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PATH INTEGRALS IN CURVED SPACE AND THE WORLDLINE FORMALISM

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Abstract

The main purpose of this thesis is the analysis of a new scheme for computing the path integral for a particle in curved spaces, which has been proposed by physicist J. Guven in the study of a scalar quantum field theory in a first quantized picture, but which has never been used for other explicit computations.

This procedure, if correct, would have the virtue of allowing the use of the flat space path integrals also in the case of a curved manifold, so that one could reproduce the coupling to gravity by using a gravitational effective scalar potential term. This effectively turns a nonlinear sigma model into a linear one, simplifying the difficulties that arise in extending the path integral formalism to curved background. It is conjectured to work in Riemann normal coordinates only.

However, a direct proof of the correctness of Guven's method is missing in the literature. In this thesis we perform a variety of checks to test the proposal. To start with, some mistakes have been found in the original proposal, which indeed contains an incorrect effective potential. We identify the correct potential so to reproduce the first two coefficients of the heat kernel expansion with the path integral, as claimed by Guven. To further test the method we check the next heat kernel coefficient. The outcome is not correct, and it signals a failure of Guven's method at higher orders. A deeper investigation is of course needed to confirm our findings.

Given our preliminary findings, we turn to a special class of curved spaces, those with maximal symmetry. In this case we find that Guven's method indeed works. As a test we compute the diagonal part of the heat kernel at order R^6, and use it for identifying the type-A trace anomalies for a scalar field in arbitrary dimensions up to $D = 12$. These results agree with expected ones, which are reproduced with great efficiency, and slightly extended. Finally, to explain this success, we prove explicitly the correctness of Guven's assumptions by using the maximal symmetry of the background.

We start this thesis with an introduction to path integrals for point particles and describe their use for evaluating perturbatively the heat kernel. Then in chapter 2 we review the known regularization schemes that are needed to treat the case of curved space in which the particle is described by the standard nonlinear sigma model action. Finally in chapter 3 we present our

original results by investigating Guven's method outlined above.

Sommario

Lo scopo primario di questa tesi è l'analisi di una nuova procedura di regolarizzazione di path integral su spazi curvi, presentata inizialmente dal fisico J. Guven e applicata al caso di una teoria di campo scalare autointeragente in formalismo di prima quantizzazione, ma mai utilizzata per svolgere ulteriori calcoli espliciti.

Questa procedura, se corretta, permetterebbe di utilizzare il formalismo di path integral su spazi piatti anche nel caso in cui la varietà di background risulti localmente curva, permettendo di inserire l'accoppiamento con la gravità in un termine scalare di potenziale efficace. Tale procedura trasforma di fatto un modello sigma non lineare in un modello efficace lineare, permettando pertanto di aggirare le usuali complicazioni dovute alla generalizzazione del concetto di path integral su spazi curvi.

Una prova diretta della correttezza della procedura di Guven sembra mancare in letteratura: per questo motivo in questa tesi verranno eseguiti vari test volti a tale verifica. Per iniziare, alcuni errori sono stati riscontrati nella proposta iniziale, tra i quali un termine di potenziale che risulta essere non corretto. Ad ogni modo siamo stati in grado di identificare un potenziale che permetta di riprodurre correttamente i primi due coefficienti dell'espansione in serie dell'heat kernel, così come dichiarato da Guven. Utilizzando lo stesso metodo abbiamo poi cercato di ottenere il successivo coefficiente dell'espansione (cubico in termini di curvatura): il risultato ottenuto non risulta essere corretto, cosa che sembrerebbe segnalare il fallimento di tale metodo ad ordini superiori, anche se ulteriori indagini sono certamente necessarie.

Visti tali risultati preliminari, siamo stati indotti a considerare una classe speciale di spazi curvi, quella degli spazi massimamente simmetrici, trovando invece che su tali spazi la procedura di Guven riproduce i risultati corretti. Come verifica abbiamo ottenuto la parte diagonale dell'heat kernel fino all'ordine R^6, che é stata poi utilizzata per riprodurre l'anomalia di traccia di tipo A per campi scalari in dimensioni arbitrarie fino a $D = 12$. Questi risultati sono in accordo con quelli attesi, che vengono riprodotti con grande efficenza. Viene pertanto fornita una prova della validità di tale procedura su questa particolare classe di varietà, utilizzando esplicitamente la simmetria di tali spazi.

Inizieremo il presente testo con un'intruzione ai path integral per particelle puntiformi, e descriveremo il loro utilizzo come metodo per ottenere un'espansione perturbativa dell'heat kernel. Nel capitolo 2 presenteremo i ben noti schemi di regolarizzazione che risultano necessari per ottenere in modo non ambiguo il formalismo di path integral per particelle che si muovono su spazi curvi, la cui azione è descritta da un modello sigma non lineare. Infine nel capitolo 3 presenteremo i risultati originali precedentemente descritti.

Contents

Chapter 1

Heat Kernel and Path integrals

1.1 Introduction

Path integrals are nowadays one of the main tools that allow the computation of loop calculations and other important quantities in quantum field theories and quantum mechanics. They were developed for the first time by the famous physicist Richard Feynman in a paper from 1948 [19], following a previous hint from P.A.M. Dirac [17], but they played no important role in particle physics until the late '70, since at first this approach seemed less promising, while today they offer a precious alternative to the classical operatorial formalism for the description of quantum mechanical systems. Althrought the equivalence of quantum mechanical path integrals and the Schrödinger operatorial picture can be proved without doubts, there are many cases in which the path integral formulation simplifies very much the calculations, and they make some concepts more intuitive, like for example the deep relation between quantum field theory and statistical mechanics. Path integrals also provide a more intuitive way to quantize classical systems, since they conceptually rely only on two pillars of quantum mechanics: the fact that a particle possesses no more a definite trajectory, and the superposition principle.

The main quantity that one wishes to compute using path integral methods is the transition amplitude (or heat kernel after a Wick rotation), that is a fundamental quantity if one needs to evaluate the correct time evolution for a quantum system. This is easily done for free theories on flat spaces using Gaussian integration and square-completion, as the Hamiltonian operator has a quadratic dependence on momenta. With the presence of

a non-vanishing potential $V(x)$ things become a little harsher, but with the aid of a perturbative expansion in terms of powers of the transition time T, one is able to solve these difficulties and obtain a definite result at any order. This perturbative expansion then generates terms which can be visualized as a set of Feynman graphs, in which one has propagators, loops and vertices, like usual QFT. These methods and techniques that one uses to define and to compute path integrals on flat spaces are briefly explained in this first chapter.
We will see in the second chapter that various subtleties arise when one passes from a flat-space theory to a curved-space one. In the usual operatorial formulation, these subtleties are due to the fact that, when one tries to obtain the correct quantum Hamiltonian from the classical one, ordering ambiguities of operators appear. In the path integral formalism similar ambiguities appear, and they can be solved using a precise regularization scheme. At present time, there are essentially three different regularization procedures which have been widely studied over the past few decades and have been shown to lead to the same expansion (as should be indeed) if one inserts the correct potential counterterm V_{ct} for each regularization scheme. These are time-slicing regularization, mode regularization and dimensional regularization. Each one of these different schemes has, of course, its own pros and cons that will be explained each time a regularization procedure is described in this text.

The purpose of this thesis, on top of reviewing the three regularization schemes mentioned above, is to test a new scheme for computing the path integral in curved space. This scheme has been proposed by the physicist J. Guven in a paper from 1988, which can be found in reference [24]. However, it seems that this regularization has never been used for explicit computations outside the work of Guven, at least to our knowledge[1]. In testing the construction of [24], at least as reported there we find that it is incorrect: (i) it contains as key element a potential V_2 that is not able to reproduce the leading term of the heat kernel expansion, (ii) the proof of a crucial statement is not reported in the paper (nor in the references). Assuming that the main idea proposed is valid, we try in thesis to fix the construction. A new form of the gravitational effective potential V_{eff}, that replaces Guven's potential V_2, is then introduced, which can be derived from basic considerations about the defining equation of the Green function.

[1]INSPIRE indicates only 1 citation of this paper, which however do not addresses the path integral construction.

Once we take in consideration the correct potential, one can show that the remaining terms containing one and two derivatives of the heat kernel erase each other, at least until fourth adiabatic order, as shown by L. Parker in [32]. Since now the kinetic term is not coupled to gravity anymore, one can define the heat kernel as a flat-space path integral where the gravitational interaction is all contained into the potential term V_{eff}, and using the standard methods described in chapter (1) one is then able to correctly reproduce the first two coefficients of the expansion. It is stated in the work of B. L. Hu and D.J. O'Connor [26] that this decoupling of the kinetic term with respect to the gravitational interaction can be proved at any order using a pretend Lorentz invariance of the momentum space representation of the heat kernel, but no explicit proof is given. Since a direct way to prove this statement beyond the fourth adiabatic order has not been found, we tried to obtain the third Seeley-DeWitt coefficient (that is of order six in the adiabatic expansion) assuming its correctness. The result obtained in that way seems to be incorrect, being inconsistent with the ones obtained with other procedures, for example by I. G. Alvramidi in [3] using a slight modification of the original DeWitt proper time expansion, or by F. Bastianelli and O. Corradini in [6] using curved-space path integrals techniques and dimensional regularization. Further investigations are of course required, but this is indeed a clue for the non-correctness of this methods to higher orders or that, at least, a non trivial extension has to be made in order to make a correction to this procedure that works for any given order of expansion.
Given the failure of this method on generic spaces we have turned our attention to maximally symmetric spaces, in which Lorentz-invariance has indeed a better chance of working. We show in fact that on these kind of spaces, using Guven's procedure, one is able to correctly obtain the diagonal part of the heat kernel expansion for spaces of arbitrary D dimensions up to order T^6. These results are obtained with increased efficiency with respect to other known regularization procedures, and in fact our results expand slightly the ones given in literature. As a further test for our results we also computed the type-A trace anomaly for specific values of D up to $D = 12$ and confronted our results with ones obtained in a different way and which are listed in reference [14]. Since the comparison is successful we conclude that Guven's method is indeed valid if one considers only maximally symmetric spaces. In the last chapter of this thesis we use quantum mechanical path integrals to formulate the quantum mechanics of relativistic particles in

first quantization (worldline formalism) and make contact with the QFT of a scalar field. In fact, all the techniques developed within the non relativistic heat kernel on curved manifolds can be readily extended, using a worldline approach, to include relativistic quantum field theories. The worldline approach can be defined as a first-quantization path integral approach in which the action functional is given as an integral over the wordline parameter s, which can be seen as the proper time which parametrises the evolution of a virtual bosonic particle along the worldline, instead than over the whole set of space-time coordinates x^μ as done in usual QFT computations. In this approach one can interpret a quantum mechanical theory as a (0+1) dimensional quantum field theory, in which all the fields depend only on time. The worldline approach can be extended in order to correctly describe particles with spin thanks to the insertion of supersymmetric conserved charges, even when the original theory is not itself supersymmetric : this case esules from the interest of this thesis an will not be treated for this reason, but extensionsto the methods that we present here can be made and can be found in [7],[20],[25]. In case a non-trivial extension of Guven's procedure could be found, an extension to particles of general spin could be made using this approach.

1.2 Development of the Path Integral Formalism

Historically, the development of the path integral formulation started as a rather new way to deduce quantum mechanics, associating a probability amplitude to each one of the possible paths, or histories, that the particle can take propagating between two fixed points of the space. Then the ordinary quantum mechanics was shown to result from the postulate that this probability amplitude has a phase proportional to the classical action. This is the road followed originally by Feynman.

In order to review the path integral formulation we will use the opposite approach: we will take quantum mechanics as granted, and we will start our derivation of the path integral formalism from quantum mechanics itself.

So, let's start considering a generic non-relativistic quantum system that is let free to evolve for a total time $T = t_f - t_i$, starting from a fixed known state. The initial state of the system can be taken to be represented, in the Dirac notation, by a ket vector $|\psi_i\rangle = |\psi(t_i)\rangle \in \mathscr{H}$[2]. The evolution of this state over time is then given by the Schrödinger equation (notice that we set here, as well as for the rest of this text, units in which $\hbar = 1$)

$$i\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle \tag{1.1}$$

This equation can be formally solved by

$$|\psi(t_f)\rangle = e^{-i\hat{H}T}|\psi(t_i)\rangle \tag{1.2}$$

where $\hat{H}(\hat{x},\hat{p})$ is the operator associated to the classical Hamiltonian function, assumed to be independent of time. The operator $e^{-i\hat{H}T}$ is then the unitary operator that makes the system evolve in time, called for that reason the time-evolution operator.
Suppose now that after a time t_f a measure of the system is performed, obtaining that the system is now in the state $|\psi_f\rangle$, and that we wish to know what's the probability amplitude of finding the system in that precise state. This amplitude is then simply given projecting the time-evolved initial state onto the measured final state, that is, taking the inner product between these two states

$$A = \langle\psi_f|\psi(t_f)\rangle = \langle\psi_f|\,e^{-i\hat{H}T}\,|\psi_i\rangle \tag{1.3}$$

We will now proceed showing how the transition amplitude can be casted into a path integral form within the mathematical framework of quantum mechanics, leaving physical meanings and intuitions for later in this chapter. As mentioned before, we will for now restrict ourselves to a non relativistic particle moving inside a flat D-dimensional space case, so that the Hamiltonian operator takes the form

$$\hat{H}(\hat{x},\hat{p}) = \frac{\hat{p}^2}{2m} + V(\hat{x}) \tag{1.4}$$

This is the Hamiltonian operator whose Legendre transform defines the action functional of the "sigma model", which is defined to be "linear" if the metric hidden inside $p^2 = g^{ij}p_ip_j$ is simply the flat metric δ^{ij}, or "non-linear" in the opposite case.

An Hamiltonian operator of this kind is what characterises a so-called "sigma model", and it is called a linear sigma model if the metric hidden inside $p^2 = g^{ij}p_ip_j$ is simply the

[2] $\mathscr{H}$ being an Hilbert space

flat metric δ^{ij}.

In order to construct the path integral we need to make use of the following identities, which are nothing but the completeness relations of position ($|x\rangle$) and momentum ($|p\rangle$) eigenstates:

$$\mathbb{1} = \int d^D x \, |x\rangle \langle x| \tag{1.5}$$

$$\mathbb{1} = \int \frac{d^D p}{(2\pi)^D} \, |p\rangle \langle p| \tag{1.6}$$

with chosen normalization

$$\langle x|x'\rangle = \delta^D(x - x') \quad , \quad \langle p|p'\rangle = 2(\pi)^D \delta^D(p - p') \tag{1.7}$$

It is now useful to insert two times the identity (1.5) into equation (1.3), obtaining

$$\begin{aligned} \langle \psi_f | \, e^{-i\hat{H}T} \, |\psi_i\rangle &= \langle \psi_f | \, \mathbb{1} e^{-i\hat{H}T} \mathbb{1} \, |\psi_i\rangle = \\ &= \int dx_i \int dx_f \;\, \psi_f^*(x_f) \, \langle x_f | \, e^{-i\hat{H}T} \, |x_i\rangle \, \psi_i(x_i) \\ &= \int dx_i \int dx_f \;\, \psi_f^*(x_f) K(x_f, x_i, T) \psi_i(x_i) \end{aligned} \tag{1.8}$$

where $\psi_i(x_i) = \langle x_i|\psi_i\rangle$ and $\psi_f(x_f) = \langle x_f|\psi_f\rangle$ are the wave functions for the initial and final states. So, if the final and initial states of the system are known, it is sufficient to calculate the matrix element of the time-evolution operator between position eigenstates to obtain the total transition amplitude. This matrix element $K(x_f, x_i, T)$ is also called the "heat kernel".

1.2.1 Phase-space path integrals

We are now ready to show that the heat kernel $K(x_f, x_i, T) = \langle x_f | \, e^{-i\hat{H}T} \, |x_i\rangle$ can be casted in a path integral form. The basic idea behind this process is to split the transition amplitude into N identical therms and inserting every time in between a completeness relation given by equation (1.5), and then let N run to infinity. In that way

$$K(x_f, x_i, T) = \langle x_f|(e^{-i\frac{T}{N}\hat{H}})^N|x_i\rangle = \langle x_f|\underbrace{e^{-i\epsilon\hat{H}}\mathbb{1}e^{-i\epsilon\hat{H}}\mathbb{1}\dots\mathbb{1}e^{-i\epsilon\hat{H}}}_{N-1\ times}|x_i\rangle =$$

$$= \lim_{N\to\infty}\int\left(\prod_{k=1}^{N-1} d^D x_k\right)\prod_{k=1}^{N}\langle x_k|e^{-i\epsilon\hat{H}}|x_{k-1}\rangle \tag{1.9}$$

where we have set $x_0^j = x_i^j$, $x_N^j = x_f^j$ and $\epsilon = T/N$. Notice that in this last expression the completeness relation has been used $(N-1)$ times in order to leave the initial and final points unchanged. But this is not yet enough. Since the aim here is to replace an operatorial expression with another one which contains no more operators, we need to insert other N completeness relations in the same form as (1.6) so that the Hamiltonian operator can be seen as acting with its momentum operator-dependent part on the right, and the position operator-dependent part on the left, where the corresponding eigenstates are. In doing that we obtain

$$K(x_i, x_f, T) = \lim_{N\to\infty}\int\left(\prod_{k=1}^{N-1} d^D x_k\right)\left(\prod_{k=1}^{N}\frac{d^D p_k}{(2\pi)^D}\right)\prod_{k=1}^{N}\langle x_k|p_k\rangle\,\langle p_k|e^{-i\epsilon\hat{H}}|x_{k-1}\rangle \tag{1.10}$$

What's left now is the evaluation of the matrix element between position and momentum eigenstates

$$\begin{aligned}
\langle p|e^{-i\epsilon\hat{H}(\hat{x},\hat{p})}|x\rangle &= \langle p|(1 - i\epsilon\hat{H} + \dots)|x\rangle = \\
&= \langle p|x\rangle - i\epsilon\,\langle p|\hat{H}(\hat{x},\hat{p})|x\rangle + \dots = \\
&= \langle p|x\rangle\,(1 - i\epsilon H(x,p) + \dots) = \\
&= \langle p|x\rangle\, e^{-i\epsilon H(x,p)+\dots}
\end{aligned} \tag{1.11}$$

in which the position and momentum operators contained in $\hat{H}(\hat{x},\hat{p})$ has been replaced by their eigenvalues, so that $H(x,p) = p^2/2m + V(x)$.
All that has been obtained up until now are exact results and no approximation was needed. We can now neglect the extra part in the exponent of the last line of previous equation and still obtain an exact relation, at least in the limit $N \to \infty$, for a vast variety of interesting physical potentials. This statement can be mathematically proven by means

of the famous "Trotter product formula" [3].

Now, recalling that

$$\langle p|x\rangle = (\langle x|p\rangle)^* = e^{-ip_i x^i} \tag{1.12}$$

and inserting it into the transition amplitude we finally arrive at an expression for the phase-space path integral

$$K(x_f, x_i, T) = \lim_{N\to\infty} \int \left(\prod_{k=1}^{N-1} d^D x_k\right)\left(\prod_{k=1}^{N} \frac{d^D p_k}{(2\pi)^D}\right) \prod_{k=1}^{N} e^{i\epsilon \sum_{k=1}^{N}\left[p_k \frac{x_k - x_{k-1}}{\epsilon} - H(x_{k-1}, p_k)\right]} \tag{1.13}$$

The exponent in the integrand function of this last equation can be seen as a discretization of the classical phase space action, recognising that

$$\frac{x_k - x_{k-1}}{\epsilon} \overset{\epsilon\to 0}{\longrightarrow} \dot{x}_k \tag{1.14}$$

and noticing that the term $p_k \cdot \dot{x}_k$ is nothing but the symplectic term, so that the term in the exponent of (1.13) can be seen as the discretized version of the Legendre transform of the classical Hamiltonian function

$$S[x,p] = \int_{t_i}^{t_f} (p\dot{x} - H(x,p))dt \quad \to \quad \epsilon \sum_{k=1}^{N} \Big[p_k \frac{x_k - x_{k-1}}{\epsilon} - H(x_{k-1}, p_k)\Big] \tag{1.15}$$

where $T = t_i - t_f = N\epsilon$ is the total time of propagation, and having the phase-space paths discretized as

$$x^j(t) \quad , \quad p_j(t) \quad \to \quad x^j_k = x^j(t_i + k\epsilon) \quad , \quad p_{k,j} = p_j(t_i + k\epsilon) \tag{1.16}$$

We finally arrive at the conclusion

$$K(x_f, x_i, T) = \int DxDp \; e^{iS[x,p]} \tag{1.17}$$

that the point-to-point transition amplitude can be represented as an integration over the whole set of phase-space trajectories, each one weighted by i times the phase

[3] A proof can be found in reference [36]

space action; this means that classical paths $x_{cl}(\tau)$, for which $\frac{\delta}{\delta x_{cl}} S[x_{cl}] = 0$, are the ones which give the biggest contribution, since they minimize the action, together with nearby paths which sum coherently with classical ones, while the ones that differ a lot from these paths give a little contribution to the integral and get usually cancelled by destructive interference.

The path integral measure

$$\lim_{N\to\infty} \int \left(\prod_{k=1}^{N-1} d^D x_k \right) \left(\prod_{k=1}^{N} \frac{d^D p_k}{(2\pi)^D} \right) = \int DxDp \tag{1.18}$$

is formally an infinite-dimensional measure, since the space of phase-space paths contains infinite possible trajectories. The exact definition of this measure has various mathematical difficulties which lie outside from the focus of our work.

We can say that what we have in equations (1.13) and (1.17) is an integration over trajectories (or paths, or histories) of the particle since what has been done in equation (1.9) can be seen as splitting the trajectory $x(t)$ connecting the initial point x_i and the final point x_f into a set of $(N-1)$ broken segments connecting each point x_k with x_{k+1}. Obviously in the limit $N \to \infty$ we would expect to recover the original trajectory. The difference between classical and quantum mechanics can be seen in the fact that, in quantum mechanics, we can not assign a definite trajectory to a particle which has been seen propagating between two points. We can see that by noticing that in this broken-line approximation the points between one segment and the other are not fixed; instead they are let free to run over the whole underling space, as well as the momentum the particle possesses along each "broken path". Then the probability transition amplitude associated with every different trajectory is summed with all the others, since in quantum mechanics we have the superposition of probability amplitudes.

1.2.2 Path integrals in configuration space

When the dependence of the action from momenta is at most quadratic, one can always derive a "configuration space path integral" by integrating out the momenta from equation (1.13). This task can be easily achieved using repeated Gaussian integrations

$$\int_{-\infty}^{\infty} \frac{d^n\phi}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}\phi^i K_{ij} \phi^j} = (\det K_{ij})^{-\frac{1}{2}} \tag{1.19}$$

$$\int_{-\infty}^{\infty} \frac{d^n\phi}{(2\pi i)^{\frac{n}{2}}} e^{-\frac{i}{2}\phi^i K_{ij}\phi^j} = (\det K_{ij})^{-\frac{1}{2}} \tag{1.20}$$

where ϕ is an n-dimensional real variable [4].

For non-trivial cases, i.e. when the potential $V(x)$ differs from 0 so that the exponent is not entirely quadratic in its arguments, an extension needs to be done, that is

$$\int_{-\infty}^{\infty} \frac{d^n\phi}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}\phi^i K_{ij}\phi^j + J_i\phi^i} = (\det K_{ij})^{-\frac{1}{2}} e^{\frac{1}{2}J_i G^{ij} J_j} \tag{1.21}$$

$$\int_{-\infty}^{\infty} \frac{d^n\phi}{(2\pi i)^{\frac{n}{2}}} e^{-\frac{i}{2}\phi^i K_{ij}\phi^j + iJ_i\phi^i} = (\det K_{ij})^{-\frac{1}{2}} e^{\frac{i}{2}J_i G^{ij} J_j} \tag{1.22}$$

where G^{ij} is the inverse matrix of K_{ij}. These last results are obtained by square completion, which means using the identity $-\frac{1}{2}K\phi^2 + J\phi = -\frac{1}{2}K(\phi - \frac{J}{K})^2 + \frac{1}{2}\frac{1}{K}J^2$ and then shifting the measure from $d\phi$ to $d(\phi - \frac{J}{K})$, which, being traslationally invariant, leaves the integrand unchanged.

This, in our case, means writing

$$p_k \frac{x_k - x_{k-1}}{\epsilon} - \frac{1}{2m}p_k^2 = -\frac{1}{2m}\Big(p_k - m\frac{x_k - x_{k-1}}{\epsilon}\Big)^2 + \frac{m}{2}\Big(\frac{x_k - x_{k-1}}{\epsilon}\Big)^2 \tag{1.23}$$

and then shifting $p_k \to p_k - m\frac{x_k - x_{k-1}}{\epsilon}$. Integrating out the newly shifted momenta one obtains

$$\begin{aligned} K(x_f, x_i, T) &= \int_{x(t_i)=x_i}^{x(t_f)=x_f} \Big(\prod_{k=1}^{N-1} d^D x_k\Big)\Big(\frac{m}{2\pi i\epsilon}\Big)^{\frac{ND}{2}} e^{i\epsilon\sum_{k=1}^{N}[\frac{m}{2}(\frac{x_k - x_{k-1}}{\epsilon})^2 - V(x_{k-1})]} \\ &= \int_{x(t_i)=x_i}^{x(t_f)=x_f} Dx\ e^{iS[x]} \end{aligned} \tag{1.24}$$

in which $S[x]$ is the configuration space classical action

$$S = \int_{t_i}^{t_f} \Big(\frac{m}{2}\dot{x}^2 - V(x)\Big)dt \quad \to \quad \sum_{k=1}^{N}\Big[\frac{m}{2}\Big(\frac{x_k - x_{k-1}}{\epsilon}\Big)^2 - V(x_{k-1})\Big] \tag{1.25}$$

[4] Notice that exponential convergence to the given value is granted if K_{ij} has a small negative imaginary part that ensures a Gaussian damping for $|\phi| \to \infty$. This is usually achieved in quantum field theory using the Feynman causal prescription. In the other case it is sufficient that K_{ij} is a positive definite matrix.

Equation (1.24) expresses the path integral formulation on configuration space.

1.2.3 One dimensional flat space free-particle case

In case of a free particle (i.e. $V(x) = 0$) the problem is exactly solvable by means of repeated Gaussian integrations. Using $N-1$ times equation (1.21) in equation (1.24) one gets

$$K(x_f, x_i, T) = \sqrt{\frac{m}{2\pi i T}} e^{im(x_f - x_i)^2/2T} \tag{1.26}$$

This result is quite suggestive: it is given, up to a prefactor, by the exponent the classical action times the imaginary unit evaluated on the classical path, the path that satisfies the classical equations of motion. This prefactor, although formally infinite in the limit $N \to \infty$, can be considered as containing the "one-loop" corrections which gives the exact results by means of the classical solution but taking in consideration also quantum effects. It can also be seen in another way: one may choose to work directly in the continuum limit, without considering the exact definition of the path integral measure, but assuming knowledge on its the formal properties like translational invariance. One then should in principle give a precise regularization in order to verify that these properties holds.

The calculation goes as follows: one considers a generic path $x(t)$ as a sum of a classical path x_{cl} plus quantum fluctuations $\phi(t)$

$$x(t) = x_{cl}(t) + \phi(t) = x_i + (x_f - x_i)\frac{t}{T} + \phi(t) \tag{1.27}$$

in which T is the total propagation time $(t_f - t_i)$. Indeed, x_{cl} solves the classical equation of motion by construction, and for this reason its explicit expression is inserted in the second equality. Quantum fluctuations are obviously such that

$$\phi(t_i) = \phi(t_f) = 0 \tag{1.28}$$

in order to preserve the fact that all paths begin at the space point x_i and ends at x_f. In that way $x_{cl}(t)$ may be interpreted as the origin of the space of paths. Then, since the free action is quadratic in the ϕ's and contains no linear terms (which indeed vanish due to x_{cl} satisfying the classical equation of motion) respect to quantum fluctuations

$$S[x] = S[x_{cl}] + S[\phi] \tag{1.29}$$

the path integral becomes

$$\begin{aligned} K(x_f, x_i, T) &= \int Dx\ e^{iS[x]} = \int D(x_{cl} + \phi)\ e^{iS[x_{cl}+\phi]} = \\ &= \int D\phi\ e^{iS[x_{cl}+\phi]} = e^{iS[x_{cl}]} \int D\phi\ e^{iS[\phi]} = \\ &= Ae^{iS[x_{cl}]} = Ae^{im\frac{(x_f - x_i)^2}{2T}} \end{aligned} \tag{1.30}$$

where translational invariance of the path integral measure has been used. The prefactor A results undetermined in that way, but can be readily obtained by taking in consideration the defining equations (1.58), (1.59).

The solution is indeed the same obtained in equation (1.26). In general for arbitrary D the result is

$$K(x_f, x_i, T) = \left(\frac{m}{2\pi i T}\right)^{\frac{D}{2}} e^{im\frac{(x_f - x_i)^2}{2T}} \tag{1.31}$$

1.3 Seeley-DeWitt Expansion Using Path Integral Methods

We have seen in the last section that when the potential term of the action vanishes, the heat kernel is easily solvable with the path integral formalism. When the case is not so, the problem has no immediate solution, and is usually solved by means of a series expansion in the form of

$$K(x_f, x_i, T) = K_0(x_i, x_f, T) \sum_{n=0}^{\infty} (iT)^n a_n(x, y) \tag{1.32}$$

where the coefficients $a_n(x_i, x_f)$ are the so-called Seeley-DeWitt coefficients and K_0 is the "free" heat kernel of equation (1.31). This series expansion is formally an asymptotic expansion in the time-parameter T, and its convergence is granted by the

Minakashisundaram-Plejel theorem, at least when T can be considered small. We will not prove this result because of the mathematical subtleties the problem hides, but a proof can be found in references [28],[22]. We will now compute this expansion and obtain the firsts coefficients (as functions of the potential $V(x)$) with the path integral approach. [5] The action we will take in consideration is (we put here the non-relativistic mass $m = 1$ for notational convenience)

$$S[x] = \int_0^T \Big(\frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j - V(x)\Big)dt \tag{1.33}$$

It is convenient to rescale the time parameter as $t = T\tau$, so that the new time variable $\tau \in [0, 1]$. In that way the rescaled action takes the form

$$S[x] = \frac{1}{T}\int_0^1 \Big(\frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j - T^2V(x)\Big)d\tau \tag{1.34}$$

where now the dot indicates differentiation respect to the new variable τ.

Now, the first step is to split this action into an exactly solvable free part, and an interaction part which is treated as a perturbation, namely

$$S[x] = S_0[x] + S_{int}[x] \tag{1.35}$$

$$S_0[x] = \frac{1}{T}\int_0^1 \frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j d\tau \quad , \quad S_{int}[x] = -T\int_0^1 V(x)d\tau \tag{1.36}$$

and subsequently splitting again the paths as previously done in (1.27). The key point in obtaining the perturbative expansion is that the average values of quantum fluctuations are computed using only the free quadratic part of the action, that is

$$A \equiv \int D\phi\ e^{iS_0[\phi]} = \frac{1}{(2\pi iT)^{\frac{D}{2}}}$$

$$\big\langle \phi^i(\tau)\big\rangle \equiv \frac{1}{A}\int D\phi\ \phi^i(\tau)e^{iS_0[\phi]}$$

$$\big\langle \phi^i(\tau)\phi^j(\tau')\big\rangle \equiv \frac{1}{A}\int D\phi\ \phi^i(\tau)\phi^j(\tau')e^{iS_0[\phi]}$$

[5] We will briefly illustrate the original method followed by DeWitt directly on curved space, at the end of chapter (2).

$$\dots \tag{1.37}$$

and in general one may define the average of an arbitrary functional $F[\phi]$ as

$$\langle F[\phi] \rangle \equiv \frac{1}{A} \int D\phi \; F[\phi] e^{iS_0[\phi]} \tag{1.38}$$

The average value $\langle \phi^i(\tau)\phi^j(\tau') \rangle$ is called 2-points correlation function and can be computed in the standard way discovered by Schwinger([37],[38]) introducing a source j which vanishes at initial and final points, that is $j(0) = j(1) = 0$: in that way

$$\begin{aligned}
< \phi^l(\tau)\phi^m(\tau') > &= \int D\phi(\tau) \; \phi^l(\tau)\phi^m(\tau') \exp\left\{ \left[i \int_0^1 \frac{\dot\phi^2}{2T} d\tilde\tau \right] \right\} \Big/ \int D\phi(\tau) \exp\left\{ \left[i \int_0^1 \frac{\dot\phi^2}{2T} d\tilde\tau \right] \right\} \\
&= \left. \frac{\frac{\delta}{i\delta j_l(\tau)} \frac{\delta}{i\delta j_m(\tau')} \int D\phi(\tau) \exp\left\{ i \int_0^1 \left(\frac{\dot\phi^2}{2T} + j_k \phi^k \right) d\tilde\tau \right\}}{\int D\phi(\tau) \exp\left\{ i \int_0^1 \frac{\dot\phi^2}{2T} d\tilde\tau \right\}} \right|_{j=0} \\
&= \left. \frac{\delta}{i\delta j_l(\tau)} \frac{\delta}{i\delta j_m(\tau')} \exp\left\{ \frac{1}{2} iT \int_0^1 d\tilde\tau d\tilde\tau' j_k(\tilde\tau) \delta^{kn} g(\tilde\tau, \tilde\tau') j_n(\tilde\tau') \right\} \right|_{j=0} \\
&= iT\delta^{lm} g(\tau, \tau')
\end{aligned} \tag{1.39}$$

where equation (1.21) has been used to obtain the last line and where $g(\tau, \tau')$ is nothing but the Green function associated to the operator ∂_τ^2, i.e. the solution to the equation

$$\frac{\partial^2}{\partial \tau^2} g(\tau, \tau') = -\delta(\tau - \tau') \tag{1.40}$$

with the boundary condition $g(\tau, \tau') = 0$ if $\tau, \tau' = 0, 1$. An explicit solution for this last equation is immediate to obtain and can be written in many different ways

$$\begin{aligned}
g(\tau, \tau') &= \tau_<(1 - \tau_>) = (1 - \tau)\tau'\theta(\tau - \tau') + (1 - \tau')\tau\theta(\tau' - \tau) \\
&= \frac{1}{2}(\tau + \tau') - \frac{1}{2}|\tau - \tau'| - \tau\tau'
\end{aligned} \tag{1.41}$$

where $\theta(\tau)$ is the usual Heaviside step function and $\tau_>$ ($\tau_<$) is the bigger (lesser) between τ and τ'. All the other average values can be readily obtained thanks to the Wick theorem ([33]), which says that any n-point correlation function of the form $\langle \phi^{i_1}(\tau_1) \dots \phi^{i_n}(\tau_n) \rangle$ is either 0, if n is odd, or, if n is even, can be expressed as a sum of products of 2-points correlation functions, that is

$$\left\langle \phi^{i_1}(\tau_1) \dots \phi^{i_n}(\tau_n) \right\rangle = \left\langle \phi^{i_1}(\tau_1) \phi^{i_2}(\tau_2) \right\rangle \dots \left\langle \phi^{i_{n-1}}(\tau_{n-1}) \phi^{i_n}(\tau_n) \right\rangle \\ + all\ possible\ products\ of\ different\ couples \tag{1.42}$$

where we have, as an example, that the 4-point correlation function is given by (we here omit the dependence on time for notational simplicity)

$$\left\langle \phi^i \phi^j \phi^k \phi^l \right\rangle = \left\langle \phi^i \phi^j \right\rangle \left\langle \phi^k \phi^l \right\rangle + \left\langle \phi^i \phi^k \right\rangle \left\langle \phi^j \phi^l \right\rangle + \left\langle \phi^i \phi^l \right\rangle \left\langle \phi^j \phi^k \right\rangle \tag{1.43}$$

Then, the path integral can be manipulated as follows

$$\begin{aligned} \int_{x(0)=x_i}^{x(1)=x_f} Dx\ e^{iS[x]} &= \int_{x(0)=x_i}^{x(1)=x_f} Dx\ e^{i(S_2[x]+S_{int}[x])} = \\ &= e^{iS_2[x_{cl}]} \int_{x(0)=x_i}^{x(1)=x_f} D\phi\ e^{iS_{int}[x_{cl}+\phi]} e^{iS_2[\phi]} = \\ &= A e^{iS_2[x_{cl}]} \left\langle e^{iS_{int}[x_{cl}+\phi]} \right\rangle = \\ &= \frac{1}{(2\pi i T)^{\frac{D}{2}}} e^{i\frac{(x_f-x_i)^2}{2T}} \left\langle \sum_{n=0}^{\infty} \frac{1}{n!} (iS_{int}[x_{cl}+\phi])^n \right\rangle \end{aligned} \tag{1.44}$$

where it is clear that the expansion of the exponential of the interaction part in the last line generates the perturbative expansion. All that's left now is to compute the various terms appearing in the average value of the last equality. Since the average value of a sum is the sum of average values we have that the first term appearing in the sum is trivial, i.e.

$$\langle 1 \rangle = 1 \tag{1.45}$$

The next term we have to consider is $\langle iS_{int}[x_{cl}+\phi] \rangle$. We can then Taylor expand the potential around the initial point x_i^j

$$
\begin{aligned}
S_{int}[x_{cl}+\phi] &= -T\int_0^1 d\tau V(x_{cl}+\phi) \\
&= -T\int_0^1 d\tau\Big(V(x_i)+[(x_f^j-x_i^j)\tau+\phi^j(\tau)]\partial_j V(x_i) \\
&+[(x_f^j-x_i^j)\tau+\phi^j(\tau)][(x_f^k-x_i^k)\tau+\phi^k(\tau)]\frac{1}{2}\partial_j\partial_k V(x_i)+\dots\Big)
\end{aligned} \tag{1.46}
$$

from which, calling $z^i = x_f^i - x_0^i$, one obtains

$$
\begin{aligned}
\langle iS_{int}[x_{cl}+\phi]\rangle &= -iTV(x_i)-\frac{iT}{2}z^i\partial_i V(x_i)-\frac{iT}{6}z^i z^j\partial_i\partial_j V(x_i) \\
&-\frac{iT}{2}\partial_i\partial_j V(x_i)\int_0^1 d\tau\ \big\langle\phi^i(\tau)\phi^j(\tau)\big\rangle+\dots
\end{aligned} \tag{1.47}
$$

The last term can be computed using equations (1.37) and (1.41)

$$
\begin{aligned}
\int_0^1 \big\langle\phi^i(\tau)\phi^j(\tau)\big\rangle d\tau &= iT\delta^{ij}\int_0^1 g(\tau,\tau)d\tau \\
&= iT\delta^{ij}\int_0^1 \tau(1-\tau)d\tau = \frac{1}{6}iT\delta^{ij}
\end{aligned} \tag{1.48}
$$

so that

$$
\begin{aligned}
\langle iS_{int}[x_{cl}+\phi]\rangle &= -iTV(x_i)-\frac{iT}{2}z^i\partial_i V(x_i)-\frac{iT}{6}z^i z^j\partial_i\partial_j V(x_i) \\
&+\frac{T^2}{12}\nabla^2 V(x_i)
\end{aligned} \tag{1.49}
$$

in which ∇^2 is the Laplacian operator $\partial_i\partial^i = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x_i}$. In the same way one can compute the following terms finding, at the lowest order

$$
\left\langle \frac{1}{2}S_{int}^2[x_{cl}+\phi]\right\rangle = +\frac{T^2}{2}V^2(x)+\dots \tag{1.50}
$$

Collecting the terms altogether one finds

$$K(x_f,x_i,T)=\frac{1}{(2\pi iT)^{\frac{D}{2}}}e^{i\frac{(x_f-x_i)^2}{2T}}\Bigg[1-iTV(x_i)-\frac{iT}{2}z^i\partial_iV(x_i)-\frac{iT}{6}z^iz^j\partial_i\partial_jV(x_i)$$
$$+\frac{(iT)^2}{2}V^2(x_i)-\frac{(iT)^2}{12}\nabla^2V(x_i)+\dots\Bigg] \quad (1.51)$$

from which one can read off the firsts Seeley-DeWitt coefficients

$$a_0(x_i,x_f)=1$$

$$a_1(x_i,x_f)=-V(x_i)-\frac{1}{2}z^i\partial_iV(x_i)-\frac{1}{6}z^iz^j\partial_i\partial_jV(x_i)+\dots$$

$$a_2(x_i,x_f)=\frac{1}{2}V^2(x_i)-\frac{1}{12}\nabla^2V(x_i)+\dots \quad (1.52)$$

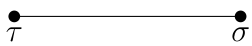
and their value for coinciding points

$$a_0(x_i,x_i)=1$$

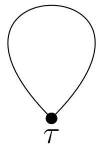
$$a_1(x_i,x_i)=-V(x_i)$$

$$a_2(x_i,x_i)=\frac{1}{2}V^2(x_i)-\frac{1}{12}\nabla^2V(x_i) \quad (1.53)$$

Terms with quantum averages obtained in this way can be visualized as a set of Feynman diagrams, where vertices are denoted by dots, the basic propagator $\langle\phi^i(\tau)\phi^j(\sigma)\rangle$ is the free 2-points correlation function and can be visualized as a line connecting the worldline points τ and σ



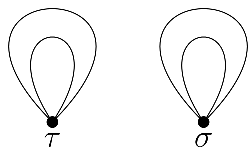
and loop graphs are of course obtained from same time contractions of propagators $(\langle\phi^i(\tau)\phi^j(\tau)\rangle)$

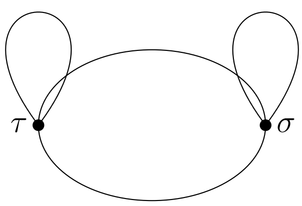


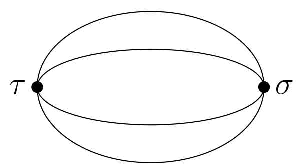
In order to make it more clear we can consider an explicit example: suppose we wish to calculate the mean value of $\langle\phi^4(\tau)\phi^4(\sigma)\rangle$. We have two vertices since this mean value actually depends on only two worldline points (τ and σ), and four lines exiting from each vertices which, in accordance with the Wick theorem, have to be joined in pairs in every possible way. Every time a line from τ is connected with a line exiting from σ we have a basic 2-point propagator, while every time two lines exiting from the same vertex are joined together we have a loop. The number of ways in which one can join different lines together and obtain the same graph is called the multiplicity of the graph. Taking all this into consideration we find at the end, in terms of Feynman diagrams

$$\left\langle\phi^4(\tau)\phi^4(\sigma)\right\rangle = 9D_1 + 72D_2 + 24D_3 \tag{1.54}$$

where

$D_1 =$  (1.55)

$D_2 =$  (1.56)

$D_3 =$  (1.57)

1.4 The Heat Kernel and the Worldline Formalism

Before going on with the dissertation, a few words on the Heat Kernel are useful. Let's start giving a more mathematical definition of the matrix element introduced above. So, let x and x' be two distinct points on an D-dimensional differentiable manifold $\mathscr{M}$, and T a real variable that ranges from 0 to ∞. Then the Heat Kernel is the unique solution of the Cauchy problem given by the Schrödinger equation

$$i\frac{\partial}{\partial T}K(x,x',T) = \hat{H}_x K(x,x',T) \tag{1.58}$$

with the boundary condition

$$\lim_{t\to 0^+} K(x,x',T) = \delta^D(x,x') \tag{1.59}$$

where $\delta^D(x,x') = g^{-\frac{1}{4}}(x)\delta^D(x-x')g^{-\frac{1}{4}}(x')$ is the scalar D-dimensional Dirac delta function on curved spaces.

