}(z) = \frac{e^z - 1}{z}, \quad E_{2,2}(z) = \frac{sinh(z^{1/2})}{z^{1/2}}. \tag{2.36}$$

We note that $E_{\alpha,\beta}(z)$is still an entire function of order $\rho = 1/\alpha$ and type 1.

2.2.1 The Mittag-Leffler integral representation and asymptotic expansions

Many of the important properties of $E_\alpha(z)$ follow from Mittag-Leffler's *integral representation*

$$E_\alpha(z) = \frac{1}{2\pi i}\int_{Ha} \frac{\zeta^{\alpha-1}e^\zeta}{\zeta^\alpha - z}d\zeta, \quad \alpha > 0, \; z \in \mathbb{C}, \tag{2.37}$$

where the path of integration Ha (the Hankel path) is a loop which starts and ends at $-\infty$ and encircles the circular disk $|\zeta| \leq |z|^{1/\alpha}$ in the positive sense: $-\pi \leq arg(\zeta) \leq \pi$ on Ha, To prove (2.37), expand the integrands in powers of ζ, integrate term-by-term, and use Hankel's integral for the reciprocal of the Gamma function.
The integrand in (2.37) has a branch-point at $\zeta = 0$. The complex ζ-plane is cut along the negative real semi-axis, and in the cut plane the integrand is single-valued: the principal branch of ζ^α is taken in the cut plane. The integrand has poles at the points $\zeta_m = z^{1/\alpha}e^{2\pi i m/\alpha}$, m integer, but only those of the poles lie in the cut plane for which $-\alpha\pi < argz + 2\pi m < \alpha\pi$.
Thus, the number of the poles inside Ha is either $[\alpha]$ or $[\alpha + 1]$, according to the value of arg z.
The most interesting properties of the Mittag-Leffler function are associated with its asymptotic developments as $z \to \infty$ in various sectors of the complex plane. These properties can be summarized as follows. For the case $0 < \alpha < 2$ we have

$$E_\alpha(z) \sim \frac{1}{\alpha}exp(z^{1/\alpha}) - \sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-\alpha k)}, \quad |z| \to \infty, \quad |argz| < \alpha\pi/2, \tag{2.38}$$

$$E_\alpha(z) \sim -\sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-\alpha k)}, \quad |z| \to \infty, \quad \alpha\pi/2 < argz < 2\pi - \alpha\pi/2. \quad (2.39)$$

For the case $\alpha \geq 2$ we have

$$E_\alpha(z) \sim \frac{1}{\alpha} \sum_m exp\big(z^{1/\alpha} e^{2\pi i m/\alpha}\big) - \sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-\alpha k)}, \quad |z| \to \infty, \qquad (2.40)$$

where m takes all integer values such that $-\alpha\pi/2 < argz + 2\pi m < \alpha\pi/2$, and arg z can assume any value from $-\infty$ to $+\infty$.

From the asymptotic properties (2.38-39-40) and the definition of the order of an entire function, we infer that the Mittag-Leffler function is an *entire function of order* $1/\alpha$ for $\alpha > 0$; in a certain sense each $E_\alpha(z)$ is the simplest entire function of its order. The Mittag-Leffler function also furnishes examples and counter- examples for the growth and other properties of entire functions of finite order.

Finally, the integral representation for the generalized Mittag-Leffler function reads

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int_{Ha} \frac{\zeta^{\alpha-\beta} e^{\zeta}}{\zeta^\alpha - z} d\zeta, \quad \alpha > 0, \; \beta \in \mathbb{C}, \; z \in \mathbb{C}. \qquad (2.41)$$

2.2.2 The Laplace transform pairs related to the Mittag-Leffler functions

The Mittag-Leffler functions are connected to the Laplace integral through the equation

$$\int_0^\infty e^{-u} E_\alpha(u^\alpha z) du \; = \; \frac{1}{z-1}, \quad \alpha > 0. \qquad (2.42)$$

The integral at the L.H.S. was evaluated by Mittag-Leffler who showed that the region of its convergence contains the unit circle and is bounded by the line $Rez^{1/\alpha} = 1$. Putting in (2.42) $u = st$ and $u^\alpha z = -at^\alpha$ with $t \leq 0$ and $\alpha \in \mathbb{C}$, and using the sign *div* for the juxtaposition of a function depending on t with its Laplace transform depending on s , we get the following Laplace transform pairs

$$E_\alpha(-at^\alpha) \div \frac{s^{\alpha-1}}{s^\alpha + a}, \quad Re(s) > |a|^{1/\alpha}. \tag{2.43}$$

More generally one can show

$$\int_0^\infty e^{-u} u^{\beta-1} E_{\alpha,\beta}(u^\alpha z) du = \frac{1}{1-z}, \quad \alpha, \beta > 0. \tag{2.44}$$

$$t^{\beta-1} E_{\alpha,\beta}(at^\alpha) \div \frac{s^{\alpha-\beta}}{s^\alpha - \alpha}, \quad Re(s) > |a|^{1/\alpha}. \tag{2.45}$$

2.2.3 Other formulas: summation and integration

For completeness hereafter we exhibit some formulas related to summation and integration of ordinary Mittag-Leffler functions (in one parameter α).
Concerning summation we outline

$$E_\alpha(z) = \frac{1}{p} \sum_{h=0}^{p-1} E_{\alpha/p}(z^{1/p} e^{i2\pi h/p}), \quad p \in \mathbb{N}, \tag{2.46}$$

from which we derive the *duplication formula*

$$E_\alpha(z) = \frac{1}{2}[E_{\alpha/2}(+z^{1/2}) + E_{\alpha/2}(-z^{1/2})]. \tag{2.47}$$

As an example of this formula we can recover, for $\alpha = 2$, the expressions of $\cosh z$ and $\cos z$ in terms of two exponential functions.
Concerning integration we outline another interesting duplication formula

$$E_{\alpha/2}(-t^{a/2}) = \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-x^2/(4t)} E_\alpha(-x^\alpha) dx, \quad x > 0, \quad t > 0. \tag{2.48}$$

It can be derived by applying a theorem of the Laplace transform theory (known as *Efros theorem*)

2.2.4 The Mittag-Leffler functions of rational order

Let us now consider the Mittag-Leffler functions of rational order $\alpha = p/q$, with $p, q \in \mathbb{N}$, relatively prime. The relevant functional relations, that we quote from Djrbashian in 1966, and Erdélyi in 1955, turn out to be

$$\left(\frac{d}{dz}\right)^p E_p(z^p) = E_p(z^p), \tag{2.49}$$

$$\frac{d^p}{dz^p} E_{p/q}(z^{p/q}) = E_{p/q}(z^{p/q}) + \sum_{k=1}^{q-1} \frac{z^{-kp/q}}{\Gamma(1-kp/q)}, \quad q = 2, 3, ... \tag{2.50}$$

$$E_{p/q}(z) = \frac{1}{p} \sum_{h=0}^{p-1} E_{1/q}(z^{1/p} e^{i2\pi h/p}), \tag{2.51}$$

and

$$E_{1/q}(z^{1/q}) = e^z \left[1 + \sum_{k=1}^{q-1} \frac{\gamma(1-k/q, z)}{\Gamma(1-k/q)}\right], \quad q = 2, 3, ... \tag{2.52}$$

where $\gamma(a, z)$ denotes the *incomplete gamma function*, defined as

$$\gamma(a, z) := \int_0^z e^{-u} u^{\alpha-1} du. \tag{2.53}$$

The relation (2.52) shows how the Mittag-Leffler functions of rational order can be expressed in terms of exponentials and incomplete gamma functions. In particular, taking in (2.52) q = 2 , we now can verify again the relation (2.34). In fact, from (2.52) we obtain

$$erf(z) = \gamma(1/2, z^2)/\sqrt{\pi} \tag{2.54}$$

See e.g. the *Handbook of Mathematical Functions* by M. Abramowitz and I.A. Stegun.

2.2.5 Some plots of the Mittag-Leffler functions

For readers' convenience we now consider the functions

$$\psi_\alpha(t) := E_\alpha(-t^\alpha), \quad t \geq 0, \quad 0 < \alpha < 1, \tag{2.55}$$

and

$$\phi_\alpha(t) := t^{-(1-\alpha)} E_{\alpha,\alpha}(-t^\alpha) = -\frac{d}{dt} E_\alpha(-t^\alpha), \quad t \geq 0, \quad 0 < \alpha < 1, \tag{2.56}$$

that play fundamental roles in fractional relaxation. The plots of $\psi_\alpha(t)$ and $\phi_\alpha(t)$ are shown for some rational values of the parameter α, by adopting linear and logarithmic scales. It is evident that for $\alpha \to 1^-$ the two functions reduce to the standard exponential function $\exp(-t)$.

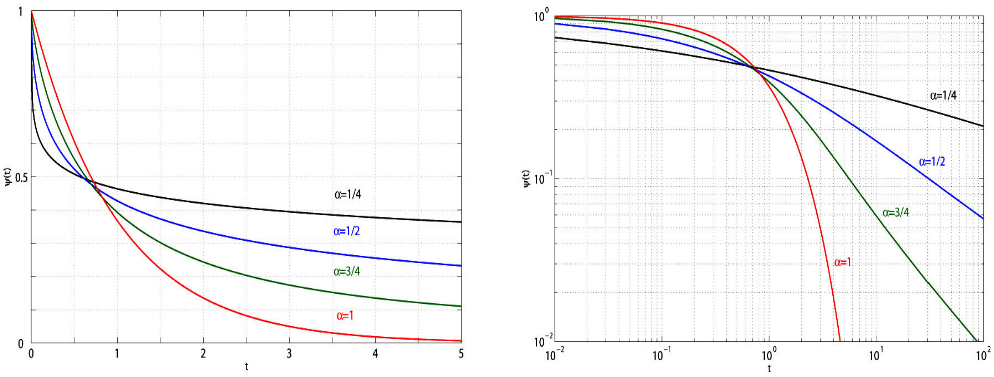


Figure 2.4: Plots of $\psi_\alpha(t)$ with $\alpha = 1/4, 1/2, 3/4, 1$ versus t; left: linear scales $(0 \leq t \leq 5)$; right: logarithmic scales $(10^{-2} \leq t \leq 10^2)$.

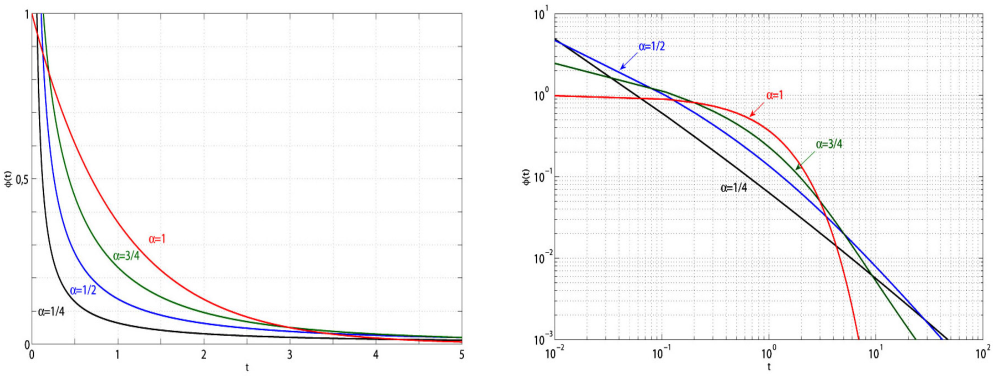


Figure 2.5: Plots of $\phi_\alpha(t)$ with $\alpha = 1/4, 1/2, 3/4, 1$ versus t; left: linear scales ($0 \le t \le 5$); right: logarithmic scales ($10^{-2} \le t \le 10^2$).

It is worth noting the algebraic decay of $\psi_\alpha(t)$ and $\phi_\alpha(t)$ as $t \to \infty$:

$$\psi_\alpha(t) \sim \frac{\sin(\alpha\pi)}{\pi} \frac{\Gamma(\alpha)}{t^\alpha}, \quad \phi_\alpha(t) \sim \frac{\sin(\alpha\pi)}{\pi} \frac{\Gamma(\alpha+1)}{t^{(\alpha+1)}}, \quad t \to \infty. \tag{2.57}$$

2.3 The Wright Functions

In this section we provide a survey of the transcendental functions known in the literature as Wright functions. We devote particular attention for two functions of the Wright type, which, in virtue of their role in applications of fractional calculus (see [14], [18]), we have called auxiliary functions.

2.3.1 The Wright function $W_{\lambda,\mu}(z)$

The Wright function, that we denote by $W_{\lambda,\mu}(z)$,is defined by the series representation, convergent in the whole complex plane,

$$\boxed{W_{\lambda,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\lambda n + \mu)} \qquad \lambda > -1, \quad \mu \in \mathbb{C}} \tag{2.58}$$

so $W_{\lambda,\mu}(z)$ is an entire function.
Originally, Wright assumed $\lambda > 0$ and, only in 1940, he considered $-1 < \lambda < 0$.
We distinguish the Wright functions in *first kind* ($\lambda \geq 0$) and *second kind* ($-1 < \lambda < 0$).

The *integral representation* reads (see [18], [20])

$$\boxed{W_{\lambda,\mu}(z) = \frac{1}{2\pi i}\int_{Ha} e^{\sigma + z\sigma^{-\lambda}} \frac{d\sigma}{\sigma^{\mu}} \qquad \lambda > -1, \quad \mu \in \mathbb{C}} \tag{2.59}$$

The equivalence between the series and integral representations is easily proven by using the Hankel formula for the Gamma function

$$\frac{1}{\Gamma(\zeta)} = \frac{1}{2\pi i}\int_{Ha} e^{u} u^{-\zeta} du, \quad \zeta \in \mathbb{C} \tag{2.60}$$

and performing a term-by-term integration. The exchange between series and integral is legitimate by the uniform convergence of the series, being $W_{\lambda,\mu}(z)$ an entire function.
We have

$$\begin{aligned} W_{\lambda,\mu}(z) &= \frac{1}{2\pi i}\int_{Ha} e^{\sigma + z\sigma^{-\lambda}} \frac{d\sigma}{\sigma^{\mu}} = \frac{1}{2\pi i}\int_{Ha} e^{\sigma}\left[\sum_{n=0}^{\infty} \frac{z^n}{n!}\sigma^{-\lambda n}\right]\frac{d\sigma}{\sigma^{\mu}} \\ &= \sum_{n=0}^{\infty}\frac{z^n}{n!}\left[\frac{1}{2\pi i}\int_{Ha} e^{\sigma}\sigma^{-\lambda n - \mu}\right] d\sigma = \sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma[\lambda n + \mu]} \end{aligned} . \tag{2.61}$$

Furthermore, it is possible to prove that the Wright function is entire of order $1/(1+\lambda)$ hence of exponential type only if $\lambda \geq 0$. The case $\lambda = 0$ is trivial since $W_{0,\mu}(z) = e^{z}/\Gamma(\mu)$.

For the detailed *asymptotic analysis* in the whole complex plane for the Wright functions, the interested reader is referred to Wong and Zhao (1999). These authors have provided asymptotic expansions of the Wright functions of the first and second kind following a new method for smoothing Stokes' discontinuities. As a matter of fact, the second kind is the most interesting for us. By setting $\lambda = -\nu \in (-1, 0)$, we recall the asymptotic expansion originally obtained byWright himself, that is valid in a suitable sector about the negative real axis as $|z| \to \infty$,

$$\begin{aligned} W_{-\nu,\mu}(z) =& Y^{1/2-\mu} e^{-Y} \left[\sum_{m=0}^{M-1} A_m Y^{-m} + O(|Y|^{-M}) \right], \\ Y = Y(z) =& (1-\nu)(-\nu^{\nu} z)^{1/(1-\nu)} \end{aligned} \tag{2.62}$$

where the A_m are certain real numbers.

The Wright functions turn out to be related to the well-known Bessel functions J_ν and I_ν for $\lambda = 1$ and $\mu = \nu + 1$. In fact, we can easily recognize the identities:

$$\begin{aligned} J_\nu(z) :=& \left(\frac{z}{2}\right)^{\nu} \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n!\Gamma(n+\nu+1)} = \left(\frac{z}{2}\right)^{\nu} W_{1,\nu+1}\left(-\frac{z^2}{4}\right), \\ W_{1,\nu+1}(-z) :=& \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!\Gamma(n+\nu+1)} = z^{-\nu/2} J_\nu(2z^{1/2}). \end{aligned} \tag{2.63}$$

and

$$\begin{aligned} I_\nu(z) :=& \left(\frac{z}{2}\right)^{\nu} \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{n!\Gamma(n+\nu+1)} = \left(\frac{z}{2}\right)^{\nu} W_{1,\nu+1}\left(\frac{z^2}{4}\right), \\ W_{1,\nu+1}(z) :=& \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(n+\nu+1)} = z^{-\nu/2} I_\nu(2z^{1/2}). \end{aligned} \tag{2.64}$$

In view of the first equation in (2.63) some authors refer to the Wright function as the *Wright generalized Bessel function* (misnamed also as the *Bessel- Maitland function*) and introduce the notation for $\lambda > 0$,

$$J_\nu^{(\lambda)}(z) := \left(\frac{z}{2}\right)^{\nu} \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n!\Gamma(\lambda n+\nu+1)} = \left(\frac{z}{2}\right)^{\nu} W_{\lambda,\nu+1}\left(-\frac{z^2}{4}\right) \tag{2.65}$$

Similar remarks can be extended to the modified Bessel functions I_ν.

Hereafter, we quote some relevant *recurrence relations* from Erdélyi et al. in 1953-1954:

$$\lambda z W_{\lambda,\lambda+\mu}(z) = W_{\lambda,\mu-1}(z) + (1-\mu) W_{\lambda,\mu}(z), \tag{2.66}$$

$$\frac{d}{dz} W_{\lambda,\mu}(z) = W_{\lambda,\lambda+\mu}(z). \tag{2.67}$$

We note that these relations can easily be derived from (2.58).

