

ALMA MATER STUDIORUM · UNIVERSITÀ DI BOLOGNA

SCUOLA DI SCIENZE

Corso di Laurea Magistrale in Matematica

CONFORMAL MAPPING
AND
BRAIN FLATTENING

Tesi di Laurea in Analisi Matematica

Relatore:

Chiar.ma Prof.

Giovanna Citti

Presentata da:

Stefano Murtagh

II Sessione

Anno Accademico 2015-2016

Abstract

In this dissertation we study some of the main results concerning conformal mappings in the complex plane and between Riemann surfaces and we apply those results to the so-called brain flattening problem. In the first part of this thesis we prove the Riemann Mapping Theorem and we provide an introduction to the Uniformization Theorem for simply connected Riemann surfaces. The second part of the thesis is focused on the brain flattening problem, which deals with how to construct a conformal mapping from the brain's cortical surface to the unitary sphere. This procedure leads to a possible definition of the discrete mean curvature on a triangulated closed surface of genus zero. This flattening method has several applications in neuroscience.

Sommario

In questa tesi studiamo alcuni dei risultati principali riguardanti la teoria delle mappe conformi nel piano complesso e tra superfici di Riemann e applichiamo tali risultati al cosiddetto problema del *brain flattening*. Nella prima parte della tesi dimostriamo il *Riemann Mapping Theorem* e forniamo un'introduzione al Teorema di Uniformizzazione per superfici di Riemann semplicemente connesse. La seconda parte della tesi è incentrata sul problema del *brain flattening*, che riguarda come costruire una mappa conforme tra la superficie corticale del cervello e la sfera unitaria. Questa procedura porta ad una possibile definizione di curvatura media discreta su una superficie triangolata chiusa e di genere zero. Tale metodo del flattening ha diverse applicazioni nelle neuroscienze.

Contents

Introduction	i
1 Review of complex analysis	1
1.1 Holomorphic functions and Cauchy's integral formula	1
1.2 Principal argument and power functions	5
1.3 A normal form for non constant holomorphic functions	6
1.4 The Residue Theorem and Hurwitz Theorem	7
1.5 Maximum Principle and Schwarz Lemma	11
1.6 Compactness of families of holomorphic functions	12
2 The Riemann Mapping Theorem	15
2.1 Conformal mapping	16
2.2 The Möbius group	20
2.3 Pick's Lemma and the hyperbolic geometry	25
2.4 The Riemann Mapping Theorem	28
3 Uniformization Theorem for Riemann surfaces	33
3.1 Riemann surfaces	34
3.1.1 Definition and examples	34
3.1.2 Basic properties of holomorphic maps between Riemann surfaces .	35
3.2 Riemann surfaces of multivalued functions	38
3.2.1 The universal covering	40
3.3 The Uniformization Theorem	41

4	Brain flattening and surface parameterization	45
4.1	The Laplace-Beltrami operator	46
4.1.1	Conformal coordinates	47
4.1.2	A particularly important elliptic PDE	48
4.2	The FEM on a triangulated compact surface	49
4.2.1	The Cotan-Laplace formula	50
4.2.2	The board vectors	52
4.2.3	Solving two sparse linear systems	53
4.3	Construction of the flattening map in Matlab	54
4.3.1	Choice of the dilation coefficient	55
4.3.2	Discrete mean curvature	58
4.4	Experimental results	59
4.4.1	How to measure the conformality of the flattening map	64
	Appendix A Matlab codes	75
	References	81

Introduction

In this dissertation we shall review some of the main results in the field of conformal geometry of two dimensional spaces, in order to analyze the so-called *brain flattening* problem from both theoretical and implementative point of view.

The problem of flattening the brain's cortical surface is motivated by the necessity to represent the cortical surface in such a way that neural analysis can be made on its flattened visualization. Behind such idea, the aim is to unfold the highly convoluted cortical surface, which is characterized by the presence of many fissures (sulci) and convolutions (gyri). Thus, a flattened or inflated version of such surface is useful for better understanding the neural activity on the sulci. Indeed, a typical application of brain flattening is the study of fMRI (functional Magnetic Resonance Imaging) data: here, a flattened version of the cortical surface is required to visualize the fMRI activation on the cortical surface and it is also a starting point for further analysis.

From a mathematical point of view, the problem of brain flattening belongs to the issue of *surface parameterization*, i.e. how to construct the parameter domain of a Riemannian surface, which is an interesting problem in many scientific fields, from texture mapping in computer graphics to cartography and, as mentioned above, in medical imaging. Flattening the cortical surface of a human brain can be achieved in different ways, depending on which geometric properties we are interested to preserve. Since both areas and angles cannot be preserved during such procedure (in view of Gauss Egregium theorem), an initial choice has to be made between mappings that aim at maintaining distances and those that aim at preserving oriented angles.

The flattening mapping that we will discuss in detail is based mainly on the work presented in [3]. It is a conformal mapping, that is, a bijective mapping which preserves

oriented angles, from a genus zero closed surface to the unitary sphere. This choice is justified by two reasons: firstly, angles can be fully preserved, while the original metric is always affected by distortion during the flattening procedure, and, furthermore, there is a rich and deep mathematical theory behind conformal mappings, developed by some of the greatest mathematicians of all times, such as Gauss, Riemann, Koebe and Poincaré.

In this dissertation we will deal mainly with conformal geometry in the complex plane and we will also present some crucial results concerning the conformal equivalence of Riemann surfaces. The first important result that we will present is the so-called Riemann mapping theorem, which was first stated by Riemann in his PhD thesis for simply connected domains with piecewise smooth boundary.

Theorem (Riemann Mapping Theorem). *Let D be a simply connected proper domain of \mathbb{C} and let $a \in D$. Let \mathbb{D} denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$. Then there exists a unique conformal mapping $f : D \rightarrow \mathbb{D}$ such that $f(a) = 0$ and $f'(a) > 0$.*

This powerful result states that every simply connected domain, independently of the regularity of its boundary and except for the whole complex plane, can be mapped biholomorphically into the unit disk.

The generalization of the previous theorem in the setting of Riemann surfaces is known as the Uniformization Theorem and it was first proved by Koebe and Poincaré.

Theorem (Uniformization Theorem). *Every simply connected Riemann surface is conformally equivalent to one of the following:*

- *The Riemann sphere $\hat{\mathbb{C}}$*
- *The complex plane \mathbb{C}*
- *The unit disk \mathbb{D}*

The above result provides the starting point for constructing the sought flattening map from the cortical brain surface S to the unit sphere. We will be able to do so in two steps: at first we will find the flattening map from $S \setminus \{p\}$ to the complex plane, by solving the following elliptic PDE on S :

$$\Delta z = \left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right) \delta_p,$$

where Δ denotes the Laplace-Beltrami operator on S and δ_p is the Dirac delta impulse at p and then we will apply the inverse stereographic projection to have the spherical representation. The solution will be achieved in the discrete setting through the Finite Element Method on the triangulated input surface and the above PDE, using the so-called discrete *Cotan-Laplace* approximation of the Laplace-Beltrami operator (see [24]), will lead to two sparse linear systems. The solution found in the first step is unique only up to scaling and translating and this implies that it has to be scaled by a proper factor in order to have the output sphere covered uniformly after the inverse stereographic projection. In order to solve this problem, we will define an *optimal coefficient of dilation*. Further, we will measure the effective conformality of the constructed map by means of a coefficient of angular distortion used in [6].

This thesis has the following structure: in Chapter 1 we will review some of the main results concerning complex analysis of one variable. We will recall the link between holomorphic and analytic functions through Cauchy's integral formula, we will discuss briefly complex power functions and we will provide a *normal form* for non constant holomorphic functions, which will lead to the open mapping theorem and to the existence of holomorphic branches of the complex logarithm. Furthermore, we will deal with the Residue Theorem and Rouché's Theorem in order to prove Hurwitz Theorem and we will recall the Maximum Principle and Schwarz Lemma. In the last section of this first chapter, we will present Montel's Theorem for the compactness of holomorphic functions.

Chapter 2 will be focused on the Riemann Mapping Theorem and its proof via the solution of a certain maximal problem. Section 2.1 will be based on conformal functions, their characterization as holomorphic functions with non vanishing derivative and some examples which will show how to map conformally some geometric domains onto the unit disk \mathbb{D} . The second section will discuss the stereographic projection and the Möbius group of linear fractional transformations, with a particular emphasis on the characterization of conformal self-maps of the unit disk. The third section will deal with Pick's Lemma and its use in defining the hyperbolic geometry on \mathbb{D} and on a general simply connected domain. The last section rotates around the proof of the Riemann Mapping Theorem using some of the results stated in the previous paragraphs.

In Chapter 3 we will present a brief introduction to Riemann surfaces, with major

emphasis on the Uniformization Theorem. In the first section we will define Riemann surfaces and discuss the basic properties of holomorphic maps between them, underlining the case of compact surfaces and the Riemann-Hurwitz relation. The second section will deal with the construction of the Riemann surface associated with a multivalued function and will provide some basic concepts of covering theory, in particular the universal covering of a Riemann surface. The last section presents the Uniformization Theorem and shows its consequences in the classification of an arbitrary Riemann surface and in the introduction of a Riemannian metric of constant curvature.

Chapter 4 will be focused on the brain flattening problem and in general on the parameterization of a closed surface of genus zero. In the first section we will define the Laplace-Beltrami operator, we will show its behaviour under conformal coordinates and we will derive the elliptic PDE mentioned earlier. In the second section we will solve such PDE via the Finite Element Method (FEM) on the triangulated input surface, we will discuss how the Laplace-Beltrami operator is discretized via the Cotan-Laplace formula and we will derive and solve two sparse linear systems which provide the sought flattening map on the complex plane. In the third section we will construct the conformal map from the input surface to the sphere using Matlab, we will define the optimal coefficient of dilation and we will show how to give a discrete measure of the mean curvature of the input surface through the above mentioned Cotan-Laplace formula. In the last section we will show some experimental results obtained on three different 3D meshes and we will measure the angular distortion introduced during the flattening and inflating procedure.

Appendix A includes the transcription of the Matlab codes we wrote to construct the flattening map.

Chapter 1

Review of complex analysis

In this chapter we will review some of the main concepts concerning complex analysis of one variable functions, which will be used intensively in Chapter 2. In Section 1.1 we will recall the link between analytic and holomorphic functions through Cauchy's integral formula. In Section 1.2 we will define the principal argument of a complex number and we will examine the behaviour of complex power functions. In Section 1.3 we will state the uniqueness principle and we will emphasize the existence of branches of the complex logarithm on simply connected domains. In Section 1.5 we will consider the maximum principle for holomorphic functions and in particular Schwarz Lemma. In Section 1.6 we will recall a version of Ascoli-Arzelà Theorem and we will state Montel's Theorem, which will be one of the main ingredients in the proof of the Riemann Mapping Theorem of Chapter 2. References for the topics developed in this chapter may be found in [1],[14], [17] and [26].

1.1 Holomorphic functions and Cauchy's integral formula

In this section we will describe the concept of holomorphic functions and we will state some of the main theorems related to this class of functions. Those results will be the starting point for the development of next sections.

Definition 1.1. Let f be a complex-valued function defined on an open subset of the complex plane \mathbb{C} . f is said to be **complex-differentiable** at z_0 if

$$\exists \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

The limit is the **complex derivative** of f at z_0 and it is referred to $f'(z_0)$ or as $\frac{df}{dz}(z_0)$.

Definition 1.2. f is **holomorphic** on the open set U if f is complex-differentiable at each point of U .

Remark 1.3. The classical definition of holomorphic function has the a priori ulterior condition on the continuity of $f'(z)$. The so-called **Goursat's Theorem** asserts that such requirement on f is redundant, since every complex-differentiable function is automatically holomorphic.

It is often convenient to write a complex-valued $f(z)$ as $f(x, y) = u(x, y) + iv(x, y)$, where u and v are real-valued functions.

Theorem 1.4. *Let f be defined on an open set U and $f = u + iv$. Then f is holomorphic on U if and only if u and v are C^1 functions satisfying the so-called **Cauchy-Riemann (CR) equations**:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (1.1)$$

In particular we have:

$$f'(z) = \frac{\partial u}{\partial x}(x, y) + i \frac{\partial v}{\partial x}(x, y)$$

Another way to express the CR equations is given by a very useful complex notation. Let $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ be defined as

$$\frac{\partial}{\partial z} = \frac{1}{2} \left[\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right], \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left[\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right].$$

Thus we have:

Proposition 1.5. *f is holomorphic at z_0 if and only if f satisfies*

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0. \quad (1.2)$$

In particular we have:

$$f'(z_0) = \frac{\partial f}{\partial z}(z_0).$$

An equivalent approach to define holomorphicity is based on considering the 1-form $f(z)dz$.

Proposition 1.6. *f is holomorphic on U if and only if the 1-form $f(z)dz$ is closed.*

Proof. Thanks to Theorem 1.4, it is sufficient to use CR equations (1.1). Adopting the usual notation $\frac{\partial u}{\partial x} = u_x$, we have:

$$f(z)dz = (u + iv)dx + (iu - v)dy.$$

Hence

$$f(z)dz \text{ is closed} \iff u_y + iv_y = -v_x + iu_x \iff \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

□

Every complex-valued function f defined on an open subset U of \mathbb{C} may be obviously seen as a map from U in the Euclidean plane \mathbb{R}^2 to \mathbb{R}^2 (identifying of course $z = x + iy$ with (x, y)). This allows us to talk about the Jacobian matrix of the function f as

$$J_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

Using CR equations (1.1) we have:

$$\det J_f = |u_x + iv_x|^2.$$

Thus we have proved the following:

Proposition 1.7. *If f is holomorphic on an open set U , then*

$$\det J_f = |f'(z)|^2. \tag{1.3}$$

One of the most important properties of holomorphic functions is contained in the so-called **Cauchy's integral formula**. Calling a domain any open connected subset of \mathbb{C} , we have the following crucial theorem.

Theorem 1.8. *Let D be a bounded domain with piecewise smooth boundary ∂D . If f is holomorphic on D and it extends smoothly on ∂D , then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\xi)}{\xi - z} d\xi, \quad \forall z \in D. \quad (1.4)$$

Theorem 1.8 shows the *rigidity* of holomorphic functions, since the behaviour of f at z depends on what happens on the boundary of any small disc centered at z_0 and contained in D .

Another crucial aspect of holomorphic maps is given by the link between them and analytic functions.

Definition 1.9. A complex-valued function f defined on an open subset U of \mathbb{C} is said to be **analytic** on U if for any point z_0 in U f can be written as a convergent power series centered at z_0 . More explicitly

$$f(z) = \sum_{k \geq 0} a_k (z - z_0)^k, \quad \text{for } |z - z_0| < \varepsilon, \quad \text{with } \varepsilon \text{ small enough.}$$

Theorem 1.10. *Suppose $\sum a_k (z - z_0)^k$ is a power series with radius of convergence $R > 0$. Then the function*

$$f(z) = \sum_{k \geq 0} a_k (z - z_0)^k, \quad |z - z_0| < R$$

is analytic and the coefficients a_k are given by

$$a_k = \frac{f^{(k)}(z_0)}{k!}, \quad k \geq 0.$$

Viceversa let f be a holomorphic function on the open ball $B(z_0, \rho)$. Then f is given by the power series

$$f(z) = \sum_{k \geq 0} a_k (z - z_0)^k, \quad |z - z_0| < \rho, \quad (1.5)$$

where for any fixed r , $0 < r < \rho$

$$a_k = \frac{1}{2\pi i} \int_{|\xi - z_0| = r} \frac{f(\xi)}{(\xi - z_0)^{k+1}} d\xi, \quad k \geq 0. \quad (1.6)$$

In particular f defined on an open subset U is holomorphic if and only if it is analytic.

As a direct consequence of (1.6), we have the famous

Theorem 1.11 (Liouville's Theorem). *Let f be an entire function, i.e. a holomorphic function on the whole complex plane, and suppose $|f|$ is bounded, then f is constant.*

1.2 Principal argument and power functions

Given a complex number $z = x + iy$, $z \neq 0$, we can consider its polar representation

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \quad (1.7)$$

where $r = \sqrt{x^2 + y^2}$ and $\theta = \arg z$ is determined only up to an integral multiple of 2π . In the following we will consider $\theta \in (-\pi, \pi]$, i.e. as the **principal argument** of z (see [14]). This choice is arbitrary, since we cannot define the argument of z in such a way that it is a continuous function on the whole punctured complex plane $\mathbb{C} \setminus \{0\}$ and thus we have to make a **branch cut** along an arbitrary half-line starting from the origin.

At this point it is useful to examine the multivalued power functions:

$$z^\alpha := e^{\alpha \log z}, \quad z \neq 0, \quad \alpha \in \mathbb{C}. \quad (1.8)$$

If we use the well-known definition of the complex logarithm, (1.8) becomes

$$z^\alpha = e^{\alpha(\log |z| + i \arg z)} e^{2\pi i \alpha m}, \quad m \in \mathbb{Z}, \quad (1.9)$$

from which it is clear that in general is not a single-valued function. As for the argument function, we have to make a cut in \mathbb{C} , in order to allow a branch (i.e. we fix $m \in \mathbb{Z}$) of the power function to be continuous (for α not an integer). This time we choose to cut along the x positive axis. Doing so, we may define on $\mathbb{C} \setminus [0, \infty)$ a continuous branch of z^α as

$$w = r^\alpha e^{i\alpha\theta}, \quad \text{for } z = r e^{i\theta}, 0 < \theta < 2\pi. \quad (1.10)$$

If we fix r and follow the behaviour of (1.10) along the circle of radius r moving counterclockwise from $\theta = 0$ to $\theta = 2\pi$, we find out that the values of z^α at the bottom of the edge of the cut are exactly $e^{2\pi i \alpha}$ times the values at the top. The same works out for a generic $m \in \mathbb{Z}$. Thus, the above reasoning shows the necessity of making a cut.

A useful and instructive example is given by the square root function \sqrt{z} . From (1.9), we notice that we have exactly two branches of \sqrt{z} , corresponding to the two 2^{nd} roots of unity. With the same reasoning as above, they are given explicitly by

$$w_1 = \sqrt{r} e^{\frac{1}{2}i\theta}, \quad w_2 = -\sqrt{r} e^{\frac{1}{2}i\theta} \quad (1.11)$$

We thus obtain two copies of the w – plane, in which w_1 and w_2 have opposite sign on the top and on the bottom of the branch cut. In Section 3.2 we will see how those two copies may be glued together according to the sign of w_1 and w_2 and the resulting object will be the Riemann surface associated with the square root function \sqrt{z} .

1.3 A normal form for non constant holomorphic functions

In this section we will state a generalization of Dini’s Inverse Function Theorem for *critical points* of holomorphic non constant functions. As a consequence we will have the Open Mapping Theorem. Furthermore, we will recall the definition of a simply connected domain and we will relate such notion to the existence of *branches* of the functions $\log z$ and $\sqrt[n]{z}$, $n \geq 1$.

Theorem 1.12 (Uniqueness Principle). *Let f be a holomorphic function on the domain D . Then the following statements are equivalent:*

- $f \equiv 0$
- for some $z_0 \in D$, $f^{(n)}(z_0) = 0$ for all $n \geq 0$
- the set $\{z \in D : f(z) = 0\}$ has an accumulation point in D .

The previous theorem is called the uniqueness principle because every two holomorphic functions f and g on a domain D , which are equal on a set with an accumulation point, are equal on the whole domain D . Further, it shows that analytic functions have isolated zeros.

We recall that *Dini’s Theorem* states that a holomorphic function f with nonzero complex-derivative at a point z_0 admits the existence of a local inverse holomorphic function around $f(z_0)$.

Given a disk D we define the punctured disk D^* as D with its center removed. The following result generalizes Dini’s Theorem even in the case where $f'(z_0) = 0$ and gives a *normal form* for holomorphic non constant functions locally around any point.

Corollary 1.13. *Let f be a holomorphic non constant function defined on a domain D . Then at every point $z_0 \in D$ there exists a positive integer n and a holomorphic function h locally at z_0 such that*

$$f(z) = f(z_0) + [(z - z_0)h(z)]^n, \quad h(z_0) \neq 0. \quad (1.12)$$

In particular there are disks $D(z_0, \rho), D(f(z_0), r)$ such that every $w \in D^(f(z_0), r)$ has precisely n pre-images under f in $D(z_0, \rho)$.*

As a direct consequence of the previous result we have

Theorem 1.14. *Every non constant holomorphic function defined on a domain is an open function.*

Definition 1.15. A closed path $\gamma(t), a \leq t \leq b$ is deformable to a point if there are paths $\gamma_s(t), a \leq t \leq b, 0 \leq s \leq 1$, in D such that $\gamma_s(t)$ depends continuously on both s and t , $\gamma_0 = \gamma$ and $\gamma_1(t) \equiv z_0$ is the constant path at $z_0 \in \mathbb{C}$. We say that the domain D is **simply connected** if every closed path in D can be deformed to a point.

Roughly speaking, we can say that D is simply connected when it has no holes or equivalently its boundary is made by a simple closed curve.

Theorem 1.16. *Let D be a simply connected domain in \mathbb{C} . Then for every f holomorphic on D such that $f \neq 0$ on D exists a holomorphic function g with $e^{g(z)} = f(z)$. In particular, if D is a simply connected domain in $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, then it exists a holomorphic function g with $e^{g(z)} = z$. Such g is called a branch of $\log z$. Similarly, there exist holomorphic branches of $\sqrt[n]{z}, n \geq 1$.*

1.4 The Residue Theorem and Hurwitz Theorem

In this paragraph we will be concerned about functions with singular points. We will state the Residue Theorem, which gives a powerful tool to calculate line integrals of functions, which are holomorphic except for a finite number of singular points. We will focus next on the logarithmic derivative and Rouché's Theorem, which are the main results to prove Hurwitz Theorem.

The following result is a generalization of Theorem 1.10 for functions which are holomorphic in an annulus.

Theorem 1.17. *Let $f(z)$ be holomorphic on the annulus*

$$C_{\rho_1, \rho_2}(z_0) := \{z \in \mathbb{C} : \rho_2 < |z - z_0| < \rho_1\} \quad \text{with } 0 < \rho_2 < \rho_1,$$

then on $C_{\rho_1, \rho_2}(z_0)$ we have

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k. \quad (1.13)$$

Further, such series converges uniformly on every compact inside $C_{\rho_1, \rho_2}(z_0)$ and the coefficients are given by

$$a_k = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\xi)}{(\xi - z_0)^{k+1}} d\xi, \quad (1.14)$$

where $k \in \mathbb{Z}$, $\rho_2 < r < \rho_1$ and γ_r is the circumference of center z_0 and radius r .

