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Open dynamics of $\mathfrak{su}(3)$ quantum systems.

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Abstract

Questo lavoro di tesi nasce con lo scopo di fornire una inquadratura generale del campo di studi relativo ad i sistemi quantistici aperti.
Esattamente come avviene nel caso della termodinamica, questo tipo di approfondimento vuole tenere conto delle interazioni che un qualunque sistema quantistico può sviluppare con l'ambiente esterno.
La prima parte del lavoro intende introdurre il lettore all'argomento; in queste prime sezioni si trattano anche alcuni argomenti più concettuali di rilevanza prettamente fisica, come ad esempio il fenomeno dell'*entenglement* o del *quatum eraser*.
La seconda parte presenta un approccio geometrico, allo scopo di chiarire come vengono a modificarsi in questo nuovo contesto le strutture geometriche entro cui si sviluppa il sistema quantistico interagente, intendendo con ciò sia le orbite unitarie, sia gli spazi formati dagli stati puri e dagli stati misti.
Infine, la parte finale della tesi sviluppa questi argomenti in due circostanze applicative, relative all'insieme delle matrici densità rispettivamente di dimensione due e tre. Nello specifico, queste due trattazioni analizzano specialmente le problematiche relative all'evoluzione temporale *aperta* , ossia quel tipo di evoluzione osservabile esclusivamente in caso di interazione del sistema quantistico con un ambiente esterno, e che per questo si discosta dalle usuali evoluzioni unitarie descrivendo invece una traiettoria che permette il passaggio dall'una all'altra di queste orbite.

Contents

Introduction

This work belongs to the recent attempts to describe open quantum systems. An open quantum system is a quantum system that is free to interact with the environment or with other systems. Because of the disturbing nature of the measuring process and because of the arising difficulties on completely isolating physical systems from the external world, this argument has lot of applications to concrete contexts. Indeed, the study of open systems is useful in fields such as quantum optics, quantum measurement theory, quantum statistical mechanics, quantum cosmology and semi-classical approximations. Moreover, the study of composite quantum systems is at the heart of quantum computation and quantum information, where concepts like entanglement can have applications in quantum teleportation or superdense coding. In fact, by the development of quantum information science, there has been a strong revival in the study of open systems aimed at further understanding the impact of decoherence phenomena on quantum information protocols. Considering the relevance of the subject, the goal of the first part of this thesis work is to summarize and clarify the general approach to the argument, from both the algebraic and the geometric points of view.

Density matrices are the most important objects in the theory of open systems, so that chapter 1, following the line of [10], will introduce these matrices and show how to manipulate them. The consistency of resolving quantum problems with this formalism will be ensured by Gleason's theorem, that will justify us to calculate quantum probabilities with the help of density operators. The Schmidt decomposition, together with its main corollary, the GHJW theorem, will provide us the necessary tools for expanding in a useful way a composite quantum state, initially expressed in the form of a direct product of density matrices of the distinct subsystems. These two theorems will allow a formal interpretation of the presence of the entanglement phenomenon between subsystems, intended as a condition over the Schmidt number of the whole state of the system. Moreover, the GHJW theorem has some important implications on the interpretation of quantum eraser phenomenon, that we will discuss.

In chapter 2 we will see how the interactions between subsystems modify the measuring process, allowing the use of Positive Operators Valued Measures, or POVMs, that are measures ruled by positive operators that have not to be orthogonal, in

contrast with the usual measures governed by orthogonal projectors, described as Positive Valued Measures, or PVMs, in standard quantum mechanics. Starting from a generic POVM , Neumark's theorem will ensure us the possibility of always reconstructing an extended system whose the POVM is the result of orthogonal measures.
In chapter 3 we will speak about open evolutions. The theoretical discussion about foundations of open time evolutions will follow the criteria used by [11], in which the open evolution is governed by linear maps, called Universal Dynamical Maps, or UDMs, that connect density matrices to density matrices. The equations and the theorems of this chapter will have the purpose of describing in detail the properties of these maps, showing also the constraints that we will need to impose to make the evolution physically consistent. Amongst these constrains, besides conditions on the possible form of the starting state, we will present the Markovian approximation, that has the objective of keeping under control the disturb coming from the environment during the evolution.
Then, in chapter 4, we will make some considerations about the geometrical nature of the spaces involved in the study of open systems. In fact, identify quantum states with density matrices instead of vectors transfers the geometrical settings of spaces of physical states from the usual Hilbert space to more complicate types of manifolds. In particular, we will see that the space of pure density matrices, as well as mixed one and their unitary orbits, become Käler manifolds, that are manifolds that admit a metric and a symplectic structure. For this first part of the chapter we will limit us to enunciate the statements of the theorems, reminding the reader to [8] for demonstrations. The end of the chapter will be reserved for an example of application to 2x2 density matrices of these geometrical concepts, taken from [8].

Finally, the last chapter of this work will present the study of 3x3 density matrices, with a special attention to the dynamical issues. Indeed, after a contextualization of the argument, we will apply a new parametrization in time for the Kraus operators in the solution of Lindblad equation, showing how the open trajectories defined by this parametrization can connect different unitary orbits in the space of 3x3 density matrices. Both analytical and numerical calculations and checking will be performed to find and verify the consistency of this parametrization. Our hope is to contribute with the intuitions applied in this particular case to the generalization of the treatment for density matrices of any dimension, that will be a natural continuation of this type of research.

Chapter 1

Open quantum systems

To avoid burdening the reading , we underling now that, for the whole chapter, we will use [10] as main reference text.

1.1 Canonical Foundations of Quantum Mechanics

Before we consider in depth the properties of open quantum systems, it can be useful to recall the main principles of quantum theory. We can summarize this by a schematic list of 3 points:

1. *States are identified by rays in a Hilbert Space H.*

   A ray is defined as an equivalence class of the vectors of the space H that differ by a multiplication of a nonzero complex scalar. Usually, to represent a ray, one selects a representative of the class that has a unit norm, like:

   $$\langle\psi|\psi\rangle = 1$$

2. *Observables are identified as self-adjoint operators on H and measures are orthogonal projections, or PVM.*

   An operator is a linear map that takes vectors into vectors on H. Because of the properties of Hilbert spaces, each self-adjoint operator has always a spectral decomposition, that means that its eigenvectors form a complete orthonormal basis for H. So, we can express a generic observable $\boldsymbol{A}$ in the following way:

   $$\boldsymbol{A} = \sum_{\mu} a_\mu \boldsymbol{P_\mu}$$

   where the $\{a_\mu\}_\mu$ are the eigenvalues of $\boldsymbol{A}$ and the $\{\boldsymbol{P_\mu}\}_\mu$ are orthogonal projectors with the canonical properties:

   $$\boldsymbol{P_\mu P_\nu} = \delta_{\mu\nu}\boldsymbol{P_\nu} \quad \text{and} \quad \sum_{\mu} \boldsymbol{P}_\mu = \boldsymbol{1}$$

Performing a measure of an observable on the quantum system means projecting the initial state into a specific orthogonal direction given by these projectors $\boldsymbol{P_\mu}$ and the possible outcomes of the measure are the different eigenvalues $\{a_\mu\}_\mu$. In fact the probability of getting the numerical outcome a_μ comes from the expression:

$$prob(a_\mu) = \langle\psi| \boldsymbol{P_\mu} |\psi\rangle$$

and the normalized state reached by the system after the measuring process is :

$$\frac{\boldsymbol{P_\mu} |\psi\rangle}{\langle\psi| \boldsymbol{P_\mu} |\psi\rangle^{\frac{1}{2}}}$$

This measure obtained by the process illustrated above if often called *Projected Valued Measure*, or briefly PVM.

3. *Evolution is unitary.*

The evolution of a quantum system is completely described by the observable $\boldsymbol{H}$ called Hamiltonian. In the Schrödinger picture, with the time dependence carried by the states, for the evolution we have the famous equation:

$$\frac{d}{dt} |\psi(t)\rangle = -i\boldsymbol{H} |\psi(t)\rangle$$

that leads to the definition of a unitary operator of evolution:

$$\boldsymbol{U(t)} = e^{-it\boldsymbol{H}}$$

satisfying:

$$|\psi(t + dt)\rangle = \boldsymbol{U(t)} |\psi(t)\rangle$$

1.2 Density matrix

The most common use of quantum mechanics is to consider features of a state in a single quantum system, typically assumed to be isolated. According to the principles of the theory, as we said, the state of the system is represented by a ray of a Hilbert space and the results of measuring correspond to eigenvalues of certain self-adjoint operators on that Hilbert space. With these assumptions, the evolution of the state, regulated by the famous Schrödinger equation, is a unitary evolution . However, if the system is not isolated anymore, but considered just part of a larger system, then (contrary to the axioms):

1. *States are not rays.*

2. *Measurements are not orthogonal projections.*

3. *Evolution is not unitary.*

Indeed, if we are looking at a subsystem of a larger quantum system, even if the state of the larger system is still a ray, the state of the subsystem needs not to be; in general, the state is represented by a density operator, called *density matrix*. In the case where the state of the subsystem is a ray, we say that the state is pure. Otherwise the state is mixed. Assuming that these basilar notions are still known by the reader, we skip the formal demonstrations and we just summarize the five properties that have to be satisfied by a pure density matrix:

1. $\boldsymbol{\rho}$ is bounded

2. $\boldsymbol{\rho}$ is self-adjoint : $\boldsymbol{\rho} = \boldsymbol{\rho}^{\dagger}$

3. $\boldsymbol{\rho}$ is positive

4. $tr(\boldsymbol{\rho}) = 1$

5. $\boldsymbol{\rho}^2 = \boldsymbol{\rho}$

Moreover, is important to recall also that, if the density matrix $\boldsymbol{\rho}$ represents a pure state, then it can be expressed as:

$$\boldsymbol{\rho} = |\psi\rangle \langle\psi| \tag{1.1}$$

where $|\psi\rangle \in H$, so that any pure density matrix can be always interpreted as a projector on the subspace of H generated by $|\psi\rangle$.
The property number 5 is the way to distinguish a pure state from a mixed one. In fact we have:

Theorem 1.2.1.
ρ is associated with a pure state iff $\boldsymbol{\rho}^2 = \boldsymbol{\rho}$

Moreover

Theorem 1.2.2.
ρ is associated with a pure state iff $tr(\boldsymbol{\rho}^2) = 1$, otherwise $tr(\boldsymbol{\rho}^2) \leq 1$

When we consider a mixed state, we may interpret $\boldsymbol{\rho}_{mixed}$ as describing an *ensemble* of pure quantum states. In future, we will refer to a general state of the larger quantum system (the system *and* the environment) with the ket $|\psi_{AB}\rangle$ that is:

$$|\psi_{AB}\rangle = \sum_{i,\mu} a_{i\mu} |i_A\rangle \otimes |\mu_B\rangle \tag{1.2}$$

where $\{|i_A\rangle\}$, $\{|\mu_B\rangle\}$ are orthonormal bases for H_A and H_B, the Hilbert spaces of systems A and B respectively. Of course we will have also: $\sum_{i,\mu} |a_{i\mu}|^2 = 1$

This means that the Hilbert space of the whole open system is $H_A \bigotimes H_B$.

It is now easy to verify what is the action of an operator related just to the system A on the state $|\psi_{AB}\rangle$; considering an operator $\boldsymbol{M} = \boldsymbol{M_A} \otimes \boldsymbol{1_B}$ we will get the expectation value:

$$\begin{aligned} \langle \boldsymbol{M} \rangle &= \langle \psi_{AB} | \, \boldsymbol{M_A} \otimes \boldsymbol{1_B} \, | \psi_{AB} \rangle \\ &= \sum_{j,\nu} a^*_{j\nu} (\langle j_A| \otimes \langle \nu_B|) \boldsymbol{M_A} \otimes \boldsymbol{1_B} \sum_{i,\mu} a_{i\mu} (|i_A\rangle \otimes |\mu_B\rangle) \\ &= \sum_{j,i,\mu} a^*_{j\mu} a_{i\mu} (\langle j_A| \, \boldsymbol{M_A} \, |i_A\rangle) \\ &= tr_A(\boldsymbol{M_A \rho_A}) \end{aligned} \tag{1.3}$$

where:

$$\boldsymbol{\rho}_A \equiv tr_B(|\psi_{AB}\rangle \langle \psi_{AB}|) \tag{1.4}$$

So we say that the density operator $\boldsymbol{\rho}_A$ for subsystem A is obtained by performing a partial trace over subsystem B of the density matrix (in this case a pure state) for the combined system AB. This way of describe the subsystem A is the most complete one we could reach; indeed we can say that ρ_A provides a complete physical description of the state of subsystem A, because, differently from a single vector of the Hilbert space H_A, it characterizes all possible states of A.

1.3 Gleason's theorem

Operating with these mathematical instruments may induce the question if the density matrix formalism is really necessary to describe quantum world, even when we are studying an isolated system. An interesting way to answer this question is given by the Gleason's theorem. This important theorem was proved by Gleason during the course of an investigation on the possible existence of new axioms for

quantum theory that would give statistical predictions different from the standard rule (1.3) illustrated in the previous section.

Gleason's theorem starts from the premise that it is the task of quantum theory to assign consistent probabilities to all possible orthogonal projections in a Hilbert space (in other words, to all possible measurements of observables). A state of a quantum system, then, is a mapping that takes each projection ($\boldsymbol{P}^2 = \boldsymbol{P}$ and $\boldsymbol{P} = \boldsymbol{P}^\dagger$) to a nonnegative real number less than one:

$$\boldsymbol{P} \Rightarrow p(\boldsymbol{P}) \text{ with } 0 \leq p(\boldsymbol{P}) \leq 1 \tag{1.5}$$

This mapping must have the properties:

1. $p(\mathbf{0}) = 0$

2. $p(\mathbf{1}) = 1$

3. if $\boldsymbol{P}_1\boldsymbol{P}_2 = 0$ then $p(\boldsymbol{P}_1 + \boldsymbol{P}_2) = p(\boldsymbol{P}_1) + p(\boldsymbol{P}_2)$

The fundamental property is the last one, that has not a trivial physical content, that can be proved experimentally by virtue of the strong superposition principle of quantum mechanics. A concrete example can be founded in [9].
Under these assumptions, the statement of the theorem is the following:

Theorem 1.3.1 (Gleason's theorem)**.**
If A is a quantum system, represented by an Hilbert space H_A *with dimensions n, then, for any possible map of type* (1.5) *it always exists a unique , hermitian, positive* $\boldsymbol{\rho}$ *with* $tr_A(\boldsymbol{\rho}) = 1$ *such that:*

$$p(\boldsymbol{P}) = tr_A(\boldsymbol{\rho}\boldsymbol{P}) \qquad \forall \boldsymbol{P} \tag{1.6}$$

iff n>2

It is important to notice that the density matrix $\boldsymbol{\rho}$ depends only on the preparation of the physical system A (it does not depend on the choice of the projectors $\boldsymbol{P}$).

Thus, the density matrix formalism is really necessary, if we want to represent observables as self-adjoint operators in Hilbert space, and to consistently assign probabilities to all possible measurement outcomes. Roughly speaking, the requirement of additivity of probabilities for mutually exclusive outcomes is so strong that we are inevitably led to the expression (1.6).

The case of the two dimensional space is more complicated because there just are not enough mutually exclusive projections in two dimensions. To better understand

the point behoves us to return to the original question hidden inside this theorem: find all the real nonnegative functions $f(u)$ such that, for any complete orthonormal basis $\{e_m\}_m$ of H_A , one has:

$$\sum_m f(e_m) = constant \tag{1.7}$$

A function of this kind was called by Gleason a *frame* function. If this summation equals to one, then the physical meaning of such a function $f(u)$ is the probability of finding a given quantum system in state u ; so, this interpretation of $f(u)$ is in agreement with one of the main postulate of quantum mechanics. Imposing the latter equals to a generic constant, like Gleason did , means giving a little different mathematical description of the quantum postulates, that directly leads to density matrix and to the statement of his theorem. Another way to express the result (1.6) of this theorem using frame functions is the following:

$$f(u) = \sum_{mn} \rho_{mn} \overline{u}_m u_n \tag{1.8}$$

where f is a frame function, u a unit vector of the Hilbert space and ρ is a non-negative matrix with unit trace.

Now we are able to prove why the two dimensional space is so special. In a two dimensional vector space unit vectors correspond to points on a unit circle, and can be denoted by an angle θ. So, the general form of a frame function in such a space is:

$$f(u) = f(\theta) + f(\theta + \frac{\pi}{2}) \tag{1.9}$$

To show the problems that appear in this space, we execute a Fourier expansion of this function:

$$f(\theta) + f(\theta + \frac{\pi}{2}) = \sum_n c_n e^{in\theta}(1 + e^{in\frac{\pi}{2}}) \tag{1.10}$$

To have a frame function, this expression must be a constant. Therefore, the only values of n allowed in the Fourier expansion are $n = 0$, and those n for which $e^{in\frac{\pi}{2}} = -1$, namely, $n = \pm 2, \pm 6, \pm 10$, etc. There is an infinity of possible forms for frames functions in a two dimensional real vector space, consequently the uniqueness does not hold.

With three or more dimensions instead, there are may more alternative ways to partition unity, but even if it can seem strange, there is less freedom, because the orthonormal bases are intertwined: a unit vector u may belong to more than one basis. Anyway, each unit vector must have a single expectation value, $f(u) = \langle uu^\dagger \rangle$,

irrespective of the choice of the basis in which it is included, that imposes severe constraints on the possible forms of $f(u)$.

For the interested reader , a complete prove of this theorem is reported in [5] .

Gleason's theorem is a powerful argument against the hypothesis that the stochastic behavior of quantum tests can be explained by the existence of a subquantum world, endowed with "hidden variables" whose values unambiguously determine the outcome of each test. If it were indeed so, then, for any specific value of the hidden variables, every elementary test (yes-no question) would have a unique, definite answer; and therefore every projector $\boldsymbol{P_u}$ would correspond to a definite value, 0 or 1. Therefore the function $f(u) = \langle \boldsymbol{P_u} \rangle$ too would everywhere be either 0 or 1 (its precise value depending on those of the hidden variables). Such a discontinuous function $f(u)$ is radically different from the smooth distribution (1.6) required by Gleason's theorem. This means that (1.6) cannot be valid, in general, for an arbitrary distribution of hidden variables; and therefore, a hidden variable theory must violate the quantum mechanics postulates that support Gleason's theorem, as long as the hidden variables have not been averaged over. This conclusion was first reached by Bell.

1.4 Schmidt decomposition

Now that we have left behind some conceptual problems well solved by Gleason's theorem, we can present a very useful tool to operate with the world of density matrices and open systems: the Schmidt decomposition. Starting from an open pure state of the standard form (1.2) we can reach, by using this decomposition, a useful way to express the density matrices of the two systems involved. So we start giving to (1.2) a new expression by a definition:

$$|\psi_{AB}\rangle = \sum_{i,\mu} a_{i\mu} |i_A\rangle \otimes |\mu_B\rangle \equiv \sum_i |i_A\rangle \otimes |\tilde{i}_B\rangle \tag{1.11}$$

Here $\{|i_A\rangle\}$ and $\{|\mu_B\rangle\}$ are orthonormal bases for H_A and H_B respectively, but to obtain the second equality we have defined:

$$|\tilde{i}_B\rangle \equiv \sum_\mu a_{i\mu} |\mu_B\rangle \tag{1.12}$$

Note that, in this new form, the set $\{|\tilde{i}_B\rangle\}$ does not need to be orthonormal.
Now let us suppose that the $\{|i_A\rangle\}$ basis is chosen to be the basis in which the density matrix $\boldsymbol{\rho_A}$ is diagonal:

$$\boldsymbol{\rho_A} = \sum_i p_i |i_A\rangle \langle i_A| \tag{1.13}$$

But, as we already know from (1.4), we can get this matrix also by performing a partial trace:

$$\begin{aligned}\boldsymbol{\rho_A} &= tr_B(|\psi_{AB}\rangle\langle\psi_{AB}|) = tr_B(\sum_i |i_A\rangle \otimes |\tilde{i}_B\rangle \sum_j \langle j_A| \otimes \langle\tilde{j}_B|) \\ &= \sum_{i,j} |i_A\rangle\langle j_A| \sum_k (\langle k|\tilde{i}_B\rangle\langle k|\tilde{j}_B\rangle) \\ &= \sum_{i,j} |i_A\rangle\langle j_A| \sum_k (\langle\tilde{i}_B|k\rangle\langle k|\tilde{j}_B\rangle) = \sum_i \sum_j |i_A\rangle\langle j_A| (\langle\tilde{i}_B|\tilde{j}_B\rangle)\end{aligned} \tag{1.14}$$

and, because the two different expressions must coincide, we must have:

$$\langle\tilde{i}_B|\tilde{j}_B\rangle = \delta_{ij} p_i \tag{1.15}$$

then we have proved that the base $\{|\tilde{i}_B\rangle\}$ is orthogonal. To get also the normalization condition we just have to rearrange the definition of the base:

$$|\tilde{i}_B\rangle \equiv \sum_\mu a_{i\mu}\sqrt{p_i}\,|\mu_B\rangle \tag{1.16}$$

this operation is always permitted because these coefficients, taken from the summation (1.13) that defines the density matrix, are all different from zero. After all we obtain a new expression for the open state:

$$|\psi_{AB}\rangle = \sum_i \sqrt{p_i}\,|i_A\rangle \otimes |\tilde{i}_B\rangle \tag{1.17}$$

given in a particular base of $H_A \bigotimes H_B$.
This form is the Schmidt decomposition of the open pure state $|\psi_{AB}\rangle$. Any open pure state can be expressed in this form, but of course the basis used depends on the pure state that is being expanded. In general, we can not simultaneously expand two different states of the whole system $H_A \bigotimes H_B$ using the same orthonormal base of H_A and H_B . Using the last equation, we can also evaluate the partial trace over H_A to obtain the density matrix of the system B:

$$\boldsymbol{\rho_B} = tr_B(|\psi_{AB}\rangle\langle\psi_{AB}|) = \sum_i p_i\,|\tilde{i}_B\rangle\langle\tilde{i}_B| \tag{1.18}$$

We see then that $\boldsymbol{\rho_A}$ and $\boldsymbol{\rho_B}$ have the same nonzero eigenvalues. Of course there is no reason for H_A and H_B to have the same dimension, so the null eigenvalues can be different in the two systems.

Anyway, in a context with no degeneration of the nonzero eigenvalues there is only an easy and unique way to construct the Schmidt decomposition of the initial state $|\psi_{AB}\rangle$: we need to diagonalize the operators $\boldsymbol{\rho_A}$ and $\boldsymbol{\rho_B}$ and to pair up the eigenstates that correspond to the same eigenvalues. But, if $\boldsymbol{\rho_A}$ has degenerate nonzero eigenvalues, then we need more information than that provided by $\boldsymbol{\rho_A}$ and $\boldsymbol{\rho_B}$ to determine the Schmidt decomposition; we need to know which $|\tilde{i}_B\rangle$ gets paired with each $|i_A\rangle$. So it still remains an ambiguity (deriving to the unitary transformations that connect the eigenstates depending on the same eigenvalues) on the possible basis used by this type of construction.

1.5 Entanglement

We consider now a quantum system A that interacts with another system B (that can be our environment). In this case A and B become *entangled*, that is, correlated. The entanglement destroys the coherence of a superposition of states of A, so that some of the phases in the superposition become inaccessible if we look at A alone. We may describe this situation by saying that the state of system A collapses (it is in one of a set of alternative states, each of which can be assigned a probability).

We mathematically define the concept of Entanglement in the following way:

Definition 1.1 (Entanglement)**.**
Given a state $|\psi_{AB}\rangle$ of the whole system A+B, one says that the subsystems A and B are *entangled* if it is not possible to express the whole state as a direct product of the states of the two subsystems:

$$|\psi_{AB}\rangle = |\phi_A\rangle \otimes |\chi_B\rangle \tag{1.19}$$

otherwise, the state is said *separable*.

The Schmidt decomposition provides an interesting mathematical criterion to characterize the delicate physical notion of *entanglement*. In fact, considering a state of the whole system (compose by A: the physical system of interest, and B: the environment) the presence of this type of connection between its subsystems can be distinguished by looking at the *Schmidt number*.

Definition 1.2 (Schmidt number)**.**
The Schmidt number is the number of nonzero eigenvalues in a Schmidt decomposition of a state.

From this definition, it follows the rigorous formalization of the criterion:

Theorem 1.5.1.
Given a state $|\psi_{AB}\rangle$ of the whole system, one can show that the subsystems A and B are entangled iff the Schmidt number of $|\psi_{AB}\rangle$ is greater than 1.

The prove of this theorem comes in a straightforward way from the above definitions.
Even if the two concepts of pureness and separability are strictly connected, it is important to not make confusion between them. Any state of the whole system AB is, by definition, a ray, and than a pure state; but the separability is just linked to the pureness of the states of his subsystems. Indeed, a separable open pure state is a direct product of pure states in H_A and H_B and can be written in the form:

$$|\psi_{AB}\rangle = |\phi_A\rangle \otimes |\chi_B\rangle \tag{1.20}$$

Thus, the reduced density matrices $\boldsymbol{\rho_A} = |\phi_A\rangle \langle\phi_A|$ and $\boldsymbol{\rho_B} = |\chi_B\rangle \langle\chi_B|$ are pure. As we said , any state that cannot be expressed as such a direct product is entangled; then $\boldsymbol{\rho_A}$ and $\boldsymbol{\rho_B}$ are mixed states.

1.6 The GHJW theorem and quantum eraser

The Schmidt decomposition is also a good starting point to achieve the GHJW theorem. In fact, the GHJW theorem is an almost trivial corollary to the Schmidt decomposition. This theorem, even if it is quite immediate in a mathematical sense, has a lot of relevant implications on the physical concept of coherence between states of a quantum system , giving a simple and illuminating explanation of the *Quantum eraser* issue. The phenomenon of quantum eraser is a critical point that could be used, as lot of scientists did, to questioning the all-pervading information content of a system enclosed inside a density matrix . This phenomenon is strictly connected with the entanglement one, and consists on the possibility of restoring coherence on a system that was initially in a incoherent superposition of states just because of entanglement with another system or environment. A clear example to consider this argument is a simple system A of a spin particle, interacting with environment. We can start with an initial state of this system that is an incoherent superposition of spin up $|\uparrow_z\rangle$ and spin down $|\downarrow_z\rangle$ along the $\hat{z}$ axis. As we know, this type of state can not be expressed in a unique vector or ray formalism, because of the stochastic nature of ensemble of pure states. What we can do to label this condition is reporting the corresponding density matrix $\boldsymbol{\rho_A} = \frac{1}{2}\mathbf{1_A}$ that, as we said in chapter 1, characterizes all the possible states of A . This last incoherent superposition is really different between a *coherent* superposition, like for example:

$$|\uparrow_x, \downarrow_x\rangle = \frac{1}{2}(|\uparrow_z\rangle \pm |\downarrow_z\rangle) \tag{1.21}$$

The main difference between them is the role played by the phase: in the case of a coherent superposition, the relative phase of the two states has observable consequences (distinguishes $|\uparrow_x\rangle$ from $|\downarrow_x\rangle$). In the case of an incoherent superposition, the relative phase is completely unobservable. The superposition becomes incoherent if the system A becomes entangled with another system B, and B is inaccessible. Heuristically, the states $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$ can interfere (the relative phase of these states can be observed) only if we have no information about whether the spin state is $|\uparrow_z\rangle$ or $|\downarrow_z\rangle$. More than that, interference can occur only if there is *in principle no possible way* to find out whether the spin is up or down along the z-axis. Entangling spin A with the system B destroys interference, (causes spin A to decohere) because it becomes possible in principle for us to determine if spin A is up or down along $\hat{z}$ by performing a suitable measurement on system B.

