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An Introduction to Credit Risk and Asset Pricing

Tesi di Laurea in Equazioni Differenziali Stocastiche

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Contents

Abstract

Into the Thesis, the author will try to give the basis of risk management and asset pricing. Both of them are fundamental elements to understand how the financial models work; this topic is judged important in the perspective of successive studies in financial math: having clear the starting point makes things easier. From the title it is clear that modern and more complex models will be only touched upon.

We decide to divide the dissertation in two different parts because, in our opinion, it is more evident that two different ways to approach at credit risk exist: on one side we try to quantify the risk deriving from giving credit, on the other we will establish a strategy that allows us to invest money with the aim to pay the other part of the agreement. Everything became more clear chapter by chapter. Financial institutions like banks are exposed at both of this type of risk.

Chapters 1 and 5 are the center of this thesis: they represent the zero point from which the modern models were originated.

Abstract

In questa tesi l'autore cercherà di affrontare i concetti della gestione del rischio e dell'asset pricing con l'obiettivo di mettere in evidenza i risultati fondamentali e lasciare al lettore una chiara idea dei processi di modellizzazione nel risk management. L'ottica con cui si consiglia di approcciarsi alle seguenti pagine è quella della ricerca dei concetti necessari per iniziare lo studio della finanza matematica, come suggerito dalla dicitura "Introduction" presente nel titolo.
Le due parti in cui è stata divisa la tesi rispecchiano l'animo della questione: da una parte si ha la necessità di quantificare quanto sia possibile perdere facendo credito, dall'altra si cerca di proteggersi dalla possibilità che, anticipando concetti che saranno chiari durante la lettura, sia il creditore stesso ad essere insolvente. Questa duplicità nasce dal tipo di operazioni che si compiono: il primo è un rischio intrinseco al credito, il secondo è conseguenza della casualità dei mercati. Istituzioni finanziarie come le banche sono un esempio in cui entrambi gli elementi sono presenti: esse prestano denaro facendosi carico di una possibile insolvenza del debitore e, al contempo, operano nel mercato azionario.
Cuore della trattazione sono i Capitoli 1 e 5 ove vengono presentati i punti di partenza delle più complesse teorie moderne.

To Laura

"What then shall we choose?
Weight or lightness?"

Milan Kundera

Ringraziamenti

Nonostante non sia coerente con la lingua dello scritto, per ringraziare chi mi è stato vicino nella stesura di questa tesi di laurea preferisco utilizzare la mia lingua madre. Principalmente per trasmettere al meglio il sentimento di gratitudine, ma anche per parlare di eventi che difficilmente troverebbero stesso respiro nello spirito di un'altra lingua. Prima di provare a dar forma a delle nuvole un po' sparse, vorrei utilizzare qualche parola per descrivere lo stato d'animo di chi scrive.
Questa tesi è certamente la conclusione formale di un'esperienza magnifica ed estremamente formativa quale è stata la laurea magistrale in matematica. Questi due anni, che mi ritrovo a dover descrivere brevemente, sono intrisi dell'animo di cui vive questa città, la nostra cara Bologna, città capace di cancellare uno ad uno ogni singolo timore di cui arrivai carico nella vecchia e cara panda nera. Qui, sebbene sia stato in grado di creare la mia indipendenza, ho potuto essere parte di qualcosa. Questa energia si è riversata in queste pagine, tra queste parole, in molte delle cose che mi è stata data la possibilità e la fortuna di fare.

Mi viene spontaneo iniziare pensando alla mia famiglia, anzitutto i miei genitori, Luisa e Franco (prima le donne!). In questi due anni, a pensarci bene, sono cambiate moltissime cose, non sono stati semplicemente la ripetizione per due di 365 giorni (in realtà il secondo anno è stato bisestile), ma qualcosa di più, forse la riscoperta del rapporto stesso. Tengo a ringraziarvi per primi perché, senza di voi, non potrei starmene qua, nella mia camera, a battere, col sorriso, questa tesi che, in un certo senso, rappresenta la miglior fine di un lungo percorso di studi. Siete condizione necessaria. Non è mai facile sostenere due figli fuorisede, ancora più difficile se lo si fa col sorriso tra le labbra ripetendo che l'obiettivo principale è permetterci (ora, piano piano, Marco sta cercando di entrare nei ringraziamenti) di studiare per iniziare a costruire, fin da subito, gli anni futuri. Forse sono stato un po' prolisso ma queste le parole sincere e d'istinto che sono scese.
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Si dice che tra amici, quelli che conosci da una vita (oramai 11 anni), bastino poche parole e qualche sguardo. Ecco, ad Andrea, Edoardo, Valerio ed Ado posso solo ricordare dei tanti sorrisi, cene o passeggiate nelle quali, in ogni occasione, la distanza fosse come azzerata. Ognuno di voi è venuto a trovarmi in questa Bologna, portando di volta in volta buon umore e nuovi stimoli, per questo vi sono grato.

Non credo siano sufficienti queste poche righe per poter far realizzare quanto sia stata e sia importante Laura. Qualsiasi cosa che scorre tra la mente porta un tuo contributo, sicuramente questa tesi ne è piena. Sei stata e sei, punto di riferimento per molte scelte, alcune delle quali non sarei stato in grado di prendere. Sorrido, dicendoti grazie, nel momento in cui penso ai pomeriggi con un caldo tè fumante a parlare di come dovessi comportarmi in quella situazione o come potessi fare per raggiungere quell’obiettivo. Dopo una laurea e una magistrale al tuo fianco e così tanta matematica ascoltata (ricordo benissimo la tua espressione quando una volta provai a ripeterti un esame) posso solo dire che senza te sarebbe stato tutto molto più complicato. Grazie, davvero. Potrei scrivere ancora, e ancora, ma i sentimenti sono un po' così, per descriverli vanno utilizzate le parole che hai nelle dita in quel momento.

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Introduction

It is commonly known that the credit risk plays a central role into firm planning and in firm models. Since the central business of the banks is either giving credit either making investments, every day, every banks exposes themselves to the intrinsic risk in the credit or in the randomness of the market. Following the numerous crisis, it does not matter if small or big, the finance corporations have tried more and more to reduce the riskiness of their operations; so, in this aim, *risk managment* was introduced. The credit risk measurement models are focused into the correlation between the elements of portfolios, into maximizing the gain by minimizing the losses and they try to indicate a "good" way to follow. These models are also made to valuate the riskiness of a certain credit exposure or of a whole exposure portfolio. First of all, we want to explain what "credit risk" exactly means.

Many different definitions of credit risk exist, nevertheless we present the only one we judge useful for our aim. Embracing the definition by [3], credit risk refers to

"the possibility that an ***unexpected change*** *in a counterparty's creditworthiness may generate a corresponding unexpected change in the* ***market value*** *of the associated credit exposure".*

In order to render this definition clear and without misunderstandings, we are going to analyze three fundamental concepts.

1. <u>Default and migration risk</u> − In short, credit risk includes the default risk and the migration risk. The first of them is associated to the insolvency of the borrower hence to the possibility that the borrower does not respect the credit line or interrupt the payments. The second one is tied to the deterioration in the borrower's credit rating.

2. <u>Risk is a random event</u> − Insomuch as this thought is trivial, we think

that it is important to stop to reflect on. The deterioration in the credit rating or the default of the borrower is both unpredictable events, so it is natural and important considering risk like a random event. Even if this words sounds obvious, it has been necessary many years to arrived to plug in risk inside firm models.

3. Credit Exposure – Last but not least it is the concept of credit exposure deriving from the other financial operation that the bank makes. So, the correct measurement of credit risk requires the valuation of the economic value of the exposure, but the most important bank's investment are connected to illiquid assets, which are not listed in the market. So internal asset-pricing model is necessary. For this reason, we are going to pledge many time to illustrate the indispensable mathematician instruments for building asset pricing models.

Chapter 1 – *The basics* is devoted to answer at "How can default risk be measured?". We will start introducing the fundamentals and step by step we will develop them up to give a stately mathematician structure as the concepts of portfolio or replicant strategy. We will conclude the chapter poring over two fundamental risk measures: the value at risk and the expected shortfall.

Chapter 2 – *Loss Distribution* generalizes the precedent treatise lifting up the lecture in a mathematician way. We will associate a distribution at the losses and after we will give the model with a random vector of default probability. It is used the word "mixture" for the two distributions because in both of them not only the losses variete but even the default probability.

Chapter 3 – *Non-Linear Portfolio* is strictly correlated at the Chapter 7. Indeed we will use the concepts introduced in *Hedging*. It would appear quite strange using tolls before introduced them, but it has been necessary for maintaining the continuity of notation and the treatise. We will touch upon only two methods in order to work with no-linear portfolio.

Chapter 4 – *Put and Call options* rapidly presents the Put options and the Call options and the relation that exists between them by the $Put - Call\, parity\, formula$.

Chapter 5 – *A Discrete World* introduces the main concepts that we will generalize in the consecutive chapter. After a quick recall at the main definition, we will immediately enter in the heart of this second part: the asset pricing in a discrete way. We will spend a bit more time on the risk-neutral probability, then in the other idea; this because when we will pass at the continuos formulation, it will be important having clear what this measure is.

Chapter 6 – *A Continuos World* is the starting point for the asset pricing. Considering the reader practical with stochastic equation, we will try to give a deepened treatise of the Black-Sholes model and of the Merton model. The chapter also contains a brief hint to the equity case.

Chapter 7 – *Hedging* tries to introduce the reader at the procedure of hedging. Nowadays, hedging becomes more and more essential in the financial models, and, more generally, one chooses a strategy which can resist at the variations in those parameters from which the model depends on. Obviously, this chapter does absolutely not go in the deep of the matter; a huge literature of the matter exists. Who reads can see this chapter like a quick and light introduction to the practice of hedging.

Part I

Credit Risk Management

Chapter 1

The Basics

In this chapter we will give some tools and the relative interpretation for the risk-management. We will start considering only one asset into our portfolio and thus we are going to introduce *the loss* derived by an exposure in the credit. After talking about *expected loss*, i.e. the amount of money that the creditor foresees to lose in case obligor defaults, we will spend time to speak about *unexpected loss*, how to calculated it and its importance into the credit risk. We will terminate the chapter giving the definition of *value at risk*, in dependence of a specific level of confidence α, and the quantities that one can derive from it: the *economic capital* and the *expected shortfall.*

1.1 Expected Loss

The expected loss is the mean value of the probability of future losses. In order to estimate the loss on a credit exposure, a bank needs three parameters:

EAD The exposure at default. *This is a random variable represented by the current exposure, called* outstandings*, plus the possible variation of the amount of the loan which may occur from now to the date of possible default, said* commitments*. The commitments indicate the further credit which the bank has decided to give to the borrower only if he claims it.*

PD The probability of defaults.

LGD The lost given default: *how many parts of the debt the bank expects to loss if the borrower defaulted.*

Therefore, it results natural to define the loss of any obligor such as

$$\tilde{\mathbf{L}} = \mathbf{EAD} \times \mathbf{LGD} \times \mathbf{L} \tag{1.1}$$

where $L = \mathbb{1}_D$, D stands for the event *"The obligor defaults after a time T"* and PD is exactly P(D).

Definition 1.1.1. *Considering the variable $\tilde{L}$ as defined in (1.1), we call its expectation*

$$\mathbf{EL} = \mathbb{E}[\tilde{\mathbf{L}}] \tag{1.2}$$

the expected loss of the underlying credit-risky asset.

When the variables present in (1.1) are independent, by the property of mean value, the above formula becomes

$$EL = \mathbb{E}(EAD) \times \mathbb{E}(LGD) \times PD$$

In the case in which EAD and LGD are also constant, (1.2) takes place

$$EL = EAD \times LGD \times PD \tag{1.3}$$

In the real life not only EAD is not constant, in fact it usually depends on uncertainties in payment and on the chosen planning horizon, but even the three variables are hardly ever independent. First of all, the greater is the probability the obligor defaults, the greater is the possibility the creditor losses the amount of money he expected to loss (LGD); hence, to cover the risk of loss, the creditor starts to sell collaterals, triggering, due to the law of supply and demand, a chain process which terminates in the devaluation of the collaterals and so in the grew of the LGD (as it happened in the crisis of sub-prime loans). At the end, in hypothetic financial stress period, that is a period in which obligors hardly pay back landers, the creditors tend to redefine a new credit line for obligors exposing themselves to a major risk of default; hereby we can note a strict link between EAD and PD.

We end this part on expected loss with a very quick discussion about PD, EAD and LGD.

1.1.1 PD

Computing default probability is the starting point in order to have a good model that works out credit risk of the obligor. The most important way for getting the PD value is the rating. Originally, rating was used to put firms on an ordinal scale by credit quality. Rating agencies do not directly assign PDs to rated clients but they assign ratings and from them they obtain the probability of default. Therefore, one should pay attention when dealing with these two different objects. For the ones interested in the parameters which fixed the rating, these sites are available:

www.moodys.com

www.standardandpoors.com

www.fitchratings.com

Summarizing, a rating grade and its assigned default probability address the creditworthiness of a client. Whereas a detailed study of ratings isn't in the aim of this thesis, in the sequel we only list the four broad categories in which the world of the rating system can be divided:

1. Casual Rating Systems
2. Balance Sheet Scorings
3. Private Client Scorings
4. Expert Rating Systems

The procedure by means of we give out a PD from a rating grade is named *calibration*.

1.1.2 EAD

The exposure at default specifies the exposure the bank does have to its borrowers. Banks stipulate with obligors the so called credit lines, which work like a credit limit for the single-obligor exposure. The part of credit line the borrower has already taken, is said *outstandigs*, instead the part the borrower can request is called *commitments*. In other words, the bank fixes a credit limit for the applicant and divides this limit in two parts: a drawn part, i.e. the amount of money the borrower will receive immediately, which is what we have called outstandings and a requesting part, namely the credit the bank has established to lend to the debtor only if he will require it. In this context, randomness doesn't get involved, fixed is fixed, and in the case obligor's default takes place, outstandings are apt to recovery and it could be lost in total. We have to consider the exposure originated from the open part of the credit line like a random variable.

1.1.3 LGD

Before all, it is important to clarify that one can see LGD either as an amount of money and as a percentage quote. Factors affecting LGD can be grouped in four main categories:

(I) The characteristic of credit exposure;

(II) The characteristic of the borrower;

(III) The peculiarities of the bank managing the recovery process;

(IV) External factors.

Despite the theme is interesting and very debated, we are not going to deepen it more: we are only introducing the mean instrument for talking about risk and leaving an idea of what we will do in the next section.

1.2 Unexpected Loss

In the previous section we have introduced the expected loss but it's likewise natural to measure potential unexpected loss that is how much the EL is trustworthy. A possible application of this value might be the valuation of the liquidity which is necessary to the bank in order to cover itself from unforeseen losses.

As a measure for the variation of EL, the standard deviation of $\tilde{L}$, defined in (1.1), is the natural first choice.

Definition 1.2.1. *We call* unexpected loss *of the underlying loan or asset the standard deviation*

$$UL = \sqrt{\mathbb{V}[\tilde{L}]} = \sqrt{\mathbb{V}[EAD \times LGD \times L]}$$

Proposition 1.2.1. *If EAD is deterministic and the LGD and the default event D are independent, the unexpected loss of a loan is given by*

$$UL = EAD \times \sqrt{\mathbb{V}[LGD] \times PD + \mathbb{E}^2 \times PD(1 - PD)}$$

Proof. Foremost, squared the UL

$$\begin{aligned} UL^2 &= \mathbb{V}[EAD \times LGD \times L] \\ &= EAD^2 \times \mathbb{V}[LGD \times L]. \end{aligned}$$

From $\mathbb{V}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ we get

$$\begin{aligned}
\mathbb{V}[LGD \times L] &= \mathbb{E}[LGD^2 \times L^2] - \mathbb{E}[LGD \times L]^2 \\
&= \mathbb{E}[LGD^2]\mathbb{E}[L^2] - \mathbb{E}[LGD]^2\mathbb{E}[L]^2 \\
&= \mathbb{E}[LGD^2]\mathbb{P}[D] - \mathbb{E}[LGD]^2\mathbb{P}[D]^2 \\
&= \mathbb{E}[LGD^2]PD - \mathbb{E}[LGD]^2 PD^2 \\
&= \mathbb{E}[LGD^2]PD - \mathbb{E}[LGD]^2 PD^2 + \mathbb{E}[LGD]^2 PD - \mathbb{E}[LGD]^2 PD \\
&= PD\big(\mathbb{E}[LGD^2] - \mathbb{E}[LGD]^2\big) + \mathbb{E}[LGD]^2 PD\big(1 - PD\big) \\
&= \mathbb{V}[LGD]PD + \mathbb{E}[LGD]^2 PD\big(1 - PD\big)
\end{aligned}$$

via the independence in hypothesis and for the definition of L.

□

Let us to consider a family of m loans

$$\tilde{L}_i = EAD_i \times LGD_i \times L_i$$

with $L_i = \mathbb{1}_{D_i}$, $\mathbb{P}(D_i) = PD_i$. This family is said *portfolio*.

Definition 1.2.2. *The portfolio loss is a random variable defined as*

$$\tilde{L}_{PF} = \sum_{i=1}^{m} \tilde{L}_i = \sum_{i=1}^{m} \Big(EAD_i \times LGD_i \times L_i\Big)$$

Definition 1.2.3. *Given a portfolio of m loss variable, the expected and unexpected loss of the portfolio are*

$$EL_{PF} = \mathbb{E}[\tilde{L}_{PF}] \qquad \textit{and} \qquad UL_{PF} = \sqrt{\mathbb{V}[\tilde{L}_{PF}]}.$$

Remark 1.2.1. For a portfolio of m loans the EL is always given by

$$EL_{PF} = \sum_{i=1}^{m} EL_i$$

where EL_i indicates the expected loss of single loss $\tilde{L}_i$.

We usually have a correlation between variables inside the same portfolio so the unexpected loss isn't linear, therefore we enunciate the following result.

Proposition 1.2.2. *Taken m loss variables with deterministic EADs, we have*

$$UL_{PF} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{m}\Big(EAD_i \times EAD_j \times Cov\big[LGD_i \times L_i, LGD_j \times L_j\big]\Big)} \tag{1.4}$$

Proof. By means of the formula

$$\mathbb{V}\Big[\sum_{i=1}^{m} c_i X_i\Big] = \sum_{i=1}^{m}\sum_{j=1}^{m} c_i c_j Cov\big[X_i, X_j\big]$$

in the case of square-integrable random variables $X_1, \ldots, X_m$ and arbitrary constants $c_1, \ldots, c_m$, we have the thesis.

