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WEAK CONVERGENCE METHODS FOR CONSTRAINT MINIMA OF FUNCTIONALS WITH CRITICAL GROWTH

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Contents

Abstract

In this thesis we describe some of the most important weak convergence techniques set forth for studying constraint minima of functionals.
In particular, we first provide theoretical backgroung regarding measure theory and Sobolev spaces; then, we concentrate on measure theoretic tools of noncompactness which allow us to understand the ways in which a weakly convergent sequence of functions may fail to be strongly convergent. In particular, we study concentartion and oscillation effects and we apply them to constraint minima of functionals in the critical growth case.

Sommario

In questa tesi presentiamo alcune delle più importanti tecniche di convergenza debole che permettono di studiare minimi vincolati di funzionali. In particolare, nella prima parte della tesi sviluppiamo le basi teoriche riguardanti la teoria della misura e degli spazi di Sobolev; poi, ci concentriamo su strumenti di misura di non compattezza che ci permettono di comprendere in che modo una successione di funzioni debolmente convergente può non essere fortemente convergente. In particolare, studiamo problemi di concentrazione e oscillazione e li applichiamo a minimi vincolati di funzionali nel caso di crescita critica.

Introduction

The aim of this thesis is to describe some of the most important weak convergence techniques set forth for studying constraint minima of functionals.
In particular we will study and provide examples in the case of critical growth.
Suppose we wish to find a minimizer for the functional

$$I(w) = \int_U F(Dw)\,dx,$$

where U is a subset of $\mathbb{R}^n$ and F is a power like behaviour function.
We look for the minimizer among all the candidate functions w in the class $\mathscr{A}$ of admissible functions:

$$\mathscr{A} := \{w \in W_0^{1,p}(U); \quad \|w\|_{L^q(U)} = 1\}$$

In case $q = p < p^*$, the critical Sobolev exponent associated to p, the existence of a minimizer $u \in \mathscr{A}$ comes directely from Sobolev imbedding Theorem; indeed, given a minimizing sequence $\{u_k\} \subset \mathscr{A}$, i.e. a sequence such that

$$I(u_k) \to \inf_{\mathscr{A}} I$$

it is possible to see that (up to a subsequence) $\{u_k\}$ is weakly convergent to a function u. Thanks to the properties of F, I is weakly lower semicontinuous, so that

$$I(u) \leq \inf_{\mathscr{A}} I.$$

Our aim is to show that $u \in \mathscr{A}$, that is $\|u\|_{L^p(U)} = 1$. Since $p < p^*$, Sobolev imbedding Theorem assures that the imbedding $W_0^{1,p}(U) \to L^p(U)$ is compact and so the L^p-norm is preserved to the limit; hence $\|u\|_{L^p(U)} = 1$.

When $q = p^*$, we are in the so called critical case for the Sobolev imbedding. In this case showing that $u \in \mathscr{A}$ requires more effort: the problem is that the imbedding $W_0^{1,p}(U) \to L^{p^*}(U)$ is not compact; as a conequence the main issue will be to find a substitute for Sobolev imbedding Theroem which allow us to deduce that $u \in \mathscr{A}$.

Chapters 1 and 2 are devoted to the construction of theoretical tools which are needed in the following part of the thesis.

In Chapter 1, we provide the main definitions and theorems concernig measure theory. After recalling the first properties of abstract measure theory, we study weak convergence in L^p spaces and in measure spaces. Finally, one Section is devoted to Riesz representation Theorem for measures.

Chapter 2 concerns Sobolev spaces $W^{m,p}$. After giving the main definitions and properties of Sobolev spaces, we turn our attention to the dual of $W^{m,p}$, denoted by $W^{-m,p'}$, where p' is the conjugate exponent of p. In particular, we identify it with a distributional space and we prove the main properties of such space. The following Section is devoted to Sobolev imbedding Theorem, which, as explained above, is one of the main tools we will use in order to find minima of functionals. Then, we prove the main theorems concerning boundary traces. Finally, we devote one Section to compactess in Sobolev spaces and we prove a refinement of Rellich Konrachov Theorem.

Chapters 3 and 4 are the core of this thesis.

In Chapter 3 we construct measure theoretic tools which allow us to understand the ways in which a weakly convergent sequence $\{u_k\}_{k=0}^{\infty}$ of functions may fail to be strongly convergent to a certain function u. As we will see, the difficulty in deducing strong convergence from weak convergence can be caused by rapid fluctuation in the functions u_k (*oscillation effect*), or because the mass of $|u_k - u|^q$ coalesces onto a set of zero Lebesgue measure (*concentration effect*).

In the first part of this Chapter, we construct appropriate methodology for characterizing concentration effects, while in the second part we turn our attention on the problem of oscillation.

Finally, Chapter 4 is organized as follows: we first prove Lagrange multiplier Theorem, which we will use to find the equation satisfied by minima of functionals both in the case of subcritical and critical growth; then, we point out the role of convexity in the calculus of variation; the last two Sections are devoted to examples. The first one is devoted to the subcritical case, which is based upon Sobolev imbedding Theorem. The last Section is devoted to the critical case: as explained above, we first need to prove theorems which allow us to deduce that the limit u of the minimizing sequence $\{u_k\}$ belongs to the class $\mathscr{A}$ of admissible functions, so that u is a minimizer. Finally, in both examples we find the equation satisfied by the constraint minimum using Lagrange multiplier Theorem.

Chapter 1

Measure Theory

We present in this chapter the basic notions of measure theory since these will be largely used in the following chapters.

1.1 Abstract measure theory

We devote the first section to the basic notions of measure theory.

Definition 1.1. Let X be a nonempty set and $\mathscr{F}$ be a collection of subsets of X. We say that $\mathscr{F}$ is an *algebra* if
$\forall F_1,\, F_2 \in \mathscr{F}$, we have $\quad \emptyset \in \mathscr{F}, \quad F_1 \cup F_2 \in \mathscr{F}, \quad X \setminus F_1 \in \mathscr{F}$.

Definition 1.2. We say that an algebra $\mathscr{F}$ is a *σ-algebra* if
for any sequence $\{F_h\}_{h=1}^{\infty} \subset \mathscr{F}$, we have $\bigcup_h F_h \in \mathscr{F}$.
Any set $F \in \mathscr{F}$ is called *measurable set*.

Definition 1.3. For any collection $\mathscr{G}$ of subset of X, the *σ-algebra generated by* $\mathscr{G}$ is the smallest σ-algebra containing $\mathscr{G}$.
If (X, τ) is a topological space, we denote by $\mathscr{B}(X)$ the σ-algebra of Borel subsets of X, that is the σ-algebra generated by open subsets of X.

Definition 1.4. A *measure space* is a couple $(X, \mathscr{F})$ where X is a nonempty set and $\mathscr{F}$ is a σ-algebra.

Remark 1. Using De Morgan laws it is not difficult to see that algebras are closed under finite intersections and σ-algebras are closed under countable intersections.

Definition 1.5. Let $(X, \mathscr{F})$ be a measure space and $\mu : \mathscr{F} \longrightarrow [0, +\infty]$.
We say that μ is *additive* if $\forall F_1,\ F_2 \in \mathscr{F}$ we have

$$F_1 \cap F_2 = \emptyset \Longrightarrow \mu(F_1 \cup F_2) = \mu(F_1) + \mu(F_2).$$

We say that μ is σ*-subadditive* if $\forall F \in \mathscr{F},\ \{F_h\}_{h=1}^{\infty} \subset \mathscr{F}$ we have

$$F \subset \bigcup_{h=0}^{\infty} F_h \Longrightarrow \mu(F) \leq \sum_{h=0}^{\infty} \mu(F_h).$$

We say that μ is σ*-additive on* $\mathscr{F}$ if, for any sequence $\{F_h\}_{h=1}^{\infty}$ of pairwise disjoint elements of $\mathscr{F}$, we have

$$\mu(\bigcup_{h=0}^{\infty} F_h) = \sum_{h=0}^{\infty} \mu(F_h).$$

Definition 1.6. We say that μ is a *positive measure* if $\mu(\emptyset) = 0$ and μ is σ-additive on $\mathscr{F}$. We say that μ is *finite* if $\mu(X) < +\infty$.
A positive meausre μ such that $\mu(X) = 1$ is called *probability measure*.

Definition 1.7. We say that a set $E \subset X$ is σ*-finte* with respect to a positive measure μ if it is the union of an increasing sequence of sets with finite measure. If X is itself σ-finite, we also say that μ is σ*-finite*.

Remark 2. Any positive measure μ is monotone with respect to set inclusion and continuous along monotone sequences.
Actually, if $\{F_h\}_{h=1}^{\infty}$ in an increasing sequence of sets, then

$$\mu(\bigcup_{h=0}^{\infty} F_h) = \lim_{h \to \infty} \mu(F_h).$$

And, if $\{F_h\}_{h=1}^{\infty}$ in a decreasing sequence of sets with $\mu(F_0) < \infty$, then

$$\mu(\bigcap_{h=0}^{\infty} F_h) = \lim_{h \to \infty} \mu(F_h).$$

Remark 3. We note also that σ-subadditivity and additivity imply σ-additivity:

$$\mu(\bigcup_{h=0}^{\infty} F_h) \leq \sum_{h=0}^{\infty} \mu(F_h) = \lim_{n\to\infty} \sum_{h=0}^{n} \mu(F_h) =$$

$$= \lim_{n\to\infty} \mu(\bigcup_{h=0}^{n} F_h) \leq \mu(\bigcup_{h=0}^{\infty} F_h).$$

Beside positive measures, it is possible to define real-valued mesaures. Notice that, according to the following definition, real measures must be finite, thus positive measures are not a particolar case of real measures.

Definition 1.8. Let $(X, \mathscr{F})$ be a measure space.
We say that $\mu : \mathscr{F} \longrightarrow [0, +\infty[$ is a *real measure* if $\mu(\emptyset) = 0$ and for any sequence $\{F_h\}_{h=1}^{\infty}$ of pairwise disjoint elements of $\mathscr{F}$ we have

$$\mu(\bigcup_{h=0}^{\infty} F_h) = \sum_{h=0}^{\infty} \mu(F_h).$$

Remark 4. The absolute convergence of the series in the above definition is a requirement on the set function μ, since the sum of the series cannot depend on the order of its terms, as the union does not.

Definition 1.9. Let μ be a real measure. We define its *total variation*, and we write $|\mu|$, the measure such that for all $F \in \mathscr{F}$,

$$|\mu|(F) := sup\left\{\sum_{h=0}^{\infty} |\mu(F_h)|;\ F_h \in \mathscr{F} \text{ pairwise disjoint},\ F = \bigcup_{h=0}^{\infty} F_h\right\}.$$

We define the *positive* and *negative parts* of meausure μ respectively as follows:

$$\mu^+ := \frac{|\mu| + \mu}{2} \qquad \text{and} \qquad \mu^- := \frac{|\mu| - \mu}{2}.$$

Theorem 1.1.1. *Let μ be a real measure on $(X, \mathscr{F})$. Then $|\mu|$ is a positive finite measure.*

For the proof, see [2] Theorem 1.6.

Remark 5. The above theorem shows that for any real measure μ its positive and negative part are positive finite measures.
It follows that the decomposition $\mu = \mu^+ + \mu^-$ holds.
This is known as *Jordan Decomposition of* μ.

Definition 1.10. Let μ be a positive measure on the measure space $(X, \mathscr{F})$.
We say that $N \subset X$ is *μ-negligible* if there exists $F \in \mathscr{F}$ such that $N \subset F$ and $\mu(F) = 0$.
We say that a property $P(x)$ depending on the point $x \in X$ *holds μ-almost-everywhere in* X if the set where P fails is a μ-negligible set.

Definition 1.11. Let μ be a positive measure on the measure space $(X, \mathscr{F})$.
Let $\mathscr{F}_\mu$ be the collection of all the subsets of X of the form $E = F \cup N$ where N is a μ-negligible set and $F \in \mathscr{F}$. Then $\mathscr{F}_\mu$ is a σ-algebra called the *μ-completion of $\mathscr{F}$* and we say that $G \in X$ is *μ-measurable* if $G \in \mathscr{F}_\mu$.
The measure μ can be extended to $\mathscr{F}_\mu$ by setting, for E as above, $\mu(E) = \mu(F)$.

Remark 6. If μ is a real measure, thanks to Theorem 1.1.1 we know that $|\mu|$ is a positive finite measure. For this reason, we legitimately call the completion of $\mathscr{F}$ with respect to $|\mu|$ the μ-completion $\mathscr{F}_\mu$ of $\mathscr{F}$.

Definition 1.12. Let $(X, \mathscr{F})$ be a measure space and (Y, d) a metric space.
A function $f : X \longrightarrow Y$ is said to be *$\mathscr{F}$-measurable* if $f^{-1}(A) \in \mathscr{F}$ for every open set $A \in Y$.
If μ is a positive measure on $(X, \mathscr{F})$ the function $f : X \longrightarrow Y$ is said to be *μ-measurable* if it is $\mathscr{F}_\mu$-measurable.

Proposition 1.1.2. *Let $(X, \mathscr{F})$ be a measure space. Then the following results hold:*

1. *if $f, g : X \longrightarrow \mathbb{R}$ are $\mathscr{F}$-measurable functions, then $\forall \alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is $\mathscr{F}$-measurable; fg is $\mathscr{F}$-measurable; $\frac{f}{g}$ is $\mathscr{F}$-measurable, provided that $g(x) \neq 0$ for any $x \in X$.*

2. *if $f, g : X \longrightarrow \overline{\mathbb{R}}$ are extended $\mathscr{F}$-measurable functions, then $\min\{f, g\}$ and $\max\{f, g\}$ are $\mathscr{F}$-measurable.*

3. *if $f_h : X \longrightarrow \overline{\mathbb{R}}$ is a squence of extended $\mathscr{F}$-measurable functions, then*
$$\inf_{h \in \mathbb{N}} f_h, \qquad \sup_{h \in \mathbb{N}} f_h, \qquad \liminf_{h \to \infty} f_h, \qquad \limsup_{h \to \infty} f_h$$
are all $\mathscr{F}$-measurable functions.

For the proof, see [3].

1.2 Integration and convergence theorems

In the first part of this section we define the integral with respect to a measure. We then introduce summable and integrable functions and we state the main convergence theorems of Levi, Fatou and Lebesgue.

Definition 1.13. Let $(X, \mathscr{F})$ be a measure space and $E \in \mathscr{F}$ the *characteristic function of E* is defined as follows:

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Definition 1.14. Let $(X, \mathscr{F})$ be a measure space. We say that $f : X \longrightarrow \mathbb{R}$ is a *simple function* if

$$f = \sum_{i=0}^{k} \lambda_i \chi_{E_i}.$$

where $E_1, ..., E_k$ are measurable sets and $\lambda_1, ..., \lambda_k \in \mathbb{R}$.

Definition 1.15. Let μ be a positive measure on $(X, \mathscr{F})$ and $f : X \longrightarrow [0, +\infty[$ a simple μ-measurable function. We define the *integral* of f on X with respect to the measure μ as:

$$\int_X f \, d\mu := \sum_{i=0}^{k} \lambda_i \mu(E_i).$$

The definition is extended to any μ-measurable function $f : X \longrightarrow [0, +\infty]$ by setting:

$$\int_X f \, d\mu := \sup\left\{ \int_X g \, d\mu : \; g \; \mu\text{-measurable, simple, } g \leq f \right\}.$$

Definition 1.16. Let μ be a positive measure on $(X, \mathscr{F})$. We say that a μ-measurable function $f : X \longrightarrow \overline{\mathbb{R}}$ is *μ-integrable* if either

$$\int_X f^+ \, d\mu < \infty \qquad \text{or} \qquad \int_X f^- \, d\mu < \infty.$$

If f is μ-integrable, we set:

$$\int_X f \, d\mu := \int_X f^+ \, d\mu - \int_X f^- \, d\mu.$$

Definition 1.17. Let μ be a positive measure on $(X, \mathscr{F})$. We say that a μ-measurable function $f : X \longrightarrow \overline{\mathbb{R}}$ is *μ-summable* if

$$\int_X |f| \, d\mu < \infty.$$

In this case we say that $f \in L^1(X, \mu)$.

Remark 7. If μ is a real measure on $(X, \mathscr{F})$, we say that a $|\mu|$-measurable function f is *μ-summable* if f is $|\mu|$-summable.

Definition 1.18. Let μ be a real measure on $(X, \mathscr{F})$ and $f : X \longrightarrow \overline{\mathbb{R}}$ a μ-summable function. We set:

$$\int_X f \, d\mu := \int_X f \, d\mu^+ - \int_X f \, d\mu^-.$$

Remark 8 (Chebyshev inequality). If $f \in L^1(X, \mu)$ is a positive function, then for any $t > 0$ we have:

$$\mu(\{x \in X : \ f(x) > t\}) \leq \frac{1}{t} \int_X f \, d\mu.$$

Indeed,

$$\frac{1}{t} \int_X f(x) \, d\mu = \int_{\{x \in X: \ \frac{f(x)}{t} > 1\}} \frac{f(x)}{t} \, d\mu + \int_{\{x \in X: \ \frac{f(x)}{t} \leq 1\}} \frac{f(x)}{t} \, d\mu \geq$$

$$\geq \int_{\{x \in X: \ \frac{f(x)}{t} > 1\}} 1 \, d\mu = \mu(\{x \in X : \ f(x) > t\}).$$

We now state the main convergence theorems of Levi, Fatou and Lebesgue.

Theorem 1.2.1 (Monotone convergence Theorem). *Let $f_h : X \longrightarrow \overline{\mathbb{R}}$ be an increasing sequence of μ-measurable functions and assume that $f_h \geq g$, with $g \in L^1(X, \mu)$, for any $h \in \mathbb{N}$. Then*

$$\lim_{h \to \infty} \int_X f_h \, d\mu = \int_X \lim_{h \to \infty} f_h \, d\mu.$$

For the proof, see [3].

Theorem 1.2.2 (Fatou's Lemma). *Let $f_h : X \longrightarrow \overline{\mathbb{R}}$ be μ-measurable functions and $g \in L^1(X, \mu)$.*

- *If $f_h \geq g$ for any $h \in \mathbb{N}$, then*

$$\int_X \liminf_{h\to\infty} f_h \, d\mu \leq \liminf_{h\to\infty} \int_X f_h \, d\mu.$$

- *If $f_h \leq g$ for any $h \in \mathbb{N}$, then*

$$\int_X \limsup_{h\to\infty} f_h \, d\mu \geq \limsup_{h\to\infty} \int_X f_h \, d\mu.$$

For the proof, see [3].

Theorem 1.2.3 (Dominated convergence Theorem)**.** *Let $(f_h)_{h\geq 1}$ a sequence of μ-measurable functions in $L^1(X,\mu)$ such that:*

1. *there exists $\lim_{h\to\infty} f_h(x) := f(x)$ for a.e $x \in X$;*

2. *there exists $g \in L^1(X,\mu)$ such that $|f_h(x)| \leq g$ for a.e $x \in X$.*

Then $f \in L^1(X,\mu)$ and

$$\lim_{h\to\infty} \int_X f_h \, d\mu = \int_X f \, d\mu.$$

For the proof, see [3].

1.3 Absolute continuity and Radon-Nikodym Theorem

We consider now the classical notions of absolute continuity, the Radon-Nikodym Theorem and Lebesgue and polar decomposition. To begin with, we introduce the measure induced by a summable distribution of mass.

Definition 1.19. Let μ be a positive measure on the measure space $(X, \mathscr{F})$ and let $f \in L^1(X,\mu)$. We define the following real measure:

$$f\mu(B) := \int_B f \, d\mu, \qquad \forall\, B \in \mathscr{F}.$$

Using the elementary properties of the integrals, it is easy to check that the above formula defines a real measure. The total variation of the real measure defined above is computed in the following proposition:

Proposition 1.3.1. *Let $f\mu$ be the real measure defined as $f\mu(B) := \int_B f\, d\mu$ $\forall\, B \in \mathscr{F}$. Then its total variation is:*

$$|f\mu|(B) := \int_B |f|\, d\mu, \qquad \forall\, B \in \mathscr{F}.$$

that is $|f\mu| = |f|\mu$.

For the proof, see [2] Proposition 1.23.

Given two measures μ and ν defined on the same measurable space $(X, \mathscr{F})$, it is interesting to find out if they can in any way be related. The following definitions and theorems concern this topic.

Definition 1.20. Let μ be a positive measure and ν a real measure on the measure space $(X, \mathscr{F})$. We say that ν *is absolutely continuous with respect to* μ, and write $\nu \ll \mu$, if for every $B \in \mathscr{F}$ the following implication holds:

$$\mu(B) = 0 \Longrightarrow |\nu|(B) = 0.$$

Definition 1.21. If μ and ν are positive measures, we say that they are *mutually singular*, and write $\nu \perp \mu$, if there exists $E \in \mathscr{F}$ such that $\mu(E) = 0$ and $\nu(X \setminus E) = 0$. If μ or ν are real measures, we say that they are *mutually singular* if $|\mu|$ and $|\nu|$ are so.

Theorem 1.3.2 (Lebesgue decomposition)**.** *Let μ be a positive measure and ν be a real measure on the measure space $(X, \mathscr{F})$. Then there exists a unique pair of real measures ν^a and ν^s such that:*

- $\nu^a \ll \mu$.
- $\nu^s \perp \mu$.
- $\nu = \nu^a + \nu^s$.

For the proof, see [13] Theorem 6.9.

Theorem 1.3.3 (Radon-Nikodym Theorem)**.** *Let μ be a positive measure and ν be a real measure on the measure space $(X, \mathscr{F})$. Assume that μ is σ-finite and $\nu \ll \mu$. Then there is a unique function $f \in L^1(X, \mathscr{F})$ such that:*

$$\nu = f\mu \qquad i.e. \qquad \nu(A) = \int_A f\,d\mu \quad \forall A \in \mathscr{F}.$$

The function f is called the density of ν with respect to μ and it is denoted by $\frac{\nu}{\mu}$.

For the proof, see [13] Theorem 6.9.

Corollary 1.3.4 (Polar decomposition)**.** *Let μ be a real measure on the measure space $(X, \mathscr{F})$. Then there exists a unique S^0-valued function $f \in L^1(X, |\mu|)$ such that $\mu = f|\mu|$.*

Proof. Since each real measure μ is absolutely continuous with respect to $|\mu|$ the Corollary follows form Radon-Nikodym Theorem 1.3.3 and Proposition 1.3.1. □

1.4 Weak convergence in L^p spaces

In this section we discuss some properties of L^p spaces. We will mainly concentrate on the notion of weak convergence of sequences of L^p functions and we will compare this kind of convergence with other notions of convergence, such as strong convergence and covergence a.e.

1.4.1 Main properties of L^p-spaces

Definition 1.22. Let μ be a positive measure on the measure space $(X, \mathscr{F})$ and $f : X \longrightarrow \overline{\mathbb{R}}$ a μ-measurable function. We set:

$$\|f\|_{L^p} := \left(\int_X |f|^p\,d\mu\right)^{\frac{1}{p}} \qquad \text{if } 1 \leq p < \infty$$

and

$$\|f\|_{L^\infty} := \inf\{C \in [0,\infty] : \ |u(x)| \leq C \quad \text{for } \mu\text{-a.e } x \in X\}.$$

We say that $f \in L^p(X, \mu)$ if $\|f\|_{L^p} \leq \infty$.

Remark 9. The set $f \in L^p(X,\mu)$ is a real vector space and $\|\cdot\|_{L^p}$ is a semi-norm; however, when dealing with measure-theoretic properties of functions, it is often convenient to consider as identical the functions that agree almost everywhere and thus, to think of the elements of L^p spaces as equivalence classes. In particular, this makes $\|\cdot\|_{L^p}$ a norm. We shall follow this path whenever our statements will depend only on the equivalence class, without further mention.

We now present some interesting properties of L^p spaces.

Theorem 1.4.1. *Let $1 \le p \le +\infty$. Then the space $L^p(X,\mu)$ with the norm $\|\cdot\|_{L^p}$ is a Banach space.*

For the proof, see [1] Theorem 2.16.

Theorem 1.4.2. *Let X be a separable measure space and $1 \le p < +\infty$. Then the space $L^p(X,\mu)$ is separable.*

For the proof, see [1] Theorem 2.21.

Definition 1.23. Let X be a Banach space. X is *uniformly convex* if, for every $\varepsilon > 0$, there exists $\delta > 0$ so that, for any two elements $x, y \in X$ with $\|x\| \le 1$, $\|y\| \le 1$, the following implication is true:

$$\|x-y\| > \varepsilon \quad \Longrightarrow \quad \left\|\frac{x+y}{2}\right\| < 1-\delta.$$

Theorem 1.4.3. *Let $1 < p < +\infty$. Then the space $L^p(X,\mu)$ is uniformly convex.*

Proof.
As $1 < p < +\infty$, the real variable function $f : t \mapsto |t|^p$ is stricty convex, so

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y) \qquad \forall x, y \in \mathbb{R},\ \lambda \in]0,1[.$$

In particular, if $\lambda = \frac{1}{2}$, $x = t+1$ and $y = t-1$ we have:

$$\left|\frac{1}{2}t + \frac{1}{2} + \frac{1}{2}t - \frac{1}{2}\right|^p < \frac{1}{2}|t+1|^p + \frac{1}{2}|t-1|^p.$$

And so,

$$F(t) := \frac{1}{2}(|t+1|^p + |t-1|^p) - |t|^p > 0, \quad \forall t \in \mathbb{R}. \tag{1.1}$$

Moreover, if $\varepsilon > 0$, F is continuous and strictly positive on every interval $[-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}]$; thus there exists $\gamma > 0$ so that

$$F(t) \geq \gamma, \quad \forall t \in [-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}]. \tag{1.2}$$

Let $h, g \in L^p(X)$ and $t = \frac{h+g}{h-g}$. Using (1.1), we have:

$$\frac{1}{2}\left(\left|\frac{h+g+h-g}{h-g}\right|^p + \left|\frac{h+g-h+g}{h-g}\right|^p\right) > \left|\frac{h+g}{h-g}\right|^p$$

$$\frac{1}{2}\left(2^p\left|\frac{h}{h-g}\right|^p + 2^p\left|\frac{g}{h-g}\right|^p\right) > \left|\frac{h+g}{h-g}\right|^p$$

$$\frac{1}{2}(|h|^p + |g|^p) > \left|\frac{h+g}{2}\right|^p, \qquad \text{for a.e. } x \in U.$$

Moreover, using (1.2) we have:
$\forall \varepsilon > 0$, $\exists \gamma > 0$ such that, if $|h-g| \geq \varepsilon|h+g|$, then

$$\frac{1}{2}\left(2^p\left|\frac{h}{h-g}\right|^p + 2^p\left|\frac{g}{h-g}\right|^p\right) - \left|\frac{h+g}{h-g}\right|^p \geq \gamma$$

$$\frac{1}{2}2^p\left(\left|\frac{h}{h-g}\right|^p + \left|\frac{g}{h-g}\right|^p\right) \geq \left|\frac{h+g}{h-g}\right|^p + \gamma$$

$$\frac{1}{2}(|h|^p + |g|^p) \geq \left|\frac{h+g}{2}\right|^p + \gamma\left|\frac{h-g}{2}\right|^p.$$

Now, fix $\varepsilon > 0$ and let $A = \{x \in X;\, |h(x) - g(x)| < \varepsilon|h(x) + g(x)|\}$ and $B = \{x \in X;\, |h(x) - g(x)| \geq \varepsilon|h(x) + g(x)|\}$.
Suppose $\left\|\frac{h+g}{2}\right\|_{L^p}^p > 1 - \delta$, where we will define $\delta > 0$ as a function of ε.
Suppose further that $\|h\|_{L^p} \leq 1$ and $\|g\|_{L^p} \leq 1$. We have:

$$1 \geq \frac{1}{2}\int_X (|h|^p + |g|^p)\, d\mu = \frac{1}{2}\int_A (|h|^p + |g|^p)\, d\mu + \frac{1}{2}\int_B (|h|^p + |g|^p)\, d\mu \geq$$

$$\geq \int_A \left|\frac{h(x)+g(x)}{2}\right|^p d\mu + \int_B \left|\frac{h(x)+g(x)}{2}\right|^p d\mu + \gamma\int_B \left|\frac{h(x)-g(x)}{2}\right|^p d\mu \geq$$

$$\geq 1 - \delta + \gamma\int_B \left|\frac{h(x)-g(x)}{2}\right|^p d\mu.$$

In particular
$$\int_B \left|\frac{h(x)-g(x)}{2}\right|^p d\mu \leq \frac{\delta}{\gamma}.$$
Thus
$$\int_X \left|\frac{h(x)-g(x)}{2}\right|^p d\mu = \int_A \left|\frac{h(x)-g(x)}{2}\right|^p d\mu + \int_B \left|\frac{h(x)-g(x)}{2}\right|^p d\mu \leq$$
$$\leq \int_A \left|\frac{h(x)-g(x)}{2}\right|^p d\mu + \frac{\delta}{\gamma} \leq$$
$$\leq \varepsilon^p \int_A \left|\frac{h(x)+g(x)}{2}\right|^p d\mu + \frac{\delta}{\gamma} \leq$$
$$\leq \varepsilon^p \frac{1}{2}\int_X |h|^p + |g|^p \, d\mu + \frac{\delta}{\gamma} \leq$$
$$\leq \varepsilon^p + \frac{\delta}{\gamma}.$$
Thus, we just proved that if $\varepsilon > 0$ and $\|h\|_{L^p} \leq 1,\ \|g\|_{L^p} \leq 1$, there exists $\delta = \gamma\varepsilon^p$ such that:
$$\left\|\frac{h+g}{2}\right\|_{L^p}^p > 1-\delta \implies \left\|\frac{h-g}{2}\right\|_{L^p}^p \leq 2\varepsilon^p.$$
Extracting the p^{th} root and using a new $\delta' := 1-(1-\delta)^{\frac{1}{p}}$, we obtain the definition of uniform convexity. □

Definition 1.24. If $1 < p < \infty$, we set $p' := \frac{p}{p-1}$ the *conjugate exponent of* p.
If $p = 1$, we set $p' := +\infty$.
If $p = +\infty$, we set $p' := 1$.

Theorem 1.4.4 (Riesz representation Theorem in L^p spaces)**.**
Suppose $1 \leq p < \infty$, μ *a finite positive measure on* X *and*
$L : L^p(X,\mu) \longrightarrow \mathbb{R}$ *a bounded linear functional on* $L^p(X,\mu)$*. Then there exists a unique function* $g \in L^{p'}(X,\mu)$*, where* p' *is the conjugate exponent of* p*, such that*
$$L(f) = \int_x fg \, d\mu \qquad \forall f \in L^p(X,\mu).$$
Moreover, $\|L\| = \|g\|_{L^{p'}(X,\mu)}$.

For the proof, see [13] Theorem 6.16.

1.4.2 Weak convergence in L^p-spaces

Definition 1.25. Let X be a Banach space and X^* the dual space of X.
Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X . We say that $\{x_n\}_{n=1}^{\infty}$ *converges weakly* to $x \in X$, and we write $x_n \rightharpoonup x$, if

$$\lim_{n\to\infty} f(x_n) = f(x) \quad \forall f \in X^*.$$

In particular, if $X = L^p$ for $1 \le p < +\infty$, thanks to Riesz representation Theorem 1.4.4, the above definition becomes

Definition 1.26. Let $1 \le p < +\infty$. We say that a sequence $(f_k)_{k=1}^{\infty} \subset L^p(X,\mu)$ *converges weakly* to $f \in L^p(X,\mu)$, and we write

$$f_k \rightharpoonup f \quad \text{in } L^p(X,\mu)$$

provided that, for each $g \in L^{p'}(X,\mu)$, we have:

$$\int_X f_k\, g\, d\mu \longrightarrow \int_X f\, g\, d\mu \quad \text{as } k \to +\infty.$$

Remark 10. In the same way we can define weak convergence in $L^\infty(X,\mu)$ checking the convergence as above for functions which belong to the dual of $L^\infty(X,\mu)$. However, usually, we are interested in checking the convegence for functions in $L^1(X,\mu)$ which is not the dual of $L^\infty(X,\mu)$. For this reason, we need to provide a different notion of weak convergence known as weak star convergence.

Definition 1.27. We say that a sequence $(f_k)_{k=1}^{\infty} \subset L^\infty(X,\mu)$ *converges weakly star* to $f \in L^\infty(X,\mu)$, and we write

$$f_k \rightharpoonup f \quad \text{in } L^\infty(X,\mu)$$

provided that, for each $g \in L^1(X,\mu)$, we have:

$$\int_X f_k\, g\, d\mu \longrightarrow \int_X f\, g\, d\mu \quad \text{as } k \to +\infty.$$

We now present some theorems concerning weak convergence in L^p spaces.

Theorem 1.4.5 (Boundedness of weakly convergent sequences).
Assume that $f_k \rightharpoonup f$ *in* $L^p(X,\mu)$*. Then*

1. $(f_k)_{k\geq 1}$ *is bounded in* $L^p(X,\mu)$*;*

2. $\|f\|_{L^p} \leq \liminf_{k\to\infty} \|f_k\|_{L^p}$ *.*

Using Theorem 1.4.5 part 1, it is possible to prove the following proposition:

Proposition 1.4.6. *If* $f_k \rightharpoonup f$ *in* $L^p(X,\mu)$ *and* $g_k \to g$ *in* $L^{p'}(X,\mu)$*, then*

$$\int_X f_k\, g_k\, d\mu \longrightarrow \int_X f\, g\, d\mu.$$

Theorem 1.4.7. *Let* X *be an uniformy convex Banach space and* $(x_n)_{n\geq 1}$ *a sequence in* X *which converges weakly to* $x \in X$*. Then:*

$$x_n \to x \;\; as \;\; n \to +\infty \qquad \Longleftrightarrow \qquad \|x_n\| \to \|x\|\,.$$

Proof.
($\Rightarrow$) Follows from the inequality: $|\|x\| - \|x_n\|| \leq \|x - x_n\|\,.$
($\Leftarrow$) If $x = 0$ then $\|x_n\| \to 0$ and the thesis is proved.
If $x \neq 0$, let $a_n := \max\{\|x\|, \|x_n\|\}$. Using the hypothesis, we have $a_n \to \|x\|$. Define now $y_n := \frac{x_n}{a_n}$ and $y := \frac{x}{\|x\|}$. As $x_n \rightharpoonup x$ and $a_n \to \|x\|$, we have $y_n \rightharpoonup y$ and so also $\frac{y_n - y}{2} \rightharpoonup y$. Moreover, $\|y_n\| = \frac{\|x_n\|}{\|a_n\|} \leq 1$ and $\|y\| = 1$.
As the norm is weak lower semicontinuous, we have:

$$1 = \|y\| \leq \liminf_{n\to+\infty} \left\|\frac{y_n + y}{2}\right\| \leq \limsup_{n\to+\infty} \left\|\frac{y_n + y}{2}\right\| \leq \limsup_{n\to+\infty} \frac{\|y_n\| + \|y\|}{2} \leq 1.$$

So,

$$\lim_{n\to+\infty} \left\|\frac{y_n + y}{2}\right\| = 1.$$

As X is uniformly convex, we deduce from the definition of uniform convexity that $\|y_n - y\| \to 0$; on the other hand, $x_n = a_n y_n$ and $x = y\,\|x\|$, and so:

$$\|x_n - x\| = \|a_n y_n - y\,\|x\|\| \leq \|a_n y_n - y_n\,\|x\|\| + \|x\|\,\|y_n - y\| =$$

$$= |a_n - \|x\||\,\|y_n\| + \|x\|\,\|y_n - y\|\,.$$

As $|a_n - \|x\|| \to 0$, $\|y_n\| \le 1$ and $\|y_n - y\| \to 0$, we have:

$$\|x_n - x\| \le |a_n - \|x\|| \, \|y_n\| + \|x\| \, \|y_n - y\| \to 0$$

and the thesis is proved.

□

Since we showed in Theorem 1.4.3 that the space $L^p(X, \mu)$ is uniformly convex, the above theorem is proved also for L^p spaces, thus the following refinement of Theorem 1.4.5 part 2 holds:

Theorem 1.4.8. *If $1 < p < \infty$ and $f_k \rightharpoonup f$ in $L^p(X, \mu)$, then:*

$$f_k \to f \text{ in } L^p(X,\mu) \quad \iff \quad \|f_k\|_{L^p} \to \|f\|_{L^p}.$$

Theorem 1.4.9 (Weak compactness)**.** *Assume $1 < p < \infty$ and the sequence $(f_k)_{k\ge1}$ is bounded in $L^p(X, \mu)$. Then there exists a subsequence $(f_{k_j})_{j=1}^{\infty} \subset (f_k)_{k=1}^{\infty}$ and a function $f \in L^p(X, \mu)$ such that $f_{k_j} \rightharpoonup f$ in $L^p(X, \mu)$.*

Remark 11. If $p = \infty$, the analogues of Theorem 1.4.5 and 1.4.9 are valid, while if $p = 1$ the weak compactness Theorem 1.4.9 is false.

A sequence which converges weakly and pointwise a.e., in general, does not converge strongly, as the following example shows.

