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EFFECTIVE POTENTIALS FOR CORPUSCULAR BLACK HOLES

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Sommario

L'approccio innovativo alla fisica dei buchi neri quantistici proposto da Dvali e Gomez, permette di trovare nuove soluzioni ad alcuni dei principali problemi dell'unione tra relatività generale e meccanica quantistica. In particolare, porta a un'interpretazione originale della geometria classica, che diventa un concetto emergente dalla descrizione della gravità attraverso uno stato quantistico di gravitoni con numero d'occupazione elevato.
Queste idee sono la principale motivazione del nostro lavoro, che chiarisce alcuni aspetti delicati stabilendo un netto collegamento tra il modello quantistico corpuscolare di cui abbiamo appena parlato e la teoria della relatività di Einstein, in approssimazione post-Newtoniana. Questo studio si basa in particolare sulla ricerca di una descrizione quantistica effettiva del potenziale gravitazionale statico, per sistemi a simmetria sferica, al primo ordine non lineare nel limite di campo debole e velocità non relativistiche. Verifichiamo esplicitamente che il nostro modello recuperi i risultati classici per due diverse distribuzioni di materia (omogenea e gaussiana). Infine, procediamo alla quantizzazione del sistema e troviamo uno stato quantistico di gravitoni (virtuali) con le correzioni necessarie per riprodurre il potenziale post-Newtoniano.

Abstract

The novel approach to quantum black holes suggested by Dvali and Gomez opens a wide range of new possibilities to the resolution of some of the main issues of the conjunction between general relativity and quantum physics. It specially leads to a creative interpretation of classical geometries as emergent descriptions of a quantum state of gravitons with large occupation number.
This work is mostly motivated by the above ideas and clarifies a number of subtle aspects, establishing a clear connection between the quantum corpuscular model previously mentioned and the greatly confirmed Einstein theory, in post-Newtonian approximation. In particular, we study an effective quantum description of the static gravitational potential for spherically symmetric systems, at the first non-linear order in weak-field limit and non-relativistic speed. We first refine the classical model watching it at work with two different compact matter distributions (homogeneous and gaussian) and explicitly showing its consistency with the expected post-Newtonian results. Then, we proceed to the quantization of this system together with the identification of a coherent quantum state of (virtual) gravitons and its quantum corrections, which account for the post-Newtonian potential.

Contents

Introduction **1**

1 General Relativity of Compact Sources **3**

1.1 Einstein Equations from an Action Principle 3
1.1.1 Einstein-Hilbert Action 3
1.1.2 Matter Action 6
1.2 The Weak-Field Approximation 6
1.2.1 Linearised Einstein Equations 7
1.2.2 Newtonian Limit 9
1.2.3 Post-Newtonian Approximation 10
1.2.4 Post-Newtonian Potential for the Schwarzschild Metric 15
1.3 Hamiltonian Formulation of General Relativity 18
1.3.1 ADM Decomposition 19
1.3.2 ADM Energy 22
1.4 York-Gibbons-Hawking Boundary Term 26
1.4.1 Non-Dynamical Term 29
1.4.2 Equivalence with ADM Energy 30
1.5 Gravitational Collapse and Singularity Theorems 31
1.5.1 Causal Structure 32
1.5.2 Definition of Singularities 34
1.5.3 Raychaudhuri Equation and the Gravitational Focusing 35
1.5.4 Penrose and Hawking-Penrose Singularity Theorems 39

2 Classicalization and Quantum Corpuscular Black-Holes **41**

2.1 Planck-Scale as a Boundary between two Worlds 42
2.2 Self-Completeness of Gravity 44

2.3 Quantum Corpuscular Black Holes 45
2.3.1 Hawking Radiation 48
2.3.2 Entropy 49
2.4 Classicalization 50

3 Quantum Corpuscular Corrections to the Newtonian Potential 53
3.1 Energy Balance 54
3.2 Effective Scalar Theory for post-Newtonian Potential 56
3.2.1 Classical Solutions 62
3.2.2 Homogeneous Matter Distribution 65
3.2.3 Gaussian Matter Distribution 68
3.3 Quantum Linear Field and Coherent Ground State 71
3.3.1 Newtonian Potential 72
3.3.2 Post-Newtonian Corrections 75

Conclusions and Outlook 79

A Hypersurfaces and embeddings 81
A.1 Normal and Tangent Vectors 82
A.2 Induced Metric and Intrinsic Geometry 83
A.3 Extrinsic Geometry 85

Introduction

Physics in the twentieth century was marked by two enormous breakthroughs: the comprehension of gravitational interactions by general relativity and the detection of a wide range of new features in the microscopic world, the quantum phenomena. Addressing the mismatch between these two theories with a quantum theory of gravity is one of the most fascinating and intriguing issues of today fundamental physics. The starting point for a solution seems to be carried by gravity itself. In particular, it resides in its main pathologies, embodied by singularities and infinities appearing respectively in the classical and quantum picture. The first arise in a situation of gravitational collapse and cosmology, as widely discussed by Penrose and Hawking in late '60s, while when trying to embed gravity in the framework of quantum field theory one inevitably ends up with non renormalizable infinities.

Black holes are systems where both gravitational and quantum effects are not negligible and are therefore a natural playground for the test of quantum gravity. The semiclassical approach of studying small fluctuations about a classical background geometry led to the famous Hawking radiation, interpreted as the creation of particle pairs from vacuum fluctuations, resulting in a completely thermal emission. The logical consequence of this thermal radiation is the information paradox, as it carries no details about what fell into the black hole. However, even with these drawbacks together with the missed resolution of the singularity problems, Hawking radiation as well as Bekenstein-Hawking entropy are still considered two main distinctive features of black holes physics.

The above problems represent the clue that black holes may not be semiclassical but actually quantum. Recently, Dvali and Gomez proposed the idea that a black hole can be modelled as a Bose-Einstein condensate of marginally bound, self-interacting gravitons offering a new perspective on the quantum aspects of these mysterious systems. Within this picture, Hawking radiation and Bekenstein-Hawking entropy acquire a geometric independent, purely quantum interpretation in terms of well known phenomena specific of condensed matter physics. The classical geometry has to be retrieved as an effective description of a quantum state with large graviton number.

The aim of the present work is to prove that a direct link between this quantum model for black holes and general relativity can be established, showing the full agreement of this new perspective with the post-Newtonian gravity. In other words, we will relate the microscopic dynamics of gravity and the macroscopic description of curved background through the identification of a quantum state for the gravitational potential as a coherent state of virtual gravitons.

In Chapter 1 we focus on those results of general relativity useful to justify the work done in last chapter. We begin by introducing the Einstein-Hilbert action, which is the starting point of the discussion on the Hamiltonian formulation of general relativity. The need to introduce this section is justified by the fact that in Chapter 3 we pick up on the Hamiltonian constraint, fundamental if one wants to deal with the quantum description of a system. We further develop the linear and post-Newtonian approximations because they precisely define the assumptions of our calculations in last chapter and the results we want to recover in the effective quantum framework. At last, we describe the singularity theorems in order to handle concretely the major problem of singularities in a gravitational collapse.

Chapter 2 starts from the limits of general relativity to introduce the novel picture of classicalization due to Dvali and Gomez. We first describe the solution proposed for the self-UV completion of Einstein gravity and then introduce the quantum corpuscular model for black holes previously named. Last section is a simple and not complete overall view of the idea of classicalization. This chapter is important as it provides the link between the first and the last part. Actually, it is one of the main motivations for this study.

Finally, in Chapter 3 we begin with the energy balance of Ref. [54] for a static and isotropic source, where both matter and gravitons self-interaction energies are considered. Then, we determine a classical effective scalar action accounting for all these contributions and evaluate the total energy for two different matter distributions. In the end, the quantum coherent state of (virtual) gravitons representing the gravitational potential of the system is found.

Chapter 1

General Relativity of Compact Sources

1.1 Einstein Equations from an Action Principle

The study of the geodesic equations reveals that the connection describes the effects of gravitational forces. Thus, thanks to the knowledge that connection is expressed in terms of the first derivatives of the metric tensor, we understand the latter as the effective gravitational potential. What is more, we know that any gravitational field is associated to a curvature tensor, which contains the square of the first derivatives of the metric. Comparison with the classical field theories, based on second-order differential equations for potentials, leads to an action principle for the metric coupled with matter fields, where the curvature tensor plays the role of an effective "kinetic term" and the matter part is coupled to gravity through the well known minimal coupling procedure. So, the aim of this section is to find an action whose variation gives the famous Einstein equations for the gravitational field coupled with matter fields. We decided to start with the gravitational part alone, leading to the Einsten equation in vacuum, and deal with the matter part later.

1.1.1 Einstein-Hilbert Action

The principle of general covariance requires an action for the metric to be a scalar under general coordinate transformations. The simplest choice we can make is writing it in terms of the scalar curvature(or Ricci scalar)

$$S_{\mathrm{EH}} = \frac{1}{16\pi G_{\mathrm{N}}} \int d^4x \sqrt{-g} R, \tag{1.1.1}$$

and it is known as the Einstein-Hilbert action [12]. The dimensional factor is fixed by the newtonian limit of the theory. We recall here and once for all the definitions of the Ricci tensor

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\lambda} + \Gamma^{\lambda}_{\lambda\rho}\Gamma^{\rho}_{\nu\mu} - \Gamma^{\lambda}_{\nu\rho}\Gamma^{\rho}_{\lambda\mu}, \tag{1.1.2}$$

and the scalar curvature $R = g^{\mu\nu} R_{\mu\nu}$, where $\Gamma^{\lambda}_{\mu\nu}$ is the affine connection.

We stress that we will ignore here total derivatives(or boundary) terms and concentrates only on the bulk Euler-Lagrange equations of motion. In fact, one usually neglects these terms choosing vanishing variations of the fields on the boundary, therefore such terms are equal to zero. We will show later that this can't be done in the present case because of the presence of both variation of the metric and its normal derivatives on the boundary and requiring both to be zero is not consistent. However, we will deal with this problem only after it will appear when trying to find an Hamiltonian formulation of general relativity in Sections 1.3 and 1.4.

Returning to the Einstein-Hilbert action we understand it as the simplest possible choice but not the only one. Actually, a curvature dependent, scalar action can be obtained by self contracting the components of the Riemann and Ricci tensors, producing terms of the form

$$R^2, \qquad R_{\mu\nu}R^{\mu\nu}, \qquad R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta}, \qquad \cdots . \tag{1.1.3}$$

Such terms invariably involve higher derivatives and higher non-linearities and are therefore irrelevant for low-energy physics.

We can now proceed to the proof that this action leads to the vacuum Einstein equations. Consider the variation of the action functional

$$\begin{aligned} \delta S_{\mathrm{EH}} &= \frac{1}{16\pi G_{\mathrm{N}}} \delta \int d^4x \sqrt{-g} g^{\mu\nu} R_{\mu\nu} \\ &= \frac{1}{16\pi G_{\mathrm{N}}} \int d^4x \left(\delta\sqrt{-g} g^{\mu\nu} R_{\mu\nu} + \sqrt{-g}\delta g^{\mu\nu} R_{\mu\nu} + \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} \right). \end{aligned} \tag{1.1.4}$$

[1] here g is the determinant of the spacetime metric, whose signature is taken to be equal to the Minkowski metric $\eta_{\mu\nu} = (-1, +1, +1, +1)$.

[2] the speed of light c is here set to 1.

The first term can be rewritten using the relation $\delta(g^{\mu\nu}g_{\nu\gamma}) = \delta(\delta^{\mu}_{\gamma}) = 0$ as

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}, \tag{1.1.5}$$

so that the total variation now reads

$$\delta S_{\mathrm{EH}} = \frac{1}{16\pi G_{\mathrm{N}}}\int d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} + \frac{1}{16\pi G_{\mathrm{N}}}\int d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}. \tag{1.1.6}$$

We only need to show that the last term is a total derivative because the first part already reproduces the product of Einstein tensor with the variation $\delta g^{\mu\nu}$. The variation $\delta R_{\mu\nu}$, will be expressed in terms of the variation of Christoffel symbols, whose relation with the metric is

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\alpha}\left(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\right), \tag{1.1.7}$$

leaving the metric variation hidden there,

$$\delta R_{\mu\nu} = \partial_\lambda\delta\Gamma^{\lambda}_{\mu\nu} - \partial_\nu\delta\Gamma^{\lambda}_{\mu\lambda} + \delta\Gamma^{\lambda}_{\lambda\rho}\Gamma^{\rho}_{\nu\mu} + \Gamma^{\lambda}_{\lambda\rho}\delta\Gamma^{\rho}_{\nu\mu} - \delta\Gamma^{\lambda}_{\nu\rho}\Gamma^{\rho}_{\lambda\mu} + \Gamma^{\lambda}_{\nu\rho}\delta\Gamma^{\rho}_{\lambda\mu}. \tag{1.1.8}$$

An important observation is now needed. We know that $\Gamma^{\lambda}_{\mu\nu}$ is not a tensor because an inhomogeneous term appear in its transformation rule under coordinate transformations. The good thing is that this term does not depend upon the metric, thus the metric variation of the Christoffel symbol is a tensor. This allows us to use the known rule for covariant differentiation of a tensor, to simplify the above expression

$$\delta R_{\mu\nu} = \nabla_\lambda\delta\Gamma^{\lambda}_{\mu\nu} - \nabla_\nu\delta\Gamma^{\lambda}_{\mu\lambda}. \tag{1.1.9}$$

We can now write the last term in the variation of S_{EH} as a boundary term (which will be discussed later)

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\lambda\left(g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu}\right) - \nabla_\nu\left(g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\lambda}\right) \tag{1.1.10}$$

$$= \nabla_\lambda\left(g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^{\nu}_{\mu\nu}\right), \tag{1.1.11}$$

and we are left with the vacuum Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \tag{1.1.12}$$

1.1.2 Matter Action

In order to obtain the non-vacuum Einstein equation we obviously need to consider the matter Lagrangian minimally coupled to the metric. A general form can be written as

$$S_{\mathrm{M}}(\phi, g_{\mu\nu}) = \int d^4x\sqrt{-g}\mathcal{L}\left(\phi, \partial_\lambda\phi, g_{\mu\nu}, \partial_\lambda g_{\mu\nu}\right). \tag{1.1.13}$$

Of course varying this action with respect to the matter fields will give the covariant equations of motion of these fields. On the other side, its variation with respect to the metric will give the matter contribution to Einstein equations. It is natural to define the source of the gravitational field to be the covariant energy-momentum tensor. Thus, we can write the variation of matter action with respect to the metric as

$$\delta S_{\mathrm{M}} = -\frac{1}{2}\int d^4x\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu} \quad \leftrightarrow \quad T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{1.1.14}$$

$T_{\mu\nu}$ can be shown to be automatically covariantly conserved on-shell as a consequence of general covariance of the matter action [1].

Now the complete action for general relativity reads

$$S[g_{\mu\nu}, \phi] = \frac{1}{16\pi G_{\mathrm{N}}}S_{\mathrm{EH}}[g_{\mu\nu}] + S_{\mathrm{M}}[g_{\mu\nu}, \phi]. \tag{1.1.15}$$

Metric variation of this action leads to the famous Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_{\mathrm{N}}T_{\mu\nu}. \tag{1.1.16}$$

1.2 The Weak-Field Approximation

The high non-linearity of Einstein field equations makes it hard, if not impossible, to find exact solutions in general, unless we consider very special symmetries. Therefore, the need to develop some methods of approximation without relying on the existence of some symmetry, naturally arises in general relativity. Fortunately, in most ordinary situations the gravitational field is very weak and this justify the study of the linearised form of Einstein equations. We show in this situation that the Newtonian limit is recovered assuming a weak and static gravitational field and non-relativistic matter. Without the last assumption, the linearised theory leads to a range of completely new features of general relativity. The most important is the prediction of gravitational waves. Anyway, we are not interested in this characteristic of the gravitational field because these are not relevant in our study. It would be therefore redundant to give a description of this situation and we refer to [1], for a detailed analysis. Intead, we focus more on the study of the approximation to further orders than the first in the weak-field, the so called *post-Newtonian approximation*.

1.2.1 Linearised Einstein Equations

We start considering a spacetime which is only slightly different from Minkowski spacetime, so that the metric $g_{\mu\nu}$ can be expanded around the Minkowski metric. Neglecting all orders higher than the first we have

$$g_{\mu\nu} \simeq \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1, \tag{1.2.1}$$

thus the tensor $h_{\mu\nu}$ represent small fluctuations of the metric. By inserting this expansion into the Einstein equations we will obtain a system of linear differential equations for $h_{\mu\nu}$. These will determine the dynamical evolution of the deviations from the Minkowski geometry.

As long as we restric ourselves to first order in $h_{\mu\nu}$, we must raise and lower indices with $\eta_{\mu\nu}$

$$h^{\mu}{}_{\nu} = g^{\mu\rho} h_{\rho\nu} = \eta^{\mu\rho} h_{\rho\nu} + \mathcal{O}(h^2). \tag{1.2.2}$$

The inverse metric is then given by

$$g^{\mu\nu} \simeq \eta^{\mu\nu} - h^{\mu\nu}. \tag{1.2.3}$$

To first order in h the affine connection is

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\eta^{\lambda\rho}\left(\partial_\nu h_{\mu\rho} + \partial_\mu h_{\nu\rho} - \partial_\rho h_{\mu\nu}\right) + \mathcal{O}(h^2), \tag{1.2.4}$$

and it is clear that at the zeroth order the affine connection vanish. Thus, the first order Ricci tensor is

$$\begin{aligned} R^{(1)}_{\mu\nu} &= \partial_\lambda \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\lambda} && (1.2.5) \\ &= \frac{1}{2}\left(\partial_\lambda \partial_\mu h^{\lambda}{}_{\nu} - \Box h_{\mu\nu} - \partial_\nu \partial_\mu h + \partial_\nu \partial_\lambda h^{\lambda}{}_{\mu}\right), && (1.2.6) \end{aligned}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$. The Ricci scalar consequently is in the form

$$R = \eta^{\mu\nu} R^{(1)}_{\mu\nu} = \partial_\mu \partial_\nu h^{\mu\nu} - \Box h, \tag{1.2.7}$$

and the Einstein field equations read

$$\partial_\lambda \partial_\mu h^\lambda{}_\nu - \Box h_{\mu\nu} - \partial_\nu \partial_\mu h + \partial_\nu \partial_\lambda h^\lambda{}_\mu \; - \eta_{\mu\nu} \partial_\lambda \partial_\sigma h^{\lambda\sigma} + \eta_{\mu\nu} \Box h = 16\pi G_{\mathrm{N}} T^{(0)}_{\mu\nu}, \tag{1.2.8}$$

or simply

$$G^{(1)}_{\mu\nu} = 16\pi G_{\mathrm{N}} T^{(0)}_{\mu\nu}, \tag{1.2.9}$$

where $G^{(1)}_{\mu\nu}$ is the Einstein tensor at first order in h. Here we wrote $T^{(0)}_{\mu\nu}$ because in order this approximation to make sense the zeroth order of the energy-momentum tensor should already be "small", i.e., of order h. This is also a consistent requirement so that $T^{(0)}_{\mu\nu}$ satisfy the standard conservation law $\partial^\mu T^{(0)}_{\mu\nu} = 0$ and is indeed compatible with the linearised Bianchi identity $\partial^\mu G^{(1)}_{\mu\nu} = 0$. We can remark that when studied in vacuum, equation (1.2.8), can be obtained from the Fierz-Pauli action [5]

$$S[h_{\mu\nu}] = \frac{1}{16\pi G_{\mathrm{N}}} \int d^4x \left(-\frac{1}{2} \partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} + \partial_\lambda h_{\mu\nu} \partial^\nu h^{\lambda\mu} + \frac{1}{2} \partial_\lambda h \partial^\lambda h - \partial_\lambda h \partial_\mu h^{\mu\lambda} \right), \tag{1.2.10}$$

and not from Einstein-Hilbert action with linearized Ricci scalar because this is linear in $h_{\mu\nu}$, by definition.

Finally, it is generally known that Einstein field equations have a local, gauge invariance under general coordinate transformations. This property translates also in the linearised picture with the requirement that the coordinate transformation leaves the field weak. The most general is of the form

$$x^\mu \longrightarrow x'^\mu = x^\mu + \epsilon^\mu(x), \tag{1.2.11}$$

where the derivatives of $\epsilon^\mu(x)$ are required to be of the same order as $h_{\mu\nu}$. Thus, if $h_{\mu\nu}$ is a solution of (1.2.8), then

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\nu \epsilon_\mu - \partial_\mu \epsilon_\nu, \tag{1.2.12}$$

will be too. This final form can be easily obtained from the general transformation law of the metric under coordinate transformations

$$g'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} g^{\alpha\beta}. \tag{1.2.13}$$

The gauge invariance helps simplifying the field equations, removing many terms with a suitable choice of coordinate system. The most convenient is expressed through the *harmonic gauge condition* (or De Donder gauge condition), for which

$$g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0 \Longrightarrow 2\partial^{\mu} h_{\mu\nu} = \partial_{\nu} h. \tag{1.2.14}$$

In this gauge the linearised Einstein equations simplify and takes the form

$$-\Box h_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\Box h = 16\pi G_{\rm N} T^{(0)}_{\mu\nu}, \tag{1.2.15}$$

or taking the trace and recognising that $\Box h = 16\pi G_{\rm N} T$,

$$-\Box h_{\mu\nu} = 16\pi G_{\rm N} \left(T^{(0)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T \right). \tag{1.2.16}$$

1.2.2 Newtonian Limit

The weak-field approximation is of course a requirement needed to recover the Newtonian results. However this situation is still too general as we anticipated at the beginning of this section. In fact, to characterise the circumstances in which we know and can trust the validity of Newton's equations, we also have to consider situations in which the field is static and the matter moves non relativistically, i.e., velocities are small with respect to the speed of light (here chosen to be 1). This implies that, among all the components of the energy-momentum tensor, only T_{00} is relevant and it represents the rest mass density. Furthermore, all time derivatives of the metric will vanish because of the static field assumption. In general, for a non-relativistic system we have $|T_{ij}| \ll T_{00}$ which implies $|G_{ij}| \ll G_{00}$. Therefore we conclude that

$$R_{ij} \simeq \frac{1}{2} g_{ij} R. \tag{1.2.17}$$

Now, only the weak field condition still has to be taken into account. Considering the last equation, we understand

$$R \simeq R^{k}{}_{k} - R_{00} \to R = 2R_{00}. \tag{1.2.18}$$

This put the 00 Einstein equation in the form

$$R_{00} = 8\pi G_{\mathrm{N}} T_{00}, \tag{1.2.19}$$

naming $T_{00} = \rho$ and thanks to equation (1.2.7) it becomes

$$\triangle h_{00} = -8\pi G_{\mathrm{N}} \rho, \tag{1.2.20}$$

allowing us to identify

$$h_{00} = -2V_{\mathrm{N}}, \tag{1.2.21}$$

in such a way to recover the standard Poisson equation for the Newtonian potential

$$\triangle V_{\mathrm{N}} = 4\pi G_{\mathrm{N}} \rho. \tag{1.2.22}$$

1.2.3 Post-Newtonian Approximation

The post-Newtonian approximation is based on the assumption of both weak field and non relativistic velocities. It was historically derived [6–8] in the framework of the study of the problem of motion, i.e in order to find the equations of motion of a system of particles in first approximation, following general relativity predictions. However, we will follow a different approach and discuss the implications of this method on the Einstein field equations. The idea is to find corrections to the metric at one higher order than what we found in the Newtonian limit. When making approximations with perturbative methods it is fundamental to bear in mind which is the small adimensional expansion parameter. In order to find it, we can start with a simple reasoning and consider the gravitational interaction between two "particles" (or planets). Newtonian mechanics tells us that for a single particle the usual kinetic energy $\frac{1}{2}mv^2$ is roughly of the same order of magnitude of potential energy $G_{\mathrm{N}}m/r^2$, therefore we find

$$v^2 \sim \frac{G_{\mathrm{N}} m}{r}. \tag{1.2.23}$$

This establishes v as our small expansion parameter and we will take trace of it by indicating with an apex (n) the terms of order v^n.

Having in mind what we made in section 1.2.1, we expect the first post-Newtonian corrections to arise from an expansion of the metric at one higher order with respect to deviation from the background Minkowski metric. Of course the Newtonian approximation will be the v^2 order and we will therefore need to go beyond it

$$g_{00} = -1 + {}^{(2)}g_{00} + {}^{(4)}g_{00} + \cdots \,, \tag{1.2.24}$$

$$g_{ij} = \delta_{ij} + {}^{(2)}g_{ij} + {}^{(4)}g_{ij} + \cdots \,, \tag{1.2.25}$$

$$g_{0i} = {}^{(3)}g_{0i} + {}^{(5)}g_{0i} + \cdots \,, \tag{1.2.26}$$

where in the last line we considered only odd powers of v because g_{0i} must change sign under time reversal. Of course this perturbative expansion will be justified a posteriori showing that it leads to consistent solutions of Einstein equations. In other words we have to check that the solution produced is actually perturbative in the sense that higher orders are always smaller. That this is not immediately verified is a direct consequence of the non-linearity of Einstein equations. The inverse metric tensor is obtained through the relation $g^{\mu\lambda}g_{\lambda\nu} = \delta^{\mu}_{\nu}$ and we find

$$g^{00} = -1 + {}^{(2)}g^{00} + {}^{(4)}g^{00} + \cdots \,, \tag{1.2.27}$$

$$g^{ij} = \delta^{ij} + {}^{(2)}g^{ij} + {}^{(4)}g^{ij} + \cdots \,, \tag{1.2.28}$$

$$g^{0i} = {}^{(3)}g^{0i} + {}^{(5)}g^{0i} + \cdots \,, \tag{1.2.29}$$

where

$${}^{(2)}g^{00} = -\,{}^{(2)}g_{00} \qquad {}^{(2)}g^{ij} = -\,{}^{(2)}g_{ij} \qquad {}^{(3)}g^{0i} = {}^{(3)}g_{0i} \qquad \cdots \tag{1.2.30}$$

Our final target is to be able to write the field equations at this order and solve them, thus our next step should necessarily be the boring evaluation of all the tensors needed to do that, i.e. the affine connection (1.1.7), the Ricci tensor (1.1.2) and the right hand side of Einstein equations related to the energy-momentum tensor. However, it would be useless to do that and we refer to [1] or [2] for an explicit analysis order by order (our conventions are the same used in [2]). We will only follow the reasoning and start by rewriting the Einstein equations in the form

$$R_{\mu\nu} = 8\pi G_{\mathrm{N}} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) = 8\pi G_{\mathrm{N}} S_{\mu\nu}, \tag{1.2.31}$$

where $T = T^{\lambda}_{\lambda}$. In computing the affine connection and the Ricci tensor we have to take into account that the space and time derivatives carry different orders, namely

$$\frac{\partial}{\partial x^i} \sim \frac{1}{r} \qquad \frac{\partial}{\partial t} \sim \frac{v}{r}. \tag{1.2.32}$$

Inserting the expansions for the metric and its inverse we obtain

$$\Gamma^{\lambda}_{\mu\nu} = {}^{(2)}\Gamma^{\lambda}_{\mu\nu} + {}^{(4)}\Gamma^{\lambda}_{\mu\nu} + \cdots \qquad \text{for } \Gamma^{i}_{00}, \Gamma^{i}_{jk}, \Gamma^{0}_{0i}, \tag{1.2.33}$$

$$\Gamma^{\lambda}_{\mu\nu} = {}^{(3)}\Gamma^{\lambda}_{\mu\nu} + {}^{(5)}\Gamma^{\lambda}_{\mu\nu} + \cdots \qquad \text{for } \Gamma^{i}_{0j}, \Gamma^{0}_{00}, \Gamma^{0}_{ij}, \tag{1.2.34}$$

where the apex (n) now stands for the order v^n/r. The Ricci tensor involves derivatives of the Christoffel symbols or quadratic dependence on them, therefore (n) will indicate terms of order v^n/r^2. We find from the previous expansions

$$R_{00} = {}^{(2)}R_{00} + {}^{(4)}R_{00} + \cdots, \tag{1.2.35}$$

$$R_{0i} = {}^{(3)}R_{0i} + {}^{(5)}R_{0i} + \cdots, \tag{1.2.36}$$

$$R_{ij} = {}^{(2)}R_{ij} + {}^{(4)}R_{ij} + \cdots. \tag{1.2.37}$$

Now, the right hand side of the Einstein equations has to be expanded similarly. We first need to expand the energy-momentum tensor. With the aim of recognizing the relevant orders, it is convenient to start from the interpretation of T^{00},T^{0i} and T^{ij} as the energy density, momentum density and momentum flux respectively. This leads to

$$T^{00} = {}^{(0)}T^{00} + {}^{(2)}T^{00} + \cdots, \tag{1.2.38}$$

$$T^{0i} = {}^{(1)}T^{0i} + {}^{(3)}T^{0i} + \cdots, \tag{1.2.39}$$

$$T^{ij} = {}^{(0)}T^{ij} + {}^{(2)}T^{ij} + \cdots \tag{1.2.40}$$

and the expansion for $S_{\mu\nu}$ straightly follows.