□

Proposition 1.2.3. *Given a portfolio as we have usually taken, with deterministic EADs and deterministic LGDs we have*

$$UL_{PF}^2 = \sum_{i,j=1}^{m}\Big(EAD_i \times EAD_j \times LGD_i \times LGD_j \times \tag{1.5}$$

$$\times\, \rho_{ij}\sqrt{PD_i(1-PD_i)PD_j(1-PD_j)}\Big) \tag{1.6}$$

Proof. The result follows directly from the (1.4), from

$$Cov[L_i, L_j] = \sqrt{\mathbb{V}[L_i]\mathbb{V}[L_j]}Corr[L_i, L_j]$$

and from $\mathbb{V}[L_i] = PD_i(1 - PD_i)$ for all $i = 1, \ldots, m$.

□

Before generalizing, we should stop and think about the sense and importance of correlation. For understanding better, we consider a portfolio in which there are only two loans with LGD $=$ 100% and EAD $= 1$. Then we work only with L_i for $i = 1, 2$ and we set $\rho = Corr[L_1, L_2]$, $p_i = PD_i$. Then, we have

$$UL_{PF}^2 = p_1(1-p_1) + p_2(1-p_2) + 2\rho\sqrt{p_1(1-p_1)}\sqrt{p_2(1-p_2)}. \tag{1.7}$$

Relative to the default correlation ρ, we may have three different situations:

- $\rho = 0$.

  In this context the third term in (1.7) vanishes. Fairly unusual in any portfolio, we can read $\rho = 0$ like an *optimal diversification*. This particular and barely feasible case refers to a situation in which the element of the portfolio are totally uncorrelated. Investing in many different assets usually the full-scale risk decreases, in fact the defaulting of a large number of assets at the same time it is pretty improbable. Seeing the UL as a substitute[1] for portfolio risk, in this case we have a minimization of the risk of simultaneous defaults.

- $\rho > 0$.

  In this case, we say that if one loan defaulted, it would increase the likelihood that the other will also default. Looking at the conditional default probability of the second counterparty and assuming that the first obligor has already defaulted:

$$\mathbb{P}[L_2 = 1 | L_1 = 1] = \frac{\mathbb{P}[L_1 = 1, L_2 = 1]}{\mathbb{P}[L_1 = 1]} = \frac{\mathbb{E}[L_1 L_2]}{p_1}$$

$$= \frac{p_1 p_2 + Cov[L_1, L_2]}{p_1} = p_2 + \frac{Cov[L_1 L_2]}{p_1}.$$

  So we see that a positive value of ρ leads the default probability of obligor 2 to increase with correlation between two loans. In other words, if $Cov[L_1, L_2] > 0$, whatever default of any assets in the portfolio has remarkable effect on the other facilities.
  The extreme case is the case of $\rho = 1$, that is the case of *perfect correlation*. If we assume even $p_1 = p_2 = p$ equation (1.7) becomes

$$UL_{PF} = 2\sqrt{p(1-p)}$$

  i.e. it is as if our portfolio had only one asset but it would be doubly risky; this situation is said *concentration risk*. Going on clarifying, the default of one obligor makes the other obligor defaulting almost surely.

[1]Mark that in comparison with the EL, the unexpected loss works out the "true" uncertainty the bank takes on when investing in a portfolio.

- $\rho < 0$.

  This is the symmetrical situation of that which came first. We only discuss the case of $\rho = -1$, that is the case of *perfect anti-correlation*. One can view an investment in asset 2 like a nearly *perfect hedge* against an investment in asset 1. Furthermore, from (1.7) the UL vanishes in the case of perfect hedge: the risk of asset 1 has been eliminated.

We are now ready for a major step forward. We consider now a discrete-market model in which we presume that all transactions, for a set time interval $[0, T]$, happened only at times

$$0 = t_0 < t_1 < \cdots < t_N = T$$

In the market we have one risk-less asset B and d risky assets $S = (S^1, \ldots, S^d)$ which are stochastic processes on a probability space $(\Omega, \mathcal{F}, P)$. In this context, Ω has a finite number of elements, $\mathcal{F} = \mathscr{P}(\Omega)$ and $P(\{\omega\}) > 0$ for any $\omega \in \Omega$. Bonds are deterministic so that we have

$$B_n = B_{n-1}(1 + r_n) \qquad n = 1, \ldots, N$$

where r_n, such that $r_n + 1 > 0$, is the risk-free rate in the period $[t_{n-1}, t_n]$ and $B_0 = 1$. On the other hand, we have

$$S_n^i = S_{n-1}^i(1 + \mu_n^i), \qquad n = 1, \ldots, N$$

where S_n^i indicates the price of i-th asset at time t_n, μ_n^i, for which $(1 + \mu_n^i) > 0$, is a real random variable that denotes the yield rate of the i-th asset in the n-period $[t_{n-1}, t_n]$.
Knowing this, $S^i = (S_n^i)_{n=0,\ldots,N}$ is a discrete stochastic process on $(\Omega, \mathcal{F}, P)$ and we name (S,B) a *discrete market* on the probability space $(\Omega, \mathcal{F}, P)$.
We set

$$\mu_n = (\mu_n^1, \ldots, \mu_n^d) \qquad 1 \leqslant n \leqslant N$$

and we introduce the filtration $(\mathcal{F}_n)$ defined by

$$\mathcal{F}_0 = \{, \Omega\} \tag{1.8}$$

$$\mathcal{F}_n = \sigma\{\mu_1, \ldots, \mu_n\} \qquad 1 \leqslant n \leqslant N \tag{1.9}$$

In this case we can interpret the σ-algebra $\mathcal{F}_n$ like the total information we have in the market at time t_n. Remark that one can also have $\mathcal{F}_n = \sigma(S_0, \ldots, S_n)$ for $0 \leqslant n \leqslant N$ and in this context we have $\mathcal{F}_N = \mathcal{F}$.

We are now ready for talking about the portfolio, the value of the portfolio and losses.

Definition 1.2.4. *We define a strategy (or portfolio) like a stochastic process in* $\mathbb{R}^{d+1}$

$$(\boldsymbol{\alpha}, \beta) = (\alpha_n^1, \ldots, \alpha_n^d, \beta_n)_{n=1,\ldots,N}$$

where α_n^i *indicates the quantity of asset* S^i *present in the portfolio and* β_n *the amount of bond during the period* $[t_{n-1}, t_n]$.

Consistently at this notation we denote the *value of portfolio* $(\boldsymbol{\alpha}, \beta)$ at time t_n by

$$V_n^{(\boldsymbol{\alpha},\beta)} = \alpha_n S_n + \beta_n B_n, \qquad n = 1, \ldots, N \tag{1.10}$$

where the product $\alpha_n S_n$ stands for the scalar product in $\mathbb{R}^d$ that is

$$\alpha_n S_n = \sum_{i=1}^{d} \alpha_n^i S_n^i \qquad n = 1, \ldots, N$$

therefore at the initial time the value of our portfolio is

$$V_0^{(\boldsymbol{\alpha},\beta)} = \sum_{i=1}^{N} \alpha_1^i S_0^i + \beta_1 B_0$$

Definition 1.2.5. *For a given time horizon* Δ*, such as 1 year or 10 days, the loss of the portfolio over the period* $[t_n, t_{n+\Delta}]$ *is a random variable and it is defined as*

$$L_{[t_n,t_{n+\Delta}]}^{(\boldsymbol{\alpha},\beta)} = L_{n+\Delta}^{(\boldsymbol{\alpha},\beta)} := -(V_{n+\Delta}^{(\boldsymbol{\alpha},\beta)} - V_n^{(\boldsymbol{\alpha},\beta)})$$

The distribution of $L_{n+\Delta}^{(\boldsymbol{\alpha},\beta)}$ is termed the loss distribution.

We now introduce some notions that we will use in the last chapter.

Definition 1.2.6. *We say* $(\boldsymbol{\alpha}, \beta)$ *to be* **self-financing** *if the following relation is valid*

$$V_{n-1}^{(\boldsymbol{\alpha},\beta)} = \alpha_n S_{n-1} + \beta_n B_{n-1} \qquad \forall\, n = 1, \ldots, N \tag{1.11}$$

Remark 1.2.2. Consider the (5.1) at time t_{n+1} and the (5.2) at the time t_n, we have the follow increment

$$V_{n+1} - V_n = \alpha_{n+1}(S_{n+1} - S_n) + \beta_{n-1}(B_{n-1} - B_n)$$

$$\Rightarrow \quad \Delta V = \boldsymbol{\alpha} \Delta S + \beta \Delta B$$

therefore the value of the portfolio depends on two things:

- the price ΔS or the bond ΔB

- the strategy (such as we can invest more money)

Then the self-financing condition imposes that we can't change the strategy, i.e. the value of the portfolio is due to the price.

Lemma 1. *Let a self-financing strategy* $(\boldsymbol{\alpha}, \beta)$ *be given; its value is work out by* V_0 *and recursively by*

$$V_n = V_{n-1}(1 + r_n) + \sum_{i=1}^{d} \alpha_n^i S_{n-1}^i (\mu_n^i - r_n) \qquad \forall\, n = 1, \ldots, N \tag{1.12}$$

Proof. From (5.3), we have

$$\begin{aligned} V_n - V_{n-1} &= \alpha_n (S_n - S_{n-1}) + \beta_n (B_n - B_{n-1}) \\ &= \sum_{i=1}^{d} \alpha_n^i S_{n-1}^i \mu_n^i + \beta_n B_{n-1} r_n \\ &= \sum_{i=1}^{d} \alpha_n^i S_{n-1}^i (\mu_n^i - r_n) + r_n V_{n-1} \end{aligned}$$

where the last parity is deduced from the definition of self-financial strategy. □

Definition 1.2.7. *A strategy* $(\boldsymbol{\alpha}, \beta)$ *is predictable if* (α_n, β_n) *is* $\mathcal{F}_n$ *measurable for every* $n = 1, \ldots, N$.

Definition 1.2.8. *We call* $\mathscr{A}$ *the family of acceptable strategies, namely the family of all self-financing and predictable strategies of the market (S,B).*

Proposition 1.2.4. *Let be* $V_0 \in \mathbb{R}$ *and* α *a predictable process; then we have only predictable process* β *such that* $(\boldsymbol{\alpha}, \beta) \in \mathscr{A}$ *and* $V_0^{(\boldsymbol{\alpha}, \beta)} = V_0$.

Proof. We define the process

$$\beta_n = \frac{V_{n-1} - \alpha_n S_{n-1}}{B_{n-1}} \qquad \forall\, n = 1, \ldots, N$$

where the process (V_n) derives from (5.3). For how we built (β_n), we have the claim. □

Definition 1.2.9. *Let* $(\boldsymbol{\alpha}, \beta) \in \mathscr{A}$ *be given, we define* the gain *of the strategy like*

$$g_n^{(\boldsymbol{\alpha}, \beta)} = \sum_{k=1}^{n} \left(\sum_{i=1}^{d} \alpha_k^i S_{k-1}^i \mu_k^i + \beta_k B_{k-1} r_k \right) \tag{1.13}$$

and we get

$$V_n = V_0 + g_n^{(\boldsymbol{\alpha}, \beta)} \tag{1.14}$$

1.3 Risk Measures

Considering the UL of a portfolio as a cushion (in terms of the amount of money) for periods of financial distress isn't the best choice: there may be a considerable probability that losses will go beyond the expected loss more than the unexpected loss. Because of that, we have to look for different ways to quantify risk capital. The most common way to estimate risk capital is *economic capital*[2].

Before continuing with the economic capital, it's important recalling and introducing some mathematical concepts. The purpose of this section is to quantify the value of the risk, so it is natural to speak about a **risk measure**

$$\rho : \{distribution\} \to \mathbb{R}$$

defined on the space of distribution and with real value. If two random variables are $law - invariant$, then they have the same risk value:

$$X \sim Y \Rightarrow \rho(X) = \rho(Y) \tag{1.15}$$

So one can define risk measure only on some set of random variables: for (1.15), starting from the space of distribution is the same as moving from a space of random variables. Yet, it is important to underline that there are some cases in which the risk measure can only be defined on the space of distributions.
Of course, ρ, should convey some notion of riskiness: a function like $\psi(X) = E[X^2]$, though being evidently law-invariant, is not a sensible choice for a risk measure.

1.3.1 Before VaR and ES

Markowitz, in his Portfolio Theory, considered **standard deviation** as a risk measure:

$$\rho(X) = \sigma(Y), \qquad X \in L^2 = \{X \mid E[X^2] < \infty\}$$

Indeed, defining the risk measure in this way respects the characteristic of being law-invariant (two random variables with same distribution have the same deviation), yet this is rather a $dispersion$ measure:

· it depends in the same way on the right and left tails of distribution:

$$\sigma(-X) = \sigma(X).$$

[2]Also said *Capital at Risk, indicated with CaR*

A financial interpretation could be that the losses have the same risk of the earnings, i.e. this measure is not coherent with the concept of risk at all.

· it does not depend on monetary shifts

$$\sigma(X+m) = \sigma(X), \quad m \in \mathbb{R}$$

that is if we add more money, we have the same risk as before.

The (lower) *standard semi-deviation* is considered a reasonable adjustment:

$$\rho(X) = \sigma_-(X) = \sqrt{E\left[X_{left}^2\right]}$$

where

$$X_{left} = (X - E[X])_- = \begin{cases} 0 & \text{if } X \leqslant E[X] \\ |\, X - E[X]\,| & \text{if } X \geqslant E[X]. \end{cases}$$

We can see that it only depends on the left tail X_{left} (i.e. the losses) and the threshold used here $x^* = E[X]$ can be changed. However, σ_- is still insensitive to monetary shifts. Another modification of the standard semi-deviation is

$$\rho(X) = -E[X] + a\,\sigma_-(X), \quad a > 0$$

As it is possible to note, by this definition, the risk measure only depends on the left tails and on monetary shifts. It also satisfies the *translation equivariance* property:

$$\rho(X+m) = \rho(X) - m, \quad m \in \mathbb{R}$$

therefore if we investe more money, the risk decreases with it.

1.3.2 VaR and ES

Definition 1.3.1. *Considered a random variable on the probability space* $(\Omega, \mathcal{F}, P)$*, an* α*-quantile is any real number* q *such that*

$$\mathbb{P}[X \leqslant q] \geqslant \alpha \qquad \textit{and} \qquad \mathbb{P}[X < q] \leqslant \alpha$$

The set of all α-quantiles of X is an interval $\left[q_\alpha^-(X), q_\alpha^+(X)\right]$, where

$$\begin{aligned} q_\alpha^-(X) &:= inf\{x \,|\, \mathbb{P}[X \leqslant x] \geqslant \alpha\} \\ q_\alpha^+(X) &:= inf\{x \,|\, \mathbb{P}[X \leqslant x] > \alpha\} \\ &= sup\{x \,|\, \mathbb{P}[X < x] \leqslant \alpha\} \end{aligned} \tag{1.16}$$

In this context, consistently with the notation that we have introduced before, our random variable is the loss of the portfolio $(\boldsymbol{\alpha}, \beta) \in \mathscr{A}$.

About Quantiles

It is best practice, if one has a random variable X with cdf $F(x) = P(X \geqslant x)$ invertible (i.e. strictly increasing and continuous), giving the definition of quantile, given a confidence level $\alpha \in (0,1)$, as

$$q_\alpha(X) = q_\alpha = F^{-1}(\alpha)$$

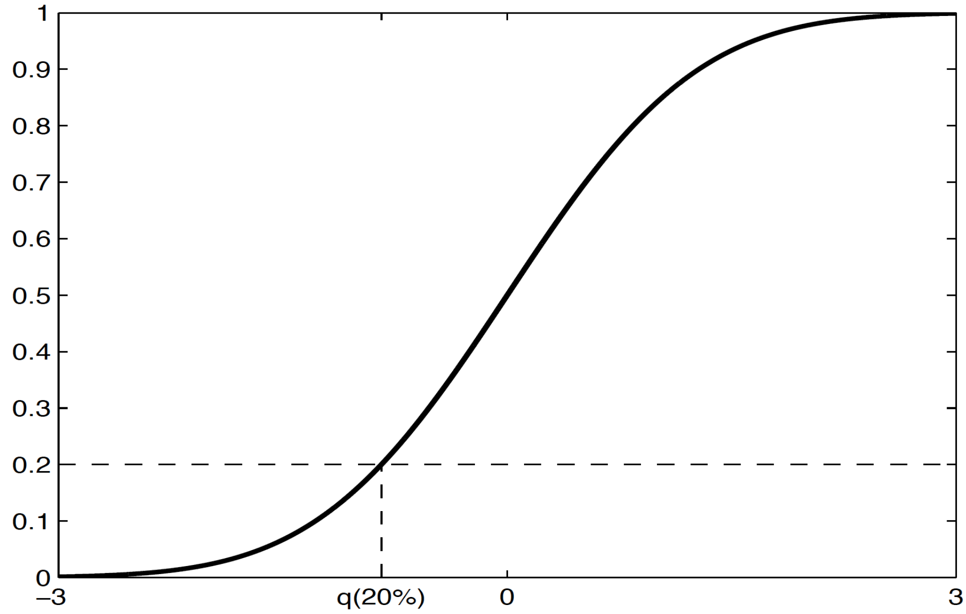
By this definition, q_α is the unique real number satisfying

$$F(q_\alpha) = P(X \leqslant q_\alpha) = \alpha$$

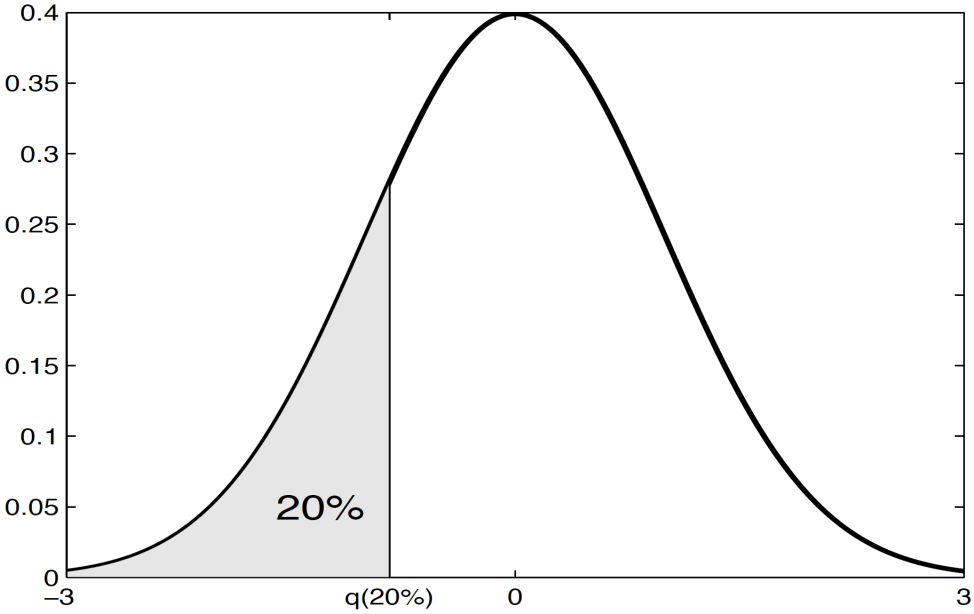
If X is continuous and its support is connected, i.e. it is an interval or a line, then F is invertible at any α. In this case the quantile satisfies

$$\int_{-\infty}^{q_\alpha} f(x)dx = \alpha$$

The figure (1.1) give us an idea of the difference existing between the two notations.