Now, considering the incoherent superposition of the spin example above, we can finally get into quantum eraser. As we already said, if an observer of the system B makes a measure of the z-spin, then, because of incoherence, the other observer in A immediately learns what is his state of spin along the z-axis. But, if the B-observer, after that z-measure, *does not look* at the result and instead makes an ulterior measure along the x-azis, communicating the results at his colleague on A,

the situation completely changes. Indeed, the measure of the spin along the x direction on B "erases" the initial state of incoherence on A, and, after this exchange of information, it becomes impossible to learn something about the z-spin of the system A by operating on the system B. Yet incoherence is not irreversible, that is, we can always create a coherent superposition of some ensemble of states in the subsystem A by projecting the whole A+B-state with a suitable measure on a pure state in the subsystem B. Thus, measuring the B spin on x-axis yields on A the state (1.21), which is a coherent superposition of spin up and spin down along z-axis. So, from this prospective, it is easy to see how this phenomenon can be used against the claim of density matrix of describing completely all the possible states of the system A. Since the information received from B-observer enables A-observer to recover a pure state from the incoherent initial mixture, how can we hold that everything can we know about A is encoded in $\boldsymbol{\rho_A}$? A possible answer to this interesting question is saying that the two following distinct settings:

- knowledge of $\boldsymbol{\rho_A}$

- knowledge of $\boldsymbol{\rho_A}$ *plus* information coming from B

are physically different.

This type of solution seems to confirm another time the concrete and physical consistency of information.

After this discussion, we are now ready to approach the GHJW theorem and see how this theorem can formalize and generalize the concepts illustrated above. As we said, any density matrix can be realized as an ensemble of pure states; for a density matrix $\boldsymbol{\rho_A}$, we consider one such realization:

$$\boldsymbol{\rho_A} = \sum_i p_i \left|\phi_i\right\rangle \left\langle\phi_i\right| \tag{1.22}$$

where, of course: $\sum_i p_i = 1$
In this equation is really not necessary to consider an orthonormal basis, and we limit to assume just the normalization condition. From this point, we can now construct a state that is often called *purification* of $\boldsymbol{\rho_A}$. To make the latter we have to associate to H_A another Hilbert space H_B of dimension at least equal to those of H_A, with one orthonormal basis, and then we can identify the purification of the density matrix with a state of the tensorial product $H_A \bigotimes H_B$, here represented by $\left|\Phi_{AB}\right\rangle_1$. Anyway this state has not to be generic, it must satisfy the following condition:

$$tr_B(\left|\Phi_{AB}\right\rangle_1 \left\langle\Phi_{AB}\right|_1) = \boldsymbol{\rho_A} \tag{1.23}$$

Starting from the same density matrix $\boldsymbol{\rho_A}$, as we will see in details, there are a lot of possible purifications, and that is way of the subscript "1" on the state. Knowing that, we can define this purification to have the following Schmidt decomposition:

$$|\Phi_{AB}\rangle_1 = \sum_i \sqrt{p_i}\,|\phi_i\rangle \otimes |\alpha_i\rangle \tag{1.24}$$

Where $\{|\alpha_i\rangle\}_i$ is an orthonormal basis of H_B.
It is easy to check that with this choice the request above is verified. So, with this purification it becomes possible to obtain an eigenstate of the system A making an orthogonal measure on B.
To generalize the notion of a quantum eraser, we wish to see that in the state $|\Phi_{AB}\rangle_1$ we can realize a different ensemble interpretation of $\boldsymbol{\rho_A}$ by performing a different measurement of B. To see that, we start choosing another basis in (1.22), like $\{|\psi_j\rangle\}_j$ (as before not necessarily orthogonal) and then we write:

$$\boldsymbol{\rho_A} = \sum_\mu p_\mu\,|\psi_\mu\rangle\,\langle\psi_\mu| \tag{1.25}$$

and this expression is nothing more than another realization of the same density matrix $\boldsymbol{\rho_A}$ as an ensemble of pure states.
With this change of basis it also changes the purification, that becomes:

$$|\Phi_{AB}\rangle_2 = \sum_\mu \sqrt{p_\mu}\,|\psi_\mu\rangle \otimes |\beta_\mu\rangle \tag{1.26}$$

Where $\{|\beta_i\rangle\}_i$ is another orthonormal basis of H_B.
Thus we have two different equations that satisfy the same trace constraint on the system B. It follows that the connection between the two states $|\Phi_{AB}\rangle_1$ and $|\Phi_{AB}\rangle_2$ must be in the form of an operator that does nothing on subsystem A and keeps the trace on the subsystem B invariant, something like:

$$|\Phi_{AB}\rangle_1 = (\mathbf{1_A} \otimes \boldsymbol{U_B})\,|\Phi_{AB}\rangle_2 \tag{1.27}$$

where $\boldsymbol{U_B}$ is an unitary operator that preserves the trace on B. It acts transforming the orthonormal basis $\{|\beta_i\rangle\}_i$ into another orthonormal one:

$$|\gamma_{\mu B}\rangle = \boldsymbol{U_B}\,|\beta_{\mu B}\rangle \tag{1.28}$$

Combining the two equations (1.27) and (1.28) one easily reaches:

$$|\Phi_{AB}\rangle_1 = \sum_\mu \sqrt{p_\mu}\,|\psi_\mu\rangle \otimes |\gamma_\mu\rangle \tag{1.29}$$

We see then, comparing this last equation with the first expression of $|\Phi_{AB}\rangle_1$, (1.24), that there is a single purification such that we can realize either the $\{|\phi_i\rangle\}_i$ ensemble or the $\{|\psi_\mu\rangle\}_\mu$ ensemble by choosing to measure the appropriate observable in system B. This final consideration is the deepest meaning of the GHJW theorem, enunciated almost at the same time by Gisin, Hughston, Jozsa and Wootters. We can know present the theorem in its formal shape:

Theorem 1.6.1 (GHJW theorem). *Let $\boldsymbol{\rho_A}$ be a density matrix in a Hilbert space H_A of dimension n, then it is possible to introduce another Hilbert space H_B with dimension at least equal to n to construct states $|\Phi_{AB}\rangle_i$ of the tensorial product $H_A \bigotimes H_B$ (called purifications of $\boldsymbol{\rho_A}$) such that:*

1. $|\Phi_{AB}\rangle_i = \sum_j \sqrt{p_j}\,|i^A{}_j\rangle \otimes |i^B{}_j\rangle$, $\forall i$

   where $\forall i$, $\{|i^A{}_j\rangle\}_j$ is a normalized basis of H_A and $\{|i^B{}_j\rangle\}_j$ is a orthogonal basis of and H_B

2. $\boldsymbol{\rho_A} = tr_B(|\Phi_{AB}\rangle_i \langle\Phi_{AB}|_i)$, $\forall i$

3. $\forall i,j\ ; i \neq j\ \exists\ \boldsymbol{U_B}$ *such that* $\boldsymbol{U_B U^\dagger{}_B} = \mathbf{1}$ *and* $|\Phi_{AB}\rangle_i = (\mathbf{1_A} \otimes \boldsymbol{U_B})\,|\Phi_{AB}\rangle_j$

As we can see, this theorem is a direct consequence of the ambiguity contained in the Schmidt decomposition, that allows to associate the same eigenvectors of the system A to different orthonormal eigenvectors of the system B, related each other by unitary transformations. In fact, the degenerate Schmidt decompositions that support the validity of the GHJW theorem are the following ones: (allowed by the same trace condition on B)

$$\begin{aligned} |\Phi_{AB}\rangle_1 &= \sum_k \sqrt{\lambda_k}\,|k\rangle \otimes |k'_1\rangle \\ |\Phi_{AB}\rangle_2 &= \sum_k \sqrt{\lambda_k}\,|k\rangle \otimes |k'_2\rangle \end{aligned} \tag{1.30}$$

where the λ_k's are the eigenvalues of $\boldsymbol{\rho_A}$, the $|k\rangle$'s are the corresponding eigenvectors and $\{|k'_1\rangle\}_k$ and $\{|k'_2\rangle\}_k$ are both orthonormal bases linked by a unitary operator $\boldsymbol{U_B}$ in the usual way. At the end of this discussion it is clear in what sense this theorem characterizes the general quantum eraser; in fact after the preparation of the state in the (1.24) conformation measuring B in the $\{|\gamma_\mu\rangle\}_\mu$ basis erases the crucial information concerning whether the state A is in the state $|\phi_i\rangle$ or $|\phi_j\rangle$ that was accessible before by appropriate measure on the system B.

Chapter 2

Generalized measurement

2.1 POVM

We are now ready to discuss one of the main important consequence of the introduction of open quantum systems. As we had already seen, even if the whole system preserves the original statements of the ordinary quantum mechanics (states are still rays, evolution is always unitary) the subsystem of our interest, often called here the A system, has different properties. At the beginning of this work, we said that on this system A *measurements are not orthogonal*. We are now showing the details about this claim; indeed we will see that in this context the measuring process is represented by a non-orthogonal *Positive Operator Valued Measure* (usually shortened by the acronym POVM) and not by the usual PVM described in section 1.1. This treatment has of course a relevant physical interest because, in a real situation of an experimental measure, it is impossible to isolate completely the system (even the same measuring set up can often play the role of an irremovable "external" font of disturb), and so it is necessary to consider the interaction between the object of the experimentation and the environment. So we start our presentation introducing as always the environment B, represented here by a quantum system in a *known* state $\boldsymbol{\rho_B}$. Likewise, the initial state of our system A is represented by the density matrix $\boldsymbol{\rho_A}^{\boldsymbol{i}}$ (the index i shows that a lot of possible configurations can be considered at this starting point and it is useful to keep track of this initial choice for the continuation of the discussion). Thus, the combined state of the original whole quantum system is $\boldsymbol{\rho_B} \otimes \boldsymbol{\rho^i}_{\boldsymbol{A}}$, that in components becomes :

$$(\boldsymbol{\rho_B} \otimes \boldsymbol{\rho^i}_{\boldsymbol{A}})_{\alpha r,\beta s} = (\boldsymbol{\rho_B})_{\alpha,\beta}(\boldsymbol{\rho^i}_{\boldsymbol{A}})_{r,s} \tag{2.1}$$

where the Greek letters refer to environment B and Latin ones to the system A.

Now we perform a measure on the whole system. From the principles of quantum mechanics we know that such a measure is represented by an orthogonal resolution of the identity. Different outcomes correspond to orthogonal projectors $\{\boldsymbol{P_\mu}\}_\mu$ which

satisfy:

$$\boldsymbol{P_\mu P_\nu} = \delta_{\mu\nu}\boldsymbol{P_\nu} \quad \text{and} \quad \sum_\mu \boldsymbol{P_\mu} = \mathbf{1} \tag{2.2}$$

Moreover, in this measure, the probability of the outcome "μ" with an initial preparation "i" of the system A, here identified by $(\boldsymbol{P})_{\mu i}$, is:

$$(\boldsymbol{P})_{\mu i} = tr_{AB}[\boldsymbol{P_\mu}(\boldsymbol{\rho_B} \otimes \boldsymbol{\rho^i}_{\boldsymbol{A}})] = \sum_{\alpha r,\beta s} (\boldsymbol{P_\mu})_{\alpha r \beta s}(\boldsymbol{\rho_B})_{\alpha,\beta}(\boldsymbol{\rho^i}_{\boldsymbol{A}})_{r,s} \tag{2.3}$$

and this can be written as:

$$(\boldsymbol{P})_{\mu i} = tr_A(\boldsymbol{A_\mu \rho^i}_{\boldsymbol{A}}) \tag{2.4}$$

where

$$(\boldsymbol{A_\mu})_{rs} \equiv \sum_{\alpha,\beta} (\boldsymbol{P_\mu})_{\alpha r \beta s}(\boldsymbol{\rho_B})_{\beta\alpha} \tag{2.5}$$

is an operator acting on the Hilbert space H_A.
So we can see from (2.4) that we have realized a way to express the outcome of a measuring process on the whole system $A + B$ just working with operators of the subsystem A, the $\{\boldsymbol{A_\mu}\}_\mu$. These hermitian matrices, which in general do not commute, satisfy:

$$\sum_\mu \boldsymbol{A_\mu} = \mathbf{1_A} \tag{2.6}$$

in fact, from (2.4), for each component rs we get:

$$(\sum_\mu \boldsymbol{A_\mu})_{rs} = \sum_{\alpha,\beta}(\sum_\mu \boldsymbol{P_\mu})_{\alpha r \beta s}(\boldsymbol{\rho_B})_{\beta\alpha} = \sum_{\alpha,\beta} \delta_{\alpha r \beta s}(\boldsymbol{\rho_B})_{\beta\alpha} = (\delta)_{rs} \sum_\alpha (\boldsymbol{\rho_B})_{\alpha\alpha} = \delta_{rs} \tag{2.7}$$

The set of $\{\boldsymbol{A_\mu}\}_\mu$ is called a *Positive Operator Valued Measure* (POVM), because each $\boldsymbol{A_\mu}$ is a positive operator. They are Hermitian and positive, but the main difference between these POVMs and the usual PVMs is that the $\{\boldsymbol{A_\mu}\}_\mu$ are not necessarily orthogonal and normalized . Concretely, if a usual projector has the form:

$$\boldsymbol{P_\mu} = |\psi_\mu\rangle \langle\psi_\mu| \tag{2.8}$$

then an operator of rank one composing a POVM is in the form:

$$\boldsymbol{A_\mu} = \lambda_\mu |\psi^A{}_\mu\rangle \langle\psi^A{}_\mu| \tag{2.9}$$

where the states $\{|\psi^A{}_\mu\rangle\}_\mu$ are not orthogonal and the λ_μ are generic complex coefficients.

This fact implies that the number of available preparations and the number of available outcomes may be different from each other, and also different from the dimensionality of Hilbert space H_A.

It can be proved that, using only POVMs composed of operators of rank one, the required number, n, of different $\boldsymbol{A_\mu}$ satisfies the inequality:

$$N \leq n \leq N^2 \tag{2.10}$$

where N is the dimensionality of the subspace of H_A spanned by the different preparations $\boldsymbol{\rho^i}_{\boldsymbol{A}}$. The prove of this claim can be founded in [4].
Another question that we can ask ourselves is: how does a general measurement ruled by a POVM affect the quantum state and the density matrix of the subsystem A? To answer, we choose a specific configuration $\boldsymbol{\rho^i}_{\boldsymbol{A}}$ and we remove the "i" index that is then superfluous. We know that, performing a non orthogonal generalized measure on this configuration, the outcome $(\boldsymbol{P})_{\mu i} \equiv p_\mu$ occurs with probability $tr_A(\boldsymbol{A_\mu \rho_A})$; then, summing over the all the possible outcomes of this kind of measure yields:

$$\begin{aligned} \boldsymbol{\rho_A} \overset{\text{measuring}}{\rightarrow} \boldsymbol{\rho_{A_{POVM}}} &= \sum_\mu |\psi^A{}_\mu\rangle \langle\psi^A{}_\mu| \lambda_\mu \langle\psi^A{}_\mu| \boldsymbol{\rho_A} |\psi^A{}_\mu\rangle \\ &= \sum_\mu \left(\sqrt{\lambda_\mu} |\psi^A{}_\mu\rangle \langle\psi^A{}_\mu| \right) \boldsymbol{\rho_A} \left(\sqrt{\lambda_\mu} |\psi^A{}_\mu\rangle \langle\psi^A{}_\mu| \right) \\ &= \sum_\mu \sqrt{\boldsymbol{A_\mu}} \boldsymbol{\rho_A} \sqrt{\boldsymbol{A_\mu}} \end{aligned} \tag{2.11}$$

and this is another relevant difference from the orthonormal projective case, where a measure encoded by the projector operators $\{\boldsymbol{P_\mu}\}_\mu$ modifies the density matrix of the system in a conventional way:

$$\boldsymbol{\rho_A} \overset{\text{measuring}}{\rightarrow} \boldsymbol{\rho_{A_{PVM}}} = \sum_\mu \boldsymbol{P_\mu \rho_A P_\mu} \tag{2.12}$$

The method just exposed is the right one to generalize the concept of measuring for open quantum systems; now we want to understand if this treatment is also invertible, or, in another words, if it is always possible, starting from a generic POVM, to construct an extended system whose the POVM is the result of orthogonal measures. The clarification of this doubt is given by *Neumark's Theorem.*

2.2 Neumark's theorem

We immediately give the statement of the theorem briefly described in the section above.

Theorem 2.2.1 (Neumark's theorem)**.**
Let H_A to be Hilbert space in which is defined a set $\{\boldsymbol{A_\mu}\}_\mu$ of positive operators such that $\sum_\mu \boldsymbol{A_\mu} = \boldsymbol{1_A}$.
Then it exists an extended Hilbert space K and on it a set of orthogonal projectors $\{\boldsymbol{P_\mu}\}_\mu$ satisfying $\sum_\mu \boldsymbol{P_\mu} = \boldsymbol{1_K}$ such that, $\forall\mu$, $\boldsymbol{A_\mu}$ is the result of projecting $\boldsymbol{P_\mu}$ from K into H_A.

Proof. We can restrict our attention to operators:

$$\boldsymbol{A_\mu} = |u_\mu\rangle \langle u_\mu| \tag{2.13}$$

that are of rank one, because any other case can be obtained from this general one. Because they are of rank 1, we know that the number N of this operators is $N \geq n$ where n is the dimension of the Hilbert space H_A (the case $N = n$ is represented by orthogonal operators). Let us now extend the original vector space H_A by introducing $N - n$ unit vectors $\{|v_s\rangle\}_{s=n+1:N}$, orthogonal to each other and to the all $|u_\mu\rangle$ from equation (2.13):

$$|w_\mu\rangle \equiv |u_\mu\rangle + \sum_{s=n+1}^{s=N} c_{\mu s} |v_s\rangle \tag{2.14}$$

it is clear that the number of these equations (N) is greater than the number of unknown complex coefficients $c_{\mu s}$ (N-n). By definition, the $\{|w_\mu\rangle\}_\mu$'s must form an orthonormal basis for an enlarged Hilbert space K of dimension N, so they must satisfy:

$$\langle w_\nu | w_\mu \rangle = \langle u_\nu | u_\mu \rangle + \sum_{s=n+1}^{s=N} \overline{c_{\nu s}} c_{\mu s} = \delta_{\nu\mu} \tag{2.15}$$

Anyway, the $\{|u_\mu\rangle\}_\mu$'s are not arbitrary: they obey the closure property $\sum_\mu \boldsymbol{A_\mu} = \boldsymbol{1_A}$, that in components it becomes explicitly:

$$\sum_\mu \overline{(|u_\mu\rangle)_i} (|u_\mu\rangle)_j = \delta_{ij} \tag{2.16}$$

where i and j run from 1 to n (the number of dimensions of the original Hilbert space, H_A). Writing also equation (2.15) in components gives:

$$\sum_i \overline{(|u_\nu\rangle)_i} (|u_\mu\rangle)_i + \sum_{s=n+1}^{s=N} \overline{c_{\nu s}} c_{\mu s} = \delta_{\nu\mu} \quad \text{with} \quad (\mu, \nu = 1, .., N) \tag{2.17}$$

We can now build up the square $N \times N$ matrix M:

$$M = \begin{bmatrix} (|u_\alpha\rangle)_1 & \dots & (|u_\alpha\rangle)_n & c_{\alpha,n+1} & \dots & c_{\alpha,N} \\ (|u_\beta\rangle)_1 & \dots & (|u_\beta\rangle)_n & c_{\beta,n+1} & \dots & c_{\beta,N} \\ \vdots & & \vdots & \vdots & & \vdots \\ (|u_N\rangle)_1 & \dots & (|u_N\rangle)_n & c_{N,n+1} & \dots & c_{N,N} \end{bmatrix} \tag{2.18}$$

The first n columns are the $(|u_\lambda\rangle)_i$, which are given, and the $N-n$ remaining columns are the unknown $c_{\lambda s}$. We can see then that equation (2.17) , in this notation, simply says that M is a unitary matrix. The first n columns, which satisfy the consistency requirement (2.16), can be considered as n orthonormal vectors in a N-dimensional space. There are then infinitely many ways of constructing $N-n$ other orthonormal vectors for the remaining columns. We thereby obtain explicitly the N orthonormal vectors $\{|w_\mu\rangle\}_\mu$ defined above. Their projections into H_A are the $\{|u_\mu\rangle\}_\mu$ of the beginning, that compose $\boldsymbol{\rho_A}$. □

At the end of this chapter we show how the possibilities of POVMs can increase the power of the already note GHJW decomposition.

2.3 GHJW with POVMs

As we know from section 1.6, if we have two Hilbert spaces , H_A and H_B, the instrument of GHJW decomposition allows us, starting from a generic open state of the form:

$$|\Phi_{AB}\rangle = \sum_{\mu=1} \sqrt{p_\mu}\, |\psi^A{}_\mu\rangle \otimes |\beta^B{}_\mu\rangle \tag{2.19}$$

to realize an ensemble of up to N pure states by measuring an appropriate observable on H_B.
Then, we represented such an ensemble as a density matrix $\boldsymbol{\rho_A}$ like the following one:

$$\boldsymbol{\rho_A} = \sum_{\mu} p_\mu\, |\psi^A{}_\mu\rangle\, \langle\psi^A{}_\mu| \tag{2.20}$$

Nevertheless , if the dimension of H_A was N, that was granted only for a Hilbert space H_B of at least the same dimension N, just because of the nature of the orthogonal measure performed on B. But, if we introduce the possibility of make a POVM on B, then we can reduce the minimum size of dimensions of H_B up to an integer number n such that $\sqrt{N} \leq n \leq N$, as we can see from the expression (2.10) .
Therefore, we may rewrite the same state $|\Phi_{AB}\rangle$ as:

$$|\Phi_{AB}\rangle = \sum_{\mu} \sqrt{p_\mu}\, |\psi^A{}_\mu\rangle \otimes |\tilde{\beta}^B_\mu\rangle \tag{2.21}$$

where $|\tilde{\beta}^B\rangle_\mu$ are the orthogonal projections of the old $|\beta^B{}_\mu\rangle$'s of equation (2.19) (those vectors that belonged to the space H_B of N dimensions) onto the new support of $\boldsymbol{\rho_B}$ of n dimensions. We may now perform the POVM on the support of $\boldsymbol{\rho_B}$ with $\boldsymbol{A_\mu} = |\tilde{\beta}^B_\mu\rangle\, \langle\tilde{\beta}^B_\mu|$, and thus prepare the state $|\psi^A{}_\mu\rangle$ with probability p_μ.

Chapter 3

Evolution of density matrix

3.1 Evolution without coupling

So far, we have not discussed the evolution in time of the density operator. At the beginning of this work we claimed that evolution of open systems is not necessarily unitary. To conclude the part of the thesis dedicated to the foundation of open quantum systems we want to present and justify this important last deviation from the usual principles of quantum mechanics. We will also see later that this type of evolution can be connected in a very interesting way to the tool of POVM just presented in the previous chapter. We begin by looking at the most simple case, concerning two non-interacting subsystems A and B. This can be useful to become familiar with the evolution of the density matrices before the situation becomes more tricky with interaction. So, considering the usual two subsystems A and B (environment) and their respective Hilbert spaces H_A and H_B, let us suppose that the Hamiltonian on $H_A \bigotimes H_B$ has the form:

$$\boldsymbol{H_{AB}} = \boldsymbol{H_A} \otimes \boldsymbol{1_B} + \boldsymbol{1_A} \otimes \boldsymbol{H_B} \tag{3.1}$$

this assumption is the mathematical condition for absence of interaction between A and B, so that each of them evolves independently. The time evolution operator for the combined system is:

$$\boldsymbol{U_{AB}}(t) = \boldsymbol{U_A}(t) \otimes \boldsymbol{U_B}(t) \tag{3.2}$$

and it decomposes into two separate unitary time evolution operators acting on each system. In the Schrödinger picture of dynamics, then, an initial pure state of the kind of equation (1.2) of the open system evolves as:

$$|\psi_{AB}(t)\rangle = \sum_{i,\mu} a_{i\mu} \, |i_A(t)\rangle \otimes |\mu_B(t)\rangle \tag{3.3}$$

where $|i_A(t)\rangle = \boldsymbol{U_A}(t) \, |i_A(0)\rangle$ and $|\mu_B(t)\rangle = \boldsymbol{U_B}(t) \, |\mu_B(0)\rangle$.

If $\boldsymbol{U_A}(t)$ and $\boldsymbol{U_B}(t)$ are unitary, $\{|i_A(t)\rangle\}_i$ and $\{|\mu_B(t)\rangle\}_\mu$ are orthogonal bases of H_A

and H_B respectively. Then, from this expression, we can obtain the law of evolution of density matrix $\boldsymbol{\rho_A}$ just taking the trace of the projector $|\psi_{AB}(t)\rangle \langle\psi_{AB}(t)|$ on the space H_B:

$$\boldsymbol{\rho_A}(t) = \sum_{\mu,i,j} a_{i\mu} a^*{}_{j\mu} \, |i_A(t)\rangle \langle j_A(t)| \tag{3.4}$$

that can be clearly simplified in the form:

$$\boldsymbol{\rho_A}(t) = \boldsymbol{U_A}(t)\boldsymbol{\rho_A(0)}\boldsymbol{U^\dagger{}_A}(t) \tag{3.5}$$

In particular, in the basis in which $\boldsymbol{\rho_A(0)}$ is diagonal, we have:

$$\boldsymbol{\rho_A}(t) = \sum_a p_a \boldsymbol{U_A}(t) \, |\psi^A{}_a(0)\rangle \langle\psi^A{}_a(0)| \, \boldsymbol{U^\dagger{}_A}(t) \tag{3.6}$$

So we can see that each state in the ensemble evolves forward in time governed by $\boldsymbol{U_A}(t)$. The equation above shows that, if on the original preparation of the system A the probability of finding the state $|\psi^A{}_a(0)\rangle$ was p_a, then, after the evolution that carries to the time t, that probability is the same of finding the state $|\psi^A{}_a(t)\rangle$. This last statement really explains well the meaning of not coupling between subsystems.

3.2 Evolution with coupling

3.2.1 Dynamical maps

In this section we will study the evolution of subsystems that are free to interchanging information with each other so that it is impossible to factorize the evolution operator in two separate parts like we have done in equation (3.2) . We will give particular attention to this argument just because this is the main object of study of this whole work.