The series given by (1.13) is called **Laurent expansion** of the function f . It allows us to give a classification into three classes of **isolated singularities**.

Definition 1.18. Let f be holomorphic on $D(z_0, r) \setminus \{z_0\}$, $z_0 \in \mathbb{C}$ and $r > 0$.

Let $f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$ its Laurent expansion. Then in z_0 there are only three mutually exclusive alternatives:

1. If $a_k = 0$ for every $k < 0$, then z_0 is a **removable singularity** and so f can be extended holomorphically on $D(z_0, r)$.
2. If it exists $m > 0$ such that $a_{-m} \neq 0$ and $a_{-n} = 0$ for every $n > m$, z_0 is a **pole of order m** .
3. If there are infinite $n > 0$ with $a_{-n} \neq 0$, z_0 is an **essential singularity**.

The following proposition gives a practical method to characterize an isolated singularity.

Proposition 1.19. *Let f be holomorphic on $D(z_0, r) \setminus \{z_0\}$, $z_0 \in \mathbb{C}$ and $r > 0$, then*

1. z_0 is a removable singularity if and only if it exists $\lim_{z \rightarrow z_0} f(z) = l \in \mathbb{C}$;

2. z_0 is a polar singularity if and only if $\lim_{z \rightarrow z_0} |f(z)| = +\infty$;

3. z_0 is an essential singularity if and only if $\lim_{z \rightarrow z_0} |f(z)|$ does not exist.

Definition 1.20. Let f be holomorphic on $D(z_0, r) \setminus \{z_0\}$, with $z_0 \in \mathbb{C}$ and $r > 0$.

Let $f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k$ be its Laurent expansion. The complex number a_{-1} is called the **residue** of f at z_0 and is denoted as $\text{Res}(f, z_0)$.

Theorem 1.21 (Residue Theorem). Let D be a bounded domain in the complex plane with a piecewise smooth boundary ∂D . Suppose f is holomorphic on $D \cup \partial D$, except for a finite number of isolated singularities z_1, \dots, z_n in D . Then

$$\frac{1}{2\pi i} \int_{\partial D} f(\xi) d\xi = \sum_{j=1}^n \text{Res}(f, z_j). \quad (1.15)$$

We say that f is a **meromorphic** function on a domain D if f is holomorphic on D except for a finite number of isolated polar singularities.

Definition 1.22. Let f be a non-vanishing meromorphic function on a domain D . We say that the **logarithmic derivative** of f is the meromorphic function $\frac{f'}{f}$ on D

The Residue Theorem applied to the logarithmic derivative gives the following result, which can be used to count zeros and poles of meromorphic functions.

Theorem 1.23. Let D be a bounded domain in the complex plane with a piecewise smooth boundary ∂D and let f be a meromorphic function on D that extends to be holomorphic on ∂D and such that $f(z) \neq a$, $a \in \mathbb{C}$, on D . Then

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z) - a} dz = N_0 - N_\infty, \quad (1.16)$$

where N_0 and N_∞ are respectively the number of zeros and the number of poles of $f(z) - a$ in D , counting multiplicities.

Theorem 1.24 (Rouché's Theorem). Let D be a bounded domain in the complex plane with a piecewise smooth boundary ∂D . Let $f(z)$ and $g(z)$ be holomorphic on $D \cup \partial D$. If $|g(z)| < |f(z)|$ for $z \in \partial D$, then f and g have the same number of zeros in D , counting multiplicities.

In order to state the next result, we recall that if a sequence of holomorphic functions converges uniformly on compact subsets of their domain to a limit function, then the limit function is holomorphic (**Weierstrass' Theorem**). From here on we would say that a sequence (f_n) on D *converges normally* to f if it converges uniformly on compact subsets of D .

Theorem 1.25 (Hurwitz Theorem). *Let D be a domain of the complex plane \mathbb{C} and $(f_n(z))$ a sequence of holomorphic functions in D that converges normally to a holomorphic function $f(z)$. Suppose $f \not\equiv 0$ and let z_0 be a zero of f . Then there exists $\rho > 0$ such that $\overline{D(z_0, \rho)} \subset D$ and $f \neq 0$ on $|z - z_0| = \rho$ and there exists a positive integer n_0 such that for $n \geq n_0$, f_n and f have the same number of zeros (counting multiplicities) in $D(z_0, \rho)$. Further, these zeros converge to z_0 as $n \rightarrow \infty$.*

Proof. Since f is a holomorphic function (Weierstrass' Theorem) and $f \not\equiv 0$, then by the Uniqueness Principle 1.12 there exists $\rho > 0$ such that $\overline{D(z_0, \rho)} \subset D$ and $f \neq 0$ on $|z - z_0| = \rho$. Therefore we have

$$\min_{|z-z_0|=\rho} |f(z)| =: \delta > 0.$$

Moreover, since f_n converges normally to f , there exists a positive integer n_0 such that

$$\sup_{|z-z_0|=\rho} |f_n(z) - f(z)| < \frac{\delta}{2}, \quad \text{for } n \geq n_0.$$

Thus on the circle $|z - z_0| = \rho$ we have

$$|f_n(z) - f(z)| < \frac{\delta}{2} < \delta \leq |f(z)|.$$

By Rouché's Theorem 1.24, f_n and f have the same number of zeros in $D(z_0, \rho)$ for $n \geq n_0$. Since the same argument works for smaller $\rho > 0$, then the zeros of f_n accumulate at z_0 . □

Corollary 1.26. *Let D be a domain of \mathbb{C} and (f_n) a sequence of holomorphic functions such that each f_n is never zero on D and suppose (f_n) converges normally to a function f . If f is ever zero on D , then $f \equiv 0$.*

Proof. If $f \not\equiv 0$, then by Hurwitz Theorem 1.25 f is never zero on D for n large, since f_n are never zero on D . □

Corollary 1.27. *Let D be a domain of \mathbb{C} and (f_n) a sequence of holomorphic **injective** functions that converges normally to a function f . Then f is either constant or injective.*

Proof. Suppose f is not constant and suppose that $z_0, \xi_0 \in D$ satisfy $f(z_0) = f(\xi_0) = w_0$. Then z_0 and ξ_0 are zeros of finite order of $f(z) - w_0$. By Theorem 1.25, there are sequences $z_n \rightarrow z_0$ and $\xi_n \rightarrow \xi_0$ such that $f_n(z_n) = f_n(\xi_n) = w_0$. Since the functions f_n are injective, we have $z_n = \xi_n$ and thus by the uniqueness of the limit $z_0 = \xi_0$. \square

1.5 Maximum Principle and Schwarz Lemma

Theorem 1.28 (Strict Maximum Principle). *Let f be a holomorphic function on a domain D . Suppose $|f|$ attains a local maximum at $z_0 \in D$, then f is constant.*

Proof. Let $D(z_0, r)$ be a neighbourhood of z_0 such that $D(z_0, r) \subset D$, then by hypothesis $|f(z_0)| \geq |f(z)|$ for all $z \in D(z_0, r)$. Suppose f is not constant, then in particular f is not constant on $D(z_0, r)$, since D is connected. By the open mapping theorem 1.14, there exists $\delta > 0$ such that

$$f(D(z_0, r)) \supseteq D(f(z_0), \delta).$$

We claim that there are points w in $D(f(z_0), \delta)$ such that $|w| > |f(z_0)|$: for example a convenient point is $w = f(z_0) + \frac{\delta}{2}e^{i \arg f(z_0)}$. Thus this fact contradicts the hypothesis $|f(z_0)| \geq |f(z)|$ for all $z \in D(z_0, r)$ \square

The following result, known as the **Maximum Principle**, is a straightforward consequence of Theorem 1.28, and states that holomorphic functions on bounded domains attain their maximum modulus on the boundary.

Corollary 1.29 (Maximum Principle). *Let D be a **bounded** domain in \mathbb{C} and f a holomorphic function on D which extends continuously on ∂D . If $|f(z)| \leq M$ for all $z \in \partial D$, then $|f(z)| \leq M$ for all $z \in D$.*

A useful corollary of the Maximum Principle is given by **Schwarz Lemma**. We will denote the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ as \mathbb{D} .

Corollary 1.30 (Schwarz Lemma). *Let $f(z)$ be a holomorphic function on \mathbb{D} . Suppose $|f(z)| \leq 1$ for all $z \in \mathbb{D}$ and $f(0) = 0$. Then*

$$|f(z)| \leq |z|, \quad |z| < 1. \quad (1.17)$$

Further, if equality holds in (1.17) at some point $z_0 \neq 0$, then $f(z) = \lambda z$ where λ is a unimodular complex constant.

We can also state an infinitesimal version of Schwarz Lemma

Corollary 1.31 (Infinitesimal Schwarz Lemma). *Let $f(z)$ be a holomorphic function on \mathbb{D} . Suppose $|f(z)| \leq 1$ for all $z \in \mathbb{D}$ and $f(0) = 0$. Then*

$$|f'(0)| \leq 1, \quad |z| < 1. \quad (1.18)$$

with equality if and only if $f(z) = \lambda z$ for some unimodular complex constant λ .

1.6 Compactness of families of holomorphic functions

In this section we will recall a version of the **Ascoli-Arzelá Theorem** in order to present the so-called **preliminary version of Montel's Theorem**.

Definition 1.32. Let E be a set of the complex plane \mathbb{C} and let \mathcal{F} be a family of complex-valued functions on E . We say that \mathcal{F} is **equicontinuous** on E if for any $\epsilon > 0$ there is $\delta > 0$ such that $|f(z) - f(w)| < \epsilon$, for all $z, w \in E$ with $|z - w| < \delta$ and for all $f \in \mathcal{F}$. We say that \mathcal{F} is **uniformly bounded** on E if there is a constant $M > 0$ such that $|f(z)| \leq M$ for all $z \in E$ and for all $f \in \mathcal{F}$.

Remark 1.33. Let \mathcal{F} be a family of holomorphic functions on a domain D . Suppose that the derivatives of the functions in \mathcal{F} are uniformly bounded on D by a positive constant M , then \mathcal{F} is equicontinuous. Infact for any $\epsilon > 0$ it is sufficient to integrate f' along a straight line segment connecting two nearby points $z, w \in D$ with $|z - w| < \frac{\epsilon}{M}$ and we have

$$|f(z) - f(w)| = \left| \int_w^z f'(\xi) d\xi \right| \leq M|z - w| \leq \epsilon, \quad \text{for all } f \in \mathcal{F}.$$

Theorem 1.34 (Ascoli-Arzelá Theorem). *Let K be a compact subset of \mathbb{C} and let \mathcal{F} be a uniformly bounded family of complex-valued continuous functions on K . Then the two following statements are equivalent:*

- *\mathcal{F} is equicontinuous.*
- *Each sequence of functions in \mathcal{F} has a subsequence that converges uniformly on K .*

The above result together with a Cantor's diagonally reasoning leads to the following result.

Theorem 1.35 (Montel's Theorem). *Suppose \mathcal{F} is a family of holomorphic functions on a domain D of \mathbb{C} such that \mathcal{F} is uniformly bounded on each compact subset of D . Then every sequence in \mathcal{F} has a subsequence that converges normally on D .*

Chapter 2

The Riemann Mapping Theorem

In this chapter we will be concerned about the famous **Riemann Mapping Theorem**. This powerful result states that every proper simply connected domain D of the complex plane can be **conformally** (i.e. preserving the angles and one-to-one) mapped onto the open unitary disk \mathbb{D} . Further, this mapping is unique up to post composition with a **conformal self-map** of \mathbb{D} . What seems to be *counterintuitive* is the fact that this conformal mapping can be achieved even if the boundary of \mathbb{D} is extremely not regular, for example a fractal. In order to fully explain the beauty of this theorem and provide a classical proof, in Section 2.1 we will define the idea of conformal maps, we will see the link between them and holomorphic functions with nowhere-vanishing derivative and we will give examples of conformal maps. In Section 2.2 we will describe the conformal self-maps of \mathbb{D} as elements of the so-called **Möbius group**. In Section 2.3 we will give a version of the Schwarz Lemma invariant under conformal self-maps of \mathbb{D} and we will see some useful results about the hyperbolic geometry of a simply connected domain of the complex plane, with particular emphasis on the unit disk \mathbb{D} . In Section 2.4 we will prove the Riemann Mapping Theorem using some results from Chapter 1 and from the previous sections of this chapter. References for this chapter can be found in [1], [14] and [26].

2.1 Conformal mapping

In this paragraph we will be concerned with the concept of **conformality**. We say that a map $f : D \rightarrow \mathbb{C}$ is conformal at $z_0 \in D$ if it preserves oriented angles between curves through z_0 . More precisely, let $\gamma(t) = x(t) + iy(t)$, $0 \leq t \leq 1$ be a smooth curve starting at $z_0 = \gamma(0)$ and let

$$\gamma'(0) = \lim_{t \rightarrow 0^+} \frac{\gamma(t) - \gamma(0)}{t} = x'(0) + iy'(0),$$

be the tangent vector to the curve γ at z_0 . We define the angle between two curves (which pass through z_0) at z_0 as the oriented angle between their tangent vectors at z_0 . Using the *chain rule* for holomorphic functions, we have

Proposition 2.1. *Let $\gamma(t) = x(t) + iy(t)$, $0 \leq t \leq 1$ be a smooth curve starting at $z_0 = \gamma(0)$ and let f be a function holomorphic at z_0 , then the tangent vector to the curve $f(\gamma(t))$ starting at z_0 is given by*

$$(f \circ \gamma)'(0) = f'(z_0)\gamma'(0). \tag{2.1}$$

Definition 2.2. Let $f(z)$ be a smooth complex-valued function. We say that f is **conformal** at z_0 if for any two smooth curves γ_0 and γ_1 starting at z_0 with non vanishing tangents at z_0 , the curves $f \circ \gamma_0$ and $f \circ \gamma_1$ have non vanishing tangents at $f(z_0)$ and the angle from $(f \circ \gamma_0)'(0)$ to $(f \circ \gamma_1)'(0)$ is the same as the angle from $(\gamma_0)'(0)$ to $(\gamma_1)'(0)$. We say that a continuously differentiable function from a domain D to the complex plane is conformal on D if it is conformal at every point in D . Moreover we say that a such f is a **conformal mapping** from the domain D to another domain E if f is conformal on D and maps D one-to-one onto E . In such case D and E are said to be **conformally equivalent**.

Remark 2.3. By equation (2.1), it follows immediately that every function holomorphic at z_0 with $f'(z_0) \neq 0$ is conformal at z_0 .

The converse of Remark 2.3 is also true and gives a characterization of a conformal function as a holomorphic function with non-zero derivative.

Theorem 2.4. *Let f be a continuously differentiable function defined on a domain D of the complex plane with its Jacobian matrix J_f nowhere zero on D . Then f is conformal on D if and only if f is holomorphic and f' is nowhere zero on D .*

We may now give some examples of conformal maps, with a particular emphasis on how to map conformally some specific classes of domains onto the open unit disk \mathbb{D} .

Example 2.5. Let \mathbb{H} denote the open upper half-plane of \mathbb{C} , more precisely

$$\mathbb{H} := \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}.$$

We claim that the function $w = (z - i)/(z + i)$ maps \mathbb{H} conformally onto \mathbb{D} . In other words \mathbb{H} and \mathbb{D} are conformally equivalent. It is a direct and easy computation to see that it is one-to-one (from $\mathbb{C} \setminus \{-i\}$) and conformal (it is a meromorphic function with a single pole at $-i$). It actually maps \mathbb{H} into \mathbb{D} : if we denote $z = a + ib$, we have

$$|w| = \left| \frac{z - i}{z + i} \right| = \frac{a^2 + (b - 1)^2}{a^2 + (b + 1)^2} \leq 1,$$

for every $z \in \mathbb{H}$ ($b > 0$). The same computation shows that the real axis is mapped onto the unit circumference. Since w is continuous and maps the boundary of \mathbb{H} onto the boundary of \mathbb{D} , we have that w is one-to-one and onto \mathbb{D} . Further, the inverse is given by $z = i(w + 1)/(1 - w)$. Both are examples of **linear fractional transformations**, which will be of extreme importance in Section 2.2.

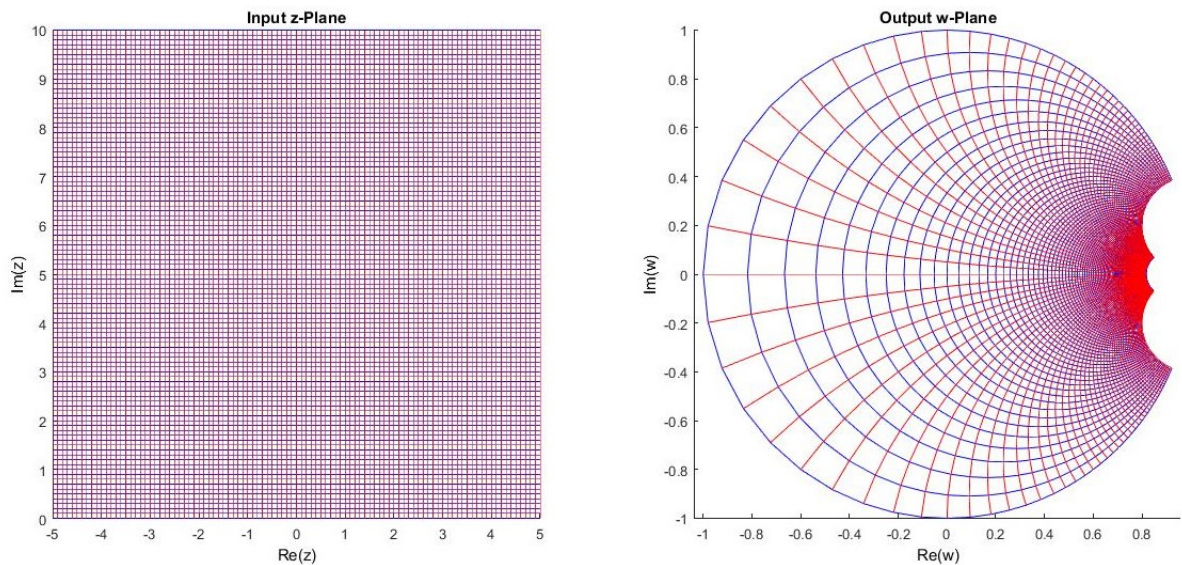


Figure 2.1: A portion of the upper plane mapped conformally to a subset of \mathbb{D}

Example 2.6. In Section 1.2 we defined the principal argument of a complex number and we examined power functions z^α , $\alpha \in \mathbb{C}$. If we consider $z \mapsto z^\beta$, $\beta \in \mathbb{R}$, by (1.9) we notice that this function acts multiplying angles by β . Thus any sector with a vertex in 0 can be rotated by a proper $z \mapsto \lambda z$, $|\lambda| = 1$ to a sector $S = \{0 < \arg z < \alpha\}$ with $\alpha \leq 2\pi$ and then S can be conformally mapped by $z^{\pi/\alpha}$ onto \mathbb{H} . Finally by Example 2.5 the function

$$w = \frac{z^{\pi/\alpha} - i}{z^{\pi/\alpha} + i}, \quad z \in S,$$

maps S conformally to \mathbb{D} .

Example 2.7. The exponential function e^z is conformal on the entire complex plane \mathbb{C} , since its derivative does not vanish at any point. However it is not a conformal mapping of \mathbb{C} onto $\mathbb{C} \setminus \{0\}$, because it is not injective. Its restriction to the horizontal strip $\{|\operatorname{Im} z| \leq \pi\}$ is, instead, a conformal mapping of the strip onto $\mathbb{C} \setminus (-\infty, 0]$. In general the exponential function conformally maps arbitrary horizontal strips such $\{\alpha < |\operatorname{Im} z| < \beta, |\beta - \alpha| \leq 2\pi\}$ into sectors of the type $\{z \in \mathbb{C} : \alpha(\bmod 2\pi) < \arg z < \beta(\bmod 2\pi)\}$. An arbitrary strip of the complex plane can be mapped conformally to \mathbb{D} . Firstly, it can be mapped onto a horizontal strip by a rotation, then it can be dilated by $z \mapsto rz$, $r > 0$ to a horizontal strip of width $\leq 2\pi$ and then by the exponential function onto a sector

with a vertex in zero. Then we may use the same method as in Example 2.6.

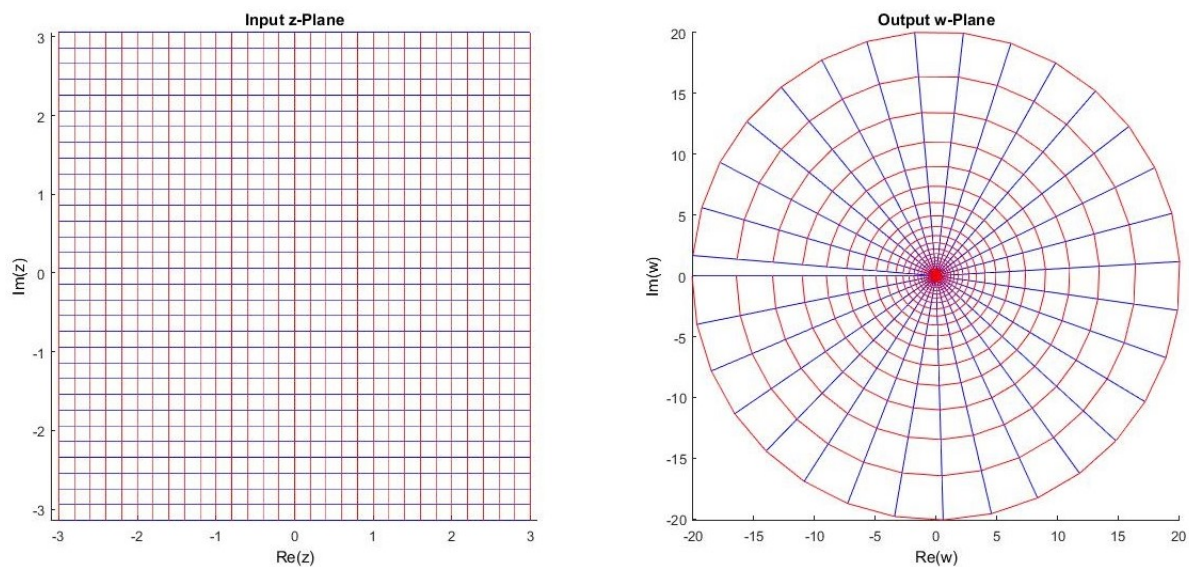


Figure 2.2: The conformal equivalence between the horizontal strip $\{|\operatorname{Im}z| \leq \pi\}$ and $\mathbb{C} \setminus (-\infty, 0]$

Example 2.8. The square function $w = z^2$ is holomorphic on \mathbb{C} and the only critical point is 0. By Remark 2.3 f is conformal at every point except the origin. However, we try to analyze better what happens at the origin. As usual it may be useful to consider f as a mapping from the z -plane to the w -plane. In the z -plane we consider two orthogonal curves starting at 0: $\gamma_x(t) = t$, $0 \leq t \leq 1$ and $\gamma_y(t) = it$, $0 \leq t \leq 1$. The first one is a horizontal ray from 0 to 1, while the second one is a vertical ray from 0 to i . Under the action of $w = z^2$, we notice that $w \circ \gamma_x$ is still a horizontal ray from 0 to 1 (but with doubled velocity) in the w -plane, while $w \circ \gamma_y$ is a horizontal ray from 0 to -1 . Thus the oriented angle from $w \circ \gamma_x$ to $w \circ \gamma_y$ is double the one between γ_x and γ_y .

