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Dynamical Localization of Abelian Gauge Fields

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Abstract

We study the possibility that a consistent description of the four-dimensional photon field can arise from a model of scalar-QED in six-dimensional space-time.

The existence of new dimensions other than the usual four represents one the most attractive approaches to adopt in the attempt to solve long lasting high energy as well as cosmological puzzles [28]. However if the introduction of extra dimensions may constitute the natural framework for the construction of new effective field theories, it also opens the crucial question of explaining why as long as we can test, the Universe appears truly four-dimensional [13; 25; 17]. Extra dimensions need to be hidden somehow and one possibility is that Standard Model particles result localized on a four-dimensional brane embedded in a higher dimensional space-time.

In the present work, we examine the possibility of localizing gauge fields by coupling the six-dimensional field to a vortex-like configuration of the charged scalar field. This represents a stable and cylindrically symmetric (along the four usual directions) solution, which carves Minkowski four dimensional space-time out of its throat.

The main result is that, the gauge field components associated with the four dimensional photon can be completely decoupled from the other fields, with a convenient choice of the gauge together with the existence of a mass hierarchy, which allows to set a cut-off for the admissible energies range, in order to exclude non physical degrees of freedom from the theory. This procedure though, doesn't leave the photon massless. We add to the six-dimensional Lagrangian a finely tuned mass term, in order to cancel the mass of the photon thus obtained. We assume this extra mass term coming from a Higgs mechanism, introducing an auxiliary scalar field with vacuum expectation value appropriately chosen.

Sommario

In questo elaborato esaminiamo la possibilità di derivare una descrizione consistente del campo di gauge associato al fotone, da un modello di QED-scalare in sei dimensioni.

L'esistenza di nuove dimensioni in aggiunta alle quattro comuni, rappresenta uno degli approcci più interessanti da adottare, nel tentativo di risolvere correnti puzzles nella fisica delle alte energie così come in cosmologia. Tuttavia se da un lato l'introduzione di extra dimensioni potrebbe rappresentare l'ambiente naturale per la costruzione di nuove teorie di campo, dall'altro fa emergere la necessità di spiegare perchè, per quanto ne sappiamo, l'Universo appaia effettivamente quadridimensionale. Le dimensioni extra devono quindi risultare nascoste in qualche modo e una delle possibilità più stimolanti è che le particelle del Modello Standard siano localizzate su una brana quadridimensionale immersa in uno spazio-tempo di dimensionalità maggiore.

Nel presente lavoro discutiamo un possibile meccanismo per la localizzazione del campo di gauge mediante l'accoppiamento con una configurazione di tipo vortice per il campo scalare carico, questa rappresenta una soluzione stabile e a simmetria cilindrica (lungo le quattro dimensioni usuali) che permette di modellare lo spazio-tempo Minkowskiano quadridimensionale, confinato all'interno del vortice.

Il risultato principale è che le componenti del campo di gauge associate al fotone quadridimensionale possono essere completamente disaccoppiate dai restanti campi, come conseguenza dell'invarianza di gauge del modello congiuntamente alla presenza di una gerarchia tra le masse delle componenti del campo di gauge sei-dimensionale, che suggerisce di fissare un cut-off al range di energie ammissibili così da escludere gradi di libertà non fisici. Questa procedura tuttavia, lascia il fotone massivo. Una possibilità è aggiungere alla Lagrangiana sei-dimensionale un termine di massa accuratamente scelto per il fotone, così da cancellare la massa ottenuta. Questo termine extra potrebbe derivare da un meccanismo di Higgs mediante l'introduzione di un campo scalare ausiliario, con valore di aspettazione del vuoto appropriatamente scelto.

Contents

Introduction

The possibility that our space has more that three spatial dimensions has been actracting the attention of physicists for many years. One of the main reasons for looking for extra-dimensions comes from the crucial help it might offer in solving long lasting puzzles like the Hierarchy Problem. This refers to the 'unnatural' huge discrepancy between the electroweak scale $m_{EM} \sim 10^3$ GeV and the Planck scale $M_{Pl} \sim 10^{19}$ GeV. Over the last few decades, explaining the smallness and radiative stability of the hierarchy $m_{EW}/M_{Pl} \sim 10^{-16}$ has been one of the greatest driving forces behind the construction of theories beyond the Standard Model. While many different specific proposals for weak and Planck scale physics have been made, the commonly held picture of the basic structure of physics beyond the Standard Model, is of a new effective field theory (like a softly broken supersymmetric theory) revealed at the weak scale, stabilizing and perhaps explaining the origin of the hierarchy [25]. On the other hand, the physics responsible for making a sensible quantum theory of gravity is revealed only at the Planck scale. The desert between the weak and Planck scales could itself be populated with towers of new effective field theories which can play a number of roles, such as triggering dynamical symmetry breakings or explaining the pattern of fermion masses and mixings.

The extra-dimensions perspective was firstly adopted in Kaluza-Klein theory [1]. In their work was postulated one extra compactified space-like dimension and assumed nothing but pure gravity in the new $(4 + 1)$-dimensional space-time. It turned out [23] that five-dimensional gravity would manifest in the observable four-dimensional space-time as gravitational, electromagnetic and scalar field. Kaluza and Klein then managed to unify gravity and electromagnetism but the theory had some inconsistencies, the calculated mass and charge of the electron, for example did not match with experimental data. Neverthless this theory was never completely abandoned. Over many decades physicists have been trying to improve the Kaluza's and Klein's concept, in the attempt of building a theory which could incorporate gravity in a reliable manner. All these resulting new field theories are consistently formulated in space-time of more than four dimensions and are commonly known as Kaluza-Klein(like) theories, independently of the specific mechanism adopted for explain how four-dimensional physics emerge.

[1] Published in 1921 by German mathematician and physicist Theodor Kaluza and extended in 1926 by Swedish theoretical physicist Oskar Klein, was one of the first attempts to create an unified field theory.

One possibility is that the extra dimensions are microscopic, compact and essentially homogeneous, ensuring that space-time is effectively four-dimensional at distances exceeding the compactification scale (size of extra-dimensions). A 'common wisdom' assumes this size to be roughly of the order of the Planck scale (although compactifications at the electroweak scale can be also considered). With the Planck length $l_{Pl} \sim 10^{-33}$ cm, probing extra dimensions within this framework appears to be hopeless.

Recently, emphasis has shifted towards non-compact extra-dimensions. The crucial ingredient is a brane on which Standard Model particles are localized. Gravity however, propagating in all dimensions as it is the dynamics of spacetime itself, could not fix in this scenario; the first successful model is due to Randall and Sundrum [2][12; 13]. They assumed a non factorizable geometry i.e. the four dimensional metrics was not independent, as it was before, of the coordinates in the extra dimensions, and managed to prove that $(4+n)$ non-compact dimensions are in perfect compatibility with experimental gravity.

In the brane world model-building the question of paramount importance is how the mechanism for implementing compactification in large extra dimensions can take place, in particular why one may ignore the massive states in lower dimensions which arise from harmonic expansions of the higher dimensional fields. These states may be ignored if they are separated from the zero modes by a well defined 'mass gap' [11]. Different approaches to realize this mechanism have been suggested, such as adopting warping factor [36] (like in the R-S model) or induced kinetic term which ensure localization on the brane [17]; another common possibility is that our Universe could be a four-dimensional topological defect dynamically occurring in a higher-dimensional space-time [25; 28; 15; 26; 27; 33]. Ordinary particles, in this approach, can be viewed as "zero modes" trapped in the core of the defect, making extra dimensions invisible for a four dimensional observer. This mechanism is referred to as dynamical compactification since the four-dimensional space-time is dynamically generated out of a higher dimensional one, through a spontaneous breaking of (a part of) the translational symmetries of the original theory.

With the present work we want to discuss explicit models for the localization of four dimensional fields in higher dimensional space-time, focusing in particular on Abelian gauge fields. The general structure is outlined briefly below.

In Chapter 1 we present some background topics, we investigate the Hierarchy Problem in more detail showing how a potential solution can naturally arise in the extra-dimensions scenario and then introduce the original Kaluza-Klein theory. As previously stated this was an attempt to derive four-dimensional theory of gravity and electromagnetism from a theory of gravity in five-dimensions alone. Despite the inconsistent results, for the first time was presented a mechanism to explain how four-dimensional physics

[2]Published in 1999 by Lisa Randall and Raman Sundrum, also known as RS2, it constituted an extension of a former model, RS1, in which the size of the extra-dimension was kept finite.

can emerge from a higher dimensional framework. From then, many different theories have been proposed, and independently of the localization procedure assumed, they all share the same Kaluza-Klein idea.

In Chapter 2 we discuss models for the localization of scalar, fermion and gauge fields in $(4+1)$ non compact dimensions. In the first two cases, compactification is dynamical ensured by a domain wall solution that shapes the four-dimensional space-time. Once Kaluza-Klein reduction is implemented, four-dimensional massless scalar and fermion fields emerge as Fourier components of localized zero modes in the extra-dimensions. In the last model a different mechanism is adopted, localization is achieved by means of a four-dimensional kinetic term for the gauge field obtained from the coupling with a localized matter current.

In Chapter 3 is presented a different procedure for the localization of Abelian gauge fields. Space-time is enlarged with two extra infinite dimensions, and the coupling is with a complex scalar field. The field equations admit a vortex-like solution for the scalar field which implements dynamical compactification once fields perturbations around this configuration are considered. The vortex solution enters the linearized gauge field equations as a mass term, function of the extra dimension, null in the core of the vortex and constant in the bulk. The main result is that the components associated to the four-dimensional photon appear completely decoupled below an energy cuf-off, set by a mass hierarchy between the six-dimensional fields. The photon thus obtained is not massless though, a fine tuning must apply so to suitably cancel the mass of the lightest Kaluza-Klein mode.

Notations

The following notations are meant to apply, unless explicitly stated.

Throughout the whole text the natural units system is adopted, as well as the Heaviside-Lorentz C. G. S. system of electromagnetic units, accordingly

$$\hbar = c = 1$$

for the reduced Planck's constant and speed of light, in this units system the Action and the Lagrangian density [3] satisfy

$$[\mathcal{S}] = 1, \qquad [\mathcal{L}] = cm^{-D}$$

D being the dimensionality of space-time, while energy and mass

$$[E] = [M] = cm^{-1}, \quad [x^0] = [\mathbf{x}] = cm$$

Finally Planck length and mass in terms of Newton constant G result

$$l_{Pl} = \sqrt{G}, \qquad M_{Pl} = \sqrt{G}^{-1}$$

Four dimensional Minkowski space-time is denoted $\mathcal{M}^{(1,3)}$, the metrics signature being mostly negative. Extension to higher dimensional manifolds will occur, $(4+n)$-dimensional space-time will then represent the Minkowskian four-dimensional one plus n

[3] Commonly referred to simply as Lagrangian.

extra-dimension, the topology of which will depend on the specific case. Capital letters and subscripts will always refer to higher dimensional quantities while Greek letters to the four-dimensional ones.

The four-dimensional D'Alambertian is defined

$$\Box_4 \equiv \partial_\mu \partial^\mu = \partial_0^2 - \nabla^2$$

so that $\Box_n$ will refer to the n-dimensional D'Alambertian.

We'll make use of cylindrical coordinates (ρ, ϕ, z), with $\rho \in [0, \infty)$, $\phi \in [0, 2\pi]$, $z \in (-\infty, +\infty)$, the metric tensor reads $g_{ab} = diag(1, \rho^2, 1)$ and the Laplacian ∇^2

$$\nabla^2 = \partial_\rho^2 + \rho^{-1}\partial_\rho + \rho^{-2}\partial_\phi^2 + \partial_z^2$$

Chapter 1

Preliminaries

1.1 The Hierarchy Problem

The Hierarchy Problem is related to the huge discrepancy between the Standard Model particles scale, the electroweak scale $m_{EW} \sim 10^3$ GeV and the scale at which gravity becomes as strong as the gauge interactions, the Planck scale $M_{Pl} \sim 10^{19}$ GeV. The central issue [1] is that the mass-squared parameter of Higgs field, which determines the electroweak scale, is quadratically sensitive to new physics at higher energies. For the Standard Model taken in isolation this would not pose any problem as this parameter could be taken as an input. One might thus suggest that there is only the Standard Model and side-step any potential issue. However gravity alone, and in particular the scale where quantum effects in gravity become important, suggests the existence of new physics at energy scales M_{Pl}, which would feed into the Higgs mass and the electroweak scale, contradicting the observed hierarchy $v = 246$ GeV [1]$\ll M_{Pl}$.

Thought as two fundamental energy scales, the smallness and radiative stability of the hierarchy $m_{EW}/M_{Pl} \sim 10^{-16}$ has represented an intriguing puzzle for many years [15]. A crucial difference emerges between these two scales though; while electroweak interactions have been probed at distances $\sim m_{EW}^{-1}$, gravitational forces have not remotely been probed at distances $\sim m_{Pl}^{-1}$: gravity has only been accurately measured in the ~ 1 mm range [42]. The interpretation of M_{Pl} as a fundamental energy scale is then based on the assumption that gravity is unmodified over the 33 orders of magnitude between where it is measured at ~ 1 mm down to the Planck length $\sim 10^{-33}$ cm.

Given that the fundamental nature of the weak scale is an experimental certainty, it has been proposed [15] to keep m_{EW} as the only fundamental short distance scale in Nature, even setting the scale for the strength of the gravitational interaction. Accordingly, Planck scale is not a fundamental scale, its enormity is a consequence of the large size of the new dimensions compared with m_{EW}. Putting m_{EW} as the new ultraviolet cut-off of the theory, the Hierarchy Problem would be then solved.

[1] Vacuum expectation value of the Higgs field.

Assuming the existence of n compact space-like new dimensions of radius $\sim R$, the Planck scale $M_{Pl(4+n)}$ of this $(4+n)$ dimensional theory is by definition $\sim m_{EW}$. At distances at which the new dimensions become manifest i.e. $r \ll R$, two test masses m_1, m_2 will feel a gravitational potential according to Gauss' law in $(4+n)$ dimensions -See Appendix A-

$$U(r) \sim \frac{m_1 m_2}{M_{Pl(4+n)}^{n+2}} \frac{1}{r^{n+1}}, \quad (r \ll R)$$

On the other hand, if the masses are placed at distances $r \gg R$, their gravitational flux lines can not continue to penetrate in the extra dimensions, and the usual $1/r$ potential is recovered

$$U(r) \sim \frac{m_1 m_2}{M_{Pl(4+n)}^{n+2} R^n} \frac{1}{r}, \quad (r \gg R)$$

From the previous expressions we can identify the effective four dimensional M_{Pl} as

$$M_{Pl}^2 \sim M_{Pl(4+n)}^{2+n} R^n$$

Assuming that m_{EW} is the only fundamental scale i.e. $M_{Pl(4+n)}^{n+2} \sim m_{EW}$, the four dimensional M_{Pl} is reproduced for

$$R \sim 10^{\frac{32}{n}-17} cm \times (\frac{1TeV}{m_{EW}})^{1+\frac{2}{n}} \tag{1.1}$$

For $n = 1$, $R \sim 10^{13}$ cm implying deviations from Newtonian gravity over solar system distances, so this case is empirically excluded. For all $n \geqslant 2$, however, the modification of gravity only becomes noticeable at distances smaller than those currently probed by experiment. The case $n = 2$ ($R \sim 100\mu m - 1mm$) appears particularly encouraging, this observation has stimulated recent activity in experimental search for deviations from Newton's gravity law at sub-millimeter distances.

While gravity has not been probed at distances smaller than a millimeter, the Standard Model gauge forces have been accurately measured at weak scale distances. Therefore, in this scenario, the Standard Model particles cannot freely propagate in the extra

n dimensions, and must be localized to a four dimensional submanifold with "thickness" $\sim m_{EW}^{-1}$ in the extra n dimensions (m_{EW} is the only short distance scale in the theory). Indeed the non-trivial task in any explicit realization of this framework is the localization of the Standard Model fields.

Independently of any specific realization, this model would have some dramatic consequences. First, gravity becomes comparable in strength to the gauge interactions at energies $m_{EW} \sim$ TeV. Future experiments performed in this range would not only probe the mechanism of electroweak symmetry breaking, but the true quantum theory of gravity. Second, for the case of 2 extra dimensions, the gravitational force law should change from $1/r^2$ to $1/r^4$ on distances $\sim 100\mu m - 1mm$, and this deviation could be observed in the next few years by the new experiments measuring gravity at sub-millimeter distances. Third, since the Standard Model fields are only localized within m_{EW}^{-1} in the extra n dimensions, in sufficiently hard collisions of energy $E_{esc} \geqslant m_{EW}$, they can acquire momentum in the extra dimensions and escape from the four-dimensional world, carrying away energy [2]. This implies a sharp upper limit to the transverse momentum which can be seen in four dimensions at $p_T = E_{esc}$.

In summary, according to [15] space-time manifold is $\mathcal{M}^{(1,3)} \times K_n$ for $n \geqslant 2$, with K_n being an n dimensional compact manifold of volume $\sim R^n$ and R given by equation (1.1). The $(4+n)$ dimensional Planck mass is $\sim m_{EW}$ thus four dimensional M_{Pl} is not a fundamental scale at all, rather, the effective four dimensional gravity is weakly coupled due to the large size R of the extra dimensions relative to the weak scale. In this framework the graviton is free to propagate in all $(4+n)$ dimensions, while the Standard Model fields must be localized on a four-dimensional submanifold of thickness m_{EW}^{-1} in the extra n dimensions. This is required because no experimental signs of the extra dimensions have been detected, despite the fact that the compactification scale, $\mu_c \sim 1/R$ would have to be much smaller than the weak scale.

However, as pointed out in [12], while this scenario does eliminate the hierarchy between the weak scale m_{EW} and the Planck scale M_{Pl}, it introduces a new hierarchy, namely that between μ_c and m_{EW}. In light of this a new alternative is explored. They show that a large mass hierarchy might originate from small extra dimensions, if the metric is assumed non-factorizable: the four-dimensional metric is multiplied by a 'warp' factor, which is a rapidly changing function of one additional dimension.

[2] Usually in theories with extra compact dimensions of size R, states with momentum in the compact dimensions are interpreted from the four-dimensional point of view as particles of mass $1/R$, but still localized in the four-dimensional world. This is because at the energies required to excite these particles, their wavelength and the size of the compact dimensions are comparable. On the contrary in this model, particles which can acquire momentum in the extra dimensions have TeV energies, and therefore wavelengths much smaller than the size of the extra dimensions $R \sim mm$. Thus, they simply escape into the extra dimensions.

$$ds^2 = e^{-2kr_c\phi}\eta_{\mu\nu}dx^\mu dx^\nu + r_c^2 d\phi^2 \tag{1.2}$$

where k is a scale of the order of M_{Pl}, r_c the compactification radius, $\eta_{\mu\nu}$ the usual flat four dimensional metric and $0 \leqslant \phi \leqslant \pi$ the coordinate for one extra dimension, which is a finite interval whose size is set by r_c. In this framework, four dimensional mass scales are related to five dimensional input mass parameters and the warp factor $e^{-2kr_c\phi}$. Thus the crucial point is that to generate a large hierarchy does not require extremely large r_c, this is because the origin of the hierarchy is an exponential function of the compactification radius, which constitutes indeed the source of the large hierarchy between the observed Planck and weak scales.

1.2 Kaluza-Klein picture

1.2.1 The original theory

Kaluza's achievement was to show that five-dimensional general relativity contains both Einstein's four-dimensional theory of gravity and Maxwell's theory of electromagnetism. He however imposed a somewhat artificial restriction (the cylinder condition) on the coordinates, essentially barring the fifth one a priori from making a direct appearance in the laws of physics. Klein's contribution was to make this restriction less artificial by suggesting a plausible physical basis for it in compactification of the fifth dimension. The key is the concept of gauge invariance, [21] which was coming to be recognized as underlying all the interactions of physics. Electrodynamics, for example, could be "derived" by imposing local $U(1)$ gauge invariance on a free particle Lagrangian. From the gauge invariant point of view, Kaluza's feat in extracting electromagnetism from five-dimensional gravity was no longer so surprising: it worked because $U(1)$ gauge invariance had been "added onto" Einstein's equations in the guise of invariance with respect to coordinate transformations along the fifth dimension. In other words, gauge symmetry had been "explained" as a geometric symmetry of spacetime [23].

