ALMA MATER STUDIORUM · UNIVERSITÀ DI BOLOGNA

FACOLTÀ DI SCIENZE MATEMATICHE, FISICHE E NATURALI
Corso di Laurea Magistrale in Matematica

ON THE SET OF OPTIMAL HOMEOMORPHISMS FOR THE NATURAL PSEUDO-DISTANCE ASSOCIATED WITH THE LIE GROUP S^1

Tesi di Laurea in Topologia Computazionale

Relatore:
Chiar.mo Prof.
Patrizio Frosini

Presentata da:
Alessandro De Gregorio

Sessione III
Anno Accademico 2015/16

Contents

Introduction

In this dissertation we will examine the concept of natural pseudo-distance, often associated with persistent homology theory. Persistent homology is an algebraic topological method used in order to study spaces of functions and their topological features. This theory has gathered the interest of the mathematical community in the last 25 years and even that of other sciences because of its usefulness in data and shape comparison problems. The value of this theory resides in its robustness to noise, discerning relevant features from trivial ones, and its ability to compress the information content of big sets of data or shapes. Persistent homology can be seen as a branch of classical homology reconsidered with some aspects deriving from Morse theory. A presentation of this theory can be found in [14].
Let us consider a topological space X and let us fix a continuous real-valued function φ defined on X, from now on called *filtering function*. It is possible to compute the homology groups of the sublevel sets of φ, looking for the evolution of the topological features of these sets. We can consider the birth and the death of each homology class, caused by the variation of the topological properties of the sublevel sets, and construct a multiset called *persistence diagram* to sum up all this information. Details regarding this construction can be found in the survey [12]. The persistence diagram reflects the main topological properties of the space X and the function φ, enabling us to associate with each encountered homology class its lifespan. Homology classes with distant birth and death time will often represent relevant topological features, while classes with shorter life are usually associated with noise and

less important features.
In this thesis we will study the concept of *natural pseudo-distance*, a dissimilarity measure between filtering functions, that works in a complementary way with respect to persistent homology. Let us take a compact topological space X and two continuous functions $\varphi, \psi : X \to \mathbb{R}$, and consider a subgroup G of the group H of all homeomorphisms from X onto itself. The natural pseudo-distance between φ and ψ associated with the group G is defined as

$$d_G(\varphi, \psi) = \inf_{g \in G} \max_{P \in X} |\varphi(P) - \psi(g(P))|. \tag{1}$$

The case in which $G = H$ has been already an object of study. In fact if X is a closed manifold it is possible to find in [9] results that relate the possible values for the natural pseudo-distance to the critical values of the filtering functions, and analogous results are presented in [10] if X is a surface. If the space X is a closed curve this concept is very similar to the Fréchet distance [1].
The natural pseudo-distance and persistent homology are linked thanks to a *stability theorem* that let us see the natural pseudo-distance as an upper bound for the *matching distance* between persistence diagrams. Analogously, the matching distance can be seen as a lower bound for the natural pseudo-distance. For a study concerning this relation we refer to [16].
In this thesis we will focus our attention on the natural pseudo-distance associated with the Lie group S^1 between filtering functions defined on S^1. Using the terms of the previous definitions we will set $X = S^1$ and the group G acting on the space of functions will be the group of rotation S^1. The motivations behind this choice are the interest in the persistent homology of closed curves and the intent to extend persistent homology to Lie groups theory.
The persistent homology of closed curves has been already object of several studies. In [6] the stability theorem of persistent homology is used to prove the length theorem for curves, a generalization of Fary's theorem. In [3] and [18] dissimilarities between closed curves are studied from the perspective of persistent homology also in order to understand how to extend results

for curves to higher dimensional manifolds. Moreover, several techniques have been developed to evaluate the natural pseudo-distance between filtering functions defined on closed curves if the group of homeomorphisms is H (cf., e.g., [8] and [11]). From an applicative point of view the ability to compare closed curves reflects the ability to find dissimilarities between waveform signals and periodic functions in one variable [19]. In these matters an important role is played by the group of homeomorphisms acting on the space of functions. While from a theoretical point of view it is possible to consider the group H of all homeomorphisms, this is often impractical from an applicative point of view. We want to consider the action of some subgroups of H that are more suitable for catching the relevant aspects in some applicative problems.

An aspect related to the natural pseudo-distance is that of *optimality*. We have recalled that the natural pseudo-distance between φ and ψ is the infimum of the function $\mathcal{L}(g) = \|\varphi - \psi \circ g\|_\infty$ where g varies in a subgroup G of H. We can consider whether it is possible or not to find a homeomorphism $\bar{g}$ in G such that $\mathcal{L}(\bar{g}) = d_G(\varphi, \psi)$. We will call such a homeomorphism *optimal*. Optimal homeomorphisms represent the best possible matching between filtering functions and give us point-to-point correspondences between two sets that are preferable to any other correspondence. In general the existence of an optimal homeomorphism is not guaranteed, as it is proven in [9]. The problem of optimality has been already object of interest for persistent homology. The results stated in [6] have been improved in [15], taking into consideration the homeomorphisms that realize the Fréchet distance between closed curves, and their existence. In [4] the problem of optimality is investigated when the considered space is a closed curve. It shows that if the filtering functions are Morse and their natural pseudo-distance associated with H is 0 then at least one optimal diffeomorphism exists.

In this thesis we will study the properties of optimal homeomorphisms for the natural pseudo-distance associated with the group S^1. The compactness of the group will ensure us the existence of at least one optimal homeomorphism.

We will find some of the properties that have to be met by any optimal homeomorphism and we will analyze their finiteness in suitable conditions.
The outline of the thesis is as follows. In Chapter 1 we will recall the mathematical background concerning the natural pseudo-distance. We will expose some of the results already found for the case in which $G = H$. We will give a brief introduction to persistent homology seeing it as an extension of simplicial homology. We will then examine the relation between the natural pseudo-distance and persistent homology.
In Chapter 2 we will show the results found regarding the natural pseudo-distance associated with the Lie group S^1. We will find the differential conditions that optimal homeomorphisms have to meet, in order to determine whether a homeomorphism may be optimal or not. We will localize the points of the two functions that have to be matched in order to minimize the pointwise distance between the two functions and we will prove that if the filtering functions are Morse the set of optimal homeomorphisms is finite.

Introduzione

Nella seguente tesi di laurea esamineremo il concetto di pseudo-distanza naturale, spesso associato alla teoria della persistenza omologica. L'omologia persistente è un metodo di tipo topologico-algebrico usato al fine di studiare spazi di funzioni e le loro caratteristiche topologiche. Tale teoria ha raccolto negli ultimi 25 anni l'interesse della comunità matematica e di svariate altre scienze grazie alla sua robustezza rispetto al rumore sui dati, riuscendo a separare le caratteristiche rispetto alla loro rilevanza, e grazie alla sua abilità di comprimere il contenuto informativo di grandi collezioni di dati o forme. L'omologia persistente può essere vista come una branca della teoria omologica classica, riconsiderata da un punto di vista derivante dalla teoria di Morse. Una presentazione di tale teoria può essere trovata in [14].
Consideriamo uno spazio topologico X e fissiamo una funzione continua a valori reali φ definita su X, d'ora in poi chiamata *funzione filtrante*. È possibile considerare i gruppi di omologia dei sottoinsiemi di livello di φ, e studiare l'evoluzione delle loro proprietà topologiche. Possiamo tenere traccia della nascita e della morte di ciascuna classe di omologia, causate dalla variazione delle proprietà topologiche dei sottoinsiemi di livello di φ, e costruire un multi-insieme chiamato *diagramma di persistenza* per riassumere tali informazioni. Si possono trovare dettagli riguardanti tale costruzione nell'articolo di rassegna [12]. Il diagramma di persistenza riflette le principali caratteristiche topologiche dello spazio X e della funzione φ, permettendoci di associare a ciascuna classe di omologia la propria durata nel tempo. Le classi di omologia con punto di nascita lontano dal punto di morte rappresenteranno

spesso delle proprietà topologiche significative. Al contrario le classi con durata breve sono spesso associate a rumore e a caratteristiche di minor rilievo.
In questa tesi studieremo il concetto di *pseudo-distanza naturale*, una misura di dissomiglianza tra funzioni filtranti, che lavora in maniera complementare rispetto all'omologia persistente. Prendiamo uno spazio topologico compatto X e due funzioni continue $\varphi, \psi : X \to \mathbb{R}$ e consideriamo un sottogruppo G del gruppo H di tutti gli omeomorfismi di X su se stesso. La pseudo-distanza naturale tra φ e ψ associata al gruppo G è definita come

$$d_G(\varphi, \psi) = \inf_{g \in G} \max_{P \in X} |\varphi(P) - \psi(g(P))|. \tag{2}$$

Il caso in cui $G = H$ è già stato oggetto di studio. Se infatti X è una varietà chiusa è possibile trovare in [9] alcuni risultati che collegano i possibili valori per la pseudo-distanza naturale ai valori critici delle funzioni filtranti, e risultati analoghi sono presentati in [10] per il caso in cui X sia una superficie. Se lo spazio X è una curva chiusa tale concetto è molto simile alla distanza di Fréchet [1].
La pseudo-distanza naturale e l'omologia persistente sono legate grazie a un *teorema di stabilità* che ci permette di vedere la pseudo-distanza naturale come una limitazione superiore per la *distanza di matching* tra diagrammi di persistenza. Analogamente, la distanza di matching può essere vista come una limitazione inferiore per la pseudo-distanza naturale. Per uno studio riguardante tale relazione rimandiamo il lettore a [16].
In questo lavoro concentreremo la nostra attenzione sulla pseudo-distanza naturale associata al gruppo di Lie S^1 tra funzioni filtranti definite su S^1. Usando i termini delle precedenti definizioni porremo $X = S^1$ e il gruppo G agente sullo spazio delle funzioni filtranti sarà il gruppo delle rotazioni S^1. Le motivazioni dietro a tale scelta sono l'interesse per l'omologia persistente delle curve chiuse e l'intento di estendere l'omologia persistente alla teoria dei gruppi di Lie.
L'omologia persistente delle curve chiuse è già stata oggetto di vari studi. In [6] il teorema di stabilità della persistenza omologica viene usato per

dimostrare il teorema della lunghezza per curve, una generalizzazione del teorema di Fary. In [3] e [18] viene studiata la dissomiglianza tra curve chiuse dal punto di vista della persistenza omologica, anche al fine di comprendere come poter generalizzare i risultati ottenuti per curve chiuse a varietà di dimensione più alta. Inoltre sono state sviluppate varie tecniche al fine di stimare la pseudo-distanza naturale tra curve chiuse nel caso in cui il gruppo degli omeomorfismi sia H (cf., e.g., [8] e [11]). Da un punto di vista applicativo il confronto curve chiuse permette di trovare diversità in segnali di tipo ondulatorio e funzioni periodiche in una variabile [19]. In tali questioni il gruppo di omeomorfismi scelto riveste un ruolo di rilievo. Mentre da un punto di vista teorico è possibile prendere in considerazione il gruppo H di tutti gli omeomorfismi, da un punto di vista applicativo ciò è spesso impraticabile. Vogliamo utilizzare alcuni sottogruppi di H che siano maggiormente adatti a catturare gli aspetti rilevanti in certi problemi applicativi.