The uniqueness of the solution to equation (1.58) is granted by the Minakshisundaram-Pleijel theorem which ensures that, at least when the underling geometrical manifold $\mathscr{M}$ is a Riemannian manifold without boundaries, only one smooth function K that matches with the previous definitions exists [6].

We can now show that, within the worldline approach, one can relate a quantum mechanical path integral with a QFT one, and how in this approach the Heat Kernel for a spinless non-relativistic particle is related with the Green function of a relativistic quantum field of the same kind. We start considering a massive scalar quantum field $\phi(x)$, defined on a general space-time manifold $\mathscr{M}$ described by the spacetime metric $g_{\mu\nu}$ $(\mu,\nu = 0,\dots,D)$. This field then satisfies the Klein-Gordon equation

$$(-\Box + m^2 + V(x))\phi(x) = \hat{H}_{KG}\phi(x) = 0 \tag{1.60}$$

where $\Box$ is the covariant D'Alambertian operator $g^{\mu\nu}\nabla_\mu\nabla_\nu$ and $V(x)$ parametrises additional couplings to external fields. The Green function, or propagator, for the scalar field ϕ on coordinate space will then satisfy

[6] See for example [10] or [28]

$$(-\Box_x + m^2 + V(x))G(x,x') = \delta^D(x,x') \tag{1.61}$$

We recall that the defining propriety of a Green function is that it verifies, in the DeWitt-Schwinger matrix-like notation [15],

$$\hat{F}\hat{G} = -\mathbb{1} \tag{1.62}$$

with $\hat{F}$ being a generic differential operator that acts on scalar functions defined on $\mathscr{M}$. We can then write

$$\hat{G} = (\hat{F})^{-1} = -\frac{1}{\hat{F}} \tag{1.63}$$

If the operator $\hat{F}$ is self-adjoint we can replace this last equation with a formal complex Laplace transform, that is[7]

$$\hat{G} = -\frac{1}{\hat{F}} = i\int_0^\infty ds\ e^{-i\hat{F}s} \tag{1.64}$$

in which s is interpreted as a fictitious proper time parameter [8] (or some equivalent affine parameter) related to the propagation of a scalar virtual particle.
The dependence on spacetime points x, x' is then obtained taking the expectation value of the operator $\hat{G}$ between the eigenstates of position operator $\langle x|$ and $|x'\rangle$

$$G(x,x') = \langle x'|\hat{G}|x\rangle = i\int_0^\infty \langle x|e^{-i\hat{F}s}|x'\rangle\, ds \tag{1.65}$$

If we now take the case of our interest, that is $\hat{F} = \hat{H}$, we can see in an heuristic way that the quantity

$$K(x,x',s) = \langle x'|e^{-i\hat{H}s}|x\rangle \tag{1.66}$$

is precisely the same quantity defined in (1.8). Then one is formally allowed to use

[7] See [12],[15]

[8] Also called worldine proper time, or proper time on the worldline since it parametrises the path of a point-like particle.

the path integral methods for the Heat Kernel evaluation we explained in the previous sections, also when one is considering a QFT theory and not a quantum mechanical one. Thous we can write

$$G(x,x') = i \int_0^{\infty} ds \int Dx \; e^{iS_{KG}[x,a,b,c]} \tag{1.67}$$

where S_{KG} is the action functional one obtains starting from the Hamiltonian given by equation (1.58), and a, b, c are the so-called "ghost fields" which are needed in order to exponentiate the non-trivial measure of the path integral, as described in a detailed way in the next chapter. The action can be decomposed as $S_{KG}[x,a,b,c] = S_{KG}[x] + S_{KG}[a,b,c]$ and its part which depends only on the variable x is given by

$$S_{KG}[x] = \frac{1}{2} \int_{\tau}^{\tau'} d\tilde{\tau} \; \big(\dot{x}^2 - V(x)\big) \tag{1.68}$$

Notice here the overall $\frac{1}{2}$ factor, which makes to scale the results for the Seeley-DeWitt coefficients we obtained in the previous section. Setting $V \to \frac{1}{2}V$ into the equations for a_1 and a_2 will recover the correct coefficients for the relativistic case. A detailed review about the worldline approach for scalar particles can be found in [4].

We said heuristic since if one enters into the defining details of the theory, things are a little different. For example, when we consider a quantum field theory the position eigenstates are no more eigenstates of the space position operator $\hat{x}^i$, instead they are now eigenstates of the spacetime position operator $\hat{x}^\mu$. For the same reason the space metric g^{ij} appearing in the action functional will be replaced with the full spacetime metric $g^{\mu\nu}$ (for example the flat-space metric δ^{ij} in equation (1.36) would be replaced with the flat-spacetime Minkowskian metric $\eta^{\mu\nu}$). Anyway it is immediate to see that the Heat Kernel defined in equation (1.65) verifies the Schrödinger equation (1.58) with $\hat{H} = \hat{H}_{KG}$ and the boundary equation (1.59), since $\langle x|x'\rangle = \delta^D(x,x')$ as defined in (2.35). This is sufficient to say that $K(x,x',s)$ is indeed the Heat Kernel defined above, since Minakshisundaram-Pleijel's theorem assures that the solution to the system of equation given by (1.58) and (1.59) is unique.
We have shown how the path integral formulation for non-relativistic spinless particle can be related to the one for a relativistic scalar field. This is the so-called worldline formalism. This relation can be extended to include also fields of spin $N/2$ with the

insertion on N extra supersymmetric conserved charges [9]: with this extension, one is formally allowed to use the worldline approach for a quantum field theory describing particles of general spin.
As a last remark we note that the Heat Kernel can also be expressed in the Euclidean formulation, in which, when one is considering QFT theories, the space-time metric has a definite positive signature. In fact we can analytically continue time to reach imaginary values, by means of what is called a "Wick rotation", that is the transformation $T \to -i\beta$, where β is the total euclidean time of propagation. Under this transformation we can see that the transition amplitude in euclidean time obeys a generalized diffusion equation

$$-\frac{\partial}{\partial\beta}K(x,x',\beta) = \hat{H}_x K(x,x',\beta) \tag{1.69}$$

and, since the first diffusing substance to be studied was historically heat, this is the origin of its name.
In order to make the notation more clear, we will indicate the proper time variable as T in the non-relativistic case, s in the relativistic case and β will always be the Euclidean time of propagation.

[9]The details will not be explained in the present text since we are interested only in scalar particles: the important thing we wish to remark is that an extension in order to include particles of different spin can be done in a standard way, as one can see in references [7], [20], [25]

Chapter 2

Path Integrals on Curved Spaces

Up until now, we only treated path integrals for particles moving on a flat manifold. When one moves into considering a curved background space, things becomes harder. In this chapter we will make the subtleties due to curved space clear step by step. We will begin with the study of the motion of a single spinless particle on a curved underlying manifold, and we will show how ambiguities, due to the non-vanishing background curvature, arise in both the usual operatorial formalism and the path integral formalism, and how they can be dealt with. We will see that curved-space path integrals require a lot of extra work when one is interested in an explicit computation of the heat kernel, like the introduction of extra degree of freedom (the ghost fields) : this is the reason that pushed us towards the study of a method in which one can use results obtained for flat spaces also when a curved background manifold is implemented, and it will be done in the final chapter. In the present chapter we will show how the time-slicing regularization give rise to a precise mapping between the operatorial and the path integral ambiguities, which is not possible with other regularization schemes. Other known regularization procedures will then be analyzed, and for each one of them we will give a sketch of the calculations that one has to make in order to obtain the correct counterterm required by the regularization, once that a renormalization condition has been chosen. We will end this chapter presenting the original method, discovered by B. S. DeWitt ([15]), that allows one to obtain the coefficients of the heat kernel expansion on curved space without the aid of path integration.

2.1 Motion of a Single Particle on Curved Space

Let's start considering the infinitesimal invariant line element for a free particle of unit mass (we set in this chapter the non-relativistic mass parameter $m = 1$ if not explicitly inserted; the dependence on mass is anyway trivial and is the same for flat spaces and curved ones) moving in a curved background D-dimensional manifold: it is given by

$$ds^2 = g_{ij} dx^i dx^j \tag{2.1}$$

where g_{ij} is the full metric tensor in some coordinate frame. Without the presence of a non-gravitational background potential the free Lagrangian contains only the kinetic term, given by

$$L(x, \dot{x}) = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j \tag{2.2}$$

where we interpret the dot as a derivative with respect of proper time (or some other affine parameter) s. The action functional is then given by

$$S[x, \dot{x}] = \int ds \; L(x(s), \dot{x}(s)) \tag{2.3}$$

If we now use Eulero-Lagrange equations we obtain an equation of motion given by

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0 \tag{2.4}$$

where the object "Γ" is the metric connection that is necessary to define a covariant derivative ∇_i so that coordinate transformation laws for tensors still hold also for their derivatives (i.e. the covariant derivative of a tensor is still a tensor). Its action on vector fields V^j and W_j is defined by the expressions

$$\nabla_i V^j = \frac{\partial V^j}{\partial x^i} + \Gamma^j_{ik} V^k \quad , \quad \nabla_i W_k = \frac{\partial W_j}{\partial x^i} - \Gamma^k_{ij} W_k \tag{2.5}$$

and, more generally, its action on tensors is the same adding a connection for each index with a + sign if it is a contravariant index, or a - sign if it is covariant. Later in this text we will employ the notation where a covariant derivative is indicated as an index which follows a ";" sign: to make it more clear [1]

[1]For the sake of simplicity we have taken a scalar quantity as an example; it is clear that the same is

$$\nabla_i A = A_{;i} \tag{2.6}$$

and when repeated derivations are implemented

$$\nabla_{i_1} \dots \nabla_{i_n} A = A_{;i_n \dots i_1} \tag{2.7}$$

Equation (2.4) is also know as the geodesic equation, since its solution parametrizes the shortest path between two points in terms of the arc lenght s (or, as stated before, of some affine parameter proportional to s). Defining a covariant derivative in the usual way of differential geometry along the path parametrized by s we can rewrite equation (2.4) as

$$\frac{D\dot{x}^i}{Ds} = 0 \tag{2.8}$$

and its interpretation being that the tangent vector remains covariantly constant along the geodesic path of the particle.
Before proceeding further lets make some notation clear. We will always take the metric connection to be the "Levi-Civita" connection, that is equivalent to the requirement of having a torsionless background (see [34]). It is symmetric in its lower indices, $\Gamma^i_{jk} = \Gamma^i_{kj}$ and is expressed in terms of the metric tensor and its derivatives as

$$\Gamma^i_{jk} = \frac{1}{2} g^{im} \Big(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m} \Big) \tag{2.9}$$

We will take the Riemann tensor as defined by the relation

$$[\nabla_i, \nabla_j] V_k = V_l R_k{}^l{}_{ij} \qquad , \qquad [\nabla_i, \nabla_j] V^k = V^l R^k{}_{lij} \tag{2.10}$$

where [,] denotes the usual commutator between operators $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$. Contractions of this tensor define the Ricci tensor and the scalar curvature as

$$R_{ij} = R^k{}_{ikj} \quad , \quad R = g^{ij} R_{ij} \tag{2.11}$$

We briefly recall the properties of the Riemann tensor [2] since we will use them later :

valid for any tensorial quantity like for example $\nabla_i A^j{}_k = A^j{}_{k;i}$ and so on.

[2]For detailed explanations and proofs see [34]

it is antisymmetric in the last two indices

$$R^i{}_{jkl} = -R^i{}_{jlk} \tag{2.12}$$

and verifies the two Bianchi identities

$$R^i{}_{jkl} + R^i{}_{ljk} + R^i{}_{klj} = 0 \tag{2.13}$$

$$\nabla_m R^i{}_{jkl} + \nabla_l R^i{}_{jmk} + \nabla_k R^i{}_{jlm} = 0 \tag{2.14}$$

By lowering an index with the metric one finds that antisymmetry is verified also for the first two indices

$$R_{ijkl} = -R_{jikl} \tag{2.15}$$

and symmetry under exchange of the two couple of indices

$$R_{ijkl} = R_{klij} \tag{2.16}$$

where we can express this last tensor in terms of the metric tensor and its derivatives as

$$R_{ijkl} = -\frac{1}{2}(\partial_l \partial_j g_{ik} - \partial_l \partial_i g_{jk} - \partial_k \partial_j g_{il} + \partial_k \partial_i g_{jl}) - g_{mn}[\Gamma^m_{ki}\Gamma^n_{jl} - \Gamma^m_{il}\Gamma^n_{jk}] \tag{2.17}$$

2.2 Hamiltonian Formalism and Curved-Space Ambiguities

In the previous section we have derived the equation of motion on curved space within the Lagrangian formalism. We can do the same in the Hamiltonian formalism, and show how ambiguities arise in the operatorial formulation. This will be used in the next section to compute a map between these operatorial ambiguities and the path integral ones, thanks to the Time Slicing regularization procedure. We start defining the Hamiltonian function as the Legendre transform of the Lagrangian of equation (2.2)

$$H(x,p) = p_i\dot{x}^i - L(x,\dot{x}) = \frac{1}{2}g^{ij}p_ip_j \tag{2.18}$$

where the momenta p_i are defined as

$$p_i \equiv \frac{\partial L}{\partial \dot{x}^i} = g_{ij}\dot{x}^j \tag{2.19}$$

Then the equations of motion are given by

$$\dot{x}^i = \{x^i, H\}_P = g^{ij}p_j \tag{2.20}$$

$$\dot{p}_i = \{p_i, H\}_P = -\frac{1}{2}(\partial_i g^{kl}p_kp_l) \tag{2.21}$$

where $\{,\}_P$ denotes Poisson brackets. The usual quantization is then accomplished reinterpreting the phase-space coordinates (x^i, p_i) as operators $(\hat{x}^i, \hat{p}_i)$ that acts on vectors defined on a suitable Hilbert space. Their action is readily obtained if one defines the commutator between these two operators as taking a value that corresponds to i times the usual Poisson bracket

$$\{x^i, p_j\}_P = \delta^i{}_j \longrightarrow [\hat{x}^i, \hat{p}_j] = i\delta^i{}_j \tag{2.22}$$

which is just canonical quantization.
Implementing the usual coordinate representation for the $\hat{p}_i$'s in which the $\hat{x}^i$ operator acts simply as a multiplicative operator $\hat{x}^i\phi(x) = x^i\phi(x)$ we immediately obtain

$$\hat{p}_i = -i\partial_i \tag{2.23}$$

The same commutation relations are also verified with a different choice of the momentum operator, which is given by

$$\hat{p}_i = -ig^{-\frac{1}{4}}\partial_i g^{\frac{1}{4}} \tag{2.24}$$

where $g = \det g_{ij}$. We actually prefer to work with this last representation since it is the same expression one gets when evaluating the expectation value of the operator $\hat{p}_i$ using the scalar product defined in equation (2.36)

$$
\begin{aligned}
\langle x|\hat{p}_i|x'\rangle &= \int \langle x|p\rangle \langle p|\hat{p}_i|x'\rangle \frac{d^D p}{(2\pi)^D} \\
&= \int \frac{e^{ip_i(x^i - x'^i)}}{g^{\frac{1}{4}}(x) g^{\frac{1}{4}}(x')} \frac{d^D p}{(2\pi)^D} \\
&= -ig^{-\frac{1}{4}}(x) g^{-\frac{1}{4}}(x') \frac{\partial}{\partial x^i} \delta^D(x - x') \\
&= -ig^{-\frac{1}{4}}(x) g^{-\frac{1}{4}}(x') \frac{\partial}{\partial x^i} g^{\frac{1}{4}}(x) g^{\frac{1}{4}}(x') \langle x|x'\rangle \\
&= -ig^{-\frac{1}{4}}(x) \frac{\partial}{\partial x^i} g^{\frac{1}{4}}(x) \langle x|x'\rangle
\end{aligned} \tag{2.25}
$$

where also equation (2.35) has been used. It is also possible to show that this choice of the momentum operator ensures the heat kernel on curved spaces to be a scalar under general coordinate transformations.

We can now see where ambiguities come from: trying to quantize the classical function

$$
H_{cl} = \frac{1}{2} g^{ij}(x) p_i p_j \tag{2.26}
$$

one faces the problem of ordering the operators since, classically, $g^{ij}(x)p_i p_j = p_i g^{ij}(x) p_j = p_i p_j g^{ij}(x)$. This is no longer true when we promote phase-space coordinates to operators, since the metric $g^{ij}(x)$ depends on the operator $\hat{x}$ which do not commute with momenta operators, as it can be seen in (2.22). A different choice on the operatorial ordering then produces different quantum Hamiltonians, leading to different quantum theories, which all reduce to the same classical theory when one takes the classical limit. The requirement of invariance of $\hat{H}$ under diffeomorphisms of the manifold (so that it describes the motion of a particle in an arbitrary coordinate frame) fixes, up to a term proportion to the curvature R, the quantum Hamiltonian to be

$$
\hat{H} = \frac{1}{2} g^{-\frac{1}{4}} \hat{p}_i g^{\frac{1}{2}} g^{ij} \hat{p}_j g^{-\frac{1}{4}} + \frac{1}{2}\xi R = -\frac{1}{2}\nabla^2 + \frac{1}{2}\xi R \tag{2.27}
$$

in which $\nabla^2 = g^{ij}\nabla_i \nabla_j$ is the covariant Laplacian operator on curved space and ξ is a non-minimal [3] dimensionless free coupling parameter to the background curvature. This term encodes all the remaining ordering ambiguities once that general coordinate

[3]Non-minimal in a sense that, explicitly reinserting the Plank constant, it is a coupling of order $\hbar^2$. The minimal coupling is then obtained setting $\xi = 0$.

invariance is required, and it arises because it is the only scalar that can be built off by at most two derivatives of the metric, and being a scalar is completely invariant under the action of a diffeomorphism. It is also related to conformal transformations of the background metric, as can be seen in the text of Birrel and Davies [11], and setting its value to $\xi = (D-2)/4(D-1)$ will make the theory conformally invariant in the massless case.

We have seen the emergence of ambiguities due to curved space[4] in the usual operatorial formulation. In the path-integral approach similar ambiguities appear. They take the form of ambiguous Feynman diagrams, which, as we stressed before, correspond to integrations of various propagators joined at vertices with some factors or derivatives. The propagators are then usually expressed as distributions whose product is in many cases ill-defined: this is in fact the reason why these ambiguities arise in the path integral formulation. These ambiguities must then be defined by some regulation procedure which makes these products well-defined, and an associated renormalization condition must be chosen in order to specify which quantum theory one is constructing from the classical one, which in the path integral approach is identified by the appearance of a precise counterterm V_{ct} that depends on the chosen regularization. These counterterms, unlike standard QFT counterterms, are finite by nature, since quantum mechanical path integral theories are super-renormalizable, as shown with the aid of power counting technique in [9].

An example in which one can see a precise mapping between the operatorial ordering ambiguity and the path integral one is the time-slicing regularization procedure, which will be explained in detail in the next section. Others known regularization procedures are mode regularization and dimensional regularization, which will be shown later in this chapter. These regularization procedures have been understood and developed in many years with the contributions of several famous physicists like B. and C. DeWitt, L. S. Schulman, I. G. Avramidi and many others, but one essentially has to rely on the work of Bastianelli and Van Nieuwenhuizen[5] to have a complete and coherent understanding

[4]Actually ambiguities arises because of the non-reversibility of the classical limit, but curved space renders these ambiguities much more severe since in non-flat spaces the metric is a local function of coordinates. As an example, take equation (1.15) in chapter 1. We have taken $H(x_{k-1}, p_k)$ to be the discretized version of $H(x,p)$. But one could in principle take that to be $H(x_k, p_k)$ or $H\big(\frac{1}{2}(x_{k-1}+x_k), p_k\big)$. At the end it makes no difference since the obtained quantum theory is still the same: this is not yet true if one works in a non-flat background.

[5][9]

of the various regularization procedures. We will essentially follow their work in the next few sections.
Another regularization scheme, which requires no counterterm and which is based on Riemann expansion, has been proposed by Guven [6], and the study of this “new” [7] proposal is the aim of the original part of this thesis. This will be explicitly done in chapter 3.

2.3 Time-Slicing Regularization

The aim of the present paragraph is to obtain a precise mapping between operatorial ambiguities and path integral ones, at least in the framework of time-slicing regularization. The time-slicing procedure is essentially a generalization of what we have done in chapter 1 while defining the path integral from usual quantum mechanics on flat space, splitting the transition amplitude into N identical terms and inserting completeness relations in between everytime. The heat kernel equation (1.58) still holds in curved space with the replacement of the flat Hamiltonian operator for a single particle $\hat{H} = -\frac{1}{2m}\delta^{ij}\partial_i\partial_j + V(\hat{x})$ with the one we obtained in the last section

$$\hat{H} = -\frac{1}{2m}\nabla^2 + V(\hat{x}) + \frac{1}{2}\xi R(\hat{x}) \tag{2.28}$$

We stress that, since quantum mechanical path integrals can be seen as a (0+1)-dimensional quantum field theory, this corresponds to the first quantization of a scalar field $\phi(x)$ that obeys the Klein-Gordon equation

$$(-\Box + m^2 + 2V(x) + \xi R(x))\phi(x) = 0 \tag{2.29}$$

and the same results we will obtain in this and the following sections can be naively obtained substituting the spatial metric g_{ij} with the space-time metric $g_{\mu\nu}$ and integrals over time with integrals over a proper time parameter. That said, when moving in considering curved spaces another generalization has to be made, that is the replacement of the usual coordinate-space measure $d^D x$ with

[6] [24]
[7] New here means that it has never been used for explicit calculations, since this proposal was made for the first time in 1988.

$$d^D x \longrightarrow \sqrt{g(x)} d^D x \tag{2.30}$$

This replacement makes the measure covariant under general coordinate transformations, since the usual measure dx^i is a vector density of weigh $-\frac{1}{2}$. This last replacement has various implications since integrations over the space-time coordinates are used to define scalar products and so on. For example, the defining relation of the delta function

$$\int \delta^D(x-y) f(y) d^D y = f(x) \tag{2.31}$$

for an arbitrary functional $f(x)$ becomes in curved space

$$\int \delta^D(x,y) f(y) \sqrt{g(y)} d^D y = f(x) \tag{2.32}$$

meaning that the scalar delta function on curved space, indicated as $\delta^D(x,y)$, is related to the usual flat delta function by

$$\delta^D(x,y) = \frac{\delta^D(x-y)}{\sqrt{g(x)}} = \frac{\delta^D(x-y)}{\sqrt{g(y)}} = \frac{\delta^D(x-y)}{g^{\frac{1}{4}}(x) g^{\frac{1}{4}}(y)} \tag{2.33}$$

Since now the completeness relation of equation (1.5) also acquires an extra factor of $\sqrt{g}$ [8] , that is

$$\mathbb{1} = \int d^D x \, |x\rangle \, \sqrt{g(x)} \, \langle x| \tag{2.34}$$

the normalization between position eigenstates becomes

$$\langle x|y\rangle = \frac{\delta^D(x-y)}{\sqrt{g(x)}} = \delta^D(x,y) \tag{2.35}$$

so that wave functions $\psi(x) = \langle x|\psi\rangle$ are scalars under general coordinate transformation, since we also recover the usual scalar product for scalar functions, namely

$$\langle \phi|\psi\rangle = \int \langle \phi|x\rangle \, \sqrt{g(x)} \, \langle x|\psi\rangle \, d^D x = \int \phi^*(x)\psi(x)\sqrt{g(x)} d^D x \tag{2.36}$$

where identity (2.34) has been used for the first equality. Also the inner product be-

[8] Notice that this is not valid for momentum completeness relation since momentum space is formally flat also when the background space-time is not.

tween position and momentum eigenstates will be slightly modified, gaining an additional factor of $g^{-1/4}(x)$

$$\langle x|p\rangle = \frac{e^{ip_i x^i}}{g^{1/4}(x)} \tag{2.37}$$

As a check, note that in that way one has $\int \langle p|x\rangle \sqrt{g(x)}\, \langle x|p'\rangle = \delta^D(p-p')$, which is in agreement with the completeness relation (1.6). Actually one could in principle choose different normalization conditions, but our choice assures that the Heat Kernel $K(x,y,T)$, as well as the the propagator $G(x,y)$, are indeed bi-scalar [9] quantities. This can be seen also in the boundary condition (1.59), where now the flat delta function has to be replaced with the scalar delta function of (2.35).
Now we can insert $(N-1)$ times the new completeness relation (2.34) and N times the relation (1.6) into the Heat Kernel and follow what we have done in chapter 1, obtaining for the phase-space path integral

$$K(x_f, x_i, T) = \int \Big(\prod_{k=1}^{N-1} d^D x_k \sqrt{g(x_k)}\Big)\Big(\prod_{k=1}^{N} \frac{d^D p_k}{(2\pi)^D}\Big) \prod_{k=1}^{N} \langle x_k|p_k\rangle \, \langle p_k|e^{i\epsilon \hat{H}}|x_{k-1}\rangle \tag{2.38}$$

We empathized before that the operatorial ordering gives rise to different quantum Hamiltonians, corresponding to different quantum theories. We stated that we would take in consideration an Hamiltonian whose kinetic term ordering would be the one given in eq. (2.27), but now we will rewrite that operator in terms of the Weyl-ordered operator $\hat{H}_W$. The Weyl-ordering of a quantum operator is the one that makes it manifestly symmetric, so that, for example, the operator $\hat{x}\hat{p}$ is rewritten, using the commutation relation given by (2.22), as $(\hat{x}\hat{p})_W = \frac{1}{2}(\hat{x}\hat{p}+\hat{p}\hat{x}+i)$. Similarly, the operator $\hat{x}^2\hat{p}$ is rewritten as $(\hat{x}^2\hat{p})_W = \frac{1}{3}(\hat{x}^2\hat{p} + \hat{x}\hat{p}\hat{x} + \hat{p}\hat{x}^2) + i\hat{x}$. In general, the Weyl-ordered form of an arbitrary operator is obtained explicitating its expansion in term of the phase-space coordinate operators $(\hat{x}^i, \hat{p}_i)$ and then rewriting it in a completely symmetric expression in $\hat{x}$ and $\hat{p}$, using the general formula for $m, n \in \mathbb{N}$

$$(\hat{x}^m \hat{p}^n)_S = \frac{1}{2^m} \sum_{l=0}^{m} \binom{m}{l} x^{m-l} p^n x^l = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} p^{n-k} x^m p^k \tag{2.39}$$

[9] With the term “bi-scalar” we indicate a quantity that depends on two space-time points and that is a scalar under reparametrization of both of its arguments.

(a derivation of this expression can be found in [9]) and making use of their commutation relation. This can be clarified by the formula

$$\hat{O} = \hat{O}_S + further\ terms = \hat{O}_W \tag{2.40}$$

where we emphasized the fact that the Weyl-ordered operator is the same operator we had initially, only rewritten in a symmetrised fashion[10]. The *further terms* appearing are obviously due to the commutation relations one has to use to symmetrize the original operator[11], and in the previous examples are given by the terms $\frac{1}{2}i$ and $i\hat{x}$ respectively. As a last thing, we note that $(\hat{x}\hat{p})_S$ is obviously equal to $(\hat{p}\hat{x})_S$, but the opposite is true for their Weyl-ordered form, that is $(\hat{x}\hat{p})_W \neq (\hat{p}\hat{x})_W$.

The importance of the Wyel-ordering is that, given a generic operator $\hat{O}(\hat{x}, \hat{p})$, we can write

$$\langle x_k | \hat{O}(\hat{x}, \hat{p}) | x_{k-1} \rangle = \int \frac{d^D p}{(2\pi)^D} \langle x_k | p \rangle \langle p | x_{k-1} \rangle O_W\Big(\frac{1}{2}(x_k + x_{k-1}), p\Big) \tag{2.41}$$

giving automatically rise to a sort of "midpoint-prescription rule". One then interprets $O_W\Big(\frac{1}{2}(x_k + x_{k-1}), p_k\Big)$ as the discretized version of the continuous function $O_W(x(t), p(t))$. This last expression can be readily proven considering that $O_S(\hat{x}, \hat{p})$ can be expressed as a sum over m, n of terms $(\hat{x}^m \hat{p}^n)_S$ plus terms which contains only one or none of the operators $\hat{x}^i, \hat{p}_i$, so that for these terms the ordering doesn't matter. In that way one has, using one time identity (1.6)

[10] To give an exact definition, we will say that an operator $\hat{O}(\hat{x}, \hat{p})$ is in a symmetrised form if all operators $\hat{x}^i$ and $\hat{p}_i$ appear in all possible ordering with equal weighs

[11] If in the *further terms* both $\hat{x}$ and $\hat{p}$ operators are present, they can also be rewritten in their symmetrised form plus other further terms until only one or none of these operators appear.

$$
\begin{aligned}
\langle z|(\hat{x}^m\hat{p}^n)_S|y\rangle &= \int \frac{d^Dp}{(2\pi)^D}\,\langle z|p\rangle\,\langle p|(\hat{x}^m\hat{p}^n)_S|y\rangle \\
&= \int \frac{d^Dp}{(2\pi)^D}\,\langle z|\frac{1}{2^m}\sum_{l=0}^{m}\binom{m}{l}\hat{x}^{m-l}\hat{p}^n|p\rangle\,\langle p|\hat{x}^l|y\rangle \\
&= \int \frac{d^Dp}{(2\pi)^D}\,\langle z|p\rangle\, z^{m-l}y^l p^n\,\langle p|y\rangle \\
&= \int \frac{d^Dp}{(2\pi)^D}\,\langle x|p\rangle\left(\frac{z+y}{2}\right)^m p^n\,\langle p|y\rangle
\end{aligned} \tag{2.42}
$$

which is exactly expression (2.41). The same expression is of course true if we take $\hat{O}$ to be the Hamiltonian operator. So, we can make use of this identity to rewrite eq. (2.38) as

$$
\begin{aligned}
K(x_f,x_i,T) = \int \Big(\prod_{k=1}^{N-1} d^Dx_k\sqrt{g(x_k)}\Big)\Big(\prod_{k=1}^{N}\frac{d^Dp_k}{(2\pi)^D}\Big)\prod_{k=1}^{N}\langle x_k|p_k\rangle\,\langle p_k|x_{k-1}\rangle\,\times \\
\times\left(e^{-i\epsilon H}\right)_W(\overline{x}_k,p)
\end{aligned} \tag{2.43}
$$

where the last round bracket indicates the arguments of the function $\left(e^{i\epsilon H}\right)_W$ and $\overline{x}_k = \frac{1}{2}(x_{k-1}+x_k)$. Note that, once we explicitate the scalar product $\langle x_k|p_k\rangle$, terms of $g^{-\frac{1}{4}}(x_k)$ appear which cancel every factor of $\sqrt{g(x_k)}$ that arise from the invariant measure, except for initial and final points so that

$$
K(x_f,x_i,T) = g^{-\frac{1}{4}}(x_i)g^{-\frac{1}{4}}(x_f)\lim_{N\to\infty}\int\Big(\prod_{k=1}^{N-1}d^Dx_k\Big)\Big(\prod_{k=1}^{N}\frac{d^Dp_k}{(2\pi)^D}\Big)e^{i\epsilon\sum_{k=1}^{N}\left[p_k\frac{x_k-x_{k-1}}{\epsilon}-H_W(\overline{x}_k,p_k)\right]} \tag{2.44}
$$

Now we can use the result that, up to terms which are higher in order ϵ, we can replace $\left(e^{-i\epsilon H}\right)_W$ with $e^{-i\epsilon H_W}$ (see [9] for a reference).
All that remains now is to compute $\hat{H}_W$ from $\hat{H} = \frac{1}{2}g^{-\frac{1}{4}}p_i g^{\frac{1}{2}}g^{ij}p_j g^{-\frac{1}{4}}$ in order to obtain the function H_W. In the first place we simply rewrite the operator by moving p_i to the left and p_j to the right. This can be done evaluating

$$
\begin{aligned}
[g^{-\frac{1}{4}}, p_i]\phi &= -ig^{-\frac{1}{2}}\partial_i(g^{\frac{1}{4}}\phi) + ig^{-\frac{1}{4}}\partial_i\phi \\
&= -ig^{-\frac{1}{2}}g^{\frac{1}{4}}\left(\frac{1}{4}\partial_i \ln g\right)\phi - ig^{-\frac{1}{4}}\partial_i\phi + ig^{-\frac{1}{4}}\partial_i\phi \\
&= -\frac{i}{4}g^{-\frac{1}{4}}(\partial_i \ln g)\phi
\end{aligned}
\tag{2.45}
$$

where the coordinate representation of equation (2.24) has been employed in the first line and the identity $g^{-\alpha} = e^{-\alpha \ln g}$ as been used to obtain the second line. The result is then

$$
\begin{aligned}
\hat{H} &= \frac{1}{2}\Big(p_i - \frac{1}{4}i\partial_j \ln g\Big)g^{-\frac{1}{4}}g^{\frac{1}{2}}g^{ij}g^{-\frac{1}{4}}\Big(p_i + \frac{1}{4}i\partial_j \ln g\Big) \\
&= \frac{1}{2}p_i g^{ij} p_j + \frac{1}{8}\partial_i(g^{ij}\partial_j \ln g) + \frac{1}{32}(\partial_i \ln g)g^{ij}(\partial_j \ln g)
\end{aligned}
\tag{2.46}
$$

We can then rewrite the first term in a Weyl-ordered form, that is

$$
\begin{aligned}
\frac{1}{2}p_i g^{ij} p_j &= \frac{1}{8}(p_i p_j g^{ij} + 2p_i g^{ij} p_j + g^{ij} p_i p_j) + \frac{1}{8}p_i[g^{ij}, p_j] + \frac{1}{8}[p_i, g^{ij}]p_j \\
&= \frac{1}{2}(p_i g^{ij} p_j)_S + \frac{1}{8}[p_i, [g^{ij}, p_j]]
\end{aligned}
\tag{2.47}
$$

and, since $[g^{ij}, p_j] = i\partial_j g^{ij}$, we obtain

$$
\begin{aligned}
\hat{H} &= \frac{1}{2}(p_i g^{ij} p_\nu)_S + \frac{1}{8}\Big[\partial_i\partial_j g^{ij} + \partial_i(g^{ij}\partial_j \ln g) + \frac{1}{4}g^{ij}(\partial_i \ln g)(\partial_j \ln g)\Big] \\
&= \frac{1}{2}(p_i g^{ij} p_j)_S + \frac{1}{8}[\partial_i\partial_j g^{ij} + g^{-\frac{1}{4}}\partial_i(g^{\frac{1}{4}}g^{ij}\partial_j \ln g)]
\end{aligned}
\tag{2.48}
$$