Example 1 (Concentration). Let $f \in C_c^\infty(\mathbb{R})$, suppose $f(0) > 0$ and let $1 < p < \infty$. Consider the sequence

$$f_{\varepsilon_n}(x) = \frac{1}{\varepsilon_n^{\frac{1}{p}}} f\left(\frac{x}{\varepsilon_n}\right).$$

We study the behaviour of this sequence as $n \to \infty$ and $\varepsilon_n \to 0$.
The sequence $\{f_{\varepsilon_n}\}$ converges pointwise to 0 almost everywhere: indeed, as $n \to \infty$ the support of f_{ε_n} concentrates around $x = 0$, hence for any fixed $\overline{x} \neq 0$, there exists $\overline{n}$ such that, for any $n \ge \overline{n}$, $\overline{x} \notin supp(f_{\varepsilon_n})$; so if n is large enough,$f_{\varepsilon_n}(\overline{x}) = 0$.
Moreover,

$$f_{\varepsilon_n}(0) = \frac{1}{\varepsilon_n^{\frac{1}{p}}} f(0) \underset{n\to\infty}{\longrightarrow} +\infty.$$

This proves that the sequence $\{f_{\varepsilon_n}\}$ converges pointwise to 0 almost everywhere.
The sequence $\{f_{\varepsilon_n}\}$ converges weakly to 0: indeed for any fixed $g \in C_c^\infty(\mathbb{R})$, using the change of variables $y = \frac{x}{\varepsilon_n}$, we get:

$$\int_{\mathbb{R}} f_{\varepsilon_n}(x)g(x)\,dx = \int_{\mathbb{R}} \frac{1}{{\varepsilon_n}^{\frac{1}{p}}} f\left(\frac{x}{\varepsilon_n}\right) g(x)\,dx =$$

$$= \int_{\mathbb{R}} \varepsilon_n^{1-\frac{1}{p}} f(y)g(\varepsilon_n y)\,dy \underset{n\to\infty}{\longrightarrow} 0.$$

This proves that the sequence $\{f_{\varepsilon_n}\}$ converges weakly to 0.
However, the sequence $\{f_{\varepsilon_n}\}$ does not converge to 0 in $L^p(\mathbb{R})$: indeed, notice that using the change of variables $y = \frac{x}{\varepsilon_n}$, we get:

$$\|f_{\varepsilon_n}\|^p_{L^p(\mathbb{R})} = \int_{\mathbb{R}} \left| \frac{1}{{\varepsilon_n}^{\frac{1}{p}}} f\left(\frac{x}{\varepsilon_n}\right)\right|^p dx = \int_{\mathbb{R}} \frac{1}{\varepsilon_n} \left| f\left(\frac{x}{\varepsilon_n}\right)\right|^p dx =$$

$$= \int_{\mathbb{R}} |f(y)|^p \, dy = \|f\|^p_{L^p(\mathbb{R})}.$$

So for any n, we have

$$\|f_{\varepsilon_n}\|_{L^p(\mathbb{R})} = \|f\|_{L^p(\mathbb{R})} > 0 \tag{1.3}$$

and so the sequence $\{f_{\varepsilon_n}\}$ does not converge to 0 in $L^p(\mathbb{R})$ because otherwise also

$$\|f_{\varepsilon_n}\|_{L^p(\mathbb{R})} \underset{n\to\infty}{\longrightarrow} 0$$

and this is a contraddiction because of (1.3).

1.4.3 Convergence of averages

We now develop some further insight into the meaning of weak convergence.
Let U be an open bounded smooth subset of $\mathbb{R}^n$, $n \geq 2$.
Assume $1 < p < \infty$ and $f_k \rightharpoonup f$ in $L^p(U)$ which, by Definition 1.26, means that for each $g \in L^{p'}(U)$, we have:

$$\int_U f_k\, g\, dx \longrightarrow \int_U f\, g\, dx \quad \text{as } k \to +\infty.$$

Then, if $E \subset U$ is a bounded measurable set, we deduce, upon setting $g := \chi_E$, that:

$$\int_E f_k\, dx \longrightarrow \int_E f\, dx \quad \text{as } k \to +\infty.$$

This implies that the averages of the functions $\{f_k\}_{k=1}^{\infty}$ over the set E converge to the average of f over E.
Conversely, we have the following statement:

Proposition 1.4.10. *If $\{f_k\}_{k=1}^{\infty}$ is bounded in $L^p(U)$ and $\int_E f_k\,dx \longrightarrow \int_E f\,dx$ as $k \to +\infty$ for each bounded measurable set $E \subset U$, then $f_k \rightharpoonup f$ in $L^p(U)$.*

Proof. We have to show that for any $g \in L^{p'}(U)$, $\int_U f_k\,g\,dx \longrightarrow \int_U f\,g\,dx$ as $k \to +\infty$. Let $g \in L^{p'}(U)$; we can choose a simple function $h = \sum_{l=1}^{N} \alpha_l \chi_{E_l}$, $E_l \subset U$ bounded measurable sets, such that:

$$\|g - h\|_{L^{p'}} \leq \varepsilon \quad \text{for some } \varepsilon > 0.$$

Thus:

$$\begin{aligned} \left|\int_U (f_k - f)\,g\,dx\right| &\leq \left|\int_U (f_k - f)\,(g-h)\,dx\right| + \left|\int_U (f_k - f)\,h\,dx\right| \leq \\ &\leq \int_U |f_k - f|\,|g-h|\,dx + \left|\sum_{l=1}^{N} \alpha_l \int_{E_l} (f_k - f)\,dx\right| \leq \\ &\leq \int_U |f_k - f|\,|g-h|\,dx + \sum_{l=1}^{N} |\alpha_l| \left|\int_{E_l} (f_k - f)\,dx\right| =: \\ &=: \qquad A \qquad + \qquad B. \end{aligned}$$

Using Hölder inequality, the fact that $\{f_k\}_{k=1}^{\infty}$ is bounded in $L^p(U)$ and the choice of h we get:

$$A \leq \|f_k - f\|_{L^p} \|g - h\|_{L^{p'}} \leq (\|f_k\|_{L^p} + \|f\|_{L^p}) \|g - h\|_{L^{p'}} \leq C\varepsilon, \quad C \in \mathbb{R}.$$

Using the hypothesis, if k il large enough, then also $B \leq \varepsilon$. Hence:

$$\left|\int_U (f_k - f)\,g\,dx\right| \leq (C+1)\varepsilon$$

and so $f_k \rightharpoonup f$ in $L^p(U)$.

□

A problem we will face in PDE application is that this convergence of averages, even if, under the above hypothesis, implies weak convergence, does not

imply norm or even a.e convergence. For example, it may be that the sequence $\{f_k\}_{k=1}^{\infty}$ does not weakly converge to f by virtue of perhaps unbounded, very high frequency and quite irregular oscillations. Such behaviour utterly excludes any simple analysis of nonlinear functional of the sequence $\{f_k\}_{k=1}^{\infty}$.
In particular, as the following example shows, $f_k \rightharpoonup f$ in $L^p(U)$ does not imply $F(f_k) \rightharpoonup F(f)$ for any nonlinear real-valued function F.

Example 2. Let $a, b, \lambda \in \mathbb{R},\ a < b$ and $0 < \lambda < 1$ so that

$$F(\lambda a + (1-\lambda)b) \neq \lambda F(a) + (\lambda)F(b).$$

Let $U =]0,1[\subset \mathbb{R}$ and we set:

$$f_k(x) = \begin{cases} a & \text{if } \frac{j}{k} \leq x \leq \frac{j+\lambda}{k}, \quad j = 0, \ldots k-1 \\ b & \text{otherwise} \end{cases}$$

Then $f_k \rightharpoonup f \equiv \lambda a + (1-\lambda)b$ in $L^\infty(U)$, that is:

$$\int_U (f_k\, g\,)dx \longrightarrow (\lambda a + (1-\lambda)b) \int_U g\, dx \quad as\, k \to \infty \qquad \forall g \in L^1(U).$$

This is can be directly checked if g is the characteristc function of measurable subsets of U; using the linearity of the integral, the result is also true if g is a simple function. Finally, let $g \in L^1(U)$ and h be a simple function such that $\|h - g\|_{L^1} \leq \varepsilon$. Then

$$\begin{aligned} \left| \int_U (f_k g\,)dx - \int_U (f g\,)dx \right| &= \left| \int_U (f_k g\, - f g\,)dx \right| = \\ &= \left| \int_U (f_k g\, - f_k h + f_k h - f h + f h - f g)dx \right| \leq \\ &\leq \left| \int_U f_k(g-h)\, dx \right| + \left| \int_U (f_k - f)h\, dx \right| + \left| \int_U f(h-g)\, dx \right| \end{aligned}$$

and, using that f_k and f are bounded and the definition of h, we get:

$$\left| \int_U f_k(g-h)\, dx \right| \leq C\, \|h-g\|_{L^1} \leq C\varepsilon, \quad C \in \mathbb{R}.$$

$$\left| \int_U f(h-g)\, dx \right| \leq D\, \|h-g\|_{L^1} \leq D\varepsilon, \quad D \in \mathbb{R}.$$

Moreover, since h is a simple function and we proved that the result is true for simple functions, if k is large enough, we have:

$$\left| \int_U (f_k - f)h \, dx \right| \leq \varepsilon.$$

This proves that

$$\int_U (f_k \, g \,) dx \longrightarrow (\lambda a + (1-\lambda)b) \int_U \, g \, dx \quad as \, k \to \infty \qquad \forall g \in L^1(U).$$

In the same way we can prove that

$$F(f_k) \rightharpoonup \overline{F} \equiv \lambda F(a) + (\lambda)F(b),$$

but, by definition of F, $\overline{F} \equiv \lambda F(a) + (\lambda)F(b) \neq F(f)$ and so $F(f_k) \not\rightharpoonup F(f)$.

Remark 12. The above calculation imply that $f_k \rightharpoonup f$ in $L^p(U)$ for any p, but $f_k \not\to f$ in $L^p(U)$. If there was strong convergence, then also $F(f_k) \to F(f)$.

1.5 Measures in metric spaces

In this section we consider only locally compact and separable metric spaces. Notice that every such space, toghether with all its open subsets, is a countable union of compact subsets. In this case, we say that the space is *σ-compact*.

Definition 1.28. Let X be a locally compact and separable metric space, $\mathscr{B}(X)$ its Borel σ-algebra and consider the measure space $(X, \mathscr{B}(X))$.
A positive measure on $(X, \mathscr{B}(X))$ is called *Borel measure*.
If a Borel measure is finite on compact sets, it is called *positive Radon measure*.

Definition 1.29. Let X be a locally compact and separable metric space, $\mathscr{B}(X)$ its Borel σ-algebra and consider the measure space $(X, \mathscr{B}(X))$. A *real-valued Radon measure* on X is a real set function defined on the relatively compact subsets of X that is a measure on $(K, \mathscr{B}(K))$ for every compact set $K \in X$.
We denote by $\mathscr{M}_{loc}(X)$ the space of real-valued Radon measures on X.
If $\mu : \mathscr{B}(X) \longrightarrow \mathbb{R}$ is a measure according to Definition 1.8, then we say that μ is a *finite Radon measure* and we denote by $\mathscr{M}(X)$ the space of finite real-valued Radon measures on X.

Let us introduce the spaces of continuous functions on X that provide the functional setting for the duality theory with Radon measures.

Definition 1.30. We denote by $C_c(X)$ the vector space of real continuous function with compact support defined in X endowed with the supremum norm
$\|u\| = \sup\{|u(x)| : \; x \in X\}$.
We denote by $C_0(X)$ the completion of $C_c(X)$ with respect to the supremum norm. $C_0(X)$ is the space of real continuous function that vanish at infinity.

The polar decomposition given by Corollary 1.3.4 leads to the following formula the total variation measure.

Proposition 1.5.1. *Let X be a locally compact and separable metric space and μ a finite real-valued Radon measure on it. Then for every open set $A \subset X$ the following equality holds:*

$$|\mu|(A) = \sup\left\{\int_X u \, d\mu : \; u \in C_c(A), \; \|u\|_\infty \leq 1\right\}.$$

Proof. Let us denote by $B := \left\{\int_X u \, d\mu : \; u \in C_c(A), \; \|u\|_\infty \leq 1\right\}$.
First of all, we show that $|\mu|(A) \geq \sup B$.
Let $f : X \longrightarrow S^0$ be given by Corollary 1.3.4 and fix $A \in X$ open. Since, from Corollary 1.3.4, we know that $\mu = f|\mu|$, we have:

$$\int_A u \, d\mu = \int_A uf \, d|\mu| \leq \|u\|_\infty \int_A |f| \, d|\mu| \leq 1 \int_A 1 \, d|\mu| = |\mu|(A).$$

Thus $|\mu|(A)$ is grater than every element in B and so $|\mu|(A) \geq \sup B$.
Now we show that $|\mu|(A) \leq \sup B$.
Since $C_c(A)$ is dense in $L^1(A, |\mu|)$, we can choose a sequence $\{u_h\}_{h=1}^\infty \subset C_c(A)$ converging to f in $L^1(A, |\mu|)$.
Moreover, by a truncation argument, we can assume that $\|u_h\|_\infty \leq 1$.
Since $\{u_h\}_{h=1}^\infty$ converges to $f\chi_A$ in $L^1(X, |\mu|)$ and using Corollary 1.3.4, we obtain

$$\lim_{h\to\infty} \int_X u_h \, d\mu = \lim_{h\to\infty} \int_X u_h f \, d|\mu|.$$

As $u_h \longrightarrow f\chi_A$ in $L^1(A,|\mu|)$, it is also true that $u_h \rightharpoonup f\chi_A$ and since $f \in L^\infty(X,|\mu|)$, using the Definition of weak convergence 1.26, we deduce that:

$$\lim_{h\to\infty}\int_X u_h f\,d|\mu| = \int_X \chi_A f^2\,d|\mu| = \int_X \chi_A 1\,d|\mu| = \int_A 1\,d|\mu| = |\mu|(A).$$

Thus, we proved that:

$$\lim_{h\to\infty}\int_X u_h\,d\mu = |\mu|(A)$$

and the inequality follows. □

Now we present a definition of outer measure in metric spaces which embodies an additivity condition on separated sets.

Definition 1.31. Let X be a metric space and $\mu : \mathscr{P}(X) \longrightarrow [0,+\infty]$, where $\mathscr{P}(X)$ denotes the set of all the subsets of X. We say that μ is an *outer measure* if $\mu(\emptyset) = 0$, μ is σ-subadditive and the following additivity condition holds:
$\forall E,\, F \in X$

$$\operatorname{dist}(E,F) > 0 \Longrightarrow \mu(E\cup F) = \mu(E) + \mu(F). \tag{1.4}$$

The main result on outer measures is the following Carathéodory Criterion.

Theorem 1.5.2 (Carathéodory Criterion)**.** *Let μ be an outer measure on the metric space X. Then μ is σ-additive on $\mathscr{B}(X)$, hence the restriction of μ to Borel sets of X is a positive measure.*

Proof. By Remark 3, since μ is σ-subadditive on $\mathscr{P}(X)$, and thus on $\mathscr{B}(X)$, in order to prove σ-additivity, it is sufficient to prove that μ is additive on $\mathscr{B}(X)$. To this aim, we set

$$\mathscr{F} = \{E \in \mathscr{B}(X) :\ \mu(B\cap E) + \mu(B\setminus E) = \mu(B) \quad \forall B \in \mathscr{B}(X)\}.$$

Notice that μ is additive on $\mathscr{F}$: given $E, F \in \mathscr{F}$ disjoint, choosing $B := E\cup F \in \mathscr{B}(X)$, we have, using the definiton of $\mathscr{F}$:

$$\mu(E\cup F) = \mu(B) = \mu(B\cap E) + \mu(B\setminus E) = \mu(E) + \mu(F).$$

Hence, in order to show that μ is σ-additive on $\mathscr{B}(X)$, it is sufficient to show that $\mathscr{F}$ is a σ-algebra containing the Borel sets.

STEP 1 The collection $\mathscr{F}$ is a σ-algebra.
$\emptyset \in \mathscr{F}$.

$E \in \mathscr{F} \Rightarrow X \setminus E \in \mathscr{F}$ since, given $B \in \mathscr{B}(X)$ and $E \in \mathscr{F}$ by definition of $\mathscr{F}$:
$\mu(B \cap (X \setminus E)) + \mu(B \setminus (X \setminus E)) = \mu(B \cap E^c) + \mu(B \setminus E^c) = \mu(B \setminus E) + \mu(B \cap E) = \mu(B)$
thus $X \setminus E \in \mathscr{F}$.

$E, F \in \mathscr{F} \Rightarrow E \cup F \in \mathscr{F}$ since, given $B \in \mathscr{B}(X)$ and $E, F \in \mathscr{F}$ we have:

$$\begin{aligned}\mu(B) &= \mu(B \cap E) + \mu(B \setminus E) = \mu(B \cap E) + \mu((B \setminus E) \cap F) + \mu((B \setminus E) \setminus F) = \\ &= \mu(B \cap E) + \mu((B \cap F) \setminus E) + \mu((B \setminus (E \cup F)) = \\ &= \mu(B \cap (E \cup F) \cap E) + \mu(B \cap (E \cup F) \setminus E) + \mu((B \setminus (E \cup F)) = \\ &= \mu(B \cap (E \cup F)) + \mu((B \setminus (E \cup F)).\end{aligned}$$

Thus $E \cup F \in \mathscr{F}$.
This shows that $\mathscr{F}$ is an algebra. Now let $\{E_h\}_{h=1}^{\infty} \subset \mathscr{F}$ and let us show that

$$E = \bigcup_{h=0}^{\infty} E_h \in \mathscr{F}.$$

Possibly replacing E_h for $h \geq 1$ by $E'_h = E_h \setminus \left(\bigcup_{j<h} E_j\right)$, we can assume that E_h are pairwise disjoint. Given $B \in \mathscr{B}(X)$ and using σ-subadditivity of μ we have:

$$\begin{aligned}\mu(B) &\leq \mu(B \setminus E) + \mu(B \cap E) \leq \mu(B \setminus E) + \sum_{h=0}^{\infty} \mu(B \cap E_h) = \\ &= \lim_{n\to\infty} \left(\mu(B \setminus E) + \mu\left(\bigcup_{h=0}^{n} (B \cap E_h) \right) \right) \leq \\ &\leq \liminf_{n\to\infty} \left(\mu(B \setminus \bigcup_{h=0}^{n} E_h) + \mu\left(B \cap \bigcup_{h=0}^{n} E_h \right) \right) \leq \mu(B).\end{aligned}$$

Thus $\mu(B \setminus E) + \mu(B \cap E) = \mu(B)$ and so $E = \bigcup_{h=0}^{\infty} E_h \in \mathscr{F}$.
This proves that the collection $\mathscr{F}$ is a σ-algebra.

STEP 2 We prove that $\mathscr{F} = \mathscr{B}(X)$.
Since $\mathscr{B}(X)$ is generated by closed subsets of X, it is sufficient to prove that $\forall C \subset X$ closed, $C \in \mathscr{F}$. Using the definition of $\mathscr{F}$, we shall prove that given, $B \in \mathscr{B}(X)$, we have $\mu(B \cap C) + \mu(B \setminus C) = \mu(B)$.

Since μ is σ-subadditive, $\mu(B) \leq \mu(B \cap C) + \mu(B \setminus C)$.
Let us show that $\mu(B) \geq \mu(B \cap C) + \mu(B \setminus C)$.
We may assume that $\mu(B) < \infty$ and let:

- $B_0 = \{x \in B : \operatorname{dist}(x, C) \geq 1\}$,
- $B_h = \left\{x \in B : \frac{1}{h+1} \leq \operatorname{dist}(x, C) < \frac{1}{h}\right\}$ for $h \geq 1$.

Using the additivity hypothesis (1.4), and bearing in mind that $\operatorname{dist}(B_{2h}, B_{2h+2}) > 0$ and $\operatorname{dist}(B_{2h+1}, B_{2h+3}) > 0$, we get $\forall n \in \mathbb{N}$:

$$\sum_{h=0}^{n} \mu(B_{2h}) = \mu\left(\bigcup_{h=0}^{n} B_{2h}\right) \leq \mu(B) \quad \text{and} \quad \sum_{h=0}^{n} \mu(B_{2h+1}) = \mu\left(\bigcup_{h=0}^{n} B_{2h+1}\right) \leq \mu(B).$$

Hence

$$\sum_{h=0}^{\infty} \mu(B_h) = \lim_{n\to\infty} \left(\sum_{h=0}^{n} \mu(B_{2h}) + \sum_{h=0}^{n} \mu(B_{2h+1})\right) \leq \lim_{n\to\infty} (2\mu(B)) = (2\mu(B)) < \infty.$$

Since $B \setminus C = \bigcup_{h=0}^{\infty} B_h$ using the σ-subadditivity of μ we get:

$$\begin{aligned}
\mu(B \cap C) + \mu(B \setminus C) &= \mu(B \cap C) + \mu\left(\bigcup_{h=0}^{\infty} B_h\right) \leq \\
&\leq \mu(B \cap C) + \mu\left(\bigcup_{h=0}^{n-1} B_h\right) + \sum_{h=n}^{\infty} \mu(B_h) = \\
&= \mu\left((B \cap C) \cup \bigcup_{h=0}^{n-1} B_h\right) + \sum_{h=n}^{\infty} \mu(B_h) \leq \\
&\leq \mu(B) + \sum_{h=n}^{\infty} \mu(B_h) \quad \forall n \in \mathbb{N}.
\end{aligned}$$

By letting $n \longrightarrow +\infty$ the inequality follows since
$\lim_{n\to\infty} \sum_{h=n}^{\infty} \mu(B_h) = 0$. □

Remark 13. The Carathéodory Criterion 1.5.2 applies as well to set functions defined only on Borel sets.

Let us introduce a construction wich leads to the definition of Lebesgue measure and that is a direct application of Carathéodory Criterion.

Example 3 (Lebesgue measure). Let $Q_r(x) = \{y \in \mathbb{R}^n : \max_i |x_i - y_i| < r\}$ the open cube with side $2r$ centered at x and for any $E \subset \mathbb{R}^n$ set

$$\mu(E) = \inf\left\{\sum_{h=0}^{\infty}(2r_h)^N : E \subset \bigcup_{h=0}^{\infty} Q_{r_h}(x_h)\right\}.$$

It can be direclty checked that μ is an outer measure. In fact, notice that dividing a big cube into smaller cubes does not affect its contribution to the sum defining $\mu(E)$, thus the cubes $Q_{r_h}(x_h)$ can be taken as small as one wants and this leads to the verification of (1.4). Moreover, we can verify the σ-subadditivity condition:
if $E \subset \bigcup_i E_i$ and for any i,

$$\mu(E_i) = \inf\left\{\sum_{h=0}^{\infty}(2r_h)^N : E_i \subset \bigcup_{h=0}^{\infty} Q^i_{r_h}(x_h)\right\},$$

then the cubes $Q^i_{r_h}(x_h)$ covering E_i can be collected to give a doubly indexed cover of $\bigcup_i E_i$, this leads to the σ-subadditivity of μ.
Thus μ is an outer measure which we call *Lebesgue outer measure* and denote by $\mathscr{L}^N$. Since it is finite on compact sets, according to Carathéodory Criterion 1.5.2, its restriction to $\mathscr{B}(X)$ is a positive Radon measure. $E \subset \mathbb{R}^n$ is said to be *Lebesgue measurable* if E belongs to the completion $\mathscr{B}_{\mathscr{L}^N}(\mathbb{R}^n)$. The σ-algebra of Lebesgue measurable sets is denoted by $\mathscr{L}_N$.

1.6 Riesz representation Theorem

In this section we prove the classical Riesz representation Theorem which states that the dual of the Banach space $C_0(X)$ is the space $\mathscr{M}(X)$ of finite real-valued measures on X.
We first recall some theorems which will be used in the proof of Riesz Theorem.

Lemma 1.6.1 (Partitions of unity). *Let X be a locally compact and separable metric space and A be an arbitrary subset of X. Let $\mathscr{P}$ be a collection of open sets in X which cover A, that is $A \subset \bigcup_{U \in \mathscr{P}} U$. Then there exists a collection Ψ of functions $\psi \in C_c(X)$ having the following properties:*

1. *For every* $\psi \in \Psi$ *and every* $x \in X$, $0 \leq \psi(x) \leq 1$.

2. *If* $K \subset\subset A$, *all but finitely many* $\psi \in \Psi$ *vanish identically on* K.

3. *For every* $\psi \in \Psi$, *there exists* $U \in \mathscr{P}$ *such that* $supp(\psi) \subset U$.

4. *For every* $x \in A$, *we have* $\sum_{\psi \in \Psi} \psi(x) = 1$.

For the proof, see [1] Theorem 3.15.

The proof of Riesz representation Theorem is based upon Carathéodory Criterion 1.5.2, Lemma 1.6.1, Theorem 1.4.4 and the following representation theorem for bounded linear and positive functionals.

Theorem 1.6.2. *Let* X *be a locally compact and separable metric space and* $L : C_c(X) \longrightarrow \mathbb{R}$ *be a functional such that:*
L is positive: $L(u) \geq 0$, *whenever* $u \geq 0,\ u \in C_c(X)$.
L is linear: $L(au + bv) = aL(u) + bL(v) \quad \forall u, v \in C_c(X), \forall a, b \in \mathbb{R}$.
L is bounded: $\|L\| = \sup\{L(u) :\ u \in C_c(X),\ |u| \leq 1\} < \infty$.
Then there is a unique positive measure μ *on* X *such that:*

$$L(u) = \int_X u\, d\mu \qquad \forall u \in C_c(X).$$

Proof. For every open set $A \subset X$ let

$$\lambda(A) := \sup\{L(u) :\ u \in C_c(X),\ 0 \leq u \leq 1,\ supp(u) \subset A\}$$

and for every $B \subset X$ let

$$\lambda(B) := \inf\{\lambda(A) : A \,\text{open},\ B \subset A\}.$$

We now show that λ is an outer measure according to Definition 1.31.

STEP 1 If σ-subadditivity and condition (1.4) hold for the open sets, then the general case readily follows.
For every $C \subset X$ let us call $P_C := \{\lambda(A) : A \,\text{open},\ C \subset A\}$.
We prove that, if σ-subadditivity holds for open sets, then, if $B \subset X$ and $B \subset \bigcup_{h=0}^{\infty} B_h$, we have $\lambda(B) \leq \sum_{h=0}^{\infty} \lambda(B_h)$.

Using the definition of λ for arbitrary subsets of X, we have that given $\varepsilon > 0$, for every $h \in \mathbb{N}$, $\quad \lambda(B_h) + \frac{\varepsilon}{2^h} \quad$ is not a lower bound for P_{B_h}. Hence, there exists an open set $A_h \subset X$, $B_h \subset A_h$ such that

$$\lambda(A_h) \leq \lambda(B_h) + \frac{\varepsilon}{2^h}.$$

And so, $A := \bigcup_{h=0}^{\infty}(A_h)$ is an open set such that $B \subset A$ and using σ-subadditivity for open sets we have:

$$\lambda(B) \leq \lambda(A) \leq \sum_{h=0}^{\infty} \lambda(A_h) \leq \sum_{h=0}^{\infty} \lambda(B_h) + 2\varepsilon.$$

Taking the limit for $\varepsilon \longrightarrow 0$ we have the thesis.
We prove that, if condition (1.4) holds for open sets, then, given $E, F \subset X$

$$dist(E, F) > 0 \Rightarrow \lambda(E \cup F) = \lambda(E) + \lambda(F).$$

Let $E, F \subset X$ such that $dist(E, F) =: 3d > 0 \quad d \in \mathbb{R},\ d > 0$.
Using σ-subadditivity we have $\lambda(E \cup F) \leq \lambda(E) + \lambda(F)$.
Now we shall prove that, given $\varepsilon > 0$, $\lambda(E \cup F) \geq \lambda(E) + \lambda(F) - \varepsilon$.
Let $P := \{x \in X :\ dist(x, E) < d\}$ and $Q := \{x \in X :\ dist(x, F) < d\}$. Notice that it easily follows from the assumption $dist(E, F) = 3d > 0$ that $dist(P, Q) > 0$.
Using the definition of λ for arbitrary subsets of X, we have that $\lambda(E \cup F) + \varepsilon$ is not a lower bound for $P_{E \cup F}$. Hence, there exists an open set $A \subset X$, $(E \cup F) \subset A$ such that $\lambda(A) \leq \lambda(E \cup F) + \varepsilon$, that is

$$\lambda(E \cup F) \geq \lambda(A) - \varepsilon.$$

Let $A_1 := P \cap A$ and $A_2 := Q \cap A$.
A_1, A_2 are open sets such that $dist(A_1, A_2) > 0$ as $dist(P, Q) > 0$. Moreover, $A_1 \cup A_2 \subset A$ and so, using (1.4) for open sets $\lambda(A) \geq \lambda(A_1 \cup A_2) = \lambda(A_1) + \lambda(A_2)$.
Finally, $E \subset A_1$ and $F \subset A_2$, thus $\lambda(A_1) \geq \lambda(E)$ and $\lambda(A_2) \geq \lambda(F)$. Hence:

$$\lambda(E \cup F) \geq \lambda(A) - \varepsilon \geq \lambda(A_1) + \lambda(A_2) - \varepsilon \geq \lambda(E) + \lambda(F) - \varepsilon.$$

Taking the limit for $\varepsilon \longrightarrow 0$ we have the thesis.

STEP 2 We prove σ-subadditivity and condition (1.4) for open sets.
Fix A and $\{A_h\}_{h\geq 0}$ open with $A \subset \bigcup A_h$; our aim is to show that $\lambda(A) \leq \sum_{h=0}^{\infty} \lambda(A_h)$, that is

$$\sup\{L(u):\ u \in C_c(X),\ 0 \leq u \leq 1,\ supp(u) \subset A\} \leq$$
$$\leq \sum_{h=0}^{\infty} \sup\{L(v):\ v \in C_c(X),\ 0 \leq v \leq 1,\ supp(v) \subset A_h\}.$$

Thus, it is sufficient to show that if $u \in C_c(X),\ 0 \leq u \leq 1,\ supp(u) \subset A$ then

$$L(u) \leq \sum_{h=0}^{\infty} \lambda(A_h) = \sum_{h=0}^{\infty} \sup\{L(v):\ v \in C_c(X),\ 0 \leq v \leq 1,\ supp(v) \subset A_h\}.$$

Then let $u \in C_c(X)$ with $0 \leq u \leq 1$ and $supp(u) \subset A \subset \bigcup A_h$. Since $supp(u) \subset X$ and $\{A_h\}$ is a collection of open subsets of X which covers $supp(u)$, using Lemma 1.6.1, there exists a collection Ψ of functions $\psi \in C_c(X)$ which verify the four properties listed in the Lemma. In particular, there exists $\psi_1, ...\psi_n$ such that $\forall\, h = 1, ...n$, $supp(\psi_h) \subset A_h$; by property 4 $\forall\, x \in \text{supp}(u)$ we have $\sum_{h=1}^{n} \psi_h(x) = \sum_{h=1}^{\infty} \psi_h(x) = 1$. So, $u = \sum_{h=1}^{n} u\psi_h(x) = \sum_{h=1}^{\infty} u\psi_h(x)$. Using the linearity of L and that $supp(u\psi_h) \subset A_h$, we have:

$$L(u) = \sum_{h=1}^{n} L(u\psi_h) = \sum_{h=1}^{\infty} L(u\psi_h) \leq \sum_{h=1}^{\infty} \lambda(A_h).$$

This prove the σ-subadditivity for open sets.

We now prove condition (1.4). Let E, F open with $dist(E, F) =: 3d > 0$, $d \in \mathbb{R},\ d > 0$. We have

$$\lambda(E \cup F) = \sup\{L(u):\ u \in C_c(X),\ 0 \leq u \leq 1,\ supp(u) \subset E \cup F\},$$
$$\lambda(E) = \sup\{L(v):\ v \in C_c(X),\ 0 \leq v \leq 1,\ supp(v) \subset E\},$$
$$\lambda(F) = \sup\{L(w):\ w \in C_c(X),\ 0 \leq w \leq 1,\ supp(w) \subset F\}.$$

Using σ-subadditivity for open sets we have $\lambda(E \cup F) \leq \lambda(E) + \lambda(F)$.
Now we shall prove that $\lambda(E \cup F) \geq \lambda(E) + \lambda(F)$.
It is sufficient to prove that, if $v \in C_c(X)$, $0 \leq v \leq 1$, $supp(v) \subset E$ and $w \in C_c(X)$, $0 \leq w \leq 1$, $supp(w) \subset F$, then

$$L(v) + L(w) \leq \lambda(E \cup F). \tag{1.5}$$

In fact, if the claim (1.5) is true, then, for a fixed w and for all v we have:

$$L(v) \leq \lambda(E \cup F) - L(w).$$

Since this is true for all v, we have, taking the supremum with respect to v:

$$\lambda(E) \leq \lambda(E \cup F) - L(w).$$

Thus, for any w we have: $L(w) \leq \lambda(E \cup F) - \lambda(E)$.
So taking the supremum with respect to w we obtain: $\lambda(F) \leq \lambda(E \cup F) - \lambda(E)$, that is $\lambda(E \cup F) \geq \lambda(E) + \lambda(F)$.

We now prove claim (1.5). Consider the function $v + w$. By definition of v and w we have $v + w \in C_c(X)$, $0 \leq v + w \leq 1$, $supp(v + w) \subset E \cup F$.
Thus using the linearity of L and the definition of $\lambda(E \cup F)$

$$L(v) + L(w) = L(v + w) \leq \lambda(E \cup F), \qquad \text{as claimed.}$$

STEP 3 We prove that $L(u) \leq 2 \int_X |u| \, d\lambda$.
To this aim we reduce to $0 \leq u \leq 1$ and for every $n \in \mathbb{N}$ and $h = 1, 2, ..., n-1$, set

$$K_h := \{x \in X : \frac{h}{n} \leq u \leq \frac{h+1}{n}\}.$$

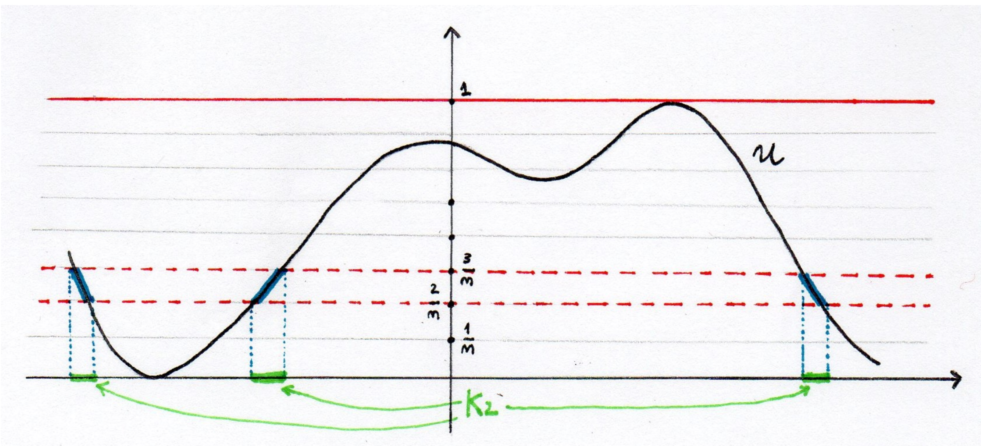

Figure 1.1: K_2.

Then, let $\{U_h\}$ open sets such that

- $K_h \subset U_h$,
- $\lambda(U_h \setminus K_h) < \frac{1}{n^2}$,
- $\forall\, x \in U_h,\ u(x) \leq \frac{h+1}{n} + \frac{1}{n^2}$.