2.3.2 The auxiliary functions $F_\nu(z)$ and $M_\nu(z)$ in $\mathbb{C}$

In his earliest analysis of the time-fractional diffusion-wave equation (see Mainardi in 1994), the author introduced the two auxiliary functions of the Wright type:

$$F_\nu(z) := W_{-\nu,0}(-z), \quad 0 < \nu < 1, \tag{2.68}$$

$$M_\nu(z) := W_{-\nu,1-\nu}(-z), \quad 0 < \nu < 1. \tag{2.69}$$

interrelated through

$$F_\nu(z) = \nu z M_\nu(z). \tag{2.70}$$

The motivation was based on the inversion of certain Laplace transforms in order to obtain the fundamental solutions of the fractional diffusion-wave equation in the space-time domain. Here we will devote particular attention to the mathematical properties of these functions limiting at the essential the discussion for the general Wright functions.

The series representations of our auxiliary functions are derived from those of $W_{\lambda,\mu}(z)$. We have:

$$F_\nu(z) := \sum_{n=1}^{\infty} \frac{(-z)^n}{n!\Gamma(-\nu n)} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{n!} \Gamma(\nu n + 1) \sin(\pi \nu n), \tag{2.71}$$

and

$$M_\nu(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma[-\nu n + (1-\nu)]} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!} \Gamma(\nu n) \sin(\pi \nu n) \tag{2.72}$$

where we have used the well-known reflection formula for the Gamma function

$$\Gamma(\zeta)\Gamma(1-\zeta) = \pi / \sin \pi\zeta.$$

We note that $F_\nu(0) = 0, M_\nu(0) = 1/\Gamma(1-\nu)$ and that the relation (2.70), consistent with the recurrence relation (2.66), can be derived from (2.71-72) arranging the terms of the series.

The *integral representations* of our auxiliary functions are derived from those of $W_{\lambda,\mu}(z)$. We have:

$$F_\nu(z) := \frac{1}{2\pi i} \int_{Ha} e^{\sigma - z\sigma^\nu} \, d\sigma, \quad z \in \mathbb{C},\ 0 < \nu < 1, \tag{2.73}$$

$$M_\nu(z) := \frac{1}{2\pi i} \int_{Ha} e^{\sigma - z\sigma^\nu} \frac{d\sigma}{\sigma^{1-\nu}}, \quad z \in \mathbb{C},\ 0 < \nu < 1. \tag{2.74}$$

We note that the relation (2.70) can be obtained directly from (2.73) and (2.74) with an integration by parts, i.e.

$$\int_{Ha} e^{\sigma - z\sigma^\nu} \frac{d\sigma}{\sigma^{1-\nu}} = \int_{Ha} e^{\sigma} \left(-\frac{1}{\nu z} \frac{d}{d\sigma} e^{-z\sigma^\nu} \right) d\sigma$$
$$= \frac{1}{\nu z} \int_{Ha} e^{\sigma - z\sigma^\nu} d\sigma.$$

The passage from the series representation to the integral representation and vice-versa for our auxiliary functions can be derived in a way similar to that adopted for the generalWright function, that is by expanding in positive powers of z the exponential function $e^{-z\sigma^\nu}$, exchanging the order between the series and the integral and using the Hankel representation of the reciprocal of the Gamma function.
Since the radius of convergence of the power series in (2.71-72) can be proven to be infinite for $0 < \nu < 1$, our auxiliary functions turn out to be entire in z and therefore the exchange between the series and the integral is legitimate[1].

Explicit expressions of $F_\nu(z)$ and $M_\nu(z)$ in terms of known functions are expected for some particular values of ν.
In Mainardi and Tomirotti (1995) the authors have shown that for $\nu = 1/q$, where $q \geq 2$ is a positive integer, the auxiliary functions can be expressed as a sum of $(q-1)$ simpler entire functions.
In the particular case $q = 2$ and $q = 3$ we find

$$M_{1/2}(z) = \frac{1}{\sqrt{\pi}} \sum_{m=0}^{\infty} (-1)^m \left(\frac{1}{2}\right)_m \frac{z^{2m}}{(2m)!} = \frac{1}{\sqrt{\pi}} e^{-z^2/4}, \tag{2.75}$$

and

$$M_{1/3}(z) = \frac{1}{\Gamma(2/3)} \sum_{m=0}^{\infty} \left(\frac{1}{3}\right)_m \frac{z^{3m}}{(3m)!} - \frac{1}{\Gamma(1/3)} \sum_{m=0}^{\infty} \left(\frac{2}{3}\right)_m \frac{z^{3m+1}}{(3m+1)!} \tag{2.76}$$
$$= 3^{2/3} Ai\big(z/3^{1/3}\big),$$

[1] The author in [Mainardi (1994)] proved these properties independently from Wright (1940), because at that time he was aware only of the work of Erdelyi et al. (1953-1955), where λ was restricted to be non-negative.

where Ai denotes the *Airy function*.
Furthermore, it can be proved that $M_{1/q}(z)$ satisfies the differential equation of order $q-1$

$$\frac{d^{q-1}}{dz^{q-1}} M_{1/q}(z) + \frac{(-1)^q}{q} z M_{1/q}(z) = 0, \tag{2.77}$$

subjected to the $q-1$ initial condition at $z=0$, derived from (2.69),

$$M_{1/q}^{(h)}(0) = \frac{(-1)^h}{\pi} \Gamma[(h+1)/q] \sin[\pi(h+1)/q], \tag{2.78}$$

with $h = 0, 1, ...q-2$.
For $q \geq 4$ Eq. (2.77) is akin to the *hyper-Airy differential equation of order* $q-1$. See e.g. Bender and Orszag in 1987. Consequently, in view of the above considerations, the auxiliary function $M_\nu(z)$ could be referred to as the *generalized hyper-Airy function*.

2.3.3 The auxiliary functions $F_\nu(x)$ and $M_\nu(x)$ in $\mathbb{R}$

We point out that the most relevant applications of Wright functions, especially our auxiliary functions, are when the independent variable is real. More precisely, in this Section we will consider functions of the variable x with $x \in \mathbb{R}^+$ or $x \in \mathbb{R}$.
When the support is all of $\mathbb{R}$ we agree to consider *even functions*; to stress the symmetry property of the function, the independent variable may be denoted by $|x|$.
We point out that in the limit $\nu \to 1^-$ the function $M_\nu(x)$, for $x \in \mathbb{R}^+$, tends to the Dirac generalized function $\delta(x-1)$.

Let us first point out the asymptotic behaviour of the function $M_\nu(x)$ as $x \to +\infty$.
Choosing as a variable x/ν rather than x, the computation of the asymptotic representation by the saddle-point approximation yields (see Mainardi and Tomirotti in 1995),

$$M_\nu(x/\nu) \sim a(\nu) x^{(\nu-1/2)/(1-\nu)} exp\left[-b(\nu) x^{1/(1-\nu)}\right], \qquad (2.79)$$

where

$$a(\nu) = \frac{1}{\sqrt{2\pi(1-\nu)}} > 0, \quad b(\nu) = \frac{1-\nu}{\nu} > 0. \qquad (2.80)$$

The above evaluation is consistent with the first term in Wright's asymptotic expansion.

We show the plots of our auxiliary functions on the real axis for some rational values of the parameter ν.
To gain more insight of the effect of the parameter itself on the behaviour close to and far from the origin, we will adopt both linear and logarithmic scale for the ordinates.
Then, we compare the plots of the $M_\nu(x)$ Wright auxiliary functions in $-5 \leq x \leq 5$ for some rational values in the ranges $\nu \in [0, 1/2]$ and $\nu \in [1/2, 1]$, respectively.

Thus in Fig 2.6 we see the transition from $exp(-|x|)$ for $\nu = 0$ to $(1/\sqrt{\pi})exp(-x^2)$ for $\nu = 1/2$, whereas in Fig 2.7 we see the transition from $(1/\sqrt{\pi})exp(-x^2)$ for $\nu = 1/2$ to the Delta Function $\delta(1-|x|)$ for $\nu = 1$.

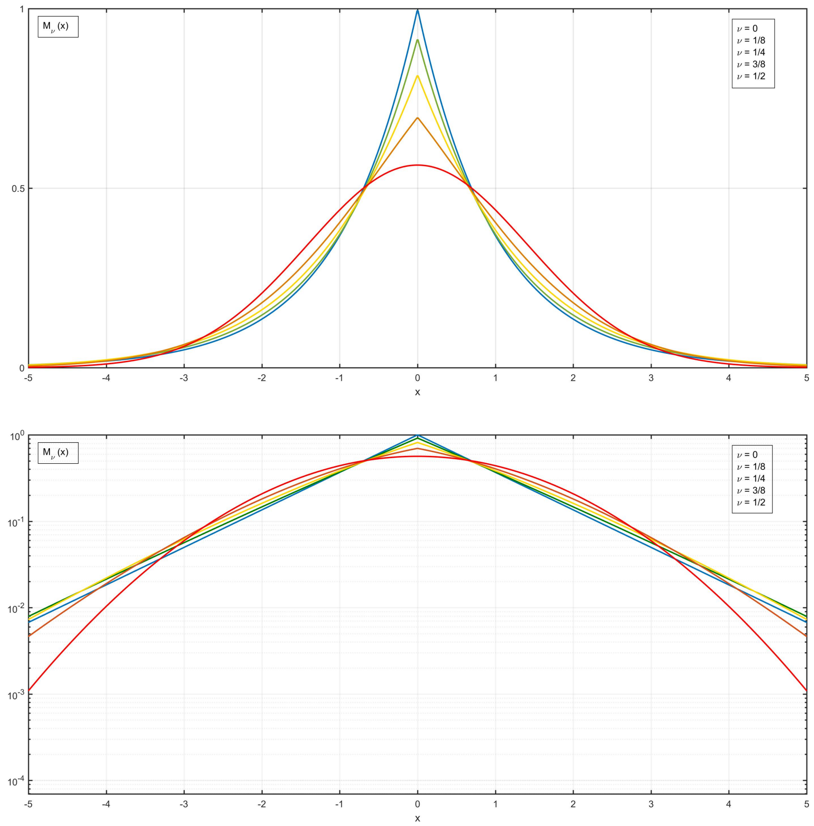


Figure 2.6: Plots of the Wright type function $M_\nu(x)$ with $\nu = 0, 1/8, 1/4, 3/8, 1/2$ for $-5 \le x \le 5$; top: linear scale, bottom: logarithmic scale.

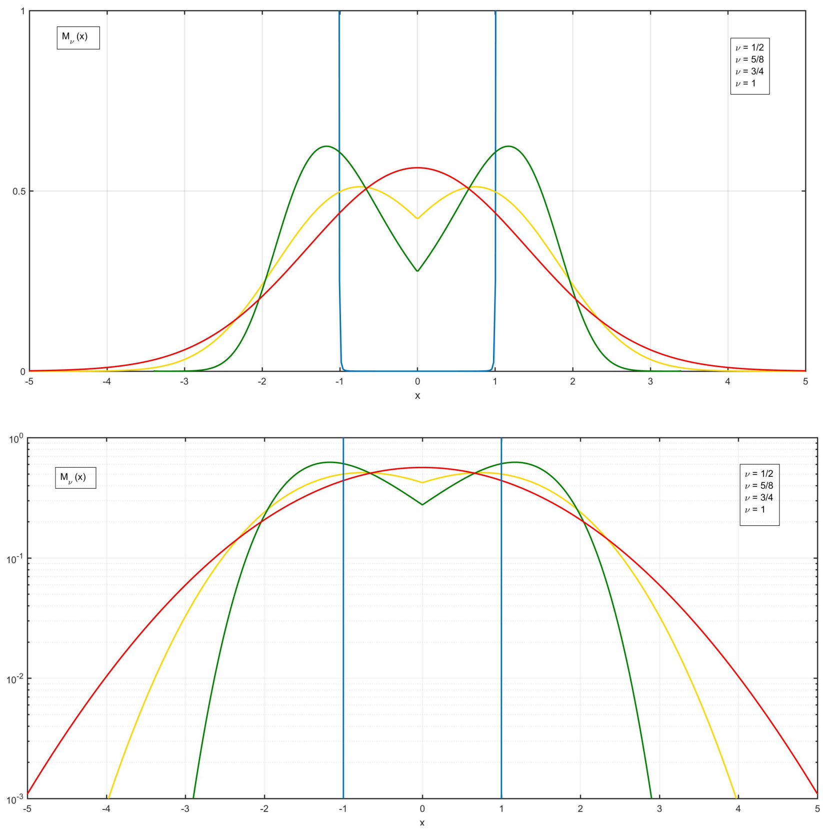


Figure 2.7: Plots of the Wright type function $M_\nu(x)$ with $\nu = 1/2, 5/8, 3/4, 1$ for $-5 \le x \le 5$; top: linear scale, bottom: logarithmic scale.

2.3.4 The Laplace Transform pair

Let us write the Laplace transform of the Wright function as

$$W_{\lambda,\mu}(\pm r) \div \mathcal{L}[W_{\lambda,\mu}(\pm r); s] := \int_0^{\infty} e^{-sr} W_{\lambda,\mu}(\pm r) dr, \qquad (2.81)$$

where r denotes a non-negative real variable, i.e. $0 \leq r < +\infty$, and s is the Laplace complex parameter.
When $\lambda > 0$ the series representation of the Wright function can be transformed term-by-term. In fact, for a known theorem of the theory of the Laplace transforms, (see e.g. Doetsch in 1974), the Laplace transform of an entire function of exponential type can be obtained by transforming term-by-term the Taylor expansion of the original function around the origin. In this case the resulting Laplace transform turns out to be analytic and vanishing at infinity. As a consequence, we obtain the *Laplace transform pair for the Wright function of the first kind* as

$$W_{\lambda,\mu}(\pm r) \div \frac{1}{s} E_{\lambda,\mu}\left(\pm \frac{1}{s}\right), \quad \lambda > 0, \quad |s| > \rho > 0, \qquad (2.82)$$

where $E_{\lambda,\mu}$ denotes the *generalized Mittag-Leffler function* in two parameters, and ρ is an arbitrary positive number. The proof is straightforward, noting that

$$\sum_{n=0}^{\infty} \frac{(\pm r)^n}{n!\Gamma(\lambda n + \mu)} \div \frac{1}{s} \sum_{n=0}^{\infty} \frac{(\pm 1/s)^n}{\Gamma(\lambda n + \mu)}, \qquad (2.83)$$

and recalling the series representation of the generalized Mittag-Leffler function,

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha > 0, \quad z \in \mathbb{C}. \qquad (2.84)$$

For $\lambda \to 0^+$ Eq. (2.79) provides the Laplace transform pair

$$W_{0^+,\mu}(\pm r) := \frac{e^{\pm r}}{\Gamma(\mu)} \div \frac{1}{\Gamma(\mu)} \frac{1}{s \mp 1}. \qquad (2.85)$$

This means

$$W_{0^+,\mu}(\pm r) \div \frac{1}{s} E_{0,\mu}\left(\pm\frac{1}{s}\right) = \frac{1}{\Gamma(\mu)s} E_0\left(\pm\frac{1}{s}\right), \quad |s| > 1, \tag{2.86}$$

where, in order to be consistent with (2.82), we have formally put

$$E_{0,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n)} = \frac{1}{\Gamma(\mu)} E_0(z) = \frac{1}{\Gamma(\mu)} \frac{1}{1-z}, \quad |z| < 1 \tag{2.87}$$

We recognize that in this limitig case the Laplace transform exhibits a simple pole at $s = \pm 1$, while for $\lambda > 0$ it exhibits an essential singularity at $s = 0$.
For $-1 < \lambda < 0$ the Wright function turns out to be an entire function of order greater than 1, so that the term-by-term transformation representation is no longer legitimate. Thus, for Wright functions of the second kind, care is required in establishing the existence of the Laplace transform, which necessarily must tend to zero as $s \to \infty$ in its half-plane of convergence.
For the sake of convenience we first derive the Laplace transform for the special case of $M_\nu(r)$; the exponential decay as $r \to \infty$ of the original function provided by (2.79) ensures the existence of the image function. From the integral representation (2.72) of the M_ν function we obtain

$$\begin{gathered} M_\nu(r) \div \frac{1}{2\pi i} \int_0^\infty e^{-sr} \left[\int_{Ha} e^{\sigma - r\sigma^\nu} \frac{d\sigma}{\sigma^{1-\nu}} \right] dr \\ \frac{1}{2\pi i} \int_{Ha} e^\sigma \sigma^{\nu-1} \left[\int_0^\infty e^{-r(s+\sigma^\nu)} dr \right] d\sigma = \frac{1}{2\pi i} \int_{Ha} \frac{e^\sigma \sigma^{\nu-1}}{\sigma^\nu + s} d\sigma. \end{gathered} \tag{2.88}$$

Then, by recalling the integral representation of the Mittag-Leffler function,

$$E_\alpha(z) = \frac{1}{2\pi i} \int_{Ha} \frac{\zeta^{\alpha-1} e^\zeta}{\zeta^\alpha - z} d\zeta, \quad \alpha > 0, \tag{2.89}$$

we obtain the Laplace transform pair

$$M_\nu(r) \div E_\nu(-s), \quad 0 < \nu < 1. \tag{2.90}$$

Although transforming the Taylor series of $M_\nu(r)$ term-by-term is not legitimate, this procedure yields a series of negative powers of s that represents the asymptotic expansion of the correct Laplace transform, $E_\nu(s)$ as $s \to \infty$ in a sector around the positive real axis. Indeed we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\int_0^\infty e^{-sr}(-r)^n dr}{n!\Gamma(-\nu n + (1-\nu))} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(-\nu n + (1-\nu))} \frac{1}{s^{n+1}} \\
&= \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\Gamma(-\nu m + 1)} \frac{1}{s^m} \sim E_\nu(-s), \quad s \to \infty
\end{aligned}$$

We note that (2.90) contains the well-known Laplace transform pair, (see e.g. in Doetsch(1974)),

$$M_{1/2}(r) := \frac{1}{\sqrt{\pi}} exp(-r^2/4) \; \div \; E_{1/2}(-s) := exp(s^2)erfc(s),$$

that is valid for all $s \in \mathbb{C}$.