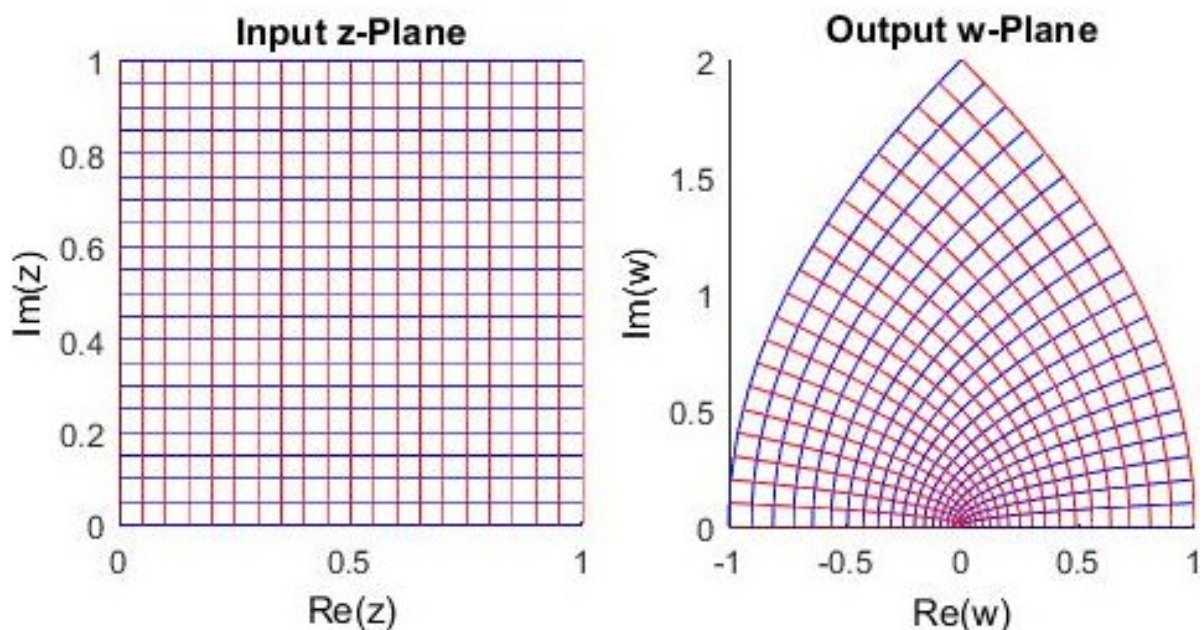


Figure 2.3: The transformation of the unit square via the square function

The behaviour of $w = z^2$ at the origin is due to the fact that $z_0 = 0$ is a zero of order 2 of f . Indeed, the following general result holds.

Theorem 2.9. *Let f be holomorphic in an open neighbourhood of a point z_0 and suppose z_0 is a zero of order p for $f(z) - f(z_0)$. Let γ_1 and γ_2 be two smooth curves starting at z_0 with non-zero tangent vectors and let θ be the oriented angle between them. Then the curves $f \circ \gamma_1$ and $f \circ \gamma_2$ have non-zero tangent vectors at $f(z_0)$ and the oriented angle between them is equal to $p\theta \pmod{2\pi}$.*

2.2 The Möbius group

The **extended complex plane** is the Alexandrov compactification of the complex plane, i.e. the complex plane with the point at infinity added. We denote it as $\hat{\mathbb{C}}$. It can be identified with the sphere \mathbb{S}^2 through the so-called **stereographic projection** and for this reason $\hat{\mathbb{C}}$ is often called the **Riemann sphere**.

We recall that any point $P = (X, Y, Z)$, except the North pole $N = (0, 0, 1)$, is mapped to a point in \mathbb{C} which is given by the intersection between the line that connects

N and P and the plane $\{Z = 0\}$. Explicitly, the stereographic projection is given by

$$\begin{aligned} P_N : \mathbb{S}^2 \setminus \{N\} &\longrightarrow \mathbb{C} \\ (X, Y, Z) &\mapsto \frac{X}{1-Z} + i \frac{Y}{1-Z}, \end{aligned} \quad (2.2)$$

which shows that it is a one-to-one map, whose inverse is given by

$$\begin{aligned} P_N^{-1} : \mathbb{C} &\longrightarrow \mathbb{S}^2 \setminus \{N\} \\ x + iy &\mapsto \left(\frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right). \end{aligned} \quad (2.3)$$

The stereographic projection maps lines of latitude on the sphere into circles centered at zero in the complex plane and, since those lines of latitude tend to the North pole N as the radii of the circles tend to ∞ , we may make correspond the North pole to the point ∞ . Doing so, we obtain an identification of the extended complex plane with the sphere \mathbb{S}^2 . Under this identification a straight line in the complex plane can be viewed as a circle on $\hat{\mathbb{C}}$ which passes through ∞ . An important feature of the stereographic projection is stated in the next result.

Proposition 2.10. *The stereographic projection maps circles on the sphere into circles in the extended complex plane.*

Before introducing the Möbius group, we recall some classical definitions about being holomorphic at ∞ .

Definition 2.11. $f : \mathbb{C} \rightarrow \hat{\mathbb{C}}$ such that for some z_0 we have $f(z_0) = \infty$ is said to be **holomorphic close to** z_0 if $1/f(z)$ is holomorphic around z_0 .

$f : \hat{\mathbb{C}} \rightarrow \mathbb{C}$ is said to be **holomorphic at** ∞ if $f(\frac{1}{z})$ is holomorphic at zero.

$f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ such that $f(\infty) = \infty$ is **holomorphic at** ∞ if $1/f(1/z)$ is holomorphic at zero.

We are now ready to take in exam some important bijective holomorphic functions of $\hat{\mathbb{C}}$ onto itself.

Definition 2.12. Any function given by

$$\phi(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C} \text{ with } ad - bc \neq 0 \quad (2.4)$$

is called a **fractional linear transformation** or **Möbius transformation**.

Those are meromorphic functions with simple poles at $z = -d/c$ (if $c \neq 0$) and thus are holomorphic except eventually at $z = -d/c$. Their complex-derivative is given by $f'(z) = (ad - bc)/(cz + d)^2$, which shows that the condition on the coefficients is meant in order to avoid constant functions. Further, by Remark 2.3 they are conformal on $\mathbb{C} \setminus \{-d/c\}$. We may see naturally those functions as transformations of the extended complex plane $\hat{\mathbb{C}}$ onto itself: if $c = 0$ we define $\phi(\infty) = \infty$, otherwise we define $\phi(-d/c) = \infty$ and $\phi(\infty) = a/c$. With those definitions and the ones given in 2.11, we see that any fractional linear transformation is a holomorphic function from $\hat{\mathbb{C}}$ to itself. Further, from (2.4) it is easy to see that every Möbius transformation has an inverse which is still a Möbius transformation and that such property is maintained even under composition. Thus the set of fractional linear transformation is a group with the usual operation of composition of functions and we will call it the **Möbius group**. It is quite useful to notice that the condition $ad - bc \neq 0$ is simply the fact that the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has nonzero determinant and that the composition of two Möbius transformations corresponds to the multiplication of their matrices. In other words, there is a group homomorphism between the group of invertible 2x2 matrices onto the Möbius group. In addition, two matrices A and B define the same Möbius transformation if and only if $A = \lambda B$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ and thus we can view A and B to be equivalent. We may summarize those considerations in the following result.

Proposition 2.13. *Every invertible matrix $A \in GL(2, \mathbb{C})$ defines a Möbius transformation*

$$\phi_A = \frac{az + b}{cz + d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

*which is holomorphic and bijective from $\hat{\mathbb{C}}$ onto itself. In other terms, ϕ_A is a **conformal self-map** of $\hat{\mathbb{C}}$. Further, the map $A \mapsto \phi_A$ depends only on the equivalence class of A given by $A \sim B$ iff $A = \lambda B$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. Thus the Möbius group is isomorphic to $SL(2, \mathbb{C})/\{\pm Id\}$.*

Since every Möbius transformation whose coefficients are multiplied by a nonzero constant leads to the same transformation, it is clear that every fractional linear transformation can be determined by three complex parameters (it suffices to adjust one of the four complex numbers a, b, c, d with a proper nonzero constant). This idea is strengthened by the next result.

Proposition 2.14. *Given two lists of distinct points z_1, z_2, z_3 and w_1, w_2, w_3 in $\hat{\mathbb{C}}$, there exists a **unique** Möbius transformation $\phi(z)$ such that $\phi(z_i) = w_i$ for $i = 1, 2, 3$.*

The following result states that the Möbius group has four generators.

Proposition 2.15. *Every Möbius transformation is given by a composition of Möbius transformation of the following kinds:*

1. $z \mapsto z + a$ (translation)
2. $z \mapsto \lambda z, \lambda > 0$ (dilation)
3. $z \mapsto e^{i\theta} z, \theta \in \mathbb{R}$ (rotation)
4. $z \mapsto \frac{1}{z}$ (inversion)

In analogy with the stereographic projection, even Möbius transformations *preserve* circles of $\hat{\mathbb{C}}$ (i.e. proper circles and lines of \mathbb{C}).

Proposition 2.16. *If we denote the class of any circle in $\hat{\mathbb{C}}$ with \mathcal{C} , then every Möbius transformation maps \mathcal{C} in \mathcal{C} .*

We return now to the unit disk \mathbb{D} and we focus mainly on its conformal self-maps, i.e. the conformal mappings from \mathbb{D} onto itself.

Proposition 2.17. *If $|a| < 1$ and $|\lambda| = 1$, then*

$$f(z) = \lambda \frac{z - a}{1 - \bar{a}z}, \quad (2.5)$$

is a conformal self-map of \mathbb{D} and it also maps the circle $|z| = 1$ to itself.

Proof. Firstly, we notice that f is a Möbius transformation, thus a conformal self-map of $\hat{\mathbb{C}}$ (in a particular a bijection). Furthermore, if $|z| = 1$ we have

$$\begin{aligned} |f(z)| &= \left| \lambda \frac{z-a}{1-\bar{a}z} \right| \\ &= \left| \frac{1}{\bar{z}} \frac{z-a}{1-\bar{a}z} \right| \\ &= \left| \frac{z-a}{\bar{z}-\bar{a}|z|^2} \right| \\ &= \left| \frac{z-a}{\bar{z}-\bar{a}} \right| \\ &= 1 \end{aligned}$$

By 2.16, f maps the circle $|z| = 1$ bijectively onto itself. Thus $f : \hat{\mathbb{C}} \setminus \{|z| = 1\} \rightarrow \hat{\mathbb{C}} \setminus \{|z| = 1\}$ is still a bijection. We notice that f is continuous and $\hat{\mathbb{C}} \setminus \{|z| = 1\}$ has two connected components (\mathbb{D} and its complement in $\hat{\mathbb{C}}$). Now $f(a) = 0 \in \mathbb{D}$, thus $f(\mathbb{D}) = \mathbb{D}$ and similarly (since $f(1/\bar{a}) = \infty$) for the complement of \mathbb{D} in $\hat{\mathbb{C}}$. \square

Corollary 2.18. *If $a, b \in \mathbb{D}$, then there exists a conformal self-map f of \mathbb{D} such that $f(a) = b$.*

Proof. Take

$$f_1 = \frac{z-a}{1-\bar{a}z}, \quad f_2 = \frac{z-b}{1-\bar{b}z},$$

then $f_1(a) = 0 = f_2(b)$. Thus the function we seek is given by $f_2^{-1} \circ f_1$. \square

The condition stated in Proposition 2.17 is not only sufficient, but it is also necessary for being a conformal self-map. In order to prove so, we first demonstrate that every conformal self-map of \mathbb{D} which fixes the origin is a rotation.

Lemma 2.19. *Let g be a conformal self-map of \mathbb{D} such that $g(0) = 0$, then $g(z) = \lambda z$ for some fixed unimodular constant λ .*

Proof. We first apply Schwarz Lemma 1.30 to g and we obtain $|g(z)| \leq |z|$. Then we apply it to $g^{-1}(w)$ and we get $|g^{-1}(w)| \leq |w|$, which for $w = g(z)$ becomes $|z| \leq |g(z)|$. Thus $|g(z)/z| = 1$ and still by Schwarz Lemma we have the thesis. \square

Theorem 2.20. Any conformal self-map of the unit disk \mathbb{D} has the form

$$f(z) = e^{i\varphi} \frac{z - a}{1 - \bar{a}z}, \quad |a| < 1 \text{ and } 0 \leq \varphi \leq 2\pi. \quad (2.6)$$

In particular a and φ are uniquely determined by

$$\begin{aligned} a &= f^{-1}(0), \\ \varphi &= \arg f'(0). \end{aligned}$$

The latest statement implies that there is a one-to-one correspondence between $\mathbb{D} \times \partial\mathbb{D}$ and the group of conformal self-maps of \mathbb{D} , which will be denoted as $Aut(\mathbb{D})$.

Proof. Let f be a conformal self-map of \mathbb{D} and by Corollary 2.18 it is not restrictive to suppose that $f(a) = 0$. Define $g(z) = (z - a)/(1 - \bar{a}z)$ and notice that by Proposition 2.17 is a conformal self of \mathbb{D} . Then $f \circ g^{-1}$ is a conformal self-map of \mathbb{D} such that fixes the origin. Thus by Lemma 2.19 we have

$$(f \circ g^{-1})(w) = e^{i\varphi} w$$

for some fixed φ , $0 \leq \varphi \leq 2\pi$. For $w = g(z)$ we have (2.6). Further, since the derivative of f is given by

$$f'(z) = e^{i\varphi} \frac{1 - |a|^2}{(1 - \bar{a}z)^2},$$

the parameter φ is uniquely specified (mod 2π) as the argument of $f'(0)$. \square

2.3 Pick's Lemma and the hyperbolic geometry

We may now give a version of Schwarz Lemma which is invariant under conformal self-maps of \mathbb{D} . The following result is called **Pick's lemma** and gives a useful hint on how to define the **hyperbolic metric** on the unit disk \mathbb{D} .

Theorem 2.21. Let $f : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic, then

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}, \quad (2.7)$$

with equality in (2.7) if and only if f is a conformal self-map of \mathbb{D} .

Suppose $w = f(z)$ is a conformal self-map of \mathbb{D} , thus there is equality in (2.7), which means in terms of differential forms

$$\frac{|dw|}{1 - |w|^2} = \frac{|dz|}{1 - |z|^2}.$$

We see then that for any piecewise smooth curve γ in \mathbb{D} , we have

$$\int_{f \circ \gamma} \frac{|dw|}{1 - |w|^2} = \int_{\gamma} \frac{|dz|}{1 - |z|^2},$$

and thus we obtain a length function which is invariant under conformal self-maps of \mathbb{D} .

Definition 2.22. The **hyperbolic metric** on the unit disk \mathbb{D} is defined as

$$\frac{2|dz|}{1 - |z|^2}. \quad (2.8)$$

Definition 2.23. We define the **length of γ in the hyperbolic metric** by

$$l(\gamma) := 2 \int_{\gamma} \frac{|dz|}{1 - |z|^2} \quad (2.9)$$

We then define the **hyperbolic distance** from z_0 to z_1 by

$$d(z_0, z_1) := \inf\{l(\gamma) : \gamma \text{ is a piecewise smooth curve joining } z_0 \text{ to } z_1 \text{ in } \mathbb{D}\} \quad (2.10)$$

Remark 2.24. The factor 2 in (2.9) is needed to have constant curvature equal to -1. Recall that the curvature K of a metric given by $\lambda(z)|dz|$ is given by

$$K = -\frac{\Delta \log \lambda}{\lambda^2} \quad (2.11)$$

In our case we have $\lambda(z) = \frac{2}{1 - |z|^2}$ and a direct computation of (2.11) gives indeed $K=-1$.

We notice that the **oriented isometries** of \mathbb{D} for the hyperbolic metric are given exactly by the conformal self-maps of \mathbb{D} . Indeed this is the result of a reformulation of Pick's Lemma 2.21 under the hyperbolic metric.

Theorem 2.25. *Every holomorphic function from \mathbb{D} to itself is a contraction mapping with respect to the hyperbolic metric*

$$d(f(z_0), f(z_1)) \leq d(z_0, z_1), \quad z_0, z_1 \in \mathbb{D}. \quad (2.12)$$

Further, equality in (2.12) holds if and only if f is a conformal self-map of D

Given the origin and a point $z \in \mathbb{D}$ we may see that the shortest path which joins them is the straight line segment from 0 to z . Given two points $z_0, z_1 \in \mathbb{D}$ it suffices to take a Möbius transformation φ which takes z_0 to the origin, in order to find the shortest curve (i.e. **hyperbolic geodesic**) linking z_0 to z_1 : indeed, the hyperbolic geodesic between z_0 and z_1 is given by the composition of the line segment joining 0 and $\varphi(z_1)$ and the inverse Möbius transformation φ^{-1} . Since we have seen in Section 2.2 that Möbius transformations preserve circles and angles, the geodesic connecting those two points is then the arc of circle passing through z_0 and z_1 that is orthogonal to the unit circle.

The distance between 0 and a point $z \in \mathbb{D}$ can be easily explicitly computed by

$$d(0, z) = \log \frac{1 + |z|}{1 - |z|}, \quad (2.13)$$

which shows that the distance tends to ∞ as z approaches the unit circle. Further, it shows that every Euclidean circle centered at the origin is a hyperbolic circle centered at the origin and viceversa. We may summarize the ideas above in the following result.

Theorem 2.26. *1. The topology induced by the hyperbolic metric is equivalent to the usual topology of \mathbb{D} .*

2. The hyperbolic geodesics are the arcs of circles (eventually portions of diameters) orthogonal to the unit circle.

3. Given any two distinct points in \mathbb{D} there is a unique geodesic linking them.

Given a simply connected domain D of the complex plane and a conformal mapping of D onto \mathbb{D} , we define the hyperbolic metric on D by

$$d_D(z) = \frac{2|g'(z)|}{1 - |g(z)|^2} |dz|, \quad z \in D. \quad (2.14)$$

This definition is independent of the choice of the conformal mapping g : in the next section we will deal with the proof of the Riemann Mapping Theorem and in Remark 2.32 we will see that any two conformal mappings of D onto \mathbb{D} can be obtained by post composing with a conformal self-map of \mathbb{D} . Together with the Pick's Lemma, the previous result will give the independence of g in (2.14).

Example 2.27. In Example 2.5 we have seen that

$$g(z) = \frac{z - i}{z + i}, \quad z \in \mathbb{H},$$

is a conformal mapping of \mathbb{H} onto \mathbb{D} . Then by (2.14), we have that the hyperbolic metric on the upper plane \mathbb{H} is given by

$$d_{\mathbb{H}}(z) = \frac{|dz|}{y}, \quad z = x + iy, y > 0 \quad (2.15)$$

We notice that (2.15) is independent of x , thus any vertical straight line in \mathbb{H} is a hyperbolic geodesic. Further, all the other geodesics are given by semicircles orthogonal to the real axis.

Definition 2.28. Let E be a Lebesgue-measurable subset of the simply connected domain D , then

$$\text{Area } D = \iint_D \lambda(z)^2 dx dy, \quad \text{where } \lambda(z) = \frac{2|g'(z)|}{1 - |g(z)|^2}. \quad (2.16)$$

A triangle whose sides are hyperbolic geodesics is called a **hyperbolic triangle**. Using the characterization of hyperbolic geodesics in the upper plane \mathbb{H} and allowing infinite triangles (whose one or more sides have infinite hyperbolic length), the following result can be proved.

Theorem 2.29. *The area of a hyperbolic triangle with angles α, β, γ is*

$$\pi - (\alpha + \beta + \gamma). \quad (2.17)$$

In particular the sum of its interior angles is $< \pi$.

2.4 The Riemann Mapping Theorem

In this section we will state and prove the famous Riemann mapping theorem.

Theorem 2.30. *Let D be a simply connected proper domain of \mathbb{C} and let $a \in D$. Then there exists a unique conformal mapping $f : D \rightarrow \mathbb{D}$ such that $f(a) = 0$ and $f'(a) > 0$.*

Remark 2.31. The hypothesis of the theorem are necessary:

- Since \mathbb{D} is a simply connected domain, then D must be as well (notice that f is in particular a continuous homeomorphism).
- D must be a proper subset of the complex plane \mathbb{C} by Liouville's Theorem 1.11.

Proof of the uniqueness part. Let f_1 and f_2 be two such conformal mappings of D onto \mathbb{D} . Then $g = f_1 \circ f_2^{-1}$ is conformal self-map of the unit disk \mathbb{D} such that it fixes the origin. Thus, by Lemma 2.19 there exists λ with $|\lambda| = 1$ such that $g(z) = \lambda z$. Further, we have

$$\lambda = g'(0) = \frac{f_1'(a)}{f_2'(a)} > 0,$$

and thus $\lambda = 1$. This implies that $g(z) = z$, which proves that $f_1 = f_2$. \square

Remark 2.32. The conformal mapping of D onto \mathbb{D} is unique up to a post composition with a conformal self-map of \mathbb{D} , i.e. given two such conformal mappings f, g then there exists a unique conformal self-map T of \mathbb{D} such that $g = T \circ f$

We are now ready to give a detailed proof of the existence part, which will rely mainly on Montel's Theorem 1.35, the corollary 1.27 of Hurwitz Theorem, the existence of a square root function on a simply connected domain not containing zero (theorem 1.16) and Pick's Lemma 2.21.

We will show that such a conformal map f exists, considering the following family of functions:

$$\mathcal{F} := \{f : D \rightarrow \mathbb{D}, f \text{ holomorphic and injective, } f(a) = 0 \text{ and } f'(a) > 0\}. \quad (2.18)$$

We proceed with our proof showing the following three claims:

1. $\mathcal{F} \neq \emptyset$
2. there is a element $f \in \mathcal{F}$ maximizing $|f'(a)|$
3. this element f maps D **onto** \mathbb{D} .