Starting from equation (2.17) and contracting that expression with $g^{ij}g^{kl}$(so that $g^{ij}g^{kl}R_{ikjl} = R$) we find that

$$
\partial_i\partial_j g^{ij} = R + g^{ij}\Gamma^k_{il}\Gamma^l_{kj} \tag{2.49}
$$

so we obtain at the end, up to a term that is a total derivative therefore can be neglected,

$$\hat{H} = \frac{1}{2}(p_i g^{ij} p_j)_S + \frac{1}{8}(R + g^{ij}\Gamma^k_{il}\Gamma^l_{kj}) \tag{2.50}$$

One can see than, in order to obtain the correct Hamiltonian in the classical limit, one has to add a counterterm that is, choosing a renormalization condition for example $\xi = 0$,

$$V_{TS} = -\frac{1}{8}(R + g^{ij}\Gamma^k_{il}\Gamma^l_{kj}) \tag{2.51}$$

Equation (2.44) can then be rewritten as

$$K(x_f, x_i, T) = \int DxDp \; e^{iS[x,p]} \tag{2.52}$$

with

$$S[x,p] = \int_0^T dt\big(p\dot{x} - H_W(x,p)\big) \tag{2.53}$$

$$H_W(x,p) = \frac{1}{2} g^{ij} p_i p_j + V_{TS} \tag{2.54}$$

This exemplifies how to deal with ambiguities in the time slicing regularization. We can now go on and obtain the configuration-space path integrals in this regularization and the perturbative expansion, like what done in the flat space case. In doing that some extra care is needed, since now every Gaussian integral of the form

$$\int \exp\Big\{ i\epsilon \Big[p_{k,i} \frac{x^i_k - x^i_{k-1}}{\epsilon} - \frac{1}{2m} g^{ij} p_{k,i} p_{k,j} \Big] \Big\} d^D p_k \tag{2.55}$$

brings down an extra $(\det g^{ij}(\overline{x}_k))^{-\frac{1}{2}} = \sqrt{\det g_{ij}(\overline{x}_k)}$ factor upon completing the square and integrating over momenta, in accordance with equation (1.21). These factors rend the measure to be no more a translational invariant, making difficult to shift the measure in order to obtain the perturbative expansion. We then use the following trick to obtain translational invariance again: we re-exponentiate these extra terms in the measure using the so-called ghosts field, adding a fictitious action for these fields that, when integrated explicitly, gives back the correct term. In that way we have

$$\int Dx\ e^{iS[x]} \prod_{k=0}^{N} \sqrt{g(\overline{x}_k)} = \int DxDaDbDc\ e^{i(S[x]+S_{gh}[a,b,c])} \tag{2.56}$$

where a, b, c are the ghost fields and as usual $Dx = \prod_{k=1}^{N} d^D x$. The ghost field a is considered a real-valued commuting field, while the fields b, c are Grassmann-valued [12] anticommuting fields. In that way integration over the variable a gives a $g^{-\frac{1}{2}}$ factor, while using Berezin integration over b and c yelds a factor of g: put together they recreate the correct factor of $\sqrt{g}$

$$\sqrt{\det g_{ij}(\overline{x}_k)} = \alpha \int d^D a_{\overline{k}} d^D b_{\overline{k}} d^D c_{\overline{k}}\ e^{i\frac{\epsilon}{2T^2} g_{ij}(\overline{x}_k)\left(b^i_{\overline{k}} c^j_{\overline{k}} + a^i_{\overline{k}} a^j_{\overline{k}}\right)} \tag{2.57}$$

Where we define the constant α in order that the integral yelds the right factor of $g^{\frac{1}{2}}(\overline{x}_k)$, and where we used the subscript $\overline{k}$ to indicate that the ghosts are related to the metric evaluated at points $\overline{x}_k$.
The procedure to follow in order to generate the perturbative expansion is still the same followed on flat-space case: we again decompose the action into a quadratic part and an interaction part

$$S = S_0 + S_{int} \tag{2.58}$$

and then we decompose the path x^i_k into a classical path $x^i_{cl,k}$ and quantum fluctuations ϕ^i_k

$$x^i_k = x^i_{cl,k} + \phi^i_k \qquad (k = 1, \ldots, N) \tag{2.59}$$

where as usual $x^j_0 = x^j_i$, $x^j_N = x^j_f$, and the classical path $x^j_{cl,k}$ satisfies the classical equations of motion and the boundary conditions $x^j_{cl,0} = x^j_i$, $x^j_{cl,N} = x^j_f$ so that $\phi^j_0 = \phi^j_N = 0$. Of course x^j_{cl} is a solution of the $N-1$ equations of motion for S_2

$$g_{jl}(x_i)(x^j_{k+1} - 2x^j_k + x^j_{k-1}) = 0 \tag{2.60}$$

[12] Since we will deal only with scalar particles and Grassmann variables are used in this text only to define ghosts, we refer to text [36] for a detailed description of how Grassmann variables work. A brief explanation is anyway given in Appendix A.

so that one can write, in the continuum limit,

$$x^j_{cl}(t) = x^j_i + \frac{t}{T}(x_f - x_i) = x^j_i + \tau(x_f - x_i) \tag{2.61}$$

where $\tau = \frac{t}{T} \in [0,1]$. Now the discretized and free part of the action is given by, after the last expansion

$$S_0[\phi, a, b, c] = \sum_{k=1}^{N} \frac{1}{2\epsilon} g_{ij}(x_i)(\phi^i_k - \phi^i_{k-1})(\phi^j_k - \phi^j_{k-1}) + \frac{\epsilon}{2T^2} g_{ij}(x_i)(b^i_{\overline{k}} c^j_{\overline{k}} + a^i_{\overline{k}} a^j_{\overline{k}}) \tag{2.62}$$

Note that the metric is evaluated at the initial point x_i: we put it that way since when computing path integrals one usually expand the metric around the origin. This of course produce a term in the interacting action of the form $[g_{ij}(\overline{x}_k) - g_{ij}(x_i)](b^i_{\overline{k}} c^j_{\overline{k}} + a^i_{\overline{k}} a^j_{\overline{k}})$ since as stated earlier the discretized ghosts fields are evaluated at midpoints. It is not mandatory to choose to evaluate the metric around the initial point x_i: we should as well have chosen another point, like for example the final point x_f or the geodesic midway point between x_i and x_f.
We then introduce sources coupled to dynamical variables in order to generate the expansion: for the non-ghosts part it reads

$$S_{(source,non\ ghosts)} = \sum_{k=1}^{N} F_{\overline{k},i} \frac{\phi^i_k - \phi^i_{k-1}}{\epsilon} + G_{\overline{k},i} \phi^i_{\overline{k}} \tag{2.63}$$

where we the coupling to $\overline{x}_k$ instead that to x_k has been preferred since the discretized action does not depend on x_k. We should now complete the squares in $S_0 + S_{(sources,non\ ghosts)}$ and then integrate over the $d^D x_k = d^D \phi_k$: the problem is that the free action is not diagonal over ϕ^i_k, since it contains terms like $\phi^i_k \phi^j_{k-1}$. We then perform an ortogonal transformation which diagonalises S_0, that is

$$\phi^i_k = \sum_{m=1}^{N-1} r^i_m \sqrt{\frac{2}{N}} \sin\Big(\frac{km\pi}{N}\Big) = \sum_{m=1}^{N-1} r^i_m O^m_k \tag{2.64}$$

where orthogonality follows from the completeness relation of the $(N-1) \times (N-1)$ real matrix $O^m_k = \sqrt{\frac{2}{N}} \sin\Big(\frac{km\pi}{N}\Big)$

$$\sum_{m=1}^{N-1} O_k^m O_{k'}^m = \delta_{k,k'} \tag{2.65}$$

as is shown in reference [9]. The orthogonality of this matrix allows us to replace $\prod_{k=1}^{N-1} d^D\phi_k$ with $\prod_{m=1}^{N-1} d^D r_m$. In this way

$$\begin{aligned} S_{(0,non\ ghost)} &= \frac{1}{2\epsilon}\sum_{k=1}^{N} g_{ij}(x_i)(\phi_k^i - \phi_{k-1}^i)(\phi_k^j - \phi_{k-1}^j) \\ &= \frac{1}{2\epsilon} g_{ij}(x_i)\sum_{k=1}^{N}\sum_{m,n=1}^{N-1}(O_k^m - O_{k-1}^m) r_m^i (O_k^n - O_{k-1}^n) r_n^j \\ &= \frac{1}{2\epsilon} g_{ij}(x_i)\sum_{k=1}^{N}\sum_{m,n=1}^{N-1}[2 r_m^i r_n^j \delta_{m,n} - O_k^m (O_{k-1}^n + O_{k-1}^n) r_m^i r_n^j] \end{aligned} \tag{2.66}$$

Now, using the relation $O_{k-1}^n + O_{k+1}^n = 2O_k^n \cos\left(\frac{n\pi}{N}\right)$[13] and the orthogonality relations of the O_k^n matrix, we obtain

$$S_{(0,non\ ghost)} = \frac{1}{\epsilon}\sum_{m=1}^{N-1} g_{ij}(x_i) r_m^i r_m^j \left(1 - \cos\frac{m\pi}{N}\right) \tag{2.67}$$

We then couple also ghosts with external sources in the same way, where we understand that the sources share the same nature with the fields they are coupled with, i.e the commuting real-valued ghost $a_{\overline{k}}^i$ will always be coupled to a real parameter $A_{\overline{k},i}$, while the anticommuting real-valued fields $b_{\overline{k}}^i$ and $c_{\overline{k}}^i$ with two Grassmann parameter, respectively $B_{\overline{k},i}$ and $C_{\overline{k},i}$

$$S_{(sources,ghosts)} = \sum_{k=1}^{N-1}(A_{\overline{k},i} a_{\overline{k}}^i + B_{\overline{k},i} b_{\overline{k}}^i + C_{\overline{k},\mu} c_{\overline{k}}^i) \tag{2.68}$$

To summarize, we have now that the full action functional is given by

$$\begin{aligned} S &= S_{(0,non\ ghost)} + S_{(0,ghost)} + S_{(source,non\ ghost)} + S_{(source,ghost)} + S_{int} \\ &= S_0 + S_{source} + S_{int} \end{aligned} \tag{2.69}$$

where, in the continuum limit, the various pieces read

[13]This follows immediately from the trigonometrical relation $\sin\alpha + \sin\beta = 2\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)$.

$$S_0[\phi,a,b,c]=\int_0^s dt\frac{1}{2}g_{ij}(x_i)(\dot{\phi}^i(t)\dot{\phi}^j(t)+a^i(t)a^j(t)+b^i(t)c^j(t)) \tag{2.70}$$

$$S_{source}[\phi,a,b,c,J,A,B,C]=\int_0^s dt(J_i(t)\phi^i(t)+A_i(t)a^i(t)+B_i(t)b^i(t)+C_i(t)c_i(t)) \tag{2.71}$$

$$S_{int}[\phi,a,b,c]=\int_0^s dt\Big[V(\phi(t))+V_{ct}+\frac{1}{2}(g_{ij}(x(t))-g^{ij}(x_i))(\dot{\phi}^i(t)\dot{\phi}^j(t)+a^i(t)a^j(t)+b^i(t)c^j(t))\Big] \tag{2.72}$$

We can now perform a square completion on the ghost sector of the path integral in the usual manner and, after integrating over a,b,c using expression (2.57), we obtain

$$\int DaDbDc\; e^{iS_{ghost[a,b,c,A,B,C]}}=g^{\frac{N}{2}}(x_i)\exp\left\{\sum_{k=1}^{N-1}\frac{iT}{\epsilon}g^{ij}(x_i)\Big(2C_{\overline{k},i}B_{\overline{k},j}+\frac{1}{2}A_{\overline{k},i}A_{\overline{k},j}\Big)\right\} \tag{2.73}$$

where the factor $g^{\frac{N}{2}}(x_i)$ is due to the integration over a,b,c and correspond to the N factors of $\sqrt{g(\overline{x}_k)}$ in expression (2.57), while the rest reproduces exactly the constant α, which has never been computed for that reason. We can now obtain the discretized propagators by twice differentiating this last expression with respect to the external sources F,G,A,B,C and then setting them to zero. We will proceed now with the aim to obtain all the possible propagators of the theory (from which all the expectation values that come from the perturbative expansion can be computed) in the discretized approach, then we will pass to the continuum limit in order to give a form to the continuum propagators which will be later compared to propagator in other regularization schemes. The first one we will compute is the propagator

$$
\begin{aligned}
\left\langle \dot\phi^i_{k+\frac12}\dot\phi^j_{k'+\frac12}\right\rangle &= \left\langle \Big(\frac{\phi^i_{k+1}-\phi^i_k}{\epsilon}\Big)\Big(\frac{\phi^j_{k'+1}-\phi^j_{k'}}{\epsilon}\Big)\right\rangle \\
&= \frac{\partial}{\partial F_{k+\frac12,i}}\frac{\partial}{\partial F_{k'+\frac12,j}}e^{iS[F,G,A,B,C]|_0} \\
&= -2i\sum_{m=1}^{N-1}\frac{\epsilon}{4(1-\cos\frac{m\pi}{N})}g^{ij}(x_i)\Big(\frac{2}{\epsilon}\sqrt{\frac{2}{N}}\sin\frac{m\pi}{2N}\Big)^2\times \\
&\times\cos\Big(k+\frac12\Big)\frac{m\pi}{N}\cos\Big(k'+\frac12\Big)\frac{m\pi}{N} \qquad (2.74)
\end{aligned}
$$

in which the subscript $k+\frac12$ stands for evaluation between k and $k+1$. Using the trigonometrical identity $2\cos\alpha\cos\beta=\cos(\alpha+\beta)+\cos(\alpha-\beta)$ we get

$$
\begin{aligned}
\left\langle \dot\phi^i_{k+\frac12}\dot\phi^j_{k'+\frac12}\right\rangle &= -\frac{i}{N\epsilon}g^{ij}(x_i)\sum_{m=1}^{N-1}\Big[\cos(k+k'+1)\frac{m\pi}{N}+\cos(k-k')\frac{m\pi}{N}\Big] \\
&= -\frac{i}{N\epsilon}g^{ij}(x_i)(-1+N\delta_{k,k'}) \qquad (2.75)
\end{aligned}
$$

Next we compute the $\phi\dot\phi$ propagator, that is

$$
\begin{aligned}
\left\langle \phi^i_{k+1/2}\dot\phi^j_{k'+1/2}\right\rangle &= \frac{\partial}{\partial G_{k+1/2,i}}\frac{\partial}{\partial F_{k'+1/2,j}}e^{iS_{source}[F,G,A,B,C]|_0} \\
&= 2i\sum_{m=1}^{N-1}\frac{\epsilon g^{ij}(x_i)}{4\Big(1-\cos\alpha_m\Big)}\frac{4\pi}{N\epsilon}\sin\frac{\alpha_m}{2}\cos\frac{\alpha_m}{2}\times \\
&\times\sin\Big[\Big(k+\frac12\Big)\alpha_m\Big]\cos\Big[\Big(k'+\frac12\Big)\alpha_m\Big] \\
&= ig^{ij}(x_i)\frac{1}{N}\sum_{m=1}^{N-1}\cos\frac{\alpha_m}{2}\left[\frac{\sin\Big(k+\frac12\Big)\alpha_m}{\sin\frac{\alpha_m}{2}}\right]\cos\left(k'+\frac12\right)\alpha_m \qquad (2.76)
\end{aligned}
$$

where we have indicated $\alpha_m=\frac{m\pi}{N}$. The evaluation of these trigonometrical series is quite convoluted, so we will give only the final result; for an exact evaluation see [9]. At the end this final result is

$$
\left\langle \phi^i_{k+1/2}\dot\phi^j_{k'+1/2}\right\rangle = ig^{ij}(x_i)\left[-\frac{\Big(k+\frac12\Big)}{N}+\frac12\delta_{k,k'}+\theta_{k,k'}\right] \qquad (2.77)
$$

where $\theta_{k,k'}$ is the discretized Heaviside step-function that is equal to unity for $k = k'$ and is 0 otherwise. Proceeding in the computation of the 2-points correlation function we have next

$$\begin{aligned}
\left\langle \phi^i_{k+1/2}\phi^j_{k'+1/2}\right\rangle &= \frac{\partial}{\partial G_{k+1/2,i}}\frac{\partial}{\partial G_{k'+1/2,j}}e^{iS_{source}[F,G,A,B,C]|_0} \\
&= 2i\sum_{m=1}^{N-1}\frac{\epsilon g^{ij}(x_i)}{4\Big(1-\cos\alpha_m\Big)}\left(\sqrt{\frac{2}{N}}\cos\frac{\alpha_m}{2}\right)^2\times \\
&\times\sin\Big(k+\frac{1}{2}\Big)\alpha_m\sin\Big(k'+\frac{1}{2}\Big)\alpha_m \\
&= \frac{i\epsilon}{2N}g^{ij}(x_i)\sum_{m=1}^{N-1}\cos^2\frac{\alpha_m}{2}\left[\frac{\sin\Big(k+\frac{1}{2}\Big)\alpha_m}{\sin\frac{\alpha_m}{2}}\right]\times \\
&\times\left[\frac{\sin\Big(k'+\frac{1}{2}\Big)\alpha_m}{\sin\frac{\alpha_m}{2}}\right] \qquad (2.78)
\end{aligned}$$

again we rely on [9] for a direct evaluation of the series, and the result is

$$\left\langle \phi^i_{k+1/2}\phi^j_{k'+1/2}\right\rangle = \frac{i\epsilon}{4N}g^{ij}(x_i)\Big[-\frac{\big(k+\frac{1}{2}\big)\big(k'+\frac{1}{2}\big)}{N}+\Big(k'+\frac{1}{2}\Big)\theta_{k,k'}+\Big(k+\frac{1}{2}\Big)\theta_{k',k}-\frac{1}{4}\delta_{k,k'}\Big] \qquad (2.79)$$

Next we have ghosts progators: from equation (2.68)

$$\begin{aligned}
\left\langle a^j_{k+1/2}a^i_{k'+1/2}\right\rangle &= \frac{\partial}{\partial A^i_{k+1/2}}\frac{\partial}{\partial A^j_{k'+1/2}}e^{iS_{source}[F,G,A,B,C]|_0} \\
&= -\frac{iT^2}{\epsilon}g^{ij}(x_i)\delta_{k,k'} \qquad (2.80)
\end{aligned}$$

$$\begin{aligned}
\left\langle b^i_{k+1/2}c^j_{k'+1/2}\right\rangle &= \frac{\partial}{\partial B^i_{k+1/2}}\frac{\partial}{\partial C^j_{k'+1/2}}e^{iS_{source}[F,G,A,B,C]|_0} \\
&= 2i\frac{T^2}{\epsilon}g^{ij}(x_i)\delta_{k,k'} \qquad (2.81)
\end{aligned}$$

This completes the evaluation of all possible propagators in the discretized approach.

If we now try to compute these propagators directly in the continuum limit we see that we obtain distributions that are ill-defined when equal-time contraction is implemented, or that do not behave well at endpoints. The discretized approach then instructs us on how to evaluate these distributions and the behaviour at these points. For example, starting from the $\phi\phi$ propagator we have, after performing a square completion on the (free+source) part of the action and integrating over the variable ϕ,

$$\begin{aligned}\left\langle \phi^i(t)\phi^l(t')\right\rangle &= \frac{\delta}{\delta J_i(t)}\frac{\delta}{\delta J_l(t')}e^{iS[J]|_0}\\ &= \frac{\delta}{\delta J_i(t)}\frac{\delta}{\delta J_l(t')}e^{\frac{i}{2}g^{il}(x_i)\int_0^T J_i(\tilde{t})g(\tilde{t},\tilde{t}')g^{il}J_l(\tilde{t}')d\tilde{t}d\tilde{t}'}\\ &= iTg^{il}(x_i)g(\tau,\tau') \end{aligned} \tag{2.82}$$

where again $g(\tau,\tau')$ is the same defined in equation (1.41) and where we have rescaled the action so that $\phi^i(t) = \phi^i(T\tau)$. The discretized approach then tells us that we have to set, in the contiuum limit, $\theta(0) = \frac{1}{2}$ since in the discretized case we have the term $\frac{1}{2}\delta_{k,k'}$ which tells us how to deal with equal-time contraction. The propagator for $\dot{\phi}\dot{\phi}$ is obtained in the continuum limit simply as

$$\begin{aligned}\left\langle \dot{\phi}^i(t)\dot{\phi}^j(t')\right\rangle &= \frac{\partial^2}{\partial t^2}\left\langle \phi^i(t)\phi^j(t')\right\rangle\\ &= -iTg^{ij}(x_i)\delta(\tau-\tau') \end{aligned} \tag{2.83}$$

since the green function $g(\tau,\tau')$ obeys $\partial_\tau^2 g(\tau,\tau') = -\delta(\tau-\tau')$: taking a confrontation with the result one obtains naively taking the continuum limit of equation (2.75), [14]

$$\left\langle \dot{\phi}^i(t)\dot{\phi}^j(t')\right\rangle = iTg^{ij}(x_i)[1-\delta(\tau,\tau')] \tag{2.84}$$

we see that one misses the term $iTg^{ij}(x_i)$. However, if one takes into account the boundary condition $\phi(0) = \phi(s) = 0$, one has to add suitable terms linear in t and t' to expression (2.83) in order to make the propagator vanish for $t,t' = 0,T$ while still maintaining $\partial^2\left\langle \phi^i(t)\phi^j(t')\right\rangle/\partial t^2 = -iTg^{ij}(x_i)\delta(\tau-\tau')$. Moreover the discrete case tells

[14] Taking also in account that $\lim_{\epsilon\to 0}\frac{1}{\epsilon}\delta_{k,k'} = \delta(t-t')$ since, as in flat space-time, we have set $\phi_k = \phi(t_i + k\epsilon)$.

us that the delta function $\delta(t-t')$ is in fact proportional to a Kronecker delta delta function: this instructs us to set $t = t'$ in the integrand where a delta function appears in the evaluation of Feynman graphs, and not to replace this delta function with some smooth function, since this would lead to an incorrect result. All the other propagators can be readily obtained directly from the continuum limit taking in account the boundary conditions and the correct rules for the evaluation of distributions obtained from the discretized case. For completeness they read

$$\left\langle \phi^i(t)\dot{\phi}^j(t') \right\rangle = iTg^{ij}(x_i)[\tau + \theta(\tau - \tau')] \tag{2.85}$$

$$\left\langle a^i(t)a^j(t') \right\rangle = -iTg^{ij}(x_i)\delta(t-t') \tag{2.86}$$

$$\left\langle b^i(t)c^j(t') \right\rangle = 2iTg^{ij}(x_i)\delta(\tau - \tau') \tag{2.87}$$

This concludes our dissertation about time-slicing regularization, since every other Feynman diagram one can get can be readily computed from these elementary propagators using the same techniques shown in section (1.4). The next few sections will be entirely finalized to the explanation of mode and dimensional regularization.

2.4 Mode Regularization

Mode regularization and dimensional regularization differ from the approach of time-slicing regularization for the reason that, instead than starting from the evaluation of the discretized transition amplitude $\langle x_k|e^{-iT\hat{H}}|x_{k-1}\rangle$, one tries to define directly the full transition amplitude as a configuration-space path integral, and then uses its formal property in order to find the propagators and the correct counterterm V_{ct}. This means

$$\langle x_f|e^{-iT\hat{H}}|x_i\rangle = \int_{BC} Dx\ e^{iS} \tag{2.88}$$

with

$$Dx = \prod_{0<t<T} \sqrt{\det g_{ij}(x(t))}d^Dx(t) \tag{2.89}$$

and where the subscript BC indicates that Dirichlet boundary conditions are implemented for initial and final time $x^j(0) = x_i^j$,$x^j(T) = x_f^j$. As remarked before this last measure is formally a scalar since is a product of scalar measures. With the definitions given in previous sections of this chapter also the action, and therefore the transition amplitude, is itself a scalar. Since the non-trivial measure given in the last equation is formally not a transitional invariant we use again the same trick introduced in the previous section, that is to re-exponentiate the factors proportional to the determinant g in the action with the aid of a commuting real-valued ghost field $a^j(t)$ and two anticommuting Grassmann-valued fields $b^j(t), c^j(t)$.

$$\prod_{0<t<T} \sqrt{\det g_{ij}(x(t))} = \int DaDbDc\ e^{iS_{gh}} \tag{2.90}$$

$$S_{gh} = \int_0^s dt\ \frac{1}{2} g_{ij}(x)(a^i a^j + b^i c^j) \tag{2.91}$$

with the transitionally-invariant ghost measure given by

$$Da = \prod_{0<t<T} d^D a(t) \quad , \quad Db = \prod_{0<t<T} d^D b(t) \quad , \quad Dc = \prod_{0<t<T} d^D c(t) \tag{2.92}$$

Now we can make use of the invariance of the measure to make again the split $S[x] = S[x_{cl} + \phi] = S[x_{cl}] + S[\phi]$, where

$$x^j(\tau) = x^j_{cl}(\tau) + \phi^j(\tau) = x_i^j + z^j\tau + \phi^j(\tau) \quad , \quad z^i = x_f^i - x_i^i \tag{2.93}$$

It is convenient to rescale the action as done on flat-space in (1.34) (except that also the ghost part of the action has now to be rescaled) from the start, and also to expand all the quantum dynamical variables into a sine series. This is always possible thanks to the boundary conditions that sets $\varphi^i = 0$ at endpoints, where φ^i indicates one of the quantum variables ϕ^i, a^i, b^i, c^i. The expansion reads

$$\varphi^i(\tau) = \sum_{m=1}^{\infty} \varphi^i_m \sin(\pi m \tau) = \lim_{M\to\infty} \sum_{m=1}^{M} \varphi^i_m \sin(\pi m \tau) \tag{2.94}$$

where φ^i_m are the Fourier coefficients of the expansion. For computational purposes the upper limit of the series will be put equal to M, restoring the right limit by letting $M \to \infty$ after the computation. This is of course analogue to a cut-off regularization.

This expansion leaves us with the measure

$$D\phi DaDbDc = \lim_{M\to\infty} A\prod_{m=1}^{M} md^D\phi_m d^D a_m d^D b_m d^D c_m \tag{2.95}$$

As done before, we now split the action (now function of only the quantum fluctuations and the ghosts since the action evaluated on classical paths can be removed from the path integral as done in (1.30)) into a free quadratic part and an interaction part

$$S = S_0 + S_{int} \tag{2.96}$$

where the pieces read now

$$S_0 = \int_0^1 d\tau \frac{1}{2} g_{ij}(x_i)(z^i z^j + \dot{\phi}^i\dot{\phi}^j + a^i a^j + b^i c^j) \tag{2.97}$$

$$S_{int} = \int_0^1 d\tau \Big(\frac{1}{2}[g_{ij}(x) - g_{ij}(x_i)](\dot{x}^i\dot{x}^j + a^i a^j + b^i c^j) + T^2[V(x) + V_{MR}]\Big) \tag{2.98}$$

in which V_{MR} is the local counterterm that is required in mode regularization. We note that, unlike time-slicing regularization, it is not possible in this approach to obtain this counterterm from the operatorial ordering, since it is unknown how to recreate this regularization starting from the discrete approach. Then one has first to evalate propagators, and then calculate the counterterm from a direct evaluation of 2-loop Feynman graphs that arise from the perturbative expansion, since this counterterm is explicitly of order T^2, as one can see from the last expression. Also note that now terms linear to $\dot{\phi}^i$ arise in S_{int}: on flat space these terms reduce to 0 since the flat metric tensor δ_{ij} does not depend on coordinates.
Inserting now the expansion (2.94) into S_0 we get

$$S_0 = \frac{1}{2} g_{ij}(x_i) + \frac{1}{4} g_{ij}(x_i)\sum_{m=1}^{M}(\pi^2 m^2 \phi^i_m \phi^j_m + a^i_m a^j_m + b^i_m c^j_m) \tag{2.99}$$

Since only the free S_0 is required when one evaluates propagators, we can immediately obtain them beginning from this last expression

$$\left\langle \phi^i(\tau)\phi^j(\tau')\right\rangle = \left\langle \sum_{m=1}^{M} \phi^i_m \sin(\pi m\tau) \sum_{n=1}^{M} \phi^j_n \sin(\pi n\tau') \right\rangle$$
$$= \sum_{m,n=1}^{M} \left\langle \phi^i_m \phi^j_n \right\rangle \sin(\pi m\tau)\sin(\pi n\tau') \tag{2.100}$$

The propagator for ϕ^i_m modes can be obtained in the usual way described in the previous chapter and the previous section, adding a coupling to a source J^m_i for every mode, and then completing squares and shifting the integration. The result is easily computed and is given by

$$\left\langle \phi^i_m \phi^j_n \right\rangle = -iTg^{ij}(x_i)\delta_{m,n}\frac{2}{\pi^2 m^2} \tag{2.101}$$

Putting together the last two expressions we get

$$\left\langle \phi^i(\tau)\phi^j(\tau')\right\rangle = -iTg^{ij}(x_i)\sum_{m=1}^{M}\frac{2}{\pi^2 m^2}\sin(\pi m\tau)\sin(\pi m\tau') = -iTg^{ij}(x_i)g^M(\tau,\tau') \tag{2.102}$$

In order to check normalization, note that in the limit $M \to \infty$ we have

$$\delta(\tau-\tau') = \sum_{m=1}^{\infty} 2\sin(\pi m\tau)\sin(\pi m\tau') \tag{2.103}$$

Since differentiating 2 times the factor in the sum of (2.102) we get exactly this expression, we recover the fact that this propagator is proportional to the green function of the operator ∂^2_τ

$$\left\langle \phi^i(\tau)\phi^j(\tau')\right\rangle = iTg^{ij}(x_i)g(\tau,\tau') \tag{2.104}$$

that is the same expression we get for time slicing regularization.
As usual we can repeat the same procedure for the ghost-dependent sector, obtaining for M finite

$$\left\langle a^i(\tau)a^j(\tau')\right\rangle = -iTg^{ij}(x_i)g^M_{gh}(\tau,\tau') \tag{2.105}$$

$$\left\langle b^i(\tau)c^j(\tau')\right\rangle = 2iTg^{ij}(x_i)g^M_{gh}(\tau,\tau') \tag{2.106}$$

where we have called the regularized green function for the ghosts

$$g^M_{gh}(\tau,\tau') = \sum_{m=1}^{M} 2\sin(\pi m\tau)\sin(\pi m\tau') \tag{2.107}$$

It is immediate to check that in the limit $M \to \infty$ we recover all the same propagators obtained in the previous section of this chapter. The importance of mode expansion is then when one evaluates Feynman graphs that involve products of various propagators, one can use the expression for finite M in order to obtain unambiguous expressions for the product of distributions, and then set again $M = \infty$ in order to recover the exact expression.

2.4.1 The counterterm V_{MR}

We will see a fundamental difference when we try to get the correct counteterms V_{ct} between time-slicing regularization and mode regularization: in the first one, we obtained the counterterm V_{TS} by choosing a definite operatorial ordering for the ambiguous quantum Hamiltonian operator. In order to apply this method, we had to evaluate the exact path integral in the discrete case, and then obtain its continuum limit. Instead, we will see that with the following procedure one can put away the subtleties due to discretized path integrals, and works directly in the continuum limit. But this of course comes with a price. In fact, with the following procedure, one is obliged to calculate Feynman graphs (that are now unambiguous thanks to the cut-off regularization given in (2.94)) in the first place in order to obtain the correct countertem at any given order T. The procedure is as follows: we require that the transition amplitude $K(x_f, x_i, T)$ should yield the correct expression for time evolution (or regression, in the present case) of an arbitrary wave function as given by

$$\begin{aligned}\psi(x_i,t_i) = \langle x_i, t_i|\psi\rangle &= \int d^Dx_f\ \sqrt{g(x_f)}\,\langle x^i_i, t_i|x^i_f, t_f\rangle\,\langle x^i_f, t_f|\psi\rangle \\ &= \int d^Dx_f\ \sqrt{g(x_f)}K(x_i,x_f,T)\psi(x_f,t_f)\end{aligned} \tag{2.108}$$

in order to get a precise evaluation, we can expand the final wave-function around the initial one: using the expansion (1.27) we get

$$\psi(x_f,t_f)=\psi(x_i,t_i)+iT\partial_t\psi(x_i,t_i)+z^i\partial_j\psi(x_i,t_i)+\frac{1}{2}z^iz^j\partial_i\partial_j\psi(x_i,t_i)+o(T^{3/2}) \tag{2.109}$$

where we counted every term z^i to be of order $T^{\frac{1}{2}}$ for reasons that will be explained later. Also the determint $g^{\frac{1}{2}}(x_f)$ can be expanded around the initial point x_i

$$\sqrt{g(x_f)}=\sqrt{g(x_i)}\Big[1+z^i\Gamma^k_{ik}+\frac{1}{2}z^iz^j\Gamma^k_{ik}\Gamma^l_{jl}+o(T^{\frac{1}{2}}\Big]\Big|_{x_i} \tag{2.110}$$

where we used the fact that $\frac{1}{\sqrt{g(x)}}\partial_i\sqrt{g(x)}=\frac{1}{2}g^{kl}\partial_i g_{kl}=\Gamma^k_{ik}$. We can expand also the interacting action S_{int} of equation (2.98) around the the initial point x_i using

$$[g_{ij}(x)-g_{ij}(x_i)]=\sum_{n=1}^{\infty}\partial_{k_n}\dots\partial_{k_1}g_{ij}(x)|_{x_i}(\phi+z\tau)^{k_1}\dots(\phi+z\tau)^{k_n} \tag{2.111}$$

It is also convenient to make an expansion in powers of the transition time T

$$S_{int}=S^{(1)}_{int}+S^{(2)}_{int}+\dots \tag{2.112}$$

In which every term S_n contributes as $(T)^{\frac{n}{2}-2}$. This expansion reads

$$S^{(1)}_{int}=\frac{1}{T}\int d\tau\Big[\frac{1}{2}\partial_k g_{ij}(\phi^k+z^k\tau)(z^iz^j+2z^i\dot{\phi}^j)+\dot{\phi}^i\dot{\phi}^j+a^ia^j+b^ic^j\Big] \tag{2.113}$$

$$\begin{aligned} S^{(2)}_{int}=&\frac{1}{T}\int d\tau\Big[\frac{1}{4}\partial_l\partial_k g_{ij}(z^kz^l\tau^2+\phi^k\phi^l+2z^k\phi^l\tau)\times\\ &\times(z^iz^j+2z^i\dot{\phi}^j+\dot{\phi}^i\dot{\phi}^j+a^\mu a^\nu+b^ic^j)+T^2(V+V_{MR})\Big] \end{aligned} \tag{2.114}$$

Inserting these last expansions and the correct expression for the Heat Kernel into equation (2.109) we get

$$\psi(x_i,t_i) = \int d^D x_f\ \sqrt{g(x_i)}[1+\dots]\Big(Ae^{i\frac{z^2}{2T}}\Big\langle 1+iS^1_{int}+iS^2_{int}-\frac{1}{2}(S^1_{int})^2\Big\rangle+\dots(\psi(x_i,t_i)+\dots) \tag{2.115}$$

We can see now that everything is evaluated on the initial point x_i, except for the factor $e^{i\frac{z^2}{2T}} = e^{i\frac{(x_i-x_f)^2}{2T}}$: integrating over the final point x_f this bring down a factor of $(2i\pi T)^{\frac{D}{2}}$. If we now take a look at the leading terms, we see that the appearance of this factor fixes the normalization constant A

$$\psi(x_i,t_i) = A(2\pi T)^{\frac{D}{2}}\psi(x_i,t_i) \ \longrightarrow\ A = \frac{1}{(2\pi T)^{\frac{D}{2}}} \tag{2.116}$$

since the first corrections arise from order T^1 and up. Explicitly evaluating terms at this given order that come from the various pieces (noticing also that terms like $\int d^D z\ z^i z^j e^{i\frac{z^2}{2T}}$ define the basic propagator $\langle z^i z^j\rangle = -iTg^{ij}$,from which all other z propagators can be obtained by means of wick contractions. Also this is the reason we stated before that every z term appearing in the expansion is of order $T^{\frac{1}{2}}$) one obtains [15]

$$T\left[\partial_t\psi(x_i) - \frac{1}{2}\nabla^2\psi(x_i) + V(x_i) + V_{MR} + \frac{1}{8}R + \frac{1}{24}g^{ij}g^{kl}g_{mn}\Gamma^m_{ik}\Gamma^n_{jl}\right] \tag{2.117}$$

From here, if we require the coupling ξ to the curvature to vanish as a renormalization condition, we immediatly get

$$V_{MR} = -\frac{1}{8}R - \frac{1}{24}g^{ij}g^{kl}g_{mn}\Gamma^m_{ik}\Gamma^n_{jl} \tag{2.118}$$

2.5 Dimensional Regularization

We begin here the discussion of another regularization scheme: dimensional regularization. The procedure is always the same we developed for mode regularization, except that now the interacting action S_{int} contains the correct counterterm V_{DR} of this regularization

[15]See reference [9]

instead that the V_{MR} one. We also use the same expansion of equation (2.94), but we do not introduce the cut-off for large modes M: instead we choose to extend the action to be defined on a manifold with extra d time-dimensions. This means that we introduce the replacement $t \to \mathbf{t} = (t^1, \ldots, t^d)$, and the integration in the action functional will be defined to be over dt^i, in which $t^\mu = (\tau, \mathbf{t})$, so that $d^{d+1}t = d\tau d\mathbf{t}$. With these formal replacements the action in $d+1$ dimensions reads

$$S = \int_\Omega d^{d+1}t \Big[\frac{1}{2} g_{ij}(\partial_\mu x^i \partial_\nu x^j + a^i a^j + b^i c^j) - T^2(V(x(t^\mu)) + V_{DR})\Big] \tag{2.119}$$

where $\Omega = [0,1] \times \mathbb{R}^D$. Note that with this extension the classical paths defined by $x^i_{cl}(\tau) = x^i_i + z^i \tau$ are left unchanged, so that the split $S = S_0 + S_{int}$ are now given by

$$S_0 = \frac{1}{2} g_{ij}(x_i) z^i z^j + \int_\Omega d^{d+1}t \ \frac{1}{2} g_{ij}(x_i)(z^i z^j + \partial_\mu \phi^i \partial_\nu \phi^j + a^i a^j + b^i c^j) \tag{2.120}$$

$$S_{int} = \int_\Omega d^{d+1}t \ \Big[\frac{1}{2}(g_{ij}(x) - g_{ij}(x_i))(\partial_\mu x^i \partial_\nu x^j + a^i a^j + b^i c^j) - T^2(V(x) + V_{DR})\Big] \tag{2.121}$$

Using the same method followed in the previous section we can obtain the propagators which, with this regularization, read

$$\big\langle \phi^i(t)\phi^j(t')\big\rangle = iT g^{ij}(x_i) g^{DR}(t,t') \tag{2.122}$$

$$\big\langle a^i(t) a^j(t')\big\rangle = -iT g^{ij} g^{DR}_{gh}(t,t') \tag{2.123}$$

$$\big\langle b^i(t) c^j(t')\big\rangle = 2iT g^{ij} g^{DR}_{gh}(t,t') \tag{2.124}$$

where we have

$$g^{DR}(t,t') = \int \frac{d^d\mathbf{k}}{(2\pi)^d} \sum_{m=1}^{\infty} \frac{2}{(\pi m)^2 + \mathbf{k}^2} \sin(\pi m \tau) \sin(\pi m \tau') e^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{t}')} \tag{2.125}$$

$$
\begin{aligned}
g_{gh}^{DR} &= \int \frac{d^d \mathbf{k}}{(2\pi)^d} \sum_{m=1}^{\infty} 2\sin(\pi m \tau)\sin(\pi m \tau') e^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{t}')} \\
&= \delta(\tau,\tau')\delta(\mathbf{t}-\mathbf{t}') = \delta^{d+1}(t,t')
\end{aligned} \tag{2.126}
$$

Taking the limit $d \to 0$ of these expression we recover the usual propagators that we computed in both time slicing and mode regularization. However, as already remarked, this limit should be formally implemented once one has already calculated the resulting Feynman graphs, so to avoid ill-defined product of distributions. The counterterm V_{DR} can also be computed in the same way used for mode regularization, taking care that with this regularization the value of some integrals do not coincide with the ones calculated using a different regularization.