Analogously, using the more general integral representation (2.59) of the Wright function, we can get the *Laplace transform pair for the Wright function of the second kind.* For the case $\lambda = -\nu \in (-1, 0)$, with $\mu > 0$ for simplicity, we obtain,

$$W_{-\nu,\mu}(-r) \div E_{\nu,\mu+\nu}(-s), \quad 0 < \nu < 1. \tag{2.91}$$

We note the minus sign in the argument in order to ensure the the existence of the Laplace transform thanks to the Wright asymptotic formula (2.62) valid in a sector about the negative real axis.

In the limit as $\lambda \to 0^-$ we formally obtain the Laplace transform pair

$$W_{0^-,\mu}(-r) := \frac{e^{-r}}{\Gamma(\mu)} \div \frac{1}{\Gamma(\mu)} \frac{1}{s+1}. \tag{2.92}$$

In order to be consistent with (2.90) we rewrite

$$W_{0^-,\mu}(-r) \div E_{0,\mu}(-s) = \frac{1}{\Gamma(\mu)} E_0(-s), \quad |s| < 1. \tag{2.93}$$

Therefore, as $\lambda \to 0^{\pm}$, we note a sort of continuity in the formal results (2.86) and (2.93) because

$$\frac{1}{(s+1)} = \begin{cases} (1/s)E_0(-1/s), & |s| > 1; \\ E_0(-s), & |s| < 1. \end{cases} \tag{2.94}$$

We now point out the relevant Laplace transform pair related to the auxiliary functions of argument $r^{-\nu}$ (proved in Mainardi (1994-1996)):

$$\frac{1}{r} F_\nu(1/r^\nu) = \frac{\nu}{r^{\nu+1}} M_\nu(1/r^\nu) \div e^{-s^\nu} \quad 0 < \nu < 1. \tag{2.95}$$

$$\frac{1}{\nu} F_\nu(1/r^\nu) = \frac{1}{r^\nu} M_\nu(1/r^\nu) \div \frac{e^{-s^\nu}}{s^{1-\nu}} \quad 0 < \nu < 1. \tag{2.96}$$

recalling that the Laplace transform pairs in (2.95) were formerly considered by Pollard in 1946, who provided a rigorous proof based on a formal result by Humbert in 1945. Later Mikusiński, in 1959, achieved a similar result based on his theory of operational calculus, and finally, albeit unaware of the previous results, Buchen and Mainardi in 1975 derived the result in a formal way; all these authors were not informed about the Wright functions. To our actual knowledge, the former author who derived the Laplace transforms pairs (2.95)-(2.96) in terms of Wright functions of the second kind was Stankovich in 1970.

2.3.5 The Wright $M-$functions in probability

The Wright M-function with support in $\mathbb{R}^+$ can be interpreted as probability density function (pdf). Consequently, extending the function in a symmetric way to all of $\mathbb{R}$ and dividing by 2 we have a symmetric pdf with support in $\mathbb{R}$. In the former case the variable is usually a time coordinate whereas in the latter the variable is the absolute value of a space coordinate. We now provide more details on these densities in the framework of the theory of probability, agreeing to denote by x and $|x|$ the variables in $\mathbb{R}^+$ and $\mathbb{R}$, respectively.

The *absolute moments* of order $\delta > -1$ of the Wright $M-$ function in $\mathbb{R}^+$ are finite and turn out to be

$$\int_0^\infty x^\delta M_\nu(x)dx = \frac{\Gamma(\delta+1)}{\Gamma(\nu\delta+1)}, \quad \delta > -1, \quad 0 \le \nu < 1. \tag{2.97}$$

In order to derive this fundamental result we proceed as follows, based on the integral representation (2.74)

$$\begin{aligned}\int_0^\infty x^\delta M_\nu(x)dx &= \int_0^\infty c^\delta \left[\frac{1}{2\pi i}\int_{Ha} e^{\sigma - x\sigma^\nu}\frac{d\sigma}{\sigma^{1-\nu}}\right]dx \\ &= \frac{1}{2\pi i}\int_{Ha} e^\sigma \left[\int_0^\infty e^{-x\sigma^\nu}x^\delta dx\right]\frac{d\sigma}{\sigma^{1-\nu}} \\ &= \frac{\Gamma(\delta+1)}{2\pi i}\int_{Ha}\frac{e^\sigma}{\sigma^{\nu\delta+1}}d\sigma = \frac{\Gamma(\delta+1)}{\Gamma(\nu\delta+1)}.\end{aligned}$$

Above we have legitimate the exchange between the two integrals and we have used the identity

$$\int_0^\infty e^{-x\sigma^\nu}x^\delta dx = \frac{\Gamma(\delta+1)}{(\sigma^\nu)^{\delta+1}},$$

In particular, for $\delta = n \in \mathbb{N}$, the above formula provides the moments of integer order that can also be computed from the Laplace transform pair as follows:

$$\int_0^{+\infty} x^n M_\nu(x)dx = \lim_{s\to 0}(-1)^n\frac{d^n}{ds^n}E_\nu(-s) = \frac{\Gamma(n+1)}{\Gamma(\nu n+1)}.$$

Incidentally, we note that the Laplace transform pair (2.91) could be obtained using the fundamental result (2.97) by developing in power series the exponential kernel of the Laplace transform and then transforming the series term-by-term.

As well-known in probability theory, the Fourier transform of a density provides the so-called *characteristic function*. In our case we have:

$$\begin{aligned}\mathcal{F}[\frac{1}{2}M_\nu(|x|)] :=& \frac{1}{2}\int_{-\infty}^{+\infty} e^{i\kappa x} M_\nu(|x|)dx \\ =& \int_0^\infty cos(\kappa x) M_{M_\nu(x)} dx = E_{2\nu}(-\kappa^2).\end{aligned} \tag{2.98}$$

For this prove it is sufficient to develop in series the cosine function and use formula (2.97),

$$\begin{aligned}\int_0^\infty cos(\kappa x) M_\nu(x)dx &= \sum_{n=0}^{\infty} (-1)^n \frac{\kappa^{2n}}{(2n)!}\int_0^\infty x^{2n} M_\nu(x)dx \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{\kappa^{2n}}{\Gamma(2\nu n+1)} = E_{2\nu}(-\kappa^2).\end{aligned}$$

2.3.6 The Wright $\mathbb{M}-$function in two variables

In view of time-fractional diffusion processes related to time-fractional diffusion equations, it is worthwhile to introduce the function in two variables

$$\mathbb{M}_\nu(x,t) := t^{-\nu} M_\nu(xt^{-\nu}), \quad 0 < \nu < 1, \quad x, t \in \mathbb{R}^+, \tag{2.99}$$

which defines a spatial probability density in x evolving in time t. Of course, for $x \in \mathbb{R}$ we have to consider the symmetric version obtained from (2.99) multiplying by 1/2 and replacing x by $|x|$.
Hereafter we provide a list of the main properties of this function, which can be derived from the Laplace and Fourier transforms for the corresponding

Wright $M-$function in one variable.
From Eq. (2.96) we derive the Laplace transform of $\mathbb{M}_\nu(x,t)$ with respect to $t \in \mathbb{R}^+$,

$$\mathcal{L}\{\mathbb{M}_\nu(x,t);\ t \to s\} = s^{\nu-1}e^{-xs^\nu}. \tag{2.100}$$

From Eq. (2.90) we derive the Laplace transform of $\mathbb{M}_\nu(x,t)$ with respect to $x \in \mathbb{R}^+$

$$\mathcal{L}\{\mathbb{M}_\nu(x,t);\ x \to s\} = E_\nu(-st^\nu). \tag{2.101}$$

From Eq. (2.98) we derive the Fourier transform of $\mathbb{M}_\nu(|x|,t)$ with respect to $x \in \mathbb{R}$,

$$\mathcal{F}\{\mathbb{M}_\nu(|x|,t);\ x \to \kappa\} = 2E_{2\nu}(-\kappa^2 t^\nu). \tag{2.102}$$

Using the Mellin transforms, Mainardi et al. in 2003, derived the following integral formula,

$$\mathbb{M}_\nu(x,t) = \int_0^\infty \mathbb{M}_\lambda(x,\tau)\mathbb{M}_\mu(\tau,t)d\tau, \quad \nu = \lambda\mu. \tag{2.103}$$

Special cases of the Wright $\mathbb{M}$-function are simply derived for $\nu = 1/2$ and $\nu = 1/3$ from the corresponding ones in the complex domain, see Eqs. (2.75-76). We devote particular attention to the case $\nu = 1/2$ for which we get from (2.75) Gaussian density in $\mathbb{R}$,

$$\mathbb{M}_{1/2}(|x|,t) = \frac{1}{2\sqrt{\pi}t^{1/2}}e^{-x^2/(4t)}. \tag{2.104}$$

For the limiting case $\nu = 1$ we obtain

$$\mathbb{M}_1(|x|,t) = \frac{1}{2}[\delta(x-t) + \delta(x+t)]. \tag{2.105}$$

2.4 The Four Sisters Functions

In subsection 1.2.5 we introduced the *3 sisters functions*, which are defined through their Laplace Transforms.
What we can note is that $\tilde{\phi}(x,s)$, $\tilde{\psi}(x,s)$ and $\tilde{\chi}(x,s)$ are particular cases of the function

$$\xi_{\mu,\nu}(x,s) = s^{-\mu}e^{-xs^{\nu}} \tag{2.106}$$

In fact, when $\nu = 1/2$, we obtain

$$\xi_{\mu,1/2}(x,s) = s^{-\mu}e^{-x\sqrt{s}} \tag{2.107}$$

Follows that:

- $\tilde{\phi}(x,s)$ is the Eq. (2.104) with $\mu = 1$
- $\tilde{\psi}(x,s)$ is the Eq. (2.104) with $\mu = 0$
- $\tilde{\chi}(x,s)$ is the Eq. (2.104) with $\mu = \nu = 1-\nu$

So, when ν is different than 1/2, we obtain the *four sisters functions*, because now $\nu \neq 1-\nu$.
From the theory of Laplace Transforms, we know that

$$\mathcal{L}^{-1}\{s^{-\mu}e^{-xs^{\nu}}\} = t^{\mu-1}W_{-\nu,\mu}\big(-xt^{-\nu}\big) \tag{2.108}$$

and so:

$$t^{\mu-1}W_{-\nu,\mu}\big(-xt^{-\nu}\big) \div s^{-\mu}e^{-xs^{\nu}} \tag{2.109}$$

Putting $z = -xt^{\nu}$ we have:

$$W_{-\nu,\mu}(-xt^{-\nu}) = \sum_{n=0}^{\infty} \frac{(-xt^{-\nu})^n}{n!\Gamma(-\nu n+\mu)}, \quad x,t \geq 0,\ 0 < \nu < 1,\ \mu > 0 \tag{2.110}$$

The *four sisters functions* are so given by

$$
\begin{aligned}
\mu = 0 \quad & e^{-xs^{\nu}} \div t^{-1} W_{-\nu,0}(-xt^{-\nu}) \\
\mu = 1-\nu \quad & \frac{e^{-xs^{\nu}}}{s^{1-\nu}} \div t^{-\nu} W_{-\nu,1-\nu}(-xt^{-\nu}) \\
\mu = \nu \quad & \frac{e^{-xs^{\nu}}}{s^{\nu}} \div t^{\nu-1} W_{-\nu,\nu}(-xt^{-\nu}) \\
\mu = 1 \quad & \frac{e^{-xs^{\nu}}}{s} \div W_{-\nu,1}(-xt^{-\nu})
\end{aligned}
\qquad (2.111)
$$

Here we show some plots of the four sister functions, both in the t and x domain, and both in linear and logarithmic scale, for some values of ν, i.e. $\nu = 1/4, 1/2, 3/4$. We will se that for $\nu = 1/2$ we will find back the three sisters functions.

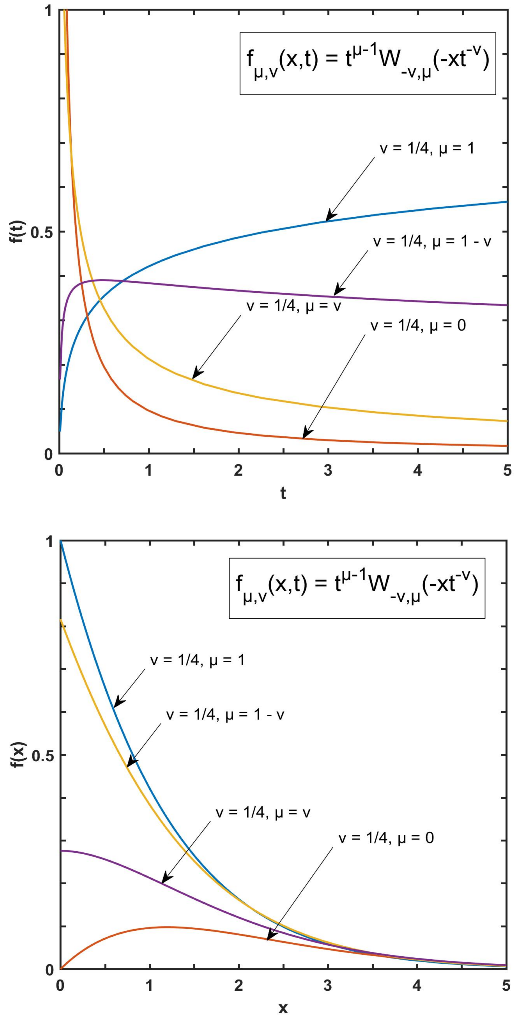


Figure 2.8: Plots of the four sisters functions in linear scale with $\nu = 1/4$; top: versus t (with $x = 1$), bottom: versus x (with $t = 1$)

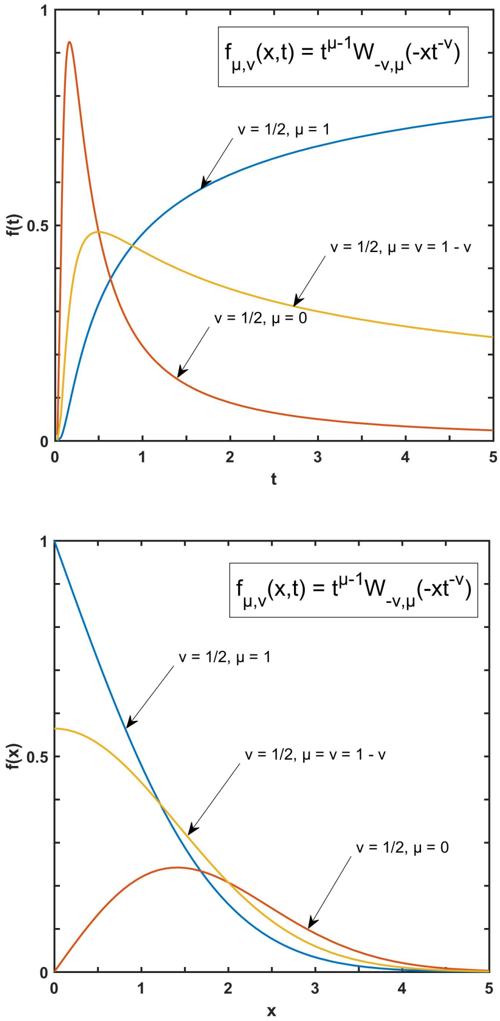


Figure 2.9: Plots of the four sisters functions in linear scale with $\nu = 1/2$; top: versus t (with $x = 1$), bottom: versus x (with $t = 1$)

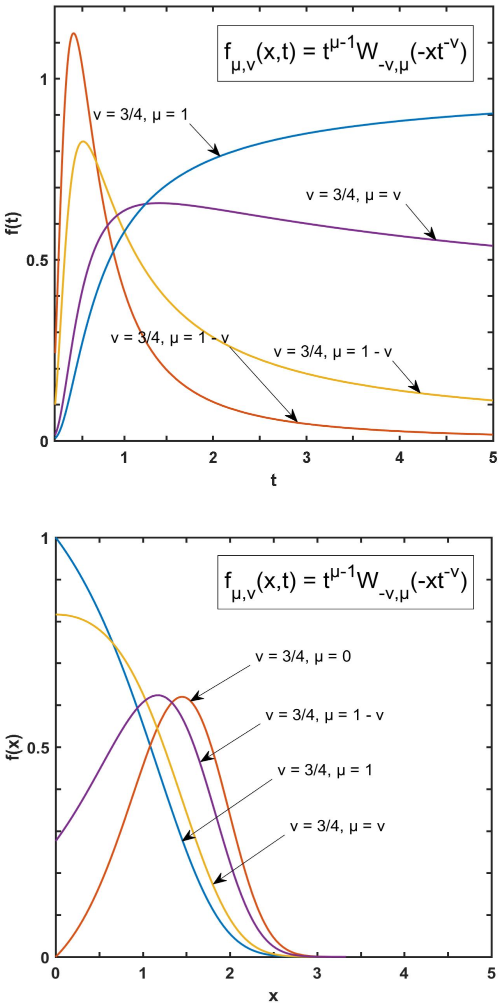


Figure 2.10: Plots of the four sisters functions in linear scale with $\nu = 3/4$; top: versus t (with $x = 1$), bottom: versus x (with $t = 1$)