Proof of 1. Given D a simply connected domain and $b \notin D$, we already know by Theorem 1.16 that there exists a holomorphic square root of $z - b$ on D (i.e. $g : D \rightarrow \mathbb{C}$ such that $g^2(z) = z - b$). By a direct and easy computation, we notice that this g is injective.

Further, this g has the property that $g(D)$ is disjoint from $-g(D)$. Indeed, if $w_0 = g(z_0)$ and $-w_0 = g(z_1)$ for $z_0, z_1 \in D$, then we would have

$$z_0 = g(z_0)^2 + b = w_0^2 + b = g(z_1)^2 + b = z_1,$$

which is impossible, since it implies that $b \in D$. By the open mapping theorem we know that such g is open and thus there exists $r > 0$ such that $D(w_0, r) \subset g(D)$. We now choose f to be given by

$$f(z) = \frac{r}{2(g(z) + w_0)}. \quad (2.19)$$

Thus f is holomorphic on D , injective and $f(D) \subset \mathbb{D}$. This argument let us construct a function f which is an element of \mathcal{F} : it is sufficient to take f as in (2.19) and compose it with a conformal-self map of \mathbb{D} that takes a to zero and choosing the parameter λ in order to have positive derivative at a . Explicitly

$$\lambda \frac{f(z) - f(a)}{1 - \overline{f(a)}f(z)},$$

for a proper choice of λ . □

Proof of 2. We now consider the following **extremal problem**: given the family \mathcal{F} as in (2.18), we seek a function $f \in \mathcal{F}$ such that maximizes $f'(a)$ among all functions in \mathcal{F} . Thus we define

$$B = \sup_{f \in \mathcal{F}} f'(a).$$

We notice that $B > 0$, since every function in \mathcal{F} is injective and as a consequence has non-zero derivative at every point. Now let (f_n) be a sequence in \mathcal{F} such that $f_n \rightarrow B$, then by Montel's Theorem 1.35 there exists a subsequence of (f_n) that converges normally in D to a holomorphic function f . By normal convergence we have $f(a) = 0$ and $f'(a) = B > 0$. Being the limit of injective holomorphic functions, by Corollary 1.27 we have that f is holomorphic and **injective** (notice that it cannot be constant, since $f'(a) > 0$). Further, by the Open Mapping Theorem 1.14 we have $f(D) \subset \mathbb{D}$. We have thus proved that $f \in \mathcal{F}$. □

Proof of 3. Let f be the limit function with maximal derivative at a as above. So far f is a conformal mapping of D onto a subdomain of \mathbb{D} . We have to prove that $f(D) = \mathbb{D}$.

In order to do so, we suppose there is $w \in \mathbb{D} \setminus f(D)$ and seek a contradiction. Let g_1 be a conformal self-map of \mathbb{D} with $g_1(w) = 0$ and let $\Omega_1 = f(D)$ and $\Omega_2 = g_1(\Omega_1)$. Then Ω_2 is a simply connected subdomain of \mathbb{D} which not contains zero, thus there exists a holomorphic square root function $\sqrt{\cdot}$. Then let g_2 be a conformal self-map of \mathbb{D} with $g_2(\sqrt{g_1(0)}) = 0$. Define

$$F = g_2 \circ \sqrt{\cdot} \circ g_1 \circ f,$$

and notice that $F(a) = 0$, F is holomorphic and injective and it is not restrictive to suppose that $F'(a) > 0$ (otherwise rotate g_2 by a proper λ , $|\lambda| = 1$). Therefore, we have that $F \in \mathcal{F}$. The inverse of $g_2 \circ \sqrt{\cdot} \circ g_1$ is the holomorphic function

$$h = g_1^{-1} \circ (g_2^{-1})^2 : \mathbb{D} \rightarrow \mathbb{D}.$$

We notice that $h(0) = 0$ and that h is not a conformal self-map of \mathbb{D} , thus by Pick's Lemma 2.21 we have

$$|h'(0)| < 1.$$

Since $h \circ F = f$, we have $(h \circ F)'(a) = f'(a)$ and thus $F'(a) > f'(a)$, which is a contradiction.

□

Thus the Riemann Mapping Theorem asserts that any simply connected domain of the complex plane is either the entire complex plane \mathbb{C} or it is conformally equivalent to the unit disk \mathbb{D} . For a simply connected domain in the Riemann sphere $\hat{\mathbb{C}}$, instead, we have three possibilities:

Corollary 2.33. *Any simply connected domain in $\hat{\mathbb{C}}$ is either the entire Riemann sphere, or it is conformally equivalent to the complex plane \mathbb{C} , or to the unit disk \mathbb{D} .*

Proof. If the domain is not the entire Riemann sphere, then we may move a point of the complement of the domain to ∞ by a Möbius transformation and then we reduce to the case of planar domains in \mathbb{C} , where we have only two possibilities, which are conformally equivalent to \mathbb{C} or \mathbb{D} . □

The result above actually covers all the possibilities for simply connected **Riemann surfaces**, which will be the main subject of study in the next chapter.

Chapter 3

Uniformization Theorem for Riemann surfaces

The Riemann Mapping Theorem, that we discussed in the previous section, can be generalized in the theory of Riemann surfaces. The main goal of this chapter is to present the **Uniformization Theorem** for simply connected Riemann surfaces:

Theorem. *Every simply connected Riemann surface is conformally equivalent to the complex plane \mathbb{C} , the Riemann sphere $\hat{\mathbb{C}}$ or the unit disk \mathbb{D} .*

A Riemann surface is a connected surface which is locally biholomorphic with an open subset of the complex plane \mathbb{C} . More precisely, it is a connected surface with a holomorphic (or conformal) atlas, which means that the transition functions are holomorphic. The first basic examples that we will discuss are the complex plane \mathbb{C} , every domain in \mathbb{C} and the Riemann sphere $\hat{\mathbb{C}}$.

The rigorous definition that we will give in Section 3.1 of a Riemann surface will allow us to define holomorphic functions between two Riemann surfaces and give some basic properties of such maps. In order to fully comprehend the theory of Riemann surfaces in Section 3.2 we will give a brief introduction to covering surfaces, with particular emphasis on the Riemann surface of a multivalued function and on the universal covering of a Riemann surface.

In Section 3.3 we will present the Uniformization Theorem and we will state some of its main consequences, such as the classification of an arbitrary Riemann surface.

References for this chapter may be found in [2],[11],[13], [14] and [26].

3.1 Riemann surfaces

In this section we will give the notion of a Riemann surface and we will provide some basic examples. Next we will compare this new concept with the one of a Riemannian surface (i.e. a 2 real-manifold with a differentiable Riemannian structure). Then we will define holomorphic and meromorphic functions between Riemann surfaces and we will discuss their basic properties.

3.1.1 Definition and examples

Definition 3.1. A Riemann surface S is a topological connected Hausdorff space equipped with an atlas $\{U_\alpha, \phi_\alpha\}$ such that

1. $\{U_\alpha : \alpha \in A\}$ is an open cover of S
2. $\phi_\alpha : U_\alpha \rightarrow \phi(U_\alpha) \subset \mathbb{C}$ is a homeomorphism for every $\alpha \in A$
3. The **transition functions** $\phi_{\alpha\beta} := \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)$ are holomorphic for each $\alpha, \beta \in A$

The atlas $\mathcal{A} = \{U_\alpha, \phi_\alpha\}$ is said holomorphic or conformal. The couple $\{U_\alpha, \phi_\alpha\}$ for a given α is said a **chart or coordinate chart**.

Remark 3.2. In the previous definition we did not require S to be second countable. Indeed, it can be proved that every Riemann surface is second countable and therefore it is equivalent to say that a Riemann surface is a connected 2 real-manifold with a holomorphic atlas.

We may see a Riemann surface as a pair (S, \mathcal{A}) where \mathcal{A} denotes a holomorphic atlas on S . Given two atlases $\mathcal{A}_1 = \{U_\alpha, \phi_\alpha\}_{\alpha \in A}$, $\mathcal{A}_2 = \{V_\beta, \varphi_\beta\}_{\beta \in B}$ we may define a partial ordering on the class of holomorphic atlases by

$$\mathcal{A}_1 < \mathcal{A}_2 \text{ if and only if for each } \alpha \in A, \text{ there is a } \beta \in B : U_\alpha \subset V_\beta \text{ and } \phi_\alpha = \varphi_\beta|_{U_\alpha}.$$

By Zorn's Lemma, it follows that an arbitrary atlas can be extended to a maximal one. Thus, it suffices to give any holomorphic atlas to define a Riemann surface, instead of specifying a maximal atlas. The surface S together with the maximal atlas determined by \mathcal{A} is called a **conformal structure** on a Riemann surface (S, \mathcal{A}) .

Remark 3.3. As we have seen in Proposition 1.7 from Section 1.1 holomorphic functions in the complex plane preserve the orientability of the plane. Therefore, a Riemann surface is an orientable 2 real-manifold.

Example 3.4. The most trivial Riemann surface is the complex plane \mathbb{C} with a single chart given by the identity function. Further, every domain D of the complex plane is still a Riemann surface with atlas given by $(D, id|_D)$. In an analogous way, every domain of a Riemann surface is still a Riemann surface.

Example 3.5. The Riemann sphere $\hat{\mathbb{C}}$ is a Riemann surface with an atlas consisting of two charts:

$$U_0 = \mathbb{C}, \phi_0(z) = z \quad \text{and} \quad U_1 = \hat{\mathbb{C}} \setminus \{0\}, \phi_1(z) = \frac{1}{z}.$$

The two transition functions are given by

$$\phi_{ij} : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C} \setminus \{0\}, i \neq j, i, j = 0, 1, \text{ with } \phi_{ij}(z) = \frac{1}{z},$$

which are indeed holomorphic. If we look at the Riemann sphere as the sphere \mathbb{S}^2 , then we use the stereographic projections from the North and the South poles as charts.

3.1.2 Basic properties of holomorphic maps between Riemann surfaces

The definition we gave in 3.1 using coordinate charts is suitable for the idea of a holomorphic map between two Riemann surfaces.

Definition 3.6. Let $f : S \rightarrow R$ be a continuous map between Riemann surfaces. f is said to be holomorphic if given any chart $\{U_\alpha, \phi_\alpha\}$ of S which contains the point p and any chart $\{V_\beta, \varphi_\beta\}$ of R containing $f(p)$ we have that

$$\varphi_\beta \circ f \circ \phi_\alpha^{-1} \tag{3.1}$$

is holomorphic near $\phi_\alpha(p)$. Further, a holomorphic map into \mathbb{C} is said to be a holomorphic **function** on S and a holomorphic map into $\hat{\mathbb{C}}$, except the constant function $f(z) = \infty$, is said to be a **meromorphic function** on S .

The set of holomorphic functions on S can be easily seen as a ring with the usual operations of sum and product of functions and we will denote it as $\mathcal{H}(S)$, while the set of meromorphic functions on S is a field denoted as $\mathcal{M}(S)$.

Definition 3.7. If $f : S \rightarrow R$ is holomorphic and bijective, then we say that S and R are **conformally equivalent**.

Since the above definitions are local, we may naturally extend all the results seen in Chapter 1 that are based on local properties. The results we will use mostly later are the following.

Theorem 3.8. *Let $f : S \rightarrow R$ be a non constant holomorphic map and $p \in S$. Then there are charts ϕ near p vanishing at p (i.e. $\phi(p) = 0$) and φ near $f(p)$ vanishing at $f(p)$ and an integer $n \geq 1$ such that*

$$\varphi \circ f \circ \phi^{-1}(z) = z^n. \tag{3.2}$$

*The integer n is called the **multiplicity** of f at p and we will denote it as $m_f(p)$.*

Theorem 3.9. *Every holomorphic non constant map between Riemann surfaces is discrete and open.*

Theorem 3.10. *Let $f : S \rightarrow \mathbb{C}$ be holomorphic and suppose $|f|$ attains its maximum on S , then f is constant.*

Theorem 3.11. *Let $f : S \rightarrow R$ be a non constant holomorphic map and S be compact. Then f is surjective and also R is compact. In particular $\mathcal{H}(S) = \mathbb{C}$.*

As corollary of the previous result, we may derive the famous Fundamental Theorem of Algebra.

Theorem 3.12. *Every non constant polynomial in \mathbb{C} has a root.*

Proof. Since every polynomial in \mathbb{C} is a holomorphic map from $\hat{\mathbb{C}}$ to itself that maps $\infty \mapsto \infty$, by 3.11 it is a surjection and in particular it has a zero. \square

We may now focus our attention on compact Riemann surfaces, in order to discuss the degree of a map and the Riemann-Hurwitz relation.

Definition 3.13. Let $f : S \rightarrow R$ be a non constant holomorphic map and let S be compact. We define the degree of f at $q \in R$ as

$$\deg_f(q) = \sum_{p: f(p)=q} m_f(p), \quad (3.3)$$

where $m_f(p)$ is defined as in (3.2).

Remark 3.14. We notice that the definition above is consistent, since S is compact and the sum in (3.3) is then finite by 3.9.

The following lemma shows that $\deg_f(q)$ does not depend on $q \in R$, thus we may define the **degree** of the map f .

Lemma 3.15. *Let $f : S \rightarrow R$ be a non constant holomorphic map and let S be compact. Then $\deg_f(q)$ is independent on $q \in R$ and we denote it as $\deg(f)$.*

Proof. Since R is connected, it suffices to show that $\deg_f(q)$ is locally constant. By 3.11 we know that f is surjective. Let now $f(p) = q$ and suppose that $m_f(p) = 1$. Then f is biholomorphic around p and so $\deg_f(q) = 1$ around $q \in R$. Suppose now that $n = m_f(p) > 1$: this means that every point q' near q , with $q' \neq q$, has exactly n pre-images near p . Let p_1, \dots, p_n those pre-images and we have that $m_f(p_j) = 1$ for $j = 1, \dots, n$. Therefore $\deg_f(q)$ is locally constant. \square

We recall that a main result in topology is the classification of orientable compact surfaces:

Theorem 3.16. *Any orientable compact surface is a sphere with g handles, where the non-negative integer g is called the **genus** of the surface.*

We recall also that the Euler-Poincaré characteristic χ of a triangulated compact surface is $\chi = \alpha_0 - \alpha_1 + \alpha_2$ where α_k is the number of k -simplices in the triangulation. For compact orientable surfaces we have $\chi = 2 - 2g$. Since compact Riemann surfaces are orientable compact surfaces, we may relate the genera of two Riemann surfaces with the degree of a nonconstant holomorphic map between them.

Theorem 3.17 (Riemann-Hurwitz relation). *Let $f : S \rightarrow R$ be a non constant holomorphic map of degree n between compact Riemann surfaces of genus g and γ respectively. Define the total branching number of f by*

$$B = \sum_{p \in S} (m_f(p) - 1).$$

Then we have

$$g = n(\gamma - 1) + 1 + B/2. \tag{3.4}$$

The following result together with the Uniformization Theorem will play a crucial role in Chapter 4.

Corollary 3.18. *If $g = 0$ then $\gamma = 0$.*

3.2 Riemann surfaces of multivalued functions

In this section we will briefly discuss another way in which Riemann surfaces arise: through the analytic continuation of a multivalued function. We will try to give only the intuition behind these ideas, while the formal aspects and proofs may be found in [13] and [17]. Further, we will recall some useful results about covering theory, with particular emphasis on the universal covering of a Riemann surface.

In Section 1.2 we described power functions and we saw that it is not possible to define them continuously on $\mathbb{C} \setminus \{0\}$ following their behaviours on different paths around 0. In general if we have an analytic function f defined on its natural domain D , we wonder whether f may be extended analytically on a bigger domain. The idea is that we may track the power series expansion of f along a path. If we start from a point $z_0 \in D$, we are interested in the *germ* of f at z_0 (i.e. its behaviour locally around z_0).

We consider a path $\gamma(t), a \leq t \leq b$ which starts at z_0 and we say that f is **analytically continuable along** γ if for each t the power series

$$f_t(z) = \sum_{n=0}^{\infty} a_n(t)(z - \gamma(t))^n,$$

is convergent for z near $\gamma(t)$ and for s near t we have $f_s(z) = f_t(z)$ in the intersection of the disks of convergence. The Uniqueness Principle 1.12 then implies that $f_b(z)$ is uniquely determined by $f_a(z)$, which in other words means that analytic continuation is unique (when it exists) and depends only on f and γ , but not on the chains of disks along γ . The main result about analytic continuation is given by the so-called **Monodromy Theorem**, which shows that the analytic continuations of a function f on two homotopic curves with the same initial and end points agree at the end point.

Theorem 3.19. *Let γ_0 and γ_1 be two homotopic curves with same initial point p and end point q . Let then U be a neighbourhood of p and let f be holomorphic on U . Suppose that f can be continued analytically on every curve of the homotopy, then the analytic continuations of f along γ_0 and γ_1 agree locally at q .*

In particular we have that on simply connected domains there is a unique analytic continuation.

Two couples (f, p) and (g, q) are direct analytic continuations if there is a path from p to q such that g is the analytic continuation of f along the path. Given a germ of a analytic function f around p , we call the **complete analytic function** of (f, p) the equivalence class given by all the direct analytic continuations of f around p , where two germs (f, p) and (g, q) are equivalent if and only if they are direct analytic continuations. We denote \mathcal{R} a complete analytic function. We underline here that \mathcal{R} is not a single-valued function in general, since for a given point it may assume different values. Our goal is to define a natural domain for it, on which it will be single-valued. It is possible to define a topology on \mathcal{R} where the open neighbourhoods U_ε of (f, p) (with ε not greater than the radius of convergence of f) are given by all the germs of type (g, q) with q close to p and g is obtained by analytic continuation of f . In this way \mathcal{R} turns out to be a topological connected Hausdorff space and most importantly a Riemann surface. We call \mathcal{R} the Riemann surface associated to f . Further, the complete analytic function (f, p) is a holomorphic function from \mathcal{R} to D , where D is the natural domain of f .

As an example, we consider the Riemann surface associated to the square root function \sqrt{z} . In Section 1.2 we saw how to define the two branches w_1, w_2 of \sqrt{z} continuously on the slit plane $\mathbb{C} \setminus [0, \infty)$. Further, we noticed that they had opposite sign on the top and on the bottom edges of the branch cut. The Riemann surface of \sqrt{z} may then be viewed as the result of gluing together along the cut the two slit planes of the two branches w_1, w_2 according to their sign.

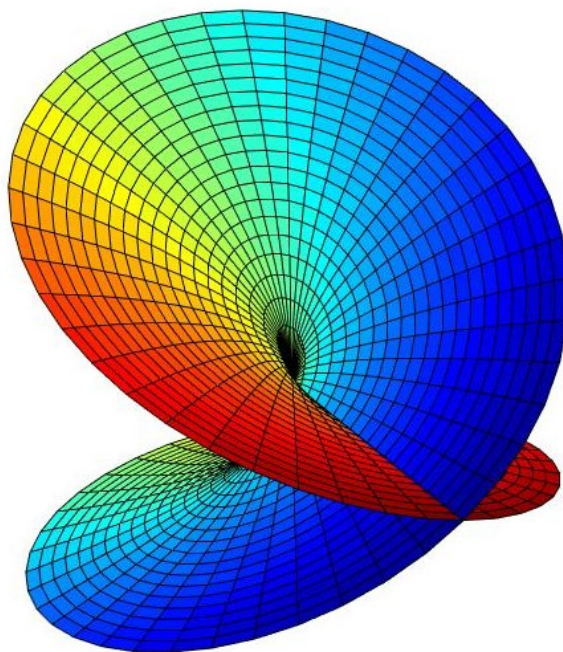


Figure 3.1: Riemann surface of \sqrt{z} .

3.2.1 The universal covering

In this paragraph we will briefly define the universal covering of a Riemann surface S . Useful references for this part are [2], [14] and [11].

Given a Riemann surface S , a smooth covering surface of S is another Riemann surface S^* such that is provided a holomorphic surjective map $\pi : S^* \rightarrow S$ with the property that for each point $P^* \in S^*$ there exist a local coordinate z^* on S^* that vanishes at P^* and a local coordinate z on S that vanishes at $\pi(P^*)$ such that π is given locally by $z = z^*$. Further, a smooth covering surface S^* is said to be unlimited if for every curve γ

on S there exists a lift of γ on S^* , i.e. for every point P^* with $\pi(P^*) = \gamma(0)$ there exists a curve γ^* on S^* starting at P^* and such that $\pi(\gamma^*) = \gamma$. For smooth unlimited covering surfaces the Monodromy Theorem 3.19 can be generalized in this new setting. What we would like to underline mostly is the close relationship between the fundamental group of S and its smooth unlimited covering surfaces.

Theorem 3.20. *S^* is a smooth unlimited covering surface of S if and only if its fundamental group $\pi_1(S^*)$ is isomorphic to a subgroup of the fundamental group $\pi_1(S)$ of S . In particular every subgroup of $\pi_1(S)$ determines a smooth unlimited covering surface S^* of S with $\pi_1(S^*)$ isomorphic to the given subgroup of $\pi_1(S)$.*

A particularly important case is the one in which the given subgroup of $\pi_1(S)$ is the trivial one: in this case the corresponding smooth unlimited covering surface is called the **universal covering** of S and it will be denoted as \tilde{S} . It is a simply consequence of 3.20 that the universal covering surface is simply connected.

3.3 The Uniformization Theorem

In this section we discuss the famous Uniformization Theorem (first proved by Koebe and Poincaré), which is a generalization to Riemann surfaces of the Riemann Mapping Theorem 2.30.

Theorem 3.21. *Every simply connected Riemann surface is conformally equivalent to one of the following:*

- *The Riemann sphere $\hat{\mathbb{C}}$*
- *The complex plane \mathbb{C}*
- *The unit disk \mathbb{D}*

We notice at first that the above three cases are mutually exclusive, since $\hat{\mathbb{C}}$ is not even topologically equivalent to the other two and, as mentioned already in Section 2.4, \mathbb{C} and \mathbb{D} are not conformally equivalent by Liouville's Theorem 1.11. In other terms,

this theorem shows that there are only three possible universal coverings of a generic Riemann surface.

A classical proof (see [11] and [14]) consists of classifying any simply connected Riemann surface depending on the existence or non-existence of a Green's function on it. This can be done via the Perron method, which is a famous procedure to solve the Dirichlet problem for planar domains and, since it is a local argument, it can be extended naturally to Riemann surfaces. In this way, a simply connected Riemann surface R is said to be:

- elliptic if and only if it is compact
- parabolic if and only if it is not compact and there is no Green's function on it
- hyperbolic if and only if there exists a Green's function on it

Up to this classification, the Riemann sphere corresponds to the elliptic case, the complex plane to the parabolic one and the unit disk to the hyperbolic one.