Un aspetto correlato alla pseudo-distanza naturale è quello dell'*ottimalità*. Abbiamo ricordato che la pseudo-distanza naturale tra φ e ψ è l'estremo inferiore del funzionale $\mathcal{L}(g) = \|\varphi - \psi \circ g\|_\infty$ dove g varia nel sottogruppo G di H. Possiamo considerare se sia possibile o meno trovare un omeomorfismo $\bar{g}$ in G per il quale $\mathcal{L}(\bar{g}) = d_G(\varphi, \psi)$. Chiameremo un tale omeomorfismo *ottimale*. Gli omeomorfismi ottimali rappresentano la miglior corrispondenza possibile tra funzioni filtranti e ci forniscono una mappa tra i due insiemi considerati che può essere preferita a tutte le altre. In generale l'esistenza di un omeomorfismo ottimale non è garantita, come provato in [9]. Il problema dell'ottimalità è già stato oggetto di interesse per gli studiosi che si occupano di persistenza omologica. I risultati riportati in [6] sono stati raffinati in [15], tenendo in considerazione gli omeomorfismi che realizzano la distanza di Fréchet tra curve chiuse e la loro esistenza. In [4] il problema dell'ottimalità viene analizzato quando lo spazio considerato è una curva chiusa. Si dimostra che se le funzioni filtranti sono di Morse e la loro pseudo-distanza naturale è zero, allora deve esistere almeno un diffeomorfismo ottimale.

In questa tesi studieremo le proprietà degli omeomorfismi ottimali per la pseudo-distanza naturale associata al gruppo S^1. La compattezza di tale gruppo ci garantirà l'esistenza di almeno un omeomorfismo ottimale. Troveremo alcune delle proprietà che devono essere soddisfatte dagli omeomorfismi ottimali e analizzeremo la loro finitezza sotto opportune ipotesi.

Il profilo della tesi è il seguente. Nel Capitolo 1 richiameremo i fondamenti matematici riguardanti la pseudo-distanza naturale. Esporremo alcuni dei risultati trovati nel caso in cui sia $G = H$. Proporremo una breve introduzione all'omologia persistente osservandola come un'estensione dell'omologia simpliciale. Esamineremo quindi la relazione tra la pseudo-distanza naturale e l'omologia persistente.

Nel Capitolo 2 mostreremo i risultati ottenuti riguardo alla pseudo-distanza naturale associata al gruppo di Lie S^1. Troveremo alcune condizioni differenziali che gli omeomorfismi ottimali devono soddisfare, al fine di determinare se un omeomorfismo possa essere ottimale o meno. Localizzeremo i punti delle due funzioni che devono essere allineati al fine di minimizzare la distanza punto a punto tra le due mappe e dimostreremo che se le funzioni filtranti sono di Morse allora l'insieme degli omeomorfismi ottimali è finito.

Chapter 1

Mathematical setting

In this chapter we introduce the mathematical tools needed to study the main problem analyzed in this thesis. We introduce persistent homology and the natural pseudo-distance, showing their main characteristics. We also see how the two subjects are related to each other.

1.1 Natural pseudo-distance

In this section we assume that X is a compact topological space and that H is the group of all homeomorphisms from X onto itself. The natural pseudo-distance is a mathematical tool developed in Size Theory in order to evaluate dissimilarities between *size pairs*, i.e. pairs (X, φ) constituted by a topological space X and real-valued function φ, called *filtering function*. It has proved itself useful in shape comparison problems where other topological or analytical methods are not able to discern the *relevant* differences between two shapes. From a topological point of view two homeomorphic spaces are one equivalent to the other, meaning that in this way we would not see any difference in two homeomorphic but different shapes. On the other hand, the analytical approach to the problem would lead to take into account all the differential properties of a space, hence being too selective in shape comparison.

The natural pseudo-distance instead tries to minimize the difference between two filtering functions with respect to the uniform norm, taking into account an invariance group of homeomorphisms that acts on the set of filtering functions. We first introduce some properties that the group of homeomorphisms H has to satisfy in order to give a good definition of the natural pseudo-distance.

Definition 1.1.1. A group $(G, \circ)$ endowed with a topology τ is called a *topological group* if the composition map $G \times G \to G : (g_1, g_2) \to g_1 \circ g_2$ and the inverse map $G \to G : g \to g^{-1}$ are continuous.

Remark 1.1.2. In the previous definition the topology considered on $G \times G$ is the product topology.

Example 1.1.3. We can give some examples of topological groups to make the concept clear.
$(\mathbb{R}, +)$ is a topological group, in fact the addition operator is a continuous function and the inverse map, $f(x) = -x$, is continuous.
The group $(S^1, +)$ is a topological group. It can be seen as the quotient $(\mathbb{R}/\sim, +_{S^1})$ where $x \sim y$ if and only if $x = y + 2k\pi$ for a suitable integer k and $+_{S^1}$ is the composition of the addition operator with the quotient map. S^1 is a topological group with respect to the quotient topology.
Any group $(G, \circ)$ is trivially a topological group if we use the discrete topology.

Given a compact metric space (X, d), we want to study the group $(H, \circ)$ of all homeomorphisms from X onto itself. We endow it with the topology induced by the metric d_U defined on H,

$$d_U(f, g) = \max_{x \in X} d(f(x), g(x)). \tag{1.1}$$

We prove that such a definition is well given.

Proposition 1.1.4. *d_U is a metric on H.*

Proof. For all $f, g, h \in H$ we have that the following properties hold:

(i) $d_U(f,g) = 0 \iff f = g$.

$$\max_{x\in X} d(f(x), g(x)) = 0 \iff \forall x \in X \quad d(f(x), g(x)) = 0 \iff$$
$$\iff f(x) = g(x) \quad \forall x \in X \iff f = g.$$

(ii) $d_U(f,g) = d_U(g,f)$. Since d is a metric and therefore it is symmetric we have

$$d_U(f,g) = \max_{x\in X} d(f(x), g(x)) = \max_{x\in X} d(g(x), f(x)) = d_U(g,f).$$

(iii) $d_U(f,h) \leq d_U(f,g) + d_U(g,h)$. We use again the properties of d:

$$d_U(f,h) = \max_{x\in X} d(f(x), h(x)) \leq \max_{x\in X} (d(f(x), g(x)) + d(g(x), h(x))) \leq$$
$$\leq \max_{x\in X} d(f(x), g(x)) + \max_{x\in X} d(g(x), h(x)) = d_U(f,g) + d_U(g,h).$$

□

We see that H is a topological group.

Theorem 1.1.5. *H is a topological group.*

Proof. We want to show that if $f = \lim_i f_i$ and $g = \lim_i g_i$ then $g \circ f = \lim_i g_i \circ f_i$ and that $\lim_i f_i^{-1} = f^{-1}$.
Let us examine the first statement. For every $i \in \mathbb{N}$ we have

$$d_U(g_i \circ f_i, g \circ f) \leq d_U(g_i \circ f_i, g \circ f_i) + d_U(g \circ f_i, g \circ f) =$$
$$= \max_{x\in X} d(g_i(f_i(x)), g(f_i(x))) + \max_{x\in X} d(g(f_i(x)), g(f(x))) =$$
$$= \max_{y\in X} d(g_i(y), g(y)) + \max_{x\in X} d(g(f_i(x)), g(f(x))).$$

For the first addend we have $lim_{i\to\infty} \max_{y\in X} d(g_i(y), g(y)) = 0$ because of the continuity of d and $\lim_i g_i = g$. For the second addend we have the same result because $\lim_i f_i = f$, d is continuous and g is uniformly continuous (because of the compactness of its domain). Therefore $g \circ f = \lim_i g_i \circ f_i$.
Now we will prove that $\lim_i f_i^{-1} = f^{-1}$. Suppose by contradiction that

$\lim_i d_U\left(f_i^{-1}, f^{-1}\right) \neq 0$, then a strictly positive constant c exists and a subsequence $\left(f_{i_j}\right)$ of (f_i) exists, such that

$$d_U\left(f_{i,j}^{-1}, f^{-1}\right) \geq c > 0 \quad \forall j \in \mathbb{N}. \tag{1.2}$$

It would still be $\lim_j d_U\left(f_{i_j}, f\right) = 0$ since $\left(f_{i_j}\right)$ is subsequence of a sequence converging to f. The inequality (1.2) let us know that for every j a point $y_j \in X$ exists such that $d_U\left(f_{i_j}^{-1}(y_j), f^{-1}(y_j)\right) \geq c > 0$. We can assume without loss of generality that a limit $\bar{y} := \lim_j y_j$ exists, we may otherwise extract a converging subsequence of (y_j) because of the compactness of X. In the same way we can assume the existence of the limit $\hat{x} = \lim_j f_{i_j}^{-1}(y_j)$. We would have

$$d\left(\hat{x}, f^{-1}(\bar{y})\right) = d\left(\lim_j f_{i,j}^{-1}(y_j), f^{-1}\left(\lim_j y_j\right)\right) = \lim_j d\left(f_{i_j}^{-1}(y_j), f^{-1}(y_j)\right) \geq c > 0, \tag{1.3}$$

hence it would be $\hat{x} \neq f^{-1}(\bar{y})$. On the other hand we have

$$d\left(f(\hat{x}), \bar{y} = d\left(f\left(\lim_j f_{i_j}^{-1}(y_j)\right), \lim_j y_j\right) = \lim_j d\left(f\left(f_{i_j}^{-1}(y_j)\right), y_j\right) = \right.$$
$$= \lim_j d\left(f\left(f_{i_j}^{-1}(y_j)\right), f_{i_j}\left(f_{i_j}^{-1}(y_j)\right)\right) \leq \lim_j d_U\left(f, f_{i,j}\right) = 0.$$

Then we would have $f(\hat{x}) = \bar{y}$. This is absurd because we know that $\hat{x} \neq f^{-1}(\bar{y})$ and f is injective. It must be then $\lim_i f_i^{-1} = f^{-1}$. □

We will take into account the right action of the group H on the space $C^0(X, \mathbb{R})$ by setting

$$\varphi.h = \varphi \circ h \quad \text{for} \quad \varphi \in C^0(X, \mathbb{R}) \quad h \in H. \tag{1.4}$$

We can prove that defined action is indeed a group action. In fact the following statements are true

$$\varphi.(h \circ g) = \varphi \circ (h \circ g) = (\varphi \circ h) \circ g = (\varphi.h).g \quad \forall h, g \in H, \forall \varphi \in C^0(X, \mathbb{R})$$
$$\varphi.\mathrm{id} = \varphi \quad \forall \varphi \in C^0(X, \mathbb{R})$$

From now on we endow $C^0(X, \mathbb{R})$ with the uniform metric.