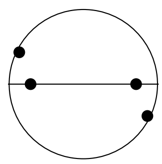
The result one finds after the calculation is ([9])

$$
V_{DR} = -\frac{1}{8} R \tag{2.127}
$$

We can see that this regularization procedure has the advantage that it does not break coordinate invariance, as one can see from the fact this counterterm does not contain the factors $\Gamma\Gamma$ that are present in the other regularization schemes, which make the counterterm to be no more coordinate invariant (since the quantity Γ is not itself a tensor).

An example of an integral which takes different values using different regularizations can be explicitly given by the graph [16]

$$
I_{10} = \int_0^1 d\tau d\sigma \big(\partial_\tau g(\tau,\sigma)\big)\big(\partial_\tau \partial_\sigma g(\tau,\sigma)\big)\big(\partial_\sigma g(\tau,\sigma)\big) = \tag{2.128}
$$



of the same reference, where explicit calculation shows that

$$
I_{10}^{MR} = -\frac{1}{12} \qquad , \qquad I_{10}^{DR} = -\frac{1}{24} \tag{2.129}
$$

[16]where here dots denotes derivatives, like for example the basic propagator $\left\langle \dot\phi^i(\tau)\dot\phi^j(\sigma)\right\rangle$ would be given by a line with two dots joining the worldline points τ and σ.

This concludes our digression about well-known path integral regularization procedures.

2.6 DeWitt Proper-time Expansion

We begin here a brief explanation of the original DeWitt's iterative procedure for the heat kernel expansion on a curved space-time manifold [15]. We will use the form of the heat kernel described here in chapter (3) when analyzing Guven's expansion, and will compare our results with the ones obtained with a generalization of this method. We therefore analyze the case of a free scalar quantum field theory obeying to the Klein-Gordon equation

$$(-\Box + m^2 + \xi R)\phi = 0 \tag{2.130}$$

Before proceeding further it is useful to spend some words about the so-called "Synge's world function" [35], or geodesic interval, $\sigma(x, x')$. It is actually a bi-scalar function (a functional which depends on two space-time points x and x' and which transforms as a scalar under a change of coordinates at both x and x') of the manifold, and geometrically corresponds to half the square of the geodesic distance evaluated between its arguments. It can be seen as a generalization of the flat term $\frac{1}{2}(x - x')^2$ on a curved space-time, and it is related to the action functional evaluated along classical paths (also called "on-shell action") as

$$S[x_{cl}] = \int_{\tau'}^{\tau} d\tilde{\tau}\ L[x_{cl}(\tilde{\tau})] = \frac{\sigma(x, x')}{\tau - \tau'} \tag{2.131}$$

To verify this last statement we recall the free action on flat spaces

$$S[x] = \int_{\tau'}^{\tau} \frac{1}{2}\eta_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\ d\tilde{\tau} \tag{2.132}$$

Setting the boundary condition $x(\tau) = x$, $x(\tau') = x'$ and recalling the usual classical solution with the chosen boundary conditions

$$x^{\mu}_{cl}(\tilde{\tau}) = x^{\mu} + (x'^{\mu} - x^{\mu})\frac{\tilde{\tau}}{(\tau - \tau')} \tag{2.133}$$

we immediately get

$$S[x_{cl}] = \int_{\tau'}^{\tau} d\tilde{\tau} \frac{1}{2} \eta_{\mu\nu} (x^\mu - x'^\mu)(x^\nu - x'^\nu) = \frac{1}{2}(x - x')^2 (\tau - \tau') \tag{2.134}$$

which of course leads to

$$\sigma(x, x') = \frac{1}{2}(x - x')^2 \tag{2.135}$$

Now that the basic definition of the world function has been given, one can obtain important relations of this function using the Hamilton-Jacobi equation on the on-shell action $S[x_{cl}]$ evaluated at initial point x'

$$\frac{\partial S}{\partial \tau} + H = 0 \tag{2.136}$$

where in our case

$$S = \frac{\sigma(x, x')}{s} \quad , \quad H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu \quad , \quad p_\mu \equiv \frac{\partial S}{\partial x^\mu} = \frac{\partial_\mu \sigma}{s} \tag{2.137}$$

where now $s = \tau - \tau'$. This equation then becomes an equation for the world function

$$g^{\mu\nu}(x) \partial_\mu \sigma(x, x') \partial_\nu \sigma(x, x') = 2\sigma(x, x') \tag{2.138}$$

which can be rewritten in a more compact expression, recalling that σ is a scalar

$$\nabla_\mu \sigma(x, x') \nabla^\mu \sigma(x, x') = 2\sigma(x, x') \tag{2.139}$$

Since the world function is a symmetric function of both its arguments $\sigma(x, x') = \sigma(x', x)$, an equal relation also holds at point x'.

We can now proceed with the description of DeWitt's procedure. It is based on an ansatz about the heat kernel, which is inspired by its form on flat manifolds (1.31) and uses separation into a leading non-analytic part and a smooth function "$\Omega(x, x', s)$" which can then be expanded into an asymptotic series as done in the flat case ((1.32)). This ansatz reads

$$K(x, x', s) = \sqrt{\frac{\Delta(x, x')}{(4\pi i s)^D}} e^{i\frac{\sigma}{2s} - im^2 s} \Omega(x, x', s) \tag{2.140}$$

Here Δ denotes the rescaled "Van Vleck-Morette" determinant, which is defined as

$$\Delta(x,x') = -\frac{\det\left[-\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x'^\beta}\sigma(x,x')\right]}{\sqrt{g(x)}\sqrt{g(x')}} \tag{2.141}$$

and is also a biscalar quantity.

We note that this ansatz contains the free-flat heat kernel $K_0 = (4\pi i s)^{-\frac{D}{2}} e^{iS_{cl}}$ (notice here the different normalization factor, which can be reabsorbed into the worldline proper time parameter employing the transformation $s' = 2s$) and the right factor $[g^{-\frac{1}{4}}(x) g^{-\frac{1}{4}}(x')]$ (as one can see from the explicit expression (2.141)) that we get for path integrals in curved space. It is immediate to see that this ansatz solves the heat kernel equation (1.58) and (1.59), provided that the smooth function Ω verifies

$$i\partial_s \Omega + \Delta^{-\frac{1}{2}}\Box(\Delta^{\frac{1}{2}}\Omega) + \frac{i}{s}\sigma^{;\alpha}\Omega_\alpha - \xi R\Omega = 0 \tag{2.142}$$

with the initial condition

$$\Omega(x,x',0) = 1 \tag{2.143}$$

This function can then be expanded in a proper-time series, obtaining the familiar expansion for the heat kernel

$$\Omega(x,x',s) = \sum_{n=0}^{\infty} \Omega_n(x,x')(is)^n \quad , \quad a_0(x,x') = 1 \tag{2.144}$$

Inserting this last expansion into equation (2.142) and equaling terms with the same power in s one obtains the recursive relations

$$\sigma^{;\alpha}\Omega_{0;\alpha} = 0 \tag{2.145}$$

$$-(n+1)\Omega_{n+1} + \Delta^{-\frac{1}{2}}\Box(\Delta^{\frac{1}{2}}\Omega_n) - \sigma^{;\alpha}\Omega_{n+1;\alpha} - \xi R\Omega_n = 0 \tag{2.146}$$

The first equation is immediately solved noting that $\Omega_0 = 1$; the second equation can be evaluated in an iterative way when one is only interested in the coincidence limit, with the aid of purely geometrical relations [17] (we follow DeWitt's notation in which square

[17] See [15] for an explicit proof

brackets around a biscalar denotes that it is evaluated for coinciding points)

$$[\sigma] = 0 \quad , \quad [\sigma_{;\alpha}] = 0 \quad , \quad [\sigma_{;\alpha\beta}] = g_{\alpha\beta} \quad , \quad [\sigma_{;\alpha\beta\gamma}] = 0$$

$$[\Delta^{\frac{1}{2}}] = 1 \quad , \quad [(\Delta^{\frac{1}{2}})_{;\alpha}] = 0 \quad , \quad [\Box\Delta^{\frac{1}{2}}] = \frac{1}{6}R$$

$$[\Box^2\Delta^{\frac{1}{2}}] = \frac{1}{30}(R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - R^{\alpha\beta}R_{\alpha\beta}) + \frac{1}{36}R^2 + \frac{1}{5}\Box R \tag{2.147}$$

These are the terms one needs for the evaluation of the coefficient Ω_2. More of these relations are necessary if one wants to get higher-order coefficients.

Now we can iteratively evaluate the coefficients Ω_n: setting $n = 0$ in (2.146)

$$-\Omega_1 + \Delta^{-\frac{1}{2}}\Box\Delta^{\frac{1}{2}} - \sigma^{;\alpha}\Omega_{1;\alpha} - \xi R = 0 \tag{2.148}$$

Therefore, taking the coincidence limit we find

$$[\Omega_1] = a_1(x', x') = [\Box\Delta^{\frac{1}{2}}] - \xi R = \left(\frac{1}{6} - \xi\right)R \tag{2.149}$$

For $n = 2$

$$-2\Omega_2 + \Delta^{-\frac{1}{2}}\Box(\Delta^{\frac{1}{2}}\Omega_1) - \sigma^{;\alpha}\Omega_{2;\alpha} - \xi R\Omega_1 = 0 \tag{2.150}$$

If we take again the coincidence limit we get

$$[\Omega_2] = a_2(x', x') = \frac{1}{2}\{[\Box(\Delta^{\frac{1}{2}}\Omega_1)] - \xi R[\Omega_1]\} \tag{2.151}$$

where

$$[\Box(\Delta^{\frac{1}{2}}\Omega_1)] = [\Omega_1][\Box\Delta^{\frac{1}{2}}] + [\Box\Omega_1] \tag{2.152}$$

If one evaluates $[\Box\Omega_1]$ by acting with $\Box$ operator on equation (2.148) and then taking the coincidence limit, one gets

$$[\Box\Omega_1] = \frac{1}{3}([\Box^2\Delta^{\frac{1}{2}}] + [\Box\Delta^{\frac{1}{2}}]^2 - \xi\Box R) \tag{2.153}$$

Now the equation for $[\Omega_2]$ contains only known quantities: substituting one finally gets the second coefficient of the expansion

$$[\Omega_2] = \frac{1}{180}(R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - R^{\alpha\beta}R_{\alpha\beta}) + \frac{1}{2}\Big(\frac{1}{6} - \xi\Big)^2 R^2 + \frac{1}{6}\Big(\frac{1}{5} - \xi\Big)\Box R \tag{2.154}$$

We will obtain again these results in the next chapter using Guven's path-integral procedure. As noticed before this results are obtained using a different regularization: using our regularization the correct results are obtained setting $s \to \frac{s}{2}$, thus getting

$$a_1(x', x') = \frac{1}{2}\Big(\frac{1}{6} - \xi\Big)R \tag{2.155}$$

and

$$a_2(x', x') = \frac{1}{720}(R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - R^{\alpha\beta}R_{\alpha\beta}) + \frac{1}{8}\Big(\frac{1}{6} - \xi\Big)^2 R^2 + \frac{1}{24}\Big(\frac{1}{5} - \xi\Big)\Box R \tag{2.156}$$

As an ultimate remark we note that we can rewrite the ansatz exponentiating the disconnected terms depending on R, so that

$$K(x, x', s) = \sqrt{\frac{\Delta(x,y)}{(4\pi i s)^D}}\, e^{i\left[\frac{\sigma}{2s} + \left(\frac{1}{6} - \xi\right)Rs - ms\right]}\tilde{\Omega}(x, y, s) \tag{2.157}$$

in which $\tilde{\Omega}(x, x', s)$ has the same expression of $\Omega(x, y, s)$ but contains no term that vanishes when one sets $R = 0$ but not its covariant derivatives. This statement was originally made as a postulate by Parker, and only successively proven in [30]. The expression given in (2.157) is usually called "R-summed form of the heat kernel".

Chapter 3

Study of a New Regularization Proposal: a Flat Space Method on Curved Manifolds

We here begin the last chapter of our dissertation. In the previous chapter we have shown how one can deal with quantum mechanical path integrals for scalar particles on curved spaces. We have seen that when one starts considering an underling curved manifold, explicit calculations in order to obtain the perturbative expansion require lot more efforts than in flat spaces, like extra ghost graphs evaluation and computation of counterterms at each order. Therefore, a procedure which allows one to use flat-space methods also on curved spaces would be very welcome. A procedure of this kind, which allows one to decouple gravitational interaction from the kinetic term, is precisely the procedure suggested by J. Guven in 1987. It makes use of the adiabatic renormalization procedure introduced by L. Parker in [32], which has shown how working in the Riemann normal coordinates frame one can obtain an equation for the propagator $G(x, x')$ of a scalar auto-interacting field in which the gravitational coupling is all expressed into a potential-like term, therefore allowing to express the heat kernel as a flat-space path integral with an extra gravitational potential term [1], at least for the firsts orders of the perturbative expansion. These extra terms in the action then will give rise to extra terms in the perturbative heat kernel expansion, proportional to products and contractions of Riemann tensors, that in other regularizations are generated by the expansion of

[1] Which can be simply treated as an extra term that appears in the interacting part of the action.

the local counterterms and ghost propagators. In his original paper and his sequent works Parker correctly obtained the first two Seely-DeWitt coefficients for coinciding points, but no effort to proceed at higher orders was done by the author. Afterwards physicists L. Hu and D. J. O'Connor generalized this result [26] at all orders, and Guven presumably used this paper as a base for defining a flat path-integral based regularization scheme. We here face two fundamental problems: the first is that it is unknown how the gravitational potential Guven uses in his paper has been found, and the second lies inside the generalization made by Hu and O'Connor. The first problem can be solved out since a slightly different potential can be obtained from basic considerations about the Green function equation, which we will show that once implemented will lead to the correct first two Seeley-DeWitt coefficients; for the second an easy solution has not been found. Since in the paper by Hu and O'Connor is stated that this method for decoupling the gravitational interaction can be generalized to all orders using recursively the Lorentz invariance of the propagator, we proceeded into the evaluation of the third coefficient $a_3(x,x)$ using the method defined by Guven: the result we obtained at the end seems to be inconsistent with the one obtained in other ways, like the one calculated via a curved path integral with dimensional regularization in ([6]) or the one calculated by Gikley ([21]) using DeWitt proper time expansion. This seems to be a clue of the non-correctness of this procedure beyond fourth order in the adiabatic expansion (or, equivalently, beyond order six of the Riemannian expansion). On the other hand, we tested this construction on maximally symmetric spaces [2] (e.g. spheres), finding that it reproduces the correct Seeley-DeWitt coefficients in arbitrary D-dimensions up to order s^3. [3]

The chapter makes use of the framework of Quantum Field Theory within the worldline approach, as described in section (1.4), and is then structured as follows: we begin obtaining the Riemann expansion for the metric, its inverse and its determinant up to sixth order. Two different ways for this expansion have been studied: the first, which rely on the Alvarez-Guamé expansion ([1]) has been shown to give up to two-hundred terms at the fifth order, becoming too cumbersome for a direct evaluation of the the expansion

[2] Where Lorentz invariance based arguments have better chances to work, and where calculations can be done in a easier way thanks to the simple form which the Riemann tensor presents on these spaces

[3] We actually performed the calculation up to order s^6, but no comparable results have been found in literature for arbitrary dimensions up to order s^3. This is because this procedure brings an undeniable simplification on the calculations needed to get the perturbative expansion. Anyway we tested our results comparing the type-A trace anomaly we get up to $D = 12$ dimensions with ones obtained by other authors using different methods, which seem to be in agreement with ours.

at the required order, while the second has been used to obtain the ultimate result. This result seems to be in accordance with the one obtained in [6]. Then we will explain how the adiabatic renormalization works, for which explicit calculation at order four has been made by the author, and how subtleties arise when one tries to generalize this result at a higher order. We will present some generalizations of the equations originally given by Parker in order to obtain an equation at sixth order for the propagator. We will then use Guven method to calculate the Seeley-DeWitt coefficient $a_3(x,x)$, showing that the result is not the same obtained by other authors. At the end of the chapter we will finally evaluate the diagonal part of the heat kernel, up to order s^6 and on arbitrary D dimensions, on maximally symmetric spaces and compare our results with known ones. What we find is that Guven's procedure is well-defined on this kind of spaces, leading to the correct result at any order of the calculation we performed. Other calculations are of course required, and also if one is not able to use Guven's method on arbitrary spaces up to fourth adiabatic order, a non-trivial extension of this method could be probably worked out.

3.1 Riemann Normal Coordinates Expansion

Riemann normal coordinates are the closest curved-space analogue of the flat Cartesian coordinates. In fact, this coordinate system is defined in such a way that geodesics that emanates from one point to another one of the manifold are mapped into a straight line. This coordinate system generically does not cover the whole manifold; instead it's defined only in the neighborhood of a given point, which we will take as the origin of the coordinate system (and we call it x'). This coordinate frame is well defined provided that geodesics do not cross, which can be always ensured by choosing a sufficiently small neighborhood.

The main idea behind Riemann normal coordinate expansion is to use geodesic through a given point to define geodesic for nearby points, in a way to recreate locally, if the manifold is smooth enough, an equation of motion which has the same form of the flat space one. Geometrically the Riemann coordinates of a generic point x are defined by the components of the tangent vector, evaluated at the origin, to the geodesic which links x and x'. Calling z^μ the components of the vector mentioned above and s the arc length of this geodesic measured between x and x' we have that the Riemann normal coordinates are defined as

$$x^{\mu} = sz^{\mu} \tag{3.1}$$

(which is of course the same equation one has for the classical path x^{μ}_{cl} as defined in (1.27) if we take x^{μ}_{i} to be represented by the null vector). Since this vector is the tangent to the geodesic, it can be expressed in an explicit way as

$$z^{\mu} = -\nabla^{\mu}\sigma(x,y) \tag{3.2}$$

We note that, since z^{μ} is a vector belonging to the tangent space which transforms as a contravariant vector, every expansion of a tensorial quantity as a power series of z^{μ} will be covariant. Equation (3.1) can be taken as the defining equation for Riemann normal coordinates. Some authors prefer to give a different (but of course equivalent) definition of these coordinates: recalling the geodesic equation on a general coordinate frame

$$\ddot{x}^{\mu} + \Gamma^{\mu}_{\rho\sigma}\dot{x}^{\rho}\dot{x}^{\sigma} = 0 \tag{3.3}$$

one can iteratively solve this equation on arbitrary coordinates[4], and then Taylor expand its solution $x^{\mu}(s)$ around $s = 0$. One then obtains

$$x^{\mu}(s) = x^{\mu}(0) + z^{\mu}s + \frac{1}{2}\Gamma^{\mu}_{\rho_1\rho_2}z^{\rho_1}z^{\rho_2}s^2 + \cdots + \frac{1}{n!}\Gamma^{\mu}_{\rho_1\cdots\rho_n}z^{\rho_1}\ldots z^{\rho_n}s^n \tag{3.4}$$

where $\Gamma^{\mu}_{\rho_1\cdots\rho_n} = \partial_{\rho_n}\ldots\partial_{\rho_3}\Gamma^{\mu}_{\rho_1\rho_2}|_{x=x'}$. Riemann normal coordinates can then be defined as the set of coordinates in which

$$\Gamma^{\mu}_{(\rho_1\cdots\rho_n)} = 0 \tag{3.5}$$

where round brackets around some indices denotes complete symmetrization of those indices. Note that all these quantities are evaluated at the origin of our coordinate system. This last equation can be taken as the defining equation of Riemann normal coordinates: as we have shown, the two formulations are of course equivalent. At the lowest order this last relation implies that

[4]Taking in consideration also the boundary conditions

$$\partial_\alpha g_{\mu\nu}(x)|_{x=x'} = 0 \tag{3.6}$$

since by construction we have $\nabla_\alpha g_{\mu\nu}(x) = 0$.

Before proceeding further we notice that a generic expansion of the metric in Riemann normal coordinates up to sixth order reads

$$\begin{aligned} g_{\mu\nu}(x = x' + z) =& \eta_{\mu\nu} + A_{\mu\alpha\nu\beta} z^\alpha z^\beta + B_{\mu\alpha\nu\beta\gamma} z^\alpha z^\beta z^\gamma + C_{\mu\alpha\nu\beta\gamma\delta} z^\alpha z^\beta z^\gamma z^\delta + \\ &+ D_{\mu\alpha\nu\beta\gamma\delta\eta} z^\alpha z^\beta z^\gamma z^\delta z^\eta + E_{\mu\alpha\nu\beta\gamma\delta\eta\theta} z^\alpha z^\beta z^\gamma z^\delta z^\eta z^\theta + o(z^7) \end{aligned} \tag{3.7}$$

since from (3.6) we know that the first partial derivative of the metric vanishes in these coordinates. The same can be done with the inverse metric $g^{\mu\nu}$, obtaining [5]

$$\begin{aligned} g^{\mu\nu}(x = x' + z) =& \eta^{\mu\nu} + A'^{\mu\ \nu}_{\ \alpha\ \beta} z^\alpha z^\beta + B'^{\mu\ \nu}_{\ \alpha\ \beta\gamma} z^\alpha z^\beta z^\gamma + C'^{\mu\ \nu}_{\ \alpha\ \beta\gamma\delta} z^\alpha z^\beta z^\gamma z^\delta + \\ &+ D'^{\mu\ \nu}_{\ \alpha\ \beta\gamma\delta\eta} z^\alpha z^\beta z^\gamma z^\delta z^\eta + E'^{\mu\ \nu}_{\ \alpha\ \beta\gamma\delta\eta\theta} z^\alpha z^\beta z^\gamma z^\delta z^\eta z^\theta + o(z^7) \end{aligned} \tag{3.8}$$

It is important to notice that all these coefficients are evaluated at the origin of coordinates x' and that they are all proportional to contractions, covariant derivatives or products of Riemann tensors, as proven explicitly in [23]. A relation between these coefficients can be readily obtained considering that, by construction, these metric tensors have to verify the relation

$$g^{\mu\rho}(x) g_{\rho\nu}(x) = \delta^\mu_{\ \nu} \tag{3.9}$$

Inserting equation (3.8) in this last expression and equating to zero terms at each order in z (since we already have $\eta^{\mu\rho}\eta_{\rho\nu} = \delta^\mu_{\ \nu}$) we obtain

$$A'^{\mu\ \nu}_{\ \alpha\ \beta} = -\eta^{\mu\rho}\eta^{\nu\sigma} A_{\rho\alpha\sigma\beta}$$

$$B'^{\mu\ \nu}_{\ \alpha\ \beta\gamma} = -\eta^{\mu\rho}\eta^{\nu\sigma} B_{\rho\alpha\sigma\beta\gamma}$$

[5] Notice that once that RNC expansion has been employed indices are raised and lowered by the flat metric $\eta^{\mu\nu}$.

$$C'^{\mu\ \nu}_{\ \alpha\ \beta\gamma\delta} = -\eta^{\mu\rho}\eta^{\nu\sigma}C_{\rho\alpha\sigma\beta\gamma\delta} - \eta^{\nu\sigma}A'^{\mu\ \rho}_{\ \alpha\ \beta}A_{\rho\gamma\sigma\delta}$$

$$D'^{\mu\ \nu}_{\ \alpha\ \beta\gamma\delta\eta} = -\eta^{\mu\rho}\eta^{\nu\sigma}D_{\rho\alpha\sigma\beta\gamma\delta\eta} - \eta^{\mu\rho}A_{\rho\alpha\sigma\beta}B'^{\sigma\ \nu}_{\ \gamma\ \delta\eta} - \eta^{\mu\rho}B_{\rho\alpha\sigma\beta\gamma}A'^{\sigma\ \nu}_{\ \delta\ \eta}$$

$$\begin{aligned} E'^{\mu\ \nu}_{\ \alpha\ \beta\gamma\delta\eta\theta} = &-\eta^{\mu\rho}\eta^{\nu\sigma}E_{\mu\alpha\nu\beta\gamma\delta\eta\theta} - \eta^{\mu\rho}A_{\rho\alpha\sigma\beta}C'^{\sigma\ \nu}_{\ \gamma\ \delta\ \eta\theta} - \eta^{\mu\rho}B_{\rho\alpha\sigma\beta\gamma}B'^{\rho\ \nu}_{\ \delta\ \eta\theta} + \\ &-\eta^{\mu\rho}C_{\rho\alpha\sigma\beta\gamma\delta}A'^{\rho\ \nu}_{\ \eta\ \theta} \end{aligned} \tag{3.10}$$

We can now proceed describing the method in which the coefficients of the expansion can be obtained. Relation (3.5) can be used to define the general Riemann coordinate expansion of a tensor. In what's next we will follow the method discovered by Alvarez-Freedman-Mukhi explained in detail in [1]. Taken a tensor quantity, for example $T_{ij}(z)$, its expansion around the origin $z = 0$ reads

$$T_{\mu\nu}(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\frac{\partial}{\partial z^{\alpha_n}} \cdots \frac{\partial}{\partial z^{\alpha_1}} T_{\mu\nu}\Big)\Big|_{z=0} z^{\alpha_1} \ldots z^{\alpha_n} \tag{3.11}$$

where the Taylor coefficients are also tensors with indices belonging to the tangent or cotangent space at the origin, since as remarked before Riemann coordinates are themselves vectors belonging to the tangent space at the origin. Symmetrised ordinary derivatives can then be expressed in terms of covariant derivative like, for example

$$\partial_\alpha T_{\mu\nu}|_{z=0} = \nabla_\alpha T_{\mu\nu}|_{z=0} \quad , \quad \partial_\alpha\partial_\beta T_{\mu\nu}|_{z=0} = \nabla_\alpha\nabla_\beta T_{\mu\nu}|_{z=0} + \Gamma^\rho_{\alpha\mu\beta}T_{\rho\nu} + \Gamma^\rho_{\alpha\nu\beta}T_{\mu\rho} \tag{3.12}$$

and so on. Using relation (3.5) one is then able to express the extra terms proportional to Γ as contractions or products of Riemann tensors. In fact evaluating the Riemann tensor at the origin one finds that

$$R^\mu_{\ \alpha\beta\gamma}(0) = \partial_\beta\Gamma^\mu_{\alpha\gamma}(0) - \partial_\gamma\Gamma^\mu_{\alpha\beta}(0) \tag{3.13}$$

and inserting this expression in equation (3.5)

$$\Gamma^{\mu}_{\alpha\beta\nu} = \partial_\nu \Gamma^{\mu}_{\alpha\beta}|_{z=0} = \frac{1}{3}(R^{\mu}{}_{\alpha\nu\beta} + R^{\mu}{}_{\beta\nu\alpha})|_{z=0} \tag{3.14}$$

so that one gets, at second order of the expansion

$$\begin{aligned} T_{\mu\nu}(z) =& T_{\mu\nu}(0) + \nabla_\alpha T_{\mu\nu}(0) z^\alpha + \frac{1}{2}\Big[\nabla_\beta \nabla_\alpha T_{\mu\nu}(0) + \frac{1}{3} R^{\rho}{}_{\alpha\mu\beta}(0) T_{\rho\nu}(0) + \\ &+ \frac{1}{3} R^{\rho}{}_{\alpha\nu\beta}(0) T_{\mu\rho}(0)\Big] z^\alpha z^\beta + o(z^3) \end{aligned} \tag{3.15}$$

The same can be done with the metric tensor, obtaining at second order

$$g_{\mu\nu}(z) = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta} z^\alpha z^\beta + o(z^3) \tag{3.16}$$

One could theoretically continue in this way in order to obtain the expansion of a general tensor, but at higher orders the relation between covariant derivatives of Riemann tensor and symmetrised derivatives of the connection becomes more and more involved and calculation then becomes very cumbersome: an explicit computation at fifth order for the metric tensor gives back up to 200 terms that one has to evaluate in order to obtain the correct coefficient of the expansion.

Anyway, if one is only interested in the expansion of the metric tensor another road can be followed, which turns out to be of simpler computation. This method makes use of an auxiliary affine parameter, $\lambda \in [0,1]$, which is defined such that for every value of the proper time τ one gets that $x^\mu(\tau,\lambda)$ is the geodesic which links the origin x' with the point $x(\tau)$

$$x^\mu(\tau, \lambda = 0) = x'^\mu \quad , \quad x^\mu(\tau, \lambda = 1) = x^\mu(\tau) \tag{3.17}$$

Then by construction the quantity $x^\mu(\tau,\lambda)$ satisfies a geodesic equation

$$\frac{D}{D\lambda}\frac{dx^\mu}{d\lambda} = \frac{d^2 x^\mu}{d\lambda^2} + \Gamma^{\mu}_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0 \tag{3.18}$$

and Riemann normal coordinates are then defined as

$$z^\mu = \frac{dx^\mu}{d\lambda}\Big|_{\lambda=0} \tag{3.19}$$

which implies that the geodetic equation (3.18) becomes, before setting $\lambda = 0$ [6]

$$\frac{D}{D\lambda} z^\mu(\tau, \lambda) = 0 \tag{3.20}$$

One then can obtain an expansion for $\lambda \to 0$ of the metric tensor expanding the scalar worldline free Lagrangian

$$L[x(\tau)] = \frac{1}{2} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu \tag{3.21}$$

where the dot indicates a derivative with respect of proper-time parameter τ. This expansion then reads

$$L[x(\tau)] = L[x(\tau, \lambda = 1)] = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n L}{d\lambda^n}\Big|_{\lambda=0} (1-0)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{D^n L}{D\lambda^n}\Big|_{\lambda=0} \tag{3.22}$$

since L is a scalar. The use of covariant derivatives instead of ordinary derivatives helps in performing calculations thanks to the identities

$$\frac{Dz^\mu}{D\lambda} = \frac{Dg_{\mu\nu}}{D\lambda} = 0 \quad , \quad \frac{D\dot{x}^\mu}{D\lambda} = \frac{Dz^\mu}{D\tau} \quad , \quad \left[\frac{D}{D\lambda}, \frac{D}{D\tau}\right] V^\mu = z^\lambda \dot{x}^\sigma R^\mu{}_{\rho\lambda\sigma} V^\rho \tag{3.23}$$

where V^μ is an arbitrary vector. The second and third identities can be verified with an explicit computation of the two sides of each equation.

We report then just a few derivatives as an example of how the calculation works

$$\frac{DL}{D\lambda} = g_{\mu\nu}(x) \frac{Dz^\mu}{D\tau} \dot{x}^\nu \tag{3.24}$$

$$\begin{aligned} \frac{D^2 L}{D\lambda^2} &= g_{\mu\nu}(x) \frac{Dz^\mu}{D\tau} \frac{Dz^\nu}{D\tau} + g_{\mu\nu}(x) \frac{D}{D\lambda} \Big(\frac{Dz^\mu}{D\tau} \Big) \dot{x}^\nu \\ &= g_{\mu\nu}(x) \frac{Dz^\mu}{D\tau} \frac{Dz^\nu}{D\tau} + g_{\mu\nu}(x) \left[\frac{D}{D\lambda}, \frac{D}{D\tau}\right] z^\mu \dot{x}^\nu = \\ &= g_{\mu\nu}(x) \frac{Dz^\mu}{D\tau} \frac{Dz^\nu}{D\tau} + g_{\mu\nu}(x) z^\lambda \dot{x}^\sigma R^\mu{}_{\rho\lambda\sigma} z^\rho \dot{x}^\nu \end{aligned} \tag{3.25}$$

[6]Note that also in the following calculations we will set $z^\mu = \frac{dx^\mu}{d\lambda}$ also when λ differs from 0.

$$\frac{D^3L}{D\lambda^3} = 3g_{\mu\nu}(x)R^{\mu}{}_{\rho\lambda\sigma}z^{\lambda}z^{\rho}\dot{x}^{\sigma}\frac{Dz^{\nu}}{D\tau} + g_{\mu\nu}(x)R^{\mu}{}_{\rho\lambda\sigma}z^{\lambda}z^{\rho}\dot{x}^{\nu}\frac{Dz^{\sigma}}{D\tau} + \dots \tag{3.26}$$

$$\frac{D^4L}{D\lambda^4} = 4g_{\mu\nu}(x)R^{\mu}{}_{\rho\lambda\sigma}z^{\lambda}z^{\rho}\frac{Dz^{\sigma}}{D\tau}\frac{Dz^{\nu}}{D\tau} + \dots \tag{3.27}$$

where dots indicates extra terms that vanish when $\lambda = 0$, since $\dot{x}(\tau, 0) = \dot{x}' = 0$. Reorganizing indices such to collect a global term $\frac{Dz^{\mu}}{D\tau}\frac{Dz^{\nu}}{D\tau}$ in the Lagrangian we immediately obtain

$$L = \frac{1}{2}\Big(g_{\mu\nu}(0) - \frac{1}{3}R_{\mu\alpha\nu\beta}(0)z^{\alpha}z^{\beta} + o(z^3)\Big)\frac{Dz^{\mu}}{D\tau}\frac{Dz^{\nu}}{D\tau} \tag{3.28}$$

which allows us to identify the expansion of the metric in Riemann normal coordinates

$$g_{\mu\nu}(z) = g_{\mu\nu}(0) - \frac{1}{3}R_{\mu\alpha\nu\beta}(0)z^{\alpha}z^{\beta} + o(z^3) \tag{3.29}$$

in accordance with the expansion found in (3.16). One can continue in this way and obtain an expansion of the metric at the desired order. We report here the result up to sixth order, which coincides with the one obtained in [6]:

$$\begin{aligned} g_{\mu\nu} =& \eta_{\mu\nu} - \frac{1}{3}R_{\mu\alpha\nu\beta}z^{\alpha}z^{\beta} - \frac{1}{6}R_{\mu\alpha\nu\beta;\gamma}z^{\alpha}z^{\beta}z^{\gamma} - \Big(\frac{1}{20}R_{\mu\alpha\nu\beta;\gamma\delta} - \frac{2}{45}R^{\lambda}{}_{\alpha\beta\mu}R_{\lambda\gamma\delta\nu}\Big)z^{\alpha}z^{\beta}z^{\gamma}z^{\delta} + \\ & - \Big(\frac{1}{90}R_{\mu\alpha\nu\beta;\gamma\delta\eta} - \frac{2}{45}R^{\lambda}{}_{\alpha\beta\mu}R_{\lambda\gamma\delta\nu;\eta}\Big)z^{\alpha}z^{\beta}z^{\gamma}z^{\delta}z^{\eta} - \Big(\frac{1}{504}R_{\mu\alpha\nu\beta;\gamma\delta\eta\theta} + \\ & - \frac{17}{1260}R^{\lambda}{}_{\alpha\beta\mu}R_{\lambda\gamma\delta\nu;\eta\theta} - \frac{11}{1008}R^{\lambda}{}_{\alpha\beta\mu;\eta}R_{\lambda\gamma\delta\nu;\theta} - \frac{1}{315}R_{\mu\alpha\beta}{}^{\lambda}R_{\lambda\gamma\delta}{}^{\kappa}R_{\kappa\eta\theta\nu}\Big)z^{\alpha}z^{\beta}z^{\gamma}z^{\delta}z^{\eta}z^{\theta} \\ & + o(z^7) \end{aligned} \tag{3.30}$$

Explicit computation of the derivatives needed at this order are reported in appendix B.