Consider Minkowski space-time plus one extra space-like dimension, the full set of coordinates being $x^M = (x^\mu, y)$, $M = 0, 1, 2, 3, 4$.
Einstein equations, with no five-dimensional energy-momentum tensor, assuming a "minimal extension" of general relativity [3], are:

$$\hat{G}_{AB} = 0$$

or, equivalently:

[3] It means that there is no modification to the mathematical structure of Einstein's theory. The only change is that tensor indices run over 0 to $(3+n)$ instead of 0 to 3.

$$\hat{R}_{AB} = 0 \tag{1.3}$$

where $\hat{G}_{AB} \equiv \hat{R}_{AB} - \hat{R}\,\hat{g}_{AB}/2$ is the Einstein tensor, $\hat{R}_{AB}$ and $\hat{R} = \hat{g}_{AB}\hat{R}^{AB}$ are the five-dimensional Ricci tensor and scalar respectively, and $\hat{g}_{AB}$ is the five-dimensional metric tensor. Hat and capital letter latin indices refer to higher dimensional quantities, greek indices to the four dimensional ones.
These equations can be derived by varying a five-dimensional version of Einstein action with respect to the five-dimensional metric:

$$\mathcal{S} = -\frac{1}{16\pi\hat{G}}\int d^4x dy\sqrt{-\hat{g}}\,\hat{R} \tag{1.4}$$

$\hat{G}$ being a "five-dimensional gravitational constant" .

The absence of matter sources in these equations reflects the attempt to explain matter (in four dimensions) as a manifestation of pure geometry (in higher ones).

The five-dimensional Ricci tensor and Christoffel symbols are defined:

$$\hat{R}_{AB} = \partial_C\hat{\Gamma}^C_{AB} - \partial_B\hat{\Gamma}^C_{AC} + \hat{\Gamma}^C_{AB}\hat{\Gamma}^D_{CD} - \hat{\Gamma}^C_{AD}\hat{\Gamma}^D_{BC}$$

$$\hat{\Gamma}^C_{AB} = \frac{1}{2}\hat{g}^{CD}(\hat{g}_{DB,A} + \hat{g}_{DA,B} - \hat{g}_{AB,D})$$

The metric tensor was suitably chosen:

$$(\hat{g}_{AB}) = \begin{pmatrix} g_{\alpha\beta} + k^2\phi^2 A_\alpha A_\beta & k\phi^2 A_\alpha \\ k\phi^2 A_\beta & \phi^2 \end{pmatrix} \tag{1.5}$$

where $g_{\alpha\beta}$ is the four dimensional metric tensor, A_α the electromagnetic potential, ϕ a scalar field and k a normalization parameter.

Kaluza then, assuming the cylinder condition [4], dropped all the derivatives with respect to the fifth coordinate, this way equations (1.3) reduce to:

$$G_{\alpha\beta} = \frac{k^2\phi^2}{2} T^{EM}_{\alpha\beta} - \frac{1}{\phi}[\nabla_\alpha(\partial_\beta\phi) - g_{\alpha\beta}\Box\phi]$$
$$\nabla^\alpha F_{\alpha\beta} = -3\frac{\partial^\alpha\phi}{\phi}F_{\alpha\beta}, \quad \Box\phi = \frac{k^2\phi^3}{4}F_{\alpha\beta}F^{\alpha\beta} \tag{1.6}$$

where $G_{\alpha\beta} \equiv R_{\alpha\beta} - R\, g_{\alpha\beta}/2$ is the Einstein tensor, $T^{EM}_{\alpha\beta} \equiv g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}/4 - F^{\gamma}_{\alpha}F_{\beta\gamma}$ the electromagnetic energy-momentum tensor and $F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha$ the strength tensor. There are a total of $10 + 4 + 1 = 15$ equations, as it has to be since a five-dimensional metric tensor has $(5 + 6)/2 = 15$ independent elements.

If ϕ is constant the first two equations of (1.6) are just the Einstein and Maxwell ones:

$$G_{\alpha\beta} = 8\pi\, G\phi^2\, T^{EM}_{\alpha\beta}, \qquad \nabla^\alpha F_{\alpha\beta} = 0 \tag{1.7}$$

with the definition $k \equiv \sqrt{16\pi G}$.

Substituting metric (1.5) into the action (1.4): [5]

$$\mathcal{S} = -\int d^4x\sqrt{-g}\phi\left(\frac{R}{16\pi G} + \frac{1}{4}\phi^2 F_{\alpha\beta}F^{\alpha\beta} + \frac{2}{3k^2}\frac{\partial^\alpha\phi\partial_\alpha\phi}{\phi^2}\right) \tag{1.8}$$

where G is defined in terms of its five-dimensional counterpart $\hat{G}$ by:

$$G \equiv \hat{G}/\int dy$$

Setting ϕ to a constant, we recover from a variational point of view Einstein-Maxwell action scaled by factors of ϕ. The fact that the action (1.4) leads to (1.8), or -equivalently- that the source-less field equations (1.3) lead to (1.7) with source matter, constitutes the

[4]Physics was a priori considered independent of the extra coordinate.

[5]The cylinder condition permits to pull $\int dy$ out of the action integral other than dropping all the derivatives with respect to y.

central miracle of Kaluza-Klein theory. Four-dimensional matter (electromagnetic radiation, at least) has been shown to arise purely from the geometry of empty five-dimensional spacetime.
The extra factor ϕ into Einstein action can be absorbed defining $\phi^2 \equiv \phi$ and operating a rescaling of the metric tensor

$$\hat{g} \rightarrow \hat{g}' \equiv \Omega^2 \hat{g}$$

with conformal factor $\Omega^2 = \phi^{-1}$.

$$\hat{g}'_{AB} = \phi^{-1} \begin{pmatrix} g_{\alpha\beta} + k^2 \phi A_\alpha A_\beta & k\phi A_\alpha \\ k\phi A_\beta & \phi \end{pmatrix} \tag{1.9}$$

This trasformation will rescale the four dimensional metric tensor as well $g_{\alpha\beta} \rightarrow g'_{\alpha\beta} = \Omega^2 g_{\alpha\beta}$ so that the new Ricci scalar will be

$$R \rightarrow R' = \Omega^{-2} \Big(R + 6 \frac{\Box \Omega}{\Omega} \Big)$$

The conformally rescaled action read:

$$\mathcal{S}' = -\int d^4x \sqrt{-g'} \Big(\frac{R'}{16\pi G} + \frac{1}{4} \phi F'_{\alpha\beta} F'^{\alpha\beta} + \frac{1}{6k^2} \frac{\partial'^\alpha \phi \, \partial'_\alpha \phi}{\phi^2} \Big) \tag{1.10}$$

where primed quantities refer to the rescaled metric i.e. $\partial'^\alpha \phi = g'^{\alpha\beta} \partial_\beta \phi$ and G, k are defined as before. The Einstein action now has the standard normalization.

In the attempt to give a physical meaning to Kaluza's original cylinder assumption, Klein postulated this dimension to be very small and gave it a circular topology S^1 so to ensure compactification. With this approach any quantity $f(x, y)$ becomes periodic in the extra coordinate i.e. $f(x, y) = f(x, y + 2\pi r)$, r being the radius of the fifth dimension and thus the scale parameter.
Fourier expanding all the fields with respect to y give

$$g_{\alpha\beta}(x, y) = \sum_{n=-\infty}^{+\infty} g^{(n)}_{\alpha\beta}(x) e^{iny/r}, \quad A_\alpha(x, y) = \sum_{n=-\infty}^{+\infty} A^{(n)}_\alpha(x) e^{iny/r},$$

$$\phi(x,y) = \sum_{n=-\infty}^{+\infty} \phi^{(n)}(x) e^{iny/r} \tag{1.11}$$

Expressed in terms of eigenfunctions of the operator $\hat{p}_y = -i\partial_y$, each of these modes carries a momentum in the y direction of order $|n|/r$, for r sufficiently small, even the $n = 1$ modes y-momenta, will be so large to fall far beyond the reach of experiments. Hence only the $n = 0$ modes, which are independent of y, will be observable, in agreement with Kaluza's theory.
In this scenario r is commonly set to the Planck length $l_{Pl} \sim 10^{-33}$ cm, which represents both a natural value and a small enough scale to guarantee that the mass of any $n \neq 0$ Fourier mode lies beyond the Planck mass.

Fourier expansion of fields also suggests a possible mechanism to explain charge quantization.

Consider the simplest kind of matter, a massless five-dimensional scalar field $\hat{\Phi}(x, y)$:

$$\mathcal{S}_{\hat{\Phi}} = -\int d^4x dy \sqrt{-\hat{g}}\, \partial^A \hat{\Phi} \partial_A \hat{\Phi} \tag{1.12}$$

This field can be expanded as above:

$$\hat{\Phi}(x,y) = \sum_{n=-\infty}^{+\infty} \hat{\Phi}^{(n)}(x) e^{iny/r}$$

substituting into (1.12) and using the rescaled metric tensor (1.9) -primes are omitted-

$$\mathcal{S}_{\hat{\Phi}} = -\sum_{n=0}^{+\infty} \int d^4x \sqrt{-g} \left[\left(\partial^\alpha + \frac{inkA^\alpha}{r}\right) \hat{\Phi}^{(n)} \left(\partial_\alpha - \frac{inkA_\alpha}{r}\right) \hat{\Phi}^{(-n)} - \frac{n^2}{\phi r^2} \hat{\Phi}^{(n)} \hat{\Phi}^{(-n)} \right]$$

We thus obtain the action for a system of charged four-dimensional scalar fields $\hat{\Phi}^{(n)}(x)$. Comparison with the minimal coupling $\partial_\alpha \to \partial_\alpha + ieA_\alpha$ of quantum electrodynamics -e being the electric charge- shows that the n-th Fourier mode quantized charge is:

$$q_n = \frac{nk}{r\sqrt{\phi}} = \frac{n\sqrt{16\pi G}}{r\sqrt{\phi}} \tag{1.13}$$

where A_α has been redefined $A_\alpha\phi^{-1/2}$ to ensure that action (1.10) has the standard normalization.

This also allows to make a rough prediction of the value of the fine structure constant identifying the charge q_1 of the first Fourier mode with the electron charge e.

Setting $r\sqrt{\phi} \sim l_{Pl} = \sqrt{G}$ [6]

$$\alpha \equiv \frac{q_1^2}{4\pi} \sim \frac{1}{4\pi}\Big(\frac{\sqrt{16\pi G}}{\sqrt{G}}\Big)^2 = 4$$

The possibility of explaining an otherwise fundamental constant would have made compactified five-dimensional Kaluza-Klein theory very attractive.

However, the masses of the scalar modes are not at all compatible with experimental data. These can be read from the quadratic term into the above action:

$$m_n = \frac{|n|}{r\sqrt{\phi}}$$

If $r\sqrt{\phi} \sim l_{Pl}$ as assumed, the electron mass m_1 would be $\sim l_{Pl}^{-1}$ i.e. of the order of the Planck mass $M_{Pl} \sim 10^{19}$ GeV, rather than ~ 0.5 MeV.

This discrepancy of some twenty-two orders of magnitude between theory and observation made the model inconsistent and thus partially rejected.

1.2.2 The core mechanism

Despite the profound gaps in the original Kaluza-Klein theory, the core idea was far to be abandoned. Extension to higher dimensional space-time is one of the building blocks for many new fundamental theories, and most of them use the Kaluza-Klein mechanism: starting from the ground-state solution of the field equations in a $(4+n)$-dimensional manifold $\mathcal{M}^{(1,3)} \times K_n$, K_n being an n dimensional non necessary compact manifold, one expands the arbitrary fluctuations around these solutions, in a complete set of eigenfunctions in K_n, identifying the coefficients of the expansion as the physical (four dimensional) fields. Substitution into the higher-dimensional equations yields the

[6] An improved determination of $r\sqrt{\phi}$ would presumably hit closer to the mark.

four-dimensional field equations and hence the spectrum.

To outline the general idea of Kaluza-Klein scenario we consider two different mechanisms by means of which four-dimensional fields can emerge out from higher dimensional ones. In both the examples space-time is assumed to be a $(4+1)$ dimensional manifold $\mathcal{M}^{(1,3)} \times K_1$, with coordinates $x^M = (x^\mu, y)$.

Following Kaluza-Klein original theory, the low energy physics will be effectively four-dimensional if the coordinate y is compact with a certain compactification radius R [25]. This means that y runs in the interval $[0, 2\pi R]$, points $y = 0$ and $y = 2\pi\, R$ being identified. The four-dimensional space is a cylider whose three dimensions x^1, x^2, x^3 are infinite, and the fourth dimension y is a circle of radius R. Assuming the cylinder homogeneous, we can write a complete set of wave functions of a free massless particle on this cylinder, i.e. the solutions to five-dimensional Klein-Gordon equation:

$$\Box_5\, \phi(x, y) = 0$$

$$\phi_{\mathbf{p},n} = e^{ip\cdot x} e^{in\frac{y}{R}}, \quad n \in \mathbb{Z}$$

the quadrimomentum p_μ and the one-dimensional angular momentum eigenvalue n being related by:

$$p^2 - \frac{n^2}{R^2} = 0$$

Hence, inhomogeneous modes with $n \neq 0$ carry energy of order $1/R$ and cannot be excited in low energy processes. Below the energy scale $1/R$, only homogeneous modes with $n = 0$ are relevant, and low energy physics is effectively four-dimensional.

Each Kaluza-Klein mode can be interpreted as a separate type of particle with mass $m_n = |n|/R$, thus resulting in a Kaluza-Klein tower of four dimensional particles with increasing masses. At low energies, only massless (on the scale $1/R$) particles can be produced, whereas at $E \sim 1/R$ extra dimensions will show up.

Since the Kaluza-Klein partners of ordinary particles (electrons, photons, etc.) have not been observed, the energy scale $1/R$ must be at least in a few hundred GeV range, so in this scenario, the size of extra dimensions must be microscopic $R \leqslant 10^{-17}$ cm.

Another possibility makes use of non factorizable metrics [22], localization is then due by gravity. Assume a five dimensional scalar field $\Phi(x^M)$ in the following geometry

$$ds^2 = e^{2\phi(y)}(dx^0)^2 - (d\mathbf{x})^2 - (dy)^2 \tag{1.14}$$

The conserved quantities associated to the cyclic coordinates $(x^0, \mathbf{x})$ are

$$E = P^A g_{AB} \, (\frac{\partial}{\partial x^0})^B = P^0 e^{2\phi}$$

$$p^i = P^A g_{AB} \, (\frac{\partial}{\partial x^i})^B = P^i$$

where

$$P^2 = e^{2\phi}(P^0)^2 - \mathbf{p}^2 - (P^5)^2 = M_5^2$$

is the five-dimensional rest mass of Φ. Then

$$P^5 = \sqrt{E^2 e^{-2\phi} - M_5^2 - \mathbf{p}^2}$$

and any classical particle will be confined in the four-dimensional space-time, bound by the potential $\phi(y)$ if

$$E^2 e^{-2\phi} - M_5^2 - \mathbf{p}^2 < 0$$

which yields

$$E < \sqrt{M_5^2 + \mathbf{p}^2}\, sup(e^{\phi})$$

The experimental non observation of any extra dimensions implies that e^{ϕ} must rise very rapidly, presumably with a length scale of the order of Planck mass.

Consider five-dimensional Klein-Gordon equation for $\Phi(x, y)$:

$$\frac{1}{\sqrt{-g}}\partial_A\Big(\sqrt{-g}\, g^{AB}\partial_B\Phi\Big) + M_5^2\,\Phi = 0$$

substituting the metric (1.14) we get:

$$\Big[e^{-2\phi}\frac{\partial^2}{\partial(x^0)^2} - \frac{\partial^2}{\partial\mathbf{x}^2} - \frac{\partial^2}{\partial y^2}\Big]\Phi - \frac{\partial\phi}{\partial y}\frac{\partial\Phi}{\partial y} + M_5^2\Phi = 0$$

Using the ansatz

$$\Phi(x, y) = exp(ip\cdot x)\, e^{-\frac{\phi}{2}}\,\tilde{\Phi}(y)$$

with $p^2 = E^2 - \mathbf{p}^2$ and $E^2 = \omega^2$, then $\tilde{\Phi}(y)$ satisfies the Schroedinger equation

$$\Big[-\frac{1}{2}\frac{\partial^2}{\partial y^2} + \frac{1}{2}\Big(\frac{1}{2}\phi'' + \frac{1}{4}\phi'\phi' - \omega^2 e^{-2\phi}\Big)\Big]\tilde{\Phi} = -\frac{1}{2}\Big[M_5^2 + \mathbf{p}^2\Big]\tilde{\Phi}$$

which leads to the following eigenvalue problem:

$$\Big[-\frac{1}{2}\frac{\partial^2}{\partial y^2} + V(\omega, y)\Big]\eta_n(\omega, y) = -\lambda_n(\omega)\,\eta_n(\omega, y)$$

Being $\lambda_n > 0$, the problem gives rise only to travelling waves in the four usual directions but bound states in the fifth coordinate

$$\Phi(x, y) = \sum_n exp(ip\cdot x)\, e^{-\phi/2}\,\eta_n(\omega, y)$$

with the excitations spectrum given by:

$$2\lambda_n(\omega) = M_5^2 + \mathbf{p}^2$$

Choosing an appropriate normalisation for x^0 we can get:

$$inf(\phi) = 0$$

Then $V(\omega, y)$ is bounded by

$$V(\omega, y) \geqslant V(\omega, y) \geqslant V(\omega = 0, y) - \frac{1}{2}\omega^2$$

Leading to the inequality

$$\lambda_n(\omega) \leqslant \lambda_0(\omega = 0) + \frac{1}{2}\omega^2$$

In the usual type of Kaluza-Klein models one obtains an infinite tower of excited states of ever increasing rest mass. For this exotic class of Kaluza-Klein theories the spectrum is significantly more complex, and the specific form will depend on the details of the model.

Chapter 2

Localization with one extra dimension

2.1 Scalar Fields

One of the first attempts to formulate a field theory postulating non-compact extra dimensions is due to Rubakov and Shaposhnikov [24]. They assumed space-time manifold to be $\mathcal{M}^{(1,3+n)}$ where $\mathcal{M}^{(1,3)}$ is the usual Minkowski space-time and n the extra spatial dimensions [1]. Ordinary particles are confined inside a potential well, sufficiently narrow along the n directions and flat along the others. The well is originated from the non-linearity of the field equations and hence is purely dynamical. Unlike Kaluza-Klein theories, this model allows particles to propagate in the $(3+n)$-dimensional flat space, provided their energy is large. Namely, if the energy of a particle created in a high energy collision exceeds the depth of the well, this particle can come out of the well and move along the extra spatial directions. This process will look for an observer living in the four-dimensional space-time, like violating energy and momentum conservation and thus from this point of view the fundamental principles of relativistic quantum theory such as $(1+3)$ unitarity and $(1+3)$ causality are correct only for particles with sufficiently low energy.

Consider a toy quantum field model describing a real scalar field in a $(1+4)$ dimensional Minkowski space-time, $x^A = (x^\mu, x^4)$, with mostly negative metrics signature, the Lagrangian being

$$\mathcal{L} = \frac{1}{2}\partial_A\phi\,\partial^A\phi + \frac{1}{2}m^2\phi^2 - \frac{1}{4}\lambda\phi^4, \qquad \lambda, m\ > 0 \tag{2.1}$$

[1] From this moment gravity effects are completely disregarded.