Proposition 1.1.6. *The group action of H on the space $C^0(X,\mathbb{R})$ is continuous.*

Proof. We will show that for any $\varepsilon > 0$ a $\delta > 0$ exists such that if $\|\varphi-\psi\|_\infty \le \delta$ and $d_U(f,g) \le \delta$ then $\|\varphi\circ f - \psi\circ g\|_\infty \le \varepsilon$.
We have

$$\|\varphi\circ f - \psi\circ g\|_\infty \le \|\varphi\circ f - \varphi\circ g\|_\infty + \|\varphi\circ g - \psi\circ g\|_\infty. \tag{1.5}$$

If $d_U(f,g) \le \delta$ and δ is small enough, since φ is uniformly continuous we have $\|\varphi\circ f - \varphi\circ g\|_\infty \le \varepsilon/2$. For the second addend we know that

$$\|\varphi\circ g - \psi\circ g\|_\infty = \|\varphi-\psi\|_\infty \le \delta \le \frac{\varepsilon}{2} \quad \text{for every } \delta \le \frac{\varepsilon}{2}. \tag{1.6}$$

Then $\|\varphi\circ f - \psi\circ g\|_\infty \le \varepsilon$ for every small enough δ. □

We are now ready to introduce the general concept of natural pseudo-distance.

Definition 1.1.7. Let us consider two functions $\varphi,\psi : X \to \mathbb{R}$ and a subgroup G of H. The *natural pseudo-distance* between φ and ψ associated with the group G is

$$d_G(\varphi,\psi) = \inf_{g\in G} \|\varphi - \psi\circ g\|_\infty \tag{1.7}$$

We can first observe that the defined function is actually a pseudo-distance.

Proposition 1.1.8. *The function d_G is a pseudo-distance.*

Proof. We will show that d_G satisfies the properties of a metric *except* for $d_G(\varphi,\psi) = 0 \implies \varphi = \psi$.

- $d_G(\varphi,\varphi) = 0$, because $\|\varphi - \varphi\circ \mathrm{id}\|_\infty = 0$.

- Since G is a group we have

$$d_G(\varphi,\psi) = \inf_{g\in G} \|\varphi - \psi\circ g\|_\infty = \inf_{g\in G} \|\varphi\circ g^{-1} - \psi\|_\infty =$$
$$= \inf_{g^{-1}\in G} \|\varphi\circ g^{-1} - \psi\|_\infty = \inf_{g\in G} \|\varphi\circ g - \psi\|_\infty =$$
$$= \inf_{g\in G} \|\psi - \varphi\circ g\|_\infty = d_G(\varphi,\psi).$$

- For every $\varphi, \psi, \chi \in C^0(X, \mathbb{R})$ and for every $f \in H$ it holds that

$$d_G(\varphi, \psi) = \inf_{g \in G} \|\varphi - \psi \circ g\|_\infty = \inf_{g \in G} \|\varphi - \psi \circ g \circ f\|_\infty \leq$$
$$\leq \inf_{g \in G} (\|\varphi - \chi \circ f\|_\infty + \|\chi \circ f - \psi \circ g \circ f\|_\infty) =$$
$$= \|\varphi - \chi \circ f\|_\infty + \inf_{g \in G} (\|\chi \circ f - \psi \circ g \circ f\|_\infty) =$$
$$= \|\varphi - \chi \circ f\|_\infty + \inf_{g \in G} (\|\chi - \psi \circ g\|_\infty).$$

  Therefore $d_G(\varphi, \psi) \leq \inf_{f \in G} \|\varphi - \chi \circ f\|_\infty + \inf_{g \in G} \|\chi - \psi \circ g\|_\infty = d_G(\varphi, \chi) + d_G(\chi, \psi)$.

□

We can see that the hypothesis of G being a group is necessary, otherwise d_G would fail to be a pseudo-distance.

Example 1.1.9. Assume that X is the unitary circle in the complex plain $X = \{e^{i\alpha} | \alpha \in \mathbb{R}\}$ and let us take $G := \{\rho_{-\pi/2}\}$ with $\rho_{-\pi/2} : X \to X$ and $\rho_{-\pi/2}(e^{i\alpha}) = e^{i(\alpha - \pi/2)}$. Consider the two functions $\varphi, \psi : X \to \mathbb{R}$, with $\varphi(e^{i\alpha}) = \sin(\alpha)$ and $\psi((e^{i\alpha})) = \cos(\alpha)$. We have that

$$d_G(\varphi, \psi) = \inf_{g \in G} \|\varphi - \psi \circ g\|_\infty = \|\sin(\alpha) - \cos\left(\alpha - \frac{\pi}{2}\right)\|_\infty = \|\sin(\alpha) - \sin(\alpha)\|_\infty = 0 \tag{1.8}$$

but at the same time

$$d_G(\psi, \varphi) = \inf_{g \in G} \|\psi - \varphi \circ g\|_\infty = \|\cos(\alpha) - \sin\left(\alpha - \frac{\pi}{2}\right)\|_\infty = \|\cos(\alpha) + \cos(\alpha)\|_\infty = 2. \tag{1.9}$$

Therefore $d_G(\varphi, \psi) \neq d_G(\psi, \varphi)$ and d_G in not a pseudo-distance.

We can give some examples to make the exposed concept clearer.

Remark 1.1.10. We want to underline that the function defined fails to be a distance because the axiom $d_G(\varphi, \psi) = 0 \implies \varphi = \psi$ is not satisfied.

Example 1.1.11. Let us take $X = [-1, 1] \subset \mathbb{R}$, $G = H$ and consider the two functions $\varphi, \psi : X \to \mathbb{R}$ with $\varphi(x) = x^4$ and $\psi(x) = x^2$. We can show that $d_G(\varphi, \psi) = 0$ since if we take the homeomorphism $h : X \to X$ defined as $h(x) = \operatorname{sgn}(x) \cdot x^2$ we have

$$d_G(\psi, \varphi) = \inf_{g \in G} \|\varphi - \psi \circ g\|_\infty = \|x^4 - \left(\operatorname{sgn}(x) \cdot x^2\right)^2\|_\infty = 0 \qquad (1.10)$$

but $\varphi \neq \psi$.

A concept strictly related to the natural pseudo-distance is that of optimality.

Definition 1.1.12. We call $\bar{g}$ an *optimal* homeomorphism between φ and ψ if

$$d_G(\varphi, \psi) = \|\varphi - \psi \circ \bar{g}\|_\infty. \qquad (1.11)$$

In general, the existence of an optimal homeomorphism is not guaranteed and we can find examples where such a homeomorphism does not exist. We will show that it is possible to find two functions φ, ψ with $d_G(\varphi, \psi) = 0$ but such that no optimal homeomorphism exists.

Example 1.1.13. Suppose $X = [0, 2\pi]$ and $G = H$. We consider the following functions, whose graphs are shown in figure 1.1:

$$\varphi(x) = \sin^2\left(\frac{x}{2}\right) \quad \text{and} \quad \psi(x) = \begin{cases} \sin^2 x & \text{if } 0 \leq x \leq \frac{\pi}{2} \\ 1 & \text{if } \frac{\pi}{2} < x < \frac{3\pi}{2} \\ \sin^2 x & \text{if } \frac{3\pi}{2} \leq x \leq 2\pi \end{cases} . \qquad (1.12)$$

It is possible to see that the natural pseudo-distance between the two functions is 0, but an optimal homeomorphism cannot exist. If by contradiction a homeomorphism $h \in H$ exists, such that $\varphi = \psi \circ h$ then it must be $\varphi^{-1}(\{1\}) = (\psi \circ h)^{-1}(\{1\})$. But on the other hand

$$\varphi^{-1}(\{1\}) = \{\pi\} \quad \text{while} \quad (\psi \circ h)^{-1}(\{1\}) = h^{-1}\left(\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\right) \qquad (1.13)$$

and since h is a homeomorphism it cannot be $\pi = h^{-1}\left(\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\right)$.

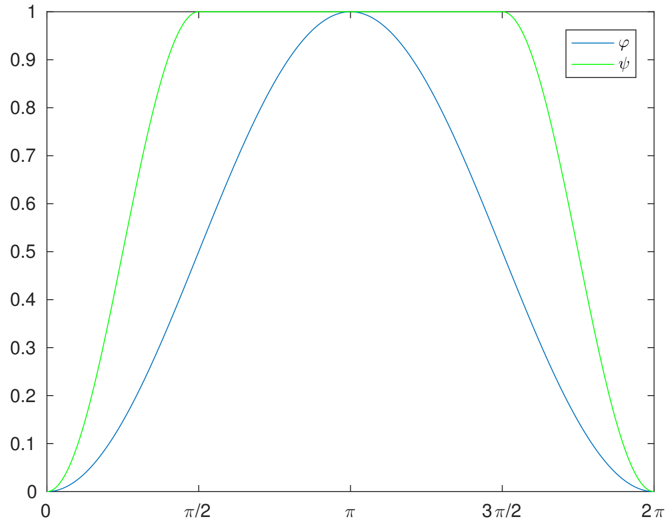

Figure 1.1: Plot of the two filtering functions defined in Example 1.1.13.

Remark 1.1.14. Let us consider the quotient space $C^0(X,\mathbb{R})/\sim$, where $\varphi \sim \psi$ if and only if a homeomorphism $g \in G$ exists such that $\varphi = \psi \circ g$. This is an equivalence relation since G is a group. We can define on this space an analogous pseudo-distance between $[f],[g] \in C^0(X,\mathbb{R})/\sim$

$$\tilde{d}_G([\varphi],[\psi]) = \inf_{g\in G} \|\varphi - \psi \circ g\|_\infty. \tag{1.14}$$

Even in this case $\tilde{d}_G$ is only a pseudo-distance and not a distance, since the last example showed it is possible to find two functions φ, ψ such that $d_G(\varphi,\psi) = 0$ but there is not any homeomorphism $g \in G$ such that $\varphi = \psi \circ g$. This matter is investigated in [2] and it is proved that for a suitable choice of the set of functions and the group G acting on it the pseudo-distance $\tilde{d}_G$ is actually a distance.

Several results have been proved to estimate the natural pseudo-distance if the group G is the group H of all homeomorphisms.

Theorem 1.1.15. *Let us assume that $\mathcal{M}$ is a closed manifold of class C^1 and $\varphi,\psi : \mathcal{M} \to \mathbb{R}$ are two functions of class C^1. Then a positive integer k exists such that one of the following statements holds:*

- *k is odd and $k \cdot d_H(\varphi, \psi)$ is equal to the distance between a critical value of φ and a critical value of ψ.*

- *k is even and $k \cdot d_H(\varphi, \psi)$ is equal to either the distance between two critical values of φ or two critical values of ψ.*

The proof is based on the concept of *train of limit d-jumps* exposed in [9]. In the hypothesis of Theorem 1.1.15 if an optimal homeomorphism exists we can assume $k = 1$.

Theorem 1.1.16. *Let $\mathcal{M}$ be a closed manifold and $\varphi, \psi : \mathcal{M} \to \mathbb{R}$ two C^1 functions. If an optimal homeomorphism exists the natural pseudo-distance between φ and ψ is equal to the distance between a critical value of φ and a critical value of ψ.*

If the considered manifold is a closed curve or a closed surface then we have some further results.