Finally, we can write the Einstein field equations which, in this approximation and with the harmonic gauge, reads [1]

$$\triangle\, {}^{(2)}g_{00} = -8\pi G_{\rm N}\, {}^{(0)}T^{00}, \tag{1.2.41}$$

$$\begin{aligned}\triangle\, {}^{(4)}g_{00} = {}& {}^{(2)}g_{00,00} + {}^{(2)}g_{ij}\, {}^{(2)}g_{00,ij} - {}^{(2)}g_{00,i}\, {}^{(2)}g_{00,i} \\ & - 8\pi G_{\rm N} \left({}^{(2)}T^{00} - 2\, {}^{(2)}g_{00}\, {}^{(0)}T^{00} + {}^{(2)}T^{ii} \right),\end{aligned} \tag{1.2.42}$$

$$\triangle\, {}^{(3)}g_{0i} = 16\pi G_{\rm N}\, {}^{(1)}T^{0i}, \tag{1.2.43}$$

$$\triangle\, {}^{(2)}g_{ij} = -8\pi G_{\rm N}\, {}^{(0)}T^{00}. \tag{1.2.44}$$

The first of these equations is identical to equation (1.2.20) and reproduces the expected Newtonian result

$$ {}^{(2)}g_{00} = -2V_{\mathrm{N}}, \tag{1.2.45} $$

where V_{N} is the Newtonian potential satisfying the usual Poisson equation as seen in the previous section. It can be written in integral form

$$ V_{\mathrm{N}}(\boldsymbol{x}, t) = -G_{\mathrm{N}} \int d^3x' \frac{{}^{(0)}T^{00}(\boldsymbol{x'}, t)}{|\boldsymbol{x} - \boldsymbol{x'}|}. \tag{1.2.46} $$

The solution to equation (1.2.44) straightly follows and can be written as ${}^{(2)}g_{ij} = -2\delta_{ij}V_{\mathrm{N}}$. On the other hand, ${}^{(3)}g_{0i}$ is a new vector potential ${}^{(3)}g_{0i}$ and requiring it to vanish at infinity we obtain

$$ \zeta_i(\boldsymbol{x}, t) = -4G_{\mathrm{N}} \int d^3x' \frac{{}^{(1)}T^{0i}(\boldsymbol{x'}, t)}{|\boldsymbol{x} - \boldsymbol{x'}|}. \tag{1.2.47} $$

At last, equation (1.2.43) can be manipulated with the identity

$$ \frac{\partial V_{\mathrm{N}}}{\partial x^i}\frac{\partial V_{\mathrm{N}}}{\partial x^i} = \frac{1}{2}\triangle V_{\mathrm{N}}^2 - V_{\mathrm{N}}\triangle V_{\mathrm{N}}, \tag{1.2.48} $$

and gives (using the solution of (1.2.41))

$$ {}^{(4)}g_{00} = -2V_{\mathrm{N}}^2 - 2\psi, \tag{1.2.49} $$

where ψ is a second potential which satisfies

$$ \triangle\psi = \frac{\partial^2 V_{\mathrm{N}}}{\partial t^2} + 4\pi G_{\mathrm{N}} \left({}^{(2)}T^{00} + {}^{(2)}T^{ii} \right), \tag{1.2.50} $$

and imposing to be vanishing at infinity can be written as

$$ \psi(\boldsymbol{x}, t) = -\int \frac{d^3x'}{|\boldsymbol{x} - \boldsymbol{x'}|} \left[\frac{1}{4\pi}\frac{\partial^2 V_{\mathrm{N}}(\boldsymbol{x'}, t)}{\partial t^2} + G_{\mathrm{N}}\, {}^{(2)}T^{00}(\boldsymbol{x'}, t) + G_{\mathrm{N}}\, {}^{(2)}T^{ii}(\boldsymbol{x'}, t) \right]. \tag{1.2.51} $$

The harmonic coordinate condition further imposes a relation between V_{N} and $\boldsymbol{\zeta}$

$$ 4\frac{\partial V_{\mathrm{N}}}{\partial t} + \nabla \cdot \boldsymbol{\zeta} = 0. \tag{1.2.52} $$

Multipole Expansion

It is interesting now to evaluate these solutions far away from an arbitrary finite distribution of energy and momentum, i.e we consider an energy-momentum tensor which vanishes for $r > R$ where $r \equiv |\boldsymbol{x}|$. Then, we can expand the denominators $|\boldsymbol{x} - \boldsymbol{x'}|$ in the above solutions with the well known multipole expansion

$$\frac{1}{|\boldsymbol{x} - \boldsymbol{x'}|} \simeq \frac{1}{r} + \frac{\boldsymbol{x} \cdot \boldsymbol{x'}}{r^3} + \cdots \,, \tag{1.2.53}$$

The second term is the dipole term which can always be made vanishing defining the coordinate system to be the one on the center of mass. Therefore, assuming an energy distribution with spherical simmetry (only the monopole term survives), we find

$$V_{\rm N} = -\frac{G_{\rm N}\, {}^{(0)}M}{r}, \tag{1.2.54}$$

$$\boldsymbol{\zeta} = -4G_{\rm N} \frac{{}^{(1)}\boldsymbol{P}}{r}, \tag{1.2.55}$$

$$\psi = -\frac{G\, {}^{(2)}M}{r}, \tag{1.2.56}$$

where

$${}^{(0)}M \equiv \int d^3x\, {}^{(0)}T^{00}, \tag{1.2.57}$$

$${}^{(1)}P^i \equiv \int d^3x\, {}^{(1)}T^{0i}, \tag{1.2.58}$$

$${}^{(2)}M \equiv \int d^3x\, {}^{(2)}T^{00}. \tag{1.2.59}$$

Further assuming that the distribution is at rest, such that ${}^{(1)}\boldsymbol{P} = 0$, we find within the accuracy of the post-Newtonian approximation

$$g_{00} \simeq -1 + \frac{2G_{\rm N}M}{r} - \frac{2G_{\rm N}^2 M^2}{r^2}, \tag{1.2.60}$$

$$g_{0i} \simeq 0, \tag{1.2.61}$$

$$g_{ij} \simeq \delta_{ij} + 2\delta_{ij}\frac{G_{\rm N}M}{r}, \tag{1.2.62}$$

where $M = {}^{(0)}M + {}^{(2)}M$. We have therefore found consistent corrections to the Newtonian potential and more in general to the spacetime metric as a solution of the Einstein equations.

1.2.4 Post-Newtonian Potential for the Schwarzschild Metric

We consider here a particular case of the above discussion, i.e that of a test particle of mass m freely falling along a radial direction in the Schwarzschild space-time around a source of mass M. Therefore the only additional assumption is that we are here explicitly choosing a static system. The results obtained here for the post-Newtonian potential will be used in comparison in Chapter 3.

The Schwarzschild metric in standard form is given by [3]

$$\mathrm{d}s^2 = -\left(1 - \frac{2\,M}{\tilde{r}}\right)\mathrm{d}\tilde{t}^2 + \left(1 - \frac{2\,M}{\tilde{r}}\right)^{-1}\mathrm{d}\tilde{r}^2 + \tilde{r}^2\,\mathrm{d}\Omega^2\ , \tag{1.2.63}$$

and the radial geodesic equation for a massive particle turns out to be

$$\frac{\mathrm{d}^2\tilde{r}}{\mathrm{d}\tau^2} = -\frac{M}{\tilde{r}^2}\ , \tag{1.2.64}$$

which looks formally equal to the Newtonian expression, but where $\tilde{r}$ is the areal radial coordinate related to the Newtonian radial distance r by

$$\mathrm{d}r = \frac{\mathrm{d}\tilde{r}}{\sqrt{1 - \frac{2\,M}{\tilde{r}}}}\ . \tag{1.2.65}$$

Moreover, the proper time τ of the freely falling particle is related to the Schwarzschild time $\tilde{t}$ by

$$\mathrm{d}\tau = \left(1 - \frac{2\,M}{\tilde{r}}\right)\frac{m}{E}\,\mathrm{d}\tilde{t}\ , \tag{1.2.66}$$

where E is the conserved energy of the particle. We thus have

$$\frac{\mathrm{d}^2\tilde{r}}{\mathrm{d}\tilde{t}^2} = -\frac{M}{\tilde{r}^2}\left(1 - \frac{2\,M}{\tilde{r}}\right)^2\left[\frac{m^2}{E^2} - 2\left(1 - \frac{2\,M}{\tilde{r}}\right)^{-3}\left(\frac{\mathrm{d}\tilde{r}}{\mathrm{d}\tilde{t}}\right)^2\right]\ . \tag{1.2.67}$$

Next, we expand the above expressions for $M/r \simeq M/\tilde{r} \ll 1$ (weak field) and $|\mathrm{d}\tilde{r}/\mathrm{d}\tilde{t}| \ll 1$ (non-relativistic regime). In order to keep track of small quantities, it is useful to introduce a parameter $\epsilon > 0$ and replace

[3] Here, we will use units with $G_\mathrm{N} = 1$ for simplicity.

$$\frac{M}{\tilde{r}} \to \epsilon\,\frac{M}{\tilde{r}}\ , \qquad \frac{\mathrm{d}\tilde{r}}{\mathrm{d}\tilde{t}} \to \epsilon\,\frac{\mathrm{d}\tilde{r}}{\mathrm{d}\tilde{t}}\ . \tag{1.2.68}$$

From the non-relativistic limit, it also follows that $E = m + \mathcal{O}(\epsilon^2)$ and any four-velocity

$$u^\mu = \left(1 + \mathcal{O}(\epsilon^2), \epsilon\,\frac{\mathrm{d}\vec{x}}{\mathrm{d}\tilde{t}} + \mathcal{O}(\epsilon^2)\right)\ , \tag{1.2.69}$$

so that the acceleration is also of order ϵ,

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = \epsilon\left(0, \frac{\mathrm{d}^2\vec{x}}{\mathrm{d}\tilde{t}^2}\right) + \mathcal{O}(\epsilon^2)\ . \tag{1.2.70}$$

We then have

$$\epsilon\,\frac{\mathrm{d}^2\tilde{r}}{\mathrm{d}\tilde{t}^2} = -\epsilon\,\frac{M}{\tilde{r}^2}\left(1 - \epsilon\,\frac{2\,M}{\tilde{r}}\right)^2\left[1 + \mathcal{O}(\epsilon^2) - 2\left(1 - \epsilon\,\frac{2\,M}{\tilde{r}}\right)^{-3}\epsilon^2\left(\frac{\mathrm{d}\tilde{r}}{\mathrm{d}\tilde{t}}\right)^2\right]\ , \tag{1.2.71}$$

and

$$r \simeq \int\left(1 + \epsilon\,\frac{M}{\tilde{r}} + \epsilon^2\,\frac{3\,M}{2\,\tilde{r}^2}\right)\mathrm{d}\tilde{r} \simeq \tilde{r}\left[1 - \epsilon\frac{M}{\tilde{r}}\log\!\left(\epsilon\,\frac{M}{\tilde{r}}\right) - \epsilon^2\frac{3\,M^2}{2\,\tilde{r}^2} + \mathcal{O}(\epsilon^3)\right] . \tag{1.2.72}$$

Since

$$r = \tilde{r} + \mathcal{O}\left(\epsilon\log\epsilon\right)\ , \tag{1.2.73}$$

it is clear that Eq. (1.2.71) to first order in ϵ reproduces the Newtonian dynamics,

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tilde{t}^2} \simeq \frac{\mathrm{d}^2\tilde{r}}{\mathrm{d}\tilde{t}^2} \simeq -\frac{M}{r^2}\ . \tag{1.2.74}$$

The interesting correction comes from including the next order. In fact, we have

$$\epsilon\,\frac{\mathrm{d}^2\tilde{r}}{\mathrm{d}\tilde{t}^2} = -\epsilon\,\frac{M}{r^2} + \epsilon^2\,\frac{4\,M^2}{r^3} + \mathcal{O}\left(\epsilon^2\log\epsilon\right)\ , \tag{1.2.75}$$

or, neglecting terms of order $\epsilon^2\log\epsilon$ and higher, and then setting $\epsilon = 1$,

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tilde{t}^2} = -\frac{M}{r^2} + \frac{4\,M^2}{r^3} = -\frac{\mathrm{d}}{\mathrm{d}r}\left(-\frac{M}{r} + \frac{2\,M^2}{r^2}\right)\ . \tag{1.2.76}$$

The correction to the Newtonian potential would therefore appear to be

$$V = \frac{2\,M^2}{r^2}\ , \tag{1.2.77}$$

but one step is stil missing.

Instead of the Schwarzschild time $\tilde{t}$, let us employ the proper time t of static observers placed along the trajectory of the falling particle, that is

$$\mathrm{d}t = \left(1 - \frac{2\,M}{r}\right)^{1/2} \mathrm{d}\tilde{t}\ . \tag{1.2.78}$$

From Eq. (1.2.66) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = \left(1 - \frac{2\,M}{r}\right)^{-1/2} \frac{E}{m}\,\frac{\mathrm{d}}{\mathrm{d}t}\ , \tag{1.2.79}$$

and Eq. (1.2.64) then becomes

$$\frac{\mathrm{d}^2\tilde{r}}{\mathrm{d}t^2} = -\frac{M}{r^2}\left(1 - \frac{2\,M}{\tilde{r}}\right)\left[\frac{m^2}{E^2} - \left(1 - \frac{2\,M}{\tilde{r}}\right)^{-2}\left(\frac{\mathrm{d}\tilde{r}}{\mathrm{d}t}\right)^2\right]\ . \tag{1.2.80}$$

Introducing like before the small parameter ϵ yields

$$\epsilon\,\frac{\mathrm{d}^2\tilde{r}}{\mathrm{d}t^2} = -\epsilon\,\frac{M}{\tilde{r}^2}\left(1 - \epsilon\,\frac{2\,M}{\tilde{r}}\right)\left[1 + \mathcal{O}(\epsilon^2) - \left(1 - \epsilon\,\frac{2\,M}{\tilde{r}}\right)^{-2}\epsilon^2\left(\frac{\mathrm{d}\tilde{r}}{\mathrm{d}t}\right)^2\right]\ , \tag{1.2.81}$$

The first order in ϵ is of course the same. However, up to second order, one obtains

$$\epsilon\,\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\epsilon\,\frac{M}{r^2} + \epsilon^2\,\frac{2\,M^2}{r^3} + \mathcal{O}\left(\epsilon^2 \log\epsilon\right)\ , \tag{1.2.82}$$

which yields the correction to the Newtonian potential

$$V = \frac{M^2}{r^2}\ . \tag{1.2.83}$$

This is precisely the expression following from the isotropic form of the Schwarzschild metric [1], and the one we will consider as our reference term in Chapter 3.

1.3 Hamiltonian Formulation of General Relativity

We now face the problem of defining energy in the framework of General Relativity (GR). It is well known that energy is the value of the Hamiltonian and this is why we need to consider the Hamiltonian formulation of GR. This discussion is of fundamental importance when one is willing to find the quantum description of a system.

Historically, Arnowitt, Deser and Misner (ADM) were the first who carried out a consistent and correct calculation of the canonical form of Einstein theory of gravitation. General coordinate invariance was the main problem in their analysis of the dynamics of gravitational field. This property plays in gravitation the same role of gauge invariance in electromagnetic theory. In fact in both cases, the effect of these invariance properties is to introduce redundant variables in the original formulation of the theory to ensure that the correct transformation properties are maintained. ADM's work concentrated in separating the metric field into the parts carrying the true dynamical information and the "gauge" ones recognising among all the equations of motion wether they were dynamical or algebraic and differential constraint equations. To achieve this result, they started with the full Einstein Lagrangian in Palatini form to find first order differential equations and singled out the time variable recasting the theory in a 3+1 dimensional form, both characteristic feature of Hamilton equations of motion. The Palatini formulation of Einstein theory consists in making the metric field and the affinity independent field variables so that the relation between the affinity and the metric results as an extra set of field equations. ADM then noticed that solving the algebraic constraint equations and deriving the Hamiltonian in this way, lead to a vanishing Hamiltonian. They overcame this problem showing that working out solutions of the differential constraints -which amounts at chosing a preferred frame- and substituting in the action lead to the true non vanishing Hamiltonian of the system. However it has later been proved by Regge and Teitelboim that the non vanishing Hamiltonian could be found under completely general assumptions, prior to fixing the coordinates or solving the constraints, including in it a surface integral.

The ADM decomposition in 3+1 dimensions plays a fundamental role in developing the Hamiltonian formulation of GR. To understand why this geometrical formulation is so important, we can analyze the dynamical informations contained in Einstein equations. One could naively think that the procedure consists in giving a distribution of mass and energy and then solving the Einstein second order equations for the geometry. What they actually tell us is instead to give the fields that generate the mass-energy, the 3-geometry of space and the time rates of change of these quantities, and only then solve the Einstein equations for the 4-geometry of spacetime. Nevertheless, this dynamical analysis is not the only reason for working on this spacetime split. In fact the split is called for also by the boundary conditions posed by the action principle itself, telling us that we need to give the 3-geometry of two successive hypersurfaces of constant t and then adjust the 4-geometry in between to extremize the action.

This is why after having described how energy of gravitational system have historically been determined, we proceed now explaining how the ADM split into 3+1 dimensions can be worked out and how this procedure can lead us to Hamiltonian formulation of GR. We make reference to the completely geometrical analysis of hypersurfaces contained in Appendix A.

1.3.1 ADM Decomposition

The main purpose of the 3+1 decomposition is to express the action in terms of the Hamiltonian and to do so, it is necessary to foliate spacetime into a family of spacelike hypersurfaces of constant "time". In order to carry out this decomposition, we describe these hypersurfaces through a scalar field $t(x^\mu)$, such that $t = const$ selects one of the hypersurfaces

$$\Sigma_t = \{x^\mu \in M; t(x^\mu) = const\}. \tag{1.3.1}$$

The only requirement we make is that this function is single-valued and that the unit normal to the hypersurface $n_\mu \propto \partial_\mu t$, is a future directed timelike vector field.

We cover each hypersurface Σ_t with a coordinate system y^i. It is not necessary to link the different systems of each hypersurface but this is convenient for us because it helps finding the decomposition of the spacetime metric. The relation between these coordinates is naurally provided if we consider a congruence of curves γ intersecting the hypersurfaces Σ_t. There is no need for this congruence to be othogonal to Σ_t. Now, choosing t as a parameter describing the curves, it is easy to see that a vector t^μ is tangent if it satisfy the relation $t^\mu \partial_\mu t = 0$. We can fix the relation between the coordinates on every hypersurface by taking a curve γ_P through a point P on Σ_t and assigning the coordinate y^i associated to this point, to all the points individuated by the intersections between this curve γ_P and all the successive hypersurfaces. In this way, we build a rigid structure of spacetime in which the coordinates y^i are held constant along a curve of the congruence. This construction furnishes the spacetime with a coordinate system (t, y^i), related to the first through a transformation $x^\mu = x^\mu(t, y^i)$. We see that with this coordinate system the tangent vector field to the congruence reads

$$t^\mu = \left(\frac{\partial x^\mu}{\partial t}\right)_{y^i} = \delta^\mu_t, \tag{1.3.2}$$

the tangent fields to Σ_t

$$e^\mu_i = \left(\frac{\partial x^\mu}{\partial y^i}\right)_t = \delta^\mu_i, \tag{1.3.3}$$

and finally the unit normal

$$n_\mu = -N\partial_\mu t \qquad n_\mu e^\mu_i = 0. \tag{1.3.4}$$

The tangent fields also have the property of vanishing Lie derivative along t, i.e.

$$\mathfrak{L}_t e^\mu_i = 0 \tag{1.3.5}$$

Now, recall we did not assume the congruence to be orthogonal to the hypersurfaces, therefore, in general the tangent vector t^μ will have both a normal and tangent part in which it can now be decomposed by means of the just written relations for normal and tangent vectors to Σ_t

$$t^\mu = Nn^\mu + N^i e^\mu_i, \tag{1.3.6}$$

where N and N^i are called the *lapse function* and *shift vector* in ADM vocabulary. They measures respectively the lapse of "time" between the hypersurfaces and how much the local spatial coordinate system shifts tangential to the "earlier" hypersurface when moving on the congruence between two successive hypersurfaces. In other words, they describe how coordinates move in time from one hypersurface to the next. Now we can use the coordinate transformation $x^\mu = x^\mu(t, y^i)$ to rewrite the metric. Let's first see how the differentials dx^μ are changed

$$dx^\mu = t^\mu dt + e^\mu_i dy^i \tag{1.3.7}$$

$$= (Ndt)\, n^\mu + \left(dy^i + N^i dt\right) e^\mu_i. \tag{1.3.8}$$

Thus the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1.3.9}$$
$$= -N^2 dt^2 + h_{ij} \left(dy^i + N^i dt\right) \left(dy^j + N^j dt\right), \tag{1.3.10}$$

where h_{ij} is the induced metric on Σ_t defined in the Appendix A. The metric of space-time now reads

$$g_{00} = N_i N^i - N^2, \qquad g_{0k} = N_k, \qquad g_{i0} = N_i, \qquad g_{ik} = h_{ik}. \tag{1.3.11}$$

We can of course find the inverse metric by calculating $g_{\alpha\beta} g^{\beta\gamma}$ and we obtain

$$g^{00} = -\frac{1}{N^2,} \qquad g^{0j} = \frac{N^j}{N^2}, \qquad g^{k0} = \frac{N^k}{N^2}, \qquad g^{kj} = h^{kj} - \frac{N^k N^j}{N^2}, \tag{1.3.12}$$

The only object we still have to define in order to complete this ADM decomposition and be able to find the Hamiltonian formulation of general relativity is the extrinsic curvature of these hypersurfaces Σ_t discussed in Appendix A. If we want to find the Hamiltonian we will have to reexpress the action in terms of the time derivatives of the induced metric. Recalling the definition (A.2.4), we find

$$\dot{h}_{ij} = \mathfrak{L}_t h_{ij} = \mathfrak{L}_t \left(g_{\mu\nu} e_i^\mu e_j^\nu\right) = \left(\mathfrak{L}_t g_{\mu\nu}\right) e_i^\mu e_j^\nu, \tag{1.3.13}$$

where use have been made of eq. (1.3.5). Now manipulating this expression with the help of relation (1.3.6), we obtain

$$\mathfrak{L}_t g_{\mu\nu} = \nabla_\mu t_\nu + \nabla_\nu t_\mu \tag{1.3.14}$$
$$= \nabla_\mu \left(N n_\nu + N_\nu\right) + \nabla_\nu \left(N n_\mu + N_\mu\right) \tag{1.3.15}$$
$$= n_\nu \partial_\mu N + n_\mu \partial_\nu N + N \left(\nabla_\mu n_\nu + \nabla_\nu n_\mu\right) + \nabla_\mu N_\nu + \nabla_\nu N_\mu, \tag{1.3.16}$$

where $N^\mu = N^i e_i^\mu$. Finally, projecting along $e_i^\mu e_j^\nu$ and recalling the definitions (A.2.12) and (A.3.3) we obtain

$$\dot{h}_{ij} = 2N K_{ij} + D_i N_j + D_j N_i, \tag{1.3.17}$$

which leads to the explicit form of the extrinsic curvature tensor in terms of the time derivative of the induced metric and of lapse and shift

$$K_{ij} = \frac{1}{2N} \left(\dot{h}_{ij} - D_i N_j - D_j N_i\right). \tag{1.3.18}$$

1.3.2 ADM Energy

We now possess all the background needed to complete the Hamiltonian formulation of GR. The Einstein-Hilbert action can now be rewritten substituting the metric (1.3.9) and neglecting surface terms it reads

$$
\begin{aligned}
S &= \int dt d^3x \mathcal{L} && (1.3.19)\\
&= \int dt d^3x \sqrt{h} N \left({}^{(3)}R + K_{ij}K^{ij} - K^2 \right), && (1.3.20)
\end{aligned}
$$

where we set $16\pi G_{\mathrm{N}} = 1$ for convenience. Here ${}^{(3)}R$ is the intrinsic curvature, K_{ij} is the just defined extrinsic curvature, (1.3.18), and $K = K^i_i = h^{ij}K_{ij}$ stands for its trace. Thus the action has been transformed from a functional of the full spacetime metric, to a functional of the 3-metric coefficients h_{ij} and the lapse and shift functions N and N_i. Recalling what we said at the beginning of this section it appear now more clear the presence in the theory of gravitation of redundant variables. Effectively, we note that the action does not depend on time derivatives of the lapse and shift and this clearly means they are not dynamical variables but appear in the action only as free variable fields whose variation leads to the hamiltonian and momentum constraints (varying N and N_i respectively). Therefore, these functions can be freely determined as they do not have to satisfy any evolution equation, and this choice amounts to a choice of coordinate frame. The ordinary procedure to cast a theory in hamiltonian form is to determine the momenta conjugate to the fields of the theory. It is obvious that no momentum will be associated to the Lagrange multipliers N and N_i as a consequence of what we have just said, but we can find the momentum conjugate to h_{ij}

$$
\pi^{ij} \equiv \frac{\delta S}{\delta \dot{h}_{ij}} = \sqrt{h}\left(K^{ij} - K h^{ij}\right). \qquad (1.3.21)
$$

This shows that π^{ij} is not a tensor because of the factor $\sqrt{h}$, it is a tensor density.