(a) Quantile of order $\alpha = 20\%$ in terms of F



(b) Quantile of order $\alpha = 20\%$ in terms of f

Figure 1.1: Two particular examples

Remark 1.3.1. We rapidly list some values of quantiles construed from the cdf of the random variables

△ **Exponential** distrution
Consider $X \sim Exp(\lambda)$, with $\lambda > 0$ then the cdf is

$$F(x) = 1 - e^{-\lambda x}, \quad x > 0$$

The cumulated function F is invertible on $[0, \infty)$ so is worth $q_\alpha = F^{-1}(\alpha)$ for any α, therefore setting

$$1 - e^{-\lambda q_\alpha} = \alpha \quad \Rightarrow \quad q_\alpha = -\frac{1}{\lambda}\log(1-\alpha)$$

△ **Uniform** distribution
If $U \sim U(0,1)$ it has as cumulated function $F(x) = x$ for every $x \in (0,1)$. So, the quantile is, trivially,

$$q_\alpha = \alpha$$

△ **Cauchy** distribution
In this case, we have

$$f(x) = \frac{1}{\pi(1+x^2)}$$

Then, resolving the integral we obtain

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{x} \frac{dy}{1+y^2} = \frac{\arctan(x)}{\pi} + \frac{1}{2}$$

from which we easily derive

$$q_\alpha = \tan\Big(\pi\big(\alpha - \frac{1}{2}\big)\Big)$$

It is a matter of fact that even when the density is known, in many cases getting the expression for the quantiles is difficult or even impossible. This is the case of two essential distributions: the <u>normal</u> and the <u>t-Student</u>.

The quantile fo the standard normal are key quantities for our purpose. They are denoted with z_α and, by definition, they satisfy

$$\Phi(z_\alpha) = P\big(Z \leqslant z_\alpha\big) = \int_{-\infty}^{z_\alpha} \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}dx = \alpha$$

where $Z \sim \mathcal{N}_{0,1}$ and Φ is the cdf function for the normal that is not known in closed form. However, the quantile z_α can be numerically approximated with great precision.

Let us peak about the t-Student distribution. Let a random variable X be given. If X has

$$f(x) = c_\nu \Big(1 + \frac{x^2}{\nu}\Big)^{-\frac{\nu+1}{2}}$$

as distribution, we will say that X has a t-Student distribution with $\nu > 0$ degrees of freedom. The positive number c_ν is a suitable normalizing constant. Note that t-Student distributions are symmetrical around 0 and have tails heavier than the normal one. The quantile of order α of the t-Student distribution with ν degrees is indicated with $t_{\nu,\alpha}$. Except for few cases, like $\nu = 1$, $\nu = 2$, $\nu = 4$, the quantile $t_{\nu,\alpha}$ cannot be written explicitly; nevertheless, like in the normal case, quantiles easily be numerically approximated. We remark that when $\nu = 1$ we recover the Cauchy distribution, indeed in this case $c_1 = 1/\pi$. So we already computed $t_{1,\alpha}$.
As we already said, the previously definition of quantile cannot be applied whenever F is not invertible, this happens when

1. F is not strictly increasing corresponding to α
2. F has a jump corresponding to α

When we cannot work with the cdf of the random variable, we get the general definition of quantile that we did in (1.16). This is the case of a discrete random variable: its cdf is a step function.

Quantiles of transformed r.v.

Let X be a random variable and h be a continuos and strictly increasing transform (hence, h is invertible).

Lemma 2. *If X and h are like above, then, for any α, the following relation holds*

$$q_\alpha(h(X)) = h\big(q_\alpha(X)\big)$$

Proof. We must divide the case when the cdf of X, F_X, is invertible from the case in which F_X is not invertible. The second case requires a bit more work than the first and it in not useful for our purpose, so we omit it. If F_X is invertible we have:

$$P(X \leqslant q_\alpha(X)) := \alpha := P\Big(h(X) \leqslant q_\alpha\big(h(X)\big)\Big) = P\Big(X \leqslant h^{-1}(q_\alpha\big(h(X)\big))\Big)$$

therefore $q_\alpha(X) = h^{-1}\big(q_\alpha(h(X))\big)$. So we have proved the claim; note that the third equation requires h increasing. □

Example 1.3.1. For instance, if we get $h(x) = e^x$ or $h(x) = \log(x)$ we have

$$q_\alpha(e^X) = e^{q_\alpha(X)} \quad \text{and} \quad q_\alpha\Big(\log(X)\Big) = \log\Big(q_\alpha(X)\Big)$$

Instead, if we consider $h(x) = |x|$ we have $q_\alpha(|X|) \neq |q_\alpha(X)|$, since h is not always increasing.

In particular, if we consider $h(x) = ax + b$ for positive a, we obtain

$$q_\alpha\Big(aX + b\Big) = aq_\alpha\big(X\big) + b \tag{1.17}$$

Consider, now, a r.v. X with finite mean μ and finite variance σ^2; its standardized version is

$$\tilde{X} = \frac{X - \mu}{\sigma} \tag{1.18}$$

with mean $E[\tilde{X}] = 0$ and variance $var(\tilde{X}) = 1$. Then for the equivalence (1.18) we have

$$X = \sigma\tilde{X} + \mu$$

therefore, for the (2), we have

$$q_\alpha(X) = \sigma q_\alpha(\tilde{X}) + \mu \tag{1.19}$$

An immediate consequence is that for computing quantile is enough to consider the standardized version of random variable.

Cornish-Fisher approximation

When we do not know the exact distribution of X but just the first four moments, we can use the Cornish-Fisher approximation of a quantile. In particular if X is standard, but not necessarily normal, with finite skewness ξ and kurtosis κ, the Cornish-Fisher approximation is

$$q_\alpha(X) \approx z_\alpha + \xi\,\frac{z_\alpha^2 - 1}{6} + \big(\kappa - 3\big)\,\frac{z_\alpha^3 - 3z_\alpha}{24} \tag{1.20}$$

where z_α is the quantile of standard normal. How we can note, the closer is the distribution of X to a standard normal, the better is the approximation; this happen when ξ is close to 0 and κ is close to 3. In the other case, the mathematic literature discourage the use. In the case in which X is not standard, we apply (1.20) to $\tilde{X}$ and then we use (1.19).

Value at Risk

Definition 1.3.2. *Given a confidence level $\alpha \in (0, 1)$, the* Value-at-Risk *of the portfolio $(\boldsymbol{\alpha}, \beta)$, denotes with VaR_α, at the confidence level α is determined by the smallest figure q with the property the likelihood that the loss $L^{(\boldsymbol{\alpha},\beta)}$ is greater than q isn't bigger than $(1 - \alpha)$. Formally*

$$\begin{aligned} VaR_\alpha &:= inf\{q \in \mathbb{R} \,|\, \mathbb{P}\big(L^{(\boldsymbol{\alpha},\beta)} > q\big) \leqslant 1 - \alpha\} \\ &= inf\{q \in \mathbb{R} \,|\, \mathbb{P}\big(L^{(\boldsymbol{\alpha},\beta)} \leqslant q\big) \geqslant \alpha\} \end{aligned}$$

In other words, VaR is a quantile of the loss distribution.
In financial terms, $VaR_\alpha(L^{(\boldsymbol{\alpha},\beta)})$ indicates the smallest quantity of capital which, if added to $L^{(\boldsymbol{\alpha},\beta)}$ and invested in a risk-free asset, holds the probability of a negative outcome under the level α. Remark that in spite of Value at Risk limits the odds of a loss, it doesn't state the amount of the loss if it happens.

Indicate with μ the mean of the loss distribution. Sometimes the statistic

$$VaR_\alpha^{mean} := VaR_\alpha - \mu$$

is used for capital-adequacy aim instead of the ordinary VaR. The difference between the two quantities is small in market risk, in which the time horizon is little and μ is near to zero. It becomes considerable in credit risk where the risk-management horizon is longer. Especially in loan pricing, one uses VaR^{mean} to work out the economic capital essential in order to cover unexpected losses in a loan portfolio.

Connected with the initial notation, we enunciate the following definition.

Definition 1.3.3. *Let a portfolio $\big(\tilde{L}_i\big)_{i=1,\ldots,m}$ be given. The* economic capital, *EC, with respect to a specify level of confidence α, is the α-quantile of the portfolio loss $\tilde{L}_{PF}$ minus the EL of the portfolio:*

$$EC_\alpha = q_\alpha - EL$$

where q_α is given as

$$q_\alpha = inf\{q > 0 \,|\, \mathbb{P}[\tilde{L}_{PF} \leqslant q] \geqslant \alpha\}.$$

We reduce q_α by the EL because is a best practice decomposing the total risk capital (that is the quantile) into a first part buffering the expected losses

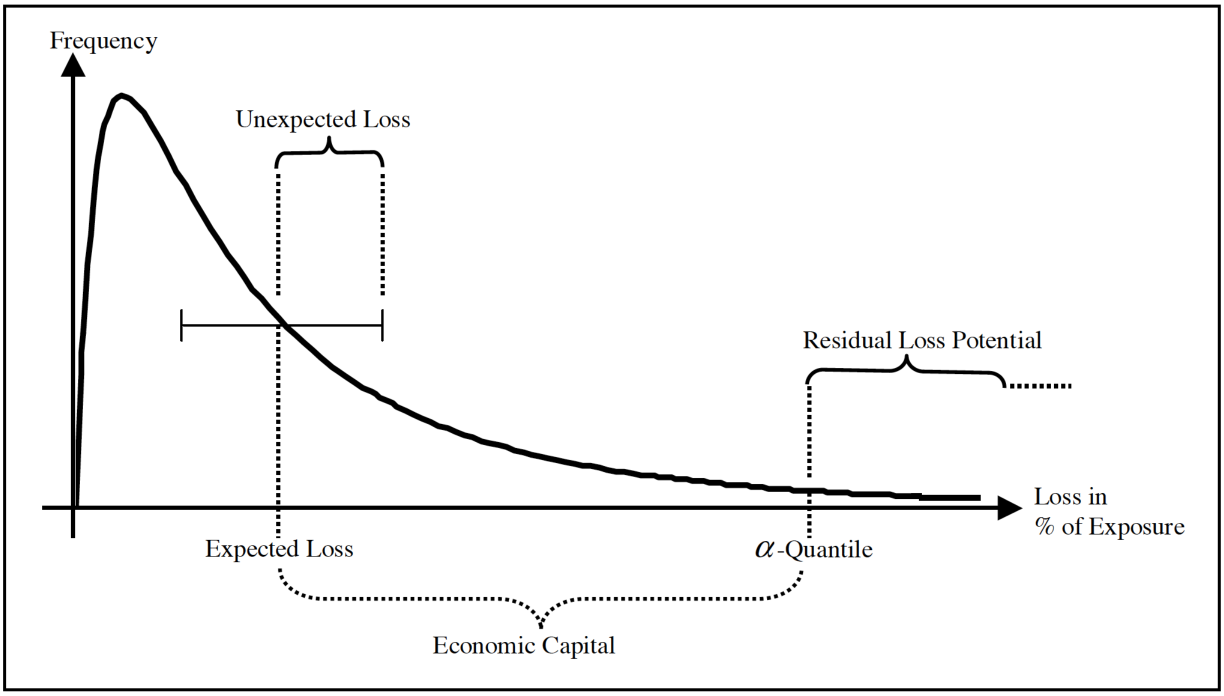


Figure 1.2: The portfolio loss distribution

and a second part covering the unexpected losses. In the figure (1.2), we sum up all quantities that we have met up until now.

Now, we list the most important criticism moved to the VaR models without deepening the single (hypothetical) shortcoming.

1. **VaR models disregard exceptional events**
   The first disapproval is intrinsic into the definition. VaR models is strictly linked to the fixed confidence level therefore it is not feasible to forecast all possible losses, that is the possible risks, that a financial institution must be able to cope with.

2. **VaR models leave out customer relations**
   A mechanical application of VaR might carry a bank to suddenly close all positions for which risk-adjusted return is inadequate. In other words, the bank may adopt a short term view that might conduce to an unawareness of long term customer relations.

3. **VaR models generate diverging results**
   Some researchers have found considerable discrepancies[3] when compar-

[3]Same discrepancy of results was noticed by the Basel Commitee during an experiment.

ing the results by different approaches. Even if these examples don't prove the unreliability of VaR, they indicate that the model should be used with caution.

4. **VaR models could penalize diversification**
Dividing the investment in two or more assets carries Var to increase and then to an increment of likelihood that something goes wrong, without considering the substantial drop of the EL. Hereby one could say that enhancing portfolio with respect to VaR may bring about concentrating portfolio in one single asset with a very small PD; in this way, the investor is exposed to big losses.

5. **VaR models amplify market instability**
In the case of all financial institutions in the financial markets adopt a VaR model, joined with any market shock, it would happen that every traders receive all the same operational signal, thus if before we had an increase in volatility, now we have a much higher volatility (it's the case of subprime loans).

6. **VaR measures "arrived when damage has already been done"**
One last criticism concerns the delay in whom VaR shows any market shock, therefore it could be difficult preventing losses.

Expected Shortfall

An alternative to EC is a risk capital based on *Expected Shortfall*, ES.

Definition 1.3.4. *Let a portfolio $(\tilde{L}_i)_{i=1,\dots,m}$ be given. The expected shortfall, EC, with respect to a specify level of confidence α, is defined as*

$$ES_\alpha(\tilde{L}_{PF}) = \mathbb{E}[\tilde{L}_{PF} \mid \tilde{L}_{PF} > VaR_\alpha(\tilde{L}_{PF})]$$

But we can give a more useful definition of ES. Instead of setting a quantile at a particular confidence level α, we calculate the Expected Shortfall like a VaR average across the entire tail specified by α:

$$ES_\alpha(\tilde{L}_{PF}) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_\beta(\tilde{L}_{PF})\, d\beta \tag{1.21}$$

We can see this quantity as a measure of a risk aversion; furthermore, note that ES focuses on the expected loss in the tail of the portfolio loss distribution, starting at VaR_α. See the figure (1.3).

For examining in depth the problem, see Beder (1995), Marshall and Siegel(1996) and Jordan and Mackay (1996)

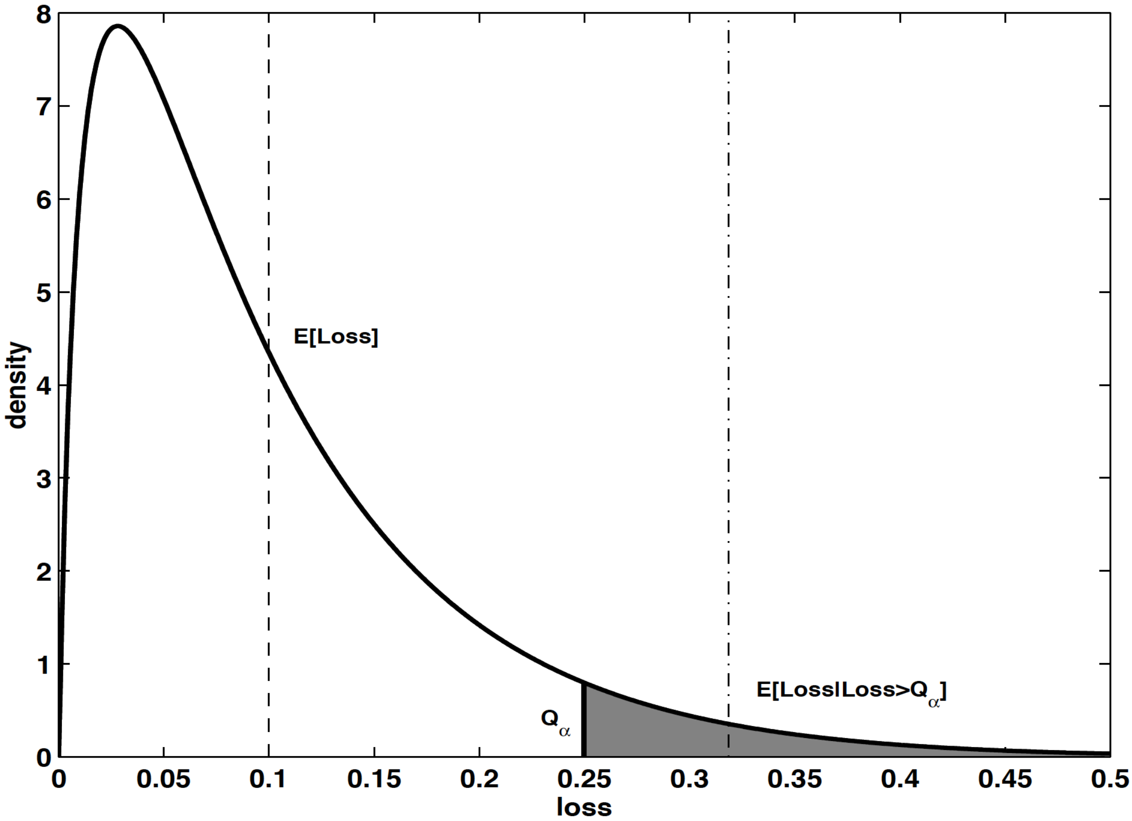


Figure 1.3: Tail conditional expectation $\mathbb{E}[\tilde{L}_{PF} \,|\, \tilde{L}_{PF} > VaR_\alpha(\tilde{L}_{PF})]$

About VaR and ES

1. Note that both the two measures consider only the left tail (losses), but ES contrarliy to VaR, depends on the entire left tail.