From this last equation one is able to read the coefficients of the expansion (3.8). We can now substitute the coefficients found in (3.7) and obtain the expansion of its inverse: in the end it reads [7]

[7] It is important to gather that these terms (and also the ones one get its inverse and their determinants) are symmetrized by the contraction with z terms, thous they can be equivalently expressed in a full symmetrized fashion simply writing the same expansion with brackets around free indices, like for example $R^{\mu\nu}{}_{\alpha\ \beta}z^{\alpha}z^{\beta} = R^{\mu\ \nu}{}_{(\alpha\ \beta)}z^{\alpha}z^{\beta}$.

$$
\begin{aligned}
g^{\mu\nu} =&\eta^{\mu\nu} + \frac{1}{3}R^{\mu\ \ \nu}_{\ \alpha\ \beta}z^\alpha z^\beta + \frac{1}{6}R^{\mu\ \ \nu}_{\ \alpha\ \beta;\gamma}z^\alpha z^\beta z^\gamma + \Big(\frac{1}{20}R^{\mu\ \ \nu}_{\ \alpha\ \beta;\gamma\delta} + \frac{1}{15}R^{\lambda\ \ \mu}_{\ \alpha\beta}R_{\lambda\gamma\delta}^{\ \ \ \ \nu}\Big)z^\alpha z^\beta z^\gamma z^\delta +\\
&+ \Big(\frac{1}{90}R^{\mu\ \ \nu}_{\ \alpha\ \beta;\gamma\delta\eta} + \frac{1}{15}R^{\lambda\ \ \mu}_{\ \alpha\beta}R_{\lambda\gamma\delta\ ;\eta}^{\ \ \ \ \nu}\Big)z^\alpha z^\beta z^\gamma z^\delta z^\eta + \Big(\frac{1}{504}R^{\mu\ \ \nu}_{\ \alpha\ \beta;\gamma\delta\eta\theta} + \frac{5}{252}R^{\lambda\ \ \mu}_{\ \alpha\beta}R_{\lambda\gamma\delta\ ;\eta\theta}^{\ \ \ \ \nu} +\\
&+ \frac{17}{1008}R^{\lambda\ \ \mu}_{\ \alpha\beta\ ;\eta}R_{\lambda\gamma\delta\ ;\theta}^{\ \ \ \ \nu} - \frac{2}{189}R^{\mu\ \ \lambda}_{\ \alpha\beta}R_{\lambda\gamma\delta}^{\ \ \ \ \kappa}R_{\kappa\eta\theta}^{\ \ \ \ \nu}\Big)z^\alpha z^\beta z^\gamma z^\delta z^\eta z^\theta + o(z^7)
\end{aligned}
\tag{3.31}
$$

In the next section we will need also the Riemann expansion of the determinant of the metric g and its inverse. The former can be found using the following identity

$$
\begin{aligned}
g = \det g_{\mu\nu} = \exp\{(\log\det g_{\mu\nu})\} = \exp\{(Tr\log g_{\mu\nu})\} =\\
= 1 + Tr(\log g_{\mu\nu}) + \frac{1}{2}(Tr\log g_{\mu\nu})^2 + \dots
\end{aligned}
\tag{3.32}
$$

and using expansion of $\log(1+x) = x - (1/2)x^2 + \dots$ we obtain

$$
\begin{aligned}
g =&1 + Az^2 + Bz^3 + \Big(C - \frac{1}{2}Tr(A_{\mu\rho}A^\rho_{\ \nu}) + \frac{1}{2}A^2\Big)z^4 + \Big(D - Tr(A_{\mu\rho}B^\rho_{\ \nu}) + AB\Big)z^5 +\\
&+ \Big(E - \frac{1}{2}Tr(B_{\mu\rho}B^\rho_{\ \nu}) + \frac{1}{3}Tr(A_{\mu\rho}A^{\rho\sigma}A_{\sigma\nu}) + \frac{1}{2}B^2 + AC + \frac{1}{3!}A^3\\
&- \frac{1}{2}ATr(A_{\mu\rho}A^\rho_{\ \nu}) - Tr(A_{\mu\rho}C^\rho_{\ \nu})\Big)z^6
\end{aligned}
\tag{3.33}
$$

where the terms of these expansion are the same of (3.7) in which we denoted $A = Tr(A_{\mu\nu})$ and so on for all the other factors (and where we also omitted indices which are contracted with z for notational convenience). Inserting these terms in the last equation one finds

$$
\begin{aligned}
g =&1 - \frac{1}{3}R_{\alpha\beta}z^\alpha z^\beta - \frac{1}{6}R_{\alpha\beta;\gamma}z^\alpha z^\beta z^\gamma + \Big(\frac{1}{18}R_{\alpha\beta}R_{\gamma\delta} - \frac{1}{20}R_{\alpha\beta;\gamma\delta} - \frac{1}{90}R^{\lambda\ \ \kappa}_{\ \alpha\beta}R_{\lambda\gamma\delta\kappa}\Big)z^\alpha z^\beta z^\gamma z^\delta +\\
&+ \Big(\frac{1}{18}R_{\alpha\beta}R_{\gamma\delta;\eta} - \frac{1}{90}R_{\alpha\beta;\gamma\delta\eta} - \frac{1}{90}R^{\lambda\ \ \kappa}_{\ \alpha\beta}R_{\lambda\gamma\delta\kappa;\eta}\Big)z^\alpha z^\beta z^\gamma z^\delta z^\eta +\\
&+ \Big(\frac{1}{72}R_{\alpha\beta;\eta}R_{\gamma\delta;\theta} - \frac{1}{504}R_{\alpha\beta;\gamma\delta\eta\theta} - \frac{1}{315}R^{\lambda\ \ \kappa}_{\ \alpha\beta}R_{\lambda\gamma\delta\kappa;\eta\theta} - \frac{1}{336}R^{\lambda\ \ \kappa}_{\ \alpha\beta\ ;\eta}R_{\lambda\gamma\delta\kappa;\theta} +\\
&+ \frac{1}{60}R_{\alpha\beta}R_{\gamma\delta;\eta\theta} + \frac{2}{2835}R_{\rho\alpha\beta}^{\ \ \ \ \lambda}R_{\lambda\gamma\delta}^{\ \ \ \ \kappa}R_{\kappa\eta\theta}^{\ \ \ \ \rho} - \frac{1}{162}R_{\alpha\beta}R_{\gamma\delta}R_{\eta\theta} +\\
&- \frac{1}{30}R_{\alpha\beta}R^{\lambda\ \ \kappa}_{\ \gamma\delta}R_{\lambda\eta\theta\kappa}\Big)z^\alpha z^\beta z^\gamma z^\delta z^\eta z^\theta + o(z^7)
\end{aligned}
\tag{3.34}
$$

Its inverse can be found considering that, since $g^{-1}g = 1$, the relation between the coefficients of the expansion of g and its inverse are the same between the metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ once we suppress the free indices μ, ν. In this way one finds, up to fifth order [8]

$$\begin{aligned} g^{-1} =& 1 + \frac{1}{3}R_{\alpha\beta}z^\alpha z^\beta + \frac{1}{6}R_{\alpha\beta;\gamma}z^\alpha z^\beta z^\gamma + \Big(\frac{1}{18}R_{\alpha\beta}R_{\gamma\delta} + \frac{1}{20}R_{\alpha\beta;\gamma\delta} + \\ &+ \frac{1}{90}R^\lambda{}_{\alpha\beta}{}^\kappa R_{\lambda\gamma\delta\kappa}\Big)z^\alpha z^\beta z^\gamma z^\delta + \Big(\frac{1}{18}R_{\alpha\beta}R_{\gamma\delta;\eta} + \frac{1}{90}R_{\alpha\beta;\gamma\delta\eta} + \\ &+ \frac{1}{90}R^\lambda{}_{\alpha\beta}{}^\kappa R_{\lambda\gamma\delta\kappa;\eta}\Big)z^\alpha z^\beta z^\gamma z^\delta z^\eta + o(z^6) \end{aligned} \tag{3.35}$$

As an extra proof, one can also obtain the same result considering

$$g^{-1} = \det g^{\mu\nu} = e^{\ln \det g^{\mu\nu}} \tag{3.36}$$

and following the same expansion of equations (3.32) and (3.33), substituting each term with the corresponding primed ones (e.g. A' instead of A). As one can explicitly see from equation (C.21), in the next section we will also need the form of g^{-2} up to fourth order in order to evaluate correctly the expansion of the potential: it can be found considering

$$\begin{aligned} g^{-2} &= (g^{-1})^2 = (1 + Az^2 + Bz^3 + Cz^4 + o(z^5)) \\ &= 1 + 2Az^2 + 2Bz^3 + (2C + A^2)z^4 + o(z^5) \\ &= 1 + \frac{2}{3}R_{\alpha\beta}z^\alpha z^\beta + \frac{1}{3}R_{\alpha\beta;\gamma}z^\alpha z^\beta z^\gamma + \\ &\quad + \Big(\frac{3}{18}R_{\alpha\beta}R_{\gamma\delta} + \frac{1}{20}R_{\alpha\beta;\gamma\delta} + \frac{1}{90}R^\lambda{}_{\alpha\beta}{}^\kappa R_{\lambda\gamma\delta\kappa}\Big) + o(z^5) \end{aligned} \tag{3.37}$$

where A, B, C are here the coefficients of the expansion of g^{-1}. Except for the expansion of the metric $g_{\mu\nu}$, the expansion of the other terms performed at this order of the expansion cannot be found anywhere in literature; we assume anyway their correctness, considering also the correctness of our calculations on maximally symmetric spaces (whose Riemann expansion is performed in the same way), as one can see from section (3.4).

[8] Since this is the order required to correctly evaluate the Seeley-DeWitt coefficient a_3.

3.2 An Analysis of Guven's Regularization Procedure

We begin here the review of Guven's method of the heat kernel expansion for a scalar particle [24]. In this section we show how this method fails to recreate correctly the first few Seeley-DeWitt coefficients, at least using the potential given by the paper cited above. Since the idea behind the method seems interesting, we tried to fix the potential term. Starting from the defining equation of the scalar propagator we have been able to find another form of the potential, which is able to recreate correctly the coefficients a_0, a_1 and a_2. Explicit calculations are shown here and the result is compared with the one obtained by other authors. We then find that Guven's method and the analysis by Parker et al. in ref. [32] are closely related, so we proceed in analyzing this last one. Using this method, and also the fact that for low-order of the expansion the propagator is proportional only to the scalar z^2, the authors were able to find the fourth-order adiabatic expansion of the Green function, which turns out to be not proportional to z^2 only : we generalize this procedure in order to find an explicit equation for the sixth-order term, which we will show to contain also the fourth-order term of the expansion. Since the presence of this non-trivial z dependence of the Green function makes impossible for us to prove or disprove one of the crucial statements made by Guven (we will explain the reasons behind this during the present analysis), we proceed calculating the next coefficient of the heat kernel expansion, supposing the correctness of Guven's method. This last calculation will be done in the next section.

Guven analyses the case of a self-interacting scalar field ϕ obeying a Klein-Gordon equation (we denoted in this section $\Box^c = g^{\mu\nu}\nabla_\mu\nabla_\nu$ the covariant box operator on curved spaces, in order not to make confusion with the flat box operator $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$)

$$(-\Box^c + \tilde{m}^2(x) + \xi R(x) - i\epsilon)\phi(x) = 0 \tag{3.38}$$

in which $\tilde{m}^2(x) = m^2 + g\phi + \frac{\lambda}{2}\phi^2$. For our purposes it will be sufficient to consider a free scalar field for which $\tilde{m}^2 = m^2$ since the gravitational-dependent part of the expansion will still be the same.

This method is in fact based on the possibility of separate, with the introduction of Riemann normal coordinates, the gravitational interaction from the kinetic term and include this coupling completely into a quadratic potential-like term. In this way the kinetic term reduces just to the flat box term $\eta^{\mu\nu}\partial_\mu\partial_\nu$, and one is able to cast the heat kernel into a flat path-integral form in which all the gravitation coupling lies into an effective potential term (that will be called V_2 in order to follow Guven's conventions) , as

$$K(x,x',s) = \int Dz\ e^{\frac{i}{s}\int_0^1 d\tau\left[\frac{\dot{z}^2}{4}-s^2m^2-s^2V_2(z(\tau))\right]} \tag{3.39}$$

where z is the same vector defined for the RNC expansion. Notice the different normalization of the path integral, which is in accordance with DeWitt's conventions explained in section (2.6): setting $s=\frac{s'}{2}$ we recover the normalization we used in order to obtain the heat kernel expansion of section (1.3), thus getting an effective potential exponentiated in the path integral which is $\frac{1}{2}V_2$. Let's see how this procedure works.

We start from the defining equation for the scalar propagator, or Green function, for a massive field in a curved background space. It is given by

$$(-\Box^c_x + m^2 + \xi R(x) - i\epsilon)G(x,x') = -\delta(x,x') \tag{3.40}$$

where x and x' are two points belonging to the underlying spacetime manifold, and which lie in a normal neighbourhood of each other. Once we employ the Riemann normal coordinates frame we will take x' to be the origin of our coordinate system. The Green function can then be related to the heat kernel K by [9]

$$G(x,x') = i\int_0^\infty K(x,x',s)ds \tag{3.41}$$

and plugging this relation into equation (3.40) one gets the defining Schrödinger equation for the heat kernel

$$(-\Box^c_x + m^2 + \xi R(x) - i\epsilon)K(x,x',s) = i\partial_s K(x,x',s) \tag{3.42}$$

We can now employ the Riemann normal coordinate expansion, taking the point x' as the origin of our coordinate system. We will call again z^μ the components of this coordinate frame. Now Guven states that, using RNC and rescaling the heat kernel as [10]

[9] See section (1.4)

[10] It is possible to argue that the transformation defined by this last equality modifies the scalar behaviour of the heat kernel: we actually remark that this rescaling can be expressed, in a generic coordinate frame, as $\overline{K}(x,x',s) = \Delta^{-\frac{1}{2}}(x,x')K(x,x',s)$, which, being $\Delta^{\frac{1}{2}}$ a bi-scalar (see (2.141)), leaves

$$\overline{K} = g^{\frac{1}{4}}(x)K(x,x',s)g^{\frac{1}{4}}(x') = g^{\frac{1}{4}}(x)K(x,x',s) \tag{3.43}$$

equation (3.42) for the newly-scaled heat kernel becomes

$$(-\eta^{\mu\nu}\partial_\mu\partial_\nu + V(z))\overline{K}(z,s) = i\partial_s\overline{K}(z,s) \tag{3.44}$$

where the potential $V(z)$ is given by

$$V(z) = m^2 + V_2(z) \tag{3.45}$$

with the "gravitational potential" (we indicate the one given by Guven as V_2 in order to follow his conventions)

$$V_2 = \xi R + 2\partial_\mu g^{\frac{1}{4}} g^{\mu\nu}\partial_\nu g^{\frac{1}{4}} \tag{3.46}$$

It is immediate to verify that, even if we suppose equation (3.44) to be true, this is not the correct form of the potential. In order to show the failure of this potential it is sufficient to analyze the zero-th order: in fact for from equation (1.53) we see that the first coefficient of the expansion is simply proportional to minus the potential (which is here $\frac{1}{2}V_2$) evaluated at the origin x', and from equation (2.155) we see that

$$a_1(x,x') = -\frac{1}{2}V(x') \tag{3.47}$$

Guven's potential, on the other heand, can be rewritten as

$$V_2 = -\frac{1}{4}g^{-\frac{3}{2}}(\partial_\mu g)g^{\mu\nu}(\partial_\nu g) + \frac{1}{2}g^{-\frac{1}{2}}(\partial_\mu g^{\mu\nu})(\partial_\nu g) + \frac{1}{2}g^{-\frac{1}{2}}g^{\mu\nu}\partial_\mu\partial_\nu g + \xi R \tag{3.48}$$

where we see explicitly that the only zero-th order term can only come from $\frac{1}{2}g^{-\frac{1}{2}}g^{\mu\nu}\partial_\mu\partial_\nu g$ since $\partial_\mu g = \partial_\mu g^{\mu\nu} = 0$ when evaluated at the origin. The value of the double derivative $\partial_\mu\partial_\nu g$ at order zero can be read off equation (C.11): at the end we find

$$V_2(z=0) = \left(\xi - \frac{1}{3}\right)R \tag{3.49}$$

---

the scalar properties of the heat kernel unchanged.

thus getting

$$a_1(x', x') = \frac{1}{2}\Big(\frac{1}{3} - \xi\Big)R \tag{3.50}$$

which is not in agreement with DeWitt's results.

Anyway, if the statement $\Box K = (\eta^{\mu\nu}\partial_\mu\partial_\nu + V_{eff})\overline{K}$ is true, one can then define a flat heat kernel with the add of a suitable effective gravitational potential V_{eff}, as

$$\overline{K}(x, x', s) = \int_{z(0)=0}^{z(s)=z} Dz\ e^{\frac{i}{s}\int_0^1 d\tau \left(\dot{z}^2 - \frac{1}{2}s^2 m^2 - \frac{1}{2}s^2 V_{eff}(z(\tau))\right)} \tag{3.51}$$

and use the techniques described in section (1.3) in order to obtain the expansion . This is therefore a crucial point. Using Guven's own words, "It is possible to cast the Schrödinger kernel [...] in a path-integral form. Unfortunately, [...], this representation appears to be intractable in a general coordinate system when the gravitational background is not trivial. However, by introducing Riemann normal coordinates, the $\Box$ appearing in equation (3.42) is reduced to the Minkowski-space form $\eta^{\mu\nu}\partial_\mu\partial_\nu$, with the effect of background geometry completely absorbed into a spacetime-dependant mass term" (i.e. a quadratic potential which could be included into the term $\tilde{m}^2(x)$).

First of all, we notice that the reference given to support this statement, [40], seems to treat completely different arguments. Nothing in that paper seems to support the statement made by Guven. Searching between all the papers cited as a reference in Guven's original paper, we found two other works which treat the argument: the first is a paper of B.L.Hu and D.J.O'Connor [26], while the second is from L. Parker and T.S.Bunch [32].

The first of these references states that equation (3.40), after rescaling of the propagator as [11]

$$\overline{G}(x, x') = g^{\frac{1}{4}}(x) G(x, x') g^{\frac{1}{4}}(x') \tag{3.52}$$

becomes

$$\hat{H}\overline{G}(x, x') = -\delta(x - x') \tag{3.53}$$

[11] Which is of course the same of equation (3.43) where, again, we have $\overline{G} = \Delta^{-\frac{1}{2}} G$ in arbitrary coordinates. Then $\overline{G}$ will be related to $\overline{K}$ in the same way G is related to K, described by equation (3.41).

where

$$\hat{H} = -\partial_\mu g^{\mu\nu}\partial_\nu + V_{eff} = -\partial_\mu g^{\mu\nu}\partial_\nu - g^{-\frac{1}{4}}\partial_\mu[g^{\frac{1}{2}}g^{\mu\nu}\partial_\nu g^{-\frac{1}{4}}] + \xi R \qquad (3.54)$$

We recognize here a different form of the gravitational potential. This form of the operator $\hat{H}$ can actually be obtained evaluating the action of $\Box$ operator on the term $G(x,x') = g^{-\frac{1}{4}}(x)\overline{G}(x,x')$ and multiplying every side of equation (3.40) by a factor of $g^{\frac{1}{4}}(x)$ so that the scalar delta function $\delta(x,x')$ becomes just the flat Dirac delta function $\delta(x-x') = \delta(z)$. This result is in fact immediate writing the scalar $\Box$ operator in the same form given by equation (2.27) once we employ the momentum representation given by (2.24) for a spacetime coordinate vector x^μ that is, written explicitly,

$$\Box^c = g^{\mu\nu}\nabla_\mu\nabla_\nu = g^{-\frac{1}{2}}\partial_\mu g^{\frac{1}{2}}g^{\mu\nu}\partial_\nu \qquad (3.55)$$

and noticing that mixed terms in which one derivative acts on the factor $g^{-\frac{1}{4}}$ and one on the scaled propagator $\overline{G}$ are equal but opposite in sign. In fact the potential term can also be written as

$$V_{eff}(x) = \xi R(x) - g^{\frac{1}{4}}(x)\Box^c g^{-\frac{1}{4}}(x) \qquad (3.56)$$

with the box operator given in the last equation. We can also write it explicitly as

$$V_{eff} = -\frac{3}{16}g^{-2}(\partial_\mu g)g^{\mu\nu}(\partial_\nu g) + \frac{1}{4}g^{-1}(\partial_\mu g^{\mu\nu})(\partial_\nu g) + \frac{1}{4}g^{-1}g^{\mu\nu}\partial_\mu\partial_\nu g + \xi R \qquad (3.57)$$

We have obtained in this way a form for the gravitational potential that can be constructed in a straightforward way by means of the simple transformation given by equation (3.52). Going on with Hu and O'Connor analysis, it is stated that "The Lorentz invariance of momentum-space representation of $\overline{G}$ implies that $g^{\mu\nu}$ in $\partial_\mu g^{\mu\nu}\partial_\nu$ becomes $\eta^{\mu\nu}$". No references or further explanations of this statement are given. It is also unclear what the authors meant saying that "$g^{\mu\nu}$ *becomes* $\eta^{\mu\nu}$". Anyway we shall see now that this is the correct form of the potential: in fact if one uses flat path-integral expansion with this form of the potential term, then one is able to obtain the correct Seeley-DeWitt coefficients at order two of the proper time expansion, which is not possible if one uses the form given by Guven. The terms that one needs to evaluate in order to obtain the firsts Seeley-DeWitt coefficients for coinciding points have been already computed in section (1.3), and can be read off from equations (1.53), where here we have to make the replacement $\nabla^2 \to \Box$. These coefficients are calculated at the space-time point x', which is taken to be the origin of our coordinate system. What is left now are the values of the RNC expansion of the potential up to second order.The various pieces one needs in order to obtain that expansion are listed in appendix C: the final results, up to second order in the RNC expansion, are

$$V_{eff}(z=0) = \Big(\xi - \frac{1}{6}\Big) R \tag{3.58}$$

$$V_{eff,\alpha}(z=0) = \Big(\xi - \frac{1}{6}\Big) R_{;\alpha} \tag{3.59}$$

$$V_{eff,\alpha\beta}(z=0) = \Big(\xi - \frac{1}{6}\Big) R_{;\alpha\beta} + 2a_{\alpha\beta} \tag{3.60}$$

where

$$a_{\alpha\beta} = -\frac{1}{40}\Box R_{\alpha\beta} + \frac{1}{120} R_{;\alpha\beta} + \frac{1}{30} R_{\alpha}{}^{\lambda} R_{\beta\lambda} + \frac{1}{60} R_{\lambda\kappa} R^{\lambda}{}_{\alpha\beta}{}^{\kappa} - \frac{1}{60} R^{\lambda\mu\kappa}{}_{\alpha} R_{\lambda\mu\kappa\beta} \tag{3.61}$$

Plugging these expressions into equations (1.53) (remembering that now we have $V(x) = \frac{1}{2} V_{eff}(x)$ as pointed out in section (1.3)) one immediately obtains the values for the Seeley-DeWitt coefficients

$$a_1(x', x') = \frac{1}{2}\Big(\frac{1}{6} - \xi\Big) R \tag{3.62}$$

$$a_2(x', x') = \frac{1}{8}\Big(\frac{1}{6} - \xi\Big)^2 R^2 + \frac{1}{24}\Big(\frac{1}{5} - \xi\Big)\Box R - \frac{1}{720} R^{\alpha\beta} R_{\alpha\beta} + \frac{1}{720} R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \tag{3.63}$$

which coincide perfectly to the results listed in reference [31], taken in consideration the different normalization of the path integral (setting $s \to 2s$ and remembering that each a_n coefficient is of order s^n will recover results listed in section (2.6) and also with the ones listed in the reference above).

What's left now is to give a precise meaning, or a proof, to the statement "$g^{\mu\nu}$ in $\partial_\mu g^{\mu\nu}\partial_\nu$ becomes $\eta^{\mu\nu}$". The only way we found to interpret this statement is by means of the work of L. Parker and T. S. Bunch [32], which is cited some lines after this statement by Hu and O'Connor and also by the same Guven, as mentioned before. They start their analysis considering a scalar field which propagator obeys an equation of the same form of (3.40): performing the transformation (3.52) and expanding everything in Riemann normal coordinates they obtain an equation (which is in agreement to what obtained by us using the same expansion) which reads, taking only terms which contains at most four derivatives of the metric

$$
\begin{aligned}
&\eta^{\mu\nu}\partial_\mu\partial_\nu\overline{G} - \Big[m^2 + \Big(\xi - \frac{1}{6}\Big)R\Big]\overline{G} - \frac{1}{3}R_\alpha{}^\nu z^\alpha\partial_\nu\overline{G} + \frac{1}{3}R^\mu{}_\alpha{}^\nu{}_\beta z^\alpha z^\beta\partial_\mu\partial_\nu\overline{G} + \\
&- \Big(\xi - \frac{1}{6}\Big)R_{;\alpha}z^\alpha\overline{G} + \Big(\frac{1}{6}R_{\alpha\beta;}{}^\nu - \frac{1}{3}R_\alpha{}^\nu{}_{;\beta}\Big)z^\alpha z^\beta\partial_\nu\overline{G} + \frac{1}{6}R^\mu{}_\alpha{}^\nu{}_{\beta;\gamma}z^\alpha z^\beta z^\gamma\partial_\mu\partial_\nu\overline{G} + \\
&- \frac{1}{2}\Big(\xi - \frac{1}{6}\Big)R_{;\alpha\beta}z^\alpha z^\beta\overline{G} + \Big(\frac{1}{40}\Box R_{\alpha\beta} - \frac{1}{120}R_{;\alpha\beta} - \frac{1}{30}R_\alpha{}^\mu R_{\beta\mu} + \\
&- \frac{1}{60}R^{\lambda\kappa}R_{\lambda\alpha\beta\kappa} + \frac{1}{60}R^\lambda{}_\alpha{}^{\mu\kappa}R_{\lambda\beta\mu\kappa}\Big)z^\alpha z^\beta\overline{G} + \\
&+ \Big(\frac{1}{10}R_{\alpha\beta;}{}^\nu{}_\gamma - \frac{3}{20}R_\alpha{}^\nu{}_{;\beta\gamma} + \frac{1}{60}R^\lambda{}_\alpha R_{\lambda\beta\gamma}{}^\nu - \frac{1}{15}R^\lambda{}_{\alpha\beta}{}^\kappa R_{\lambda\gamma}{}^\nu{}_\kappa\Big)z^\alpha z^\beta z^\gamma\partial_\nu\overline{G} + \\
&+ \Big(\frac{1}{20}R^\mu{}_\alpha{}^\nu{}_{\beta;\gamma\delta} + \frac{1}{15}R^\lambda{}_{\alpha\beta}{}^\mu R_{\lambda\gamma\delta}{}^\nu\Big)z^\alpha z^\beta z^\gamma z^\delta\partial_\mu\partial_\nu\overline{G} = -\delta
\end{aligned}
\tag{3.64}
$$

This result can be readily obtained from equation (3.53) inserting the RNC expansion of the metric tensor, its inverse and their determinant obtained in the last section, and some of the derivatives listed in appendix C. Once that one has obtained equation (3.64), one can obtain a local momentum-space representation by means of a local D-dimensional Fourier transformation

$$
\overline{G}(x, x') = \int \frac{d^D k}{(2\pi)^D} e^{ik_\mu z^\mu}\overline{G}(k) \tag{3.65}
$$

in which one can think at $\overline{G}(k)$ as the Fourier transform of a function which coincides with a solution of equation (3.64) in an open set containg the origin x', and having a compact support on a normal neighborhood of the same point. This construction does not interfere with the structure of the singularity for $x \to x'$, so that the Fourier transform (3.65) is sufficient to study the present case, although failing in describing the global behaviour of the Green function. The authors then employ an adiabatic regularization, discovered by Parker himself (see [31],[32]), that consists in an expansion of the Green function as a sum of terms

$$\overline{G}(k) = \overline{G}_0(k) + \overline{G}_1(k) + \overline{G}_2(k) + \dots \tag{3.66}$$

where every $\overline{G}_i(k)$ has a geometrical coefficient involving i derivatives of the metric. Indeed this adiabatic expansion and RNC expansion are closely related, since every term in the expansion of the metric in RNC corresponds to an expansion in derivatives of the metric tensor, that is an adiabatic expansion. Using a dimensional analysis it turns out that every of these coefficients is of order $k^{-(2+i)}$, such that this last expansion can be seen as an expansion for large k, which turning back to its coordinate-space representation corresponds to an expansion for $z \to 0$. If we now perform the Fourier transform and use adiabatic expansion into equation (3.64) we can compare terms with the same adiabatic order, remembering that each Riemann tensor (or Ricci tensor, or scalar curvature) are of order 2 in this expansion, as one can immediately see from equation (3.8).

It is immediate to obtain that, at the lowest order, the only remaining terms are

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\overline{G}_0(x,x') - m^2\overline{G}_0(x,x') = -\delta(x-x') \tag{3.67}$$

and, employing the Fourier transform,

$$-k^2\overline{G}_0(k) - m^2\overline{G}_0(k) = -1 \tag{3.68}$$

we see that it is nothing but the familiar solution for a scalar field on Minkowsky space

$$\overline{G}_0(k) = \frac{1}{k^2+m^2} \tag{3.69}$$

while, having no first-order terms, one obtains

$$\overline{G}_1(k) = 0 \tag{3.70}$$

We then get, at second adiabatic order,

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\overline{G}_2(x,x') - m^2\overline{G}_2(x,x') - \Big(\xi - \frac{1}{6}\Big)R\overline{G}_0 - \frac{1}{3}R_\alpha{}^\nu z^\alpha\partial_\nu\overline{G}_0(x,x')$$
$$+\frac{1}{3}R^\mu{}_\alpha{}^\nu{}_\beta z^\alpha z^\beta\partial_\mu\partial_\nu\overline{G}_0 = 0 \tag{3.71}$$

It is important to get that the term $\overline{G}_0(x,x')$ depends manifestly only on the scalar $z^2 = \eta_{\mu\nu}z^\mu z^\nu$: for this kind of functions this last equation can be simplified thanks to the use of the identity

$$-\frac{1}{3}R_\alpha{}^\nu z^\alpha\partial_\nu\overline{G}_0 + \frac{1}{3}R^\mu{}_\alpha{}^\nu{}_\beta z^\alpha z^\beta\partial_\mu\partial_\nu\overline{G}_0 = 0 \tag{3.72}$$

obtaining

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\overline{G}_2(x,x') - m^2\overline{G}_2(x,x') - \Big(\xi - \frac{1}{6}\Big)R\overline{G}_0 = 0 \tag{3.73}$$

from which, inserting explicit expression for $\overline{G}_0$ and employing the Fourier transform, we get

$$\overline{G}_2(k) = \frac{\left(\xi - \frac{1}{6}\right)R}{(k^2+m^2)^2} \tag{3.74}$$

It is important to notice now that one could rewrite identity (3.72) as the second adiabatic order expansion of

$$(\partial_\mu g^{\mu\nu})\partial_\nu\overline{G}(x,x') + (g^{\mu\nu} - \eta^{\mu\nu})\partial_\mu\partial_\nu\overline{G}(x,x') = 0 \tag{3.75}$$

If this is true at all orders, we have found a way to explain why the term $\partial_\mu g^{\mu\nu}\partial_\nu$ *becomes* $\eta^{\mu\nu}\partial_\mu\partial_\nu$. In fact, equations like this last one can be found at any order, supposed that the Green function is always a function only of the scalar z^2. [12] We will obtain an original proof of this statement at the end of this section.

Proceeding now with Parker's iterative procedure, we can obtain the explicit form of the propagator $\overline{G}$ at higher orders in adiabatic expansion. The third order adiabatic

[12] We remark here that in Parker's work this property is never taken as granted nor proven: they instead obtain an expression for i-th order of the adiabatic expansion of the propagator, then verify that it depends only on z^2 and iteratively employ this condition on the equation than one obtains for the $(i+1)$-th order of the same expansion. In paper [32] and subsequent works this recursive relation is used up to order $i = 3$, and nothing is said about the next leading order. Also note that the authors have never given a proof of identity (3.72), nor of the higher-order ones (3.76), (3.77).

equation can also be simplified, considering that also the $\overline{G}_2$ is a function of the scalar quantity z^2 only, so that equation (3.72) continues to hold also for this term. Furthermore, the z^2 dependence of the function $\overline{G}_0$ leads to further simplifications on the third- and fourth-adiabatic order equations, since we also have

$$\left(\frac{1}{6}R_{\alpha\beta;}{}^{\nu} - \frac{1}{3}R_{\alpha}{}^{\nu}{}_{;\beta}\right)z^\alpha z^\beta \partial_\nu \overline{G}_0 + \frac{1}{6}R^\mu{}_\alpha{}^\nu{}_{\beta;\gamma} z^\alpha z^\beta z^\gamma \partial_\mu \partial_\nu \overline{G}_0 = 0 \tag{3.76}$$

$$\begin{aligned}
&\left(\frac{1}{10}R_{\alpha\beta;}{}^{\nu}{}_{\gamma} - \frac{3}{20}R_{\alpha}{}^{\nu}{}_{;\beta\gamma} + \frac{1}{60}R^\lambda{}_\alpha R_{\lambda\beta\gamma}{}^\nu - \frac{1}{15}R^\lambda{}_{\alpha\beta}{}^\kappa R_{\lambda\gamma}{}^\nu{}_\kappa\right) z^\alpha z^\beta z^\gamma \partial_\nu \overline{G}_0 + \\
&+ \left(\frac{1}{20}R^\mu{}_\alpha{}^\nu{}_{\beta;\gamma\delta} + \frac{1}{15}R^\lambda{}_{\alpha\beta}{}^\mu R_{\lambda\gamma\delta}{}^\nu\right) z^\alpha z^\beta z^\gamma z^\delta \partial_\mu \partial_\nu \overline{G}_0 = 0
\end{aligned} \tag{3.77}$$

Notice that also these last identities can be seen respectively as the second and third adiabatic order of the expansion of equation (3.75). After the previous simplifications the equation for the third-order term $\overline{G}_3$ becomes

$$\eta^{\mu\nu}\partial_\mu \partial_\nu \overline{G}_3 - m^2 \overline{G}_3 - \left(\xi - \frac{1}{6}\right) R_{;\alpha} z^\alpha \overline{G}_0 = 0 \tag{3.78}$$

Noticing now that

$$\begin{aligned}
z^\alpha \overline{G}(x, x') &= z^\alpha \int \frac{d^D k}{(2\pi)^D} e^{ikz} \overline{G}(k) = \int \frac{d^D k}{(2\pi)^D} \left(-i \frac{\partial}{\partial k_\alpha} e^{ikz}\right) \overline{G}(k) \\
&= \int \frac{d^D k}{(2\pi)^D} e^{ikz} i \partial^\alpha \overline{G}(k)
\end{aligned} \tag{3.79}$$

we obtain

$$(k^2 + m^2)\overline{G}_3 - i\left(\xi - \frac{1}{6}\right) R_{;\alpha} \partial^\alpha (k^2 + m^2)^{-1} = 0 \tag{3.80}$$

that is

$$\overline{G}_3(k) = i \frac{\left(\xi - \frac{1}{6}\right) R_{;\alpha}}{k^2 + m^2} \partial^\alpha (k^2 + m^2)^{-1} \tag{3.81}$$

Turning back to coordinate space again, we see that also $\overline{G}_3(x, x')$ is identically vanishing, having a dependence which is a odd power of k. At order four we are left with

$$
\begin{aligned}
&\eta^{\mu\nu}\partial_\mu\partial_\nu\overline{G}_4 - m^2\overline{G}_4 - \Big(\xi - \frac{1}{6}\Big)R\overline{G}_2 - \frac{1}{2}\Big(\xi - \frac{1}{6}\Big)R_{;\alpha\beta}z^\alpha z^\beta\overline{G}_0 \\
&+ \Big(\frac{1}{40}\Box R_{\alpha\beta} - \frac{1}{120}R_{;\alpha\beta} - \frac{1}{30}R_\alpha^{\ \mu}R_{\beta\mu} - \frac{1}{60}R^{\lambda\kappa}R_{\lambda\alpha\beta\kappa} + \frac{1}{60}R^\lambda_{\ \alpha}{}^{\mu\kappa}R_{\lambda\beta\mu\kappa}\Big)z^\alpha z^\beta\overline{G}_0 = 0
\end{aligned}
\tag{3.82}
$$

so that we get the fourth-order term as

$$
\begin{aligned}
\overline{G}_4(k) = &\Big(\xi - \frac{1}{6}\Big)^2 R^2(k^2+m^2)^{-3} + \frac{1}{2}\Big(\xi - \frac{1}{6}\Big)\frac{R_{;\alpha\beta}}{k^2+m^2}\partial^\alpha\partial^\beta(k^2+m^2)^{-1} \\
&- a_{\alpha\beta}(k^2+m^2)^{-1}\partial^\alpha\partial^\beta(k^2+m^2)^{-1}
\end{aligned}
\tag{3.83}
$$

where $a_{\alpha\beta}$ is the same quantity of equation (3.61). This last term can be rewritten, using identity

$$
(k^2+m^2)^{-1}\partial^\alpha\partial^\beta(k^2+m^2)^{-1} = \frac{1}{3}\partial^\alpha\partial^\beta(k^2+m^2)^{-2} - \frac{2}{3}\eta^{\alpha\beta}(k^2+m^2)^{-3} \tag{3.84}
$$

as

$$
\begin{aligned}
\overline{G}_4(k) = &\frac{1}{3}\Big[-a_{\alpha\beta} + \frac{1}{2}\Big(\xi - \frac{1}{6}\Big)R_{;\alpha\beta}\Big]\partial^\alpha\partial^\beta(k^2+m^2)^{-2} \\
&+ \Big[\Big(\xi - \frac{1}{6}\Big)^2 R^2 + \frac{2}{3}a^\alpha_{\ \alpha} - \frac{1}{3}\Big(\xi - \frac{1}{6}\Big)\Box R\Big](k^2+m^2)^{-3}
\end{aligned}
\tag{3.85}
$$

This is the highest-order term obtained by the authors. One could theoretically extend this procedure at higher orders, provided that one is able to get simplifications like the ones brought by identities (3.72), (3.76), (3.77) also for the terms $\overline{G}_i(x,x')$ for $i \geq 4$, which appears in the equations for higher-order adiabatic expansion of the Green function. Notice that it is immediate to see that this is the first term which is explicitly not proportional to the scalar z^2 only. In fact performing an anti-fourier transform on $\overline{G}_4$ will transform the terms $a_{\alpha\beta}\partial^\alpha\partial^\beta(k^2+m^2)^{-}2\ a_{\alpha\beta}z^\alpha z^\beta$; the appearance of this kind of terms makes the Green function to have a non-trivial dependence on z, thus making equation (3.75) much more difficult to verify. If the presence of this new term makes this equation to be no longer true, it is obvious that Guven's procedure cannot be extended at arbitrary orders, since it is based on the fact that in Riemann normal coordinates one has $\partial_\mu g^{\mu\nu}\partial_\nu \overline{G} = 0$, where derivatives acts through all that follows. Since we have not been able to proof, or disproof, that last equation with this non-trivial dependence we calculated the next Seeley-DeWitt coefficient a_3, which is of order 6 in the adiabatic expansion, supposing anyway its correctness also when $\overline{G}(x, x') \neq \overline{G}(z^2)$, as a test for Guven's procedure beyond fourth adiabatic order. This will be done in the next section.

We end this section calculating explicitly the Green function for the next two adiabatic orders, in order to show the failure of Guven's procedure at higher orders. This calculation is original and cannot be found in literature. It is convenient now to express the inverse metric $g^{\mu\nu}$ as

$$g^{\mu\nu}(x) = \eta^{\mu\nu} + h^{\mu\nu}(x) \tag{3.86}$$

where $h^{\mu\nu}$ contains all the terms of the Riemann expansion. It can also be expressed as $h^{\mu\nu} = (h^{\mu\nu})_2 + (h^{\mu\nu})_3 + (h^{\mu\nu})_2 + \ldots$ in which the number at the bottom of round brackets indicates the corresponding order of the adiabatic expansion. Then, from equations (3.53) and (3.54), we get that the generic form of expression (3.64) is given by

$$[\eta^{\mu\nu}\partial_\mu\partial_\nu + h^{\mu\nu}(x)\partial_\mu\partial_\nu + (\partial_\mu h^{\mu\nu}(x))\partial_\nu - V_{eff}(x) - m^2]\overline{G}(x, x') = \delta(x - x') \tag{3.87}$$

Expanding this last equation at fifth adiabatic order we get

$$\begin{aligned}
&\eta^{\mu\nu}\partial_\mu\partial_\nu\overline{G}_5(x, x') + (h^{\mu\nu}(x))_5\partial_\mu\partial_\nu\overline{G}_0(x, x') + (h^{\mu\nu}(x))_3\partial_\mu\partial_\nu\overline{G}_2(x, x')+\\
&+ (\partial_\mu h^{\mu\nu}(x))_5\partial_\mu\partial_\nu\overline{G}_0(x, x') + (\partial_\mu h^{\mu\nu}(x))_3\partial_\mu\partial_\nu\overline{G}_2(x, x') - (V_{eff}(x))_5\overline{G}_0(x, x')+\\
&- (V_{eff}(x))_3\overline{G}_2(x, x') - m^2\overline{G}_5(x, x') = 0
\end{aligned} \tag{3.88}$$

since $\overline{G}_1(x, x') = 0 = \overline{G}_3(x, x')$. Using the z^2 dependence of $\overline{G}_0(x, x')$ and $\overline{G}_2(x, x')$ we have that $(h^{\mu\nu}(x))_5\partial_\mu\partial_\nu\overline{G}_0 = -(\partial_\mu h^{\mu\nu}(x))_5\partial_\mu\partial_\nu\overline{G}_0(x, x')$ and $(h^{\mu\nu}(x))_3\partial_\mu\partial_\nu\overline{G}_2(x, x') = -(\partial_\mu h^{\mu\nu}(x))_3\partial_\mu\partial_\nu\overline{G}_2(x, x')$, thous we are left with

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\overline{G}_5(x,x') - m^2\overline{G}_5(x,x') = (V_{eff}(x))_5\overline{G}_0(x,x') + (V_{eff}(x))_3\overline{G}_2(x,x') \tag{3.89}$$

and, turning to momentum space,

$$\overline{G}_5(k) = -[(V_{eff})_5\overline{G}_0(k) + (V_{eff})_3\overline{G}_2(k)](k^2+m^2)^{-1} \tag{3.90}$$

the value of $(V_{eff})_3$ and $(V_{eff})_5$ can be read off appendix C, remembering that for the potential the i-th order of the adiabatic expansion corresponds to the $(i-2)$-th order of the RNC expansion. The final expression then reads

$$\begin{aligned}\overline{G}_5(k) =& i\Big[\frac{1}{3!}\Big(\xi-\frac{1}{6}\Big)R_{;\alpha\beta\gamma\delta}(k^2+m^2)^{-1}\partial^\alpha\partial^\beta\partial^\gamma(k^2+m^2)^{-1}+\\ &+a_{\alpha\beta\gamma}(k^2+m^2)^{-1}\partial^\alpha\partial^\beta\partial^\gamma(k^2+m^2)^{-1} - \Big(\xi-\frac{1}{6}\Big)^2 RR_{;\alpha}\partial^\alpha(k^2+m^2)^{-2}\Big]\end{aligned} \tag{3.91}$$

This term is once again vanishing when turning back to coordinate space, due to its odd-power dependence on k. In order to get something "new" we have to evaluate the next adiabatic order expansion. This is where the problems begin. In fact we have that the equation for the sixth adiabatic order of the Green function reads

$$\begin{aligned}&\eta^{\mu\nu}\partial_\mu\partial_\nu\overline{G}_6(x,x') + (h^{\mu\nu}(x))_6\partial_\mu\partial_\nu\overline{G}_0(x,x') + (h^{\mu\nu}(x))_4\partial_\mu\partial_\nu\overline{G}_2(x,x')+\\ &+(h^{\mu\nu}(x))_2\partial_\mu\partial_\nu\overline{G}_4(x,x') + (\partial_\mu h^{\mu\nu}(x))_6\partial_\mu\partial_\nu\overline{G}_0(x,x') + (\partial_\mu h^{\mu\nu}(x))_4\partial_\mu\partial_\nu\overline{G}_2(x,x')+\\ &+(\partial_\mu h^{\mu\nu}(x))_2\partial_\mu\partial_\nu\overline{G}_4(x,x') - (V_{eff}(x))_6\overline{G}_0(x,x') - (V_{eff}(x))_4\overline{G}_2(x,x')+\\ &-(V_{eff}(x))_2\overline{G}_4(x,x') - m^2\overline{G}_5(x,x') = 0\end{aligned} \tag{3.92}$$

We see here explicitly the appearance of the term $\overline{G}_4$ into this expression. But, as we remarked before, this is the first term of the adiabatic expansion which contains a non-trivial dependence on z. This makes the kinetic term to be no more the trivial minkowskian box operator, since we have no way to prove identities like (3.72), (3.76), (3.77) if the Green function is not only proportional to powers of z^2.