So, what we really want to find is a *dynamical map* acting on a generic density matrix defined on H_A which connects the density matrix of the subsystem A at times t_0 and t_1:

$$\clubsuit_{(t_0,t_1)} : \boldsymbol{\rho_A}(t_0) \to \boldsymbol{\rho_A}(t_1) \tag{3.7}$$

We know that a similar type of map has to depend not only on the unitary evolution operator of the whole system $\boldsymbol{U_{AB}}(t_1, t_0)$ but also on the properties of the system B and A themselves. So, it seems convenient to divide , for the initial state of the whole system A+B , the correlated and the uncorrelated part [13] :

$$\boldsymbol{\rho_{AB}}(t_0) = \boldsymbol{\rho_A}(t_0) \otimes \boldsymbol{\rho_B}(t_0) + \boldsymbol{\rho_{corr}}(t_0) \tag{3.8}$$

where the $\boldsymbol{\rho_{corr}}(t_0)$ symbolizes all the possible starting interactions between A and B but it has no physical meaning for the subsystems considered separately, so that [13] :

$$tr_A(\boldsymbol{\rho_{corr}}(t_0)) = tr_B(\boldsymbol{\rho_{corr}}(t_0)) = 0 \tag{3.9}$$

About the presence of the correlation term, we have the following theorem:

Theorem 3.2.1.
Let the density matrix of subsystem A to be pure: $\boldsymbol{\rho_A} = |\psi_A\rangle \langle\psi_A|$*, then, in the whole system density matrix,* $\boldsymbol{\rho_{AB}} = \boldsymbol{\rho_A} \otimes \boldsymbol{\rho_B} + \boldsymbol{\rho_{corr}}$*, the term* $\boldsymbol{\rho_{corr}} = 0$

Proof. We can consider the density operator $\boldsymbol{\rho_{AB}}$. If this density matrix is pure, then it has to be of the form:

$$\boldsymbol{\rho_{AB}} = |\psi_A\rangle \langle\psi_A| \otimes |\phi_B\rangle \langle\phi_B| \tag{3.10}$$

and so $\boldsymbol{\rho_{corr}} = 0$. So, assume that $\boldsymbol{\rho_{AB}}$ is mixed. Thus, we can write it as a convex combination of pure matrices:

$$\boldsymbol{\rho_{AB}} = \sum_\mu p_\mu \, |{\alpha^{AB}}_\mu\rangle \langle{\alpha^{AB}}_\mu| \tag{3.11}$$

but, because we know from hypothesis that $\boldsymbol{\rho_A}$ is pure, then we must have:

$$tr_B[\sum_\mu p_\mu \, |{\alpha^{AB}}_\mu\rangle \langle{\alpha^{AB}}_\mu|] \equiv \sum_\mu p_\mu \boldsymbol{\rho^A_\mu} \tag{3.12}$$

to be pure.
(Here we have defined the $\boldsymbol{\rho^A_\mu} = tr_B[|{\alpha^{AB}}_\mu\rangle \langle{\alpha^{AB}}_\mu|]$, that are operators acting on H_A).
The only two possibilities to satisfy this request are:

1. $p_\mu = 1$ for $\mu = \mu'$ and $p_\mu = 0$ for each other $\mu \neq \mu'$

2. $\boldsymbol{\rho^A_\mu} = \boldsymbol{\rho^A}$ for all μ

For the case (1) we immediately get, from equation (3.11) :

$$\boldsymbol{\rho_{AB}} = |{\alpha^{AB}}_{\mu'}\rangle \langle{\alpha^{AB}}_{\mu'}| \tag{3.13}$$

that proves the theorem. Then, the case (2) remains.
In that case we can expand each $|{\alpha^{AB}}_\mu\rangle$ with a Schmidt decomposition $|{\alpha^{AB}}_\mu\rangle = \sum_i \lambda_{\mu,i} \, |{u^A}_{\mu,i}\rangle \otimes |{v^B}_{\mu,i}\rangle$, and then obtain:

$$\begin{aligned} \boldsymbol{\rho_A} = \boldsymbol{\rho^A_\mu} &= tr_B[|{\alpha^{AB}}_\mu\rangle \langle{\alpha^{AB}}_\mu|] \\ &= \sum_i {\lambda_{\mu,i}}^2 \, |{u^A}_{\mu,i}\rangle \langle{u^A}_{\mu,i}| \quad \forall\mu \end{aligned} \tag{3.14}$$

and then the last equation can be reformulated to be independent from μ:

$$\boldsymbol{\rho_A} = \sum_i {\lambda_i}^2 \, |{u^A}_i\rangle \langle{u^A}_i| \tag{3.15}$$

Finally, because $\boldsymbol{\rho_A}$ has to be pure, we must have $\lambda_i = 1$ for $i = i'$ and $\lambda_i = 0$ for all $i \neq i'$. Therefore we finally get:

$$\boldsymbol{\rho_{AB}} = |u^A{}_{i'}\rangle \otimes |v^B{}_{i'}\rangle \tag{3.16}$$

□

Anyway, we already know from the previous section that the evolution of the whole system must satisfy:

$$\boldsymbol{\rho_{AB}}(t_1) = \boldsymbol{U_{AB}}(t_1, t_0)\boldsymbol{\rho_{AB}}(t_0)\boldsymbol{U^\dagger{}_{AB}}(t_1, t_0) \tag{3.17}$$

Thus, as we have always done, if we want to reach the expression of $\boldsymbol{\rho_A}(t_1)$ it will be sufficient to perform the trace of this last expression on the subsystem B :

$$\boldsymbol{\rho_A}(t_1) = tr_B[\boldsymbol{U_{AB}}(t_1, t_0)\boldsymbol{\rho_{AB}}(t_0)\boldsymbol{U^\dagger{}_{AB}}(t_1, t_0)] \tag{3.18}$$

Therefore we can substitute the explicit form of $\boldsymbol{\rho_{AB}}(t_0)$ provided by equation (3.8) in the last expression , obtaining:

$$\begin{aligned}
\boldsymbol{\rho_A}(t_1) &= tr_B\{\boldsymbol{U_{AB}}(t_1, t_0)[\boldsymbol{\rho_A}(t_0) \otimes \boldsymbol{\rho_B}(t_0) + \boldsymbol{\rho_{corr}}(t_0)]\boldsymbol{U^\dagger{}_{AB}}(t_1, t_0)\} \\
&= \sum_i \lambda_i \, tr_B[\boldsymbol{U_{AB}}(t_1, t_0)(\boldsymbol{\rho_A}(t_0) \otimes |\psi^B{}_i\rangle \langle\psi^B{}_i|)\boldsymbol{U^\dagger{}_{AB}}(t_1, t_0)] \\
&+ tr_B[\boldsymbol{U_{AB}}(t_1, t_0)(\boldsymbol{\rho_{corr}}(t_0))\boldsymbol{U^\dagger{}_{AB}}(t_1, t_0)] \\
&= \sum_\alpha \boldsymbol{K}_\alpha(t_1, t_0)\boldsymbol{\rho_A}(t_0)\boldsymbol{K}_\alpha{}^\dagger(t_1, t_0) + \delta\boldsymbol{\rho}(t_1, t_0) \\
&\equiv \clubsuit(\boldsymbol{\rho_A}(t_0))
\end{aligned} \tag{3.19}$$

where α is a double index : $\alpha = i, j$ so that:

$$\boldsymbol{K}_{i,j}(t_1, t_0) = \sqrt{\lambda_i}\, \langle\psi^B{}_j| \, \boldsymbol{U_{AB}}(t_1, t_0) \, |\psi^B{}_i\rangle \tag{3.20}$$

and we have used the spectral decomposition $\boldsymbol{\rho_B}(t_0) = \sum_i \lambda_i \, |\psi^B{}_i\rangle \langle\psi^B{}_i|$.
The term $\delta\boldsymbol{\rho}(t_1, t_0)$ is defined as:

$$\delta\boldsymbol{\rho}(t_1, t_0) \equiv tr_B[\boldsymbol{U_{AB}}(t_1, t_0)(\boldsymbol{\rho_{corr}}(t_0))\boldsymbol{U^\dagger{}_{AB}}(t_1, t_0)] \tag{3.21}$$

The positivity requirement of density operators forces each term in the dynamical map to be interconnected and dependent on the state upon which it acts; this means that a dynamical map with some values of $\boldsymbol{K}_{i,j}(t_1, t_0)$ and $\delta\boldsymbol{\rho}(t_1, t_0)$ may describe a physical evolution for some states $\boldsymbol{\rho_A}(t_0)$ and an unphysical evolution for others. This characteristic makes working with this type of evolution very delicate and complex. Anyway, to try to give some regularity and some more general rules to this maps D. Salgado and D. M. Tong reached independently an important theorem that gives a useful description of these mathematical objects:

Theorem 3.2.2.
Any kind of time evolution of a quantum state $\boldsymbol{\rho_A}(t_0)$ *can always be written in the form:*

$$\boldsymbol{\rho_A}(t_1) = \sum_{\alpha} \boldsymbol{K_\alpha}(t_1, t_0, \boldsymbol{\rho_A}(t_0))\boldsymbol{\rho_A}(t_0)\boldsymbol{K_\alpha}^{\dagger}(t_1, t_0, \boldsymbol{\rho_A}(t_0)) \tag{3.22}$$

Proof. Let us give a very simple proof; we can imagine an operator that we will call $\boldsymbol{U_{change}}$ that can act on tensor product in the following way:

$$\begin{aligned} &\boldsymbol{U_{change}}(H_A \bigotimes H_B) = H_B \bigotimes H_A \\ &\boldsymbol{U_{change}}(\boldsymbol{\rho_A} \otimes \boldsymbol{\rho_B})\boldsymbol{U_{change}}^{\dagger} = \boldsymbol{\rho_B} \otimes \boldsymbol{\rho_A} \end{aligned} \tag{3.23}$$

Thus, given an original density matrix at time t_0 of the system A, $\boldsymbol{\rho_A}(t_0)$, and a final matrix $\boldsymbol{\rho_A}(t_1)$ that we want to get at time t_1 after the evolution , we construct the state of $H_A \bigotimes H_A$:

$$\boldsymbol{\rho_A}(t_0) \otimes \boldsymbol{\rho_A}(t_1) \tag{3.24}$$

So, we can see that the final operator at time t_1 can be also written as:

$$\boldsymbol{\rho_A}(t_1) = tr_2[\boldsymbol{U_{change}}(\boldsymbol{\rho_A}(t_0) \otimes \boldsymbol{\rho_A}(t_1))\boldsymbol{U_{change}}^{\dagger}] \tag{3.25}$$

where tr_2 denotes the partial trace with respect to the second member of the composed state in the tensor product. Finally, by taking the spectral decomposition of $\boldsymbol{\rho_A}(t_1)$ in the central term of the above equation we obtain an expression of the form (3.22) □

Note that this decomposition is clearly not unique.
This theorem shows that, reducing the domain of a dynamical map, it is always possible to obtain another dynamical map without the inhomogeneous term $\delta\boldsymbol{\rho}(t_1, t_0)$.

3.2.2 UDM: Universal Dynamical Maps

Using the concept of dynamical maps, what we really want to reach is a way of treating the evolution of a quantum system, let us say A, independently from the particular starting configuration $\boldsymbol{\rho_A}(t_0)$ in which it is prepared. So, the tool that we are searching for is a linear, positive defined map $\clubsuit_{(t_1,t_0)}$ called a *Universal Dynamical Map* or *UDM* that could satisfy:

$$\clubsuit_{(t_1,t_0)}\boldsymbol{\rho_A}(t_0) \equiv \sum_{\alpha} \boldsymbol{K_\alpha}(t_1, t_0)\boldsymbol{\rho_A}(t_0)\boldsymbol{K^{\dagger}}_{\boldsymbol{\alpha}}(t_1, t_0) = \boldsymbol{\rho_A}(t_1) \tag{3.26}$$

where the operators $\boldsymbol{K_\alpha}(t_1, t_0)$ do not depend here from the initial matrix $\boldsymbol{\rho_A}(t_0)$. In addition we observe that the normalization condition of the trace $tr_A[\boldsymbol{\rho_A}(t_1)] = 1$ imposes that:

$$\sum_{\alpha} \boldsymbol{K^{\dagger}}_{\boldsymbol{\alpha}}(t_1, t_0)\boldsymbol{K_\alpha}(t_1, t_0) = \boldsymbol{1_A} \tag{3.27}$$

So, the following important theorem can inform us on the specific starting conditions that could realize our goal:

Theorem 3.2.3. *A dynamical map $\clubsuit_{(t_1,t_0)}$ is a UDM if and only if it is induced from an extended system A+B with the initial condition $\boldsymbol{\rho_{AB}}(t_0) = \boldsymbol{\rho_A}(t_0) \otimes \boldsymbol{\rho_B}(t_0)$ where $\boldsymbol{\rho_B}(t_0)$ is fixed for any $\boldsymbol{\rho_A}(t_0)$.*

Proof. The first implication is really straightforward; indeed the condition fixed by the hypothesis implies directly the dynamical map loses its dependence on $\boldsymbol{\rho_A}(t_0)$ and so becomes an UDM. For the other implication we start considering for hypothesis the existence of a UDM as defined in (3.26). Then, we suppose the original density matrix of the whole system A+B to be of the form:

$$\boldsymbol{\rho_{AB}}(t_0) = \boldsymbol{\rho_A}(t_0) \otimes |\psi_B\rangle \langle\psi_B| \tag{3.28}$$

for each possible $\boldsymbol{\rho_A}(t_0)$; and try to verify if it is allowed.
(If we consider $\boldsymbol{\rho_B}(t_0)$ as a mixed density matrix we can proceed with the same treatment, just by the use of the ensemble representation of any mixed operator).
To prove that let us look at the definition of the operators $\{\boldsymbol{K_\alpha}(t_1,t_0)\}_\alpha$ given in (3.20) ; with this hypothesis, they become:

$$\boldsymbol{K_\alpha}(t_1,t_0) = \langle {\phi_\alpha}^B| \boldsymbol{U}(t_1,t_0) |\psi^B\rangle \tag{3.29}$$

where $\{\langle {\phi_\alpha}^B|\}_\alpha$ is a generic basis for H_B.
We can see that this condition fixes only a few elements of $\boldsymbol{U}(t_1,t_0)$ and the following equation shows also that it preserves the necessary unitary constraint, in fact:

$$\begin{aligned}\sum_\alpha {\boldsymbol{K_\alpha}}^\dagger(t_1,t_0)\boldsymbol{K_\alpha}(t_1,t_0) &= \sum_\alpha \langle\psi^B| \boldsymbol{U}^\dagger(t_1,t_0) |{\phi_\alpha}^B\rangle \langle {\phi_\alpha}^B| \boldsymbol{U}(t_1,t_0) |\psi^B\rangle \\ &= \langle\psi^B| \boldsymbol{U}^\dagger(t_1,t_0)\boldsymbol{U}(t_1,t_0) |\psi^B\rangle \\ &= \boldsymbol{1_A}\end{aligned} \tag{3.30}$$

So, it is always possible, for an UDM of this type, to define an $\boldsymbol{U}(t_1,t_0)$ such that the original density matrix of the whole system can be: $\boldsymbol{\rho_{AB}}(t_0) = \boldsymbol{\rho_A}(t_0) \otimes |\psi_B\rangle \langle\psi_B|$ □

We can summarize and clarify what we have said up to now by means of the following diagram:

$$\begin{array}{ccc} \boldsymbol{\rho_A}(t_0) \otimes \boldsymbol{\rho_B}(t_0) & \xrightarrow{\boldsymbol{U}(t_1,t_0)} & \boldsymbol{\rho_{AB}}(t_1) \\ {\scriptstyle tr_B}\downarrow & & \downarrow{\scriptstyle tr_B} \\ \boldsymbol{\rho_A}(t_0) & \xrightarrow[\clubsuit_{(t_1,t_0)}]{} & \boldsymbol{\rho_A}(t_1) \end{array}$$

It is now the moment to explain why UDMs are so important in physic and what kind of properties they have.

Linearity and complete positivity

As we already said, UDMs are requested to be linear and positive. Linearity is fundamental because of the possibility of treating any mixed density matrix that evolves

in time as the sum of the evolutions of its pure parts in the ensemble interpretation. Positivity is fundamental and necessary to preserve the positivity condition on any density matrix. However, as we will see, a normal concept of positivity for an operator is really not sufficient for a UDM. One has to resort to the new concept of *complete positivity*. To make clear this argument we present a mental experiment. Suppose that, in addition of our usual two systems A and B, another system C is involved in our discussion. The system C is completely invisible to our eyes, that means that it never interacts neither with A nor with B. So, we can write the evolution operator of the whole system (A+B+C) as:

$$\boldsymbol{U}_{ABC} = \boldsymbol{U}_{AB} \otimes \boldsymbol{U}_{C} \tag{3.31}$$

Thus, considering at t_0 an initial density matrix of the whole system $\boldsymbol{\rho_{ABC}}(t_0)$ it evolves until time t_1 like:

$$\boldsymbol{\rho_{ABC}}(t_1) = \boldsymbol{U}_{AB}(t_1,t_0) \otimes \boldsymbol{U}_{C}(t_1,t_0)\boldsymbol{\rho_{ABC}}(t_0)\boldsymbol{U}^{\dagger}{}_{AB}(t_1,t_0) \otimes \boldsymbol{U}^{\dagger}{}_{C}(t_1,t_0) \tag{3.32}$$

We are now interested in the density operator concerning only the subsystem A+B, then we perform on this last expression a trace on the space H_C, getting:

$$\begin{aligned} \boldsymbol{\rho_{AB}}(t_1) &= tr_C[\boldsymbol{U}_{AB}(t_1,t_0)\boldsymbol{U}_{C}(t_1,t_0)\boldsymbol{\rho_{ABC}}(t_0)\boldsymbol{U}^{\dagger}{}_{AB}(t_1,t_0)\boldsymbol{U}^{\dagger}{}_{C}(t_1,t_0)] \\ &= \boldsymbol{U}_{AB}(t_1,t_0) tr_C[\boldsymbol{\rho_{AB}}(t_0)]\boldsymbol{U}^{\dagger}{}_{AB}(t_1,t_0) \end{aligned} \tag{3.33}$$

Therefore, as expected, we can see that the presence of the system C does not perturb the dynamics in any way.
Nevertheless, if we suppose that the evolution of the density matrix $\boldsymbol{\rho_{AB}}(t_0)$ is ruled by a UDM, then we know from theorem 3.2.3 that the initial density operator must be in the form:

$$\boldsymbol{\rho_{AB}}(t_0) = \boldsymbol{\rho_{A}}(t_0) \otimes \boldsymbol{\rho_{B}}(t_0) \quad \forall \boldsymbol{\rho_{A}}(t_0) \tag{3.34}$$

and the introduction of the system C unavoidably transforms the last equation in:

$$\boldsymbol{\rho_{ABC}}(t_0) = \boldsymbol{\rho_{AC}}(t_0) \otimes \boldsymbol{\rho_{B}}(t_0) \quad \forall \boldsymbol{\rho_{AC}}(t_0) \tag{3.35}$$

We can now study the evolution of the system A+C performing the trace of the last equation on H_B:

$$\begin{aligned} \boldsymbol{\rho_{AC}}(t_1) &= tr_B[\boldsymbol{U}_{AB}(t_1,t_0)\boldsymbol{U}_{C}(t_1,t_0)\boldsymbol{\rho_{AC}}(t_0)\boldsymbol{\rho_{B}}(t_0)\boldsymbol{U}^{\dagger}{}_{AB}(t_1,t_0)\boldsymbol{U}^{\dagger}{}_{C}(t_1,t_0)] \\ &= \sum_{\alpha} \boldsymbol{K}_{\alpha}(t_1,t_0)\boldsymbol{U}_{C}(t_1,t_0)\boldsymbol{\rho_{AC}}(t_0)\boldsymbol{U}^{\dagger}{}_{C}(t_1,t_0)\boldsymbol{K}^{\dagger}{}_{\alpha}(t_1,t_0) \end{aligned} \tag{3.36}$$

where we have used the spectral decomposition of $\boldsymbol{\rho_{B}}(t_0)$ and then the $\{\boldsymbol{K}_{\alpha}(t_1,t_0)\}_{\alpha}$ in the same usual way of equations (3.19) and (3.20).
Looking to this last expression it becomes clear that:

$$\boldsymbol{\rho_{AC}}(t_1) = \clubsuit^{A}{}_{(t_1,t_0)} \otimes \uplus_{(t_1,t_0)}[\boldsymbol{\rho_{AC}}(t_0)] \tag{3.37}$$

where $\uplus_{(t_1,t_0)}[\cdot] \equiv \boldsymbol{U}_C(t_1,t_0)[\cdot]\boldsymbol{U}^{\dagger}{}_C(t_1,t_0)$ is the normal unitary evolution of the system C and $\clubsuit^A{}_{(t_1,t_0)}$ is an UDM on the system A.

Making some considerations about this last equation we can conclude that the temporal evolution of the two subsystems A and C happens undisturbed, as a tensor product, independently from the dimensions of H_C and from the characteristics of C . However, the requirement of positivity must hold if we want that $\boldsymbol{\rho_{AC}}(t_1)$ can be a density matrix. Then the following operator:

$$\clubsuit^A{}_{(t_1,t_0)} \otimes \uplus_{(t_1,t_0)} = [\clubsuit^A{}_{(t_1,t_0)} \otimes \mathbf{1}_A] \otimes [\mathbf{1}_C \otimes \uplus_{(t_1,t_0)}] \tag{3.38}$$

must be positive; but because $\mathbf{1}_C \otimes \uplus_{(t_1,t_0)}$ is a unitary operator, what we are asking for is the positivity of:

$$\clubsuit^A{}_{(t_1,t_0)} \otimes \mathbf{1}_A \tag{3.39}$$

This is what we call the requirement of *complete positivity*, and it is really much stronger than the usual positivity. Since we can never dismiss the possible existence of some extra system C, out of our control, a UDM must always be completely positive. Therefore, a formal definition of a UDM is a (trace-preserving) linear map which is completely positive. The reverse statement, that claims that any possible complete positive linear map can always be written as (3.26) has been proved by Kraus in [7]. That is why sometimes the expansion of a generic UDM in the form of (3.26) is called *Kraus decomposition* and the operators $\{\boldsymbol{K_\alpha}(t_1,t_0)\}_\alpha$ are called *Kraus operators.*

UDMs as Contractions

In this section we will see an interesting property of UDMs that will allow us to distinguish the maps with a concrete physical meaning from the others. To discuss about this property, we have to consider a generic self-adjoint operator T acting on the Hilbert space H_A as an element of the Banach space $\mathbb{B}$ of the trace-class operators, with the norm is defined as:

$$||\boldsymbol{T}|| = tr[\sqrt{\boldsymbol{T}\boldsymbol{T}^{\dagger}}] = tr[\sqrt{\boldsymbol{T}^2}] \tag{3.40}$$

We know that, amongst these operators, the physical density matrices are the operators $\boldsymbol{\rho_A}$ which are also positive-semidefinite and with $tr(\boldsymbol{\rho_A}) = 1$, so that $||\boldsymbol{\rho_A}|| = 1$; we indicate this subspace of $\mathbb{B}$ with the symbol $\mathbb{B}^+{}_1$.
So then, to look for dynamic maps we can focus our attention on the dual space $\mathbb{B}^*$ composed by all the possible linear applications of the kind $\clubsuit(\mathbb{B}) \to \mathbb{B}$.
This space is a Banach space too, with the induced norm:

$$||\clubsuit|| = sup_{\substack{x\in\mathbb{B} \\ ||x||\neq 0}} \frac{||\clubsuit(x)||}{x} = sup_{\substack{x\in\mathbb{B} \\ ||x||=1}} ||\clubsuit(x)|| \tag{3.41}$$

In particular we are interested in the ones that send the subspace $\mathbb{B}^+{}_1$ into itself; indeed these ones preserve the physics of the system, as a UMD has to do. Before introducing the main theorem that enforces this requirement, we recall the definition of a *contraction* on a Banach space.

Definition 3.1 (contraction)**.** A linear operator $\clubsuit$ on a Banach space $\mathbb{B}$, is said to be a contraction if:

$$||\clubsuit(x)|| \leq ||x|| \quad \forall x \in \mathbb{B} \tag{3.42}$$

Then we can present the important theorem:

Theorem 3.2.4. *A linear map $\clubsuit \in \mathbb{B}^*$ leaves invariant $\mathbb{B}^+{}_1$ if and only if it preserves the trace and is a contraction on $\mathbb{B}$, so that*

$$||\clubsuit|| \leq 1. \tag{3.43}$$

Proof. If $\clubsuit \in \mathbb{B}^*$ leaves invariant $\mathbb{B}^+{}_1$, it means that it conserves the trace of its arguments. Moreover, from the definition of the trace norm (3.40), we can write:

$$||\clubsuit(x)|| = ||x|| \quad \forall x | x \text{ is positive} \in \mathbb{B} \tag{3.44}$$

that proves that, for the positive operators, $\clubsuit$ is a contraction. To prove definitively the first implication of the theorem we have now to consider the action of $\clubsuit$ on negative operators. Anyone of these operators, that we call σ, can be always described by:

$$\sigma = \sigma^+ - \sigma^- \tag{3.45}$$

where we have:

$$\begin{cases} \sigma^+ = \sum_i \lambda_i \left|\psi_i\right\rangle \left\langle\psi_i\right| \quad with\ \lambda_i \geq 0 \\ \sigma^- = -\sum_j \lambda_j \left|\psi_j\right\rangle \left\langle\psi_j\right| \quad with\ \lambda_j < 0 \end{cases} \tag{3.46}$$

$\{|\psi_\alpha\rangle\}_{\alpha=i+j}$ is an orthogonal basis for the Hilbert space H_A considered for the definition of $\mathbb{B}$.

We note that both the operators σ^+ and σ^- are positive defined. Moreover, because of the definition of the norm trace and the orthogonality of $\{|\psi_\alpha\rangle\}_\alpha$ we obtain:

$$||\sigma|| = \sum_\alpha |\lambda_\alpha| = ||\sigma^+|| + ||\sigma^-|| \tag{3.47}$$

and these considerations allow us to achieve the following result:

$$||\clubsuit(\sigma)|| = ||\clubsuit(\sigma^+ - \sigma^-)|| \leq \clubsuit(||\sigma^+||) + \clubsuit(||\sigma^-||) = ||\sigma^+|| + ||\sigma^-|| = ||\sigma|| \tag{3.48}$$

that reproduces the definition of a contraction. Then we have proved the first implication of the theorem.