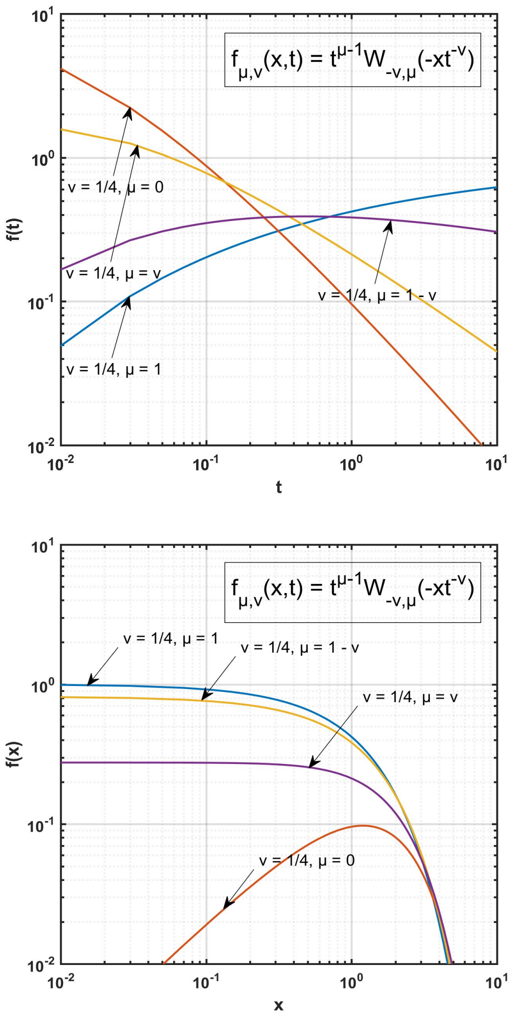


Figure 2.11: Plots of the four sisters functions in log-log scale with $\nu = 1/4$; top: versus t (with $x = 1$), bottom: versus x (with $t = 1$)

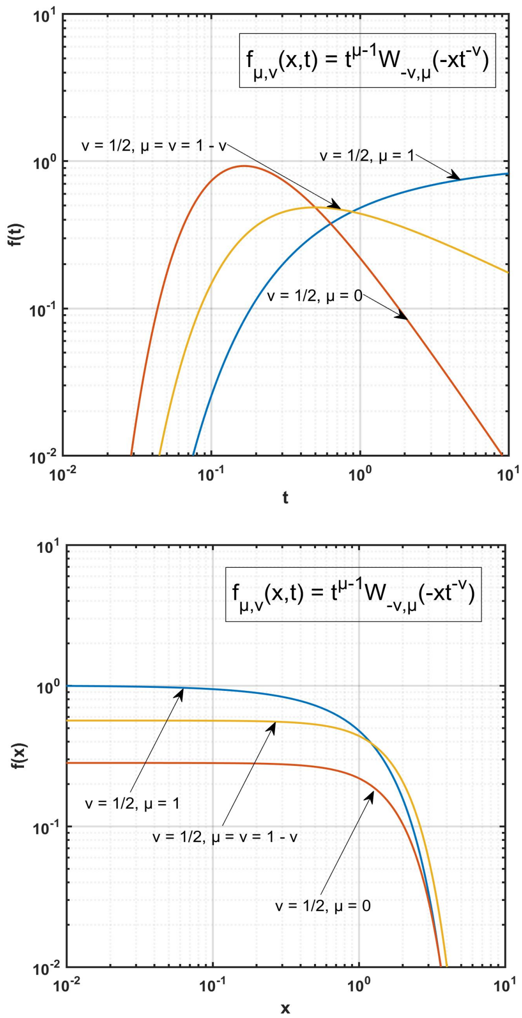


Figure 2.12: Plots of the four sisters functions in log-log scale with $\nu = 1/2$; top: versus t (with $x = 1$), bottom: versus x (with $t = 1$)

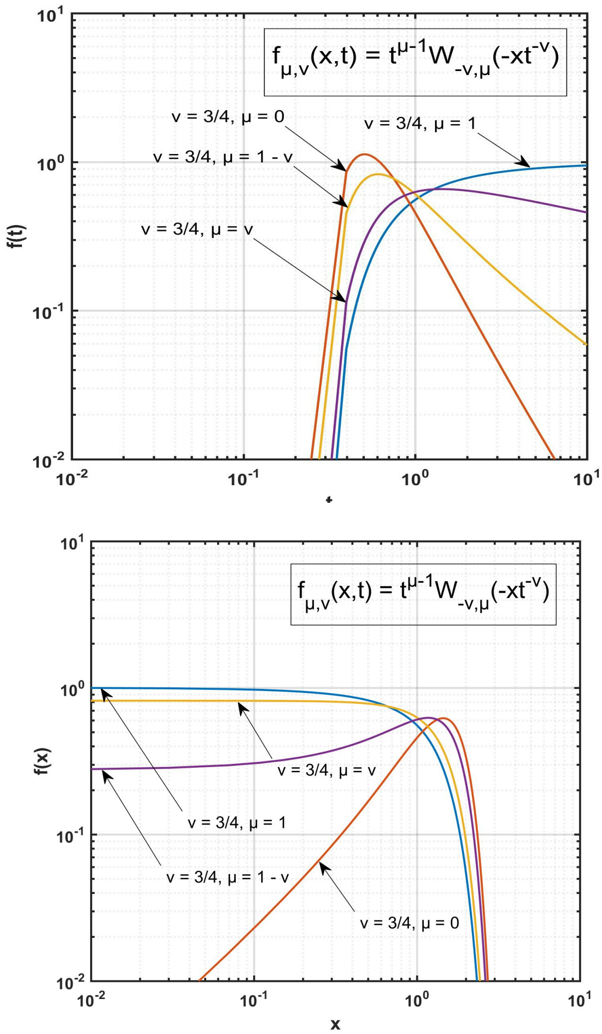


Figure 2.13: Plots of the four sisters functions in log-log scale with $\nu = 3/4$; top: versus t (with $x = 1$), bottom: versus x (with $t = 1$)

Chapter 3

Fractional Calculus in Diffusion-Wave Problems

Fractional calculus is a natural generalization of calculus, and in many applications amount to replacing the time derivative in an evolution equation with a derivative of fractional order.
The time-fractional diffusion equation is obtained from the standard diffusion equation (or the D'Alembert wave equation) by replacing the first-order (or the second-order) time derivative by a fractional derivative (of order $0 < \beta \leq 2$, in Riemann-Liouville or Caputo sense) (see [16], [19])

$$\boxed{\frac{\partial^\beta w}{\partial t^\beta} = a \frac{\partial^2 w}{\partial x^2}, \quad a > 0, \quad 0 < \beta \leq 2} \tag{3.1}$$

It is well known that diffusion and wave equations behave quite differently regarding their response to a localized disturbance: whereas the diffusion equation describes a process where a disturbance spreads infinitely fast, the propagation speed of the disturbance is constant for the wave equation. In a certain sense, the time-fractional diffusion-wave equation interpolates between these two different responses.

3.1 Basic definitions of Fractional Calculus

In this section we follow the approach of the textbook of R. Hilfer [5], which provides a methodical treatment of the theory.
From a historical point of view, the first serious attempt to give a logical definition of a fractional derivative is due to Liouville: he started in 1832 with the well known result $D^n e^{ax} = a^n e^{ax}$, $n \in \mathbb{N}$, and extended it at first in the particular case $\nu = 1/2, a = 2$, and then to arbitrary order $\nu \in \mathbb{R}^+$ by

$$D^\nu e^{ax} = a^\nu e^{ax}. \tag{3.2}$$

He assumed the series representation for $f(x)$ as $f(x) = \sum_{k=0}^{\infty} c_k a_k^\nu e^{a_k x}$ and derived the derivative of arbitrary order ν by

$$D^\nu f(x) = \sum_{k=0}^{\infty} c_k a_k^\nu e^{a_k x}. \tag{3.3}$$

Whereas this was Liouville's first approach, his second method was applied to the explicit function x^{-a}. He considered the integral $\int_0^\infty u^{a-1} e^{-xu} du$. Substituting $xu = t$ gives the result $I = x^{-a} \int_0^\infty t^{a-1} e^{-t} dt = x^{-a}\Gamma(a)$ (for $Re(a) > 0$). Operating on both sides of $x^{-a} = I/\Gamma(a)$ with D^ν, he obtained, using $D^\nu(e^{-xu}) = (-1)^\nu u^\nu e^{-xu}$,

$$D^\nu x^{-a} = (-1)^\nu \frac{\Gamma(a+\nu)}{\Gamma(a)} x^{-a-\nu}. \tag{3.4}$$

Liouville used the latter in his investigations of potential theory.
Since the ordinary differential equation $d^n y/dx^n = 0$ of n-th order has the complementary (general) solution $y = c_0 + c_1 x + ... + c_{n-1} x^{n-1}$, Liouville considered that the fractional order equation $d^\alpha y/dx^\alpha = 0,\ \alpha \in \mathbb{R}^+$, should have a suitable corresponding complementary solution, too. In this respect Riemann added a $\psi(x)$ to (3.6) below as the complementary function (shown to be of indeterminate nature by Cayley in 1880).

Not like classical Newtonian derivatives, a fractional derivative is defined via a fractional integral.
One development for the discussion begins with a generalization of repeated integration. Thus, if f is locally integrable on (a, ∞), then the n-fold iterated integral is given by

$$\begin{aligned} {}_aI_x^n f(x) :=& \int_a^x du_1 \int_a^{u_1} du_2 ... \int_a^{u_{n-1}} f(u_n) du_n \\ =& \frac{1}{(n-1)!} \int_a^x (x-u)^{n-1} f(u) du \end{aligned} \tag{3.5}$$

for almost all x with $-\infty \leq a < x < \infty$ and $n \in \mathbb{N}$. Writing $(n-1)! = \Gamma(n)$, an immediate generalization is the integral of f of fractional order $\alpha > 0$,

$${}_aI_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-u)^{\alpha-1} f(u) du \tag{3.6}$$

and similarly for $-\infty < x < b \leq \infty$

$${}_xI_b^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (u-x)^{\alpha-1} f(u) du \tag{3.7}$$

both being defined for suitable f. The subscripts in I denote the terminals of integration (in given order).
We can observe that (3.6) for $\alpha = n$ can be shown to be the unique solution of the initial value problem

$$y^{(n)}(x) = f(x), y(a) = y'(a) = ... = y^{(n-1)}(a) = 0. \tag{3.8}$$

When $a = -\infty$, equation (3.6) is equivalent to Liouville's definition, and when $a = 0$ we have Riemann's definition (without the complementary function). One so generally speaks of ${}_aI_x^\alpha f$ as the *Riemann-Liouville* fractional integral of order α of f.
Concerning existence of fractional integrals, let $f \in L^1_{loc}(a, \infty)$. Then, if

$a > -\infty$, ${}_aI_x^\alpha f(x)$ is finite almost everywhere on (a, ∞) and belongs to $L^1_{loc}(a, \infty)$. If $a = -\infty$, it is assumed that f behaves at $-\infty$ such that the integral (3.6) converges.
Under this assumptions, the fractional integrals satisfy the additive index law (or semigroup [1] property)

$$ {}_aI_x^\alpha \, {}_aI_x^\beta f = {}_aI_x^{\alpha+\beta} \quad (\alpha, \beta > 0). \tag{3.9} $$

Indeed, by the change of the order of integration, we have

$$ \begin{aligned} {}_aI_x^\alpha \, {}_aI_x^\beta f(x) =& \frac{1}{\Gamma(\alpha)} \int_a^x (x-u)^{\alpha-1} du \frac{1}{\Gamma(\beta)} \int_a^u (u-t)^{\beta-1} f(t) dt \\ =& \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x f(t) dt \int_t^x (x-u)^{\alpha-1} (u-t)^{\beta-1} du \end{aligned} \tag{3.10} $$

The second integral on the right equals, under the substitution $y = \frac{u-t}{x-t}$,

$$ \begin{aligned} (x-t)^{\alpha+\beta-1} \int_0^1 (1-y)^{\alpha-1} y^{\beta-1} dy =& B(\alpha, \beta)(x-t)^{\alpha+\beta-1} \\ =& \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} (x-t)^{\alpha+\beta-1}, \end{aligned} \tag{3.11} $$

$B(\alpha, \beta)$ being the Beta function.
When this is substituted into the above the result follows. In particular, we have

$$ {}_aI_x^{n+\alpha} f = {}_aI_x^n \, {}_aI_x^\alpha f \quad (n \in \mathbb{N}, \ \alpha > 0) \tag{3.12} $$

which implies by n-fold differentiation

$$ \frac{d^n}{dx^n} \, {}_aI_x^{n+\alpha} f(x) = {}_aI_x^\alpha f(x) \quad (n \in \mathbb{N}, \ \alpha > 0) \tag{3.13} $$

[1] A *semigroup* is a pair $(X, *)$ in which X is a non-empty set and $*$ is a binary associative operation on X; the equation $(x * y) * z = x * (y * z)$ holds for all $x, y, z \in X$.

for almost all x.
The above results also hold for complex parameters α, if the condition $a > 0$ is replaced by $Re(a) > 0$; then the operation ${}_aI_x^\alpha$ may be considered as a holomorphic function of α for $Re(\alpha) > 0$, which can be extended to the whole complex plane by analytic continuation, if f is sufficiently smooth.
To understand and establish this fact we assume for convenience that f is an infinitely differentiable function defined on $\mathbb{R}$ with compact support contained in $[a, \infty)$ if $a > -\infty$, implying that $f^{(n)}(a) = 0$ for $n = 0, 1, 2, ...$
Then for any fixed $x > a$ the integral (3.6) is a holomorphic function of α for $Re(\alpha) > 0$. Now, integration by parts n-times yelds

$$ {}_aI_x^\alpha f(x) = {}_aI_x^{n+\alpha} f^{(n)}(x) \quad (Re(\alpha) > 0,\ n \in \mathbb{N}). \tag{3.14}$$

Applying the semigroup property (3.12) to the expression on the right in (3.14) and differentiating the result n-times with respect to x, we obtain

$$\frac{d^n}{dx^n}\, {}_aI_x^\alpha f(x) = {}_aI_x^{n+\alpha} f^{(n)}(x) \quad (Re(\alpha) > 0,\ n \in \mathbb{N}) \tag{3.15}$$

showing that under the hypotheses assumed on f the operations of integration of fractional order α and differentiation of integral order n commute.
Returning to formula (3.14), we now realize that its right-hand side is a holomorphic function of α in the wider domain $\{\alpha \in \mathbb{C};\ Re(\alpha) > -n\}$ and even equals there $\frac{d^n}{dx^n}\, {}_aI_x^{n+\alpha} f(x)$, by (3.15). Thus we can extend ${}_aI_x^\alpha f(x)$ to the domain $\{\alpha \in \mathbb{C};\ Re(\alpha) \le 0\}$ analytically, defining for $\alpha \in \mathbb{C}$ with $Re(\alpha) \le 0$)

$$ {}_aI_x^\alpha f(x) := {}_aI_x^{n+\alpha} f^{(n)}(x) = \frac{d^n}{dx^n}\, {}_aI_x^{n+\alpha} f(x) \tag{3.16}$$

with any integer $n > -Re(\alpha)$. In particular, we obtain

$$ {}_aI_x^0 f(x) = f(x),\ {}_aI_x^{-n} f(x) = f^{(n)}(x) \quad (n \in \mathbb{N}). \tag{3.17}$$

The expressions occurring in formula (3.16) are meaningful for much more general classes of functions, and thus give rise to the following definitions of

fractional derivatives which go back to Liouville.
Let α be a complex number with $Re(\alpha) > 0$ and $n = [Re(\alpha)] + 1$, where $[Re(\alpha)]$ denotes the integral part of $Re(\alpha)$. Then the right-handed fractional derivative of order α is defined by

$$_aD_x^0 f(x) = \frac{d^n}{dx^n}\, {}_aI_x^{n-\alpha} f(x) \quad (n = [Re(\alpha)] + 1) \tag{3.18}$$

for any $f \in L^1_{loc}(a, \infty)$ for which the expression on the right exists.