Notice that it is possible to find $\{U_h\}$ open sets that satisfy those properties since by definition $\lambda(K_h) = \inf\{\lambda(A) : A\,\text{open},\ K_h \subset A\}$.
We now show that

$$|L(u)| \leq 2\int_X u\, d\lambda + \frac{1}{n^2}\lambda(X) + \frac{\|L\|}{n} + O(\frac{1}{n}).$$

Write $u = \frac{1}{n} + (u - \frac{1}{n}) = \frac{1}{n} - (u - \frac{1}{n})^- + (u - \frac{1}{n})^+$. Hence

$$\begin{aligned}|L(u)| &= \left| L\left(\frac{1}{n} - (u - \frac{1}{n})^-\right) + L\left((u - \frac{1}{n})^+\right)\right| \leq \\ &\leq \left| L\left(\frac{1}{n} - (u - \frac{1}{n})^-\right)\right| + \left| L\left((u - \frac{1}{n})^+\right)\right|.\end{aligned}$$

Since $\left|\frac{1}{n} - (u - \frac{1}{n})^-\right| \leq \frac{1}{n}$ we have:

$$\left| L\left(\frac{1}{n} - (u - \frac{1}{n})^-\right)\right| \leq \|L\| \left\|\frac{1}{n} - (u - \frac{1}{n})^-\right\|_\infty \leq \frac{\|L\|}{n}.$$

Thus we proved that

$$|L(u)| \leq \frac{\|L\|}{n} + \left| L\left((u - \frac{1}{n})^+\right)\right|.$$

Now we estimate $\left| L\left((u - \frac{1}{n})^+\right)\right|$.
If $x \in K_h$ then by the definition of K_h we have: $\frac{h-1}{n} \leq u(x) - \frac{1}{n} \leq \frac{h}{n}$.
Moreover by definition of U_h

$$supp((u - \frac{1}{n})^+) \subset \bigcup_{h=1}^{n-1} K_h \subset \bigcup_{h=1}^{n-1} U_h.$$

Since $supp((u-\frac{1}{n})^+) \subset X$ and $\{U_h\}_{h=1}^{\infty}$ is a collection of open subsets of X which covers $supp((u-\frac{1}{n})^+)$, using Lemma 1.6.1, there exists a collection Ψ of functions $\psi \in C_c(X)$ which verify the four properties listed in the Lemma. In particular, there exists $\psi_1, ...\psi_{n-1}$ such that $\forall\, h = 1, ...n-1$, $supp(\psi_h) \subset U_h$; by property 4 $\forall\, x \in\ supp((u-\frac{1}{n})^+)$ we have $\sum_{h=1}^{n-1} \psi_h(x) = 1$.
So, $(u-\frac{1}{n})^+ = \sum_{h=1}^{n-1}((u-\frac{1}{n})^+\psi_h(x))$.
Since $supp((u-\frac{1}{n})^+\psi_h(x)) \subset U_h$ and for any $x \in U_h$

$$0 \leq (u-\frac{1}{n})^+\psi_h(x) \leq (u-\frac{1}{n})^+ \leq \frac{h}{n} + \frac{1}{n^2}$$

we obtain

$$\left|L\left((u-\frac{1}{n})^+\right)\right| = \left|\sum_{h=1}^{n-1} L\left((u-\frac{1}{n})^+\psi_h(x)\right)\right| \leq \sum_{h=1}^{n-1}\left(\frac{h}{n}+\frac{1}{n^2}\right)\lambda(U_h) \leq$$
$$\leq \sum_{h=1}^{n-1}\left(\frac{h}{n}+\frac{1}{n^2}\right)\left(\lambda(K_h)+\lambda(U_h \setminus K_h)\right).$$

By definition of U_h, we have $\lambda(U_h \setminus K_h) < \frac{1}{n^2}$, thus:

$$\left|L\left((u-\frac{1}{n})^+\right)\right| \leq \sum_{h=1}^{n-1}\left(\frac{h}{n}+\frac{1}{n^2}\right)\left(\lambda(K_h)+\frac{1}{n^2}\right) \leq$$
$$\leq \sum_{h=1}^{n-1}\frac{h}{n}\lambda(K_h) + \frac{1}{n^2}\sum_{h=1}^{n-1}\lambda(K_h) + \sum_{h=1}^{n-1}\frac{h}{n}\frac{1}{n^2} + \frac{1}{n^2}\sum_{h=1}^{n-1}\frac{1}{n^2}.$$

Since $\sum_{h=1}^{n-1}\frac{h}{n}\lambda(K_h)$ is the integral of a simple function less or equal than u and taking into account that the sets K_h may not be disjoint, by Definition 1.15, we have:

$$\sum_{h=1}^{n-1}\frac{h}{n}\lambda(K_h) \leq 2\int_X u\, d\lambda.$$

Moreover, $\sum_{h=1}^{n-1}\lambda(K_h) \leq \lambda(X)$. And so:

$$\left|L\left((u-\frac{1}{n})^+\right)\right| \leq 2\int_X u\, d\lambda + \frac{1}{n^2}\lambda(X) + \frac{n-1}{2n^2} + \frac{n-1}{n^4} =$$
$$= 2\int_X u\, d\lambda + \frac{1}{n^2}\lambda(X) + O(\frac{1}{n}).$$

In conclusion, if $0 \leq u \leq 1$ we proved that

$$|L(u)| \leq 2\int_X u\, d\lambda + \frac{1}{n^2}\lambda(X) + \frac{\|L\|}{n} + O(\frac{1}{n}).$$

In the general case $u \in C_c(X)$, we notice that there exists $M \in \mathbb{R}$ such that $-M \leq u \leq M$ and we reduce to the previous case considering the positive and negative part of u. In this way we obtain:

$$L(u) \leq |L(u)| \leq |L(u^+)| + |L(u^-)| \leq 2\bigg(\int_X |u|\, d\lambda + \frac{1}{n^2}\lambda(X) + \frac{\|L\|}{n}\bigg) + O(\frac{1}{n})$$

and passing to the limit as $n \longrightarrow +\infty$ we have:

$$L(u) \leq 2\int_X |u|\, d\lambda.$$

STEP 4 Using Riesz representation Theorem 1.4.4 in L^2 and the Hölder inequality, we extend L to $L^2(X,\lambda)$ and we construct μ.
First of all, notice that by Carathéodory Criterion 1.5.2 the measure λ is a positive measure, thus the Hilbert space $L^2(X,\lambda)$ is well defined according to Definition 1.22. If $u \in C_0(X)$, which is dense in $L^2(X,\lambda)$, using Hölder inequality, we have:

$$|L(u)| \leq 2\int_X |u|\, d\lambda \leq 2\bigg(\int_X |u|^2\, d\lambda\bigg)^{\frac{1}{2}} \lambda(X)^{\frac{1}{2}} = c\, \|u\|_{L^2(X,\lambda)}.$$

Thus L can be extended to a linear and continuous functional, which we still call L, such that $L : L^2(X,\lambda) \longrightarrow \mathbb{R}$. By Riesz representation Theorem 1.4.4 in L^2 there exists a unique $v \in L^2(X,\lambda)$ such that $\forall\, u \in L^2(X,\lambda)$ we have:

$$L(u) = \int_X uv\, d\lambda =: \int_X u\, d\mu,$$

where, if $A \in \mathscr{B}(X)$, $\quad \mu(A) = \int_A v\, d\lambda$.
This proves the existence of the measure $\mu := v\lambda$.
Now we claim that the function v found above is $v = 1$ λ-a.e. In order to prove our claim, let $U \subset X$ be an open set; then for any function $u \in C_c(X)$,
$0 \leq u \leq 1,\ supp(u) \subset U$ we have:

$$L(u) = \int_U u\, v\, d\lambda \leq \int_U v\, d\lambda.$$

Hence, by definition of λ we have also:

$$\lambda(U) = \int_U d\lambda = \sup\{L(u) :\ u \in C_c(X),\ 0 \leq u \leq 1,\ supp(u) \subset U\} \leq \int_U v\, d\lambda.$$

Now consider a sequence $\{u_n\}_{n=1}^{\infty}$ such that, for any n, $u_n \in C_c(X)$, $0 \leq u_n \leq 1$, $supp(u_n) \subset U$ and $\{u_n\} \uparrow \chi_U$ in $L^2(X, \lambda)$. Using the definition of λ and monotone convergence Theorem 1.2.1 we get:

$$\lambda(U) \geq L(u_n) = \int_U u_n \, v \, d\lambda \longrightarrow \int_U v \, d\lambda \qquad \text{as } n \longrightarrow +\infty.$$

This proves that for any open set $U \subset X$, $\lambda(U) = \int_U d\lambda = \int_U v \, d\lambda$ and so $v = 1$ λ-a.e. In conclusion $\mu = \lambda$.

STEP 5 Now we prove the uniqueness.
Let μ_1, μ_2 be positive measures on X as in the thesis of the theorem; set $\nu = \mu_1 - \mu_2$. ν is a real-valued finite Radon measure on X and suppose $\int_X u \, d\nu = 0$, $\forall u \in C_c(X)$. By Polar decomposition Theorem 1.3.4, there exists a unique S^0-valued function $f \in L^1(X, |\nu|)$ such that $\nu = f|\nu|$. Since $C_c(X)$ is dense in $L^1(X, |\nu|)$, we can find a sequence $\{h_n\} \subset C_c(X)$ which converges to f in $L^1(X, |\nu|)$.
Moreover, using the above representation for ν and bearing in mind that $\{h_n\} \subset C_c(X)$ we have:

$$\int_X (f - h_n) f \, d|\nu| = \int_X f^2 \, d|\nu| - \int_X h_n f \, d|\nu| = \int_X 1 \, d|\nu| - \int_X h_n \, d\nu = \\ = |\nu|(X) - 0 = |\nu|(X).$$

Thus, using that $\{h_n\}$ converges to f in $L^1(X, |\nu|)$, we get:

$$|\nu|(X) = \int_X (f - h_n) f \, d|\nu| \leq \int_X |f - h_n| \, d|\nu| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

So $|\nu|(X) = 0$, hence $\nu = 0$.
This proves uniqueness and concludes the proof.

□

Now we prove Riesz representation theorem which asserts that the dual of the Banach space $C_0(X)$ is the space $\mathscr{M}(X)$ of finite real-valued measures on X.

Theorem 1.6.3. *Let X be a locally compact and separable metric space and $L : C_0(X) \longrightarrow \mathbb{R}$ be a functional such that:*
L is linear: $L(au + bv) = aL(u) + bL(v) \quad \forall u, v \in C_0(X), \forall a, b \in \mathbb{R}$.
L is bounded: $\|L\|_{C_0(X)} = \sup\{L(u) : u \in C_0(X), \ |u| \leq 1\} < \infty$.
Then there is a unique real Radon finite measure μ on X such that:

$$L(u) = \int_X u \, d\mu \qquad \forall u \in C_0(X). \tag{1.6}$$

Moreover,

$$\|L\|_{C_0(X)} = |\mu|(X). \tag{1.7}$$

Proof.
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Let μ_1, μ_2 real Radon finite measures on X as in the thesis of the theorem;
set $\nu = \mu_1 - \mu_2$.
ν is a real Radon finite measure on X and suppose $\int_X u \, d\nu = 0, \quad \forall u \in C_0(X)$.
By Polar decomposition Theorem 1.3.4, there exists a unique S^0-valued function $f \in L^1(X, |\nu|)$ such that $\nu = f|\nu|$. Since $C_0(X)$ is dense in $L^1(X, |\nu|)$, we can find a sequence $\{h_n\} \subset C_0(X)$ which converges to f in $L^1(X, |\nu|)$.
Moreover, using the above representation for ν and bearing in mind that $\{h_n\} \subset C_0(X)$ we have:

$$\int_X (f - h_n) f \, d|\nu| = \int_X f^2 \, d|\nu| - \int_X h_n f \, d|\nu| = \int_X 1 \, d|\nu| - \int_X h_n \, d\nu =$$
$$= |\nu|(X) - 0 = |\nu|(X).$$

Thus, using that $\{h_n\}$ converges to f in $L^1(X, |\nu|)$, we get:

$$|\nu|(X) = \int_X (f - h_n) f \, d|\nu| \leq \int_X |f - h_n| \, d|\nu| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

So $|\nu|(X) = 0$, hence $\nu = 0$. This proves the uniqueness.

EXISTENCE
Consider a given bounded and linear functional L on $C_0(X)$. Without loss of generality we can assume $\|L\|_{C_0(X)} = 1$. In fact, if $\|L\|_{C_0(X)} = 0$ the only measure which satisfies (1.6) is $\mu = 0$; while, if $\|L\|_{C_0(X)} > 1$, we can consider $\tilde{L} := \frac{L}{\|L\|_{C_0(X)}}$.

The main idea of the proof is to construct a positive linear functional Λ on $C_c(X)$ such that

$$|L(u)| \le \Lambda(|u|) \le \|u\|_\infty \qquad \forall u \in C_c(X). \tag{1.8}$$

Once we have this Λ, we associate with it a positive Borel measure λ as in Theorem 1.6.2. Since $\lambda(X) = \sup\{\Lambda(u) : \ u \in C_c(X),\ 0 \le u \le 1\}$, and since any $u \in C_c(X),\ 0 \le u \le 1$ is such that $\|u\|_\infty \le 1$, using (1.8), we get:

$$\Lambda(u) = \Lambda(|u|) \le \|u\|_\infty \le 1$$

and we see that actually $\lambda(X) \le 1$. We deduce from (1.8) that

$$|L(u)| \le \Lambda(|u|) = \int_X |u|\, d\lambda = \|u\|_{L^1(X,\lambda)} \qquad \forall u \in C_c(X).$$

Thus L is a linear functional on $C_c(X)$ of norm at most 1 with respect to the $L^1(X,\lambda)$-norm. Since $L^1(X,\lambda)$ is dense in $C_c(X)$, there exists a norm preserving extension of L to a linear functional on $L^1(X,\lambda)$ (which we still call L). Therefore, by Theorem 1.4.4 (case $p = 1$) there exists a unique function $g \in L^\infty(X,\lambda)$ with $\|g\|_\infty = \|L\|_{L^1(X,\lambda)}$ such that

$$L(u) = \int_X u\, g\, d\lambda, \qquad \forall u \in C_c(X)\ (\subset L^1(X,\lambda)\ ). \tag{1.9}$$

Since the norm is preserved by the extension, $\|g\|_\infty = \|L\|_{L^1(X,\lambda)} \le 1$
and so $|g| \le 1$ $\lambda-$almost everywhere. Each side of (1.9) is a continuous functional on $C_0(X)$ and since $C_c(X)$ is dense in $C_0(X)$, (1.9) holds for any $u \in C_0(X)$. Hence we obtain (1.6) with $\mu := g\lambda$.
Moreover, since $\|L\|_{C_0(X)} = 1$, (1.9) shows that

$$\int_X |g|\, d\lambda \ge \sup\{|L(u)|; \quad u \in C_0(X)\ \|u\|_\infty \le 1\} = \|L\|_{C_0(X)} = 1.$$

The above inequality is true since, for any $u \in C_0(X)$, $\|u\|_\infty \le 1$, we have

$$|L(u)| = \left|\int_X f g\, d\lambda\right| \le \int_X |f|\,|g|\, d\lambda \le \int_X |g|\, d\lambda.$$

So, to recap, we know that $\lambda(X) \le 1$ and $|g| \le 1$ λ-almost everywhere. These facts are compatible only if $\lambda(X) = 1$ and $|g| = 1$ λ-a.e. because

$$1 = \|L\|_{C_0(X)} = \int_X |g|\, d\lambda \le \|g\|_\infty\, \lambda(X) \le \|g\|_\infty = \|L\|_{L^1(X,\lambda)} \le 1.$$

Hence, by Theorem 1.3.1, we get $d|\mu| = |g|d\lambda$ and so

$$|\mu|(X) = \lambda(X) = 1 = \|L\|_{C_0(X)}$$

and this proves (1.7).
Since $\lambda(X) = 1$ and $|g| = 1$ λ-a.e., then $g \in L^1(X, \lambda)$.
Now we shall construct the positive functional Λ which satisfies (1.8). First of all, we define its values on the functions $u \in C_c^+(X)$, where we denote by $C_c^+(X)$ the space of continuous and positive functions on X with compact support. We define

$$\Lambda(u) := \sup\{|L(h)|; \quad h \in C_c(X),\, |h| \leq u\}, \qquad \forall u \in C_c^+(X). \tag{1.10}$$

It follows that Λ is positive and satisfies (1.8); in fact the first inequality in (1.8) is trivial, since $|u| = u$, while the second one follows from the boundedness of L because $\forall h \in C_c(X),\ |h| \leq u$ we have

$$|L(h)| \leq \|L\|_{C_0(X)} \|h\|_\infty = \|h\|_\infty \leq \|u\|_\infty ,$$

and so Λ is bounded with norm at most 1. Moreover Λ satisfies the following properties:

1. For any $u \in C_c^+(X)$ and $c \geq 0$, we have

$$\Lambda(cu) = c\Lambda(u).$$

2. For any u_1, $u_2 \in C_c^+(X)$ such that $0 \leq u_1 \leq u_2$ we have

$$\Lambda(u_1) \leq \Lambda(u_1).$$

Property 1. follows form the linearity of L, while 2. is true since

$$|h| \leq u_1 \Rightarrow |h| \leq u_2 \Rightarrow \sup_{|h|\leq u_1} |L(h)| \leq \sup_{|\tilde{h}|\leq u_2} |L(\tilde{h})|.$$

To conclude that Λ is linear we shall prove that

$$\Lambda(u + v) = \Lambda(u) + \Lambda(v), \qquad \forall u,\, v \in C_c^+(X) \tag{1.11}$$

and then we will extend Λ to a linear and positive functional on all of $C_c(X)$.
To this aim, let u, $v \in C_c^+(X)$ and $\varepsilon > 0$. By (1.10), there exist h_1, $h_2 \in C_c(X)$ such that

$$|h_1| \leq u, \ |h_2| \leq v \quad \text{and} \quad \Lambda(u) \leq |L(h_1)| + \varepsilon, \ \Lambda(v) \leq |L(h_2)| + \varepsilon.$$

Moreover, we can assume that $L(h_i) \geq 0$ otherwise we consider $-h_i$ for $i = 1, 2$. We have:

$$\Lambda(u) + \Lambda(v) \leq L(h_1) + L(h_2) + 2\varepsilon = L(h_1 + h_2) + 2\varepsilon \leq \Lambda(u + v) + 2\varepsilon$$

and this proves $(\geq)$ in (1.11).
Now choose $h \in C_c(X)$ such that $|h| \leq u + v$ and let $V := \{x; \quad u(x) + v(x) > 0\}$ and define

$$\begin{cases} h_1(x) = \frac{u(x)h(x)}{u(x)+v(x)} & \text{if } x \in V \\ h_1(x) = 0 & \text{if } x \notin V \end{cases} \qquad \begin{cases} h_2(x) = \frac{v(x)h(x)}{u(x)+v(x)} & \text{if } x \in V \\ h_2(x) = 0 & \text{if } x \notin V \end{cases}$$

V is open and h_1 is continuous on V, while on V^c h_1 vanishes identically. Notice that on X we have $|h_1| \leq |h|$. Now let $x_0 \in \partial V$; since $\partial V \subset V^c$ and h is continuous on X, we have

$$|h_1(x)| \leq |h(x)| \longrightarrow 0 \quad \text{as} \quad x \longrightarrow x_0$$

and it follows that $h_1 \in C_c(X)$ and the same holds for h_2.
Since $h_1 + h_2 = h$ and $|h| \leq u + v$, we have that $|h_1| \leq u$ and $|h_2| \leq v$, hence

$$|L(h)| = |L(h_1) + L(h_2)| \leq |L(h_1)| + |L(h_2)| \leq \Lambda(u) + \Lambda(v).$$

Passing to the supremum on h such that $|h| \leq u + v$, we get

$$\Lambda(u + v) \leq \Lambda(u) + \Lambda(v)$$

and this proves $(\leq)$ in (1.11); thus Λ is linear.
Now we extend Λ to a linear and positive functional on all of $C_c(X)$.
Let $u \in C_c(X)$. Since $u = u^+ + u^-$ where u^+, $u^- \in C_c^+(X)$, we define

$$\Lambda(u) = \Lambda(u^+) + \Lambda(u^-) \qquad \forall u \in C_c(X).$$

This extension preserves linearity and this completes the proof. $\square$

1.7 Weak* convergence

In this section we define weak star convergence for sequences of Radon measures, we prove the classical De La Vallée Poussin compactness criterion and we provide some useful properties of weak star convergence for measures.

Riesz Theorem can be restated by saying that the dual of the Banach space $C_0(X)$ is the space $\mathscr{M}(X)$ of finite real-valued measures on X, under the pairing

$$(u, \mu) = \int_X u \, d\mu.$$

Moreover by Proposition 1.5.1, $|\mu|(X)$ is the dual norm.
Analogously, $\mathscr{M}_{loc}(X)$ can be identified with the dual of the locally convex space $C_c(X)$. Accordingly, two different notions of weak* convergence of Radon measures are defined.

Definition 1.32. Let $\mu \in \mathscr{M}_{loc}(X)$ and let $\{\mu_h\}_{h=1}^{\infty} \subset \mathscr{M}_{loc}(X)$. We say that $\{\mu_h\}_{h=1}^{\infty}$ *locally weakly* converges* to μ if

$$\lim_{h\to\infty} \int_X u \, d\mu_h = \int_X u \, d\mu \qquad \forall u \in C_c(X).$$

Definition 1.33. Let $\mu \in \mathscr{M}(X)$ and let $\{\mu_h\}_{h=1}^{\infty} \subset \mathscr{M}(X)$. We say that $\{\mu_h\}_{h=1}^{\infty}$ *weakly* converges* to μ, and we write $\mu_h \rightharpoonup \mu$ if

$$\lim_{h\to\infty} \int_X u \, d\mu_h = \int_X u \, d\mu \qquad \forall u \in C_0(X).$$

Now we prove the classical De La Vallée Poussin compactness criterion for finite Radon measures.

Theorem 1.7.1 (Weak* compactness)**.** *Let X be a locally compact and separable metric space and $\{\mu_h\}_{h=1}^{\infty} \subset \mathscr{M}(X)$ a sequence of finite Radon measures. Assume $\{\mu_h\}_{h=1}^{\infty}$ bounded in $\mathscr{M}(X)$, that is* $\sup\{|\mu_h|(X) : h \in \mathbb{N}\} < \infty$*. Then, there exists a subsequence $\{\mu_{h_j}\}_{j=1}^{\infty} \subset \{\mu_h\}_{h=1}^{\infty}$ and a measure $\mu \in \mathscr{M}(X)$ with $\mu_{h_j} \rightharpoonup \mu$ in $\mathscr{M}(X)$.*

Proof. Assume that $|\mu_h|(X) \leq 1$ $\forall h \in \mathbb{N}$ and let $\{u_k\}_{k=1}^{\infty} \subset C_0(X)$ be a sequance such that $\|u_k\|_\infty \leq 1$ and $G := \mathrm{span}\{u_k,\, k \in \mathbb{N}\}$ is dense in $C_0(X)$. Then using a diagonal argument, it is possible to find a subsequence $\{\mu_{h_j}\}_{j=1}^{\infty}$ such that $\forall k \in \mathbb{N}$ the sequence $\{u_k, \mu_{h_j}\}_{j=1}^{\infty}$ has a limit as $j \longrightarrow +\infty$ whose absolute value does not exceed 1. That is, using the notations above,

$$\int_X u \, d\mu_{h_j} \longrightarrow l_k \qquad \text{as } j \longrightarrow +\infty$$

and $|l_k| \leq 1$ since $\|u_k\|_\infty \leq 1$ and $|\mu_{h_j}|(X) \leq 1$.
The above limit exists in the whole $G := \text{span}\{u_k,\, k \in \mathbb{N}\}$ and so if $u \in G$ then there exists the limit

$$\lim_{j\to\infty} \int_X u \, d\mu_{h_j} =: L(u) \tag{1.12}$$

and L is a linear and continuous functional on G whose norm does not exceed 1. Since G is dense in $C_0(X)$, L can be extended to a linear continuous functional whose norm is less or equal than 1 on $C_0(X)$. This means, thanks to Riesz theorem 1.6.3, that a measure $\mu \in \mathscr{M}(X)$ with $|\mu|(X) = \|L\| \leq 1$ is defined.
Now we prove that $\mu_{h_j} \rightharpoonup \mu$. According to Definition 1.33, let $w \in C_0(X)$. Our aim is to prove that $\lim_{j\to\infty} \int_X w \, d\mu_{h_j} = \int_X w \, d\mu$. Let $\varepsilon > 0$. Since G is dense in $C_0(X)$, there exists $v \in G$ such that $\|w - v\|_\infty \leq \varepsilon$. Thus:

$$\left|\int_X w \, d\mu_{h_j} - \int_X w \, d\mu\right| \leq \left|\int_X |w - v| \, d\mu_{h_j}\right| + \left|\int_X |w - v| \, d\mu\right| + \left|\int_X v \, d\mu_{h_j} - \int_X v \, d\mu\right| \leq$$

$$\leq \|w - v\|_\infty |\mu_{h_j}|(X) + \|w - v\|_\infty |\mu|(X) + \left|\int_X v \, d\mu_{h_j} - \int_X v \, d\mu\right|.$$

Since $v \in G$ and using (1.12) we have

$$\lim_{j\to\infty} \int_X v \, d\mu_{h_j} = L(v) = \int_X v \, d\mu,$$

thus for j lare enough we have

$$\left|\int_X v \, d\mu_{h_j} - \int_X v \, d\mu\right| \leq \varepsilon.$$

Finally, since $\|w - v\|_\infty \leq \varepsilon, \quad |\mu_{h_j}|(X) \leq 1 \quad$ and $\quad |\mu|(X) \leq 1$, we have

$$\left|\int_X w \, d\mu_{h_j} - \int_X w \, d\mu\right| \leq$$

$$\leq \|w - v\|_\infty |\mu_{h_j}|(X) + \|w - v\|_\infty |\mu|(X) + \left|\int_X v \, d\mu_{h_j} - \int_X v \, d\mu\right| \leq 3\varepsilon.$$

So that $\{\mu_{h_j}\}_{j=1}^{\infty}$ weakly* converges to μ. □

In the following lemma we give an approximation theorem for semicontinuous functions through Lipschitz continuous ones.

Lemma 1.7.2. *Let* $c \in \mathbb{R}$, $u : X \longrightarrow [c, +\infty]$ *not identically equal to* $+\infty$, d *the distance function on* X*. Define for* $t > 0$*:*

$$u_t(x) := \inf\{u(w) + td(x, w) : \ w \in X\}.$$

Then $Lip(u_t) \leq t$, $u_t \leq u$ *and, if* x *is a lower semicontinuity point of* u*, then* $u_t(x) \uparrow u(x) \quad as \quad t \uparrow +\infty$.

Proof. Let us call $B_{x,t} := \{u(w) + td(x, w) : \ w \in X\}$.
First of all we prove that u_t is Lipschitz continuous.
Let us call $f_y(w) := u(w) + td(y, w)$ for $y \in X$. Then

$$u_t(y) := \inf B_{y,t} = \inf\{f_y(w) : \ w \in X\}.$$

Let $\{w_k\}_{k=1}^{\infty} \subset X$ a minimizing senquence such that

$$u_t(y) = \inf\{f_y(w) : \ w \in X\} = \lim_{k \to \infty} f_y(w_k).$$

So, $\forall k$ we have:

$$u_t(x) - f_y(w_k) \leq f_x(w_k) - f_y(w_k) = u(w_k) + td(x, w_k) - u(w_k) - td(y, w_k) \leq td(x, y).$$

Thus, passing to the limit for $k \longrightarrow \infty$ we have: $u_t(x) - u_t(y) \leq t\, d(x, y)$.
The above calculation apply in the same way if we exchange x and y and so $|u_t(x) - u_t(y)| \leq t\, d(x, y)$. It follows that $Lip(u_t) \leq t$.
Moreover, $u_t \leq u$ since $u(x) = u(x) + t\, d(x, x) \in B_{x,t}$ and $u_t(x) := \inf B_{x,t}$.
Let x be a lower semicontinuity point of u, so x is such that $\liminf_{y \to x} u(y) \geq u(x)$.
If $u_t(x) \uparrow \infty \quad \text{as} \quad t \uparrow +\infty$, we are done because $u_t \leq u$.
Otherwise, suppose that $u_t(x)$ converges to a finite limit. Let $x_t \in X$ be such that

$$u(x_t) + td(x, x_t) < u_t(x) + 2^{-t}.$$

Then, using the definition of u,

$$td(x, x_t) \leq u_t(x) + 2^{-t} - u(x_t) \leq u_t(x) + 2^{-t} - c$$

and so $t\,d(x,x_t)$ is bounded, hence x_t converge to x as $t \uparrow +\infty$.
Passing to the limit in the inequality $u(x_t) < u_t(x) + 2^{-t}$, and using that x is a lower semicontinuity point of u, we have:

$$u(x) \le \liminf_{t\to\infty} u(x_t) \le \liminf_{t\to\infty}(u_t(x) + 2^{-t}) \le \lim_{t\to\infty} u_t(x).$$

On the other hand, since $u_t \le u$, $\lim_{t\to\infty} u_t(x) \le u(x)$ and then

$$\lim_{t\to\infty} u_t(x) = u(x),$$

as claimed. □

Proposition 1.7.3. *Let X be a locally compact and separable metric space and $\{\mu_h\}_{h=1}^{\infty}$ be a sequence of positive Radon measures on X such that $\mu_h \rightharpoonup \mu$. Then: for every lower semicontinuous function $u :\longrightarrow [0,+\infty]$, we have:*

$$\liminf_{h\to\infty} \int_X u\,d\mu_h \ge \int_X u\,d\mu$$

and for every upper semicontinuous function $v :\longrightarrow [0,+\infty[$ with compact support, we have:

$$\limsup_{h\to\infty} \int_X v\,d\mu_h \le \int_X v\,d\mu.$$

Proof. Let $u :\longrightarrow [0,+\infty]$ be a lower semicontinuous function and, excluding the trivial case $u \equiv \infty$, let u_t be as in Lemma 1.7.2.
Let $\psi \in C_c(X)$ such that $0 \le \psi \le 1$.
Then, using that $\mu_h \rightharpoonup \mu$ and that $u_t\psi \in C_c(X)$, we get:

$$\int_X u_t\psi\,d\mu = \lim_{h\to\infty} \int_X u_t\psi\,d\mu_h \le \liminf_{h\to\infty} \int_X u_t\,d\mu_h$$

and so:

$$\sup_{\psi} \int_X u_t\psi\,d\mu \le \liminf_{h\to\infty} \int_X u_t\,d\mu_h.$$

Since u is lower semicontinuous, by Lemma 1.7.2, $u_t(x) \uparrow u(x)$ $\forall x \in X$, hence by monotone convergence Theorem 1.2.1, we get:

$$\sup_{\psi} \int_X u_t\psi\,d\mu = \int_X u_t\,d\mu \longrightarrow \int_X u\,d\mu \quad \text{as } t \longrightarrow \infty$$

and

$$\liminf_{h\to\infty}\int_X u_t\,d\mu_h \longrightarrow \liminf_{h\to\infty}\int_X u\,d\mu_h.$$

So

$$\liminf_{h\to\infty}\int_X u\,d\mu_h \geq \int_X u\,d\mu.$$

If $v :\longrightarrow [0,+\infty[$ with compact support is upper semicontinuous, we follow a similar argument using $v_t(x) = \sup\{v(y) - t\,d(x,y) : \ x \in X\}$. Since the support of v is compact and v is bounded, there exists a relatively compact neighbourhood U of $supp(v)$ which contains the support of v_t for any t sufficiently large.

Example 4. Interesting particular cases of Proposition 1.7.3 are obtained for characteristic function of compact and open sets. If $\{\mu_h\}_{h=1}^{\infty}$ is a sequence of positive Radon measures on X such that $\mu_h \rightharpoonup \mu$, then:

1. if K is compact we have: $\mu(K) \geq \limsup_h \mu_h(K)$.

2. if A is open we have: $\mu(A) \leq \liminf_h \mu_h(A)$.

Chapter 2

Sobolev spaces

In this chapter we introduce Sobolev spaces and establish some of their most important properties. Sobolev spaces are function spaces whose elements are functions whose partial derivatives satisfy certain integrability conditions.

Definition 2.1. We call *multi-index* an n-tuple of nonnegative integers $\alpha = (\alpha_1, ..., \alpha_n)$. We define the *lenght* of the multi-index α as $|\alpha| := \sum_{i=1}^{n} \alpha_i$.
If $x \in \mathbb{R}^n$, we denote by x^α the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.
If α and β are two multi-indices, we say that $\beta \leq \alpha$ provided that $\beta_j \leq \alpha_j$ for $1 \leq j \leq n$. In this case also $\alpha - \beta$ is a multi-index.
We also denote $\alpha! = \alpha_1! \cdots \alpha_n!$, and if $\beta \leq \alpha$,

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}.$$

If $D_j = \frac{\partial}{\partial x_j}$, then we denote a *differential operator of order* $|\alpha|$ by

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}.$$

Proposition 2.0.4 (Leibniz formula)**.** *Let $x \in \mathbb{R}^n$ and u and v functions which are $|\alpha|$ times continuously differentable near x. Then:*

$$D^\alpha(uv)(x) = \sum_{\beta \leq \alpha} (D^\beta u(x))(D^{\alpha-\beta} v(x)).$$

Definition 2.2. Let Ω be an arbitrary nonempty open set in $\mathbb{R}^n$. Let m be a positive integer and $1 \leq p \leq \infty$. We define a functional $\|\cdot\|_{m,p}$ as follows:

$$\|u\|_{m,p} = \left(\sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{L^p}^p \right)^{\frac{1}{p}} \quad \text{if } 1 \leq p < \infty. \tag{2.1}$$

$$\|u\|_{m,\infty} = \max_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{L^\infty}. \tag{2.2}$$

for any function u for which the right side makes sense.
(2.1) or (2.2) defines a norm on any vector space of functions on which the right side takes finite values provided that functions are identified in the space if they are equal almost everywhere in Ω.

We now define three vector spaces on which $\|\cdot\|_{m,p}$ is a norm.

Definition 2.3. For any positive integer m and $1 \leq p \leq \infty$ we define:

1. $H^{m,p}(\Omega) \equiv$ the completion of $\{u \in C^m(\Omega); \quad \|u\|_{m,p} < \infty\}$ with respect to the norm $\|\cdot\|_{m,p}$.

2. $W^{m,p}(\Omega) \equiv \{u \in L^p(\Omega); \quad D^\alpha u \in L^p(\Omega) \text{ for } 0 \leq |\alpha| \leq m\}$.

3. $W_0^{m,p}(\Omega) \equiv$ the closure of $C_c^\infty(\Omega)$ in the space $W^{m,p}(\Omega)$.

Equipped with the appropriate norm (2.1) or (2.2) they are called
Sobolev spaces over Ω.

Remark 14. Obviously $W^{0,p}(\Omega) = L^p(\Omega)$, and since $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$, if $1 \leq p < \infty$, also $W_0^{0,p}(\Omega) = L^p(\Omega)$.
Moreover, for any m, we have the following chain of imbeddings:

$$W_0^{m,p}(\Omega) \to W^{m,p}(\Omega) \to L^p(\Omega).$$

Theorem 2.0.5. *$W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ endowed with the norm $\|\cdot\|_{m,p}$ are Banach spaces.*

For the proof, see [1] Theorem 3.3.

Corollary 2.0.6. $H^{m,p}(\Omega) \subset W^{m,p}(\Omega)$.

For the proof, see [1] Corollary 3.4.

Theorem 2.0.7. *$W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ are separable if $1 \leq p < \infty$ and they are uniformly convex and reflexive if $1 < p < \infty$.*
In particular $W^{m,2}(\Omega)$ is a separable Hilbert space with inner product:

$$(u,v)_m = \sum_{0 \leq |\alpha| \leq m} (D^\alpha u, D^\alpha v),$$

where $(u,v) = \int_\Omega u(x)\overline{v(x)}dx$ is the inner product in $L^2(\Omega)$.

Proof. Several important properties of the spaces $W^{m,p}(\Omega)$ can be easily obtained by regarding $W^{m,p}(\Omega)$ as a closed subspace of an L^p space on a disjoint union of copies of Ω.
Let $n, m \in \mathbb{N}$, $n \geq 1$, $m \geq 0$ and $N \equiv N(n,m)$ the number of multi-indices $\alpha = (\alpha_1, ..., \alpha_n)$ such that $|\alpha| \leq m$. For each α let Ω_α be a copy of Ω in a different copy of $\mathbb{R}^n$ so that the N domains Ω_α are, by construction, disjoint. Let $\Omega^{(m)}$ be the union of these domains, that is $\Omega^{(m)} = \bigcup_{|\alpha| \leq m} \Omega_\alpha$.
Given $u \in W^{m,p}(\Omega)$, let U be the function on $\Omega^{(m)}$ that coincides with $D^\alpha u$ on Ω_α.
It is easy to check that the map

$$P : W^{m,p}(\Omega) \longrightarrow L^p(\Omega^{(m)}) \tag{2.3}$$

taking u to U is an isometry. Since $W^{m,p}(\Omega)$ is a complete space, also the range W of the isometry P is complete and thus W is a closed subspace of $L^p(\Omega^{(m)})$. Hence W is separable if $1 \leq p < \infty$ and is uniformly convex and reflexive if $1 < p < \infty$. Since $W^{m,p}(\Omega) = P^{-1}(W)$ the same conclusions hold for $W^{m,p}(\Omega)$. In particular $W^{m,2}(\Omega)$ is a separable Hilbert space, since this is true for $L^2(\Omega^{(m)})$; moreover, if u, $v \in W^{m,2}(\Omega)$, then u, $D^\alpha u$, v, $D^\alpha v \in L^2(\Omega)$ and $L^2(\Omega)$ is a separable Hilbert space with the inner product $(u,v) = \int_\Omega u(x)\overline{v(x)}\,dx$ and $\|u\|_{L^2} = (u,u)^{\frac{1}{2}}$; so we define in $W^{m,2}(\Omega)$ the inner product:

$$(u,v)_m = \sum_{0 \leq |\alpha| \leq m} (D^\alpha u, D^\alpha v) \quad \text{and} \quad \|u\|_{W^{m,2}} = ((u,v)_m)^{\frac{1}{2}}.$$

□

We now state the classical Mayers and Serrin theorem $W = H$.

Theorem 2.0.8 (W=H)**.** *If $1 \leq p < \infty$, then $H^{m,p}(\Omega) = W^{m,p}(\Omega)$.*

For the proof, see [1] Theorem 3.17.