Theorem 1.1.17. *Let $\mathcal{M}$ be a closed curve of class C^1 and $\varphi, \psi : \mathcal{M} \to \mathbb{R}$ two functions of class C^1. Then one of the following statements holds:*

- *$d_H(\varphi, \psi)$ is equal to the distance between a critical value of φ and a critical value of ψ.*

- *$d_H(\varphi, \psi)$ is equal to the distance between two critical values of φ.*

- *$d_H(\varphi, \psi)$ is equal to the distance between two critical values of ψ.*

Theorem 1.1.18. *Let $\mathcal{M}$ be a closed surface of class C^1 and let $\varphi, \psi : \mathcal{M} \to \mathbb{R}$ two C^1 functions. Then one of the following statements holds:*

- *$d_H(\varphi, \psi)$ is equal to the distance between a critical value of φ and a critical value of ψ.*

- *$d_H(\varphi, \psi)$ is equal to one half of the distance between two critical values of φ or one half of the distance between two critical values of ψ.*

- *$d_H(\varphi, \psi)$ is equal to one third of the distance between a critical value of φ and a critical value of ψ.*

Computing the natural pseudo-distance is often difficult and costly from a computational point of view. If the group G acting on the space of functions is not the whole group H the theorems above stated cease to be true and we have no clue about the possible value of the natural pseudo-distance.

Proposition 1.1.19. *Let G_1, G_2 be two subgroups of the group H, with $G_1 \subseteq G_2$. Then*

$$d_{G_2}(\varphi, \psi) \leq d_{G_1}(\varphi, \psi) \quad \forall \varphi, \psi \in C^0(X, \mathbb{R}). \tag{1.15}$$

Proof. It follows immediately from this property of the infimum:

$$\text{if } A \subseteq B \text{ then } \inf A \geq \inf B. \tag{1.16}$$

□

We need to develop techniques to estimate the natural pseudo-distance in this latter case. One possible way is to find an upper bound and a lower bound for its value. In the following section we describe persistent homology and find a lower bound for the pseudo-distance d_H by means of a stability theorem [16].

1.2 Persistent homology

Persistent homology is an algebraic topological method used for analyzing spaces of real functions and their topological features. It is largely used in computer vision and shape comparison problems thanks to its robustness to noise and the ability to reduce the dimension of the analyzed data.
In this thesis we will expose this subject as an extension of the *simplicial homology theory* with coefficients in $\mathbb{Z}_2$. We will focus on the $0-$dimensional persistent homology, developed as Size Theory, in order to introduce the thematics relative to the stability of this theory. We will expose a brief

summary of simplicial homology with coefficients in $\mathbb{Z}_2$ to better understand the processes related to persistent homology.

1.2.1 Simplicial homology

In this section the space V will be the vector space $\mathbb{R}^d$ of real $d-$tuples.

Definition 1.2.1. We say that $k+1$ elements of V, $\{x_0, \ldots, x_k\}$ are *affinely independent*, if the vectors $\{x_i - x_0 \mid 1 \leq i \leq k\}$ are linearly independent. An *n-simplex* is the convex hull of an n-tuple $\alpha = (x_0, \ldots, x_n)$ of $n+1$ affinely independent points of V. The elements of the n-tuple are called *vertices* of the simplex. A *face* of a n-simplex σ with vertices $\{x_0, \ldots, x_n\}$ is the convex hull of a k-tuple $(x_{i_1}, \ldots, x_{i_k})$ with $\{x_{i_1}, \ldots, x_{i_k}\} \subset \{x_0, \ldots, x_n\}$.

We can define a particular structure of simplices.

Definition 1.2.2. An *simplicial complex* K is a finite collection of simplices with the following property:

- for any simplex α in K if β is a face of α then β is a simplex of K.
- if σ_1 and σ_2 are simplices in K then $\sigma_1 \cup \sigma_2$ is either empty or a face of both σ_1 and σ_2

We can now give an algebraic structure to algebraic complexes.

Definition 1.2.3. A *simplicial* $p-$*chain* is a linear combination of p-simplices

$$c = \sum_{i=1}^{n} \lambda_i \sigma_i \tag{1.17}$$

where λ_i is an element of $\mathbb{Z}_2$ for all i and $\sigma_i \in K$ is a p-simplex for all i. The set $C_p(K)$ of p-chains in K is then a $\mathbb{Z}_2$-vector space.

We will now define a boundary map that maps each p-simplex of $C_p(K)$, to the $(p-1)$-chain of $C_{p-1}(K)$ composed by its faces.

Definition 1.2.4. Given the two vector spaces $C_p(K)$ and $C_{p-1}(K)$ we define the *boundary map* $\partial_p : C_p(K) \to C_{p-1}(K)$ for each simplex $\sigma = (x_0, \ldots, x_p)$ as

$$\partial_p(\sigma) = \sum_{i=0}^{p} (x_0, \ldots, \hat{x}_i, \ldots, x_n) \tag{1.18}$$

where $(x_0, \ldots, \hat{x}_i, \ldots, x_n)$ is the $(p-1)$-face of σ generated by the vertices $\{x_o, \ldots, x_p\} \setminus \{x_i\}$. We obtain a homomorphism of vector spaces by linearly extending this map.

Definition 1.2.5. A *chain complex* $(\mathcal{C}, d)$ is a sequence of abelian groups C_p with boundary maps $d_p : C_p \to C_{p-1}$ such that the composition $d_{p-1} \circ d_p = 0$ for all p.

Lemma 1.2.6. *The collection of $\mathbb{Z}_2$-vector spaces $C_p(K)_{p\in\mathbb{N}}$ connected by boundary maps ∂_p forms a chain complex $(\mathcal{C}(K), \partial)$.*

Proof. If we show that $\partial_{p-1}(\partial_p(\sigma)) = 0$ for any p-simplex $\sigma = (x_0, \ldots, x_n)$, then $\partial_{p-1}(\partial_p(c)) = 0$ for any p-chain $c \in C_p(K)$. We see that

$$\partial_{k-1}(\partial_k(\sigma)) = \partial_{k-1}\left(\sum_{i=0}^{p} (x_0, \ldots, \hat{x}_i, \ldots, x_p)\right) =$$

$$= \sum_{i=0}^{p} \partial_{k-1}((x_0, \ldots, \hat{x}_i, \ldots, x_p)) = \sum_{i=0}^{p} \sum_{j=0, j\neq i}^{p} (x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_p) =$$

$$\sum_{0\leq i<j\leq p} (x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_p) + \sum_{0\leq j<i\leq p} (x_0, \ldots, \hat{x}_j, \ldots, \hat{x}_i, \ldots, x_p).$$

The two summations are constituted by the same $(p-1)$-simplices. Therefore

$$\partial_{k-1}(\partial_k(\sigma) = \sum_{0\leq i<j\leq p} [2]_2 (x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_p) = 0, \tag{1.19}$$

because the coefficients of the simplices are element of $\mathbb{Z}_2$, that has characteristic equal to 2. □

Corollary 1.2.7. *For every p we have* $\mathrm{im}\partial_p \subseteq \ker\partial_{p-1}$.

Proof. Let c be a $(p-1)$-chain belonging to $\mathrm{im}\partial_p$. Then a p-chain $\tilde{c}$ must exist such that $c = \partial_p(\tilde{c})$. Then because of lemma 1.2.6 we have

$$\partial_{p-1}(c) = \partial_{p-1}(\partial_p(\tilde{c}) = 0 \tag{1.20}$$

for every $c \in \mathrm{im}\partial_p$. □

Definition 1.2.8. We call $Z_p = \ker\partial_p$ the group of p-cycles and $B_p = \mathrm{im}\partial_{p+1}$ the group of p-boundaries.

From Corollary 1.2.7 we can see that for every p the set B_p is a vector subspace of Z_p. We are ready to define the simplicial homology groups.

Definition 1.2.9. The p^{th}-homology group H_p is the quotient group

$$H_p = \frac{Z_p}{B_p} \quad \text{for all } p \text{ in } \mathbb{Z}. \tag{1.21}$$

We call $\beta_p = \mathrm{rank} H_p$ the p^{th}-Betti number of K.

The p^{th}-homology group represents the classes of p-cycles that are not boundaries of $(p+1)$-chains, therefore the p^{th}-Betti number indicates the number of p-dimensional holes present in the simplicial complex K.

1.2.2 Persistence

Let X be a topological space and φ a continuous real-valued function defined on X. It is possible to consider the sublevel sets of X with respect to the function φ

$$X_u = \{x \in X | \varphi(x) \leq u\} \quad u \in \mathbb{R}. \tag{1.22}$$

If $u < v$ we can consider the inclusion map $i_{u,v} : X_u \to X_v$.

Definition 1.2.10. The family $\{X_u\}_{u \in \mathbb{R}}$ is called the *filtration* on X induced by the *filtering function* φ.

In an analogous way it is possible to define a filtration on a simplicial complex K.

A filtration on a simplicial complex K is a non-decreasing family $\{K_u\}$ of subcomplexes, with K_u subcomplex of K_v if $u < v$. For every $u < v$ an injective simplicial map $i_{u,v} : K_u \to K_v$ is defined such that $i_{u,w} = i_{u,v} \circ i_{v,w}$ for every $u < v < w$. Each of these simplicial maps induces a homomorphism between the homology groups of the subcomplexes

$$i^*_{u,v} : H_p(K_u) \to H_p(K_v) \quad \text{for every integer } p. \tag{1.23}$$

Further details may be found in [13].

Definition 1.2.11. The p^{th} persistent homology group calculated at the point (u, v) is the group

$$PH_p(u, v) = i^*_{u,v}(H_p(K_u)) \subseteq H_p(K_v). \tag{1.24}$$

The number $\beta_p(u, v) = \text{rank} PH_p(u, v)$ is called p^{th} *persistent Betti number* calculated at (u, v).

Persistent homology groups enable us to distinguish homological features (i.e. homology classes) that arise at sublevel K_u and vanish at a higher level K_v. The difference $v - u$ is called *persistence* of the homology class. If a homology class never vanishes its persistence is set equal to infinity. Generally speaking, a class with high persistence reflects a relevant topological feature of the considered complex, while classes with small persistence are often considered less significant.

We will focus on the concept of reduced size function [7], equivalent to 0^{th}-degree persistent homology, to link the natural pseudo-distance with persistent homology. Analogous results can be given in higher degree. Let X be a compact locally contractible space and φ a filtering function defined on it.

Definition 1.2.12. Let v be a real number. We say that two points $P, Q \in X$ are $\langle \varphi \leq v \rangle$-connected if and only if a connected component C of $X_v = \{x \in X | \varphi(x) \leq v\}$ exists, such that P and Q belong to it.

We define $\Delta^+ = \{(x, y) \in \mathbb{R}^2 | x < y\}$.

Definition 1.2.13. We call *reduced size function* associated with the size pair (X, φ) the function $l_{(X,\varphi)^*:\Delta^+ \to \mathbb{N}}$ defined by setting $l^*_{(X,\varphi)}(u, v)$ equal to the number of equivalence classes of X_u with respect to the relation of $\langle \varphi \leq v \rangle$-connectedness.