3.1.1 The Time Fractional Derivatives

For the aim of this work, we will consider the *Caputo derivative*.
For a sufficiently well-behaved function $f(t)$ $(t \in \mathbb{R}^+)$ and for any positive number μ this one, togheter with the *Riemann-Liouville derivative*, are so defined through the Riemann-Liouville fractional integral of ordere $\mu > 0$

$$\boxed{J_t^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau) d\tau.} \tag{3.19}$$

The fractional derivative of order $\mu > 0$ in the *Riemann-Liouville* sense is defined as the operator D_t^μ which is the left inverse of the Riemann-Liouville integral of order μ (in analogy with the ordinary derivative), that is

$$D_t^\mu J_t^\mu = I, \ \ \mu > 0. \tag{3.20}$$

If m denotes the positive integer such that $m - 1 < \mu \leq m$, we obtain

$$D_t^\mu f(t) := D_t^m J_t^{m-\mu} f(t), \tag{3.21}$$

hence for $m - 1 < \mu < m$,

$$\boxed{D_t^\mu f(t) = \frac{d^m}{dt^m} \left[\frac{1}{\Gamma(m-\mu)} \int_0^t \frac{f(\tau) d\tau}{(t-\tau)^{\mu+1-m}} \right],} \tag{3.22}$$

and for $\mu = m$

$$D_t^{\mu} f(t) = \frac{d^m}{dt^m} f(t). \tag{3.23}$$

For completion we define $D_t^0 = I$.
On the other hand, the fractional derivative of order $\mu > 0$ in the *Caputo sense* is defined as the operator ${}_*D_t^{\mu}$ such that

$${}_*D_t^{\mu} f(t) := {}_tJ^{m-\mu} \, {}_tD^m f(t) \tag{3.24}$$

hence for $m-1 < \mu < m$,

$${}_*D_t^{\mu} f(t) = \frac{1}{\Gamma(m-\mu)} \int_0^t \frac{f^{(m)}(\tau)d\tau}{(t-\tau)^{\mu+1-m}}, \tag{3.25}$$

and for $\mu = m$

$${}_*D_t^{\mu} f(t) = \frac{d^m}{dt^m} f(t) \tag{3.26}$$

This definition requires for non-integer μ the absolute integrability of the derivative of order m. Whenever we use the operator ${}_*D_t^{\mu}$ we (tacitly) assume that this condition is met. We easily recognize that in general the two fractional derivative differ for non integer orders unless the function $f(t)$ along with its first $m-1$ derivatives vanishes at $t = 0^+$. In fact, assuming that the passage of the $m-$derivative under the integral is legitimate, we have

$$\boxed{D_t^{\mu} f(t) = {}_*D_t^{\mu} f(t) + \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^{k-\mu}}{\Gamma(k-\mu+1)},} \tag{3.27}$$

and therefore, recalling the fractional derivative of the power functions

$$\boxed{{}_*D_t^{\mu} f(t) = D_t^{\mu} \left(f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^k}{k!} \right).} \tag{3.28}$$

From (3.28) we recognize that the Caputo fractional derivative represents a sort of regularization in the time origin for the Riemann-Liouville fractional derivative. We also note that for its existence all the limiting values $f^{(k)(0^+)} := lim_{t\to 0^+} D_t^k f(t)$ are required to be finite for $k = 0, 1...m-1$.
In the special case $f^{(k)}(0^+) = 0$ for $k = 0, 1, ...m-1$, we recover the identity between the two fractional derivatives.
We now explore the most relevant differences between the two fractional derivatives. We first observe their different behaviour at the end points of the interval $(m-1, m)$, namely when the order is any positive integer. For $\mu \to m^-$ both derivatives reduce to D_t^m, due to the fact that the operator $J_t^0 = I$ commutes with D_t^m. However, for $\mu \to (m-1)^+$ we have:

$$\begin{cases} D_t^\mu f(t) \to D_t^m J_t^1 f(t) = D_t^{m-1} f(t) = f^{(m-1)}(t), \\ {}_*D_t^\mu f(t) \to J_t^1 D_t^m f(t) - f^{(m-1)}(0^+). \end{cases} \tag{3.29}$$

As a consequence, roughly speaking, we can say that D_t^μ is, with respect to its order μ, an operator continuous at any positive integer, whereas ${}_*D_t^\mu$ is an operator only left-continuous.
We point out the major utility of the Caputo fractional derivative in treating initial-value problems for physical and engineering applications where initial conditions are usually expressed in terms of integer-order derivatives. This can be easily seen using the Laplace transformation. For the Caputo derivative of order μ, with $m-1 < \mu \le m$ we have

$$\begin{gathered} \mathcal{L}\{{}_*D_t^\mu f(t); s\} = s^\mu \tilde{f}(s) - \sum_{k=0}^{m-1} s^{\mu-1-k} f^{(k)}(0^+), \\ f^{(k)}(0^+) := \lim_{t\to 0^+} D_t^k f(t). \end{gathered} \tag{3.30}$$

The corresponding rule for the Riemann-Liouville derivative of order μ is

$$
\begin{aligned}
\mathcal{L}\{D_t^{\mu} f(t); s\} &= s^{\mu} \tilde{f}(s) - \sum_{k=0}^{m-1} s^{m-1-k} g^{(k)}(0^+), \\
g^{(k)}(0^+) &:= \lim_{t \to 0^+} D_t^k g(t), \; g(t) := J_t^{m-\mu} f(t).
\end{aligned}
\tag{3.31}
$$

Thus the rule (3.31) is more cumbersome to be used than (3.30) since it requires initial values concerning an extra function $g(t)$ related to the given $f(t)$, however, when all the limiting values $f^{(k)}(0^+)$ are finite and the order is not integer, we can prove by that all $g^{(k)}(0^+)$ vanish so that the formula (3.31) simplifies into

$$
\mathcal{L}\{D_t^{\mu} f(t); s\} = s^{\mu} \tilde{f}(s), \; m-1 < \mu < m. \tag{3.32}
$$

In the special case $f^{(k)}(0^+) = 0$ for $k = 0, 1, ...m-1$, e recover the identity between the two fractional derivatives.
We note that the Laplace transform rule (3.30) was the starting point of Caputo, see Caputo (1967), Caputo (1969), for defining his generalized derivative in the late sixties.

3.2 The Time Fractional Diffusion-Wave Equation

Here we finally provide the fundamental solutions of the basic *Cauchy* and *Signalling* problems for the following equation

$$
\boxed{\frac{\partial^{2\beta} u}{\partial t^{2\beta}} = D \frac{\partial^2 u}{\partial x^2}, \quad 0 < \beta \leq 1, \quad D > 0}, \tag{3.33}
$$

where D is a positive constant with the dimensions $L^2 T^{-2\beta}$, and $u = u(x, t, \beta)$ is the field variable, which is assumed to be a *causal function* of time. The time derivative of (real) order $\alpha = 2\beta$ denotes a generalized derivative defined in the framework of the so-called *Riemann - Liouville fractional* *calculus.*

Adopting the definition of Caputo (1969):

$$\boxed{\frac{d^\alpha}{dt^\alpha} f(t) := \begin{cases} f^{(n)}(t) & if \quad \alpha = n \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau & if \quad n-1 < \alpha < n \end{cases}} \tag{3.34}$$

where $f^{(n)}(t)$ denotes the ordinary derivative of order n and Γ the Gamma function. We are going to apply the technique of the Laplace transform according to which

$$\boxed{\mathcal{L}\left\{ \frac{d^\alpha}{dt^\alpha} f(t) \right\} = s^\alpha \tilde{f}(s) - \sum_{k=0}^{n-1} s^{\alpha-1-k} f^{(k)}(0^+), \quad n-1 < \alpha \leq n.} \tag{3.35}$$

In the problems which we consider here, it is $n = 1$ when $0 < \beta \leq 1/2$ and $n = 2$ when $1/2 < \beta \leq 1$; consequently, Eq. (3.33) will be considered as a *time fractional diffusion equation* in the former case, while as a *time fractional wave equation* in the latter. In the general case $0 < \beta \leq 1$ we agree to refer to Eq.(3.33) as the *time fractional diffusion-wave equation (TFDWE).*

The basic problems for the TFDWE can be formulated basing on the corresponding problems for the ordinary diffusion and wave equations. Denoting by $f(x)$ and $g(t)$ two given functions, we define

a) *Cauchy problem*

$$\begin{cases} u(x, 0^+; \beta) = f(x), \quad -\infty < x < +\infty; \\ u(\pm\infty, t; \beta) = 0, \quad t > 0. \end{cases} \tag{3.36}$$

b) *Signalling problem*

$$\begin{cases} u(x,0^+;\beta) = 0, \quad 0 < x < +\infty; \\ u(0^+,t;\beta) = g(t), \quad u(+\infty,t;\beta) = 0 \quad t > 0. \end{cases} \tag{3.37}$$

If $1/2 < \beta \leq 1$, we must add in (3.36-37) the initial values of the first time derivative of the field variable, $u_t(x,0^+;\beta)$ since in this case (3.33) turns out to be akin to the wave equation and consequently two linearly independent solutions are to be determined. However, to ensure the continuous dependence of the solutions to our basic problems on the parameter β in the transition from $\beta = (1/2)^-$ to $\beta = (1/2)^+$, we agree to assume $u_t(x,0^+;\beta) = 0$.

For both problems we shall determine the corresponding Green functions by the LT technique extending the results obtained previously for the ordinary diffusion equation.
For the Cauchy problem the Eq. (1.45) turns out to be generalized into

$$s^{2\beta}\tilde{\mathcal{G}}_c - D\frac{d^2\tilde{\mathcal{G}}_c}{dx^2} = \delta(x)s^{2\beta-1}, \quad -\infty < x < +\infty, \tag{3.38}$$

and imposing the necessary boundary conditions we obtain

$$\boxed{\tilde{\mathcal{G}}_c(x,s;\beta) = \frac{1}{2\sqrt{D}s^{1-\beta}}e^{-(|x|/\sqrt{D})s^\beta}}. \tag{3.39}$$

For the *Signalling problem* the corresponding Eq.(1.54) turns out to be generalized into

$$s^{2\beta}\tilde{\mathcal{G}}_s - D\frac{d^2\tilde{\mathcal{G}}_s}{dx^2} = 0, \quad 0 \leq x < +\infty \tag{3.40}$$

and also we have

$$\boxed{\tilde{\mathcal{G}}_s(x,s;\beta) = e^{-(x/\sqrt{D})s^\beta}}, \tag{3.41}$$

We then got a generalization of the previous equations (1.50-57).
We easily recognize

$$\frac{d}{ds}\tilde{\mathcal{G}}_s = -2\beta x\tilde{\mathcal{G}}_c, \quad x > 0, \tag{3.42}$$

which implies for the original Green functions the following *reciprocity relation*

$$\boxed{x\mathcal{G}_c(x,t;\beta) = \frac{t}{2\beta}\mathcal{G}_s(x,t,;\beta), \quad x > 0} \tag{3.43}$$

which generalizes (1.29).

The limiting case $\beta = 1$ in (3.33) yields the ordinary wave equation for which it is well known that the corresponding Green functions are, for $x > 0$ and putting $c = \sqrt{D}$,

$$\mathcal{G}_c^w(x,t) = \frac{1}{2}\delta(x - ct) = \frac{1}{2c}\delta(t - x/c), \quad \mathcal{G}_s^w(x,t) = \delta(t - x/c). \tag{3.44}$$

For $0 < \beta < 1$ we can introduce the appropriate *similarity variable*

$$\boxed{z = \frac{|x|}{\sqrt{D}t^\beta}, \quad 0 < \beta < 1}, \tag{3.45}$$

which allows us to generalize Eq.(1.30) proving that, for $x > 0$, a certain function $M(z;\beta)$ exist such that

$$\boxed{x\mathcal{C}_c(x,t;\beta) = \frac{z}{2}M(z;\beta), \quad 0 < \beta < 1}. \tag{3.46}$$

The function $M(z;\beta)$, referred to as *the auxiliary function for the TFDWE (3.33)*, will be proved to be an entire function of z, related to the Wright function, which satisfies the condition

$$\boxed{\int_0^{+\infty} M(z;\beta)dz = 1}. \tag{3.47}$$

The interested reader can study in deep the argument in the appropriate sections of Chapter 2.

Here we plots the function $M(z;\beta)$ in $0 \leq z \leq 4$ for some rational values of $\beta \neq 1/2$, and we compare them with $M(z;1/2)$ (the Gaussian, in dashed lines).

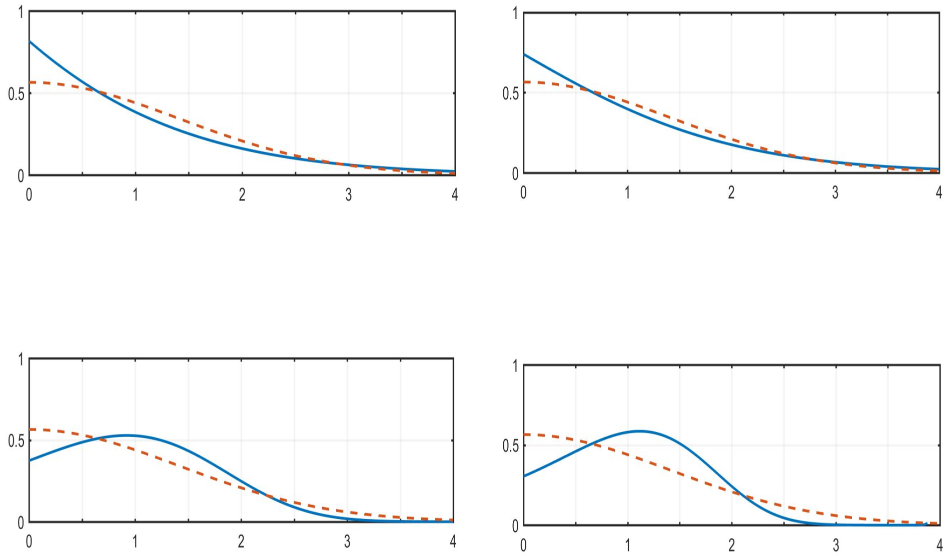


Figure 3.1: Comparison of $M(z;\beta)$ (continuous line) with $M(z;1/2)$ (dashed line) in $0 \leq z \leq 4$, for various values of β: (a) 1/4, (b) 1/3, (c) 2/3, (d) 3/4.

For the cases (a) $\beta = 1/4$ and (b) $\beta = 1/3$ we get some comparison between the fractional diffusion and the ordinary diffusion. We recognize the similarity of the behaviours except for small z; this is consistent with a *superslow process*.

For the cases (c) $\beta = 2/3$ and (d) $\beta = 3/4$ the comparison with the ordinary diffusion exhibits a dramatic difference in that a transition from the gaussian function, centred at $z = 0$, (pure diffusion) to the delta function, centred at $z = 1$, (pure wave-propagation) appears. This is consistent with an *intermediate process* between diffusion and wave- propagation.

In order to gain a better insight of the above phenomena, we find it worthwhile to illustrate the evolution of a simple initial signal provided by the box-function $f(x) = H(1 - |x|)$. Using non dimensional variables $x' = x/x_0, t' = t/t_0$ where $x_0 = \sqrt{D}t_0^\beta$, we exhibit plots of the solution versus x' $(0 \leq x' \leq 3)$, at fixed t' (t' = 0, 0.5, 1) for some fractional values of β. We have chosen the same values of β as before $(\beta = 1/4, 1/3, 1/2, 2/3, 3/4)$ adding $\beta = 1$, which corresponds the classical wave phenomenon.
In Fig.(3.2) we compare the cases concerning the fractional diffusion equation $(0 < \beta \leq 1/2)$, while in Fig.(3.3) we compare those concerning the fractional wave equation $(1/2 < \beta \leq 1)$. We easily recognize the "slow" character for the cases with $0 < \beta < 1/2$ and the "intermediate" character for those with $1/2 < \beta < 1$.

By performing relatively simple mathematical computations, we have been able to provide a reasonably self-contained treatment of the fundamental solutions for the evolution equation of fractional order, known as the *fractional diffusion-wave equation.* This equation is expected to govern generalized processes which interpolate or extrapolate the classical phenomenon of diffusion. The solutions have been shown to be expressed in terms of an auxiliary function of the similarity variable, and turn out to be essential in providing the evolution of any initial signal by a proper convolution. In particular, the transition from diffusion to wave can be outlined.

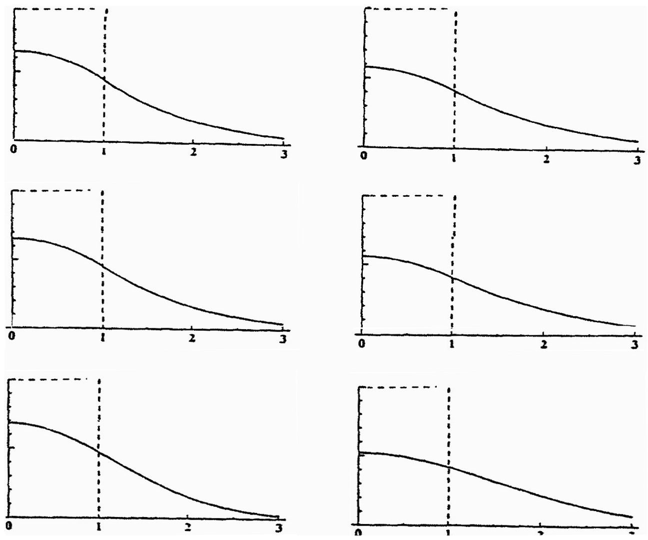


Figure 3.2: Evolution of the initial box-signal (dashed line) at $t' = 0.5$ (left) and $t' = 1.$ (right), versus x', for various values of β: from top to bottom $\beta = 1/4,\ 1/3,\ 1/2.$

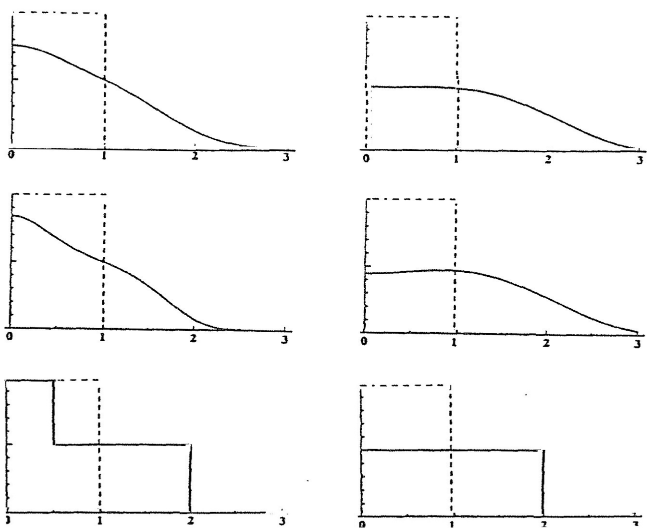


Figure 3.3: Evolution of the initial box-signal (dashed line) at $t' = 0.5$ (left) and $t' = 1.$ (right), versus x', for various values of β: from top to bottom $\beta = 2/3,\ 3/4,\ 1.$

Chapter 4

Time Fractional Cable Equation and Applications in Physics

4.1 Introduction

In this section, and in the others, we refer to the textbooks of Johnston and Wu [6], and to the textbooks of Magin [9] for mathematical details.
The nervous system plays a fundamental role in communication and process information. It is thanks to it that animals, including humans, perceive, learn, think and are aware of themselves and the outside world. The nervous system is formed by more basic structural units, that are individual neurons, which convey neural information by virtue of electrical and chemical signals; communication between neurons occurs through specialized contact zones, the synapses.
Synapses can be of two types, i.e. electrical (which provides a current flow by means of intercellular bridging pores) or chemical (which release chemical messengers or neurotransmitters).
Both types of synapses can be observed with the electron microscope.
The location of the electrical synapses is normally at the *gap junctions*, which

consist of specialized proteins that allow current flow from one neuron to the other, having established channels bridging the interiors of two neurons. A chemical synapse consists of synaptic vesicles in the presynaptic neuron and membrane thickening in the presynaptic and postsynaptic membranes (active zones).