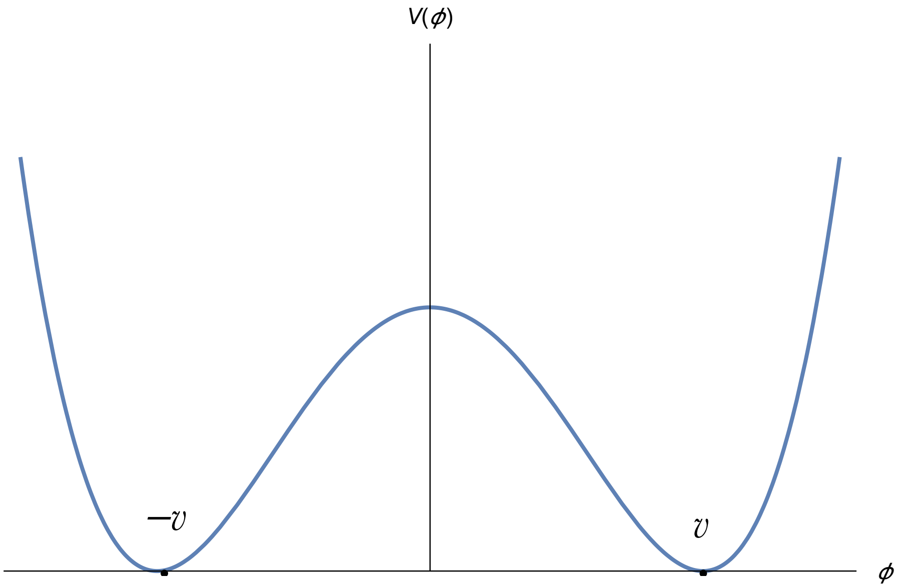

Figure 2.1: Sketch of $\mathbb{Z}_2$ symmetry breaking potential, the configuration $\phi = 0$ represents a local maximum, $V[0] = 0$, the minimum occurs at $\phi_0 = \pm v$, $V[\pm v] = -\lambda v^2/2$.

The scalar potential

$$V[\phi] = -\frac{1}{2}m\phi^2 + \frac{1}{4}\lambda\phi^4$$

has a double-well shape with two degenerate minima -Figure.2.1- occurring at

$$\phi_0 = \pm\, m/\sqrt{\lambda} \equiv \pm\, v$$

The Euler-Lagrange field equation

$$\{\Box_5 - m^2 + \lambda\,\phi^2\}\,\phi(x) = 0 \tag{2.2}$$

admits a domain wall solution $\phi^{cl}(x^4)$ -Figure 2.2- independent of the three spatial coordinates (x^1, x^2, x^3) and of time x^0 i.e.

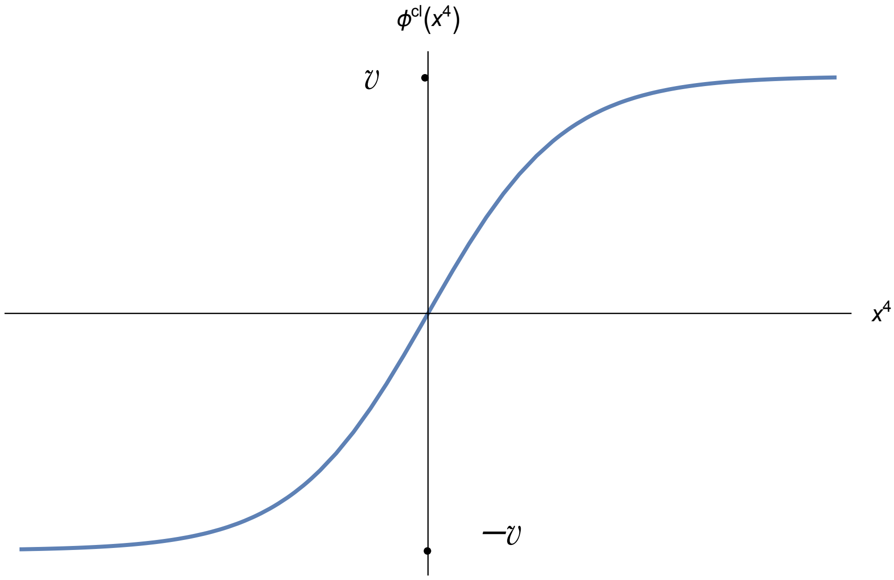

Figure 2.2: Sketch of the solution $\phi^{cl} = v\, tgh(m/\sqrt{2}\, x^4)$. It describes a domain wall of thickness $\sim m^{-1}$ and asymptotes $\pm v$.

$$\phi^{cl}(x^4) = \frac{m}{\sqrt{\lambda}}\, tgh(\frac{m}{\sqrt{2}}\, x^4) \tag{2.3}$$

this provides a potential wall separating the two vacua of the model, narrow in the fourth direction if m is sufficiently large. The spectrum of perturbations can be obtained linearizing equation (2.2) around ϕ^{cl}

$$\phi(x^A) = \phi^{cl} + \phi'$$

Thus

$$\{\Box_5 - m^2 + 3\lambda\, (\phi^{cl})^2\}\phi' = 0 \tag{2.4}$$

They found three types of perturbations:

(a.)

$$\phi'(x^\mu, x^4) = \frac{d\phi^{cl}}{dx^4}\, e^{ik_\mu x^\mu} \tag{2.5}$$

which yields $E^2 = \mathbf{k}^2$.

Being $d\phi^{cl}/dx^4$ localized around $x^4 = 0$, (2.5) corresponds to four-dimensional massless scalar particles confined inside the wall.

(b.)

$$\phi'(x^\mu, x^4) = u(x^4)\, e^{ik_\mu x^\mu} \tag{2.6}$$

with $E^2 = \mathbf{k}^2 + \frac{3}{2}m^2$ so that $u(x^4)$ represents a normalizable solution of

$$\{-\frac{d^2}{d(x^4)^2} - m^2 + 3\lambda(\phi^{cl})^2\}u = \frac{3}{2}m^2 u$$

these perturbations are also confined inside the wall.

(c.)

There also exist solutions non localized in the extra dimension.

From

$$\lim_{|x^4|\to\infty} tgh(\frac{m}{\sqrt{2}}\, x^4) = 1$$

follows that at large $|x^4|$

$$\phi'(x^\mu, x^4) \sim e^{ik_\mu x^\mu - ik^4 x^4} \tag{2.7}$$

with $E^2 = \mathbf{k}^2 + (k^4)^2 + (2m)^2$

These represent five-dimensional massive particles freely moving along all the four spatial dimensions.

Considering the (1 + 3) dimensional space-time (interior of the domain wall) as a toy model of our world, the extra dimension become accessible at high energies: in this scenario a collision of particles of type (*a*.) with center of mass energy exceeding $2\sqrt{2}m$ can result in creation of particles of type (*c*.) which can leave the domain wall.

2.2 Fermion Fields

The model introduced in the previous section can be extended to include massless fermion fields living in (1 + 3) dimensions. Coupling the scalar field $\phi(x^A)$ with a fermion field $\Psi(x^A)$ via Yukawa interaction, the following Lagrangian must be added to (2.1):

$$\mathcal{L}_\psi = i\bar{\Psi}\Gamma^A\partial_A\Psi - h\phi\bar{\Psi}\Psi \tag{2.8}$$

Ψ is a four components spinor and the (1 + 4) dimensional γ matrices are

$$\Gamma^\mu = \gamma^\mu, \qquad \mu = 0,..,3$$

$$\Gamma^4 \equiv -i\gamma^5$$

γ^μ, γ^5 being the standard Dirac matrices.

Let's note that in each of the classical vacua, $\phi_0 = \pm m/\sqrt{\lambda}$, the Yukawa coupling reduces to a mass term for the five dimensional fermions, the mass being

$$m_5 = \frac{hm}{\sqrt{\lambda}}$$

In the presence of the domain wall (2.3), the five dimensional Dirac equation

$$\{i\Gamma^A\partial_A - h\phi^{cl}\}\Psi = 0 \tag{2.9}$$

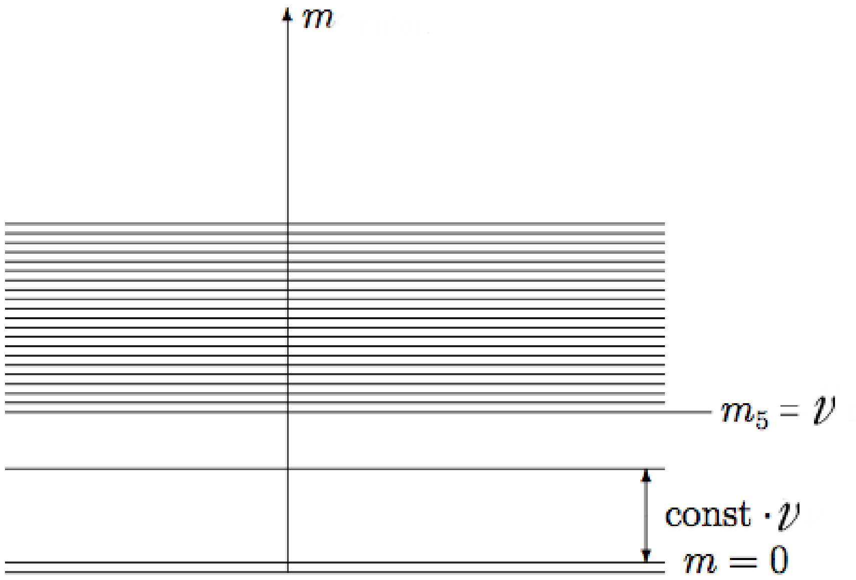

Figure 2.3: Four-dimensional spinor field mass spectrum, besides the massless zero mode, there could or not exist other bound states with $m = k\,v$, $k < 1$; the continuum spectrum starts at $m = m_5 = v$. Picture taken from [25].

has a solution

$$\Psi^{(0)}(x^\mu, x^4) = exp\Big(- h \int_0^{x^4} \phi^{cl}(x^{4'})dx^{4'} \Big) \times \psi(x^\mu) \tag{2.10}$$

where $\psi(x^\mu)$ is a massless left-handed $(1 + 3)$ dimensional spinor

$$i\gamma^\mu \partial_\mu \psi = 0, \qquad \gamma^5 \psi = -\psi$$

This solution is localized inside the wall and at large value of $|x^4|$ decays exponentially

$$\Psi^{(0)} \propto e^{-m_5 |x^4|}$$

Besides the chiral zero mode [25; 24], there may or may not exist bound states, but in any case the masses of the latter are proportional to $|\phi_0|$ and are large for large $|\phi_0|$.

There is also a continuum part of the spectrum starting at $m = m_5$, Figure 2.3; these states correspond to five-dimensional fermions which are not bound to the domain wall and escape to $|x^4| = \infty$.

The existence of massless modes for the four dimensional spinor field, which are meant to mimic ordinary matter, represents the central point of this model. They propagate with the speed of light along the domain wall, but do not move along the extra-dimension and in realistic theories they should acquire small masses by one or another mechanism. At low energies, their interactions can produce only zero modes again, so physics is effectively four-dimensional. Zero modes interacting at high energies, however, will produce continuum modes, the extra dimension will open up, and particles will be able to leave the brane, escape to $|x^4| = \infty$ and literally disappear from our world. For a four dimensional observer, made by particles stuck into the wall, these high energy processes will look like $e^+e^- \to$ nothing, or $e^+e^- \to \gamma +$ nothing.

2.3 Abelian Gauge Fileds

In [17] is presented a model for the localization of Abelian Gauge Fields on a three-brane embedded in five-dimensional space-time.
The four coordinates of the Minkowski space-time $\mathcal{M}^{(1,3)}$ are x^μ, with $\mu = 0, .., 3$; the extra coordinate is y. Capital letters and subscripts are used for five-dimensional quantities, Greek letters for the corresponding four-dimensional ones. The metric convention is mostly negative.
In simplest scenario, the four-dimensional brane has the form of a delta-function placed in $y = 0$. Let's $\mathcal{A}_C(x^M)$, denote a gauge field living in the bulk of the five-dimensional space-time, the Lagrangian being

$$\mathcal{L}_1 = -\frac{1}{4g^2}\mathcal{F}^2_{AB} \tag{2.11}$$

where the strength tensor is defined $\mathcal{F}_{AB} \equiv \partial_A \mathcal{A}_B - \partial_B \mathcal{A}_A$ and g is a coupling constant with dimensionality

$$[g^2] = cm$$

hence the gauge field $\mathcal{A}_C$ has canonical dimension

$$[\mathcal{A}_C] = cm^{-1}$$

The three-dimensional brane action reads:

$$\mathcal{S}_{3-brane} = -\tau \int d^4x \sqrt{\bar{g}}$$

where τ is the brane tension and $\bar{g}_{\mu\nu}$ the induced metric on the brane i.e.

$$\bar{g}_{\mu\nu}(x) = g_{\mu\nu}(x,y)|_{y=0}$$

In general there will be localized matter fields on the brane world-volume. Accounting for them, the additional term must be added to the previous action

$$\tilde{\mathcal{S}}_{3-brane} = \mathcal{S}_{3-brane} + \int d^4x \sqrt{\bar{g}}\, \mathcal{L}(\psi)$$

These fields will give rise to a localized current:

$$J_A(x,y) = J_\mu(x)\delta(y)\delta^\mu_A \tag{2.12}$$

Five-dimensional current conservation is guaranteed by five-dimensional gauge invariance and because of the vanishing of the fifth component of $J_A(x,y)$, four-dimensional current conservation is also satisfied:

$$\partial_A J^A(x,y) = \partial_\mu J^\mu(x) = 0$$

This current will interact with bulk field according to

$$\mathcal{L}_{int} = \int d^4x\, dy\, J_C(x,y)\mathcal{A}^C(x,y) = \int d^4x J_\mu(x)\mathcal{A}^\mu(x,0) \tag{2.13}$$

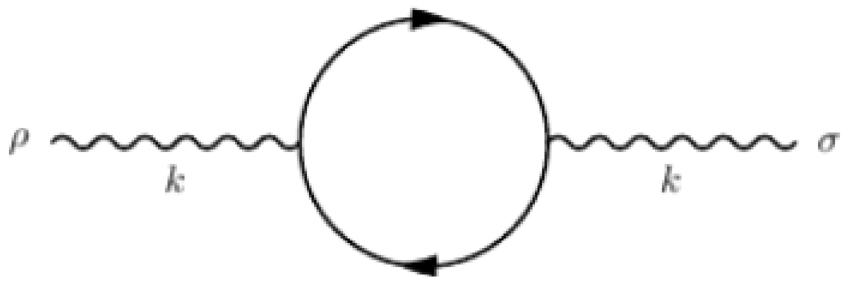

Figure 2.4: The one-loop diagram generating the kinetic term for $\mathcal{A}_\mu(x)$. Wave lines denote photon propagators while solid lines fermion ones.

thus the effective interaction is between the four-dimensional current and the gauge field $\mathcal{A}_\mu(x)$:

$$\mathcal{A}_\mu(x) \equiv \mathcal{A}_\mu(x, y = 0)$$

Due to the interaction (2.13) a kinetic term for $\mathcal{A}_\mu(x)$ is induced on the brane world-volume, this emerges from one-loop diagrams with two external legs of $\mathcal{A}_\mu(x)$ and localized matter running into the loop -Figure 2.4. As a result the following Lagrangian should be added to (2.11)

$$\mathcal{L}_2 = -\frac{1}{4e^2} F_{\mu\nu}^2 + higher\ derivatives$$

with

$$e^{-2} = \frac{2N_f}{3\pi}\, ln(\frac{\Lambda}{\mu})$$

where N_f (number of flavours) accounts for the different kinds of fermions running into the loop and Λ, μ are the ultraviolet and infrared cut-offs. The induced term has the correct negative sign, were the loops generated by localized bosonic fields the sign would be wrong (positive).

The total five-dimensional low energy Lagrangian for the gauge fields results

$$\mathcal{L} = -\frac{1}{4g^2}\mathcal{F}_{AB}^2 - \frac{1}{4e^2}F_{\mu\nu}^2\delta(y) - \mathcal{A}_B J^B$$

where a source term has been added to account for the effects of the induced kinetic term on the Coulomb potential.

The Euler-Lagrange equations read

$$\frac{1}{g^2}(\partial_C\partial^C\mathcal{A}_B + \partial_B\partial^C\mathcal{A}_C) + \delta(y)\frac{1}{e^2}(\partial_\mu\partial^\mu\mathcal{A}_\nu + \partial_\nu\partial^\mu\mathcal{A}_\mu)\delta_{\nu B} = J_B(x,y)$$

Choosing the Lorentz gauge $\partial_C\mathcal{A}^C = 0$ and assuming the source to be of the form

$$J_B(x,y) = J_\mu(x)\,\delta(y)\,\delta^\mu_B$$

we get

$$\partial_C\partial^C\mathcal{A}_\mu + \frac{g^2}{e^2}\delta(y)\left(\partial_\beta\partial^\beta\mathcal{A}_\mu + \partial_\mu\partial^\beta\mathcal{A}_\beta\right) = g^2 J_\mu(x)\delta(y) \tag{2.14}$$

$$\partial_C\partial^C\mathcal{A}_y = 0 \tag{2.15}$$

Fourier transforming with respect to x^μ [2]

$$\mathcal{A}_\mu(x,y) = \int \frac{d^4p}{(2\pi)^4}\,\tilde{\mathcal{A}}_\mu(p,y)\,e^{ip\cdot x}$$

[2]Wick rotation to Euclidean space is performed, so that $p^2 = p_4^2 + p_1^2 + p_2^2 + p_3^2$ with $p_4 = ip_0$.

$$J_\mu(x,y) = \int \frac{d^4p}{(2\pi)^4}\, \tilde{J}_\mu(p,y)\, e^{ip\cdot x}$$

yields

$$(p^2 - \partial_y^2)\tilde{\mathcal{A}}_\mu(p,y) + \frac{g^2}{e^2}\delta(y)\left(p^2\tilde{\mathcal{A}}_\mu(p,y) + ip_\mu\partial^\beta\tilde{\mathcal{A}}_\beta(p,y)\right) = g^2\tilde{J}_\mu(p)\delta(y) \qquad (2.16)$$

$$(p^2 - \partial_y^2)\tilde{\mathcal{A}}_y(p,y) = 0 \qquad (2.17)$$

Let's multiply both sides of equation (2.16) by $\tilde{J}_\mu(p)$, using four-dimensional transversality $p_\mu\tilde{J}^\mu = 0$ we obtain

$$(p^2 - \partial_y^2)f(y,p) + \frac{g^2p^2}{e^2}\delta(y)f(y,p) = g^2\tilde{J}^2\delta(y) \qquad (2.18)$$

where $f(y,p) \equiv \tilde{A}_\mu(y,p)\tilde{J}^\mu(p)$.

For $y < 0$ we get

$$f(y,p) = A(p)e^{py}, \qquad y < 0$$

for $y > 0$

$$f(y,p) = B(p)e^{-py}, \qquad y > 0$$

and from the continuity of the solution in $y = 0$ follows $A(p) = B(p)$. Integrating equation (2.18) from $-\epsilon$ to $+\epsilon$ and then taking the limit for $\epsilon \to 0$

$$B(p)(2p + \frac{g^2p^2}{e^2}) = g^2\tilde{J}^2$$

and hence

$$B(p) = e^2 \tilde{J}^2 \frac{1}{p^2 + 2pe^2/g^2}$$

the solution to equation (2.16) then reads

$$\tilde{A}_\mu(y,p)\tilde{J}^\mu(p) = e^2 \tilde{J}^2 \frac{1}{p^2 + 2pe^2/g^2} e^{-p|y|} \tag{2.19}$$

Taking the four-divergence of equation (2.14) and using $\partial_\mu J^\mu = 0$ we get

$$\partial_\mu \mathcal{A}^\mu = 0$$

which, together with $\partial_C \mathcal{A}^C = 0$ entails

$$\partial_y \mathcal{A}_y = 0$$

from equation (2.15) follows

$$\Box \mathcal{A}_y = 0$$

which describes a four-dimensional massless scalar field, decoupled from the matter fields localized on the brane. Hence from the point of view of a four-dimensional observer we can forget about $\mathcal{A}_y$ and focus only on $\mathcal{A}_\mu$.