Now define the Hamiltonian as the Legendre transform of the Lagrangian

$$
\begin{aligned}
H &= \int d^3x \left(\pi^{ij}\dot{h}_{ij} - \mathcal{L} \right) && (1.3.22)\\
&= \int d^3x \sqrt{h} \left(N\mathcal{H} + N^i \mathcal{H}_i \right), && (1.3.23)
\end{aligned}
$$

obtained after integrating by parts and neglecting surface terms. Here,

$$\mathcal{H} = -\,^{(3)}R + h^{-1}\pi^{ij}\pi_{ij} - \frac{1}{2}h^{-1}\pi^2 \tag{1.3.24a}$$

$$\mathcal{H}_i = -2h_{ik}D_j\left(h^{-1/2}\pi^{jk}\right), \tag{1.3.24b}$$

with $\pi = h^{ij}\pi_{ij}$. In the Hamiltonian formalism it is now clear that h_{ij} and π^{ij} are the dynamical variables while the N and N^i only play the role of Lagrange multipliers leading to the Hamiltonian and momentum constraints, i.e., we require $\delta H/\delta N = \delta H/\delta N^i = 0$, which straightly gives $\mathcal{H} = \mathcal{H}_i = 0$. The corresponding Hamilton's equations of motion read

$$\dot{h}_{ij} = \frac{\delta H}{\delta \pi^{ij}} \qquad \dot{\pi}^{ij} = -\frac{\delta H}{\delta h_{ij}}. \tag{1.3.25}$$

We are not interested in the explicit form of these equations because our aim is to define energy for a gravitational system but we can observe that the first of these equation only reproduces the definition of the conjugate momentum π^{ij} while the second one is a complicate relation.

Now that we have determined the Hamiltonian for GR it is straightforward to define energy of a solution of equations of motion as the value of the Hamiltonian. But what happens is quite strange, the Hamiltonian vanishes for any solution of the constraints equations.

From now on our analysis moves away from ADM's work, [9]. We get to the same result following an easier and more general reasoning first outlined by T. Regge and C. Teitelboim, [10]. As we already pointed in the introduction to this section, ADM now proceeded solving the constraint equations to find the true canonical variables and then substituted these solutions in the Hamiltonian to find the non-vanishing one. The procedure of solving constraint equations was thus equivalent to the choice of a preferred frame which they called "radiation frame", in analogy with the electromagnetic case. However, Regge and Teitelboim later pointed that there was no need for this particular choice and that a true non-vanishing Hamiltonian for the system could be found before fixing the spacetime coordinates. They started from the fundamental consistency requirement that a well defined phase space of a dynamical system must contain all physical solutions of the equations of motion. Otherwise, the variational problem would have no solution because the extremal trajectories wouldn't have been admitted from the beginning. The solution of this problem was found in the addition of a surface integral of the form

$$\int dA\, n_i \left(\partial_j h_{ij} - \partial_i h_{jj}\right). \tag{1.3.26}$$

This is the same result obtained by ADM but Regge and Teitelboim pointed out that the way they derived it made clear that its inclusion in the Hamiltonian was found "...by a fundamental reason and not by some ad hoc considerations...". We will now describe how this surface integral comes out of the theory using Regge and Teitelboim's simple reasoning. Actually, the problem is evident from equations (1.3.25). There, we didn't stress that to calculate those functional derivatives we need to integrate by parts in order to remove derivatives form the variations $\delta\pi^{ij}$ and δh_{ij} and obtain the variation of the Hamiltonian in the following form

$$\delta H = \int d^3x \left(A^{ij}\delta h_{ij} + B_{ij}\delta\pi^{ij}\right), \tag{1.3.27}$$

otherwise no Hamilton equations can be defined at all. The problem now arises that this form can not be reached starting with the Hamiltonian (1.3.23), because among the surface terms coming out in this variation, there is a non-vanishing one. Thus, the question is which of these integrals can be neglected legitimately.

Let us assume the case in which our surfaces of constant t are asymptotically flat and that our time foliation is chosen so that our spacetime coordinates become Minkowskian at large r. Then, every solution of Einstein's equations representing an asymptotically flat spacetime behaves at spatial infinity as

$$h_{ij} = \delta_{ij} + \mathcal{O}\left(1/r\right) \tag{1.3.28a}$$

$$\pi^{ij} = \mathcal{O}\left(1/r^2\right) \tag{1.3.28b}$$

$$\delta h_{ij} = \mathcal{O}\left(1/r\right) \tag{1.3.28c}$$

$$\delta\pi^{ij} = \mathcal{O}\left(1/r^2\right) \tag{1.3.28d}$$

$$N = 1 + \mathcal{O}\left(1/r\right) \tag{1.3.28e}$$

$$N^i = \mathcal{O}\left(1/r\right) \tag{1.3.28f}$$

$$N_{,i} = \mathcal{O}\left(1/r^2\right) \tag{1.3.28g}$$

$$N^i_{,j} = \mathcal{O}\left(1/r^2\right). \tag{1.3.28h}$$

We consider the boundary of our constant t hypersurface to be a 2-sphere of constant r, S_r.

Now we have all what we need to determine which boundary terms can be neglected and which are relevant. Let's start taking into account for the variation with respect π^{ij}. We understand, looking at equation (1.3.24b), that the only surface term arising here is

$$\int_{S_r} dA \left(-2N^i h_{ik} n_j h^{-1/2} \delta\pi^{jk}\right), \tag{1.3.29}$$

where we named the area element and its outward unit normal dA and n^j respectively. Clearly $dA = \mathcal{O}\left(r^2\right)$ and with our asymptotic conditions (1.3.28), this expression vanishes as $r \to \infty$. No relevant surface term arises when we vary π^{ij}.

Things are different when we vary h_{ij} because there are two ways in which boundary terms come out. The first is when we take the variation of $h^{-1/2}$ in (1.3.24b), but this clearly generates a surface integral very similar to the one just considered for the variation of π^{ij} and it vanishes in the same way as $r \to \infty$. Then, we have the variation of the term ${}^{(3)}R$ in (1.3.24a), which is identical to the variation of the 4-dimensional Ricci scalar calculated to obtain the Eintein's equations from the Einstein-Hilbert action, a part from that we are here varying the 3-d Ricci scalar

$$\delta\, {}^{(3)}R = -R^{ij}\delta h_{ij} + D^i D^j(\delta h_{ij}) - D^k D_k((h^{ij}\delta h_{ij}) \tag{1.3.30}$$

Two derivatives appear on the variations, this means we will have to integrate by parts twice leading to two surfaces terms

$$S_1 = -\int_{S_r} dAN \left[n^i D^j(\delta h_{ij}) - n^k D_k(h^{ij}\delta h_{ij})\right] \tag{1.3.31}$$

$$S_2 = \int_{S_r} dA \left(n^j \delta h_{ij} D^i N - h^{ij}\delta h_{ij} n^k D_k N\right) \tag{1.3.32}$$

using the boundary conditions (1.3.28), we immediately see that $S_2 \to 0$ when $r \to \infty$. The first one instead is non vanishing in this limit and thus it is the relevant integral we have to be careful with and not discard. It can be simplified using the fact that $h_{ij} \to \delta_{ij}$

$$\lim_{r\to\infty} S_1 = -\lim_{r\to\infty} \int_{S_r} dA\; n_i \left(\partial_j \delta h_{ij} - \partial_i \delta h_{jj}\right) = -\delta E_{ADM} \tag{1.3.33}$$

where

$$E_{ADM} = \lim_{r\to\infty} \int_{S_r} dA\; n_i \left(\partial_j h_{ij} - \partial_i h_{jj}\right) \tag{1.3.34}$$

If we now add this surface term to the Hamiltonian (1.3.23), we note that when we vary the Hamiltonian with respect to h_{ij} the boundary term S_1 will be cancelled by the variation of the surface term E_{ADM}. Hence no surface terms arise in this way and the variation of the Hamiltonian can finally be written in the form (1.3.27). Thus Hamilton's equations are now well defined and we understand that for asymptotically flat spacetimes the correct Hamiltonian for general relativity is

$$H' = H + E_{ADM} \tag{1.3.35}$$

Now that we have a satisfactory variational principle, we can evaluate the Hamiltonian on a solution in order to find energy. Since H vanishes for every solution, it is trivial to see that energy will be E_{ADM}, which is called *Arnowitt-Deser-Misner energy.* It is an easy exercise to show that E_{ADM} for the Schwarzschild line element gives [11]

$$E_{ADM} = M \tag{1.3.36}$$

and thanks to the knowledge that in a time independent spacetime metric the total momentum vanishes, we can identify this with M_{ADM}.

A final remark can be made. We can justify the fact we did not consider at all the matter part, thinking about how fast matter fields decrease at infinity. Of course they will approach to zero faster than the metric which means that every surface integral containing these fields vanish. For this reason we restricted ourselves to the pure gravitational field without matter.

1.4 York-Gibbons-Hawking Boundary Term

We have clearly shown the importance of boundary terms in general relativity, proceeding in a somewhat historical manner. Thus, we introduced this problem in the context of the Hamiltonian formulation. Anyway, as anticipated in Section 1.1.1, the question of boundary terms arise at the stage of the action principle itself. We report here the result (1.1.10) we found varying the Einstein-Hilbert action. It will be useful to have the boundary term contributing to the gravitational action explicitly written here

$$\delta S_{\mathrm{B}} = \frac{1}{16\pi G_{\mathrm{N}}} \int_M d^4x \sqrt{-g} \nabla_\lambda \left(g^{\mu\nu} \delta\Gamma^\lambda_{\mu\nu} - g^{\mu\lambda} \delta\Gamma^\nu_{\mu\nu} \right). \tag{1.4.1}$$

We now explicitly show that this term can not be neglected. First, apply the Gauss theorem to express the above term as an integral over the boundary Σ of the considered spacetime M

$$\delta S_{\mathrm{B}} = \frac{1}{16\pi G_{\mathrm{N}}} \int_{\Sigma} dS_{\lambda} \sqrt{-g} \left(g^{\mu\nu} \delta \Gamma^{\lambda}_{\mu\nu} - g^{\mu\lambda} \delta \Gamma^{\nu}_{\mu\nu} \right) \tag{1.4.2}$$

$$= \frac{1}{16\pi G_{\mathrm{N}}} \int_{\Sigma} d^3y \sqrt{h} n_{\lambda} \left(g^{\mu\nu} \delta \Gamma^{\lambda}_{\mu\nu} - g^{\mu\lambda} \delta \Gamma^{\nu}_{\mu\nu} \right), \tag{1.4.3}$$

where we introduced the foliation of spacetime discussed in Appendix A and we therefore understand h as the determinant of the induced metric on the hypersurface Σ. Let's now evaluate this variational contribution, after having imposed Dirichlet boundary conditions on the metric, i.e. $\delta g_{\mu\nu}|_{\Sigma} = 0$, and using the definition of the Christoffel connection

$$n_{\lambda} \left(g^{\mu\nu} \delta \Gamma^{\lambda}_{\mu\nu} - g^{\mu\lambda} \delta \Gamma^{\nu}_{\mu\nu} \right) = \frac{1}{2} n^{\rho} g^{\mu\nu} \left(\partial_{\mu} \delta g_{\rho\nu} + \partial_{\nu} \delta g_{\rho\mu} - \partial_{\rho} \delta g_{\mu\nu} \right) \tag{1.4.4}$$

$$- \frac{1}{2} n^{\mu} g^{\nu\rho} \left(\partial_{\mu} \delta g_{\rho\nu} + \partial_{\nu} \delta g_{\rho\mu} - \partial_{\rho} \delta g_{\mu\nu} \right) \tag{1.4.5}$$

$$= n^{\rho} g^{\mu\nu} \partial_{\mu} g_{\rho\nu} - n^{\rho} g^{\mu\nu} \partial_{\rho} \delta g_{\mu\nu} \tag{1.4.6}$$

$$= \left(n^{\mu} g^{\rho\nu} - n^{\rho} g^{\mu\nu} \right) \partial_{\rho} \delta g_{\mu\nu}. \tag{1.4.7}$$

In order to separate the metric gradients along tangential and normal directions with respect to Σ, we will use the decomposition of the metric $g^{\mu\nu} = h^{\mu\nu} + \epsilon n^{\mu} n^{\nu}$, inverse of relation (A.2.5), found in Appendix A

$$\left(n^{\mu} g^{\rho\nu} - n^{\rho} g^{\mu\nu} \right) \partial_{\rho} \delta g_{\mu\nu} = \left[n^{\mu} \left(h^{\rho\nu} + \epsilon n^{\rho} n^{\nu} \right) - n^{\rho} \left(h^{\mu\nu} + \epsilon n^{\mu} n^{\nu} \right) \right] \partial_{\rho} \delta g_{\mu\nu} \tag{1.4.8}$$

$$= n^{\mu} h^{\rho\nu} \partial_{\rho} \delta g_{\mu\nu} - n^{\rho} h^{\mu\nu} \partial_{\rho} \delta g_{\mu\nu}. \tag{1.4.9}$$

The first term in this expression is understood as the tangentially projected -by the induced metric- part of the gradient and it vanishes as a consequence of our choice of boundary condition. The other one instead is projected in the normal direction in the same way and it gives a non-vanishing contribute. We are left with the true boundary term coming from the variation of the Ricci tensor in the Einstein-Hilbert action

$$\frac{1}{16\pi G_{\mathrm{N}}} \int_{M} d^4x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} = - \frac{1}{16\pi G_{\mathrm{N}}} \int_{\Sigma} d^3y \sqrt{h} h^{\mu\nu} n^{\rho} \partial_{\rho} \delta g_{\mu\nu}. \tag{1.4.10}$$

Therefore, as anticipated, the functional variation of Einstein-Hilbert action does not lead to the expected Einstein equations because of this non-vanishing boundary term.

Actually, that action is not even differentiable. We understand now the link with the discussion made in the framework of the Hamiltonian formulation. In fact, we already noticed the problem of non-definitness of the variational problem. In a similar manner, we can resolve this issue adding a suitable boundary term to the action. Here, the word suitable means whose variation leads to exactly the same variational contribute of the Ricci tensor, apart from terms proportional to $\delta g^{\mu\nu}$ or its tangential derivatives which vanish under the boundary conditions we imposed. Among all the candidates, the most common choice is the York-Gibbons-Hawking term, [12], [13], thanks to its simple covariant formulation and clear geometric interpretation

$$S_{\mathrm{YGH}} = \frac{1}{8\pi G_{\mathrm{N}}} \int_{\Sigma} d^3y \sqrt{h} K, \tag{1.4.11}$$

where K here is the trace of the extrinsic curvature tensor which already appeared in the context of Hamiltonian formulation. It can be explicitly demonstrated, that variation of this term exactly cancels the boundary term coming out of the variation of the Einstein-Hilbert action. In fact, thanks to equations (A.3.3) and (A.3.5) we can rewrite the trace of the extrinsic curvature tensor as

$$K = h^{\mu\nu} K_{\mu\nu} = h^{\mu\nu} \left(\partial_\mu n_\nu - \Gamma^\rho_{\mu\nu} n_\rho \right), \tag{1.4.12}$$

and performing the corresponding variation, where we neglect all terms whose variational contribution vanishes as a consequence of our boundary conditions, we obtain

$$\begin{aligned}
\delta \left(\sqrt{h} 2K \right) &= 2\sqrt{h} h^{\mu\nu} \left(\partial_\mu \delta n_\nu - n_\rho \delta \Gamma^\rho_{\mu\nu} \right) && (1.4.13)\\
&= -2\sqrt{h} h^{\mu\nu} n_\rho \delta \Gamma^\rho_{\mu\nu} && (1.4.14)\\
&= -2\sqrt{h} h^{\mu\nu} n^\alpha \frac{1}{2} \left(\partial_\mu \delta g_{\nu\alpha} + \partial_\nu \delta g_{\mu\alpha} - \partial_\alpha \delta g_{\mu\nu} \right) && (1.4.15)\\
&= \sqrt{h} h^{\mu\nu} n^\alpha \partial_\alpha \delta g_{\mu\nu}. && (1.4.16)
\end{aligned}$$

It is now very easy to see that this equals the term due to the Ricci tensor variation (1.4.10)

Finally, we can properly formulate the variational principle leading to the Einstein equations of motion (in vacuum) as

$$\delta_g \left(S_{\mathrm{EH}} + S_{\mathrm{YGH}} \right) = 0 \quad \Longrightarrow \quad G_{\mu\nu} = 0. \tag{1.4.17}$$

1.4.1 Non-Dynamical Term

The action found so far is well defined only if we consider a compact spacetime, while it diverges for non-compact ones [14]. However, Hawking and Horowitz, [15], found a way to solve this problem, based on the choice of a reference background which is required to be a static solution of the field equations. The physical action is then found subtracting the background action S_0 to the previously obtained one

$$S_{\text{phys}}[g, \phi] = S[g, \phi] - S[g_0, \phi_0]. \tag{1.4.18}$$

It is now finite if g and ϕ approaches g_0 and ϕ_0 asymptotically, this means they have to induce the same fields on the boundary. In particular, if one is interested in asymptotically flat spacetimes, then the appropriate reference background metric is just the flat Minkowski metric, and the action takes the simple form

$$S_{\text{phys}} = \frac{1}{16\pi G_{\text{N}}} \int d^4x \sqrt{-g}\, (R + \mathcal{L}_{\text{M}}) + \frac{1}{8\pi G_{\text{N}}} \int d^3y \sqrt{h}\, (K - K_0)\,, \tag{1.4.19}$$

where K_0 is the trace of the extrinsic curvature of the boundary embedded in Minkowski spacetime. This procedure is however general and one could consider spacetimes which are not asymptotically flat by taking the proper background.

This result is reflected in the Hamiltonian derived from this action. Therefore, the physical Hamiltonian is given by the difference between the one computed with $S[g, \phi]$ and the one computed from the background. To cast the action in hamiltonian form and the recognise the physical Hamiltonian, the procedure is very similar to the one explained in section 1.3.2 (see [15] and [18]) and we only report the result

$$H = \frac{1}{16\pi G_{\text{N}}} \int_{\Sigma} d^3y \left(N\mathcal{H} + N^i \mathcal{H}_i \right) - \frac{1}{8\pi G_{\text{N}}} \int_{S_\infty} dA \left(N^{2\,(2)}K - N^i \pi_{ij} n^j \right) \tag{1.4.20a}$$

$$H_0 = -\frac{1}{8\pi G_{\text{N}}} \int_{S_\infty} dA \; N^{2\,(2)}K_0 \tag{1.4.20b}$$

where n^i is the unit normal to the hypersurface of constant t at infinity, ${}^{(2)}K$ is the trace of the extrinsic curvature of the surface S^∞ in Σ and π^{ij} the momentum conjugate to g_{ij} as in the previous sections. Thus, energy reads

$$E = -\frac{1}{8\pi G_{\text{N}}} \int_{S_\infty} dA \left[\left(N^{2\,(2)}K - N^{2\,(2)}K_0 \right) - N^i \pi_{ij} n^j \right]. \tag{1.4.21}$$

1.4.2 Equivalence with ADM Energy

Of course, we hope that the total energy just calculated directly from the action with the York-Gibbons-Hawking boundary term is equivalent to the one found by ADM. We demonstrate in particular that the latter is contained as a special case because, as we stated before, this procedure is general and valid not only for asymptotically flat spacetimes. Thus, in order to show the equivalence we have to restrict equation (1.4.21), to the case of an asymptotically flat spacetime and only then compare it with equation (1.3.34). The energy (1.4.21) depends on the choice of lapse and shift which, as already seen in Section 1.3.2, are $N = 1$ and $N^i = 0$ in the case of asymptotically flat spacetimes. Therefore equation (1.4.21) now reads

$$E = -\frac{1}{8\pi G_{\rm N}} \int_{S_\infty} dA \left(N^{2\,(2)}K - N^{2\,(2)}K_0 \right). \tag{1.4.22}$$

To show the equality we will take S^∞ here to be the same surface $\lim_{r\to\infty} S_r$ as in equation (1.3.34). Now, it is convenient to choose an appropriate set of coordinate so that the metric on the sphere reads, using the same notation as in [15],

$$ds^2 = dr^2 + q_{ij} dx^i dx^j, \tag{1.4.23}$$

and similarly for the background metric

$$ds^2 = d\rho^2 + q^0_{ij} dy^i dy^j, \tag{1.4.24}$$

where the i, j run over the angle coordinates and $q(q^0)$ is a function of $r(\rho)$ and $x^i(y^i)$. In order to have the same unit normal in the two metrics on S_∞ we make the diffeomorphism from the original spacetime to the background which gives us the relations $r = \rho$ and $x^i = y^i$. The initial assumption that the two metric must agree on the boundary (remember when we said g and ϕ needs to approach g_0 and ϕ_0 asymptotically) now translates in the concrete relation that $h_{ij} = q_{ij} - q^0_{ij} = 0$ on S_∞. Therefore in these coordinates we have

$$E = -\frac{1}{8\pi G_{\rm N}} \int_{S_\infty} dA \left({}^{(2)}K - {}^{(2)}K_0 \right) = -\frac{1}{16\pi G_{\rm N}} \int_{S_\infty} dA \; q^{ij} \partial_r h_{ij} \tag{1.4.25}$$

Now turn to the ADM's expression (1.3.34) and rewrite $n_i \partial_j h_{ij} = \partial_j (n_i h_{ij}) - h_{ij} \partial_j n_i$. The first term vanishes because of the orthogonality between the induced metric and the normal vector. The second one vanishes because h_{ij} does so on S_∞. Thus for the ADM energy we obtain

$$E_{\mathrm{ADM}} = -\frac{1}{16\pi G_{\mathrm{N}}} \int_{S_\infty} dA\ \partial_r h = -\frac{1}{16\pi G_{\mathrm{N}}} \int_{S_\infty} dA\ q^{ij} \partial_r h_{ij}. \tag{1.4.26}$$

We have therefore shown that the two expressions are indeed equivalent.

1.5 Gravitational Collapse and Singularity Theorems

An important feature of gravity is that under certain conditions, a system with sufficiently large concentration of mass is instable, essentially due to the r^{-2} attractive behaviour. The work of Chandrasekhar [19], first showed that a star of mass greater than 1.3 solar masses, which has exhausted its thermal and nuclear energy resources, cannot sustain its own gravitational pull, leading to a *gravitational collapse*. Situations in which the instability of gravity is manifest are present also at cosmological scale, in those models involving a contraction of the entire Universe or in the time reverse situation of the big bang. General relativity is able to make predictions about the ultimate fate of a system undergoing a gravitational collapse. The first simple example was illustrated by Oppenheimer and Snyder long years ago [20], in the case of the gravitational collapse of a model with exact spherical symmetry, which unavoidably shrinks in a spacetime singularity. Later, many authors started questioning whether this only was a special feature related to the symmetry of the system. Penrose [21] and only a few later Hawking [22], contributed showing that the evolution to a singularity of a gravitational system, is a result that general relativity provides without assuming any exact simmetry and only imposing reasonable conditions such as the positivity of energy, a suitable causality assumption, and a condition such as the existence of trapped surfaces. The first focused on the case of a gravitational collapse and the second on a cosmological case. They finally reached together few years later [23], the most general statement that can be done in Einstein relativity about the presence of a singularity whcih included both gravitational collapse and cosmology. The major drawback is that their theorems tell nothing about the nature of the singularity, they only state very accurately when it happens to exist.

In the following we introduce the basic knowledge to understand their results which by no way intend to be a complete description. We refer to [24,25] for a detailed analysis.

1.5.1 Causal Structure

The universe is described through the mathematical model of the *spacetime*, defined as a connected four dimensional Hausdorff C^∞ manifold M, together with a Lorentz metric(i.e. with signature $+2$) g on M. These are all conditions inferred only by physical reasonability. One of these is that the spacetime should have a regular causal structure which has to be added formally. In the case of Minkowski spacetime, this translates in the request that no material particle can travel faster than light but in general relativity this is true only locally and things can be very different when we consider the global causal structure. For example the singularities occurring in a gravitational collapse mentioned before, could have heavy implications on the causality of the spacetime. This is why we have to discuss at least the main ingredients of the causal structure needed to handle the question of singularities.

Consider a time orientable manifold, so that we do not need a two fold covering. We will mainly deal with *timelike curves* and *non-spacelike curves*, where the first are choosed to be smooth and with timelike future directed tanget vectors , while the second have timelike or null tangent vectors. It is known that a curve with end points can be extended into the past or future (through its affine parameter), but if it has no end point in the past, i.e. it continues indefinitely, it is called *past-inextendible* and analogously for the future case. If it is both past and future inextendible it is only called *inextendible.*

Now, if $p, q \in M$, we will write

- $p \ll q$ if there is a timelike curve with past end point p and future end point q;
- $p < q$ if either $p = q$ or there is a non-spacelike curve from p to q.

We can therefore define the so called *chronological future* and *chronological past* of a point p as

$$I^+(p) = \{q \in M : p \ll q\}, \tag{1.5.1}$$

$$I^-(p) = \{q \in M : q \ll p\}, \tag{1.5.2}$$

and equivalently the causal future and causal past for p

$$J^+(p) = \{q \in M : p < q\}, \tag{1.5.3}$$

$$J^-(p) = \{q \in M : q < p\}. \tag{1.5.4}$$

It is then immediate to define the same for a set $S \subset M$ as the union, e.g $I^+(S) = \bigcup_{p\in S} I^+(p)$.