2. For quantile definition, VaR_α and ES_α are increasing in α.

3. If X has unbounded below support (i.e. it permits arbitrary losses) we have
$$VaR_\alpha(X),\, ES_\alpha(X) \to \infty \quad \text{as} \searrow 0$$

4. From Var and ES definitions we obtain
$$ES_\alpha(X) \geqslant VaR_\alpha(X)$$
that is ES is more conservative than VaR.

5. The ES is coherent while Var it is not:

$$ES_\alpha(\tilde{L}_{PF_1} + \tilde{L}_{PF_2}) \leqslant ES_\alpha(\tilde{L}_{PF_1}) + ES_\alpha(\tilde{L}_{PF_2})$$

   and, since we have obtained a bond for ES, this inequality is reassuring.

6. VaR is defined for all distribution, while the ES requires that the left tail is integrable; for instance ES is not defined for Cauchy distributions. Some distributions used in operative models or in insurance risk pose this problem while this is not generally the case in market or credit risk.

7. Computationally, Var is more easy to compute than ES that involves a bit more work.

8. From (1.17), it immediately follows

$$\begin{aligned} VaR_\alpha(aX + b) &= aVaR_\alpha(X) - b \\ ES_\alpha(aX + b) &= aES_\alpha(X) - b \end{aligned}$$

   Therefore, both risk measures are positively homogeneous, i.e. $\rho(aX) = a\rho(X)$, and translation equivariant, i.e. $\rho(X + b) = \rho(X) - b$. In particular, if X has finite mean μ and finite variance σ^2 with $\tilde{X}$ its standardized version, then

$$VaR_\alpha = \sigma VaR_\alpha(\tilde{X}) - \mu \qquad ES_\alpha(X) = \sigma ES_\alpha(\tilde{X}) - \mu.$$

Chapter 2

Loss Distribution

In this chapter we are going to give to default model a probabilistic approach. In Chapter 1 we have defined loss variables like a litmus paper for default events; hence now we try to bestow a distribution on this random variables. We also attempt to furnish a financial interpretation with our treatise. We are going to focus on three different well note distribution: *Bernoulli* and *Poisson* distribution. Depending on what distribution we choose, we reap a different model, into the set of industry models: Bernoulli's distribution is associated to models by Moody's KMV, RiskMetrics Group and more bank-internal models. CreditRisk$^+$ is based on Poisson's distribution. At the end we compare two models trying to give an overview on pro and cons of Bernoulli and Poisson distribution. Since in the real life, almost surely, the underlying are correlated one other, we will directly explain those models so called *mixture models*, i.e. those which introduce correlations between assets. We assume we have m counterparties and, for simplicity, we take on a *two-state* approach, that is only default or survival is considered.

2.1 Bernoulli Mixture Model

A vector of r.v. $\mathbf{L} = (L_1, \ldots, L_m)$ is named a Bernoulli *loss* statistics if

$$L_i \sim B(1, P_i)$$

where P_i is an element of $\mathbf{P} = (P_1, \ldots, P_m)$, the vector of loss probabilities which are random variables with some distribution $\mathbf{F}$ with support in $[0, 1]^m$. We define the *loss* of $\mathbf{L}$ as

$$L = \sum_{i=1}^{m} L_i.$$

Moreover, we take on the independence of $L_1, \ldots, L_m$ conditioning on a particular sequence of value for $\mathbf{P}$, i.e. $\mathbf{p} = (p_1, \ldots, p_m)$:

$$L_{i_{|\,P_i=p_i}} \sim B(1, p_i), \qquad (L_{i_{|\,\mathbf{P}=\mathbf{p}}})_{i=1,\ldots,m} \text{independent}$$

From the theory of probability we have

$$\mathbb{P}[L_1 = l_1, \ldots, L_m = l_m] = \int_{[0,1]^m} \prod_{i=1}^{m} p_i^{l_i}(1-p_i)^{1-l_i}\, d\mathbf{F}(p_1, \ldots, p_m)$$

in which $l_i \in \{0, 1\}$. Furthermore the mean and the variance of the single losses are worked out by

$$E[L_i] = E[P_i], \qquad var(L_i) = E[P_i](1 - E[P_i])$$

where the proof of the first equality is trivial while for the second one we must do more little work:

$$\begin{aligned} var(L_i) &= var(E[L_i \,|\, \mathbf{P}]) + E[var(L_i \,|\, \mathbf{P})] = var(P_i) + E[P_i(1 - P_i)] = \\ &= E[P_i](1 - E[P_i]). \end{aligned} \tag{2.1}$$

Note that the series of equalities directly derive from the definition and property of conditional variance. By means the definition of covariance we have

$$Cov(L_i, L_j) = E[L_i\, L_j] - E[L_i]E[L_j] = Cov(P_i,\, P_j)$$

therefore, we are ready to compute the default correlation between single losses:

$$Corr(L_i,\, L_j) = \frac{Cov(L_i,\, L_j)}{\sqrt{var(L_i)var(L_j)}} \tag{2.2}$$

then, by (2.1)

$$Corr(L_i,\, L_j) = \frac{Cov(L_i,\, L_j)}{\sqrt{E[P_i](1 - E[P_i])}\sqrt{E[P_j](1 - E[P_j])}}. \tag{2.3}$$

Hence from (2.3), we deduce that the covariance structure of $\mathbf{F}$ and $\mathbf{P}$ capture the dependence between losses in the portfolio.

2.1.1 Uniform Portfolio

Retail portfolios and portfolios of smaller banks have frequently fairly homogeneous composition so it makes sense speak about portfolios with unvarying PD and uniform correlation; this type of portfolios are said *uniform* *portfolio*. This category of strategy[1] are characterized by having all exposures around the same size and type in terms of risk. Thanks to uniformity, we are be able to talk about the *changeableness* between the variable L_i: $(L_1, \ldots, L_m) \sim (L_{\pi(1),\ldots,\pi(m)})$ for any permutations π. The variable $L_i \sim B(1; P)$ with $P \sim F$, where F is a distribution function with support in $[0, 1]$. The conditional independence is assumed like in the general case. By means the following equality

$$\mathbb{P}[L_1 = l_1, \ldots, L_m = l_m] = \int_0^1 p^k (1-p)^{m-k}\, dF(p) \tag{2.4}$$

getting

$$k = \sum_{i=1}^{k} l_i \quad \text{and} \quad l_i \in \{0, 1\}.$$

we can work out the probability that exactly k defaults occurs:

$$\mathbb{P}[L = k] = \binom{m}{k} \int_0^1 p^k (1-p)^{m-k}\, dF(p). \tag{2.5}$$

Since we are also in a two-state context and by (2.5), we have

$$\overline{p} = \mathbb{P}[L_i = 1] = E[L_i] = \int_0^1 p\, dF(p)$$

therefore, under the independence hypothesis, we are able to obtain

$$\mathbb{P}[L_i = 1, L_j = 1] = \int_0^1 p^2\, dF(p). \tag{2.6}$$

Hence, utilizing (2.6) and the definition (2.2), we obtain the uniform default correlation:

$$\rho = \frac{\mathbb{P}[L_i = 1,\, L_j = 1] - \overline{p}^2}{\overline{p}(1 - \overline{p})}. \tag{2.7}$$

Examining the equality (2.7), we note that the ratio take account of the big influence of volatility[2] on correlation into the loss statistics: a growth on volatility produced bigger default correlation. Moreover, seeing as how the volatility cannot be negative, the model does not consider negative dependencies among the risks into the portfolio. We have two extreme cases:

[1]We remind the parallelism between portfolio and strategy which is defined in (1.2.4)

[2]We remind that var(P) $= \int_0^1 p^2\, dF(p) - \overline{p}^2$ and it is worth $\sigma^2 = \sqrt{var}$, i.e. variance and volatility are strictly linked.

1. The correlation is none that is to say the dearth of randomness. Furthermore, in aforementioned case, the distribution F is all concentred in $\overline{p}$ so $L \sim B(1, \overline{p})$.

2. The second case is about $Corr[L_i, L_j] = 1$, namely, all assets have the same behavior at the same time: portfolio's component either default or survive concurrently. This rigidity is perfect when all obligors survive; it happens with probability $(1-\overline{p})$. On the other hand, with probability $\overline{p}$, all obligors default at same time.

2.2 Poisson Mixture Model

The loss statistics **L'** $= (L'_1, \ldots, L'_m)$ is such that $L'_i \sim Pois(\Lambda_i)$ where $\boldsymbol{\Lambda} = (\Lambda_1, \ldots, \Lambda_m)$ is a random vector with distribution function $\mathbf{F}$ with support in $[0, \infty)^m$. As in the Bernoulli's case, if we take $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$ like realization of $\boldsymbol{\Lambda}$, we have

$$L'_{i_{|\,\Lambda_i=\lambda_i}} \sim Pois(\lambda_i), \qquad (L'_{i_{|\,\boldsymbol{\Lambda}=\boldsymbol{\lambda}}})_{i=1,\ldots,m} \quad \text{independent}$$

By the definition of Poisson distribution, the joint distribution is given by

$$\mathbb{P}[L'_1 = l'_1, \ldots, L'_m = l'_m] =$$

$$= \int_{[0,\infty)^m} e^{-(\lambda_1+\ldots+\lambda_m)} \prod_{i=1}^{m} \frac{\lambda_i^{l'_i}}{l'_i!} \, d\mathbf{F}(\lambda_1, \ldots, \lambda_m)$$

where, instead of what we have seen previously, $l'_i \in \{0, 1, 2, \ldots\}$. The mean and variance of Poisson's variables is

$$E[L'_i] = E[\Lambda_i] \quad \text{and} \quad var(L'_i) = var(\Lambda_i) + E[\Lambda_i] \tag{2.8}$$

for every i. Another time, since $Cov(L'_i, L'_j) = Cov(\Lambda_i, \Lambda_j)$, we have this close formula for the correlation:

$$Corr[L'_i, L'_j] = \frac{Cov[\Lambda_i, \Lambda_j]}{\sqrt{var(\Lambda_i) + E[\Lambda_i]}\sqrt{var(\Lambda_j) + E[\Lambda_j]}}. \tag{2.9}$$

Like in (2.3) the correlation derive from the distribution $\mathbf{F}$. Into this type of approach the default probability of obligor i is given by

$$\mathbb{P}[L'_i \geqslant 1].$$

Note that this form of PD permits multiple defaults of a single obligor. The probability that obligor i defaults more than once is

$$\mathbb{P}[L_i' \geqslant 2] = 1 - e^{-\lambda_i}(1 + \lambda_i)$$

which is typically a small number. A cause of the cumulated function of a Poisson's distribution, we have

$$p_i = \mathbb{P}[L_i' \geqslant 1] = 1 - e^{-\lambda_i} \approx \lambda_i$$

which shows that the likelihood that an *exponential waiting time* with intensity λ_i takes place in the first years is equals to the one-year default probability.

2.2.1 Uniform Portfolio

Similarly to the Bernoulli model, it makes sense speaking about a model in which, by means the restriction to one uniform intensity and one uniform correlation, the portfolio is uniform with respect to the risk of singular assets. In other words we have $L_i' \sim Pois(\Lambda)$ with $\Lambda \sim F$. In this case we have

$$\mathbb{P}[L_1' = l_1', \ldots, L_m' = l_m'] = \int_0^\infty e^{-m\lambda} \frac{\lambda^{(l_1' + \ldots + l_m')}}{l_1'! \cdots l_m'!} \, dF(\lambda)$$

and going on with to readapt the earlier general case, we obtain

$$\mathbb{P}[L' = k] = \int_0^\infty \mathbb{P}[L' = k \mid \Lambda = \lambda] \, dF =$$

$$= \int_0^\infty e^{m\lambda} \frac{m^k \lambda^k}{k!} \, dF(\lambda)$$

Remark 2.2.1. Seeing as now the support of the distribution F is not bounded, if we use the Poisson distributions in order to model our portfolio we include the possibility that the absolute loss L' can exceed the number of "physically" possible defaults.

In this context the PD is defined by

$$\overline{p} = \mathbb{P}[L_i' \geqslant 1] = \int_0^\infty \mathbb{P}[L_i' \geqslant 1 \mid \Lambda = \lambda] \, dF(\lambda)$$

$$= \int_0^\infty (1 - e^{-\lambda}) \, dF(\lambda)$$

We carry on with giving the equivalent face of (2.9):

$$Corr[L_i', L_j'] = \frac{var(\Lambda)}{var(\Lambda) + E[\Lambda]}, \qquad (i \neq j) \tag{2.10}$$

Call *dispersion* of a distribution of a certain r.v. X, the ratio

$$D_X = \frac{var(X)}{E[X]}$$

Then, from (2.8), we can observe that Poisson mixture models are overdispersed, i.e. with dispersion greater than one[3]. The meaning of the formula (2.10) is more clear if we interpret it in the sense of dispersion. This characteristic of Poisson mixture models could be used for understand if we really can apply it to the credit risk measurement: if the data about the underlying do not contemplate overdispertion, it does not make sense using this type of models. In order to interpret correlation in terms of dispersion, we indicate with D_Λ the dispersion of the random intensity Λ. Rewriting (2.8) by D_Λ we achieve

$$Corr[L_i', L_j'] = \frac{D_\Lambda}{D_\Lambda + 1}$$

therefore the correlation between variables is directly proportional to dispersion of intensity Λ, i.e. the higher dispersion, the greater dependence between obligor's default.

2.3 Bernoulli against Poisson Mixture

We want very rapidly compare the two models; if one is interested on, he can be found more details on [1]. Let us rewrite the (2.3) in the other and equivalent form:

$$Corr[L_i, L_j] =$$

$$= \frac{Cov(P_i, P_j)}{\sqrt{var(P_i) + E[P_i(1-P_i)]}\sqrt{var(P_j) + E[P_j(1-P_j)]}}. \tag{2.11}$$

If for a moment we assume that the variable P_i presents in the Bernoulli models and Λ_i have the same mean and variance, by means noting that

[3]Note that the Poisson distribution has dispersion equal to one, so it is a reference point to estimate if the distribution is overdispersed or underdispersed.

the argument of the square in (2.9) and (2.11) are respectively one greater than the other, we get that the level of default correlation in the Bernoulli models are higher than those in Poisson's models. In other words, under this assumption on the first and second moment of the variables into the two models, we have that $var(L'_i)$ is always exceed the variance of L_i.
We can close this chapter saying that *if we model a portfolio with Bernoulli model we foresee fatter tail than a comparably calibrated Poisson model.*

Chapter 3

Non-Linear Portfolio

First of all, for simplifying the treatise, we consider the *Profit&Loss*, PL, instead of the losses. It is simply a change of sign: $PL = -L$. Working with PL is typical of the insurance models, see [4]. Sometimes, into this chapter, we reveal mathematical concepts that we will take on the chapter 6. If one is not practical with the following concepts, we suggest to go to the Chapter 6 and to pay serious attention to the section (6.2) and to the chapter 7.
Frequently, in real portfolios there are non-linear products, whose value depends on a non-linear way on the underlying risk factors. In these cases, the total PL is a non-linear function of risk factors.

Example 3.0.1. Consider a portfolio made by π_S shares and π_C Call options (written on the same underlying asset). So, if we see the PL like a function of the stock return R, by the Black-Scholes formula (6.7), we have

$$PL = \pi_s R + \pi_C\big(c_{BS}(S_0 e^R) - c_{BS}(S_0)\big)$$

where c_{BS} is the Black-Sholes function. In this example we are simplifying the matter: we are supposing that the time interval is very short, that is not always true, especially in credit risk management. However, even in this case one can note that the function c_{BS} is not linear, therefore combination of different option prices gives rise to highly non-linear portfolios.

Evidently, even if the distribution of risk factors has a simple form, when the portfolio is no-linear, the distribution of PL can be difficult to derive. We go beyond this obstacle in three possible and different ways:

1. Delta approximation

2. Delta-Gamma approximation

3. Full evaluation

What Delta and the Gamma are is clearly explained into the chapter (7). The last way, that we do not debate, concerns the used of MonteCarlo or historical method, without the employ of approximations. The full evaluation approach is the most accurate one, even if it involves repeated computations of non-linear functions.

3.1 Delta approximation

If the time interval is short (for the general and more complex case, we remand at the bibliography), we have a small value of R, then we can use a Delta approximation. Consider a very simple portfolio made by one Call option. Using the Black-Scholes model we know that the PL is given by

$$PL = c_{BS}\big(S_0 e^R\big) - c_{BS}\big(S_0\big). \tag{3.1}$$

So, using the Taylor linear approximation around $R = 0$, we reach

$$PL \approx S_0 R \Delta \tag{3.2}$$

See the figure (3.1).

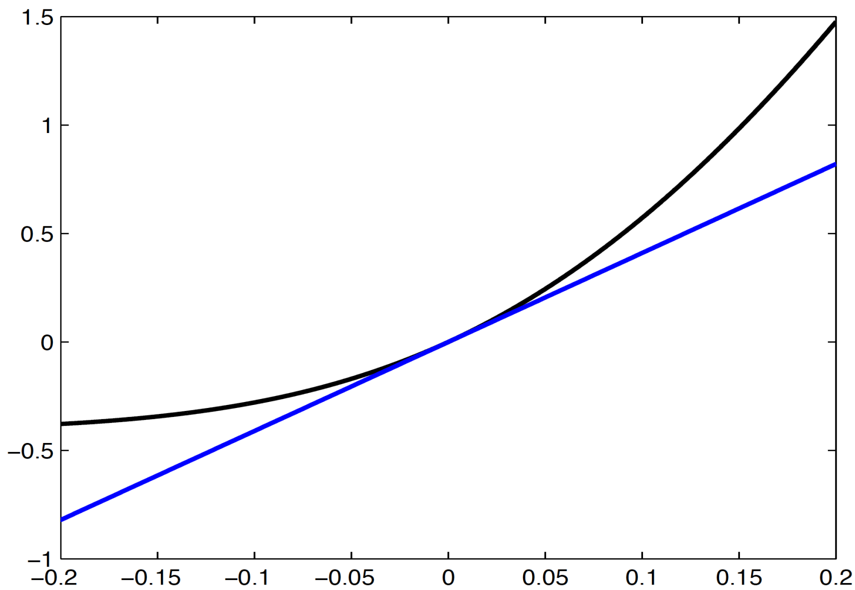


Figure 3.1: We put in blue the exact value of PL and, in black, the reader find the value of Delta approximation.