Conversely, if we assume that $\clubsuit$ is a contraction and preserves the trace, then for $\rho \in \mathbb{B}^+{}_1$ we have the following chain of inequalities:

$$||\rho|| = tr(\rho) = tr(\clubsuit(\rho)) \leq ||\clubsuit(\rho)|| \leq ||\rho|| \tag{3.49}$$

So it follows that the inequalities become equalities : $||\clubsuit(\rho)|| = ||\rho|| = tr(\clubsuit(\rho)) = ||\clubsuit(\rho)||$. Since $\rho \in \mathbb{B}^+{}_1$ if and only if $||\rho|| = tr(\rho) = 1$, the last equality implies that $\clubsuit(\rho) \in \mathbb{B}^+{}_1$ for any $\rho \in \mathbb{B}^+{}_1$. □

Inverse of a UDM

Another important question that one can ask about UDMs is the following: is that kind of time evolution reversible ? We are wondering if, given an UDM like $\clubsuit_{(t_1,t_0)}$ is it always possible to find another UDM $\clubsuit_{(t_0,t_1)}$ of the same physical system A such that:

$$\clubsuit_{(t_1,t_0)}\clubsuit_{(t_0,t_1)} = \clubsuit_{(t_1,t_0)}\clubsuit_{(t_1,t_0)}{}^{-1} = \mathbf{1}_{\boldsymbol{A}} \tag{3.50}$$

The answer of such a doubt is provided by the following theorem:

Theorem 3.2.5. *If $\clubsuit_{(t_1,t_0)}$ is an UDM, then it can be inverted by another UDM if and only if it is unitary.*

Proof. To start the demonstration we recall what we have said in the previous section: any UDM has to be a contraction on the Banach space of trace-class operators defined over the Hilbert space H_A. So, given $\clubsuit_{(t_1,t_0)}$, if it really exists its inverse, we must have:

$$||\boldsymbol{\sigma}|| = ||\clubsuit_{(t_1,t_0)}{}^{-1}\clubsuit_{(t_1,t_0)}(\boldsymbol{\sigma})|| \leq ||\clubsuit_{(t_1,t_0)}(\boldsymbol{\sigma})|| \leq ||\boldsymbol{\sigma}|| \tag{3.51}$$

where $\boldsymbol{\sigma}$ is a generic element $\in \mathbb{B}$.
As we can see from this chain of inequalities , it follows that, for any invertible UDM one finds:

$$||\clubsuit_{(t_1,t_0)}(\boldsymbol{\sigma})|| = ||\boldsymbol{\sigma}|| \tag{3.52}$$

From this condition it is easy to prove also that an invertible UDM must transform pure matrices in pure matrices. Suppose that $|\psi\rangle\langle\psi|$ is a pure density operator and $\clubsuit_{(t_1,t_0)}(|\psi\rangle\langle\psi|)$ is not. Then it is always possible to write $\clubsuit_{(t_1,t_0)}(|\psi\rangle\langle\psi|)$ as a convex combination of pure matrices:

$$\clubsuit_{(t_1,t_0)}(|\psi\rangle\langle\psi|) = p\ \ \boldsymbol{\rho}_1 + (1-p)\boldsymbol{\rho}_2 \tag{3.53}$$

but then considering the inverse of the above expression, we would get:

$$|\psi\rangle\langle\psi| = p\ \ \clubsuit_{(t_1,t_0)}{}^{-1}(\boldsymbol{\rho}_1) + (1-p)\clubsuit_{(t_1,t_0)}{}^{-1}(\boldsymbol{\rho}_2) \tag{3.54}$$

Since $\clubsuit_{(t_1,t_0)}$ is considered to be a bijective operator and $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ are pure, we would have $\clubsuit_{(t_1,t_0)}{}^{-1}(\boldsymbol{\rho}_1) \neq \clubsuit_{(t_1,t_0)}{}^{-1}(\boldsymbol{\rho}_2)$ and then we would reach an expression of the pure density operator $|\psi\rangle\langle\psi|$ as a convex combination of other density matrices. We know that this situation is unacceptable. So the hypothesis we have made is wrong, and pure matrices are always sent in other pure ones by an invertible UDM. Now we take a specific expression of an element $\boldsymbol{\sigma}$ of $\mathbb{B}$:

$$\boldsymbol{\sigma} = \frac{1}{2}(|\psi_1\rangle\langle\psi_1| - |\psi_2\rangle\langle\psi_2|) \tag{3.55}$$

where $|\psi_1\rangle\langle\psi_1|$ and $|\psi_2\rangle\langle\psi_2|$ are two arbitrary pure density operators.
Because we have just proved that the invertible $\clubsuit_{(t_1,t_0)}$ transforms pure operators in pure ones, we can write:

$$\clubsuit_{(t_1,t_0)}(\boldsymbol{\sigma}) = \clubsuit_{(t_1,t_0)}[\frac{1}{2}(|\psi_1\rangle\langle\psi_1| - |\psi_2\rangle\langle\psi_2|)] = \frac{1}{2}(|\tilde{\psi_1}\rangle\langle\tilde{\psi_1}| - |\tilde{\psi_2}\rangle\langle\tilde{\psi_2}|) \tag{3.56}$$

where $|\tilde{\psi_1}\rangle\langle\tilde{\psi_1}| = \clubsuit_{(t_1,t_0)}(|\psi_1\rangle\langle\psi_1|)$ and $|\tilde{\psi_2}\rangle\langle\tilde{\psi_2}| = \clubsuit_{(t_1,t_0)}(|\psi_2\rangle\langle\psi_2|)$ are two pure density operators too.

The eigenvalues equation for $\boldsymbol{\sigma}$ reads:

$$\boldsymbol{\sigma}(\alpha|\psi_1\rangle + \beta|\psi_2\rangle) = \lambda(\alpha|\psi_1\rangle + \beta|\psi_2\rangle) \tag{3.57}$$

where α and β are generic complex coefficients.
Replacing $\boldsymbol{\sigma}$ with the expression of (3.55) and projecting this last equation into the two subspaces spanned by $|\psi_1\rangle$ and $|\psi_2\rangle$ we get respectively :

$$\begin{cases} \lambda = +\frac{1}{2}\sqrt{1 - |\langle\psi_2|\psi_1\rangle|^2} \\ \lambda = -\frac{1}{2}\sqrt{1 - |\langle\psi_2|\psi_1\rangle|^2} \end{cases} \tag{3.58}$$

At this point it is immediate to calculate the trace norm of this operator:

$$||\boldsymbol{\sigma}|| = \sum_i |\lambda_i| = \sqrt{1 - |\langle\psi_2|\psi_1\rangle|^2} \tag{3.59}$$

Thus, using equation (3.52), valid for any element of $\mathbb{B}$, we get:

$$||\clubsuit_{(t_1,t_0)}(\boldsymbol{\sigma})|| = \sqrt{1 - |\langle\tilde{\psi_2}|\tilde{\psi_1}\rangle|^2} = ||\boldsymbol{\sigma}|| = \sqrt{1 - |\langle\psi_2|\psi_1\rangle|^2} \tag{3.60}$$

Therefore:

$$|\langle\tilde{\psi_2}|\tilde{\psi_1}\rangle| = |\langle\psi_2|\psi_1\rangle| \tag{3.61}$$

Finally, exploiting the Wigner's theorem [2], we can affirm that a transformation that satisfies the above condition must be of the form:

$$\clubsuit_{(t_1,t_0)}(\boldsymbol{\sigma}) = \boldsymbol{V}\boldsymbol{\sigma}\boldsymbol{V}^\dagger \tag{3.62}$$

where $\boldsymbol{V}$ has to be a unitary or anti-unitary operator. Because the anti-unitary case is incompatible with the complete positive condition, only the unitary one remains. Extending this type of demonstration to a $\boldsymbol{\sigma}$ composed by more than two pure density matrices is really straightforward, and requires the same treatment. □

Anyway, the exclusive use of unitary UDM is really unnecessary; the reversible maps represent indeed the most conventional type of evolution, and often the most interesting and realistic properties are involved when one considers the not unitary case.

3.2.3 POVM as a UDM

The last property we presented allows us to connect the concepts of POVM and UDM. As we have seen, these two mathematical instruments become necessary in the study of open systems. We start by considering the time evolution of the state $|\psi^A\rangle$ included in an initial state $|\psi^A\rangle \otimes |\phi^B\rangle$ of the open system A+B:

$$|\psi^A\rangle |\phi^B\rangle \to_t \sum_\alpha \boldsymbol{K}_\alpha |\psi^A\rangle |\alpha^B\rangle \tag{3.63}$$

where $\{|\alpha^B\rangle\}_\alpha$ is an orthonormal basis of the subsystem B and the $\{\boldsymbol{K}_\alpha\}_\alpha$ are the Kraus operators governing the transformations of density matrices for the subsystem A in such a unitary evolution of the whole system.

This is a correct way of writing an evolution for the states of subsystem A. Indeed, replacing here the expression of the $\{\boldsymbol{K}_\alpha\}_\alpha$ of equation (3.29), (we know, by means of theorem 3.2.3, that (3.29) is the right definition for the Kraus operators of any UDM) we obtain:

$$\begin{aligned} |\psi^A\rangle |\phi^B\rangle &\to_t \sum_\alpha \langle\alpha^B| \left(\boldsymbol{U_{AB}}(t_1,t_0)[|\phi^B\rangle \otimes |\psi^A\rangle]\right) |\alpha^B\rangle \\ &= tr_B\Big(\boldsymbol{U_{AB}}(t_1,t_0)[|\phi^B\rangle \otimes |\psi^A\rangle]\Big) \end{aligned} \tag{3.64}$$

Thus, once we reach the state (3.63) , we can perform an orthogonal measure onto the basis $\{|\alpha^B\rangle\}$ on the system B. From this measure, we can get the outcome α with probability $p(\alpha)$:

$$p(\alpha) = \langle\psi^A| \left(\boldsymbol{K}_\alpha{}^\dagger \boldsymbol{K}_\alpha\right) |\psi^A\rangle \tag{3.65}$$

and this last equation can be rearranged in the form:

$$p(\alpha) = tr_A(\boldsymbol{A}_\alpha \boldsymbol{\rho_A}) \tag{3.66}$$

where $\boldsymbol{\rho_A} = |\psi^A\rangle \langle\psi^A|$ and $\boldsymbol{A}_\alpha = \boldsymbol{K}_\alpha{}^\dagger \boldsymbol{K}_\alpha$.
We can now note that from the definition of the $\{\boldsymbol{K}_\alpha\}_\alpha$ it follows that the $\{\boldsymbol{A}_\alpha\}_\alpha$ do not necessarily commute for different values of α. Therefore, equation (3.66) could be the result of a POVM, as we can see comparing the latter with equation (2.4). Indeed, the $\{\boldsymbol{A}_\alpha\}$'s are evidently positive, and the necessary condition for a POVM:

$$\sum_\alpha \boldsymbol{A}_\alpha = \boldsymbol{1_A} \tag{3.67}$$

is provided by the Kraus normalization condition:

$$\sum_\alpha \boldsymbol{K}_\alpha{}^\dagger \boldsymbol{K}_\alpha = \boldsymbol{1_A} \tag{3.68}$$

So, we have just realized a POVM that, according with the equation (2.11) of chapter 2, can modify the starting density matrix $\boldsymbol{\rho_A}$ once that the measurement is

performed, in the following way:

$$\boldsymbol{\rho_A} \overset{\rightarrow}{\text{measuring}} \boldsymbol{\rho_{APOVM}} = \sum_{\alpha} \sqrt{\boldsymbol{A_\alpha}} \boldsymbol{\rho_A} \sqrt{\boldsymbol{A_\alpha}} \tag{3.69}$$

but the transformation above of the density matrix $\boldsymbol{\rho_A}$ is really the same that we could reach with the help of Kraus operators $\{\boldsymbol{K}_\alpha\}_\alpha$:

$$\boldsymbol{\rho_A} \overset{\rightarrow}{\text{time evolving}} \clubsuit_{(t_0,t)}(\boldsymbol{\rho_A}) = \sum_{\alpha} \boldsymbol{K_\alpha}(t_0,t) \boldsymbol{\rho_A} \boldsymbol{K_\alpha}^\dagger(t_0,t) \tag{3.70}$$

Therefore, at the end we can claim that every POVM has a *unitary representation*, which means that one can always obtain the POVM on the subsystem A by making evolve in time the whole system with a unitary evolution and then performing an orthogonal measurement on B. We can summarize the previous statement saying that, as we can directly see from (3.69) and (3.70), the act of a POVM on A always corresponds to the evolution ruled by a certain UDM on this subsystem.

3.3 Lindblad equation

At this point we know a lot of details about time evolution of density matrices in open systems; but a key question is still unresolved: is it possible to formalize this evolution with the help of a differential equation? What is the form of such differential equation? Trying to answer to these questions brings out some critical problems hidden inside the mathematical structure and definition of UDMs. We know that, between two time coordinates $t_2 \geq t_1$, while the whole system is transforming in a usual unitary way, the starting density matrix $\boldsymbol{\rho}_A(t_1)$ of the subsystem A evolves as:

$$\boldsymbol{\rho}_A(t_2) = \clubsuit_{(t_2,t_1)}(\boldsymbol{\rho}_A(t_1)) = \sum_{\alpha} \boldsymbol{K}_\alpha(t_2,t_1) \boldsymbol{\rho}_A(t_1) \boldsymbol{K}^\dagger{}_\alpha(t_2,t_1) \tag{3.71}$$

and we have also just learned, by means of theorem (3.2.3), that this type of evolution is granted only if, at initial time t_1, the density operator of the whole system A+B is in the form of a tensor product like:

$$\boldsymbol{\rho_{AB}}(t_1) = \boldsymbol{\rho_A}(t_1) \otimes \boldsymbol{\rho_B}(t_1) \tag{3.72}$$

where $\boldsymbol{\rho_B}(t_1)$ is fixed for any possible $\boldsymbol{\rho_A}(t_1)$.

Up to here, there is no evidence of a problem. Nevertheless, looking more carefully at the process of evolution, one may ask himself: what happens if the state of the system at time t_1 is the result of another, previous, evolution started at $t_0 \leq t_1$? That is the most appropriate doubt that may come out when one is trying to describe correctly time evolution. If this hypothesis is true, and we think that also the first evolution was ruled by an UDM, then we have to assume, in the same way as

we have done before, that $\boldsymbol{\rho_{AB}}(t_0)$ was in the form of equation (3.72).
Therefore we have to write, for the density operator of the whole system at time t_1:

$$\boldsymbol{\rho_{AB}}(t_1) = \boldsymbol{U}(t_1,t_0)\boldsymbol{\rho_{AB}}(t_0)\boldsymbol{U}^\dagger(t_1,t_0) = \boldsymbol{U}(t_1,t_0)\big[\boldsymbol{\rho_A}(t_0)\otimes\boldsymbol{\rho_B}(t_0)\big]\boldsymbol{U}^\dagger(t_1,t_0) \quad (3.73)$$

and so emerges clearly the problem: from the above equation there is no warranty for this last density operator at time t_1 to be in the correct form of (3.72). Thus, at this time, it falls the main requirement of theorem (3.2.3). This fact means that, starting from an UDM type of evolution (on our example, $t_0 \to t_1$), in general, is not always permitted to perform another one immediately consecutive ($t_1 \to t_2$). Notice that, instead, what is always allowed is the direct evolution from $t_0 \to t_2$, governed by an unique UDM $\clubsuit_{(t_2,t_0)}$; indeed, for this type of evolution, the initial condition (3.72) is fully satisfied.

Therefore, as a difference with the dynamics of isolated systems, it is clear that these difficulties arising from the continuity of time for UDMs makes impossible to formulate the general dynamics of open quantum systems by means of differential equations which generate families of UDMs. That was the mathematical explanation, but it is possible to justify the arising of these troubles also in a more physical, intuitive and informal way. Moreover, spending some words on this argument can be really instructive to understand also the formal solution that we will adopt to solve these problems. In fact we can claim that at the origin of the problem is the fact that the two subsystems develop correlations $\boldsymbol{\rho}_{corr}$ during the evolution. The key point is that, differently from what happens in isolated systems, where the Shrödinger equation needs only the initial condition to be solved, here, the environment (or, equivalently, the other system B) retains a memory of informations for a while, and can transfer it back to subsystem A. This implies the leak of linearity and the impossibility of find a general equation.
Anyway, to untie these delicate and overwhelming complications, we can make a reasonable approximation, that allows us to manage and control the general dynamic in a linear and useful way. Once that we have understood the root of the problem, we can fix it assuming that the *memory* of the environment is sufficiently short to forget all the past informations before they can flow again into the dynamic. Heuristically, we are limiting $\boldsymbol{\rho}_{corr}$ to be insignificant at any time of the evolution. Mathematically speaking, this assumption goes under the name of *Markovian evolution*. A more detailed and formal treatment of this argument can be found in the appendix , at the end of this work. Anyway, if we embrace this approximation, we recover the composition law for the UDM:

$$\clubsuit_{(t_2,t_0)} = \clubsuit_{(t_2,t_1)}\clubsuit_{(t_1,t_0)} \quad \forall t_2 \geq t_1 \geq t_0 \quad (3.74)$$

and this condition directly leads us to the possibility of determining the $\clubsuit_{(t,t')}$ by resolving a differential equation, often called *master equation*.

The derivation of such an equation that we present on this section is more physical; a more mathematical approach is given in the appendix.
So we start considering a UDM-evolution of a density matrix $\boldsymbol{\rho}$ expressed in its Kraus decomposition:

$$\boldsymbol{\rho}(t) = \sum_{\alpha} \boldsymbol{K}_{\alpha}(t)\boldsymbol{\rho}(0){\boldsymbol{K}^{\dagger}}_{\alpha}(t) \tag{3.75}$$

Now we consider only the first order,that means:

$$\boldsymbol{\rho}(t) = \boldsymbol{\rho}(0) + O(dt) \tag{3.76}$$

Then, looking at equation (3.75), we can try to make a useful choice of the Kraus operators that satisfy the above equality. We notice that the following assumptions:

$$\begin{aligned} \boldsymbol{K}_0 &\equiv \mathbf{1} + (-i\boldsymbol{H} + \boldsymbol{R})dt \\ \boldsymbol{K}_{\alpha} &\equiv \sqrt{dt}\boldsymbol{L}_{\alpha} \quad with \;\; \alpha > 0 \end{aligned} \tag{3.77}$$

where $\boldsymbol{R}$ and $\boldsymbol{H}$ (that is the canonical Hamiltonian of the system considered) are both Hermitian operators, are a perfect consistent choices.
We see that the operator $\boldsymbol{R}$ is completely determined by the Kraus normalization condition; indeed, with this choice of Kraus operators we get:

$$\mathbf{1} = \sum_{\alpha} \boldsymbol{K}_{\alpha}(t){\boldsymbol{K}^{\dagger}}_{\alpha}(t) = \mathbf{1} + dt\big(2\boldsymbol{R} + \sum_{\alpha>0} {\boldsymbol{L}^{\dagger}}_{\alpha}\boldsymbol{L}_{\alpha}\big) \quad \forall t \tag{3.78}$$

and then :

$$\boldsymbol{R} = -\frac{1}{2}\sum_{\alpha>0} {\boldsymbol{L}^{\dagger}}_{\alpha}\boldsymbol{L}_{\alpha} \tag{3.79}$$

at this point we can substitute this expression of the Kraus operators into equation (3.75) and, by writing $\boldsymbol{\rho}(t) = \boldsymbol{\rho}(0) + \dot{\boldsymbol{\rho}}dt$, we can identify the derivate in time of the density matrix evolution with:

$$\dot{\boldsymbol{\rho}} \equiv \boldsymbol{\mathcal{L}}_t(\boldsymbol{\rho}) = -i[\boldsymbol{H}, \boldsymbol{\rho}] + \sum_{\alpha>0} \big(\boldsymbol{L}_{\alpha}\boldsymbol{\rho}{\boldsymbol{L}^{\dagger}}_{\alpha} - \frac{1}{2}\{{\boldsymbol{L}^{\dagger}}_{\alpha}\boldsymbol{L}_{\alpha}, \boldsymbol{\rho}\}\big) \tag{3.80}$$

This is the *master equation* of the dynamic, and the symbol $\boldsymbol{\mathcal{L}}_t$ is called *Linbladian*. Looking at the equation it is fairly clear that $tr(\boldsymbol{\mathcal{L}}_t)(\boldsymbol{\rho}) = 0$, so that the trace of the density operators is always preserved. The complete positivity is less evident, but it follows directly from the Kraus representation that has led to the final expression. Thus, the process described by this master equation correctly transforms density matrices in density matrices, and it expresses a UDM. We can try to enter a bit more deeply into the physical meaning of this master equation. The first term, $-i[\boldsymbol{H}, \boldsymbol{\rho}]$ expresses the canonical unitary evolution of operators, completely described by the initial condition $\boldsymbol{\rho}$ and by the Hamiltonian of the system involved. So, the presence of all the other terms that we find in the sum depends directly from the interaction that our subsystem develops with the environment during the evolution. In particular, each contribution $\boldsymbol{L}_{\alpha}\boldsymbol{\rho}{\boldsymbol{L}^{\dagger}}_{\alpha}$ represents the probability that

occurs a *quantum jump* that leads the environment from the initial fixed state to the state $|\alpha\rangle$ of a certain orthonormal basis. These operators, $\boldsymbol{L}_\alpha$, that here are time-independent because of our cutting off at the first order, are often called *Lindblad operators*. Notice that here it is crucial to assume the Markovian approximation, that makes sure that this probability is proportional to dt, so that it can increase *linearly* in time. The final term $\frac{1}{2}\{\boldsymbol{L}^\dagger{}_\alpha \boldsymbol{L}_\alpha, \boldsymbol{\rho}\}$ is needed to keep satisfied the normalization condition in case of no quantum jumps occurs in the environment.

Commonly , a family of UDMs that satisfies the Markovian requirement in the form of equation (3.74) is called a *Quantum Dynamical Semigroup*. The formal definition is:

Definition 3.2 (Quantum Dynamical Semigroup)**.** A Quantum Dynamical Semigroup is a family of linear maps $\{\clubsuit_t,\ t \geq 0\}$ such that:

1. $\clubsuit_t$ is a UDM

2. $\clubsuit_s \circ \clubsuit_t = \clubsuit_{t+s}$

3. $Tr[\clubsuit_t(\boldsymbol{\rho})\boldsymbol{A}]$ is a continuous function of t for any density matrix $\boldsymbol{\rho}$ of the system (without the environment) and for any hermitian and bounded operator $\boldsymbol{A}$ defined on the Hilbert space of the system.

Finally, now that we have presented some of the fundamentals properties of open systems, concerning measuring and time evolution, in the following sections we will apply this treatment to concrete cases, the world of 2x2 and 3x3 density operators , trying to individuate the correct and explicit expressions of the Kraus decomposition and the Lindbladian for this type of systems.
Nevertheless, before we can analyzing the concrete examples, we have to present some of the geometric implications that the universe of open systems induces. Indeed, observing by a geometrical point of view, it is possible to formalize the spaces of density matrices in a really instructive way, that, characterizing the possible orbits of the dynamics, could shed light on the deeper meanings and distinctions between unitary and open evolutions.

Chapter 4

Geometrical approach

Before getting into the discussion, we premise that this geometric overture, on our purpose, has the goal of clarify some features that will be used to study the concrete examples in the next following sections. Thus, the attention of the reader is directed towards the most important results about the description of the orbits, and the most formal mathematical details, as well as the demonstrations of the theorems, are here omitted, in order to simplify the presentation. A more detailed description of these arguments can be found in [8].

4.1 Projective Hilbert space

We begin to discuss what are the geometrical consequences of working with equivalence classes on a Hilbert space, not with vectors, which means working with rays. When we do that, we are considering the *Projective Hilbert space*.
In the following, we will work with finite dimensional Hilbert spaces, which are enough for our future discussions.

Definition 4.1 (Projective Hilbert space)**.** The Projective Hilbert space is defined as:

$$\mathcal{P}(\mathcal{H}) = \{[|\Psi\rangle] : |\Psi\rangle, |\Phi\rangle \in [|\Psi\rangle] \Leftrightarrow |\Psi\rangle = \lambda |\Phi\rangle \ ; \ |\Psi\rangle, |\Phi\rangle \in H_0, \lambda \in \mathbb{C}_0\} \quad (4.1)$$

where $\mathbb{C}_0$ is defined as $\mathbb{C} - \{0\}$.

This mathematical object is obtained starting from an Hilbert space H of dimensions n, quotienting it by the two multiplication operations, respectively by a norm of a complex number and by a phase, so that to implement the equivalence of the states belonging to the same ray. The procedure can be schematically illustrated in

the following way:

$$\begin{array}{ccc} \mathbb{R}_+ & \longrightarrow & H_0 = H - \{\vec{0}\} \\ & & \Big\downarrow \\ U(1) & \longrightarrow & \mathbb{S}^{2n-1} \\ & & \Big\downarrow \\ & & \mathcal{P}(\mathcal{H}) \end{array}$$

With this definition, the original Hilbert space gains a structure of a principal fiber bundle, with $\mathcal{P}(\mathcal{H})$ as a base space, and a typical fiber $\mathbb{C}_0 = U(1) \times \mathbb{R}_+$.

Definition 4.2 (Principal fiber bundle)**.** A principle fiber bundle is composed by:

1. a t-dimensional manifold T, called total space.

2. an m-dimensional manifold M, called base space.

3. a map $\pi : T \to M$, called projection, such that π is a surjective and continuous function.

4. a topological space $\mathcal{F}$, called typical fiber, such that $\mathcal{F}$ is omeomorphic to all the spaces $\pi^{-1}(m), \forall m \in M$, that are called fibers.

The base can also be part of the total space, i.e $T = M \times N$, and the latter is an important particular case of principle fiber.
The Hermitian structure of H allows the association of the equivalence class $[|\Psi\rangle]$ with the rank-one projector defined in the previous sections.

4.2 Kähler Manifold

This new framework carries the scaffolding of Quantum mechanics from the usual Hilbert space, that was a linear vector space, to a new geometric context that is no more linear, because is described by manifolds. Then, it becomes necessary to update all the Quantum quantities to a tensorial formulation. Even if this work has not the purpose of study all the details of this updating process, there is a concept we really need to define.

Definition 4.3 (Kähler Manifold)**.** Let K be a real, even dimensional, manifold on which we define a *complex structure*.
A complex structure J on a manifold is a map on the tangent bundle TK such that:

$$J : TK \to TK; \quad J_0(v) \equiv iv \quad \forall v \in TK \tag{4.2}$$

so that :

$$J^2 = -1_n \tag{4.3}$$

Then, let ω be a closed two-form, called *symplectic form*, satisfying the compatibility condition:

$$\omega(x, Jy) + \omega(Jx, y) = 0; \;\; \forall x, y \in TK \tag{4.4}$$

We notice that, starting from this two-form, we can define another one, that we call g :

$$g(x, y) \equiv \omega(x, Jy) \quad \rightarrow \omega(x, y) = -g(x, Jy) \;\; \forall x, y \in TK \tag{4.5}$$

It is easy to see that g is symmetric and, iff ω is non-degenerate, also g is. Thus, in this case, g represents a *metric* for TK.
The analog of equation (4.4) for g becomes:

$$g(x, Jy) + g(Jx, y) = 0; \;\; \forall x, y \in TK \tag{4.6}$$

Also , $J^2 = -1_n$ implies:

$$\omega(Jx, Jy) = \omega(Jx, y); \; g(Jx, Jy) = g(x, y) \; \forall x, y \in TK \tag{4.7}$$

A tensorial triple (g, J, ω) , with g a metric, that satisfies the above requirements, is said an *admissible triple*.
Finally, we can affirm that K is a *Käler Manifold* if it admits an admissible triple with a positive metric.