The fundamental difference between neurons and most other cells in the body is that neurons can generate and transmit neural signals; this neural signals, of electrical or chemical type, are the messengers used by the nervous system for all its functions. So, what is important is to understand the principles and mechanisms of neural signals.

Despite the large variety and complexity of neuronal morphology and synaptic connectivity, the nervous system it's based on a number of basic principles of signaling for all neurons and synapses.

The electrical signals are carried primarily by transmembrane ion currents in neurons or other excitable cells, and result in changes in transmembrane voltage.

In the nervous system are involved four ion species in transmembrane currents: sodium (Na^+), potassium (K^+), calcium (Ca^{2+}), and chloride (Cl^-), with the first three carrying positive charges (cations) and the fourth carrying negative charges (anions); The movement of these ions is governed by physical laws.

The ionic concentration gradient across the membrane is the energy source of ion movement, which is maintained by ion pumps whose energy is derived from hydrolysis of ATP molecules. The electrochemical potential across the membrane is so set up by these concentration gradients, which drives ion flow in accordance with the laws of diffusion and drift (Ohm's law).

The energy sources and ion species involved in electrical signals are relatively simple, but the control mechanisms for the passage of ions across the membrane are quite complicated.

Thanks to the ion channels, that are aqueous pores formed by transmembrane protein molecules, the ions flow across the membrane.

Most neurons are not of regular shapes, but exhibit complex morphology. The cell body is excited differently from a signal generated at the tip of a dendrite rather than a signal generated at the cell body itself, because the dendritic signal must travel along the thin dendritic processes to the cell body. It is important, therefore, the incorporation of morphology and shape of the neuron into any analysis of neural signaling. Under conditions where the membrane is linear, electrical signals in one part of a neuron diffuse passively (down the electrochemical gradient) to other parts of the neuron. Principles dealing with such signal spreading are embodied in what is called *Linear Cable Theory*.

The cable model is widely used in several fields of science to describe the propagation of signals. A relevant medical and biological example is the anomalous subdiffusion in spiny neuronal dendrites observed in several studies of the last decade.

4.2 Standard Neuronal Cable Theory

The one dimensional cable model is treated to model the electrical conduction of non-isopotential excitable cells (see [25] [24]). In particular it describes the spatial and the temporal dependence of *trans-membrane potential* $V_m(x,t)$ along the axial x direction of a cylindrical nerve cell segment; it can be derived directly from the *Nernst-Planck* equation for electro-diffusive motion of ions, see Qian and Sejnowski [21].

Transmembrane potential is generated by ionic concentration gradient across the membrane, and is maintained non null at rest (no current) by a combination of passive and active cell mechanisms.

Biological membranes exhibit properties similar to those in electric circuits,and so it is customary in membrane physiology to describe the electrical behavior of biological membranes in terms of electric circuits. In this case, membrane behaviour is summarized by an electrical circuit with an axial internal resistance r_i , a transmembrane capacitance c_m and a transmembrane resistance

r_m in parallel, connecting the inner part to the outside.

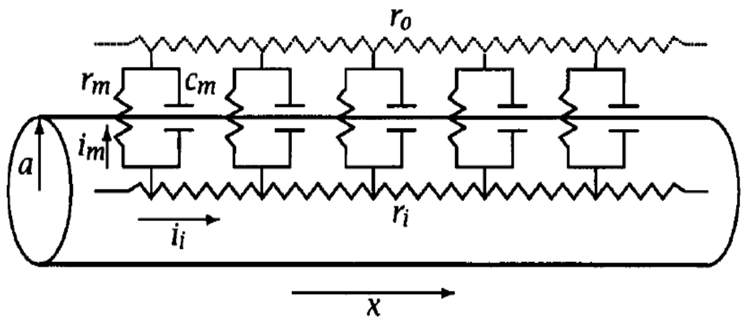


Figure 4.1: Cable Model for a nerve cell

The resulting differential equation for the trans-membrane potential takes the form of a standard diffusion equation with an extra a term to account leakage of ions out of the membrane, and it results in a decay of the electric signal in space and in time

$$\lambda^2 \frac{\partial^2 V_m(x,t)}{\partial x^2} - \kappa \frac{\partial V_m(x,t)}{\partial t} - V_m(x,t) = 0, \tag{4.1}$$

where λ and γ are space and time constants related to the membrane resistance and capacitance per unit length, i.e. $\lambda = \sqrt{r_m/r_i}$ and $\kappa = r_m c_m$. For simplicity, we will use the dimensionless scaled variables $X = x/\lambda$ and $T = t/\kappa$, so that we consider the equation

$$\frac{\partial^2 V_m(x,t)}{\partial X^2} - \frac{\partial V_m(x,t)}{\partial T} - V_m(x,t) = 0, \tag{4.2}$$

Some interesting quantities to neurophysiology are connected to the *Signal Problem* and the *Second Kind boundary condition problem.* Signal Problem is interesting to understand how the system evolves when excited at one end with a specific potential profile, instead the other one is interesting because can be related to the profile of a current injected across the membrane.

In signalling problems the cable is considered of semi-infinite length ($0 \leq X \leq \infty$), initially quiescent for $T < 0$ and excited for $T \geq 0$ at the accessible end ($X = 0$) with a given input in membrane potential $V_m(0,T) = g(t)$. The solution can be derived via the Laplace Transform (LT) approach:

$$\frac{\partial^2 V_m(X,T)}{\partial X^2} = (s+1)V_m(X,T), \tag{4.3}$$

and the LT of the solution results

$$\tilde{V}_m(X,s) = g(s)e^{-\sqrt{s+1}X}. \tag{4.4}$$

Relevant cases are impulsive input $g(t) = \delta(t)$ and unit step input $g(t) = H(t)$, where $\delta(t)$ and $H(t)$ denote the Dirac and the Heaviside functions, respectively. The solutions corresponding to these inputs can be obtained by LT inversion and read in our notation

$$\mathcal{G}_s(X,T) = \frac{X}{\sqrt{4\pi T^3}} e^{-\left(\frac{X^2}{4T}+T\right)} \tag{4.5}$$

and

$$\mathcal{H}_s(X,T) = \int_0^T \mathcal{G}_s(X,T')dT'. \tag{4.6}$$

We refer to $\mathcal{G}_s$ to as the fundamental solution or the Green function for the signalling problem of the (linear) cable equation, whereas to $\mathcal{H}_s$ as the step response. As known, the Green function is used in the time convolution integral to represent the solution corresponding to any given input $g(T)$ as follows

$$V_m(X,T) = \int_0^T g(T-T')\mathcal{G}_s(X,T')dT'. \tag{4.7}$$

The spatial variance associated to this model is known to evolve linearly in time.

If we consider an impulse or a step current injected at some point X the problem is subjected to the following boundary conditions, specifically

$$I = I_0\delta(T) = \frac{-1}{r_i\lambda}\frac{\partial V_m(X,T)}{\partial X}, \tag{4.8}$$

or

$$I = I_0 H(T) = \frac{-1}{r_i\lambda}\frac{\partial V_m(X,T)}{\partial X}. \tag{4.9}$$

We consider the adimensional current $I = I_0 r_i \lambda$ and put it to unity for convenience. Appling the impulse in $X = 0$ the LT reduces to

$$\tilde{V}_m(X,s) = \frac{1}{\sqrt{s+1}}e^{-\sqrt{s+1}X} \tag{4.10}$$

the Green function and the step response function (when a step current is applied in $X = 0$) reads, respectively,

$$\mathcal{G}_m(X,T) = \frac{1}{\sqrt{\pi T}}e^{-\left(\frac{X^2}{4T}+T\right)}, \tag{4.11}$$

and

$$\mathcal{H}_m(X,T) = \int_0^T \frac{1}{\sqrt{\pi T'}}e^{-\left(\frac{X^2}{4T'}+T'\right)}dT' \tag{4.12}$$

We emphasize that in this standard case the Green function $\mathcal{G}_m(X,T)$ is equal to the Green function for the Cauchy problem, name it $\mathcal{G}_c(X,T)$, for an infinite cable up to constant coefficients.
The motion of ions along the nerve cells is conditionated by this model, that predicts a mean square displacement of diffusing ions that scales linearly with time.
By the way significative deviations from linear behaviour have been measured by experiments, of which a relevant medical and biological example is the

anomalous subdiffusion in neuronal dendritic spines, which it is the subject of the next section.

4.3 Fractional Neuronal Cable Theory

As seen in [8], fractional calculus can provide a concise model for the description of the dynamic events that occur in biological tissues. Such a description is important for gaining an understanding of the underlying multiscale processes that occur when, for example, tissues are electrically stimulated or mechanically stressed.
The mathematics of fractional calculus has been applied successfully in physics, chemistry, and materials science to describe dielectrics, electrodes and viscoelastic materials.
Anomalous subdiffusion can be modelled introducing some fractional component into the classical cable model. The fractional cable model developed in this section is defined by replacing the first order time derivative in (4.2) with a fractional derivative of order $\alpha \in (0,1)$ of Caputo type

$$\frac{\partial^2 V_m(x,t)}{\partial X^2} - \frac{\partial^\alpha V_m(x,t)}{\partial T^\alpha} - V_m(x,t) = 0, \tag{4.13}$$

The solutions of the most relevant boundary problems (Signal Problem, Cauchy Problem, Second Kind Boundary Problem) are explicitly calculated in integral form containing Wright functions. Thanks to the variability of the parameter α, the corresponding solutions are expected to better describe the qualitative behaviour of the membrane potential observed in experiments respect to the standard case $\alpha = 1$. From a mathematical point of view this model is a simple extension to fractional behaviour of the Neuronal Cable Model.

The solution of the Signalling Problem can be derived via Laplace Transform, however the inversion of the LT solution for Eq.(4.13) requires special effort because of the term $V_m(x,t)$; when this term is not present, the resulting equation is the well known time fractional diffusion equation:

$$\frac{\partial^2 V_m^*(x,t)}{\partial X^2} - \frac{\partial^\alpha V_m^*(x,t)}{\partial T^\alpha} = 0, \tag{4.14}$$

Specifically for the signalling problem the general solution provided by Mainardi in integral convolution form reads

$$V_m^*(X,T) = \int_0^T g(T-T')\mathcal{G}_{\alpha,s}^*(X,T')dT', \quad \mathcal{G}_{\alpha,s}^*(X,T) = \frac{1}{T}W_{-\alpha/2,0}(-X/T^{\alpha/2}), \tag{4.15}$$

where $\mathcal{G}_{\alpha,s}^*$ denotes the Green function of the signalling problem of the fractional time diffusion equation (4.14) and $W_{-\alpha/2,0}(\cdot)$ is a particular case of the Wright function.

Then the Green function for the Signalling Problem of the time fractional diffusion equation (4.14)can be written as

$$\mathcal{G}_{\alpha,s}^*(X,T) = \frac{1}{T}F_{\alpha/2}(X/T^{\alpha/2}) = \frac{\alpha}{2}\frac{X}{T^{\alpha/2+1}}M_{\alpha/2}(X/T^{\alpha/2}) \tag{4.16}$$

Applying the Laplace transform to Eq. (4.13) with the boundary conditions required by the signalling problem, that is $V_m(X,0^+) = 0$, $V_m(0,T) = g(t)$, we have

$$(s^\alpha + 1)\tilde{V}_m(X,s) - \frac{\partial^2 \tilde{V}_m(X,s)}{\partial X^2} = 0, \tag{4.17}$$

which is a second order equation in the variable X with solution:

$$\tilde{V}_m(X,s) = \tilde{g}_(s)e^{-\sqrt{s^\alpha+1}X}. \tag{4.18}$$

Because of the shift constant in the square root of the Laplace transform in Eq.(4.18), the inversion is no longer straightforward with the Wright functions as it is in the time fractional diffusion equation (4.14).

Consequently, the difficulty is overcomed recurring to the application of the *Efros theorem*, that generalizes the well known convolution theorem for Laplace transforms.
We obtain then:

$$G(\tau, T) = \int_0^T \frac{g(T-T')}{T'} W_{-\alpha,0}(-\tau/T'^{\alpha}) dT' \tag{4.19}$$

.

The general solution for the signal problem can be written in terms of known functions:

$$\begin{aligned} V_m(X,T) &= \int_0^\infty \frac{X}{\sqrt{4\pi\tau^3}} e^{-\left(\frac{X^2}{4\tau}+\tau\right)} \left[\int_0^\infty \frac{g(T-T')}{T'} W_{-\alpha,0}(-\tau/T'^{\alpha}) dT' \right] d\tau \\ &= \int_0^T g(T-T') \left[\int_0^\infty \frac{X}{\sqrt{4\pi\tau^3}} e^{-\left(\frac{X^2}{4\tau}+\tau\right)} \frac{1}{T'} F_\alpha\left(\frac{\tau}{T'^{\alpha}}\right) d\tau \right] dT'. \end{aligned} \tag{4.20}$$

Substituting $g(T) = \delta(T)$ in the general solution (4.20) we obtain the Green function for the fractional model (4.13):

$$\begin{aligned} V_m(X,T) := \mathcal{G}_{\alpha,s}(X,T) &= \int_0^\infty \mathcal{G}_s(X,\tau) \frac{1}{T} F_\alpha\left(\frac{\tau}{T^\alpha}\right) d\tau \\ &= \int_0^\infty \mathcal{G}_s(X,\tau) \mathcal{G}^*_{2\alpha,s}(\tau,T) d\tau \end{aligned} \tag{4.21}$$

When $g(T) = H(T)$ we obtain the step response of our fractional cable equation:

$$\begin{aligned} V_m(X,T) := \mathcal{H}_{\alpha,s}(X,T) &= \int_0^\infty \mathcal{G}_s(X,\tau) \left[\int_0^T \mathcal{G}^*_{2\alpha,s}(\tau,T') dT' \right] d\tau \\ &= \int_0^\infty \mathcal{G}_s(X,\tau) \mathcal{H}^*_{2\alpha,s}(\tau,T) d\tau \end{aligned} \tag{4.22}$$

After some manipulations including the change of variable $z = \tau/T'^{\alpha}$ and integrating by parts after using the recurrence relation of Wright functions:

$$\frac{dW_{\lambda,\mu}(z)}{dz} = W_{\lambda,\lambda+\mu}(z)$$

and the relation between the auxiliary functions:

$$F_\nu(z) = \nu z M_\nu(z)$$

we may rewrite the step-response solution as:

$$\begin{aligned} V_m(X,T) := \mathcal{H}_{\alpha,s}(X,T) &= \int_0^\infty \mathcal{H}_s(X,\tau)\frac{1}{T^\alpha}M_\alpha\big(\frac{\tau}{T^\alpha}\big)d\tau \\ &= \int_0^\infty \mathcal{H}_s(X,\tau)\mathcal{G}^*_{2\alpha,c}(\tau,T)d\tau \end{aligned} \tag{4.23}$$

where $\mathcal{H}_{\alpha,s}(X,T)$ is the step response function for the standard cable model and $\mathcal{G}^*_{2\alpha,c}(\tau,T)$ is the fundamental solution of the time fractional diffusion equation for the Cauchy Problem; the same expression can easier be derived by direct application of the Efros theorem.

For the *Cauchy problem*, we consider an infinite cabe with boundary conditions $V_m(\pm\infty,T)=0$ and initial condition $V_m(X,0)=f(X)$. The general solution of the Cauchy problem is related to the Green function $\mathcal{G}_{\alpha,c}(X,T)$ through the following relation:

$$V_m(X,T) = \int_{-\infty}^{+\infty} f(x-\xi)\mathcal{G}_{\alpha,c}(\xi,T)d\xi. \tag{4.24}$$

$\mathcal{G}_{\alpha,c}(X,T)$ can be derived via Laplace Transform:

$$(s^\alpha+1)\tilde{\mathcal{G}}_{\alpha,c}(X,s) - \frac{\partial^2\tilde{\mathcal{G}}_{\alpha,c}}{\partial X^2} = \delta(X)s^{\alpha-1}, \tag{4.25}$$

boundary conditions imposes:

$$\tilde{\mathcal{G}}_{\alpha,c}(X,s) = \begin{cases} c_1(s)e^{-X\sqrt{s^\alpha+1}}, \text{ if } & X>0 \\ c_2(s)e^{+X\sqrt{s^\alpha+1}}, \text{ if } & X<0 \end{cases} \tag{4.26}$$

Imposing $\tilde{\mathcal{G}}_{\alpha,c}(0^-, s) = \tilde{\mathcal{G}}_{\alpha,c}(0^+, s)$ leads to $c_1(s) = c_2(s)$. Integrating Eq.(4.13) over X from 0^- to 0^+ we have:

$$\frac{\partial \tilde{\mathcal{G}}_{\alpha,c}(0^+, s)}{\partial X} - \frac{\partial \tilde{\mathcal{G}}_{\alpha,c}(0^-, s)}{\partial X} = -s^{\alpha-1} \tag{4.27}$$

The coefficients result:

$$c_1(s) = c_2(s) = \frac{1}{2s^{1-\alpha}\sqrt{s^\alpha + 1}} \tag{4.28}$$

the resulting LT of the Green function reads:

$$\tilde{\mathcal{G}}_{\alpha,c}(X, s) = \frac{1}{2s^{1-\alpha}\sqrt{s^\alpha + 1}} e^{-X\sqrt{s^\alpha+1}} \tag{4.29}$$

The inversion can be easily performed for $X > 0$, thanks again to the Efros theorem, and extended by symmetry respect to the X-axes for $X < 0$. The inverse LT for the Green function reads:

$$\begin{aligned} \mathcal{G}_{\alpha,c}(X, T) &= \int_0^\infty \frac{1}{\sqrt{4\pi\tau}} e^{-\left(\frac{X^2}{4\tau}+\tau\right)} \frac{1}{T^\alpha} M_\alpha(\tau/T^\alpha) d\tau \\ &= \int_0^\infty \mathcal{G}_c(X, \tau) \mathcal{G}^*_{2\alpha,c}(\tau, T) d\tau \end{aligned} \tag{4.30}$$