We are now ready to give a proof to equation (3.75) which works at all orders of expansion, if the Green function can be entirely expressed in terms of powers of the scalar z^2. In fact, if this is the case, we can expand the propagator as

$$\overline{G}(z^2) = \sum_{n=0}^{\infty} a_n (z^2)^n \tag{3.93}$$

(a_n being constant complex coefficients with respect to the variable z). Then his derivatives read

$$\partial_\nu \overline{G}(z^2) = \sum_{n=1}^{\infty} 2na_n (z^2)^{n-1} z_\nu \tag{3.94}$$

$$\begin{aligned}\partial_\mu \partial_\nu \overline{G}(z^2) &= \sum_{n=2}^{\infty} 4n(n-1)a_n (z^2)^{n-2} z_\mu z_\nu + \sum_{n=1}^{\infty} 2na_n (z^\lambda z_\lambda)^{n-1} \delta_{\mu\nu} = \\ &= \sum_{n=1}^{\infty} [4n(n+1)a_{n+1} z_\mu z_\nu + 2na_n \delta_{\mu\nu}](z^2)^{n-1}\end{aligned} \tag{3.95}$$

In the same way we can make a generic expansion of the metric in the form

$$g^{\mu\nu} = \eta^{\mu\nu} + \sum_{n=0}^{\infty} h^{\mu\ \nu}_{\ \alpha\ \beta\rho_1\ldots\rho_n} z^\alpha z^\beta z^{\rho_1} \ldots z^{\rho_n} \tag{3.96}$$

where we have the following properties for the coefficients which are inherited form the properties of Riemann tensors [13]

$$h^{\mu\ \nu}_{\ \alpha\ \beta\rho_1\ldots\rho_n} = -h^{\ \mu\nu}_{\alpha\ \ \beta\rho_1\ldots\rho_n} = -h^{\mu\ \ \nu}_{\ \alpha\beta\ \rho_1\ldots\rho_n} \tag{3.97}$$

$$h^{\mu\ \nu}_{\ \mu\ \alpha\rho_1\ldots\rho_n} = 0 = h^{\mu\ \nu}_{\ \alpha\ \nu\rho_1\ldots\rho_n} \tag{3.98}$$

together with the property, which is an immediate consequence of (3.97)

$$h^{\mu\ \nu}_{\ \alpha\ \beta\rho_1\ldots\rho_n} z^\alpha z_\mu = 0 = h^{\mu\ \nu}_{\ \alpha\ \beta\rho_1\ldots\rho_n} z^\beta z_\nu \tag{3.99}$$

[13]We recall here that the coefficients $h^{\mu\ \nu}_{\ \alpha\ \beta\rho_1\ldots\rho_n}$ are proportional only to products and contractions of various Riemann tensors. Calling α the index that every time is antisymmetric with respect to the exchange of μ, and β the index which has the same relation with the free index ν we obtain this expansion.

In order to verify equation (3.75) we also need the general structure of of one derivative of the inverse metric tensor $g^{\mu\nu}$. Recalling that all these coefficients are calculated for $z = 0$ we get

$$\partial_\mu g^{\mu\nu} = \sum_{n=0}^{\infty}\left[h^{\mu\ \nu}_{\ \alpha\ \mu\rho_1\dots\rho_n} z^\alpha z^{\rho_1}\dots z^{\rho_n} + \sum_{k=1}^{n} h^{\mu\ \nu}_{\ \alpha\ \beta\rho_1\dots\rho_{k-1}\mu\rho_{k+1}\dots\rho_n} z^\alpha z^\beta z^{\rho_1}\dots z^{\rho_{k-1}} z^{\rho_{k+1}}\dots z^{\rho_n}\right] \tag{3.100}$$

Putting it all together we obtain

$$(g^{\mu\nu} - \eta^{\mu\nu})\partial_\mu\partial_\nu\overline{G} = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}[4m(m-1)a_{m-1}h^{\mu\ \nu}_{\ \alpha\ \beta\rho_1\dots\rho_n} z^\alpha z^\beta z^{\rho_1}\dots z^{\rho_n}(z^2)^{m-1}z_\mu z_\nu + \\ + 2ma_m h^{\mu}_{\ \alpha\mu\beta\rho_1\dots\rho_n} z^\alpha z^\beta z^{\rho_1}\dots z^{\rho_n}(z^2)^{m-1}] \tag{3.101}$$

$$(\partial_\mu g^{\mu\nu})\partial_\nu\overline{G} = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left[\sum_{k=1}^{n} 2ma_m h^{\mu\ \nu}_{\ \alpha\ \beta\rho_1\dots\rho_{k-1}\mu\rho_{k+1}\dots\rho_n} z^\alpha z^\beta z^{\rho_1}\dots z^{\rho_{k-1}} z^{\rho_{k+1}}\dots z^{\rho_n}(z^2)^{m-1}z_\nu + \\ + 2ma_m h^{\mu}_{\ \alpha\nu\mu\rho_1\dots\rho_n} z^\alpha z^\nu z^{\rho_1}\dots z^{\rho_n}(z^2)^{m-1}\right] \tag{3.102}$$

We recognize here that the last term is the same in both the expansions, but with an opposite sign: the other terms are instead identically vanishing in virtue of property (3.99). We have shown now that, if the rescaled propagator $\overline{G}(x, x')$ depends actually only on the scalar z^2, an equation of the same form of (3.44) can be found, and using the gravitational potential given in (3.57) one can use flat path integral methods in order to evaluate the coefficients of the heat kernel expansion. We have seen explicitly that this is in fact true up to order three of the adiabatic expansion (performing an anti-Fourier transform on $\overline{G}_0$ and $\overline{G}_2$ will show the correctness of this statement), but is not so in general, since in $\overline{G}_4$ terms like $a_{\alpha\beta}z^\alpha z^\beta$ appear: while terms like that are still Lorentz invariant (since they are scalars under general coordinate transformations), they are not proportional to z^2, at least on a generically curved manifold. One could theoretically insert these extra terms into equation (3.93) in order to verify if identity (3.75) still holds, but this would produce an equation which is of difficult evaluation. In fact, writing these extra terms as an extra $\sum_{n=1}^{\infty} a_{\alpha_1 \ldots \alpha_n} z^{\alpha_1} \ldots z^{\alpha_n}$, one faces the problem of finding a closed-form expression for the evaluation of its derivatives, which would appear in the expansion of $\partial_\nu \overline{G}$ and $\partial_\mu \partial_\nu \overline{G}$ and which would require the knowledge of the exact form of $\overline{G}_i$ at every order of the expansion. However, if we had started from the symmetrized expression for the metric $g^{\mu\nu}$, we would have obtained the coefficients $a_{\alpha_1 \ldots \alpha_n}$ in a symmetrized way. With these factors written in an explicit symmetrized form we are able to obtain a closed expression which both the terms $a_{(\alpha_1 \ldots \alpha_n)}$ and the expansion of the inverse metric in Riemann normal coordinates $h^{\mu\nu}_{(\rho_1 \ldots \rho_n)}$ have to satisfy if equation (3.75) has to be true, which is

$$\sum_{n,m=1}^{\infty} (n+1) a_{(\alpha_1 \ldots \alpha_m \nu)} h^{\mu\nu}_{(\rho_1 \ldots \rho_n \mu)} = \sum_{n,m=1}^{\infty} (m+2) a_{(\alpha_1 \ldots \alpha_m \mu\nu)} h^{\mu\nu}{}_{(\rho_1 \ldots \rho_n)} \tag{3.103}$$

This result can be obtained expressing the metric as

$$g^{\mu\nu} = \eta^{\mu\nu} + \sum_{n=0}^{\infty} h^{\mu\nu}{}_{(\rho_1 \ldots \rho_n)} z_1^{\alpha} \ldots z_n^{\alpha} \tag{3.104}$$

which makes his derivative to be

$$\partial_\mu g^{\mu\nu} = \sum_{n=1}^{\infty} (n+1) h^{\mu\nu}{}_{(\rho_1 \ldots \rho_n \mu)} z_1^{\rho} \ldots z_n^{\rho} \tag{3.105}$$

and the propagator and his derivatives as

$$\overline{G}(z) = \sum_{n=0}^{\infty} a_n z^{2n} + \sum_{n=1}^{\infty} a_{(\alpha_1 \ldots \alpha_n)} z_1^{\alpha} \ldots z^{\alpha_n} \tag{3.106}$$

$$\begin{aligned}
\partial_\nu \overline{G}(z) &= \sum_{n=1}^{\infty} 2n a_n z^{2(n-1)} z_\nu + \sum_{m=2}^{\infty} n a_{(\alpha_1 \ldots \alpha_{n-1} \mu)} z^{\alpha_1} \ldots z^{\alpha_n} \\
&= \sum_{n=1}^{\infty} [2n a_n z^{2(n-1)} z_\nu + (n+1) a_{(\alpha_1 \ldots \alpha_n \mu)} z^{\alpha_1} \ldots z^{\alpha_n}]
\end{aligned} \tag{3.107}$$

$$\partial_\mu\partial_\nu\overline{G}(z) = \sum_{n=1}^{\infty}[4n(n+1)a_{n+1}z^{2(n-1)}z_\mu z_\nu + 2na_n z^{2(n-1)}\delta_{\mu\nu}+$$
$$+ (m+1)(m+2)a_{(\alpha_1\ldots\alpha_n\mu\nu)}z^{\alpha_1}\ldots z^{\alpha_n}] \tag{3.108}$$

Putting all this together and considering the validity of equation (3.75) without the presence of these extra terms, we obtain the result (3.103). Since a way to prove or disprove this relation cannot be found by us, we instead test the validity of (3.75) in the presence of these extra terms by directly evaluating the third heat kernel coefficient. This is done in the following section.

3.3 The a_3 Coefficient

This section is dedicated to the evaluation of the coefficient of order s^3 (a_3) into the proper time expansion of the heat kernel, using the same procedure described in the previous section. This is of course a way to test Guven's regularization scheme on spaces of arbitrary curvature. We will then suppose that the non-linear sigma model given by equation (3.42) can be casted into the effective linear sigma model of equation (3.44) at any order of the perturbative expansion (using the effective potential we found in the last section and which is given by equation (3.56)), we will put the heat kernel in a flat path integral form and then use the standard methods described in section (1.3) to obtain the ultimate result. The computation is straightforward, but nevertheless extremely laborious.

Starting from the heat kernel in the form given by equation (3.51) and manipulating the path integral as done in section (1.3) we get the expression

$$\overline{K}(x,x',s) = \frac{1}{(2\pi s)^{\frac{D}{2}}}e^{\frac{i}{s}\left(\frac{\dot{z}^2}{2}-\frac{1}{2}m^2s^2-\frac{1}{2}s^2V_{eff}(0)\right)}\tilde{\Omega}(z,s) \tag{3.109}$$

where

$$\begin{aligned}\tilde{\Omega}(z,s) &= \int_{\phi(0)=0}^{\phi(1)=0} D\phi(\tau)\ e^{i\int_0^1 d\tau\left[\frac{\dot{\phi}^2}{2s}-\frac{s}{2}V_{eff}(0)\right]} \Big/ \int_{\phi(0)=0}^{\phi(1)=0} D\phi(\tau)\ e^{i\int_0^1 d\tau\frac{\dot{\phi}^2}{2s}} \\ &\equiv \left\langle e^{-\frac{is}{2}\int_0^1 d\tau[V_{eff}(\phi(\tau))-V_{eff}(0)]}\right\rangle \end{aligned} \tag{3.110}$$

We recognize here the R-summed form of the heat kernel given by equation (2.157): in fact we have that $-\frac{1}{2}V_{eff}(0) = \frac{1}{2}\Big(\frac{1}{6}-\xi\Big)R$ which, taken in consideration the different normalization of the path integral, is the same factor exponentiated out of Ω in that equation. The missing of the factor $\Delta^{\frac{1}{2}}(x,x)$ is due to the fact that we are calculating the scaled heat kernel $\overline{K}$ instead then K: as pointed out in the previous section, the two terms differs precisely by this factor.

We can now evaluate $\tilde{\Omega}$ perturbatively, expressing equation (3.110) as a power series

$$\left\langle e^{-\frac{is}{2}\int_0^1 d\tau[V(\phi(\tau))-V(0)]}\right\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\Big(-\frac{is}{2}\Big)^n\left\langle\Big[\int_0^1 d\tau(V_{eff}(\phi(\tau))-V_{eff}(0))\Big]^n\right\rangle \tag{3.111}$$

and then Taylor expand the potential around the origin $z=0$

$$\tilde{\Omega}(0,s) = \sum_{n=0}^{\infty}\frac{1}{n!}\Big(-\frac{is}{2}\Big)^n\Big[\sum_{m=0}^{\infty}\frac{1}{m!}V_{eff,\alpha_1\dots\alpha_m}(0)\int_0^1 d\tau\,\langle\phi^{\alpha_1}\dots\phi^{\alpha_m}\rangle\Big]^n \tag{3.112}$$

We then get that, up to third order in s (remembering that each mean value $\left\langle\phi^\alpha\phi^\beta\right\rangle$ is of order s as one can see from (1.39) and that mean values of an odd number of fields ϕ is vanishing),

$$\begin{aligned}\tilde{\Omega}(0,s) =& 1-\frac{is}{4}V_{eff,\alpha\beta}(0)\int_0^1 d\tau\left\langle\phi^\alpha(\tau)\phi^\beta(\tau)\right\rangle-\frac{is}{2\cdot 4!}V_{eff,\alpha\beta\gamma\delta}(0)\times\\ &\times\int_0^1 d\tau\left\langle\phi^\alpha(\tau)\phi^\beta(\tau)\phi^\gamma(\tau)\phi^\delta(\tau)\right\rangle+\frac{(is)^2}{8}V_{eff,\alpha}(0)V_{eff,\beta}(0)\times\\ &\times\int_0^1 d\tau d\sigma\left\langle\phi^\alpha(\tau)\phi^\beta(\sigma)\right\rangle+o(s^4)\end{aligned} \tag{3.113}$$

Evaluating all the mean values appearing in the previous equation we get

$$\tilde{\Omega}(0,s) = 1 - \frac{(is)^2}{4} V_{eff,\alpha}{}^{\alpha} \int_0^1 d\tau \; g(\tau,\tau) - \frac{(is)^3}{2 \cdot 4!} (V_{eff,\alpha}{}^{\alpha}{}_{\beta}{}^{\beta} + V_{eff,\alpha\beta}{}^{\alpha\beta} + V_{eff,\alpha\beta}{}^{\beta\alpha}) \times$$
$$\times \int_0^1 d\tau g^2(\tau\tau) + \frac{(is)^3}{8} V_{eff,\alpha} V_{eff}{}^{\alpha} \int_0^1 d\tau d\sigma g(\tau,\sigma) \tag{3.114}$$

Notice here that $V_{eff,\alpha}{}^{\alpha}{}_{\beta}{}^{\beta} + V_{eff,\alpha\beta}{}^{\alpha\beta} + V_{eff,\alpha\beta}{}^{\beta\alpha} \neq 3 V_{eff,\alpha}{}^{\alpha}{}_{\beta}{}^{\beta}$ since we chose to work with the expression of the metric which is not symmetrized: the symmetrization is recovered here taking in consideration all the different contractions. We have then

$$\int_0^1 d\tau \; g(\tau,\tau) = \frac{1}{6}$$

$$\int_0^1 d\tau \; g^2(\tau,\tau) = \frac{1}{30}$$

$$\int_0^1 d\tau d\sigma \; g(\tau,\sigma) = \frac{1}{12} \tag{3.115}$$

Thus obtaining for the third heat kernel coefficient

$$\tilde{a}_3(z=0) = \tilde{a}_3(x',x') = -\frac{1}{60 \cdot 4!} [V_{eff,\alpha}{}^{\alpha}{}_{\beta}{}^{\beta} + V_{eff,\alpha\beta}{}^{\alpha\beta} + V_{eff,\alpha\beta}{}^{\beta\alpha}] + \frac{1}{96} V_{eff,\alpha} V_{eff}{}^{\alpha} \tag{3.116}$$

in which the tilde means that these coefficients are obtained from the R-summed form of the heat kernel, so that a whole hierarchy of terms (which vanish when putting $R = 0$ but not its derivatives) is missing. They can anyway be recovered considering $\sum_{n=0}^{\infty} (is)^n a_n(x',x') = e^{i(\xi - 1/6)Rs} \sum_{n=0}^{\infty} (is)^n \tilde{a}_n(x',x')$.

The expansion of the gravitational effective potential can be read off appendix C, and obtaining the fourth-order term of this expansion is the most difficult task of this calculation. Since its evaluation takes really big efforts occasional mistakes could always occur, so a computational way to evaluate these terms of the expansion would be very welcome. Anyway, taking the expansion of the potential from appendix C, one is able to obtain the final result, which reads, in terms of the basis of curvature invariants cubic in the curvature expressed in [6]:

$$
\begin{aligned}
a_3(x',x') =& -\frac{1}{4050}K_2 + \frac{31}{21600}K_3 + \frac{1}{3360}K_4 - \frac{143}{37800}K_5 - \frac{19}{12600}K_6 + \\
& - \frac{1}{453600}K_7 - \frac{17}{113400}K_8 - \frac{43}{75600}K_{10} + \frac{23}{50400}K_{11} - \frac{13}{10080}K_{12} + \\
& + \frac{1}{672}K_{13} - \frac{11}{6720}K_{14} + \frac{3}{8960}K_{15} - \frac{11}{30240}K_{16} + \frac{1}{2440}K_{17} + \\
& + \xi\Big(\frac{1}{720}K_{12} + \frac{1}{240}K_{16} - \frac{1}{480}K_{17} + \frac{1}{96}\xi K_{16}\Big) \qquad (3.117)
\end{aligned}
$$

This result can be compared with $\overline{a}_3$ in section (3.6) of [31]. The comparison in unsuccessful: taking in consideration the different normalization of the path integral, we recognize that the only terms which are correct are the ones proportional to the parameter ξ and the coefficient of K_{17}: this is because they are the only terms which do not get any contribution from the terms which are supposed to vanish in Guven's procedure (in fact the terms in ξ come from the product $V_{eff,\alpha}V_{eff}{}^{\alpha}$ of equation (3.116), where $V_{eff,\alpha}$ is a term of order 3 in the adiabatic expansion, so it gets no contribution from these terms which of course vanish up to fourth adiabatic order as explicitly shown by Parker *et al.*. In addiction, these extra terms contains always at least a product of two R since they always came from a product of terms like the ones pointed out in equation (3.103).

3.4 Heat Kernel's RNC Perturvative Expansion on Maximally Symmetric Spaces and the Type-A Trace Anomaly

We have shown in the last section how Guven's path integral regularization seems to fail on arbitrarily curved spaces up to third perturbative order in the proper time expansion. In this section instead we wish to test the same flat path integral construction for non-relativistic particles of unit mass on maximally symmetric spaces. The reasons that pushed us towards considering this kind of spaces are fundamentally two: first, symmetry-based arguments have indeed a better chance to work on a maximally symmetric space; second, it is claimed in paper [29] that Parker's adiabatic regularization procedure on $(A)dS$ spaces was shown to give the same perturbative result of the original DeWitt's proper time expansion also up to six-th order of the adiabatic expansion (order s^3) , and no extension has to be made. This would of course imply that the same relation should hold also for Guven's path integral construction.

We will then suppose that a path integral construction, on the same form of the one given by equation (3.39) and with the same potential given by equation (3.56), can be done at all orders. We will use the Euclidean formulation of quantum mechanical path integrals ($s \to -i\beta$) in order to use the same conventions of our paper [8]. Here we x^i the Riemann normal coordinates defining the space point x, while earlier we used z^i.

Thous, we will consider

$$\overline{K}(x,x',\beta) = \int Dx\ e^{-S[x]} \qquad , \qquad S[x] = \frac{1}{\beta}\int_0^1 d\tau\ \left(\frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j + \beta^2 V_{eff}\right) \tag{3.118}$$

where now $V_{eff} = -\frac{1}{2}g^{\frac{1}{4}}\Box^c g^{-\frac{1}{4}}$ and where the bar over K indicates that the transformation given by equation (3.43) has been implemented.

Since, as pointed out in chapter's introduction, a direct confrontation with our results in arbitrary D dimensions cannot be made we tested the trace (conformal) anomaly we obtain with this procedure at specific values of D (up to $D = 12$) with the one obtained using different computational methods (see [14]).

Using the same conventions employed before in the present text, we take the Ricci scalar R to be positive on spheres. In maximally symmetric spaces the Riemann tensor is related to the metric tensor by the simple expression (see [41] as a reference)

$$R_{mnab} = M^2(g_{ma}g_{nb} - g_{mb}g_{na}) \tag{3.119}$$

where M^2 is a constant that can be either positive, negative, or vanishing (flat space). The Ricci tensors are then defined by

$$R_{mn} = R^a{}_{man} = M^2(D-1)g_{mn}$$

$$R = R_m{}^m = M^2(D-1)D \tag{3.120}$$

so that the constant M^2 is related to the constant Ricci scalar R by

$$M^2 = \frac{R}{(D-1)D} \tag{3.121}$$

which is positive on a sphere. The expansion of the metric in RNC on a maximally symmetric space can be obtained using the same method we used in section (3.1) and explicitly inserting the form of the Riemann tensor on these kind of spaces given by equation (3.119) and reads

$$g_{mn}(x) = \delta_{mn} + (\delta_{mn} - \hat{x}_m \hat{x}_n)\Big(-\frac{1}{3}(Mx)^2 + \frac{32}{6!}(Mx)^4 - \frac{16}{7!}(Mx)^6 + \cdots\Big) \tag{3.122}$$

where

$$x = \sqrt{\vec{x}^{\,2}} \qquad , \qquad \hat{x}^m = \frac{x^m}{x} \tag{3.123}$$

One may compute all terms of the series recursively, and sum the series to get (see [5])

$$\begin{aligned} g_{mn}(x) &= \delta_{mn} + P_{mn} \sum_{n=1}^{\infty} \frac{2(-1)^n}{(2n+2)!}(2Mx)^{2n} \\ &= \delta_{mn} + P_{mn}\, \frac{1 - 2(Mx)^2 - \cos(2Mx)}{2(Mx)^2} \end{aligned} \tag{3.124}$$

where the projector P_{mn} is defined by

$$P_{mn} = \delta_{mn} - \hat{x}_m \hat{x}_n \tag{3.125}$$

Defining the auxiliary functions

$$f(x) = \frac{1 - 2(Mx)^2 - \cos(2Mx)}{2(Mx)^2} \qquad , \qquad h(x) = -\frac{f(x)}{1 + f(x)} \tag{3.126}$$

allows to write the metric, its inverse, and the metric determinant in Riemann normal coordinates as

$$
\begin{aligned}
g_{mn}(x) &= \delta_{mn} + f(x)P_{mn} \\
g^{mn}(x) &= \delta^{mn} + h(x)P^{mn} \\
g(x) &= (1+f(x))^{D-1}
\end{aligned}
\tag{3.127}
$$

Now we are ready to find a RNC expansion of the gravitational potential V_{eff}. Using the previous equations (3.126) and (3.127) in (3.56) (evaluated in the minimal coupling $\xi = 0$ for simplicity) we obtain a closed form the RNC expansion of the potential, which reads (we here set $M = 1$ for notational convenience, as M can be reintroduced by dimensional analysis)

$$
V_{eff}(x) = \frac{(D-1)}{8}\left[\frac{(D-5)}{4}\left(\frac{f'}{1+f}\right)^2 + \frac{1}{1+f}\left(\frac{(D-1)}{x}f' + f''\right)\right] \tag{3.128}
$$

which is evaluated to

$$
V_{eff}(x) = \frac{D-D^2}{12} + \frac{(D-1)(D-3)}{48}\frac{\left(5x^2 - 3 + (x^2+3)\cos(2x)\right)}{x^2\sin^2(x)} \tag{3.129}
$$

and which expands to

$$
\begin{aligned}
V_{eff}(x) = \frac{D-D^2}{12} &+ (D-1)(D-3)\left(\frac{x^2}{120} + \frac{x^4}{756} + \frac{x^6}{5400} + \frac{x^8}{41580} + \right. \\
&+ \left. \frac{691x^{10}}{232186500} + \frac{x^{12}}{2806650} + O\left(x^{14}\right)\right) = \sum_{m=0}^{\infty} k_{2m}x^{2m}
\end{aligned}
\tag{3.130}
$$

Using the same techniques described in section (1.3) one obtains that the heat kernel for coinciding points $x = x'$ is given by

$$
\overline{K}(x, x', \beta) = \frac{\left\langle e^{-S_{int}}\right\rangle}{(2\pi\beta)^{\frac{D}{2}}} \tag{3.131}
$$

where here

$$S_{int}[x] = \beta \int_0^1 d\tau \; V_{eff}(x) \tag{3.132}$$

In order to evaluate the mean value of the exponential of (3.131) it is convenient to express the interacting action as

$$S_{int} = \sum_{m=0}^{\infty} S_{2m} \tag{3.133}$$

where S_{2m} is the term containing the power $(x^2)^m$, with $x^2 = \vec{x}^{\,2} = x^i x_i$. For simplicity we denote them by

$$S_{2m} = \beta k_{2m} \int_0^1 \; d\tau \, (x^2)^m \; . \tag{3.134}$$

where the numerical coefficients k_{2m} are read off from (3.130). We here consider only connected terms (which are denoted as $\langle \ldots \rangle_c$ when also disconnected terms are present): the full perturbative expansion can then be obtained expanding the exponential which contains only the connected terms. This expansion then reads

$$\begin{aligned} K(x,x',\beta) =& \frac{1}{(2\pi\beta)^{\frac{D}{2}}} \exp\Bigg[- \langle S_2 \rangle - \langle S_4 \rangle - \langle S_6 \rangle - \langle S_8 \rangle - \langle S_{10} \rangle + \\ & + \frac{1}{2}\left\langle S_2^2 \right\rangle_c + \langle S_2 S_4 \rangle_c + \frac{1}{2}\left\langle S_4^2 \right\rangle_c + \langle S_2 S_6 \rangle_c - \frac{1}{3!}\left\langle S_2^3 \right\rangle_c + o(\beta^7) \Bigg] \end{aligned} \tag{3.135}$$

The various pieces in the exponential can be evaluated as previously done in section (1.3) using Wick contractions and the basic propagators of equation (1.39). We list here their value, divided by the order β which they contribute. We denoted here the Green function of equation (1.41) as $g(\tau_1, \tau_2) = -\Delta_{12}$ for notational convenience.

Order β

There is only a constant term that does not require any Wick contraction

$$-S_0 = \beta \frac{D(D-1)}{12} \tag{3.136}$$

Order β^2

$$-\langle S_2\rangle = \beta^2 k_2 D \underbrace{\int_0^1 d\tau_1\, \Delta_{11}}_{-\frac{1}{6}} \tag{3.137}$$

<u>Order β^3</u>

$$-\langle S_4\rangle = -\beta^3 k_4 D(D+2) \underbrace{\int_0^1 d\tau_1\, \Delta_{11}^2}_{\frac{1}{30}} \tag{3.138}$$

<u>Order β^4</u>

$$-\langle S_6\rangle = \beta^4 k_6 \underbrace{(D^3 + 6D^2 + 8D)}_{D(D+2)(D+4)} \underbrace{\int_0^1 d\tau_1\, \Delta_{11}^3}_{-\frac{1}{140}} \tag{3.139}$$

$$\frac{1}{2}\langle S_2^2\rangle_c = \beta^4 k_2^2 D \underbrace{\int_0^1 d\tau_1 \int_0^1 d\tau_2\, \Delta_{12}^2}_{\frac{1}{90}} \tag{3.140}$$

<u>Order β^5</u>

$$-\langle S_8\rangle = -\beta^5 k_8 \underbrace{\Big(D^4 + 12D^3 + (12+32)D^2 + 48D\Big)}_{D(D+2)(D+4)(D+6)} \underbrace{\int_0^1 d\tau_1\, \Delta_{11}^4}_{\frac{1}{630}} \tag{3.141}$$

$$\langle S_2 S_4\rangle_c = -\beta^5 k_2 k_4 (4D^2 + 8D) \underbrace{\int_0^1 d\tau_1 \int_0^1 d\tau_2\, \Delta_{12}^2 \Delta_{22}}_{-\frac{1}{420}} \tag{3.142}$$

<u>Order β^6</u>

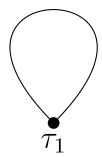
$$-\langle S_{10}\rangle = \beta^6 k_{10} \underbrace{\Big(D^5 + 20D^4 + (80+60)D^3 + (240+160)D^2 + 384D\Big)}_{D(D+2)(D+4)(D+6)(D+8)} \underbrace{\int_0^1 d\tau_1\, \Delta_{11}^5}_{-\frac{1}{2772}} \tag{3.143}$$

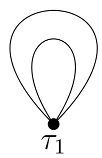
$$-\frac{1}{3!}\langle S_2^3\rangle_c = \frac{\beta^6}{3!}\;k_2^3\;8D\underbrace{\int_0^1 d\tau_1\int_0^1 d\tau_2\int_0^1 d\tau_3\,\Delta_{12}\Delta_{23}\Delta_{31}}_{-\frac{1}{945}} \tag{3.144}$$

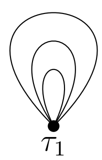
$$\langle S_2 S_6\rangle_c = \beta^6 k_2 k_6 6D(D^2+6D+8)\underbrace{\int_0^1 d\tau_1\int_0^1 d\tau_2\,\Delta_{12}^2\Delta_{22}^2}_{\frac{1}{1890}} \tag{3.145}$$

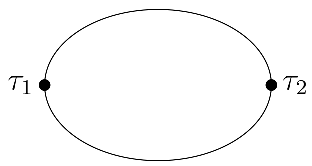
$$\frac{1}{2}\langle S_4^2\rangle_c = \frac{\beta^6}{2}\;k_4^2\left(8D(D+2)\underbrace{\int_0^1 d\tau_1\int_0^1 d\tau_2\,\Delta_{12}^4}_{\frac{1}{3150}}+8d(D^2+4D+4)\underbrace{\int_0^1 d\tau_1\int_0^1 d\tau_2\,\Delta_{11}\Delta_{12}^2\Delta_{22}}_{\frac{13}{25200}}\right)\;. \tag{3.146}$$

These terms can then be expressed in terms of Feynman graphs as

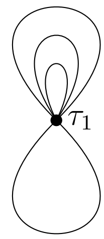
$\langle S_2\rangle =$ 

$\langle S_4\rangle =$ 

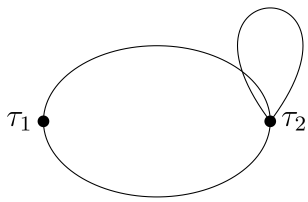
$\langle S_6\rangle =$ 

$\left\langle S_2^2\right\rangle_c =$ 

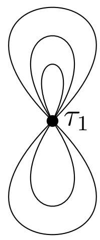
$\langle S_8 \rangle =$



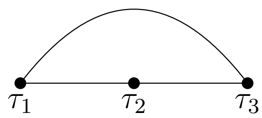
$\langle S_2 S_4 \rangle_c =$



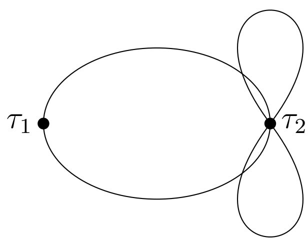
$\langle S_{10} \rangle =$



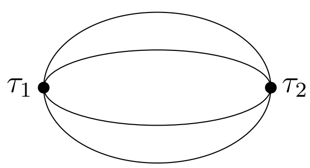
$\left\langle S_2^3 \right\rangle_c =$



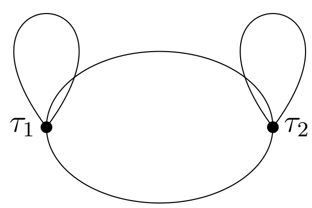
$\langle S_2 S_6 \rangle_c =$



$\left\langle S_4^2 \right\rangle =$



$+$



We are now ready to give the final result, which, putting all the pieces together, reads

$$
\begin{aligned}
\overline{K}(x,x',\beta) = & \frac{1}{(2\pi\beta)^{\frac{D}{2}}} \exp\Bigg[\beta\frac{D(D-1)}{12} - (D-1)(D-3)\Big(\beta^2\frac{D}{720} + \beta^3\frac{D(D+2)}{22680} + \\
& + \beta^4\frac{D(D^2+20D+15)}{1814400} - \beta^5\frac{D(D+2)(D^2-12D-9)}{14968800} + \\
& - \beta^6\frac{D(1623D^4-716D^3-65930D^2-123572D-60165)}{245188944000}\Big) + o(\beta^7)\Bigg]
\end{aligned}
\tag{3.147}
$$

or, written in terms of $R = D(D-1)$ (for $M^2 = 1$)

$$
\begin{aligned}
\overline{K}(x,x',\beta) = & \frac{1}{(2\pi\beta)^{\frac{D}{2}}} \exp\Bigg[\frac{\beta R}{12} - \frac{(\beta R)^2}{6!}\frac{(D-3)}{D(D-1)} - \frac{(\beta R)^3}{9!}\frac{16(D-3)(D+2)}{D^2(D-1)^2} + \\
& - \frac{(\beta R)^4}{10!}\frac{2(D-3)(D^2+20D+15)}{D^3(D-1)^3} + \\
& + \frac{(\beta R)^5}{11!}\frac{8(D-3)(D+2)(D^2-12D-9)}{3D^4(D-1)^4} + \\
& + \frac{(\beta R)^6}{13!}\frac{8(D-3)(1623D^4-716D^3-65930D^2-123572D-60165)}{315D^5(D-1)^5} + \\
& + O(\beta^7)\Bigg]
\end{aligned}
\tag{3.148}
$$

with the exponential that can be expanded to identify the first six heat kernel coefficients. Amazingly, it compares successfully with eq. (16) of ref. [6] (taking into account that $\xi = 0$ and that the sign of R has been reversed). In that reference the calculation was performed up to order $(\beta R)^3$. In the present case those results are reproduced almost trivially, and in fact we have been able to push the calculation to higher orders. These higher orders are new, as far as we know, and we assume them to be correct, given that lower orders have been reproduced exactly. Anyway, another way to test these results exists and consists in evaluating the trace anomaly one obtains for specific values of D in the conformal coupling $\xi = \frac{1}{6}$. These results can then be compared by ones found by Copeland and Toms in [14] using a zeta-function regularization procedure. In fact, trace anomalies characterize conformal field theories. They amount to the fact that the trace of the energy-momentum tensor for conformal fields, which vanishes at the classical level, acquires anomalous terms at the quantum level. These terms depend on the background geometry of the spacetime on which the conformal fields are coupled to, and they are captured by the appropriate Seeley–DeWitt coefficient sitting in the heat kernel expansion of the associated conformal operator. The relation between trace anomaly and the heat kernel is given by[14]

$$\langle T^m_{\ m}(x')\rangle = \lim_{\beta\to 0} K(x',x',\beta) \tag{3.149}$$

where it is understood that the limit picks up just the β-independent term—divergent terms are removed by QFT renormalization. This procedure selects the appropriate Seeley–DeWitt coefficient sitting in the expansion of $K(x',x';\beta)$. In fact, for even dimensions, the beta term that comes from the normalization factor $A = (2\pi\beta)^{\frac{D}{2}}$ selects only one of the terms of the heat kernel expansions once that a specific value for D has been chosen (for example, in $D = 12$ the only surviving term of the heat kernel expansion in the limit $\beta \to 0$ would be the one of order β^6). Results are reported in the following table, where the second form is written in terms of $a^2 = \frac{1}{M^2} = \frac{D(D-1)}{R}$ to directly compare with the results tabulated in [14].