We have introduced this notion because it is possible to show that the space of pure density matrices, as well as the unitary orbits of each mixed matrix, are Kähler Manifold. Physically speaking, the Kähler manifold is the perfect setting in which one can operate, because of the possibility of defining a metrical vector space and a symplectic structure, from which can follow the definition of an Hamiltonian function and, consequently, the birth of a sensible physical system. Therefore, what we will do in the following sections of this chapter is to construct the conditions to affirm that the spaces of density matrices, pure or mixed ones, as well as their unitary orbits, present this type of structure.

4.3 Realification of the Hilbert space

We want now to study a procedure that allows to transform the original Hilbert space of n dimension into a $2n$ vector space that admits a structure of Kähler manifold. This procedure is ofter called *realification* , and the vector space obtained, $H_{\Re}$, that is also a Kähler manifold, is called the *realified* of H.
The latter is a real vector space that coincides with H as a group, (Abelian group under addition) but in which only multiplication by real scalars is allowed. Skipping the formalities, we can say that, if we choose a basis on $H \simeq \mathbb{C}^n$, like $\{\vec{e_1}, \vec{e_2}, ..., \vec{e_n}\}$

the corresponding basis in $H_{\Re}$ becomes: $\{\vec{e_1}, \vec{e_2}, ..., \vec{e_n}, i\vec{e_1}, i\vec{e_2}, ..., i\vec{e_n}\}$ and so $H \simeq \mathbb{R}^{2n}$.

Once the basis is chosen, the realification of a generic vector in H is given by:

$$\vec{x} = (u^k + iv^k)\vec{e}_k \in H \rightarrow \vec{y} = (u^1, u^2, .., u^n, v^1, v^2, .., v^n) \in H_{\Re} \tag{4.8}$$

Of course, this transformation involves also all the observables and operators. Briefly, we can say that if , chosen a basis, a generic Hermitian operator $\boldsymbol{A}$ acting on H can be represented as:

$$A = \alpha + i\beta \tag{4.9}$$

where α and β are real $n \times n$ matrices; then we can present the corresponding realified operator $\boldsymbol{A_{\Re}}$ as the $2n \times 2n$ matrix:

$$A_{\Re} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \tag{4.10}$$

The properties $(\boldsymbol{A} + \boldsymbol{B})_{\Re} = \boldsymbol{A_{\Re}} + \boldsymbol{B_{\Re}}$ and $(\boldsymbol{AB})_{\Re} = \boldsymbol{A_{\Re}B_{\Re}}$ can be easily checked, so that the set of all the linear operators that are realifications of complex operators on H is both a subspace of the vector space of all the linear operators on $H_{\Re}$ as well as a subalgebra of the associative algebra $\boldsymbol{\mathcal{GL}}(2n, \Re)$.

Finally, the operation of multiplication by the imaginary unit on H is interpreted by the linear operator $\boldsymbol{J}$, whose representation in any basis of $H_{\Re}$ is:

$$J = \begin{pmatrix} 0_{n\times n} & -1_{n\times n} \\ 1_{n\times n} & 0_{n\times n} \end{pmatrix} \tag{4.11}$$

with the property:

$$\boldsymbol{J}^2 = -\boldsymbol{1_{2n}} \tag{4.12}$$

The choice of this particular letter to symbolize this operator is not casual, in fact it establishes a complex structure on $H_{\Re}$, considered as a manifold. Making an explicit choice of coordinates, such that two generic vectors of H becomes: $\vec{x} = (u, v)$ and $\vec{y} = (u', v')$, we can operatively introduce the quantities:

$$g(x, y) = uu' + vv' \text{ and } \omega(x, y) = uv' - vu' \tag{4.13}$$

The two corresponding differential forms, g and ω, together with the already discussed complex structure, constitute an admissible triple for $H_{\Re}$, with a positive metric.

4.4 Geometry of pure states

Operating, as we said, in the projective Hilbert space, in this section we want to understand what are the properties of the space just composed by the pure density matrices and their unitary orbits. In the previous chapters we have defined a pure

density matrix as a ray, that could be associated with a rank-one projector, like $|\psi\rangle\langle\psi|$. Now, we need a way to characterize the set of these projectors with all their unitary orbits , that will be the space of pure density matrices. To give a geometrical description to the latter, we have to find a way to portray the unitary evolution, so we want to define the *co-adjoint action of unitary group U(n)*. We start recalling the definition of the Lie algebra $\mathfrak{u}(\mathfrak{n})$:

Definition 4.4. The Lie algebra of U(n) can be defined as $\mathfrak{u}(\mathfrak{n}) := T_eU(n)$, that is the tangent space to U(n) at the identity $e \in U(n)$, equipped with a bilinear operation $[\cdot,\cdot] : \mathfrak{u}(\mathfrak{n}) \times \mathfrak{u}(\mathfrak{n}) \to \mathfrak{u}(\mathfrak{n})$, called *Lie bracket* that satisfies the following axioms:

- Alternativity:

$$[x, x] = 0 \quad \forall x \in \mathfrak{u}(\mathfrak{n})$$

- Jacobi identity:

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \quad \forall x, y, z \in \mathfrak{u}(\mathfrak{n})$$

From this definition we immediately note that the dimension of $\mathfrak{u}(\mathfrak{n})$ is the same of U(n).
We now present a useful connection between $\mathfrak{u}(\mathfrak{n})$ and U(n) , claiming that, for every element $U \in U(n)$, it is always possible to write:

$$U = e^u \quad \text{for some } u \in \mathfrak{u}(\mathfrak{n}); \tag{4.14}$$

more details in theory of Lie groups , Lie algebras and their representations can be found in [14]. Therefore because any element $U \in U(n)$ acting on a Hilbert space is a unitary operator , from the equation above it follows that every element of the Lie algebra $\mathfrak{u}(\mathfrak{n})$ has to be anti-Hermitian.
We can now define the adjoint action of the group as the map:

Definition 4.5 (Adjoint action of U(n))**.**

$$\begin{aligned} Ad :& \mathfrak{u}(\mathfrak{n}) \to \mathfrak{u}(\mathfrak{n}) \\ & Ad(T) \equiv UTU^\dagger \end{aligned} \tag{4.15}$$

where $U \in U(n)$ and $T \in \mathfrak{u}(\mathfrak{n})$.

The co-adjoint action of $U(n)$ is the same concept, considered from the starting point of the dual space of the Lie algebra, $\mathfrak{u}(\mathfrak{n})^*$. This dual space represents the space of all the n-dimensional Hermitian operators, thus the co-adjoint action, that is the geometrical transposition of the concept of an unitary evolution, is the main object of our interest. Moreover, the adjoint action, as well as the co-adjoint action

of U(n), because of the properties of the group, are actions that preserve the metric and the symplectic form, so that the triple and the Kähler structure remain valid.
According to the pairing of spaces and dual spaces, between $\mathfrak{u}(\mathfrak{n})$ and $\mathfrak{u}(\mathfrak{n})^*$ there is an isomorphisfm mediated by a scalar product that connects anti-Hermitian operators with Hermitian ones:

$$\begin{aligned} &\langle A, T\rangle = \frac{i}{2} Tr(AT) \quad \forall A \in \mathfrak{u}(\mathfrak{n})^*, T \in \mathfrak{u}(\mathfrak{n}) \\ &\mathfrak{u}(\mathfrak{n}) \ni T \underset{iso}{\rightarrow} iT \in \mathfrak{u}(\mathfrak{n})^* \end{aligned} \tag{4.16}$$

With this isomorphisfm, it is easy to check that also $\mathfrak{u}(\mathfrak{n})^*$ is a Lie algebra, and that the co-adjoint action of the $U(n)$ group can be defined in the same way of the adjoint one:

Definition 4.6 (Co-adjoint action of U(n))**.**

$$\begin{aligned} CoAd :&\mathfrak{u}(\mathfrak{n})^* \rightarrow \mathfrak{u}(\mathfrak{n})^* \\ &CoAd(T) \equiv UTU^\dagger \end{aligned} \tag{4.17}$$

where $U \in U(n)$ and $T \in \mathfrak{u}(\mathfrak{n})^*$.

So, now that we know how to operate for implementing the concept of unitary evolution, we want to operate with the co-adjoint action of U(n) over the set of projectors of rank one, that we denote by $\mathcal{W}(\mathcal{H})^1 = \{|x\rangle \langle x| \, | \, |x\rangle \in H - \{\vec{0}\}\}$.
The co-adjoint action over $\mathcal{W}(\mathcal{H})^1$ foliates the latter into the spaces:

$$\mathcal{W}_r{}^1 = \{|x\rangle \langle x| \quad | \quad \langle x|x\rangle = r, |x\rangle \in H - \{\vec{0}\}\} \tag{4.18}$$

one for each possible value of the norm r.
This result is completely consistent, because we know that a unitary transformation does not change the norm of vectors.
Then, at the end, if we want to consider just the space of pure density matrices, inclusive of all their unitary orbits, we only have to put $r = 1$ and we get the space: $\mathcal{W}_1{}^1$.
At this final point, defining the correct tensorial quantities that could build up the appropriate admissible triple, is possible to demonstrate this last theorem, that here we just limit to present :

Theorem 4.4.1. *The space of pure states $\mathcal{W}_1{}^1$ is a Kähler manifold.*

The prove of the latter can be found in [8].
In particular, the Kähler structure of $\mathcal{W}_1{}^1$ comes from the structure of the realification $H_{\Re}$ of the original Hilbert space H.

4.5 Geometry of mixed states

Our goal in this section is to extend the treatment we have used for pure states to mixed ones. As we will see, the procedure is here much trickier.
We start by introducing the space of non-negatively defined operators $\mathcal{P}(\mathcal{H})_n$:

$$\mathcal{P}(\mathcal{H})_n = \{\rho | \rho = TT^\dagger \;\; T \in gl(H)\} \tag{4.19}$$

where the subscript n represents the dimension of H.
We note that: $\rho \in \mathcal{P}(\mathcal{H})_n \rightarrow \rho \in u(H)^*$.
From this definition, we can separate the spaces of mixed matrices with the same rank. That means defining:

$$\mathcal{P}(\mathcal{H})_n^{\;k} = \{\rho | \rho = TT^\dagger,\; Rank(\rho) = k,\; T \in gl(H)\} \tag{4.20}$$

Adding to these definition the condition on the trace gives back the corresponding spaces for density matrices:

$$\mathcal{D}(\mathcal{H})_n = \{\rho | \rho \in \mathcal{P}(\mathcal{H})_n | Tr(\rho) = 1\} \;\text{ and }\; \mathcal{D}(\mathcal{H})_n^{\;k} = \{\rho | \rho \in \mathcal{P}(\mathcal{H})_n^{\;k} \, | Tr(\rho) = 1\} \tag{4.21}$$

Notice that the space of density matrices $\mathcal{D}(\mathcal{H})_n$ is a convex cone in $u(H)^*$. In fact, every matrix in $\mathcal{D}(\mathcal{H})_n$ can be written as a convex combination of pure states, then the pure states are the extreme points of $\mathcal{D}(\mathcal{H})_n$. The details about these considerations can be founded in [1].
As we have done for the space of pure states, we now report a theorem that clarifies the geometrical nature of these spaces $\mathcal{D}(\mathcal{H})_n^{\;k}$ for each possible value of $k \leq n$:

Theorem 4.5.1. *The spaces $\mathcal{D}(\mathcal{H})_n^{\;k}$ of density states of rank $k \leq n$ are smooth and connected submanifolds in $u(H)^*$. Moreover the stratification into submanifolds of $\mathcal{D}(\mathcal{H})_n^{\;k}$ is maximal; i.e. every smooth curve in the space of Hermitian matrices which lies entirely in $\mathcal{D}(\mathcal{H})_n$ is such that:*

$$\gamma(t) \in \mathcal{D}(\mathcal{H})_n^{\;k} \;\; \rightarrow \;\; \gamma(t) \in T_{\gamma(t)} \mathcal{D}(\mathcal{H})_n^{\;k} \tag{4.22}$$

The interested reader can find the demonstration in [6].
Now that we have understood the properties of the starting spaces, we need a feature to characterize the unitary orbits that arise from these spaces, and that is the purpose of the next two fundamental theorems:

Theorem 4.5.2. *Let ρ_1 and ρ_2 two density matrices on $H \simeq \mathbb{C}^n$, then ρ_1 and ρ_2 are unitarly equivalent (i.e. $\rho_2 = U\rho_1 U^\dagger$ for some unitary matrix U) if and only if ρ_1 and ρ_2 have the same spectrum; that is the same eigenvalues including multiplicity. Moreover we have: $Tr[(\rho_2)^r] = Tr[(\rho_1)^r] \;\; \forall r = 1, 2, ..., n$*

This theorem has a really deep implication. In fact this statement means that every unitary orbit can be identified by a unique diagonal matrix like:

$$\rho_{rappresentative} = diag(\lambda_1 1_{n_1 \times n_1}, \lambda_2 1_{n_2 \times n_2}, ..., \lambda_r 1_{n_r \times n_r}) \tag{4.23}$$

where a particular criterion of ordering is chosen to dispose the eigenvalues on the diagonal, so as to remove any ambiguity.
Therefore, all the possible orbits are distinguished by the possible values of these eigenvalues, that, as we already know, have to satisfy the constraints : $\lambda_i \in [0,1] \ \forall i = 1,2,..,r$ and $\sum_i^r \lambda_i = 1$. Anyhow, there is still an infinity of possible combinations, so that we can conclude that $U(n)$ partitions the set $\mathcal{D}(\mathcal{H})_n$ into infinite (an uncountable) family of orbits or strata.
The second theorem that we present clarifies the dimensions and the nature of these strata, and it will be essential later for our studies about concrete examples of 2x2 and 3x3 density matrices.

Theorem 4.5.3. *Let $U(n)$ act on $\mathcal{D}(\mathcal{H})_n$ by the co-adjoint action and let ρ a density matrix with $r \geq 1$ eigenvalues λ_i with multiplicity n_i ; then the orbit of ρ is homeomorphic to the manifold:*

$$U(n)/[U(n_1) \times U(n_2) \times \ \times U(n_r)] \tag{4.24}$$

of real dimension $n^2 - \sum_i^r {n_i}^2$.

As consequence of this last theorem we have for the orbit of a pure matrix of a n-level quantum system:

$$U(n)/[U(1) \times U(n-1)] = \mathcal{D}(\mathcal{H})_n{}^1 = \mathcal{W}(\mathcal{H})_1{}^1 \tag{4.25}$$

as we expected.
The last remaining question to clarify is if these strata could be considered as valid physical settings, that means, if they admit a structure of Kähler manifold.
As we expected, the answer to this question is affirmative, and a complete prove can be found in [8] . We just present the result, the final theorem that concludes this chapter .

Theorem 4.5.4. *For the orbit O_ρ of every density matrix $\rho \in \mathcal{D}(\mathcal{H})_n{}^k$ with $k \leq n$, is always possible to find an admissible triple $(J^{O_\rho}, \eta^{O_\rho}, \gamma^{O_\rho})$ such that O_ρ is a Kähler manifold.*

We can now finally translate the theoretical structure examined up to here in a practical example: the density matrices 2x2, related to two level quantum systems, or Q-bits.

4.6 Q-bits

Because we know from the last section that each unitary orbit can be identified with a representative diagonal density matrix, we can always report a generic 2x2 density

matrix at his unitarly equivalent form:

$$\boldsymbol{\rho} = \begin{pmatrix} p_0 & 0 \\ 0 & 1-p_0 \end{pmatrix} \tag{4.26}$$

Where p_0 is a probability, so that $0 \leq p_0 \leq 1$.

4.6.1 The Bloch sphere

The first issue we need to deepen the argument is a tool to represent these matrices in a more useful way, the so called *Bloch Sphere*. We know that a generic 2x2 matrix can always be expressed as a combination of the Pauli matrices and the identity matrix. The representation that we will use is the following:

$$\boldsymbol{\rho(\vec{R})} = \frac{1}{2}(\mathbf{1_2} + R_1\boldsymbol{\sigma_1} + R_2\boldsymbol{\sigma_2} + R_3\boldsymbol{\sigma_3}) \tag{4.27}$$

that in a matrix form becomes:

$$\boldsymbol{\rho(\vec{R})} = \frac{1}{2}\begin{pmatrix} 1+R_3 & R_1 - iR_2 \\ R_1 + iR_2 & 1-R_3 \end{pmatrix} \tag{4.28}$$

In this way we have established a connection between the $\Re^3$ vectors $\vec{R} = (R_1, R_2, R_3)$, usually called *Bloch vectors* and the density matrices for Q-bits. The explicit form of this connection is:

$$R_i = Tr(\boldsymbol{\sigma_i \rho(\vec{R})}) \tag{4.29}$$

To make the latter physically consistent we have to impose the positivity of matrix (4.28), so that:

$$det\boldsymbol{\rho(\vec{R})} \geq 0 \rightarrow \frac{1}{4}(1 - |\vec{R}|^2) \rightarrow |\vec{R}|^2 \leq 1 \tag{4.30}$$

This condition is really sufficient to satisfy the positivity request, because the possibility of two negative eigenvalues is excluded by the constraint $tr[\boldsymbol{\rho(\vec{R})}] = 1$. Then, we have limited the possible region of this biunique connection to a unit 3-ball on $\Re^3$, that is traditionally called *Bloch sphere*, even if it is actually a ball. The edge of this ball, given by the condition $det\boldsymbol{\rho(\vec{R})} = 0 \rightarrow |\vec{R}|^2 = 1$ consists of the 2x2 density matrices with one eigenvalue equal to zero and the other equal to one, that are the pure states. Therefore the edge corresponds to the space $\mathcal{W}(\mathcal{H})_1{}^1$ previously defined in section 4.4.
Another interesting area of this ball is the center. The center of the Bloch Sphere

is, by definition, given by $\vec{R} = (0,0,0)$ so that $det\boldsymbol{\rho}(\vec{\mathbf{0}}) = \frac{1}{4}$. Applying the notation used in the equation (4.26) and the trace one condition we get:

$$p_0 q_0 = \frac{1}{4} \quad \text{and} \quad p_0 + q_0 = 1 \rightarrow p_0 = q_0 = \frac{1}{2} \tag{4.31}$$

Thus, the center of the Bloch Sphere corresponds to the density matrix:

$$\boldsymbol{\rho}(\vec{\mathbf{0}}) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \tag{4.32}$$

the latter is called *the maximum entangled state* that, being proportional to the identity matrix, is invariant under the change of basis defined by the adjoint action of an unitary matrix $\boldsymbol{U}$, hence it is a fixed point of the co-adjoint action.
Choosing the particular basis $\{|0\rangle, |1\rangle\}$ we can identify the pure state , $\boldsymbol{\rho_0}$, corresponding to the 3-vector $\vec{R}_{\boldsymbol{\rho_0}} = (0,0,1) \equiv \boldsymbol{P_0}$ as the matrix:

$$\boldsymbol{\rho_0} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.33}$$

and then, to clarify what we have said up to here , we can represent the Bloch sphere in fig. 4.1.

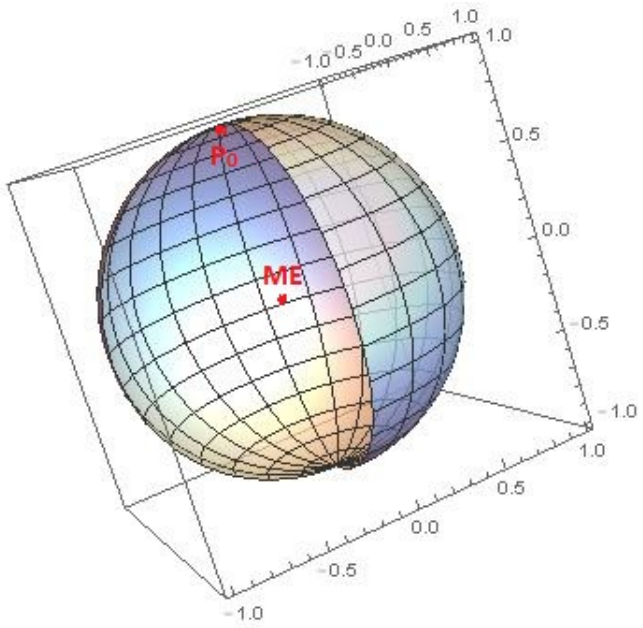

Figure 4.1: *Bloch sphere with the maximum entangled state and the pure state* $\boldsymbol{\rho_0}$. *The coloration has just an aesthetic purpose.*

4.6.2 Unitary evolution on the Bloch sphere

Now that we have got a way to clearly visualize the density matrices we can analyze the time evolution observing the paths on the Bloch sphere. As we know, concerning these quantum objects, there are two different types of possible evolutions, the unitary and the open one. Starting just considering the unitary one, we can apply the

important notions coming from the theorem (4.5.3) to categorize all the possible unitary obits.

Because the distinct eigenvalues p_0 and $p_0 - 1$ are at maximum two, we have only two distinct cases:

- $p_0 = \frac{1}{2}$

  then the orbit is homeomorphic to:

  $$U(2)/U(2)$$

  so that the orbit has the dimension of a point, and concretely there is not an evolution: the system keeps staying in the ME, blocked at the center of the sphere.

- $p_0 \neq \frac{1}{2}$

  then we have an orbit homeomorphic to:

  $$U(2)/[U(1) \times U(1)] \sim S^2$$

Thus, interpreting the orbit of the first illustrated case as a 2-sphere of null radius, we can claim that all the orbits for the group $U(2)$ are homeomorphic to a 2-sphere of fixed radius, concentric to the Bloch sphere. Furthermore, it is easy to see that, varying the value of p_0 from 0 to 1 we can reconstruct the full Bloch sphere. Indeed, we know that co-adjoint action of the unitary group $U(2)$ on our starting density matrix $\boldsymbol{\rho_0}$ preserves the original determinant , and so, because of the mentioned connection between determinant and the modulus of 3-vectors on the Bloch sphere, we immediately note that , starting any unitary evolution from a point of the sphere with modulus $R = |\vec{R}|$, we will remain on a sphere centered on the origin of the sphere and with radius R. Consequently, the set of all the unitary orbits, for all the possible starting points inside the sphere, reconstructs the whole sphere.
By analogy with the considerations of section 4.5 the set composed by all the unitary orbits excluding the edge of the Bloch sphere represents the space $\mathcal{D}(\mathcal{H})_2^2$.

As we have done before, we graphically conclude this section showing in fig. 4.2 an example of the described unitary evolution on the Bloch sphere, where it has the form of a rotation around the ME state, starting from the already considered point $\boldsymbol{P_0}$.

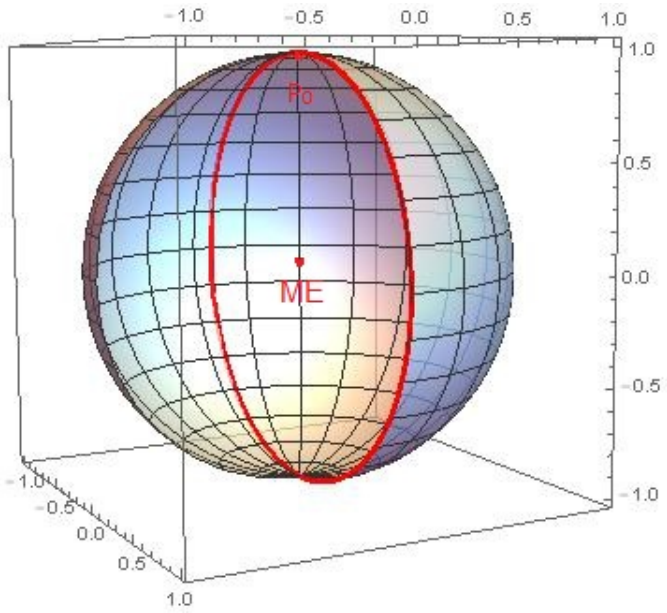

Figure 4.2: *Unitary evolution of a pure state* $\boldsymbol{\rho_0}$ *on the Bloch sphere*

The direction of the rotation depends on the particular unitary operator $\boldsymbol{U}$ used to perform the evolution.

4.6.3 Open evolution on the Bloch sphere

More complicated than the unitary one, the open evolution has to be determined trying to evaluate the correct UDM for the system. So, what we want to do is to implement a time evolution that satisfies all the theoretical requests of UDMs that we have showed in the previous chapter. What we know, from these properties, is that the open evolutions[1] connect pure states to mixed ones and so they are not invertible. To make general this treatment we start from a generic initial state $\boldsymbol{\rho}(0)$ of the form of the equation (4.26). Then we can imagine a general UDM ♣ that operates in the following way:

$$\boldsymbol{\rho}(0) = \begin{pmatrix} p_0 & 0 \\ 0 & 1-p_0 \end{pmatrix} \xrightarrow[\clubsuit_{(t,t_0)}]{} \boldsymbol{\rho}(t) = \begin{pmatrix} p(t) & 0 \\ 0 & 1-p(t) \end{pmatrix} \tag{4.34}$$

where $p(t)$ is a function of t that gives $p(0) = p_0$ and that always satisfies $0 \leq p(t) \leq 1$ in such a way that $\boldsymbol{\rho}(t)$ remains a density matrix at any time.

We now show one of the possible consequent choices of the Kraus operators, taken from the thesis work in [8], where it has been used for solving the Lindblad equation. This choice is the following:

[1] From now on, using this word in this context, we refer to the evolutions that are specifically not unitary, just because the unitary ones has been already treated

$$\boldsymbol{K_1}(t) = \sqrt{\frac{1}{2} + \frac{e^{-t}}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \;\; ; \;\; \boldsymbol{K_2}(t) = \sqrt{\frac{1}{2} - \frac{e^{-t}}{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.35}$$

In fact, with straightforward calculation one can prove that:

$$\begin{aligned} &\boldsymbol{K_1}^\dagger(t)\boldsymbol{K_1}(t) + \boldsymbol{K_2}^\dagger(t)\boldsymbol{K_2}(t) = \mathbf{1_2} \quad \forall t \\ &\boldsymbol{K_1}(t)\boldsymbol{\rho_0}\boldsymbol{K_1}^\dagger(t) + \boldsymbol{K_2}(t)\boldsymbol{\rho_0}\boldsymbol{K_2}^\dagger(t) = \boldsymbol{\rho}(t) \end{aligned} \tag{4.36}$$

With this choice , the function $p(t)$ introduced in the equation (4.34) becomes:

$$p(t) = \frac{1}{2}e^{-t}(-1 + e^t + 2p_0) = \frac{1}{2} + e^{-t}(p_0 - \frac{1}{2}) \tag{4.37}$$

It is now interesting looking at the Bloch vectors. Starting from the Bloch vector corresponding to $\boldsymbol{\rho}(0)$, $\vec{R}(0) = (0, 0, 2p_0 - 1)$, we get the Bloch vector at time t : $\vec{R}(t) = (0, 0, 2p(t) - 1)$ and, comparing the modulus of these two different $\Re^3$ vectors we note that, iff $p_0 \neq \frac{1}{2}$:

$$|\vec{R}(0)| = 2p_0 - 1 > 2p(t) - 1 = e^{-t}(-1 + e^t + 2p_0) - 1 = |\vec{R}(t)| \quad \forall t > 0 \tag{4.38}$$

The latter is a really significant inequality because it mathematically expresses the irreversibility nature of this type of evolution. Indeed, increasing in time, the Bloch vectors can only become nearer and nearer to the center of the sphere , never coming back, converging to the ME state.
Anyway, with this parametrization of time t the ME state is only an asymptotic state, as we can see from the following limit:

$$\lim_{t\to\infty} \boldsymbol{\rho}(t) = \begin{pmatrix} \frac{1}{2} + e^{-t}(p_0 - \frac{1}{2}) & 0 \\ 0 & \frac{1}{2} - e^{-t}(p_0 - \frac{1}{2}) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \tag{4.39}$$

Instead, if we think of starting from this final state, as we can see from the inequality (4.38) inserting the value $p_0 = \frac{1}{2}$, we get $|\vec{R}(t)| = 0$ for any instant t, so that there is no more evolution and the quantum system keeps staying in that fixed state exactly like the unitary case.