A main consequence of theorem 3.21 is given by the fact that every simply connected Riemann surface admits a Riemannian metric of constant curvature. In particular, elliptic surfaces have constant curvature $+1$, parabolic surfaces 0 and hyperbolic ones -1 . This follows from the definition of spherical, Euclidean and hyperbolic metrics respectively:

- The spherical metric is $\frac{2}{1+|z|^2}|dz|$
- The Euclidean metric is $|dz|$
- The hyperbolic metric is $\frac{2}{1-|z|^2}|dz|$

Another crucial result is achieved by putting together the Uniformization theorem 3.21 and the topological theory of coverings (see [11],[2] and [14]) that we briefly introduced in Section 3.2.1. The above result concerning metrics of constant curvature on simply connected Riemann surfaces can be thus extended to arbitrary ones, through their universal coverings. The following theorem is a generalization of Uniformization Theorem for arbitrary Riemann surfaces.

Theorem 3.22. *The only Riemann surface whose universal covering is the Riemann sphere is the Riemann sphere itself. The only Riemann surface whose universal covering is \mathbb{C} are the complex plane itself, the punctured complex plane $\mathbb{C} \setminus \{0\}$ and conformal tori. All other Riemann surfaces have \mathbb{D} as universal covering. In particular we can introduce a Riemannian metric of constant curvature which is: positive only for the Riemann sphere $\hat{\mathbb{C}}$, zero for those Riemann surfaces with \mathbb{C} as universal covering and negative for all the others.*

The above result shows that, apart from a few exceptional Riemann surfaces, most Riemann surfaces are hyperbolic, in the sense that their universal covering is the unit disk.

Chapter 4

Brain flattening and surface parameterization

The Uniformization Theorem that we discussed in the previous part states, in particular, that every closed Riemann surface of genus zero is conformally equivalent to the Riemann sphere \mathbb{S}^2 . The aim of this chapter is indeed to construct a conformal mapping from any compact surface of genus zero to the sphere. The approach we will use is based mainly on the work in [3]. The procedure to produce such a mapping is based on the idea of computing a solution of a certain elliptic PDE on the input surface S via the Finite Element Method (FEM). More explicitly, we will work with a triangulated surface and we will compute a mapping from $S \setminus \{p\}$ to \mathbb{C} , by solving two sparse linear systems. Then, via the inverse stereographic projection to the North pole, we will obtain the sought mapping to the sphere.

This type of problem is often related to the one of **surface parameterization** [12], which means finding a one-to-one mapping from a certain surface to a suitable parameter domain (in our case the complex plane). Parameterizations have various applications in many scientific fields, such as for example texture mapping in computer graphics, cartography and medical image processing and analysis. The latter involves the so-called **brain flattening** problem, i.e. how to give flattened representations of the brain cortical surface. Since the human cortex is a highly convoluted surface with many folds (gyri) and fissures (sulci), flattening the cortex may give an easier and more useful way to examine

neural activities (such as fMRI data). Obviously, such a flattening approach produces geometric distortions and both lengths and angles cannot be preserved (otherwise we would have an isometry between a non-constant Gaussian curvature surface and the plane, which is impossible due to Gauss Egregium theorem). Thus, there is a first division in methods which try to preserve lengths and the ones which try to preserve angles. We will be interested in the latter class, whose methods are called conformal and whose mathematical background is more solid.

In Section 4.1 we will define the Laplace-Beltrami operator and derive the elliptic PDE we mentioned before. In Section 4.2 we will briefly review the Finite Element Method and we will examine the so-called Cotan-Laplacian formula (see [24] and [4]), which gives a discrete version of the Laplace-Beltrami operator for triangulated surfaces. In Section 4.3 we will examine further some practical aspects of the construction of the flattening map, in particular how to choose a proper coefficient for the dilation of the flattened surface before applying the inverse stereographic projection, and we will see how solving the discrete laplacian problem leads to a possible way to define the discrete mean curvature ([4] and [20]). In Section 4.4 we will see some experimental results on both the cortical and the white matter surfaces of a human brain and also on a different 3D model of *The Stanford 3D Scanning Repository*. We will show many images of the input meshes and their spherical and flattened versions along with a measure of angle distortion introduced during the flattening procedure.

4.1 The Laplace-Beltrami operator

Given a Riemannian surface is possible to define the so-called **Laplace-Beltrami operator** [5], which is the extension of the usual laplacian operator in \mathbb{R}^n . In this section we will consider the first fundamental form of a given Riemannian surface in local coordinates (x, y) as

$$ds^2 = E(x, y)dx^2 + 2F(x, y)dx dy + G(x, y)dy^2,$$

where the coefficients E, F, G have to satisfy the following conditions (in order to have a positive definite metric):

$$E > 0 \quad \text{and} \quad EG - F^2 > 0.$$

Definition 4.1. Let (S, g) be a Riemannian surface with g denoting the metric and let $\phi \in C^\infty(S)$. The Laplace-Beltrami operator is defined as

$$\begin{aligned} \Delta_S : C^\infty(S) &\longrightarrow C^\infty(S) \\ \phi &\mapsto \operatorname{div}_g \circ \nabla_g(\phi). \end{aligned}$$

In local coordinates (x, y) is given by

$$\Delta_S = \frac{1}{\sqrt{\det g}} \left(\frac{\partial}{\partial x} \left(\frac{G}{\sqrt{\det g}} \frac{\partial}{\partial x} - \frac{F}{\sqrt{\det g}} \frac{\partial}{\partial y} \right) + \frac{\partial}{\partial y} \left(\frac{E}{\sqrt{\det g}} \frac{\partial}{\partial y} - \frac{F}{\sqrt{\det g}} \frac{\partial}{\partial x} \right) \right).$$

4.1.1 Conformal coordinates

We would like now to introduce new local coordinates (u, v) such that the first fundamental form becomes

$$ds^2 = \lambda(u, v)(du^2 + dv^2), \quad \lambda(u, v) > 0. \quad (4.1)$$

In other words, we would like to diagonalize the first fundamental form and get $E = G = \lambda, F = 0$. If we accomplish such task, then we would have a **conformal metric** on the surface S . It is important to notice that (4.1) implies that the new metric is the usual euclidean metric multiplied by a positive factor, which in general depends on the point chosen on the surface. Local coordinates such as (u, v) above are called **conformal (or isothermal) coordinates**. Their existence was first proved by Gauss for the case of real analytic coefficients E, F, G and later by Korn and Lichtenstein for coefficients in the Hölder class of functions. The proof given by Gauss may be found in [2] and [22]. In other terms the result obtained by Gauss can be stated as

Theorem 4.2. *Every real-analytic conformal metric on an orientable surface induces a conformal structure.*

Given a Riemannian surface (S, g) we choose local conformal coordinates (u, v) and, as a direct consequence of the existence of conformal coordinates, we may give a much easier formulation of the Laplace-Beltrami operator:

$$\Delta_S = \frac{1}{\lambda(u, v)} \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right). \quad (4.2)$$

4.1.2 A particularly important elliptic PDE

We consider a Riemannian surface S and we choose conformal coordinates (u, v) near a point p such that $u = v = 0$ in p . Up to scaling if necessary, we may always have $\lambda(p) = 1$. We are now ready to consider the main equation of this chapter, which is an elliptic PDE defined on S :

$$\Delta z = \left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right) \delta_p, \quad (4.3)$$

where δ_p is the Dirac delta impulse at p . Since we chose conformal coordinates, Δ_S reduces to the much easier formula (4.2). In the rest of this paragraph we are concerned with showing that a solution of (4.3) exists and most importantly how such a solution yields the desired conformal mapping from $S \setminus \{p\}$ to the complex plane \mathbb{C} .

For the existence part it suffices (see [25]) to find out whether the integral on S of the right hand side vanishes. But this actually happens, since

$$\iint_S \left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right) \delta_p dS \stackrel{(\star)}{=} - \left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right) (1) \Big|_p = 0,$$

where (\star) comes from integration by parts.

At this point we deal with deriving (4.3) by means of the properties we would like z to have. We are seeking a conformal mapping z , thus it must be in particular injective. We then define $w = u + iv$ and this implies that

$$z(w) = \frac{a_{-1}}{w} + \sum_{n=0}^{\infty} a_n w^n, \quad (4.4)$$

i.e. the Laurent expansion of z must have a simple pole in p . This comes from the fact that there must be a singularity at p in order to obtain a Dirac delta centered at p and this singularity cannot be neither essential due to the Casorati-Weierstrass Theorem

neither removable. Then we may apply the usual laplacian operator to both sides of (4.4) and, since most of the terms of the right hand side are harmonic, we obtain

$$\begin{aligned}
\Delta z &= a_1 \Delta \left(\frac{1}{w} \right) \\
&= a_1 \Delta \left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right) \log |w| \\
&= a_1 \left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right) \Delta \log |w| \\
&\stackrel{(*)}{=} a_1 \left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right) (2\pi \delta_p(w)),
\end{aligned} \tag{4.5}$$

where $(*)$ comes from the crucial fact that $\frac{1}{2\pi} \log |w|$ is the fundamental solution of the laplacian in \mathbb{R}^2 . It is important to highlight at this point that the solution z is not unique: every other z_1 of the form $z_1 = \gamma_1 + \gamma_2 z$ with $\gamma_1, \gamma_2 \in \mathbb{C}$ is still a solution of (4.3). In other words, the solution is unique up to a conformal self-map of the complex plane. This fact will be decisive in the implementative construction of z . The above considerations lead us to choose $a_1 = 1/2\pi$ and this implies that (4.5) turns out to be

$$\left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right) \delta_p,$$

as we wanted.

4.2 The FEM on a triangulated compact surface

We consider now how to solve (4.3) via the Finite Element Method (FEM) (see [10] and [16] for more details). FEM is based on a weak formulation of the PDE we are willing to solve, on the choice of a finite vector space of functions which will approximate the solution and on the discretization of the domain. From now on we consider S as a triangulated surface with N_v vertices $\{v_i\}$ and N_f faces $\{f_i\}$. We choose the finite vector space of piecewise linear functions $PL(S)$, whose base is given by the *hat functions*:

$$\phi_i(v_j) = \delta_{ij} \tag{4.6}$$

ϕ linear on each triangle

The weak formulation of (4.3) is given by

$$\iint_S \nabla z \cdot \nabla f \, dS = - \iint_S f \left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right) \delta_p \, dS \tag{4.7}$$

where $z, f \in PL(S)$. We look for z to solve (4.7) for every $f \in PL(S)$. Since (4.7) is linear in f , it suffices that it holds whenever $f = \phi_j$ for every $j = 1, \dots, N_v$. Further we may represent z in terms of the basis hat functions as $z = (z_1, \dots, z_{N_v})$ and thus (4.7) becomes

$$\sum_{i=1}^{N_v} z_i \iint_S \nabla \phi_i \cdot \nabla \phi_j dS = - \iint_S \phi_j \left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right) \delta_p dS \quad (4.8)$$

4.2.1 The Cotan-Laplace formula

We introduce at this point the key ingredient for a discrete version of the Laplace-Beltrami operator. From the previous considerations we have transformed Δz into a linear combination of surface integrals of the dot product of the gradients of the basis hat functions. We thus define the matrix $D = (D_{i,j})$ by

$$D_{i,j} = \iint_S \nabla \phi_i \cdot \nabla \phi_j dS \quad (4.9)$$

In FEM theory the matrix which represents the discrete version of the operator of the PDE is usually called *stiffness matrix*. We notice from (4.6) and (4.9) that the actual support of the surface integral on the right hand-side of (4.9) is not the entire surface S , but it is only the intersection of the 1-ring of vertex v_i (i.e. all triangles whose edges contain v_i) and the 1-ring of vertex v_j . Thus, $D_{i,j}$ is non-zero if and only if v_i and v_j are connected by an edge in the triangulation of S . Further, this implies that D is symmetric and sparse.

Instead of using some quadrature formulae to compute (4.9), we adopt the **Cotan-Laplace** formula (see [24]):

$$D_{i,j} = -\frac{1}{2}(\cot \alpha + \cot \beta), \quad i \neq j \quad (4.10)$$

where α and β are the opposite angles of the two triangular faces which share the edge connecting v_i and v_j .

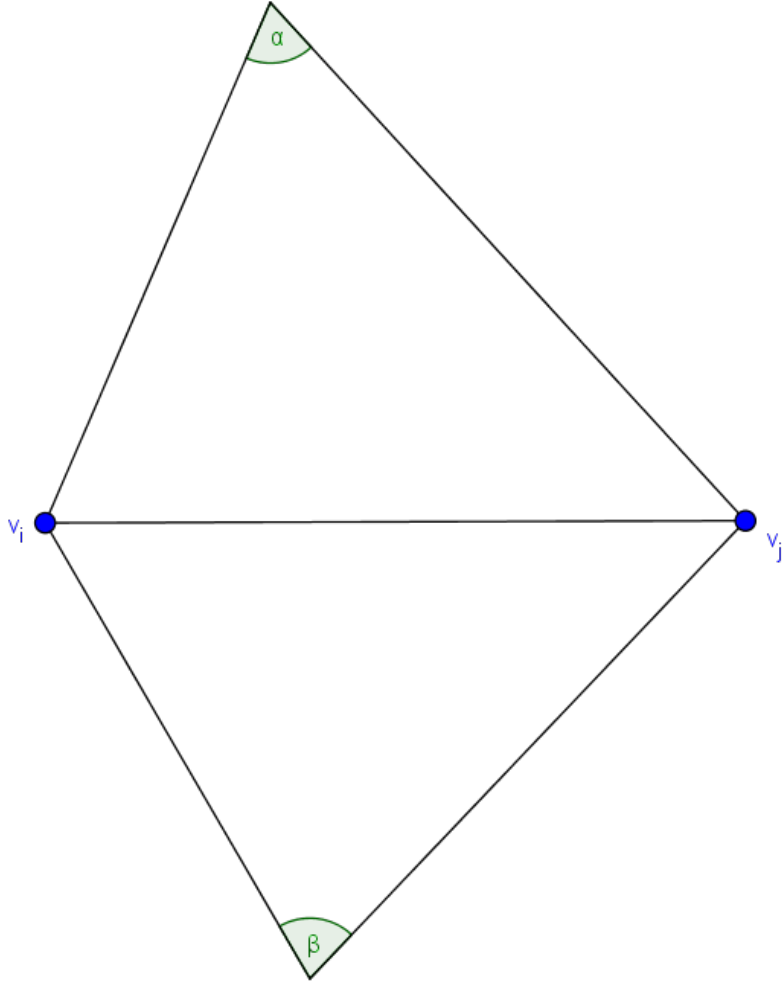


Figure 4.1: Angles α and β used in (4.10)

The diagonal elements of D can be computed by

$$D_{i,i} = - \sum_{j=1, j \neq i}^{N_v} D_{i,j}, \quad (4.11)$$

since

$$\sum_{j=1}^{N_v} D_{i,j} = \sum_{j=1}^{N_v} \iint_S \nabla \phi_i \cdot \nabla \phi_j dS = \iint_S \nabla \phi_i \cdot \nabla 1 dS = 0.$$

4.2.2 The board vectors

So far we managed to give a discrete formulation of left-hand side of (4.8). Now we would like to deal with its right-hand side. We denote with $\{A, B, C\}$ the vertices of the face in whose interior lies the point p

We consider then a generic basis hat function $\phi_i \in PL(S)$ and by an integration by parts and by the behaviour of Dirac delta, we have

$$\iint_S \phi_i \left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right) \delta_p dS = - \left(\frac{\partial \phi_i}{\partial u} - i \frac{\partial \phi_i}{\partial v} \right) \Big|_p \quad (4.12)$$

Since ϕ_i is piecewise linear, (4.12) is completely determined by the values of ϕ_i on the three vertices A, B, C . This implies immediately that the right-hand side of (4.12) is non-zero only for three basis hat functions, explicitly those who equal 1 in A, B or C . With a little abuse of notation, we denote them ϕ_A, ϕ_B, ϕ_C . Choosing the u, v axis as in Figure 4.2, those derivatives may be computed by

$$\begin{aligned} \frac{\partial \phi_A}{\partial u} &= -\frac{1}{\|B - A\|}, & \frac{\partial \phi_A}{\partial v} &= \frac{\theta - 1}{\|C - D\|} \\ \frac{\partial \phi_B}{\partial u} &= \frac{1}{\|B - A\|}, & \frac{\partial \phi_B}{\partial v} &= \frac{-\theta}{\|C - D\|} \\ \frac{\partial \phi_C}{\partial u} &= 0, & \frac{\partial \phi_C}{\partial v} &= \frac{1}{\|C - D\|} \end{aligned} \quad (4.13)$$

where

$$\begin{aligned} \theta &= \frac{(C - A) \cdot (B - A)}{\|B - A\|^2} \\ D &= A + \theta(B - A) \end{aligned}$$

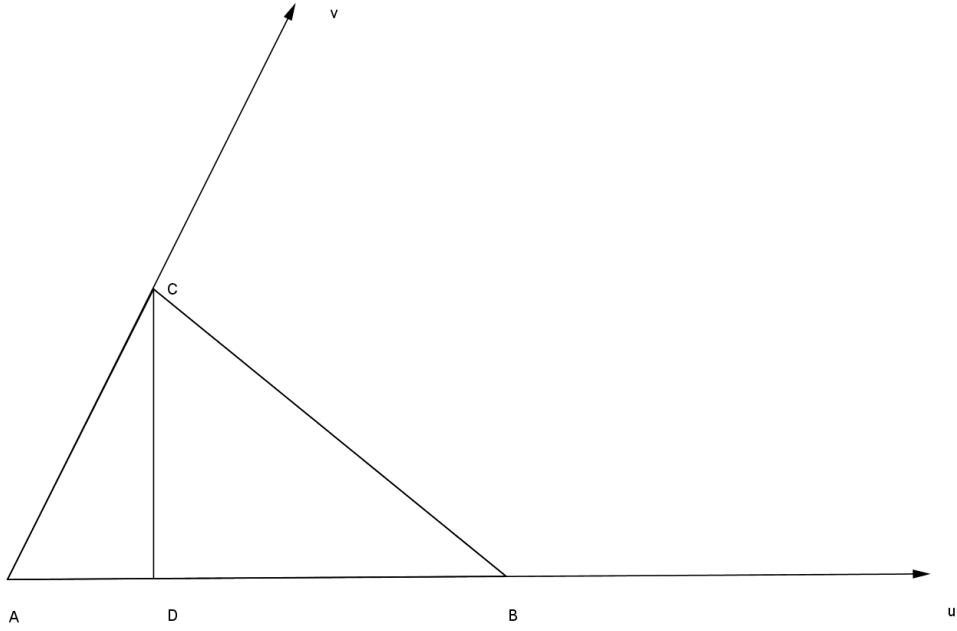


Figure 4.2: Triangle $\{A, B, C\}$ with axis u and v .

4.2.3 Solving two sparse linear systems

Putting together (4.9),(4.12) and (4.13) into (4.8) we get a reformulation in matrix terms of the main elliptic PDE (4.3):

$$Dz = w \tag{4.14}$$

where $w = a - ib$ is given by

$$a_i = \begin{cases} 0 & \text{if } i \notin \{A, B\} \\ \frac{-1}{\|B-A\|} & \text{if } i = A \\ \frac{1}{\|B-A\|} & \text{if } i = B \end{cases} \tag{4.15}$$

and

$$b_i = \begin{cases} 0 & \text{if } i \notin \{A, B, C\} \\ \frac{-1+\theta}{\|C-D\|} & \text{if } i = A \\ \frac{-\theta}{\|C-D\|} & \text{if } i = B \\ \frac{1}{\|C-D\|} & \text{if } i = C \end{cases} \tag{4.16}$$

Using $z = x + iy$, we need to solve the two linear systems

$$Dx = a \tag{4.17}$$

$$Dy = -b \tag{4.18}$$

Therefore, we managed to transform the continuous elliptic PDE (4.3) into the two sparse linear systems (4.17) and (4.18). We recall that D is a $N_v \times N_v$ real, symmetric, sparse matrix and by construction D is also diagonally dominant with real positive diagonal elements, thus it is positive semidefinite. Further we notice that D is singular, since the sum of the elements of every row is zero. Thus it is not obvious that (4.17) and (4.18) are solvable. However it is shown in [3] that the kernel of D is given by constant vectors only, therefore (4.17) and (4.18) are solvable if and only if the sum of the elements of a and b respectively are zero, which is indeed the case by (4.15) and (4.16). Further, the solutions are unique up to addition of a constant vector to a and b .

In view of the properties of D the solutions x and y of (4.17) and (4.18) may be computed efficiently using the Cholesky factorization of D , which gives $D = U^t U$, with U an upper triangular matrix. In this way x and y can be found solving the two following systems

$$\begin{cases} U^t \xi &= a \\ Ux &= \xi \end{cases} \quad \text{and} \quad \begin{cases} U^t \zeta &= -b \\ Uy &= \zeta \end{cases} \tag{4.19}$$

4.3 Construction of the flattening map in Matlab

In this paragraph we take in exam the implementative aspects of the construction of a flattening map from $S \setminus \{p\}$ to \mathbb{C} . The first practical problem is which triangle $\{A, B, C, \}$ that contains p has to be taken on the triangulated surface S . Since the inverse stereographic projection will produce some distortion from an angle perspective when computed numerically, we choose p to be *further* away from the area we are most interested to see inflated. In addition, we try to pick a regular triangle in order to minimize the angular distortion produced during the inflation process (we will relate to this aspect better in 4.4.1).

Then we need to compute the stiffness matrix D in an efficient way. The crucial point is how to construct a sparse matrix in Matlab in order to have little computational

times. For this purpose, we were helped by [15]. The algorithm we created to build D manages to deal with very refined meshes, whose vertices may be hundreds of thousands, in few seconds.

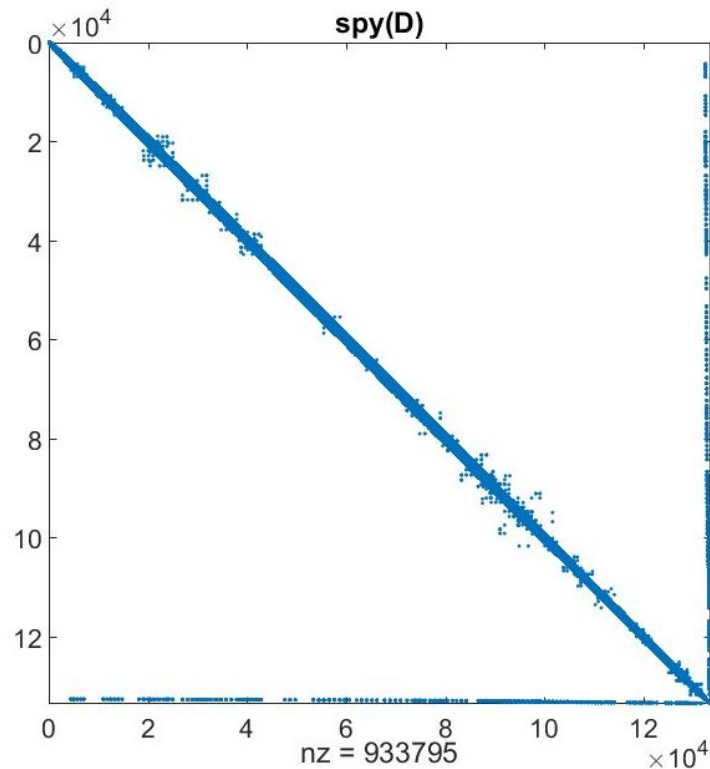


Figure 4.3: Sparsity pattern for the left hemisphere of a human brain with 133401 vertices

Computing the board vectors, instead, is much less complicated, since the only triangular face that contributes to non-zero elements of a and b is $\{A, B, C\}$.