General Delta approximation

In general, a portfolio is made by several different options or more general derivatives on the same underlying. If there are π_S units of the underlying and π_k units of a certain type of derivative, for $k = 1, \ldots, K$, the *Delta* approximation becomes

$$PL \approx \pi_S S_0 R + \sum_k \pi_k S_0 R \Delta_k = \left(\pi_S + \sum_k \pi_k \Delta_k \right) S_0 R = \Delta_{port} S_0 R.$$

where Δ_k is the *Delta* of the derivative of type k. It is also possible having N underlying stocks and the K derivatives could depend on one or more of these underlyings.
Let $\Delta_{k,n}$ be the derivative of the pricing function for the k-th derivative with respect to S_n, the underlying n. If the derivative k does not have S_n like underlying, then $\Delta_{k,n} = 0$. Said that, we introduce

$$\Delta_{port,n} = \pi_{S,n} + \sum_{k=1}^{K} \pi_k \Delta_{k,n}$$

where $\pi_{S,n}$ are the units of the stock n. Therefore the Delta approximation for a general no-linear portfolio is given by

$$PL \approx \sum_{n=1}^{N} S_{0,n} R_n \Delta_{port,n} \tag{3.3}$$

With *Delta* approximation, we can state the PL like a linear function of the underlying returns. At least two observations are necessary:

1. Since the *Delta* depends on the underlying, its values are strictly linked with the volatility, then the value of the *Delta* must be worked out day by day.

2. The calculus of *Delta* not always is simply to make: it is more probable that the pricing formula by means of *Delta* must be computed, it is very far to be simply.

3.2 Delta-Gamma approximation

Consider again the context already introduced in section (3.1). The quadratic Taylor approximation of (3.1) gives us the $Delta - Gamma$ approximation:

$$PL \approx S_0 R \Delta + \frac{S_0^2}{2} R \Gamma \tag{3.4}$$

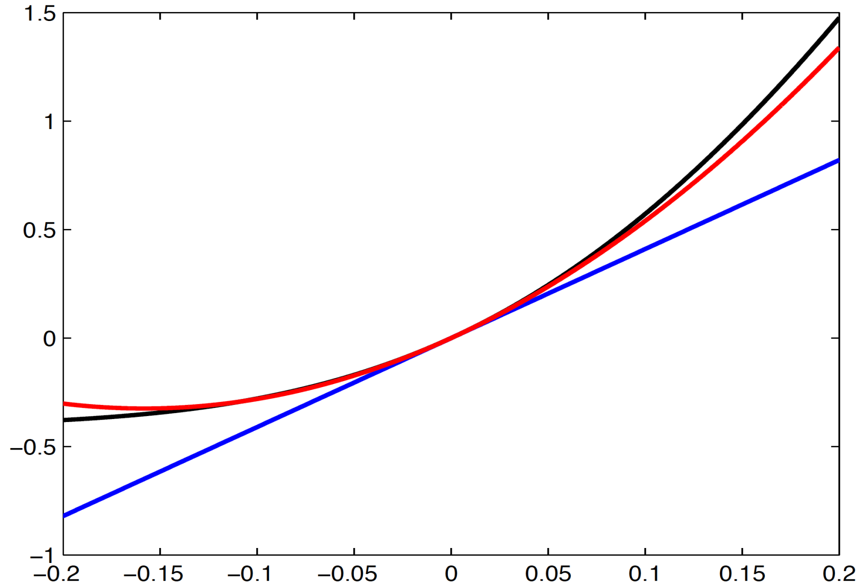


Figure 3.2: We set the exact PL in black, the *Delta* approximation in blue and the *Delta* − *Gamma* approximation in red.

General Delta-Gamma approximation

If there are π_s units of the underlying and π_k units of a certain type of derivative, for $k = 1, \ldots, K$, then the *Delta* − *Gamma* approximation is

$$PL \approx \Big(\pi_s + \sum_k \pi_k \Delta_k\Big) S_0 R + \frac{\sum_k \pi_k \Gamma_k}{2} S_0^2 R^2 = S_0 R \Delta_{port} + \frac{S_0^2 R^2}{2} \Gamma_{port} \quad (3.5)$$

More in general, consider the case of N underlyings and K type of derivatives like in earlier section, then the *Delta* − *Gamma* approximation becomes

$$PL \approx \sum_n \Big(S_{0,n} R_n \Delta_{port,n} + \frac{S_{0,n}^2 R_n^2}{2} \Gamma_{port,n}\Big) \quad (3.6)$$

where

$$\Delta_{port,n} = \pi_{S,n} + \sum_k \pi_k \Delta_{k,n}, \qquad \Gamma_{port,n} = \pi_k \Gamma_{k,n}$$

Like it is common in practice, we do not take into consideration the mixed second derivatives of a price.

Part II

Pricing Models

Chapter 4

Put and Call options

For completeness we give the definition of option

Definition 4.0.1. *An `option` is a contract stipulated between an `option seller` (said also "option writer") and an `option buyer` (even called "option holder") that it gives at the second the `right` but not the obligation to sell or buy the pointed out asset at some stipulated time for some precise price.*

We have also the following notation

Maturity It indicates the time when the option buyer can be exercised the option. This date is also indicates with either exercise date or expiration date. The time before the maturity is said *time to maturity*.

Strike price It is the price at which the option can be exercised, it is written in the option. It is also called exercise price.

Exist two types of options: `call` and `put`. The first of this grants the right to *buy* the underlying asset for the strike price, while the right to *sell* the asset presents in the contract at the strike price is given by the put option. If the option can be exercised before the maturity, it is said *American* option, otherwise we will say it *European*. At the end we explain the difference between *short position* and *long position*. If one sells an asset without possessing it, he assumes a short position. On the other hand, we said that one is in a long position if he has bought the option. Let us go to analyze the payoff of an European Call option with strike K, maturity T; indicate with S_T the price of the underlying asset at time T. At time T we have two possibilities:

1. $S_T > K$
   The gain obtained buying the underlying asset at price K and selling it at market price S_T is $S_T - K$.

2. $S_T \leqslant K$
   In this case it is not convenient, for the option holder, exercising the option so the payoff is zero; the price of the option is the loss of the investor.

By means that, we have that the payoff of an European Call option is

$$(S_T - K)^+ = \max\{S_T - K, 0\}$$

Proceeding at the same way, the payoff of an European Put option is given by

$$(K - S_T)^+ = \max\{K - S_T, 0\}$$

Combining Call and Put options, it arises a lot of other derivatives. In the figure (4.1) we can see summarized the characteristic of call and put options related at the long and short position.

The derivatives can be used principally in two way:

i) hedging the risk;

ii) speculation.

Starting from we do not know the price of the asset exchanged in the contract at time T, the first main problem is the pricing of the options, i.e. establishing an fair price for the options. Another important problem is related to the hedging. We can note the payoff of a Call option can grow indefinitely therefore the option writer exposes itself at potentially unlimited loss; for this reason born the problem of finding a strategy that can make the payoff of the option at the maturity.

Remark 4.0.1. Long position have a *limited downside risk* inasmuch for the option buyer the worst case is the loss of the invested money. On the other hand option buyer has *unlimited upside chance*. About the option writer we can see *unlimited downside risk* therefore the best case for him is when the option holder does not exercise the option.

Proposition 4.0.1. ***(No-arbitrage principle)***
Let $\mathsf{X}=(X_t)_{t\geq 0}$ and $\mathsf{Y}=(Y_t)_{t\geq 0}$ denote two different risky assets such as $X_T \leqslant Y_T$. If the market is arbitrage-free it happens that

$$X_t \leqslant Y_t \quad \text{for } t \leqslant T. \tag{4.1}$$

Proof. Suppose for absurd that $X_t > Y_t$ then we can do the following investment strategy:

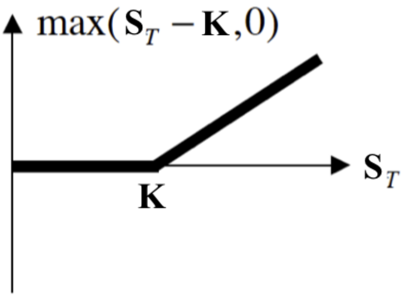
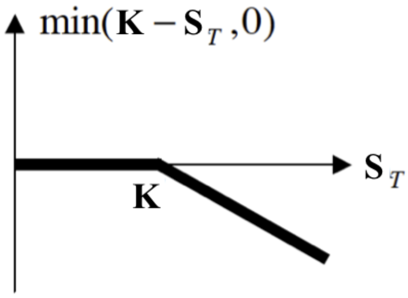
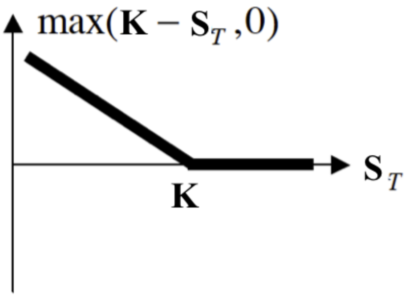
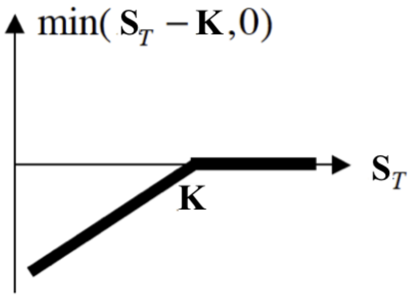
	LONG	SHORT
CALL	▪ buyer/holder of option ▪ payer of option price ▪ option to buy the asset ▪ payoff: 	▪ seller/writer of option ▪ receiver of option price ▪ obligation upon request of option holder to deliver the asset ▪ payoff: 
PUT	▪ buyer/holder of option ▪ payer of option price ▪ option to sell the asset ▪ payoff: 	▪ seller/writer of option ▪ receiver of option price ▪ obligation upon request of option holder to buy the asset ▪ payoff: 

Figure 4.1

- short selling of X_t
- long buying of Y_t
- investing the residual $X_t - Y_t > 0$ in the riskless bond.

At time T, first of all, we receive the payoff $(X_t - Y_t)e^{r(T-t)}$ deriving from the investment in riskless bond. Since $X_T = Y_T$, we can cover the cost of short selling only with the money that the investment in Y has given back. Therefore, we have made a riskless gain i.e. we have made an arbitrage but this negates the hypothesis of no-arbitrage of the market. □

Corollary 1. *From (4.0.1) follows*

$$X_T = Y_T \qquad \Rightarrow \qquad X_t = Y_t, \quad t \geq T \tag{4.2}$$

Proposition 4.0.2. ***(Put-Call parity)***
Consider a Call option `c` *and a Put option* `p`*, both of European type with maturity* T *and strike* k*. Assuming the no-arbitrage principle holds we have*

$$c_t + k\,e^{-r(T-t)} = p_t + S_t, \qquad t \in [0,T]. \tag{4.3}$$

where r *is the risk-free rate.*

Proof. Consider two investments

$$X_t = c_t + \frac{k}{B_t}B_t \qquad and \qquad Y_t = p_t + S_t$$

note that the value at time T are the same:

$$X_T = Y_T = \max\{K, S_T\}$$

So, from (4.0.1), the claim follows. □

Just for completeness, if the asset pays a dividend D at date between t and T the previous formula becames

$$c_t = p_t + S_t - D - k\,e^{-r(T-t)}.$$

Proposition 4.0.3. *For European options hold the following inequality*

$$\left(S_t - k\,e^{-r(T-t)}\right)^+ < c_t < S_t,$$
$$\left(k\,e^{-r(T-t)} - S_t\right)^+ < p_t < k\,e^{-r(T-t)}$$

where $t \leq T$*.*

Proof. Via (4.1)

$$c_t, p_t > 0$$

by (4.3) we get

$$c_t > S_t - k\, e^{-r(T-t)}$$

therefore, joint to the first inequality, we have the estimate from below. At the end, from $c_T < S_T$ and from (4.1) we obtain the estimate from above. Similarly, the other estimate can be proved. □

Proposition 4.0.4. *If a European and an American Call option are written on the same underlying asset with the same maturity T and strike price k, then the respective prices of two Call are equal.*

Proof. Consider two portfolio

- one American Call option and an amount of money of size $k\, e^{-rT}$
- one share of the underlying asset S_t

If we exercise the call before the expiration date, the portfolio value is

$$S_t - k + k\, e^{-r(T-t)} < S_t \qquad for\ \ t < T$$

Otherwise, exercising the Call at the maturity the total value of the first portfolio is

$$\max\{S_T, k\}.$$

In other word, this show that in a nondivident-paying context never should exercised before the maturity. □

Chapter 5

A Discrete World

The purpose of this chapter is to introduce some of the most important instruments for the price of options and derivatives in a discrete market. We will also try to give an economical interpretation of the mathematical objects with whome we will work. One can see the credit risk like the risk originated from the possible change in the value of portfolio, due to unexpected change in the credit quality. Into the chapter, we will speak about discrete-time models.

5.1 Recall

For a major compactness of the treatise, we now give a summarize of the definition that we have done in the first chapter, but we will use them more intensely in the sequel. Furthermore, we will introduce the definition of *arbitrage*.
In the context of a discrete-market model, we denote the price of our d-risky assets with $S = (S^1, \dots, S^d)$ and the price fo the bond with B. Through (S, B) we indicate the market in which we are. By means of $\mu_n = (\mu_n^1, \dots, \mu_n^d)$ we refer to the yield rate of S in the n period and by means of r_n we indicates the risk-free rate in the n period.

Definition 5.1.1. *We define a strategy (or portfolio) like a stochastic process in* $\mathbb{R}^{d+1}$

$$(\boldsymbol{\alpha}, \beta) = (\alpha_n^1, \dots, \alpha_n^d, \beta_n)_{n=1,\dots,N}$$

where α_n^i *indicates the quantity of asset* S^i *present in the portfolio and* β_n *the amount of bond during the period* $[t_{n-1}, t_n]$.

Consistently at this notation, we denote the *value of portfolio* $(\boldsymbol{\alpha}, \beta)$ at time t_n by

$$V_n^{(\boldsymbol{\alpha},\beta)} = \alpha_n S_n + \beta_n B_n, \qquad n = 1, \ldots, N. \tag{5.1}$$

At the initial time the value of our portfolio is

$$V_0^{(\boldsymbol{\alpha},\beta)} = \sum_{i=1}^{N} \alpha_1^i S_0^i + \beta_1 B_0$$

Definition 5.1.2. *We say $(\boldsymbol{\alpha}, \beta)$ to be* self-financing *if the following relation is valid*

$$V_{n-1}^{(\boldsymbol{\alpha},\beta)} = \alpha_n S_{n-1} + \beta_n B_{n-1} \qquad \forall\, n = 1, \ldots, N \tag{5.2}$$

Remark 5.1.1. The variation, from time t_{n-1} to t_n, of the value of a self-financing strategy $(\boldsymbol{\alpha}, \beta)$ is given by:

$$V_n^{(\boldsymbol{\alpha},\beta)} - V_{n-1}^{(\boldsymbol{\alpha},\beta)} = \alpha_n(S_n - S_{n-1}) + \beta_n(B_n - B_{n-1}) \tag{5.3}$$

and, as we can note, this change is caused by the variation of S and B and not because we have invested more money.

Definition 5.1.3. *A strategy $(\boldsymbol{\alpha}, \beta)$ is predictable if (α_n, β_n) is $\mathcal{F}_n$ measurable for every $n = 1, \ldots, N$.*

Definition 5.1.4. *We call $\mathscr{A}$ the family of acceptable strategies, namely the family of all self-financing and predictable strategies of the market (S,B).*

Definition 5.1.5. *Given $(\boldsymbol{\alpha}, \beta) \in \mathscr{A}$, we say that it is an* arbitrage strategy *if the value $V = V^{(\boldsymbol{\alpha},\beta)}$ is such that*

a) $V_0 = 0$;

and it exists $n \geqslant 1$ for that

b) $V_n \geqslant 0$ P-a.s.;

c) $P(V_n > 0) > 0$

Moreover, if $\mathscr{A}$ does not have into itself arbitrage strategies, we say that the market (S, B) is arbitrage-free.

Remark 5.1.2. Summarizing, an arbitrage strategy costs 0 at the beginning; it is probably that its price increases; sooner or later the strategy lets me give a positive net gain.

The arbitrage-free condition is the base for the most important models; we are in this case in dependence on the probabilistic model taken, that is on $(\Omega,\, \mathcal{F},\, P)$ and $(S,\, B)$. Instead of verifying if $\mathscr{A}$ does not have arbitrage strategy, one can work with *martingale measure*; the existence of this measure proved the absence of arbitrage. In this thesis we will not deep more.
We now want strengthen the concept of admissible-strategy.

Definition 5.1.6. *The couple* $(\boldsymbol{\alpha},\, \beta) \in \mathscr{A}$ *is said admissible if*

$$V_n^{(\boldsymbol{\alpha},\, \beta)} \geqslant 0 \quad P - a.s. \quad \forall\, n \leqslant N$$

In a discrete market, it is possible to modify the strategy for making it admissible, for this reason the arbitrage condition includes the admissibility.

Proposition 5.1.1. *A discrete market is arbitrage free if and only if there admissible arbitrage strategies do not exist .*

Proof. Suppose by absurd that $(\boldsymbol{\alpha},\, \beta)$ is an arbitrage strategy; we want to construct an admissible arbitrage strategy $(\boldsymbol{\alpha}',\, \beta')$. In this case we have

$S_0^{(\boldsymbol{\alpha},\, \beta)} \,=\, 0$

and we can find an n for wich

$\alpha_n S_n \,+\, \beta_n B_n \,\geqslant\, 0$ a.s.

$P(\alpha_n S_n \,+\, \beta_n B_n \,>\, 0) \,>\, 0.$

In the case of $(\boldsymbol{\alpha},\, \beta)$ is not admissible then it exists $k \,<\, N$ and $F\, in\, \mathcal{F}$ with $P(F) \,>\, 0$ for which

$\alpha_k S_k \,+\, \beta_k B_k \,<\, 0$ on F

$\alpha_n S_n \,+\, \beta_n B_n \,\geqslant\, 0$ a.s. for $k \,<\, n \,<\, N$.