Analyzing the evolution at different times, that means evaluating the succession:

$$\clubsuit_{(t,0)}(\boldsymbol{\rho}(0)) = \boldsymbol{\rho}(t), \clubsuit_{(t+\epsilon,0)}(\boldsymbol{\rho}(0)) = \boldsymbol{\rho}(t+\epsilon), \boldsymbol{\rho}(t+2\epsilon), \ldots, \boldsymbol{\rho}(t+n\epsilon) \tag{4.40}$$

where ϵ is a real number > 0 and n is an integer sufficiently big to get a sizable number of points; we can draw the open evolution on the Bloch sphere as shown in fig. 4.3.

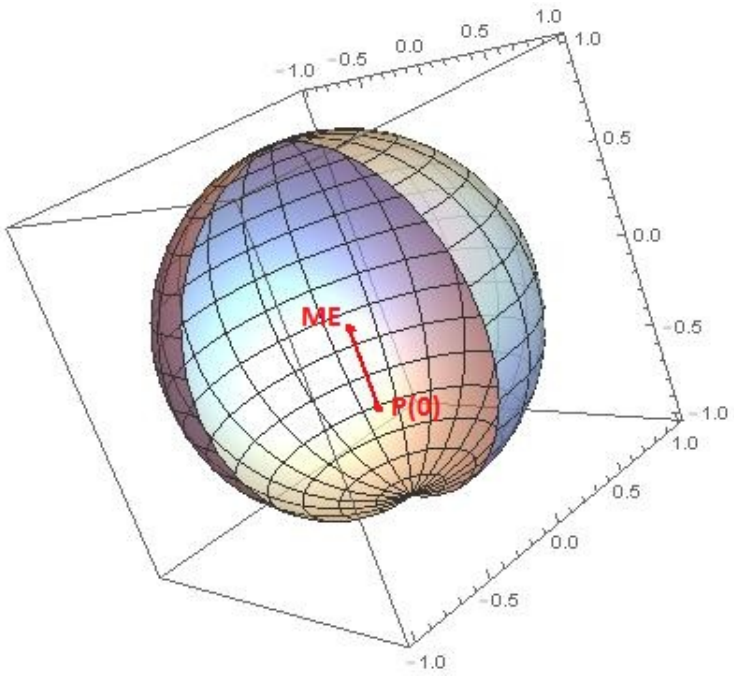

Figure 4.3: *Open evolution of a generic initial state $P(0)$ on the Bloch sphere*

Nevertheless, we have not considered yet the Markovian requirement. This interpretation, as we have said before, is the one with more intuitive physical properties, and it is the only one that allows to describe this open evolution with a Master differential equation. The latter will allow us to analyze the general form of the "derivative" $\dot{\boldsymbol{\rho}} \equiv \boldsymbol{\mathcal{L}}_t$. Anyway, with a direct calculation one can see that the fundamental key at the heart of the Markovian approximation , in form of the following equation:

$$\clubsuit_{(s,t)}\clubsuit_{(t,0)} = \clubsuit_{(s,0)} \quad \forall s \geq t \geq 0 \tag{4.41}$$

is satisfied by the supposed time dependence of our open evolution, in fact:

$$\begin{aligned}
\clubsuit_{(s,t)} \circ \clubsuit_{(t,0)} &= \clubsuit_{(s,t)}(\boldsymbol{K_1}(t)\boldsymbol{\rho(0)}\boldsymbol{K_1}^{\dagger}(t) + \boldsymbol{K_2}(t)\boldsymbol{\rho(0)}\boldsymbol{K_2}^{\dagger}(t)) = \\
&\clubsuit_{(s,t)} \begin{pmatrix} \frac{1}{2}e^{-t}\left(2p_0 + e^t - 1\right) & 0 \\ 0 & \frac{1}{2}e^{-t}\left(-2p_0 + e^t + 1\right) \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}e^{-s-t}\left(2p_0 + e^{s+t} - 1\right) & 0 \\ 0 & \frac{1}{2}e^{-s-t}\left(-2p_0 + e^{s+t} + 1\right) \end{pmatrix} \\
&= \clubsuit_{(s,0)} \quad \forall s \geq t \geq 0
\end{aligned} \tag{4.42}$$

Graphically speaking, on the Bloch sphere, this means that the action of the composition of two consecutive open evolutions has the form shown in fig. 4.4

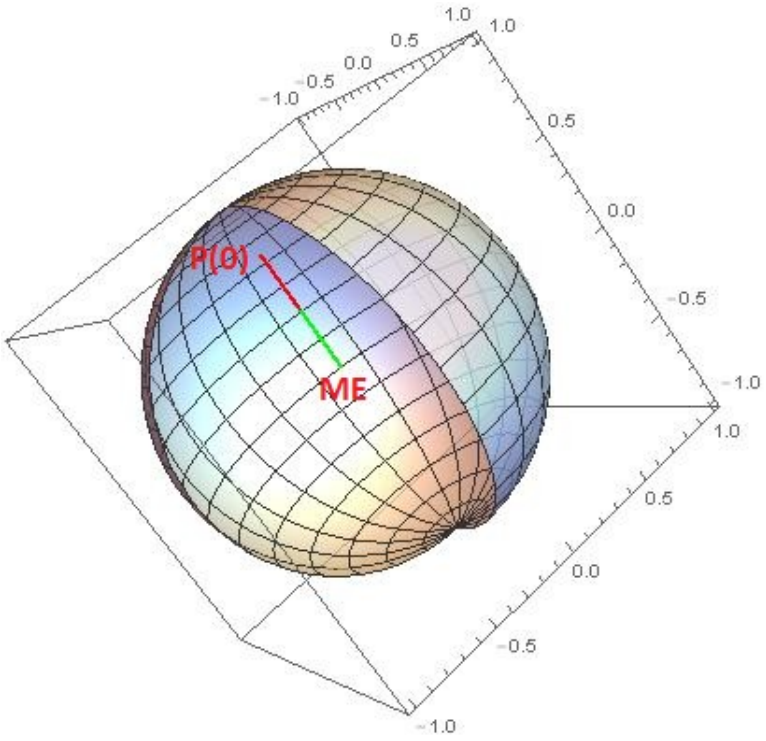

Figure 4.4: *The starting point* $P(0)$ *has* $p_0 = 0.86$ *and we have considered* $t = 0.6$ *,*$s = 2.4$

We can also think of mixing open and unitary evolutions. The latter has relevant differences depending on the order of the combination between the two evolutions. Indeed, if we first perform the unitary evolution on a generic starting density matrix $\boldsymbol{\rho}_0$ of the form of equation (4.26), and then the open evolution to the result, to be consistent with the other inverted combination we need to modify the basis in which the Kraus operators are written, just because , as we know, the unitary evolution corresponds to a change of basis.
To prove this fact we show separately the two distinct cases and then we demonstrate that they are equivalent.

- *Open evolution before, unitary evolution after*
  For the open evolution we maintain the same Kraus operators previously defined in equation (4.35) ; for the unitary one, we use a generic unitary operator $\boldsymbol{U}$. So we have:

$$\boldsymbol{\rho_1} = \boldsymbol{U}(\boldsymbol{K_1}(t)\boldsymbol{\rho}_0\boldsymbol{K_1}^\dagger(t) + \boldsymbol{K_2}(t)\boldsymbol{\rho}_0\boldsymbol{K_2}^\dagger(t))\boldsymbol{U}^\dagger = \boldsymbol{U}\boldsymbol{\rho}(t)\boldsymbol{U}^\dagger \tag{4.43}$$

- *Unitary evolution before, open evolution after*
  As we said, we define the changed Kraus operators:

$$\tilde{\boldsymbol{K_1}}(t) \equiv \boldsymbol{U}\boldsymbol{K_1}(t)\boldsymbol{U}^\dagger \;\; ; \;\; \tilde{\boldsymbol{K_2}}(t) \equiv \boldsymbol{U}\boldsymbol{K_2}(t)\boldsymbol{U}^\dagger \tag{4.44}$$

Thus, we perform the same unitary evolution on $\boldsymbol{\rho_0}$, followed by an open evolution ruled by the new Kraus operators :

$$\boldsymbol{\rho}_2 = \tilde{\boldsymbol{K_1}}(t)\boldsymbol{U}\boldsymbol{\rho}_0\boldsymbol{U}^\dagger\tilde{\boldsymbol{K_1}}^\dagger(t) + \tilde{\boldsymbol{K_2}}(t)\boldsymbol{U}\boldsymbol{\rho}_0\boldsymbol{U}^\dagger\tilde{\boldsymbol{K_2}}^\dagger(t) \tag{4.45}$$

Replacing the definitions on the last equation it becomes clear that , only with these prescriptions, the two different procedures are equivalent, in fact we get:

$$\begin{aligned}\boldsymbol{\rho}_2 &= \boldsymbol{U}(\boldsymbol{K_1}(t)\boldsymbol{\rho}_0\boldsymbol{K_1}^\dagger(t) + \boldsymbol{K_2}(t)\boldsymbol{\rho}_0\boldsymbol{K_2}^\dagger(t))\boldsymbol{U}^\dagger \\ &= \boldsymbol{U}\boldsymbol{\rho}(t)\boldsymbol{U}^\dagger = \boldsymbol{\rho}_1\end{aligned} \tag{4.46}$$

We can now present in fig. 4.5 a graphic summary of these considerations on the Bloch sphere, where the starting point is the same considered for the previous picture:

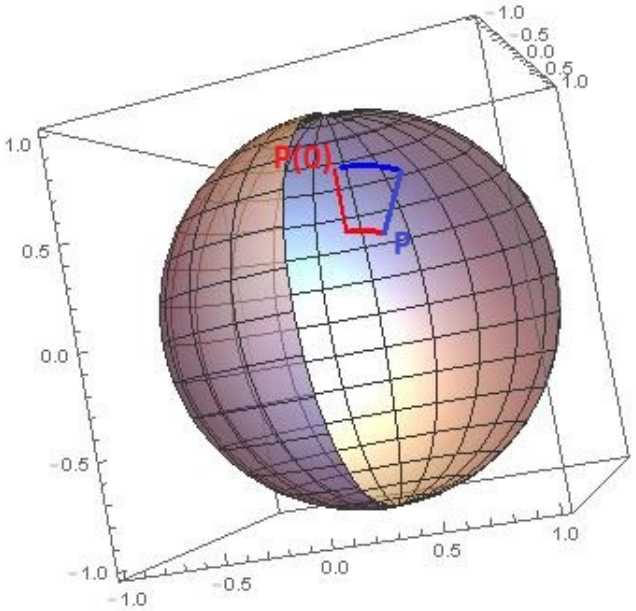

Figure 4.5: *The blue and the red lines represent the two equivalent types of composite evolutions.*

At the end of these argumentations it is clear that, combining open and unitary evolutions, it is always possible to reach any point of the Bloch sphere with the modulus less then the modulus of the Bloch vector of the arbitrary starting point. The latter give us the opportunity to construct infinite patterns on the sphere, with the only constraint to maintain the modulus of the Bloch vectors decreasing in time.

The last remaining question to deepen for the Q-bits case is the form of the Lindbladian and the Lindblad operators for the open evolution. As we have done in a previous chapter, we analyze the first order in time, for an infinitesimal time variation dt:

$$\boldsymbol{\rho}(dt) = \begin{pmatrix} \frac{1}{2} + e^{-dt}(p_0 - \frac{1}{2}) & 0 \\ 0 & \frac{1}{2} - e^{-dt}(p_0 - \frac{1}{2}) \end{pmatrix} \tag{4.47}$$

Hence , the first order in a Taylor expansion is:

$$\boldsymbol{\rho}(dt) = \frac{1}{2}\begin{pmatrix} 2p_0 - 2p_0 dt + dt & 0 \\ 0 & 2 - 2p_0 + 2p_0 dt - dt \end{pmatrix} \tag{4.48}$$

and then we finally get the Lindbladian:

$$\dot{\boldsymbol{\rho}} = \boldsymbol{\mathcal{L}}_t(\boldsymbol{\rho}) = \frac{\boldsymbol{\rho}(dt) - \boldsymbol{\rho}(0)}{dt} = \frac{1}{2}(1 - 2p_0)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}(1 - 2p_0)\boldsymbol{\sigma_3} \tag{4.49}$$

To conclude , we want to determinate the expression of Lindblad operators. From equation (4.48) we can see that:

$$\boldsymbol{\rho}(dt) = (1 - \frac{dt}{2})\mathbf{1_2}\boldsymbol{\rho_0} + \frac{dt}{2}\boldsymbol{\sigma_1}\boldsymbol{\rho_0}\boldsymbol{\sigma_1} \tag{4.50}$$

Hence, comparing the latter with the result of equation (3.80) we deduce that the first term in the sum corresponds to the normalization term proportional to the identity matrix, and we remain with only one Lindblad operator:

$$\boldsymbol{L_1} = \sqrt{\frac{1}{2}}\boldsymbol{\sigma_1} \tag{4.51}$$

that correctly agrees with its definition from equation (3.77) and that returns, applying the Master equation without considering the part proportional to the identity, the previous expression for the derivative of the open evolution:

$$\dot{\boldsymbol{\rho}} \equiv \boldsymbol{\mathcal{L}}_t(\boldsymbol{\rho}) = \boldsymbol{L_1}\boldsymbol{\rho_0}\boldsymbol{L_1}^\dagger - \frac{1}{2}\{\boldsymbol{L_1}^\dagger\boldsymbol{L_1}, \boldsymbol{\rho_0}\} = \frac{1}{2}(1 - 2p_0)\boldsymbol{\sigma_3} \tag{4.52}$$

Chapter 5

Q-trits

Finally, the last content of this work is the study of the density matrices 3x3, related to open quantum systems with 3 levels, often called *Q-trits*. The generic diagonal density matrix in this context has the form:

$$\boldsymbol{\rho_0} = \begin{pmatrix} p_0 & 0 & 0 \\ 0 & q_0 & 0 \\ 0 & 0 & 1-p_0-q_0 \end{pmatrix} \tag{5.1}$$

where $p_0, q_0 \in [0,1]$ and $p_0 + q_0 \leq 1$ so that the trace condition is always respected. Anyhow, the addition of this new degree of freedom implies the origin of further complications concerning the unitary orbits, compared to the previous U(2) case, that can be immediately understood applying to this new context the results of theorem (4.5.3). In fact, because of the increased number of eigenvalues of $\boldsymbol{\rho_0}$, the theorem yields three distinct types of orbit:

- $p_0 = q_0 = \frac{1}{3}$

  then the orbit is homeomorphic to:

  $$U(3)/U(3)$$

  so, we recover the orbit with the dimension of a point.

- $\boldsymbol{\rho_0}$ has two equal eigenvalues that differ from the last remaining one (like, for example $p_0 = q_0 = \frac{1}{2}$)

  then we have a co-adjoint orbit homeomorphic to:

  $$U(3)/[U(2) \times U(1)]$$

  which is a 4-dimension manifold.

- $\boldsymbol{\rho_0}$ has every eigenvalue different from each other, that means 3 distinct eigenvalues.

  Therefore the co-adjoint orbit is homeomorphic to:

$$U(3)/[U(1) \times U(1) \times U(1)]$$

  which is a 6-dimension manifold.

At the end of this list, it becomes evident that, almost about visualizing the orbits, we are not able to represent them in an intuitive way anymore. We have to leave the useful resource of the Bloch sphere, that can contain in a unique three dimensional picture all the possible evolutions of the system, because the dimensions of these new orbits are not accessible anymore. However, the intricacy is not confined to our limits of visualization, as we will see with some sly manipulations.
In fact, also in this case, we can define a connection between matrices and vectors, building an extension of the concept of Bloch vectors. Any 3x3 density matrix can be expanded on the algebra of Lie generators of $SU(3)$ plus the identity matrix:

$$\boldsymbol{\rho} = \frac{1}{3}\mathbf{1_3} + \frac{1}{2}\sum_{k=1}^{8} s_k \boldsymbol{t_k} \tag{5.2}$$

where the $\{s_k\}_k$ are real parameters and the 8 matrices $\{\boldsymbol{t_k}\}_k$ that constitute the basis of the Lie algebra $\mathfrak{su}(3)$ can be chosen ad lib.
This procedure allows to associate to each matrix $\boldsymbol{\rho}$ a vector $\vec{S}(\boldsymbol{\rho}) \in \Re^8$ such that $\vec{S}(\boldsymbol{\rho}) = (s_1, s_2, .., s_8)$.
In the reference [12] it is shown that each unitary orbit leaves invariant the modulus of this vector, so that the distance from the center of $\Re^8$ of each vector related to different matrices in the same unitary orbit remains unchanged.
Moreover, we know that the surface on $\Re^8$ constituted by the vectors with the same modulus is a 7-sphere. Nevertheless, taking as a starting point for the unitary orbit a pure density matrix, that must have one eigenvalues equals to one and the other two null, the theorem above claims that the co-adjoint orbit has to be omeomorphic to a 4-dimension manifold. Therefore, we clearly infer that, unlike the showed $U(2)$ case, the unitary orbit of pure matrices ruled by the $U(3)$ group is not a sphere, but only a submanifold of a sphere.
This consideration, here naively solved by words, exemplifies effectively the complexity of this setting, not only related to our difficulties on figuring spaces with dimension higher then 3.

However, we can select a particular way of visualizing the space of density matrices that, even if it is just a section of the whole space of all the possible evolutions, it is cleverly focused on the open scenery at the expense of the unitary one.

If we choose for the basis the Gell-Mann matrices:

$$t_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\ t_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\ t_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

$$t_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix};\ t_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix};\ t_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \tag{5.3}$$

$$t_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix};\ t_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

we note that any diagonal 3x3 density matrix in the form of (5.1) can be represented just by the use of the identity matrix and the last two t_3 and t_8, so that:

$$\boldsymbol{\rho}_0 = \frac{1}{3}\mathbf{1_3} + \frac{p_0 - q_0}{2}\boldsymbol{t_3} + \frac{p_0 + q_0 - 2(1 - p_0 - q_0)}{2\sqrt{3}}\boldsymbol{t_8} =$$

$$= \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{p_0 - q_0}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{3p_0 + 3q_0 - 2}{6}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{5.4}$$

and then, because the coefficient of the identity is always the same for all the matrices, we can think of reducing the set of all the possible diagonal density matrices to a 2-dimensional subspace of $\Re^8$, spanned by all the vectors of the form:

$$\vec{S}(\boldsymbol{\rho}_0) = (p_0 - q_0, \frac{p_0 + q_0 - 2(1 - p_0 - q_0)}{\sqrt{3}}) \tag{5.5}$$

If we draw a graphic of this subspace, respecting the conditions $p_0, q_0 \in [0, 1]$ and $p_0 + q_0 \leq 1$, we get the triangle shown in fig. 5.1.

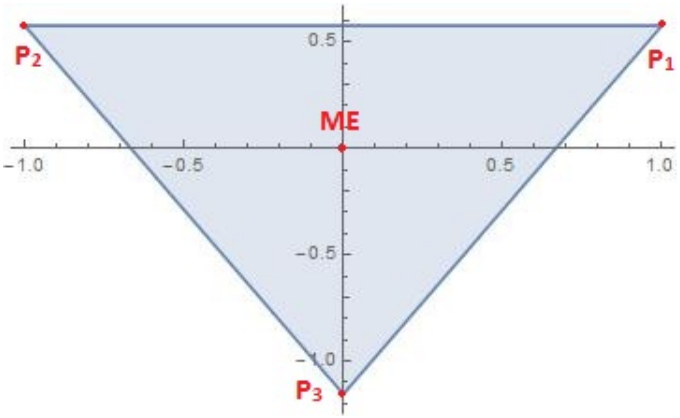

Figure 5.1: *the space of diagonal density matrices*

The origin of this space is the maximum entangled state, (ME in fig. 5.1) :

$$\vec{S}_{ME} \equiv \vec{S}(\boldsymbol{\rho}_{ME}) = (0,0) \quad \rightarrow \quad \boldsymbol{\rho}_{ME} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.6}$$

the other three points pictured in fig. 5.1, P_1 , P_2 and P_3 are the three extremal pure density matrices:

$$\begin{aligned} \vec{S}_{P_1} \equiv \vec{S}(\boldsymbol{\rho}_{P_1}) = (1, \frac{1}{\sqrt{3}}) \quad \rightarrow \quad \boldsymbol{\rho}_{P_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \vec{S}_{P_1} \equiv \vec{S}(\boldsymbol{\rho}_{P_2}) = (-1, \frac{1}{\sqrt{3}}) \quad \rightarrow \quad \boldsymbol{\rho}_{P_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \vec{S}_{P_3} \equiv \vec{S}(\boldsymbol{\rho}_{P_3}) = (0, -\frac{2}{\sqrt{3}}) \quad \rightarrow \quad \boldsymbol{\rho}_{P_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{5.7}$$

It is now important to dwell on what we have really done, from a conceptual point of view, introducing this representation. As we know from the previous geometrical treatment of chapter 4, each diagonal matrix corresponds to a unitary orbit, and the orbits differ depending on the spectrum, i.e. on its eigenvalues and their multiplicities. Therefore, parameterizing only the diagonal matrices, what we are doing is preparing a setting that could ignore the ambiguity caused by the equivalence of unitary orbits. Such a setting is the perfect surrounding for studying the open evolutions, that can be thought exactly like the transformations that the system performs passing from an unitary orbit to another. Of course, the illustrated one is not the only way to parametrize the diagonal density matrices, and other solutions are possible. Nevertheless, this particular choice is suited to our purposes, as we will see soon.

Anyway, the careful reader could object to these considerations the incontrovertible observation that the three pure density matrices associated to the points showed above belong to the same unitary orbit. That is true; in fact, even with these precautions, the unitary ambiguity is not completely removed, but just reduced to a set of three points. To explain better this statement, it is sufficient to underline that, considering all possible different diagonal density matrices, we are always including the three admissible permutations of the three distinct matrices with the same spectrum. Nevertheless, we know that these three matrices belong to a unique unitary orbit, even if they produce three separate points on the set of fig. 5.1.
Observing these particular three points, we can note that the unitary matrices involved in the the transformations are:

$$\boldsymbol{U_{12}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} ; \ \boldsymbol{U_{13}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} ; \ \boldsymbol{U_{23}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{5.8}$$

such that:

$$\boldsymbol{\rho}_{P_2} = \boldsymbol{U_{12}}\boldsymbol{\rho}_{P_1}\boldsymbol{U_{12}}^{\dagger} \ ; \ \boldsymbol{\rho}_{P_3} = \boldsymbol{U_{13}}\boldsymbol{\rho}_{P_1}\boldsymbol{U_{13}}^{\dagger} = \boldsymbol{U_{23}}\boldsymbol{\rho}_{P_2}\boldsymbol{U_{23}}^{\dagger} \ ; \tag{5.9}$$

Thus, considering the form of these matrices and the structure of the graphic, we can conclude that, if we really want to remove the unitary ambiguity, we have to consider only one of the three possible sections A_1 , A_2 and A_3 of our triangle, as we can see from fig. 5.2.

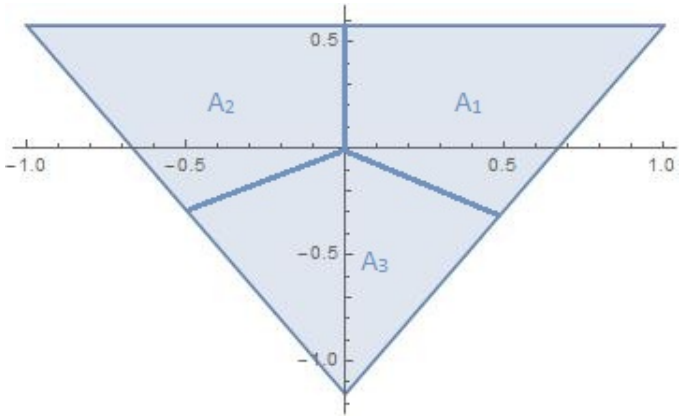

Figure 5.2: *the three areas unitarly equivalent*

Anyway, working with the whole triangle is just easier, and we can do it without problems, just remembering these crucial considerations.
Finally, after these preparatory contextualizations, we are able to study the open evolution. Because we are limiting ourselves to diagonal matrices, for the Kraus operators we choose matrices that maintain the diagonal structure during the evolution. Starting from this assumption, we pick up all the rotation matrices plus the

identity:

$$R_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \; R_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}; \; R_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \\ R_4 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \; R_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{5.10}$$

and we equip them with a suitable temporal dependence, as a unique multiplicative function for all the operators, trying to retrace the same steps as for the $U(2)$ case. Nevertheless, because we do not really want to lose generality, we introduce a vector $\vec{a} = (a_0, a_1, a_2, a_3, a_4, a_5) \in \Re^6$ that represents a degree of freedom for the coefficients that could multiply these rotation matrices involved in the definition of Kraus operators. Thus, each component a_i of this vector , with $i = 0, 2, .., 5$, is associated to the corresponding matrix R_i. The a_0 term is the one in front of the identity matrix. Introducing this vector of coefficients $\vec{a}$ gives us the possibility of analyzing and discern in the whole set of plausible combinations of coefficients the correct ones that respect all the UDMs conditions. Defining the normalization parameter $n = \sum_{i=0}^{5} a_i$ we can clarify what we have explained in words showing the selected Kraus operators:

$$\boldsymbol{K_1}(t) = \sqrt{\frac{a_1}{n+1}(1-e^{-t})}R_1 = \sqrt{\frac{a_1}{n+1}(1-e^{-t})} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \\ \boldsymbol{K_2}(t) = \sqrt{\frac{a_2}{n+1}(1-e^{-t})}R_2 = \sqrt{\frac{a_2}{n+1}(1-e^{-t})} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}; \\ \boldsymbol{K_3}(t) = \sqrt{\frac{a_3}{n+1}(1-e^{-t})}R_3 = \sqrt{\frac{a_3}{n+1}(1-e^{-t})} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \\ \boldsymbol{K_4}(t) = \sqrt{\frac{a_4}{n+1}(1-e^{-t})}R_4 = \sqrt{\frac{a_4}{n+1}(1-e^{-t})} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \\ \boldsymbol{K_5}(t) = \sqrt{\frac{a_5}{n+1}(1-e^{-t})}R_5 = \sqrt{\frac{a_5}{n+1}(1-e^{-t})} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{5.11}$$

Moreover, the Kraus operator connected to the identity matrix is the following one:

$$\boldsymbol{K_0}(t) = \sqrt{\frac{1 + a_0 + (a_0 - n)e^{-t}}{n+1}} 1_{3\times 3} \tag{5.12}$$

but it could be expressed in a more intuitive way separating the part assimilable to the other Kraus operators from the part connected with the normalization:

$$\boldsymbol{K_0}^1(t) + \boldsymbol{K_0}^2(t) = \sqrt{\frac{a_0}{n+1}(1-e^{-t})}1_{3\times3} + \sqrt{\frac{1}{n+1}(1+ne^{-t})}1_{3\times3} \tag{5.13}$$

Because all the operations that involve Kraus operators are quadratic, the two different expressions are really equivalent, and we preferably use the last one.
With this specific choice of the operators, it is straightforward to verify that, for all the possible values of the real vector $\vec{a} \in \Re^6$, we always have:

$$\sum_{i=0}^{5} \boldsymbol{K}_i^\dagger(t)\boldsymbol{K}_i(t) = \mathbf{1_3} \quad \forall t \tag{5.14}$$

The last equality means that starting the evolution with these Kraus operators from a generic diagonal density matrix $\boldsymbol{\rho_0}$ of the form (5.1), we have:

$$Tr\left(\boldsymbol{\rho}(t)\right) = 1 \quad \forall t, \vec{a} \tag{5.15}$$

The above equality is the deepest reason that led us to this particular definition, and all the normalization conditions have been imposed to this purpose.