This definition improves that of *size function*, c.f. [17], since it is possible to prove that the function $l_{(X,\varphi)^*(u,v)}$ is right-continuous in both u and v. From our definitions of persistent Betti numbers follows also that $l^*_{(X,\varphi)}(u, v) = \beta_0(u, v)$, since the reduced size function counts the number of connected components of X_v that contain at least one point of X_u. The reduced size function then summarizes all the characteristics of the 0^{th}-degree persistent homology group. We will now examine a way to compare different reduced size functions, and therefore to compare two different size pairs.

Definition 1.2.14. Assume $p \in \Delta^+$. We define the *multiplicity* of p for $l^*_{(X,\varphi)}$, $\mu(p)$, as the minimum over all the positive real numbers ε, with $u + \varepsilon < v - \varepsilon$, of

$$l^*_{(X,\varphi)}(u+\varepsilon, v-\varepsilon - l^*_{(X,\varphi)}(u - \varepsilon, v - \varepsilon) - l^*_{(X,\varphi)}(u + \varepsilon, v + \varepsilon) + l^*_{(X,\varphi)}(u - \varepsilon, v + \varepsilon). \quad (1.25)$$

We will call *proper cornerpoint* for $l^*_{(X,\varphi)}$ any point $p \in \Delta^+$ that has strictly positive multiplicity, i.e. $\mu(p) > 0$.

Definition 1.2.15. Assume that r is the vertical line $x = k$. We identify r with the pair (k, ∞), and we define the *multiplicity* of r for $l^*_{(X,\varphi)}$, $\mu(r)$, as the minimum over all the positive real ε, with $k + \varepsilon < 1/\varepsilon$, of

$$l^*_{(X,\varphi)}(k + \varepsilon, 1/\varepsilon) - l^*_{(X,\varphi)}(k - \varepsilon, 1/\varepsilon). \quad (1.26)$$

We call *cornerpoint at infinity* for $l_{(X,\varphi)^*}$ any line r such that $\mu(r) > 0$.

Let $\bar{\mathbb{R}}$ be the extended real line $\mathbb{R} \cap \{\infty\}$. We can now associate each reduced size function with a formal sum of corner points. It will be possible to define a metric on the set of these formal sums that will make it possible to compare reduced size functions, and in the same way persistent Betti numbers of higher degree.

Definition 1.2.16. The 0^{th}-*degree persistence diagram* of φ is the linear combination

$$\sum_{u<v\in\bar{\mathbb{R}}} \mu\left((u,v)\right)\cdot(u,v)\,. \tag{1.27}$$

Persistence diagrams can be seen equivalently as formal sums or multisets of cornerpoints counted with their multiplicity. We can now introduce a distance between persistence diagrams.

Definition 1.2.17. Let l_1^* and l_2^* be two reduced size functions. Let C_1 and C_2 be their respective persistence diagrams (seen as multisets of cornerpoints) augmented with a countably infinite collection of points of the diagonal $\{(x,y)\in\mathbb{R}^2|x=y\}$. The *matching distance* between them is defined as

$$d_{match}\left(l_1,l_2\right)=\inf_{\sigma}\sup_{p\in C_1}\delta\left(p,\sigma\left(p\right)\right) \tag{1.28}$$

where σ varies among the bijections between C_1 and C_2 and δ is defined as

$$\delta\left(\left(p_x,p_y\right),\left(q_x,q_y\right)\right)=\min\left\{\max\{|p_x-q_x|,|p_y-q_y|\},\max\left\{\frac{p_y-p_x}{2},\frac{q_y-q_x}{2}\right\}\right\}. \tag{1.29}$$

Example 1.2.18. In Figure 1.2 we can see a closed curve and the filtering function φ defined on it as the height of each point of the curve. The 0^{th}-degree persistent diagram associated with it is drawn in Figure 1.3.

We can now state the theorem that links the natural pseudo-distance and persistent homology.

Theorem 1.2.19 (Stability theorem)**.** *Let φ and ψ be two filtering functions on X. The following inequality holds*

$$d_{match}\left(l^*_{(X,\varphi)},l^*_{(X,\psi)}\right)\leq d_H\left(\varphi,\psi\right). \tag{1.30}$$

Thanks to this theorem we have a way to establish a lower-bound for the natural pseudo-distance, that would be otherwise difficult to compute.

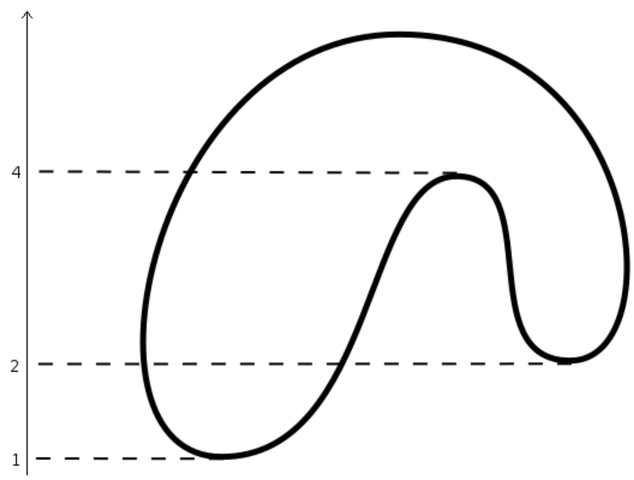

Figure 1.2: The filtering function φ, defined in Example 1.2.18, is represented as the height function in the figure.

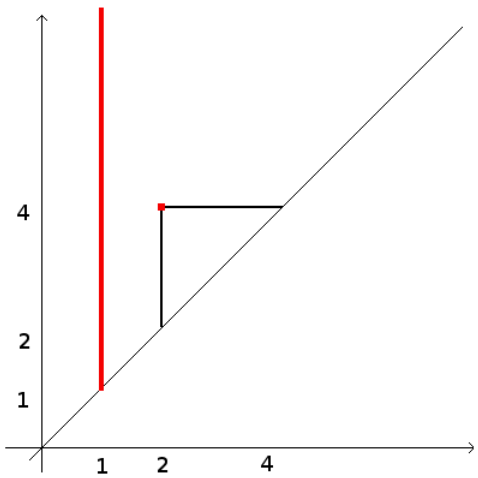

Figure 1.3: Persistence diagram of the 0^{th}-degree for the function φ.

Chapter 2

Natural pseudo-distance associated with the Lie group S^1

2.1 Natural pseudo-distance and optimality

Let us consider the Lie group S^1. For the sake of simplicity we will see the manifold S^1 as the quotient space $\mathbb{R}/\sim$, were $x \sim y$ if and only if $x - y$ is a multiple of 2π. Every element $\alpha \in S^1$ can also be identified with the homeomorphism $\rho_\alpha : S^1 \to S^1$ defined by setting $\rho_\alpha(\theta) = \theta + \alpha$. The set of filtering functions will be the space $\mathcal{M}$ of all Morse functions from S^1 to $\mathbb{R}$. If in definition 1.1.7 we set $G = S^1$ we get the following *natural pseudo-distance* on $\mathcal{M}$:

$$d_{S^1}(\varphi, \psi) = \inf_{\rho_\alpha \in S^1} ||\varphi - \psi \circ \rho_\alpha||_\infty. \tag{2.1}$$

We recall the definition of optimality since it will be fundamental in this chapter.

Definition 2.1.1. We say that $\rho_{\bar{\alpha}}$ is an *optimal* homeomorphism between φ and ψ if

$$d_{S^1}(\varphi, \psi) = \max_{\theta \in S^1} |\varphi(\theta) - \psi(\rho_{\bar{\alpha}}(\theta))|. \tag{2.2}$$

Whilst in general the existence of an optimal homeomorphism is not guaranteed, if the group of homeomorphisms is compact we can always find at least one optimal homeomorphism.

Proposition 2.1.2. *An optimal homeomorphism always exists in S^1 for every pair of filtering functions φ and ψ.*

Proof. Since the group action of S^1 is continuous on $\mathcal{M}$ we can find a sequence of homeomorphisms (ρ_{α_i}) such that

$$\lim_{i\to\infty} \|\varphi - \psi \circ \rho_{\alpha_i}\|_\infty = d_{S^1}(\varphi, \psi). \tag{2.3}$$

Therefore we can extract a converging subsequence of (ρ_{α_i}), because of the compactness of S^1. Then we have a sequence $(\rho_{\alpha_{i_j}})$ and a homeomorphism $\rho_{\bar{\alpha}}$ such that

$$\lim_{j\to\infty} \rho_{\alpha_{i_j}} = \rho_{\bar{\alpha}}. \tag{2.4}$$

We have that $\rho_{\bar{\alpha}}$ is an optimal homeomorphism because

$$\lim_{j\to\infty} \|\varphi - \psi \circ \rho_{\alpha_{i_j}}\|_\infty = \|\varphi - \psi \circ \rho_{\bar{\alpha}}\|_\infty = d_{S^1}(\varphi, \psi). \tag{2.5}$$

□

In order to make our calculations clearer we introduce the following functions that we will often use in this section:

Definition 2.1.3. Given $\varphi, \psi \in \mathcal{M}$ we will set

$$F : S^1 \times S^1 \to \mathbb{R}$$

$$F(\theta, \alpha) = |\varphi(\theta) - \psi(\theta + \alpha)|$$

and

$$f_\alpha : S^1 \to \mathbb{R} \qquad \text{for each fixed } \alpha \in S^1$$

$$f_\alpha(\theta) = F(\theta, \alpha).$$

If $\rho_{\bar{\alpha}}$ is an optimal homeomorphism, we say that a point $(\bar{\theta}, \bar{\alpha}) \in S^1 \times S^1$ *realizes the natural pseudo-distance* if

$$d_{S^1}(\varphi, \psi) = F(\bar{\theta}, \bar{\alpha}).$$

We can see that the functions F and f_α, for any α in S^1 are uniformly continuous, since they are given by the composition of uniformly continuous functions. Whilst they cannot be globally C^1 functions, because of the absolute value, in a neighborhood of the points (θ, α) where $F(\theta, \alpha) \neq 0$ we have that F is at least of class C^1. We have similarly that f_α is C^1 in a neighborhood of the points θ where $f_\alpha(\theta) \neq 0$. In this section we will confine ourselves to the study of filtering functions φ, ψ with $d_{S^1}(\varphi, \psi) = 0$, since this latter case can be treated easily in the next section. In such a hypothesis the function F will be C^1 in a neighborhood of the points that realize the natural pseudo-distance. We will write $d_{S^1}(\varphi, \psi) = \bar{d}$ too lighten our notation.