The two sparse linear systems (4.17) and (4.18) are solved via (4.19) using the Matlab function *chol*, which deals even with sparse matrices.

4.3.1 Choice of the dilation coefficient

In the paragraph 4.1.2 we mentioned that the conformal mapping z that we find via FEM on the input triangulated surface is unique, but only up to a conformal-self map of \mathbb{C} , i.e. up to translating and scaling. This facet is extremely important if we would like to have a good output on the sphere. As a matter of fact, when we find $x + iy$, we

usually have a result as in Figure 4.4, in which the *big triangle* is the image via z of the triangle $\{A, B, C\}$ on S .

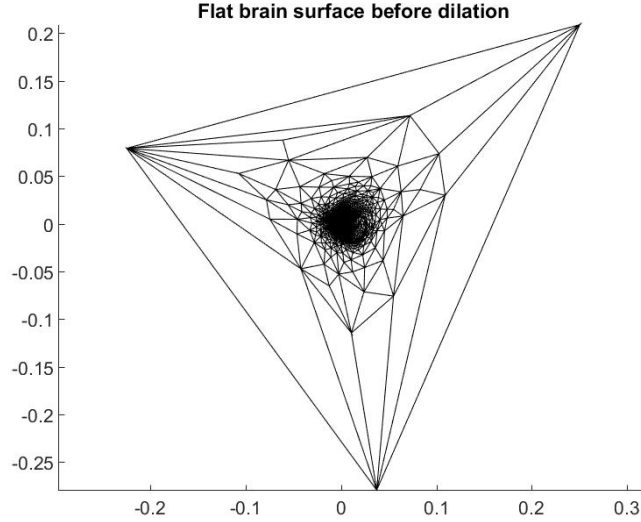


Figure 4.4: Flattened left hemisphere before dilation

We notice that the big triangle is inside the unit circle and thus, after the inverse stereographic projection, all the triangles will accumulate around the South pole, without covering the whole sphere. Therefore, it is necessary to dilate $x + iy$ in a proper way such that the triangles cover uniformly the resulting triangulated sphere.

We call T the big triangle and t the smallest triangle (i.e. the one which contains the origin) obtained after flattening S . Then we denote with R and r the radii of their respective circumscribed circumferences. By Proposition 2.10 we know that the stereographic projection maps circles into circles and a straightforward computation shows that the inverse stereographic projection given by (2.3) sends a circle centered at the origin of radius γ into a circle of radius $\frac{2\gamma}{\gamma^2 + 1}$. We need the circumscribed circumferences of T and t to be mapped by P_N^{-1} into circles of the same radius, in order to have uniformity of triangles on the sphere. In formulae we have

$$\frac{2R}{R^2 + 1} = \frac{2r}{r^2 + 1}. \quad (4.20)$$

If we denote $l := R/r$, then (4.20) becomes

$$l = \frac{R^2 + 1}{r^2 + 1} \quad (4.21)$$

Since l is invariant under dilation, the optimal coefficient of dilation may be found by solving the following equation for k

$$l = \frac{(Rk)^2 + 1}{(rk)^2 + 1},$$

which actually leads to

$$k = \frac{1}{\sqrt{rR}}.$$

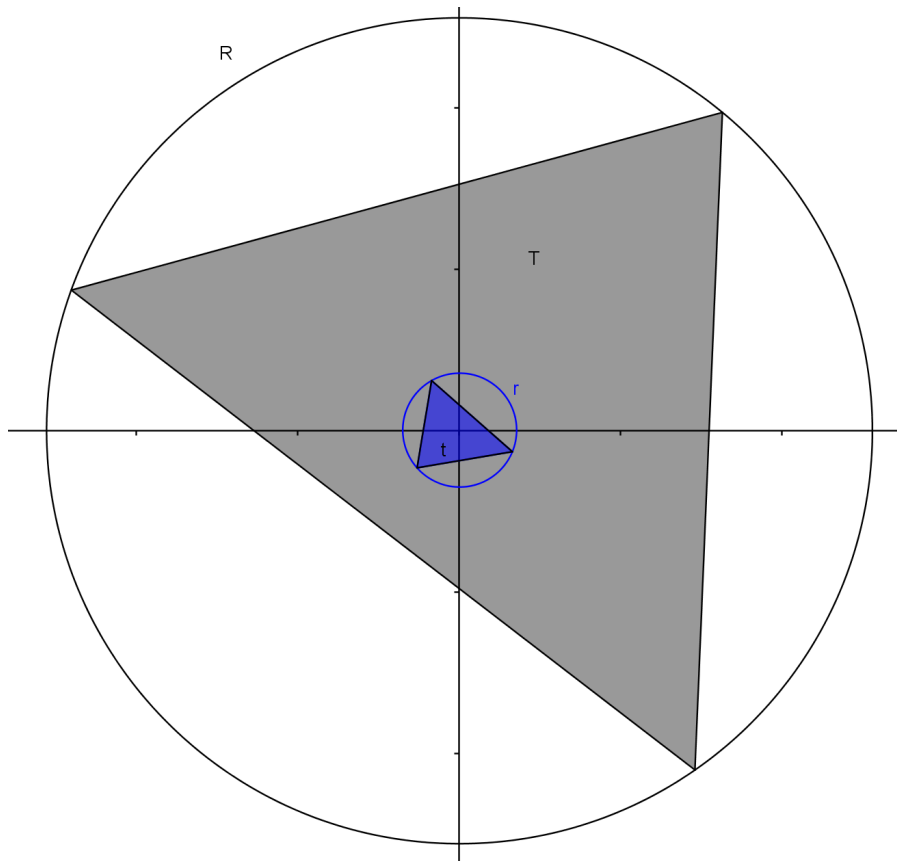


Figure 4.5: The triangles T and t with their respective circumscribed circumferences.

Definition 4.3. We define the optimal coefficient of dilation as

$$k = \frac{1}{\sqrt{rR}} \quad (4.22)$$

During implementation of (4.22), we need to recall that the respective radii of triangles T and t with edges $\{F, G, H\}$ and $\{f, g, h\}$ may be computed through the following classical Euclidean geometry formula:

$$R = \frac{FGH}{4\text{Area}(T)}, \quad r = \frac{fgh}{4\text{Area}(t)}. \quad (4.23)$$

4.3.2 Discrete mean curvature

The discrete version of the Laplace-Beltrami operator leads us to the definition of a discrete measure of mean curvature at a vertex of the given triangulation. This can be done, exploiting the relation between the continuous Laplace-Beltrami operator and the mean curvature at a point x on a surface S embedded in the euclidean three-dimensional space (see [28]) given by

$$\Delta r(x) = -2HN(x),$$

where H is the mean curvature at x , $N(x)$ is the unit outward normal vector at x and r is the restriction of the identity map to S in \mathbb{R}^3 . In an analogous discrete way, we define the **discrete mean curvature vector at vertex** v_i as

$$\vec{H}(v_i) = -\frac{1}{2} \sum_{j=1}^{N_v} D_{i,j} v_j \quad (4.24)$$

It is important to underline that (4.24) can be easily computed by

$$\vec{H} = -\frac{1}{2} D(v_1, \dots, v_{N_v})^t. \quad (4.25)$$

Once we have obtained (4.25), it suffices to calculate its norm as a row vector for any vertex and this computation, after having decided the sign of the computed norm according to the direction of the normal vector at each vertex, gives the **mean scalar curvature** \mathbf{H} at every vertex.

We may summarize the algorithm to construct the desired flattening map as:

Table 4.1: Flattening algorithm

Input	Triangulate surface S
1.	Choose a triangular face $\{A, B, C\}$ in whose interior lies the point p
2.	Compute D, a and b using (4.10),(4.11),(4.15) and (4.16)
3.	Solve (4.19) to obtain x and y
4.	Compute the optimal dilation coefficient k with (4.22)
5.	Compute the inverse stereographic projection P_N^{-1} using (2.3)
6.	Calculate the mean scalar curvature H at each vertex
Output	Flattened mesh surface on the plane and on the sphere. Input mesh coloured in base of its discrete mean curvature. Spherical mesh coloured like the input one

4.4 Experimental results

In this section we take in exam experimental results relative to the flattening and inflating procedure discussed in the previous paragraphs. We have used three main models to test our algorithm:

1. The cortical surface mesh with 133401 vertices and 266798 faces of a left hemisphere of a human brain.
2. The white matter surface mesh with 133299 vertices and 266594 faces of a right hemisphere of a human brain.
3. A 3D model mesh of an armadillo with 50002 vertices and 100000 faces

Triangular meshes 1. and 2. come from the open source software *FreeSurfer*, which is suite for processing and analyzing MRI images of the brain; the model of the armadillo, instead, belongs to *The Stanford 3D Scanning Repository*.

In addition to the discrete mean scalar curvature, we used another algorithm to compute discrete mean curvature, which is described in [7] and implemented in Matlab inside the *Toolbox Graph* by Gabriel Peyre. This different method performs a smoothing

of the mesh and thus the computed mean curvature is less local than the one we discussed in 4.3.2. We will refer to it as C_{mean} .

In Figure 4.6 we see the cortical surface of a left hemisphere of a human brain coloured with C_{mean} and its spherical parameterization obtained via our algorithm 4.1. We can notice that C_{mean} outlines very well the gyri and the sulci of the cortical surface and those may be seen as well on the inflated surface on the sphere.

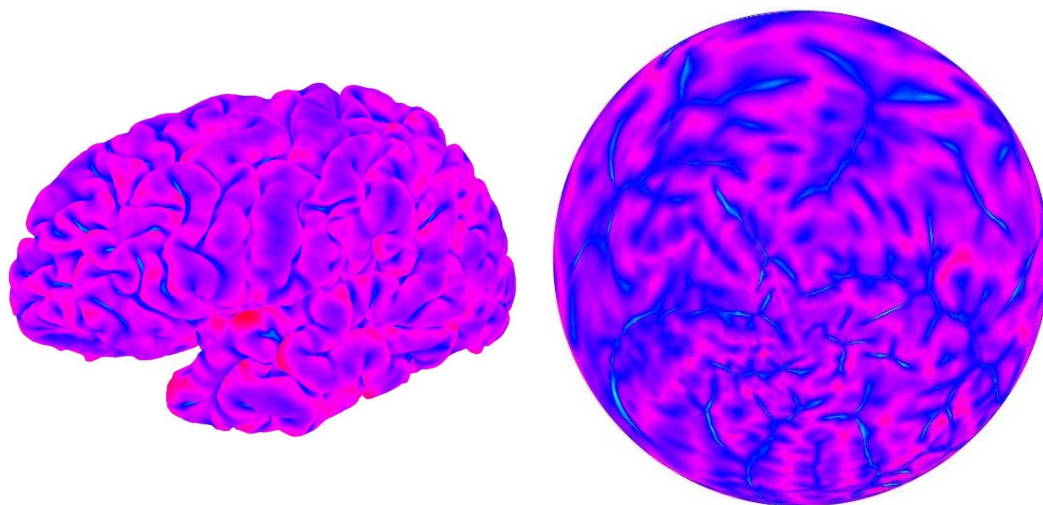


Figure 4.6: Cortical surface and its spherical representation with C_{mean} .

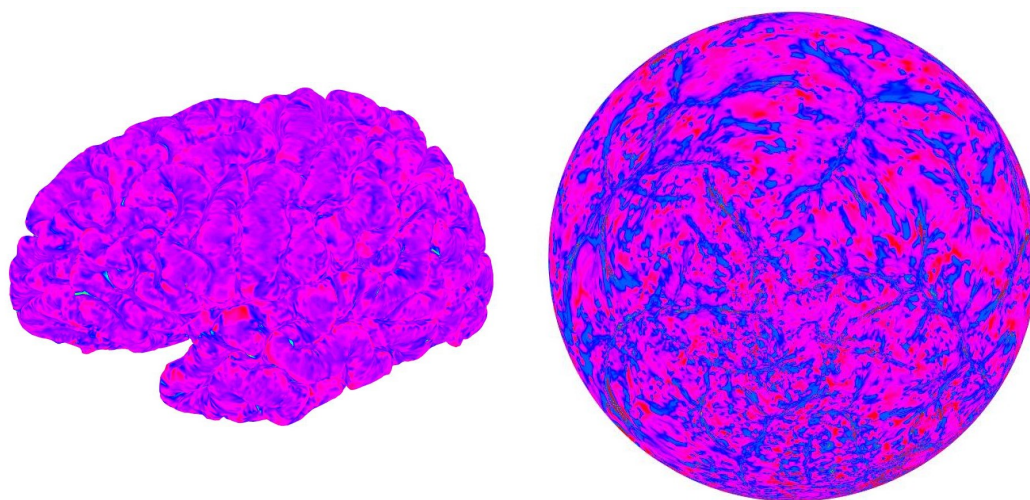


Figure 4.7: Cortical surface and its spherical representation with discrete mean curvature H .

In Figure 4.7 we see the same triangulated surfaces coloured, instead, using the discrete mean scalar curvature H . If we compare those new images with the two above, it is clear that our result is more difficult to understand from a graphical point of view. Nonetheless, H shows the typical highly convoluted aspect of a cortical surface.

We made the same type of analysis with the white matter surface of a right hemisphere of a human brain.

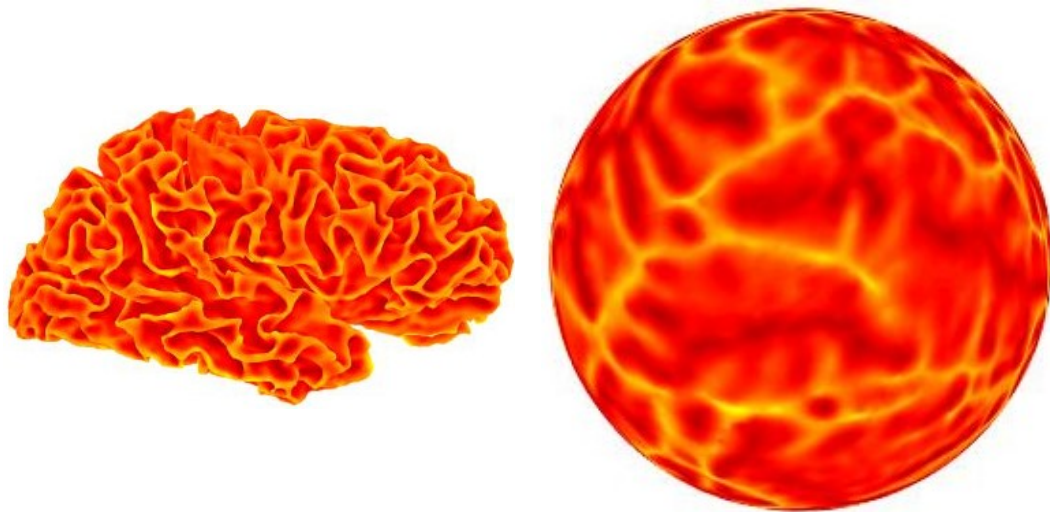


Figure 4.8: White matter surface and its spherical representation with C_{mean}

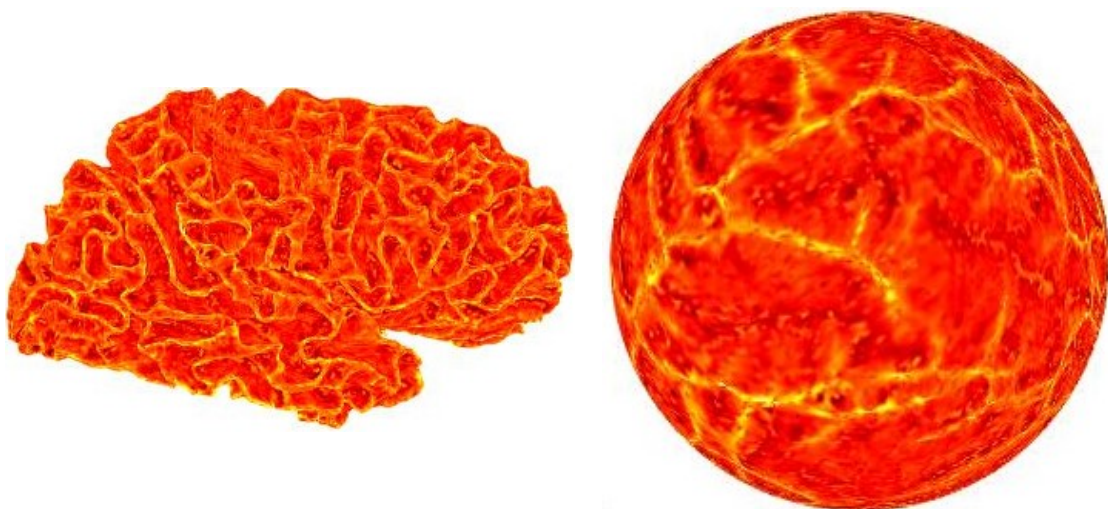


Figure 4.9: White matter surface and its spherical representation with discrete mean curvature H

In this case we notice a much better graphical result of H , as we see that both spherical representations display very similar patterns.

However, in the previous spherical images it is still difficult to recognize immediately their corresponding input surfaces due to the geometric nature of the cortical and white matter surfaces. The case of the armadillo from *The Stanford 3D Scanning Repository* is more instructive from this point of view, since we may identify the original armadillo in its inflated spherical representation. In Figure 4.10, for example, we are able to recognize the armadillo's abdomen.

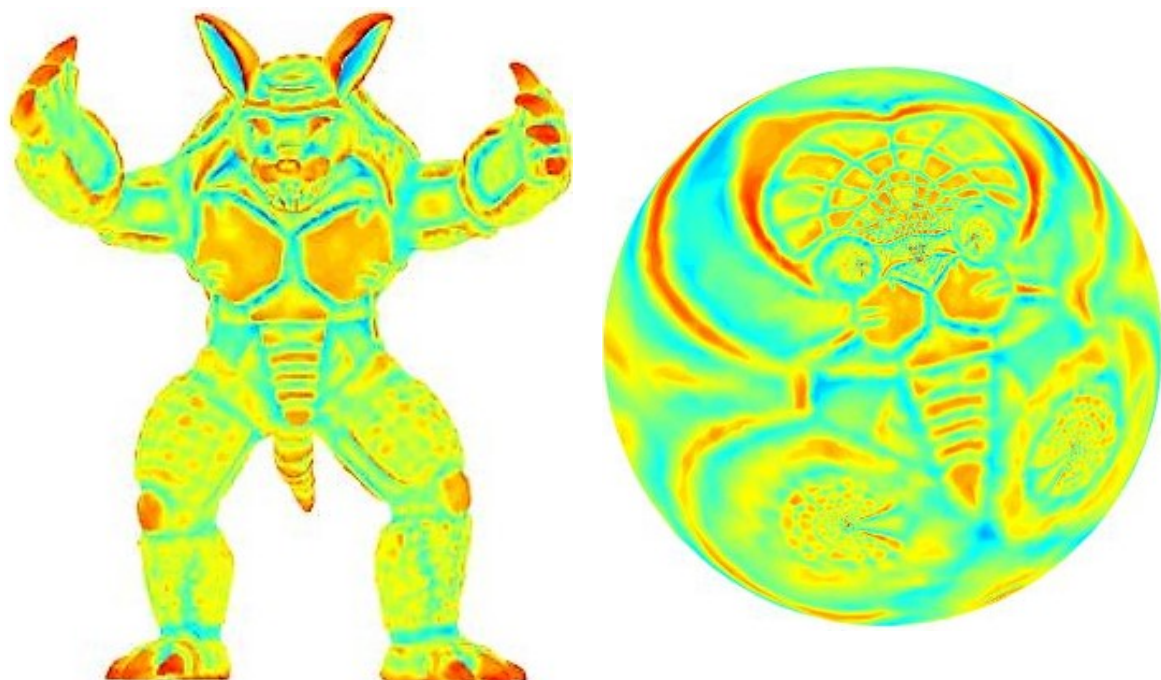


Figure 4.10: Armadillo and its spherical representation with Cmean

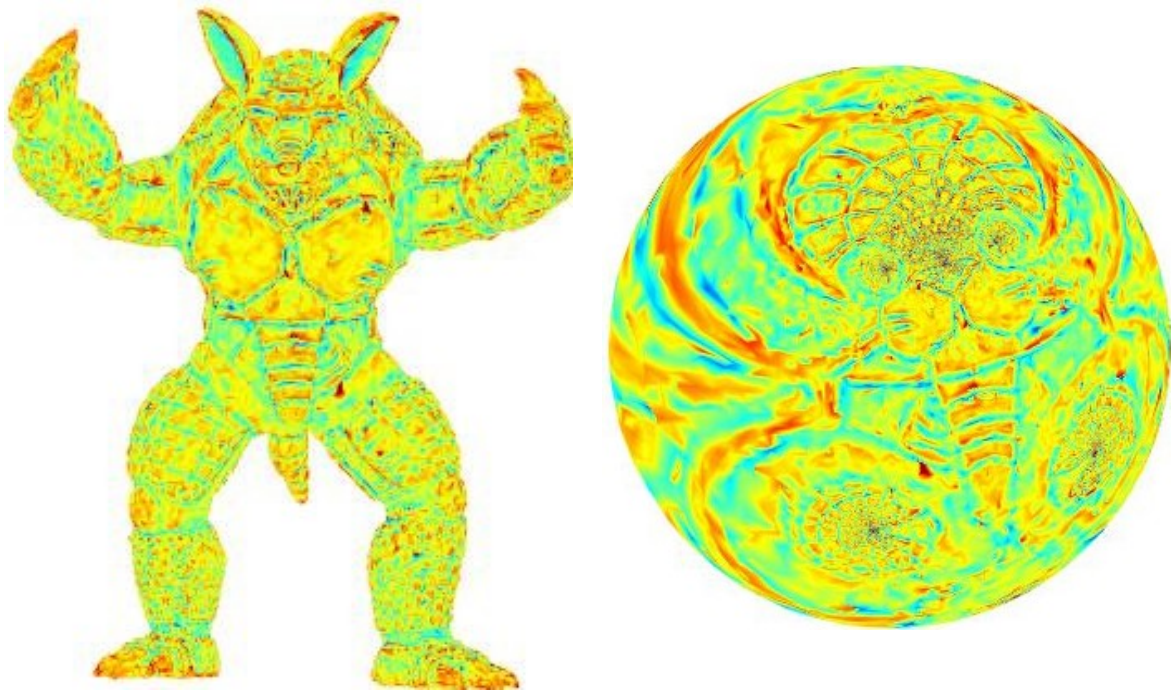


Figure 4.11: Armadillo and its spherical representation with discrete mean curvature H

The next figures shows the armadillo seen from its back.