From (2.19), the propagator of the gauge field on the brane world-volume at $y = 0$ takes the form:

$$\mathcal{D}_{\mu\nu} = \frac{\eta_{\mu\nu}}{p^2 + 2e^2 p/g^2}[1 + O(p)] \tag{2.20}$$

Thus there are two distinct regimes with respect to the critical value of momentum $p_* = 2e^2/g^2$. For small momenta, $p \ll p_*$ (2.20) resembles the propagator of the five-dimensional theory, in the limit of large momenta, $p \gg p_*$ the four-dimensional photon propagator -in Feynman gauge- is recovered. It gets clearer to turn to coordinate space; the effect of the propagator (2.20) on the interaction between two static probe charges results [16]:

$$V(r) \propto \frac{1}{r}\left(sin(\frac{r}{r_*})Ci(\frac{r}{r_*}) + \frac{1}{2}cos(\frac{r}{r_*})\left[\pi - 2Si(\frac{r}{r_*})\right]\right) \tag{2.21}$$

where $r = |\mathbf{x}|$ and

$$Ci(z) \equiv \gamma + ln(z) + \int_0^z dt\,(cos(t) - 1)/t, \qquad Si(z) \equiv \int_0^z dt\,sin(t)/t$$

$\gamma \sim 0.577$ is the Euler-Mascheroni constant and the critical distance scale r_* is defined:

$$r_* \equiv g^2/2e^2$$

At short distances $r \ll r_*$

$$V(r) \propto \frac{1}{r}\left(\frac{\pi}{2} + \left[-1 + \gamma + ln(\frac{r}{r_*})\right]\frac{r}{r_*} + O(r^2)\right)$$

As previously stated for $p \gg p_*$ the potential has the correct four-dimensional $1/r$ scaling. At intermediate distances it is modified by a logarithmic repulsion term.

For $r \gg r_*$ the (2.21) becomes

$$V(r) \propto \frac{1}{r}\left(\frac{r_*}{r} + O(\frac{1}{r^2})\right)$$

Thus, the large-distances potential scales as $1/r^2$, in accordance with five-dimensional theory laws. The result has been interpreted as follows: a gauge field emitted by a source

localized on the brane propagates along the brane but gradually dissipates in the bulk, however the lower the frequency of the signal the faster it leaks in the extra space, this phenomenon is referred to as 'infrared transparency'.
Consequently, r_* represents the crossover scale between four dimensional and five dimensional theories. At first sight, the bound on its value is expected to be very severe, at least comparable to the present Hubble size [3]. This is due to the fact that the electromagnetic waves propagating over the cosmic distances, constantly detected, seem to behave in a perfectly four-dimensional way. Surprisingly enough, the actual bound on r_* is rather mild; this is because of the phenomenon of 'infrared transparency' according to which the large wavelength radiation penetrates easier in extra dimensions. To see this, consider an electromagnetic wave produced by a monochromatic source l_μ located on the brane:

$$J_\mu(x,y) \sim l_\mu\, \delta(y)\, \delta^{(3)}(x) exp(i\omega t) \tag{2.22}$$

the corresponding wave equation is given in (2.14) with the right-handed side substituted by (2.22) and $\mathcal{A}_y$ set to zero. The induced world-volume contribution (the second left-hand side term into equation(2.14)) is responsible for the existence of the two regimes, without this the wave would behave as five-dimensional:

$$\mathcal{A}_\mu \sim \epsilon_\mu \sqrt{\omega} \frac{e^{i\omega(t-R)}}{R^{3/2}} \tag{2.23}$$

R being the five-dimensional radial coordinate. On the other hand, a four-dimensional wave which propagates in the world-volume would be described by:

$$\mathcal{A}_\mu \sim \epsilon_\mu \frac{e^{i\omega(t-R)}}{R} \delta(y) \tag{2.24}$$

The crossover between the four-dimensional and five-dimensional behaviour can be estimated, looking at the distance at which the two regimes become comparable i.e.

$$\frac{g^2}{e^2}\delta(y)\partial_\mu\partial^\mu\Big(\sqrt{\omega}\frac{e^{i\omega(t-R)}}{R^{3/2}}\Big)|_{r_\omega} = \partial_\mu\partial^\mu\Big(\frac{e^{i\omega(t-R)}}{R}\delta(y)\Big)|_{r_\omega}$$

[3] The radius of a Hubble sphere (known as the Hubble length) is $c/H_0 \sim 14 \times 10^{12}\, ly$ where c is the speed of light $c \sim 3 \times 10^8 m/s$ and $H_0 \sim 71.9$ (Km/s)/MPc is the Hubble constant. The surface of a Hubble sphere is called the microphysical horizon.

which yields:

$$r \sim r_\omega \sim \omega\, r_*^2 \tag{2.25}$$

Thus r_ω sets the distance below which the wave propagates as four-dimensional. The waves with the frequency $\omega \gg r_*^{-1}$ will propagate as four-dimensional waves over the distances much larger than r_*. This suggests that even if the Coulomb law gets modified at relatively short distances, propagation of visible light at larger scales will still look perfectly four-dimensional. For instance, assuming that the Coulomb law breaks down beyond the solar system size, that is $r_* \sim 10^{15}$ cm and considering the largest wave-length radiation propagating over cosmic distances which has been detected so far, i.e. the radio waves in a meter wave-length range $\omega \sim 10^{-2} cm^{-1}$, such radiation would propagate according to the laws of four-dimensional physics over the distance scale

$$r_{radio} \sim 10^{28}\ cm$$

which is comparable to the size of the Universe [4]. Thus, the Coulomb law might break down at a scale of the solar system size and we would not even notice it!

[4]The diameter of the Universe is $\sim 91 \times 10^{10}\ ly \sim 10^{29} cm$.

Chapter 3

Localization with two extra dimensions

In this chapter we present a possible model for the localization of Abelian Gauge Fields in $(4+2)$ dimensions. Compactification is dynamical, due to the existence of a topological defect, a vortex solution, which models four-dimensional space-time. The approach partially follows the one described in [15].

The main result is that, the field components associated with the four-dimensional photon can be completely decoupled from the other fields, thanks to the gauge invariance of the field equations together with the appearance of a mass hierarchy between the fields, which sets a cut-off for the admissible energies range. This procedure though, will not leave the photon massless. We add to the six-dimensional Lagrangian a finely tuned mass term for the photon, in order to cancel the mass of the photon thus obtained. We assume this extra mass term coming from Higgs mechanism, introducing an auxiliary scalar field with vacuum expectation value appropriately chosen.

3.1 The vortex solution

Consider six dimensional Minkowski space-time $\mathcal{M}^{(1,5)}$ [1]. A charged scalar field

$$\Phi(x^M) = \phi_1(x^M) + i\phi_2(x^M)$$

coupled, via the free parameter g, to an Abelian gauge field $A^M(x^N)$, is described by the following Lagrangian:

[1] throughout the entire section $M = 0, 1, 2, 3, 5, 6$ with $x^M = (x^\mu, y^a)$ and $a = 1, 2$.

$$\mathcal{L} = (D_M\Phi)(D^M\Phi)^* - \frac{\lambda}{2}\Big(|\Phi|^2 - \frac{v^2}{2}\Big)^2 - \frac{1}{4}F_{MN}F^{MN} \tag{3.1}$$

λ, v are free positive parameters and

$$F_{MN} \equiv \partial_M A_N - \partial_N A_M, \qquad D_M \equiv \partial_M - ig\, A_M$$

define, respectively, the six-dimensional Abelian strength tensor and the gauge covariant derivative, so to ensure local $U(1)$ gauge invariance

$$\Phi \rightarrow \Phi' = e^{ig\chi}\,\Phi$$

$$A_M \rightarrow A'_M = A_M + \partial_M\chi$$

and

$$\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L}$$

for arbitrary choice of $\chi(x^M)$.

According to the units system conventions, the following relations hold:

$$[\Phi] = [v] = cm^{-2}, \quad [A^2] = cm^{-4} \quad [g] = cm, \quad [\lambda] = cm^2$$

The Euler-Lagrange field equations result:

$$\begin{cases} D_M(D^M\Phi) + \lambda(|\Phi|^2 - \frac{v^2}{2})\Phi = 0 \\ D^*_M(D^M\Phi)^* + \lambda(|\Phi|^2 - \frac{v^2}{2})\Phi^* = 0 \\ \partial_L F^{LM} + ig(\Phi^* D^M\Phi - (D^M\Phi)^*\Phi) = 0 \end{cases} \tag{3.2}$$

Alternatively equations (3.2) could have been derived varying the Action

$$\mathcal{S} = \int_{-\infty}^{+\infty} d^4x \int_{-\infty}^{+\infty} d^2y \ \mathcal{L} \tag{3.3}$$

with respect to the fields Φ, Φ^* and A_M.

Because of the 'wrong' mass sign in (3.1) [2], the minimum of the scalar potential is reached for a non null value of the field Φ, namely

$$|\Phi|^2 = \frac{v^2}{2} \tag{3.4}$$

which defines a $U(1)$ invariant set of degenerate vacua, with topology S^1 -Figure 3.1.

The vacuum configuration thus results

$$|\Phi|^2 = \frac{v^2}{2} \quad \vee \quad A_M = 0$$

We could try to investigate the spectrum of this model, looking for stable time-independent solution, with finite energy, other than the vacuum one [31; 32]. If this exist, it will represent a lump of energy density known as a soliton Figure 3.2.

Let's use the following ansatz for the scalar field, independent of x^μ, and thus cylindrically symmetric [30]:

$$\Phi_0(\rho, \phi) = \frac{v}{\sqrt{2}} f(m\rho)\, e^{in\phi}, \qquad n \in \mathbb{Z}/\{0\} \tag{3.5}$$

where m is a parameter with dimension of a mass and $f(m\rho)$ cannot be expressed in terms of known functions but has the following behaviour -Figure 3.3-

[2] Expansion of Scalar Potential in (3.1) yields $\lambda v^2/2|\Phi|^2 - \lambda/2|\Phi|^4 + \lambda v^4/8$, thus we see that the mass term appears with the unconventional plus sign.

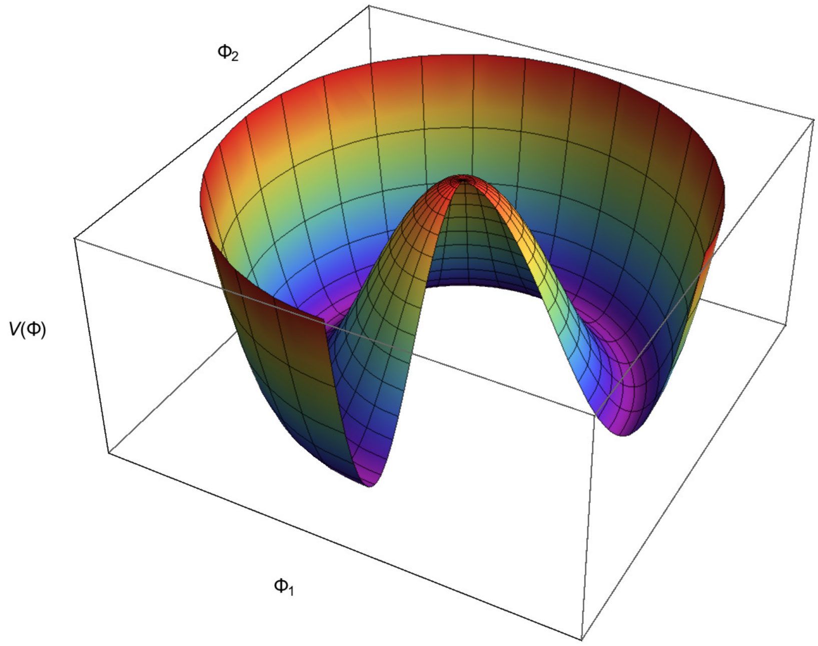

Figure 3.1: Sketch of the Potential functional $V[\Phi]$, the vacuum manifold is described by $|\Phi|^2 = v^2/2$, configuration $\Phi = 0$ represents a local maximum.

$$\begin{cases} \rho \to 0, & f \to 0 \\ \rho \to \infty, & f \to 1 \end{cases} \tag{3.6}$$

The configuration (3.5) cannot be continuously deformed to the trivial vacuum one because of the nonzero winding number n; an integer cannot change continuously and thus the winding number must be preserved by smooth deformations of the fields that keep the energy finite: it represents a topological invariant -see Appendix B. Starting from a configuration with nonzero winding number we cannot reach the vacuum, which has $n = 0$, by means of a continuous transformation; since time evolution is continuous, the winding number must be a constant of the motion. This constitutes a 'topological conservation law' because it's not associated with any symmetry of the Lagrangian.
Solution (3.5) represents a string along the x^μ direction and it's known as Nielsen-Olesen vortex. The lowest energy configuration corresponds to $n = 1$ and represents a lump of energy density surrounding the origin in the plane y^1y^2.

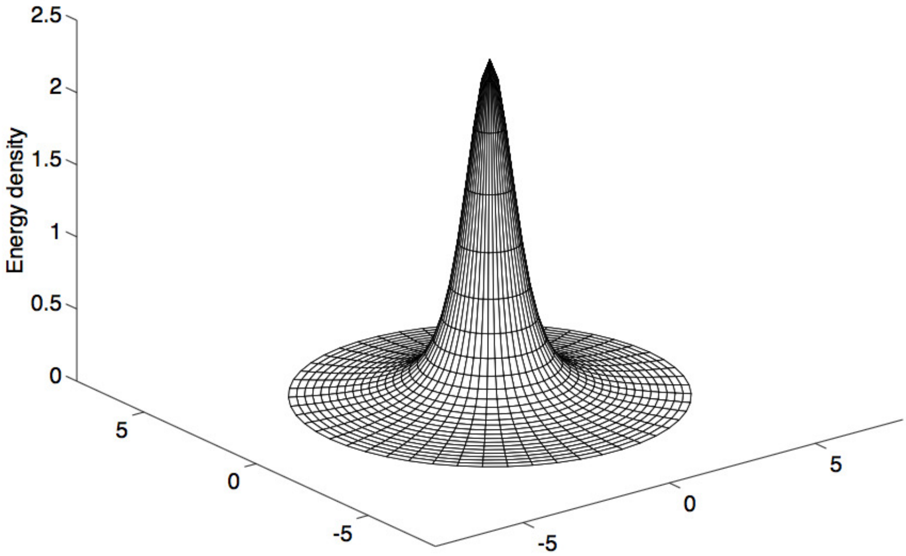

Figure 3.2: Energy density distribution for a soliton-like configuration. Picture taken from [30].

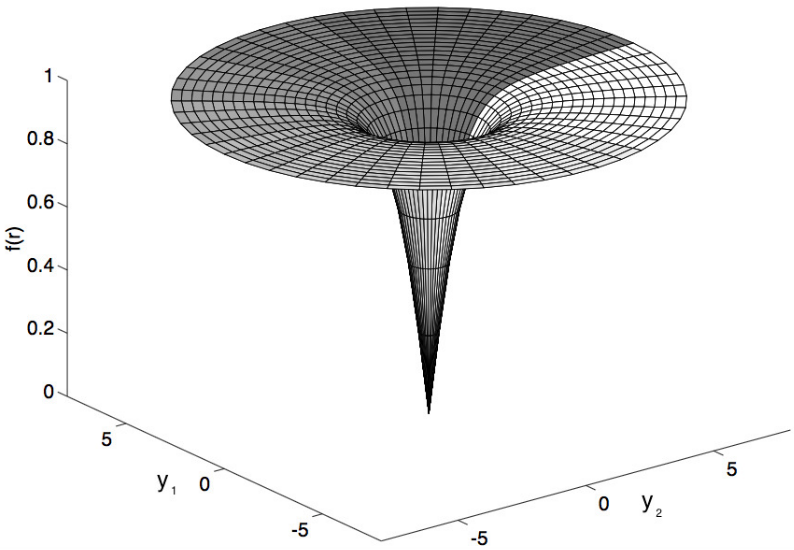

Figure 3.3: Sketch of the radial part of the vortex solution. Picture taken from [30].

Having switched coordinates of the extra-dimensional space

$$(x^\mu, y^1, y^2) \ \rightarrow (x^\mu, \rho, \phi)$$

with $\rho \in [0, \infty)$, $\phi \in [0, 2\pi]$, and

$$\begin{cases} \rho = \sqrt{(y^1)^2 + (y^2)^2} \\ \phi = arctg(\frac{y^2}{y^1}) \end{cases}$$

the new metrics

$$g_{MN} = \begin{pmatrix} +1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & -1 & & \\ & & & & -1 & \\ & & & & & -\rho^2 \end{pmatrix}$$

leads to the following non null affine connection elements:

$$\Gamma^\rho_{\phi\phi} = -\rho \qquad \Gamma^\phi_{\phi\rho} = \Gamma^\phi_{\rho\phi} = 1/\rho$$

The generalization of Euler-Lagrange equations (3.2) to arbitrary coordinate systems is:

$$\begin{cases} \mathcal{D}_M(\mathcal{D}^M\Phi) + \lambda(|\Phi|^2 - \frac{v^2}{2})\Phi = 0 \\ \mathcal{D}^*_M(\mathcal{D}^M\Phi)^* + \lambda(|\Phi|^2 - \frac{v^2}{2})\Phi^* = 0 \\ \nabla_M F^{MN} = -\mathcal{J}^N \end{cases}$$

the metric-covariant derivative ∇_M, the metric-gauge-covariant derivative $\mathcal{D}_M$ and the gauge-covariant current $\mathcal{J}^M$ being defined by:

$$\nabla_M V^N \equiv \partial_M V^N + \Gamma^N_{ML} V^L, \qquad \mathcal{D}_M \equiv \nabla_M - igA_M$$

$$\mathcal{J}^M \equiv ig(\Phi^* \mathcal{D}^M \Phi - (\mathcal{D}^M \Phi)^* \Phi)$$

Thus we have

$$\{\nabla_M \nabla^M + \lambda(|\Phi|^2 - v^2/2) - 2igA^M \partial_M - g^2 A^2 - ig\nabla_M A^M\}\Phi = 0 \tag{3.7}$$

$$\{\nabla_M \nabla^M + \lambda(|\Phi|^2 - v^2/2) + 2igA^M \partial_M - g^2 A^2 + ig\nabla_M A^M\}\Phi^* = 0 \tag{3.8}$$

$$\{\nabla_M \nabla^M + 2g^2|\Phi|^2\}A^\mu - \partial^\mu \, \nabla_M A^M = -j^\mu \tag{3.9}$$

$$\{\nabla_M \nabla^M + \rho^{-2} + 2g^2|\Phi|^2\}A^\rho + 2\rho^{-1}\partial_\phi A^\phi - \partial^\rho \, \nabla_M A^M = -j^\rho \tag{3.10}$$

$$\{\nabla_M \nabla^M - 2\rho^{-1}\partial_\rho + 2g^2|\Phi|^2\}A^\phi - 2\rho^{-3}\partial_\phi A^\rho - \partial^\phi \, \nabla_M A^M = -j^\phi \tag{3.11}$$

where

$$\partial_M \partial^M \equiv \partial_\mu \partial^\mu - \partial_\rho^2 - \rho^{-2}\partial_\phi^2, \qquad \nabla_M \nabla^M \equiv \partial_M \partial^M - \frac{1}{\rho}\partial_\rho$$

$$\nabla_M A^M = \partial_M A^M + \rho^{-1} A^\rho$$

and

$$j^M \equiv ig\, \Phi^* \overset{\leftrightarrow}{\partial^M} \Phi$$

Substituting the vortex solution (3.5) into equations (3.7)-(3.11) leads to:

$$-m^2 f'' - m\rho^{-1} f' + n^2\rho^{-2} f + m_f^2(f^2-1)f - ig(\partial_M A_0^M + \rho^{-1}A_0^\rho)f + \\ - 2igmA_0^\rho f' + 2gnA_0^\phi f - g^2 A_0^2 f = 0 \tag{3.12}$$

$$-m^2 f'' - m\rho^{-1} f' + n^2\rho^{-2} f + m_f^2(f^2-1)f + ig(\partial_M A_0^M + \rho^{-1}A_0^\rho)f + \\ + 2igmA_0^\rho f' + 2gnA_0^\phi f - g^2 A_0^2 f = 0 \tag{3.13}$$

$$\{\Box_4 - \partial_\rho^2 - \rho^{-1}\partial_\rho - \rho^{-2}\partial_\phi^2 + m_0^2 f^2\}A_0^\mu - \partial^\mu(\partial_M A_0^M + \rho^{-1}A_0^\rho) = 0 \tag{3.14}$$

$$\{\Box_4 - \partial_\rho^2 - \rho^{-2}\partial_\phi^2 + m_0^2 f^2\}A_0^\rho + \partial_\rho\partial_M A_0^M + 2\rho^{-1}\partial_\phi A_0^\phi = 0 \tag{3.15}$$

$$\{\Box_4 - \partial_\rho^2 - 3\rho^{-1}\partial_\rho - \rho^{-2}\partial_\phi^2 + m_0^2 f^2\}A_0^\phi - 2\rho^{-3}\partial_\phi A_0^\rho + \\ + \rho^{-2}\partial_\phi(\partial_M A_0^M + \rho^{-1}A_0^\rho) = -n\rho^{-2}\, gv^2 f^2 \tag{3.16}$$

Prime indicates derivative with respect to the argument and the following definitions have been adopted: $m_f^2 \equiv \lambda v^2/2$ and $m_0^2 \equiv g^2 v^2$.