Example 5. Theorem 2.0.8 can not be extended to the case $p = \infty$. For instance, let $\Omega = \{x \in \mathbb{R},\ -1 < x < 1\}$ and $u(x) = |x|$.
Then $u'(x) = \frac{x}{|x|}$ for $x \neq 0$ and so $u \in W^{1,\infty}(\Omega)$, but $u \notin H^{1,\infty}(\Omega)$ since, if $0 < \varepsilon < \frac{1}{2}$, there exists no function $\psi \in C^1(\Omega)$ such that $\|\psi' - u'\|_\infty \leq \varepsilon$.

We now formulate a condition on a domain Ω that guarantees that for any k and m, $C^k(\overline{\Omega})$ is dense in $W^{m,p}(\Omega)$ provided that $1 \leq p < \infty$.

Definition 2.4 (Segment Conditon)**.** A domain Ω satisfies the *segment condition* if every $x \in \partial\Omega$ has a neighbourhood U_x and a nonzero vector y_x such that if $z \in \overline{\Omega} \cap U_x$, then $z + ty_x \in \Omega$ for $0 < t < 1$.
If nonempty, the boundary of Ω satisfying this condition must be $(n-1)$-dimensional and must lie on only one part of its boundary.

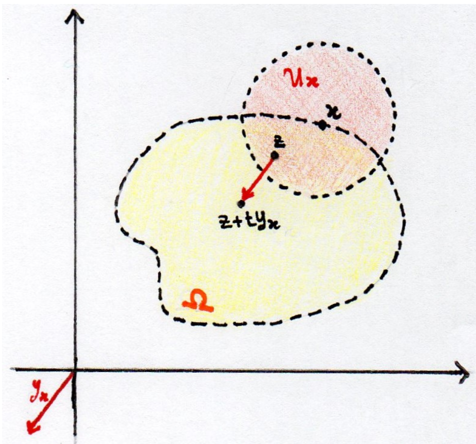

Figure 2.1: Segment condition.

Theorem 2.0.9. *If Ω satisfies the segment condition, then the set of restrictions to Ω of functions in $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{m,p}(\Omega)$ for $1 \leq p < \infty$.*

For the proof, see [1] Theorem 3.22.

2.1 Duality and the spaces $W^{-m,p'}(\Omega)$

Let us fix Ω, m, p, the number N, the space W, the spaces $L^p(\Omega^{(m)})$ and the operator P defined in (2.3). We also define:

$$\langle u, v\rangle := \int_\Omega u(x)v(x)\,dx,$$

for any u, v for which the right side makes sense.
Finally for a given p, let p' denote the conjugate exponent of p.

In this section we first extend the Riesz representation Theorem to the space $W^{m,p}(\Omega)$. Then, we identify the dual of $W_0^{m,p}(\Omega)$ with a subspace of $\mathscr{D}'(\Omega)$. Finally we show that if $1 < p < \infty$, the dual of $W_0^{m,p}(\Omega)$ can also be indentified with the completion of $L^{p'}(\Omega)$ with respect to a norm weaker than the usual $L^{p'}$ norm.

Theorem 2.1.1 (The dual of $L^p(\Omega^{(m)})$)**.** *Let $1 \le p < \infty$.*
Then to every $L \in \left(L^p(\Omega^{(m)})\right)'$, there corresponds a unique $v \in L^{p'}(\Omega^{(m)})$ such that, for every $u \in L^p(\Omega^{(m)})$

$$L(u) = \int_{\Omega^{(m)}} u(x)v(x)\,dx = \sum_{|\alpha|\le m}\int_{\Omega_\alpha} u_\alpha(x)v_\alpha(x)\,dx = \sum_{|\alpha|\le m}\langle u_\alpha v_\alpha\rangle,$$

where u_α, v_α are the restrictions of u, v respectively to Ω_α.
Moreover, $\|L\|_{\left(L^p(\Omega^{(m)})\right)'} = \|v\|_{L^{p'}(\Omega^{(m)})}$, thus $\left(L^p(\Omega^{(m)})\right)' \equiv L^{p'}(\Omega^{(m)})$.

Proof. This is valid because $L^p(\Omega^{(m)})$ is an L^p-space for which Riesz representation Theorem holds.

Theorem 2.1.2 (The dual of $W^{m,p}(\Omega)$)**.** *Let $1 \le p < \infty$.*
Then for every $L \in \left(W^{m,p}(\Omega)\right)'$ there exist elements $v \in L^{p'}(\Omega^{(m)})$ such that, if v_α is the restriction of v to Ω_α, we have, for any $u \in W^{m,p}(\Omega)$,

$$L(u) = \sum_{0\le|\alpha|\le m}\langle D^\alpha u, v_\alpha\rangle. \tag{2.4}$$

Moreover,

$$\|L\|_{\left(W^{m,p}(\Omega)\right)'} = \inf\|v\|_{L^{p'}(\Omega^{(m)})} = \min\|v\|_{L^{p'}(\Omega^{(m)})}\,, \tag{2.5}$$

where the infimum is taken over, and attained, on the set of all functions $v \in L^{p'}(\Omega^{(m)})$ *for which* (2.4) *holds for every* $u \in W^{m,p}(\Omega)$.

Proof. Let W be the range of $P : W^{m,p}(\Omega) \longrightarrow L^p(\Omega^{(m)})$ and define a functional $L^* : W \longrightarrow \mathbb{R}$ such that for any $u \in W^{m,p}(\Omega)$, $L^*(Pu) = L(u)$. By definition of P, Pu coincides with $D^\alpha u$ on Ω_α and since P is an isometric isomorphism, $\|Pu\|_{L^p(\Omega^{(m)})} = \|u\|_{W^{m,p}(\Omega)}$. Thus, since $L^* \in W'$ and using that P is an isometric isomorphism, we have:

$$\|L^*\|_{W'} = \sup\{L^*(Pu);\ \|Pu\|_{L^p(\Omega^{(m)})} = 1\} =$$

$$= \sup\{L(u);\ \|u\|_{W^{m,p}(\Omega)} = 1\} = \|L\|_{\left(W^{m,p}(\Omega)\right)'}.$$

By Hahn-Banach Theorem, there exists a norm preserving extension $\hat{L}$ of L^* to all of $L^p(\Omega^{(m)})$ and by Theorem 2.1.1 there exists $v \in L^{p'}(\Omega^{(m)})$ such that if $u \in L^p(\Omega^{(m)})$, then

$$\hat{L}(u) = \sum_{0 \le |\alpha| \le m} \langle u_\alpha, v_\alpha \rangle.$$

Hence, if $u \in W^{m,p}(\Omega)$, using the definition of $\hat{L}$ and bearing in mind that by definition of P, $(Pu)_\alpha = D^\alpha u$, we obtain:

$$L(u) = L^*(Pu) = \hat{L}(Pu) = \sum_{0 \le |\alpha| \le m} \langle (Pu)_\alpha, v_\alpha \rangle = \sum_{0 \le |\alpha| \le m} \langle D^\alpha u, v_\alpha \rangle.$$

Moreover,

$$\|L\|_{\left(W^{m,p}(\Omega)\right)'} = \|L^*\|_{W'} = \left\|\hat{L}\right\|_{\left(L^p(\Omega^{(m)})\right)'} = \|v\|_{L^{p'}(\Omega^{(m)})}.$$

Now (2.5) must hold since any element $v \in L^{p'}(\Omega^{(m)})$ for which (2.4) holds for any $u \in W^{m,p}(\Omega)$, defines a linear and continuous functional, which we call L. This functional extends L^* since on W it operates as L^*. Moreover $\|v\|_{L^{p'}(\Omega^{(m)})} \ge \|L\|_{\left(W^{m,p}(\Omega)\right)'}$ because the norm of the extension is not less than the norm of the operator which is extended.

Finally, the minimum is attained when v is associated with a norm preserving Hahn-Banach extension. □

Theorem 2.1.3. *If $1 \leq p < \infty$, then every element L of $\left(W^{m,p}(\Omega)\right)'$ is an extension to $(W^{m,p}(\Omega))$ of a distribution $T \in \mathscr{D}'(\Omega)$.*

Proof. Let L be the operator given by (2.4) for some $v \in L^{p'}(\Omega^{(m)})$ and define T, $T_{v_\alpha} \in D'(\Omega)$ such that:

$$T_{v_\alpha}(\phi) = \langle \phi, v_\alpha \rangle \qquad \forall \phi \in \mathscr{D}(\Omega), \quad 0 \leq |\alpha| \leq m$$

and

$$T = \sum_{0 \leq |\alpha| \leq m} (-1)^{|\alpha|} D^\alpha T_{v_\alpha}. \tag{2.6}$$

For any $\phi \in \mathscr{D}(\Omega) \subset W^{m,p}(\Omega)$, we have:

$$\begin{aligned} T(\phi) &= \sum_{0 \leq |\alpha| \leq m} (-1)^{|\alpha|} \langle D^\alpha v_\alpha, \phi \rangle = \sum_{0 \leq |\alpha| \leq m} \langle v_\alpha, D^\alpha \phi \rangle = \\ &= \sum_{0 \leq |\alpha| \leq m} T_{v_\alpha}(D^\alpha \phi) = L(\phi), \end{aligned}$$

hence L is clearly an extension of T. Moreover, using (2.5) we have:

$$\|L\|_{\left(W^{m,p}(\Omega)\right)'} = \min\{\|v\|_{L^{p'}(\Omega^{(m)})},\ L \text{ extends } T \text{ given by (2.6)}\}.$$

□

Remark 15. The conclusion of the above theorem is also true for $L \in \left(W_0^{m,p}(\Omega)\right)'$. In fact, any of such functionals possesses a norm preserving extension to $(W^{m,p}(\Omega))$ and for this extension the thesis of the theorem holds.

Let $T \in \mathscr{D}'(\Omega)$ having the form (2.6) for some $v \in L^{p'}(\Omega^{(m)})$, $1 \leq p' \leq \infty$. Then we proved that T possesses a, possibly non unique, continuous extension to $W^{m,p}(\Omega)$. However it possesses a unique continuous extension to $W_0^{m,p}(\Omega)$, as the following theorem states.

Theorem 2.1.4. *Let $T \in \mathscr{D}'(\Omega)$ having the form (2.6) for some $v \in L^{p'}(\Omega^{(m)})$. Then T possesses a unique continuous extension to $W_0^{m,p}(\Omega)$.*

Proof. Let $u \in W_0^{m,p}(\Omega)$. By definition of $W_0^{m,p}(\Omega)$, it is possible to find a sequence $\{\phi_n\}_{n=0}^{\infty}$ in $C_c^{\infty}(\Omega) \equiv \mathscr{D}(\Omega)$ which converges to u in norm in $W_0^{m,p}(\Omega)$. Thus, using the definition of T and Hölder inequality, we get:

$$\begin{aligned} T(\phi_k) - T(\phi_n) &\leq \sum_{0\leq|\alpha|\leq m} |T_{v_\alpha}(D^\alpha \phi_k - D^\alpha \phi_n)| = \sum_{0\leq|\alpha|\leq m} |L(\phi_k - \phi_n)| = \\ &= \sum_{0\leq|\alpha|\leq m} |\langle D^\alpha(\phi_k - \phi_n), v_\alpha\rangle| \leq \\ &\leq \sum_{0\leq|\alpha|\leq m} \int_{\Omega_\alpha} |D^\alpha(\phi_k - \phi_n)|\,|v_\alpha|\,dx \leq \\ &\leq \sum_{0\leq|\alpha|\leq m} \|D^\alpha(\phi_k - \phi_n)\|_p \|v\|_{p'} \leq \\ &\leq \|\phi_k - \phi_n\|_{m,p} \|v\|_{p'} \longrightarrow 0 \qquad \text{as } k,\, n \longrightarrow \infty. \end{aligned}$$

Hence $\{T(\phi_n)\}_{n=0}^{\infty}$ is a Cauchy sequence in $\mathbb{C}$ and so it converges to a limit which we call $L(u)$ and which does not depend on the choice of the sequence $\{\phi_n\}_{n=0}^{\infty}$. Actually, if also $\{\psi_n\}_{n=0}^{\infty}$ is a sequence in C_c^{∞} which converges to u in norm in $W_0^{m,p}(\Omega)$, with the same passages as above, we can to prove that

$$T(\phi_n) - T(\psi_n) \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$

The functional L thus defined is linear because T is linear:
if $\psi_k \longrightarrow u$ and $\phi_k \longrightarrow v$, then $T(\psi_k) \longrightarrow L(u)$ and $T(\phi_k) \longrightarrow L(v)$, moreover $T(\psi_k + \phi_k) \longrightarrow L(u+v)$, but also $T(\psi_k + \phi_k) = T(\psi_k) + T(\phi_k) \longrightarrow L(u) + L(v)$; and so L is linear. Finally, $L \in \left(W_0^{m,p}(\Omega)\right)'$ since, if
$u = \lim_{n\to\infty} \phi_n$, the L is continuous because

$$|L(u)| = \lim_{n\to\infty} |T(\phi_n)| \leq \lim_{n\to\infty} \|\phi_n\|_{m,p} \|v\|_{p'} = \|u\|_{m,p} \|v\|_{p'} = c\,\|u\|_{m,p}\,.$$

□

Thanks to Theorems 2.1.3 and 2.1.4, we have therefore proved the following theorem:

Theorem 2.1.5 (The normed dual of $W_0^{m,p}(\Omega)$)**.** *If* $1 \leq p < \infty$*,* p' *is the conjugate exponent of* p *and* $m \geq 1$*, then the dual space* $\left(W_0^{m,p}(\Omega)\right)'$ *is isometrically isomorph* *to the Banach space, which we call $W^{-m,p'}(\Omega)$, consisting of those distributions $T \in \mathscr{D}'(\Omega)$ which satisfy* (2.6) *and have norm:*

$$\|T\| = \min\{\|v\|_{L^{p'}(\Omega^m)}\,;\;\; v \text{ satisfy } (2.6)\}$$

Remark 16. $W^{-m,p'}(\Omega)$ is a complete space thanks to the isometric isomorphism. Moreover it is separable and reflexive if $1 < p < \infty$.

Remark 17. When $W_0^{m,p}(\Omega)$ is a proper subset of $W^{m,p}(\Omega)$, continuous linar functionals on $W^{m,p}(\Omega)$ are not fully determined by their restriction to $C_c(\Omega)$ and so are not determined by distributions T given by (2.6).

2.1.1 The $(-m, p')$ norm on $L^{p'}(\Omega)$

There is another way of characterizing the dual of $W_0^{m,p}(\Omega)$ if $1 < p < \infty$.
Eache element $v \in L^{p'}(\Omega)$ determines an element $L_v \in \left(W_0^{m,p}(\Omega)\right)'$ by means of $L(u) = \langle u, v\rangle$; in fact L_v is linear and continuous, since by Hölder inequality:

$$|L_v(u)| = |\langle u, v\rangle| \le \|v\|_{p'} \|u\|_p \le \|v\|_{p'} \|u\|_{m,p}\,.$$

Definition 2.5. The $(-m, p')$-norm of $v \in L^{p'}(\Omega)$ is the norm of the functional L_v, that is:

$$\|v\|_{-m,p'} = \|L_v\|_{\left(W_0^{m,p}(\Omega)\right)'} = \sup_{u\in W_0^{m,p}(\Omega),\, \|u\|_{m,p}\le 1} |\langle u, v\rangle|.$$

Remark 18. Clearly $\|v\|_{-m,p'} \le \|v\|_{p'}$ since by Hölder inequality:

$$\|v\|_{-m,p'} = \sup_{u\in W_0^{m,p}(\Omega),\, \|u\|_{m,p}\le 1} |\langle u, v\rangle| \le \|u\|_{m,p} \|v\|_{p'} \le 1\,\|v\|_{p'} = \|v\|_{p'}\,.$$

Remark 19 (Generalization of Hölder inequality)*.* For any $u \in W_0^{m,p}(\Omega)$ and
$v \in L^{p'}(\Omega)$ we have:

$$|\langle u, v\rangle| = \|u\|_{m,p} \left|\left\langle \frac{u}{\|u\|_{m,p}}, v\right\rangle\right| \le \|u\|_{m,p} \|v\|_{-m,p'}\,.$$

Proposition 2.1.6. *Let $V := \{L_v;\; v \in L^{p'}(\Omega)\}$ which is a vector subspace of $W_0^{m,p}(\Omega)$. Then V is dense in $W_0^{m,p}(\Omega)$.*

Proof. It is sufficient to show that if $F \in \left(W_0^{m,p}(\Omega)\right)''$ such that $F(L_v) = 0$ for any $L_v \in V$, then $F = 0$ in $\left(W_0^{m,p}(\Omega)\right)''$. Since $W_0^{m,p}(\Omega)$ is reflexive, there exists $f \in W_0^{m,p}(\Omega)$ which corresponds to $F \in \left(W_0^{m,p}(\Omega)\right)''$ such that
$\langle f, v\rangle = L_v(f) = F(L_v) = 0$ for any $v \in L^{p'}(\Omega)$, that is:

$$\int_\Omega f(x)\, v(x)\, dx = 0 \qquad \forall v \in L^{p'}(\Omega).$$

So $f(x)$ must be zero almost everywhere in Ω; thus $f = 0$ in $W_0^{m,p}(\Omega)$ and $F = 0$ in $\left(W_0^{m,p}(\Omega)\right)''$. □

Let $H^{-m,p'}(\Omega)$ denote the completion of $L^{p'}(\Omega)$ with respect to the norm $\|\cdot\|_{-m,p'}$. Then we have

$$H^{-m,p'}(\Omega) = \left(W_0^{m,p}(\Omega)\right)' \equiv W^{-m,p'}(\Omega).$$

In particular, corresponding to any $v \in H^{-m,p'}(\Omega)$, there exists a distribution $T_v \in W^{-m,p'}(\Omega)$ such that, for any $\phi \in \mathscr{D}(\Omega)$ and sequence $\{v_n\}_{n=0}^{\infty}$ such that $\lim_{n\to\infty} \|v_n - v\|_{-m,p'} = 0$, we have:

$$T_v(\phi) = \lim_{n\to\infty} \langle \phi, v_n\rangle.$$

Conversely, any $T \in W^{-m,p'}(\Omega)$ satisfies $T = T_v$ for some v of that kind; moreover by Remark 19, $|T_v(\phi)| \le \|\phi\|_{m,p} \|v\|_{-m,p'}$.
This shows that the dual space of $W_0^{m,p}(\Omega)$ can be characterized for $1 < p < \infty$ as the completion of $L^{p'}(\Omega)$ with respect to the norm $\|\cdot\|_{-m,p'}$.

Remark 20. A similar argument as the one provided above shows that the dual space of $W^{m,p}(\Omega)$ can be characterized for $1 < p < \infty$ as the completion of $L^{p'}(\Omega)$ with respect to the following norm:

$$\|v\|_{-m,p'}^{*} = \sup_{u \in W^{m,p}(\Omega),\ \|u\|_{m,p} \le 1} |\langle u, v\rangle|.$$

2.2 Sobolev imbedding Theorem

The imbedding characteristics of Sobolev spaces are essential in their use in analysis, especially in the study of differential an integral operators. The most important imbedding results for Sobolev spaces are gathered into a single theorem, called the *Sobolev imbedding theorem*, although they are of different types and can require different methods of proof. Most of the imbeddings hold for domains $\Omega \in \mathbb{R}^n$ satisfying the "cone condition" which enables us to derive pointwise estimates for the values of a function at the vertex of a truncated cone from suitable averages of the values of the function and its derivatives over the cone. Some of the imbeddings require stronger geometric hypothesis which force Ω to have a $(n-1)$-dimensional boundary which is the graph of a Lipschitz continuous function and to lie on only one side of its boundary. In this section we will first discuss these geometric properties of domains, then we will state the Sobolev imbedding Theorem.

2.2.1 Geometric properties of domains

Imbedding properties of Sobolev spaces depend on regularity properties of the domain Ω. Such properties are normally expressed in terms of geometric or analytic conditions that may or may not be satisfied by a given domain. We specify below some of these conditions.

Definition 2.6. Let v be a nonzero vector in $\mathbb{R}^n$, and for each $x \neq 0$ let $\hat{xv}$ be the angle between the position vector x and v. For given such v, $\rho > 0$ and k such that $0 < k \leq \pi$, we call a *finite cone* of height ρ, axis direction v and aperture k, with vertex at the origin, the following set:

$$C = \{x \in \mathbb{R}^n : \ x = 0 \text{ or } 0 < |x| \leq \rho, \ \hat{xv} \leq \frac{k}{2}\}.$$

Note that $x + C = \{x + y : \ y \in C\}$ is a finite cone with vertex at x but the same dimensions and axis direction as C and it is obtained by parallel traslation of C.

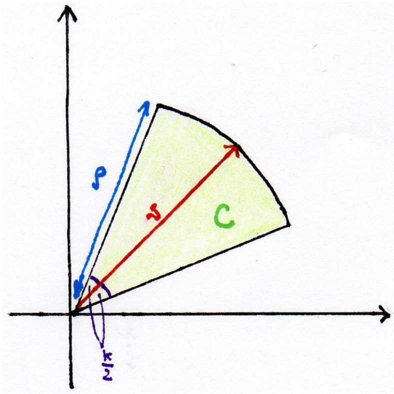

Figure 2.2: Finite cone of height ρ, axis direction v and aperture k, with vertex at the origin.

Definition 2.7. An open cover $\mathscr{P}$ of a set $S \in \mathbb{R}^n$ is said to be *locally finite* if any compact set in $\mathbb{R}^n$ can intersect at most finitely many members of $\mathscr{P}$.
Moreover, such locally finite collection of sets must be countable.
If S is closed, then any open cover of S by sets with a uniform bound on their diameters possesses a locally finite subcover.

We now specify some geometric properties that a domain $\Omega \in \mathbb{R}^n$ may possess and which will be necessary in the following part of this section.
We denote by $\partial\Omega$ the boundary of the domain Ω and by Ω_δ the set of points in Ω within distance δ of the boundary of Ω.

$$\Omega_\delta = \{x \in \Omega : \ dist(x, \partial\Omega) < \delta\}.$$

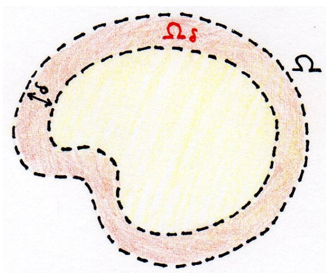

Figure 2.3: Ω_δ.

Definition 2.8 (The Cone Condition)**.** Ω satisfies the *cone condition* if there exists a finite cone C such that each $x \in \Omega$ is a vertex of a finite cone C_x contained in Ω and congruent to C. Note that C_x does not need to be obtained from C by parallel traslation, but simply by rigid motion.

Definition 2.9 (The Strong Local Lipschitz Condition)**.** Ω satisfies the *strong local Lipschitz condition* if there exist positive numbers δ and M and a locally finite open cover $\{U_j\}_{j=0}^{\infty}$ of $\partial\Omega$, and for each j a real-valued function f_j of $n-1$ variables, such that the following conditions hold:

1. For some finite R, every collection of $R+1$ of the sets U_j has empty intersection.

2. For every pair of points x, $y \in \Omega_\delta$ such that $|x-y| < \delta$, there exists j such that
$$x,\, y \in V_j \equiv \{x \in U_j : \ \ dist(x, \partial U_j) > \delta\}.$$

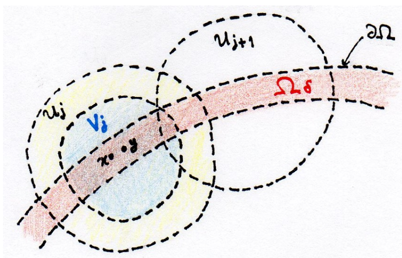

Figure 2.4: Strong local Lipschitz condition, condition 2.

3. Each function f_j satisfies a Lipschitz condition with constant M; that is, if $\xi = (\xi_1, ..., \xi_{n-1})$, $\rho = (\rho_1, ..., \rho_{n-1}) \in \mathbb{R}^{n-1}$, then
$$|f(\xi) - f(\rho)| \leq M|\xi - \rho|.$$

4. For some cartesian coordinate system $(\gamma_{j,1}, ..., \gamma_{j,n})$ in U_j, then $\Omega \cap U_j$ is represented by the inequality:
$$\gamma_{j,n} < f_j(\gamma_{j,1}, ..., \gamma_{j,n1}).$$

Remark 21. If Ω is bounded, the set of condition above reduce to the simple condition that Ω should have a locally Lipschitz boundary, that is each point x on the boundary $\partial\Omega$ should have a neighbourhood W_x whose interection with $\partial\Omega$ should be the graph of a Lipschitz continuous function.

Definition 2.10. Let ϕ be a one-to-one trasformation of a domain $\Omega \in \mathbb{R}^n$ onto a domain $G \in \mathbb{R}^n$ having inverse $\psi = \phi^{-1}$. We say that ϕ is *m-smooth* if, when we write: $y = \phi(x)$ and $x = \psi(y)$ in the form:

$$\begin{aligned} y_1 &= \phi_1(x_1, ..., x_n), \qquad & x_1 &= \psi_1(y_1, ..., y_n), \\ y_2 &= \phi_2(x_1, ..., x_n), \qquad & x_2 &= \psi_2(y_1, ..., y_n), \\ &\vdots \\ y_n &= \phi_n(x_1, ..., x_n), \qquad & x_n &= \psi_n(y_1, ..., y_n). \end{aligned}$$

then $\phi_1, ..., \phi_n$ belong to $C^m(\overline{\Omega})$ and $\psi_1, ..., \psi_n$ belong to $C^m(\overline{G})$.

Definition 2.11 (The Uniform C^m-Regularity Condition)**.** Ω satisfies the *uniform C^m-regularity condition* if there exists a locally finite open cover$\{U_j\}_{j=0}^{\infty}$ of $\partial\Omega$ and a corresponding sequence $\{\phi_j\}_{j=0}^{\infty}$ of m-smooth transformations with ϕ_j taking U_j onto the ball $B = \{y \in \mathbb{R}^n : \ |y| < 1\}$ and having inverse $\psi_j = \phi_j^{-1}$, such that:

1. For some finite R, every collection of $R + 1$ of the sets U_j has empty intersection.

2. For some $\delta > 0$, $\quad \Omega_\delta \subset \bigcup_{j=1}^{\infty} \psi_j\Big(\{y \in \mathbb{R}^n : \ |y| < \frac{1}{2}\}\Big)$.

3. For each j, $\quad \phi_j(U_j \cap \Omega) = \{y \in B : \ y_n > 0\}$.

4. If $(\phi_{j,1}, ..., \phi_{j,n})$ and $(\psi_{j,1}, ..., \psi_{j,n})$ are the components of ϕ_j and ψ_j, then there is a constant M such that for every α with $0 < |\alpha| \leq m$, every i such that $1 \leq i \leq n$, and every j, we have

$$\begin{aligned} |D^\alpha \phi_{j,i}(x)| &\leq M \qquad \text{for } x \in U_j, \\ |D^\alpha \psi_{j,i}(y)| &\leq M \qquad \text{for } y \in B. \end{aligned}$$

Remark 22. Except for the cone condition, the other conditions defined above require the boundary $\partial\Omega$ to be $(n-1)$-dimensional and that Ω lies on only one side of its boundary.
Tipically, most of the imbeddings of $W^{m,p}(\Omega)$ have been proven for domains satisfying the cone condition. However, the imbeddings into spaces $C^j(\overline{\Omega})$ and $C^{j,\lambda}(\overline{\Omega})$ of uniformly continuous functions, require that Ω lies on one side of its boundary. These imbeddings are usually proven for domain satisfying the strong local Lipschitz condition. Finally we note that Ω does not need to satisfy any of these conditions for appropriate imbedding of $W_0^{m,p}(\Omega)$ to be valid.

2.2.2 Sobolev imbedding Theorem

In the following theorem we indicate by:

- $C_B^j(\Omega)$, the space of function having bounded, continuous derivatives up to order j on Ω normed by

$$\|u\|_{C_B^j(\Omega)} = \max_{0\le|\alpha|\le j} \sup_{x\in\Omega} |D^\alpha u(x)|.$$

- $C^j(\overline{\Omega})$, the closed subspace of $C_B^j(\Omega)$ consisting of function having bounded, uniformly continuous derivatives up to order j on Ω with the same norm as $C_B^j(\Omega)$:

$$\|\phi\|_{C^j(\overline{\Omega})} = \max_{0\le|\alpha|\le j} \sup_{x\in\Omega} |D^\alpha \phi(x)|.$$

  This space is smaller than $C_B^j(\Omega)$ in that its elements must be uniformly continuous in Ω.

- $C^{j,\lambda}(\overline{\Omega})$, the closed subspace of $C^j(\overline{\Omega})$ consisting of functions whose derivatives up to order j satisfy Hölder conditions of exponent λ in Ω. The norm on $C^{j,\lambda}(\overline{\Omega})$ is:

$$\|\phi\|_{C^{j,\lambda}(\overline{\Omega})} = \|\phi\|_{C^j(\overline{\Omega})} + \max_{0\le|\alpha|\le j} \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|D^\alpha\phi(x) - D^\alpha\phi(y)|}{|x-y|^\lambda}.$$

Theorem 2.2.1 (Sobolev imbedding Theorem)**.** *Let Ω be a domain in $\mathbb{R}^n$ and for $1 \le k \le n$ let Ω_k be the intersection of Ω with a plane of dimension k in $\mathbb{R}^n$. (If $k = n$, then $\Omega_k = \Omega$.) Let $j \ge 0$ and $m \ge 1$ be integers and let $1 \le p < \infty$. Denote by p^* the critical Sobolev exponent associated with p.*
PART I*: Suppose Ω satisfies the Cone Condition 2.8.*

Case A *If either $mp > n$ or $m = n$ and $p = 1$, then:*

$$W^{j+m,p}(\Omega) \longrightarrow C_B^j(\Omega). \tag{2.7}$$

Moreover, if $1 \le k \le n$, then:

$$W^{j+m,p}(\Omega) \longrightarrow W^{j,q}(\Omega) \qquad \text{for } p \le q \le \infty, \tag{2.8}$$

and, in particular,

$$W^{m,p}(\Omega) \longrightarrow L^q(\Omega) \qquad \text{for } p \le q \le \infty.$$

Case B *If $1 \le k \le n$ and $mp = n$, then:*

$$W^{j+m,p}(\Omega) \longrightarrow W^{j,q}(\Omega_k) \qquad \text{for } p \le q < \infty, \tag{2.9}$$

and, in particular,

$$W^{m,p}(\Omega) \longrightarrow L^q(\Omega) \qquad \text{for } p \le q < \infty.$$

Case C *If $mp < n$ and either $n - mp < k \le n$, or $p = 1$ and $n - m \le k \le n$, then:*

$$W^{j+m,p}(\Omega) \longrightarrow W^{j,q}(\Omega_k) \qquad \text{for } p \le q \le p* = \frac{kp}{(n-mp)}. \tag{2.10}$$

In particular,

$$W^{m,p}(\Omega) \longrightarrow L^q(\Omega) \qquad \text{for } p \le q \le p* = \frac{np}{(n-mp)}. \tag{2.11}$$

The imbedding constants for the imbeddings above depend only on n, m, p, q, j, k *and the dimensions of the cone* C *in the cone condition.*

PART II: *Suppose* Ω *satisfies the Strong Local Lipschitz condition 2.9. Then the target space* $C^j_B(\Omega)$ *of the imbedding* (2.7) *can be replaced with the smaller space* $C^j(\overline{\Omega})$ *and the imbedding can be further refined as follows:*
if $mp > n > (m-1)p$, *then:*

$$W^{j+m,p}(\Omega) \longrightarrow C^{j,\lambda}(\overline{\Omega}) \qquad \textit{for } 0 < \lambda \leq (m - \frac{n}{p}), \tag{2.12}$$

and if $n = (m-1)p$, *then:*

$$W^{j+m,p}(\Omega) \longrightarrow C^{j,\lambda}(\overline{\Omega}) \qquad \textit{for } 0 < \lambda < 1. \tag{2.13}$$

Moreover, if $n = m - 1$ *and* $p = 1$ *then* (2.13) *also holds for* $\lambda = 1$.

PART III: *All of the imbeddings in Part A and Part B are valid for arbitrary domains* Ω *if the* W*-space undergoing the imbedding is replaced with the corresponding* W_0*-space.*

For the proof, see [1] Theorem 4.12.

Targets of the imbeddings

The Sobolev imbedding theorem asserts the existence of imbeddings of $W^{m,p}(\Omega)$ (or $W_0^{m,p}(\Omega)$) into Banach spaces of the following types:

(1) $W^{j,q}(\Omega)$, where $j \leq m$ and in particular in $L^q(\Omega)$.

(2) $W^{j,q}(\Omega_k)$, where for $1 \leq k < n$, Ω_k is the intersection of Ω with a plane of dimension k in $\mathbb{R}^n$.

(3) $C^j_B(\Omega)$.

(4) $C^j(\overline{\Omega})$.

(5) $C^{j,\lambda}(\overline{\Omega})$.

Since elements of $W^{m,p}(\Omega)$ are not functions defined everywhere on Ω but rather equivalence classes of such functions defined and equal up to sets of measure zero, we must clarify what is meant by imbeddings of types $(2)-(5)$.

What is intended for imbedding into the contiuous functions spaces, types $(3)-(5)$ is that the equivalence class $u \in W^{m,p}(\Omega)$ should contain an element that belongs to the continuous function space that is the target of the imbedding and is bounded in that space by a constant times $\|u\|_{W^{m,p}(\Omega)}$. For example, existence of the imbedding

$$W^{m,p}(\Omega) \longrightarrow C^j_B(\Omega)$$

means that each $u \in W^{m,p}(\Omega)$, when considered as a function, can be redefined on a subset of Ω, which has measure zero, to produce a new function $v \in C^j_B(\Omega)$ such that u and v belong to the same equivalence class in $W^{m,p}(\Omega)$ and

$$\|v\|_{C^j_B(\Omega)} \leq K \|u\|_{W^{m,p}(\Omega)} \qquad K \text{ independent of } u.$$

Even more care is necessary to interpret imbedding of type (2).

We should clarify in which way elements in $W^{m,p}(\Omega)$ are observed in $W^{j,q}(\Omega_k)$. The intuitive idea could be to consider the "restriction" to Ω_k of $u \in W^{m,p}(\Omega)$, but given the nature of the elements of $W^{m,p}(\Omega)$, the classical idea of restriction does not make sense for such u. However, the restriction to Ω_k does make sense for functions $u \in C^\infty(\Omega) \cap W^{m,p}(\Omega)$. The imbedding of type (2) is initially proved for functions $u \in C^\infty(\Omega) \cap W^{m,p}(\Omega)$, hence there exists $C > 0$ such that for any $u \in C^\infty(\Omega) \cap W^{m,p}(\Omega)$ we have:

$$\left\| u\big|_{\Omega_k} \right\|_{W^{j,q}(\Omega_k)} \leq C \|u\|_{W^{m,p}(\Omega)}. \tag{2.14}$$

Thanks to (2.14) it is possible to extend the idea of "restriction" for functions $u \in W^{m,p}(\Omega)$. Let $u \in W^{m,p}(\Omega)$; by Theorem 2.0.8 there exists a sequence $\{u_i\}_{i=0}^\infty$ in $C^\infty(\Omega) \cap W^{m,p}(\Omega)$ such that

$$\|u_i - u\|_{W^{m,p}(\Omega)} \longrightarrow 0 \qquad \text{as } i \longrightarrow +\infty. \tag{2.15}$$

For such u_i (2.14) is true; thus, using (2.14) and (2.15), the sequence $\left\{u_i\Big|_{\Omega_k}\right\}_{i=0}^{\infty}$ is a Cauchy sequence in $W^{j,q}(\Omega_k)$:

$$\left\|u_i\Big|_{\Omega_k} - u_s\Big|_{\Omega_k}\right\|_{W^{j,q}(\Omega_k)} \le C\,\|u_i - u_s\|_{W^{m,p}(\Omega)} \longrightarrow 0 \qquad \text{as } i,\, s \longrightarrow +\infty.$$

Hence $\left\{u_i\Big|_{\Omega_k}\right\}_{i=0}^{\infty}$ converges in $W^{j,q}(\Omega_k)$ to a function $v \in W^{j,q}(\Omega_k)$.
This v is independent of the sequence $\{u_i\}_{i=0}^{\infty}$ which converges to u in $W^{m,p}(\Omega)$ because if $\{w_i\}_{i=0}^{\infty} \subset C^{\infty}(\Omega) \cap W^{m,p}(\Omega)$ is another sequence such that

$$\|w_i - u\|_{W^{m,p}(\Omega)} \longrightarrow 0 \qquad \text{as } i \longrightarrow +\infty;$$

then:

$$\left\|u_i\Big|_{\Omega_k} - w_i\Big|_{\Omega_k}\right\|_{W^{j,q}(\Omega_k)} \le C\,\|u_i - w_i\|_{W^{m,p}(\Omega)} \le$$
$$\le C\,\|u_i - u\|_{W^{m,p}(\Omega)} + C\,\|u - w_i\|_{W^{m,p}(\Omega)} \longrightarrow 0 \qquad \text{as } i \longrightarrow +\infty.$$

The function $v \in W^{j,q}(\Omega_k)$ is called *trace* of u on Ω_k and formalizes the intuitive idea of "restriction" to Ω_k of $u \in W^{m,p}(\Omega)$.