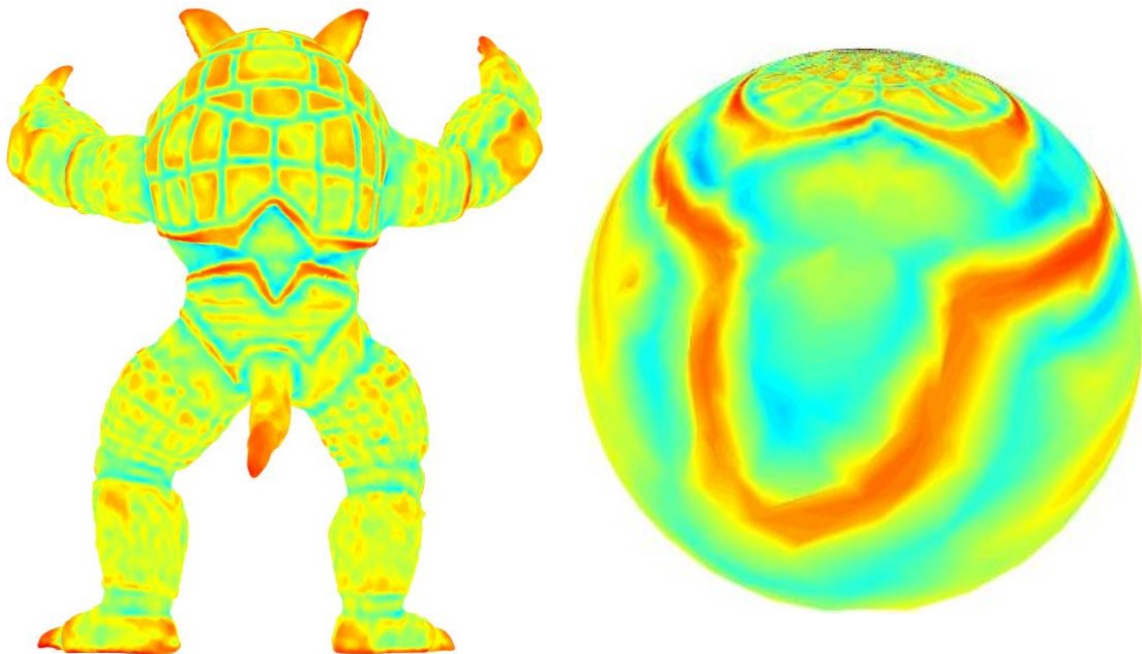


Figure 4.12: Armadillo and its spherical representation from the back with C_{mean}

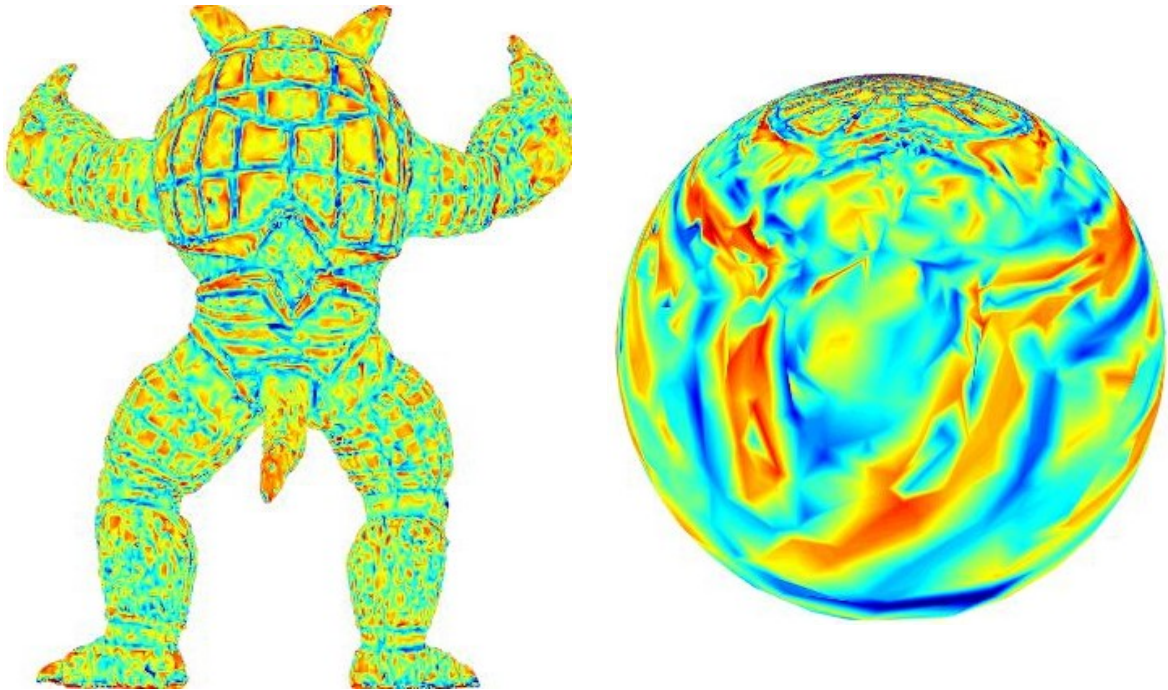


Figure 4.13: Armadillo and its spherical representation from the back with discrete mean curvature H

The point p was chosen on its back, between the two red opposite triangles in Figure 4.12.

4.4.1 How to measure the conformality of the flattening map

The discrete process we used to construct the flattening map from S to the sphere \mathcal{S}^2 , compared to the continuous case, suffers surely from numerical errors during the solution of the two sparse linear systems and from the geometrical approximations introduced by the triangulation of a smooth surface. Thus, the resulting conformality of the flattening map cannot be fully respected in the discrete setting. Therefore, it is necessary to introduce a measure of how angles are distorted during this procedure, in order to understand the quality of our flattening algorithm from a conformal position. We may measure this angular distortion A_d as (see [6]):

$$A_d := \frac{|\alpha - \alpha'| + |\beta - \beta'| + |\gamma - \gamma'|}{2\pi}, \quad (4.26)$$

where α, β, γ denote the three internal angles of a generic triangle T in S and α', β', γ' are the corresponding angles of T after the flattening map and the inverse stereographic projection. In view of (4.26), A_d is a coefficient whose range is $[0, 1]$. We expect that the major distortion is introduced near the point p , due to the nature of the inverse stereographic projection. This motivates the choice of the point p to be *as further as possible* from the region of interest on S . Our expectations are confirmed experimentally by the following images.

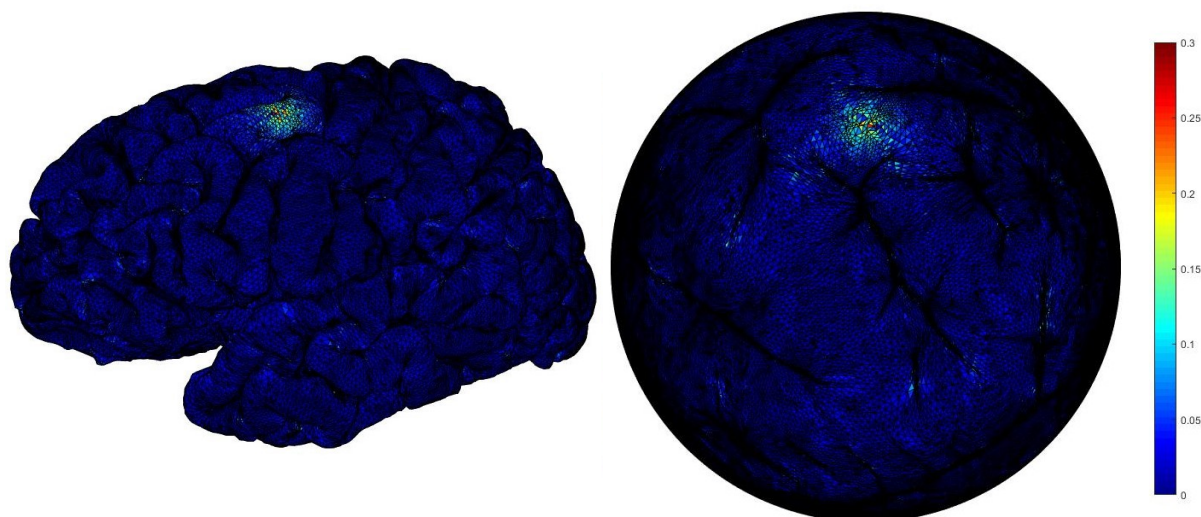


Figure 4.14: Left hemisphere cortex and its spherical representation with angular distortion A_d

Figure 4.14 clearly points out where the point p has been chosen on the cortical surface. Further, we notice how in general the other areas have a deep blue colour, which indicates that there was very little conformal distortion during the flattening procedure. Same considerations hold for the spherical representation in Figure 4.14, in which the North pole shows the major angular distortion as expected. Figure 4.15 explicates further the above reasonings.

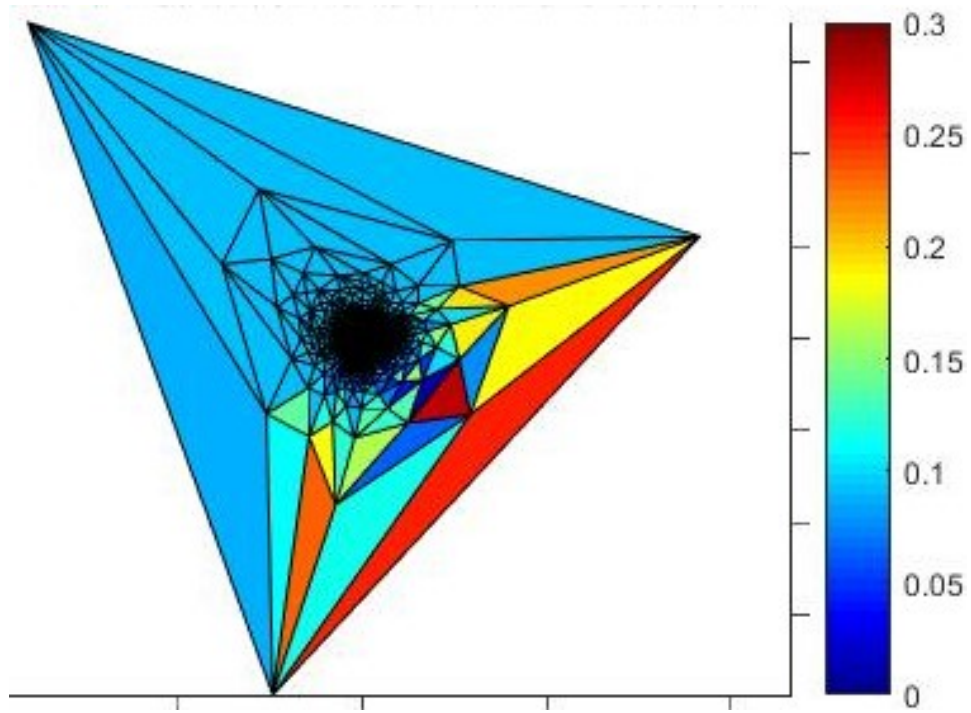


Figure 4.15: The flattened brain surface: the area near the *big triangle* shows a great angle distortion

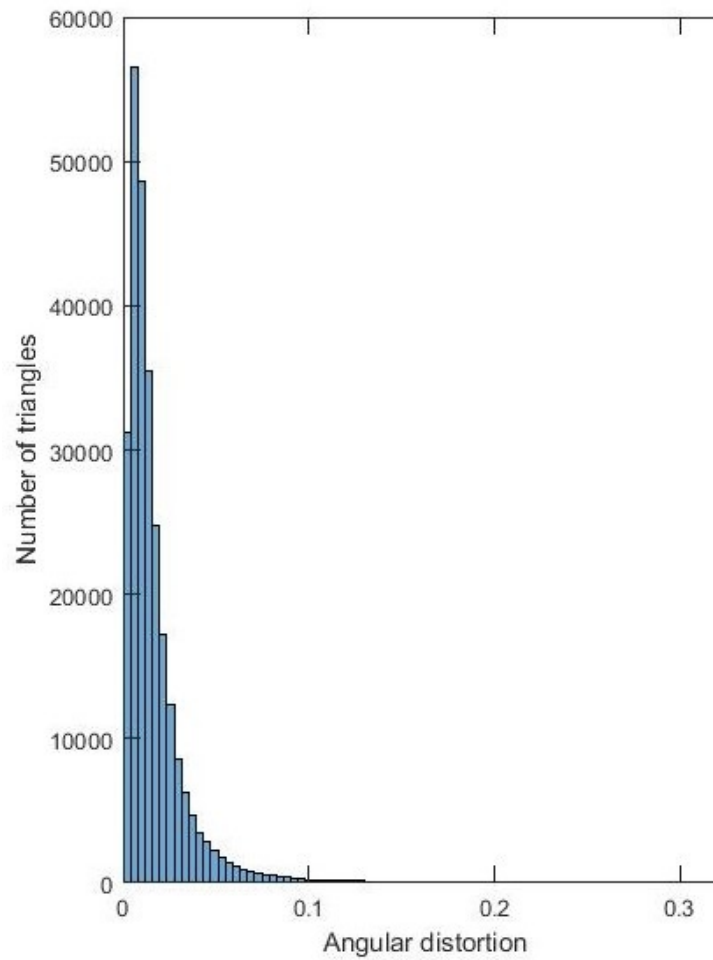


Figure 4.16: Histogram which shows the number of triangles counted in base of angular distortion for the cortical surface

The effective conformality of the flattening map may be understood and visualized better by the histogram in Figure 4.16.

We tested the conformality of the flattening map also on the white matter surface and we obtained similar results as for the cortical surface.

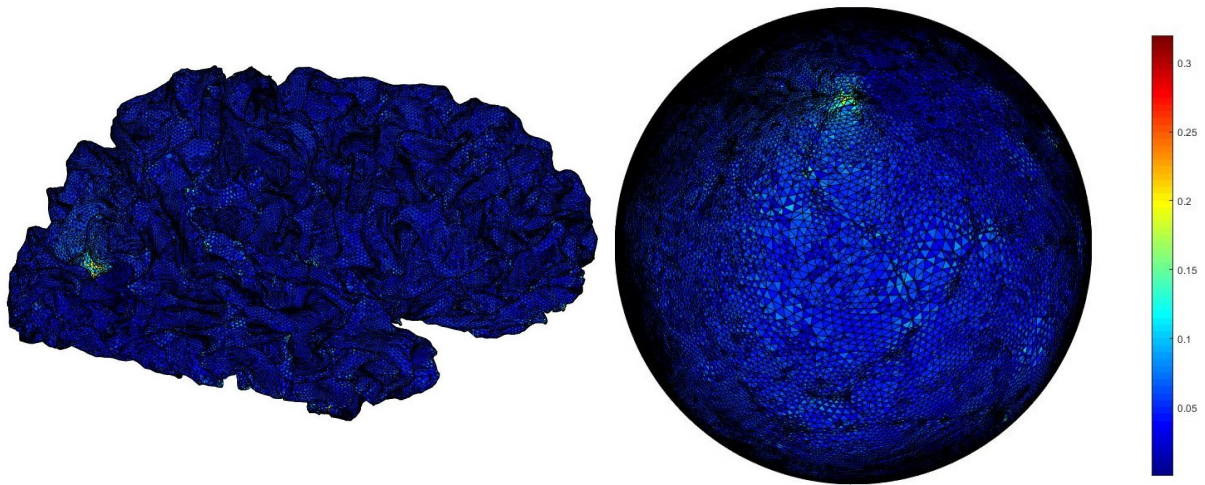


Figure 4.17: Right hemisphere white matter and its spherical representation with angular distortion A_d

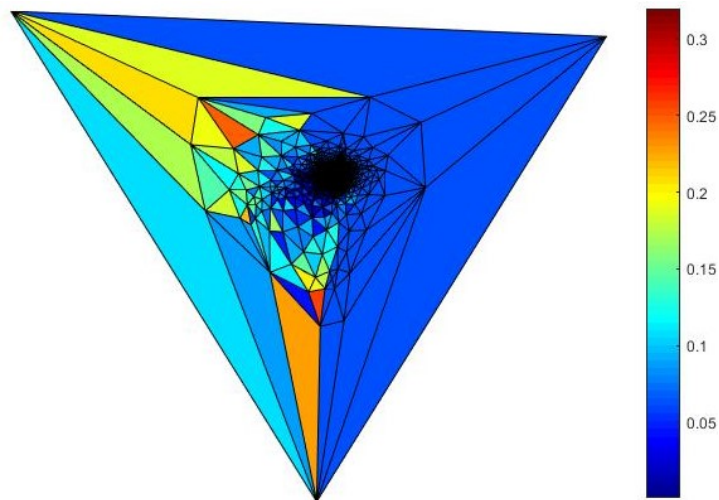


Figure 4.18: The flattened white matter surface: the area near the *big triangle* shows a great angle distortion

Compared to Figure 4.16 where the majority of the triangles had nearly zero angular distortion, Figure 4.19 shows a slightly different behaviour, since the peak of the histogram is shifted a little to a greater angular distortion. This reflects the fact that the white matter surface is a much more convoluted surface than the cortical one.

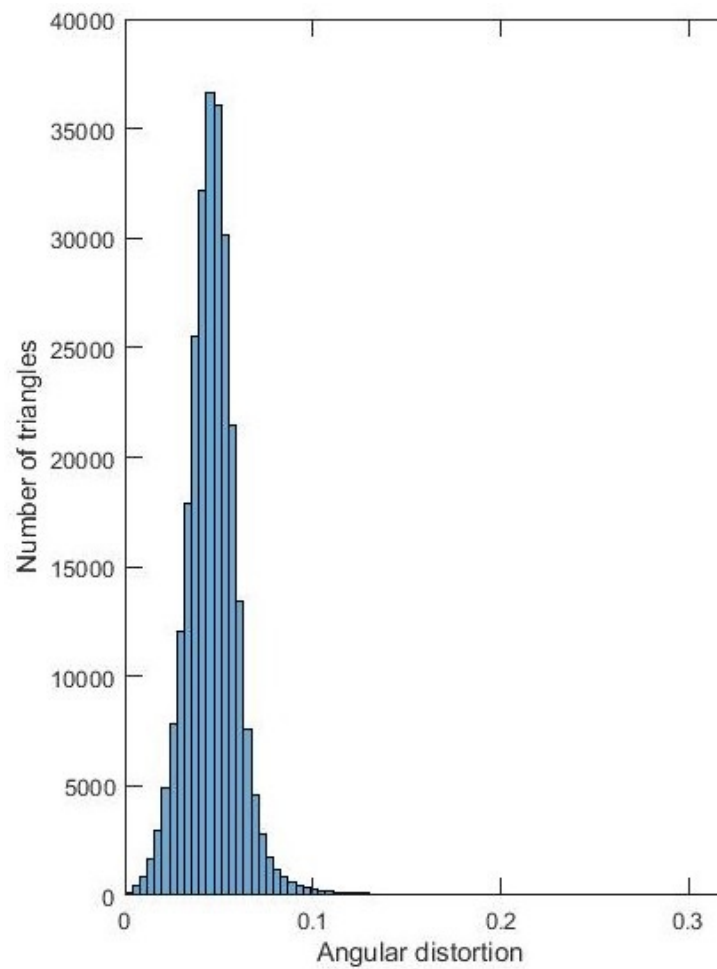


Figure 4.19: Histogram which shows the number of triangles counted in base of angular distortion for the white matter surface

As in Section 4.4, the armadillo provides instructive results from a basic graphical point of view.

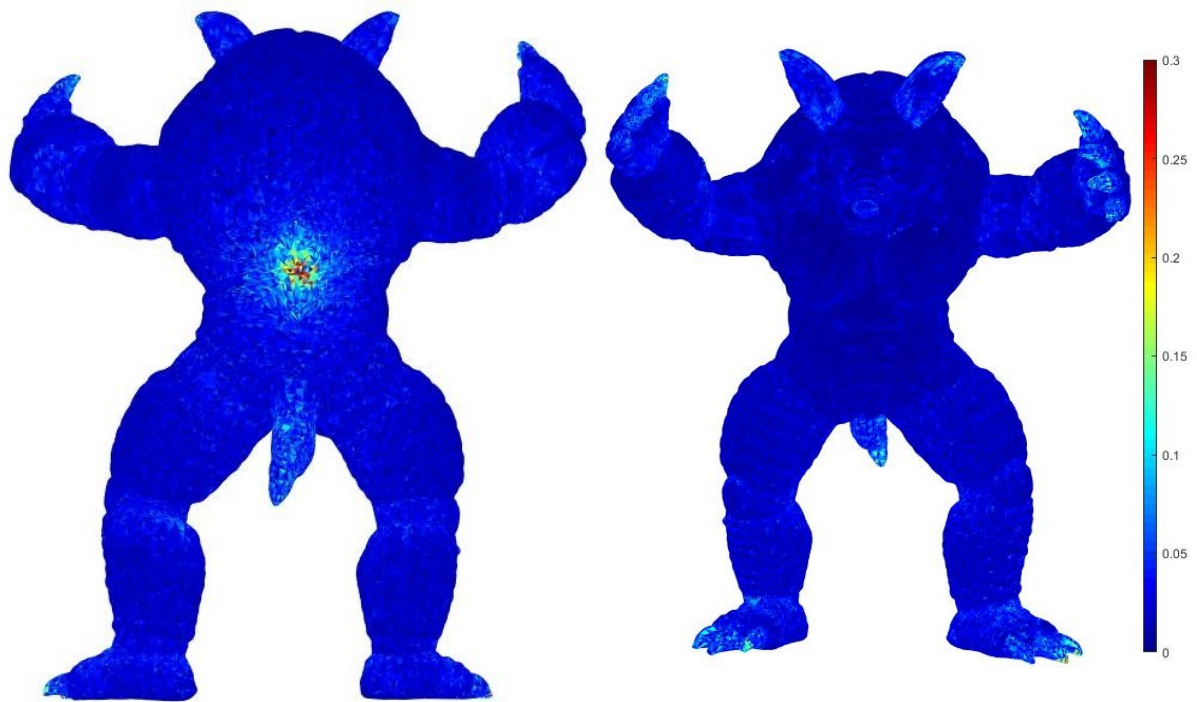


Figure 4.20: Back and front views of the armadillo with angular distortion: the point p is pointed out through its higher A_d .