Because of the source term into equation (3.16) there exists a non null component of the gauge field A_0^M, the solution takes the form:

$$A_0^M(\rho) = -\frac{n}{\rho^2 g}\,\eta(m\rho)\,\delta^{M\phi} \tag{3.17}$$

where $\eta(m\rho)$ has not an analytic expression but is defined by:

$$\eta'' - \frac{\eta'}{m\rho} + \frac{m_0^2}{m^2} f^2 (1-\eta) = 0 \tag{3.18}$$

We can understand the asymptotic behaviour of $\eta(m\rho)$, taking the limit for small and large values of $\xi \equiv m\rho$, we also set $m = m_0$:

$$\xi \to 0, \qquad \eta'' - \frac{\eta'}{\xi} \sim 0 \to \qquad \eta \sim a\,\xi^2$$

$$\xi \to \infty, \qquad \eta'' + (1-\eta) \sim 0 \to \qquad \eta \sim 1$$

Where we have used the (3.6).

With A_0^M given by (3.17), the (3.12) now reads:

$$f'' + \frac{1}{\xi} f' - \frac{n^2}{\xi^2}(1-\eta)^2 f - \beta(f^2 - 1) f = 0 \tag{3.19}$$

in the previous equation we have defined $\beta \equiv m_f^2/m_0^2 = \lambda/2g^2$.

Hence:

$$\xi \to 0, \qquad f'' + \frac{1}{\xi} f' - \frac{n^2}{\xi^2} f \sim 0 \to \qquad f \sim b\,\xi^{|n|} \tag{3.20}$$

Numerical evaluation of equations (3.18) and (3.19) for the case $n = 1$, is shown in Figure 3.4.

The asymptotic behaviour $f \sim 1$, $\eta \sim 1$ can be made more precise [57].

Writing for $\xi \to \infty$:

$$f \to 1 + \delta f$$

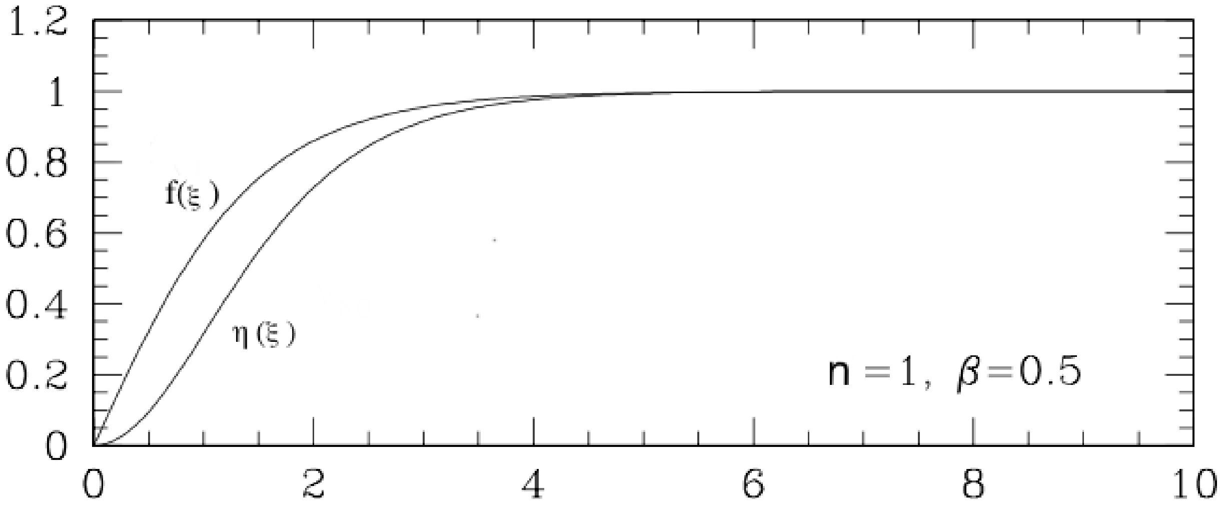

Figure 3.4: Numerical solutions of eq. (3.18) and (3.19). Picture taken from [32].

$$\eta \rightarrow 1 + \delta\eta$$

equations (3.18), (3.19) ignoring quadratic terms in δf, read:

$$\delta\eta'' - \frac{\delta\eta'}{\xi} - \delta\eta = 0 \tag{3.21}$$

$$\delta f'' + \frac{\delta f'}{\xi} - \frac{n^2}{\xi^2}\delta\eta^2 - 2\beta\delta f = 0 \tag{3.22}$$

Using the following ansatz:

$$\delta\eta = e^{-\gamma\xi}\,\xi^{\alpha}(c_1^v + \frac{c_2^v}{\xi})$$

with c_1^v and c_2^v arbitrary constants, the (3.21), collecting terms $\sim \xi^{\alpha}$ and $\sim \xi^{\alpha-1}$ (the $\sim \xi^{\alpha-2}$ and $\sim \xi^{\alpha-3}$ ones being negligible for $\xi \rightarrow \infty$) becomes:

$$\xi^{\alpha}\{c_1^v(\gamma^2 - 1)\} = 0$$

$$\xi^{\alpha-1}\{c_1^v\gamma(1-2\alpha)+c_2^v(\gamma-1)\}=0$$

which yields:

$$\gamma=1 \qquad \vee \qquad \alpha=\frac{1}{2}$$

and c_2^v can be set to 0.

$$\delta\eta=e^{-\xi}\,\xi^{1/2}c^v \tag{3.23}$$

Substituting (3.23) into (3.22):

$$\delta f''+\frac{\delta f'}{\xi}-2\beta\delta f=n^2\frac{e^{-2\xi}}{\xi}(c^v)^2$$

The solution to the associated homogeneous equation can be obtained using the same ansatz for δf:

$$\delta f=e^{-\gamma\xi}\,\xi^{\alpha}(c_1^s+\frac{c_2^s}{\xi^2})$$

Thus:

$$\xi^{\alpha}\{c_1^s(\gamma^2-2\beta)\}=0$$

$$\xi^{\alpha-1}\{c_1^s\gamma(1+2\alpha)+c_2^s(2\beta-\gamma^2)\}=0$$

now:

$$\gamma=\sqrt{2\beta} \qquad \vee \qquad \alpha=-\frac{1}{2}$$

and

$$\delta f_0 = e^{-\sqrt{2\beta}\xi}\,\xi^{-1/2} c^s$$

The particular solution takes the form:

$$\delta f_p = c_p\, n^2 \frac{e^{-2\xi}}{\xi} (c^v)^2$$

where the constant $c_p = \frac{1}{4-2\beta}$. The general solution reads:

$$\delta f = \frac{e^{-\sqrt{2\beta}\xi}}{\sqrt{\xi}} c^s - (c^v)^2\, n^2 \frac{e^{-2\xi}}{2(\beta-2)\xi}$$

with:

$$\beta \equiv \frac{m_f^2}{m_0^2} = \frac{1}{2}\frac{\lambda v^2}{g^2 v^2}$$

Thus:

$$\beta \lesssim 2, \qquad \delta f \to \frac{e^{-\sqrt{2\beta}\xi}}{\sqrt{\xi}} c^s$$

$$\beta > 2, \qquad \delta f \to -(c^v)^2\, n^2 \frac{e^{-2\xi}}{2(\beta-2)\xi}$$

For $\beta = 2$, c^v must be zero for δf being finite, this constitutes the threshold for the production of a pair of scalar particles with mass m_0^2 starting from a scalar particle with mass $m_s^2 = \lambda v^2/2$ i.e $\beta \sim 2 \to m_s^2 = 2m_0^2$.

The total asymptotic behaviour results:

$\xi \to 0$

$$f \sim b\,\xi^{|n|}, \qquad \eta \sim a\,\xi^2, \qquad a,\, b > 0 \tag{3.24}$$

$\xi \to \infty$

$$f \sim 1 + \frac{e^{-\sqrt{2\beta}\,\xi}}{\xi} c^s - (c^v)^2\, n^2 \frac{e^{-2\xi}}{2(\beta-2)\xi}, \qquad \eta \sim 1 + e^{-\xi}\,(\xi)^{1/2} c^v$$

Further considerations on the stability of vortex solution (3.5) and (3.17) in terms of the winding number and the value of β can be found in [32], here we only stress that for the lowest energy configuration $n = 1$ stability is guaranteed for every β.

3.2 Linearization of the field equations

Turning back to the equations (3.12)-(3.16) we define the background configuration to be the Nielsen-Olesen vortex solution:

$$\Phi_0 = \frac{v}{\sqrt{2}} f(m_0\rho)\, e^{in\phi} \qquad \vee \qquad A_0^M = -\frac{n}{\rho^2 g}\,\eta(m_0\rho)\,\delta^{M\phi}$$

Considering small fluctuations around this configuration, we can linearize the field equations (3.7)-(3.11)

$$\Phi(x, \rho, \phi) = \Phi_0 + \tilde{\varphi}(x.\rho, \phi)$$

$$A^M(x, \rho, \phi) = A_0^M + \mathcal{A}^M(x.\rho, \phi)$$

with terms $\sim O(\tilde{\varphi}^2)$, $\sim O(\mathcal{A}^2)$ and $\sim O(\tilde{\varphi}\mathcal{A})$ neglected, thus we get

$$\{\Box_4 - \partial_\rho^2 - \rho^{-1}\partial_\rho - \rho^{-2}\partial_\phi^2 + m_f^2(2f^2 - 1) + n^2\eta^2\rho^{-2}\}\tilde{\varphi} + 2in\rho^{-2}\eta\partial_\phi\tilde{\varphi} + m_f^2 f^2 e^{2in\phi}\tilde{\varphi}^* =$$
$$= i\,2^{-1/2} m_0 f(\partial_M\mathcal{A}^M + \rho^{-1}\mathcal{A}^\rho)e^{in\phi} + i\sqrt{2}m_0\mathcal{A}^\rho\partial_\rho f e^{in\phi} + \sqrt{2}nm_0 f(\eta - 1)\mathcal{A}^\phi e^{in\phi} \tag{3.25}$$

⋆ ⋆ ⋆

$$\{\Box_4 - \partial_\rho^2 - \rho^{-1}\partial_\rho - \rho^{-2}\partial_\phi^2 + m_f^2(2f^2-1) + n^2\eta^2\rho^{-2}\}\tilde{\varphi}^* - 2in\rho^{-2}\eta\partial_\phi\tilde{\varphi}^* + m_f^2 f^2 e^{-2in\phi}\tilde{\varphi} =$$
$$= -i\,2^{-1/2}m_0 f(\partial_M\mathcal{A}^M + \rho^{-1}\mathcal{A}^\rho)e^{-in\phi} - i\sqrt{2}m_0\mathcal{A}^\rho\partial_\rho f e^{-in\phi} + \sqrt{2}nm_0 f(\eta-1)\mathcal{A}^\phi e^{-in\phi} \tag{3.26}$$

⋆ ⋆ ⋆

$$\{\Box_4 - \partial_\rho^2 - \rho^{-1}\partial_\rho - \rho^{-2}\partial_\phi^2 + 2g^2|\Phi_0|^2\}\mathcal{A}^\mu - \partial^\mu(\partial_M\mathcal{A}^M + \rho^{-1}\mathcal{A}^\rho) = -ig\{\Phi_0^*\partial^\mu\tilde{\varphi} - \Phi_0\partial^\mu\tilde{\varphi}^*\} \tag{3.27}$$

⋆ ⋆ ⋆

$$\{\Box_4 - \partial_\rho^2 - \rho^{-1}\partial_\rho - \rho^{-2}\partial_\phi^2 + \rho^{-2} + 2g^2|\Phi_0|^2\}\mathcal{A}^\rho + 2\rho^{-1}\partial_\phi\mathcal{A}^\phi +$$
$$+ \partial_\rho(\partial_M\mathcal{A}^M + \frac{1}{\rho}\mathcal{A}^\rho) = -ig\{[\Phi_0^*\partial^\rho\tilde{\varphi} - \Phi_0\partial^\rho\tilde{\varphi}^*] + [\tilde{\varphi}^*\partial^\rho\Phi_0 - \tilde{\varphi}\partial^\rho\Phi_0^*]\} \tag{3.28}$$

⋆ ⋆ ⋆

$$\{\Box_4 - \partial_\rho^2 - 3\rho^{-1}\partial_\rho - \rho^{-2}\partial_\phi^2 + 2g^2|\Phi_0|^2\}\mathcal{A}^\phi - 2\rho^{-3}\partial_\phi\mathcal{A}^\rho +$$
$$+ \rho^{-2}\partial_\phi(\partial_M\mathcal{A}^M + \frac{1}{\rho}\mathcal{A}^\rho) = -ig\{[\Phi_0^*\partial^\phi\tilde{\varphi} - \Phi_0\partial^\phi\tilde{\varphi}^*] + [\tilde{\varphi}^*\partial^\phi\Phi_0 - \tilde{\varphi}\partial^\phi\Phi_0^*]\} \tag{3.29}$$

The first two equations suggest to set:

$$\tilde{\varphi}(x,\rho,\phi) = \varphi(x,\rho,\phi)e^{in\phi}$$

$$\tilde{\varphi}^*(x,\rho,\phi) = \varphi^*(x,\rho,\phi)e^{-in\phi}$$

so equations (3.25), (3.26) now read:

$$e^{in\phi}\{\Box_4 - \partial_\rho^2 - \rho^{-1}\partial_\rho - \rho^{-2}\partial_\phi^2 + n^2\rho^{-2}(\eta^2-1)^2 + m_f^2(2f^2-1) + 2in\rho^{-2}(\eta-1)\partial_\phi\}\,\varphi +$$
$$+ m_f^2 f^2\varphi^* e^{in\phi} = \{i\,2^{-1/2}m_0 f(\partial_M\mathcal{A}^M + \rho^{-1}\mathcal{A}^\rho) + i\sqrt{2}m_0\mathcal{A}^\rho\partial_\rho f + \sqrt{2}nm_0 f(\eta-1)\mathcal{A}^\phi\}e^{in\phi} \tag{3.30}$$

⋆ ⋆ ⋆

$$e^{-in\phi}\{\Box_4 - \partial_\rho^2 - \rho^{-1}\partial_\rho - \rho^{-2}\partial_\phi^2 + n^2\rho^{-2}(\eta-1)^2 + m_f^2(2f^2-1) - 2in\rho^{-2}(\eta-1)\partial_\phi\}\varphi^*$$
$$+m_f^2 f^2 \varphi\, e^{-in\phi} = \{-i\,2^{-1/2}m_0 f(\partial_M\mathcal{A}^M + \rho^{-1}\mathcal{A}^\rho) - i\sqrt{2}m_0\mathcal{A}^\rho\partial_\rho f + \sqrt{2}nm_0 f(\eta-1)\mathcal{A}^\phi\}e^{-in\phi} \tag{3.31}$$

Consider the $U(1)$ gauge trasformation:

$$\Phi \to \Phi' = \Phi\, e^{ig\chi}$$

$$A_M \to A'_M = A_M + \partial_M\chi$$

where $\chi(x,\rho,\phi)$ is a small real scalar field of the same order of φ, so that we can drop $O(\chi^2)$ terms in the defining series of the exponential:

$$\Phi' \approx (\Phi_0 + \tilde{\varphi} + ig\,\Phi_0\,\chi) = (\,|\Phi_0| + \varphi + i\,g|\Phi_0|\,\chi)\, e^{in\phi}$$

$$A'_M = A_{0\,M} + \mathcal{A}_M + \partial_M\chi$$

From the previous relations we see that the infinitesimal gauge transformation will act on φ and $\mathcal{A}_M$ in the following way:

$$\begin{cases} \varphi \to \varphi + ig|\Phi_0|\,\chi \\ \mathcal{A}_M \to \mathcal{A}_M + \partial_M\chi \end{cases} \tag{3.32}$$

and because of gauge invariance of the Lagrangian, the field equations need to read the same after this transformation is implemented.. To check this out, we must compare the equations (3.25)-(3.29) with the ones transformed according to (3.32), for gauge invariance to apply the difference between the two, has to vanish.

From equation (3.25), after setting $\mu \equiv vg/\sqrt{2}$ we obtain:

$$\{\,\Box_6 + n^2\rho^{-2}(\eta-1)^2 + m_f^2(f^2-1) + 2in\rho^{-2}(\eta-1)\partial_\phi\}\,i\mu f\chi =$$
$$= i\,\mu f(\partial_M\partial^M\chi + \rho^{-1}\partial^\rho\chi) + i\sqrt{2}m_0\partial^\rho\chi\,\partial_\rho f - \rho^{-2}\sqrt{2}nm_0 f(\eta-1)\partial_\phi\chi$$

linear terms in ∂_ϕ cancel and we are left with

$$i\mu f\,\Box_6\,\chi - i\mu\chi\{\partial_\rho^2 f + \rho^{-1}\partial_\rho f - n^2\rho^{-2}(\eta-1)^2 f - m_f^2(f^2-1)f\}+$$
$$-2i\mu\partial_\rho f\,\partial_\rho\chi = i\mu f\,\Box_6\chi - i\sqrt{2}m_0\partial_\rho\chi\,\partial_\rho f$$

Recognizing between brackets the (3.19) -the background equation satisfied by f- the previous equality holds for arbitrary fields χ.

Let's check the remaining equations, from the (3.27) we get:

$$\{\partial_M\partial^M - \rho^{-1}\partial_\rho + 2g^2|\Phi_0|^2\}\partial^\mu\chi - \partial^\mu(\partial_M\partial^M\chi + \rho^{-1}\partial^\rho\chi) = 2\mu g f|\Phi_0|\partial^\mu\chi$$

which is again true. Taking the (3.28):

$$\{\Box_4 - \partial_\rho^2 - \rho^{-1}\partial_\rho - \rho^{-2}\partial_\phi^2 + \rho^{-2} + 2g^2|\Phi_0|^2\}\partial^\rho\chi + 2\rho^{-1}\partial_\phi\partial^\phi\chi+$$
$$+\partial_\rho(\partial_M\partial^M\chi + \rho^{-1}\partial^\rho\chi) = 2g^2|\Phi_0|\,\partial^\rho(|\Phi_0|\chi) - 2g^2|\Phi_0|\,\chi\,\partial^\rho|\Phi_0|$$

and then:

$$-\partial_\rho\{\Box_4 - \partial_\rho^2 - \rho^{-2}\partial_\phi^2\}\chi + \partial_\rho(\rho^{-1}\partial_\rho\chi) + \rho^{-2}\partial_\rho\chi + 2\rho^{-3}\partial_\phi^2\chi+$$
$$+\rho^{-2}\partial^\rho\chi + 2g^2|\Phi_0|^2\partial^\rho\chi - 2\rho^{-3}\partial_\phi^2\chi + \partial_\rho\partial_M\partial^M\chi + \partial_\rho(\rho^{-1}\partial^\rho\chi) = 2g^2|\Phi_0|^2\partial^\rho\chi$$

Recalling that $\partial_M\partial^M = \Box_4 - \partial_\rho^2 - \rho^{-2}\partial_\phi^2$ we see that all terms vanish.