Therefore we are able to construct a new arbitrage strategy:

- $\alpha'_n = 0$; $\beta'_n = 0$ on $\Omega \setminus F$ for all n
- Instead on F

$$\alpha'_n = \begin{cases} 0 & n \leqslant k, \\ \alpha_n & n > k. \end{cases}, \quad \beta'_n = \begin{cases} 0 & n \leqslant k, \\ \alpha_n - (\alpha_k S_k + \beta_k B_k) & n > k. \end{cases}$$

It is simple proved that $(\boldsymbol{\alpha}',\, \beta')$ it is an admissible arbitrage strategy. □

5.2 Asset Pricing

Definition 5.2.1. *Let us fix an asset $Y = (Y_n)$ we define $(\tilde{S},\, \tilde{B})$ as* ***the normalized market*** *with respect to Y with $\tilde{S} = (\tilde{S}^i_1,\, \ldots,\, \tilde{S}^i_N)$, $\tilde{B} = (\tilde{B}_1,\, \ldots,\, \tilde{B}_N)$ and where*

$$\tilde{S}^i_n = \frac{S^i_n}{Y_n}, \qquad \tilde{B}_n = \frac{B_n}{Y_n}$$

Remark 5.2.1. By means of units of Y, we gauge the prices of the other assets; for this reason Y is said numeraire. Very often, B is taken as numeraire and in this case $\tilde{S}^i$ denotes the *discounted price* of the i-th asset.

Got a strategy $(\boldsymbol{\alpha},\, \beta)$, we have

$$\tilde{V}^{(\boldsymbol{\alpha},\beta)}_n = \frac{V^{(\boldsymbol{\alpha},\beta)}_n}{B_n}$$

In this new path, the self-financing condition becomes

$$\tilde{V}^{(\boldsymbol{\alpha},\beta)}_n = \alpha_n \tilde{S}_{n-1} + \beta_n.$$

where $n = 1,\, \ldots,\, N$.

Remark 5.2.2. We can readapt the preceding results like following:

1. The discounted value of $(\boldsymbol{\alpha},\, \beta)$, self-financing strategy, is uniquely determined by V_0 and

$$\tilde{V}^{(\boldsymbol{\alpha},\beta)}_n = \tilde{V}^{(\boldsymbol{\alpha},\beta)}_{n-1} + \sum_{i=1}^{d} \alpha^i_n \left(\tilde{S}^i_n - \tilde{S}^i_{n-1}\right)$$

   with $n = 1,\, \ldots,\, N$.

2. The successive formula holds:

$$\tilde{V}^{(\boldsymbol{\alpha},\beta)}_n = V_0 + G^{(}_n\alpha) \tag{5.4}$$

   where

$$G^{(\alpha)}_n = \sum_{k=1}^{n} \alpha_k \left(\tilde{S}_k - \tilde{S}_{k-1}\right)$$

   is the *normalized gain* related to the predictable process α; it does not depend on β. We point out that this formula implies that $(\boldsymbol{\alpha},\, \beta)$ *is self-financing <u>if and only if</u> $\tilde{V}^{(\boldsymbol{\alpha},\beta)}$ is the transform of $\tilde{S}$ by $\boldsymbol{\alpha}$.*

5.2.1 Risk-neutral Probability

At the beginning of this chapter we recalled the concept of arbitrage and we have said that a discrete market (S, B) is arbitrage-free if a strategy $(\boldsymbol{\alpha}, \beta)$ with the characteristic into the definition does not exist. Trying to verify the non-existence of that strategy is very far to possible, so, to make more operative the concept of absence of arbitrage, we introduce the risk neutral probability or, more usually, the `equivalent martingale measure` (in short EMM or simply MM).

Definition 5.2.2. *An MM with numeraire Y is a probability measure Q on $(\Omega, \mathcal{F})$ such that*

1. *Q is equivalent to P*
2. *the Y-normalized prices are Q-martingales:*

$$\frac{S_{n-1}}{Y_{n-1}} = E^Q\left[\frac{S_n}{Y_n} \,|\, \mathcal{F}_{n-1}\right], \qquad \frac{B_{n-1}}{Y_{n-1}} = E^Q\left[\frac{B_n}{Y_n} \,|\, \mathcal{F}_{n-1}\right]$$

for every $n = 1, \ldots, N$.

Remark 5.2.3. In the case of $Y = B$, the martingale measure related to this numeraire gives

$$\tilde{S}_k = E^Q\left[\tilde{S}_n \,|\, \mathcal{F}_k\right], \qquad 0 \leqslant k < n \leqslant N,$$

and for the martingale property

$$E^Q\left[\tilde{S}_n\right] = E^Q\left[E^Q\left[\tilde{S}_n \,|\, \mathcal{F}_0\right]\right] = \tilde{S}_0, \qquad n \leqslant N \tag{5.5}$$

The last formula can be interpreted as: *"the expectations of the future normalized prices are equal to the current ones"*.

Since Q is equivalent to P, in the market (S, B) the absence of arbitrage under P is equivalent at the arbitrage-free respectively at Q.

Theorem 5.2.1. ***(First Fundamental Theorem of asset pricing)*** *A discrete market is arbitrage-free <u>if and only if</u> at least one MM exists.*

For the proof we remand at [2].
The following theorem, whose enunciate we are just going to give, permit us to pass from a MM with numeraire Y to a MM with another numeraire. Basically, to make this change is useful when in the market different currencies for prices are used. In a theoretical way, the change of numeraire might be used to semplify computations.

Theorem 5.2.2. *Let a discrete market (S, B) and a MM Q with numeraire Y be given. Let X be a positive adapted process such that $\left(\frac{X_n}{Y_n}\right)$ is a Q-martingale. The new measure Q^X, defined by*

$$\frac{d\,Q^X}{d\,Q} = \frac{X_N}{X_0}\left(\frac{Y_N}{Y_0}\right)^{-1}$$

has the property

$$Y_n E^Q\left[\frac{Z}{Y_N} \mid \mathcal{F}_n\right] = X_n E^{Q^X}\left[\frac{Z}{X_N} \mid \mathcal{F}_n\right], \qquad n \leqslant N, \tag{5.6}$$

for every random variable Z.

Remark 5.2.4. Relatively at the theorem, we can make two different observations.

1. X represents the value process of another asset or strategy and it is considered the new numeraire
2. Subsequently, Q^X is a EMM with numeraire X.

Proposition 5.2.1. *1. Let Y be a numeraire of an MM Q and $(\boldsymbol{\alpha}, \beta) \in \mathscr{A}$. Then $\tilde{V}^{(\boldsymbol{\alpha}, \beta)}$ is a Q-martingale and holds the risk-neutral pricing formula:*

$$\tilde{V}_0^{(\boldsymbol{\alpha}, \beta)} = E^Q\left[\tilde{V}_n^{(\boldsymbol{\alpha}, \beta)}\right], \qquad n \leqslant N$$

2. If Q is an equivalent measure to P and $\tilde{V}^{(\boldsymbol{\alpha}, \beta)}$ is a martingale with respect to Q for every $(\boldsymbol{\alpha}, \beta) \in \mathscr{A}$, then Q is an MM with numeraire Y.

Proof. For simplicity, we only consider the case $Y = B$. By means of (5.2.2), starting from another numeraire, it is possibile bringing back to the case that we take into account. For hypothesis, $(\boldsymbol{\alpha}, \beta)\, in\, \mathscr{A}$ so

$$\tilde{V}_n^{(\boldsymbol{\alpha}, \beta)} = \tilde{V}_{n-1}^{(\boldsymbol{\alpha}, \beta)} + \alpha_n(\tilde{S}_n - \tilde{S}_{n-1})$$

by that we get

$$\begin{aligned} E^Q\left[\tilde{V}_n^{(\boldsymbol{\alpha}, \beta)} \mid \mathcal{F}_{n-1}\right] &= \tilde{v}_{n-1}^{(\boldsymbol{\alpha}, \beta)} + E^Q\left[\alpha_n(\tilde{S}_n - \tilde{S}_{n-1}) \mid \mathcal{F}_{n-1}\right] = \\ &= \tilde{V}_{n-1}^{(\boldsymbol{\alpha}, \beta)} + \alpha_n E^Q\left[\tilde{S}_n - \tilde{S}_{n-1} \mid \mathcal{F}_{n-1}\right] = \\ &= \tilde{V}_{n-1}^{(\boldsymbol{\alpha}, \beta)} \end{aligned}$$

by (5.4). □

Proposition 5.2.2. ***(No arbitrage principle)***
If (S, B) is a arbitrage-free market, $(\boldsymbol{\alpha}, \beta)$, $(\boldsymbol{\alpha}', \beta') \in \mathscr{A}$ and

$$V_N^{(\boldsymbol{\alpha}, \beta)} = V_N^{(\boldsymbol{\alpha}', \beta')} \quad P - a.s.$$

then

$$V_n^{(\boldsymbol{\alpha}, \beta)} = V_n^{(\boldsymbol{\alpha}', \beta')} \quad P - a.s., \qquad n \leqslant N$$

Proof. Through the assumption of arbitrage-free market, we get the existence of a MM Q with a certain numeraire Y. From the definition of EMM and by the fact that $\tilde{V}^{(\boldsymbol{\alpha}, \beta)}$ and $\tilde{V}^{(\boldsymbol{\alpha}', \beta')}$ are Q-martingale, the claim follows. □

Let X be the payoff of an option with maturity T. One of the purposes for the option writer is to be able to cover the value of the option at the maturity; then it is necessary to replicate the derivative.

Definition 5.2.3. *A strategy $(\boldsymbol{\alpha}, \beta) \in \mathscr{A}$ is said replicating strategy if it assumes the value of the derivative X at the time T*

$$V_N^{(\boldsymbol{\alpha}, \beta)} = X \quad a.s.$$

If the aforementioned strategy exists, then X is called *replicable.*

Theorem 5.2.3. *Consider a replicable derivative X in (S, B), a arbitrage-free market. Then for any EMM Q with numeraire B and for any replicating strategy $(\boldsymbol{\alpha}, \beta) \in \mathscr{A}$, we have*

$$E^Q\left[\frac{X}{B_N} \,|\, \mathcal{F}_n\right] = \frac{V_n^{(\boldsymbol{\alpha}, \beta)}}{B_n} \qquad n = 0 \ldots, N.$$

The process $V^{(\boldsymbol{\alpha}, \beta)}$ is called risk-neutral price of X.

Definition 5.2.4. *Given a market, if we are able to replicate every European derivative, then this market is named complete.*

Theorem 5.2.4. ***(Second Fundamental Theorem of asset pricing)***
An arbitrage-free market (S,B) is complete if and only if a unique EMM with numeraire B exists.

By means of the first and second fundamental theorems, for studying the completeness and the absence of arbitrage in a market, we reduce the claim to prove the existence and the unicity of an EMM with numeraire the free-risk bond.

5.2.2 Binomial Model

Everyone interested can go on studying the binomial model and, sending the number of periods of the model to infinity, one is able to obtain the $Black - Scholes$ formula. For more deepen treatise, we refer to [?]. We list rapidly the main steps to reach the formula. Consider a binomial model with $N \in \mathbb{N}$ periods. We denote the interest rate, the increase and decrease factors (synthetically they are the parameters which indicate the intensity of ascent and descent in the binomial model) by r_N, u_N, d_N. Let $T > 0$ the maturity of derivative X and we get

$$\delta_N = \frac{T}{N}$$

so we obtain

$$1 + r_N = e^{r\delta_N}, \qquad u_N = e^{\sigma\sqrt{\delta_N}+\alpha\delta_N}, \qquad d_N = e^{-\sigma\sqrt{\delta_N}+\beta\delta_n}$$

with σ the volatility, α, β real constants and r the annual risk-free rate.

Proposition 5.2.3. *It is worth that*

$$\lim_{N\to\infty} E^{Q_N}\big[X_N\big] = \left(r - \frac{\sigma^2}{2}\right)T \tag{5.7}$$

$$\lim_{N\to\infty} var^{Q_N}\big(X_N\big) = \sigma^2 T \tag{5.8}$$

It is possibile to show that X_N converges in distribution to a normally distributed variable X and then, through (5.7) and (5.8), we obtain

$$X \sim \mathcal{N}_{\left(r-\frac{\sigma^2}{2}\right)T,\, \sigma^2 T} \tag{5.9}$$

Theorem 5.2.5. *Consider a N-period binomial model with parameter u_N, d_N, r_N like above and an European Put option with strike K, maturity T and price $P_0^{(N)}$. Let X be as (5.9). Then the following limit exists*

$$\lim_{N\to\infty} P_0^{(N)} = P_0 = e^{-rT}\, E\left[\left(K - S_0 E^X\right)^+\right]$$

Definition 5.2.5. *The price P_0 is named* Black-Scholes price *of an European Put Option with strike K and maturity T*

Proposition 5.2.4. ***(Black-Scholes formula)***
The following equality is worth

$$P_0 = Ke^{-rT}\Phi(-d_2) - S_0\Phi(-d_1) \tag{5.10}$$

where

$$\begin{aligned} d_1 &= \frac{\log(\frac{S_0}{K}) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \\ d_2 &= \frac{\log(\frac{S_0}{K}) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \end{aligned} \tag{5.11}$$

The greek letter Φ indicates the standard normal distribution function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\, dy \qquad x \in \mathbb{R}$$

Remark 5.2.5. By the Put-Call parity formula (4.0.2) and (5.2.4), we are able to achieve the $Black - Scholes$ price for an European Call option with strike K and maturity T:

$$C_0 = S_0\Phi(d_1) - Ke^{-rT}\Phi d_2 \tag{5.12}$$

Chapter 6

A Continuos World

The Black and Scholes equation and the Merton's model are two classical instruments used in the credit risk management. First of all, we introduce the theoretical elements of the stochastic processes, that are normally used in continuos-time financial models. After this short introduction, we will present the Black-Scholes formula and at the end we will try to analyze the pricing of a derivative, whose underlying asset is not exchanged on the market like the case of a derivative on the temperature.

6.1 Stochastic Process

First and foremost, consider a probability space $(\Omega, \mathcal{F}, P)$ and a real interval $I \subseteq \mathbb{R}_{\geq 0}$.

Definition 6.1.1. *A collection $(X_t)_{t \in I}$ of random variables with values in $\mathbb{R}^N$ is said a* **measurable stochastic process** *if the map*

$$X : I \times \Omega \to \mathbb{R}^N, \qquad X(t, \omega) = X_t(\omega)$$

is $\mathscr{B}(I) \otimes \mathcal{F}$-measurable. The process X is **integrable** *if $X_t \in L^1(\Omega, P)$ for every $t \in I$*

Definition 6.1.2. *A stochastic process X is* **continuos** *a.s. if*

$$t \to X_t(\omega)$$

are continuos function $\forall \, \omega \in \Omega$.

Definition 6.1.3. *A* **filtration** *$(\mathcal{F}_t)_{t \geq 0}$ in $(\Omega, \mathcal{F}, P)$ is an increasing family of sub-σ-algebras of $\mathcal{F}$. Taken a stochastic process X, we can define its natural filtration as*

$$\tilde{\mathcal{F}}_t^X = \sigma(X_s \,|\, 0 \leqslant s \leqslant t)$$

Definition 6.1.4. *Gotten a filtration* $(\mathcal{F}_t)_{t\geq 0}$*, we define a stochastic process* X ***adapted*** *as a process for which* $\tilde{\mathcal{F}}_t^X \subseteq \mathcal{F}_t$*, i.e.* X *is adapted if* X_t *is measurable with respect to the* σ*-algebra* $\mathcal{F}_t$ *for every* t*.*

Definition 6.1.5. *A stochastic process* $W = (W_t)_{t\geq 0}$ *is called a real* ***Brownian motion*** *if it has real value and satisfies the following three properties*

1. $W_0 = 0$ *a.s.*
2. W *is adapted with respect to the filtration and it is continuos*
3. $W_t - W_s \sim \mathcal{N}_{0,t-s}$ *and the random variable is independent of* $\mathcal{F}_s$*, for* $0 \leqslant s < t$*.*

Like a direct consequence of (6.1.5) we know

$$W_t \sim \mathcal{N}_{0,t}$$

Example 6.1.1. A model for the price of a risky asset S is

$$S_t = S_0\left(1 + \mu t\right) + \sigma W_t$$

where S_0 is the initial price of the asset, μ is the expected rate of return and σ is the volatility of the asset. If $\sigma > 0$, the process $S = (S_t)_{t\geq 0}$ is a gaussian

$$S_t \sim \mathcal{N}_{S_0(1+\mu t),\sigma^2 t}$$

and so

$$E[S_t] = S_0(1 + \mu t)$$

therefore we can observe that the Brownian motion introduces the "noise" without modifying the mean.
It is important saying that this model is not used for a lack of continuity in the rate and because we would have positive value of the likelihood that S_t is negative.