Starting from $\boldsymbol{\rho_0}$ at time $t = 0$, we are now able to calculate the open evolution at a generic time t, and this operation will depend on the components of $\vec{a}$ considered from time to time:

$$\clubsuit^{\vec{a}}_{(t,0)}(\boldsymbol{\rho_0}) \equiv \boldsymbol{\rho}^{\vec{a}}(t) = \sum_{i=0}^{5} \boldsymbol{K}_i(t)\boldsymbol{\rho_0}\boldsymbol{K}_i^\dagger(t) \tag{5.16}$$

Anyhow, as we know, (5.14) is not the only condition that the open evolution has to satisfy, and, the composition requirement, in the form of the known equation:

$$\clubsuit_{(s,t)}\clubsuit_{(t,0)} = \clubsuit_{(s,0)} \quad \forall s \geq t \geq 0 \tag{5.17}$$

appears to be really constraining, so that it considerably reduces the possible range of the coefficients in $\vec{a}$. These calculations are really tiresome and voluminous and can be solved numerically imposing the equivalence:

$$\clubsuit_{(s,t)}\clubsuit_{(t,0)} - \clubsuit_{(s,0)} = 0 \ \ \forall t, s \geq 0, \ \forall p_0, q_0 \in [0,1] \, , p_0 + q_0 \leq 1 \tag{5.18}$$

for each one of the three diagonal term of the resulting matrix.

We have performed these calculations with the help of the software *Mathematica*. These procedure yields seven different non-trivial combinations of coefficients and some of them are really complex. When we mention the term *combination* we mean

a relation between the components of $\vec{a}$ that could satisfy (5.18). Anyway, it is possible to show that, for the majority of these combinations, it does not exist a numeric range for the coefficients in which the positivity of the eigenvalues is preserved for any time and any starting point inside the triangle. Therefore, they are solutions that we have to reject. As an example of these considerations, we report the combination:

$$\vec{a} = \left(\frac{1}{2}(4a_1 + a_2 + 3a_3 - 2),\ a_1,\ a_2,\ a_3,\ a_1 - a_2 + a_3,\ \frac{1}{2}(-2a_1 - a_2 - a_3)\right) \quad (5.19)$$

As we can see, in this particular solution of (5.18), the only coefficients that are free are a_1 , a_2 and a_3 and the others are linear combinations of these three.
Nevertheless, evaluating the limit for $t \to \infty$ of the resulting open evolution $\clubsuit^{\vec{a}}_{(t,0)}(\boldsymbol{\rho_0})$, we get:

$$\lim_{t\to\infty} \clubsuit^{\vec{a}}_{(t,0)}(\boldsymbol{\rho_0}) = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{(2a_1-a_2+a_3)(3p_0-1)}{6(a_1+a_3)} + q_0 & 0 \\ 0 & 0 & -\frac{-3p_0a_2+a_2+6a_1(p_0-1)+a_3(3p_0-5)}{6(a_1+a_3)} - q_0 \end{pmatrix} \quad (5.20)$$

and the system:

$$\begin{cases} \frac{(2a_1-a_2+a_3)(3p_0-1)}{6(a_1+a_3)} + q_0 \geq 0 \\ -\frac{-3p_0a_2+a_2+6a_1(p_0-1)+a_3(3p_0-5)}{6(a_1+a_3)} - q_0 \geq 0 \qquad \forall p_0, q_0 \in [0,1]\ , p_0 + q_0 \leq 1 \\ a_1 + a_3 \neq 0 \end{cases} \quad (5.21)$$

has no real solutions in a_0, a_1 and a_2 .
The trajectory of this evolution is depicted in fig. 5.3.

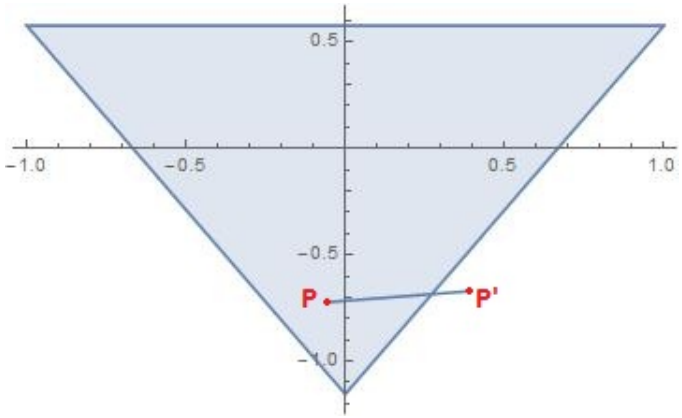

Figure 5.3: *The starting point P corresponds to the density matrix with eigenvalues: $p_0 = 0.1$, $q_0 = 0.15$; the arrival point P' , that is the result of the open evolution, has instead $p_0 = 0.333$, $q_0 = -0.0542$*

Therefore, as we can see, the breaking of the positivity condition reflects into the overtaking of the borders of our triangle representing density matrices.

Thus, the evaluation of systems of inequalities like (5.21) for all the seven combinations leads us to just four combinations that satisfy also the positivity condition for all the possible starting points in a certain range of the components of $\vec{a}$:

$$\begin{aligned}
&\vec{a}^1 = (2a_1 + a_3 - 1, a_1, a_1, a_3, -a_1, -a_1) \quad \text{with} \quad a_1 \neq -a_3 \\
&\vec{a}^2 = (2a_1 + a_4 - 1, a_1, a_1, -a_1, a_4, -a_1) \quad \text{with} \quad a_1 \neq -a_4 \\
&\vec{a}^3 = (2a_1 + a_5 - 1, a_1, a_1, -a_1, -a_1, a_5) \quad \text{with} \quad a_1 \neq -a_5 \\
&\vec{a}^4 = (a_1 - 1, a_1, a_1, a_3, a_3, a_3) \quad \forall a_1, a_3 \in \Re
\end{aligned} \tag{5.22}$$

Moreover, we can ascertain that, for all these four combinations, what really matters is the relation between the components, that satisfies all the above constraints, and not the specific value of the coefficients, that does not infer the dynamic. To clarify this fact we can consider the example of the evolution related to the combination $\vec{a}^1$:

$$\boldsymbol{\rho}^{\vec{a}^1}(t) = \begin{pmatrix} \frac{1}{2}e^{-t}\left(2p_0 - e^t(q_0 - 1) + q_0 - 1\right) & 0 & 0 \\ 0 & q_0 & 0 \\ 0 & 0 & -\frac{1}{2}e^{-t}\left(2p_0 + e^t(q_0 - 1) + q_0 - 1\right) \end{pmatrix} \tag{5.23}$$

this evolution is calculated keeping a generic expression of a_1 and a_3 but, as we can clearly see, the components of $\vec{a}^1$ are not involved in the expression.
Therefore, it seems logical and useful to simplify the combinations above in (5.22), choosing specific values for the coefficients and preserving effectively the same dynamic. Then, at the end of this procedure, we come to the final four correct open evolutions, in the following form:

$$\begin{aligned}
&\vec{a}^1 = (0,0,0,1,0,0) \ ; \ \vec{a}^2 = (0,0,0,0,1,0) \ ; \ \vec{a}^3 = (0,0,0,0,0,1) \\
&\vec{a}^4 = (0,1,1,0,0,0)
\end{aligned} \tag{5.24}$$

So, now that we have found the right evolutions, we can analyze the details of the dynamics. We can immediately see that the first three evolutions are related to the three rotation matrices R_3, R_4 and R_5 respectively, matrices that keep one eigenvalue fixed and mix the other two. Indeed, the corresponding open evolutions maintain unchanged in time one eigenvalue, and they describe straight orbits that are always parallel to the sides of the triangle. We can now study this three evolutions separately one by one :

- $\vec{a}^1 = (0,0,0,1,0,0)$:
  This particular combination is related to just one Kraus operator :

$$\boldsymbol{K_3}(t) = \sqrt{\frac{1}{2}(1 - e^{-t})}R_3 \tag{5.25}$$

so that the orbit is described by the equation:

$$\boldsymbol{\rho}^{\vec{a}^1}(t) = \boldsymbol{K_0}^2(t)\boldsymbol{\rho_0}\boldsymbol{K_0}^{2\dagger}(t) + \boldsymbol{K_3}(t)\boldsymbol{\rho_0}\boldsymbol{K_3}^{\dagger}(t) =$$
$$= \begin{pmatrix} \frac{1}{2}e^{-t}\left(2p_0 - e^t(q_0-1) + q_0 - 1\right) & 0 & 0 \\ 0 & q_0 & 0 \\ 0 & 0 & -\frac{1}{2}e^{-t}\left(2p_0 + e^t(q_0-1) + q_0 - 1\right) \end{pmatrix} \tag{5.26}$$

Expanding the Kraus operator to the first order around $t = 0$ we get the Lindblad operator :

$$\boldsymbol{L_3} = \sqrt{\frac{1}{2}}R_3 \tag{5.27}$$

that leads us to the expression of the derivative of the evolution:

$$\boldsymbol{\mathcal{L}}_t(\boldsymbol{\rho}) = \dot{\boldsymbol{\rho}}^{\vec{a}^1} = \boldsymbol{L_3}\boldsymbol{\rho_0}\boldsymbol{L_3}^{\dagger} - \frac{1}{2}\{\boldsymbol{L_3}^{\dagger}\boldsymbol{L_3}, \boldsymbol{\rho_0}\} =$$
$$= \frac{1}{2}\begin{pmatrix} -2p_0 - q_0 + 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2p_0 + q_0 - 1 \end{pmatrix} \tag{5.28}$$

We can show these results with the graphic of fig. 5.4.

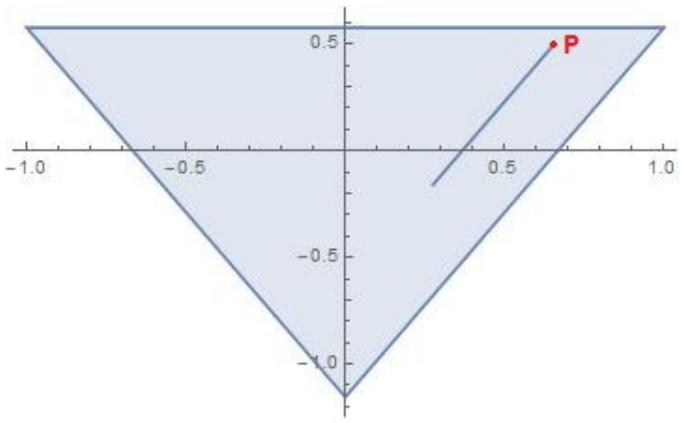

Figure 5.4: *Open evolution using* $\vec{a}^1$ *. The starting point P corresponds to the density matrix with eigenvalues:* $p_0 = 0.8$, $q_0 = 0.15$

The study of the possible ending points of trajectories is showed later.

- $\vec{a}^2 = (0, 0, 0, 0, 1, 0)$:
  This particular combination is related to just one Kraus operator :

$$\boldsymbol{K_4}(t) = \sqrt{\frac{1}{2}(1 - e^{-t})}R_4 \tag{5.29}$$

so that the orbit is described by the equation:

$$\boldsymbol{\rho}^{\vec{a}^2}(t) = \boldsymbol{K_0}^2(t)\boldsymbol{\rho_0}\boldsymbol{K_0}^{2\dagger}(t) + \boldsymbol{K_4}(t)\boldsymbol{\rho_0}\boldsymbol{K_4}^{\dagger}(t) =$$
$$= \begin{pmatrix} \frac{1}{2}e^{-t}\left(p_0 - q_0 + e^t(p_0+q_0)\right) & 0 & 0 \\ 0 & \frac{1}{2}e^{-t}\left(-p_0 + q_0 + e^t(p_0+q_0)\right) & 0 \\ 0 & 0 & -p_0 - q_0 + 1 \end{pmatrix} \tag{5.30}$$

Expanding the Kraus operator to the first order around $t = 0$ we get the Lindblad operator :

$$\boldsymbol{L_4} = \sqrt{\frac{1}{2}}R_4 \tag{5.31}$$

that leads us to the expression of the derivative of the evolution:

$$\boldsymbol{\mathcal{L}}_t(\boldsymbol{\rho}) = \dot{\boldsymbol{\rho}}^{\vec{a}^1} = \boldsymbol{L_4}\boldsymbol{\rho_0}\boldsymbol{L_4}^{\dagger} - \frac{1}{2}\{\boldsymbol{L_4}^{\dagger}\boldsymbol{L_4}, \boldsymbol{\rho_0}\} =$$
$$= \frac{1}{2}\begin{pmatrix} q_0 - p_0 & 0 & 0 \\ 0 & p_0 - q_0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.32}$$

We can show these results with the graphic in fig. 5.5.

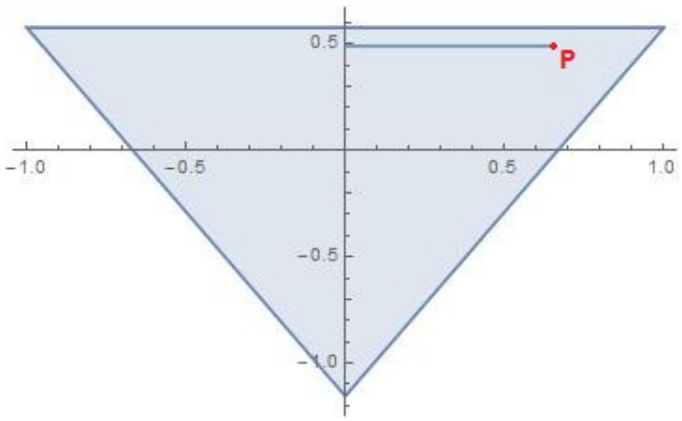

Figure 5.5: *Open evolution using $\vec{a}^2$. The starting point P corresponds to the density matrix with eigenvalues: $p_0 = 0.8$, $q_0 = 0.15$*

- $\vec{a}^4 = (0,0,0,0,0,1)$:
  This particular combination is related to just one Kraus operator :

$$\boldsymbol{K_5}(t) = \sqrt{\frac{1}{2}(1 - e^{-t})}R_5 \tag{5.33}$$

so that the orbit is described by the equation:

$$\boldsymbol{\rho}^{\vec{a}^4}(t) = \boldsymbol{K_0}^2(t)\boldsymbol{\rho_0}\boldsymbol{K_0}^{2\dagger}(t) + \boldsymbol{K_5}(t)\boldsymbol{\rho_0}\boldsymbol{K_5}^{\dagger}(t) =$$
$$= \begin{pmatrix} p_0 & 0 & 0 \\ 0 & \frac{1}{2}e^{-t}\left(-e^t(p_0-1) + p_0 + 2q_0 - 1\right) & 0 \\ 0 & 0 & -\frac{1}{2}e^{-t}\left(e^t(p_0-1) + p_0 + 2q_0 - 1\right) \end{pmatrix} \tag{5.34}$$

Expanding the Kraus operator to the first order around $t = 0$ we get the Lindblad operator :

$$\boldsymbol{L_5} = \sqrt{\frac{1}{2}} R_5 \tag{5.35}$$

that leads us to the expression of the derivative of the evolution:

$$\begin{aligned} \boldsymbol{\mathcal{L}}_t(\boldsymbol{\rho}) = \dot{\boldsymbol{\rho}}^{\vec{a}^3} = \boldsymbol{L_5 \rho_0 L_5}^\dagger - \frac{1}{2}\{\boldsymbol{L_5}^\dagger \boldsymbol{L_5}, \boldsymbol{\rho_0}\} = \\ = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -p_0 - 2q_0 + 1 & 0 \\ 0 & 0 & p_0 + 2q_0 - 1 \end{pmatrix} \end{aligned} \tag{5.36}$$

We can show these results with the graphic in fig. 5.6.

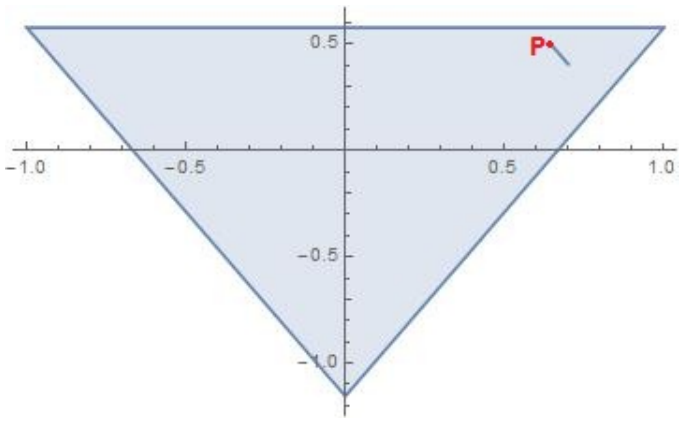

Figure 5.6: *Open evolution using* $\vec{a}^3$ *. The starting point P corresponds to the density matrix with eigenvalues:* $p_0 = 0.8$, $q_0 = 0.15$

Evaluating the limit for $t \to \infty$ for the orbit of each one of these three combinations we can easily draw the graphic of the three *asymptotes* in fig. 5.7.

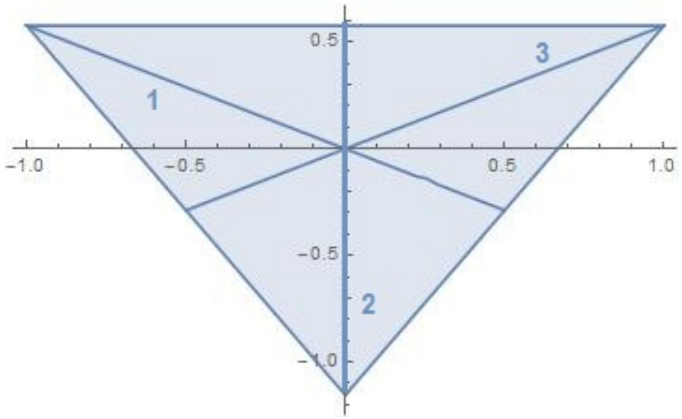

Figure 5.7: *with the number j we have denoted the asymptote corresponding to the combination* $\vec{a}^j$

We call them asymptotes because, starting from a generic point inside the triangle,

the trajectory continues straight parallel to one of the sides converging to the reachable point belonging to one of the blue lines drawn in fig. 5.7 (which one depends on the combination chosen for the evolution, that, as we have already seen, determines also the side of the triangle to follow) where the derivative of this specific evolution goes to zero, and then the dynamic stops. The asymptotes, that are the bisectors of the triangle, are composed by all the diagonal density matrices with two degenerate eigenvalues. The sets of density matrices that compose these special lines are:

$$\vec{a}^1 \to \begin{pmatrix} \frac{1-q_0}{2} & 0 & 0 \\ 0 & q_0 & 0 \\ 0 & 0 & \frac{1-q_0}{2} \end{pmatrix}; \quad \vec{a}^2 \to \begin{pmatrix} \frac{p_0+q_0}{2} & 0 & 0 \\ 0 & \frac{p_0+q_0}{2} & 0 \\ 0 & 0 & -p_0-q_0+1 \end{pmatrix}$$

$$\vec{a}^3 \to \begin{pmatrix} p_0 & 0 & 0 \\ 0 & \frac{1-p_0}{2} & 0 \\ 0 & 0 & \frac{1-p_0}{2} \end{pmatrix} \tag{5.37}$$

$$\forall p_0, q_0 \in [0,1] \, , p_0 + q_0 \leq 1$$

Starting the open evolution from one of these points only two of the three directions showed are accessible.
We can also claim that the treatment is completely specular in each of the three unitarly equivalent sections of the triangle, in fact we can see that, after a change of basis ruled by the unitary matrices in (5.8) , the transformed Kraus operators become:

$$\begin{aligned} &\boldsymbol{U_{12}}\boldsymbol{K_3}(t)\boldsymbol{U_{12}}^\dagger = \boldsymbol{K_5}(t) \; ; \; \boldsymbol{U_{12}}\boldsymbol{K_4}(t)\boldsymbol{U_{12}}^\dagger = \boldsymbol{K_4}(t) \; ; \; \boldsymbol{U_{12}}\boldsymbol{K_5}(t)\boldsymbol{U_{12}}^\dagger = \boldsymbol{K_3}(t) \\ &\boldsymbol{U_{13}}\boldsymbol{K_3}(t)\boldsymbol{U_{13}}^\dagger = \boldsymbol{K_3}(t) \; ; \; \boldsymbol{U_{13}}\boldsymbol{K_4}(t)\boldsymbol{U_{13}}^\dagger = \boldsymbol{K_5}(t) \; ; \; \boldsymbol{U_{13}}\boldsymbol{K_5}(t)\boldsymbol{U_{13}}^\dagger = \boldsymbol{K_4}(t) \\ &\boldsymbol{U_{23}}\boldsymbol{K_3}(t)\boldsymbol{U_{23}}^\dagger = \boldsymbol{K_4}(t) \; ; \; \boldsymbol{U_{23}}\boldsymbol{K_4}(t)\boldsymbol{U_{23}}^\dagger = \boldsymbol{K_3}(t) \; ; \; \boldsymbol{U_{23}}\boldsymbol{K_5}(t)\boldsymbol{U_{23}}^\dagger = \boldsymbol{K_5}(t) \end{aligned} \tag{5.38}$$

Then, the ambiguity of the unitary orbits is reduced to an exchange of role amongst these three operators, so that, for example, starting from the vertix P_1 the $\vec{a}^1$ evolution is the unitary equivalent of starting from P_2 the $\vec{a}^3$ evolution.

The last combination remained, $\vec{a}^4$, determines instead a completely different type of evolution.
This particular combination is related to the two Kraus operators :

$$\boldsymbol{K_1}(t) = \sqrt{\frac{1}{3}(1-e^{-t})}R_1 \;\text{ and }\; \boldsymbol{K_2}(t) = \sqrt{\frac{1}{3}(1-e^{-t})}R_2 \tag{5.39}$$

so that the orbit is described by the equation:

$$\boldsymbol{\rho}^{\vec{a}^4}(t) = \boldsymbol{K_0}^2(t)\boldsymbol{\rho_0}\boldsymbol{K_0}^{2^\dagger}(t) + \boldsymbol{K_1}(t)\boldsymbol{\rho_0}\boldsymbol{K_1}^\dagger(t) + \boldsymbol{K_2}(t)\boldsymbol{\rho_0}\boldsymbol{K_2}^\dagger(t) =$$
$$= \begin{pmatrix} \frac{1}{3}e^{-t}\left(3p_0 + e^t - 1\right) & 0 & 0 \\ 0 & \frac{1}{3}e^{-t}\left(3q_0 + e^t - 1\right) & 0 \\ 0 & 0 & \frac{1}{3}e^{-t}\left(-3p_0 + e^t - 3q_0 + 2\right) \end{pmatrix} \tag{5.40}$$

Expanding the Kraus operators to the first order around $t = 0$ we get the Lindblad operators :

$$\boldsymbol{L_1} = \sqrt{\frac{1}{3}}R_1 \;\; ; \;\; \boldsymbol{L_2} = \sqrt{\frac{1}{3}}R_2 \tag{5.41}$$

that lead us to the expression of the derivative of the open evolution:

$$\boldsymbol{\mathcal{L}}_t(\boldsymbol{\rho}) = \dot{\boldsymbol{\rho}}^{\vec{a}^4} = \boldsymbol{L_1}\boldsymbol{\rho_0}\boldsymbol{L_1}^\dagger - \frac{1}{2}\{\boldsymbol{L_1}^\dagger\boldsymbol{L_1}, \boldsymbol{\rho_0}\} + \boldsymbol{L_2}\boldsymbol{\rho_0}\boldsymbol{L_2}^\dagger - \frac{1}{2}\{\boldsymbol{L_2}^\dagger\boldsymbol{L_2}, \boldsymbol{\rho_0}\} =$$

$$= \begin{pmatrix} \frac{1}{3} - p_0 & 0 & 0 \\ 0 & \frac{1}{3} - q_0 & 0 \\ 0 & 0 & p_0 + q_0 - \frac{2}{3} \end{pmatrix} \tag{5.42}$$

As we can see, this is the only combination that changes all three the eigenvalues during the evolution.
We can show these results with the graphic in fig. 5.8.

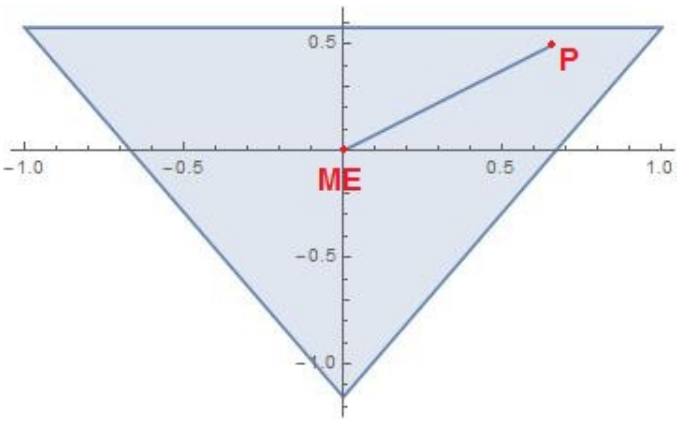

Figure 5.8: *Open evolution using $\vec{a}^4$. The starting point P corresponds to the density matrix with eigenvalues:* $p_0 = 0.8$, $q_0 = 0.15$

This type of evolution always converges toward the Maximum Entangled State, that is the asymptotic limit for every possible starting point.
The invariant nature of this evolution is evident also under unitary transformation,

in fact we have:

$$\boldsymbol{U_{12}K_1}(t)\boldsymbol{U_{12}}^\dagger = \boldsymbol{U_{13}K_1}(t)\boldsymbol{U_{13}}^\dagger = \boldsymbol{U_{23}K_1}(t)\boldsymbol{U_{23}}^\dagger = \boldsymbol{K_2}(t)$$
$$\boldsymbol{U_{12}K_2}(t)\boldsymbol{U_{12}}^\dagger = \boldsymbol{U_{13}K_2}(t)\boldsymbol{U_{13}}^\dagger = \boldsymbol{U_{23}K_2}(t)\boldsymbol{U_{23}}^\dagger = \boldsymbol{K_1}(t) \tag{5.43}$$

We always have the possibility to mix ad lib all the four open evolutions showed up to here. By way of example, in fig. 5.9 we show the resulting orbit obtained by composing all the three progressions parallel to the sides, from one asymptote to another one, with this very last evolution directed to the center.