Finally from equation (3.29):

$$\{\Box_4 - \partial_\rho^2 - 3\rho^{-1}\partial_\rho - \rho^{-2}\partial_\phi^2 + \rho^{-2} + 2g^2|\Phi_0|^2\}\partial^\phi\chi - 2\rho^{-3}\partial_\phi\partial^\rho\chi +$$
$$+\rho^{-2}\partial_\phi(\partial_M\partial^M\chi + \rho^{-1}\partial^\rho\chi) = 2g^2|\Phi_0|\,\partial^\phi(|\Phi_0|\chi) - 2g^2|\Phi_0|\,\chi\,\partial^\phi|\Phi_0|$$

which gives

$$-\rho^{-2}\partial_\phi\{\Box_4 - \partial_\rho^2 - \rho^{-2}\partial_\phi^2\}\chi + 2g^2|\Phi_0|^2\partial^\phi\chi - 2\rho^{-3}\partial_\phi\partial^\rho\chi +$$
$$+\rho^{-2}\partial_\phi(\partial_M\partial^M\chi + \rho^{-1}\partial^\rho\chi) = 2g^2|\Phi_0|\,\partial^\phi(|\Phi_0|\chi) - 2g^2|\Phi_0|\,\chi\,\partial^\phi|\Phi_0|$$

again satisfied. We'll use the trasformation (3.32) to simplify the field equations.

Let's write equations (3.25)-(3.29) explicitly in terms of the real fields φ_1 and φ_2

$$\varphi(x,\rho,\phi) = \varphi_1 + i\,\varphi_2$$

$$\{\Box_4 - \partial_\rho^2 - \rho^{-1}\partial_\rho - \rho^{-2}\partial_\phi^2 + n^2\rho^{-2}(\eta-1)^2 + m_f^2(3f^2-1)\}\varphi_1 +$$
$$- 2n(\eta-1)\rho^{-2}\partial_\phi\,\varphi_2 = \sqrt{2}nm_0 f(\eta-1)\mathcal{A}^\phi \tag{3.33}$$

$$\{\Box_4 - \partial_\rho^2 - \rho^{-1}\partial_\rho - \rho^{-2}\partial_\phi^2 + n^2\rho^{-2}(\eta-1)^2 + m_f^2(f^2-1)\}\varphi_2 +$$
$$+ 2n(\eta-1)\rho^{-2}\partial_\phi\,\varphi_1 = \sqrt{2}m_0\mathcal{A}^\rho\partial_\rho f + 2^{-1/2}m_0 f(\partial_M\mathcal{A}^M + \rho^{-1}\mathcal{A}^\rho) \tag{3.34}$$

$$\{\Box_4 - \partial_\rho^2 - \rho^{-1}\partial_\rho - \rho^{-2}\partial_\phi^2 + m_0^2 f^2\}\mathcal{A}^\mu - \partial^\mu(\partial_M\mathcal{A}^M + \rho^{-1}\mathcal{A}^\rho) = \sqrt{2}m_0 f g^{\mu\nu}\partial_\nu\,\varphi_2 \tag{3.35}$$

$$\{\Box_4 - \partial_\rho^2 - \rho^{-1}\partial_\rho - \rho^{-2}\partial_\phi^2 + \rho^{-2} + m_0^2 f^2\}\mathcal{A}^\rho + 2\rho^{-1}\partial_\phi\mathcal{A}^\phi +$$
$$+ \partial_\rho(\partial_M\mathcal{A}^M + \rho^{-1}\mathcal{A}^\rho) = \sqrt{2}m_0\{\varphi_2\,\partial_\rho f - f\,\partial_\rho\,\varphi_2\} \tag{3.36}$$

$$\{\Box_4 - \partial_\rho^2 - 3\rho^{-1}\partial_\rho + \rho^{-2}\partial_\phi^2 + m_0^2 f^2\}\mathcal{A}^\phi - 2\rho^{-3}\partial_\phi\mathcal{A}^\rho +$$
$$+ \rho^{-2}\partial_\phi(\partial_M\mathcal{A}^M + \rho^{-1}\mathcal{A}^\rho) = -\sqrt{2}m_0 f\rho^{-2}\{\partial_\phi\varphi_2 + 2n\,\varphi_1\} \quad (3.37)$$

Equation (3.33) can be rearranged, for $\rho \neq 0$

$$\{\Box_4 - \partial_\rho^2 - \rho^{-1}\partial_\rho - \rho^{-2}\partial_\phi^2 + n^2\rho^{-2}(\eta-1)^2 + m_f^2(3f^2-1)\}\varphi_1 =$$
$$= -\sqrt{2}n(\eta-1)fm_0\rho^{-2}\{\mathcal{A}_\phi - \mu^{-1}\partial_\phi(\varphi_2/f)\}$$

while from equation (3.34) we get:

$$(\partial_M\partial^M + \rho^{-1}\partial^\rho)\varphi_2 + \{n^2\rho^{-2}(\eta-1)^2 + m_f^2(f^2-1)\}\,\varphi_2 + 2n\rho^{-2}(\eta-1)\partial_\phi\varphi_1 =$$
$$= \sqrt{2}m_0\mathcal{A}^\rho\partial_\rho f + 2^{-1/2}m_0 f(\partial_M\mathcal{A}^M + \rho^{-1}\mathcal{A}^\rho)$$

Using again the (3.19) for terms between brackets

$$(\partial_M\partial^M + \rho^{-1}\partial^\rho)\varphi_2 - (\partial_M\partial^M f + \rho^{-1}\partial^\rho f)\varphi_2/f + 2n\rho^{-2}(\eta-1)\partial_\phi\varphi_1 =$$
$$= \sqrt{2}m_0\mathcal{A}^\rho\partial_\rho f + 2^{-1/2}m_0 f(\partial_M\mathcal{A}^M + \rho^{-1}\mathcal{A}^\rho)$$

and dividing both members by $m_0 f/2^{1/2} \equiv \mu f$

$$\frac{1}{\mu f}\partial_M\partial^M\varphi_2 + \frac{1}{\mu f}\frac{1}{\rho}\partial^\rho\varphi_2 - \frac{1}{\rho^2\mu}\varphi_2(\partial_M\partial^M f + \frac{1}{\rho}\partial^\rho f) + \frac{2n}{\mu f}\frac{1}{\rho^2}(\eta-1)\partial_\phi\varphi_1 =$$
$$= \frac{2}{f}\mathcal{A}^\rho\partial_\rho f + (\partial_M\mathcal{A}^M + \frac{1}{\rho}\mathcal{A}^\rho)$$

little manipulation yields

$$\partial_M(\mathcal{A}^M - \mu^{-1}\partial^M(\varphi_2/f)) - \frac{2}{\mu f^2}\partial^M\varphi_2\,\partial_M f + \frac{2}{\mu f^3}\partial^M f\,\partial_M f\varphi_2 + \frac{1}{\rho}\mathcal{A}^\rho +$$
$$-\frac{1}{\mu f}\frac{1}{\rho}\partial^\rho\varphi_2 + \frac{1}{\mu f^2}\frac{1}{\rho}(\partial^\rho f)\,\varphi_2 - \frac{2n}{\mu\rho^2}\frac{1}{f}(\eta-1)\partial_\phi\varphi_1 + \frac{2}{f}\mathcal{A}^\rho\partial_\rho f = 0$$

and finally

$$\partial_M(\mathcal{A}^M - \mu^{-1}\partial^M(\varphi_2/f)) + \frac{1}{\rho}(\mathcal{A}^\rho - \mu^{-1}\partial^\rho(\varphi_2/f)) +$$
$$+\frac{2}{f}\partial_\rho f\{\mathcal{A}^\rho - \mu^{-1}\partial^\rho(\varphi_2/f)\} - \frac{2n}{\mu f}\frac{1}{\rho^2}(\eta-1)\partial_\phi\varphi_1 = 0$$

We see that φ_2 and $\mathcal{A}^M$ appear always in the same combination $\mathcal{A}^M - \mu^{-1}\partial^M(\varphi_2/f)$, this is another statement of gauge invariance of the field equations, by means of which we can make the choice

$$\chi = -\mu^{-1}(\varphi_2/f) \tag{3.38}$$

and drop explicit dependence on the field φ_2.

$$\begin{cases} \varphi \to \varphi' = \varphi_1 \\ \mathcal{A}_M \to \mathcal{A}'_M = \mathcal{A}_M - \mu^{-1}\partial_M(\varphi_2/f) \end{cases}$$

Note that in the bulk, for $\xi \gg 1$ and $f(\xi) \sim 1$, the previous transformation is equivalent to the Unitary gauge of the Higgs model with φ_2 being the massless Goldstone boson arising from the global $U(1)$ symmetry breaking in the vacuum [58]; on the other hand, the choice (3.38) is not well defined at the origin $\xi = 0$ and hence we must assume φ_2 to vanish in the core of the vortex.

In terms of the new fields

$$\begin{cases} \varphi' = \varphi_1 = \varphi_1' \\ \varphi_2' = 0 \\ \mathcal{A}'^{\mu} = \mathcal{A}^{\mu} - (\mu f)^{-1}\,\partial^{\mu}\varphi_2 \\ \mathcal{A}'^{\phi} = \mathcal{A}^{\phi} - (\mu f)^{-1}\,\partial^{\phi}\varphi_2 \\ \mathcal{A}'^{\rho} = \mathcal{A}^{\rho} - \mu^{-1}\,\partial^{\rho}(\varphi_2/f) \end{cases}$$

equations (3.33)-(3.37) now read (neglecting primes):

$$\{\Box_4 - \partial_\rho^2 - \rho^{-1}\partial_\rho - \rho^{-2}\partial_\phi^2 + n^2\rho^{-2}(\eta-1)^2 + m_f^2(3f^2-1)\}\varphi = \\ = -\sqrt{2}n(\eta-1)fm_0\rho^{-2}\mathcal{A}_\phi \tag{3.39}$$

$$\partial_M\mathcal{A}^M + \frac{1}{\rho}\mathcal{A}^\rho + \frac{2}{f}\partial_\rho f\mathcal{A}^\rho - \frac{2n}{\mu f}\frac{1}{\rho^2}(\eta-1)\partial_\phi\varphi = 0 \tag{3.40}$$

$$\{\Box_4 - \partial_\rho^2 - \rho^{-1}\partial_\rho - \rho^{-2}\partial_\phi^2 + m_0^2f^2\}\mathcal{A}^\mu - \partial^\mu(\partial_M\mathcal{A}^M + \rho^{-1}\mathcal{A}^\rho) = 0 \tag{3.41}$$

$$\{\Box_4 - \partial_\rho^2 - \rho^{-1}\partial_\rho - \rho^{-2}\partial_\phi^2 + \rho^{-2} + m_0^2f^2\}\mathcal{A}^\rho + 2\rho^{-1}\partial_\phi\mathcal{A}^\phi + \\ + \partial_\rho(\partial_M\mathcal{A}^M + \rho^{-1}\mathcal{A}^\rho) = 0 \tag{3.42}$$

$$\{\Box_4 - \partial_\rho^2 - 3\rho^{-1}\partial_\rho - \rho^{-2}\partial_\phi^2 + m_0^2f^2\}\mathcal{A}^\phi - 2\rho^{-3}\partial_\phi\mathcal{A}^\rho + \\ + \rho^{-2}\partial_\phi(\partial_M\mathcal{A}^M + \rho^{-1}\mathcal{A}^\rho) = -2nm_0f\sqrt{2}\rho^{-2}\varphi \tag{3.43}$$

Thus we have obtained four dynamical equations for the fields φ_1, $\mathcal{A}^\mu$, $\mathcal{A}^\rho$ and $\mathcal{A}^\phi$ and a constraint relation from the equation for φ_2.

3.3 Possible solutions

Let's decompose the spatial components of the four-vector field $\mathcal{A}^\mu$ into an irrotational and solenoidal part -Appendix C-

$$\mathcal{A}^M = (\mathcal{A}^0, \partial^i \mathcal{A} + \mathcal{A}^{i\,T}, \mathcal{A}^\rho, \mathcal{A}^\phi)$$

where the transverse component satisfies by definition

$$\partial_i \mathcal{A}^{i\,T} = 0$$

using the 'constraint equation' (3.40), we are left with

$$\{\Box_4 - \partial_\rho^2 - \rho^{-1}\partial_\rho - \rho^{-2}\partial_\phi^2 + n^2\rho^{-2}(\eta-1)^2 + m_f^2(3f^2-1)\}\varphi = \\ = -\sqrt{2}n(\eta-1)fm_0\rho^{-2}\mathcal{A}_\phi \tag{3.44}$$

$$\partial_0\mathcal{A}^0 - \nabla^2\mathcal{A} + \partial_\rho\mathcal{A}^\rho + \partial_\phi\mathcal{A}^\phi + \frac{1}{\rho}\mathcal{A}^\rho + \\ + \frac{2}{f}\partial_\rho f\mathcal{A}^\rho - \frac{2n}{\mu f}\frac{1}{\rho^2}(\eta-1)\partial_\phi\varphi = 0 \tag{3.45}$$

$\mu = 0$

$$\{\Box_4 - \partial_\rho^2 - \rho^{-1}\partial_\rho - \rho^{-2}\partial_\phi^2 + m_0^2 f^2\}\mathcal{A}^0 + \\ + \frac{2}{f}\partial_\rho f\,\partial_0\mathcal{A}^\rho - \frac{2n}{\mu f}\frac{1}{\rho^2}(\eta-1)\partial_0\partial_\phi\varphi = 0 \tag{3.46}$$

$\mu = i$

$$\{\Box_4 - \partial_\rho^2 - \rho^{-1}\partial_\rho - \rho^{-2}\partial_\phi^2 + m_0^2 f^2\}\mathcal{A}^{i\,T} = 0 \tag{3.47}$$

$$\begin{aligned}\{\Box_4 - \partial_\rho^2 - \rho^{-1}\partial_\rho - \rho^{-2}\partial_\phi^2 + m_0^2 f^2\}\mathcal{A}+ \\ + \frac{2}{f}\partial_\rho f \mathcal{A}^\rho - \frac{2n}{\mu f}\frac{1}{\rho^2}(\eta - 1)\partial_\phi \varphi = 0\end{aligned} \tag{3.48}$$

Comparison between (3.46) and (3.48) shows that $\mathcal{A}^0$ and $\partial_0\mathcal{A}$ solve the same equation, thus we identify

$$\mathcal{A}^0 = \partial_0 \mathcal{A} \tag{3.49}$$

Indeed equation (3.45) can be inverted in terms of $\mathcal{A}$, and once this is substituted into (3.48) we loose any explicit dependence on it. This shows that $\mathcal{A}$ is not independent grade of freedom, its dynamics can be derived from that of $\mathcal{A}^0$ and can be ignored.

The last equations for $\mathcal{A}^\rho$ and $\mathcal{A}^\phi$ are

$$\begin{aligned}\{\Box_4 - \partial_\rho^2 - \rho^{-1}\partial_\rho - \rho^{-2}\partial_\phi^2 + \rho^{-2} + m_0^2 f^2\}\mathcal{A}^\rho + 2\rho^{-1}\partial_\phi \mathcal{A}^\phi + \\ - \partial_\rho \{\frac{2}{f}\partial_\rho f \mathcal{A}^\rho - \frac{2n}{\mu f}\frac{1}{\rho^2}(\eta - 1)\partial_\phi \varphi\} = 0\end{aligned} \tag{3.50}$$

$$\begin{aligned}\{\Box_4 - \partial_\rho^2 - 3\rho^{-1}\partial_\rho - \rho^{-2}\partial_\phi^2 + m_0^2 f^2\}\mathcal{A}^\phi - 2\rho^{-3}\partial_\phi \mathcal{A}^\rho + \\ - \rho^{-2}\partial_\phi \{\frac{2}{f}\partial_\rho f \mathcal{A}^\rho - \frac{2n}{\mu f}\frac{1}{\rho^2}(\eta - 1)\partial_\phi \varphi\} = -2n m_0 f \sqrt{2}\rho^{-2}\varphi\end{aligned} \tag{3.51}$$

The Hamiltonian reads

$$\mathcal{H} = \int d^3x \int \rho \, d\rho \, d\phi \, T^{00}$$

with

$$
\begin{aligned}
T^{00} = (\partial_0\varphi)^2 + |\nabla\varphi|^2 + [\partial_\rho(\frac{v}{\sqrt{2}}f + \varphi)]^2 + \frac{\lambda}{2}[(\frac{vf}{\sqrt{2}} + \varphi)^2 - \frac{v^2}{2}]^2 + \frac{1}{\rho^2}(\partial_\phi\varphi)^2 + \\
-\{g^2\mathcal{A}_0^2 - g^2\mathcal{A}^2 - g^2\mathcal{A}^{\rho\,2} - \frac{1}{\rho^2}[n(1-\eta) + g\mathcal{A}_\phi]^2\}(\frac{vf}{\sqrt{2}} + \varphi)^2 \\
- F^{0M}\partial^0 A_M + \frac{1}{4}F_{MN}F^{MN} \qquad (3.52)
\end{aligned}
$$

For $\rho \to 0$ terms proportional to $\partial_\phi\varphi$, $\partial_\phi\mathcal{A}^M$ will increase the value of the Hamiltonian; being interested in the configurations of minimal energy we can drop the angular dependence on ϕ, this is equivalent to Kaluza ansatz -see Chapter 1- on Fourier expanding the fields into a complete set of eigenfunctions in the domain $[0, 2\pi]$, and keep only the zero modes; we also set $n = 1$:

$$
\{\Box_4 - \partial_\rho^2 - \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}(\eta-1)^2 + m_f^2(3f^2-1)\}\varphi = -\frac{\sqrt{2}}{\rho^2}(\eta-1)fm_0\mathcal{A}_\phi \qquad (3.53)
$$

$$
\{\Box_4 - \partial_\rho^2 - \frac{1}{\rho}\partial_\rho + m_0^2 f^2\}\mathcal{A}^{i\,T} = 0 \qquad (3.54)
$$

$$
\{\Box_4 - \partial_\rho^2 - \frac{1}{\rho}\partial_\rho + m_0^2 f^2\}\mathcal{A}^0 + \frac{2}{f}\partial_\rho f\partial_0\mathcal{A}^\rho = 0 \qquad (3.55)
$$

$$
\{\Box_4 - \partial_\rho^2 - \frac{1}{\rho}\partial_\rho + \rho^{-2} + m_0^2 f^2\}\mathcal{A}^\rho - \partial_\rho(\frac{2}{f}\partial_\rho f\mathcal{A}^\rho) = 0 \qquad (3.56)
$$

$$
\{\Box_4 - \partial_\rho^2 - \frac{3}{\rho}\partial_\rho + m_0^2 f^2\}\mathcal{A}^\phi = -\frac{2\sqrt{2}m_0}{\rho^2}f\,\varphi \qquad (3.57)
$$

The first and fifth equations above are decoupled from the other ones, and describe the fluctuations of the background string solution.

The second, third and fourth describe, from the four-dimensional point of view, the vector and scalar components of the photon and a scalar $\mathcal{A}^\rho$ which acts as a source for the scalar photon through the term $2f^{-1}\partial_\rho f \partial_0 \mathcal{A}^\rho$.