We can already introduce the notion of causality in this picture starting from the local causality principle which states that over small regions of spacetime the causal structure is essentially the same as in special relativity. As already noticed at the beginning of this section, on a larger scale global pathological features could arise. These can be ruled out introducing some physiscally reasonable topological assumptions, first of all we demand the *chronology condition* to hold, i.e. (M, g) does not contain any closed timelike curve. Einstein theory does not provide us with this information, it does not exclude the presence a priori of such curves, in fact there are models consistent with general relativity where closed timelike curves are admitted. Therefore, this assumption puts its ground on the fact that it is easier to think about a spacetime with this feature than without it. Another condition is the so called *causality condition* which holds if there are no closed non-spacelike curves. It is demonstrated [24], that under "physically realistic" situations (based on some energy considerations which will be studied later) these two conditions are actually equivalent. However, we need to keep in mind the possibility that there might be points of the spacetime in which these conditions fail. For example, another pathology of the spacetime arises even if spacetime is causal (causality condition holds), in situations in which non-spacelike curves turn arbitrarily close to the starting point p and others similar limiting processes. We can rule out such situations with a whole class of causality conditions, one of which is the *strong causality condition*. It states that for every $p \in M$, every neighbourhood of p contains a neighbourhood of p which no non-spacelike curves intersect more than once.

Yet, even imposition of strong causality does not ensure that all the causal pathologies are removed as we could still have a spacetime at the verge of violating the chronological condition for which a slight variation of the metric can lead to timelike closed curves. But if we think general relativity as a classical limit of a more general quantum theory, in such a theory the Uncertainty Principle would prevent the metric from having a precise value. Hence, always for the sake of physical reasonability, we would like the causality of the spacetime to be preserved under small fluctuations of the metric. This can be achieved imposing the *stable causality condition*, which holds on M if the spacetime metric g has an open neighbourhood in the C^0 topology such that there are no closed timelike curves in any metric belonging to the neighbourhood. Roughly speaking this means that we can slightly expand the light cones at every point without introducing closed timelike curves.

The concept of causality in relativity has some implications on the causal connection between two events at different points. We state that these two points are causally related only if a non-spacelike curve connecting them exists. Thus, events in the future will be determined by the knowledge of data on an initial surface, namely, if we have a set $S \subset M$, we can determine its *future Cauchy development* $D^+(S)$, defined as the set of all $p \in M$ such that every past-inextendible non-spacelike curve through p meets S. Intrinsic in this reasoning, there will be a limit on the region that can be predicted from the data on S. This zone will be called *future Cauchy horizon* $H^+(S)$

$$H^+(S) = \{p \in M : p \in D^+(S), I^+(p) \cap D^+(S) = 0\}. \tag{1.5.5}$$

Obviously we can define both for the past case in a straightforward manner. It can be shown [24] that when S is a closed achronal set then $H^+(S)$ is too and this is the case of interest (roughly speaking we want S to be a spacelike surface). Finally, we define the *edge* of an achronal set S to be the set of points $p \in S$ such that if $r \ll p \ll q$, with a timelike curve γ from r to q containing p, then every neighbourhood of γ contains a timelike curve from r to q which does not meet S.

Closely related to the future and past Cauchy developments is the concept of global hyperbolicity which we report for completeness. A spacetime (M, g) is said *globally hyperbolic* if the strong causality condition holds on it and if for any two points $p, q \in M$, $J^+(p) \cap J^-(q)$ is compact. The most fundamental consequence of this property is that a globally hyperbolic spacetime is foliated by spacelike hypersurfaces Σ, or rather $M = \Sigma \times R$, allowing to identify a global time function on it.

1.5.2 Definition of Singularities

Before being able to state the theorems which tells us under which circumstances a singularity appears, we need to precisely define what do we mean with the world singularity. There are many explicit examples of singularities in gravity. One show up in the famous Schwarzschild solution [26]. In this well known case there are two singularities at first sight, one at $r = 2m$ and the other at $r = 0$. However, it is known that only in $r = 0$ there is a true curvature singularity, while in the other situation it can be shown to be only a mathematical pathology due to a not so proper choice of the coordinate system [27]. Therefore we require a criterion to determine whether the singularity is a true one or not. In other words, we need a clear indication that a singular point has been cut out. To start with, we state that a spacetime (M, g) is *geodesically complete* if every geodesic can be extended to arbitrary values of its affine parameter. Of course we can divide this reasoning for timelike, null and spacelike geodesics. The opposite feature is of course named geodesic incompleteness. The case of timelike geodesic incompleteness has a clear physical meaning in that it opens the possibility of the existence of freely moving observers whose stories are cut at some point in the past or in the future. A similar reasoning can be made for null geodesic incompleteness identifying it as the corresponding case of zero rest mass particles (observers) while the spacelike case has no physical significance for us. Therefore, we are led to the minimum condition (sufficient for our purposes) that a spacetime can be considered singularity free if it is timelike and null geodesically complete. Obviously, a singularity will occur otherwise.

1.5.3 Raychaudhuri Equation and the Gravitational Focusing

We now concentrate on the evolution in time of a congruence of timelike geodesics (the null case straightly follows). We will show that the effect of a positive energy density matter field is the focusing of non-spacelike trajectories, which is strictly related to the problem of singularities. Thus, consider a congruence of timelike geodesics, defined as a family of timelike curves over an open region of spacetime and such that through every point of this region, one and only one geodesic of the family passes there. In other words, curves of a congruence never cross each other. If the curves are smooth, the congruence defines a smooth timelike vector field v^μ on the spacetime, tangent to itself. Of course, it can be normalized to be the unit tangent vector $v^\mu v_\mu = -1$ and in an analogous manner with what done in Appendix A, we can define the purely spatial metric as the induced metric on the hypersurface orthogonal to v^μ, $h_{\mu\nu}$. Now, take the covariant derivative of the tangent vector $\nabla_\nu v_\mu$ and decompose it into its trace, symmetric trace-free part, antisymmetric part

$$\theta = \nabla_\mu v^\mu, \tag{1.5.6}$$

$$\sigma_{\mu\nu} = \nabla_{(\nu} v_{\mu)} - \frac{1}{3}\theta h_{\mu\nu}, \tag{1.5.7}$$

$$\omega_{\mu\nu} = \nabla_{[\nu} v_{\mu]}, \tag{1.5.8}$$

which are respectively called *expansion scalar*, *shear tensor* and *rotation tensor*. Therefore the covariant derivative of the tangent vector field reads

$$\nabla_\nu v_\mu = \frac{1}{3}\theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}. \tag{1.5.9}$$

We want to find an evolution equation for θ because it is the parameter related to the expansion, to the change of "volume" and its evolution will tell us if the geodesics are converging or diverging. In order to find it we start with

$$\begin{aligned} v^\sigma \nabla_\sigma \nabla_\nu v_\mu &= v_{\mu;\nu\sigma} v^\sigma = \left(v_{\mu;\sigma\nu} - R_{\mu\rho\nu\sigma} v^\rho \right) v^\sigma \\ &= \left(v_{\mu;\sigma} v^\sigma \right)_{;\nu} - v_{\mu;\sigma} v^\sigma_{;\nu} - R_{\mu\rho\nu\sigma} v^\rho v^\sigma \\ &= - - v_{\mu;\sigma} v^\sigma_{;\nu} - R_{\mu\rho\nu\sigma} v^\rho v^\sigma , \end{aligned} \tag{1.5.10}$$

where in the second line we used the property that $v_{\mu;\sigma\nu} - v_{\mu;\nu\sigma} = R_{\mu\rho\nu\sigma} v^\rho$ and in the last line the geodesic equation $v^\mu_{;\nu} v^\nu = 0$. The evolution equation for θ is obtained by taking the trace of the above expression

$$\begin{aligned} \frac{d\theta}{d\tau} &= -\nabla^\nu v^\mu \nabla_\nu v_\mu - R_{\mu\nu} v^\mu v^\nu \\ &= -R_{\mu\nu} v^\mu v^\nu - \frac{1}{3}\theta^2 - \sigma^{\mu\nu}\sigma_{\mu\nu} + \omega^{\mu\nu}\omega_{\mu\nu} \\ &= -R_{\mu\nu} v^\mu v^\nu - \frac{1}{3}\theta^2 - 2\sigma^2 + 2\omega^2 . \end{aligned} \tag{1.5.11}$$

This is the *Raychaudhuri equation* for a congruence of timelike geodesics and describes the rate of change of the volume expansion as the curves are moved along.

Energy Conditions

The first term in the Raychaudhuri equation opens another argument which plays a fundamental role in the singularity theorems. In fact, rewriting it using the Einstein equations we obtain

$$R_{\mu\nu} v^\mu v^\nu = 8\pi G_{\rm N} \left[T_{\mu\nu} v^\mu v^\nu + \frac{1}{2} T \right] . \tag{1.5.12}$$

We recognize that the term $T_{\mu\nu} v^\mu v^\nu$ is the energy density as measured by an observer moving on a timelike geodesic with four velocity v^μ. It seems that, in order to determine the occurrence of singularities, we need to know the exact energy-momentum tensor but this is obviously too much complicated and too restrictive as pointed in [22]. Fortunately, the singularity theorems only need some physically reasonable energy conditions on the term (1.5.12) as they will be independent on the exact form of the energy-momentum tensor. We can list some of them on which those theorems are based. For any timelike vector field we have

- *weak energy condition* $T_{\mu\nu}v^\mu v^\nu \geq 0$.

  This is a reasonable assumption as it is equivalent to saying that the energy measured by any physical observer is non-negative.

- *strong energy condition* $T_{\mu\nu}v^\mu v^\nu \geq -\frac{1}{2}T$.

  This condition is stronger because it states that the matter stresses are not large enough to make the right hand side of (1.5.12) negative.

- *dominant energy condition* $T_{\mu\nu}v^\mu v^\nu \geq 0$ and $T^{\mu\nu}v_\mu$ is a non-spacelike vector.

  It can be equivalently stated that the energy density is non-negative and dominates the other components of the energy-momentum tensor (pressure).

The first two conditions can be extended to the case of null vectors.

Conjugate Points and Gravitational Focusing

The strong energy condition inserted in the Raychaudhuri equation (1.5.11), shows intuitively that the effect of matter on the spacetime curvature causes a focusing effect in the congruence of timelike geodesics due to the attractive nature of the gravitational interaction. We now study in more detail this effect starting with the general analysis of the separation between two neighbouring geodesics of the congruence. If we name the separation z^μ, this follows the geodesic deviation equation [1]

$$\frac{D^2 z^\mu}{D\tau^2} = -R^\mu_{\nu\rho\sigma}v^\nu z^\rho v^\sigma. \tag{1.5.13}$$

The solutions of this equation are called *Jacobi fields*. We will say that two points p and q along a non-spacelike geodesic γ are *conjugate points* if a Jacobi field exist along γ which does not identically vanish but which is zero both at p and q. In other words, the two points are conjugate if there is a neighbouring geodesic to γ which meets it at p, q. Thus, the occurrence of such points on a geodesic is a clear signal that $\theta \to -\infty$. In order to understand this close relation lets go back to Raychaudhuri equation where we now consider an irrotational (i.e. $\omega_{\mu\nu} = 0$) congruence of timelike geodesics and a spacetime satisfying the strong energy condition. It follows that

$$\frac{d\theta}{d\tau} \leq -\frac{\theta^2}{3}, \tag{1.5.14}$$

whence it is clear that the expansion scalar is decreasing in this case along the geodesic and we can integrate the above expression to find

$$\theta^{-1} \geq \theta_0^{-1} + \frac{\tau}{3}, \tag{1.5.15}$$

where θ_0 is the initial value of the expansion. Therefore, if the congruence is initially converging with θ_0 negative, then $\theta \to -\infty$ in a finite proper time interval $\tau \leq 3/|\theta_0|$. This makes more clear why we said that the occurrence of conjugate points is related to the divergence of the expansion scalar. In fact, the following holds

Proposition 1.5.1.
If at some point p of the timelike geodesic γ the expansion θ has a negative value $\theta_0 < 0$ and if the strong energy condition holds $R_{\mu\nu}v^\mu v^\nu \geq 0$ everywhere, then there will be a point q conjugate to p along γ between $\gamma(\tau_0)$ and $\gamma(\tau_0 - 3/\theta_0)$, provided γ can be extended to this parameter value.

These results holds similarly for null geodesics (see [24]). The basic implication of the above achievements is that, once a convergence occurs in a congruence of the timelike geodesics, the conjugate points must develop in the spacetime. These will be interpreted as the singularities of the congruence.

Generic Condition

We can now prove that the existence of conjugate points in a timelike geodesic still holds under weaker conditions. In fact, the assumption of the strong energy condition together with negativity of the expansion scalar at some point, is equivalent to saying that we must have $R_{\mu\nu}v^\mu v^\nu > 0$ at a point $r \in \gamma$. Anyway, in all our analysis of convergence of the geodesics, we left apart the fact that also the shear tensor $\sigma_{\mu\nu}$ contributes at the decreasing behaviour of the expansion θ. Therefore, we understand that even if $R_{\mu\nu}v^\mu v^\nu = 0$ everywhere, we can still draw the above conclusions provided that at some point the shear tensor is different from zero. Actually, it can be shown, [18], that this condition is verified if

$$R_{\mu\nu\alpha\beta}v^\nu v^\beta \neq 0, \tag{1.5.16}$$

at at least one point of the geodesic. That it contributes as a negative term is guaranteed by the fact that it appear as $-\sigma^2$ in equation (1.5.11). We will say that a spacetime satisfy the *timelike generic condition* if each geodesic has some point at which the above requirement is satisfied. Hence, all what is necessary for the existence of conjugate points on the timelike geodesic γ, is that $R_{\mu\nu}v^\mu v^\nu \geq 0$ and $R_{\mu\nu\alpha\beta}v^\nu v^\beta \neq 0$ at one point at least (the proof of this statement can be found in [24]).

Similarly in the case of null geodesic congruence, we can define the *null generic condition* saying that a spacetime satisfies it if every null geodesic possesses at least one point where

$$k_{[\gamma}R_{\mu]\nu\alpha[\beta}k_{\delta]}k^{\nu}k^{\alpha}, \tag{1.5.17}$$

where k^{μ} is the null tangent vector. The results are the same as in the timelike case with this condition. Systems which does not satisfy the generic conditions are very special and can be considered as not physiscally relevant.

1.5.4 Penrose and Hawking-Penrose Singularity Theorems

We have developed so far the minimum objects needed to understand the singularity theorems. Here, we will only focus on the Penrose singularity theorem [21] and the Hawking-Penrose singularity theorem [23], which are respectively the first to be obtained without assumption of symmetry and the last and more general one.

The Penrose theorem applies to spacetime singularities due to a gravitational collapse. It shows that once a star passes inside its Schwarzschild surface it can not come out again. Of course the Schwarzschild surface is defined only in the case of exact spherical symmetry, but Penrose introduced a more general criterion applicable also in more general situations without exact symmetry. It is the existence of a *closed trapped surface* T, defined as a closed C^2 spacelike two-surface, such as both "ingoing" and "outgoing" future directed null geodesics orthogonal to T are converging at T. The last property can be restated as the requirement that the expansion scalar θ of those geodesics is everywhere negative. Roughly speaking, we can think about this definition as a situation in which T is in a such strong gravitational field that even the outgoing light rays are converging on it, therefore the matter inside this surface is trapped in a succession of always smaller surfaces. This is stated rigorously in Penrose theorem

Theorem 1.5.2 (Penrose 1965)**.**
A spacetime (M, g) cannot be null geodesically complete if the following holds:

1. *The strong energy condition is satisfied for every null vector k^{μ}, i.e. $R_{\mu\nu}k^{\mu}k^{\nu} \geq 0$;*
2. *There is a non-compact Cauchy surface Σ in M (which implies that M is globally hyperbolic);*
3. *There is a closed trapped surface T in M.*

The statement that the spacetime cannot be geodesically complete can be translated in terms of the expansion parameter [18], by saying that if $\theta_0 < 0$ is the maximum value of θ for both null geodesics, then at least one inextendible future directed orthogonal null geodesic from T, has affine lenght no greater than $2/|\theta_0|$.

This theorem tells us that in a collapsing star either a singularity or a Cauchy horizon occur, but assumes the very restrictive condition of global hyperbolicity, which does not seem to be realistic in many cases (such as an everywhere expanding universe). In order to describe a more general situation we will state the Hawking-Penrose theorem

Theorem 1.5.3 (Hawking-Penrose 1970)**.**
A spacetime (M, g) is not timelike and null geodesically complete if the following conditions hold:

1. $R_{\mu\nu}k^\mu k^\nu \geq 0$ *for every non-spacelike vector* k^μ*;*

2. *The generic condition* (1.5.17) *is satisfied for every non-spacelike geodesic where* k^μ *is the tangent vector to the geodesic;*

3. *There are no timelike closed curves (i. e. the chronology condition holds);*

4. *There exists at least one of the following:*

   - *a compact achronal set without edge;*
   - *a closed trapped surface;*
   - *a point p such that on every past(or future) null geodesic from p, the divergence θ of the null geodesics from p becomes negative (which means as already seen that the geodesics start converging, they are focused by matter).*

We immediately notice that this statement is not restricted to the existence of a trapped surface but include other situations and what is more, it is not grounded on the assumption of global hyperbolicity but rather subtitutes it with the requirement that the generic condition must be satisfied.

Chapter 2

Classicalization and Quantum Corpuscular Black-Holes

Since the time of Rutherford's experiments, the guiding principle of fundamental physics has been the understanding of nature at different length scales, directly related to some energy scales. The problem of ultraviolet divergences of the theory naturally arises in this perspective and their cancellation is essential for a theory to yield any physical prediction. The renormalization group approach is based on the study of ultraviolet divergences by isolating the dependence of the theory on the short-distance degrees of freedom of the field. In other words, we have to carry out the integration over high-momentum degrees of freedom leading to an effective Lagrangian, whose coupling parameters will contain corrections coming out from the integration, showing the scale-dependence of the parameters of the theory. This is an iterative procedure which stops when the energy scale of the experiment we are thinking about is reached. Renormalizable theories usually possess a fixed point in this iteration represented by the free field theory. When we have completed this operation, we will be left with some new, weakly coupled degrees of freedom, i.e. an effective Lagrangian. These theories will be referred to as UV-complete in a Wilsonian way, [29]. This brief introduction to renormalization only means to stress the key points of how the UV divergences of well defined theories can be cured through the standard Wilsonian method. A full description can be found in [30].

Unfortunately, as it is well known, Einsteinian gravity is not renormalizable within the framework of Wilsonian renormalization. However, as recently pointed out by Dvali et al. [31], there is a possibility for gravity to be self UV-complete through the phenomenon of *classicalization* [31–34], whose central idea is that some theories prevent themselves form entering into the strong coupling regime, by sharing the total energy among many weakly-interacting soft quanta, represented by the IR degrees of freedom of the theory itself. The underlying concept which paved the way to this novel approach was the statement that in Einstein's gravity, the Planck length $\ell_{\rm p}$ represents an absolute lower bound, any distance $\ell < \ell_{\rm p}$ can never be resolved, in principle. This argumentation also goes under the name of *generalized uncertainty principle* [47–50].

Within this picture, Dvali and Gomez introduced a new quantum model for black holes described as Bose-Einstein condensates of weakly interacting quanta [35–39]. Their existence will be justified in the later sections but the significance of a quantum model for black holes is already clear. In fact, the singularity theorems described in the previous chapter (Section 1.5), show that in classical general relativity the inevitable end of a gravitational collapse is the central singularity. The following historical step is the semiclassical model based on the quantization on curved spacetime [28], where one studies small fluctuations about a background geometry, which is treated as an intrinsically classical entity. It is well known that in this semiclassical framework, some new features of the black holes arise, such as the Hawking radiation [44] and the Bekenstein entropy [45, 46], but it unavoidably carries some pathologies, i.e. the information paradox and the ambiguity in the origin of the Hawking radiation. What is more, it does not cure the problem of the singularity. The completely quantum model by Dvali and Gomez provides a frame in which black holes have no central singularity and in which Hawking radiation acquires a clear quantum origin in terms of the known phenomenon of quantum depletion of a Bose-Einstein condensate.

In the following we will neglect all constant terms of order one to avoid unnecessary complications.

2.1 Planck-Scale as a Boundary between two Worlds

We are now going to study in more detail the statement that the Planck length, defined as

$$\ell_{\rm p} = \sqrt{\hbar G_{\rm N}}, \tag{2.1.1}$$

is the fundamental quantum scale in gravity. It is sometimes understood as the length scale at which quantum fluctuations of the spacetime metric cannot be ignored. The corresponding mass scale is obviously the Planck mass

$$m_{\mathrm{p}} = \frac{\hbar}{\ell_{\mathrm{p}}} = \sqrt{\frac{\hbar}{G_{\mathrm{N}}}}. \tag{2.1.2}$$

It is necessary to introduce the two others lenght scales associated to a particle of mass M, namely the Compton wavelength

$$\ell_c = \frac{\hbar}{M}, \tag{2.1.3}$$

and the gravitational(Schwarzschild) radius

$$R_{\mathrm{H}} = 2G_{\mathrm{N}}M. \tag{2.1.4}$$

The first one is an intrinsic quantum scale, as the Planck lenght is, in fact it vanishes in the classical limit $\hbar \to 0$. It sets the length scale at which the energy of quantum fluctuations exceeds the mass(energy) of the particle. On the other hand, the gravitational radius is clearly independent of $\hbar$ and it is thus a classical length scale. It represents the distance at which the gravitational effects of a localized source become strong. Actually, it is well known form the classical theory of general relativity that a source localized within its Schwarzschild radius becomes a black hole.

Having these definitions in mind, we will argue that the Planck mass represents the boundary between the elementary particle world and black holes. Consider two different regimes, one in which the particle is heavier than the Planck mass and the opposite. We now try to understand which are the first dominant effects we deal with when approaching to the source of mass M.

- $M < m_{\mathrm{p}}$

  Of course this case implies that $\ell_c > \ell_{\mathrm{p}} > R_{\mathrm{H}}$. Therefore the Compton length represents the leading scale. This means that the gravitational effects are shielded by the Compton wavelength, i.e. if we consider the Newtonian potential of a mass $M \ll m_{\mathrm{p}}$ at a distance $r \sim \ell_c$, we find that its absolute value satisfies

  $$V_{\mathrm{N}} \simeq \frac{G_{\mathrm{N}}M}{\ell_c} \simeq \frac{R_{\mathrm{H}}}{\ell_c} \ll 1. \tag{2.1.5}$$

  Hence, the quantum effects become important way before we can approach the Schwarzschild radius of the source. This is why we talk about elementary particles in this regime.

- $M > m_{\rm p}$

  Now the situation completely switch as we have $\ell_c < \ell_{\rm p} < R_{\rm H}$ and the prevailing lenght scale is the gravitational radius. The same arguments of the previous point lead us to the conclusion that we are facing a (semi)classical black hole

Equal conclusions are reached through a thought experiment. Suppose we want to resolve sub-planckian distances $\ell \ll \ell_{\rm p}$, in a scattering experiment. Such a measurement involve localizing at least a total amount of energy $\hbar/\ell$ in a box of size ℓ. If we were able to do this, the Scharzschild radius of such a localized source would be

$$R_{\rm H} = 2G_{\rm N}M \simeq \frac{\ell_{\rm p}}{m_{\rm p}}\frac{\ell_{\rm p} m_{\rm p}}{\ell} \simeq \ell_{\rm p}\frac{\ell_{\rm p}}{\ell} \gg \ell_{\rm p}. \tag{2.1.6}$$

Hence, any attempt to resolve distances shorter than $\ell_{\rm p}$ require a localization of energy in a region whose dimension is much smaller than the corresponding Schwarzschild radius. Therefore, the measurement will inevitably lead to the formation of a classical black hole and we will "detect" it much before being able to probe the distance ℓ.

2.2 Self-Completeness of Gravity

The conclusions of last section open the possibility of a completely new and non-Wilsonian UV-completion for gravity, i.e. they suggest the idea that gravity is *self-complete* in deep UV [31]. In other words, no new propagating degrees of freedom are needed to describe the UV behaviour of gravity. Indeed, we have just shown that the classical black holes represent an obstacle to the existence of propagating quantum states above the Planck energy scale. Even if we admit their presence, the corresponding distances could never be probed due to the insuperable barrier produced by the black hole and these states would have no physical meaning. In particular, all the information about trans-planckian ($M \gg m_{\rm p}$) gravity is actually encoded by the semiclassical macroscopic black holes of the same mass M and any attempt to resolve physics at distances shorter than the Planck length inevitably bounces back to much larger distances,

$$\ell \leftrightarrow \frac{\ell_{\rm p}^2}{\ell}. \tag{2.2.1}$$

This relation shows that our attempts to go deeper in the UV domain have the only result of giving back a heavier(and hence larger thanks to definition (2.1.6)) black hole. Therefore, the more we try to probe UV physics the more we end up describing (semi)classical IR gravity, represented by the black hole behaviour. We are here implicitly assuming that Einsteinian gravity is a fully consistent theory in the IR, whose only propagating degree of freedom is the massless graviton, otherwise we would encounter inconsistencies even at low energies which would require the introduction of new light degrees of freedom. A close connection between the two energy domains can thus be established in gravity

$$\text{Deep-UV Gravity} \quad \Longleftrightarrow \quad \text{Deep-IR Gravity.} \tag{2.2.2}$$

The fact that trans-planckian gravity is fully described by its light degrees of freedom seems to be consistent as it resists to many deformations of the theory, i.e. to different attempts to introduce trans-planckian poles in the graviton propagator (see, e.g. [34]).

2.3 Quantum Corpuscular Black Holes

In the previous sections we intentionally left apart one crucial aspect intrinsic in this argumentation, the existence of purely quantum black holes. Actually, their attendance is a built in property within this framework as it straightly comes out when one wonder if the above connection between IR and UV gravity still holds near the Planck scale. Let's go back to the example of the mass in the two regimes (elementary particle and black hole) and try to understand which should be the degrees of freedom at the Planck scale. In order to achieve this, we start with a classical black hole of mass $M \gg m_{\mathrm{p}}$ and take advantage of the evaporation property of the black holes to see it shrinking down to the Planck scale. A quantum state of mass $M \sim m_{\mathrm{p}}$ at this point should be reached since it is impossible to cross the two regimes of semiclassical black holes and elementary particles without passing through an intermediate quantum state (a sharp quantum resonance). In the light of the three fundamental lenght scales defined before, we see that in this case $\ell_c = \ell_{\mathrm{p}} = R_{\mathrm{H}}$ which shows that states with $M \sim m_{\mathrm{p}}$ are on the boundary of the two worlds. They are strongly gravitating objects at a distance $\ell \sim \ell_c$, namely we cannot consider them as semiclassical black holes because quantum fluctuations are fully important nor elementary particles because gravitational effects are fundamental too, this is why we call them quantum black holes. The aim of this discussion is only to stress the existence of quantum states of mass around m_{p} as a built in property of Einstein's gravity spectrum. This is the main reason for trying to find a completely quantum model describing black holes. In fact, this framework paved the way to the new perspective on the quantum aspects of black holes physics recently proposed by Dvali and Gomez [36–39]. Their picture describes the black hole as a Bose-Einstein condensate of N very weakly interacting soft gravitons and does not rely at all on the classical geometric features such as horizon. This very simple idea leads to some really fascinating conclusions, first of all the fact that this model will be fully characterized by the number N.