2.3 Boundary traces

Of importance in the study of boundary value problems for differential operators defined on a domain Ω is the determination of spaces of functions defined on the boundary of Ω which contain the traces $u\Big|_{\partial\Omega}$ of functions $u \in W^{m,p}(\Omega)$.
In this section we will prove an L^q-imbedding result for such traces which can be obtained for domains with suitably smooth boundaries as a corollary of Sobolev imbedding Theorem 2.2.1 using an extension operator. Then we will prove that functions in $W^{m,p}(\Omega)$ belong to $W_0^{m,p}(\Omega)$ if and only if they have suitably trivial boundary traces.

Definition 2.12. Let Ω be a domain in $\mathbb{R}^n$ satisfying the Uniform C^m-Regularity Condition 2.11. Hence, there exists a locally finite open cover $\{U_j\}_{j=0}^{\infty}$ of $\partial\Omega$ and a corresponding sequence $\{\psi_j\}_{j=0}^{\infty}$ of m-smooth transformations with ψ_j taking the ball $B = \{y \in \mathbb{R}^n : \ |y| < 1\}$ onto U_j, such that, defined $B_0 := \{y \in B; \ \ |y_n = 0\}$,

we have $U_j \cap \partial\Omega = \psi_j(B_0)$.
Let f be a function such that $supp(f) \in U_j$, $d\sigma$ be the $(n-1)$-volume element on $\partial\Omega$, $y' = (y_1, ..., y_{n-1})$ and, if $x = \psi_j(y)$, then let

$$J_j(y') = \left[\sum_{k=1}^{n} \left(\frac{\partial(x_1, ..., \hat{x_k}, ..., x_n)}{\partial(y_1, ..., y_{n-1})} \right)^2 \right]^{\frac{1}{2}} \Bigg|_{y_n=0}.$$

We define the *integral* of f over $\partial\Omega$ as follows:

$$\int_{\partial\Omega} f(x)\, d\sigma = \int_{U_j \cap \partial\Omega} f(x)\, d\sigma \int_{B_0} f \circ \psi_j(y', 0) J_j(y')\, dy'. \tag{2.16}$$

If f is an arbitrary funcion defined on $\mathbb{R}^n$ and $\{v_j\}_{j=0}^{\infty}$ is a partition of unity subordinate to U_j, we set:

$$\int_{\partial\Omega} f(x)\, d\sigma = \sum_j \int_{\partial\Omega} f(x) v_j(x)\, d\sigma. \tag{2.17}$$

Definition 2.13. Let Ω be a domain in $\mathbb{R}^n$. For given m and p, let E be a linear operator mapping $W^{m,p}(\Omega)$ into $W^{m,p}(\mathbb{R}^n)$.
E is called a *simple (m,p)-extension operator for* Ω if there exists a constant $K = K(m,p)$ such that for evey $u \in W^{m,p}(\Omega)$ the following conditions hold:

1. $Eu(x) = u(x)$ a.e in Ω,
2. $\|Eu\|_{W^{m,p}(\mathbb{R}^n)} \leq K \|u\|_{W^{m,p}(\Omega)}$.

Theorem 2.3.1 (A Boundary Trace Imbedding Theorem)**.** *Let Ω be a domain in $\mathbb{R}^n$ satisfying the uniform C^m-regularity condition 2.11 and suppose there exists a simple (m,p)-extension operator E for Ω. Suppose also that $mp < n$ and $p \leq q \leq p^* = \frac{(n-1)p}{(n-mp)}$. Then*

$$W^{m,p}(\Omega) \longrightarrow L^q(\partial\Omega). \tag{2.18}$$

If $mp = n$, then imbedding (2.18) *holds for $p \leq q < \infty$.*

Remark 23. Imbedding (2.18) should be interpreted in the following sense.
Let U_j be one of the open sets which cover $\partial\Omega$ and ψ_j the corresponding m-smooth transformation such that $U_j \cap \partial\Omega = \psi_j(B_0)$.

If $u \in W^{m,p}(\Omega)$, then $Eu =: v$ has a trace on $\partial\Omega$ in the sense described in Section 2.2.2. In fact, for $y \in B$ let $w(y) = v(\psi(y))$, that is $w = v \circ \psi$; then $w \in W^{m,p}(B)$. Let $\tilde{w}$ the trace of w on $\{y_n = 0\}$ as defined in Section 2.2.2. We define $\tilde{v} = \tilde{w} \circ \psi^{-1}$; $\tilde{v}$ is defined on $U_j \cap \partial\Omega$ and is the *trace* of v (thus of u) on $U_j \cap \partial\Omega$.
Moreover $\|Eu\|_{W^{0,q}(\partial\Omega)} \leq K \|u\|_{W^{m,p}(\Omega)}$ with K independent of u.
Note that, since $C_c(\mathbb{R}^n)$ is dense in $W^{m,p}(\Omega)$, $\|Eu\|_{W^{0,q}(\partial\Omega)}$ is independent of the particular extension operator E used.

Proof. We prove the case $mp < n$ and $q = p^* = \frac{(n-1)p}{(n-mp)}$; the other cases are similar. By definition of E, there exists a constant K_1 such that

$$\|Eu\|_{W^{m,p}(\mathbb{R}^n)} \leq K_1 \|u\|_{W^{m,p}(\Omega)}.$$

By the Uniform C^m-Regularity Condition 2.11 there exists a constant K_2 such that for each j and every $y \in B$, we have

$$x = \psi_j(y) \in U_j, \qquad |J_j(y')| \leq K_2, \qquad \left|\frac{\partial(y_1, ..., y_n)}{\partial(x_1, ..., x_n)}\right| \leq K_2.$$

Since $0 \leq v_j(x) \leq 1$ on $\mathbb{R}^n$, and using the imbedding (2.10) of Sobolev imbedding Theorem 2.2.1 applied over B, we have, for $u \in W^{m,p}(\Omega)$,

$$\begin{aligned}
\int_{\partial\Omega} |Eu(x)|^q \, d\sigma &\leq \sum_j \int_{U_j \cap \partial\Omega} |Eu(x)|^q \, d\sigma \leq \\
&\leq K_2 \sum_j \|Eu \circ \psi_j\|^q_{W^{0,q}(B_0)} \leq \\
&\leq K_3 \sum_j \left(\|Eu \circ \psi_j\|^p_{W^{p,m}(B)}\right)^{\frac{q}{p}} \leq \\
&\leq K_4 \sum_j \left(\|Eu\|^p_{W^{p,m}(U_j)}\right)^{\frac{q}{p}}.
\end{aligned}$$

Using the finite intersection property possessed by the cover $\{U_j\}_{j=0}^{\infty}$ we get:

$$K_4 \sum_j \left(\|Eu\|^p_{W^{p,m}(U_j)}\right)^{\frac{q}{p}} \leq K_4 R \|Eu\|^q_{W^{p,m}(\mathbb{R}^n)}.$$

Note that the constant K_4 is independent of j because, if $\psi_j = (\psi_{j,1}, ...\psi_{j,n})$, then $|D^\alpha \psi_{j,i}(y)| \leq c$ for all i, j where c is a constant.

In conclusion, using the definition of E, we obtain:

$$\int_{\partial\Omega} |Eu(x)|^q \, d\sigma \leq K_4 R \, \|Eu\|^q_{W^{p,m}(\mathbb{R}^n)} \leq$$
$$\leq K_5 \, \|u\|^q_{W^{p,m}(\Omega)} \, .$$

This completes the proof. □

Now we will prove that functions in $W^{m,p}(\Omega)$ belong to $W_0^{m,p}(\Omega)$ if and only if they have suitably trivial boundary traces. First of all, we state some theorems which will be needed in the proof.

Definition 2.14. If function u is defined on Ω , we denote by $\tilde{u}$ the *zero extension* of u to the complement Ω^c of Ω in $\mathbb{R}^n$:

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \Omega^c \end{cases}$$

The following lemma shows that the mapping $u \longmapsto \tilde{u}$ maps $W_0^{m,p}(\Omega)$ isometrically into $W^{m,p}(\mathbb{R}^n)$

Lemma 2.3.2. *Let $u \in W_0^{m,p}(\Omega)$. If $|\alpha| \leq m$, then $D^\alpha \tilde{u} = \widetilde{D^\alpha u}$ in the distributional sense in $\mathbb{R}^n$. Hence $\tilde{u} \in W^{m,p}(\mathbb{R}^n)$.*

For the proof, see [1] Lemma 3.27.

Now we state a theorem which gives a characterization of $W_0^{m,p}(\Omega)$ by exterior extension.

Theorem 2.3.3. *Suppose that Ω satisfies the segment condition.*
Then a function u on Ω belongs to $W_0^{m,p}(\Omega)$ if and only if the zero extension $\tilde{u}$ of u belongs to $W^{m,p}(\mathbb{R}^n)$.

For the proof, see [1] Theorem 5.29.

Remark 24. Note that Lemma 2.3.2 shows, with no hypotesis on Ω, that
if $u \in W_0^{m,p}(\Omega)$, then $\tilde{u} \in W^{m,p}(\mathbb{R}^n)$.

Theorem 2.3.4 (Trivial traces)**.** *Under the same hypothesis as Theorem 2.3.1, a function $u \in W_0^{m,p}(\Omega)$ belongs to $W_0^{m,p}(\Omega)$ if and only if the boundary traces of its derivatives of order less than m all coincide with the 0-function.*

Proof. ($\Rightarrow$) First of all, notice that every function in C_0^∞ has trivial boundary trace and so do all derivatives of such functions. Since the trace mapping is a linear and continuous operator from $W^{m,p}(\Omega)$ to $W^{m-1,p}(\partial\Omega)$, all functions in $W_0^{m,p}(\Omega)$ have trivial boundary traces, and so do their derivatives of order less then m.
($\Leftarrow$) Let $u \in W^{m,p}(\Omega)$ such that u and all its derivatives of order less than m have trivial boundary traces. We can reduce, using localization and a suitable change of variables, to the case where $\Omega = \{x \in \mathbb{R}^n; \quad x_n > 0\}$.
Now we will show that the zero-extension $\tilde{u}$ of u belongs to $W^{m,p}(\mathbb{R}^n)$ and so, by Theorem 2.3.3, we will conclude that $u \in W_0^{m,p}(\Omega)$. To this aim, we claim that, if $u \in W^{m,p}(\Omega)$ such that u and all its derivatives of order less than m have trivial boundary traces, then the distributional derivatives $D^\alpha\tilde{u}$ of order at most m coincide with the zero-extension $\widetilde{D^\alpha u}$. In fact, we first approximate the integrals

$$\int_{\mathbb{R}^n} \tilde{u}(x)\, D^\alpha\phi(x)\, dx \qquad \text{and} \qquad (-1)^\alpha \int_{\mathbb{R}^n} \widetilde{D^\alpha u}(x)\ \phi(x)\, dx \tag{2.19}$$

by approximating u with functions $v_j \in C^\infty(\overline{\Omega})$ without requiring that these functions have trivial traces. Since $v_j \in C^\infty(\overline{\Omega})$, we can integrate by parts with respect to $x_1, ..., x_{n-1}$ and then with respect to x_n. In this way, denoting by e_n the unit vector $(0, ..., 0, 1)$, we show that the difference

$$\left(\int_{\mathbb{R}^n} \tilde{v}_j(x)\, D^\alpha\phi(x)\, dx\right) - \left((-1)^\alpha \int_{\mathbb{R}^n} \widetilde{D^\alpha v_j}(x)\ \phi(x)\, dx\right)$$

is a finite alternating sum of integrals of the form

$$\int_{\mathbb{R}^{n-1}} \left(D^{\alpha-ke_n} v_j(x_1, ..., x_{n-1}, 0)\, D_n^{k-1}\phi(x_1, ..., x_{n-1}, 0)\right) dx_1...dx_{n-1}, \quad k > 0 \tag{2.20}$$

By Theorem 2.0.9 we can choose the sequence $\{v_j\}_{j=0}^\infty$ to converge to u in $W^{m,p}(\Omega)$. Thus, for each multi-index β such that $\beta < \alpha$, the trace of $D^\beta v_j$ will converge in $L^p(\mathbb{R}^{n-1})$ to the trace of $D^\beta u$, which is 0 in that space. Since the restriction of $D_n^{k-1}\phi$ to $\mathbb{R}^{n-1}$ belongs to $L^{p'}(\mathbb{R}^{n-1})$, each of the integrals (2.20) tends to 0 as $j \longrightarrow +\infty$. If follows that the two integrals (2.19) are equal and so $\tilde{u}$ belongs to $W^{m,p}(\mathbb{R}^n)$ as desired. □

2.4 Compactness in Sobolev spaces

In this section we prove some useful compactness theorems for measures and we provide a refinement of Rellich Kondrachov Theorem.
We denote by U an open, bounded, smooth subset of $\mathbb{R}^n$, $n \geq 2$.

A consequence of Definition 1.25 and of representation theorems for $W^{1,p}$, is the following definition of weak convergence in $W^{1,p}(U)$.

Definition 2.15. Let $1 \leq p < \infty$. We say that a sequence $\{f_k\}_{k=0}^{\infty} \subset W^{1,p}(U)$ *converges weakly to* $f \in W^{1,p}(U)$, and we write $f_k \rightharpoonup f$, provided that
$f_k \rightharpoonup f$ in $L^p(U)$ and $Df_k \rightharpoonup Df$ in $L^p(U; \mathbb{R}^n)$.

For later reference we state the Gagliardo-Nirenberg-Sobolev inequality.

Theorem 2.4.1 (Gagliardo-Nirenberg-Sobolev Inequality)**.** *If* $1 \leq p < n$ *and* $p^* = \frac{pn}{(n-p)}$ *is the critical Sobolev exponent associated with* p*, then*

$$\|f\|_{L^{p^*}(\mathbb{R}^n)} \leq C_p \|Df\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}, \tag{2.21}$$

for any function $f \in C_c^1(\mathbb{R}^n)$*.*
The optimal constant C_p *depends only on* p *and* n*.*

For the proof, see [1] Theorem 4.31.

Remark 25. Invoking usual approximations, the above estimate is also true provided $f \in L^{p^*}$ and $Df \in L^p$.
Moreover, it follows from (2.4.1) and standard extension theorems that, for each $1 \leq q \leq p^*$ and $f \in W^{1,p}(U)$, since U is bounded,

$$\|f\|_{L^q(U)} \leq C \|f\|_{W^{1,p}(U)}.$$

The constant C depends only on p, n and U.

Now we state the Rellich-Kondrachov Theorem which asserts that the imbedding $W^{1,p}(U) \subset L^q(U)$ is in fact compact if $1 \leq p \leq n$ and $1 \leq q < p^*$. In the following chapters we will devote considerable effort to understand how compactness fails for the critical case $q = p^*$.

Theorem 2.4.2 (Rellich-Kondrachov Theorem).
Let $1 \le p < n$ and $p^* = \frac{pn}{(n-p)}$ is the Sobolev conjugate of p.
Then the Sobolev space $W^{1,p}(U)$ is continuously embedded in the space $L^{p^*}(U)$ and it is compactly embedded in $L^q(U)$ for any q such that $1 \le q < p^*$.

For the proof, see [9].

Theorem 2.4.3 (Compactness for measures). *Assume that the sequence $\{\mu_k\}_{k=0}^\infty$ is bounded in $\mathscr{M}(U)$. Then $\{\mu_k\}_{n=0}^\infty$ is precompact in $W^{-1,p}(U)$ for each $1 \le p < 1^*$ where $1^* = \frac{n}{(n-1)}$.*

Proof. Since $\{\mu_k\}_{k=0}^\infty$ is bounded in $\mathscr{M}(U)$, by Theorem 1.7.1, we may extract a subsequence $\{\mu_{k_j}\}_{j=0}^\infty \subset \{\mu_k\}_{k=0}^\infty$ such that $\mu_{k_j} \rightharpoonup \mu$ in $\mathscr{M}(U)$, for some $\mu \in \mathscr{M}(U)$. Let $p' = \frac{p}{(p-1)}$ and B the closed unit ball in $W_0^{1,p'}(U)$. Using that $1 \le p < 1^*$, we have $p' > n$; in fact

$$p' > n \Leftrightarrow \frac{p}{(p-1)} > n \Leftrightarrow p > np - n \Leftrightarrow p(1-n) > -n \Leftrightarrow$$
$$p(n-1) > n \Leftrightarrow p < \frac{n}{(n-1)} = 1^* \quad \text{and this is true by hypothesis.}$$

Thus, by Sobolev imbedding Theorem 2.2.1 PART II, B is compact in $C_0(\overline{U})$. Since B is compact, it is closed and totally bounded, and so given $\varepsilon > 0$, there exist functions $\{\phi_i\}_{i=1}^{N(\varepsilon)} \subset C_0(\overline{U})$ such that

$$\min_{1 \le i \le N(\varepsilon)} \|\phi - \phi_i\|_{C(\overline{U})} < \varepsilon \qquad \forall\, \phi \in B.$$

Hence, if $\phi \in B$, for some i such that $1 \le i \le N(\varepsilon)$, we have:

$$\left| \int_U \phi \, d\mu_{k_j} - \int_U \phi \, d\mu \right| = \left| \int_U (\phi - \phi_i) \, d\mu_{k_j} + \int_U \phi_i \, d\mu_{k_j} - \int_U (\phi - \phi_i) \, d\mu - \int_U \phi_i \, d\mu \right| \le$$

$$\le \int_U |\phi - \phi_i| \, d\mu_{k_j} + \int_U |\phi - \phi_i| \, d\mu + \left| \int_U \phi_i \, d\mu_{k_j} - \int_U \phi_i \, d\mu \right|.$$

- Using that $\{\mu_{k_j}\}_{j=0}^\infty$ is bounded and that, for a suitable choice of i, we have $|\phi - \phi_i| < \varepsilon$, we get:

$$\int_U |\phi - \phi_i| \, d\mu_{k_j} \le \varepsilon \sup_j |\mu_{k_j}|(U).$$

- As above, for a suitable choice of i, we have $|\phi - \phi_i| < \varepsilon$, thus:

$$\int_U |\phi - \phi_i|\, d\mu \le \varepsilon|\mu|(U) \le \varepsilon \sup_j |\mu_{k_j}|(U).$$

- Since $\mu_{k_j} \rightharpoonup \mu$ in $\mathscr{M}(U)$ and $\{\phi_i\}_{i=1}^{N(\varepsilon)} \subset C_0(\overline{U})$, if j is large enough, we have:

$$\left|\int_U \phi_i \, d\mu_{k_j} - \int_U \phi_i \, d\mu\right| \le \varepsilon,$$

and this is true for any finite nuber of i.

In conclusion:

$$\begin{aligned}
&\left|\int_U \phi \, d\mu_{k_j} - \int_U \phi \, d\mu\right| \le \\
&\le \int_U |\phi - \phi_i|\, d\mu_{k_j} + \int_U |\phi - \phi_i|\, d\mu + \left|\int_U \phi_i \, d\mu_{k_j} - \int_U \phi_i \, d\mu\right| \le \\
&\le 2\varepsilon \sup_j |\mu_{k_j}|(U) + \varepsilon.
\end{aligned}$$

Hence,

$$\lim_{j\to+\infty} \sup_{\phi\in B} \left|\int_U \phi \, d\mu_{k_j} - \int_U \phi \, d\mu\right| = \lim_{j\to+\infty} \left\|\mu_{k_j} - \mu\right\|_{W^{-1,p}(U)} = 0.$$

And so $\mu_{k_j} \longrightarrow \mu$ in $W^{-1,p}(U)$. □

Corollary 2.4.4. *Let $\{f_k\}_{k=1}^{\infty}$ be a bounded sequence in $W^{-1,p}(U)$, for some $p > 2$, $\{g_k\}_{k=1}^{\infty}$ be a precompact sequence in $W^{-1,2}(U)$ and $\{h_k\}_{k=1}^{\infty}$ be a bounded sequence in $\mathscr{M}(U)$. Suppose further that $f_k = g_k + h_k$, $(k = 1, ...)$.*
Then $\{f_k\}_{k=1}^{\infty}$ is precompact in $W^{-1,2}(U)$.

Proof. For $k = 1, 2...$ we claim that it is possible to find a function u_k which is the weak solution of

$$\begin{cases} -\Delta u_k = f_k & \text{in } U \\ u_k = 0 & \text{in } \partial U \end{cases} \tag{2.22}$$

In fact, $f_k \in W^{-1,p}(U)$, $p > 2$ and we look for $u_k \in W_0^{1,2}(U)$ which satisfies (2.22). Recall that $W_0^{1,2}(U)$ is an Hilbert space with inner product given by:

$$\langle u, v\rangle_{W_0^{1,2}(U)} = \langle Du, Dv\rangle_{L^2(U)}.$$

The condition $-\Delta u_k = f_k$ in $W^{-1,p}(U)$ means that, for any $\psi \in W^{1,p'}(U)$

$$\langle \psi, -\Delta u_k \rangle = \langle \psi, f_k \rangle, \qquad \text{that is} \qquad \langle D\psi, Du_k \rangle = \langle \psi, f_k \rangle.$$

Consider now the functional $L : W^{1,p'}(U) \longrightarrow \mathbb{R}$ such that $L(\psi) = \langle \psi, f_k \rangle$.
L is linear and, using that $\{f_k\}_{k=0}^{\infty}$ is bounded in $W^{-1,p}(U)$, L is bounded:

$$|L(\psi)| = |\langle \psi, f_k \rangle| \leq \|f_k\|_{W^{-1,p}(U)} \|\psi\|_{W^{1,p'}(U)} \leq C_1 \|\psi\|_{W^{1,p'}(U)}.$$

Since $p > 2$, we have $p' < 2$ and so $W_0^{1,2}(U) \subset W^{1,p'}(U)$; hence:

$$|L(\psi)| \leq C_1 \|\psi\|_{W^{1,p'}(U)} \leq C_2 \|\psi\|_{W_0^{1,2}(U)}.$$

So the functional, which we still call L, $L : W_0^{1,2}(U) \longrightarrow \mathbb{R}$ such that $L(\psi) = \langle \psi, f_k \rangle$ is linear and bounded, and by Riesz representation Theorem in Hilbert spaces, there exists a unique $u_k \in W_0^{1,2}(U)$ such that, for any $\psi \in W_0^{1,2}(U)$,

$$L(\psi) = \langle \psi, u_k \rangle_{W_0^{1,2}(U)}.$$

This prove the existence of the function u_k, the weak solution (2.22).
Now write $u_k = v_k + w_k$ where

$$\begin{cases} -\Delta v_k = g_k & \text{in } U \\ v_k = 0 & \text{in } \partial U \end{cases} \tag{2.23}$$

and

$$\begin{cases} -\Delta w_k = h_k & \text{in } U \\ w_k = 0 & \text{in } \partial U \end{cases} \tag{2.24}$$

$\{g_k\}$ is a precompact sequence in $W^{-1,2}(U)$ and so it is bounded in $W^{-1,2}(U)$. As done above, it is possible to find a function v_k which is the weak solution of (2.23). Moreover, $\{h_k\}_{k=0}^{\infty}$ is a bounded sequence in $\mathscr{M}(U)$ and so the functional $M : C_0(U) \longrightarrow \mathbb{R}$ such that $M(\phi) = \langle \phi, h_k \rangle$ is linear and bounded and

$$|M(\phi)| = |\langle \phi, h_k \rangle| \leq C_1 \|\phi\|_{L^1(U)} \leq C_2 \|\phi\|_{L^2(U)} \leq C_3 \|\phi\|_{W_0^{1,2}(U)},$$

and so, as done above it is possible to find a function w_k which is the weak solution of (2.24). Notice that $\langle u_k, u_k\rangle_{W_0^{1,2}(U)} = \|u_k\|^2_{W_0^{1,2}(U)} = L(u_k)$ and, since L is bounded, we have $L(u_k) \leq C\,\|u_k\|_{W_0^{1,2}(U)}$.
Hence $\|u_k\|_{W_0^{1,2}(U)} \leq C = C_4\,\|f_k\|_{W^{-1,p}(U)}$ and using that

$$p > 2 \Rightarrow \Big(W^{-1,2}(U) \subset W^{-1,p}(U)\Big),$$

we get:

$$\|u_k\|_{W_0^{1,2}(U)} \leq C_5\,\|f_k\|_{W^{-1,2}(U)}\,.$$

Similar inequalities are true for v_k and w_k.
Thus $\{v_k\}_{k=0}^{\infty}$ is precompact in $W_0^{1,2}(U)$, since by hypothesis $\{g_k\}_{k=1}^{\infty}$ is precompact in $W^{-1,2}(U)$ and

$$\|v_k - v_h\|_{W_0^{1,2}(U)} \leq C_6\,\|g_k - g_h\|_{W^{-1,2}(U)}\,.$$

As a consequence $\{v_k\}$ is precompact in $W_0^{1,q}(U)$ for any q such that $q < 2$. Moreover, by hypothesis $\{h_k\}_{k=0}^{\infty}$ is bounded in $\mathscr{M}(U)$ and so by Theorem 2.4.3 $\{h_k\}_{k=0}^{\infty}$ is precompact in $W^{-1,q}(U)$ for any q, $1 \leq q < 1^*$; hence, as done above, $\{w_k\}_{k=0}^{\infty}$ is precompact in $W_0^{1,q}(U)$ for any q such that $1 < q < 1^*$.
The above conclusions imply that it is possible to find q, $1 < q < 2$, such that $u_k = v_k + w_k$ is precompact in $W_0^{1,q}(U)$ and so $\{f_k\}_{k=0}^{\infty}$ is precompact in $W^{-1,q}(U)$. Since $q < 2$, we have $W^{-1,q}(U) \subset W^{-1,2}(U)$ and so

$$\|f_k\|_{W^{-1,2}(U)} \leq C_7\,\|f_k\|_{W^{-1,q}(U)}\,.$$

Hence, $\{f_k\}_{k=1}^{\infty}$ is precompact in $W^{-1,2}(U)$. □

Our intention now is to prove a theorem which refines Rellich-Kondrachov Theorem 2.4.2 and which asserts that a bounded sequence in $W^{1,q}(U)$ has a subsequence which converges uniformly except for a very small set. The idea is that the set on which the uniform convergence fails has not only small Lebesgue measure, but also small capacity.

Definition 2.16. Let $1 \leq p < n$ and, given a set M, we denote by M^0 the interior of M. For each $A \subset \mathbb{R}^n$ we define the *p-capacity of A* as follows:

$$Cap_p(A) = \inf\left\{\int_{\mathbb{R}^n} |Df|^p\,dx;\quad f \in L^{p^*}(\mathbb{R}^n),\ Df \in L^p(\mathbb{R}^n),\ A \subset \{f \geq 1\}^0\right\}.$$

Definition 2.17. If $f \in L^1_{loc}(\mathbb{R}^n)$, we define its *precise representative* to be

$$f^*(x) := \begin{cases} \lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f\,dy & \text{if this limit exists} \\ 0 & \text{otherwise} \end{cases}$$

Note that $f^* = f$ a.e.

Remark 26. If $f \in W^{1,p}_{loc}(\mathbb{R}^n)$, then the limit in the definition above exists except for points x in a set of p-capacity zero. Note that when we speak of pointwise properties of a function, we always mean the pointwise properties of its precise representative.

Now we state a proposition which gives an estimate of the p-capacity of $\{f \geq 1\}$.

Proposition 2.4.5. *For each $f \in L^{p^*}(\mathbb{R}^n)$ such that $Df \in L^p(\mathbb{R}^n; \mathbb{R}^n)$, there exists a constant C depending only on p and n such that:*

$$Cap_p(\{f \geq 1\}) \leq C \int_{\mathbb{R}^n} |Df|^p\,dx.$$

For the proof, see [8].

Theorem 2.4.6 (Refinement of Rellich Theorem). *Let $\{f_k\}_{k=1}^\infty$ be a bounded sequence in $W^{1,q}(U)$. Then there exists a subsequence $\{f_{k_j}\}_{j=1}^\infty \subset \{f_k\}_{k=1}^\infty$ and a function $f \in W^{1,q}(U)$ such that for each p, $1 \leq p < q$ and each $\delta > 0$, there exists a relatively closed set $E_\delta \subset U$ with*

1. *$Cap_p(U \setminus E_\delta) \leq \delta$,*

2. *$f_{k_j} \longrightarrow f$ uniformly on E_δ.*

Proof. We my suppose that $f_k \in W^{1,q}_0(U)$ for $k = 1, 2, ...$ since, otherwise, we choose V such that $U \subset\subset V$ and extend each f_k to belong to $W^{1,q}_0(V)$.
Passing, if necessary, to a subsequence, we may assume that $f_k \longrightarrow f$ in $L^q(U)$ for some $f \in W^{1,q}_0(U)$. In fact:

- $\{f_k\}_{k=0}^\infty$ is bounded in $W^{1,q}_0(U)$ and so there exists a subsequence $\{f_{k_s}\}_{s=0}^\infty$ which converges weakly to $f \in W^{1,q}_0(U)$.

- $\{f_{k_s}\}_{s=0}^{\infty}$ is bounded in $W_0^{1,q}(U)$ and by Rellich-Kondachov Theorem 2.4.2, $W_0^{1,q}(U)$ is compactly embedded in L^p for any p such that $q \le p < q^*$. In particular, since $q \le q^*$, $W_0^{1,q}(U)$ is compactly embedded in $L^q(U)$: Thus there exists a subsequence $\{f_{k_{s_r}}\} \subset \{f_{k_s}\}$ which converges strongly in $L^q(U)$ to a function $g \in L^q(U)$.

- $\{f_{k_{s_r}}\}_{r=0}^{\infty}$ also converges weakly to f in $W_0^{1,q}(U)$.

- Thanks to the uniqueness of weak limit we conclude that $f = g$ and so $f_k \longrightarrow f$ in $L^q(U)$ for some $f \in W_0^{1,q}(U)$.

Fix δ, $\varepsilon > 0$ and let

$$E_\varepsilon^k := \{x \in U; \quad |f_k(x) - f(x)| \ge \varepsilon\}$$

and

$$g_\varepsilon^k := \frac{2}{\varepsilon} \max\{|f_k - f| - \frac{\varepsilon}{2}, 0\}.$$

Since f_k, $f \in W_0^{1,q}(U)$, also $g_\varepsilon^k \in W_0^{1,q}(U)$ and by definition of E_ε^k, $g_\varepsilon^k \ge 1$ on E_ε^k. Hence using Proposition 2.4.5 and the definition of g_ε^k,

$$\begin{aligned} Cap_p(E_\varepsilon^k) &= Cap_p(\{g_\varepsilon^k \ge 1\}) \le C \int_U |Dg_\varepsilon^k|^p \, dx \le \\ &\le C \left(\frac{2}{\varepsilon}\right)^p \int_{\{x;\ |f_k(x) - f(x)| - \frac{\varepsilon}{2} > 0\}} \Big| D(\,|f_k(x) - f(x)|\,) \Big|^p \, dx. \end{aligned}$$

Notice that, since f_k, $f \in W_0^{1,q}(U)$, we have $|D(\,|f_k(x) - f(x)|\,)\Big|^p \in L^{\frac{q}{p}}(U)$; moreover the set $\{x;\ |f_k(x) - f(x)| - \frac{\varepsilon}{2} > 0\} = E_{\frac{\varepsilon}{2}}^k$.
By Hölder inequality and denoting by $\mathscr{L}^n$ the n-dimensional Lebesgue measure, we obtain:

$$\begin{aligned} &C \left(\frac{2}{\varepsilon}\right)^p \int_{E^k_{\frac{\varepsilon}{2}}} 1 \cdot \Big| D(\,|f_k(x) - f(x)|\,) \Big|^p \, dx \le \\ &\le C \left(\frac{2}{\varepsilon}\right)^p \left(\mathscr{L}^n \left(E^k_{\frac{\varepsilon}{2}}\right)\right)^{1 - \frac{p}{q}} \left(\int_{E^k_{\frac{\varepsilon}{2}}} \Big| D(\,|f_k(x) - f(x)|\,) \Big|^q \, dx \right)^{\frac{p}{q}}. \end{aligned}$$

Now, recall the following elementary inequality:

$$a,\, b \in \mathbb{R},\; a,\, b > 0 \Rightarrow |a+b|^q \leq C_q(|a|^q + |b|^q)$$

and notice that since f_k is bounded in $W_0^{1,q}(U)$, we have $|Df_k|^q \leq C_1$ and $|Df_k|^q \leq C_2$. Hence we obtain:

$$\begin{aligned}
&C\left(\frac{2}{\varepsilon}\right)^p \left(\mathscr{L}^n\left(E^k_{\frac{\varepsilon}{2}}\right)\right)^{1-\frac{p}{q}} \left(\int_{E^k_{\frac{\varepsilon}{2}}} \left|D(\,|f_k(x)-f(x)|\,)\right|^q dx\right)^{\frac{p}{q}} \leq \\
&\leq C\left(\frac{2}{\varepsilon}\right)^p \left(\mathscr{L}^n\left(E^k_{\frac{\varepsilon}{2}}\right)\right)^{1-\frac{p}{q}} \left(C_q\int_U |Df_k|^q + |Df|^q\right)^{\frac{p}{q}} \leq \\
&\leq C(\varepsilon)\left(\mathscr{L}^n\left(E^k_{\frac{\varepsilon}{2}}\right)\right)^{1-\frac{p}{q}}.
\end{aligned}$$

Moreover,

$$\|f_k - f\|^{q-p}_{L^q(U)} = \left(\int_U |f_k - f|^q\, dx\right)^{\frac{1}{q}(q-p)} = \left(\int_U |f_k - f|^q\, dx\right)^{1-\frac{p}{q}}$$

and

$$C(\varepsilon)\int_U |f_k - f|^q\, dx \geq C(\varepsilon)\int_{E^k_{\frac{\varepsilon}{2}}} \left(\frac{2}{\varepsilon}\right)^q dx = C(\varepsilon)\mathscr{L}^n\left(E^k_{\frac{\varepsilon}{2}}\right).$$

In conclusion we obtain:

$$Cap_p(E^k_\varepsilon) \leq C(\varepsilon)\left(\mathscr{L}^n\left(E^k_{\frac{\varepsilon}{2}}\right)\right)^{1-\frac{p}{q}} \leq C(\varepsilon)\,\|f_k - f\|^{q-p}_{L^q(U)}. \tag{2.25}$$

Now choose a subsequence $\{f_{k_j}\}_{j=0}^\infty \subset \{f_k\}_{k=0}^\infty$ such that

$$\sum_{j=1}^\infty \left\|f_{k_j} - f\right\|^{q-p}_{L^q(U)} < +\infty.$$

Notice that it is possible to find such subsequence since $f_k \longrightarrow f$ in $L^q(U)$.
Write

$$F_i^l := \bigcup_{j=l}^\infty E^{k_j}_{\frac{1}{i}} = \bigcup_{j=l}^\infty \{x \in U; \quad |f_{k_j}(x) - f(x)| \geq \frac{1}{i}\}.$$

Thus, using the subadditivity of the p-capacity and inequality (2.25) we have

$$Cap_p(F_i^l) \leq \sum_{j=l}^\infty Cap_p\left(E^{k_j}_{\frac{1}{i}}\right) \leq C(i)\sum_{j=l}^\infty \left\|f_{k_j} - f\right\|^{q-p}_{L^q(U)}$$

and so, if $l = l(i)$ is large enough we obtain:

$$Cap_p(F_i^l) \leq \frac{\delta}{2^{i+1}}.$$

Moreover, $Cap_p(E) = \inf\{Cap_p(V);\ V \text{ open},\ E \subset V\}$ and so

$$Cap_p(F_i^l) = \inf\{Cap_p(G_i^l);\ G_i^l \text{ open},\ F_i^l \subset G_i^l\}.$$

Call $G := \{Cap_p(G_i^l);\ G_i^l \text{ open},\ F_i^l \subset G_i^l\}$. Since $Cap_p(F_i^l) \leq \frac{\delta}{2^{i+1}} < \frac{\delta}{2^i}$, we have $\frac{\delta}{2^i} > \inf G$ and so $\frac{\delta}{2^i}$ is not a lower bound for G; hence, for any i, there exists $\tilde{g} \in G$ such that $\tilde{g} < \frac{\delta}{2^i}$, that is: $\forall i,\ \exists G_i^l$ open, such that $F_i^l \subset G_i^l$ and

$$Cap_p(G_i^l) < \frac{\delta}{2^i}.$$

Finally, let

$$E_\delta := U \setminus \bigcup_{i=1}^{\infty} G_i^{l(i)};$$

we have $(U \setminus E_\delta) \subset \bigcup_{i=1}^{\infty} G_i^{l(i)}$ and so:

$$Cap_p\left(U \setminus E_\delta\right) \leq Cap_p\left(\bigcup_{i=1}^{\infty} G_i^{l(i)}\right) \leq \sum_{i=1}^{\infty} Cap_p\left(G_i^{l(i)}\right) \leq \delta$$

and $f_{k_j} \longrightarrow f$ uniformly on E_δ because we showed that $|f_{k_j} - f| > \frac{1}{i}$ only on F_i^l, while on $\left(F_i^l\right)^c$ (and thus on E_δ), $|f_{k_j} - f| \leq \frac{1}{i}$, and so we proved that for any i, there exists j_i such that for any $j \geq j_i$ and for any $x \in E_\delta$ we have $|f_{k_j} - f| \leq \frac{1}{i}$ and this completes the proof. $\square$

Chapter 3

Measures of concentration and measures of oscillation

This chapter is devoted to the construction of measure-theoretic tools to allow us understand the ways in which a weakly convergent sequence of functions can fail to be strongly convergent.