Another interesting biological problem is to consider an injected current in the sys- tem. Transmembrane potential is related to the transmembrane current through the relation $-I = \frac{\partial^2 V - m(X,T)}{\partial X^2}$, where the minus sign is due to the direction of the current, in this case flowing inside the cell. Let's consider a singular point injected current in $X = 0$, it takes the form $I(X, T) = I_0 \delta(X) f(T)$. Integrating from 0^- to 0^+ we obtain the relation

$$-I_0 f(T) = \frac{\partial V_m(X, T)}{\partial X}\Big|_{X=0^+} - \frac{\partial V_m(X, T)}{\partial X}\Big|_{X=0^-} \tag{4.31}$$

We recall the LT for the semi-infinite cable for an initially undisturbed cable:

$$\tilde{V}_m(X,s) = \tilde{V}_m(0,s)e^{-X\sqrt{s^\alpha+1}}. \tag{4.32}$$

At the boundary condition we have:

$$I_0\tilde{f}(s) = -\frac{\partial\tilde{V}_m(X,s)}{\partial X}\Big|_{X=0^+}, \tag{4.33}$$

if we consider an impulse injection of current in $X = 0$ we have $I_0\delta(T) = -\frac{\partial V_m(X,T)}{\partial X}\big|_{0^+}$. Applying this condition to the LT we obtain:

$$\tilde{V}_m(0^+,s) = \frac{I_0}{\sqrt{s^\alpha+1}} \tag{4.34}$$

leading to the following Laplace Transformed solution:

$$\tilde{\mathcal{G}}_{\alpha,m}(X,s) = \frac{I_0}{\sqrt{s^\alpha+1}}e^{-X\sqrt{s^\alpha+1}} \tag{4.35}$$

According to the previous derivations it is then straightforward that the inverse LT takes the form:

$$\begin{aligned}\mathcal{G}_{\alpha,m}(X,T) &= \int_0^\infty \frac{I_0}{\sqrt{\pi\tau}}e^{-\left(\frac{X^2}{4\tau}+\tau\right)}\frac{1}{T}W_{-\alpha,0}(-\tau/T^\alpha)d\tau \\ &= \int_0^\infty \mathcal{G}_m(X,\tau)\mathcal{G}^*_{2\alpha,s}(\tau,T)d\tau,\end{aligned} \tag{4.36}$$

For a generic boundary $I_0\tilde{f}(s)$ we obtain:

$$\tilde{V}_m(X,s) = \frac{I_0 f(s)}{\sqrt{s^\alpha+1}}e^{-X\sqrt{s^\alpha+1}} \tag{4.37}$$

The general solution becomes:

$$V_m(X,T) = \int_0^T f(T-T')\mathcal{G}_{\alpha,m}(X,T')dT' \tag{4.38}$$

The solution is symmetric respect to X, the problem can be then extended to the infinite cable introducing a factor 1/2: $\mathcal{G}^\infty_{\alpha,m}(X,T) = \frac{1}{2}\mathcal{G}_{\alpha,m}(X,T)$.

The extension to the infinite cable case admits also the following generalization, current injection in $X_0 \neq 0$ is equivalent to shift the cable of the same value X_0, then:

$$V^{\infty}_{X_0,m}(X,T) = \int_0^T f(T-T')\mathcal{G}^{\infty}_{\alpha,m}(X-X_0,T')dT' \tag{4.39}$$

When the injected current is a step function we obtain the following LT solution:

$$\tilde{\mathcal{H}}_{\alpha,m}(X,s) = \frac{I_0}{s\sqrt{s^\alpha+1}}e^{-X\sqrt{s^\alpha+1}} = \frac{I_0}{s^{1-\alpha}s^\alpha\sqrt{s^\alpha+1}}e^{+X\sqrt{s^\alpha+1}} \tag{4.40}$$

and then:

$$\begin{aligned}\mathcal{H}_{\alpha,m}(X,T) &= \int_0^\infty \mathcal{H}_m(X,\tau)\frac{1}{T^\alpha}M_\alpha(\tau/T^\alpha)d\tau \\ &= \int_0^\infty \mathcal{H}_m(X,\tau)\mathcal{G}^*_{2\alpha,c}(\tau,T)d\tau,\end{aligned} \tag{4.41}$$

4.4 Experimental Evidence of Anomalous Diffusion

In an experiment made by F. Santamaria, S. Wils, E. D. Schutter, and G. J. Augustine (see [22]), are combined local photolysis of caged compounds with fluorescence imaging, in order to visualize molecular diffusion within dendrites of cerebellar *Purkinje cells.*
What they saw is that the diffusion of a volume marker, fluorescein dextran, within spiny dendrites was remarkably slow in comparison to its diffusion in smooth dendrites.
Anomalous diffusion is yielded by a transient trapping of molecules within dendritic spines, as it is shown by computer simulations.
The non-linear behavior of spatial variance over time is also reflected in a

time-dependent reduction in D_{app}, that is the apparent diffusion coefficient, which continuously decreased over time and did not reach a stable value within 1s.

Purkinje cells (PCs) are a good system in which to study this question because their dendrites contain both sections without spines, as well as branches with high densities of spines; they found that a biologically inert compound, *fluorescein dextran* (FD), exhibits different diffusion properties in these two types of dendrites, with diffusion being slower and non-linear within spiny dendrites. The latter behaviour emerges because spines act as temporary traps for molecules as they move along spiny dendrites. In fact, using a combination of optical experiments and computer simulations, is determined that dendritic spines act as traps to produce anomalous diffusion of molecules along dendrites of cerebellar PCs.

The results indicate that spines influence dendritic structure in a way that limits the spread of intracellular chemical signals.

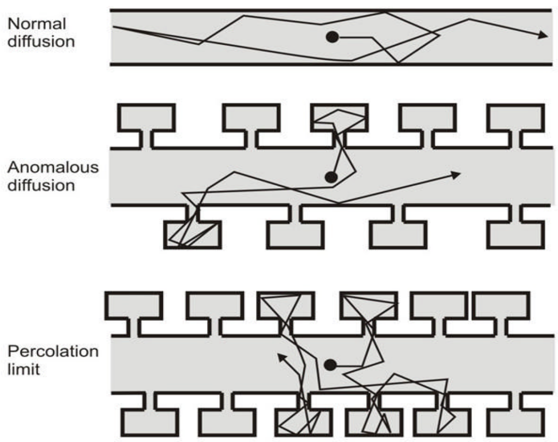


Figure 4.2: Figure taken from F. Santamaria, S. Wils, E. D. Schutter, and G. J. Augustine, Neuron 52 (2006). Fig. 3C.

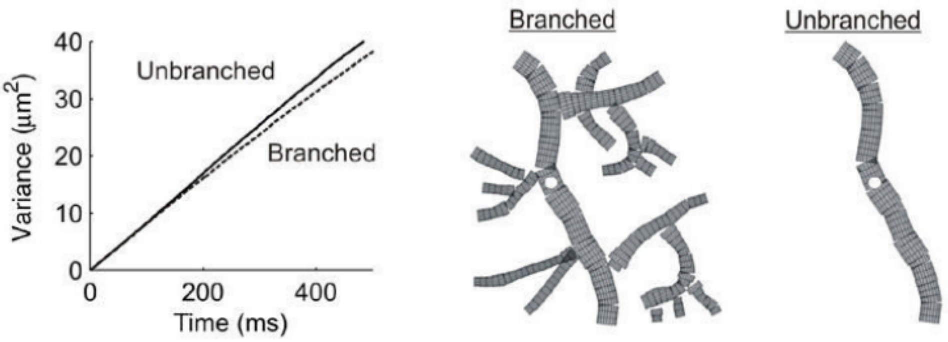


Figure 4.3: Figures taken from F. Santamaria, S. Wils, E. D. Schutter, and G. J. Augustine, Neuron 52 (2006). Fig. 7B. We can see that there is no significant effect of branching on diffusion.

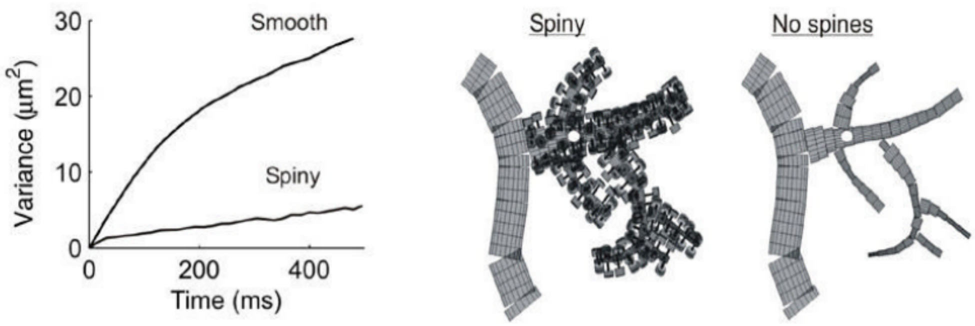


Figure 4.4: Figures taken from F. Santamaria, S. Wils, E. D. Schutter, and G. J. Augustine, Neuron 52 (2006). Fig. 7D. Spine density has more impact then presence of branches over the diffusion regime.

4.5 Conclusion

The cable model, fractional or linear, is used to describe subthreshold potentials, or passive potentials, associated to dendritic processes in neurons. The travelling potential is summed up in the center of the cell, called soma, and an action potential is produced when a threshold is exceeded. Anomalous regimes of diffusion can then have a deep impact on the communication strength. Diffusion results more anomalous, i.e. the fractional exponent α decreases, with increasing spine density.
Decreasing spine density is characteristic of aging, pathologies as neurological disorders and Down's sindrome, then subdiffusive regimes are in some sense associated to a healthy condition. It has been suggested that increasing spine density should serve to compensate time delay of postsynaptic potentials along dendrites and to reduce their long time temporal attenuation.
We now look at our plotted solutions for the fractional cable equation when an impulsive potential is applied at the accessible end: it can be noted from Fig. (4.5) that peak high decreases more rapidly with decreasing α at early times, viceversa is less suppressed at longer times, and the cross over time increases with decreasing α.

Looking at the potential versus time it can also be noted that potential functions associated to lower α last for longer time at appreciable intensity and arrive faster at early times with respect to the normal diffusion case ($\alpha = 1$).
By the way, when a constant potential is applied at the accessible end, as we can note from Fig. (4.6), the exponential suppression of the potential along the dendrite is reduced for high X values with respect to normal diffusion. Instead for small X the potential results just slightly more suppressed in the sub-diffusion process.

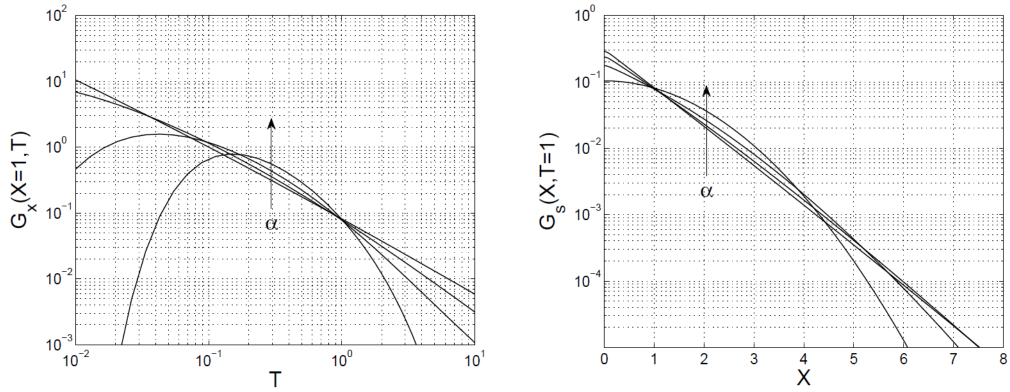


Figure 4.5: Green function for Signal Problem is calculated and plotted for $X = 1$ as function of time T (left panel) and for $T = 1$ as function of X (right panel). Several values of parameter α are compared: 0.25, 0.5, 0.75, 1.00.

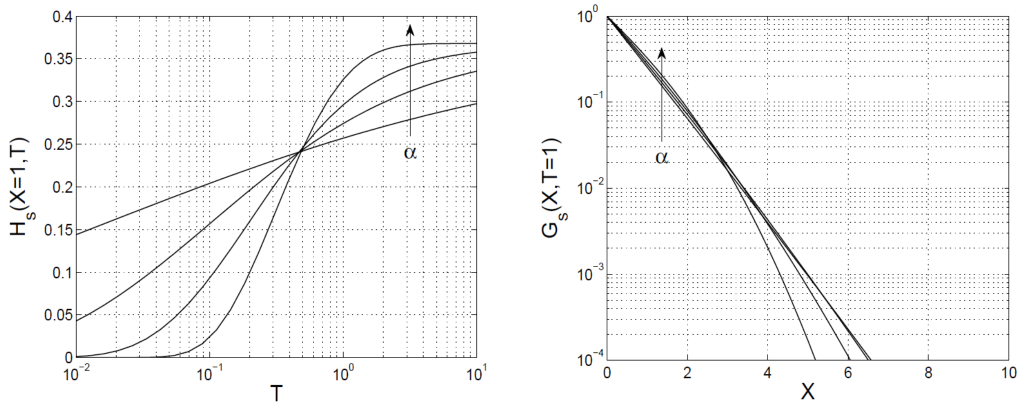


Figure 4.6: Step response function for Signal Problem is calculated and plotted for $X = 1$ as function of time T (left panel) and for $T = 1$ as function of X (right panel). Several values of parameter α are compared: 0.25, 0.5, 0.75, 1.00.

These behaviours can be noticed also for the other cases in Fig. (4.7) and Fig. (4.8-9).

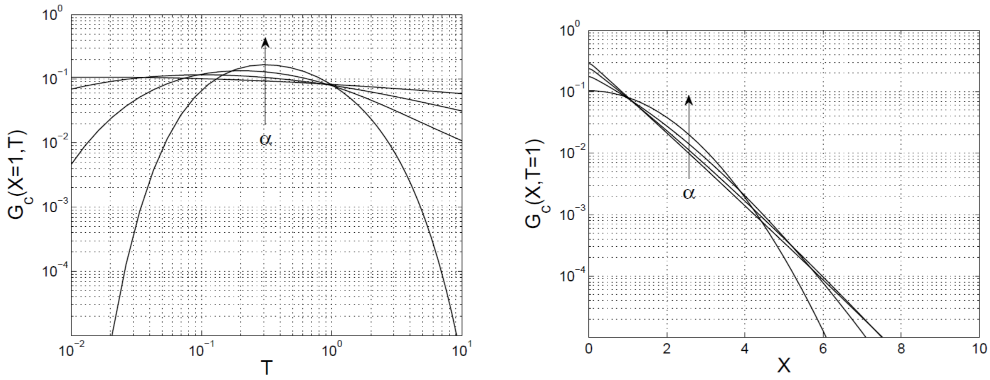


Figure 4.7: Green function for Cauchy Problem is calculated and plotted for $X = 1$ as function of time T (left panel) and for $T = 1$ as function of X (right panel). Several values of parameter α are compared: 0.25, 0.5, 0.75, 1.00.

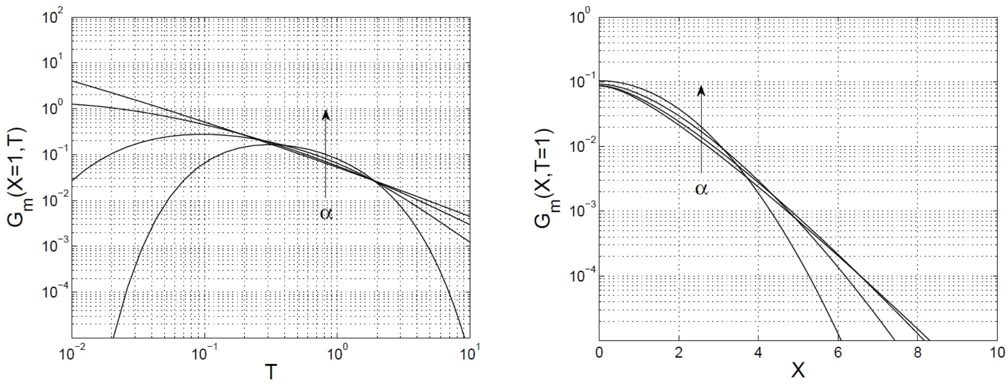


Figure 4.8: Green function for Second Kind Boundary Problem is calculated and plotted for $X = 1$ as function of time T (left panel) and for $T = 1$ as function of X (right panel). Several values of parameter α are compared: 0.25, 0.5, 0.75, 1.00.

From a mathematical point of view the Efros theorem extends the concept of convolution as an integral form that is consistent with a subordination-type integral. However such integral form does not necessary connote a subordi-

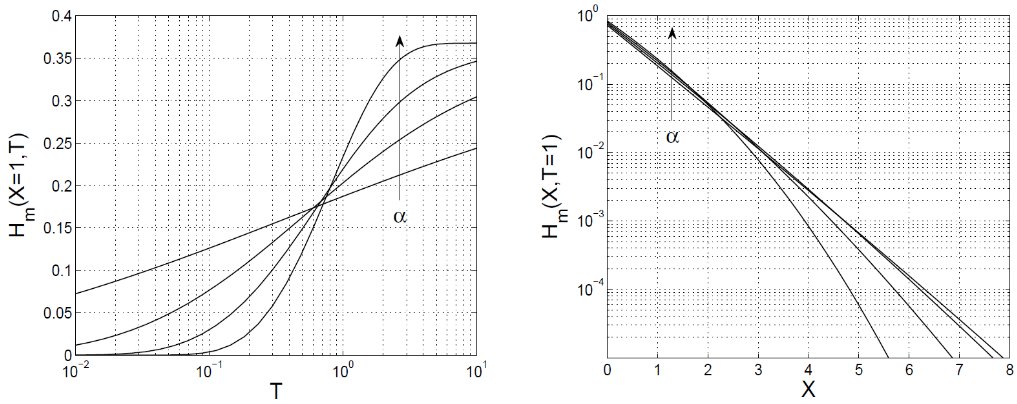


Figure 4.9: Step response function for Second Kind Boundary Problem is calculated and plotted for $X = 1$ as function of time T (left panel) and for $T = 1$ as function of X (right panel). Several values of parameter α are compared: 0.25, 0.5, 0.75, 1.00.

nated process, but could also be interpreted as a consequence of the random nature of the media in which particles are diffusing. This model can be also read as a generalization of time fractional diffusion processes where mass is not conserved due to leakage. This approach naturally recover the solution for the time-fractional case in the limit in which the leakage is put to zero in the integral forms.