Lemma 2.1.4. *Let $\rho_{\bar{\alpha}}$ be an optimal homeomorphism and $\bar{\alpha}$ be the element of S^1 associated with it. Then there exist two points $\bar{\theta}_1$ and $\bar{\theta}_2$ of S^1 and two sequences (θ_i^u, α_i^u), (θ_i^d, α_i^d) in $S^1 \times S^1$ with the following properties:*

$$\alpha_i^u \xrightarrow{i \to \infty} \bar{\alpha} \qquad \alpha_i^u > \bar{\alpha}$$

$$\theta_i^u \xrightarrow{i \to \infty} \bar{\theta}_1 \ \textit{with}$$

$$f_{\alpha_i^u}(\theta_i^u) = \max_{\theta \in S^1} f_{\alpha_i^u}(\theta) \qquad f_{\bar{\alpha}}(\bar{\theta}_1) = \max_{\theta \in S^1} f_{\bar{\alpha}}(\theta)$$

and

$$\alpha_i^d \xrightarrow{i \to \infty} \bar{\alpha} \qquad \alpha_i^d < \bar{\alpha}$$

$$\theta_i^d \xrightarrow{i \to \infty} \bar{\theta}_2 \ \textit{with}$$

$$f_{\alpha_i^d}(\theta_i^d) = \max_{\theta \in S^1} f_{\alpha_i^d}(\theta) \qquad f_{\bar{\alpha}}(\bar{\theta}_2) = \max_{\theta \in S^1} f_{\bar{\alpha}}(\theta).$$

Proof. We will prove the existence of the first sequence as for the other one the same arguments can be used. We can take a sequence (α_i^u) such that $\lim_{i\to\infty} \alpha_i^u = \bar{\alpha}$ and $\alpha_i^u > \bar{\alpha}$ for every index i. Since every function f_α is defined on S^1, and therefore has a compact domain, for each i it is possible to choose a point θ_i^u of global maximum for the function $f_{\alpha_i^u}$. The resulting sequence (θ_i^u) is defined in a compact space, hence we can take a convergent subsequence $(\theta_{i_j}^u)$ with limit a point $\bar{\theta}_1$. Associating with each $(\theta_{i_j}^u)$ the corresponding $\alpha_{i_j}^u$ we obtain a sequence $(\theta_{i_j}^u, \alpha_{i_j}^u)$ that converges to $(\bar{\theta}_1, \bar{\alpha})$. It remains to prove that $\bar{\theta}_1$ is a point of global maximum for the function $f_{\bar{\alpha}}$. By contradiction suppose that $\bar{\theta}_1$ is not a point of global maximum. The function $f_{\bar{\alpha}}$ must have at least one point of absolute maximum, let us call it $\tilde{\theta}$. Then, since F is a uniformly continuous function, there must exist a neighborhood U of $(\tilde{\theta}, \bar{\alpha})$ and a neighborhood W of $(\bar{\theta}_1, \bar{\alpha})$ such that on U the function F takes values strictly greater than the ones taken on W. Then for some of the $f_{\alpha^u_{i_j}}$ we can find some points $\tilde{\theta}^u_{i_j}$ such that $f_{\alpha^u_{i_j}}(\tilde{\theta}^u_{i_j}) > f_{\alpha^u_{i_j}}(\theta^u_{i_j})$ and that is absurd for how we chose the sequence $(\theta^u_{i_j})$. $\square$

We can now give conditions that the points $(\bar{\theta}, \bar{\alpha})$ realizing the natural pseudo-distance have to satisfy.

Theorem 2.1.5. *Let us take an optimal homeomorphism $\rho_{\bar{\alpha}}$ and assume that the function $f_{\bar{\alpha}}$ has only one point of absolute maximum $\bar{\theta}$, i.e. only one point $\bar{\theta}$ in S^1 such that $f_{\bar{\alpha}}(\bar{\theta}) = d_{S^1}(\varphi, \psi)$. Then $(\bar{\theta}, \bar{\alpha})$ is a critical point for the function F:*

$$\frac{\partial F}{\partial \theta}(\bar{\theta}, \bar{\alpha}) = 0 \qquad \textit{and} \qquad \frac{\partial F}{\partial \alpha}(\bar{\theta}, \bar{\alpha}) = 0. \tag{2.6}$$

Proof. At first we see that the function F is at least C^1 in a neighborhood of $(\bar{\theta}, \bar{\alpha})$, because otherwise we would have $F(\bar{\theta}, \bar{\alpha}) = 0$ and then $f_{\bar{\alpha}}$ would be the constant function equal to 0. This is absurd since $f_{\bar{\alpha}}$ has only one point of absolute maximum. We immediately see that it must be $\frac{\partial F}{\partial \theta}(\bar{\theta}, \bar{\alpha}) = \frac{d}{d\theta} f_{\bar{\alpha}}(\bar{\theta}) = 0$, since $\bar{\theta}$ is a point of global maximum for the function $f_{\bar{\alpha}}$. Now, let us consider the two sequences (θ^u_i, α^u_i), (θ^d_i, α^d_i) in $S^1 \times S^1$, whose existence is guaranteed by Lemma 2.1.4. Since $\bar{\theta}$ is the only point of absolute maximum for the function $f_{\bar{\alpha}}$ it must be

$$(\theta^u_i, \alpha^u_i) \xrightarrow{i \to \infty} (\bar{\theta}, \bar{\alpha}) \quad \text{and}$$
$$(\theta^d_i, \alpha^d_i) \xrightarrow{i \to \infty} (\bar{\theta}, \bar{\alpha}).$$

Therefore, since for all i we have $F(\theta^u_i, \alpha^u_i) \geq F(\bar{\theta}, \bar{\alpha}) \geq F(\theta^u_i, \bar{\alpha})$ and similarly $F(\theta^d_i, \alpha^d_i) \geq F(\bar{\theta}, \bar{\alpha}) \geq F(\theta^d_i, \bar{\alpha})$, we see that

$$\lim_{i\to\infty} \frac{F(\theta_i^u, \alpha_i^u) - F(\theta_i^u, \bar{\alpha})}{\alpha_i^u - \bar{\alpha}} \geq 0$$
$$\lim_{i\to\infty} \frac{F(\theta_i^d, \alpha_i^d) - F(\theta_i^d, \bar{\alpha})}{\alpha_i^d - \bar{\alpha}} \leq 0$$

and since both the limits converge to $\frac{\partial F}{\partial \alpha}(\bar{\theta}, \bar{\alpha})$ it must be $\frac{\partial F}{\partial \alpha}(\bar{\theta}, \bar{\alpha}) = 0$. □

Example 2.1.6. As an example illustrating the statement of Theorem 2.1.5 we can consider the two functions $\varphi(\theta) = \frac{1}{2}\sin^2(\frac{\theta}{2})$ and $\psi(\theta) = \sin^2(\frac{\theta}{2})$. In this case the natural pseudo-distance is $\frac{1}{2}$ and it is realized by the point $(0, \pi)$, which is a critical point for $F(\theta, \alpha)$.

We can also find examples where ρ_α is an optimal homeomorphism and the points (θ, α) that realize the natural pseudo-distance are not critical points for F.

Example 2.1.7. Consider the two functions φ and ψ drawn in Figure 2.1. We see that the optimal homeomorphism is the identity and that the points that realize the natural pseudo-distance are $(\frac{\pi}{2}, 0)$ and $(\frac{3\pi}{2}, 0)$, which are not critical points for the function F.

The following proposition cover the case in which the points that realize the natural pseudo-distance are not critical points for F.

Proposition 2.1.8. *Let us assume that $\bar{d} \neq 0$ and take an optimal homeomorphism $\rho_{\bar{\alpha}}$. We define $\Theta = \{\theta \in S^1 | f_{\bar{\alpha}}(\theta) := F(\theta, \bar{\alpha}) = \bar{d}\}$ and suppose that none of the points $(\theta, \bar{\alpha}), \theta \in \Theta$ is a critical point for F. Then:*

$$\exists \bar{\theta}_1, \bar{\theta}_2 \in \Theta \text{ such that } \frac{\partial F}{\partial \alpha}(\bar{\theta}_1, \bar{\alpha}) \frac{\partial F}{\partial \alpha}(\bar{\theta}_2, \bar{\alpha}) < 0. \tag{2.7}$$

Proof. Since $\bar{d} \neq 0$, the function F is at least C^1 in a neighborhood of the set Θ. We can consider the sequences defined in Lemma 2.1.4. We have that $(\theta_i^u, \alpha_i^u) \xrightarrow{i\to\infty} (\bar{\theta}_1, \bar{\alpha})$. Then since $F(\theta_i^u, \alpha_i^u) \geq F(\bar{\theta}_1, \bar{\alpha}) \geq F(\theta_i^u, \bar{\alpha})$,

$$\frac{\partial F}{\partial \alpha}(\bar{\theta}_1, \bar{\alpha}) = \lim_{i\to\infty} \frac{F(\theta_i^u, \alpha_i^u) - F(\theta_i^u, \bar{\alpha})}{\alpha_i^u - \bar{\alpha}} \geq 0. \tag{2.8}$$

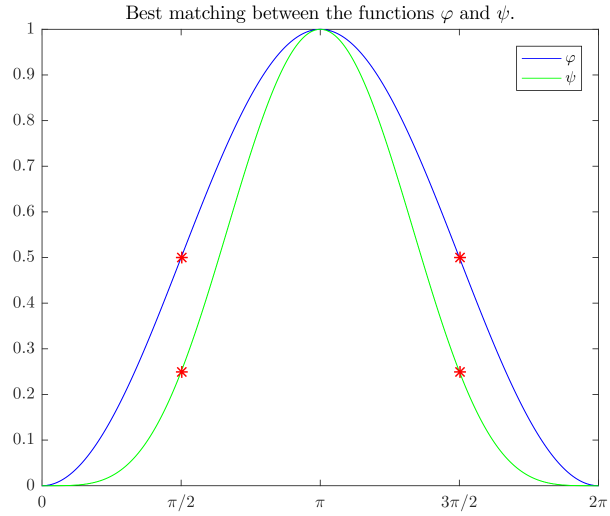

Figure 2.1: Best matching between two Morse functions. The red dots represent the points at which the maximum pointwise distance between the two functions is attained. Their abscissas are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

This limit cannot be equal to 0 because otherwise the point $(\bar{\theta}_1, \bar{\alpha})$ would be a critical point for the function F, against our assumptions. Similarly, we have that $F(\theta_i^d, \alpha_i^d) \geq F(\bar{\theta}_2, \bar{\alpha}) \geq F(\theta_i^d, \bar{\alpha})$,

$$(\theta_i^d, \alpha_i^d) \xrightarrow{i \to \infty} (\bar{\theta}_2, \bar{\alpha})$$

$$\frac{\partial F}{\partial \alpha}(\bar{\theta}_2, \bar{\alpha}) = \lim_{i \to \infty} \frac{F(\theta_i^d, \alpha_i^d) - F(\theta_i^d, \bar{\alpha})}{\alpha_i^d - \bar{\alpha}} \leq 0.$$

Also in this case the limit cannot be equal to 0 because of our hypothesis, so that $\bar{\theta}_1 \neq \bar{\theta}_2$. Therefore, $\frac{\partial F}{\partial \alpha}(\bar{\theta}_1, \bar{\alpha}) \frac{\partial F}{\partial \alpha}(\bar{\theta}_2, \bar{\alpha}) < 0$. $\square$

This proposition concerns cases similar to the one treated in the Example 2.1.7. Let us consider the optimal homeomorphism $\rho_0 = id$, associated with $\bar{\alpha} = 0$. If we take ρ_ε, with an arbitrary small $\varepsilon > 0$, then in a neighborhood of the point $\frac{\pi}{2}$ the values taken by the function f_ε are smaller than the ones taken by the function f_0, but at the same time in a neighborhood of the point $\frac{3\pi}{2}$ the values taken by the function f_ε are greater than the ones taken by the function f_0. We get an analogous result if we consider $\rho_{-\varepsilon}$, so that $\max_\theta f_\varepsilon(\theta) > \max_\theta f_0(\theta) = \bar{d}$ for any non-null ε whose absolute value is small enough.