Therefore a key ingredient of this black hole quantum portrait is its understanding in terms of the graviton number N. In order to do so, consider a pure gravitating source of mass M. In this picture the role of matter is completely neglected and only serves to understand how the compact source formed but it will play no role for the time being (we will come back on this feature in Chapter 3). Suppose that such a source is spherical and of radius R well above its Schwarzschild radius, $R \gg m_{\mathrm{p}}$. The starting point of all the construction is that the gravitational field produced by M is well described by the Newtonian potential

$$V_{\mathrm{N}}(r) = -\frac{G_{\mathrm{N}} M}{r}, \tag{2.3.1}$$

viewed as a superposition of non propagating gravitons(which we can think as a Bose-Einstein condensate). We now show, in line with [36], that the situation is very different if we consider $R \gg R_{\mathrm{H}}$ or if R approaches and eventually crosses the gravitational radius. In fact, until $R \gg R_{\mathrm{H}}$ the gravitons have very long wavelengths and are thus weakly interacting. This allows us to completely neglect interactions between them and also the interaction of one graviton with the collective gravitational energy. In other words, we are assuming that in this regime all gravitational self-interactions can be discarded. Therefore, in this case the condensate obviously cannot be self-sustained and it will dissipate. Actually, unless we consider some matter contribution, there is no reason why it should even form. On the other hand, it seems quite reasonable that when R approaches R_{H}, we can still ignore the interactions between individual gravitons but the gravitational energy becomes much bigger and we have to take the self-sourcing due to the collective gravitational energy into account. The fundamental assumption is that this interaction is strong enough to confine the gravitons inside a finite volume, i.e. the condensate is self-sustained at this point. In order to show how this construction works, it is usefull to make energy considerations about this system. First, we can associate an effective mass m to the gravitons via the Compton wavelength $\lambda = \hbar/m = \ell_{\mathrm{p}} m_{\mathrm{p}}/m$, thanks to their localization. The total energy will therefore be written as $M = Nm$. We can now find another key ingredient of this picture, namely the effective gravitational coupling of the collective interaction with one graviton

$$\alpha = \frac{|V_{\rm N}(R = \lambda)|}{N} = \frac{\ell_{\rm p}^2}{\lambda^2} = \frac{m^2}{m_{\rm p}^2}, \tag{2.3.2}$$

this allows to find the average potential energy per graviton as

$$U \simeq m V_{\rm N}(\lambda) \simeq -N\alpha m. \tag{2.3.3}$$

We imagine the black hole will form when the energy of the graviton $E_K \simeq m$ is just below the amount needed to escape the potential well, this yields the condition

$$E_K + U = 0, \tag{2.3.4}$$

and translates into

$$N\alpha = 1, \tag{2.3.5}$$

called by the authors [36], *maximal packing*. It is now immediate to find the scaling relations of all the parameters of the theory in terms of N. First of all, the mass of the gravitons can be written as $m = m_{\rm p}/\sqrt{N}$ then the total mass and the gravitons wavelength

$$M = \sqrt{N} m_{\rm p}, \tag{2.3.6}$$

$$\lambda = \sqrt{N} \ell_{\rm p}. \tag{2.3.7}$$

An important property we will rely on in the following is that now the horizon's size, represented by the Schwarzschild radius is clearly of the same order as the Compton wavelength of the gravitons $\lambda \simeq R_{\rm H}$.

These relations show the striking result that the emerging quantum picture of the black hole is fully parametrized by N. The connection with the Bose-Einstein condensate is better established in [39] in comparison with the studies on quantum phase transitions in cold atomic systems with attractive interaction [41–43]. Actually, the results of these papers demonstrate that in presence of an attractive interaction, a BEC of fixed size undergoes a phase transition above a critical value of the occupation number N (given by a condition equivalent to the maximal packing), resulting in an instability of the system. At this point, the Bogoliubov modes become almost degenerate with the ground state, as the energy gap falls down as $1/N$. Making a direct comparison between these systems and the just defined model for quantum black holes, we are able to show that the most mysterious properties of black holes (we will mainly focus on Hawking radiation [44] and Bekenstein entropy [45, 46]), are spontaneous consequences of well known phenomenons from condensed matter physics. The central point of this similarity, is that the maximal packing condition sets the equivalence with BEC's at the critical point of a quantum phase transition and quantum black holes.

2.3.1 Hawking Radiation

The above results now help us finding a truly quantum origin of Hawking radiation, identified in an effect analogous to the quantum depletion of the Bose-Einstein condensate, according to which in a BEC of interacting bosons there are always some particles with energies above the ground state. In fact, in our case the black hole is a leaky bound-state, as the escape energy is just above the energy of the condensed quanta. Hence, a graviton can obtain such an amount of energy by scattering with the background potential. The underlying dominant process is the $2 \to 2$ graviton scattering as it is the most probable one. These gravitons will therefore leak out and join the continuum spectrum (a quantum state for the black hole describing this situation and the emergent Hawking radiation has been found in [52, 53]). At first order, these scatterings give rise to a depletion rate,

$$\Gamma \simeq \frac{1}{N^2} N^2 \frac{\hbar}{\sqrt{N}\ell_{\mathrm{p}}} + \mathcal{O}(N^{-1}), \tag{2.3.8}$$

where the first factor comes from the interaction strength ($N^{-2} = \alpha^2$), the second factor N^2 is combinatoric since we have N gravitons interacting with $N - 1 \simeq N$ gravitons and the third one comes from the characteristic energy of the process. This rate sets the time scale $\triangle t = \hbar \Gamma^{-1}$ during which a graviton with escape energy is emitted from the black hole and allows us to find the law which describes the decrease of the graviton number due to quantum depletion, which at first order is given by the $2 \to 2$ scatterings,

$$\dot{N} \simeq -\hbar^{-1}\Gamma = -\frac{1}{\sqrt{N}\ell_{\mathrm{p}}} + \mathcal{O}(N^{-1}). \tag{2.3.9}$$

This again shows that during a time $\triangle t = \sqrt{N}\ell_{\mathrm{p}}$ the condensate emits one graviton. As explained in [36], this emission reproduces the Hawking radiation, or at least its purely gravitational part, and accordingly leads to the standard decrease in the black hole mass

$$\dot{M} \simeq \frac{m_{\mathrm{p}}}{\sqrt{N}}\dot{N} = -\frac{m_{\mathrm{p}}}{N\ell_{\mathrm{p}}} = -\frac{m_{\mathrm{p}}^3}{\ell_{\mathrm{p}}}\frac{1}{M^2}. \tag{2.3.10}$$

Now, defining the temperature as

$$T = \frac{\hbar}{\sqrt{N}\ell_{\mathrm{p}}} = \frac{m_{\mathrm{p}}}{\sqrt{N}}, \tag{2.3.11}$$

it can be identified as the Hawking temperature as it puts the above equations in the usual Hawking expression for the evaporation rate

$$\dot{M} = -\frac{T^2}{\hbar}. \tag{2.3.12}$$

Therefore, we have seen that the Hawking temperature and radiaton emerge as a direct consequence of the phenomenon of quantum depletion of a Bose-Einstein condensate, applied to the case where the bosons are weakly coupled gravitons, without relying at all on classical notions of geometry. To be more precise, the Hawking radiation is recovered only in the semiclassical limit

$$N \to \infty, \quad \ell_{\mathrm{p}} \to 0, \quad \lambda = \sqrt{N}\ell_{\mathrm{p}} = \text{finite}, \quad \hbar = \text{finite}. \tag{2.3.13}$$

In fact, the number of gravitons N measures the level of classicality of the source as pointed in [36, 39].

As a final remark to this feature of the quantum black hole model, we wish to stress that even if the depletion obviously leads to a decrease in the number of gravitons N, this does not take the system out of the critical point, as it would instead be for a BEC of cold atoms, because the coupling parameter α depends on N in a way that ensures the maximal packing condition is preserved.

2.3.2 Entropy

As anticipated at the beginning of this section, we wish to interpret the Bekenstein entropy of the black hole within this model. This allows us to write it in terms of the number N of soft gravitons in the condensate. It is convenient to work with the analogy discussed in [39], where the black hole is described in complete similarity with the Bose-Einstein condensate of cold atoms as already explained before. In this point of view, the entropy of the system is of course related to the number of states in which the N gravitons can be. Regarding this, we already mentioned that at the critical point, the Bogoliubov modes of the condensate become almost degenerate with the ground state as their energy gap falls as $1/N$. A similar behaviour is in fact found also in [52, 53] where the quantum state of the system is written explicitly, splitting it in a discrete specrtum part (ground state) and a continuum spectrum part (Bogoliubov modes). These are the modes responsible for the quantum depletion (leading to the Hawking radiation in the semiclassical limit) and the entropy of the system, understood in terms of the degeneracy of the BEC state. With this situation in mind, it is easy to see qualitatively that the number of states will have an exponential dependence on N. In fact, at the critical point the number of quasi degenerate states is of order N, this means we can define nearly N Bogoliubov quasi zero modes. The number of states for the N graviton will therefore be somewhat exponential in N and the entropy

$$S \simeq \log n_{\text{states}} \simeq N. \tag{2.3.14}$$

It can be easily shown, that the square Schwarzschild radius has linear dependence on N, hence this result is in qualitative agreement with the Bekenstein formula where entropy scales with the horizon area. A more quantitative result again has been shown in [52].

2.4 Classicalization

After having discussed the quantum corpuscular black hole model, we can finally understand the idea of self-completeness of Einstein gravity by classicalization in a quantum field theoretic point of view. We already said at the beginning of this chapter that classicalization is a process with which a theory prevents itself from entering the strong coupling regime by redistributing the energy among many weakly interacting quanta. This would open a new possibility for UV-completion and to understand the deep difference with other scenarios, we can take as example the QCD. First, remember that in quantum field theory the strength of the interaction between elementary particles is encoded in a quantum coupling $\alpha(E)$ which depends on the energy scale fixed by the experiments. This means that the perturbative approach is valid until $\alpha \ll 1$ but at a certain energy scale Λ, the interaction becomes strong and scattering amplitudes violate unitarity. In QCD this situation is solved by introducing completely new degrees of freedom (the quarks) that help finding a consistent description.

Classicalization works in a completely different way, i.e. the theory employes the same IR degrees of freedom,which we call X, used in the weakly coupled domain. In the case of gravity this means that the theory will still be understood in terms of gravitons alone. As already discussed in section 2.1 and 2.2, the role of Λ in gravity is taken up by the Planck mass m_p which separates the two worlds of elementary particles and classical states such as black holes. This already shows a striking difference with respect to the QCD case : the macroscopic black holes are not independent quantum particles, but rather multi-gravitons states as shown in section 2.3. However, there is no reason why classicalization should be characteristic of gravity alone. In fact, Dvali and Gomez recently proposed [32] that the same construction can be generalized to other field theories extending the reasoning applied to general relativity. This is why we are going to describe how it works in general terms, without referring explicitly to gravitons and black-holes. Consider a scattering experiments involving the collision of two X quanta with center of mass energy $\sqrt{s} \gg \Lambda$. The coupling at such energy will be strong $\alpha(\sqrt{s}) \gg 1$ so unitarity seems to be violated in this process and the perturbation theory in α breaks down. The problem comes from the fact that the energy exchange per quanta is too high. The proposed idea of classicalization is that the system itself solves this problem by turning the two particle scattering into a multi-particle process

$$2X \to 2X \quad \Longrightarrow \quad 2X \to NX. \tag{2.4.1}$$

In other words, the system replaces the elementary two particle scattering in a process composed of many elementary scatterings where the momentum exchange per-quantum is small enough to make the coupling weak. This is what we meant by saying that the system prevents itself from entering in the strong regime. The total energy $\sqrt{s}$ will be distributed among N quanta instead of two, where the number N must satisfy the condition

$$\frac{\sqrt{s}}{N} < \Lambda \quad \Longleftrightarrow \quad \alpha\left(\frac{\sqrt{s}}{N}\right) < 1. \tag{2.4.2}$$

It follows that the higher $\sqrt{s}$ is, the more quanta will be needed in order for classicalization to work as UV-completion.

In the case of gravity, this mechanism seems to work well because high energy scatterings in gravity are dominated by black holes which we described as many gravitons states. Various works on trans-Planckian scattering followed the pioneering papers [55–58] and even if we will not enter into the technical subtleties of this problem, it is important to have in mind that black holes can in principle be produced by particle collisions. We can only qualitatively describe the process considering a two particles collision with trans-Planckian center of mass energy, $\sqrt{s} \gg m_\mathrm{p}$. If there are no long range repulsive forces acting between the particles and if the impact parameter is less than the gravitational radius corresponding to the center of mass energy, $R_{\rm H} = 2G_{\rm N}\sqrt{s}$, the initial energy will be localized within a region whose radius is smaller than $R_{\rm H}$. Therefore, a black hole is expected to form. As shown in [40], classicalization perfectly fits this process and in particular the process of black holes production in high energy scattering can be understood to be the effect of classicalization.

Chapter 3

Quantum Corpuscular Corrections to the Newtonian Potential

In the Newtonian theory, energy is a well-defined quantity and is conserved along physical trajectories (barring friction), which ensures the existence of a scalar potential for the gravitational force. However, we showed in Section 1.3.2 that in general relativity the very concept of energy becomes much more problematic (see also, e.g. [59] and References therein for a recent discussion) and there is no invariant notion of a scalar potential. Even if one just considers the motion of test particles, the existence of conserved quantities along geodesics requires the presence of Killing vector fields. In sufficiently symmetric space-times, one may therefore end up with equations of motion containing potential terms, whose explicit form will still depend on the choice of observer (time and spatial coordinates). Overall, such premises allow for a "Newtonian-like" description of gravitating systems with strong space-time symmetries, like time-independence and isotropy, which can in turn be quantised by standard methods [60, 61].

We are here particularly interested in static and isotropic compact sources, for which one can indeed determine an effective theory for the gravitational potential, up to a certain degree of confidence. As explained in Section 1.2, when the local curvature of space-time is weak and test particles propagate at non-relativistic speed, non-linearities are suppressed. The geodesic equation of motion thereby takes the form of the standard Newtonian law with a potential determined by the Poisson equation (1.2.22), and Post-Newtonian corrections can be further obtained by including non-linear interaction terms (see Section 1.2.3). The inclusion of these non-linear terms in the quantum effective description of the gravitational potential is precisely what we are going to address in this work, following on the results of Ref. [54].

One of the motivations for this study is provided by the corpuscular model of gravity recently theorised by Dvali and Gomez [35–39] and outlined in Chapter 2. We recall that according to this model, a black hole is described by a large number of gravitons in the same (macroscopically large) state, thus realising a Bose-Einstein condensate at the critical point [51–53]. In particular, the constituents of such a self-gravitating object are assumed to be marginally bound in their gravitational potential well, whose size is given by the characteristic Compton-de Broglie wavelength $\lambda \sim r_g$, where we rewrite the Schwarzschild radius of the black hole of mass M as [1] $R_H = 2\ell_p M/m_p$, and whose depth is proportional to the very large number $N_G \sim M^2/m_p^2$ of soft quanta in this condensate. In the original proposal depicted in Chapter 2, the role of matter was argued to be essentially negligible by considering the number of its degrees of freedom subdominant with respect to the gravitational ones, especially when representing black holes of astrophysical size.

When the contribution of gravitons is properly related to the necessary presence of ordinary baryonic matter, not only the picture enriches, but it also becomes clearly connected to the post-Newtonian approximation [54]. We will start with a resume of the energy balance of Ref. [54] and then refine those findings, by first deriving the effective action for a static and spherically symmetric potential from the Einstein-Hilbert action in the weak field and non-relativistic approximations. We shall then show that including higher order terms yields classical results in agreement with the standard post-Newtonian expansion of the Schwarzschild metric (see Section 1.2.4) and a quantum picture overall consistent with the one recalled above from Ref. [54]. We remark once more this picture is based on identifying the quantum state of the gravitational potential as a coherent state of (virtual) soft gravitons, which provides a link between the microscopic dynamics of gravity, understood in terms of interacting quanta, and the macroscopic description of a curved background.

3.1 Energy Balance

The basic idea is very easy to explain. Suppose we consider a spherically symmetric and compact stellar object composed of N baryons of rest mass μ. We have shown in Section 1.3.2 that in asymptotically flat spacetime, the Hamiltonian constraint associated with

[1] We shall mostly use units with $c = 1$ and the Newton constant $G_N = \ell_p/m_p$, where ℓ_p is the Planck length and m_p the Planck mass (so that $\hbar = \ell_p\, m_p$).

the freedom of time reparametrization leads to energy conservation. If we name $H_{\rm B}$ and $H_{\rm M}$ the Hamiltonian of matter and pure gravity obtained by varying the action with respect to the lapse function (see Section 1.3.2), the Hamiltonian constraint then takes the form

$$H \equiv H_{\rm B} + H_{\rm G} = M, \tag{3.1.1}$$

where M is the ADM energy of the system which emerges from boundary terms. Now, if the N baryons are initially very far apart, their total ADM energy is simply given by

$$M \simeq N\,\mu = E_{\rm B}, \tag{3.1.2}$$

As these baryons fall towards each other, while staying inside a sphere of radius R, their energy $E_{\rm B}$ will be decreased by the negative interaction potential energy $U_{\rm BG}$ and will acquire a kinetic energy $K_{\rm B}$, so that

$$E_{\rm B} = M + K_{\rm B} + U_{\rm BG} + U_{\rm BB}. \tag{3.1.3}$$

We included here the term $U_{\rm BB} \geq 0$ that account for a repulsive interaction among baryons responsible for the pression required to reach a static configuration. Therefore, from a purely classical point of view, we would obtain the classical equations of motion of the baryons $K_{\rm B}(R)+U_{\rm BG}(R)+U_{\rm BB}(R) = 0$, where we wrote the explicit R dependence. However, our aim is to find a quantum description of this model. In order to do so, we start evaluating their (negative) gravitational energy in Newtonian form

$$U_{\rm BG} \sim N\,\mu\,V_N \sim -\frac{\ell_{\rm p}\,M^2}{m_{\rm p}\,R}\ , \tag{3.1.4}$$

where $V_{\rm N} \sim -\ell_{\rm p}\,M/m_{\rm p}\,R$ is the (negative) Newtonian potential. In terms of quantum physics, this gravitational potential can be represented by the expectation value of a scalar field $\hat{\Phi}$ over a coherent state $|\,g\,\rangle$ of (virtual) gravitons,

$$\langle\,g|\,\hat{\Phi}\,|g\,\rangle \sim V_{\rm N}\ . \tag{3.1.5}$$

The immediate aftermath (see Ref. [54]) is that the graviton number $N_{\rm G}$ generated by matter inside the sphere of radius R is determined by the normalisation of the coherent state and reproduces Bekenstein's area law [45, 46], that is

$$N_{\rm G} \sim \frac{M^2}{m_{\rm p}^2} \sim \frac{R_{\rm H}^2}{\ell_{\rm p}^2}\ , \tag{3.1.6}$$

where $R_{\rm H}$ is now the gravitational radius (2.1.6) of the sphere of baryons.

Since M is conserved, the number of gravitons should be conserved too, so that $U_{\rm BG} \sim N_{\rm G}\epsilon_{\rm G}$. Therefore, assuming most gravitons have the same wave-length $\lambda_{\rm G}$ (see Section 2.3), the (negative) energy of each single graviton is correspondingly given by

$$\epsilon_{\rm G} \sim \frac{U_{\rm BG}}{N_{\rm G}} \sim -\frac{m_{\rm p}\,\ell_{\rm p}}{R} \ , \tag{3.1.7}$$

which yields the typical Compton-de Broglie length $\lambda_{\rm G} \sim R$. Now, since the gravitons self-interact, we should add this contribution to our energy balance ,

$$U_{\rm GG}(R) \sim N_{\rm G}\,\epsilon_G \,\langle\, g|\,\hat{\Phi}\,|g\,\rangle \sim \frac{\ell_{\rm p}^2\,M^3}{m_{\rm p}^2\,R^2} \ . \tag{3.1.8}$$

Hence, the gravitons self-interaction energy reproduces the (positive) post-Newtonian energy. This view is consistent with the standard lore, since the $U_{\rm GG} \ll |U_{\rm BG}|$ for a star with size $R \gg R_{\rm H}$. Furthermore, for $R \simeq R_{\rm H}$, one has

$$U(R_{\rm H}) \equiv U_{\rm BG}(R_{\rm H}) + U_{\rm GG}(R_{\rm H}) \simeq 0 \ , \tag{3.1.9}$$

which is precisely the "maximal packing" condition found in Section 2.3.

We will now proceed constructing the effective action for the gravitational potential up to the first post-Newtonian correction and study a few solutions of the corresponding classical field equation. The analogous quantum picture will then be given, analysing in details the coherent state and estimate its post-Newtonian corrections.

3.2 Effective Scalar Theory for post-Newtonian Potential

It is well known that a scalar field can be used as the potential for the velocity of a classical fluid [62]. We will show here that it can also be used in order to describe the usual post-Newtonian correction that appears in the weak field expansion of the Schwarzschild metric. It is important to recall that this picture implicitly assumes the choice of a specific reference frame for static observers (for more details, see Section 1.2.4)

As widely discussed in Chapter 1, the Einstein-Hilbert action (1.1.1) in the weak-field approximation for static fields and in the case of non-relativistic matter, leads to the well known Poisson equation (1.2.22) for the gravitational potential, i.e. to the Newtonian potential identified in $h_{00} = -2V_{\rm N}$. It is then straightforward to find an effective scalar field theory for this gravitational potential. First of all, we shall just consider (static) spherically symmetric systems, so that $\rho = \rho(r)$ and $V_{\rm N} = V_{\rm N}(r)$, correspondingly. The starting point will then be the Fierz-Pauli action (1.2.10), adapted to our scope, i.e. evaluated in the de Donder gauge (1.2.14) and in the case of a static spherically symmetric system

$$\begin{aligned}
L_{\mathrm{FP}} &= \frac{m_{\mathrm{p}}}{16\,\pi\,\ell_{\mathrm{p}}}\int \mathrm{d}^3x \left(\frac{1}{2}\,\partial_\mu h\,\partial^\mu h - \frac{1}{2}\,\partial_\mu h_{\nu\sigma}\,\partial^\mu h^{\nu\sigma} + \partial_\mu h_{\nu\sigma}\,\partial^\nu h^{\mu\sigma} - \partial_\mu h\,\partial_\sigma h^{\mu\sigma}\right) \\
&= \frac{m_{\mathrm{p}}}{16\,\pi\,\ell_{\mathrm{p}}}\int \mathrm{d}^3x \left(\partial_\mu h_{\nu\sigma}\,\partial^\nu h^{\mu\sigma} - \frac{1}{2}\,\partial_\mu h_{\nu\sigma}\,\partial^\mu h^{\nu\sigma}\right) \\
&\simeq -\frac{m_{\mathrm{p}}}{32\,\pi\,\ell_{\mathrm{p}}}\int \mathrm{d}^3x\,\partial_\mu h_{00}\,\partial^\mu h_{00} \\
&= -4\,\pi\int_0^\infty r^2\,\mathrm{d}r\,\frac{m_{\mathrm{p}}}{8\,\pi\,\ell_{\mathrm{p}}}\left(V'\right)^2 \ .
\end{aligned} \tag{3.2.1}$$

As already noticed in Section 1.2.1, this is one order higher than the corresponding equation of motion we wish to obtain and to be consistent it should be one order higher than the matter part too. In order to obtain the matter Lagrangian, we recall (see Section 1.2.2) that the energy momentum tensor is determined solely by the energy density in this non-relativistic regime

$$T_{\mu\nu}(\boldsymbol{x}) \simeq u_\mu u_\nu \rho(\boldsymbol{x}), \tag{3.2.2}$$

where $u^\mu = \delta^\mu_0$ is the four-velocity of the static source fluid. Note further that the above stress-energy tensor follows from the simple matter Lagrangian density

$$\mathcal{L}_{\mathrm{M}} \simeq -\rho(\boldsymbol{x}), \tag{3.2.3}$$

as one can see from the variation of the baryonic matter density [63]

$$\delta\rho = \frac{1}{2}\,\rho\,(g_{\mu\nu} + u_\mu\,u_\nu)\,\delta g^{\mu\nu} \ , \tag{3.2.4}$$

and the well-known formula (1.1.5). This is indeed the case of interest to us here, since we do not consider explicitly the matter dynamics but only how (static) matter generates the gravitational field in the non-relativistic limit, in which the matter pressure is negligible [62] [2].

Therefore, the matter Lagrangian reads

[2] A non-negligible matter pressure usually complicates the system significantly and is left fro a separate work

$$
\begin{aligned}
L_{\mathrm{M}} &= \int \mathrm{d}^3x \left(\sqrt{-g}\,\mathcal{L}_{\mathrm{M}}\right)_{(1)} \\
&\simeq 4\,\pi \int_0^\infty r^2\,\mathrm{d}r\,\frac{h_{00}}{2}\,\rho \\
&= -4\,\pi \int_0^\infty r^2\,\mathrm{d}r\,V\,\rho\;.
\end{aligned}
\tag{3.2.5}
$$

Finally, putting the two pieces together yields

$$
\begin{aligned}
L[V_{\mathrm{N}}] &\simeq 4\,\pi \int_0^\infty r^2\,\mathrm{d}r \left(\frac{m_{\mathrm{p}}}{32\,\pi\,\ell_{\mathrm{p}}}\,h_{00}\,\triangle h_{00} + \frac{h_{00}}{2}\,\rho\right) \\
&= 4\,\pi \int_0^\infty r^2\,\mathrm{d}r \left(\frac{m_{\mathrm{p}}}{8\,\pi\,\ell_{\mathrm{p}}}\,V_{\mathrm{N}}\,\triangle V_{\mathrm{N}} - \rho\,V_{\mathrm{N}}\right) \\
&= -4\,\pi \int_0^\infty r^2\,\mathrm{d}r \left[\frac{m_{\mathrm{p}}}{8\,\pi\,\ell_{\mathrm{p}}}\left(V_{\mathrm{N}}'\right)^2 + \rho\,V_{\mathrm{N}}\right]\;,
\end{aligned}
\tag{3.2.6}
$$

where we integrated by parts [3] and $f' \equiv \mathrm{d}f/\mathrm{d}r$. Varying this Lagrangian with respect to V_{N}, we obtain Eq.(1.2.22) straightforwardly [4].