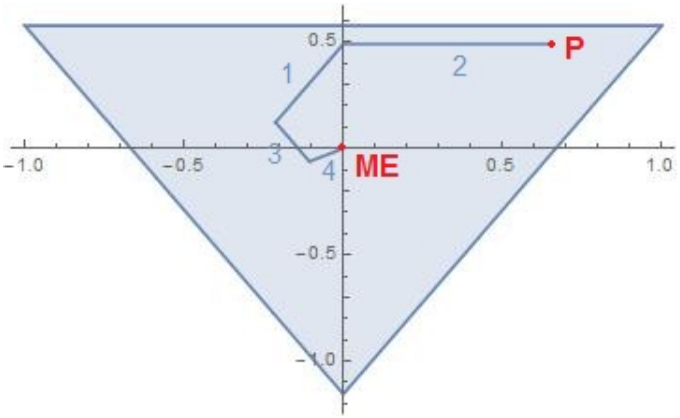

Figure 5.9: *Open mixed evolution using the coefficients $\vec{a}^i$ in each stretch denoted by the number i . The starting point P corresponds to the density matrix with eigenvalues: $p_0 = 0.8$, $q_0 = 0.15$*

This procedure allows us to discover one relevant property that, similarly to the $U(2)$ case, is an interesting behavior of any possible open motion. Indeed, the modulus of the vector associated to each density matrix in the representation chosen always decreases , regardless of the orbit constructed mixing the four correct evolutions.

Regarding the definition of the components of the vector (5.5), we note that the modulus, that has his maximum on the three vertices :

$$|\vec{S}_{P_1}| = |\vec{S}_{P_2}| = |\vec{S}_{P_3}| = \sqrt{\frac{4}{3}} \tag{5.44}$$

and his minimum on the Maximum Entangled State:

$$|\vec{S}_{ME}| = 0 \tag{5.45}$$

can be expressed in a useful way:

$$|\vec{S}(\boldsymbol{\rho})| = 2\sqrt{\frac{1}{3} - b_{\boldsymbol{\rho}}} = 2\sqrt{max_{\boldsymbol{\rho}} b_{\boldsymbol{\rho}} - b_{\boldsymbol{\rho}}} \tag{5.46}$$

where $b_{\boldsymbol{\rho}}$ is the quadratic coefficient of the characteristic polynomial $p_{\boldsymbol{\rho}}(x)$ of the diagonal density matrix $\boldsymbol{\rho}$, defined as:

$$p_{\boldsymbol{\rho}}(x) = -x^3 + b_{\boldsymbol{\rho}} x^2 - a_{\boldsymbol{\rho}} x + c_{\boldsymbol{\rho}} \tag{5.47}$$

with :

$$\begin{aligned} c_{\boldsymbol{\rho}} &\equiv det(\boldsymbol{\rho}) \\ a_{\boldsymbol{\rho}} &\equiv tr(\boldsymbol{\rho}) \\ b_{\boldsymbol{\rho}} &\equiv \rho^{(1,1)}\rho^{(2,2)} + \rho^{(1,1)}\rho^{(3,3)} + \rho^{(2,2)}\rho^{(3,3)} \end{aligned} \tag{5.48}$$

In a unitary evolution, because the spectrum remains the same, the three coefficients of this polynomial do not change; instead, in an open evolution, even if the value of the trace is constricted to 1, the other two coefficients vary in time, and, writing their expressions for all these four evolutions, it is possible to ascertain that they are increasing during the motion; always increasing also considering mixed evolutions from one asymptote to the other like the example showed in fig. 5.9.

Thus, from the expression in equation (5.46), we immediately deduce the correct decreasing behavior of the modulus of the vectors corresponding to density matrices, that constitutes a similarity with the $U(2)$ case.

Conclusions

To briefly summarize what we have done in each chapter of this work we present a schematic list :

1. Following the line of [10] we have discussed the context of open quantum systems from an algebraic point of view, introducing the concepts of density matrices, entanglement and quantum eraser. With Schmidt decomposition and GHJW theorem we have studied how to express open states and how to relate the different parts that compose an open system.

2. We reflected upon the measuring process in an open context, understanding the necessity of a new treatment that led us to the definition of a POVM.

3. We have studied the open evolution of subsystems , coherently with the approach used by [11], discussing the requirements and the implications of working with UDMs. The Markovian approximation has been illustrated with its physical meaning, and by means of this approximation it has been possible to present the Lindblad equation. We have also noticed an interesting connection between UDMs and POVMs.

4. We have recollected some geometrical results from [8] concerning the Kähler structure of the space of pure states, mixed states, and the realification of Hilbert space. We have also showed an important way to describe the unitary orbits and their geometrical properties, allowing to geometrically understand the meaning of open evolutions. In the second part of this chapter we have applied the theory for the concrete example of 2x2 density matrices, connected with the Q-bits world.

5. Checking the analytical calculations with the help of the software *Mathematica* we have found a new explicit formula to parametrize the Kraus operators for the 3x3 density matrices. It has been showed that this particular formula satisfies all the UDMs conditions, so that it has led us to correct solutions of the Lindblad equation in that specific case. These solutions have been

studied in detail and the corresponding open trajectories have been illustrated in graphics with the help of the same software.

Open questions and the U(n) case

Just before concluding this work, we want to underline some implications on generalizing the treatment used for the $U(3)$ case for the $U(n)$ problem.
Because of the generality of this procedure, and the self-normalizing nature of the Kraus operators showed, the idea of using the rotation matrices equipped with opportune coefficients that could satisfy all the UDMs properties can always be applied in a generic dimension. The structure of the time-depending part of the Kraus operators could remain completely unchanged and it continues to appear valid, satisfying automatically the trace one requirement at any time. The only obstacle to mechanically follow the same procedure is the complexity of the numeric calculation, that could require softwares more and more performing. Indeed, in a n-dimensional context we have $n!$ permutation of the eigenvalues of density matrices, so that our vector of coefficients for Kraus operators becomes $\vec{a} \in \Re^{n!}$ and, just to selecting all the admissible combinations of the components that could satisfy the n-dimension analogous of equation (5.18), we need a huge computing power.

Anyhow, resolving the $U(3)$ case ensures us the existence of some known correct motions also in higher dimensions. In fact, the four evolutions founded could be reproduced almost identical when some eigenvalues of the n-dimensional density matrix are kept fixed. Indeed, as we have seen previously, the first three evolutions connected with $\vec{a}^1$, $\vec{a}^2$ and $\vec{a}^3$ resolve completely the eventuality of an evolution characterized by only two eigenvalues free to change. Therefore, in a n-dimension context, we can always think of using a Kraus operator like the following one:

$$\boldsymbol{K_i}(t) = \sqrt{\frac{1}{2}(1-e^{-t})} \begin{pmatrix} 1_{(n-3)\times(n-3)} & 0 \\ 0 & R_i \end{pmatrix} \quad \text{with} \;\; i = 1,2,3 \tag{5.49}$$

that, combined with the usual identity Kraus operator:

$$\boldsymbol{K_0}^2(t) = \sqrt{\frac{1}{2}(1+e^{-t})} 1_{n\times n} \tag{5.50}$$

gives us a correct open evolution.
Because all three R_3 , R_4 and R_5 has a 1 on the main diagonal, the number of these types of evolutions is : $\frac{n!}{(n-2)!2}$.
At the same way, we can extend to n dimensions the last remaining $U(3)$ evolution, that moves three eigenvalues keeping the others fixed and that is related to the

coefficient vector $\vec{a}^4$. The latter can be done using the two Kraus operators:

$$\begin{aligned} \boldsymbol{K_1}(t) &= \sqrt{\frac{1}{3}(1-e^{-t})} \begin{pmatrix} 1_{(n-3)\times(n-3)} & 0 \\ 0 & R_1 \end{pmatrix} \\ \boldsymbol{K_2}(t) &= \sqrt{\frac{1}{3}(1-e^{-t})} \begin{pmatrix} 1_{(n-3)\times(n-3)} & 0 \\ 0 & R_2 \end{pmatrix} \end{aligned} \tag{5.51}$$

that, combined with the usual identity Kraus operator:

$$\boldsymbol{K_0}^2(t) = \sqrt{\frac{1}{3}(1+2e^{-t})} 1_{n\times n} \tag{5.52}$$

gives us a correct open evolution.
The number of this type of evolutions is : $\frac{n!}{(n-3)!3!}$.

Therefore, by solving the $U(3)$ case we have guaranteed

$$\frac{n!}{(n-2)!2} + \frac{n!}{(n-3)!3!} = \frac{1}{6}n(n+1)(n-1) \tag{5.53}$$

valid solutions of Lindblad equation in any dimension.

Thus, our hope is that the basis lied by this work could help for the building of an advantageous iterative method for a complete theorization of the generic $U(n)$ case that could curtail the burdensome amount of calculations.

Appendix A

Quantum Markov process: mathematical structure

We premise that, the attitude taken in treating these mathematical arguments is not so rigorous here, but it pursues the objective of giving some brief but useful clarifications just about the tools that have been used on this work. For a more depth approach to Quantum Markov process, the interested reader could consult [3]. Before treating the quantum topic, we remind briefly the concept of classical Markovian process. We define a *stochastic process* as a variable X that takes random values depending from a parameter t (for our purpose, t can be considered as the usual time):

Definition A.1 (Stochastic process)**.**

$$\{X(t); t \in I \subset \Re\} \tag{A.1}$$

If we suppose that the I set is limited, and its elements can be labeled with integer numbers, then we affirm that the stochastic process X is a Markovian process if the value assumed by X at any arbitrary time t_n depends only on the value that X took at time t_{n-1} , forgetting all the other values taken before that. Mathematically speaking, in terms of conditional probabilities, we have:

Definition A.2 (Markovian process)**.**

$$p(x_n, t_n | x_{n-1}, t_{n-1}; \ldots; x_0, t_0) = p(x_n, t_n | x_{n-1}, t_{n-1}) \quad \forall t_n \in I \tag{A.2}$$

Now we can imagine that the range I is continuous, so that $x_{n-1}, t_{n-1} \equiv x', t'$ can be considered as infinitesimally close to $x_n, t_n \equiv x, t$. At this point, just from the definition of conditional probability, we obtain:

$$p(x, t \surd x', t') = p(x, t | x', t') p(x', t') \tag{A.3}$$

where we used the symbol $\surd$ to represent the joint probability that the random variable takes both the values x at time t and x' at time t'. Therefore, to calculate

the probability of $p(x,t)$ we can just integrate the last expression with respect to x' :

$$p(x,t) = \int dx' p(x,t|x',t')p(x',t') \equiv \int dx' K(x,t|x',t')p(x',t') \tag{A.4}$$

where we have defined $p(x,t|x',t') \equiv K(x,t|x',t')$.
Then we can give the definition of an homogeneous Markovian process:

Definition A.3 (Homogeneous Markovian process)**.** A Markovian process is said to be homogeneous iff $K(x,t|x',t')$ is only a function of the difference between the two time parameters involved, so that:

$$K(x,t|x',t') = K_{t-t'}(x|x') \tag{A.5}$$

Moreover, we can repeat the same process but involving three different times : $t_3 > t_2 > t_1$ and three corresponding different values of the random variable. Applying the usual definition of conditional probability yields :

$$\begin{aligned} p(x_3,t_3\sqrt{x_2},t_2\sqrt{x_1},t_1) &= p(x_3,t_3|x_2,t_2\sqrt{x_1},t_1)p(x_2,t_2\sqrt{x_1},t_1) \\ &= p(x_3,t_3|x_2,t_2\sqrt{x_1},t_1)p(x_2,t_2|x_1,t_1)p(x_1,t_1) \end{aligned} \tag{A.6}$$

But the Markov condition (A.2) implies that:

$$p(x_3,t_3|x_2,t_2\sqrt{x_1},t_1) = p(x_3,t_3|x_2,t_2) \tag{A.7}$$

so that we have:

$$\begin{aligned} &p(x_3,t_3\sqrt{x_2},t_2\sqrt{x_1},t_1) = p(x_3,t_3|x_2,t_2)p(x_2,t_2|x_1,t_1)p(x_1,t_1) \\ &p(x_3,t_3\sqrt{x_2},t_2|x_1,t_1)p(x_1,t_1) = p(x_3,t_3|x_2,t_2)p(x_2,t_2|x_1,t_1)p(x_1,t_1) \\ &p(x_3,t_3\sqrt{x_2},t_2|x_1,t_1) = p(x_3,t_3|x_2,t_2)p(x_2,t_2|x_1,t_1) \end{aligned} \tag{A.8}$$

Now we can integrate the last equation with respect to x_2, obtaining:

$$\begin{aligned} p(x_3,t_3|x_1,t_1) &= \int dx_2\ p(x_3,t_3|x_2,t_2)p(x_2,t_2|x_1,t_1) \\ &= \int dx_2\ K(x_3,t_3|x_2,t_2)K(x_2,t_2|x_1,t_1) \end{aligned} \tag{A.9}$$

and this equation is called the *Chapman-Kolmogorov equation*:

Definition A.4 (Chapman-Kolmogorov equation)**.**

$$K(x_3,t_3|x_1,t_1) = \int dx_2\ K(x_3,t_3|x_2,t_2)K(x_2,t_2|x_1,t_1) \tag{A.10}$$

We can see that, all that this equation claims is that, under Markovian hypothesis, the evolution in time of the $K(x,t|x',t')$ is governed by a composition rule. So it becomes clear that this classical treatment can be easily extended to a quantum setting. In fact, looking at the equation (A.4) , it can be easy to imagine the $K(x,t|x',t')$ as the propagators of time evolutions from time t' to time t. Indeed,

since they are conditional probabilities , one can say that $K(x,t|x',t')$ connects the probability $p(x',t')$ to $p(x,t)$. Moreover, because they are probabilities , they are always positive and normalized. So it is really intuitive to identify, in a quantum context, these $K(x,t|x',t')$ with the UDMs $\clubsuit_{(t,t')}$. Therefore, in such a parallelism , the role played here by the probabilities $p(x,t)$ is played by the density operators $\boldsymbol{\rho}(t)$.

Once we get into this quantum extension of the concepts above, we can try to observe the behavior of the time evolution of density operators under Markovian assumptions. To do that, for positive ϵ, we express the difference :

$$\boldsymbol{\rho}(t+\epsilon)-\boldsymbol{\rho}(t)=[\clubsuit_{(t+\epsilon,0)}-\clubsuit_{(t,0)}]\boldsymbol{\rho}(0)=[\clubsuit_{(t+\epsilon,t)}-\mathbf{1}]\clubsuit_{(t,0)}\boldsymbol{\rho}(0)=[\clubsuit_{(t+\epsilon,t)}-\mathbf{1}]\boldsymbol{\rho}(t) \tag{A.11}$$

Then, assuming that the time evolution is sufficiently smooth, we can perform the limit for $\epsilon\to 0$ of the above equation, trying to reach a linear differential equation for $\boldsymbol{\rho}(t)$:

$$\frac{d\boldsymbol{\rho}(t)}{dt}=\lim_{\epsilon\to 0}\frac{\boldsymbol{\rho}(t+\epsilon)-\boldsymbol{\rho}(t)}{\epsilon}=\lim_{\epsilon\to 0}\frac{\clubsuit_{(t+\epsilon,t)}-\mathbf{1}}{\epsilon}\boldsymbol{\rho}(t)\equiv\boldsymbol{\mathcal{L}}_t\boldsymbol{\rho}(t) \tag{A.12}$$

This equation is often called *master equation* and the operator $\boldsymbol{\mathcal{L}}_t$ is the *generator* of the evolution, or *Lindbladian*, defined by:

Definition A.5 (Lindbladian)**.**

$$\boldsymbol{\mathcal{L}}_t=\lim_{\epsilon\to 0}\frac{\clubsuit_{(t+\epsilon,t)}-\mathbf{1}}{\epsilon} \tag{A.13}$$

Under Markovian assumption , the form of the master equation is unique and it is defined by the following, important, theorem, that concludes this appendix:

Theorem A.0.1.
A differential equation is a Markovian master equation if and only if it can be written in the form:

$$\frac{d\boldsymbol{\rho}(t)}{dt}=-i[\boldsymbol{H}(t),\boldsymbol{\rho}(t)]+\sum_k\gamma_k(t)\Big[\boldsymbol{L_k}(t)\boldsymbol{\rho}(t)\boldsymbol{L}^\dagger{}_k(t)-\frac{1}{2}\{\boldsymbol{L}^\dagger{}_{\boldsymbol{k}}(t)\boldsymbol{L_k}(t),\boldsymbol{\rho}(t)\}\Big] \tag{A.14}$$

where $\boldsymbol{H}(t)$ and $\boldsymbol{L_k}(t)$ are time-dependent operators and $\boldsymbol{H}(t)$ is self-adjoint. The $\gamma_k(t)$ are complex coefficients that satisfy: $\gamma_k(t)\geq 0\quad\forall t,k$.

Proof. We start considering a generic UDM, $\clubsuit_{(t_2,t_1)}$, with $t_2>t_1$. We know that it can expressed in a Kraus decomposition:

$$\clubsuit_{(t_2,t_1)}(\boldsymbol{\rho})=\sum_\alpha\boldsymbol{K_\alpha}(t_2,t_1)\boldsymbol{\rho}\boldsymbol{K_\alpha}^\dagger(t_2,t_1) \tag{A.15}$$

Now we choose a basis $\{\boldsymbol{F_j};j=1,2,\cdots,N^2\}$ for the space of the operators acting on density matrices, that have dimension N^2 , where N is the dimension of the

density matrices. We want this basis to be orthonormal, and then we choose an inner product between these operators, that we call the Hilbert-Schmidt (HS) inner product:

$$(\boldsymbol{F_i}, \boldsymbol{F_j})_{HS} \equiv tr(\boldsymbol{F_i}\boldsymbol{F_j}) \equiv \delta_{ij} \tag{A.16}$$

We make another particular choice : $\boldsymbol{F_{N^2}} \equiv \frac{1}{\sqrt{N}}\boldsymbol{1_{N^2}}$; so that, because of the orthonormality of the basis, we get $tr(\boldsymbol{F_j}) = 0 \quad \forall j \neq N^2$. With these premises, expanding equation (A.15) in such a basis yields:

$$\clubsuit_{(t_2,t_1)}(\boldsymbol{\rho}) = \sum_{j,k} c_{jk}(t_2,t_1)\boldsymbol{F_j}\boldsymbol{\rho}\boldsymbol{F_k}^\dagger \tag{A.17}$$

where the coefficients :

$$c_{jk}(t_1,t_2) = \sum_\alpha \big((\boldsymbol{F_j}, \boldsymbol{K_\alpha}(t_2,t_1))_{HS}\big)\big((\boldsymbol{F_k}, \boldsymbol{K_\alpha}(t_2,t_1))_{HS}\big)^* \tag{A.18}$$

form a matrix $N^2 \times N^2$ that we call $\boldsymbol{c}(t_2,t_1)$.
$\boldsymbol{c}(t_2,t_1)$ is a semi-positive matrix; in fact, for every N^2-vector v, we have:

$$(v, \boldsymbol{c}(t_2,t_1)v) = \sum_{j,k} v^*{}_j c_{jk}(t_1,t_2)v_k = \sum_\alpha \left| \sum_k v_k (\boldsymbol{F_k}, \boldsymbol{K_\alpha}(t_2,t_1))_{HS} \right|^2 \tag{A.19}$$

Since we said that $t_2 > t_1$ we can pick $t_2 = t + \epsilon$ and $t_1 = t$, with ϵ real positive number. Let us now calculate the expression of the Lindbladian in the usual way, but expressing the UDM $\clubsuit_{(t+\epsilon,t)}$ in the Kraus decomposition with the basis showed above:

$$\boldsymbol{\mathcal{L}}_t = \lim_{\epsilon\to 0} \sum_{j,k} \frac{c_{jk}(t+\epsilon,t)\boldsymbol{F_j}\boldsymbol{\rho}\boldsymbol{F_k}^\dagger - \boldsymbol{1}}{\epsilon} \tag{A.20}$$

and now we can make use of the useful definition of the basis chosen, separating and making explicit the different components of this sum:

$$\begin{aligned}\boldsymbol{\mathcal{L}}_t(\boldsymbol{\rho}) = \lim_{\epsilon\to 0} \Bigg[& \frac{1}{N}\frac{c_{N^2N^2}(t+\epsilon,t) - N}{\epsilon}\boldsymbol{\rho} + \\ & + \frac{1}{\sqrt{N}} \sum_{k=1}^{N^2-1} \Big(\frac{c_{kN^2}(t+\epsilon,t)}{\epsilon}\boldsymbol{F_k}\boldsymbol{\rho} + \frac{c_{N^2k}(t+\epsilon,t)}{\epsilon}\boldsymbol{\rho}\boldsymbol{F_k}^\dagger \Big) + \\ & + \sum_{j,k=1}^{N^2-1} \frac{c_{jk}(t+\epsilon,t)}{\epsilon}\boldsymbol{F_j}\boldsymbol{\rho}\boldsymbol{F_k}^\dagger \Bigg]\end{aligned} \tag{A.21}$$

and this allows us to define some new time-dependents coefficients $a_{jk}(t)$:

$$\begin{aligned} a_{N^2N^2}(t) &\equiv \lim_{\epsilon\to 0} \frac{c_{N^2N^2}(t+\epsilon,t) - N}{\epsilon} \\ a_{kN^2}(t) &\equiv \lim_{\epsilon\to 0} \frac{c_{kN^2}(t+\epsilon,t)}{\epsilon} \quad for \;\; k = 1,2,\cdots,N^2-1 \\ a_{jk}(t) &\equiv \lim_{\epsilon\to 0} \frac{c_{jk}(t+\epsilon,t)}{\epsilon} \quad for \;\; j,k = 1,2,\cdots,N^2-1 \end{aligned} \tag{A.22}$$

and then some useful operators:

$$\begin{aligned}
\boldsymbol{M}(t) &= \frac{1}{\sqrt{N}} \sum_{k=1}^{N^2-1} a_{kN^2}(t) \boldsymbol{F_k \rho} \\
\boldsymbol{G}(t) &= \frac{1}{N} \frac{a_{N^2N^2}(t)}{2N} \mathbf{1}_{N^2} + \frac{1}{2}[\boldsymbol{M}^\dagger(t) + \boldsymbol{M}(t)] \\
\boldsymbol{H}(t) &= \frac{1}{2i}[\boldsymbol{M}^\dagger(t) - \boldsymbol{M}(t)]
\end{aligned} \tag{A.23}$$

where the last one operator $\boldsymbol{H}(t)$ is clearly self-adjoint.
At this point, we can notice that, by means of these definitions, it is possible to rearrange equation (A.21) in the form:

$$\boldsymbol{\mathcal{L}}_t(\boldsymbol{\rho}) = -i[\boldsymbol{H}(t), \boldsymbol{\rho}] + \{\boldsymbol{G}(t), \boldsymbol{\rho}\} + \sum_{j,k=1}^{N^2-1} a_{jk}(t) \boldsymbol{F_j \rho F_k}^\dagger \tag{A.24}$$

We can get an important improvement in this equation above by imposing some properties of UDMs on the evolution. In fact, if we want that this Lindbladian governs a UDM type of evolution, we have to ask $tr(\boldsymbol{\mathcal{L}}_t) = 0$, so that the trace of density matrices does not change during the time evolution. Therefore we can write:

$$tr\big(\boldsymbol{\mathcal{L}}_t(\boldsymbol{\rho})\big) = tr\left[\big(2\boldsymbol{G}(t) + \sum_{j,k=1}^{N^2-1} a_{jk}(t) \boldsymbol{F_k}^\dagger \boldsymbol{F_j}\big)\boldsymbol{\rho}\right] = 0 \tag{A.25}$$

and from this expression immediately follows:

$$\boldsymbol{G}(t) = -\frac{1}{2} \sum_{j,k=1}^{N^2-1} a_{jk}(t) \boldsymbol{F_k}^\dagger \boldsymbol{F_j} \tag{A.26}$$

so that (A.24) becomes:

$$\boldsymbol{\mathcal{L}}_t(\boldsymbol{\rho}) = -i[\boldsymbol{H}(t), \boldsymbol{\rho}] + \sum_{j,k=1}^{N^2-1} a_{jk}(t) \left[\boldsymbol{F_j \rho F_k}^\dagger - \frac{1}{2}\{\boldsymbol{F_k}^\dagger \boldsymbol{F_j}, \boldsymbol{\rho}\}\right] \tag{A.27}$$

At the end we notice that the semi-positive nature of the coefficients $c_{jk}(t+\epsilon, t)$ ensures the same nature for the $(N^2-1) \times (N^2-1)$ square matrix $a_{jk}(t)$ in such a way that it can be diagonalized , for any time t, by the action of one time-dependent unitary matrix $u(t)$; this is:

$$\sum_{j,k=1}^{N^2-1} u_{mj}(t) a_{jk}(t) u^*{}_{nk}(t) = \gamma_m(t) \delta_{mn} \quad \forall t \tag{A.28}$$

where each eigenvalue is positive or null: $\gamma_m(t) \geq 0 \quad \forall t, m$.
Extending the same unitary transformation to the operators too:

$$\boldsymbol{L}_k(t) \equiv \sum_{j=1}^{N^2-1} u^*{}_{kj}(t) \boldsymbol{F_j} \tag{A.29}$$

finally leads to the desired result:

$$\boldsymbol{\mathcal{L}}_t(\boldsymbol{\rho}) = \frac{d\boldsymbol{\rho}(t)}{dt} = -i[\boldsymbol{H}(t), \boldsymbol{\rho}(t)] + \sum_k \gamma_k(t)\left[\boldsymbol{L_k}(t)\boldsymbol{\rho}(t)\boldsymbol{L}^\dagger{}_k(t) - \frac{1}{2}\{\boldsymbol{L}^\dagger{}_{\boldsymbol{k}}(t)\boldsymbol{L_k}(t), \boldsymbol{\rho}(t)\}\right] \tag{A.30}$$

□

Since this differential equation is linear, there exists a continuous family of propagators $\clubsuit_{(t,t')}$ satisfying:

$$\begin{aligned} &\clubsuit_{(t_2,t_0)} = \clubsuit_{(t_2,t_1)}\clubsuit_{(t_1,t_0)} \quad \forall t_2 \geq t_1 \geq t_0 \\ &\textit{and} \\ &\clubsuit_{(t,t)} = \mathbf{1}_{N^2} \quad \forall t \end{aligned} \tag{A.31}$$

Even if we have currently used in the demonstration above the fact that these operators are UDMs, formally prove that they *really* are UDMs needs not only the trace preserving requirement (at first imposed in the demonstration and then correctly satisfied) but also the complete positivity one. Prove this last requirement rigorously is a bit tricky, and we refer the interested reader to [11].

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