In conclusion the presented fractional cable model satisfies the main biological features of the dendritic cell signalling problem. With respect to models solved as Cauchy problem, our approach could include specific time dependent boundary conditions, which will allow to reconstruct with accuracy the expected signal at the soma if the model will result capable to predict real data behaviour.

Appendix A

The Laplace Transform

A.1 Introduction

The Laplace Transform (or L-Transform or LT) is a kind of integral transforms, i.e. a transform of the following form

$$F(z) = \int K(z, z')f(z')dz' \tag{A.1}$$

where $z \in \mathbb{C}$ and K is a kernel, which give us a correspondences between functions through integrals; in this case associates $f(z')$ with $F(z)$.
The reasoning that leds to the introduction of integral transform is that they are useful in turn a complicated problem into a simpler one.
In this and in the next section about Laplace transform we refer to [10] and [23].
Given a causal and locally summable[1] function $f(t)$, with $t \in \mathbb{R}$, the *Laplace Transform* $\mathcal{L}[f(t)]$ of $f(t)$ is defined by the following integral (if it exists)

$$\boxed{\mathcal{L}[f(t)] = \tilde{f}(s) = \int_0^\infty e^{-st} f(t)dt = \lim_{T\to\infty} \int_0^T e^{-st} f(t)dt} \tag{A.2}$$

where $s \in \mathbb{C}$.
The common notation to synthetically point out a function and its Laplace

[1]that is $\int_0^T |f(t)| < \infty$

Transform is: $f(t) \div \tilde{f}(s)$.

One may demonstrate that, if the integral (A.2) is convergent for s_0, then it is convergent for every s whose real part is greater than the real part of s_0; so the Laplace Transform is defined in a half plane.

We are so saying that exist an abscissa of convergence σ_c, such that $\mathcal{L}[f(t)]$ is convergent only when $Re(s) > \sigma_c$. When $Re(s) < \sigma_c$ we have a divergence, and when $Re(s) = \sigma_c$ we can't say nothing: follow that the LT is defined in an open half-plane.

Moreover, we say that $f(t)$ is absolutely $\mathcal{L}$ transformable when

$$\int_0^{\infty} e^{-st}|f(t)|dt < \infty \tag{A.3}$$

Besides, we may say that the integral (A.2) is *uniformly convergent* if

$$\mathcal{L}[f(t)] = \lim_{T\to\infty} \int_0^{T} e^{-st} f(t)dt \tag{A.4}$$

is indipendent from s.

A notable theorem (*the initial and final values theorem*) relates $f(t)$ and its L-Transform

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\tilde{f}(s) \tag{A.5}$$

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} s\tilde{f}(s) \tag{A.6}$$

Evidently when one of the four limits does not exist, the theorem does not hold.

Laplace Transform has a number of properties which allow to get L-Transform of more complex functions. The most significant ones are (let $\sigma_c(\tilde{f}(s)) = \lambda$)

1. **linearity**

$$\sum_{k=1}^{N} c_k f_k(t) \div \sum_{k=1}^{N} c_k \tilde{f}_k(s) \tag{A.7}$$

with $\sigma_c = max\{\lambda_k\}$

2. **scale change** ($a \in \mathbb{R}^+$)

$$f(at) \div \frac{1}{a}\tilde{f}\left(\frac{s}{a}\right) \tag{A.8}$$

with $\sigma_c = a\lambda$

3. **translation** to the right ($a \in \mathbb{R}^+$ and $H(t-a)$ is the Heaviside function)

$$f(t-a)H(t-a) \div e^{as}\tilde{f}(s) \tag{A.9}$$

with $\sigma_c = \lambda$

4. **translation** to the left ($a \in \mathbb{R}^+$)

$$f(t+a) \div e^{as}\left[\tilde{f}(s) - \int_0^a e^{-st}f(t)dt\right] \tag{A.10}$$

5. **multiplication** by $e^{\alpha t}$ ($\alpha \in \mathbb{C}$)

$$f(t)e^{\alpha t} \div \tilde{f}(s-\alpha) \tag{A.11}$$

with $\sigma_c = \lambda + Re(\alpha)$

6. **multiplication** by t^n

$$t^n f(t) \div (-1)^n \frac{d^n}{ds^n}\tilde{f}(s) \tag{A.12}$$

with $\sigma_c = \lambda$

7. **division** by t

$$\frac{f(t)}{t} \div \int_s^\infty \tilde{f}(s)ds \tag{A.13}$$

8. **derivation**

$$\frac{d^n f}{dt^n} \div s^n \tilde{f}(s) - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0^+) \tag{A.14}$$

with $\sigma_c = max(\sigma_c(\mathcal{L}\left(\frac{df}{dt}\right)), 0)$

9. **integration**

$$\int_0^t f(t')dt' \div \frac{\tilde{f}(s)}{s} \tag{A.15}$$

with $\sigma_c = max(\lambda, 0)$

10. **convolution**

$$f * g \div \tilde{f}(s)\tilde{g}(s) \tag{A.16}$$

11. **causal periodic functions**

$$f(t) = f(t + kT) \div \frac{\int_0^T e^{-st} f(t)dt}{1 - e^{-sT}} \tag{A.17}$$

A.2 The Inverse Laplace Transform: Bromwich and Titchmarsh Formulas

Given $\tilde{f}(s)$, it is possible to go back to $f(t)$: one may demonstrate that (if $\tilde{f}(s)$ is analytical and $\lim_{s\to\infty} \tilde{f}(s) = 0$)

$$f(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{st} \tilde{f}(s) ds \tag{A.18}$$

with $\sigma \in \mathbb{R}, \sigma > \sigma_c$. The function $f(t)$ must satisfy the same conditions that are valid for (A.2).

To calculate the above integral, is convenient using the following path

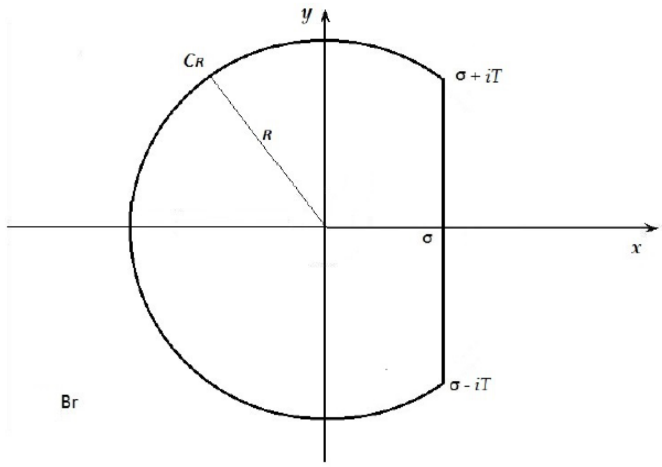


Figure A.1: The *Bromwich contour.*

In practice, the integral in (A.18) is evaluated by considering the contour integral

$$\frac{1}{2\pi i}\oint_C e^{st}\tilde{f}(s)ds \tag{A.19}$$

where C is the *Bromwich contour*, composed of a line and the arc of a circle of radius R, with the center at the origin O.
If we represent the arc by Γ, it follows from (A.18) that, since $T = \sqrt{R^2 - \sigma^2}$,

$$\begin{aligned} f(t) &= \lim_{R\to\infty} \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} e^{st}\tilde{f}(s)ds \\ &= \lim_{R\to\infty}\left[\frac{1}{2\pi i}\oint_C e^{st}\tilde{f}(s)ds - \frac{1}{2\pi i}\int_\Gamma e^{st}\tilde{f}(s)ds\right] \end{aligned} \tag{A.20}$$

When the function $\tilde{f}(s)$ has at least one singularity, to compute the integral in eq. (A.18) one may use a simple theorem, the *Bromwich formula*, known as *Heaviside formula* too, if is true that.

- The only singularities of $f(s)$ are poles, all of which lie to the left of the line $s = \sigma$, for some real constant σ.

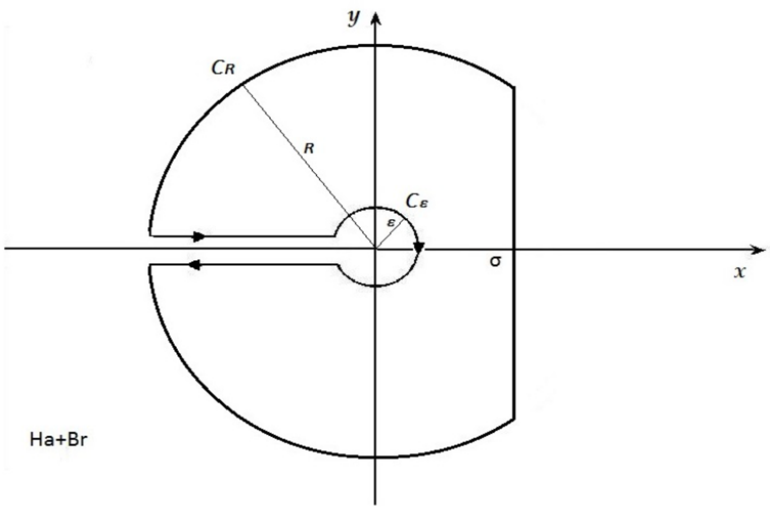


Figure A.2: The *Hankel Path* combinated with the *Bromwich Path.*

- The integral around Γ in (A.20) approaches zero as $R \to \infty$.

Then, by the residue theorem we can write (A.20) as

$$f(t) = \sum_k Res_k(e^{st}\tilde{f}(s)) \tag{A.21}$$

where the sum is made over all finite singularities.

Extending the above result considering $\tilde{f}(s)$ as a *polidrome function*, the Bromwich contour is suitable modified for the presence of some branch points. For example, if $\tilde{f}(s)$ has only one branch point at $s = 0$, the correct contour to use is a combination of *Bromwich contuor* with the *Hankel Path.*

The Hankel Path is a loop which starts from $-\infty$ along the lower side of negative real axis, encircles a small circle in the positive direction and ends at $-\infty$ along the upper side of the negative real axis, as in Fig. 2. . Considering the closed path in Fig. A.2 :

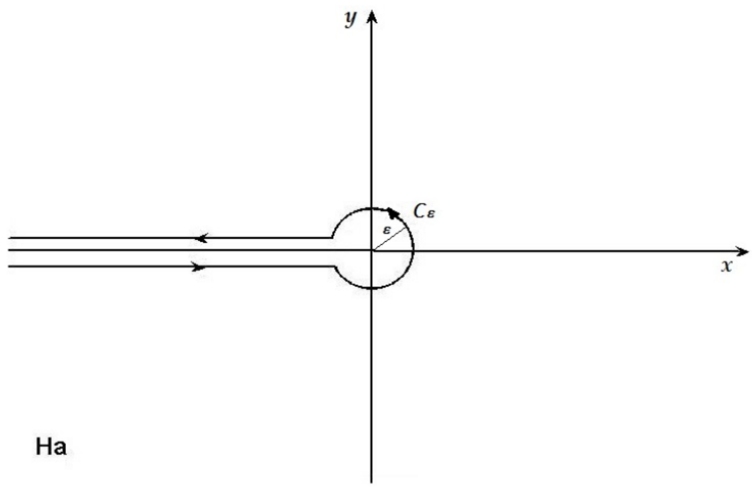


Figure A.3: The *Hankel Path* combinated with the *Bromwich Path.*

$$\begin{aligned} &\frac{1}{2\pi i}\oint e^{st}\tilde{f}(s)ds = \\ &\frac{1}{2\pi i}\left[\int_{Br} e^{st}\tilde{f}(s)ds + \int_{Ha} e^{st}\tilde{f}(s)ds + \right. \\ &\left. + \int_{C_R} e^{st}\tilde{f}(s)ds + \int_{C_\epsilon} e^{st}\tilde{f}(s)ds\right] = 0 \end{aligned} \tag{A.22}$$

The integral is equal to 0, by *Cauchy's Theorem*, since the singularity ($s = 0$) is not inside the closed path.
Also, the integrals on C_R and C_ϵ approch zero by the *Jordan's Lemma* and the *Little Circle Lemma.*

Follow that

$$\int_{Br} e^{st}\tilde{f}(s)ds = -\int_{Ha} e^{st}\tilde{f}(s)ds \tag{A.23}$$

Note that the above equality is a particular case in which

$$\sum_k Res_k\big(e^{st}\tilde{f}(s)\big)= 0; \tag{A.24}$$

So, in presence of poles the general relation is:

$$\int_{Br} e^{st}\tilde{f}(s)ds = -\left(\int_{Ha} e^{st}\tilde{f}(s)ds + \sum_k Res_k\big(e^{st}\tilde{f}(s)\big)\right) \tag{A.25}$$

An observation: multiplying $\tilde{f}(s)$ by e^{st} does not affect the poles of $\tilde{f}(s)$. Continue analyzing the Hankel Path contribute by a variable change to extract the result:

$$\begin{aligned} & s = re^{(\mp i\pi)}, \quad ds = dr; \\ f(t) &= -\frac{1}{2\pi i}\int_{\infty}^{0} e^{-rt}\tilde{f}(re^{-i\pi})dr + \int_{0}^{\infty} e^{-rt}\tilde{f}(re^{i\pi})dr = \\ &= -\frac{1}{2\pi i}\int_{0}^{\infty} e^{-rt}\big(\tilde{f}(re^{i\pi}) - \tilde{f}(re^{-i\pi})\big)dr = \\ &= -\frac{1}{\pi}\int_{0}^{\pi} e^{-rt}\left(\pm Im[\tilde{f}(s)]|_{s=re^{\mp i\pi}}\right)dr \end{aligned} \tag{A.26}$$

The last step is made considering the relation $Im(z) = \frac{z-\bar{z}}{2i}$.
The final result is:

$$\begin{aligned} f(t) &= \frac{1}{2\pi i}\int_{Br} e^{st}\tilde{f}(s)ds = \int_{0}^{\infty} e^{-rt}K(r)dr, \quad where \\ K(r) &= -\frac{1}{\pi}\left(\pm Im[\tilde{f}(s)]|_{s=re^{\mp i\pi}}\right) \end{aligned} \tag{A.27}$$

$K(r)$ take the name of *Kernel* and correspond to the *Titchmarsh Formula.*
To find the Inverse Laplace Transform of functions which have infinitely many isolated singularities, the above methods can be applied; the curved portion of Bromwich contour is chosen to be of such radius R_m so as to enclose only

a finite number of the singularities and so as not to pass through any singularity. The Inverse Laplace Transform is the found by taking an appropriate limit as $m \to \infty$.

A.3 The Efros Theorem

It is known that the Laplace transform technique plays a fundamental role in finding the solution of linear differential equations and linear integral equations of convolution type; as a consequence, it turns out to be of great importance in the solution of problems involving fractional derivatives and integrals. A noteworthy applications of this technique in the framework of the fractional calculus concern the diffusion-wave phenomena.
In this section we recall a theorem due to the russian mathematician A. *Efros* in 1935, which is also referred to as *generalized multiplication theorem* (GMT). This theorem, almost unknown in the western literature, just provides an interesting generalization of the well-known convolution or multiplication theorem for Laplace transforms.

A.3.1 Statement of the Efros Theorem (GMT)

In this section we may refer to [3].
Let be given analytic functions G(s) and q(s) and the relations

$$F(s) = \mathcal{L}[f(t)], \quad G(s)e^{-\tau q(s)} = \mathcal{L}[g(t,\tau)], \tag{A.28}$$

then it holds

$$G(s)F(q(s)) = \mathcal{L}\left[\int_0^\infty f(\tau)g(t,\tau)d\tau\right]. \tag{A.29}$$

For the *proof*, we start with the right hand side of (A.29), that is:

$$\mathcal{L}\left[\int_0^\infty f(\tau)g(t,\tau)d\tau\right] = \int_0^\infty e^{-st}\int_0^\infty f(\tau)g(t,\tau)d\tau dt \\ = \int_0^\infty f(\tau)\int_0^\infty g(t,\tau)e^{-st}dtd\tau, \tag{A.30}$$

provided we can reverse the order of integration. But the inner integral in the last one is the Laplace transform of $g(t,\tau)$ and so we can write

$$\mathcal{L}\left[\int_0^\infty f(\tau)g(t,\tau)d\tau\right] = G(s)\int_0^\infty f(\tau)e^{-q(s)\tau}d\tau = G(s)F[q(s)], \tag{A.31}$$

that completes the proof.

If in particular, we take $q(s) = s$, then

$$\mathcal{L}[g(t,\tau)] = e^{-s\tau}G(s) \tag{A.32}$$

and by the shift theorem $g(t,\tau) = g(t-\tau)$. Hence formula (A.29) becomes

$$F(s)G(s) = \mathcal{L}\left[\int_0^\infty f(\tau)g(t-\tau)d\tau\right] = \mathcal{L}\left[\int_0^t f(\tau)g(t-\tau)d\tau\right] \tag{A.33}$$

since for original functions we have $g(t-\tau) = 0$ for $\tau > t$.

The last formula shows that Efros' theorem is a generalization of the convolutional theorem for the Laplace transform.

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