Definition 6.1.6. *Fixed* (t, x)*, we define*

$$W_T^{t,x} = W_T - W_t + x, \qquad T \geqslant t$$

a Brownian motion starting from x *at time* t*. It also has a normal distribution but with different mean and variance:*

$$W_T^{t,x} \sim \mathcal{N}_{x,T-t}$$

Definition 6.1.7. *We called* ***transition density*** *of* $W_T^{t,x}$*, the function*

$$\Gamma(t,x;T,y) = \frac{1}{\sqrt{2\pi(T-t)}} \exp\Big(-\frac{(x-y)^2}{2(T-t)}\Big)$$

where the couple (t,x) *is the starting point and* (T,y) *is the end one.*

Remark 6.1.1. (**Link with heat equation**)
Let us to consider the two forms of the heat equation:

1. backward operator
$$L_{B=}\frac{1}{2}\partial_{xx} + \partial_t$$
2. forward operator
$$L_F = \frac{1}{2}\partial_{yy} - \partial_T$$

It is easy showing that $\Gamma(t,x;T,y)$ is a solution for heating operator:

$$\Big(\frac{1}{2}\partial_{xx} + \partial_t\Big)\,\Gamma(t,x;T,y) = 0$$
$$\Big(\frac{1}{2}\partial_{yy} - \partial_T\Big)\,\Gamma(t,x;T,y) = 0$$

Starting from the forward operator we can pose the following Cauchy's problem

$$\begin{cases} L_F\, u(T,y) = 0 & T > t \\ u(t,y) = \phi(y) & y \in \mathbb{R} \end{cases}$$

For what we have said, it is worth

$$u(T,y) = \int_{\mathbb{R}} \Gamma(t,x;T,y)\phi(x)dx$$

Consider now $v(t,x) = u(T-t+t_0,x)$ and so

$$\begin{cases} L_B v(t,x) = 0 & t < T,\, x \in \mathbb{R} \\ v(T,x) = \phi(x) \end{cases}$$

where $v(T,x) = u(t_0,x) = \phi(x)$ and $\partial_t v(t,x) = -\partial_T u$, that is, with the backward operator, we are going back and this is the reason why the above Cauchy problem finds a natural application in finance: I know the payoff at the final time and I want to determine the initial price of the derivative. However, we have

$$v(t,x) = \int_{\mathbb{R}} \Gamma(t,x;T,y)\phi(y)dy.$$

Since Γ is the density of $W_T^{t,x}$, in a probabilistic view making the integral with respect to the variable x means making an expected value of the random variable

$$v(t,x) = \int_{\mathbb{R}} \Gamma(t,x;T,y)\phi(y)dy = E[\phi(W_T^{t,x})].$$

Therefore we have reached that *working out a backward Cauchy problem gives us the risk-neutral price; indeed the function $v(t,x)$ is the expected value of the payoff function.*

Definition 6.1.8. *Given a probability space $\big(\Omega, \mathcal{F}, P, (\mathcal{F}_t)\big)$ we introduce*

$$\mathbb{L}^2 = \left\{ \alpha \,|\, \alpha \textit{ is adapted and } E\left[\int_0^T \alpha_t^2 dt\right] < \infty \right\}$$

Definition 6.1.9. *Taken u,$v \in \mathbb{L}^2$, we define an* **Itô process** *like:*

$$X_t = X_0 + \int_0^t u_s\,ds + \int_0^t v_s\,dW_s$$

NOTATION: the *Itô* process can be indicate as

$$dX_t = u_t\,dt + v_t\,dW_t.$$

Proposition 6.1.1. *(***Formula di Itô per moto browniano** *)*
Consider a Brownian motion W_t and a function $F = F(t,x) \in \mathcal{C}^{1,2}(\mathbb{R}^2)$. Then

1. *the stochastic process*

$$Y_t = F(t, W_t)$$

is an Itô process

2. *we also have*

$$F(t,W_t) - F(0,W_0) = \int_0^t (\partial_t F)(s,W_s)\,ds + \int_0^t (\partial_x F)(s,W_s)\,dW_s + \frac{1}{2}\int_0^t (\partial_{xx} F)(s,W_s)\,ds$$

Proposition 6.1.2. *Consider an Itô process and a function $f = f(t,x) \in \mathcal{C}^{1,2}(\mathbb{R}^2)$. Then*

1. the stochastic process

$$Y_t = f(t, X_t)$$

is an Itô process

2. we also have

$$df(t, X_t) = \partial_t(t, X_t)dt + \partial_x f(t, X_t)dX_t + \frac{1}{2}\partial_{xx} f(t, X_t)d\langle X\rangle_t \qquad (6.1)$$

where

$$\langle X\rangle_t = \int_0^t v_s^2\, ds$$

Example 6.1.2. (**An application**)
Consider $u, v, \in L^2([0, T])$ and an *Itô* process with deterministic coefficients:

$$X_t = X_0 + \int_0^t u(s)\, ds + \int_0^t v(s)\, dW_s$$

We want to show that X_t has a normal distribution. First of all, we calculate mean and variance of the process: since the stochastic integral is a martingale, we have

$$E[X_t] = X_0 + \int_0^t u(s)\, ds.$$

Moreover, by the *Itô* isometry we have

$$var(X_t) = E\Big[\big(X_t - E[X_t]\big)^2\Big] = E\Big[\big(\int_0^t v(s)\, dW_s\big)^2\Big] = E\Big[\int_0^t v^2(s)\, ds\Big] = \int_0^t v^2(s)\, ds.$$

For showing the process that we have considered has normal distribution, we calculate its characteristic function:

$$\varphi_{X_t}(\xi) = E[e^{i\xi X_t}] = e^{i\xi X_0} + \int_0^t E[e^{i\xi X_s}](i\xi u(s) - \frac{\xi}{2}s^2(s))\, ds \qquad (6.2)$$

where, setting $Y_t = e^{i\xi X_t}$, the third equality directly cames from

$$\begin{aligned} dY_t &= i\xi Y_t\, dX_t + \frac{1}{2}(i\xi)^2 Y_t\, d\langle X_t\rangle = \\ &= (i\xi Y_t u(t) + \frac{1}{2}(i\xi)^2 Y_t v^2(t))\, dt + i\xi Y_t v(t) dW_t \end{aligned}$$

$$\Rightarrow e^{i\xi X_t} = e^{i\xi X_0} + \int_0^t Y_s\big(i\xi u(s) - \frac{\xi}{2}v^2(s)\big)\, ds + \int_0^t i\xi e^{i\xi X_s} v(s)\, dW_s.$$

so one can observe that the first integral is deterministic, while the second integral has only Y_t like stochastic term. Coming back to characteristic function, the (6.2) can be read as

$$\varphi(t) = e^{i\xi X_0} + \int_0^t \varphi(s)a(s)\,ds$$

putting

$$a(s) = i\xi u(s) - \frac{\xi}{2}s^2(s)$$

that is like having

$$\begin{cases} \varphi'(t) = a(t)\varphi(t) \\ \varphi(0) = e^{i\xi X_0} \end{cases}$$

which solution is

$$\varphi(t) = \exp\big(i\xi X_0 + \int_0^t a(s)\,ds\big)$$

$$\Rightarrow \varphi_{X_t}(\xi) = \exp\Big(i\xi X_0 + \int_0^t \big(i\xi u(s) - \frac{\xi}{2}v^2(s)\big)\,ds\Big)$$

Hence, comparing the distribution function with mean and variance of the process that we have considered, we can say that X_t is normally distributed.

6.2 Black-Scholes Model

As we have already said, the market is a couple (S, B) where the first represents a risky asset, while the second is simply a bond. Now, we impose that the bond satisfies

$$\begin{cases} dB_t = rB_t dt \\ B_0 = 1 \end{cases} \qquad \Rightarrow B_t = e^{rt}.$$

and we consider an asset S_t such that

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where μ is the *expected rate of return* of S and σ is the volatility. From the stochastic differential equation, we know that

$$S_t = S_0 e^{\sigma W_t + \alpha t}$$

Let we determine α:

$$dS_t = \alpha S_t dt + \sigma S_t dW_t + \frac{1}{2}\sigma^2 dt = (\alpha S_t + \frac{1}{2}\sigma^2 S_t)dt + \sigma S_t dW_t$$

$$\Rightarrow \mu = \alpha + \frac{\sigma^2}{2} \qquad \Rightarrow \alpha = \mu - \frac{\sigma^2}{2}$$

therefore the process S_t has the form

$$S_t = S_0\, e^{\sigma W_t + (\mu - \frac{\sigma^2}{2})t}$$

We can note that

$$E[S_t] = S_0 e^{\mu t}$$

so the parameter μ is coherent with the meaning of expected rate of return. It is important to underline that, written in this form, the process S_t has log-normal distribution:

$$P(S_t \in [a,b]) = P(\log S_t \in [\log a, \log b]) = P\big([\log S_0 + \sigma W_t + (\mu - \frac{\sigma^2}{2})t] \in [\log a, \log b]\big).$$

Definition 6.2.1. *Given a probability space $(\Omega,\, \mathcal{F},\, P\,(\mathcal{F}_t)_t)$, a **strategy** is a couple $(\alpha_t,\, \beta_t)$ of stochastic processes with the following properties:*

1. *they are adapted processes at the filtration $(\mathcal{F}_t)_t$ with respect to the measure P*

2. *the couple is self-financing i.e.*

$$dV_t = \alpha_t dS_t + \beta_t dB_t$$

where V_t is the stochastic process of the strategy's values, S_t represents the risky asset and B_t is the process of bonds.

Since the Call options depend only on the final price, we can consider a different type of strategies.

Definition 6.2.2. *Let $\alpha(t,x)$, $\beta(t,x)$ be two regular functions, a strategy $(\alpha_t,\, \beta_t)$ is called **markovian** if*

$$\alpha_t = \alpha(t, S_t) \qquad \beta_t = \beta(t, S_t).$$

Remark 6.2.1. 1. Comparing markovian strategies with adapted ones, we can note that saying $(\alpha_t,\, \beta_t)$ is adapted, i.e. $\mathcal{F}_t$ measurable, is a weaker condition than being markovian. Indeed the term *adapted* indicates a dependence of the strategy on all prices until the time t, while the markovian condition conveys a dependence of $(\alpha_t,\, \beta_t)$ only on the price at time t.

2. The stochastic process β_t should be also dependant on the process B_t, that is $\beta_t = \beta(t, S_t, B_t)$ but $B_t = e^{rt}$ so the link with B_t is already expressed from the time t.

If the strategy is markovian, the self-financing condition becomes

$$V_t = \alpha_t S_t + \beta_t B_t = f(t, S_t)$$

which expresses in terms of stochastic differential turns into

$$dV_t = \alpha_t dS_t + \beta_t dB_t.$$

The function $f(t, s)$ can be introduced just only because the strategy is markovian. Let us now apply the *Itô* formula at f

$$df(t, S_t) = \partial_t f(t, S_t) + \partial_s f(t, S_t) dS_t + \frac{1}{2} \partial_{ss} f(t, S_t) d\langle S \rangle_t \tag{6.3}$$

It is a trivial calculation that $\langle S \rangle_t = \sigma^2 S_t^2 dt$. From self-financing condition we also have

$$df(t, S_t) = \alpha_t dS_t + \beta_t r B_t dt = \alpha_t dS_t + r(\alpha_t S_t - f(t, S_t))dt \tag{6.4}$$

Comparing the (6.3) to (6.4) we have

- ⊙
$$\alpha_t = (\partial_s f)(t, S_t)$$
  and since the strategy is markovian we have that the function α is
$$\alpha = (\partial_s f) \tag{6.5}$$
  That, in other words, means *"if the strategy is self-financing, the numbers of risky assets must be equals to $\partial_s f$ "* where f is the function which represents the value of portfolio.

- ⊙ Matching the two parts in dt, we get
$$(\partial_t f)(t, S_t) + \frac{\sigma^2 S_t^2}{2} (\partial_{ss} f)(t, S_t) = r\big(f(t, S_t) - S_t (\partial_s f)(t, S_t)\big) \tag{6.6}$$
  As S_t is log-normally distributed, it can assume any positive real value and since (6.6) is an equality between aleatory quantities, we have the famous ***Black-Scholes differential equation***
$$\partial_t f(t, s) + \frac{\sigma^2 s^2}{2} \partial_{ss} f(t, s) + rs \partial_s f(t, s) - rf(t, s) = 0 \tag{6.7}$$

Definition 6.2.3. *We called Black-Scholes differential operator*

$$L_{BS} = \partial_t + \frac{\sigma^2 s^2}{2}\partial_{ss} + rs\partial_s - r. \tag{6.8}$$

Then, from (6.7) we deduce that the function f, i.e. the value of the portfolio, is a solution for $L_{BS} = 0$.

Right now, we want to construct a self-financing strategy (α_t, β_t), which replies the payoff of the derivative: taken a Call Option, we ask that (α_t, β_t) is such as the value of the portfolio at maturity T is $V_T = (S_t - k)^+ = \varphi(S_t)$ where $\varphi(s) = (s-k)^+$. In this way, we obtain the following Cauchy problem

$$\begin{cases} (L_{BS}f)(t,s) = 0 & t < T,\ s > 0 \\ f(T,s) = \varphi(s) & s > 0 \end{cases} \tag{6.9}$$

With a bit of work, it is possible showing that the problem (6.9) has solution, then the price of **not arbitrage** is

$$V_0 = f(0, S_0) \tag{6.10}$$

Remark 6.2.2.

1. One is able to note that the price V_0 does not depend on the parameter μ
2. Making in (6.8) the variable change $x = \log s$, the problem (6.9) becomes the classical *heat equation*, then the solution is given by

$$f(t, e^x) \approx \int_{\mathbb{R}} Gauss\,(e^x - k)^+\,dx \tag{6.11}$$

   where $Gauss$ states for the fundamental solution of the heat equation; it is multiplied for $\varphi(e^x)$, i.e. the payoff function evaluates in e^x.

6.2.1 Many ways lead to BS

At least other two different ways to amount to the Black-Scholes model exist, and we are going to speak about them. The first one is not very mathematically correct, instead the second is very coherent.

Heuristic method

We suppose that a portfolio V_t exists, and it mades by certain quantities of risky assets and by a Call option with price f:

$$V_t = \alpha_t S_t - f(t, S_t). \tag{6.12}$$

We want to neutralize the risk deriving from this portfolio, i.e. we want that the portfolio does not depend on the variation of the underlying assets. The natural way to work out this variations is posing the derivative of V_t respect to S equal to 0:

$$\partial_S V = 0.$$

So, making the derivative on (6.12), we obtain

$$0 = \alpha_t - \partial_S f(t, S_t) \qquad \Rightarrow \qquad \alpha_t = \partial_s f(t, S_t) \tag{6.13}$$

but we care about underlining that this equation does not make sense: why do not we derive even α? However the heuristic method goes on passing at stochastic differential equation:

$$dV_t = \alpha_t dS_t - df(t, S_t)$$

and, applying the *Itô* formula on $df(t, S_t)$, it becomes

$$dV_t = \alpha_t dS_t - \partial_t f dt - \partial_s f dS_t - \frac{1}{2}\sigma^2 S^2 \partial_{ss} f dt$$

then, from (6.13), we have

$$dV_t = -\partial_t f - \frac{1}{2}\sigma^2 S^2 \partial_{ss} f dt = -(\partial_t f + \frac{1}{2}\sigma^2 S^2 \partial_{ss} f) dt$$

that is we have obtained that the stochastic differential of V is exclusively deterministic, i.e. without risk. Comparing

$$dV_t = -(\partial_t f + \frac{1}{2}\sigma^2 S^2 \partial_{ss} f) dt \tag{6.14}$$

with the condition of no arbitrage

$$dV_t = rV_t dt \tag{6.15}$$

we obtain

$$-(\partial_t f + \frac{1}{2}\sigma^2 S^2 \partial_{ss} f) = rV_t \tag{6.16}$$

which is exactly the Black-Scholes differential equation.

A third approach

From the remark (6.2.2), one could decide to replace μ with any other parameter. We chose r. Then we have

$$\tilde{S} = e^{-rt} S_t \quad \Rightarrow \quad d\tilde{S}_t = \sigma \tilde{S}_t dW_t$$

where W_t is brownian motion. But it is also true that, by *Itô* formula

$$d\big(e^{-rt}f(t,S_t)\big) = e^{-rt}(L_{BS}f)(t,S_t)dt + e^{-rt}\sigma S_t(\partial_s f)(t,S_t)dW_t \qquad (6.17)$$

then if f is the solution of (6.8), the product $e^{-rt}f(t,S_t)$ is a martingale. It is essential noting that we have (6.17) only because we have chose to replace μ with r, otherwise we would have obtained another form for the *Itô* formula development. So, starting from (6.17) we have

$$f(0,S_0) = e^{-rt}E[f(T,S_T)] = e^{-rt}E[\varphi(S_T)]$$

then

$$\log\frac{S_t}{S_0} \sim \mathcal{N}_{(r-\frac{\sigma^2}{2})t,\sigma^2 t} \qquad (6.18)$$

This shows us that the initial price of derivative is the discounted mean value of the payoff of the option.

6.3 Risk Neutral Probability

In the section (6.2.1) we have taken the possibility to replace μ with r for granted, but this possibility is guaranteed by the Girsanov Theorem. We will see that changing the drift coefficients is equivalent to make a changing in measure terms.

Theorem 6.3.1. ***(The Girsanov Theorem)***
Let W be a Brownian motion on $(\Omega,\mathcal{F},P,(\mathcal{F})_t)$ and we consider a stochastic process $\lambda = (\lambda_t)_t$ such as $\lambda \in \mathbb{L}^2$ and $E[e^{\int_0^t \lambda_s^2 ds}] < \infty$. Getting a stochastic process $W_t^\lambda = W_t + \int_0^t \lambda_s dt$ (note that W_t^λ is not a Brownian motion) then a measure Q exists such as

1. $Q \sim P$
2. *W_t^λ is Brownian motion on $(\Omega,\mathcal{F},Q,(\mathcal{F}_t))$*
3. *it is worth*
$$\frac{dQ}{dP} = \exp\Big(-\int_0^T \lambda_s ds - \frac{1}{2}\int_0^T \lambda_s^2 ds\Big)$$

Remark 6.3.1. The stochastic process W_t^λ has $\mathcal{N}_{0,t}$ as distribution with respect to Q, i.e. under the Q-measure W_t^λ became a *standard* Brownian motion.

We now try to modify BS model for the asset S in order to have an expression of S in which appear the process W_t^λ. Start with the BS model:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{6.19}$$

with W Brownian motion on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ where P is the real measure. We transform (6.19) in the following way

$$dS_t = rS_t dt + \sigma S_t \Big(dW_t + \frac{\mu - r}{\sigma} dt \Big)$$

therefore posing

$$\lambda = \frac{\mu - r}{\sigma}, \qquad W_t^\lambda = dW_t + \frac{\mu - r}{\sigma}$$

we obtain

$$dS_t = rS_t\, dt + \sigma S_t\, dW_T^\lambda \tag{6.20}$$

where W_t^λ is a Brownian motion with respect to Q, that we well know it exists from Girsanov's theorem.
So, if we replace μ with r, the asset S follows the dynamics explaind by (6.20) and therefore $\tilde{S}_t$ is an Q-martingale:

$$\tilde{S}_t = E^Q[\tilde{S}_t \,|\, \mathcal{F}_t].$$

We want now to work out the evolution of the discounted price f.

$$d\big(e^{-rt} f(t, S_t)\big) = e^{-rt}\big(L_{BS} f\big)(t, S_t) dt + e^{-rt} \sigma S_t \big(\partial_s f\big)(t, S_t)\, dW_t^\lambda$$

If f is the price of our derivative, it satisfiea (6.7) and then the drift is cancelled. Therefore it remains only the stochastic terms, so the discounted price

$$e^{-rt} f(t, S_t)$$

is a martingale. Morover, if the strategy is replicant we have $f(T, S_T) = \varphi(S_T)$ therefore

$$f(t, S_t) = e^{-r(T-t)} E^Q[f(T, S_T) \,|\, \mathcal{F}_t] = e^{-r(T-t)} E^Q[\varphi(S_T) \,|\, \mathcal{F}_t] \tag{6.21}$$

Before going on, it is important to stop and reflect that the distribution of $\frac{S_T}{S_0}$ depends on which measure we consider:

$$\log\left(\frac{S_T}{S_0}\right) \sim \mathcal{N}_{(r - \frac{\sigma^2}{2})T, \sigma^2 T} \qquad \text{w.r.t Q}$$

$$\log\left(\frac{S_T}{S_0}\right) \sim \mathcal{N}_{(\mu-\frac{\sigma^2}{2})T,\sigma^2 T} \qquad \text{w.r.t P}$$

Then the calculus of the mean value in (6.21) has to be done relatively to the parameters of the normal on the measure Q.
So, it is clear that when we replace μ with r, we make a change of drift that produces the neutral price to risk. By this point of view, we are not more able to have informations about μ: using Q we make the price respect to martingale measure that does not consider what happens in the reality.