2.2 Localization of points that realize the natural pseudo-distance and finiteness of the set of optimal homeomorphisms

We are now ready to introduce the main results studied in this thesis. In Theorem 2.2.1 we gather all the results of the previous section and we analyze also the case in which the natural pseudo-distance between the filtering functions is 0. We will then be able to localize certain particular points for the two functions that have to be "aligned" in order to obtain a correspondence between the two functions that minimize the pointwise distance. In this way we well be also able to estimate the possible values for the natural pseudo-distance and the correspondent optimal homeomorphisms. We will see how we can use these results in the Example 2.2.3. In Theorem 2.2.2 we will study the problem of the finiteness of the set of the optimal homeomorphisms. We will see that if the filtering functions are Morse there may be only a finite number of optimal homeomorphisms. We will see in fact that if an infinite number of optimal homeomorphisms exist then we can find an infinite sequence of critical point for F. This sequence will let us find a degenerate point for the function F, i.e. a critical point at which the Hessian matrix of F has not maximum rank. Studying the determinant of the Hessian matrix at such a point we will be able to see that at least one of the filtering functions must have a degenerate point. This will be contradictory because we assumed that the filtering functions are Morse.

Theorem 2.2.1. *Let $\varphi, \psi \in \mathcal{M}$ and $\bar{d} = d_{S^1}(\varphi, \psi)$. At least one of the following statements holds:*

1. *There exist θ_1 critical point for φ and θ_2 critical point for ψ such that*

$\bar{d} = |\varphi(\theta_1) - \psi(\theta_2)|$;

2. *There exist* θ_1, θ_2, $\tilde{\theta}_1$ *and* $\tilde{\theta}_2$ *such that* $\bar{d} = |\varphi(\theta_1) - \psi(\theta_2)| = |\varphi(\tilde{\theta}_1) - \psi(\tilde{\theta}_2)|$ *with*

$$\begin{cases} \varphi'(\theta_1) = \psi'(\theta_2) \text{ and } \varphi'(\tilde{\theta}_1) = \psi'(\tilde{\theta}_2) \\ \theta_1 - \theta_2 = \tilde{\theta}_1 - \tilde{\theta}_2 \\ \varphi'(\theta_1)\varphi'(\tilde{\theta}_1) < 0 \quad (\text{or equivalently} \quad \psi'(\theta_2)\psi'(\tilde{\theta}_2) < 0) \end{cases}$$

$$\text{if} \qquad (\varphi(\theta_1) - \psi(\theta_2)) \cdot (\varphi(\tilde{\theta}_1) - \psi(\tilde{\theta}_2)) > 0$$

or

$$\begin{cases} \varphi'(\theta_1) = \psi'(\theta_2) \text{ and } \varphi'(\tilde{\theta}_1) = \psi'(\tilde{\theta}_2) \\ \theta_1 - \theta_2 = \tilde{\theta}_1 - \tilde{\theta}_2 \\ \varphi'(\theta_1)\varphi'(\tilde{\theta}_1) > 0 \quad (\text{or equivalently} \quad \psi'(\theta_2)\psi'(\tilde{\theta}_2) > 0) \end{cases}$$

$$\text{if} \qquad (\varphi(\theta_1) - \psi(\theta_2)) \cdot (\varphi(\tilde{\theta}_1) - \psi(\tilde{\theta}_2)) < 0.$$

Proof. If $\bar{d} \neq 0$ the result follows from Theorem 2.1.5 and Proposition 2.1.8, by setting $\bar{\alpha} = \theta_1 - \theta_2$. If $\bar{d} = 0$ there must exist a $\rho \in S^1$ such that $\varphi = \psi \circ \rho$ and therefore both the statements of the theorem are trivially true. □

Our last result concerns the finiteness of the set of optimal homeomorphisms.

Theorem 2.2.2. *The number of optimal homeomorphisms between two functions* $\varphi, \psi \in \mathcal{M}$ *is finite.*

Proof. We will see that if there is an infinite number of optimal homeomorphisms then at least one of the two functions φ, ψ has a degenerate point. Suppose there is an infinite family of optimal homeomorphisms between φ and ψ. For the compactness of the Lie group S^1 we can construct a sequence (ρ_{α_i}) of optimal homeomorphisms different from each other, which converges to an optimal homeomorphism $\rho_{\bar{\alpha}}$. By possibly extracting a subsequence, we can assume either $\alpha_i > \bar{\alpha}$ for all i or $\alpha_i < \bar{\alpha}$ for all i. We will confine ourselves to examine just the first case, since the other can be managed in a completely analogous way. Once again for the compactness of S^1, we can find a sequence (θ_i) that converges to a point $\bar{\theta}$, such that

$$f_{\alpha_i}(\theta_i) = \max_{\theta} f_{\alpha_i}(\theta) = \bar{d} \quad \text{and} \quad f_{\bar{\alpha}}(\bar{\theta}) = \max_{\theta} f_{\bar{\alpha}}(\theta) = \bar{d}.$$

The sequence $((\theta_i, \alpha_i))$ converges to $(\bar{\theta}, \bar{\alpha})$. We see that the following inequalities hold for all i:

$$F(\theta_i, \alpha_i) - F(\theta_i, \bar{\alpha}) \geq F(\theta_i, \alpha_i) - F(\bar{\theta}, \bar{\alpha}) = \bar{d} - \bar{d} = 0 \tag{2.9}$$

$$F(\bar{\theta}, \alpha_i) - F(\bar{\theta}, \bar{\alpha}) \leq F(\theta_i, \alpha_i) - F(\bar{\theta}, \bar{\alpha}) = \bar{d} - \bar{d} = 0. \tag{2.10}$$

Now we will consider the case in which $\bar{d} > 0$ and therefore the function F is a C^1 function near the points that realize the natural pseudo-distance. We have that

$$\frac{\partial F}{\partial \alpha}(\bar{\theta}, \bar{\alpha}) = \lim_{i \to \infty} \frac{F(\theta_i, \alpha_i) - F(\theta_i, \bar{\alpha})}{\alpha_i - \bar{\alpha}} \geq 0 \tag{2.11}$$

$$\frac{\partial F}{\partial \alpha}(\bar{\theta}, \bar{\alpha}) = \lim_{i \to \infty} \frac{F(\bar{\theta}, \alpha_i) - F(\bar{\theta}, \bar{\alpha})}{\alpha_i - \bar{\alpha}} \leq 0 \tag{2.12}$$

and hence $(\bar{\theta}, \bar{\alpha})$ is a critical point for F. Now we want to show there is an infinite sequence of critical points for F, proving that F is not a Morse function and that a degenerate critical point exists.
Let us consider the continuous function $g(\alpha) = \max_{\theta \in S^1} F(\theta, \alpha)$. We know that $g(\alpha_i) = \bar{d}$ for every index i, and that the function g must attain its maximum value over every compact interval $[\alpha_i, \alpha_{i+1}]$. Let us call $\tilde{\alpha}_i$ the point at which the restriction of g to the set $[\alpha_i, \alpha_{i+1}]$ takes its maximum value. We can now consider the point $\tilde{\theta}_i$ of global maximum for the function $f_{\tilde{\alpha}_i}$. The pair $(\tilde{\theta}_i, \tilde{\alpha}_i)$ is a maximum point for the function F, and hence also a critical point, since F is at least C^1 in a neighbourhood of that pair.

Since there exists a point $(\bar{\theta}, \bar{\alpha})$ that is an accumulation point of critical points for the function F, $(\bar{\theta}, \bar{\alpha})$ is also a degenerate point for the function

F. This means that the Hessian matrix at $(\bar\theta, \bar\alpha)$ has determinant equal to 0. The Hessian matrix is

$$\operatorname{sgn}(\varphi(\bar\theta) - \psi(\bar\theta + \bar\alpha)) \cdot \begin{pmatrix} \varphi''(\bar\theta) - \psi''(\bar\theta + \bar\alpha) & -\psi''(\bar\theta + \bar\alpha) \\ -\psi''(\bar\theta + \bar\alpha) & -\psi''(\bar\theta + \bar\alpha) \end{pmatrix} \tag{2.13}$$

(where we use the notation: $\varphi''(x) = \frac{d^2\varphi}{dx^2}(x)$ and the same for ψ). If $\psi''(\bar\theta+\bar\alpha) \neq 0$, it must be $\varphi''(\bar\theta) - \psi''(\bar\theta + \bar\alpha) = -\psi''(\bar\theta + \bar\alpha)$, and hence $\varphi''(\bar\theta) = 0$. Since $(\bar\theta, \bar\alpha)$ is a critical point for F it must also be:

$$\frac{\partial F}{\partial \theta}(\bar\theta, \bar\alpha) = 0 \qquad \text{and} \qquad \frac{\partial F}{\partial \alpha}(\bar\theta, \bar\alpha) = 0. \tag{2.14}$$

The equality $\frac{\partial F}{\partial \alpha}(\bar\theta, \bar\alpha) = 0$ implies that $-\psi'(\bar\theta + \bar\alpha) = 0$. Since$\frac{\partial F}{\partial \theta}(\bar\theta, \bar\alpha) = 0$ it follows that $\varphi'(\bar\theta) - \psi'(\bar\theta, \bar\alpha) = 0$, and it must be $\varphi'(\bar\theta) = 0$. Then $\bar\theta$ is a degenerate point for the function φ, which cannot be a Morse function.
Analogously, in the case $\psi''(\bar\theta + \bar\alpha) = 0$, we have that $\psi'(\bar\theta + \bar\alpha) = 0$ and hence $\bar\theta + \bar\alpha$ is a degenerate point for the function ψ, which cannot be a Morse function.
We eventually consider the case in which $\bar d = 0$. If φ, ψ are Morse functions and $\bar d = 0$, there cannot exist an infinite number of homeomorphisms ρ_α such that $\varphi = \psi \circ \rho_\alpha$, since these homeomorphisms must match the critical points of the two functions, and the number of these points is finite. □

Example 2.2.3. The use of Theorems 2.2.1 and 2.2.2 can be clarified by the following example, illustrating the computation of the natural pseudo-distance between the two functions $\varphi(\theta) = \frac{1}{2}\sin(2\theta)$ and $\psi(\theta) = \sin(\theta)$. Since φ and ψ are Morse, Theorem 2.2.2 guarantees that the number of optimal homeomorphisms between φ and ψ is finite. We can apply Theorem 2.2.1 to find these homeomorphisms. Since the difference between the maximum values of the two functions is $\frac{1}{2}$, the inequality $\bar d \geq \frac{1}{2}$ holds. Let us look for the points described in the second statement of Theorem 2.2.1. For any $\alpha \in S^1$ we want to find the corresponding θs that satisfy the equation $\varphi'(\theta) - \psi'(\theta + \alpha)$, i.e.