Let's recall that $f(\xi)$ and $\eta(\xi)$ have the following behaviours at small and large values of the variable $\xi = m_0\rho$:

$$f(\xi) \underset{\xi \to 0}{\sim} b\,\xi \qquad \vee \qquad \eta(\xi) \underset{\xi\to 0}{\sim} a\,\xi^2$$

$$f(\xi) \underset{\xi \gg 1}{\sim} 1 \qquad \vee \qquad \eta(\xi) \underset{\xi\gg 1}{\sim} 1$$

and consider equations (3.54)-(3.56) only.

In the region $\xi \ll 1$ these read

$$\{\frac{\Box_4}{m_0^2} - \partial_\xi^2 - \frac{1}{\xi}\partial_\xi + b^2\,\xi^2\}\mathcal{A}^{i\,T} = 0 \tag{3.58}$$

$$\{\frac{\Box_4}{m_0^2} - \partial_\xi^2 - \frac{1}{\xi}\partial_\xi + b^2\,\xi^2\}\mathcal{A}^0 + \frac{2}{m_0\xi}\partial_0\mathcal{A}^\rho = 0 \tag{3.59}$$

$$\{\frac{\Box_4}{m_0^2} - \partial_\xi^2 - \frac{3}{\xi}\partial_\xi + \frac{3}{\xi^2} + b^2\xi^2\}\mathcal{A}^\rho = 0 \tag{3.60}$$

While for $\xi \gg 1$ we have

$$\{\frac{\Box_4}{m_0^2} - \partial_\xi^2 - \frac{1}{\xi}\partial_\xi + 1\}\mathcal{A}^{i\,T} = 0 \tag{3.61}$$

$$\{\frac{\Box_4}{m_0^2} - \partial_\xi^2 - \frac{1}{\xi}\partial_\xi + 1\}\mathcal{A}^0 = 0 \tag{3.62}$$

$$\{\frac{\Box_4}{m_0^2} - \partial_\xi^2 - \frac{1}{\xi}\partial_\xi + \frac{1}{\xi^2} + 1\}\mathcal{A}^\rho = 0 \tag{3.63}$$

Let's now consider equations (3.60) and (3.63) first, writing

$$\mathcal{A}^\rho(x,\rho) = \Xi(\xi)\hat{\mathcal{A}}^\rho(x,\rho)$$

and substituting into (3.60)

$$q(\xi)\,(\frac{\Box_4}{m_0^2} - \partial_\xi^2)\hat{\mathcal{A}}^\rho - \hat{\mathcal{A}}^\rho\,\frac{d^2}{d\xi^2}\,q(\xi) - 2\,\frac{d}{d\xi}q(\xi)\,\partial_\xi\hat{\mathcal{A}}^\rho - \frac{3}{\xi}\hat{\mathcal{A}}^\rho\,\frac{d}{d\xi}q(\xi) +$$
$$-q(\xi)\frac{3}{\xi}\,\partial_\xi\hat{\mathcal{A}}^\rho + (\frac{3}{\xi^2} + b^2\,\xi^2)q(\xi)\,\hat{\mathcal{A}}_I^\rho = 0$$

where $q(\xi) \equiv \lim_{\xi\to 0}\Xi(\xi)$ need to cancel terms propotional to $\partial_\xi\hat{\mathcal{A}}^\rho$

$$\frac{d}{d\xi}q(\xi) + \frac{3}{2\xi}q(\xi) = 0 \;\leftrightarrow\; q(\xi) = \xi^{-3/2}$$

and

$$\xi^{-3/2}\,(\frac{\Box_4}{m_0^2} - \partial_\xi^2 + \frac{15}{4\xi^2} + b\xi^2)\hat{\mathcal{A}}^\rho = 0$$

Thus for $0 < \xi \ll 1$ have:

$$(\frac{\Box_4}{m_0^2} - \partial_\xi^2 + V_{eff}^\rho(\xi))\hat{\mathcal{A}}^\rho = 0 \tag{3.64}$$

with

$$V_{eff}^\rho(\xi) = \frac{15}{4}\frac{1}{\xi^2} + b^2\,\xi^2$$

Similarly, from equation (3.63) we get:

$$q'(\xi)\Big(\frac{\Box_4}{m_0^2} - \partial_\xi^2\Big)\hat{\mathcal{A}}^\rho - \hat{\mathcal{A}}^\rho \frac{d^2}{d\xi^2} q'(\xi) - 2\frac{d}{d\xi} q'(\xi)\partial_\xi \hat{\mathcal{A}}^\rho - \hat{\mathcal{A}}^\rho \frac{d}{d\xi} q'(\xi) +$$

$$-q'(\xi)\partial_\xi \hat{\mathcal{A}}^\rho + q'(\xi)(1 + \frac{1}{\xi})\hat{\mathcal{A}}^\rho = 0$$

where $q'(\xi) = \lim_{\xi\to\infty} \Xi(\xi)$. Hence

$$q'(\xi) = \xi^{-1/2}$$

and we obtain, for $\xi \gg 1$:

$$\{\frac{\Box_4}{m_0^2} - \partial_\xi^2 + V_{eff}^\rho\}\hat{\mathcal{A}}^\rho = 0 \tag{3.65}$$

with

$$V_{eff}^\rho = 1 + \frac{3}{4}\frac{1}{\xi^2}$$

Factoring out the eigenfunctions of $\Box_4$ into (3.64) and (3.65), with free parameter m_1^2 we get

$$\hat{\mathcal{A}}^\rho \sim e^{ip'\cdot x} w_{m_1}(\xi)$$

where $p'^2 = m_1^2$.

We define $k_1 \equiv m_1^2/m_0^2$, and arrive to the following Schroedinger problem:

$$\Big(-\frac{d^2}{d\xi^2}+V^{\rho}_{eff}(\xi)\Big)w_{m_1}=k_1 w_{m_1} \tag{3.66}$$

with:

$$V^{\rho}_{eff}(\xi)\sim\begin{cases}\frac{15}{4}\frac{1}{\xi^2}+b^2\xi^2, & \xi\ll 1\\ \\ \frac{3}{4}\frac{1}{\xi^2}+1, & \xi\gg 1\end{cases}$$

The effective potential diverges for $\xi\rightarrow 0$ -Figure 3.5, hence forcing the solution to vanish at the origin. Furthermore we see that $V^{\rho}_{eff}(\xi)$ is always positive and bounded by below by 1, thus from

$$k_1 > Min(V^{\rho}_{eff})$$

we can set $k_1 > 1$.

Let's write the equations explicitly:

$$\frac{d^2}{d\xi^2}w^I_{m_1}-\Big(\frac{15}{4\xi^2}+b^2\xi^2\Big)w^I_{m_1}+k_1 w^I_{m_1}=0,\quad \xi\ll 1 \tag{3.67}$$

$$\frac{d^2}{d\xi^2}w^{II}_{m_1}-\Big(\frac{3}{4}\frac{1}{\xi^2}+1\Big)w^{II}_{m_1}+k_1 w^{II}_{m_1}=0,\quad \xi\gg 1 \tag{3.68}$$

Where we have labeled the respective asymptotic solutions in the two regimes in order to avoid confusion.

The general solution of equation (3.67) [3], $w^I_{m_1}(\rho)$ is a linear combination of Hypergeometric and Laguerre functions [9], the former diverge for $\xi\rightarrow 0$ and must be ignored, then:

$$w^I_{m_1}=D_1\,e^{-b\,\xi^2/2}\,\xi^{-3/2}L\left[\alpha,-2,b\,\xi^2\right]$$

[3] All computational analysis have been obtained with Wolfram Mathematica 11.

with $\alpha = \frac{k_1-2b}{4b}$. Thus considering only the ξ depndece of the field $\mathcal{A}^\rho$:

$$\mathcal{A}^\rho \propto D_1\, e^{-b\,\xi^2/2}\,\xi^{-3} L\,[\alpha, -2, b\,\xi^2], \qquad \xi \ll 1$$

For $\xi \to \infty$ from (3.68), recalling that $k_1 > 1$ we have:

$$\{\frac{d^2}{d\xi^2} + (k_1 - 1)\}\, w^{II}_{m_1} \approx 0$$

thus the solution is expressed in terms of plane waves

$$w^{II}_{m_1} \sim D_3\; exp(i\gamma\,\xi), \qquad \gamma \equiv \sqrt{k_1 - 1}$$

and

$$\mathcal{A}^\rho \propto D_3\,\xi^{-1/2}\; exp(i\gamma\,\xi), \qquad \xi \gg 1$$

It turns out that $\mathcal{A}^\rho$ describes, from the four-dimensional point of view, a massive scalar field, non localized in the extra-dimensions and vanishing at the origin, with four-dimensional mass $m_1^2 > m_0^2$. This allows to say that for energies lower than m_0 which can be assumed to be of the order of m_{EW}, $\mathcal{A}^\rho$ is not dynamical and can be ignored in the equation (3.59), where it appears as a source term for the scalar component $\mathcal{A}^0$. Under this condition, the gauge field components associated to the four-dimensional photon i.e. $\mathcal{A}^\mu(x,\rho)$, are completely decoupled and the corresponding equations potentially solvable.

Let's then consider equations (3.59) and (3.62) assuming a cut-off $\sim m_0$ for the energy range.

$$\{\frac{\Box_4}{m_0^2} - \partial_\xi^2 - \frac{1}{\xi}\partial_\xi + b^2\,\xi^2\}\mathcal{A}^0 = 0, \quad \xi \ll 1$$

$$\{\frac{\Box_4}{m_0^2} - \partial_\xi^2 - \frac{1}{\xi}\partial_\xi + 1\}\mathcal{A}^0 = 0, \quad \xi \gg 1$$

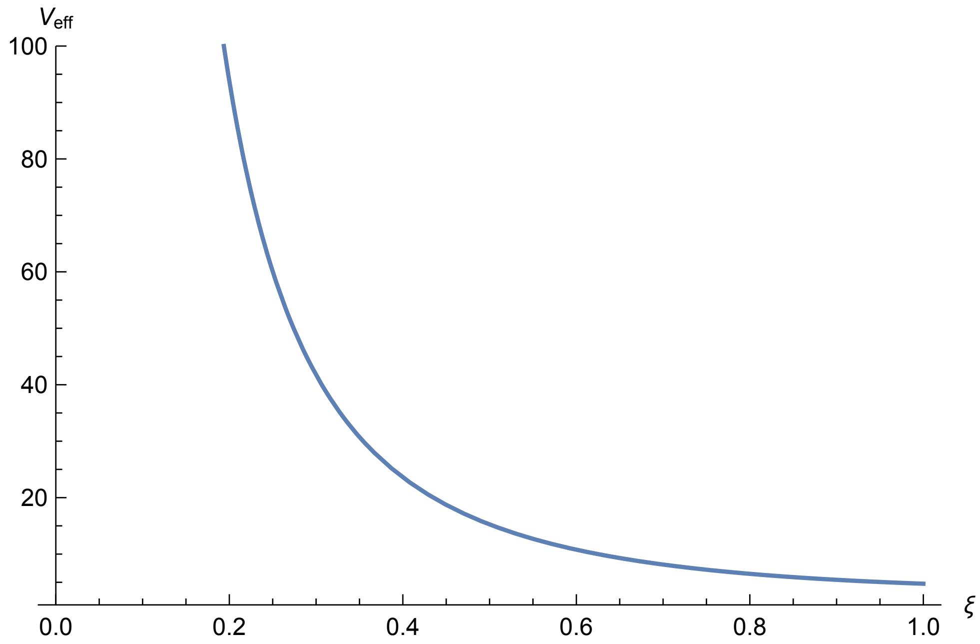

Figure 3.5: Effective potential for $w_{m_1}(\rho)$, in the region $\xi < 1$.

Being separable with respect to the variables (x^μ, ρ) we can factor out the solutions of Klein-Gordon equation with free parameter m_2^2:

$$\{\Box_4 + m_2^2\}\, \alpha(x) = 0$$

which gives

$$\mathcal{A}^0(x,\rho) \sim e^{iqx}\, \mathcal{R}(\rho)$$

with $q^2 = m_2^2$.

Setting $k \equiv m_2^2/m_0^2$ we obtain the following asymptotic equations for $\mathcal{R}(\rho)$:

$$\{-\frac{d^2}{d\xi^2} - \frac{1}{\xi}\frac{d}{d\xi} + b^2\,\xi^2 - k\}\mathcal{R} = 0, \quad \xi \ll 1 \tag{3.69}$$

$$\{-\frac{d^2}{d\xi^2} - \frac{1}{\xi}\frac{d}{d\xi} - (k-1)\}\mathcal{R} = 0, \quad \xi \gg 1 \tag{3.70}$$

We recognize the (3.69) to be the Schroedinger equation of an isotropic two-dimensional harmonic oscillator in polar coordinates [10], with unitary mass, frequency $\omega \equiv b$, energy $E \equiv k/2$ and null angular momentum eigenvalue $m = 0$ (from the absence of ϕ dependence); the energy spectrum is given by

$$E_n = b(n+1), \qquad n \in \mathbb{N}_0$$

and hence

$$k_n = 2b\,(n+1), \qquad n \in \mathbb{N}_0 \tag{3.71}$$

with

$$\mathcal{R}_n(\xi) = \mathcal{N}_n\, e^{-b\frac{\xi^2}{2}}\, L^{1/2}_{\frac{n}{2}}\,(b\xi^2) \tag{3.72}$$

where $L^p_q(w)$ are the Laguerre polynomials and $\mathcal{N}_n$ a normalization constant.

We have to select the admissible values for k_n. Let's set

$$\mathcal{R}_n = h(\xi)\, r_n(\xi) \tag{3.73}$$

equation (3.69) becomes

$$h\, r_n''^I + r_n^I\, h'' + 2\, h'\, r_n'^I + \frac{1}{\xi} r_n^I\, h' + h\frac{1}{\xi} r_n'^I - b^2\,\xi^2 h\, r_n^I + k_n\, h\, r_n^I = 0 \tag{3.74}$$

where $h(\xi)$ is chosen so to cancel terms proportional to $r_n^I(\xi)$

$$h'(\xi) + \frac{1}{2\xi} h(\xi) = 0 \ \leftrightarrow \ h(\xi) = \xi^{-1/2}$$

then equation (3.74) yields

$$\xi^{-1/2}\{\frac{d^2}{d\xi^2} + \frac{1}{4\xi^2} - b^2\,\xi^2 + k_n\}\, r_n^I = 0 \tag{3.75}$$

Analogously, substituting (3.73) into (3.70) gives

$$h(\xi) = \xi^{-1/2}$$

and

$$\xi^{-1/2}\{\frac{d^2}{d\xi^2} + \frac{1}{4\xi^2} - (1 - k_n)\}\, r_n^{II} = 0 \tag{3.76}$$

Hence we arrive to the following Schroedinger problem for $r_n(\xi)$, $\xi \neq 0$:

$$\Big(- \frac{d^2}{d\xi^2} + V_{eff}(\xi)\Big)\, r_n = k_n\, r_n \tag{3.77}$$

The effective potential cannot be expressed in terms of any analytic function, all we know are the asymptotic behaviours for very small and very large value of the variable ξ, namely:

$$V_{eff}(\xi) \sim \begin{cases} V_{eff}^I = -\frac{1}{4\xi^2} + b^2\,\xi^2, & \xi \ll 1 \\ \\ V_{eff}^{II} = -\frac{1}{4\xi^2} + 1, & \xi \gg 1 \end{cases}$$

It turns out that bound states, if any, are possible only provided $k_n < V_{eff}(\infty)$, from the previous relations we should expect:

- $k_n < 1$, the problem might give rise to bound states
- $k_n > 1$, only continuum states are allowed

This is clear if we consider the limit $\xi \to \infty$ of equation (3.77)

$$\{\frac{d^2}{d\xi^2} - (1 - k_n)\}\, r_n^{II} \approx 0$$

the solution falls down exponentially for $k_n < 1$ but results in plane waves for $k_n > 1$

$$\begin{cases} r_n^{II}(\xi) \sim exp(-\beta_n \xi), & k_n < 1 \\ r_n^{II}(\xi) \sim exp(i\beta_n \xi), & k_n > 1 \end{cases}$$

with $\beta_n \equiv \sqrt{|1 - k_n|}$.

However having set m_0 as the energies cut-off, it follows that condition $k_n > 1$ is automatically excluded and only $k_n < 1$ may apply.

Turning to the spectrum (3.71) we find the following condition for the lowest energy level to be a bound state:

$$k_0 = 2b < 1 \leftrightarrow b < 1/2$$

The corresponding wave function

$$\mathcal{R}_0(\xi) \sim \mathcal{N}_0\, e^{-b\frac{\xi^2}{2}}$$

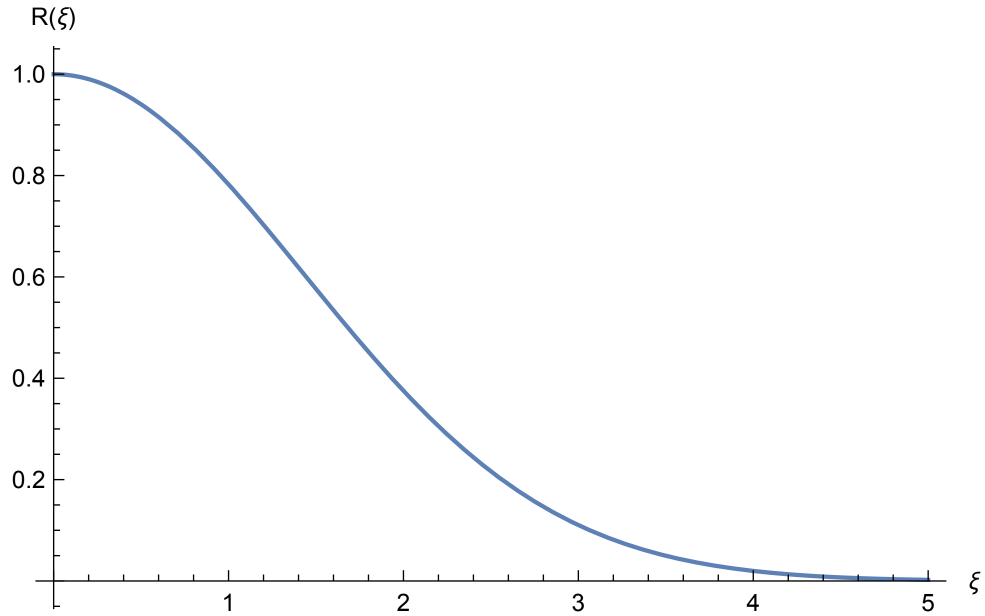

Figure 3.6: Sketch of $R_0(\xi)$ for $b = 0.4$.

is sketched in Figure (3.6).

Accordingly higher energy states with $n \neq 0$ will be confined in the vortex core iff

$$k_n = 2\,b(1+n) < 1 \tag{3.78}$$

The complete solution for $\mathcal{A}^0$, in absence of ϕ dependence reads:

$$\mathcal{A}^0(x,\rho) = \sum_{n=0}^{\bar{n}} \alpha^{(n)}(x)\,\mathcal{R}_n(\rho)$$

where $\bar{n}$ is the biggest value of n that satisfies (3.78) and:

$$\mathcal{R}_n(\rho) \approx \begin{cases} e^{-b\frac{\xi^2}{2}}\,L^{1/2}_{\frac{n}{2}}(b\xi^2), & \xi \ll 1 \\ \\ e^{-\sqrt{1-k_n}\,\xi} & \xi \gg 1 \end{cases}$$

Thus we have found a set of $\bar{n}$ Kaluza-Klein modes localized in the extra-dimensions, these are Gaussian like at small value of the variable ξ and exponentially vanishing at infinity. Requiring $b < 1/2$ guarantees that exists at least one bound state, namely $\bar{n} = 0$, where other values are admitted according to (3.78).