Remark 6.3.2. We want to make some trivial but outstanding observations.

1. When we use in the treatise the measure P, with f we compute the value of replicant and auto-financial strategy.

2. In terms of Q, we work out the risk neutral price.

3. The ratio
$$\lambda = \frac{\mu - r}{\sigma}$$
is called *the market price of the risk*. It indicates how much we gain investing in risky assets (it is natural considering $\mu - r > 0$), knowing that a risk σ exists.

6.4 Implicit Volatility

For what we have said in section (6.3), the price of a Call option depends on the value of underlying asset S_0, on the maturity T, on the strike k, on the volatility σ and on the free-risk rate r:

$$\overline{C} = CALL_{BS}\big(r, S_0, T, k, \sigma\big) \tag{6.22}$$

For arriving where we want to, we fix all parameters but the volatility σ: $\overline{C} = C_{BS}(\sigma)$. Then the price is an increasing function in σ, therefore it is invertible: from the price $\overline{C}$ we can determine which is the volatility, such as, if we insert it into the BS formula namely C_{BS}, from (6.22) we can obtain exactly $\overline{C}$. In other words, we can obtain $\overline{\sigma}$ such as

$$\overline{C} = C_{BS}(r, S_0, T, k, \overline{\sigma}). \tag{6.23}$$

This volatility, that is unique thanks to the invertibility, is said ***implicit volatility***. So, by (6.23) we price the derivative by means $\overline{\sigma}$. Black-Scholes model is an instrument to express the price, not for pricing. It is a sort of *"language"* used by the market, not a model for the market.

A meaningful question could be: *"Why do we have to express the price by the volatility?"* Cannot we use simply the *price*?
Fix for a moment the value of r and S_0 but leave free k and T. So, we have an implicit volatility that changes in dependence on k and T. If we want to compare two derivatives with different maturity or different strike (actually we have even the possibility of same maturity and different strike or same strike and different maturity) we knock our head against a wall. It turns in an easy way when we compare the implicit volatilities: it is a relative informer. This variability of the implicit volatility generates the so called *volatility surface.*
If we used the BS model for pricing, we would have a flat surface. Many other models can be given working on no constant volatility:

1. Local volatility models.

2. Stochastic volatility models.

3. Jump models.

If one is interested in one of this models can be look up in [2].

6.5 The Merton Model

Merton proposed his model in 1974 and in a little time it became very popular; it is still used. Over the time, the model has been developed and adapted to the various cases. The asset values are represented as a stochastic process (V_t). We take no-arbitrage principle for granted and the markets are supposed frictionless, i.e. without taxes, transaction costs, bankruptcy costs. The model assumes that the dividends cannot be paid out and the debts cannot be released. In Merton's model, the debt consists of one single zero coupon bond with value B and maturity T. Therefore, the value of the firm's asset is

$$V_t = E_t + B_t, \qquad t \leqslant T$$

where $(E_t)_t$ is the stochastic process associated to the equity and $(B_t)_t$ is relative to the bond. The firm's default happens only at T and if debt holders cannot be paid. At time T two situations can occur:

1. $\mathbf{V_T > B}$ In this case the liabilities are greater than the value of the assets of the firm. In this case the debtholders receive B, the shareholders receive $E_T = V_T - B$ and default does not occur.

2. $\mathbf{V_T} \leqslant \mathbf{B}$: the situation gets complicated. The liabilities exceed the value of the assets of the firm and the company is not able to abide by the financial duty. The shareholders, who do not receive something, leave the control of the firm to bondholders. Therefore we have $B_T = V_T$ and $E_T = 0$.

In short

$$E_T = \max\big\{V_T - B, 0\big\} = \big(V_T - B\big)^+ \tag{6.24}$$

$$B_T = \min\big\{V_T, B\big\} = B - \big(B - V_T\big)^+ \tag{6.25}$$

A typical strategy of debt holders is to try to neutralize the credit risk taking a short position in a Put option on V with strike B and maturity T: in this way they have bought a credit protection against the default risk of the firm. Putting together the (6.25) and the last consideration we have the following observation.

Remark 6.5.1. From the side of the company, the debt obligation of the firm can be described by having a long position in a Put option, while for the debt holders it is similar to write a Put option to the firm.

The shareholders of the firm have the right to liquidate the company that, jointed with (6.24), led us to the successive observation.

Remark 6.5.2. By the firm's point of view, equity and writing a Call option have the same meaning. Then again, shareholders take on a long position in a Call option on the firm's asset values.

We want to draw attention to the contrary risk preferences between share and debtholders; therefore an increasing volatility is

· good for shareholders, indeed they have a long position in a Call option (the value of the call is naturally pushed up by increasing volatility)

· bad for debtholders inasmuch they have pledged a short Put.

Merton's model is fairly simplistic: it does not think about the possibility that the default can occur in any and different dates, not only at a fixed time T. Formally, the default time is a random variable called *stopping time* and defined as

$$\tau = \inf\{t \geqslant 0 \,|\, V_t \leqslant B\}$$

that is the first time in which the default occurs. The process of the stopping time renews itself after each default. Furthermore, nowadays bankruptcy is not automatically implied by the default. Other developments consider stochastic interest rates and jumps for the process $(V_t)_t$.

6.5.1 From Equity to Asset Values

In this paragraph we want to generalize the Black-Scholes model to the case with *dividends*. Let us consider the process of a risky-asset $S = (S_t)_t$ and suppose it follows a geometric Brownian motion. We also wish that S_t satisfies the stochastic differential equation with the insertion of dividend payments:

$$dS_t = (\mu_S S_t - C_t)dt + \sigma_S S_t dB_t \tag{6.26}$$

where C_t is the dividend paid by the firm at time t, μ is the expected rate of return of the asset S and σ the volatility. Since right now we have supposed that the market value of the debt D_t at time t is a nonstochastic exponential function

$$D_t = D_0 e^{\mu_D t}.$$

We take into consideration a smooth function $E = E(t, x, y) \in \mathcal{C}^{2,1,1}$ then, by *Itô* formula (6.1.2), the process $(E_t)_t$ which is represented by

$$E_t = E(t, S_t, D_t)$$

and solves

$$\begin{aligned} dE_t = \Big[&(\partial_t E)(t, S_t, D_t) + \big(\mu_S S_t - C_t\big)(\partial_x E)(t, S_t, D_t) + \\ &+ \mu_D D_t (\partial_y E)(t, S_t, D_t) + \frac{1}{2}\sigma_S^2 S_t^2 (\partial_{xx} E)(t, S_t, D_t)\Big] dt + \\ &+ \sigma_S A_t (\partial_x E)(t, S_t, D_t) dB_t \end{aligned}$$

The process E represents the *value of the firm's equity*. Like we have done in the section (6.2) (starting from (6.3) to the end), considering a *self-financing* condition for the strategy, imposing $C_t = \delta S_t$ and equaling two equations, with the same transitions, we reach:

$$\begin{aligned} \partial_t E(t, x, y) + (rx - \delta x)(\partial_x E)(t, x, y) + \mu_D D_t (\partial_y E)(t, x, y) + \\ + \frac{1}{2}\sigma_A^2 x^2 (\partial_{xx} E)(t, x, y) - rE(t, x, y) = 0 \end{aligned} \tag{6.27}$$

If we put $\delta = 0$ and $D_0 = 0$, we have again the Black-Scholes equation, already seen in (6.7).

Reflect about the failure of a company. First of all, it is necessary to say that at the moment in which the ratio $\frac{S_t}{D_t}$ amounts to some critical level γ, the firm is considered to be in bankruptcy. This awkward level is chosen by the equity-holders: the firm will go on working until equity holders are reluctant to have more losses than the ones that already occurred. It is possibile to demonstrate that with the boundary condition

$$E(S,D)|_{|S/D=\gamma} = 0$$

$$\lim_{\frac{S}{D}\to\infty} E(S,D) = S - \frac{\delta}{r}D$$

the (6.27) admits solution and it is possible demonstrating that it is given by

$$E(S,D) = D\left[\frac{S}{D} - \frac{\delta}{r-\mu_D} - \left(\gamma - \frac{\delta}{r-\mu_D}\right)\left(\frac{S/D}{\gamma}\right)^{\gamma}\right] \tag{6.28}$$

where

$$\lambda = \frac{1}{\sigma_S^2}\left[\left(\frac{\sigma_S^2}{2} + \delta + \mu_D - r\right) - \sqrt{\left(r - \frac{\sigma_S^2}{2} - \delta - \mu_D\right)^2 + 2\sigma_S^2\left(r-\mu_D\right)}\right]$$

As we have said, the level γ is chosen by the investor so it is possible determining it by the first order condition $\partial_\gamma E = 0$ therefore we have

$$\gamma = \frac{\lambda}{\lambda - 1}\frac{\delta}{r-\mu_D}$$

Chapter 7

Hedging

The fundamental question that we have tried to explain in the earlier chapter is "What is the price of a derivative?". When a bank sells a derivative, it must decide a price for it, so, knowing what is the right price is a matter of life and death: if the price is too elevate, anyone will buy it, if instead the price is not coherent with the derivative, the bank exposes itself at arbitrage, i.e. it permits to make money at someone else with its own detriment. After pricing, understand how the price change plays a central role in a financial point of view: it is important to know if the price is quite enough sensitive at the variation of volatility, of the asset value or of the short term rate. The natural sensitivity indicators are the partial derivatives of the value of the portfolio with respect to the corresponding risk factors. Since at each derivative we associate a greek letter, they are called as ***The Greeks***. In this chapter we will give an expression for each of Greeks. They are so important because, putting one of this derivatives equal to zero, we calibrate our strategy in order that the price of the option is insensitive to the variations in the parameter with respect to we have made the derivative. This is called *hedging*. If we want to defend from the variations of the asset price, we will make a Delta-hedging; likewise, if we want to be indifferent to the volatility, we will make a Vega-hedging. It is possible making a combination of this hedging: Delta-Vega-hedging, Gamma-Vega-hedging, and so on.

7.1 The Greeks

We have seen that the price of the derivatives depends on the price of underlying asset S, on the short-term rate r, on the volatility σ and on the time t: $f = f(t, s, \sigma, r)$. So, for understanding its sensitivity at the variation of those parameters we introduce:

$$\Delta = \partial_s f \qquad \textbf{\textit{Delta}}$$

$$\Gamma = \partial_{ss} f \qquad \textbf{\textit{Gamma}}$$

$$\mathcal{V} = \partial_\sigma f \qquad \textbf{\textit{Vega}}$$

$$\varrho = \partial r f \qquad \textbf{\textit{Rho}}$$

$$\Theta = \partial_t f \qquad \textbf{\textit{Theta}}$$

We have an explicit expression for the Greeks of European Put and Call options, simply differentiating Black-Scholes formula (6.7). We will only talk about Call options. We remind to the reader of, at time t, the price of an European Call with maturity T and strike K is:

$$C_t = g(d_1)$$

where

$$d_1 = \frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \tag{7.1}$$

and

$$g(d) = S_t \Phi(d) - K e^{-r(T-t)} \Phi(d - \sigma\sqrt{T-t}), \quad d \in \mathbb{R} \tag{7.2}$$

with

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\, dy$$

A few times, it is convenient to use

$$d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\log\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

Proposition 7.1.1. *It is worth that*

$$g'(d_1) = 0 \tag{7.3}$$

Proof. One can observe that

$$\Phi'(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

Therefore

$$g'(d) = S_t \frac{e^{-\frac{d^2}{2}}}{\sqrt{2\pi}} - Ke^{-r(T-t)} \frac{e^{\frac{(d-\sigma\sqrt{T-t})^2}{2}}}{\sqrt{2\pi}} =$$

$$= \frac{e^{-\frac{d^2}{2}}}{\sqrt{2\pi}} \Big(S_t - Ke^{-\left(r+\frac{\sigma^2}{2}\right)(T-t)} e^{d\sigma\sqrt{T-t}} \Big).$$

By (7.1), we have the claim. □

Corollary 2. *As consequence of (7.1.1) we have*

$$S_t \Phi'(d_1) = Ke^{-r(T-t)} \Phi'(d_1 - \sigma\sqrt{T-t}) \tag{7.4}$$

Now we are going to examinate each Greek of a Call option.

7.1.1 Delta

By (7.1.1), we have:

$$\Delta = \Phi(d_1) \tag{7.5}$$

then the values of the Δ is included into the interval $]0,1[$:

$$0 < \Delta < 1$$

Remark 7.1.1. We can note that when the option is deep *in the money*, i.e. the value of the payoff is major than the strike, the value of Δ tends to 1; that is because, almost surely, we exercise the option. When the option is *at the money*, i.e. the value of the payoff is equal to the strike, Δ is equal to 0.5: there are many uncertainties (we do not know if exercising the option is the right thing), so, the Δ is more sensitive to the variations in the price of underlying asset.

The Δ can be also interpreted like the amount of risky asset that has to hold in the Delta-hedging portfolio. According to (7.5), the Call option is treated as an equivalent long position in Δ units of the underlying. By

$$\lim_{s\to 0^+} d_1 = -\infty, \qquad \lim_{s\to +\infty} d_1 = +\infty \tag{7.6}$$

we reach

$$\lim_{s\to 0+} C_t = 0, \qquad \lim_{s\to +\infty} C_t = +\infty$$
$$\lim_{s\to 0^+} \Delta = 0, \qquad \lim_{s\to +\infty} \Delta = 1$$

Remark 7.1.2. Using the Put-Call parity formula (4.0.2), we immediately find that

$$\Delta_{put} = \Delta_{call} - 1$$

It follows that $\Delta_{put} \in [-1, 0]$, so that a Put option is equivalent to a short position in the underlying.

7.1.2 Gamma

Since

$$\Gamma = \partial_{ss} g = \partial_s \Delta = \Phi'(d_1)\partial_s d_1$$

we get

$$\Gamma = \frac{\Phi'(d_1)}{\sigma S_t \sqrt{T-t}} \tag{7.7}$$

We can note that the Γ is a positive function, so, by its definition, we have that price g is a convex function, and the *Delta* is a increasing function; both of them with respect to the underlying asset. By means of (7.6), we arrive at

$$\lim_{s\to 0+} \Gamma = \lim_{s\to +\infty} \Gamma = 0$$

Remark 7.1.3. Being the derivative of *Delta*, the function *Gamma* assumes big values when the option is at the money; this confirm the elevate sensibility of Δ in such case. Furthermore, when the option is at the money, the greater is the value of Γ, the greater is the distance between the price of the option and the discounted strike. Intuitively, this reflects the matter that the impact of small variations in the value of the asset on the price of the option is more significant when the option is at the money.

7.1.3 Vega

We have

$$\mathcal{V} = S_t \sqrt{T-t}\,\Phi'(d_1) \tag{7.8}$$

Indeed

$$\mathcal{V} = \partial_\sigma C_t = g'(d_1)\partial_\sigma d_1 + Ke^{-r(T-t)}\Phi'(d_1 - \sigma\sqrt{T-t})\sqrt{T-t}$$

then, by (7.3) and (7.4), we get

$$\mathcal{V} = S_t \sqrt{T-t}\,\Phi'(d_1)$$

The $Vega$ is positive, therefore the price is a strictly increasing function of the volatility. We remind that this important matter has been used into the section (6.4), speaking about the implicit volatility. It is possible to show that

$$\lim_{\sigma\to 0^+} C_t = \left(S_t - Ke^{-r(T-t)}\right)^+, \qquad \lim_{\sigma\to +\infty} C_t = S_t.$$

Since the proof is only a sequence formal calculus, we are not going to make it. We only give the hint to put

$$\lambda = \log\big(\frac{S_t}{K}\big) + r(T-t)$$

and to divide the proof in dependence on the possible values of λ. From those limits, it follows that

$$\left(S_t - Ke^{-r(T-t)}\right)^+ < C_t < S_t.$$

7.1.4 Theta

Since

$$\Theta = \partial_t C_t = g'(d_1)\partial_t d_1 - rKe^{-r(T-t)}\Phi(d_2) - Ke^{-r(T-t)}\Phi'(d_2)\frac{\sigma}{2\sqrt{T-t}}$$

by (7.4), we reach

$$\Theta = -rKe^{-r(T-t)}\Phi(d_2) - \frac{\sigma S_t}{2\sqrt{T-t}}\Phi'(d_1) \tag{7.9}$$

We can note that $\Theta < 0$, that is the price of a Call option is a decreasing function with respect to the time; when we are no far off the maturity T, the price is smallest than the beginning. It makes sense, indeed at the moment in which we are closed to being in T, the effects of volatility are fairly small and, with them, even the possibility of profit.

7.1.5 Rho

Due to

$$\varrho = \partial_r C_t = g'(d_1)\partial_r d_1 + K(T-t)e^{-r(T-t)}\Phi(d_2)$$

we have

$$\varrho = K(T-t)e^{-r(T-t)}\Phi(d_2) \tag{7.10}$$

Since the factor $\exp(-r(T-t))$ decreases when the r increases, the payment of discounted strike K is inversely proportionate to r; therefore, the price of a Call increases when the risk-free rate does so. Indeed $\varrho > 0$.

The Greeks of an European Put Option

We just give, without the proof, the equation for every Greeks.

$$\Delta = \partial_s P_t = \Phi(d_1) - 1 \qquad < 0$$

$$\Gamma = \partial_{ss} P_t = \frac{\Phi'(d_1)}{\sigma S_t \sqrt{T-t}} \qquad > 0$$

$$\mathcal{V} = \partial_\sigma P_t = S_t \sqrt{T-t}\,\Phi'(d_1) \qquad > 0$$

$$\Theta = \partial_t P_t = rKe^{-r(T-t)}\big(1 - \Phi'(d_2)\big) - \frac{\sigma S_t}{2\sqrt{T-t}}\Phi'(d_1) \qquad \in \mathbb{R}$$

$$\varrho = \partial_r P_t = K(T-t)e^{-r(T-t)}\big(\Phi(d_2) - 1\big) \qquad < 0$$

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