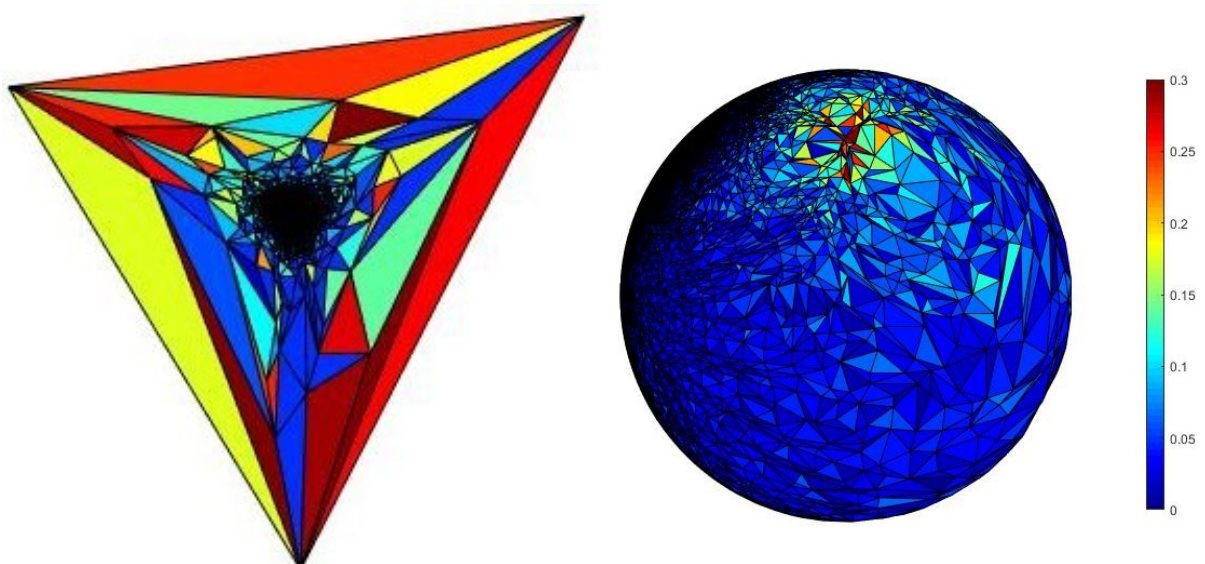


Figure 4.21: Flattened and spherical armadillo with angular distortion.

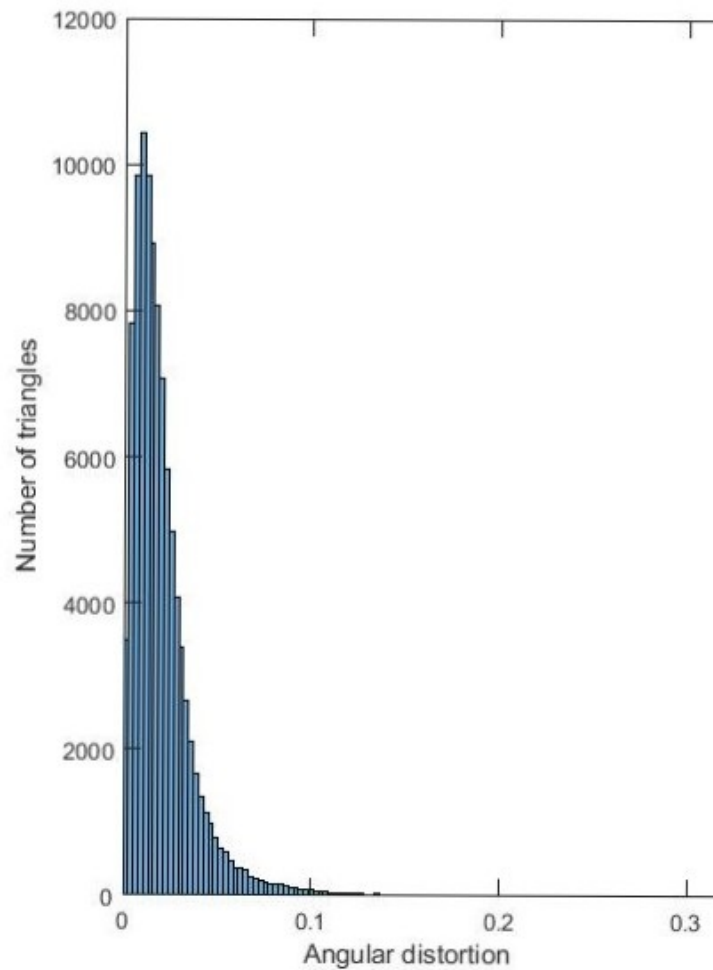


Figure 4.22: Histogram which shows the number of triangles counted based on angular distortion for the armadillo

Another way to visualize how much angles have been distorted may be achieved in the following manner. Given the spherical coordinates of the spherical mesh of the cortical surface, we may draw a chessboard on it and then use the same colour map for the input cortical surface. In this way, we can understand if the orthogonal squares of the chessboard are still orthogonal on the chessboard resulting on the input surface.

In Figure 4.23 we see the results of the chessboard texture mapping discussed above. We notice that the black and white squares are still orthogonal on the cortical surface. This is quite clear in both images and in particular in the one on the right, which shows less convoluted and more flat areas.

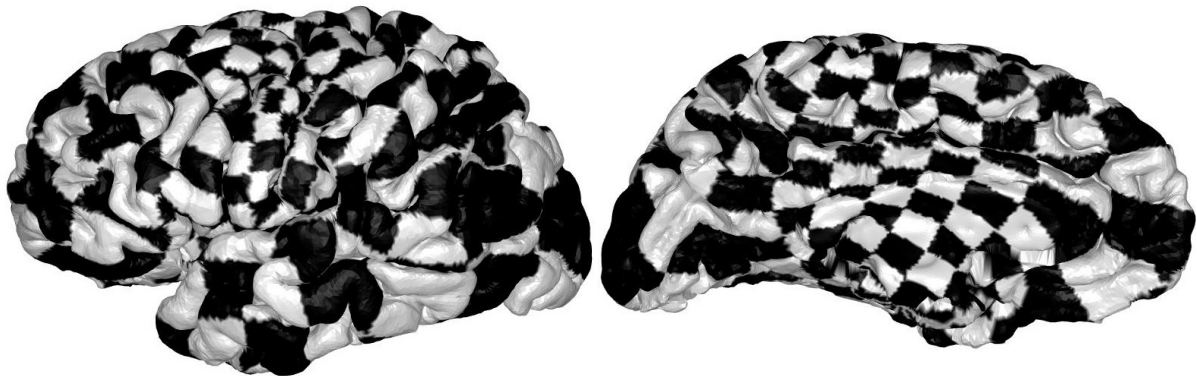


Figure 4.23: Cortical surface with chessboard texture mapping from two opposite views.

We tried to draw a chessboard also on the white matter surface.

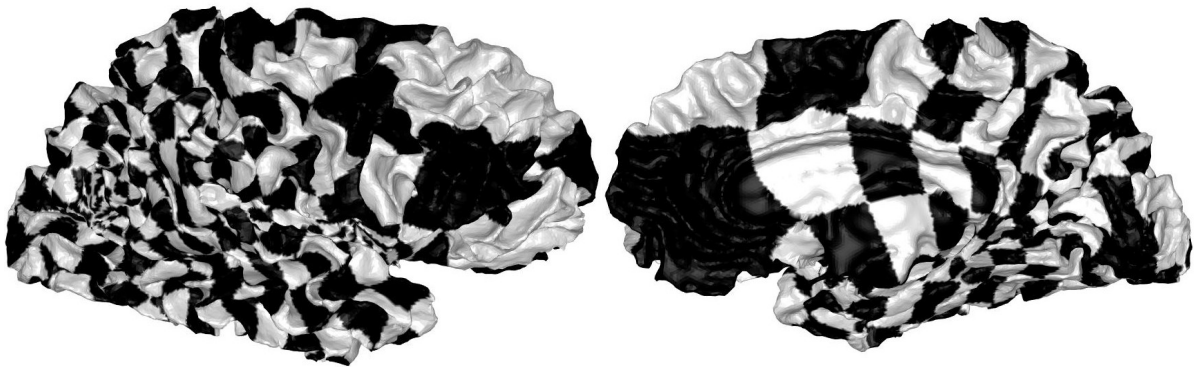


Figure 4.24: White matter surface with chessboard texture mapping from two opposite views.

We tested this idea of a chessboard texture mapping also on the mesh of a head of a statue (downloaded at scanify.fuel-3d.com/it/portfolio), which is surely *more spherical* and less convoluted than the two surfaces we tested before.

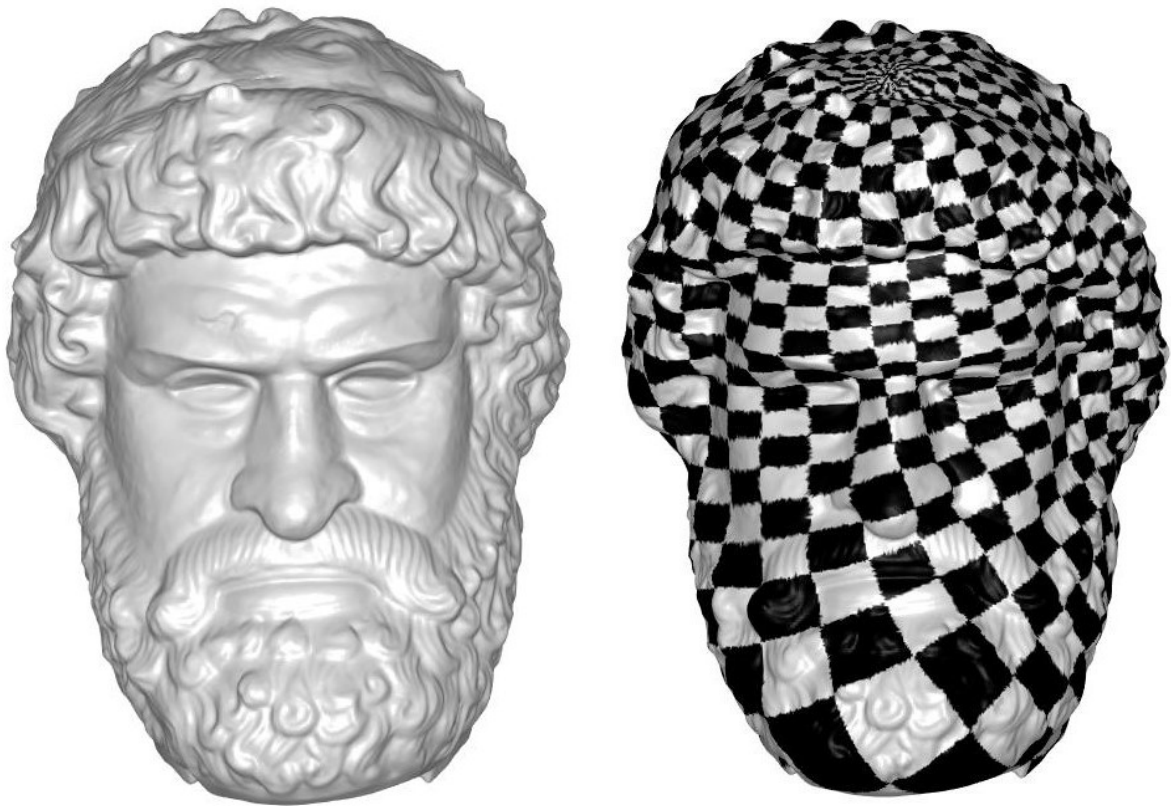


Figure 4.25: Chessboard texture mapping on the head of a statue

Figure 4.25 shows a very nice result: we may clearly notice that orthogonality between white and black pieces of the chessboard holds and that the North pole is shown on the top of the head.

Appendix A

Matlab codes

```
function [a,b]=bord_vectors(vertices ,faces ,indf)
%This function calculates the board vectors a and b
%Input:vertices=array with n1 X 3 elements
%       :faces=array with m1 X 3 elements
%       :indf=index of the {A,B,C} in which lies the point p

n=sizeof(vertices);
n1=n(1);
m=sizeof(faces);
m1=m(1);
F_R=faces(indf ,:);
indA=F_R(1);
indB=F_R(2);
indC=F_R(3);
A=vertices(indA ,:);
B=vertices(indB ,:);
C=vertices(indC ,:);
nab=norm(A-B);
teta=dot(C-A,B-A)/(nab)^2;
```

```

D=A+teta*(B-A);
ncd=norm(C-D);
a=sparse(n1,1);
b=sparse(n1,1);
for i=1:n1
    if i==indA
        a(i)=-1/nab;
        b(i)=(teta-1)/(ncd);
    elseif i==indB
        a(i)=1/nab;
        b(i)=-teta/ncd;
    elseif i==indC
        b(i)=1/ncd;
    end;
end;
end;

```

```

function conf_dist=conformal_distortion(vertices,...
sphere_vertices,faces)
%It computes the conformal distortion for every triangular face
%in faces for a given face F with angle a,b,c and as,bs,cs
% respectively on the input surface and on the sphere
%conf_dist(F)=(|a-as|+|b-bs|+|c-cs|)/(2*pi)
m=size(faces);
m1=m(1); %number of faces
conf_dist=zeros(m1,1);
for i=1:m1
    indP=faces(i,1);
    indQ=faces(i,2);
    indR=faces(i,3);
    P=vertices(indP,:);
    Q=vertices(indQ,:);

```



```

R=vertices(indR,:);
PR=P-R;
PQ=P-Q;
QR=Q-R;
Ps=sphere_vertices(indP,:);
Qs=sphere_vertices(indQ,:);
Rs=sphere_vertices(indR,:);
PRs=Ps-Rs;
PQs=Ps-Qs;
QRs=Qs-Rs;
tetaR=atan2(norm(cross(PR,QR)),dot(PR,QR));
tetaP=atan2(norm(cross(-PR,-PQ)),dot(-PR,-PQ));
tetaQ=atan2(norm(cross(PQ,-QR)),dot(PQ,-QR));
tetaRs=atan2(norm(cross(PRs,QRs)),dot(PRs,QRs));
tetaPs=atan2(norm(cross(-PRs,-PQs)),dot(-PRs,-PQs));
tetaQs=atan2(norm(cross(PQs,-QRs)),dot(PQs,-QRs));
conf_dist(i)=(abs(tetaR-tetaRs)+... \
abs(tetaP-tetaPs)+abs(tetaQ-tetaQs))/(2*pi);

```

end

```

function D=matrix_stiffness4(vertices,faces)
%This function computes the stiffness matrix D
%Input: vertices=array with n1 X 3 elements
%      : faces=array with m1 X 3 elements
%Output: sparse matrix D with n1 X n1 elements
n=size(vertices);
n1=n(1);
m=size(faces);
m1=m(1);
nnz=6*m1; %maximum number of nonzero elements
IndRow=zeros(nnz,1);

```

```

IndCol=zeros(nnz,1);
Val=zeros(nnz,1);
last_ind=0;
for i=1:m1
    indP=faces(i,1);
    indQ=faces(i,2);
    indR=faces(i,3);
    P=vertices(indP,:);
    Q=vertices(indQ,:);
    R=vertices(indR,:);
    PR=P-R;
    PQ=P-Q;
    QR=Q-R;
    tetaR=atan2(norm(cross(PR,QR)),dot(PR,QR));
    tetaP=atan2(norm(cross(-PR,-PQ)),dot(-PR,-PQ));
    tetaQ=atan2(norm(cross(PQ,-QR)),dot(PQ,-QR));
    cotangR=cot(tetaR);
    cotangP=cot(tetaP);
    cotangQ=cot(tetaQ);
    IndthisRow=[indP indP indQ];
    IndthisCol=[indQ indR indR];
    thisVal=[-0.5*cotangR -0.5*cotangQ -0.5*cotangP];
    IndRow(last_ind+1:last_ind+3)=IndthisRow';
    IndCol(last_ind+1:last_ind+3)=IndthisCol';
    Val(last_ind+1:last_ind+3)=thisVal';
    last_ind=last_ind+3;
end;
D0=sparse(IndRow(1:last_ind),IndCol(1:last_ind),...
Val(1:last_ind),n1,n1);
D1=D0+D0';
val_diag=zeros(n1,1);

```

```

val_diag(1:n1)=-sum(D1(1:n1,:),2);
indR1=1:n1;
D2=sparse(indR1',indR1',val_diag);
D=D1+D2;

```

```

function k=optimal_dilation(indf,faces,planevertices)
%It computes the optimal dilation coeff k
%Input:planevertices=array with n1 X 2 elements
%      :faces=array with m1 X 3 elements
%      :indf=index of the triangular face {A,B,C} in which lies
%the point p
%Output: scalar k
tr=triangulation(faces,planevertices); %struct triangulation
indt=pointLocation(tr,[0 0]);
[P,A,R]=perim_area_raggiocirc(indf,faces,planevertices);
[p,a,r]=perim_area_raggiocirc(indt,faces,planevertices);
k=1/sqrt(r*R);

```

```

function [p2,a,R]=perim_area_raggiocirc(indf,faces,vertices)
%This function calculates perimeter,area and radius of
%circumscribed circle to the triangular face F which
%contains the point p
%Input:vertices=array with n1 X 3 elements
%      :faces=array with m1 X 3 elements
%      :indf=index of the triangular face {A,B,C} in which lies
%the point p
%Output: perimeter p2, area a, radius R

F=faces(indf,:);
A=vertices(F(1,:),:);

```

```
B=vertices(F(2),:);
C=vertices(F(3),:);
AB=norm(A-B);
AC=norm(A-C);
BC=norm(C-B);
p2=AB+BC+AC;
a=sqrt(0.5*p2*(0.5*p2-AB)*(0.5*p2-AC)*(0.5*p2-BC));
R=AB*AC*BC/(4*a);
```

Bibliography

- [1] Lars V Ahlfors. Complex analysis: an introduction to the theory of analytic functions of one complex variable. *New York, London*, 1953.
- [2] Lars Valerian Ahlfors and Leo Sario. *Riemann surfaces*. Princeton university press, 2015.
- [3] Sigurd Angenent, Steven Haker, Allen R Tannenbaum, and Ron Kikinis. On the laplace–beltrami operator and brain surface flattening. *Georgia Institute of Technology*, 1999.
- [4] Alexander I Bobenko and Boris A Springborn. A discrete laplace–beltrami operator for simplicial surfaces. *Discrete & Computational Geometry*, 38(4):740–756, 2007.
- [5] Yaiza Canzani. Analysis on manifolds via the laplacian. *Lecture Notes, Harvard University*, <http://www.math.harvard.edu/canzani/math253.html>, 2013.
- [6] Pui Tung Choi, Ka Chun Lam, and Lok Ming Lui. Flash: Fast landmark aligned spherical harmonic parameterization for genus-0 closed brain surfaces. *SIAM Journal on Imaging Sciences*, 8(1):67–94, 2015.
- [7] David Cohen-Steiner and Jean-Marie Morvan. Restricted delaunay triangulations and normal cycle. In *Proceedings of the nineteenth annual symposium on Computational geometry*, pages 312–321. ACM, 2003.
- [8] Manfredo Perdigao Do Carmo and Manfredo Perdigao Do Carmo. *Differential geometry of curves and surfaces*, volume 2. Prentice-hall Englewood Cliffs, 1976.

- [9] Manfredo Perdigao do Carmo Valero. *Riemannian geometry*. Birkhäuser Basel, 1992.
- [10] Howard C Elman, David J Silvester, and Andrew J Wathen. *Finite elements and fast iterative solvers: with applications in incompressible fluid dynamics*. Oxford University Press (UK), 2014.
- [11] Hershel M Farkas and Irwin Kra. *Riemann surfaces*. Springer, 1992.
- [12] Michael S Floater and Kai Hormann. Surface parameterization: a tutorial and survey. In *Advances in multiresolution for geometric modelling*, pages 157–186. Springer, 2005.
- [13] Eberhard Freitag. *Complex analysis 2: Riemann surfaces, several complex variables, Abelian functions, higher modular functions*. Springer Science & Business Media, 2011.
- [14] Theodore Gamelin. *Complex analysis*. Springer Science & Business Media, 2003.
- [15] John R Gilbert, Cleve Moler, and Robert Schreiber. Sparse matrices in matlab: Design and implementation. *SIAM Journal on Matrix Analysis and Applications*, 13(1):333–356, 1992.
- [16] Mark S Gockenbach. *Understanding and implementing the finite element method*. Siam, 2006.
- [17] Roger Godement. *Analysis III: analytic and differential functions, manifolds and Riemann surfaces*. Springer, 2002.
- [18] Xianfeng Gu, Yalin Wang, Tony F Chan, Paul M Thompson, and Shing-Tung Yau. Genus zero surface conformal mapping and its application to brain surface mapping. *IEEE Transactions on Medical Imaging*, 23(8):949–958, 2004.
- [19] Steven Haker, Sigurd Angenent, Allen R Tannenbaum, Ron Kikinis, Guillermo Sapiro, and Michael Halle. Conformal surface parameterization for texture mapping. *Georgia Institute of Technology*, 2000.

- [20] Klaus Hildebrandt and Konrad Polthier. Anisotropic filtering of non-linear surface features. In *Computer Graphics Forum*, volume 23-3, pages 391–400. Wiley Online Library, 2004.
- [21] Monica K Hurdal and Ken Stephenson. Discrete conformal methods for cortical brain flattening. *Neuroimage*, 45(1):S86–S98, 2009.
- [22] Yoichi Imayoshi and Masahiko Taniguchi. *An introduction to Teichmüller spaces*. Springer Science & Business Media, 2012.
- [23] Frank Morgan. *Riemannian geometry: a beginner’s guide*. Jones and Bartlett Publishers, 1998.
- [24] Ulrich Pinkall and Konrad Polthier. Computing discrete minimal surfaces and their conjugates. *Experimental mathematics*, 2(1):15–36, 1993.
- [25] Jeffrey Rauch. *Partial Differential Equations*. Springer, 1991.
- [26] Wilhelm Schlag. *A course in complex analysis and Riemann surfaces*, volume 154. American Mathematical Society, 2014.
- [27] Olga Sorkine. Laplacian mesh processing. *STAR report*, 2006.
- [28] Thomas James Willmore. *Riemannian geometry*. Oxford Science Publications, 1996.