Following the reasoning of [54], the post-Newtonian corrections come from non-linearities, therefore it seems reasonable to think that the corresponding effective action comes from improving the linear expansion to a further order. We essentially follow the same procedure of Section 1.2.1, the only differences are that we will explicitly keep track of the different orders through a parameter ϵ and we will consider also the next-to-leading order (NLO) in the weak-field expansion. Hence, we rewrite eq. (1.2.1) as

$$
g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu} \tag{3.2.7}
$$

which gives

$$
g^{\mu\nu} = \eta^{\mu\nu} - \epsilon\,h^{\mu\nu} + \epsilon^2 h^{\mu\lambda}\,h_\lambda^\nu + \mathcal{O}(\epsilon^3)\;, \tag{3.2.8}
$$

the integration measure reads

[3] The boundary conditions that ensure vanishing of boundary terms will be explicitly shown when necessary.

[4] Were one to identify the Lagrangian density in Eq.(3.2.6) with the pressure p_{N} of the gravitational field, it would appear the Newtonian potential has the equation of state $p_{\mathrm{N}} = -\rho_{\mathrm{N}}/3$ [62].

$$\sqrt{-g} = 1 + \frac{\epsilon}{2}\, h + \frac{\epsilon^2}{8} \left(h^2 - 2\, h_\mu{}^\nu\, h_\nu{}^\mu \right) + \mathcal{O}(\epsilon^3) \ , \tag{3.2.9}$$

and the scalar curvature R is obtained from the Ricci tensor (1.1.2) provided one has computed the Christoffel symbols

$$\Gamma^\lambda_{\mu\nu} \simeq \frac{\epsilon}{2} \left(\eta^{\lambda\rho} - \epsilon\, h^{\lambda\rho} + \epsilon^2\, h^{\lambda\sigma}\, h^\rho_\sigma \right) \left(\partial_\mu h_{\rho\nu} + \partial_\nu h_{\rho\mu} - \partial_\rho h_{\mu\nu} \right) \ . \tag{3.2.10}$$

Therefore, in order to have second-order corrections to the equations of motion, we must compute the Ricci scalar up to the third order and consequently consider the second order for the matter part. After some tedious algebra, one finds

$$\begin{aligned} \left(\sqrt{-g}\, R \right)_{(3)} &= h^\mu_\nu \left(\partial_\mu h^\lambda_\rho\, \partial^\nu h^\rho_\lambda - \partial^\lambda h^\nu_\mu\, \partial_\lambda h \right) + 2\, h^\mu_\nu\, \partial_\lambda h^\rho_\mu \left(\partial^\lambda h^\nu_\rho - \partial^\nu h^\lambda_\rho \right) \\ &\quad - \frac{1}{2}\, h\, \partial^\mu h^\lambda_\nu\, \partial_\mu h^\nu_\lambda + \frac{1}{4}\, h\, \partial_\mu h\, \partial^\mu h \\ &\simeq -h_{00} \left(\partial_r h_{00} \right)^2 \\ &\simeq V \left(V' \right)^2 \ , \end{aligned} \tag{3.2.11}$$

and

$$\left(\sqrt{-g}\, \mathcal{L}_\mathrm{M} \right)_{(2)} = \frac{1}{8}\, h^2_{00}\, T_{00} = \frac{1}{2}\, V^2\, \rho \ . \tag{3.2.12}$$

Adding all the contributions, and explicitly rescaling $m_\mathrm{p}/(8\,\pi\,\ell_\mathrm{p})$ by a factor of ϵ^{-1},one obtains the action

$$S[V] = 4\,\pi \int \epsilon\, \mathrm{d}t \int_0^\infty r^2\, \mathrm{d}r \left\{ \frac{m_\mathrm{p}}{8\,\pi\,\ell_\mathrm{p}}\, V \triangle V - \rho\, V + \frac{\epsilon}{2} \left[\frac{m_\mathrm{p}}{4\,\pi\,\ell_\mathrm{p}} \left(V' \right)^2 + V\, \rho \right] V \right\} \ . \tag{3.2.13}$$

A few remarks are now in order. First of all, we have derived Eq. (3.2.13) in the de Donder gauge (1.2.14), which explicitly reads

$$\partial_t h_{00} = 0 \tag{3.2.14}$$

for static configurations $h_{00} = h_{00}(r)$, and is therefore automatically satisfied in our case. This means that the above action can be used for describing the gravitational potential $V = V(r)$ measured by any static observer placed at constant radial coordinate r (provided test particles move at non-relativistic speed). In fact, there remains the ambiguity in the definition of the observer time t, which in turn determines the value of ϵ in Eq. (3.2.13), as can be seen by the simple fact that the time measure is $\epsilon\,\mathrm{d}t$. On the other hand, changing ϵ, and therefore the time (albeit in such a way that motions remain non-relativistic) does not affect the dynamics of the Newtonian part of the potential, whereas the post-Newtonian part inside the curly brackets acquires a different weight. This is completely consistent with the expansion of the Schwarzschild metric described in Section 1.2.4, in which we showed that the Newtonian potential is uniquely defined by choosing a static observer, whereas the form of the first post-Newtonian correction varies with the specific choice of time.

At this point, it is convenient to introduce the (dimensionless) matter coupling $q_{\rm B}$, originating from the stress-energy tensor, by formally rescaling $\rho \to q_{\rm B}\,\rho$ in the above expressions. Likewise, the "self-coupling" q_Φ will designate terms of higher order in ϵ. In particular, we set $\epsilon = 4\,q_\Phi$ so that the post-Newtonian potential (1.2.83) is recovered for $q_\Phi = 1$ [5]. With these definitions, the above action yields the total Lagrangian for a new field V, namely

$$
\begin{aligned}
L[V] &= 4\,\pi \int_0^\infty r^2\,\mathrm{d}r \left[\frac{m_{\rm p}}{8\,\pi\,\ell_{\rm p}}\,V \triangle V - q_{\rm B}\,\rho\,V + 2\,q_\Phi\left(q_{\rm B}\,V\,\rho - 2\,J_V\right)V\right] \\
&= 4\,\pi \int_0^\infty r^2\,\mathrm{d}r \left[\frac{m_{\rm p}}{8\,\pi\,\ell_{\rm p}}\,V \triangle V - q_{\rm B}\,V\,\rho\left(1 - 2\,q_\Phi\,V\right) + \frac{q_\Phi\,m_{\rm p}}{2\,\pi\,\ell_{\rm p}}\,V\left(V'\right)^2\right] \\
&= -4\,\pi \int_0^\infty r^2\,\mathrm{d}r \left[\frac{m_{\rm p}}{8\,\pi\,\ell_{\rm p}}\left(1 - 4\,q_\Phi\,V\right)\left(V'\right)^2 + q_{\rm B}\,V\,\rho\left(1 - 2\,q_\Phi\,V\right)\right]\ ,
\end{aligned}
\tag{3.2.15}
$$

As a final remark, we can now show that the above non linear correction (3.2.11) to the purely gravitational part can be interpreted as the self-interaction of the gravitational field with its own energy density, in full agreement with the results of [54]. To show this, we start by computing the Hamiltonian

$$
H[V_{\rm N}] = -L[V_{\rm N}] = 4\,\pi \int_0^\infty r^2\,\mathrm{d}r \left(-\frac{m_{\rm p}}{8\,\pi\,\ell_{\rm p}}\,V_{\rm N} \triangle V_{\rm N} + \rho\,V_{\rm N}\right)\ , \tag{3.2.16}
$$

as follows from the static approximation. If we evaluate this expression on-shell by means of Eq.(1.2.22), we get the Newtonian potential energy

$$
U_{\rm N}(r) = 2\,\pi \int_0^r \bar r^2\,\mathrm{d}\bar r\,\rho(\bar r)\,V_{\rm N}(\bar r)\ , \tag{3.2.17}
$$

which one can view as given by the interaction of the matter distribution enclosed in a sphere of radius r with the gravitational field. Following Ref. [54], we could then define a self-gravitational source J_V given by the gravitational energy $U_{\rm N}$ per unit volume. We first note that

[5]The post-Newtonian correction (1.2.77) can instead be obtained for $q_\Phi = 2$.

$$\begin{aligned} U_{\mathrm{N}}(r) &= \frac{m_{\mathrm{p}}}{2\,\ell_{\mathrm{p}}} \int_0^r \bar{r}^2\,\mathrm{d}\bar{r}\, V_{\mathrm{N}}(\bar{r}) \triangle V_{\mathrm{N}}(\bar{r}) \\ &= -\frac{m_{\mathrm{p}}}{2\,\ell_{\mathrm{p}}} \int_0^r \bar{r}^2\,\mathrm{d}\bar{r}\, \left[V_{\mathrm{N}}'(\bar{r})\right]^2 \ , \end{aligned} \tag{3.2.18}$$

where we used Eq.(1.2.22) and then integrated by parts discarding boundary terms. The corresponding energy density is then given by

$$J_V(r) = \frac{1}{4\,\pi\,r^2}\,\frac{\mathrm{d}}{\mathrm{d}r} U_{\mathrm{N}}(r) = -\frac{m_{\mathrm{p}}}{8\,\pi\,\ell_{\mathrm{p}}} \left[V_{\mathrm{N}}'(r)\right]^2 \ . \tag{3.2.19}$$

This contribution is in fact proportional to the one found above in eq. (3.2.11).

The Euler-Lagrange equation for V is given by

$$\begin{aligned} 0 &= \frac{\delta\mathcal{L}}{\delta V} - \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\delta\mathcal{L}}{\delta V'}\right) \\ &= 4\,\pi\,r^2 \left[-q_{\mathrm{B}}\,\rho + 4\,q_{\mathrm{B}}\,q_\Phi\,\rho\,V + \frac{q_\Phi\,m_{\mathrm{p}}}{2\,\pi\,\ell_{\mathrm{p}}}\left(V'\right)^2\right] \\ &\quad + \frac{m_{\mathrm{p}}}{\ell_{\mathrm{p}}}\left[r^2\,V'\,(1-4\,q_\Phi\,V)\right]' \ , \end{aligned} \tag{3.2.20}$$

and, on taking into account that $r^2 \triangle f(r) = \left(r^2\,f'\right)'$ for spherically symmetric functions, we obtain the field equation

$$(1-4\,q_\Phi\,V)\triangle V = 4\,\pi\,q_{\mathrm{B}}\,\frac{\ell_{\mathrm{p}}}{m_{\mathrm{p}}}\,\rho\,(1-4\,q_\Phi\,V) + 2\,q_\Phi\left(V'\right)^2 \ . \tag{3.2.21}$$

This differential equation is obviously hard to solve analytically for a general source. We will therefore expand the field V up to first order in the coupling q_Φ [6],

$$V(r) = V_{(0)}(r) + q_\Phi\,V_{(1)}(r) \ , \tag{3.2.22}$$

and solve Eq.(3.2.21) order by order. In particular, we have

$$\triangle V_{(0)} = 4\,\pi\,q_{\mathrm{B}}\,\frac{\ell_{\mathrm{p}}}{m_{\mathrm{p}}}\,\rho \ , \tag{3.2.23}$$

which, when $q_B = 1$, is just the Poisson Eq.(1.2.22) for the Newtonian potential and

$$\triangle V_{(1)} = 2\left(V_{(0)}'\right)^2 \ , \tag{3.2.24}$$

[6] Since Eq. (3.2.21) is obtained from a Lagrangian defined up to first order in q_Φ, higher-order terms in the solution would not be meaningful.

which gives the correction at first order in q_Φ.

To linear order in q_Φ, the on-shell Hamiltonian (3.2.16) is also replaced by

$$\begin{aligned} H[V] &= -L[V] \\ &\simeq 4\pi \int_0^\infty r^2 \,\mathrm{d}r \left\{ -\frac{V}{2} \left[q_\mathrm{B}\,\rho + \frac{q_\Phi\, m_\mathrm{p}}{2\pi\,\ell_\mathrm{p}} \left(V'\right)^2 \right] + q_\mathrm{B}\,\rho\, V - \frac{q_\Phi\, m_\mathrm{p}}{2\pi\,\ell_\mathrm{p}} V \left(V'\right)^2 \right\} \\ &\simeq 2\pi \int_0^\infty \mathrm{d}r\, r^2 \left[q_\mathrm{B}\,\rho\, V \left(1 - 4\, q_\Phi\, V\right) - q_\Phi \frac{3\, m_\mathrm{p}}{2\pi\,\ell_\mathrm{p}} V \left(V'^2\right) \right] , \end{aligned} \tag{3.2.25}$$

where we used Eq. (3.2.21). In the following, we will still denote the on-shell contribution containing the matter density ρ with

$$U_\mathrm{BG} = 2\pi\, q_\mathrm{B} \int_0^\infty r^2\,\mathrm{d}r\,\rho \left[V_{(0)} + q_\Phi \left(V_{(1)} - 4\, V_{(0)}^2 \right) \right] + \mathcal{O}(q_\Phi^2) , \tag{3.2.26}$$

which reduces to the Newtonian U_N in Eq. (3.2.17) for $q_\mathrm{B} = 1$ and $q_\Phi = 0$, and the rest as

$$U_\mathrm{GG} = -3\, q_\Phi \frac{\ell_\mathrm{p}}{m_\mathrm{p}} \int_0^\infty r^2\,\mathrm{d}r\, V_{(0)} \left(V'_{(0)} \right)^2 + \mathcal{O}(q_\Phi^2) . \tag{3.2.27}$$

3.2.1 Classical Solutions

We will now study the general classical solutions to Eqs. (3.2.23) and (3.2.24). Since we are interested in static and spherically symmetric sources, it is convenient to consider eigenfunctions of the Laplace operator,

$$\triangle j_0(k\, r) = -k^2\, j_0(k\, r) , \tag{3.2.28}$$

that is, the spherical Bessel function of the first kind

$$j_0(k\, r) = \frac{\sin(k\, r)}{k\, r} , \tag{3.2.29}$$

which enjoys the normalisation

$$4\pi \int_0^\infty r^2\,\mathrm{d}r\, j_0(p\, r)\, j_0(k\, r) = \frac{2\,\pi^2}{k^2}\,\delta(p - k) . \tag{3.2.30}$$

Assuming the matter density is a smooth function of the radial coordinate, we can project it on the above modes,

$$\tilde{\rho}(k) = 4\,\pi \int_0^\infty r^2\,\mathrm{d}r\,j_0(k\,r)\,\rho(r) \ , \tag{3.2.31}$$

and likewise

$$\tilde{V}_{(n)}(k) = 4\,\pi \int_0^\infty r^2\,\mathrm{d}r\,j_0(k\,r)\,V_{(n)}(r) \ . \tag{3.2.32}$$

Inverting these expressions, one obtains the expansions in Laplacian eigenfunctions,

$$f(r) = \int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\,j_0(k\,r)\,\tilde{f}(k) \ , \tag{3.2.33}$$

in which we used

$$\int \frac{\mathrm{d}^3k}{(2\,\pi)^3} = \int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2} \ , \tag{3.2.34}$$

since all our functions only depend on the radial momentum $k \geq 0$.

The zero-order Eq. (3.2.23) in momentum space reads

$$\tilde{V}_{(0)}(k) = -4\,\pi\,q_{\mathrm{B}}\,\frac{\ell_{\mathrm{p}}\,\tilde{\rho}(k)}{m_{\mathrm{p}}\,k^2} \ , \tag{3.2.35}$$

which can be inverted to yield the solution

$$V_{(0)}(r) = -2\,q_{\mathrm{B}}\,\frac{\ell_{\mathrm{p}}}{m_{\mathrm{p}}} \int_0^\infty \frac{\mathrm{d}k}{\pi}\,j_0(k\,r)\,\tilde{\rho}(k) \ . \tag{3.2.36}$$

The r.h.s. of Eq. (3.2.24) can then be written as

$$2\left(V_{(0)}'(r)\right)^2 = q_{\mathrm{B}}^2\,\frac{8\,\ell_{\mathrm{p}}^2}{m_{\mathrm{p}}^2}\left(\int_0^\infty \frac{k\,\mathrm{d}k}{\pi}\,j_1(k\,r)\,\tilde{\rho}(k)\right)^2 \ , \tag{3.2.37}$$

where we used Eq. (3.2.35) and

$$[j_0(k\,r)]' = -k\,j_1(k\,r) \ . \tag{3.2.38}$$

The first-order Eq. (3.2.24) is however easier to solve directly in coordinate space usually.

For example, for a point-like source of mass M_0, whose density is given by

$$\rho = M_0\,\delta^{(3)}(\boldsymbol{x}) = \frac{M_0}{4\,\pi\,r^2}\,\delta(r) \ , \tag{3.2.39}$$

one finds

$$\tilde{\rho}(k) = M_0 \int_0^\infty \mathrm{d}r\,j_0(k\,r)\,\delta(r) = M_0 \ , \tag{3.2.40}$$

and Eq. (3.2.35) yields the Newtonian potential outside a spherical source of mass M_0 (for $q_{\rm B} = 1$), that is

$$V_{(0)}(r) = -2\, q_{\rm B}\, \frac{\ell_{\rm p}\, M_0}{m_{\rm p}\, r} \int_0^\infty \frac{{\rm d}z}{\pi}\, j_0(z) = -q_{\rm B}\, \frac{\ell_{\rm p}\, M_0}{m_{\rm p}\, r} \ . \tag{3.2.41}$$

Note that this solution automatically satisfies the regularity condition

$$\lim_{r\to\infty} V_{(0)}(r) = 0 \ . \tag{3.2.42}$$

Next, for $r > 0$, one has

$$\begin{aligned} 2\left(V'_{(0)}(r)\right)^2 &= q_{\rm B}^2\, \frac{8\,\ell_{\rm p}^2\, M_0^2}{m_{\rm p}^2\, r^4} \left(\int_0^\infty \frac{z\,{\rm d}z}{\pi}\, j_1(z)\right)^2 \\ &= q_{\rm B}^2\, \frac{2\,\ell_{\rm p}^2\, M_0^2}{m_{\rm p}^2\, r^4} \ , \end{aligned} \tag{3.2.43}$$

and Eq. (3.2.24) admits the general solution

$$V_{(1)} = A_1 - q_{\rm B}\, \frac{\ell_{\rm p}\, M_1}{m_{\rm p}\, r} + q_{\rm B}^2\, \frac{\ell_{\rm p}^2\, M_0^2}{m_{\rm p}^2\, r^2} \ . \tag{3.2.44}$$

On imposing the same boundary condition (3.2.42) to $V_{(1)}$, one obtains $A_1 = 0$. The arbitrary constant M_1 results in a (arbitrary) shift of the ADM mass,

$$M = M_0 + q_\Phi\, M_1 \ , \tag{3.2.45}$$

and one is therefore left with the potential

$$V = -q_{\rm B}\, \frac{\ell_{\rm p}\, M}{m_{\rm p}\, r} + q_\Phi\, q_{\rm B}^2\, \frac{\ell_{\rm p}^2\, M^2}{m_{\rm p}^2\, r^2} + \mathcal{O}(q_\Phi^2) \ . \tag{3.2.46}$$

This expression matches the expected post-Newtonian form (1.2.83) at large r for $q_{\rm B} = q_\Phi = 1$. It also clearly shows the limitation of the present approach: at small r, the post-Newtonian correction $V_{(1)}$ grows faster than $V_{(0)} = V_N$ and our perturbative approach will necessarily break down.

We can also evaluate the potential energy (3.2.25) generated by the point-like source. The baryon-graviton energy (3.2.26) of course diverges, but we can regularise the matter density (3.2.39) by replacing $\delta(r) \to \delta(r - r_0)$, where $0 < r_0 \ll \ell_{\rm p}\, M_0/m_{\rm p}$. We then find

$$U_{\rm BG} \simeq -q_{\rm B}^2\, \frac{\ell_{\rm p}\, M_0\, M}{2\, m_{\rm p}\, r_0} - q_{\rm B}^3\, q_\Phi\, \frac{3\,\ell_{\rm p}^2\, M^3}{2\, m_{\rm p}^2\, r_0^2} \ . \tag{3.2.47}$$

With the same regularisation, we obtain the graviton-graviton energy

$$U_{\rm GG} \simeq -3\, q_\Phi\, \frac{\ell_{\rm p}}{m_{\rm p}} \int_{r_0}^{\infty} r^2\, {\rm d}r\, V_{(0)} \left(V'_{(0)}\right)^2 = q_{\rm B}^3\, q_\Phi\, \frac{3\,\ell_{\rm p}^2\, M^3}{2\, m_{\rm p}^2\, r_0^2}\ , \tag{3.2.48}$$

which precisely cancels against the first order correction to U_{BG} in Eq. (3.2.47), and

$$U = U_{\rm BG} + U_{\rm GG} = -q_{\rm B}^2\, \frac{\ell_{\rm p}\, M_0\, M}{2\, m_{\rm p}\, r_0}\ . \tag{3.2.49}$$

Of course, for $r \simeq r_0 \ll \ell_{\rm p}\, M_0/m_{\rm p}$, the post-Newtonian term in Eq. (3.2.46) becomes much larger than the Newtonian contribution, which pushes the above $U_{\rm BG}$ and $U_{\rm GG}$ beyond the regime of validity of our approximations. Nonetheless, it is important to notice that, given the effective Lagrangian (3.2.15), the total gravitational energy (3.2.49) for a point-like source will never vanish and the maximal packing condition (3.1.9) cannot be realised. This is consistent with the concept of corpuscular black holes as quantum objects with a (very) large spatial extensions $R \sim R_{\rm H}$.

For the reasons above, we shall next study extended distributions of matter, which will indeed lead to different, more sensible results within the scope of our approach.

3.2.2 Homogeneous Matter Distribution

For an arbitrary matter density, it is hopeless to solve the equation (3.2.24) for $V_{(1)}$ analytically. Let us then consider the very simple case in which ρ is uniform inside a sphere of radius R,

$$\rho(r) = \frac{3\, M_0}{4\,\pi\, R^3}\, \Theta(R - r)\ , \tag{3.2.50}$$

where Θ is the Heaviside step function and

$$M_0 = 4\,\pi \int_0^{\infty} r^2\, {\rm d}r\, \rho(r)\ . \tag{3.2.51}$$

For this matter density, we shall now solve Eqs. (3.2.23) and (3.2.24) with boundary conditions that ensure V is regular both at the origin $r = 0$ and infinity, that is

$$V'_{(n)}(0) = \lim_{r\to\infty} V_{(n)}(r) = 0\ , \tag{3.2.52}$$

and smooth across the border $r = R$,

$$\lim_{r\to R^-} V_{(n)}(r) = \lim_{r\to R^+} V_{(n)}(r)\ , \quad \lim_{r\to R^-} V'_{(n)}(r) = \lim_{r\to R^+} V'_{(n)}(r)\ . \tag{3.2.53}$$

The solution to Eq. (3.2.23) inside the sphere is then given by

$$V_{(0)\mathrm{in}}(r) = q_\mathrm{B}\,\frac{\ell_\mathrm{p}\,M_0}{2\,m_\mathrm{p}\,R^3}\left(r^2 - 3\,R^2\right) \tag{3.2.54}$$

while outside

$$V_{(0)\mathrm{out}}(r) = -q_\mathrm{B}\,\frac{\ell_\mathrm{p}\,M_0}{m_\mathrm{p}\,r}\ , \tag{3.2.55}$$

which of course equal the Newtonian potential for $q_\mathrm{B} = 1$.

At first order in q_Φ we instead have

$$V_{(1)\mathrm{in}}(r) = q_\mathrm{B}^2\,\frac{\ell_\mathrm{p}^2\,M_0^2}{10\,m_\mathrm{p}^2\,R^6}\left(r^4 - 15\,R^4\right) \tag{3.2.56}$$

and

$$V_{(1)\mathrm{out}}(r) = q_\mathrm{B}^2\,\frac{\ell_\mathrm{p}^2\,M_0^2}{5\,m_\mathrm{p}^2\,R}\,\frac{5\,R - 12\,r}{r^2}\ . \tag{3.2.57}$$

The complete outer solution to first order in q_Φ is thus given by

$$V_\mathrm{out}(r) = -q_\mathrm{B}\,\frac{\ell_\mathrm{p}\,M_0}{m_\mathrm{p}\,r}\left(1 + q_\Phi\,q_\mathrm{B}\,\frac{12\,\ell_\mathrm{p}\,M_0}{5\,m_\mathrm{p}\,R}\right) + q_\mathrm{B}^2\,q_\Phi\,\frac{\ell_\mathrm{p}^2\,M_0^2}{m_\mathrm{p}^2\,r^2} + \mathcal{O}(q_\Phi^2)\ . \tag{3.2.58}$$

From this outer potential, we see that, unlike for the point-like source, we are left with no arbitrary constant and the ADM mass is determined as

$$M = M_0\left(1 + q_\Phi\,q_\mathrm{B}\,\frac{12\,\ell_\mathrm{p}\,M_0}{5\,m_\mathrm{p}\,R}\right) + \mathcal{O}(q_\Phi^2)\ , \tag{3.2.59}$$

and, replacing this expression into the solutions, we finally obtain

$$\begin{aligned} V_\mathrm{in}(r) &= q_\mathrm{B}\frac{\ell_\mathrm{p}\,M}{2\,m_\mathrm{p}\,R^3}\left(r^2 - 3\,R^2\right) \\ &\quad + q_\mathrm{B}^2\,q_\Phi\,\frac{\ell_\mathrm{p}^2\,M^2}{10\,m_\mathrm{p}^2\,R^6}\left(r^4 - 12\,R^2\,r^2 + 21\,R^4\right) + \mathcal{O}(q_\Phi^2)\ , \end{aligned} \tag{3.2.60}$$

$$V_\mathrm{out}(r) = -q_\mathrm{B}\,\frac{\ell_\mathrm{p}\,M}{m_\mathrm{p}\,r} + q_\mathrm{B}^2\,q_\Phi\,\frac{\ell_\mathrm{p}^2\,M^2}{m_\mathrm{p}^2\,r^2} + \mathcal{O}(q_\Phi^2)\ . \tag{3.2.61}$$

We can now see that the outer field again reproduces the first post-Newtonian result (1.2.83) of Section 1.2.4 when $q_\mathrm{B} = q_\Phi = 1$ (see Figs. 3.1 and 3.2 for two examples).