$$\cos(2\theta) = \cos(\theta + \alpha). \tag{2.15}$$

We obtain the following solutions depending on α:

$$\bar{\theta}_1 = \alpha \quad \bar{\theta}_2 = -\frac{\alpha}{3} \quad \bar{\theta}_3 = -\frac{\alpha}{3} + \frac{2\pi}{3} \quad \bar{\theta}_4 = -\frac{\alpha}{3} + \frac{4\pi}{3}.$$

We insert these values in the equation $|\varphi(\theta_1) - \psi(\theta_2)| = |\varphi(\tilde{\theta}_1) - \psi(\tilde{\theta}_2)|$ (see Theorem 2.2.1) to find the possible α associated with the optimal homeomorphisms. If ρ_α is an optimal homeomorphism, two indexes i and j must exist such that

$$\bar{d} = \left|\frac{1}{2}\sin(2\bar{\theta}_i) - \sin(\bar{\theta}_i + \alpha)\right| = \left|\frac{1}{2}\sin(2\bar{\theta}_j) - \sin(\bar{\theta}_j + \alpha)\right|.$$

This leads to consider these possible values for α: $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$. With some calculations we see that

$$\max_{\theta \in S^1} f_\alpha(\theta) = \frac{3\sqrt{3}}{4} \qquad \text{for } \alpha \in \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\} \tag{2.16}$$

$$\max_{\theta \in S^1} f_\alpha(\theta) = \frac{3}{2} \qquad \text{for } \alpha \in \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}. \tag{2.17}$$

Hence the natural pseudo-distance is $\frac{3\sqrt{3}}{4}$ and the points $\theta_1 = \theta_2 = \frac{2\pi}{3}$ and $\tilde{\theta}_1 = \tilde{\theta}_2 = \frac{4\pi}{3}$ satisfy the conditions of the second statement of Theorem 2.2.1.

Chapter 3

Conclusions

In this thesis we have studied the main properties of the natural pseudo-distance associated with the Lie group S^1 between two functions defined on the closed curve S^1. We have seen how it differs from the natural pseudo-distance associated with the group of all homeomorphisms. In the latter case we have seen that the possible values of $d_H(\varphi, \psi)$ are strictly related to the critical values of the functions φ and ψ. In fact in this case the optimal homeomorphisms, when they exist, align the critical points of the two functions. If the two functions are Morse they have only a finite number of critical values, hence we can consider a finite number of possible values for $d_H(\varphi, \psi)$, as seen in Theorem 1.1.17. If we consider $d_{S^1}(\varphi, \psi)$, as we have seen in the examples 2.1.7 and 2.2.3, its possible values are not related to the critical values of the two functions. Therefore, in this case we do not have any a priori way to estimate the value of the natural pseudo-distance based on the critical values of the functions taken into account. We can certainly combine the Stability Theorem 1.2.19 and Proposition 1.1.19 to obtain

$$d_{match}\left(l^*_\varphi, l^*_\psi\right) \leq d_H\left(\varphi, \psi\right) \leq d_{S^1}\left(\varphi, \psi\right) \tag{3.1}$$

where l^*_φ and l^*_ψ are the reduced size functions for the pairs (S^1, φ) and (S^1, ψ). To estimate the value of $d_{S^1}\left(\varphi, \psi\right)$ Theorem 2.2.1 may be useful. If the first statement of the theorem is satisfied it basically resembles Theorem 6.3 in [9], and an optimal homeomorphism is given by a suitable alignment between two critical points of the functions. If the second statement is satisfied we can find an estimate of the natural pseudo-distance studying the derivatives of the two functions. The theorem let us see that finding the possible values for $d_{S^1}(\varphi, \psi)$ is strictly related to finding particular pairs (θ_1, θ_2) and $\left(\tilde{\theta}_1, \tilde{\theta}_2\right)$ of points for the two functions with $\theta_1 - \theta_2 = \tilde{\theta}_1 - \tilde{\theta}_2$ and such that $\varphi(\theta_1) = \psi(\theta_2)$ and $\varphi(\tilde{\theta}_1) = \psi(\tilde{\theta}_2)$. If we are able to recognize these pairs then the problem of optimality is also solved. If we find two pairs as described in the second statement of Theorem 2.2.1 a candidate to be an optimal homeomorphism is the one that maps θ_1 to θ_2 and therefore $\tilde{\theta}_1$ to $\tilde{\theta}_2$. The strength of this theorem resides in the possibility to translate the problem of evaluating the natural pseudo-distance to that of analyzing only certain structures of points that arise between the two functions. However without any other hypothesis the number of these pairs may not be finite, and we would have to analyze an infinite number of possible values for $d_{S^1}(\varphi, \psi)$. This situation is where Theorem 2.2.2 comes into play. It assures us that if the filtering functions are Morse then there are only a finite number of optimal homeomorphisms. This fact simplifies the search for optimal homeomorphisms. Hence the finiteness of the set of optimal homeomorphisms plays a key role in this scenario. We can see that the desired result is yielded for Morse functions. This fact does not represent a strong drawback, because it is well known that the set of Morse functions is dense in the set of smooth functions and that smooth functions are dense in $C^k(\mathcal{M}, \mathbb{R})$ for any k, if $\mathcal{M}$ is a closed C^k manifold (cf. [20] and [21]). Then assuming of working with Morse functions is not a strong restriction since they can be approximated by continuous functions.

It would be interesting to extend these results to higher dimensional cases. This could mean considering higher dimensional closed manifolds on which a group of homeomorphisms isomorphic to S^1 acts, or considering the action of higher dimensional Lie groups on themselves. Similar techniques to that used to prove Theorem 2.2.1 could be used to find the structures of points that arise between the two filtering functions. At the same time, we would like to find conditions that assure the finiteness of the set of optimal homeomorphisms for the considered natural pseudo-distances. It is possible to prove that Theorem 2.2.2 ceases to hold if the manifold taken into account is not S^1.

Example 3.0.1. Let us consider the sphere $X = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. We define the two functions $\varphi, \psi : X \to \mathbb{R}$ as

$$\varphi(x, y, z) = z \quad \text{and} \quad \psi(x, y, z) = \frac{1}{2} z. \tag{3.2}$$

The group of homeomorphisms acting on the functions will be the group G of rotations around the z-axis. We can clearly find a diffeomorphism between S^1 and G since we can write G as

$$G = \left\{ \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \alpha \in \mathbb{R} \right\}. \tag{3.3}$$

In this case it is easy to see that $d_G(\varphi, \psi) = \frac{1}{2}$ but every homeomorphism $g \in G$ is optimal since $\|\varphi - \psi \circ g\|_\infty = \frac{1}{2}$, because the action of G leaves both φ and ψ unvaried.

There should be then a certain kind of coherence between the dimension of the manifold and that of the group of homeomorphisms. Another possibility could be to consider multidimensional functions, that have already been taken into account by persistent homology [5]. We would then have to use a suitable definition for Morse functions with values in $\mathbb{R}^d$, with $d > 1$, for which a result similar to Theorem 2.2.2 could exist.

Bibliography

[1] Helmut Alt and Maike Buchin. Can we compute the similarity between surfaces? *Discrete Comput. Geom.*, 43(1):78–99, 2010.

[2] Francesca Cagliari, Barbara Di Fabio, and Claudia Landi. The natural pseudo-distance as a quotient pseudo-metric, and applications. *Forum Math.*, 27(3):1729–1742, 2015.

[3] Gunnar Carlsson. Topological pattern recognition for point cloud data. *Acta Numer.*, 23:289–368, 2014.

[4] Andrea Cerri and Barbara Di Fabio. On certain optimal diffeomorphisms between closed curves. *Forum Math.*, 26(6):1611–1628, 2014.

[5] Andrea Cerri, Barbara Di Fabio, Massimo Ferri, Patrizio Frosini, and Claudia Landi. Betti numbers in multidimensional persistent homology are stable functions. *Math. Methods Appl. Sci.*, 36(12):1543–1557, 2013.

[6] David Cohen-Steiner and Herbert Edelsbrunner. Inequalities for the curvature of curves and surfaces. *Found. Comput. Math.*, 7(4):391–404, 2007.

[7] Michele d'Amico, Patrizio Frosini, and Claudia Landi. Natural pseudo-distance and optimal matching between reduced size functions. *Acta Appl. Math.*, 109(2):527–554, 2010.

[8] Barbara Di Fabio and Claudia Landi. Reeb graphs of curves are stable under function perturbations. *Math. Methods Appl. Sci.*, 35(12):1456–1471, 2012.

[9] Pietro Donatini and Patrizio Frosini. Natural pseudodistances between closed manifolds. *Forum Math.*, 16(5):695–715, 2004.

[10] Pietro Donatini and Patrizio Frosini. Natural pseudodistances between closed surfaces. *J. Eur. Math. Soc. (JEMS)*, 9(2):331–353, 2007.

[11] Pietro Donatini and Patrizio Frosini. Natural pseudo-distances between closed curves. *Forum Math.*, 21(6):981–999, 2009.

[12] Herbert Edelsbrunner and John Harer. Persistent homology—a survey. In *Surveys on discrete and computational geometry*, volume 453 of *Contemp. Math.*, pages 257–282. Amer. Math. Soc., Providence, RI, 2008.

[13] Herbert Edelsbrunner and John L. Harer. *Computational topology, An introduction.* American Mathematical Society, Providence, RI, 2010.

[14] Herbert Edelsbrunner and Dmitriy Morozov. Persistent homology: theory and practice. In *European Congress of Mathematics*, pages 31–50. Eur. Math. Soc., Zürich, 2013.

[15] Brittany Terese Fasy. The difference in length of curves in $\mathbb{R}^n$. *Acta Sci. Math. (Szeged)*, 77(1-2):359–367, 2011.

[16] Patrizio Frosini and Grzegorz Jabłoński. Combining persistent homology and invariance groups for shape comparison. *Discrete Comput. Geom.*, 55(2):373–409, 2016.

[17] Patrizio Frosini and Claudia Landi. Size theory as a topological tool for computer vision. *Pattern Recognition And Image Analysis*, 9(4):596–603, 1999.

[18] Patrizio Frosini and Claudia Landi. Uniqueness of models in persistent homology: the case of curves. *Inverse Problems*, 27(12):124005, 14, 2011.

[19] Hee Il Hahn. Application of persistent homology to topological analysis of waveform signals. *GSTF Journal of Mathematics, Statistics and Operations Research (JMSOR)*, 3(2):77–83, 2016.

[20] John Milnor. *Morse theory.* Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J., 1963.

[21] Jacob Palis, Jr. and Welington de Melo. *Geometric theory of dynamical systems.* Springer-Verlag, New York-Berlin, 1982. An introduction, Translated from the Portuguese by A. K. Manning.