The lightest mode coefficient i.e. $\alpha^{(0)}(x)$ is identified with the scalar potential of the four-dimensional photon field, this however is not massless as we would wish, the mass being:

$$m_{2,0}^2 = 2\, b\, m_0^2$$

The same analysis repeated for equations (3.58) and (3.61) shows that we can expand $\mathcal{A}^{i\,T}(x,\rho)$ in terms of localized functions in the ξ dimension as well obtaining

$$\mathcal{A}^{i\,T}(x,\rho) = \sum_{n=0}^{\bar{n}} \alpha^{i\,(n)}(x) \mathcal{V}_n(\rho)$$

with:

$$\alpha^{i\,(n)}(x) \sim \epsilon^i(p)\, e^{ipx}$$

where $p^2 = m_{p,n}^2$ and $p_i \cdot \epsilon^i = 0$.

The lowest energy mode coefficient $\alpha^{i\,(0)}(x)$ describes a three-dimensional vector field, with two independent degrees of freedom because of the condition $\partial_i \mathcal{A}^{i\,T} = 0$, decoupled from the other components of the gauge field $\mathcal{A}^M(x^N)$ as well as from $\varphi(x^M)$ and localized in the core of the vortex; this is thus identified with the vector field describing the photon in the Coulomb gauge. As before, however, this mode has a non null value of the four-dimensional mass $m_{p,0}^2 = 2\, b\, m_0^2$.

A possible way to recover the correct description of the photon field, could be the introduction of a fine tuning mass that exactly cancel the zero mode one [20]. We can assume this extra term coming from a Higgs mechanism, coupling the gauge field $A^M(x^N)$ to an auxiliary complex scalar field $\psi(x^M)$. If the Higgs vacuum expectation value is suitably chosen, the gauge field can acquire the precise mass term needed. We leave the details of proving that such a mechanism may occur, to a later work. Here we just point out that the auxiliary field need to be a ghost, it must occur with a negative kinetic term in order for the gauge field to acquire the correct mass with a 'wrong' minus sing. The introduction of a ghost however, should't rise new issues if its mass falls far outside the theory energy range, and the instabilities can't set and propagate.

Once the extra mass term is added, it will have the effect to decrease the fields potentials of the same amount so that the hierarchy between the $\mathcal{A}^\rho$ and $\mathcal{A}^0, \mathcal{A}^{i\,T}$ will not be changed.

Equations (3.69) and (3.70) thus read

$$\{-\frac{d^2}{d\xi^2} - \frac{1}{\xi}\frac{d}{d\xi} + b^2\,\xi^2 - (k_n + 2b)\}\mathcal{R}_n = 0, \quad \xi \ll 1$$

$$\{-\frac{d^2}{d\xi^2} - \frac{1}{\xi}\frac{d}{d\xi} - (k_n + 2b - 1)\}\mathcal{R}_n = 0, \quad \xi \gg 1$$

which together with (3.71), give

$$k_n = 2b\,n, \qquad n \in \mathbb{N}_0$$

hence ensuring that the lightest mode with $n = 0$ is actually massless.

Conclusions

Theories with large and infinite extra dimensions look rather exotic, at least for the moment. Nevertheless the help they might offer to long lasting puzzles as well as to the construction of new effective field theories, makes them far to be abandoned. In the present work we have discussed different approaches for the localization of four-dimensional fields in higher dimensional space-time, showing that under appropriate conditions the results can be really encouraging. The main purpose however, has been to investigate a possible mechanism for the localization of Abelian gauge fields, different from the one exposed in [17].

Our starting point has been a model of scalar-QED in six dimensional space-time. The emerging picture is that four-dimensional space-time, can be build in the core of a thin six dimensional vortex, a topological defect, symmetric along the four usual coordinates, which represents a stable solution of the scalar field equations, narrow in the extra dimensions $\{y^a\}$ with characteristic width $\sim m_0^{-1}$. At perturbation level the scalar field only interacts with the ϕ component of the gauge field, namely $\mathcal{A}^\phi(x^N)$, the corresponding equations describe the fluctuations of the background vortex configuration; the remaining gauge field components describe the scalar and vector six-dimensional photon $\mathcal{A}^0(x^N)$, $\mathcal{A}^{i\,T}(x^N)$ together with a scalar field $\mathcal{A}^\rho(x^N)$ which acts as a source for $\mathcal{A}^0$. We found that it is possible to define a low energy regime, where $\mathcal{A}^\rho$ is not dynamical, while a localized spectrum for $\mathcal{A}$ and $\mathcal{A}^{i\,T}$ is possible; this possibility is closely related to exact form of the radial function entering the vortex solution $f(\xi)$, namely the value of b which gives the slope of $f(\xi)$ in proximity of the vortex core. The loosen condition for having just one bound state reads $b < 1/2$, where tighter values of b would allow to accommodate more bound states. Conversely, $\mathcal{A}^\rho$ doesn't admit bound states in the extra dimensions and represents from the four dimensional point of view a massive scalar field, the mass being greater than m_0. Setting m_0 as an energy cut-off for the theory, permits thus to exclude this non physical degree of freedom from the effective spectrum. The four dimensional photon obtained with this procedure results localized, decoupled but massive though. We suggest that a fine tuning, to exactly cancel this mass term, might occur, coupling the six dimensional gauge field to an auxiliary scalar field in order to implement Higgs mechanism. We don't give any detail and leave the proof that such an occurrence may take place for a future work.

Appendix A

Gauss' law in (4+n) dimensions

Consider an N-dimensional flat and infinite space-like manifold. The divergence theorem, for any vector field $\mathbf{g}$, reads:

$$\int_\Omega d^N x \, div \, \mathbf{g} = \int_\Sigma d^{N-1}x \, < \mathbf{g}, \hat{n} > \tag{A.1}$$

where Ω is an arbitrary N-dimensional volume, Σ the enclosing surface and $\hat{n}$ the unit vector normal to the surface and pointing outward.

Assume $\mathbf{g}(r)$ to be a radial function, $r = \sqrt{x^i x^i}$ and $i = 1, .., N$, obeying Gauss' law:

$$div \, \mathbf{g} = -4\pi G_N \, \rho \tag{A.2}$$

ρ is the mass density and G_N the N-dimensional generalization of Newton constant.

Choosing $\Omega = \mathcal{B}^N(r)$ and $\Sigma = S^{N-1}(r)$, respectively the N-dimensional ball of radius r and the enclosing $N-1$ sphere and combining equations (A.1) and (A.2), we get [56]:

$$Vol \, (S^{N-1}(r)) \, \mathbf{g}(r) = -4\pi G_N \, M \, \hat{u}_r$$

from the relation

$$Vol \, (S^{N-1}(r)) = \frac{2\pi^{N/2}}{\Gamma(N/2)} r^{N-1}$$

where M is the mass enclosed by $\mathcal{B}^N$, hence it follows

$$\mathbf{g}(r) = -\mathcal{K}\, G_N \, \frac{M}{r^{N-1}} \, \hat{u}_r$$

with $\mathcal{K} = \frac{1}{\Gamma(N/2)} 8\pi^{(N/2+1)}$.

From $\mathbf{g} = -grad(V)$ we find the gravitational potential:

$$V(r) \sim G_N \frac{M}{r^{N-2}}$$

For the three dimensional ordinary space, $N = 3$, we recover

$$\mathbf{g}(r) = -G_3 \, \frac{M}{r^2} \, \hat{u}_r$$

together with

$$V(r) = G_3 \, \frac{M}{r}$$

Suppose the N-dimensional space-like manifold to be made by three flat and infinite dimensions while the remaining $n = N - 3$ extra-dimensions result compactified into an n-dimensional sphere S^n of radius R [55]. The complete set of coordinates is $\{x^i, y^a\}, \; i = 1, 2, 3$ and $a = 4, ..., n$:

$$\sum_{a=4}^{n} (y^a)^2 = R^2$$

At scales much smaller than the compactification radius R, we don't resolve the curved nature of the extra-coordinates, and space looks flat along all the N dimensions Figure A.1. All coordinates are on an equal footing and the compact nature of the extra

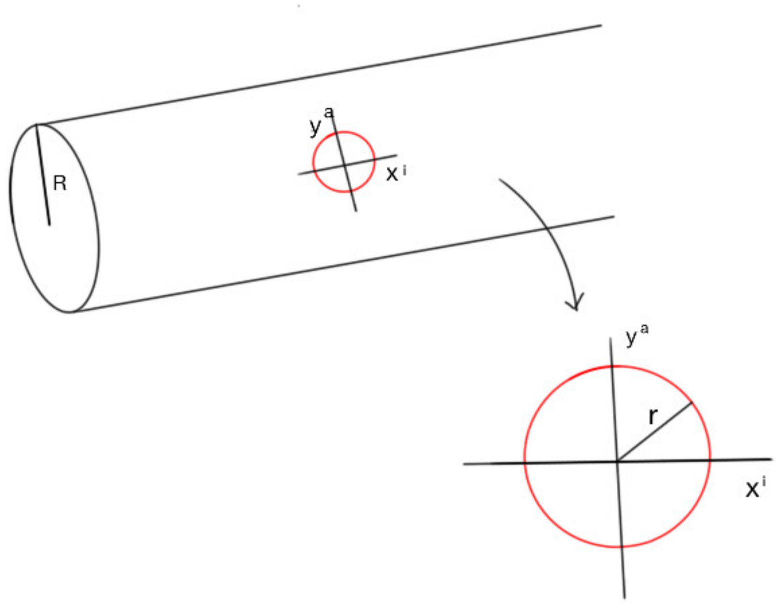

Figure A.1: Space schematically represented as mixture of three flat and n compact dimensions, $i = 1, 2, 3$ and $a = 4, .., n$; the total dimensionality of the bulk is N. Flux line is showed in red.

dimensions doesn't affect the form of the gravitational potential.

From the previous calculations, it follows

$$V(r) = G_{(3+n)} \frac{M}{r^{n+1}}, \qquad r \leqslant R$$

where

$$r = \sqrt{x^i x^i + y^a y^a}$$

At distances much larger than the characteristic scale of the extra-dimensions, Figure A.2, the flux lines run predominantly along the flat dimensions, and we recover the usual three-dimensional inverse distance law:

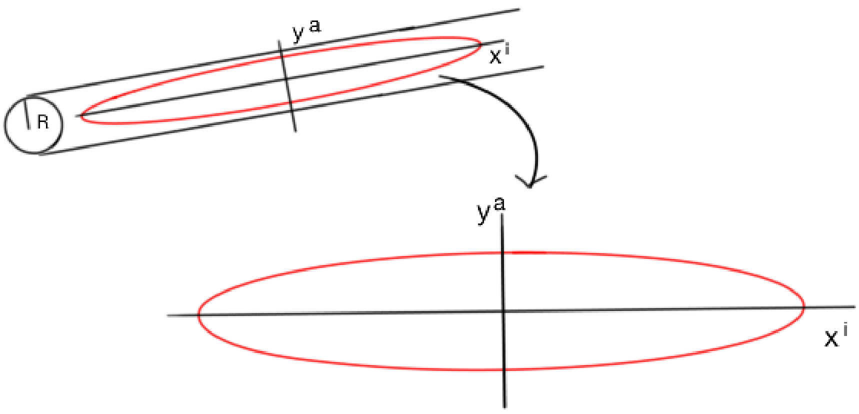

Figure A.2: For $x^i \gg y^a$, contribution to the flux lines (in red) coming from the extra-dimensions, can be ignored and volume and surface area are governed only by x^i.

$$V(r) = G_3 \frac{M}{r}, \qquad r \gg R$$

where now

$$r = \sqrt{x^i x^i + y^a y^a} \approx \sqrt{x^i x^i}$$

Appendix B

Topological Defects

Consider a classical field theory with energy density $T_{00} \geqslant 0$, such that $T_{00} = 0$ everywhere for the ground states (or 'vacua') of the theory. A solution of the classical equations of motion is said to be dissipative if [32]

$$\lim_{t->\infty} max_{\mathbf{x}} T_{00}(t, \mathbf{x}) = 0 \tag{B.1}$$

If the vacuum manifold $\mathcal{V}$, the set of minima of the potential, contains non-contractible n-spheres, then field configurations in $(n+1)$-spatial dimensions whose asymptotic values as $r \to \infty$ 'wrap around' those spheres are necessarily non-dissipative, since continuity of the scalar field guarantees that, at all times, at least in one point in space the scalar potential (and thus the energy) will be non-zero. The region in space where energy is localized is referred to as a topological defect. In three spatial dimensions, it is customary to use the names monopole, string [1] and domain wall to refer to defects that are poin-tlike, one-dimensional or two-dimensional respectively.

In order to decide whether a theory, possesses non dissipative solutions, we need to examine the topology of the vacuum manifold.

Let G be the Lagrangian symmetry group, and ϕ_0 a point on $\mathcal{V}$. Then

$$\forall g \in G, \quad g\phi_0 \in \mathcal{V}$$

[1]The names cosmic string and vortex are also common. Usually, 'vortex' refers to the configuration in two spatial dimensions, and 'string' to the corresponding configuration in three spatial dimensions; the adjective 'cosmic' helps to distinguish them from the so-called fundamental strings or superstrings.

Since many different elements g may yield the same point, it is convenient to introduce the isotropy group H of ϕ_0, as the set of all elements $h \in G$ such that

$$h\phi_0 = \phi_0$$

Then

$$g\phi_0 = g'\phi_0 \quad \leftrightarrow \quad g' = gh, \qquad h \in H$$

We say that the points of $\mathcal{V}$ are in one to one correspondence with the left cosets of H in G i.e.

$$\mathcal{V} = G/H$$

In the case of vortex configuration, the asymptotic solution

$$\bar{\phi}(\varphi) = \lim_{\rho \to \infty} \phi(\rho, \varphi)$$

build on ϕ_0, i.e. $\bar{\phi}(0) = \phi_0$, must define a non-contractible loop in $\mathcal{V}$, a map from S^1 into $\mathcal{V}$ based at ϕ_0, that cannot be shrunk to a point, so that the energy carried is always non zero. Non-contractible loops are classified by the elements of the fundamental group, or first homotopy group of $\mathcal{V}$, denoted $\pi_1(\mathcal{V}, \phi_0)$. Two loops based at ϕ_0 are homotopic if one can be smoothly deformed into the other without leaving $\mathcal{V}$. This is an equivalence relation; the equivalence classes, or homotopy classes of loops, are the elements of $\pi_1(\mathcal{V}, \phi_0)$. These classes have a group structure: the identity is the class of contractible loops, homotopic to the trivial loop which remains at ϕ_0. The inverse is the class comprising the same loops traversed in the reverse sense and the product is defined by traversing two loops in succession.

If $\mathcal{V}$ is connected, then $\pi_1(\mathcal{V}, \phi_0)$ does not depend on the base point ϕ_0 and the first homotopy group is often denoted simply by $\pi_1(\mathcal{V})$. If $\pi_1(\mathcal{V})$ is trivial, comprising the identity element only, then $\mathcal{V}$ is said to be simply connected. A necessary, but not sufficient, condition for the existence of stable vortices is $\pi_1(\mathcal{V})$ to be non-trivial.

If G is chosen to be simply connected (which can always be done by going to the 'universal covering group'), then an equivalent condition is that H contains disconnected pieces. In the case of the Abelian model seen in Chapter 3, we must replace $U(1)$ by its covering group $\mathbb{R}$, then the isotropy subgroup is the group $\mathbb{Z}$ of integers (transformations with phase equal to a multiple of 2π) and

$$\mathcal{V} = \mathbb{R}/\mathbb{Z} = S^1$$

which is not simply connected. This suggests the existence of non dissipative solutions, but there is certainly no guarantee. For example, all loops with non-zero winding number n are non-contractible, but for $\beta > 1$ and $|n| > 1$, there are no stable vortices.

Cosmic strings and other topological defects are particularly interesting also from a cosmological point of view because they might have been formed at phase transitions in the very early history of the Universe. Like transitions in condensed matter systems, these may have led to the formation of defects of one kind or another. In many cases, such defects were stable for topological or other reasons and may therefore have survived, a few of them even to the present day. If such defects existed, they would constitute a uniquely direct connection to the highly energetic events of the early Universe [30].

Appendix C

Decomposition of a vector field

Every vector field $\mathbf{A}(x) = \mathbf{A}(\mathbf{r}, t)$ can be uniquely decomposed into longitudinal $\mathbf{A}_{||}(\mathbf{r}, t)$ and transverse $\mathbf{A}_{\perp}(\mathbf{r}, t)$ components, as a consequence of Helmhotz theorem [60; 59] with respectively:

$$\nabla \times \mathbf{A}_{||} = 0, \qquad \nabla \cdot \mathbf{A}_{\perp} = 0$$

Starting from the tautology:

$$\mathbf{A}(\mathbf{r}, t) = \int dV' \, \mathbf{A}(\mathbf{r}', t)\delta(\mathbf{r} - \mathbf{r}')$$

together with the identity:

$$\delta(\mathbf{r} - \mathbf{r}') = -\nabla^2(1/4\pi|\mathbf{r} - \mathbf{r}'|)$$

we get:

$$\mathbf{A}(\mathbf{r}, t) = -\int dV' \, \mathbf{A}(\mathbf{r}', t) \, \nabla^2(1/4\pi|\mathbf{r} - \mathbf{r}'|)$$

Being the Laplacian a function of $\mathbf{r}$ we can invert the order and write:

$$\mathbf{A}(\mathbf{r},t) = -\int dV'\,\nabla^2\Big(\mathbf{A}(\mathbf{r}',t)\,/4\pi|\mathbf{r}-\mathbf{r}'|\Big)$$

and from the general relation valid for any vector field $\mathbf{E}$

$$\nabla^2\mathbf{E} = -\nabla\times\nabla\times\mathbf{E} + \nabla(\nabla\cdot\mathbf{E})$$

we arrive to:

$$\mathbf{A}(\mathbf{r},t) = \mathbf{A}_{||}(\mathbf{r}.t) + \mathbf{A}_{\perp}(\mathbf{r},t)$$

where:

$$\mathbf{A}_{||}(\mathbf{r},t) = -\nabla\int dV'\,\nabla\cdot\Big[\mathbf{A}(\mathbf{r}',t)/4\pi|\mathbf{r}-\mathbf{r}'|\Big] \tag{C.1}$$

$$\mathbf{A}_{\perp}(\mathbf{r},t) = \nabla\times\int dV'\,\nabla\times\Big[\mathbf{A}(\mathbf{r}',t)/4\pi|\mathbf{r}-\mathbf{r}'|\Big] \tag{C.2}$$

using Fourier expansion:

$$\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} = \frac{1}{(2\pi)^{3/2}}\int d^3p\,\frac{1}{\mathbf{p}^2}\,exp\Big(i\mathbf{p}(\mathbf{r}-\mathbf{r}')\Big)$$

and substituting into (C.1), (C.2):

$$\mathbf{A}_{||}(\mathbf{r},t) = -\nabla^2\int\frac{d^3p}{(2\pi)^{3/2}}\,\frac{\tilde{\mathbf{A}}(\mathbf{p},t)}{\mathbf{p}^2}\,e^{i\mathbf{pr}}$$

$$\mathbf{A}_\perp(\mathbf{r},t) = \nabla \times \nabla \times \int \frac{d^3p}{(2\pi)^{3/2}} \, \frac{\tilde{\mathbf{A}}(\mathbf{p},t)}{\mathbf{p}^2} \, e^{i\mathbf{pr}}$$

This decomposition could have been obtained as well, making use of the longitudinal and transverse delta functions $\delta_{||}^{ij}(\mathbf{r}-\mathbf{r}')$ and $\delta_{\perp}^{ij}(\mathbf{r}-\mathbf{r}')$, which project out the respective vector field components. In momentum space these read:

$$\delta_{||}^{ij}(\mathbf{p}) = p^i p^j/\mathbf{p}^2, \qquad \delta_{\perp}^{ij}(\mathbf{p} = \delta_{ij} - p^i p^j/\mathbf{p}^2$$

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