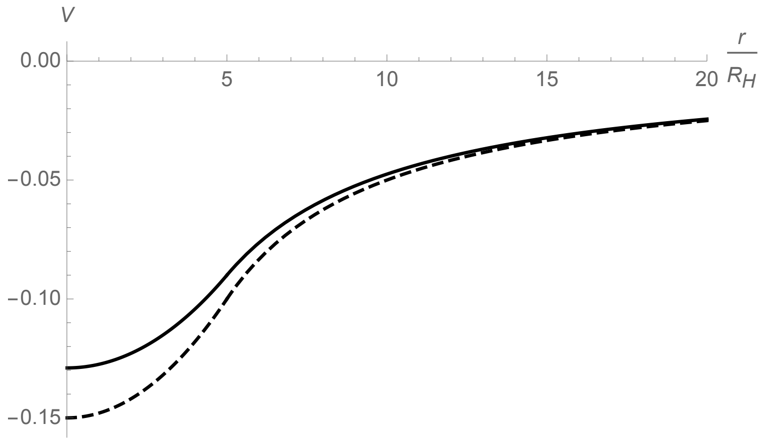

Figure 3.1: Potential to first order in q_Φ (solid line) vs Newtonian potential (dashed line) for $R = 10\,\ell_{\rm p}\,M/m_{\rm p} \equiv 5\,R_{\rm H}$ and $q_{\rm B} = q_\Phi = 1$.

Since the density (3.2.50) is sufficiently regular, we can evaluate the corresponding gravitational energy (3.2.25) without the need of a regulator. The baryon-graviton energy (3.2.26) is found to be

$$\begin{aligned} U_{\rm BG}(R) &= 2\,\pi\,q_{\rm B}\int_0^R r^2\,{\rm d}r\,\rho\left[V_{(0)\rm in} + q_\Phi\left(V_{(1)\rm in} - 4\,V_{(0)\rm in}^2\right)\right] + \mathcal{O}(q_\Phi^2) \\ &= -q_{\rm B}^2\,\frac{3\,\ell_{\rm p}\,M^2}{5\,m_{\rm p}\,R} - q_{\rm B}^3\,q_\Phi\,\frac{267\,\ell_{\rm p}^2\,M^3}{350\,m_{\rm p}^2\,R^2} + \mathcal{O}(q_\Phi^2) \\ &\equiv U_{(0)\rm BG}(R) + q_\Phi\,U_{(1)\rm BG}(R) + \mathcal{O}(q_\Phi^2)\ , \end{aligned} \tag{3.2.62}$$

where $U_{(0)\rm BG}$ is just the Newtonian contribution (for $q_{\rm B} = 1$) and $U_{(1)\rm BG}$ the post-Newtonian correction. Analogously, the self-sourcing contribution (3.2.27) gives

$$\begin{aligned} U_{\rm GG}(R) &= -3q_\Phi\,\frac{m_{\rm p}}{\ell_{\rm p}}\left[\int_0^R r^2\,{\rm d}r\,V_{(0)\rm in}\left(V'_{(0)\rm in}\right)^2 + \int_R^\infty r^2\,{\rm d}r\,V_{(0)\rm out}\left(V'_{(0)\rm out}\right)^2\right] + \mathcal{O}(q_\Phi^2) \\ &= q_{\rm B}^3\,q_\Phi\,\frac{153\,\ell_{\rm p}^2\,M_0^3}{70\,m_{\rm p}^2\,R^2} + \mathcal{O}(q_\Phi^2)\ . \end{aligned} \tag{3.2.63}$$

Since now $U_{\rm GG} > q_\Phi\,|U_{(1)\rm BG}|$, adding the two terms together yields the total gravitational energy

$$U(R) = -q_{\rm B}^2\,\frac{3\,\ell_{\rm p}\,M^2}{5\,m_{\rm p}\,R} + q_{\rm B}^3\,q_\Phi\,\frac{249\,\ell_{\rm p}^2\,M^3}{175\,m_{\rm p}^2\,R^2} + \mathcal{O}(q_\Phi^2)\ , \tag{3.2.64}$$

which appears in line with what was estimated in Ref. [54]: the (order q_Φ) post-Newtonian energy is positive, and would equal the Newtonian contribution for a source

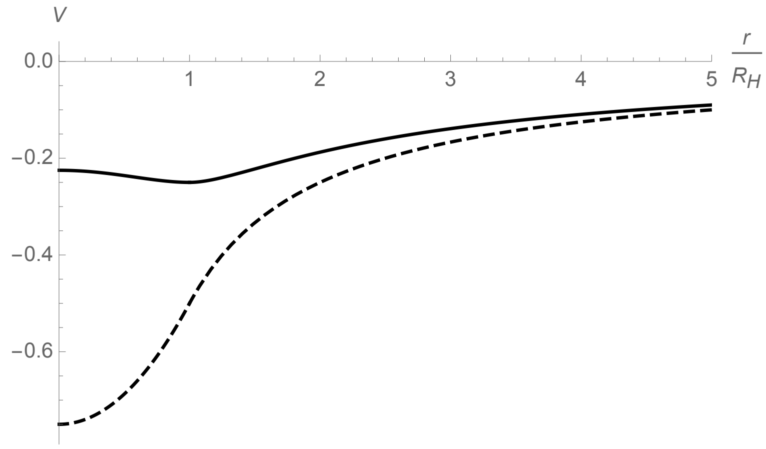

Figure 3.2: Potential to first order in q_Φ (solid line) vs Newtonian potential (dashed line) for $R = 2\,\ell_\mathrm{p}\,M/m_\mathrm{p} \equiv R_\mathrm{H}$ and $q_\mathrm{B} = q_\Phi = 1$.

of radius

$$R \simeq \frac{83\,\ell_\mathrm{p}\,M}{35\,m_\mathrm{p}} \simeq 1.2\,R_\mathrm{H} \ , \tag{3.2.65}$$

where se wet $q_\mathrm{B} = q_\Phi = 1$. One has therefore recovered the "maximal packing" condition (3.1.9) of Section 2 in the limit $R \sim R_\mathrm{H}$ from a regular matter distribution. However, note that, strictly speaking, the above value of R falls outside the regime of validity of our approximations.

3.2.3 Gaussian Matter Distribution

As an example of even more regular matter density, we can consider a Gaussian distribution of width σ,

$$\rho(r) = \frac{M_0\,e^{-\frac{r^2}{\sigma^2}}}{\pi^{3/2}\,\sigma^3} \ , \tag{3.2.66}$$

where again

$$M_0 = 4\,\pi \int_0^\infty r^2\,\mathrm{d}r\,\rho(r) \ . \tag{3.2.67}$$

Let us remark that the above density is essentially zero for $r \gtrsim R \equiv 3\,\sigma$, which will allow us to make contact with the previous case.

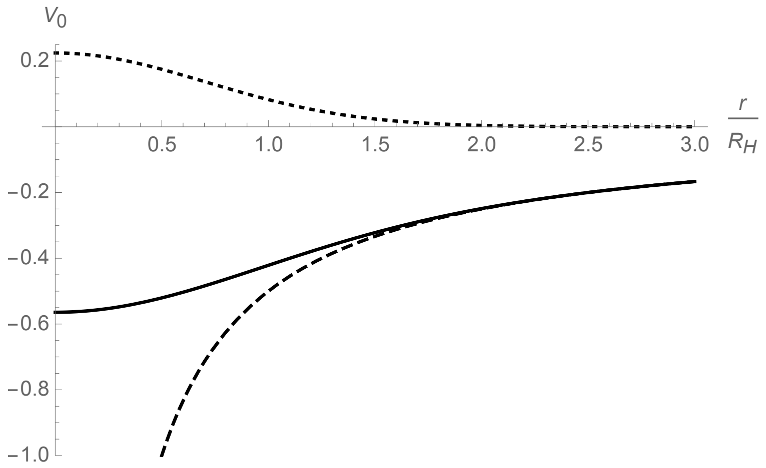

Figure 3.3: Newtonian potential (solid line) for Gaussian matter density with $\sigma = 2\,\ell_{\rm p}\,M_0/m_{\rm p}$ (dotted line) vs Newtonian potential (dashed line) for point-like source of mass M_0 (with $q_{\rm B} = 1$).

For this matter density, we shall now solve Eqs. (3.2.23) and (3.2.24) with the boundary conditions (3.2.52) that ensure V is regular both at the origin $r = 0$ and at infinity. We first note that Eq. (3.2.31) yields

$$\tilde{\rho}(k) = M_0\, e^{-\frac{\sigma^2\,k^2}{4}} \ , \tag{3.2.68}$$

from which

$$\begin{aligned} V_{(0)}(r) &= -2\,q_{\rm B}\,\frac{\ell_{\rm p}\,M_0}{m_{\rm p}} \int_0^\infty \frac{{\rm d}k}{\pi}\, j_0(k\,r)\, e^{-\frac{\sigma^2\,k^2}{4}} \\ &= -q_{\rm B}\,\frac{\ell_{\rm p}\,M_0}{m_{\rm p}\,r}\,{\rm Erf}(r/\sigma) \ . \end{aligned} \tag{3.2.69}$$

For a comparison with the analogous potential generated by a point-like source with the same mass M_0, see Fig. 3.3. For $r \gtrsim R = 3\,\sigma = 3\,R_{\rm H}/2$, the two potentials are clearly indistinguishable, whereas $V_{(0)}$ looks very similar to the case of homogeneous matter for $0 \le r < R$ (see Fig. 3.1).

The first-order equation (3.2.24) now reads

$$\triangle V_{(1)} = 2\,q_{\rm B}^2\,\frac{\ell_{\rm p}\,M_0^2}{m_{\rm p}^2\,r^4} \left[{\rm Erf}(r/\sigma) - \frac{2\,r}{\sqrt{\pi}\,\sigma}\, e^{-\frac{r^2}{\sigma^2}}\right]^2 \equiv 2\,q_{\rm B}^2\,\frac{\ell_{\rm p}\,M_0^2}{m_{\rm p}^2}\,G(r) \ , \tag{3.2.70}$$

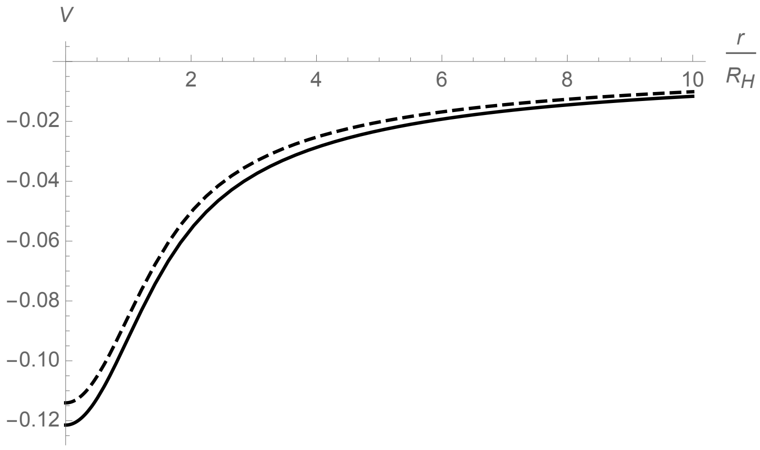

Figure 3.4: Potential up to first order in q_Φ (solid line) vs Newtonian potential (dashed line) for Gaussian matter density with $\sigma = 2\,\ell_\mathrm{p}\,M/m_\mathrm{p} \equiv R_\mathrm{H}$ (with $q_\mathrm{B} = q_\Phi = 1$).

and we note that

$$G(r) \simeq \begin{cases} \dfrac{16\,r^2}{9\,\pi\,\sigma^6} & \text{for} \quad r \to 0 \\ \\ \dfrac{1}{r^4} & \text{for} \quad r \to \infty\ , \end{cases} \tag{3.2.71}$$

which are the same asymptotic behaviours one finds for a homogenous source of size $R \sim \sigma$. We can therefore expect the proper solution to Eq. (3.2.70) behaves like Eq. (3.2.56) for $r \to 0$ and (3.2.57) for $r \to \infty$. In fact, one finds

$$\begin{aligned} V_{(1)} \;&=\; 2\,q_\mathrm{B}^2\,\frac{\ell_\mathrm{p}^2\,M_0^2}{m_\mathrm{p}^2}\left\{\frac{\left[\mathrm{erf}\left(\frac{r}{\sigma}\right)\right]^2 - 1}{\sigma^2} - \frac{\sqrt{2}\,\mathrm{erf}\left(\sqrt{2}\,\frac{r}{\sigma}\right)}{\sqrt{\pi}\,\sigma\,r}\right. \\ &\quad \left. + \frac{\left[\mathrm{erf}\left(\frac{r}{\sigma}\right)\right]^2}{2\,r^2} + \frac{2\,e^{-\frac{r^2}{\sigma^2}}\,\mathrm{erf}\left(\frac{r}{\sigma}\right)}{\sqrt{\pi}\,\sigma\,r}\right\}\ , \end{aligned} \tag{3.2.72}$$

in which we see the second term in curly brackets again leads to a shift in the ADM mass,

$$M = M_0\left(1 + q_\mathrm{B}\,q_\Phi\,\frac{2\,\sqrt{2}\,\ell_\mathrm{p}\,M_0}{\sqrt{\pi}\,m_\mathrm{p}\,\sigma}\right)\ , \tag{3.2.73}$$

while the third term reproduces the usual post-Newtonian potential (1.2.83) for $r \gg \sigma$. For an example of the complete potential up to first order in q_Φ, see Fig. 3.4. Note that for the relatively small value of σ used in that plot, the main effect of $V_{(1)}$ in Eq. (3.2.72) is to increase the ADM mass according to Eq. (3.2.73), which lowers the total potential significantly with respect to the Newtonian curve for $M = M_0$ shown in Fig. 3.3.

3.3 Quantum Linear Field and Coherent Ground State

We are now going to see how one can reproduce the previous classical results in a quantum theory. We will proceed by canonically quantising a suitably rescaled potential field, and then identifying the quantum state which yields expectation values close to the classical expressions.

A canonically normalised scalar field Φ has dimensions of $\sqrt{\text{mass/length}}$, while the potential V is dimensionless. We therefore define

$$\Phi = \sqrt{\frac{m_{\rm p}}{\ell_{\rm p}}}\, V \ , \qquad J_{\rm B} = 4\,\pi \sqrt{\frac{\ell_{\rm p}}{m_{\rm p}}}\, \rho \ , \tag{3.3.1}$$

and replace these new quantities in Eq. (3.2.15). After rescaling the whole Lagrangian (3.2.15) by a factor of $4\,\pi$, in order to have a canonically normalised kinetic term, we obtain the scalar field Lagrangian

$$\begin{aligned} L[\Phi] \;=\; 4\,\pi \int_0^\infty r^2\, \mathrm{d}r \Bigg[&\frac{1}{2}\, \Phi\, \Box \Phi - q_{\rm B}\, J_{\rm B}\, \Phi \left(1 - 2\, q_\Phi \sqrt{\frac{\ell_{\rm p}}{m_{\rm p}}}\, \Phi \right) \\ &+ 2\, q_\Phi \sqrt{\frac{\ell_{\rm p}}{m_{\rm p}}}\, \left(\partial_\mu \Phi\right)^2 \Phi \Bigg] \ , \end{aligned} \tag{3.3.2}$$

where we again assumed $\Phi = \Phi(t, r)$.

As usual, we define the quantum field operators starting from the "free" theory, corresponding to $q_{\rm B} = q_\Phi = 0$, that is we will employ normal modes of the equation

$$\Box \Phi = 0 \ . \tag{3.3.3}$$

In particular, since we are interested in static and spherically symmetric states, we can again employ the eigenfunctions (3.2.28) of the Laplace operator, and define the time-dependent modes

$$u_k(t, r) = j_0(k\, r)\, e^{i\,\omega\, t} \ , \tag{3.3.4}$$

which satisfy

$$4\,\pi \int_0^\infty r^2\, \mathrm{d}r\, u_p^*(t, r)\, u_k(t, r) = \frac{2\,\pi^2}{k^2}\, \delta(p - k) \ . \tag{3.3.5}$$

Upon replacing (3.3.4) into Eq. (3.3.3), one of course obtains the mass-shell relation $\omega = k$, so that the field operator and its conjugate momentum are respectively given by

$$\hat{\Phi}(t,r) = \int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\sqrt{\frac{\ell_\mathrm{p}\,m_\mathrm{p}}{2\,k}}\,(k\,r)\left(\hat{a}_k\,e^{i\,k\,t} + \hat{a}_k^\dagger\,e^{-i\,k\,t}\right) \ , \tag{3.3.6}$$

and

$$\hat{\Pi}(t,r) = i\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\sqrt{\frac{\ell_\mathrm{p}\,m_\mathrm{p}\,k}{2}}\,(k\,r)\left(\hat{a}_k\,e^{i\,k\,t} - \hat{a}_k^\dagger\,e^{-i\,k\,t}\right) \ , \tag{3.3.7}$$

where the creation and annihilation operators satisfy

$$\left[\hat{a}_h, \hat{a}_k^\dagger\right] = \frac{2\,\pi^2}{k^2}\,\delta(h-k) \ , \tag{3.3.8}$$

and we again used Eq. (3.2.34).

3.3.1 Newtonian Potential

Let us now turn to Eq. (3.2.21), and look for a quantum state $|\,g\,\rangle$ of Φ which reproduces the classical solution. First of all, we will consider the Newtonian case, that is we set $q_\Phi = 0$ and find a solution for Eq. (3.2.23). In terms of the new variables Φ and J_B, this equation reads

$$\triangle\Phi_c(r) = q_\mathrm{B}\,J_\mathrm{B}(r) \tag{3.3.9}$$

where we emphasised that we shall only consider static currents $J_\mathrm{B} = J_\mathrm{B}(r)$ and correspondingly static fields. Upon expanding (3.3.9) on the modes (3.2.29), one finds the classical solution in momentum space is of course given by Eq. (3.2.35), which now reads

$$\tilde{\Phi}_c(k) = -q_\mathrm{B}\,\frac{\tilde{J}_\mathrm{B}(k)}{k^2} \ , \tag{3.3.10}$$

with $\tilde{J}_\mathrm{B}^*(k) = \tilde{J}_\mathrm{B}(k)$ from the reality of $J_\mathrm{B}(r)$, and analogously $\tilde{\Phi}_c^*(k) = \tilde{\Phi}_c(k)$. We then define the coherent state

$$\hat{a}_k|\,g\,\rangle = e^{i\,\gamma_k(t)}\,g_k|\,g\,\rangle \ , \tag{3.3.11}$$

where

$$g_k = \sqrt{\frac{k}{2\,\ell_\mathrm{p} m_\mathrm{p}}}\,\tilde{\Phi}_c(k) = -q_\mathrm{B}\,\frac{\tilde{J}_\mathrm{B}(k)}{\sqrt{2\,\ell_\mathrm{p} m_\mathrm{p}\,k^3}} \ . \tag{3.3.12}$$

Acting on such a state with the operator $\hat{\Phi}$ yields the expectation value

$$\begin{aligned}
\langle\, g\,|\hat{\Phi}(t,r)|\, g\,\rangle &= \langle\, g\,| \int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\sqrt{\frac{\ell_\mathrm{p}\,m_\mathrm{p}}{2\,k}}\, j_0(k\,r)\left(\hat{a}_k\, e^{i\,k\,t} + \hat{a}_k^\dagger\, e^{-i\,k\,t}\right)|\, g\,\rangle \\
&= \langle\, g\,|\, g\,\rangle \int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\sqrt{\frac{\ell_\mathrm{p}\,m_\mathrm{p}}{2\,k}}\, j_0(k\,r) \\
&\qquad \times \left[g_k\, e^{i\,k\,t+i\,\gamma_k(t)} + g_k^*\, e^{-i\,k\,t-i\,\gamma_k(t)}\right] \\
&= -q_\mathrm{B}\,\langle\, g\,|\, g\,\rangle \int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\,\frac{j_0(k\,r)}{2\,k^2}\,\tilde{J}_\mathrm{B}(k) \\
&\qquad \times \left[e^{i\,k\,t+i\,\gamma_k(t)} + e^{-i\,k\,t-i\,\gamma_k(t)}\right]
\end{aligned} \tag{3.3.13}$$

Now, assuming $\langle\, g\,|\, g\,\rangle = 1$ and $\gamma_k(t) = -k\,t$, we finally obtain

$$\begin{aligned}
\langle\, g\,|\hat{\Phi}(t,r)|\, g\,\rangle &= -q_\mathrm{B} \int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\, j_0(k\,r)\,\frac{\tilde{J}_\mathrm{B}(k)}{k^2} \\
&= \int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\, j_0(k\,r)\,\tilde{\Phi}_c(k) = \Phi_c(r)\ ,
\end{aligned} \tag{3.3.14}$$

which is exactly the classical solution to Eq. (3.3.9).

It is particularly important to study the normalisation of $|\, g\,\rangle$. One can explicitly write this state in terms of the true vacuum $|\, 0\,\rangle$ as

$$|\, g\,\rangle = e^{-\frac{N_\mathrm{G}}{2}} \exp\left\{\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\, g_k\,\hat{a}_k^\dagger\right\}|\, 0\,\rangle\ , \tag{3.3.15}$$

where N_G is just a normalisation factor for now. By making use of the commutation relation (3.3.8) and the well-known Baker-Campbell-Hausdorff formulas, one then obtains

$$\begin{aligned}
\langle\, g\,|\, g\,\rangle &= e^{-N_\mathrm{G}}\langle\, 0\,|\exp\left\{\int_0^\infty \frac{p^2\,\mathrm{d}p}{2\,\pi^2}\, g_p^*\,\hat{a}_p\right\}\exp\left\{\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\, g_k\,\hat{a}_k^\dagger\right\}|\, 0\,\rangle \\
&= e^{-N_\mathrm{G}}\langle\, 0\,|\exp\left\{\int_0^\infty \frac{p^2\,\mathrm{d}p}{2\,\pi^2}\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\, g_p\, g_k\,\left[\hat{a}_p, \hat{a}_k^\dagger\right]\right\}|\, 0\,\rangle \\
&= e^{-N_\mathrm{G}} \exp\left\{\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\, g_k^2\right\}\ ,
\end{aligned} \tag{3.3.16}$$

so that

$$N_\mathrm{G} = \int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\, g_k^2 = \langle\, g\,|\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\,\hat{a}_k^\dagger\,\hat{a}_k\,|\, g\,\rangle\ , \tag{3.3.17}$$

and N_G is shown to precisely equal the total occupation number of modes in the state $|\, g\,\rangle$. This quantity typically diverges, as we can show with a simple example.

Let us consider the point-like source (3.2.39), for which we have

$$g_k = -q_{\rm B}\,\frac{4\,\pi\,M_0}{m_{\rm p}\,\sqrt{2\,k^3}}\ . \tag{3.3.18}$$

The general treatment above shows that the zero-order field will exactly equal the (suitably rescaled) Newtonian potential (3.2.41), and

$$N_{\rm G} = q_{\rm B}^2\,\frac{4\,M_0^2}{m_{\rm p}^2}\int_{k_0}^{\Lambda}\frac{{\rm d}k}{k} = q_{\rm B}^2\,\ln\!\left(\frac{\Lambda}{k_0}\right)\frac{4\,M_0^2}{m_{\rm p}^2}\ , \tag{3.3.19}$$

where we introduced both a infrared cut-off k_0 and a ultraviolet cut-off Λ to regularise the divergences. The latter originates from the source being point-like, which allows for modes of infinitely large momentum, and is usually not present when one considers regular matter densities. The former is instead due to assuming the source lives in an infinite volume or, equivalently, is eternal so that its static gravitational field extends to infinite distances

Had we considered a source of mass M with finite size R, we can anticipate that one would typically find

$$N_{\rm G} \sim \frac{M^2}{m_{\rm p}^2}\,\ln\!\left(\frac{R_\infty}{R}\right)\ , \tag{3.3.20}$$

where $R_\infty = k_0^{-1} \gg R$ denotes the size of the universe within which the gravitational field is static. It is of paramount importance to note that $N_{\rm G}$ depends on R much less than it does on the mass M, since

$$\frac{{\rm d}N_{\rm G}}{N_{\rm G}} \sim 2\,\frac{{\rm d}M}{M} - \frac{1}{\ln(R_\infty/R)}\,\frac{{\rm d}R}{R}\ , \tag{3.3.21}$$

and the effect of the variation in the source size R can be made arbitrarily small by simply choosing a very large R_∞. This results can in fact be confirmed explicitly by employing the Gaussian source (3.2.66), that is

$$\tilde{J}_{\rm B}(k) = 4\,\pi\,M_0\,\sqrt{\frac{\ell_{\rm p}}{m_{\rm p}}}\,e^{-\frac{\sigma^2\,k^2}{4}}\ , \tag{3.3.22}$$

from which

$$N_{\rm G} = q_{\rm B}^2\,\frac{4\,M_0^2}{m_{\rm p}^2}\int_{k_0}^{\infty}\frac{{\rm d}k}{k}\,e^{-\frac{\sigma^2\,k^2}{4}} = q_{\rm B}^2\,\frac{2\,M_0^2}{m_{\rm p}^2}\,\Gamma\!\left(0,\frac{\sigma^2}{R_\infty^2}\right)\ , \tag{3.3.23}$$

where we again introduced a cut-off $k_0 = 1/(2\,R_\infty)$ and

$$\Gamma(0,x) = \int_x^\infty \frac{\mathrm{d}t}{t}\,e^{-t} \tag{3.3.24}$$

is the (lower) incomplete gamma function. The relative variation,

$$\frac{\mathrm{d}N_\mathrm{G}}{N_\mathrm{G}} = 2\,\frac{\mathrm{d}M_0}{M_0} - \frac{2\,e^{-\sigma^2/R_\infty^2}}{\Gamma\left(0,\sigma^2/R_\infty^2\right)}\,\frac{\mathrm{d}\sigma}{\sigma}\ , \tag{3.3.25}$$

shows once more that the number of quanta in the coherent state is much more influenced by changes in the bare mass of the source than it is by changes in the width σ, for the arbitrary cut-off R_∞ may be taken much larger than σ. Moreover, since

$$\Gamma\left(0,\frac{\sigma^2}{R_\infty^2}\right) \simeq 2\,\ln\left(\frac{R_\infty}{\sigma}\right)\ , \tag{3.3.26}$$

we see that the estimate in Eq. (3.3.20) is actually confirmed by taking $R \simeq \sigma$.