Let U be an open, bounded and smooth subset of $\mathbb{R}^n$, $1 < q < \infty$. Let $\{f_k\}_{k=0}^{\infty}$ be a sequence in $L^q(U)$ and assume that

$$\begin{aligned} f_k \rightharpoonup f \quad & \text{weakly in} \quad L^q(U), \\ & \text{but} \\ f_k \not\rightarrow f \quad & \text{strongly in} \quad L^q(U). \end{aligned}$$

There are several distinct ways which can cause this breakdown of strong convergence.

First observe that, even if we know that the functions $\{f_k\}_{k=0}^{\infty}$ are bounded in the L^∞-norm, so that f_k converges weakly to f in $L^p(U)$ for all $1 \leq p < \infty$, we still cannot deduce strong convergence in $L^p(U)$ for any p, $1 \leq p < \infty$.

The difficulty can be caused by very rapid fluctuations in the functions f_k. This is the problem of *oscillation*. See Example 2.

Secondly, observe that, even if we know additionally that:

$$f_k \to f \quad \text{a.e. in } U, \tag{3.1}$$

so that wild oscillations are excluded, we still cannot deduce strong convergence in $L^q(U)$. The problem is that the mass of $|f_k - f|^q$ may coalesce onto a set of zero Lebesgue measure. This is the problem of *concentration.* See Example 1. Finally, both oscillation and concentration effects can occur simultaneously, creating thereby the problem of *oscillation/concentration.*

3.1 Measures of concentration

In this section we construct appropriate methodology for characterizing concentration effects.

3.1.1 Defect measures

We introduce certain measures which record the failure of weak convergence in $L^q(U)$ to imply strong convergence in $L^q(U)$.

Definition 3.1. Let $\{f_k\}_{k=0}^{\infty}$ be a sequence in $L^q(U)$ and assume that

$$f_k \rightharpoonup f \quad \text{weakly in} \quad L^q(U); \tag{3.2}$$

we define the measures

$$\theta_k := |f_k - f|^q, \qquad k = 1, 2, ...$$

Hence for any Borel set $E \subset U$,

$$\theta_k(E) = \int_E |f_k - f|^q \, dx.$$

Remark 27. This measure controls how close is f_k to f in the L^q-norm restricted to the Borel set E. Consequently, we can expect that the limiting behaviour of the measures $\{\theta_k\}_{k=1}^{\infty}$ reflects the possible failure of strong convergence in $L^q(U)$.

Definition 3.2. For each Borel set $E \subset U$ we define the *reduced defect measure* θ associated with the weak convergence (3.2), as follows:

$$\theta(E) := \limsup_{k \to +\infty} \int_E |f_k - f|^q \, dx.$$

This measure encodes the information about the extent to which strong convergence fails.

Proposition 3.1.1. *If* $\mathscr{L}^n\left((E \cup F) \setminus (E \cap F)\right) = 0$, *then* $\theta(E) = \theta(F)$.

Proof. Note that by hypothesis

$$\mathscr{L}^n\left((E \cup F) \setminus (E \cap F)\right) = \mathscr{L}^n\left((E \setminus F) \cup (F \setminus E)\right) = 0,$$

and in particular

$$\mathscr{L}^n\left((E \setminus F)\right) = \mathscr{L}^n\left((F \setminus E)\right) = 0.$$

Hence,

$$\begin{aligned}\int_E |f_k - f|^q\, dx &= \int_{E \setminus F} |f_k - f|^q\, dx + \int_{E \cap F} |f_k - f|^q\, dx = \\ &= \int_{E \cap F} |f_k - f|^q\, dx = \int_{E \cap F} |f_k - f|^q\, dx,\end{aligned}$$

and

$$\begin{aligned}\int_F |f_k - f|^q\, dx &= \int_{F \setminus E} |f_k - f|^q\, dx + \int_{E \cap F} |f_k - f|^q\, dx = \\ &= \int_{E \cap F} |f_k - f|^q\, dx = \int_{E \cap F} |f_k - f|^q\, dx.\end{aligned}$$

Taking the $\limsup_{k \to \infty}$ we obtain $\theta(E) = \theta(F)$. □

Remark 28. Note that f_k converges strongly to f in $L^q(E) \Leftrightarrow \theta(E) = 0$. Moreover θ is only a finitely-additive outer measure.

Now we will study the set upon which the measure θ is concentrated inasmuch as the non vanishing of θ on a certain set signals the failure of strong convergence on such set. We will present two alternative ways to assess the smallness of the set onto which the measure coalesces: the p-capacity (see Definition 2.16) and Hausdorff measure.

Definition 3.3. If $0 \leq s < \infty$ and $0 < \delta \leq \infty$, we define the *s-dimensional Hausdorff premeasure* H^s_δ by setting, for each $A \subset \mathbb{R}^n$,

$$H^s_\delta(A) := \left\{ \sum_{j=1}^{\infty} \alpha(s) \left(\frac{\mathrm{diam} C_j}{2} \right)^s ; \quad A \subset \bigcup_{j=1}^{\infty} C_j,\ \mathrm{diam} C_j \leq \delta \right\},$$

where

$$\alpha(s) := \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2}+1)}.$$

Definition 3.4. The *s-dimensional Hausdorff measure* H^s is given by:

$$H^s(A) = \lim_{\delta \to 0} H^s_\delta(A) = \sup_{\delta > 0} H^s_\delta(A),$$

for each $A \subset \mathbb{R}^n$.

Definition 3.5. We say that θ is concentrated on a set of p-capacity zero if there exist open sets $\{V_i\}_{i=1}^{\infty}$ in U such that

- $\theta(U \setminus V_i) = 0,\ i = 1, 2, ...$
- $Cap_p(V_i) \longrightarrow 0 \quad \text{as} \quad i \longrightarrow \infty.$

Definition 3.6. We say that θ is concentrated on a set of Hausdorff H^s-measure zero if there exist open sets $\{V_i\}_{i=1}^{\infty}$ in U and a sequence $\{\delta_i\}_{i=1}^{\infty} \subset]0, +\infty[$ such that

- $\theta(U \setminus V_i) = 0,\ i = 1, 2, ...$
- $\delta_i \longrightarrow 0 \quad \text{as} \quad i \longrightarrow \infty,$
- $H^s_{\delta_i}(V_i) \longrightarrow 0 \quad \text{as} \quad i \longrightarrow \infty.$

Remark 29. The idea is that the measure θ is concentrated on the set

$$C = \bigcap_{i=1}^{\infty} V_i,$$

with either $Cap_p(C) = 0$ or $H^s(C) = 0$.
Since, however, θ is only finitely subadditive, we cannot deduce that
$\theta(U \setminus C) = 0$ as the following example shows.

Example 6. Let $U =]0, 1[$ and $f \equiv 0$ and

$$f_k(x) \equiv \begin{cases} k & \text{if} \quad \frac{1}{2} - \frac{1}{2k} \le x \le \frac{1}{2} + \frac{1}{2k} \\ 0 & \text{otherwise} \end{cases}$$

Then θ is concentrated on $C = \left\{\frac{1}{2}\right\}$ and for any open set V such that $C \subset V$ we have $\theta(U \setminus V) = 0$, but $\theta(U \setminus C) = 1$.

3.1.2 A refinement of Fatou's Lemma

As already explained, concentration phenomena will arise in situations in which we have both weak convergence and pointwise convergence of a sequence $\{f_k\}_{k=0}^{\infty}$. By Fatou's Lemma 1.2.2 and thanks to the pointwise convergence of the sequence $\{f_k\}_{k=0}^{\infty}$, we have that:

$$\|f\|_{L^q(U)} \leq \liminf_{k\to\infty} \|f_k\|_{L^q(U)} .$$

However, this conclusion could also be deduced from the weak convergence of the sequence $\{f_k\}_{k=0}^{\infty}$ in $L^q(U)$ and Theorem 1.4.5.
Brezis and Lieb have examined this situation more carefully and they established the following sharp assertion, known as Brezis and Lieb Lemma.

Theorem 3.1.2 (Brezis and Lieb Lemma)**.** *Let $1 \leq q < \infty$, $\{f_k\}_{k=0}^{\infty}$ be a sequence in $L^q(U)$ and assume that*

$$f_k \rightharpoonup f \quad \textit{weakly in} \quad L^q(U)$$

and

$$f_k \to f \quad \textit{a.e. in}\, U.$$

Then

$$\lim_{k\to\infty}\left(\|f_k\|^q_{L^q(U)} - \|f_k - f\|^q_{L^q(U)}\right) = \|f\|^q_{L^q(U)} . \tag{3.3}$$

Remark 30. The main point of the theorem is that f_k decouples in the limit as measured in L^q-norm, into $(f_k - f)$ and f.

Proof. First of all we recall the following elementary inequality:
$\forall a,\, b\, \in \mathbb{R}$ and $\varepsilon > 0$ we have:

$$|\,|a+b|^q - |a|^q\,| \leq \varepsilon|a|^q + C(\varepsilon)|b|^q, \tag{3.4}$$

where $C(\varepsilon)$ is a constant depending only on ε and q.
Let $g_k^{\varepsilon} := \left(\Big|\,|f_k|^q - |f_k - f|^q - |f|^q\,\Big| - \varepsilon|f_k - f|^q\right)^+$.
Since by hypothesis $f_k \to f$ a.e. in U, we have $g_k^{\varepsilon} \longrightarrow 0$ as $k \longrightarrow \infty$. Moreover,

$$\begin{aligned} g_k^{\varepsilon} &\leq \left(\Big|\,|f_k|^q - |f_k - f|^q\,\Big| + |f|^q - \varepsilon|f_k - f|^q\right)^+ \leq \\ &\leq \left|\Big|\,|f_k|^q - |f_k - f|^q\,\Big| + |f|^q - \varepsilon|f_k - f|^q\right|. \end{aligned}$$

Now, we use inequality (3.4) with $a = f_k - f$ and $b = f$ and obtain:

$$\begin{aligned} g_k^\varepsilon &\le \Big| \big| |f_k|^q - |f_k - f|^q \big| + |f|^q - \varepsilon |f_k - f|^q \Big| \le \\ &\le \Big| |f|^q + \varepsilon |f_k - f|^q + C(\varepsilon)|f|^q - \varepsilon |f_k - f|^q \Big| = \\ &= |f|^q (1 + C(\varepsilon)). \end{aligned}$$

Since $f \in L^q(U)$, we have $|f|^q \in L^1(U)$, and we proved that
$g_k^\varepsilon \le |f|^q(1 + C(\varepsilon)) \in L^1(U)$ and the right hand side is independet of k. Moreover $g_k^\varepsilon \longrightarrow 0$ pointwise. Thus by dominated convergence Theorem 1.2.3 we have:

$$\lim_{k\to\infty} \int_U g_k^\varepsilon \, dx = 0.$$

But then

$$\begin{aligned} &|f_k|^q - |f_k - f|^q - |f|^q = |f_k|^q - |f_k - f|^q - |f|^q - \varepsilon |f_k - f|^q + \varepsilon |f_k - f|^q \le \\ &\le \Big(\big| |f_k|^q - |f_k - f|^q - |f|^q \big| - \varepsilon |f_k - f|^q \Big)^+ + \varepsilon |f_k - f|^q \le \\ &\le g_k^\varepsilon + \varepsilon |f_k - f|^q, \end{aligned}$$

and so

$$\int_U \big| |f_k|^q - |f_k - f|^q - |f|^q \big| \, dx \le \int_U g_k^\varepsilon \, dx + \varepsilon \int_U |f_k - f|^q \, dx.$$

In conclusion

$$\limsup_{k\to\infty} \int_U \big| |f_k|^q - |f_k - f|^q - |f|^q \big| \, dx \le \varepsilon \limsup_{k\to\infty} \int_U |f_k - f|^q \, dx = O(\varepsilon).$$

and this concludes the proof. □

3.1.3 Concentration and Sobolev inequalities

An important instance in which concentration phenomena occur concerns the lack of compactness of the injection of $W^{1,q}(U)$ into $L^{q^*}(U)$ for $1 \le q < n$. We single out for attention the case $U = \mathbb{R}^n$ for which the following characterization of noncompactness is available. For simplicity we only consider the case $q = 2$.

Theorem 3.1.3. *Assume* $n \geq 3$ *and let* $\{f_k\}_{k=0}^{\infty}$ *be a sequence such that:*

(i) $f_k \to f$ *strongly in* $L^2_{loc}(\mathbb{R}^n)$,

(ii) $Df_k \rightharpoonup Df$ *weakly in* $L^2(\mathbb{R}^n; \mathbb{R}^n)$,

(iii) $|Df_k|^2 \rightharpoonup \mu$ *in* $\mathscr{M}(\mathbb{R}^n)$,

(iv) $|f_k|^{2^*} \rightharpoonup \nu$ *in* $\mathscr{M}(\mathbb{R}^n)$.

Then:

1. *There exists an almost countable index set* J *and a set of distinct points* $\{x_j\}_{j\in J} \subset \mathbb{R}^n$ *and nonnegative weights* $\{\mu_j, \nu_j\}_{j\in J}$ *such that*

$$\nu = |f|^{2^*} + \sum_{j\in J} \nu_j \delta_{x_j} \tag{3.5}$$

and

$$\mu \geq |Df|^2 + \sum_{j\in J} \mu_j \delta_{x_j}. \tag{3.6}$$

2. *Moreover, denoting by* C_2 *the optimal constant for the Gagliardo-Niremberg-Sobolev inequality 2.4.1,*

$$\nu_j \leq C_2^{2^*} + \mu_j^{\frac{2}{2^*}}, \quad j \in J. \tag{3.7}$$

3. *If* $f \equiv 0$ *and* $\nu(\mathbb{R}^n)^{\frac{1}{2^*}} \geq C_2 \mu(\mathbb{R}^n)^{\frac{1}{2}}$, *then* ν *is concentrated at a single point.*

Proof. We first assume that $f \equiv 0$ and let $\phi \in C_c(\mathbb{R}^n)$. Using the Gagliardo-Niremberg-Sobolev inequality 2.4.1, we have:

$$\left(\int_{\mathbb{R}^n} |\phi f_k|^{2^*} \, dx\right)^{\frac{1}{2^*}} \leq C_2 \left(\int_{\mathbb{R}^n} |D(\phi f_k)|^2 \, dx\right)^{\frac{1}{2}},$$

that is

$$\left(\int_{\mathbb{R}^n} |\phi|^{2^*} |f_k|^{2^*} \, dx\right)^{\frac{1}{2^*}} \leq C_2 \left(\int_{\mathbb{R}^n} |(D\phi) f_k + \phi(Df_k)|^2 \, dx\right)^{\frac{1}{2}}.$$

Since $f_k \to f \equiv 0$ strongly in $L^2_{loc}(\mathbb{R}^n)$ and using hypothesis **(iii)** and **(iv)** we obtain:

$$\left(\int_{\mathbb{R}^n} |\phi|^{2^*} \, d\nu\right)^{\frac{1}{2^*}} \leq C_2 \left(\int_{\mathbb{R}^n} |\phi|^2 \, d\mu\right)^{\frac{1}{2}}. \tag{3.8}$$

Now we substitute ϕ with functions in $C_c(\mathbb{R}^n)$ which approximate χ_E, where E is a Borel set, and we get:

$$(\nu(E))^{\frac{1}{2^*}} \leq C_2 \left(\mu(E)\right)^{\frac{1}{2}}. \tag{3.9}$$

Since μ is a finite measure by hypothesis the set

$$D := \{x \in \mathbb{R}^n; \quad \mu(\{x\}) > 1\}$$

is at most countable. Hence there exists an almost countable index set J such that we can write $D = \{x_j\}_{j\in J}$ and $\mu_j \equiv \mu(\{x_j\})$, $j \in J$ so that

$$\mu \geq \sum_{j\in J} \mu_j \delta_{x_j}. \tag{3.10}$$

In fact, if $E \subset \mathbb{R}^n$ is a Borel set and $E' = E \cap \{x_j,\ j \in J\} \subset E$ we have

$$\mu(E) \geq \mu(E') = \sum_{x_j \in E} x_j \mu_j = \sum_{x_j\in E} \langle \mu_j \delta_{x_j},\, \chi_E \rangle.$$

Moreover, (3.9) implies $\nu \ll \mu$ and, if $E \subset \mathbb{R}^n$ is a Borel set, we have

$$\nu(E) = \int_E D_\mu \nu \, d\mu, \qquad \text{where} \qquad D_\mu \nu(x) \equiv \lim_{r\to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))} \tag{3.11}$$

and this limit exists for μ-a.e $x \in \mathbb{R}^n$. (This is a consequence of theory of symmetric derivatives of Radon measures; see Federer [F] pag. 152-169).
Now, by (3.9), we deduce that $\nu(E) \leq C_2^{2^*} \left(\mu(E)\right)^{\frac{2^*}{2}}$ and so, if $\mu(B(x,r)) \neq 0$ we obtain:

$$\frac{\nu(B(x,r))}{\mu(B(x,r))} \leq C_2^{2^*} \left(\mu(B(x,r))\right)^{\frac{2^*}{2}-1} = C_2^{2^*} \left(\mu(B(x,r))\right)^{\frac{n}{n-2}-1} = C_2^{2^*} \left(\mu(B(x,r))\right)^{\frac{2}{n-2}}. \tag{3.12}$$

But then

$$D_\mu \nu = 0 \qquad \mu\text{-a.e. on } \mathbb{R}^n \setminus D. \tag{3.13}$$

Now define $\nu_j := D_\mu \nu(x_j)\mu_j$. Using (3.11) and (3.13) we get

$$\nu = \sum_{j\in J} D_\mu \nu(x_j)\mu_j = \sum_{j\in J} \nu(x_j)\delta_{x_j},$$

and by (3.10) we get $\mu \geq \sum_{j\in J} \mu_j \delta_{x_j}$. So we obtain thesis 1. for $f \equiv 0$. Moreover, also thesis 2. is proved. In fact

$$\nu_j = D_\mu \nu(x_j)\mu_j = \lim_{r\to 0} \frac{\nu(B(x_j, r))}{\mu(B(x_j, r))}\mu_j,$$

and, as shown in (3.12),

$$\frac{\nu(B(x_j, r))}{\mu(B(x_j, r))} \leq C_2^{2^*} \left(\mu(B(x_j, r))\right)^{\frac{2^*}{2}-1}$$

and passing to the limit for $r \to 0$ we get:

$$\nu_j = \lim_{r\to 0} \frac{\nu(B(x_j, r))}{\mu(B(x_j, r))}\mu_j \leq C_2^{2^*} \lim_{r\to 0} \left(\mu(B(x_j, r))\right)^{\frac{2^*}{2}-1} \mu_j =$$

$$= C_2^{2^*} \mu_j^{\frac{2^*}{2}-1} \mu_j = C_2^{2^*} \mu_j^{\frac{2^*}{2}}.$$

Now, suppose the hypothesis of assertion 3. is satisfied, that is $f \equiv 0$ and $\nu(\mathbb{R}^n)^{\frac{1}{2^*}} \geq C_2 \mu(\mathbb{R}^n)^{\frac{1}{2}}$. Since we showed that, for any $E \subset \mathbb{R}^n$ Borel set, we have $(\nu(E))^{\frac{1}{2^*}} \leq C_2 \left(\mu(E)\right)^{\frac{1}{2}}$, in particular this is true for $\mathbb{R}^n$, and so

$$\nu(\mathbb{R}^n)^{\frac{1}{2^*}} = C_2 \mu(\mathbb{R}^n)^{\frac{1}{2}}. \tag{3.14}$$

Moreover, by (3.8) and Hölder inequality of exponents $p = \frac{n}{n-2}$ and $q = \frac{n}{2}$ we get

$$\int_{\mathbb{R}^n} |\phi|^{2^*} d\nu \leq C_2^{2^*} \left(\int_{\mathbb{R}^n} |\phi|^2 d\mu\right)^{\frac{2^*}{2}} = C_2^{2^*} \left(\int_{\mathbb{R}^n} |\phi|^2 d\mu\right)^{\frac{n}{n-2}} \leq$$

$$\leq C_2^{2^*} \mu(\mathbb{R}^n)^{\frac{2}{n-2}} \left(\int_{\mathbb{R}^n} |\phi|^{\frac{2n}{n-2}} d\mu\right) = C_2^{2^*} \mu(\mathbb{R}^n)^{\frac{2}{n-2}} \left(\int_{\mathbb{R}^n} |\phi|^{2^*} d\mu\right).$$

Hence, we obtain

$$\left(\int_{\mathbb{R}^n} |\phi|^{2^*} d\nu\right)^{\frac{1}{2^*}} \leq C_2 \mu(\mathbb{R}^n)^{\frac{1}{n}} \left(\int_{\mathbb{R}^n} |\phi|^{2^*} d\mu\right)^{\frac{1}{2^*}}$$

and we deduce that

$$\nu = C_2^{2^*} \mu(\mathbb{R}^n)^{\frac{2}{n-2}} \mu. \tag{3.15}$$

Consequently (3.8) becomes:

$$\left(\int_{\mathbb{R}^n} |\phi|^{2^*} d\nu\right)^{\frac{1}{2^*}} \nu(\mathbb{R}^n)^{\frac{1}{n}} \leq \left(\int_{\mathbb{R}^n} |\phi|^2 d\nu\right)^{\frac{1}{2}}.$$

This is true because, by (3.14),

$$C_2 = \frac{\nu(\mathbb{R}^n)^{\frac{1}{2^*}}}{\mu(\mathbb{R}^n)^{\frac{1}{2}}},$$

and so (3.8) reads

$$\left(\int_{\mathbb{R}^n} |\phi|^{2^*}\, d\nu\right)^{\frac{1}{2^*}} \le \frac{\nu(\mathbb{R}^n)^{\frac{1}{2^*}}}{\mu(\mathbb{R}^n)^{\frac{1}{2}}} \left(\int_{\mathbb{R}^n} |\phi|^2\, d\mu\right)^{\frac{1}{2}}.$$

And so using the formula (3.15) we obtain

$$\left(\int_{\mathbb{R}^n} |\phi|^{2^*}\, d\nu\right)^{\frac{1}{2^*}} \le \frac{\nu(\mathbb{R}^n)^{\frac{1}{2^*}}}{\mu(\mathbb{R}^n)^{\frac{1}{2}}} \left(\frac{1}{C_2^{2^*}\,\mu(\mathbb{R}^n)^{\frac{2}{n-2}}} \int_{\mathbb{R}^n} |\phi|^2\, d\mu\right)^{\frac{1}{2}} =$$

$$= \nu(\mathbb{R}^n)^{-\frac{1}{n}} \left(\int_{\mathbb{R}^n} |\phi|^2\, d\mu\right)^{\frac{1}{2}},$$

that is

$$\left(\int_{\mathbb{R}^n} |\phi|^{2^*}\, d\nu\right)^{\frac{1}{2^*}} \nu(\mathbb{R}^n)^{\frac{1}{n}} \le \left(\int_{\mathbb{R}^n} |\phi|^2\, d\nu\right)^{\frac{1}{2}},$$

as claimed. As a consequence, for any $E \subset \mathbb{R}^n$ Borel set,

$$\nu(E)^{\frac{1}{2^*}}\nu(\mathbb{R}^n)^{\frac{1}{n}} \le \nu(E)^{\frac{1}{2}}.$$

This is a contraddiction if ν is concentrated at more than one point.
This proves 3.
Now suppose $f \not\equiv 0$ and let $g_k = f_k - f$. The calculation in the proofs of 1. and 2. apply to $\{g_k\}_{k=1}^{\infty}$. Moreover

$$|Dg_k|^2 = |Df_k - Df|^2 = |Df_k|^2 - 2\,Df_k \cdot Df + |Df|^2 \rightharpoonup \mu - |Df|^2 \quad \text{in } \mathscr{M}(\mathbb{R}^n).$$

Finally, according to Brezis-Lieb Lemma 3.1.2,

$$|g_k|^{2^*} = |f_k - f|^{2^*} \rightharpoonup \nu - |f|^{2^*} \quad \text{in } \mathscr{M}(\mathbb{R}^n).$$

□

3.2 Measures of oscillation

In this section we turn our attention to the problem of oscillation.
As mentioned at the beginning of this chapter, we expect such difficulties to arise when we have weak convergence, but we do not know the convergence is almost everywhere as well. On the other hand, we now suppose that we have enough good estimates that concentration problems do not occur. The technical difficulties therefore concern the possibility of wild, but suppose bounded, oscillations which may be present in our weakly convergent sequence. The main idea, as in the previous section, it to construct certain measures which appropriately encode the persistent, limiting structure of such oscillations.
In this section we will work with vector-valued mappings from U into $\mathbb{R}^m$, $m \geq 1$.

3.2.1 Slicing measures

Definition 3.7. Let μ be a finite nonnegative Radon measure on $\mathbb{R}^{n+m}$.
We denote by σ the *canonical projection* of μ onto $\mathbb{R}^n$, that is

$$\sigma(E) := \mu(E \times \mathbb{R}^n), \qquad \text{for each Borel subset } \; E \subset \mathbb{R}^n.$$

We now recall Stone-Weierstrass Theorem, which will be used in the proof of the first theorem of this section.

Theorem 3.2.1. *Let X be a compact Hausdorff space. Suppose A is an algebra of continuous real-valued functions on X which contains the constant functions and which separates points in X, that is, for any x, $y \in X$ there exists a function $f \in A$ such that $f(x) \neq f(y)$. Then A is dense in $C(X)$.*

For the proof, see [12] Theorem 12.3.

Theorem 3.2.2. *For σ-a.e. point $x \in \mathbb{R}^n$ there exists a Radon probability measure ν_x on $\mathbb{R}^m$ such that, for each function $f \in C_B(R^n \times R^m)$,*

1. *the mapping $x \mapsto \int_{\mathbb{R}^m} f(x, y)\, d\nu_x(y)$ is σ-measurable*

2. $\int_{\mathbb{R}^{n+m}} f(x, y)\, d\mu(x, y) = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f(x, y)\, d\nu_x(y) \right) d\sigma(x)$

Proof. **STEP 1**: Let $\{f_k\}_{k=0}^{\infty}$ be a countable dense subset of $C_c(\mathbb{R}^m)$. For any Borel set $E \subset \mathbb{R}^m$, define the signed Radon measures

$$\gamma^k(E) := \int_{E\times\mathbb{R}^m} f_k(y)\, d\mu(x,y), \qquad k = 1, 2, ...$$

Clearly $\gamma^k \ll \sigma$, since if $\sigma(E) = 0$, then, by definition of σ, $\mu(E \times \mathbb{R}^m) = 0$ and using the properties of integrals, also $\gamma^k(E) = 0$. Hence, again invoking the theory of symmetric derivatives of Radon measures (see Federer [F] pag. 152-169),
for σ-a.e. point $x \in \mathbb{R}^n$, the limits

$$D_\sigma\gamma^k(x) = \lim_{r\to 0} \frac{\gamma^k(B(x,r))}{\sigma(B(x,r))}, \qquad k = 1, 2, ... \tag{3.16}$$

exist and the mappings $x \mapsto D_\sigma\gamma^k(x)$ are σ-measurable and bounded because f_k are bounded and

$$\left|\frac{\gamma^k(B(x,r))}{\sigma(B(x,r))}\right| = \left|\frac{\int_{B(x,r)\times\mathbb{R}^m} f_k(y)\, d\mu(x,y)}{\mu(B(x,r)\times\mathbb{R}^m)}\right| \le C_{f_k}.$$

Moreover, for any Borel set $E \subset \mathbb{R}^m$,

$$\int_{E\times\mathbb{R}^m} f_k(y)\, d\mu(x,y) = \gamma^k(E) = \int_E D_\sigma\gamma^k(x)\, d\sigma(x), \qquad k = 1, 2, ... \tag{3.17}$$

STEP 2: Now we prove the theorem for functions $f(x,y) = h(y)\chi_E(x)$ where $h \in C_c(\mathbb{R}^m)$ and E as above. Since $\{f_k\}_{k=0}^{\infty}$ is a countable dense subset of $C_c(\mathbb{R}^m)$ we can choose a subsequence $\{f_{k_j}\}_{j=0}^{\infty} \subset \{f_k\}_{k=0}^{\infty}$ such that $f_{k_j} \longrightarrow h$ uniformly on $\mathbb{R}^m$. Then for each point $x \in \mathbb{R}^n$ for which (3.16) holds, the limit

$$\Gamma_x(h) := \lim_{j\to\infty} D_\sigma\gamma^{k_j}(x)$$

exists and does not depend on the subsequence approximating h. Since the mapping $h \mapsto \Gamma_x(h)$ is a bounded and linear functional on $C_c(\mathbb{R}^m)$, by Riesz representation Theorem 1.6.3, there exists a unique Radon measure ν_x on $\mathbb{R}^m$ such that, for any $h \in C_c(\mathbb{R}^m)$

$$\Gamma_x(h) = \int_{\mathbb{R}^m} h(y)\, d\nu_x(y). \tag{3.18}$$

Additionally, $x \mapsto \Gamma_x(h)$ is bounded and σ-measurable, and from (3.17) we deduce

$$\int_{E\times\mathbb{R}^m} f_{k_j}(y)\,d\mu(x,y) = \int_E D_\sigma\gamma^{k_j}(x)\,d\sigma(x). \tag{3.19}$$

By uniform convergence we can pass to the limit for $j\to\infty$ in the left side of (3.19) and we obtain

$$\lim_{j\to\infty}\int_{E\times\mathbb{R}^m} f_{k_j}(y)\,d\mu(x,y) = \int_{E\times\mathbb{R}^m} h(y)\,d\mu(x,y).$$

While in the right side we can pass to the limit for $j\to\infty$ using dominated convergence Theorem 1.2.3; in fact:

- By definition of $\Gamma_x(h)$, we have

$$D_\sigma\gamma^{k_j}(x) \longrightarrow \Gamma_x(h), \qquad \text{as } \; j\to\infty.$$

- $|D_\sigma\gamma^{k_j}(x)| \le C$, where C is a constant which does not depend on j and x and is summable with respect to the finite measure σ.
  This claim is true since by definition:

$$D_\sigma\gamma^{k_j}(x) = \lim_{r\to 0}\frac{\int_{B(x,r)\times\mathbb{R}^m} f_{k_j}(y)\,d\mu(x,y)}{\mu(B(x,r)\times\mathbb{R}^m)},$$

  but $f_{k_j} \longrightarrow h$ uniformly on $\mathbb{R}^m$, and so $\forall\varepsilon>0,\ \exists j_0$ such that $\forall j\ge j_0$, $|f_{k_j}-h|\le\varepsilon,\ \ \forall y\in\mathbb{R}^m$, hence
  $|f_{k_j}(y)| \le |f_{k_j}(y)-h(y)| + |f(y)| \le \varepsilon + C_1$ and the right hand side is independent from j
  and so $|D_\sigma\gamma^{k_j}(x)| \le \varepsilon + C_1 =: C$ as claimed.

Passing to the limit in the right side we get:

$$\lim_{j\to\infty}\int_E D_\sigma\gamma^{k_j}(x)\,d\sigma(x) = \int_E\left(\int_{\mathbb{R}^m} h(y)\,d\nu_x(y)\right)\,d\sigma(x).$$

Finally, passing to the limit for $j\to\infty$ in (3.19) we obtain

$$\int_{E\times\mathbb{R}^m} h(y)\,d\mu(x,y) = \int_E\left(\int_{\mathbb{R}^m} h(y)\,d\nu_x(y)\right)\,d\sigma(x), \tag{3.20}$$

for E and f as above.

STEP 3: Now we prove the theorem for functions $f(x,y) = g(x)h(y)$, where $g \in C_c(\mathbb{R}^n)$ and $h \in C_c(\mathbb{R}^m)$. By approximation, we can represent g as $g = \lim_{k\to\infty} \psi_k$, where $\psi_k = \sum_{i=1}^{n_k} y_{i,k}\chi_{E_{i,k}}$, using (3.20), we get:

$$\begin{aligned}
&\int_{\mathbb{R}^{n+m}} g(x)h(y)\,d\mu(x,y) = \lim_{k\to\infty}\sum_{i=1}^{n_k} y_{i,k}\int_{E_{i,k}\times\mathbb{R}^m} h(y)\,d\mu(x,y) = \\
&= \lim_{k\to\infty}\sum_{i=1}^{n_k} y_{i,k}\int_{E_{i,k}}\left(\int_{\mathbb{R}^m} h(y)\,d\nu_x(y)\right)d\sigma(x) = \\
&= \lim_{k\to\infty}\int_{\mathbb{R}^n}\psi_k(x)\left(\int_{\mathbb{R}^m} h(y)\,d\nu_x(y)\right)d\sigma(x) = \\
&= \int_{\mathbb{R}^n} g(x)\left(\int_{\mathbb{R}^m} h(y)\,d\nu_x(y)\right)d\sigma(x).
\end{aligned} \tag{3.21}$$

Setting $h \equiv 1$ we deduce from (3.20) that $\mu(E\times\mathbb{R}^m) = \nu_x(\mathbb{R}^m)\sigma(E)$ but $\mu(E\times\mathbb{R}^m) = \sigma(E)$ and so we have $\nu_x(\mathbb{R}^m) = 1$.

STEP 4: Finally, by Stone-Weierstrass Theorem 3.2.1, any continuous and bounded function f on $\mathbb{R}^{m+n}$ can be locally uniformly approximated by finite sum of the form $\sum_{i=1}^N g_i(x)h_i(y)$ where g_i, h_i are bounded continuous functions, $i = 1,...,N$, and so we have:

- the mapping $x \mapsto \sum_{i=1}^N g_i(x)\int_{\mathbb{R}^m} h_i(y)\,d\nu_x(y) \to \int_{\mathbb{R}^m} f(x,y)\,d\nu_x(y)$ is σ-measurable because g_i is σ-measurable for any i and $\int_{\mathbb{R}^m} h_i(y)\,d\nu_x(y)$ is σ-measurable because of (3.18). This proves 1.

- By (3.21) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{n+m}}\sum_{i=1}^N g_i(x)h_i(y)\,d\mu(x,y) = \sum_{i=1}^N\int_{\mathbb{R}^n} g_i(x)\left(\int_{\mathbb{R}^m} h_i(y)\,d\nu_x(y)\right)d\sigma(x) = \\
&= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^m}\sum_{i=1}^N g_i(x)h_i(y)\,d\nu_x(y)\right)d\sigma(x) \underset{N\to\infty}{\longrightarrow} \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^m} f(x,y)\,d\nu_x(y)\right)d\sigma(x).
\end{aligned}$$

  This proves 2.

□

3.2.2 Young measures

We now use Theorem 3.2.2 to provide a theoretic characterization on the incompatibility of weak convergence and nonlinear composition.

Theorem 3.2.3. *Let* $\{f_k\}_{k=1}^{\infty}$ *be a bounded sequence in* $L^{\infty}(U;\mathbb{R}^m)$.
Then there exists a subsequence $\{f_{k_j}\}_{j=1}^{\infty} \subset \{f_k\}_{k=1}^{\infty}$ *and, for a.e.* $x \in U$*, a Borel probability measure* ν_x *on* $\mathbb{R}^m$*, such that, for each* $F \in C(\mathbb{R}^m)$*, we have:*

$$F(f_{k_j}) \rightharpoonup \overline{F} \quad in \quad L^{\infty}(U), \tag{3.22}$$

where

$$\overline{F}(x) = \int_{\mathbb{R}^m} F(y)\, d\nu_x(y) \quad a.e. \quad x \in U.$$

Remark 31. By weak compactness Theorem 1.4.9, since $\{f_k\}_{k=1}^{\infty}$ is a bounded sequence in $L^{\infty}(U;\mathbb{R}^m)$, there exists a subsequence $\{f_{k_j}\}_{j=1}^{\infty} \subset \{f_k\}_{k=1}^{\infty}$ and a function $f \in L^{\infty}(U;\mathbb{R}^m)$ such that $f_{k_j} \rightharpoonup f$ in $L^{\infty}(U)$. The above theorem points out that, in general, given $F \in C(\mathbb{R}^m)$, $F(f_{k_j}) \not\rightharpoonup F(f)$. See Example 2.

Definition 3.8. We call $\{\nu_x\}_{x\in U}$ the family of *Young measures* associated with the subsequence $\{f_{k_j}\}_{j=1}^{\infty} \subset \{f_k\}_{k=1}^{\infty}$.