D	$\langle T^\mu{}_\mu\rangle$	$\langle T^\mu{}_\mu\rangle$
2	$\frac{R}{24\,\pi}$	$\frac{1}{12\,\pi a^2}$
4	$-\frac{R^2}{34\,560\,\pi^2}$	$-\frac{1}{240\,\pi^2 a^4}$
6	$\frac{R^3}{21\,772\,800\,\pi^3}$	$\frac{5}{4\,032\,\pi^3 a^6}$
8	$-\frac{23\,R^4}{339\,880\,181\,760\,\pi^4}$	$-\frac{23}{34\,560\,\pi^4 a^8}$
10	$\frac{263\,R^5}{2\,993\,075\,712\,000\,000\,\pi^5}$	$\frac{263}{506\,880\,\pi^5 a^{10}}$
12	$-\frac{133\,787\,R^6}{1\,330\,910\,037\,208\,675\,123\,200\,\pi^6}$	$-\frac{133\,787}{251\,596\,800\,\pi^6 a^{12}}$

Table 3.1: The type-A trace anomaly of a scalar field

The comparison is successful, except at $D = 12$, where our respective coefficients differ by a number of the order of 10^{-13}. Our result is correct, as using the zeta function

[14]Since we use here Weyl anomaly as a test for our results only, and that this relation is all that one needs in order to obtain the anomaly from heat kernel calculations, a detailed description on the subject is not given here but can be found in reference [18].

approach employed in [14], see also [13], we reproduce our result[15]. All that is left now is an analytic proof of the correctness of this method which makes explicit use of the maximal symmetry of the background: we anyway remark that in maximally symmetric spaces all curvature tensors are given algebraically in terms of the metric and of the constant scalar curvature R, see eqs. (3.119), (3.120), so that by symmetry arguments the quantity $\overline{K}(x,x',\beta)$ can only depend from the coordinates through the scalar function $x^2 = \delta_{ij}x^i x^j$. But this means that the proof we given in section (3.2) for the statement $\partial_\mu g^{\mu\nu}\partial_\nu \overline{G} + (g^{\mu\nu} - \eta^{\mu\nu})\partial_\mu\partial_\nu \overline{G} = 0$ when $\overline{G} = \overline{G}(x^2)$ is sufficient, since $\overline{G}$ has the same dependence on coordinates of $\overline{K}$. We anyway present here a simpler proof which makes explicit use of the expressions given in this section. In order to prove that using RNC expansion the heat kernel $\overline{K}$ verifies the equation

$$-\frac{\partial}{\partial\beta}\overline{K}(x,x',\beta) = \Big(-\frac{1}{2}\delta_{ij} + V_{eff}(x)\Big)\overline{K}(x,x',\beta) \tag{3.150}$$

we need to show that the "curved" differential operator $g^{\mu\nu}\nabla_\mu\nabla_\nu$ acts on $\overline{K}$ simply as the "flat" box operator given in the previous equation. This means

$$(\partial_i g^{ij}\partial_j - \delta^{ij}\partial_i\partial_j)\overline{K}(x,x',\beta) = 0 \tag{3.151}$$

Taking as usual x' to be the origin of our coordinate frame, and using equations (3.125) and (3.127), the left hand side of the previous equation reduces to

$$\big[P^{ij}(x)\partial_i\partial_j + \partial_i(h(x)P^{ij}(x))\partial_j\big] \tag{3.152}$$

Using now the dependence of $\overline{K}$ from x^2 and β only, so that $\partial_i\overline{K}(x,x',\beta) = \frac{\partial x^2}{\partial x^i}\frac{\partial}{\partial x^2}K(x,x',\beta)$, and using the orthogonality condition $P^{ij}x_j = 0$, we get

$$\partial_i(h(x)P^{ij}(x))\partial_j\overline{K}(x,x',\beta) = -2(d-1)h(x)\frac{\partial}{\partial x^2}\overline{K}(x,x',\beta) \tag{3.153}$$

and

[15] The mismatch could perhaps have happened due to some inappropriate rounding of the exact number, occasionally introduced by calculators.

$$
\begin{aligned}
P^{ij}(x)\partial_i\partial_j\overline{K}(x,x',\beta) &= 2h(x)\delta_{ij}P^{ij}(x)\frac{\partial}{\partial x^2}\overline{K}(x,x',\beta) \\
&= 2(d-1)h(x)\frac{\partial}{\partial x^2}K(x,x',\beta) \qquad (3.154)
\end{aligned}
$$

We have in this way analytically verified that Guven's procedure indeed works on maximally symmetric spaces at any given order of the perturbative expansion.

Chapter 4

Conclusions

We are now ready to sum up the results obtained in the present text. In the first chapter of this thesis we have reviewed the quantum-mechanical path integral and applied it to evaluate perturbatively the heat kernel, both on flat and curved (but torsionless) background manifold. We have indicated how the path integral for point particles can be used to describe perturbatively quantum field theories with the method known as the worldline formalism. When the path integral is defined on curved spaces regularizations are needed, and in chapter 2 we have reviewed three well-known regularization schemes that allow to compute unambiguosly the path integral. This puts Guven's proposal, that we presented in chapter 3, in the right perspective.

The proposal, put forward by Guven in ref. [23], describes a procedure that transforms the non-linear sigma model action of the particle into an effective linear sigma model, by using Riemann normal coordinates. Then Guven used the linear sigma model to construct a path integral, and applied it to study a self-interacting scalar field theory. However this method has never been used for explicit calculations outside the work of that author. The method, if correct, is potentially useful, and for this reason we decided to analyze it further in this thesis.

We found out that the proposal contained an effective potential which is not able to correctly reproduce the first Seeley-DeWitt coefficients of the heat kernel expansion. It is anyway possible to obtain an effective potential from basic considerations about the defining equation of the propagator, following indications by Parker *et al.* in [32] that used a different formalism. With this new effective potential we have been able to correctly reproduce the Seeley-DeWitt coefficients up to order s^2, where s is the proper time.

Anyway, the lack of a proof of a crucial statement made by Guven himself makes it impossible to determine whether the procedure extends to higher orders. Since a proof (or disproof) of this crucial statement could not be found in the literature nor by us, we decided to calculate the next heat kernel coefficient, of order s^3, on arbitrarily curved spaces, supposing the correctness of this statement. We found that the result for a_3 is not consistent with the the well-known one obtained previously by various authors and identified using different methods. This result signals a failure of Guven's regularization scheme at higher orders. Anyway we remark that, due to the difficulty of the calculation, a re-analysis of our findings would be desired.

Given our preliminary (negative) findings on arbitrary geometries, we decided to consider a particular class of curved spaces: those with maximal symmetry, namely de Sitter or Anti de Sitter spacetimes or spheres and hyperbolic spaces of constant negative curvature for metrics of euclidean signature. We found that the procedure works well on this kind of spaces, bringing an undeniable simplification on the calculations. Indeed we have been able to evaluate the heat kernel coefficients in arbitrary D dimensions up to order s^6, which agree with the one given in [6] that reported them up to order s^3. To further check our results we used a conformal coupling in the heat kernel to extract the type-A trace anomaly for specific values of D, up to $D = 12$. The comparison of our results with known ones is successful. This success has pushed us to find a proof of of the validity of the conjectured path integral on maximally symmetric spaces, which we have found indeed by using the maximal symmetry of the background.

To conclude we have found that Guven's method is correct on maximally symmetric spaces, and we have provided a proof of its validity. On the other hand we have found indications that the method fails on arbitrary geometries, though we would like to re-analyze our calculations, which are indeed very lengthy, to confirm our preliminary findings.

For further applications, it would be interesting to test Guven's procedure and the flat path integral construction also in the case of particles with spin.

Appendices

Appendix A

Grassmann Variables

Grassmann variables, also known as "anticommuting numbers", allows one to reproduce fermionic degrees of freedom which are associated to spin, also at the classical level, since they allow to describe "classical modes" whose quantization produces spin-like degrees of freedom. In the worldline approach one may describe relativistic point-like particles with spin by their worldline coordinates, which are their position in space-time and Grassman variables to take in account for their additional spin degrees of freedom.

Grassmann variables are in fact a representation of the n-dimensional Grassmann algebra $\mathscr{G} = \{\theta_i\}$ which is formed by generators θ_i with $i = 1 \dots n$ that satisfy

$$\theta_i\theta_j + \theta_j\theta_i = 0 \tag{A.1}$$

or, equivalently,

$$\{\theta_i, \theta_j\} = 0 \tag{A.2}$$

and, in particular, the square of every Grassmann number is always vanishing

$$\theta_i^2 = 0 \tag{A.3}$$

General functions of Grassmann variables can be defined multiplying these generators and their products by real or complex numbers: for example, for $n = 1$ the only possible class of functions one can create is

$$f(\theta) = f_0 + f_1\theta \tag{A.4}$$

with f_0 and f_1 real or complex numbers. In case $n = 2$ one has

$$f(\theta) = f_0 + f_1\theta_1 + f_2\theta_2 + f_3\theta_1\theta_2 \tag{A.5}$$

and so on. A generic function of these variables is always defined by its Taylor expansion, since this expansion always contains a finite number of terms: for example the exponential function e^θ can be expressed as $1 + \theta$ since $\theta^2 = 0$.

Derivatives operators with respect of Grassmann variables can be defined in a very simple way, since a generic function can have at most a linear dependence to a fixed Grassmann variable, and one has just to keep track of the signs. Since Grassmann variables do not commute, one can define two types of differentiation, depending on which side one chooses to remove the given variable during differentiation. One then defines "left derivatives" of a Grassmann-valued function by removing the variable from the left of its Taylor expansion: for example for the function $f(\theta_1, \theta_2)$ defined above one has

$$\frac{\partial_L}{\partial\theta_1} f(\theta_1, \theta_2) = f_1 + f_3\theta_2 \tag{A.6}$$

and in the same way one defines right-derivatives by removing the variable from the right, that is

$$\frac{\partial_R}{\partial\theta_1} f(\theta_1, \theta_2) = f_1 - f_3\theta_2 \tag{A.7}$$

where the minus sign comes from the commutation relations of θ_1 and θ_2. Equivalently, using the infinitesimal Grassmann increment $\delta\theta$ one has

$$\delta f = \delta\theta \frac{\partial_L f}{\partial\theta} = \frac{\partial_R f}{\partial\theta} \delta\theta \tag{A.8}$$

If not specified otherwise we will take all derivatives with respect to Grassmann variables to be left-derivatives.

Integration over Grassmann variables can be defined, according to Berezin, to be equivalent to differentiation:

$$\int d\theta \equiv \frac{\partial_L}{\partial\theta} \tag{A.9}$$

This definition has the virtue to produce a translational invariant measure

$$\int d\theta f(\theta+\eta) = \int d\theta f(\theta) \tag{A.10}$$

which can be proven by direct evaluation. This property is indeed essential when one has to generate the perturbative expansion of the Grassmann-dependent part of the heat kernel.

One can define Grassmann variables to be either real or complex. For a real Grassmann variable one has

$$\overline{\theta} = \theta \tag{A.11}$$

where the bar indicates complex conjugation. For products of Grassmann variables the complex conjugate is defined to include an exchange in position

$$\overline{\theta_1\theta_2} = \overline{\theta_2\theta_1} \tag{A.12}$$

so that the product of two real Grassmann variables is purely imaginary

$$\overline{\theta_1\theta_2} = -\theta_1\theta_2 \tag{A.13}$$

while taking i times the same product will produce a formally real object

$$\overline{i\theta_1\theta_2} = i\theta_1\theta_2 \tag{A.14}$$

We remark that a complex Grassmann variable η can always be decomposed into two real Grassmann variables θ_1 and θ_2 by

$$\eta = \frac{1}{\sqrt{2}}(\theta_1 + i\theta_2) \quad , \quad \overline{\eta} = \frac{1}{\sqrt{2}}(\theta_1 - i\theta_2) \tag{A.15}$$

where as usual one can choose η and $\overline{\eta}$ to be independent variables, or equivalently θ_1 and θ_2.

We wish now to consider more in detail gaussian integration over Grassmann variables, since gaussian integration is the heart of the path integral formalism. The case in which we have only one Grassmann variable is trivial,

$$e^{-a\theta^2} = 1 \tag{A.16}$$

since θ has a vanishing anticommutator with itself. One then needs at least two real Grassmann variables θ_1 and θ_2 to have a non-trivial exponential which is quadratic with respect to Grassmann variables: in that case one has

$$e^{-a\theta_1\theta_2} = 1 - a\theta_1\theta_2 \tag{A.17}$$

Using the expression above it is immediate to get the result of the gaussian integral

$$\int d\theta_1 d\theta_2 \; e^{-a\theta_1\theta_2} = a \tag{A.18}$$

One can also rewrite this last equation defining $\theta^i = (\theta_1, \theta_2)$ the 2x2 antisymmetric matrix A^{ij}

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \tag{A.19}$$

obtaining the result

$$\int d\theta_1 d\theta_2 \; e^{-\frac{1}{2}\theta_i A^{ij}\theta_j} = \det^{\frac{1}{2}} A \tag{A.20}$$

notice that this expression is well defined since the determinant is always positive definite for every antisymmetric real matrix (and by analytic extension for every complex antisymmetric matrix). The formula presented above is readily generalized for an even number $n = 2m$ of real Grassmann variables

$$\int d^n\theta \; e^{-\frac{1}{2}\theta_i A^{ij}\theta_j} = \det^{\frac{1}{2}} A \tag{A.21}$$

In a similar way one can obtain the result of gaussian integration over a complex Grassmann variable, which is

$$\int d^n\overline{\eta} d^n\eta \; e^{-\overline{\eta}_i A^{ij}\eta_j} = \det A \tag{A.22}$$

Notice that for physical applications to dynamical models one needs an infinite-dimensional Grassmann algebra: this can be achieved defining a Grassmann-valued function of time, that is $\theta_i \to \theta(t)$, where at different times corresponds a different generator

$$\theta^2(t) = 0 \quad , \quad \theta(t_1)\theta(t_2) = -\theta(t_2)\theta(t_1). \tag{A.23}$$

Appendix B

Expansion of the Lagrangian in RNC

We list here all the derivatives of the Lagrangian that are necessary to obtain the sixth-order term of the metric expansion in RNC. Here we denoted with q^μ the components of a generic coordinate system.

$$\frac{DL}{D\lambda} = g_{\mu\nu}(q)\frac{Dz^\mu}{D\tau}\dot{q}^\nu \tag{B.1}$$

$$\frac{D^2L}{D\lambda} = g_{\mu\nu}(q)\frac{Dz^\mu}{D\tau}\frac{Dz^\nu}{D\tau} + g_{\mu\nu}(q)z^\lambda\dot{q}^\sigma R_{\lambda\sigma}{}^\mu{}_\rho z^\rho\dot{q}^\nu \tag{B.2}$$

$$\frac{D^3L}{D\lambda^3} = 3g_{\mu\nu}(q)R_{\lambda\sigma}{}^\mu{}_\rho z^\lambda z^\rho\dot{q}^\sigma\frac{Dz^\nu}{D\tau} + g_{\mu\nu}(q)R_{\lambda\sigma}{}^\mu{}_\rho z^\lambda z^\rho\dot{q}^\nu\frac{Dz^\sigma}{D\tau} + g_{\mu\nu}(q)R_{\lambda\sigma}{}^\mu{}_{\rho;\eta}z^\lambda z^\rho z^\eta\dot{q}^\nu\dot{q}^\sigma \tag{B.3}$$

$$\begin{aligned}\frac{D^4L}{D\lambda^4} =& 4g_{\mu\nu}(q)R_{\lambda\sigma}{}^\mu{}_\rho z^\lambda z^\rho\frac{Dz^\sigma}{D\tau}\frac{Dz^\nu}{D\tau} + 4g_{\mu\nu}(q)R_{\lambda\sigma}{}^\mu{}_{\rho;\eta}z^\lambda z^\rho z^\eta\dot{q}^\sigma\frac{Dz^\nu}{D\tau} + \\ &+ 3g_{\mu\nu}(q)R_{\lambda\sigma}{}^\mu{}_\rho R_{\alpha\beta}{}^\nu{}_\gamma z^\lambda z^\rho z^\alpha z^\gamma\dot{q}^\sigma\dot{q}^\beta + g_{\mu\nu}(q)R_{\alpha\beta}{}^\sigma{}_\gamma R_{\lambda\sigma}{}^\mu{}_\rho z^\lambda z^\rho z^\alpha z^\gamma\dot{q}^\beta\dot{q}^\nu \\ &+ 2g_{\mu\nu}(q)R_{\lambda\sigma}{}^\mu{}_{\rho;\eta}z^\lambda z^\rho z^\eta\dot{q}^\nu\frac{Dz^\sigma}{D\tau} + g_{\mu\nu}(q)R_{\lambda\sigma}{}^\mu{}_{\rho;\eta\kappa}z^\lambda z^\rho z^\eta z^\kappa\dot{q}^\sigma\dot{q}^\nu\end{aligned} \tag{B.4}$$

$$
\begin{aligned}
\frac{D^5L}{D\lambda^5} =&10g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta}z^\lambda z^\rho z^\eta\frac{Dz^\sigma}{D\tau}\frac{Dz^\nu}{D\tau}+5g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta\kappa}z^\lambda z^\rho z^\eta z^\kappa\dot{q}^\sigma\frac{Dz^\nu}{D\tau}+\\
&+3g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta\kappa}z^\lambda z^\rho z^\eta z^\kappa\dot{q}^\nu\frac{Dz^\sigma}{D\tau}+\\
&+5g_{\mu\nu}(q)R_{\alpha\beta}{}^{\sigma}{}_{\gamma}R_{\lambda\sigma}{}^{\mu}{}_{\rho}z^\alpha z^\gamma z^\lambda z^\rho\dot{q}^\beta\frac{Dz^\nu}{D\tau}+g_{\mu\nu}(q)R_{\alpha\beta}{}^{\sigma}{}_{\gamma}R_{\lambda\sigma}{}^{\mu}{}_{\rho}z^\alpha z^\gamma z^\lambda z^\rho\dot{q}^\nu\frac{Dz^\beta}{D\tau}+\\
&+g_{\mu\nu}(q)R_{\alpha\beta}{}^{\sigma}{}_{\gamma;\eta}R_{\lambda\sigma}{}^{\mu}{}_{\rho}z^\alpha z^\gamma z^\eta z^\lambda z^\rho\dot{q}^\beta\dot{q}^\nu+3g_{\mu\nu}(q)R_{\alpha\beta}{}^{\sigma}{}_{\gamma}R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta}z^\alpha z^\gamma z^\lambda z^\rho z^\eta\dot{q}^\beta\dot{q}^\nu+\\
&+7g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\nu}{}_{\gamma}z^\lambda z^\rho z^\alpha z^\gamma\dot{q}^\beta\frac{Dz^\sigma}{D\tau}+3g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\nu}{}_{\gamma}z^\lambda z^\rho z^\alpha z^\gamma\dot{q}^\sigma\frac{Dz^\beta}{D\tau}+\\
&+7g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta}R_{\alpha\beta}{}^{\nu}{}_{\gamma}z^\lambda z^\rho z^\eta z^\alpha z^\gamma\dot{q}^\beta\dot{q}^\sigma+3g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\nu}{}_{\gamma;\eta}z^\lambda z^\rho z^\alpha z^\gamma z^\eta\dot{q}^\beta\dot{q}^\sigma+\\
&+R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta\kappa\xi}z^\lambda z^\rho z^\eta z^\kappa z^\xi\dot{q}^\sigma\dot{q}^\nu
\end{aligned}
\tag{B.5}
$$

$$
\begin{aligned}
\frac{D^6L}{D\lambda^6} =&18g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta\kappa}z^\lambda z^\rho z^\eta z^\kappa\frac{Dz^\sigma}{D\tau}\frac{Dz^\nu}{D\tau}+6g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta\kappa\xi}z^\lambda z^\rho z^\eta z^\kappa z^\xi\dot{q}^\sigma\frac{Dz^\nu}{D\tau}+\\
&+4g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta\kappa\xi}z^\lambda z^\rho z^\eta z^\kappa z^\xi\dot{q}^\nu\frac{Dz^\sigma}{D\tau}+\\
&+6g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\sigma}{}_{\gamma}z^\lambda z^\rho z^\alpha z^\gamma\frac{Dz^\beta}{D\tau}\frac{Dz^\nu}{D\tau}+10g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\nu}{}_{\gamma}z^\lambda z^\rho z^\alpha z^\gamma\frac{Dz^\beta}{D\tau}\frac{Dz^\sigma}{D\tau}+\\
&+24g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta}R_{\alpha\beta}{}^{\nu}{}_{\gamma}z^\lambda z^\rho z^\eta z^\alpha z^\gamma\dot{q}^\beta\frac{Dz^\sigma}{D\tau}+18g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta}R_{\alpha\beta}{}^{\sigma}{}_{\gamma}z^\lambda z^\rho z^\eta z^\alpha z^\gamma\dot{q}^\beta\frac{Dz^\nu}{D\tau}+\\
&+10g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta}R_{\alpha\beta}{}^{\nu}{}_{\gamma}z^\lambda z^\rho z^\eta z^\alpha z^\gamma\dot{q}^\sigma\frac{Dz^\beta}{D\tau}+4g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta}R_{\alpha\beta}{}^{\sigma}{}_{\gamma}z^\lambda z^\rho z^\eta z^\alpha z^\gamma\dot{q}^\nu\frac{Dz^\beta}{D\tau}+\\
&+6g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\sigma}{}_{\gamma;\eta}z^\lambda z^\rho z^\alpha z^\gamma z^\eta\dot{q}^\beta\frac{Dz^\nu}{D\tau}+2g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\sigma}{}_{\gamma;\eta}z^\lambda z^\rho z^\alpha z^\gamma z^\eta\dot{q}^\nu\frac{Dz^\beta}{D\tau}+\\
&+10g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\nu}{}_{\gamma;\eta}z^\lambda z^\rho z^\alpha z^\gamma z^\eta\dot{q}^\beta\frac{Dz^\sigma}{D\tau}+6g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\nu}{}_{\gamma;\eta}z^\lambda z^\rho z^\alpha z^\gamma z^\eta\dot{q}^\sigma\frac{Dz^\beta}{D\tau}+\\
&+10g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta}R_{\alpha\beta}{}^{\nu}{}_{\gamma;\kappa}z^\lambda z^\rho z^\eta z^\alpha z^\gamma z^\kappa\dot{q}^\beta\dot{q}^\sigma+4g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta}R_{\alpha\beta}{}^{\sigma}{}_{\gamma;\kappa}z^\lambda z^\rho z^\eta z^\alpha z^\gamma z^\kappa\dot{q}^\beta\dot{q}^\nu+\\
&+12g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta\kappa}R_{\alpha\beta}{}^{\nu}{}_{\gamma}z^\lambda z^\rho z^\eta z^\kappa z^\alpha z^\gamma\dot{q}^\beta\dot{q}^\sigma+6g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta\kappa}R_{\alpha\beta}{}^{\sigma}{}_{\gamma}z^\lambda z^\rho z^\eta z^\kappa z^\alpha z^\gamma\dot{q}^\beta\dot{q}^\nu+\\
&+3g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\nu}{}_{\gamma;\eta\kappa}z^\lambda z^\rho z^\alpha z^\gamma z^\eta z^\kappa\dot{q}^\beta\dot{q}^\sigma+g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\sigma}{}_{\gamma;\eta\kappa}z^\lambda z^\rho z^\alpha z^\gamma z^\eta z^\kappa\dot{q}^\beta\dot{q}^\nu+\\
&+12g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\sigma}{}_{\gamma}R_{\tau\phi}{}^{\nu}{}_{\psi}z^\lambda z^\rho z^\alpha z^\gamma z^\tau z^\psi\dot{q}^\beta\dot{q}^\phi+\\
&+g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\sigma}{}_{\gamma}R_{\tau\phi}{}^{\beta}{}_{\psi}z^\lambda z^\rho z^\alpha z^\gamma z^\tau z^\psi\dot{q}^\nu\dot{q}^\phi+\\
&+3g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\nu}{}_{\gamma}R_{\tau\phi}{}^{\beta}{}_{\psi}z^\lambda z^\rho z^\alpha z^\gamma z^\tau z^\psi\dot{q}^\sigma\dot{q}^\phi+g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta\kappa\xi\theta}z^\lambda z^\rho z^\eta z^\kappa z^\xi z^\theta\dot{q}^\sigma\dot{q}^\nu
\end{aligned}
\tag{B.6}
$$

$$
\begin{aligned}
\frac{D^7L}{D\lambda^7} &= 28g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta\kappa\xi}z^\lambda z^\rho z^\eta z^\kappa z^\xi\frac{Dz^\sigma}{D\tau}\frac{Dz^\nu}{D\tau}+7g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta\kappa\xi\theta}z^\lambda z^\rho z^\eta z^\kappa z^\xi z^\theta\dot q^\sigma\frac{Dz^\nu}{D\tau}+\\
&+5g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta\kappa\xi\theta}z^\lambda z^\rho z^\eta z^\kappa z^\xi z^\theta\dot q^\nu\frac{Dz^\sigma}{D\tau}+g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta\kappa\xi\theta\iota}z^\lambda z^\rho z^\eta z^\kappa z^\xi z^\theta z^\iota\dot q^\nu\dot q^\sigma+\\
&+28g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta}R_{\alpha\beta}{}^{\sigma}{}_{\gamma}z^\lambda z^\rho z^\eta z^\alpha z^\gamma\frac{Dz^\beta}{D\tau}\frac{Dz^\nu}{D\tau}+14g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\sigma}{}_{\gamma;\eta}z^\lambda z^\rho z^\alpha z^\gamma z^\eta\frac{Dz^\beta}{D\tau}\frac{Dz^\nu}{D\tau}+\\
&+44g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta}R_{\alpha\beta}{}^{\nu}{}_{\gamma}z^\lambda z^\rho z^\alpha z^\gamma\frac{Dz^\beta}{D\tau}\frac{Dz^\sigma}{D\tau}+26g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\nu}{}_{\gamma;\eta}z^\lambda z^\rho z^\alpha z^\gamma\frac{Dz^\beta}{D\tau}\frac{Dz^\sigma}{D\tau}+\\
&+42g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta\kappa}R_{\alpha\beta}{}^{\sigma}{}_{\gamma}z^\lambda z^\rho z^\eta z^\kappa z^\alpha z^\gamma\dot q^\beta\frac{Dz^\nu}{D\tau}+54g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta\kappa}R_{\alpha\beta}{}^{\nu}{}_{\gamma}z^\lambda z^\rho z^\eta z^\kappa z^\alpha z^\gamma\dot q^\beta\frac{Dz^\sigma}{D\tau}+\\
&+22g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta\kappa}R_{\alpha\beta}{}^{\nu}{}_{\gamma}z^\lambda z^\rho z^\eta z^\kappa z^\alpha z^\gamma\dot q^\sigma\frac{Dz^\beta}{D\tau}+10g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta\kappa}R_{\alpha\beta}{}^{\sigma}{}_{\gamma}z^\lambda z^\rho z^\eta z^\kappa z^\alpha z^\gamma\dot q^\nu\frac{Dz^\beta}{D\tau}+\\
&+44g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta}R_{\alpha\beta}{}^{\nu}{}_{\gamma;\kappa}z^\lambda z^\rho z^\eta z^\alpha z^\gamma z^\kappa\dot q^\beta\frac{Dz^\sigma}{D\tau}+28g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta}R_{\alpha\beta}{}^{\sigma}{}_{\gamma;\kappa}z^\lambda z^\rho z^\eta z^\alpha z^\gamma z^\kappa\dot q^\beta\frac{Dz^\nu}{D\tau}+\\
&+26g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta}R_{\alpha\beta}{}^{\nu}{}_{\gamma;\kappa}z^\lambda z^\rho z^\eta z^\alpha z^\gamma z^\kappa\dot q^\sigma\frac{Dz^\beta}{D\tau}+10g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta}R_{\alpha\beta}{}^{\sigma}{}_{\gamma;\kappa}z^\lambda z^\rho z^\eta z^\alpha z^\gamma z^\kappa\dot q^\nu\frac{Dz^\beta}{D\tau}+\\
&+9g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\nu}{}_{\gamma;\eta\kappa}z^\lambda z^\rho z^\alpha z^\gamma z^\eta z^\kappa\dot q^\sigma\frac{Dz^\beta}{D\tau}+3g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\sigma}{}_{\gamma;\eta\kappa}z^\lambda z^\rho z^\alpha z^\gamma z^\eta z^\kappa\dot q^\nu\frac{Dz^\beta}{D\tau}+\\
&+13g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\nu}{}_{\gamma;\eta\kappa}z^\lambda z^\rho z^\alpha z^\gamma z^\eta z^\kappa\dot q^\beta\frac{Dz^\sigma}{D\tau}+7g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\sigma}{}_{\gamma;\eta\kappa}z^\lambda z^\rho z^\alpha z^\gamma z^\eta z^\kappa\dot q^\beta\frac{Dz^\nu}{D\tau}+\\
&+22g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta\kappa}R_{\alpha\beta}{}^{\nu}{}_{\gamma;\xi}z^\lambda z^\rho z^\eta z^\kappa z^\alpha z^\gamma z^\xi\dot q^\beta\dot q^\sigma+13g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta}R_{\alpha\beta}{}^{\nu}{}_{\gamma;\kappa\xi}z^\lambda z^\rho z^\eta z^\alpha z^\gamma z^\kappa z^\xi\dot q^\beta\dot q^\sigma+\\
&+10g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta\kappa}R_{\alpha\beta}{}^{\sigma}{}_{\gamma;\xi}z^\lambda z^\rho z^\eta z^\kappa z^\alpha z^\gamma z^\xi\dot q^\beta\dot q^\nu+5g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta}R_{\alpha\beta}{}^{\sigma}{}_{\gamma;\kappa\xi}z^\lambda z^\rho z^\eta z^\alpha z^\gamma z^\kappa z^\xi\dot q^\beta\dot q^\nu+\\
&+18g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta\kappa\xi}R_{\alpha\beta}{}^{\nu}{}_{\gamma}z^\lambda z^\rho z^\eta z^\kappa z^\xi z^\alpha z^\gamma\dot q^\beta\dot q^\sigma+10g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta\kappa\xi}R_{\alpha\beta}{}^{\sigma}{}_{\gamma}z^\lambda z^\rho z^\eta z^\kappa z^\xi z^\alpha z^\gamma\dot q^\beta\dot q^\nu+\\
&+3g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\nu}{}_{\gamma;\eta\kappa\xi}z^\lambda z^\rho z^\alpha z^\gamma z^\eta z^\kappa z^\xi\dot q^\beta\dot q^\sigma+g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\sigma}{}_{\gamma;\eta\kappa\xi}z^\lambda z^\rho z^\alpha z^\gamma z^\eta z^\kappa z^\xi\dot q^\beta\dot q^\nu+\\
&+28g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\sigma}{}_{\gamma}R_{\tau\phi}{}^{\nu}{}_{\psi}z^\lambda z^\rho z^\alpha z^\gamma z^\tau z^\psi\dot q^\phi\frac{Dz^\beta}{D\tau}+7g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\sigma}{}_{\gamma}R_{\tau\phi}{}^{\beta}{}_{\psi}z^\lambda z^\rho z^\alpha z^\gamma z^\tau z^\psi\dot q^\phi\frac{Dz^\nu}{D\tau}+\\
&+13g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\nu}{}_{\gamma}R_{\tau\phi}{}^{\beta}{}_{\psi}z^\lambda z^\rho z^\alpha z^\gamma z^\tau z^\psi\dot q^\phi\frac{Dz^\sigma}{D\tau}+3g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\nu}{}_{\gamma}R_{\tau\phi}{}^{\beta}{}_{\psi}z^\lambda z^\rho z^\alpha z^\gamma z^\tau z^\psi\dot q^\sigma\frac{Dz^\phi}{D\tau}+\\
&+12g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\sigma}{}_{\gamma}R_{\tau\phi}{}^{\nu}{}_{\psi}z^\lambda z^\rho z^\alpha z^\gamma z^\tau z^\psi\dot q^\beta\frac{Dz^\phi}{D\tau}+g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\sigma}{}_{\gamma}R_{\tau\phi}{}^{\beta}{}_{\psi}z^\lambda z^\rho z^\alpha z^\gamma z^\tau z^\psi\dot q^\nu\frac{Dz^\phi}{D\tau}+\\
&+54g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta}R_{\alpha\beta}{}^{\nu}{}_{\gamma}R_{\tau\phi}{}^{\sigma}{}_{\psi}z^\lambda z^\rho z^\eta z^\alpha z^\gamma z^\tau z^\psi\dot q^\beta\dot q^\phi+\\
&+22g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\nu}{}_{\gamma;\eta}R_{\tau\phi}{}^{\sigma}{}_{\psi}z^\lambda z^\rho z^\alpha z^\gamma z^\eta z^\tau z^\psi\dot q^\beta\dot q^\phi+\\
&+18g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\nu}{}_{\gamma}R_{\tau\phi}{}^{\sigma}{}_{\psi;\eta}z^\lambda z^\rho z^\alpha z^\gamma z^\tau z^\psi z^\eta\dot q^\beta\dot q^\phi+\\
&+13g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta}R_{\alpha\beta}{}^{\nu}{}_{\gamma}R_{\tau\phi}{}^{\beta}{}_{\psi}z^\lambda z^\rho z^\eta z^\alpha z^\gamma z^\tau z^\psi\dot q^\sigma\dot q^\phi+\\
&+9g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\nu}{}_{\gamma;\eta}R_{\tau\phi}{}^{\beta}{}_{\psi}z^\lambda z^\rho z^\alpha z^\gamma z^\eta z^\tau z^\psi\dot q^\sigma\dot q^\phi+\\
&+3g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\nu}{}_{\gamma}R_{\tau\phi}{}^{\beta}{}_{\psi;\eta}z^\lambda z^\rho z^\alpha z^\gamma z^\tau z^\psi z^\eta\dot q^\sigma\dot q^\phi+\\
&+5g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho;\eta}R_{\alpha\beta}{}^{\sigma}{}_{\gamma}R_{\tau\phi}{}^{\beta}{}_{\psi}z^\lambda z^\rho z^\eta z^\alpha z^\gamma z^\tau z^\psi\dot q^\nu\dot q^\phi+\\
&+3g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\sigma}{}_{\gamma;\eta}R_{\tau\phi}{}^{\beta}{}_{\psi}z^\lambda z^\rho z^\alpha z^\gamma z^\eta z^\tau z^\psi\dot q^\nu\dot q^\phi+\\
&+g_{\mu\nu}(q)R_{\lambda\sigma}{}^{\mu}{}_{\rho}R_{\alpha\beta}{}^{\sigma}{}_{\gamma}R_{\tau\phi}{}^{\beta}{}_{\psi;\eta}z^\lambda z^\rho z^\alpha z^\gamma z^\tau z^\psi z^\eta\dot q^\nu\dot q^\phi
\end{aligned}
\tag{B.7}
$$

And reporting only terms which differ from zero when setting $\lambda = 0$ we get, up to eighth-order

$$\begin{aligned}
\frac{D^8 L}{D\lambda^8} =&40 g_{\mu\nu}(q) R_{\lambda\sigma\ \ \rho;\eta\kappa\xi\theta}^{\ \ \ \mu} z^\lambda z^\rho z^\eta z^\kappa z^\xi z^\theta \frac{Dz^\sigma}{D\tau}\frac{Dz^\nu}{D\tau} + \\
&+ 120 g_{\mu\nu}(q) R_{\lambda\sigma\ \ \rho;\eta\kappa}^{\ \ \ \mu} R_{\alpha\beta\ \ \gamma}^{\ \ \ \nu} z^\lambda z^\rho z^\eta z^\kappa z^\alpha z^\gamma \frac{Dz^\beta}{D\tau}\frac{Dz^\sigma}{D\tau} + \\
&+ 80 g_{\mu\nu}(q) R_{\lambda\sigma\ \ \rho;\eta\kappa}^{\ \ \ \mu} R_{\alpha\beta\ \ \gamma}^{\ \ \ \sigma} z^\lambda z^\rho z^\eta z^\kappa z^\alpha z^\gamma \frac{Dz^\beta}{D\tau}\frac{Dz^\nu}{D\tau} + \\
&+ 140 g_{\mu\nu}(q) R_{\lambda\sigma\ \ \rho;\eta}^{\ \ \ \mu} R_{\alpha\beta\ \ \gamma;\kappa}^{\ \ \ \nu} z^\lambda z^\rho z^\eta z^\alpha z^\gamma z^\kappa \frac{Dz^\beta}{D\tau}\frac{Dz^\sigma}{D\tau} + \\
&+ 80 g_{\mu\nu}(q) R_{\lambda\sigma\ \ \rho;\eta}^{\ \ \ \mu} R_{\alpha\beta\ \ \gamma;\kappa}^{\ \ \ \sigma} z^\lambda z^\rho z^\eta z^\alpha z^\gamma z^\kappa \frac{Dz^\beta}{D\tau}\frac{Dz^\nu}{D\tau} + \\
&+ 48 g_{\mu\nu}(q) R_{\lambda\sigma\ \ \rho}^{\ \ \ \mu} R_{\alpha\beta\ \ \gamma;\eta\kappa}^{\ \ \ \nu} z^\lambda z^\rho z^\alpha z^\gamma z^\eta z^\kappa \frac{Dz^\beta}{D\tau}\frac{Dz^\sigma}{D\tau} + \\
&+ 24 g_{\mu\nu}(q) R_{\lambda\sigma\ \ \rho}^{\ \ \ \mu} R_{\alpha\beta\ \ \gamma;\eta\kappa}^{\ \ \ \sigma} z^\lambda z^\rho z^\alpha z^\gamma z^\eta z^\kappa \frac{Dz^\beta}{D\tau}\frac{Dz^\nu}{D\tau} + \\
&+ 40 g_{\mu\nu}(q) R_{\lambda\sigma\ \ \rho}^{\ \ \ \mu} R_{\alpha\beta\ \ \gamma}^{\ \ \ \nu} R_{\tau\phi\ \ \psi}^{\ \ \ \sigma} z^\lambda z^\rho z^\alpha z^\gamma z^\tau z^\psi \frac{Dz^\phi}{D\tau}\frac{Dz^\beta}{D\tau} + \\
&+ 16 g_{\mu\nu}(q) R_{\lambda\sigma\ \ \rho}^{\ \ \ \mu} R_{\alpha\beta\ \ \gamma}^{\ \ \ \nu} R_{\tau\phi\ \ \psi}^{\ \ \ \beta} z^\lambda z^\rho z^\alpha z^\gamma z^\tau z^\psi \frac{Dz^\phi}{D\tau}\frac{Dz^\sigma}{D\tau} + \\
&+ 8 g_{\mu\nu}(q) R_{\lambda\sigma\ \ \rho}^{\ \ \ \mu} R_{\alpha\beta\ \ \gamma}^{\ \ \ \sigma} R_{\tau\phi\ \ \psi}^{\ \ \ \beta} z^\lambda z^\rho z^\alpha z^\gamma z^\tau z^\psi \frac{Dz^\nu}{D\tau}\frac{Dz^\phi}{D\tau}
\end{aligned} \tag{B.8}$$

From here, reorganizing indices such to collect a global term $\frac{Dz^\mu}{D\tau}\frac{Dz^\nu}{D\tau}$ and setting to 0 every term which contains a $\dot{q}$ (which is vanishing at the origin) we obtain the expansion of the metric in Riemann normal coordinates, as given by equation (3.30).