3.3.2 Post-Newtonian Corrections

Having established that

$$\sqrt{\frac{\ell_\mathrm{p}}{m_\mathrm{p}}}\langle\, g\,|\hat{\Phi}(t,r)|\,g\,\rangle = V_\mathrm{N}(r) = V_{(0)}(r)\ , \tag{3.3.27}$$

solves Eq. (3.2.23), we can tackle Eq. (3.2.24), which we now rewrite as

$$\triangle V_{(1)} = 2\,\frac{\ell_\mathrm{p}}{m_\mathrm{p}}\,\langle\, g\,|\left(\hat{\Phi}'\right)^2|\,g\,\rangle\ . \tag{3.3.28}$$

In the above,

$$\hat{\Phi}'(t,r) = -\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\sqrt{\frac{\ell_\mathrm{p}\,m_\mathrm{p}}{2\,k}}\,k\,j_1(k\,r)\left(\hat{a}_k\,e^{i\,k\,t} + \hat{a}_k^\dagger\,e^{-i\,k\,t}\right)\ , \tag{3.3.29}$$

so that

$$\begin{aligned}
2\,\frac{\ell_\mathrm{p}}{m_\mathrm{p}}\langle\, g\,|\left(\hat{\Phi}'\right)^2|\,g\,\rangle \;&=\; 2\,\ell_\mathrm{p}^2\int_0^\infty \frac{p^{5/2}\,\mathrm{d}p}{2\,\sqrt{2}\,\pi^2}\int_0^\infty \frac{k^{5/2}\,\mathrm{d}k}{2\,\sqrt{2}\,\pi^2}\,j_1(p\,r)\,j_1(k\,r) \\
&\qquad \times\langle\, g\,|\left(\hat{a}_p\,e^{i\,p\,t} + \hat{a}_p^\dagger\,e^{-i\,p\,t}\right)\left(\hat{a}_k\,e^{i\,k\,t} + \hat{a}_k^\dagger\,e^{-i\,k\,t}\right)|\,g\,\rangle \\
&=\; 2\,\ell_\mathrm{p}^2\int_0^\infty \frac{p^{5/2}\,\mathrm{d}p}{2\,\sqrt{2}\,\pi^2}\int_0^\infty \frac{k^{5/2}\,\mathrm{d}k}{2\,\sqrt{2}\,\pi^2}\,j_1(p\,r)\,j_1(k\,r) \\
&\qquad \times\left(4\,g_p\,g_k + \left[\hat{a}_p,\hat{a}_k^\dagger\right]e^{i\,(p-k)\,t}\right) \\
&=\; 8\,\ell_\mathrm{p}^2\left[\int_0^\infty \frac{k^{5/2}\,\mathrm{d}k}{2\,\sqrt{2}\,\pi^2}\,j_1(k\,r)\,g_k\right]^2 + 2\,\ell_\mathrm{p}^2\int_0^\infty \frac{k^3\,\mathrm{d}k}{4\,\pi^2}\,\left[j_1(k\,r)\right]^2 \\
&\equiv\; J_g + J_0\ .
\end{aligned} \tag{3.3.30}$$

Note that the (diverging) term denoted by J_0 is a purely vacuum contribution independent of the quantum state and we can simply discard it by imposing the normal ordering in the expectation value above. From the expression (3.3.12) of the eigenvalues g_k, with the rescaling (3.3.1) for the matter density, one can immediately see that J_g equals the classical expression (3.2.37), that is

$$2\,\frac{\ell_{\rm p}}{m_{\rm p}}\langle\, g \mid \left(\hat{\Phi}'\right)^2 \mid g\,\rangle = 2\,\left(V'_{(0)}\right)^2 \ , \tag{3.3.31}$$

for any matter distribution. This shows that the coherent state $\mid g\,\rangle$ obtained from the Newtonian potential is indeed a very good starting point for our perturbative quantum analysis.

We should now determine a modified coherent state $\mid g'\,\rangle$, such that

$$\sqrt{\frac{\ell_{\rm p}}{m_{\rm p}}}\,\langle\, g' \,|\hat{\Phi}|\, g'\,\rangle \simeq V_{(0)} + q_\Phi\, V_{(1)} \ , \tag{3.3.32}$$

where all expressions will be given to first order in q_Φ from now on. Like we expanded the classical potential in Eq. (3.2.22), we can also write

$$\mid g'\,\rangle \simeq \mathcal{N}\left(\mid g\,\rangle + q_\Phi \mid \delta g\,\rangle\right) \ , \tag{3.3.33}$$

with

$$\hat{a}_k \mid g'\,\rangle \simeq g_k \mid g\,\rangle + q_\Phi\, \delta g_k \mid \delta g\,\rangle \ , \tag{3.3.34}$$

and the normalisation constant

$$|\mathcal{N}|^2 \simeq 1 - 2\,q_\Phi\,{\rm Re}\,\langle\, \delta g \mid g\,\rangle \ . \tag{3.3.35}$$

Upon replacing these expressions, we obtain

$$\begin{aligned}
\langle\, g' \,|\hat{\Phi}|\, g'\,\rangle &\simeq \left(1 - 2\,q_\Phi\,{\rm Re}\,\langle\, \delta g \mid g\,\rangle\right)\Big(\langle\, g \,|\hat{\Phi}|\, g\,\rangle + 2\,q_\Phi\,{\rm Re}\,\langle\, \delta g \,|\hat{\Phi}|\, g\,\rangle\Big) \\
&\simeq \langle\, g \,|\hat{\Phi}|\, g\,\rangle + 2\,q_\Phi\,{\rm Re}\,\langle\, \delta g \,|\hat{\Phi}|\, g\,\rangle - 2\,q_\Phi\,\langle\, g \,|\hat{\Phi}|\, g\,\rangle\,{\rm Re}\,\langle\, \delta g \mid g\,\rangle \ ,
\end{aligned} \tag{3.3.36}$$

and Eq (3.3.32) yields

$$\mathrm{Re}\langle\, \delta g \,|\hat{\Phi}|\, g\,\rangle - \langle\, g \,|\hat{\Phi}|\, g\,\rangle\,\mathrm{Re}\langle\, \delta g \mid g\,\rangle = \sqrt{\frac{m_\mathrm{p}}{\ell_\mathrm{p}}}\,\frac{V_{(1)}}{2}\ . \tag{3.3.37}$$

By applying the Laplacian operator on both sides, we finally get

$$\frac{\triangle\left(\mathrm{Re}\langle\, \delta g \,|\hat{\Phi}|\, g\,\rangle\right)}{\mathrm{Re}\langle\, \delta g \mid g\,\rangle} = \triangle\langle\, g \,|\hat{\Phi}|\, g\,\rangle + \sqrt{\frac{\ell_\mathrm{p}}{m_\mathrm{p}}}\,\frac{\langle\, g \mid \left(\hat{\Phi}'\right)^2 \mid g\,\rangle}{\mathrm{Re}\langle\, \delta g \mid g\,\rangle}\ , \tag{3.3.38}$$

where we used Eqs. (3.3.28).

The above equation relates each eigenvalue δg_k to all of the g_p's in the Newtonian coherent state, which obviously makes solving it very complicated. We will instead estimate the solution by following the argument of Ref. [54] that was summarised in the Introduction. Namely, we assume most of the N_G gravitons are in one mode of wavelength $\lambda_\mathrm{G} \simeq R$(see Chapter 2), so that

$$\hat{\Phi} \simeq \sqrt{\ell_\mathrm{p}\, m_\mathrm{p}}\,\bar{k}^{3/2}\,\Delta\bar{k}\, j_0(\bar{k}\, r)\,\left(\hat{a}_{\bar{k}} + \hat{a}^\dagger_{\bar{k}}\right)\ , \tag{3.3.39}$$

where $\bar{k} \simeq R^{-1} \simeq \Delta\bar{k}$, and we neglect numerical factors of order one. In particular, we have

$$\begin{aligned} &\simeq\ \sqrt{\ell_\mathrm{p}\, m_\mathrm{p}}\,\bar{k}^{3/2}\,\Delta\bar{k}\, j_0(\bar{k}\, r)\, g_{\bar{k}} && (3.3.40)\\ \langle\, \delta g \,|\hat{\Phi}|\, g\,\rangle\ &\simeq\ \langle\, \delta g \mid g\,\rangle\,\sqrt{\ell_\mathrm{p}\, m_\mathrm{p}}\,\bar{k}^{3/2}\,\Delta\bar{k}\, j_0(\bar{k}\, r)\,(g_{\bar{k}} + \delta g_{\bar{k}})\ , && (3.3.41)\end{aligned}$$

and

$$\langle\, g \mid \left(\hat{\Phi}'\right)^2 \mid g\,\rangle \simeq \ell_\mathrm{p}\, m_\mathrm{p}\,\bar{k}^5\,(\Delta\bar{k})^2\, j_1^2(\bar{k}\, r)\, g_{\bar{k}}^2\ , \tag{3.3.42}$$

where we again subtracted the vacuum term J_0 from Eq. (3.3.30). Plugging these results into Eq.(3.3.38) finally yields

$$\delta g_{\bar{k}} \simeq -\ell_\mathrm{p}\,\bar{k}^{3/2}\,\Delta\bar{k}\, g_{\bar{k}}^2 \sim -\ell_\mathrm{p}\,\bar{k}^{5/2}\, g_{\bar{k}}^2\ . \tag{3.3.43}$$

For instance, for the point-like source (3.2.39), one obtains

$$\delta g_{\bar{k}} \sim -\frac{\ell_{\rm p}\, M_0^2}{m_{\rm p}^2\, \bar{k}^{1/2}} \sim \frac{\ell_{\rm p}\, M_0}{m_{\rm p}\, r_0}\, g_{\bar{k}} \sim \frac{R_{\rm H}}{r_0}\, g_{\bar{k}} \ , \tag{3.3.44}$$

where we set the characteristic size of the source $R \sim r_0$, the latter being the same ultra-violet cut-off we introduced for computing the (diverging) classical gravitational energy (3.2.49). For $r_0 \ll R_{\rm H}$, this result clearly falls outside the range of our approximations, since $\delta g_{\bar{k}} \gg g_{\bar{k}}$ (of course, we assume $q_{\rm B} \sim q_\Phi \sim 1$). For the Gaussian source (3.2.66), we instead obtain

$$\delta g_{\bar{k}} \sim \frac{\ell_{\rm p}\, M_0^2}{m_{\rm p}^2\, \bar{k}^{1/2}}\, e^{-\frac{\sigma^2\, \bar{k}^2}{2}} \sim \frac{R_{\rm H}}{\sigma}\, g_{\bar{k}} \ , \tag{3.3.45}$$

having set $\bar{k} \simeq R^{-1} \sim \sigma^{-1}$. We then see the perturbation $\delta g_{\bar{k}} \ll g_{\bar{k}}$ when the source is much more extended than its gravitational radius, which is indeed consistent with the classical results we are trying to reproduce quantum mechanically.

Conclusions and Outlook

Starting from the Einstein-Hilbert action in weak-field and non-relativistic approximations we first derived an effective scalar action for the Newtonian potential, which sets the ground to the addition of non-linearities. Actually, after having ensured that our theory correctly reproduced the Poisson equation (1.2.22), we were able to find a more general and interesting effective action. We added the next-to-leading order terms in the above approximation and found out that those contributions exactly accounted for the post-Newtonian corrections expected by general relativity.

We obtained the above corrections in two different cases. First, we considered the simple, although unrealistic, situation of a uniform and compact distribution of matter. Then, the more regular and less trivial choice of a gaussian matter distribution was taken into account. Of course, in both instances, the equations of motion could not be solved exactly, thus we used perturbative methods. It was found a posteriori, as in every non linear theory, that this expansion is allowed unless the observer approach the Schwarzschild radius of the source.

In the second part of our work, we reproduced the above results in a quantum theory. In particular, we canonically quantized a suitably dimensioned potential field whose action was found simply rescaling the effective classical one previously obtained. Finally, we identified an appropriate quantum coherent state and reproduced the classical results through the expectation values of the field.

These detailed calculations substantially support the energy balance outlined in Ref. [54] and the consequent derivation of the maximal packing condition (3.1.9), which is a crucial ingredient for corpuscular models of black holes, therefore establishing a surprising, though expected, connection between the quantum corpuscular model and post-Newtonian gravity.

We did not mention before that we were also able to estimate the number of gravitons

$N_{\rm G}$ in that state. We can make some conclusive considerations about this result and the others.

Although it remains true $N_{\rm G}$ mainly depends on the mass of the static source, we found it also (weakly) depends on the ratio R/R_∞ between the size of the source and the size of the region within which the gravitational potential is static. Such a dependence becomes negligible for an ideal static system (with $R_\infty \to \infty$), but could play a much bigger role in a dynamical situation when the source evolves in time and the extension of the outer region of static potential is comparable to R. In fact, the number $N_{\rm G}$ in Eq. (3.3.20) vanishes for $R_\infty \simeq R$ and grows logarithmically with R_∞, meaning that the (Newtonian) coherent state $|\, g\,\rangle$ becomes (logarithmically) more and more populated as the region of static potential extends further and further away from the source.

It would also be tempting to consider the case $R = R_{\rm H}$ and relate the second term in Eq. (3.3.21) to logarithmic corrections for the Bekenstein-Hawking entropy of black holes. We have however noted repeatedly that a source of size $R \lesssim R_{\rm H}$ usually falls outside the regime of validity of our approximations. Nonetheless, from the classical point of view, nothing particularly wrong seems to happen in the limiting case $R \simeq R_{\rm H}$, except the very equality (3.1.9) that gives support to the corpuscular model of black holes now occurs precisely in this borderline condition. That $R \simeq R_{\rm H}$ becomes critical for our description is further made clear by the estimate (3.3.45) of quantum corrections $|\, \delta g\,\rangle$ to the coherent state $|\, g\,\rangle$ that reproduces the Newtonian potential, since the corrections must become comparable to the Newtonian part for $\sigma \sim R \to R_{\rm H}$. Whether this is in full agreement with the post-Newtonian description of General Relativity or it instead signals a breakdown of the classical picture near the threshold of black hole formation will require a much more careful analysis. We leave this seemingly very relevant topic of quantum perturbations, along with the role of matter pressure (which we totally neglected here), for future works.

Appendix A

Hypersurfaces and embeddings

The aim of this appendix is to define the notions of embedding, embedded hypersurface and the related intrinsic and extrinsic curvature. All of these definitions will turn very useful when describing the ADM decomposition of spacetime and therefore to the hamiltonian formulation of general relativity. We start by defining a hypersurface Σ as an n dimensional submanifold of a $n+1$ dimensional manifold M, where the definition of a manifold is assumed. There are two ways to think about a hypersurface.

1. Embedding

   Here we define the hypersurface through an embedding map

   $$\Phi : \Sigma \to M. \tag{A.0.1}$$

   This definition can be made more concrete describing M with coordinates x^μ and Σ with coordinates y^i. Then the embedding Φ is given specifying the coordinate of the point in M which correspond to a point in Σ, with respect to the coordinates of Σ. Thus, Φ gives a system of parametric equations

   $$\Phi : y^i \to x^\mu(y^i), \tag{A.0.2}$$

   where Φ must be at least injective so that distinct points in Σ are mapped in distinct points in M.

2. Embedded hypersurface

   Now we consider the hypersurface really as a submanifold of M, defined by putting a restriction on the coordinates

$$\Sigma = \{x \in M, S(x) = 0\}, \tag{A.0.3}$$

for some real valued function S on M.

These two definitions can easily be shown to be equivalent but we will use both because sometimes one is easier than the other. Their equivalence is established using S as one of the coordinate of the manifold M, so that $x^\mu \to (S, x^i)$. Now the embbedding become

$$x^\mu(y) \to \quad S(y) = 0, \quad x^i(y) = y^i \tag{A.0.4}$$

It is very useful for our purposes to consider the case in which we have a spacetime manifold in which we can identify a "time coordinate". Here $x^\mu = (t, x^i)$ and the hypersurface will be described by $t(y) = t_0$, $x^i(y) = y^i$ using the first definition and $S(t, x^i) = t - t_0 = 0$ with the other one.

Our principal interest will be the description of the so called "intrinsic" and "extrinsic" geometry. The first is provided by the metric tensor induced on the hypersurface, therefore it is related to the properties of measuring lengths, areas, volumes, angles etc on the hypersurface. The latter instead is strictly related to the embedding and contains informations about how the hypersurface is embedded in the M manifold.

A.1 Normal and Tangent Vectors

Now, we need to define normal and tangent vectors to the hypersurface. When we think at the hypersurface as the embedding it is easy to see that the tangent vectors are

$$\partial_i \to \frac{\partial x^\mu}{\partial y^i}\partial_\mu = e_i^\mu \partial_\mu, \tag{A.1.1}$$

thus every tangent vector can be written as

$$v^\mu = v^i e_i^\mu, \tag{A.1.2}$$

where v^i is the tangent vector on Σ corresponding to the tangent vector v^μ on M. Since the $e_i^\mu = \frac{\partial x^\mu}{\partial y^i}$ are linearly independent tangent vectors to the image of Σ in M , normal vectors ξ^μ to are characterised by

$$g_{\mu\nu} e_i^\mu \xi^\nu = 0. \tag{A.1.3}$$

Of course they can be normalised to have

$$n_\mu n^\mu = \epsilon \tag{A.1.4}$$

where ϵ is +1 if the hypersurface is timelike an -1 when it is spacelike. The latter will be our case when we study the hamiltonian formulation of general relativity with hypersurfaces of constant t. This definition of normal vectors is somewhat implicit. We can give a more concrete definition for them thinking the hypersurface as an embedded hypersurface. In fact this characterization imply that $S(x) \neq 0$ when one moves away of Σ, therefore $\partial_\mu S \neq 0$ on it . That gradient can then be identified with the normal vector to Σ, thinking that S varies only in the orthogonal direction of the hypersurface. What is more, it can be normalised in a straightforward manner to give (A.1.4).

A.2 Induced Metric and Intrinsic Geometry

In the case of a spacelike (or timelike) hypersurface Σ endowed with normalised normal vectors n^μ,we can construct the induced metric out of the full metric $g_{\mu\nu}$ of the entire manifold M where Σ is embedded. It will be very usefull to take both points of view introduced at the beginning of the appendix into account. The first will directly lead to the induced metric as a restriction on Σ of metric of the ambient space M. On the other hand, the second approach will provide us with projectors which allow us to project tensors defined on M and restricted to Σ into tangent directions on Σ. These will be very useful when trying to describe the extrinsic curvature of Σ.

Lets start describing how the parametrized form of the hypersurface naturally lead to the induced metric. Actually, this is simply achieved by restricting the metric and the displacements to Σ,

$$\begin{aligned} ds^2|_\Sigma &= g_{\mu\nu}(x)dx^\mu dx^\nu|_\Sigma && \text{(A.2.1)} \\ &= g_{\mu\nu}(x(y))\frac{\partial x^\mu}{\partial y^i}\frac{\partial x^\nu}{\partial y^j}dy^i dy^j && \text{(A.2.2)} \\ &= h_{ij}(y)dy^i dy^j. && \text{(A.2.3)} \end{aligned}$$

Thus the induced metric reads

$$h_{ij}(y) = g_{\mu\nu}(x(y))\frac{\partial x^\mu}{\partial y^i}\frac{\partial x^\nu}{\partial y^j} = g_{\mu\nu}e^\mu_i e^\nu_j. \tag{A.2.4}$$

The difference between the two metric is evident from these relations.That is, $g_{\mu\nu}$ is a (0,2) tensor under spacetime coordinate transformation and a scalar under transformations of coordinate on Σ, while h_{ij} enjoy the opposite properties.

Now, lets turn to the other point of view. With the normal unit vectors $n^\mu n_\mu = \epsilon = \pm 1$ we can build up the tensor $h_{\mu\nu}$, defined on Σ

$$h_{\mu\nu} = g_{\mu\nu} - \epsilon n_\mu n_\nu, \tag{A.2.5}$$

it is easy to see that it is orthogonal to the vectors n^μ

$$h_{\mu\nu} n^\nu = g_{\mu\nu} n^\nu - \epsilon n_\mu n_\nu n^\nu = n_\mu - \epsilon^2 n_\mu = 0, \tag{A.2.6}$$

and that for vectors V^μ orthogonal to the n^μ, the scalar product with $h_{\mu\nu} n^\nu$ is identical to that with $g_{\mu\nu} n^\nu$

$$g_{\mu\nu} n^\mu V^\nu = 0 \quad \rightarrow \quad g_{\mu\nu} V^\nu = h_{\mu\nu} V^\nu. \tag{A.2.7}$$

These two properties ensure that $h_{\mu\nu}$, restricted to Σ, is the induced metric on that hypersurface. We can see the precise relation making this restriction and this also unify the two points of view. In fact, recalling equation (A.2.4)

$$h_{ij} = g_{\mu\nu} e^\mu_i e^\nu_j = h_{\mu\nu} e^\mu_i e^\nu_j, \tag{A.2.8}$$

where in the second equality we took advantage of the orthogonality between n^μ and e^μ_i. The differences between $h_{\mu\nu}$ and h_{ij} are substantially that the first is a (0,2) tensor on M and a scalar on Σ and it is degenerate while the second enjoys the opposite features.

We can now build the projection operators

$$h^\mu{}_\nu = g^{\mu\gamma} h_{\gamma\nu} = \delta^\mu{}_\nu - \epsilon n^\mu n_\nu, \tag{A.2.9}$$

which allows to project every covariant or contravariant spacetime tensor field onto tangent vectors to Σ. This can be easily demonstrated by direct calculation.

Given the induced metric, one could now start constructing all the objects already know from differential geometry such as the covariant derivative of arbitrary tensors on Σ. We will denote with an index (Σ) the objects built in this way.

$${}^{(\Sigma)}\nabla_i v^j = \partial_i v^j + {}^{(\Sigma)}\Gamma^j_{ik} v^k = D_i v^j, \qquad {}^{(\Sigma)}\nabla_i v_j = \partial_i v^j - {}^{(\Sigma)}\Gamma^k_{ij} v_k = D_i v_j, \tag{A.2.10}$$

where ${}^{(\Sigma)}\Gamma^{j}_{ik}$ satisfy the usual relation but with the metric h_{ij} and lead to the usual definition of Riemann and Ricci tensors and curvature scalar, of course intrinsic to Σ. On the other hand we can think at a more general procedure to do so. Namely, we could start considering a tangent vector to Σ in the spacetime, take his projected covariant derivative using the projectors (A.2.9) and then "pull back" to the Σ coordinate system. It can be demonstrated, [14], that this operation will lead to the same result, i.e.

$$D_i v_j = ^{(\Sigma)} \nabla_i v_j = e^{\mu}_i e^{\nu}_j h_{\mu}{}^{\gamma} h_{\nu}{}^{\delta} \nabla_{\gamma} v_{\delta} \tag{A.2.11}$$

$$= e^{\gamma}_i e^{\delta}_j \nabla_{\gamma} v_{\delta}. \tag{A.2.12}$$

Thus for projected tangent vectors the projected covariant derivative is equal to the intrinsic covariant derivative of the hypersurface Σ. The next step will be the understanding of what happens to normal vectors to Σ and to do so we will need the extrinsic geometry, something strictly related to the embedding of the hypersurface in the ambient manifold.

A.3 Extrinsic Geometry

As mentioned before, intrinsic geometry is only one aspect of an hypersurface. In particular, we showed it is related to everything which can be defined on the hypersurface alone, with no concern to the fact that this hypersurface is embedded in an ambient space. Extrinsic geometry will provide us with the missing part of this study, telling how the hypersurface is embedded in the ambient space. To detect the presence of this embedding, one need to move off the hypersurface and this is why it can't be understood by means of intrinsic measurements on Σ alone. To make it clearer, consider a circle S^1 embedded in R^2. Of course the intrinsic geometry of the circle is flat because the Riemann tensor identically vanish in one dimensional spaces. At the same time, it is easy to understand that this circle bend around in R^2 and this feature will be captured only studying the extrinsic curvature of S^1. No information about this bending will never be given by intrinsic features of the circle.

These arguments let us understand that the defining relation of the extrinsic curvaure is the change in the normal vectors to Σ, expressed through their covariant derivative and then projected onto Σ. Namely, the extrinsic curvature of Σ in M is

$$K_{\mu\nu} = h_{\mu}{}^{\gamma} h_{\nu}{}^{\delta} \nabla_{\gamma} n_{\delta}, \tag{A.3.1}$$

where we made use of the tangential projectors to Σ, (A.2.9). This simplify if we extend the normal vectors outside Σ in a way that $n^\mu n_\mu = \epsilon$. Actually, the second projection will be useless, a part from the delta factor it provides, and we can write

$$K_{\mu\nu} = h_\mu{}^\gamma \nabla_\gamma n_\nu = \nabla_\mu n_\nu - \epsilon n_\mu n^\gamma \nabla_\gamma n_\nu. \tag{A.3.2}$$

Having this in mind and with a parametrised form of the hypersurface $x^\mu(y^k)$, then we can "pull back" this tensor as we did for the intrinsic geometry before and write the extrinsic curvature tensor as

$$K_{ij} = e^\mu_i e^\nu_j K_{\mu\nu} = e^\mu_i e^\nu_j \nabla_\mu n_\nu. \tag{A.3.3}$$

Then, taking the relation $e^\nu_j n_\nu = 0$ into account we can rewrite it in the form

$$e^\mu_i e^\nu_j \nabla_\mu n_\nu = -e^\mu_i n_\nu \nabla_\mu e^\nu_j = -\left(\partial_i e^\nu_j + \Gamma^\nu_{\mu\gamma} e^\mu_i e^\gamma_j\right) n_\nu. \tag{A.3.4}$$

This shows that the extrinsic curvature tensor is symmetric as a consequence of the definition of e^ν_j and the simmetry of the Christoffel symbols. Last equation help us rewriting the extrinsic curvature tensor in a useful form

$$K_{ij} = e^\mu_i e^\nu_j \left(\partial_\mu n_\nu - \Gamma^\rho_{\mu\nu} n_\rho\right) \tag{A.3.5}$$

Now we understand the anticipated relation between the extrinsic curvature tensor and the normal components of the connection. In general the covariant derivative of a tangent vector won't be again a tangent vector, it will rather have a normal component. In fact consider the expression $e^\mu_i \nabla_\mu v^\nu$, where v^ν is a tangent vector and manipulate it in the following way

$$\begin{aligned} e^\mu_i \nabla_\mu v^\nu &= g^{\nu\gamma} e^\mu_i \nabla_\mu v_\gamma = \left(h^{\nu\gamma} + \epsilon n^\nu n^\gamma\right) e^\mu_i \nabla_\mu v_\gamma && \text{(A.3.6)} \\ &= \left(h^{jk} e^\nu_j e^\gamma_k + \epsilon n^\nu n^\gamma\right) e^\mu_i \nabla_\mu v_\gamma && \text{(A.3.7)} \\ &= h^{jk} e^\nu_j D_i v_k + \epsilon n^\nu e^\mu_i v^j n^\gamma h_{\gamma\delta} \nabla_\mu e^\delta_j && \text{(A.3.8)} \\ &= \left(D_i v^j\right) e^\nu_j - \epsilon \left(v^j K_{ij}\right) n^\nu, && \text{(A.3.9)} \end{aligned}$$

where we made use of the definitions (A.2.12), (A.3.3) and of the decomposition of the metric (A.2.5). It is now clear that the extrinsic geometry embodies the normal components of the connection. In fact the normal component of the parallely transported vector v^ν vanishes only when K_{ij} does so.

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