Proof. For each Borel set $E \subset \mathbb{R}^n \times \mathbb{R}^m$, define

$$\mu_k(E) := \int_U \chi_E(x, f_k(x))\, dx, \qquad k = 1, 2, ...$$

$\{\mu_k\}_{k=1}^{\infty}$ is a bounded sequence of nonnegative measures in $\mathscr{M}(U \times \mathbb{R}^m)$ because

$$\sup_k \mu_k(U \times \mathbb{R}^m) = \sup_k \int_U \chi_{U\times\mathbb{R}^m}(x, f_k(x))\, dx = \int_U 1\, dx = \mathscr{L}^n(U) < +\infty.$$

Thus by weak compactness Theorem for measures 1.7.1, there exists a subsequence $\{\mu_{k_j}\}_{j=1}^{\infty} \subset \{\mu_k\}_{k=1}^{\infty}$ and a nonegative measure μ such that $\mu_{k_j} \rightharpoonup \mu$ in $\mathscr{M}(U \times \mathbb{R}^m)$. Now, denoting by $\mathscr{L}^n\Big|_U$ the n-dimensional Lebesgue measure restricted to U, we claim that the projection of μ onto $\mathbb{R}^n$ is $\sigma = \mathscr{L}^n\Big|_U$.

To prove this claim, note that if $V \subset U$ is open, Example 4 implies

$$\sigma(V) = \mu(V \times \mathbb{R}^m) \leq \liminf_{j\to\infty} \mu_{k_j}(V \times \mathbb{R}^m) =$$

$$= \liminf_{j\to\infty} \int_U \chi_{V\times\mathbb{R}^m}(x, f_{k_j}(x))\, dx =$$

$$= \liminf_{j\to\infty} \int_U \chi_V\, dx = \mathscr{L}^n(V).$$

Thus $\sigma \leq \mathscr{L}^n\Big|_U$.

On the other hand, let $K \subset U$ be compact. Since $\{f_{k_j}\}_{j=1}^\infty$ is bounded sequence in $L^\infty(U;\mathbb{R}^m)$, there exists $R > 0$ such that $supp(\mu), supp(\mu_{k_j}) \subset U \times B(0,\, R)$. Hence, by Example 4 we obtain

$$\sigma(K) = \mu(K \times \mathbb{R}^m) = \mu(K \times B(0,\, R)) \geq \limsup_{j\to\infty} \mu_{k_j}(K \times B(0,\, R)) = \mathscr{L}^n(K).$$

And so, $\sigma \geq \mathscr{L}^n\Big|_U$. This proves the claim: $\sigma = \mathscr{L}^n\Big|_U$.

Now from Theorem 3.2.2 we deduce that there exists for a.e. $x \in U$ a Borel probability measure ν_x such that, for any bounded and continuous function f,

$$\int_{\mathbb{R}^{n+m}} f(x,y)\, d\mu(x,y) = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f(x,y)\, d\nu_x(y)\right) d\sigma(x),$$

but, since $\sigma = \mathscr{L}^n\Big|_U$, the above equality becomes

$$\int_{\mathbb{R}^{n+m}} f(x,y)\, d\mu(x,y) = \int_U \left(\int_{\mathbb{R}^m} f(x,y)\, d\nu_x(y)\right) dx. \tag{3.23}$$

Set $f(x,y) = \zeta(x)F(y)$ where $\zeta \in C_c(U)$ and $F \in C_c(\mathbb{R}^m)$.

Our aim is to show that $F(f_{k_j}) \rightharpoonup \overline{F}$ in $L^\infty(U)$, that is

$$\int_U F(f_{k_j}(x))g(x)\, dx \longrightarrow \int_U \overline{F}(x)g(x)\, dx \qquad \text{for any } g \in L^1(U),$$

but since $C_c(U)$ is dense in $L^1(U)$, it is sufficent to prove that

$$\int_U \zeta(x)F(f_{k_j}(x))\, dx \longrightarrow \int_U \zeta(x)\overline{F}(x)\, dx \qquad \text{for any } \zeta \in C_c(U).$$

By definition of μ_{k_j}, we have:

$$\lim_{j\to\infty} \int_U \zeta(x)F(f_{k_j}(x))\, dx = \lim_{j\to\infty} \int_{\mathbb{R}^{m+n}} f(x,y)\, d\mu_{k_j}(x,y);$$

since $\mu_{k_j} \rightharpoonup \mu$ in $\mathscr{M}(U \times \mathbb{R}^m)$ and using (3.23) we obtain:

$$\lim_{j\to\infty} \int_U \zeta(x) F(f_{k_j}(x))\, dx = \lim_{j\to\infty} \int_{\mathbb{R}^{m+n}} f(x,y)\, d\mu_{k_j}(x,y) =$$
$$= \int_{\mathbb{R}^{m+n}} f(x,y)\, d\mu(x,y) = \int_U \zeta(x) \left(\int_{\mathbb{R}^m} F(y)\, d\nu_x(y) \right) dx =$$
$$= \int_U \zeta(x) \overline{F}(x)\, dx.$$

A similar calculation obtains if only $F \in C(\mathbb{R}^m)$, since we can approximate F with continuous functions with an increasing sequence of compact supports. □

Structure of Young measures

By making specific choices of F in Theorem 3.2.3 we can deduce certain information regarding the structure of the Young measures.

Example 7. If there exists $C \subset \mathbb{R}^m$ closed such that $f_k \in C$ a.e. for $k = 1, 2, ...,$ then $supp \nu_x \subset C$ a.e. $x \in U$.
In fact, if we consider F such that F vanishes on C, we have $F(f_k) = 0$ for any $k = 1, 2, ...$ because $f_k \in C$ a.e. for $k = 1, 2, ...$; hence

$$F(f_k) = 0 \rightharpoonup \overline{F} = 0,$$

and so

$$0 = \overline{F}(x) = \int_{\mathbb{R}^m} F(y)\, d\nu_x(y), \qquad \forall F \text{ such that } F = 0 \text{ on } C.$$

Thus, $supp \nu_x \subset C$ a.e. $x \in U$.

Example 8. If ν_x is a unit point mass for a.e. $x \in U$, then, passing if necessary to a further subsequence, we have $f_{k_j} \to f$ a.e. in U.
Indeed, since $\{f_k\}_{k=1}^{\infty}$ is a bounded sequence in $L^{\infty}(U; \mathbb{R}^m)$, by weak compactness Theorem 1.4.9, there exists a subsequence $\{f_{k_j}\}_{j=1}^{\infty} \subset \{f_k\}_{k=1}^{\infty}$ and a function $f \in L^{\infty}(U; \mathbb{R}^m)$ such that $f_{k_j} \rightharpoonup f \quad \text{in} \quad L^{\infty}(U)$.
Furthermore, if we choose $F \equiv id$ we have

$$F(f_{k_j}) = f_{k_j} \rightharpoonup \overline{F};$$

thanks to the uniqueness of the weak limit, it must be $\overline{F} = f$;
this means that:

$$\overline{F}(x) = f(x) = \int_{\mathbb{R}^m} F(y)\, d\nu_x(y) = \int_{\mathbb{R}^m} y\, d\nu_x(y).$$

Since ν_x is a unit point mass for a.e. $x \in U$, we necessarily have $\nu_x = \delta_{f(x)}$.
Moreover, $f_{k_j} \longrightarrow f \quad \text{in} \quad L^2(U;\mathbb{R}^m)$ because

$$\left|\int_U \left(|f_{k_j}(x)|^2 - \int_{\mathbb{R}^m} |y|^2\, d\nu_x(y)\right)\, dx\right| = \left|\int_U \left(|f_{k_j}(x)|^2 - |f(x)|^2\right)\, dx\right| \leq$$

$$\leq \int_U \left|f_{k_j}(x)^2 - f(x)^2\right|\, dx = \int_U \left|f_{k_j}(x) - f(x)\right|\left|f_{k_j}(x) + f(x)\right|\, dx;$$

since f_{k_j}, f are bounded, $\left|f_{k_j}(x) + f(x)\right| \leq C$ where C is a constant; moreover, notice that, since U is bounded, the function $1 \in L^1(U)$ and so using that $f_{k_j} \rightharpoonup f \quad \text{in} \quad L^\infty(U)$, we get

$$\left|\int_U \left(|f_{k_j}(x)|^2 - \int_{\mathbb{R}^m} |y|^2\, d\nu_x(y)\right)\, dx\right| \leq \int_U \left|f_{k_j}(x) - f(x)\right|\left|f_{k_j}(x) + f(x)\right|\, dx \leq$$

$$\leq C \int_U \left|f_{k_j}(x) - f(x)\right|\, dx \longrightarrow 0.$$

In conclusion, as $j \to \infty$ we have

$$\left\|f_{k_j}\right\|^2_{L^2(U)} = \int_U |f_{k_j}|^2(x)\, dx \longrightarrow \int_U \left(\int_{\mathbb{R}^m} |y|^2\, d\nu_x(y)\right)\, dx = \|f\|^2_{L^2(U)},$$

and so $f_{k_j} \longrightarrow f \quad \text{in} \quad L^2(U;\mathbb{R}^m)$ as claimed.

Chapter 4

Constraint minima of functionals

In this chapter we provide examples of partial differential equations solved by minima of functionals. The main tool we will use to find such minima is Lagrange multiplier method. We will analyze separately the case of critical and subcritical exponent since they require different approaches.

4.1 Lagrange multipliers method

In this section we prove Lagrange multiplier Theorem.
If X, Y are Banach spaces, $x_0 \in X$ and $r > 0$ such that $B(x_0, r) \subset X$,
$F : B(x_0, r) \longrightarrow \mathbb{R}$ and $G : B(x_0, r) \longrightarrow Y$ continuous and differentiable, we want to find the minimum of F when x varies in $\{G(x) = 0\}$.
Before stating Lagrange multiplier Theorem, we prove some propositions which we will need in its proof.

Notation 1. Given X, Y Banach spaces, we denote by $L(X, Y)$ the *space of linear functionals* from X to Y.
Given a functional $T \in L(X, Y)$, we denote by $R(T)$ the *range* of T and by $N(T)$ the *kernel* of T.
We denote by $\rho(\cdot, \cdot)$ the distance between a point and a set or between two sets.

Proposition 4.1.1. *Let X, Y be Banach spaces, $T \in L(X, Y)$ such that $R(T) = Y$, $x_0 \in X$ and $r > 0$ such that $B(x_0, r) \subset X$ and*

$G : B(x_0, r) \subset X \longrightarrow Y$ such that $G(x_0) = 0$ and, for any x, $\overline{x} \in B(x_0, r)$,

$$|G(x) - G(\overline{x}) - T(x - \overline{x})| \leq \delta |x - \overline{x}|, \tag{4.1}$$

where δ such that $\delta\gamma < \frac{1}{2}$ with γ such that $\rho(x, N(T)) \leq \gamma |T(x)|$ on X.
Then there exist $s \in]0,\, r]$, $\alpha > 0$ and a function $g : B(x_0, s) \longrightarrow X$ such that

$$G(x + g(x)) = 0 \quad \textit{and} \quad |g(x)| \leq \alpha |G(x)| \;\; \textit{on} \;\; B(x_0, s).$$

Proof. Since, in general, the functional T is not invertible, for any $z \in B(x_0, s)$ and any $x \in B(x_0, r)$, with $s \leq r$ appropriate, the set

$$\Psi_z(x) = I(x) - T^{-1} G(z + x)$$

contains more than one element.
For x, $\overline{x} \in B(x_0, r) - z$, there exist $u \in \Psi_z(x)$ and $\overline{u} \in \Psi_z(\overline{x})$ such that

$$\rho(\Psi_z(x), \Psi_z(\overline{x})) = \rho(u + N(T), \overline{u} + N(T));$$

moreover, since $u \in \Psi_z(x) = I(x) - T^{-1}G(z + x)$, we have $x - u \in T^{-1}G(z + x)$ and, in the same way, $\overline{x} - \overline{u} \in T^{-1}G(z + \overline{x})$, and so

$$T(x - u) = G(z + x) \quad \text{and} \quad T(\overline{x} - \overline{u}) = G(z + \overline{x}).$$

Thus, using the fact that $\rho(x, N(T)) \leq \gamma |T(x)|$, we have

$$\begin{aligned}
&\rho(\Psi_z(x), \Psi_z(\overline{x})) = \rho(u - \overline{u}, N(T)) \leq \gamma |T(u - \overline{u})| = \\
&= \gamma |T(x - u) - T(\overline{x} - \overline{u}) - T(x - \overline{x})| = \gamma |G(z + x) - G(z + \overline{x}) - T(x - \overline{x})| \leq \\
&\leq \gamma\delta |x - \overline{x}|, \qquad \text{with } \; \delta\gamma < \frac{1}{2}.
\end{aligned}$$

Since $N(T)$ is closed, also $\Psi_z(x)$ is closed and if we choose $u \in T^{-1}G(z)$, we get:

$$\rho(0, \Psi_z(0)) = \rho(0, u + N(T)) \leq \gamma |T(u)| = \gamma |G(z)|.$$

Thus, if $z \in B(x_0, s)$ with $s < \frac{r}{2}$ and s is such that $\gamma |G(z)| < \frac{r}{4} < \frac{r}{2}(1 - \gamma\delta)$ on $B(x_0, s)$, then there exists $g(z) \in B(0, \frac{r}{2})$ satisfying

$$|g(z)| \leq \frac{2}{1 - \gamma\delta} \, \rho(0, \Psi_z(0)) \leq \alpha |G(z)|, \quad \text{with } \; \alpha = 4\gamma.$$

□

Definition 4.1. Let X, Y be Banach spaces, $x_0 \in X$ and $r > 0$ such that $B(x_0, r) \subset X$ and $G : B(x_0, r) \longrightarrow Y$ continuous and differentiable. Let

$$D := \{x \in B(x_0, r); \quad G(x) = 0\}, \quad \text{and } x_0 \in D.$$

Let $h \in D$. We say that h is a *tangent vector* to D at x_0, and we write $h \in T_D(x_0)$, if there exists $\delta > 0$ and a function $v : [0, \delta] \longrightarrow D$ such that

- $v(0) = x_0$,
- $v'(0) = h$.

Proposition 4.1.2. *Let D as above and $G : B(x_0, r) \longrightarrow Y$ differentiable. If $G'(x_0) : X \longrightarrow Y$ is onto, then $N(G'(x_0)) = T_D(x_0)$.*

Proof. Since $T_D(x_0)$ is the set containing all tangent vectors to D in x_0, an element $h \in T_D(x_0)$ must be such that

$$x_0 + th + w(t) \in D \;\text{ for } t > 0 \qquad \text{and} \qquad \frac{w(t)}{t} \to 0 \;\text{ as } t \to 0^+,$$

where $w(t) = v(t) - x_0 - th$ with $w(0) = 0$ and

$$\frac{w(t)}{t} = \frac{v(t) - v(0)}{t} - h \to v'(0) - h = 0 \;\text{ as } t \to 0^+.$$

Hence $h \in N(G'(x_0))$ because by Taylor expansion we have

$$\begin{aligned} 0 = G(x_0 + th + w(t)) &= G(x_0) + G'(x_0)th + G'(x_0)w(t) + o(th + w(t)) = \\ &= 0 + G'(x_0)th + G'(x_0)w(t) + o(t), \end{aligned}$$

that is $G'(x_0)th + G'(x_0)w(t) + o(t) = 0$. And so:

$$G'(x_0)h = -G'(x_0)\frac{w(t)}{t} + o(1) \to 0 \;\text{ as } t \to 0^+;$$

hence $G'(x_0)h = 0$ and $h \in N(G'(x_0))$. This proves $T_D(x_0) \subset N(G'(x_0))$.
Now we prove $N(G'(x_0)) \subset T_D(x_0)$. Let $h \in N(G'(x_0))$, we shall find w such that:

(i) $G(x_0 + th + w(t)) = 0$,

(ii) $w(t) = o(t)$ as $t \to 0^+$.

Notice that condition **(ii)** is true if we find w such that $|w(t)| \leq M|G(x_0 + th)|$ for some M and for any small t. Since G is differentiable and condition (4.1) is satisfied, by Proposition 4.1.1, there exists $g(z) \in B(x_0, s)$ for any $z \in B(x_0, s)$ such that

$$G(z + g(z)) = 0 \quad \text{and} \quad |g(z)| \leq \alpha|G(z)| \ \text{ on } \ B(x_0, s).$$

Thus, if we choose $w(t) := g(x_0 + th)$, for a small t, we have

$$|w(t)| \leq \alpha|G(x_0 + th)| \quad \text{and} \quad G(x_0 + th + w(t)) = 0,$$

and so $h \in T_D(x_0)$ and this concludes the proof. □

Now we are ready to prove Lagrange multiplier Theorem

Theorem 4.1.3 (Lagrange multiplier Theorem)**.** *Let X, Y be Banach spaces, $x_0 \in X$ and $r > 0$ such that $B(x_0, r) \subset X$, $F : B(x_0, r) \longrightarrow \mathbb{R}$ and $G : B(x_0, r) \longrightarrow Y$ continuous and differentiable such that $G(x_0) = 0$ and $R(G'(x_0))$ closed. Suppose also that*

$$F(x_0) = \min\{F(x); \quad x \in B(x_0, r),\ G(x) = 0\}.$$

Then, there exist Lagrange Multipliers $\lambda \in \mathbb{R}$ and $y^* \in Y^*$ not all zero such that

$$\lambda F'(x_0) + (G'(x_0))^* y^* = 0.$$

Moreover, if $R(G'(x_0)) = Y$, then $\lambda \neq 0$.

Proof. Let $Y_0 = R(G'(x_0))$.

- Suppose $Y_0 \neq Y$, then the proof is easy: Y_0 is closed, hence there exists $y^* \in Y^* \setminus \{0\}$ such that $y^*(Y_0) = 0$. By definition of Y_0, $y^*(G'(x_0)(X)) = 0$ and so $(G'(x_0))^* y^* = 0$, which is the thesis if $\lambda = 0$.

- Suppose $Y_0 = Y$, that is $G'(x_0) : X \longrightarrow Y$ is onto. The hypothesis on $F(x_0)$ implies that the function

$$\psi(t) := F(x_0 + th + w(t))$$

has a minimum in $t = 0$, and so passing to the limit for $t \to 0^+$ in

$$\frac{\psi(t) - \psi(0)}{t} = F'(x_0)h + F'(x_0)\frac{w(t)}{t} + \frac{o(|th + w(t)|)}{t}$$

gives $0 = \psi'(0) = F'(x_0)h$; hence $F'(x_0)h = 0$.
Moreover, since $R(G'(x_0)) = Y$, by Proposition 4.1.2, $N(G'(x_0)) = T_D(x_0)$.
As $F'(x_0)h = 0$ for any $h \in T_D(x_0)$, then

$$F'(x_0) \in (N(G'(x_0)))^{\perp} = R(G'(x_0)^*);$$

so $F'(x_0) = -G'(x_0)^* y^*$ for some $y^* \in Y^*$
Thus $F'(x_0) + G'(x_0)^* y^* = 0$.

□

Corollary 4.1.4. *Let X be a Banach space, $x_0 \in X$ and $r > 0$ such that $B(x_0, r) \subset X$, $F : B(x_0, r) \longrightarrow \mathbb{R}$ and $G : B(x_0, r) \longrightarrow \mathbb{R}$ continuous and differentiable such that $G(x_0) = 0$ and $G'(x_0) \neq 0$. Then, there exist $\lambda \in \mathbb{R}$, $\lambda \neq 0$ such that*

$$F'(x_0) + \lambda(G'(x_0)) = 0.$$

4.2 Convexity

Convexity is the simplest structural condition which is, at least, partially compatible with weak convergence. In this section we explain the role of convexity in the calculus of variation.

4.2.1 Calculus of variation

The calculus of variation provides one of the most important instances where weak convergence methods were successfully applied to nonlinear problems.
Our aim is to find a minimizer for the functional:

$$I(w) = \int_U F(Dw)\, dx \tag{4.2}$$

among all candidate functions w lying in $\mathscr{A}$, which is the class of admissible functions. We take

$$\mathscr{A} = \{w \in W^{1,q}(U); \quad w = g \text{ on } \partial U\},$$

for $1 < q < \infty$, where the boundary values are assumed in the trace sense (see Subsection 2.2.2) for some function $g : \partial U \longrightarrow \mathbb{R}$.
Given a smooth function $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ we want to establish conditions ensuring the existence of a minimizer. To this aim, let $\{u_k\}_{k=1}^{\infty} \subset \mathscr{A}$ be a minimizing sequence,

$$I(u_k) \longrightarrow \inf_{w \in \mathscr{A}} I(w), \qquad \text{as } k \to \infty$$

and suppose this infimum to be finite.

Definition 4.2. Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth function, $\alpha > 0$, $\beta \geq 0$ constants. We say that F satisfies the *coercivity condition* if, for any $p \in \mathbb{R}^n$,

$$F(p) \geq \alpha |p|^q - \beta.$$

Assume that F satiesfies the coercivity condition according to the above definition; this implies that the sequence $\{u_k\}_{k=1}^{\infty}$ is bounded in $W^{1,q}(U)$:
since $\{u_k\}_{k=1}^{\infty}$ is a minimizing sequence,

$$\int_U F(Du_k)\, dx \longrightarrow \inf_{w \in \mathscr{A}} I(w) =: m$$

and by coercivity condition

$$\int_U F(Du_k)\,dx \geq \int_U (\alpha|Du_k|^q - \beta)\,dx = -\beta\mathscr{L}^n(U) + \alpha\int_U |Du_k|^q\,dx =$$
$$= C + \alpha\,\|u_k\|^q_{W^{1,q}(U)}\,;$$

hence, $\{u_k\}_{k=1}^{\infty}$ is bounded in $W^{1,q}(U)$.
Thus, there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty} \subset \{u_k\}_{k=1}^{\infty}$ and $u \in \mathscr{A}$ such that

$$u_{k_j} \rightharpoonup u \qquad \text{as } \; j \to \infty, \;\; \text{in } \; W^{1,q}(U).$$

Now what is left to show is that the minimizer we are looking for is $u \in \mathscr{A}$.
To do so, we must show that:

$$u_{k_j} \rightharpoonup u \;\Rightarrow I(u) \leq \liminf_{j\to\infty} I(U_{k_j}), \tag{4.3}$$

that is, I is lower semicontinuous with respect to weak convergence in $W^{1,q}(U)$.
In this way we would have:

$$\inf_{w\in\mathscr{A}} I(w) \leq I(u) \leq \liminf_{j\to\infty} I(u_{k_j}) = \inf_{w\in\mathscr{A}} I(w),$$

and so, since $u \in \mathscr{A}$ we would deduce:

$$I(u) = \inf_{w\in\mathscr{A}} I(w) = \min_{w\in\mathscr{A}} I(w).$$

Since I depends on F, to conclude we must understand what structural assumption we may need on F to deduce (4.3). The natural assumption will be that F is convex. Indeed the following Proposition holds:

Proposition 4.2.1. *A convex function F is a convex hull of linear functions. More precisely, if F is convex, we can approximate it with a monotone increasing sequence of maximums of affine functions, that is:*

$$F(p) = \lim_{m\to\infty} F^m(p) \quad \textit{with} \quad F^m(p) = \max_{j=1,\dots m} (b_j \cdot p + c^j), \quad p \in \mathbb{R}^n.$$

Since weak convergence is defined in terms of linear functionals, an affine function is weakly continuous and, by Proposition 4.2.1, a convex function is a convex hull of affine functions, we expect that the structural assumption we may need on F to deduce (4.3) is that F is convex, that is

$$\xi^T D^2 F(p)\xi \geq 0, \quad p,\, \xi \in \mathbb{R}^n. \tag{4.4}$$

4.2.2 Weak lower semicontinuity

The aim of this paragraph is to verify that the intuition we had in the above paragraph, that is to assume that F is convex, is indeed the proper structural hypothesis in order to show that I is lower semicontinuous with respect to weak convergence in $W^{1,q}(U)$.

Theorem 4.2.2. *The functional I is lower semicontinuous with respect to weak convergence in $W^{1,q}(U)$ if and only if F is convex.*

Proof. ($\Rightarrow$) Fix a vector $p \in \mathbb{R}^n$ and suppose that $U = Q$, where Q is the unit open cube in $\mathbb{R}^n$. Fix any $v \in C_c^\infty(Q)$ and for any $k \in \mathbb{N}_0$, we subdivide Q into subcubes $\{Q_l\}_{l=1}^{2^{kn}}$ of side lenght $\frac{1}{2^k}$.
Then denote by x_l the center of the cube Q_l and define

$$u_k(x) = \frac{1}{2^k} v(2^k(x - x_l)) + p \cdot x, \quad x \in Q_l$$

and let $u(x) = p \cdot x$. Then, $u_k \rightharpoonup u$ in $W^{1,q}(U)$; indeed

$$\left| \frac{1}{2^k} v(2^k(x - x_l)) \right| \leq \frac{\sup |v|}{2^k} \leq \frac{C}{2^k} \to 0, \quad \text{as } k \to \infty$$

and so

$$u_k \longrightarrow u \quad \text{uniformly} \quad \text{as } k \to \infty.$$

Moreover,

$$\frac{\partial u_k}{\partial x_i}(x) = \frac{\partial v}{\partial x_i}(2^k(x - x_l)) + p_i,$$

$$\frac{\partial u_k}{\partial x_i} \rightharpoonup w \quad \text{in } L^q(U) \text{ for some function } w, \text{ as } k \to \infty,$$

but $w = \frac{\partial u}{\partial x_i}$ because for any $\phi \in C_c^\infty(U)$, integrating by parts,

$$\int_Q u_k(x) \partial_{x_i} \phi(x)\, dx = -\int_Q \partial_{x_i} u_k \phi \, dx$$

and passing to the limit for $k \to \infty$ we get

$$\int_Q u(x) \partial_{x_i} \phi(x)\, dx = -\int_Q w\phi \, dx;$$

so, by definition of weak derivative, $w = \frac{\partial u}{\partial x_i}$.
This shows that $u_k \rightharpoonup u$ in $W^{1,q}(U)$.
Finally, since by hypothesis the functional I is lower semicontinuous with respect to weak convergence in $W^{1,q}(U)$ and noting that
$F(Du) = F(D(p \cdot x)) = F(p)$, we have

$$\mathscr{L}^n(Q)F(p) = \int_Q F(Du)\,dx = I(u) \leq \liminf_{k\to\infty} I(u_k) = \int_Q F(p + Dv)\,dx.$$

Thus, in particular, the function $u(x) = p \cdot x$ is a minimizer subject to its own boundary values on ∂Q; consequently, inequality (4.4), asserts that F is convex.
($\Leftarrow$) Now suppose that $u_k \rightharpoonup u$ in $W^{1,q}(U)$ and assume for the moment that F is the maximum of finitely many affine functions (and so F is convex), that is

$$F(p) = \max_{j=1,\dots,m} (b_j \cdot p + c^j), \quad p \in \mathbb{R}^n. \tag{4.5}$$

Moreover, let $E_j = \{x \in U; \quad F(Du(x)) = b_j \cdot Du(x) + c^j\}$ and suppose E_j to be disjoint for $j = 1, ..., m$. Then $U = \bigcup_{j=1}^m E_j$.
Now, since weak convergence implies the convergence of averages (see Subsection 1.4.3), we have

$$I(u) = \int_U F(Du)\,dx = \sum_{j=1}^m \int_{E_j} (b_j \cdot Du(x) + c^j)\,dx =$$

$$= \lim_{k\to\infty} \sum_{j=1}^m \int_{E_j} (b_j \cdot Du_k(x) + c^j)\,dx.$$

Now by the max representation formula (4.5), we have

$$\lim_{k\to\infty} \sum_{j=1}^m \int_{E_j} (b_j \cdot Du_k(x) + c^j)\,dx \leq \liminf_{k\to\infty} \sum_{j=1}^m \int_{E_j} F(Du_k)\,dx = \liminf_{k\to\infty} I(u_k).$$

This estabilishes the weak lower semicontinuity of I when F is the maximum of finitely many planes.
In the general case with F convex, using Proposition 4.2.1, we can approximate F with a monotone increasing sequence of maximums of affine functions and we conclude applying monotone convergence Theorem 1.2.1. $\square$

Example 9 (Euler-Lagrange equation). Under appropriate growth hypothesis on F, so that u is a minimizer, we have that the function

$$i(t) = I(u + tv) = \int_U F(Du + tDv)\,dx$$

attains its minimum in $t = 0$, hence $i'(0) = 0$; in particular:

$$0 = i'(0) = \int_U \sum_{j=1}^{n} \frac{\partial F}{\partial p_j}(Du) D_j v\,dx = -\int_U \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial p_j}\right) v\,dx =$$

$$= -\int_U div(DF(Du)) v\,dx,$$

so the minimizer u is a weak solution of the *Euler-Lagrange equation*:

$$\begin{cases} -div(DF(Du)) = 0 & \text{in } U \\ u = g & \text{in } \partial U \end{cases}$$

This is an example of a nonlinear PDE solved by weak convergence methods.

4.2.3 Convergence of energies and strong convergence

In this paragraph we show that under appropriate structural hypothesis on F and if $I(u_k) \to I(u)$, the the full minimizing sequence converges strongly to u in $W^{1,2}(U)$.

Theorem 4.2.3. *Let $q = 2$ and suppose that F satisfies*

1. $|F(p)| \le C(1 + |p|^2), \quad p \in \mathbb{R}^n$.

2. *F is uniformly strictly convex, that is: $\exists \gamma > 0$ such that $\forall p, \xi \in \mathbb{R}^n$,*

$$\xi^T D^2 F(p) \xi \ge \gamma.$$

Suppose also that $u_k \rightharpoonup u$ in $W^{1,2}(U)$ and that $I(u_k) \to I(u)$.
Then $u_k \longrightarrow u$ strongly in $W^{1,2}(U)$.

Remark 32. This theorem asserts that the convergence of the energies $I(u_k)$ improves weak to strong convergence. The uniform convexity of F is necessary because it damps out wild oscillations in $\{Du_k\}_{k=1}^{\infty}$.

Proof. Using Taylor expansion, for any $p,\, q \in \mathbb{R}^n$,

$$F(q) = F(p) + DF(p) \cdot (q - p) + \frac{1}{2}(q - p)^T D^2 F(\tilde{p})(q - p).$$

Hence by 2., we have

$$\begin{aligned} F(q) &= F(p) + DF(p) \cdot (q - p) + \frac{1}{2}(q - p)^T D^2 F(\tilde{p})(q - p) \geq \\ &\geq F(p) + DF(p) \cdot (q - p) + \frac{\gamma}{2}|q - p|^2. \end{aligned}$$

Set $p = Du$ and $q = Du_k$ and integrating over U, we obtain

$$I(u_k) \geq I(u) + \int_U DF(Du) \cdot (Du_k - Du)\, dx + \frac{\gamma}{2} \int_U (|Du_k - Du|^2)\, dx,$$

that is

$$I(u_k) - I(u) - \int_U DF(Du) \cdot (Du_k - Du)\, dx \geq \frac{\gamma}{2} \int_U (|Du_k - Du|^2)\, dx. \tag{4.6}$$

Now using Taylor expansion, the convexity of F and 1. we get
$|DF(p)| \leq C(1 + |p|)$ which means $|DF(Du)| \leq C(1 + |Du|)$.
Since $u \in W^{1,2}(U)$, then $Du \in W^{1,2}(U)$ and so, thanks to the above inequality, also $DF(Du) \in W^{1,2}(U)$.
Since by hypothesis $u_k \rightharpoonup u$ in $W^{1,2}(U)$, then $Du_k \rightharpoonup Du$ in $L^2(U, \mathbb{R}^n)$, so

$$\int_U DF(Du) \cdot (Du_k - Du)\, dx \to 0, \quad \text{as } k \to \infty.$$

Moreover, $I(u_k) \to I(u)$ and so $I(u_k) - I(u) \to 0$. Thus by (4.6),

$$Du_k \to Du \quad \text{in } \; L^2(U, \mathbb{R}^n),$$

so $u_k \longrightarrow u$ strongly in $W^{1,2}(U)$. □

4.3 Examples with subcritical exponent

Given a nonlinear functional I, our aim is to find a minimizer for I in a certain class of admissible functions $\mathscr{A}$ such that

$$\mathscr{A} = \{w \in W_0^{1,s}(U); \quad \|w\|_{L^s} = 1\}, \quad \text{with } s < s^*.$$

Since the exponent s is subcritical because is $s < s^*$, Sobolev imbedding Theorem assures that if u such that $I(u) = \inf_{w\in\mathscr{A}} I(w)$, then u is a minimum. In fact we will prove that $u \in \mathscr{A}$, that is $\|u\|_{L^s} = 1$, using that if $1 \leq q < s^*$, then the imbedding $W^{1,s}(U) \longrightarrow L^q(U)$ is compact.
Then we will find the equation satisfied by the minimizer using the Corollary to Lagrange multipliers Theorem 4.1.4.

Example 10. Let U be a smooth bounded subset of $\mathbb{R}^n$. We will find a minimizer for the functional

$$I(w) = \int_U |Dw|^2\, dx$$

among all candidate functions w lying in $\mathscr{A}$ where

$$\mathscr{A} = \{w \in W_0^{1,2}(U); \quad \|w\|_{L^2} = 1\} \quad \text{and } 2 < 2^* = \frac{2n}{n-2}.$$

Then, we will find the equation solved by the minimizer.
Let $\{u_k\}_{k=1}^{\infty} \subset \mathscr{A}$ be a minimizing sequence, that is

$$I(u_k) \longrightarrow \inf_{w\in\mathscr{A}} I(w) \qquad \text{as } k \to \infty.$$

Then $\{u_k\}_{k=1}^{\infty}$ is bounded in $W_0^{1,2}(U)$ because there exists M such that for any $k \in \mathbb{N}$,

$$M \geq |I(u_k)| = \|Du_k\|^2_{L^2(U)} = \|u_k\|^2_{W_0^{1,2}(U)}$$

and so there exists a subsequence $\{u_{k_l}\}_{l=1}^{\infty}$ which weakly converges in $W_0^{1,2}(U)$. Moreover, by Sobolev imbedding Theorem 2.2.1, since $2 < 2^*$, the imbedding $W_0^{1,2}(U) \longrightarrow L^2(U)$ is compact and so there exists a subsubsequence, which we still call $\{u_{k_l}\}_{l=1}^{\infty}$, which strongly converges to u in $L^2(U)$ and $\|u\|_{L^2(U)} = 1$ because

$\|u_{k_l}\|_{L^2(U)} = 1\ \forall l$. This function u is the minimizer we were looking for: indeed, as $\{u_{k_l}\}_{l=1}^{\infty}$ weakly converges in $W_0^{1,2}(U)$,

$$\|u\|^2_{W_0^{1,2}(U)} \leq \liminf_{l\to\infty} \|u_{k_l}\|^2_{W_0^{1,2}(U)}$$

and so

$$\inf_{w\in\mathscr{A}} I(w) \leq I(u) = \|u\|^2_{W_0^{1,2}(U)} \leq \liminf_{k\to\infty} \|u_{k_l}\|^2_{W_0^{1,2}(U)} = \liminf_{l\to\infty} I(u_{k_l}) =$$

$$= \inf_{w\in\mathscr{A}} I(w).$$

Since using Sobolev imbedding Theorem we showed that $\|u\|_{L^2(U)} = 1$, $u \in \mathscr{A}$ is a minimizer for I.

Remark 33. In this case, the reason why we could show that $u \in \mathscr{A}$ is that we look for functions w such that $\|w\|_{L^2} = 1$ and the exponent is $2 < 2^*$. In fact, Sobolev imbedding Theorem assures that the imbedding $W_0^{1,2}(U) \longrightarrow L^q(U)$ is compact for any $1 \leq q < 2^*$ and so $W_0^{1,2}(U) \longrightarrow L^2(U)$ is compact. Thanks to this compactness, the L^2-norm is preserved passing to the limit and this allowed us to deduce that $\|u\|_{L^2(U)} = 1$. In the following part of this chapter we will study the case with critical exponent 2^* and the main problem we will face will be to find a substitute for Sobolev imbedding Theorem which allows us to deduce that $u \in \mathscr{A}$.