Appendix C

RNC Expansion of the Effective Potential

In this appendix we list the RNC expansion of one and two derivatives of the inverse metric $g^{\mu\nu}$ and the metric determinant g, evaluated at the origin of the coordinate frame, that one needs to obtain the RNC expansion of the potential V_2 up to the order needed to obtain the correct Seeley-DeWitt coefficient a_3. The number on bottom of round brackets indicates the order of the term in RNC expansion. We remark that all the free indices in these expansions can be rearranged at will, since they are always contracted with an equal number of z terms. All of the following calculations are original.

$$(\partial_\mu g^{\mu\nu})_0 = 0 \tag{C.1}$$

$$(\partial_\mu g^{\mu\nu})_1 = -\frac{1}{3} R^{\nu}{}_{\alpha} \tag{C.2}$$

$$(\partial_\mu g^{\mu\nu})_2 = \frac{1}{6} R_{\alpha\beta;}{}^{\nu} - \frac{1}{3} R_{\alpha}{}^{\nu}{}_{;\beta} \tag{C.3}$$

$$(\partial_\mu g^{\mu\nu})_3 = \frac{1}{10} R_{\alpha\beta;}{}^{\nu}{}_{\gamma} - \frac{3}{20} R_{\alpha}{}^{\nu}{}_{;\beta\gamma} + \frac{1}{60} R_{\alpha\lambda} R^{\lambda}{}_{\beta\gamma}{}^{\nu} - \frac{1}{15} R^{\lambda}{}_{\alpha\beta}{}^{\kappa} R_{\lambda\gamma}{}^{\nu}{}_{\kappa} \tag{C.4}$$

$$(\partial_\mu g)_0 = 0 \tag{C.5}$$

$$(\partial_\mu g)_1 = -\frac{2}{3}R_{\alpha\mu} \tag{C.6}$$

$$(\partial_\mu g)_2 = -\frac{1}{3}R_{\alpha\mu;\beta} - \frac{1}{6}R_{\alpha\beta;\mu} \tag{C.7}$$

$$(\partial_\mu g)_3 = -\frac{1}{10}R_{\alpha\mu;\beta\gamma} - \frac{1}{10}R_{\alpha\beta;\mu\gamma} + \frac{2}{9}R_{\alpha\beta}R_{\gamma\mu} - \frac{1}{10}R_{\alpha\lambda}R^{\lambda}{}_{\beta\gamma\mu} - \frac{2}{45}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\gamma\mu\kappa} \tag{C.8}$$

$$\begin{aligned}(\partial_\mu g)_4 = &-\frac{1}{45}R_{\alpha\mu;\beta\gamma\delta} - \frac{1}{30}R_{\alpha\beta;\mu\gamma\delta} + \frac{1}{9}R_{\alpha\mu}R_{\beta\gamma;\delta} + \frac{1}{9}R_{\alpha\beta}R_{\gamma\mu;\delta} + \frac{1}{18}R_{\alpha\beta}R_{\gamma\delta;\mu} + \\ &- \frac{1}{15}R_{\alpha\lambda;\beta}R^{\lambda}{}_{\gamma\delta\mu} - \frac{1}{90}R_{\alpha\beta;\lambda}R^{\lambda}{}_{\gamma\delta\mu} - \frac{2}{45}R_{\alpha\lambda}R^{\lambda}{}_{\beta\gamma\mu;\delta} - \frac{1}{45}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\gamma\mu\kappa;\delta} + \\ &- \frac{1}{45}R^{\lambda}{}_{\alpha\mu}{}^{\kappa}R_{\lambda\beta\gamma\kappa;\delta} - \frac{1}{90}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\gamma\delta\kappa;\mu}\end{aligned} \tag{C.9}$$

$$\begin{aligned}(\partial_\mu g)_5 = &-\frac{1}{252}R_{\alpha\mu;\beta\gamma\delta\eta} - \frac{1}{126}R_{\alpha\beta;\mu\gamma\delta\eta} + \frac{1}{36}R_{\alpha\beta;\gamma}R_{\delta\eta;\mu} + \frac{1}{18}R_{\alpha\beta;\gamma}R_{\delta\mu;\eta} + \frac{1}{30}R_{\alpha\beta}R_{\gamma\delta;\mu\eta} + \\ &+ \frac{1}{30}R_{\alpha\beta}R_{\gamma\mu;\delta\eta} + \frac{1}{30}R_{\alpha\mu}R_{\beta\gamma;\delta\eta} - \frac{2}{315}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\gamma\mu\kappa;\delta\eta} - \frac{2}{315}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\gamma\delta\kappa;\mu\eta} + \\ &- \frac{2}{315}R^{\lambda}{}_{\alpha\mu}{}^{\kappa}R_{\lambda\beta\gamma\kappa;\delta\eta} - \frac{1}{84}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}{}_{;\gamma}R_{\lambda\delta\mu\kappa;\eta} - \frac{1}{168}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}{}_{;\gamma}R_{\lambda\delta\eta\kappa;\mu} - \frac{1}{27}R_{\alpha\beta}R_{\gamma\delta}R_{\eta\mu} + \\ &- \frac{2}{15}R_{\alpha\beta}R^{\lambda}{}_{\gamma\delta}{}^{\kappa}R_{\lambda\eta\mu\kappa} - \frac{1}{15}R_{\alpha\mu}R^{\lambda}{}_{\beta\gamma}{}^{\kappa}R_{\lambda\delta\eta\kappa} + \frac{4}{945}R_{\lambda\alpha\beta}{}^{\kappa}R_{\kappa\gamma\delta}{}^{\rho}R_{\rho\eta\mu}{}^{\lambda} - \frac{1}{126}R_{\alpha\beta;\gamma\lambda}R^{\lambda}{}_{\delta\eta\mu} + \\ &- \frac{1}{42}R_{\alpha\lambda;\beta\gamma}R^{\lambda}{}_{\delta\eta\mu} + \frac{1}{84}R_{\alpha\lambda}R^{\lambda}{}_{\beta\gamma}{}^{\kappa}R_{\kappa\delta\eta\mu} - \frac{2}{63}R_{\alpha\lambda;\beta}R^{\lambda}{}_{\gamma\delta\mu;\eta} - \frac{1}{84}R_{\alpha\lambda}R^{\lambda}{}_{\beta\gamma\mu;\delta\eta} + \\ &- \frac{1}{252}R_{\alpha\beta;\lambda}R^{\lambda}{}_{\gamma\delta\mu;\eta} + \frac{1}{30}R_{\alpha\beta}R_{\gamma\lambda}R^{\lambda}{}_{\delta\eta\mu} + \frac{2}{315}R_{\lambda\alpha\beta\kappa}R^{\lambda}{}_{\gamma}{}^{\rho\kappa}R_{\rho\delta\eta\mu} + \frac{2}{315}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\gamma\delta\rho}R^{\rho}{}_{\kappa\eta\mu}\end{aligned} \tag{C.10}$$

$$(\partial_\mu\partial_\nu g)_0 = -\frac{2}{3}R_{\mu\nu} \tag{C.11}$$

$$(\partial_\mu\partial_\nu g)_1 = -\frac{1}{3}R_{\alpha\mu;\nu} - \frac{1}{3}R_{\alpha\nu;\mu} - \frac{1}{3}R_{\mu\nu;\alpha} \tag{C.12}$$

$$
\begin{aligned}
(\partial_\mu\partial_\nu g)_2 = & -\frac{1}{10}R_{\alpha\beta;\mu\nu} - \frac{1}{5}R_{\alpha\nu;\mu\beta} - \frac{1}{5}R_{\alpha\mu;\nu\beta} - \frac{1}{10}R_{\mu\nu;\alpha\beta} - \frac{1}{5}R_{\alpha\lambda}R^{\lambda}{}_{\nu\beta\mu} + \\
& - \frac{1}{5}R_{\alpha\lambda}R^{\lambda}{}_{\mu\beta\nu} - \frac{1}{10}R_{\lambda\mu}R^{\lambda}{}_{\beta\gamma\nu} - \frac{1}{10}R_{\lambda\nu}R^{\lambda}{}_{\beta\gamma\mu} + \frac{2}{9}R_{\mu\nu}R_{\alpha\beta} + \frac{4}{9}R_{\alpha\nu}R_{\beta\mu} + \\
& - \frac{2}{45}R^{\lambda}{}_{\mu\nu}{}^{\kappa}R_{\lambda\alpha\beta\kappa} - \frac{2}{45}R^{\lambda}{}_{\alpha\nu}{}^{\kappa}R_{\lambda\mu\beta\kappa} - \frac{2}{45}R^{\lambda}{}_{\alpha\nu}{}^{\kappa}R_{\lambda\beta\mu\kappa}
\end{aligned} \tag{C.13}
$$

$$
\begin{aligned}
(\partial_\mu\partial_\nu g)_3 = & -\frac{1}{15}R_{\alpha\beta;\gamma\nu\mu} - \frac{1}{15}R_{\alpha\mu;\nu\beta\gamma} - \frac{1}{15}R_{\alpha\nu;\mu\beta\gamma} - \frac{1}{45}R_{\mu\nu;\alpha\beta\gamma} - \frac{2}{15}R_{\alpha\lambda;\beta}R^{\lambda}{}_{\mu\gamma\nu} + \\
& - \frac{4}{45}R_{\alpha\lambda}R^{\lambda}{}_{\nu\beta\mu;\gamma} - \frac{2}{45}R_{\lambda\nu}R^{\lambda}{}_{\alpha\beta\mu;\gamma} - \frac{1}{15}R_{\lambda\nu;\alpha}R^{\lambda}{}_{\beta\gamma\mu} - \frac{1}{15}R_{\lambda\mu;\alpha}R^{\lambda}{}_{\beta\gamma\nu} + \\
& - \frac{1}{45}R_{\alpha\nu;\lambda}R^{\lambda}_{\beta\gamma\mu} - \frac{1}{45}R_{\alpha\mu;\lambda}R^{\lambda}{}_{\beta\gamma\nu} - \frac{2}{45}R_{\lambda\mu}R^{\lambda}{}_{\alpha\beta\nu;\gamma} + \frac{1}{15}R_{\alpha\lambda;\nu}R^{\lambda}{}_{\beta\gamma\mu} + \\
& + \frac{4}{45}R_{\alpha\lambda}R^{\lambda}{}_{\beta\gamma\nu;\mu} - \frac{4}{45}R_{\alpha\lambda}R^{\lambda}{}_{\mu\beta\nu;\gamma} - \frac{1}{90}R_{\alpha\beta;\lambda}R^{\lambda}{}_{\gamma\mu\nu} - \frac{1}{90}R_{\alpha\beta;\lambda}R^{\lambda}{}_{\mu\gamma\nu} + \\
& + \frac{1}{30}R_{\alpha\beta;\lambda}R^{\lambda}{}_{\nu\gamma\mu} + \frac{1}{15}R_{\alpha\lambda;\mu}R^{\lambda}_{\beta\gamma\nu} + \frac{1}{9}R_{\mu\nu}R_{\alpha\beta;\gamma} + \frac{2}{9}R_{\alpha\nu}R_{\beta\mu;\gamma} + \frac{2}{9}R_{\alpha\mu}R_{\beta\nu;\gamma} + \\
& + \frac{1}{9}R_{\alpha\nu}R_{\beta\gamma;\mu} + \frac{1}{9}R_{\alpha\mu}R_{\beta\gamma;\nu} + \frac{1}{9}R_{\alpha\beta}R_{\gamma\nu;\mu} + \frac{1}{9}R_{\alpha\beta}R_{\gamma\mu;\nu} + \frac{1}{9}R_{\alpha\beta}R_{\mu\nu;\gamma} + \\
& - \frac{1}{45}R^{\lambda}{}_{\mu\nu\kappa}R_{\lambda\alpha\beta}{}^{\kappa}{}_{;\gamma} - \frac{1}{45}R^{\lambda}{}_{\alpha\nu\kappa}R_{\lambda\beta\gamma}{}^{\kappa}{}_{;\mu} - \frac{1}{45}R^{\lambda}{}_{\alpha\mu\kappa}R_{\lambda\beta\gamma}{}^{\kappa}{}_{;\nu} - \frac{1}{45}R^{\lambda}{}_{\alpha\beta\kappa}R_{\lambda\mu\nu}{}^{\kappa}{}_{;\gamma} + \\
& - \frac{1}{45}R^{\lambda}{}_{\alpha\nu\kappa}R_{\lambda\beta\mu}{}^{\kappa}{}_{;\gamma} - \frac{1}{45}R^{\lambda}{}_{\alpha\nu\kappa}R_{\lambda\mu\beta}{}^{\kappa}{}_{;\gamma} - \frac{1}{45}R^{\lambda}{}_{\alpha\mu\kappa}R_{\lambda\beta\nu}{}^{\kappa}{}_{;\gamma} - \frac{1}{45}R^{\lambda}{}_{\alpha\mu\kappa}R_{\lambda\nu\beta}{}^{\kappa}{}_{;\gamma} + \\
& - \frac{1}{45}R^{\lambda}{}_{\alpha\beta\kappa}R_{\lambda\gamma\nu}{}^{\kappa}{}_{;\mu} - \frac{1}{45}R^{\lambda}{}_{\alpha\beta\kappa}R_{\lambda\gamma\mu}{}^{\kappa}{}_{;\nu}
\end{aligned} \tag{C.14}
$$

Since the next (and last) term of the expansion is too long to fit in a single page we have separated it arbitrarily into two terms, the first containing only terms with products of at most two Riemann (or Ricci) tensors, and one containing only products of three tensors.

$$
\begin{aligned}
(\partial_\mu\partial_\nu g)_4 = &-\frac{1}{252}R_{\mu\nu;\alpha\beta\gamma\delta}-\frac{1}{63}R_{\alpha\mu;\nu\beta\gamma\delta}-\frac{1}{63}R_{\alpha\nu;\mu\beta\gamma\delta}-\frac{1}{42}R_{\alpha\beta;\gamma\delta\nu\mu}+\frac{1}{42}R_{\alpha\lambda;\beta\gamma}R^{\lambda}{}_{\mu\delta\nu}+\\
&+\frac{1}{42}R_{\alpha\lambda;\beta\gamma}R^{\lambda}{}_{\nu\delta\mu}+\frac{1}{84}R_{\alpha\lambda}R^{\lambda}{}_{\mu\beta\nu;\gamma\delta}+\frac{1}{84}R_{\alpha\lambda}R^{\lambda}{}_{\nu\beta\mu;\gamma\delta}+\frac{1}{42}R_{\lambda\nu;\alpha\beta}R^{\lambda}{}_{\gamma\delta\mu}+\\
&+\frac{1}{42}R_{\lambda\mu;\alpha\beta}R^{\lambda}{}_{\gamma\delta\nu}+\frac{2}{63}R_{\lambda\mu;\alpha}R^{\lambda}{}_{\beta\gamma\nu;\delta}+\frac{2}{63}R_{\lambda\nu;\alpha}R^{\lambda}{}_{\beta\gamma\mu;\delta}+\frac{1}{84}R_{\lambda\mu}R^{\lambda}{}_{\alpha\beta\nu;\gamma\delta}+\\
&+\frac{1}{84}R_{\lambda\nu}R^{\lambda}{}_{\alpha\beta\mu;\gamma\delta}+\frac{1}{63}R_{\alpha\mu;\beta\lambda}R^{\lambda}{}_{\gamma\delta\nu}+\frac{2}{63}R_{\alpha\lambda;\beta}R^{\lambda}{}_{\mu\gamma\nu;\delta}+\frac{2}{63}R_{\alpha\lambda;\beta}R^{\lambda}{}_{\nu\gamma\mu;\delta}+\\
&+\frac{1}{63}R_{\alpha\nu;\beta\lambda}R^{\lambda}{}_{\gamma\delta\mu}+\frac{1}{126}R_{\alpha\mu;\lambda}R^{\lambda}{}_{\beta\gamma\nu;\delta}+\frac{1}{126}R_{\alpha\nu;\lambda}R^{\lambda}{}_{\beta\gamma\mu;\delta}-\frac{1}{42}R_{\alpha\lambda}R^{\lambda}{}_{\mu\beta\kappa}R^{\kappa}{}_{\gamma\delta\nu}+\\
&+\frac{1}{84}R_{\lambda\mu}R^{\lambda}{}_{\alpha\beta\kappa}R^{\kappa}{}_{\gamma\delta\nu}+\frac{1}{84}R_{\lambda\nu}R^{\lambda}{}_{\alpha\beta\kappa}R^{\kappa}{}_{\gamma\delta\mu}-\frac{1}{21}R_{\alpha\lambda;\nu\beta}R^{\lambda}{}_{\gamma\delta\mu}-\frac{1}{21}R_{\alpha\lambda;\mu\beta}R^{\lambda}{}_{\gamma\delta\nu}+\\
&-\frac{4}{63}R_{\alpha\lambda;\beta}R^{\lambda}{}_{\gamma\delta\nu;\mu}-\frac{1}{63}R_{\alpha\lambda\nu}R^{\lambda}{}_{\beta\gamma\mu;\delta}-\frac{1}{63}R_{\alpha\lambda;\mu}R^{\lambda}{}_{\beta\gamma\nu;\delta}-\frac{1}{42}R_{\alpha\lambda}R^{\lambda}{}_{\beta\gamma\nu;\mu\delta}+\\
&-\frac{5}{252}R_{\alpha\beta;\lambda}R^{\lambda}{}_{\gamma\delta\nu;\mu}-\frac{1}{63}R_{\alpha\beta;\nu\lambda}R^{\lambda}{}_{\gamma\delta\mu}-\frac{1}{63}R_{\alpha\beta;\mu\lambda}R^{\lambda}{}_{\gamma\delta\nu}+\frac{1}{126}R_{\alpha\beta;\gamma\lambda}R^{\lambda}{}_{\mu\delta\nu}+\\
&-\frac{1}{42}R_{\alpha\beta;\gamma\lambda}R^{\lambda}{}_{\nu\delta\mu}+\frac{1}{126}R_{\alpha\beta;\gamma\lambda}R^{\lambda}{}_{\delta\mu\nu}+\frac{1}{42}R_{\alpha\lambda}R^{\lambda}{}_{\beta\gamma\kappa}R^{\kappa}{}_{\delta\mu\nu}+\frac{1}{252}R_{\alpha\beta;\lambda}R^{\lambda}{}_{\mu\gamma\nu;\delta}+\\
&-\frac{1}{126}R_{\alpha\beta;\lambda}R^{\lambda}{}_{\nu\gamma\mu;\delta}+\frac{1}{252}R_{\alpha\beta;\lambda}R^{\lambda}{}_{\gamma\mu\nu;\delta}+\frac{1}{84}R_{\alpha\lambda}R^{\lambda}{}_{\nu\beta}{}^{\kappa}R_{\kappa\gamma\delta\mu}+\\
&+\frac{1}{18}R_{\alpha\beta;\gamma}R_{\delta\nu;\mu}+\frac{1}{18}R_{\alpha\beta;\gamma}R_{\delta\mu;\nu}+\frac{1}{36}R_{\alpha\beta;\nu}R_{\gamma\delta;\mu}+\frac{1}{18}R_{\alpha\nu;\beta}R_{\gamma\delta;\mu}+\\
&+\frac{1}{18}R_{\alpha\beta;\gamma}R_{\mu\nu;\delta}+\frac{1}{18}R_{\alpha\beta;\nu}R_{\gamma\mu;\delta}+\frac{1}{9}R_{\alpha\nu;\beta}R_{\gamma\mu;\delta}+\frac{1}{30}R_{\alpha\beta}R_{\gamma\delta;\nu\mu}+\\
&+\frac{1}{15}R_{\alpha\beta}R_{\gamma\mu;\nu\delta}+\frac{1}{15}R_{\alpha\mu}R_{\beta\gamma;\nu\delta}+\frac{1}{15}R_{\alpha\beta}R_{\gamma\nu;\mu\delta}+\frac{1}{30}R_{\alpha\beta}R_{\mu\nu;\gamma\delta}+\frac{1}{15}R_{\alpha\mu}R_{\beta\nu;\gamma\delta}+\\
&+\frac{1}{15}R_{\alpha\nu}R_{\beta\gamma;\mu\delta}+\frac{1}{15}R_{\alpha\nu}R_{\beta\mu;\gamma\delta}+\frac{1}{30}R_{\mu\nu}R_{\alpha\beta;\gamma\delta}-\frac{4}{315}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\gamma\nu\kappa;\mu\delta}+\\
&-\frac{2}{315}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\mu\nu\kappa;\gamma\delta}-\frac{2}{315}R^{\lambda}{}_{\alpha\mu}{}^{\kappa}R_{\lambda\beta\nu\kappa;\gamma\delta}-\frac{2}{315}R^{\lambda}{}_{\mu\alpha}{}^{\kappa}R_{\lambda\beta\nu\kappa;\gamma\delta}+\\
&-\frac{2}{315}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\gamma\delta\kappa;\nu\mu}-\frac{4}{315}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\gamma\mu\kappa;\nu\delta}-\frac{4}{315}R^{\lambda}{}_{\alpha\mu}{}^{\kappa}R_{\lambda\beta\gamma\kappa;\nu\delta}+\\
&-\frac{4}{315}R^{\lambda}{}_{\alpha\nu}{}^{\kappa}R_{\lambda\beta\gamma\kappa;\mu\delta}-\frac{2}{315}R^{\lambda}{}_{\alpha\nu}{}^{\kappa}R_{\lambda\beta\mu\kappa;\gamma\delta}-\frac{2}{315}R^{\lambda}{}_{\nu\alpha}{}^{\kappa}R_{\lambda\beta\mu\kappa;\gamma\delta}+\\
&-\frac{2}{315}R^{\lambda}{}_{\mu\nu}{}^{\kappa}R_{\lambda\alpha\beta\kappa;\gamma\delta}-\frac{1}{84}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}{}_{;\gamma}R_{\lambda\delta\mu\kappa;\nu}-\frac{1}{84}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}{}_{;\gamma}R_{\lambda\mu\nu\kappa;\delta}+\\
&-\frac{1}{84}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}{}_{;\nu}R_{\lambda\gamma\mu\kappa;\delta}-\frac{1}{84}R^{\lambda}{}_{\alpha\nu}{}^{\kappa}{}_{;\beta}R_{\lambda\gamma\mu\kappa;\delta}-\frac{1}{84}R^{\lambda}{}_{\nu\alpha}{}^{\kappa}{}_{;\beta}R_{\lambda\gamma\mu\kappa;\delta}+\\
&-\frac{1}{84}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}{}_{;\gamma}R_{\lambda\delta\nu\kappa;\mu}-\frac{1}{168}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}{}_{;\nu}R_{\lambda\gamma\delta\kappa;\mu}-\frac{1}{84}R^{\lambda}{}_{\alpha\nu}{}^{\kappa}{}_{;\beta}R_{\lambda\gamma\delta\kappa;\mu}
\end{aligned}
\tag{C.15}
$$

$$
\begin{aligned}
(\partial_\mu\partial_\nu g)_4 =& -\frac{1}{27}R_{\alpha\beta}R_{\gamma\delta}R_{\mu\nu} - \frac{4}{27}R_{\alpha\beta}R_{\gamma\mu}R_{\delta\nu} - \frac{2}{15}R_{\alpha\beta}R^{\lambda}{}_{\gamma\delta}{}^{\kappa}R_{\lambda\mu\nu\kappa} - \frac{2}{15}R_{\alpha\beta}R^{\lambda}{}_{\gamma\mu}{}^{\kappa}R_{\lambda\delta\nu\kappa}+\\
&-\frac{2}{15}R_{\alpha\beta}R^{\lambda}{}_{\mu\gamma}{}^{\kappa}R_{\lambda\delta\nu\kappa} - \frac{4}{15}R_{\alpha\mu}R^{\lambda}{}_{\beta\gamma}{}^{\kappa}R_{\lambda\delta\nu\kappa} - \frac{4}{15}R_{\alpha\nu}R^{\lambda}{}_{\beta\gamma}{}^{\kappa}R_{\lambda\delta\mu\kappa}+\\
&-\frac{1}{15}R_{\mu\nu}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\gamma\delta\kappa} - \frac{1}{15}R_{\alpha\nu}R_{\beta\lambda}R^{\lambda}{}_{\gamma\delta\mu} - \frac{1}{30}R_{\alpha\beta}R_{\lambda\nu}R^{\lambda}_{\gamma\delta\mu}+\\
&-\frac{1}{30}R_{\alpha\beta}R_{\gamma\lambda}R^{\lambda}{}_{\mu\delta\nu} - \frac{1}{30}R_{\alpha\beta}R_{\gamma\lambda}R^{\lambda}{}_{\nu\delta\mu} - \frac{1}{30}R_{\alpha\beta}R_{\lambda\mu}R^{\lambda}{}_{\gamma\delta\nu}+\\
&-\frac{1}{15}R_{\alpha\mu}R_{\beta\lambda}R^{\lambda}{}_{\gamma\delta\nu} - \frac{1}{30}R_{\alpha\beta}R_{\gamma\lambda}R^{\lambda}{}_{\delta\mu\nu} + \frac{2}{315}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\gamma\nu\rho}R^{\rho}{}_{\kappa\delta\mu}+\\
&+\frac{4}{315}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\gamma\rho\kappa}R^{\rho}{}_{\mu\delta\nu} + \frac{2}{315}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\rho\nu\kappa}R^{\rho}{}_{\gamma\delta\mu} + \frac{2}{315}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\nu\gamma\rho}R^{\rho}{}_{\kappa\delta\mu}+\\
&+\frac{2}{315}R^{\lambda}{}_{\alpha\nu}{}^{\kappa}R_{\lambda\beta\gamma\rho}R^{\rho}{}_{\kappa\delta\mu} + \frac{2}{315}R^{\lambda}{}_{\alpha\nu}{}^{\kappa}R_{\lambda\beta\rho\kappa}R^{\rho}{}_{\gamma\delta\mu} + \frac{2}{315}R^{\lambda}{}_{\alpha\nu}{}^{\kappa}R_{\lambda\rho\beta\kappa}R^{\rho}{}_{\gamma\delta\mu}+\\
&+\frac{2}{315}R^{\lambda}{}_{\nu\alpha}{}^{\kappa}R_{\lambda\beta\gamma\rho}R^{\rho}{}_{\kappa\delta\mu} + \frac{2}{315}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\mu\rho\kappa}R^{\rho}{}_{\gamma\delta\nu} + \frac{2}{315}R^{\lambda}{}_{\alpha\mu}{}^{\kappa}R_{\lambda\beta\rho\kappa}R^{\rho}{}_{\gamma\delta\nu}+\\
&+\frac{2}{315}R^{\lambda}{}_{\mu\alpha}{}^{\kappa}R_{\lambda\beta\rho\kappa}R^{\rho}{}_{\gamma\delta\nu} + \frac{2}{315}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\gamma\delta\rho}R^{\rho}{}_{\nu\mu\kappa} - \frac{2}{945}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\gamma\delta\rho}R^{\rho}{}_{\mu\nu\kappa}+\\
&+\frac{2}{315}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\gamma\mu\rho}R^{\rho}{}_{\nu\delta\kappa} - \frac{2}{945}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\gamma\mu\rho}R^{\rho}{}_{\delta\nu\kappa} + \frac{2}{315}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\mu\gamma\rho}R^{\rho}{}_{\nu\delta\kappa}+\\
&-\frac{2}{945}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\mu\gamma\rho}R^{\rho}{}_{\delta\nu\kappa} + \frac{2}{315}R^{\lambda}{}_{\alpha\mu}{}^{\kappa}R_{\lambda\beta\gamma\rho}R^{\rho}{}_{\nu\delta\kappa} - \frac{2}{945}R^{\lambda}{}_{\alpha\mu}{}^{\kappa}R_{\lambda\beta\gamma\rho}R^{\rho}{}_{\delta\nu\kappa}+\\
&+\frac{2}{315}R^{\lambda}{}_{\mu\alpha}{}^{\kappa}R_{\lambda\beta\gamma\rho}R^{\rho}{}_{\nu\delta\kappa} - \frac{2}{945}R^{\lambda}{}_{\mu\alpha}{}^{\kappa}R_{\lambda\beta\gamma\rho}R^{\rho}{}_{\delta\nu\kappa} - \frac{1}{28}R_{\alpha\lambda}R^{\lambda}{}_{\beta\nu}{}^{\kappa}R_{\kappa\gamma\delta\mu}+\\
&+\frac{1}{28}R_{\alpha\lambda}R^{\lambda\kappa}{}_{\beta\nu}R_{\kappa\gamma\delta\mu} + \frac{1}{14}R_{\lambda\kappa}R^{\lambda}{}_{\alpha\beta\mu}R^{\kappa}{}_{\gamma\delta\nu}
\end{aligned} \tag{C.16}
$$

Plugging the value of the derivatives listed above into equation (3.57) and using the expansion given by equations (3.31) and (3.35) we obtain the expansion of the effective potential at any order given by, up to order four in the RNC expansion (which is of order six in the adiabatic expansion)

$$
\begin{aligned}
V_{eff}(0) &= \frac{1}{4}(g^{-1})_0(g^{\mu\nu})_0(\partial_\mu\partial_\nu g)_0 + \xi R\\
&= \left(\xi - \frac{1}{6}\right)R
\end{aligned} \tag{C.17}
$$

$$
\begin{aligned}
V_{eff,\alpha}(0) &= \frac{1}{4}(g^{-1})_0(g^{\mu\nu})_0(\partial_\mu\partial_\nu g)_1 + \xi R_{;\alpha}\\
&= \left(\xi - \frac{1}{6}\right)R_{;\alpha}
\end{aligned} \tag{C.18}
$$

$$
\begin{aligned}
\frac{1}{2}V_{eff,\alpha\beta}(0) = &-\frac{3}{16}(g^{-2})_0(g^{\mu\nu})_0(\partial_\mu g)_1(\partial_\nu g)_1 + \frac{1}{4}(g^{-1})_0(\partial_\mu g^{\mu\nu})_1(\partial_\nu g)_1 + \\
&+ \frac{1}{4}(g^{-1})_0(g^{\mu\nu})_0(\partial_\mu\partial_\nu g)_2 + \frac{1}{4}(g^{-1})_0(g^{\mu\nu})_2(\partial_\mu\partial_\nu g)_0 + \\
&+ \frac{1}{4}(g^{-1})_2(g^{\mu\nu})_0(\partial_\mu\partial_\nu g)_0 + \frac{1}{2}\xi R_{;\alpha\beta} \\
= &\frac{1}{2}\Big(\xi - \frac{1}{6}\Big)R_{;\alpha\beta} + a_{\alpha\beta}
\end{aligned} \tag{C.19}
$$

$$
\begin{aligned}
\frac{1}{3!}V_{eff,\alpha\beta\gamma}(0) = &-2\cdot\frac{3}{16}(g^{-2})_0(g^{\mu\nu})_0(\partial_\mu g)_1(\partial_\nu g)_2 + \frac{1}{4}(g^{-1})_0(\partial_\mu g^{\mu\nu})_1(\partial_\nu g)_2 + \\
&+ \frac{1}{4}(g^{-1})_0(\partial_\mu g^{\mu\nu})_2(\partial_\nu g)_1 + \frac{1}{4}(g^{-1})_0(g^{\mu\nu})_0(\partial_\mu\partial_\nu g)_3 + \\
&+ \frac{1}{4}(g^{-1})_0(g^{\mu\nu})_2(\partial_\mu\partial_\nu g)_1 + \frac{1}{4}(g^{-1})_2(g^{\mu\nu})_0(\partial_\mu\partial_\nu g)_1 + \frac{1}{3!}\xi R_{;\alpha\beta\gamma} \\
= &\frac{1}{3!}\Big(\xi - \frac{1}{6}\Big)R_{;\alpha\beta\gamma} + a_{\alpha\beta\gamma}
\end{aligned} \tag{C.20}
$$

$$
\begin{aligned}
\frac{1}{4!}V_{eff,\alpha\beta\gamma\delta}(0) = &-2\cdot\frac{3}{16}(g^{-2})_0(g^{\mu\nu})_0(\partial_\mu g)_1(\partial_\nu g)_3 - \frac{3}{16}(g^{-2})_0(g^{\mu\nu})_0(\partial_\mu g)_2(\partial_\nu g)_2 + \\
&- \frac{3}{16}(g^{-2})_0(g^{\mu\nu})_2(\partial_\mu g)_1(\partial_\nu g)_1 - \frac{3}{16}(g^{-2})_2(g^{\mu\nu})_0(\partial_\mu g)_1(\partial_\nu g)_1 + \\
&+ \frac{1}{4}(g^{-1})_0(\partial_\mu g^{\mu\nu})_1(\partial_\nu g)_3 + \frac{1}{4}(g^{-1})_0(\partial_\mu g^{\mu\nu})_2(\partial_\nu g)_2 + \\
&+ \frac{1}{4}(g^{-1})_0(\partial_\mu g^{\mu\nu})_3(\partial_\nu g)_1 + \frac{1}{4}(g^{-1})_2(\partial_\mu g^{\mu\nu})_1(\partial_\nu g)_1 + \\
&+ \frac{1}{4}(g^{-1})_0(g^{\mu\nu})_0(\partial_\mu\partial_\nu g)_4 + \frac{1}{4}(g^{-1})_0(g^{\mu\nu})_2(\partial_\mu\partial_\nu g)_2 + \\
&+ \frac{1}{4}(g^{-1})_0(g^{\mu\nu})_3(\partial_\mu\partial_\nu g)_1 + \frac{1}{4}(g^{-1})_0(g^{\mu\nu})_4(\partial_\mu\partial_\nu g)_0 + \\
&+ \frac{1}{4}(g^{-1})_2(g^{\mu\nu})_0(\partial_\mu\partial_\nu g)_2 + \frac{1}{4}(g^{-1})_2(g^{\mu\nu})_2(\partial_\mu\partial_\nu g)_0 + \\
&+ \frac{1}{4}(g^{-1})_3(g^{\mu\nu})_0(\partial_\mu\partial_\nu g)_1 + \frac{1}{4}(g^{-1})_4(g^{\mu\nu})_0(\partial_\mu\partial_\nu g)_0 + \frac{1}{4!}\xi R_{;\alpha\beta\gamma\delta} \\
= &\frac{1}{4!}\Big(\xi - \frac{1}{6}\Big)R_{;\alpha\beta\gamma\delta} + a_{\alpha\beta\gamma\delta}
\end{aligned} \tag{C.21}
$$

where

$$a_{\alpha\beta} = -\frac{1}{40}\Box R_{\alpha\beta} + \frac{1}{120}R_{;\alpha\beta} + \frac{1}{30}R_{\alpha}{}^{\lambda}R_{\beta\lambda} + \frac{1}{60}R_{\lambda\kappa}R^{\lambda}{}_{\alpha\beta}{}^{\kappa} - \frac{1}{60}R^{\lambda\mu\kappa}{}_{\alpha}R_{\lambda\mu\kappa\beta} \tag{C.22}$$

$$\begin{aligned}
a_{\alpha\beta\gamma} = & -\frac{1}{60}\Box R_{\alpha\beta;\gamma} + \frac{1}{180}R_{;\alpha\beta\gamma} + \frac{1}{36}RR_{\alpha\beta;\gamma} + \frac{1}{6}R_{\alpha}{}^{\mu}R_{\beta\mu;\gamma} + \frac{1}{60}R_{\alpha}{}^{\mu}R_{\beta\gamma;\mu} + \\
& - \frac{11}{180}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\gamma;\kappa} + \frac{13}{360}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\kappa;\gamma} + \frac{1}{36}R_{\lambda\kappa}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}{}_{;\gamma} - \frac{1}{60}R^{\lambda}{}_{\alpha}{}^{\mu\kappa}R_{\lambda\beta\mu\kappa;\gamma} + \\
& - \frac{1}{90}R^{\lambda}{}_{\alpha}{}^{\mu\kappa}R_{\lambda\beta\gamma\kappa;\mu}
\end{aligned} \tag{C.23}$$

$$\begin{aligned}
a_{\alpha\beta\gamma\delta} = & \frac{1}{504}R_{\alpha\beta\gamma\delta} - \frac{1}{168}\Box R_{\alpha\beta;\gamma\delta} + \frac{3}{56}R_{\alpha}{}^{\mu}R_{\beta\mu;\gamma\delta} + \frac{1}{504}R_{\alpha}{}^{\mu}R_{\beta\gamma;\delta\mu} + \frac{5}{144}R_{\alpha}{}^{\mu}{}_{;\beta}R_{\gamma\mu;\delta} + \\
& + \frac{17}{1008}R_{\alpha}{}^{\mu}{}_{;\beta}R_{\gamma\delta;\mu} - \frac{1}{4032}R_{\alpha\beta;}{}^{\mu}R_{\gamma\delta;\mu} + \frac{71}{2520}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\kappa;\gamma\delta} + \frac{1}{90}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\gamma;\delta\kappa} + \\
& + \frac{1}{2520}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\gamma\delta;\lambda\kappa} + \frac{1}{63}R_{\lambda\kappa}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}{}_{;\gamma\delta} + \frac{13}{336}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}{}_{;\gamma}R_{\lambda\kappa;\delta} + \frac{1}{56}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}{}_{;\gamma}R_{\lambda\delta;\kappa} + \\
& - \frac{1}{210}R^{\lambda}{}_{\alpha}{}^{\mu\kappa}R_{\lambda\beta\mu\kappa;\gamma\delta} - \frac{2}{315}R^{\lambda}{}_{\alpha}{}^{\mu\kappa}R_{\lambda\beta\gamma\kappa;\delta\mu} - \frac{1}{168}R^{\lambda}{}_{\alpha}{}^{\mu\kappa}{}_{;\beta}R_{\lambda\beta\gamma\kappa;\mu} + \\
& - \frac{1}{112}R^{\lambda}{}_{\alpha}{}^{\mu\kappa}{}_{;\beta}R_{\lambda\beta\mu\kappa;\gamma} - \frac{1}{672}R^{\lambda}{}_{\alpha\beta}{}^{\kappa}{}_{;}{}^{\mu}R_{\lambda\gamma\delta\kappa;\mu} + \frac{1}{30}R_{\alpha\beta}R_{\lambda\kappa}R^{\lambda}{}_{\gamma\delta}{}^{\kappa} + \\
& - \frac{7}{120}R_{\alpha\beta}R^{\lambda}{}_{\gamma}{}^{\mu\kappa}R_{\lambda\delta\mu\kappa} - \frac{1}{54}RR^{\lambda}{}_{\alpha\beta}{}^{\kappa}R_{\lambda\gamma\delta\kappa} - \frac{253}{7560}R_{\alpha\lambda}R^{\lambda}{}_{\beta\gamma}{}^{\kappa}R_{\kappa\delta} + \\
& + \frac{137}{7560}R_{\rho}{}^{\lambda}R_{\lambda\alpha\beta}^{\kappa}R_{\kappa\gamma\delta}{}^{\rho} - \frac{899}{7560}R_{\alpha\mu}R^{\lambda}{}_{\beta}{}^{\mu\kappa}R_{\lambda\gamma\delta\kappa} + \frac{1}{126}R^{\rho}{}_{\alpha\beta}{}^{\sigma}R_{\lambda\rho\sigma\kappa}R^{\lambda}{}_{\gamma\delta}{}^{\kappa} + \\
& - \frac{8}{945}R^{\rho}{}_{\alpha\beta}{}^{\sigma}R_{\lambda\rho\gamma\kappa}R^{\lambda}{}_{\delta\sigma}{}^{\kappa} + \frac{1}{135}R^{\rho}{}_{\alpha\beta}{}^{\sigma}R_{\lambda\gamma\rho\kappa}R^{\lambda}{}_{\delta\sigma}{}^{\kappa}
\end{aligned} \tag{C.24}$$

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