Now we find the equation satiesfied by u using the Corollary to Lagrange multiplier Theorem 4.1.4, with $F(w) = I(w) = \int_U |Dw|^2\,dx$ and with the constraint $G(w) = 0$, where $G(w) = \|w\|^2_{L^2} - 1$.
Let $h \in W_0^{1,2}(U)$ and we calculate $dF(w)(h)$ and $dG(w)(h)$.

$$dF(w)(h) = \lim_{t\to 0}\left[\frac{F(w+th) - F(w)}{t}\right] =$$

$$= \lim_{t\to 0}\left[\frac{1}{t}\left(\int_U |D(w+th)|^2\,dx - \int_U |Dw|^2\,dx\right)\right] =$$

$$= \lim_{t\to 0}\left[\frac{1}{t}\left(\langle w+th, w+th\rangle_{W_0^{1,2}} - \langle w, w\rangle_{W_0^{1,2}}\right)\right] =$$

$$= \lim_{t\to 0}\left[2\langle w, h\rangle_{W_0^{1,2}} + t\right] = 2\langle w, h\rangle_{W_0^{1,2}}$$

$$= 2\int_U Dw\cdot Dh\,dx = -2\int_U \Delta w\, h\,dx,$$

and, denoting by $g(w) = |w|^2$, we have

$$dG(w)(h) = \lim_{t\to 0}\left[\frac{G(w+th)-G(w)}{t}\right] =$$

$$= \lim_{t\to 0}\left[\frac{1}{t}\int_U (g(w+th)-g(w))\,dx\right] =$$

$$= \lim_{t\to 0}\left[\frac{1}{t}\int_U (g(w)+g'(w)th+o(t)-g(w))\,dx\right] =$$

$$= \lim_{t\to 0}\left[\int_U (g'(w)h+o(1))\,dx\right] = 2\int_U w\,h\,dx.$$

Now we apply Corollary 4.1.4: since u is the minimizer there exist $\lambda \in \mathbb{R}$ such that for any h as above,

$$dF(u)(h) = \lambda dG(u)(h) \Rightarrow -2\int_U \Delta u\,h\,dx = 2\lambda\int_U u\,h\,dx$$

and so the equation satisfied by the minimizer u is $-\Delta u = \lambda u$.
Using that $dF(u)(h) = \lambda dG(u)(h)$ is true also for $h = u$, we can find λ:

$$I(u) = \min_{w\in\mathscr{A}} I(w) = \int_U |Du|^2\,dx = \frac{1}{2}dF(u)(u) = \frac{\lambda}{2}\,dG(u)(u) = \lambda\int_U |u|^2\,dx = \lambda.$$

In conclusion the minimizer u satisfies

$$-\Delta u = \left(\min_{w\in\mathscr{A}} I(w)\right) u.$$

We now provide another example with subcritical exponent.

Example 11. Let U be a smooth bounded subset of $\mathbb{R}^n$. We will find a minimizer for the functional

$$I(w) = \int_U |Dw|^p\,dx$$

among all candidate functions w lying in $\mathscr{A}$ where

$$\mathscr{A} = \{w \in W_0^{1,p}(U);\quad \|w\|_{L^p} = 1\}\quad \text{and } p < p^* = \frac{pn}{n-p}.$$

Then, we will find the equation solved by the minimizer.
Let $\{u_k\}_{k=1}^{\infty} \subset \mathscr{A}$ be a minimizing sequence,

$$I(u_k) \longrightarrow \inf_{w\in\mathscr{A}} I(w) \qquad \text{as } k\to\infty.$$

Then $\{u_k\}_{k=1}^{\infty}$ is bounded in $W_0^{1,p}(U)$ because there exists M such that for any $k \in \mathbb{N}$,

$$M \geq |I(u_k)| = \|Du_k\|_{L^p(U)}^p = \|u_k\|_{W_0^{1,p}(U)}^p ,$$

so there exists a subsequence $\{u_{k_l}\}_{l=1}^{\infty}$ which weakly converges in $W_0^{1,p}(U)$.
Moreover, by Sobolev imbedding Theorem 2.2.1, since $p < p^*$, the imbedding $W_0^{1,p}(U) \longrightarrow L^p(U)$ is compact and so there exists a subsubsequence, which we still call $\{u_{k_l}\}_{l=1}^{\infty}$, which strongly converges to u in $L^p(U)$ and $\|u\|_{L^p(U)} = 1$ because $\|u_{k_l}\|_{L^p(U)} = 1$ $\forall l$. This function u is the minimizer we were looking for: indeed, as $\{u_{k_l}\}_{l=1}^{\infty}$ weakly converges in $W_0^{1,p}(U)$,

$$\|u\|_{W_0^{1,p}(U)}^p \leq \liminf_{k\to\infty} \|u_{k_l}\|_{W_0^{1,p}(U)}^p$$

and so

$$\inf_{w\in\mathscr{A}} I(w) \leq I(u) = \|u\|_{W_0^{1,p}(U)}^p \leq \liminf_{k\to\infty} \|u_{k_l}\|_{W_0^{1,p}(U)}^p = \liminf_{k\to\infty} I(u_{k_l}) =$$

$$= \inf_{w\in\mathscr{A}} I(w).$$

Since, using Sobolev imbedding Theorem, we showed that $\|u\|_{L^p(U)} = 1$, $u \in \mathscr{A}$ is a minimizer for I. As in Remark 33 notice the importance of working with subcritical exponent $p < p^*$.
Now we find the equation satiesfied by u using Corollary to Lagrange multiplier Theorem 4.1.4, with $F(w) = I(w) = \int_U |Dw|^p\, dx$ and with the constraint $G(w) = 0$, where $G(w) = \|w\|_{L^p}^p - 1$. Arguing as in Example 10, let $h \in W_0^{1,p}(U)$ and we calculate $dF(w)(h)$ and $dG(w)(h)$.

$$dF(w)(h) = \lim_{t\to 0}\left[\frac{1}{t}\int_U \Big(p|Dw|^{p-2}Dw\cdot tDh + o(t)\Big)\, dx\right] =$$

$$= \lim_{t\to 0}\int_U \Big(p|Dw|^{p-2}Dw\cdot Dh + o(1)\Big)\, dx =$$

$$= p\int_U \Big(|Dw|^{p-2}Dw\cdot Dh\Big)\, dx =$$

$$= -p\int_U div(|Dw|^{p-2}Dw)h\, dx,$$

and, denoting by $g(w) = |w|^p$, we have

$$dG(w)(h) = \lim_{t\to 0}\left[\int_U \left(g'(w)h + o(1)\right)\, dx\right] =$$

$$= \lim_{t\to 0}\left[\int_U \left(p|w|^{p-2}w\, h + o(1)\right)\, dx\right] =$$

$$= \int_U \left(p|w|^{p-2}w\, h\right)\, dx.$$

Now we apply Corollary 4.1.4: since u is the minimizer there exist $\lambda \in \mathbb{R}$ such that for any h as above,

$$dF(u)(h) = \lambda dG(u)(h) \Rightarrow -p\int_U div(|Du|^{p-2}Du)h\, dx = \lambda \int_U \left(p|u|^{p-2}u\, h\right)\, dx,$$

so the equation satisfied by the minimizer u is

$$-div(|Du|^{p-2}Du) = \lambda\left(p|u|^{p-2}u\right).$$

Using that $dF(u)(h) = \lambda dG(u)(h)$ is true also for $h = u$, we can find λ:

$$I(u) = \min_{w\in\mathscr{A}} I(w) = \int_U |Du|^p\, dx = \frac{1}{p}dF(u)(u) = \frac{\lambda}{p}\, dG(u)(u) = \lambda\int_U |u|^p\, dx = \lambda.$$

In conclusion the minimizer u satisfies

$$-div(|Du|^{p-2}Du) = \left(\min_{w\in\mathscr{A}} I(w)\right)\left(p|u|^{p-2}u\right).$$

4.4 Examples with critical exponent

We now turn our attention to variational problems which involve critical growth nonlinearities and which fail to satisfy usual compactness criteria. As in the previous section, we are interested in discussing the problem of minimizing a certain functional I over an admissible set $\mathscr{A}$; however in this case we will deal with the critical exponent case. The main problem will be to show that the infimum over $\mathscr{A}$ is actually a minimizer because in this case we cannot apply Sobolev imbedding Theorem.

Example 12 (Critical Sobolev nonlinearities). We will discuss the problem of minimizing the functional

$$I(w) = \int_{\mathbb{R}^n} |Dw|^2 \, dx \quad n \geq 3$$

over the admissible set

$$\mathscr{A} = \{w \in L^{2^*}(\mathbb{R}^n); \quad \|w\|_{L^{2^*}(\mathbb{R}^n)} = 1, \ Dw \in L^2(\mathbb{R}^n, \mathbb{R}^n)\}.$$

Notice that in this case we look for a minimizer u such that $\|w\|_{L^{2^*}} = 1$, where 2^* is the crtitical exponent.

Denoting by C_2 the optimal constant for $q = 2$ in Gagliardo-Nirenberg-Sobolev inequality 2.4.1, for any $w \in \mathscr{A}$, we have

$$I(w) = \int_{\mathbb{R}^n} |Dw|^2 \, dx = \|Dw\|^2_{L^2(\mathbb{R}^n)} \geq C_2^{-2} \, \|w\|^2_{L^{2^*}(\mathbb{R}^n)} = C_2^{-2},$$

so $I = \inf_{w \in \mathscr{A}} I(w) = C_2^{-2}$.

Now we inquire as to whether this infimum is obtained, and so choose a minimizing sequence $\{u_k\}_{k=1}^{\infty} \subset \mathscr{A}$ such that

$$I(u_k) \longrightarrow I, \quad \text{as } \ k \to \infty. \tag{4.7}$$

Since (4.7) means that $\|Du_k\|^2_{L^2(\mathbb{R}^n)} \longrightarrow I$ as $k \to \infty$, $\{Du_k\}_{k=1}^{\infty}$ is bounded in $L^2(\mathbb{R}^n)$ and so we may assume that $Du_k \rightharpoonup Du$ in $L^2(\mathbb{R}^n, \mathbb{R}^n)$.

Moreover, for any k, $u_k \in \mathscr{A}$ and so $\|u_k\|_{L^{2^*}(\mathbb{R}^n)} = 1$; hence $\{u_k\}_{k=1}^{\infty}$ is bounded in $L^{2^*}(\mathbb{R}^n)$ and we may assume that $u_k \rightharpoonup u$ in $L^{2^*}(\mathbb{R}^n)$.

As the integrand of I is convex, we infer from Theorem 4.2.2 that

$$I(u) \leq \liminf_{k \to \infty} I(u_k) = \inf_{w \in \mathscr{A}} I(w).$$

Thus, provided that $u \in \mathscr{A}$, u is the minimizer we were looking for.
Since by Theorem 1.4.5,

$$\|u\|_{L^{2^*}(\mathbb{R}^n)} \leq 1, \tag{4.8}$$

the real question is whether or not

$$\|u\|_{L^{2^*}(\mathbb{R}^n)} = 1.$$

because this would imply $u \in \mathscr{A}$. Notice here the main difference with the sub-critical exponent case, where, by Sobolev imbedding Theorem, we could directly deduce that $u \in \mathscr{A}$.
Now there are two possible failures of compactness here.

1. It can happen that strict inequality obtains in (4.8) because some of the mass of the approximations leaks out to infinity. In other words, it may be that the family of measures $\{\nu_k\}_{k=1}^{\infty}$, $\nu_k = |u_k|^{2^*}$, $k = 1, 2, \ldots$ is not tight.

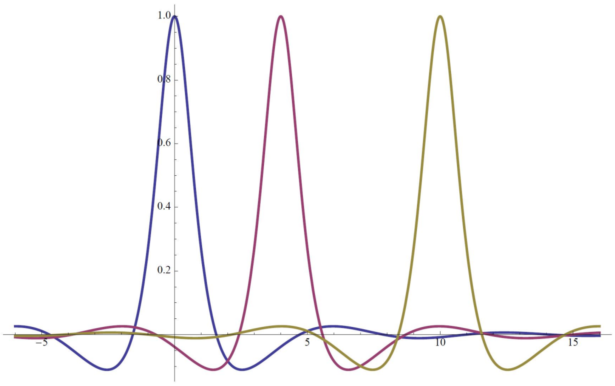

Figure 4.1: The mass of $\{\nu_k\}_{k=1}^{\infty}$ leaks out at infinity.

2. A second and rather more troublesome prospect is that, even if $\{\nu_k\}_{k=1}^{\infty}$ is tight, so that we may suppose $\nu_k \longrightarrow \nu$ in $\mathscr{M}(\mathbb{R}^n)$ and $\nu(\mathbb{R}^n) = 1$, then it can happen that
$$1 = \nu(\mathbb{R}^n) \neq \int_{\mathbb{R}^n} |u|^{2^*} \, dx.$$

This possibility arises if ν has a singular part, that is if the measures $\{\nu_k\}_{k=1}^{\infty}$ concentrate in the limit some of their mass into a set of Lebesgue measure zero.

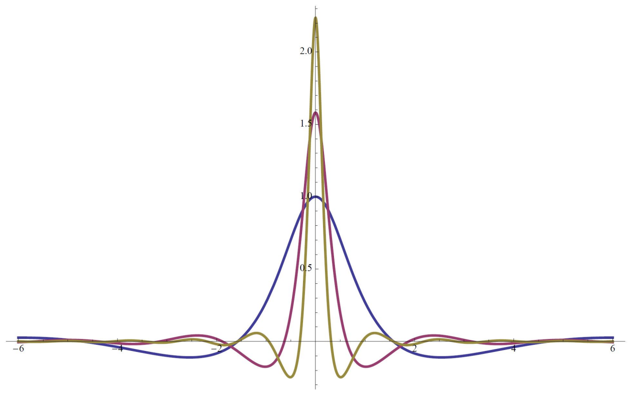

Figure 4.2: The mass of $\{\nu_k\}_{k=1}^{\infty}$ concentrates around $x = 0$.

Each of these unpleasant possibilities can definitely happen if we are unfortunate with the choice of the minimizing sequence. In fact, with appropriate translations and dilatations we can effect either of the failures of compactness recalled above. To see this, let $v \in \mathscr{A}$, $y \in \mathbb{R}^n$ and $s > 0$ and consider the rescaled function

$$v^{y,s}(x) = s^{\frac{-(n-2)}{2}} v\left(\frac{(x-y)}{s}\right), \quad x \in \mathbb{R}^n.$$

Using the change of variables $\left(\frac{(x-y)}{s}\right) = z$, $dx = s^n\, dz$, $v^{y,s}(x)$ satisfies

$$\begin{aligned}\int_{\mathbb{R}^n} |v^{y,s}(x)|^{2^*}\, dx &= \int_{\mathbb{R}^n} s^{\frac{-(n-2)}{2}2^*} \left| v\left(\frac{(x-y)}{s}\right)\right|^{2^*} dx = \\ &= \int_{\mathbb{R}^n} s^{-n+n} |v(z)|^{2^*}\, dz = \int_{\mathbb{R}^n} |v(z)|^{2^*}\, dz = \\ &= 1.\end{aligned}$$

Moreover

$$\frac{\partial v^{y,s}(x)}{\partial x_i} = s^{\frac{-(n-2)}{2}} \frac{\partial v}{\partial x_i}\left(\frac{(x-y)}{s}\right) s^{-1} = s^{-\frac{n}{2}} \frac{\partial v}{\partial x_i}\left(\frac{(x-y)}{s}\right)$$

and

$$Dv^{y,s}(x) = s^{-\frac{n}{2}} Dv\left(\frac{(x-y)}{s}\right).$$

Hence,

$$\int_{\mathbb{R}^n} |Dv^{y,s}(x)|^2 \, dx = \int_{\mathbb{R}^n} s^{-n+n} |Dv(z)|^2 \, dz = \int_{\mathbb{R}^n} |Dv(z)|^2 \, dz.$$

Consequently, given any minimizing sequence $\{u_k\}_{k=1}^{\infty}$, we could consider a new minimizing sequence $\{u_k^{y_k,s_k}\}_{k=1}^{\infty}$ for which by appropriate choices of translations $\{y_k\}_{k=1}^{\infty}$ and dilatation $\{s_k\}_{k=1}^{\infty}$, we could effect either of the failure of compactness recalled above.

A far more interesting prospect is designing translations and dilatatons to exclude the failure of compactness. The following theorem will provide the existence of the minimizer $u \in \mathscr{A}$.

Theorem 4.4.1. *Let* $\{u_k\}_{k=1}^{\infty} \subset \mathscr{A}$ *be a minimizing sequence satisfying*

$$I(u_k) \longrightarrow I \quad \text{as } k \to \infty.$$

Then there exist translations $\{y_k\}_{k=1}^{\infty} \subset \mathbb{R}^n$ *and dilatations* $\{s_k\}_{k=1}^{\infty} \subset]0,\infty[$, *such that the rescaled family* $\{u_k^{y_k,s_k}\}_{k=1}^{\infty} \subset \mathscr{A}$ *is strongly precompact in* $L^{2^*}(\mathbb{R}^n)$.
In particular, there exists a minimizer $u \in \mathscr{A}$.

Theorem 4.4.2. *Given that a minimizer exists, a careful analysis using symmetrization and ODE theory shows that any minimizer is of the form*

$$u_{y,\varepsilon}(x) = \frac{C_\varepsilon}{(\varepsilon + |x-y|^2)^{\frac{(n-2)}{2}}}, \qquad x \in \mathbb{R}^n; \tag{4.9}$$

for $\varepsilon > 0$, $y \in \mathbb{R}^n$ *and an appropriate normalization constant* C_ε.
(See [10], [11]).

Proof of Theorem 4.4.1. For $t > 0$ and $k = 1, 2, \ldots$, define *Lévy concentration functions* as follows:

$$Q_k(t) := \sup_{z \in \mathbb{R}^n} \int_{B(z,t)} |u_k|^{2^*} \, dx.$$

Denote by $Q_k^{y,s}(t)$ the concentration function of $u_k^{y,s}$, then by a change of variables we obtain $Q_k^{y,s}(t) = Q_k^{y,1}\left(\frac{t}{s}\right)$. Moreover, since $\{u_k\}_{k=1}^{\infty} \subset \mathscr{A}$,

$$\lim_{t\to\infty} Q_k(t) = \lim_{t\to\infty}\left[\sup_{z\in\mathbb{R}^n}\int_{B(z,t)}|u_k|^{2^*}\,dx\right] = \int_{\mathbb{R}^n}|u_k|^{2^*}\,dx = 1$$

and so we can choose dilatations $\{s_k\}_{k=1}^{\infty} \subset]0,\infty[$ such that $Q_k^{y,s_k}(1) = \frac{1}{2}$ for any $y \in \mathbb{R}^n$, $k = 1, 2, ...$ This done, it is now possible to select translations $\{y_k\}_{k=1}^{\infty} \subset \mathbb{R}^n$ such that the measures $\nu_k^{y_k,s_k} = |u_k^{y_k,s_k}|^{2^*}$, $k = 1, 2, ...$ are tight in $\mathscr{M}(\mathbb{R}^n)$.
The idea is that using appropriate translations we can shift so that at least half of the mass of $\nu_k^{y_k,s_k}$ remains in the unit ball.
If, however, part of the mass escapes to infinity, then our minimization problem splits into two parts, the sum of whose energies turns out to be strictly greater than the energy we would obtain if splitting did not occur.
(See [11] for a more detailed explanation).
To simplify notation we assume that dilatations and translations were unnecessary so that $Q_k(1) = \frac{1}{2}$, $k = 1, 2, ...$ and measures $\{\nu_k\}$ are tight.
Passing, if needed, to a subsequence, we can assume that

$$\nu_k \rightharpoonup \nu \quad \text{in } \mathscr{M}(\mathbb{R}^n) \quad \text{and} \quad \nu(\mathbb{R}^n) = 1. \tag{4.10}$$

Moreover, defined $\mu_k := |Du_k|^2$, $k = 1, 2, ...$, we may also suppose that

$$\mu_k \rightharpoonup \mu \quad \text{in } \mathscr{M}(\mathbb{R}^n).$$

Now we claim that $u \neq 0$. To prove this claim we note that by hypothesis on $\{u_k\}_{k=1}^{\infty}$, $\mu_k(\mathbb{R}^n) \longrightarrow I$ and

$$\mu(\mathbb{R}^n) = \int_{\mathbb{R}^n}|Du|^2\,dx = I(u) \leq \inf_{w\in\mathscr{A}} I(w) = C_2^{-2}.$$

But then if $u \equiv 0$, thanks to (4.10) we can invoke thesis 3. of Theorem 3.1.3 and find that ν is concentrated at a single point x_0. But then

$$\frac{1}{2} = Q_k(1) = \sup_{z\in\mathbb{R}^n}\int_{B(z,1)}|u_k|^{2^*}\,dx \geq \int_{B(x_0,1)}|u_k|^{2^*}\,dx =$$

$$= \int_{\mathbb{R}^n}\chi_{B(x_0,1)}\,d\nu_k \longrightarrow \int_{\mathbb{R}^n}\chi_{B(x_0,1)}\,d\nu = \nu(B(x_0,1)) = 1.$$

This is a contraddiction, hence $u \neq 0$.
Now we claim that $u \in \mathscr{A}$. If not, then $\|u\|^{2^*}_{L^{2^*}(\mathbb{R}^n)} = \alpha$, $0 < \alpha < 1$.
Define

$$\mathscr{A}_\alpha := \{w \in L^{2^*}(\mathbb{R}^n); \quad \|w\|^{2^*}_{L^{2^*}(\mathbb{R}^n)} = \alpha,\ Dw \in L^2(\mathbb{R}^n, \mathbb{R}^n)\}$$

and set $I_\alpha = \inf_{w \in \mathscr{A}_\alpha} I(w)$.
Then $I_\alpha = I\alpha^{\frac{2}{2^*}} = C_2^{-2}\alpha^{\frac{2}{2^*}}$ because C_2 is the optimal constant of Gagliardo-Nirenberg-Sobolev inequality 2.4.1 and if $w \in \mathscr{A}_\alpha$,

$$I(w) = \int_{\mathbb{R}^n} |Dw|^2\,dx \geq C_2^{-2}\,\|w\|^2_{L^{2^*}(\mathbb{R}^n)} = C_2^{-2}\alpha^{\frac{2}{2^*}} = I\alpha^{\frac{2}{2^*}}.$$

Finally by thesis 1. and 2. of Theorem 3.1.3, for some countable set of points $\{x_j\}_j \in J$ and positive weights $\{\mu_j,\, \nu_j\}_j \in J$ we have

a) $\nu = |u|^{2^*} + \sum_{j \in J} \nu_j\, \delta_{x_j}$,

b) $\mu \geq |Du|^2 + \sum_{j \in J} \mu_j\, \delta_{x_j}$.

moreover the positive weights $\{\mu_j,\, \nu_j\}_j \in J$ satisfy $\nu_j \leq C_2^{2^*}\mu_j^{\frac{2^*}{2}}$; hence

$$\mu_j \geq C_2^{-2^*\,\frac{2}{2^*}}\nu_j^{\frac{2}{2^*}} = C_2^{-2}\nu_j^{\frac{2}{2^*}} = I\nu_j^{\frac{2}{2^*}}.$$

Moreover,

$$1 = \nu(\mathbb{R}^n) = \alpha + \sum_{j \in J} \nu_j.$$

Consequently, since $u \in \mathscr{A}_\alpha$ we obtain a contraddiction because

$$I \geq \mu(\mathbb{R}^n) \geq \int_{\mathbb{R}^n} |Du|^2\,dx + \sum_{j \in J} \mu_j \geq I_\alpha + \sum_{j \in J} \mu_j \geq$$

$$\geq I\alpha^{\frac{2}{2^*}} + \sum_{j \in J} I\nu_j^{\frac{2}{2^*}} = I(\alpha^{\frac{2}{2^*}} + \sum_{j \in J} \nu_j^{\frac{2}{2^*}}) > I.$$

□

Now we find the equation satiesfied by u using Corollary to Lagrange Multiplier Theorem 4.1.4 with $F(w) = I(w) = \int_{\mathbb{R}^n} |Dw|^2\,dx$ and with the constraint $G(w) = 0$, where $G(w) = \|w\|^{2^*}_{L^{2^*}(\mathbb{R}^n)} - 1$.

Arguing as in Example 10, let $h \in L^{2^*}(\mathbb{R}^n)$ and we calculate $dF(w)(h)$ and $dG(w)(h)$.

$$dF(w)(h) = -2\int_{\mathbb{R}^n} \Delta wh\,dx,$$

and, denoting by $g(w) = |w|^{2^*}$, we have

$$dG(w)(h) = \lim_{t\to 0}\left[\int_{\mathbb{R}^n} (g'(w)h + o(1))\; dx\right] =$$

$$= 2^*\int_{\mathbb{R}^n} |w|^{\frac{4}{(n-2)}} w\,h\,dx.$$

Now we apply Corollary 4.1.4: since u is the minimizer there exist $\lambda \in \mathbb{R}$ such that for any h as above,

$$dF(u)(h) = \lambda dG(u)(h) \Rightarrow -2\int_U \Delta u\,h\,dx = \lambda 2^*\int_{\mathbb{R}^n} |u|^{\frac{4}{(n-2)}} u\,h\,dx,$$

so the equation satisfied by the minimizer u is $-\Delta u = \lambda\frac{2^*}{2}|u|^{\frac{4}{(n-2)}}u$.
Using that $dF(u)(h) = \lambda dG(u)(h)$ is true also for $h = u$, we can find λ:

$$I(u) = \min_{w\in\mathscr{A}} I(w) = \int_{\mathbb{R}^n} |Du|^2\,dx = \frac{1}{2}dF(u)(u) = \frac{\lambda}{2}\,dG(u)(u) =$$

$$= \frac{\lambda}{2}2^*\int_{\mathbb{R}^n} |u|^{2^*}\,dx = \lambda\frac{2^*}{2} \quad\Rightarrow\quad \lambda = I(u)\frac{2}{2^*}.$$

In conclusion the minimizer u satisfies

$$-\Delta u = \left(\min_{w\in\mathscr{A}} I(w)\right)|u|^{\frac{4}{(n-2)}}u.$$

Example 13 (Strong convergence of minimizing sequences).
We now study the problem of strong convergence of minimizing sequences for critical growth variational problem set in a bounded smooth domain $U \subset \mathbb{R}^n$, $n \geq 3$. For $\gamma > 0$, we are interested in minimizing the functional

$$I^\gamma(w) = \int_U \left(|Dw|^2 - \gamma w^2\right)\,dx$$

over the admissible set

$$\mathscr{A} = \{w \in W_0^{1,2}(U); \quad \|w\|_{L^{2^*}(U)} = 1, \}.$$

This problem resembles the one just treated, in particular we still work with the critical exponent 2^*; however, in this case

1. A lower order perturbation γw^2 occurs in the energy functional.

2. We work on a bounded domain U.

We will show that 1. and 2. together restore strong convergence for minimizing sequences. In other words, we will use the structure of the nonlinearity to argue directly that concentration does not occur.
So, choose a minimizing sequence $\{u_k\}_{k=1}^{\infty} \subset \mathscr{A}$ such that

$$I^\gamma(u_k) \longrightarrow \inf_{w\in\mathscr{A}} I^\gamma(w) =: I_\gamma, \quad \text{as } \ k \to \infty. \tag{4.11}$$

We also denote by

$$I^0(w) = \int_U |Dw|^2 \, dx$$

and

$$I_0 := \inf_{w\in\mathscr{A}} I^0(w).$$

Lemma 4.4.3. *If $\lambda > 0$ and $n \geq 4$, then $I_\gamma \leq I_0$.*

Proof. If we were working on $\mathbb{R}^n$, we would know (see Theorem 4.4.2) that the infimum of I^0 over $\mathscr{A}$ is obtained by functions $u_{y,\varepsilon}$ of the form

$$u_{y,\varepsilon}(x) = \frac{C_\varepsilon}{(\varepsilon + |x-y|^2)^{\frac{(n-2)}{2}}} \qquad x \in \mathbb{R}^n, \tag{4.12}$$

for $\varepsilon > 0$, $y \in \mathbb{R}^n$ and an appropriate normalization constant C_ε.
In the case we are studying, we work on $U \subset \mathbb{R}^n$ bounded and assume for simplicity that $0 \in U$. From what remarked above, the function $u_{0,\varepsilon}(x)$ is a good candidate for a minimum of I^0, except that $u_{0,\varepsilon}(x) \neq 0$ on ∂U.
To repair this defect, given $\zeta \in C_c^\infty(U)$ such that $\zeta \equiv 1$ near 0, for any $x \in U$ define

$$v^\varepsilon(x) = \zeta(x) u_{0,\varepsilon}(x)$$

and modify the normalization constant C_ε so that $\|v^\varepsilon\|_{L^{2^*}(U)} = 1$.
In this way $v^\varepsilon \in \mathscr{A}$. A careful analysis carried out in [4] shows that, for some constant $K > 0$,

$$I^\gamma(v^\varepsilon) = \begin{cases} I_0 + O(\varepsilon^{\frac{(n-2)}{2}}) - \gamma K \varepsilon & \text{if } n \geq 5 \\ I_0 + O(\varepsilon)\gamma K |\log \varepsilon| & \text{if } n = 4 \end{cases}$$

Hence, if we choose ε small enough, $I_\gamma \le I^\gamma(v^\varepsilon) < I_0$. □

Theorem 4.4.4. Let $0 < \gamma < \gamma_1$, $n \ge 4$ and $\{u_k\}_{k=1}^\infty \subset \mathscr{A}$ be a minimizing sequence for I^γ. Then there exists a subsequence $\{u_{k_j}\}_{j=1}^\infty \subset \{u_k\}_{k=1}^\infty$ and a function $u \in \mathscr{A}$ such that

$$u_{k_j} \longrightarrow u \quad \textit{strongly in } W_0^{1,2}(U).$$

In particular $u \in \mathscr{A}$ is a minimizer for I^γ.

Proof. Since $\{u_k\}_{k=1}^\infty \subset \mathscr{A}$ is a minimizing sequence for I^γ,

$$I^\gamma(u_k) \longrightarrow I_\gamma \quad \text{as} \ k \to \infty,$$

so $I^\gamma(u_k) = I_\gamma + o(1)$ and, as $\{u_k\}_{k=1}^\infty \subset \mathscr{A}$, $\|u_k\|_{L^{2^*}(U)} = 1$.
Since $2 < 2^*$ and U is bounded, $L^{2^*}(U) \subset L^2(U)$ and so
$\|u_k\|_{L^2(U)} \le C \|u_k\|_{L^{2^*}(U)} = C$.
This means that the sequence $\{u_k\}_{k=1}^\infty$ is bounded in $L^2(U)$ and so there exists a subsequence $\{u_{k_j}\}_{j=1}^\infty \subset \{u_k\}_{k=1}^\infty$ such that $u_{k_j} \rightharpoonup u$ in $L^2(U)$.
Moreover,

$$I^\gamma(u_k) = \|Du_k\|_{L^2(U)}^2 - \gamma \|u_k\|_{L^2(U)}^2,$$

so $\|Du_k\|_{L^2(U)}^2 \le \tilde{C}$ because $\{u_k\}_{k=1}^\infty$ is bounded in $L^2(U)$ and $I^\gamma(u_k)$ converges by hypothesis. Thus $Du_k \rightharpoonup Du$ in $L^2(U)$ and so $u_{k_j} \rightharpoonup u$ in $W_0^{1,2}(U)$.
Furthermore, since $\{u_k\}_{k=1}^\infty$ is bounded in $W_0^{1,2}(U)$, by Rellich-Kondrachov Theorem 2.4.2,

$$u_{k_j} \to u \quad \text{a.e and strongly in } L^p(U),\ 1 \le p < 2^*.$$

Now, set $v_{k_j} := u_{k_j} - u$, $j = 1, 2, ...$ Then $v_{k_j} \rightharpoonup 0$ in $W_0^{1,2}(U)$, hence

$$I^0(v_{k_j}) = \int_U |Dv_{k_j}|^2 \, dx \longrightarrow 0 \quad \text{as} \ j \to \infty.$$

This implies that $\{I^0(v_{k_j})\}_j$ is bounded and so

$$I^\gamma(u) + I^0(v_{k_j}) = I_\gamma + o(1).$$

Using Brezis-Lieb Lemma 3.1.2, we deduce:

$$1 = \int_U |u_{k_j}|^{2^*} \, dx = \int_U |u|^{2^*} + |u_{k_j} - u|^{2^*} \, dx + o(1).$$

But then, using the elementary inequality $(a+b)^\theta \leq a^\theta + b^\theta$, with $0 \leq \theta \leq 1$, a, $b > 0$, and since $\frac{2}{2^*} \leq 1$, we get:

$$1 + o(1) \leq \left(\int_U |u|^{2^*} + |v_{k_j}|^{2^*}\, dx \right)^{\frac{2}{2^*}} \leq \|u\|^2_{L^{2^*}(U)} + \left\| v_{k_j} \right\|^2_{L^{2^*}(U)}.$$

Thus $1 \leq \|u\|^2_{L^{2^*}(U)} + \left\| v_{k_j} \right\| |^2_{L^{2^*}(U)} + o(1)$ and so

$$I^\gamma(u) + I^0(v_{k_j}) = I_\gamma + o(1) \leq I_\gamma \left(\|u\|^2_{L^{2^*}(U)} + \left\| v_{k_j} \right\| |^2_{L^{2^*}(U)} \right) + o(1). \tag{4.13}$$

Now set $w := \frac{u}{\|u\|_{L^{2^*}(U)}}$. We have that $\|w\|_{L^{2^*}(U)} = 1$ so that $w \in \mathscr{A}$. Moreover

$$\begin{aligned} I_\gamma \leq I^\gamma(w) &= \|Dw\|^2_{L^2(U)} - \gamma \|w\|^2_{L^2(U)} = \\ &= \frac{1}{\|u\|^2_{L^{2^*}(U)}} \left(\|Du\|^2_{L^2(U)} - \gamma \|u\|^2_{L^2(U)} \right) = \\ &= \frac{1}{\|u\|^2_{L^{2^*}(U)}} I^\gamma(u). \end{aligned} \tag{4.14}$$

Hence we obtain $I_\gamma \|u\|^2_{L^{2^*}(U)} \leq I^\gamma(u)$. But then, using (4.13)

$$\begin{aligned} I^\gamma(u) + I^0(v_{k_j}) &\leq I_\gamma \|u\|^2_{L^{2^*}(U)} + I_\gamma \left\| v_{k_j} \right\|^2_{L^{2^*}(U)} + o(1) \leq \\ &\leq I^\gamma(u) + I_\gamma \left\| v_{k_j} \right\|^2_{L^{2^*}(U)} + o(1). \end{aligned}$$

So: $I^0(v_{k_j}) \leq I_\gamma \left\| v_{k_j} \right\|^2_{L^{2^*}(U)} + o(1)$.
In the same way as in (4.14) we get $I_0 \left\| v_{k_j} \right\|^2_{L^{2^*}(U)} \leq I^0(v_{k_j})$ and we deduce

$$I_0 \left\| v_{k_j} \right\|^2_{L^{2^*}(U)} \leq I^0(v_{k_j}) \leq I_\gamma \left\| v_{k_j} \right\|^2_{L^{2^*}(U)} + o(1).$$

Consequently,

$$(I_0 - I_\gamma) I^0(v_{k_j}) \leq o(1), \quad \text{that is} \quad (I_0 - I_\gamma) \left\| v_{k_j} \right\|^2_{W^{1,2}_0(U)} \leq o(1)$$

and since by Lemma 4.4.3, $(I_0 - I_\gamma) > 0$, we have

$$\left\| v_{k_j} \right\|^2_{W^{1,2}_0(U)} = \left\| u_{k_j} - u \right\|^2_{W^{1,2}_0(U)} \longrightarrow 0, \quad \text{as } j \to \infty.$$

This proves that $u_{k_j} \longrightarrow u$ strongly in $W^{1,2}_0(U)$. □

Now we find the equation satiesfied by u using the Corollary to Lagrange Multiplier Theorem 4.1.4 with $F(w) = I^\gamma(w) = \int_U (|Dw|^2 - \gamma w^2)\, dx$ and with the constraint $G(w) = 0$, where $G(w) = \|w\|_{L^{2^*}(U)}^{2^*} - 1$.
Arguing as in Example 10, let $h \in W_0^{1,2}(U)$ and we calculate $dF(w)(h)$ and $dG(w)(h)$.

$$dF(w)(h) = 2\int_U (Dw \cdot Dh)\, dx - 2\gamma \int_U w\, h\, dx =$$
$$= -2\int_U \Delta w h\, dx - 2\gamma \int_U w\, h\, dx =$$
$$= -2\int_U (\Delta w + \gamma w)\, h\, dx,$$

and, denoting by $g(w) = |w|^{2^*}$, we have

$$dG(w)(h) = \lim_{t\to 0}\left[\int_U (g'(w)h + o(1))\, dx\right] =$$
$$= 2^* \int_U |w|^{\frac{4}{(n-2)}} w\, h\, dx.$$

Now we apply Corollary 4.1.4: since u is the minimizer there exist $\lambda \in \mathbb{R}$ such that for any h as above,

$$dF(u)(h) = \lambda dG(u)(h) \Rightarrow -2\int_U (\Delta u\, + \gamma u)\, h\, dx = \lambda 2^* \int_U |u|^{\frac{4}{(n-2)}} u\, h\, dx,$$

so the equation satisfied by the minimizer u is $-(\Delta u + \gamma u) = \lambda \frac{2^*}{2} |u|^{\frac{4}{(n-2)}} u$.
Using that $dF(u)(h) = \lambda dG(u)(h)$ is true also for $h = u$, we can find λ:

$$I^\gamma(u) = \min_{w\in\mathscr{A}} I^\gamma(w) = \int_U \left(|Du|^2 - \gamma u^2\right)\, dx = \frac{1}{2} dF(u)(u) = \frac{\lambda}{2}\, dG(u)(u) =$$
$$= \frac{\lambda}{2} 2^* \int_U |u|^{2^*}\, dx = \lambda \frac{2^*}{2} \quad \Rightarrow \quad \lambda = I^\gamma(u)\frac{2}{2^*}.$$

In conclusion the minimizer u satisfies

$$-(\Delta u + \gamma u) = \left(\min_{w\in\mathscr{A}} I^\gamma(w)\right) |u|^{\frac{4}{(n-2)}} u.$$

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