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On the trace anomaly of a Weyl fermion

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Abstract

La seguente tesi si sviluppa in tre parti: un'introduzione alle simmetrie conformi e di scala, una parte centrale dedicata alle anomalie quantistiche ed una terza parte dedicata all'anomalia di traccia per fermioni. Nella seconda parte in particolare si introduce il metodo di calcolo alla Fujikawa e si discute la scelta di regolatori adeguati ed un metodo per ottenerli, si applicano poi questi metodi ai campi, scalare e vettoriale, per l'anomalia di traccia in spazio curvo. Nell'ultimo capitolo si calcolano le anomalie di traccia per un fermione di Dirac e per uno di Weyl; la motivazione per calcolare queste anomalie nasce dal fatto che recenti articoli hanno suggerito che possa emergere un termine immaginario proporzionale alle densità di Pontryagin nell'anomalia di Weyl. Noi non abbiamo trovato questo termine e il risultato è che l'anomalia di traccia risulta essere metà di quella per il caso di Dirac.

Contents

Introduction

Trace anomalies were first discovered in 1973 by D. Capper and M. J. Duff. Massless fields of spin 0, $\frac{1}{2}$ and 1 in interaction with gravity classically show invariance under a Weyl rescaling of the metric. This invariance does not occur once we pass to a quantum theory. This manifests itself in the fact that the trace of the energy-momentum tensor of these theories vanishes at the classical level, but acquires anomalous terms at the quantum level. These terms depend on the background geometry of the spacetime on which the conformal field theories are coupled to. The trace (or Weyl) anomaly has a big importance in physics (for a review see [1]) and historically was first connected to the Hawking effect and to gravitational instantons. The Weyl anomaly has also found use in cosmology, supersymmetry and string theory. In cosmology the Weyl anomaly has been connected to inflation, the cosmological constant, particle production and wormholes. For example in the early universe the trace anomaly on a de Sitter space determines the energy momentum tensor and in this way it is related to inflation. In string theory the preservation of Weyl invariance is connected to the critical dimension, to the central charge of the Virasoro algebra and when background fields are present it is possible to obtain the Einstein field equations coupled to matter (and stringy extensions) from the trace anomaly. The Weyl anomaly is also broadly studied today, for example for its connection to the renormalization group or in supersymmetry, where the trace of the stress tensor, the divergence of the axial current and the gamma trace of the spinor currents form a scalar supermultiplet. We are in particular interested in a recent result [2, 3], which analyses the trace anomaly for a Weyl fermion and claims that it contains a purely imaginary term proportional to the Pontryagin density. Such a result would have big consequences since it may imply a unitarity problem at one loop. Moreover it could become a selective criterion for consistent theories. In the following our final aim is to calculate such anomaly with a different method to check the results, on the way we introduce the general topics of anomalies and relevant symmetries of field theories coupled to gravity. In particular in the first chapter we introduce both the conformal invariance and Weyl rescaling, and we show the effect of the latter on massless scalar, vector and Dirac fields. In chapter two we introduce the tools that will be used for the calculation. We will review a path integral approach leading to the Fujikawa method of computing anomalies, and then we will discuss how to choose a consistent regulator that must be introduced in that method. To do so we use a comparison with Pauli Villars regularization, and by using this parallelism we will establish a method to calculate anomalies. Then we will apply this method to the Weyl anomaly for the scalar and Maxwell bosonic fields. Finally in chapter three we calculate the trace anomaly for both Dirac and Weyl fermions. Our final result unfortunately does not match the one mentioned above. We find that the trace anomaly for a Weyl fermion is half the trace anomaly of a Dirac one which does not present a Pontryagin term nor an imaginary one.

Chapter 1

Classical Models, conformal symmetry and Weyl invariance

1.1 Conformal and Weyl symmetry

In this chapter we shall give an introduction to conformal symmetry and Weyl invariance. Afterwards we will show these symmetries in three classical models: scalar (spin 0), fermionic (spin $\frac{1}{2}$), and Maxwell (spin 1). To start off we consider conformal invariance. A conformal transformation can be described as the most general coordinate transformation that preserves angles. To be more specific we say that a transformation $x'^{\mu} = \phi^{\mu}(x)$ is conformal if it satisfies the condition

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x). \tag{1.1}$$

If we restrict to a four dimensional euclidean space, starting from the above condition it is possible to calculate the transformations that respect it, and one finds in cartesian coordinates

$$\begin{aligned} x'^{\mu} =& x^{\mu} + a^{\mu} \\ x'^{\mu} =& \alpha x^{\mu} \\ x'^{\mu} =& M^{\mu}{}_{\nu} x^{\nu} \\ x'^{\mu} =& \frac{x^{\mu} - b^{\mu} x^2}{1 - 2b^{\nu} x_{\nu} + b^2 x^2} \end{aligned} \tag{1.2}$$

where from top to bottom we have translations, dilation, rigid rotations and special conformal transformations; in particular $M^{\mu\nu}$ is an orthogonal matrix belonging to the group $SO(4)$. For a more intuitive expression we can rewrite the SCT (Special Conformal Transformation) in the form

$$\frac{x'^{\mu}}{x'^{2}} = \frac{x^{\mu}}{x^{2}} - b^{\mu} \tag{1.3}$$

which let us understand that SCT are an inversion followed by a translation and then followed by another inversion. We show it in a more explicit way writing

$$x \to \frac{x^{\mu}}{x^{2}} \to \frac{x^{\mu}}{x^{2}} - b^{\mu} \to \frac{\frac{x^{\mu}}{x^{2}} - b^{\mu}}{(\frac{x^{\mu}}{x^{2}} - b^{\mu})^{2}} = \frac{x^{\mu} - b^{\mu}x^{2}}{1 - 2b^{\nu}x_{\nu} + b^{2}x^{2}}. \tag{1.4}$$

We notice that the Poincaré group is a subgroup of the conformal one. The generators of this group acting on the coordinates are easily found to be the following ones

$$\begin{aligned} P_{\mu} =& -i\partial_{\mu} \\ D =& -ix^{\mu}\partial_{\mu} \\ L_{\mu\nu} =& i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}) \\ K_{\mu} =& -i(2x_{\mu}x^{\nu}\partial_{\nu} - x^{2}\partial_{\mu}) \end{aligned} \tag{1.5}$$

where the order is the same as before. The Lie algebra associated to the conformal algebra can now be evaluated, and one finds

$$\begin{aligned} [D, P_{\mu}] =& iP_{\mu} \\ [D, K_{\mu}] =& iK_{\mu} \\ [K_{\mu}, P_{\nu}] =& 2i(\eta_{\mu\nu}D - L_{\mu\nu}) \\ [K_{\rho}, L_{\mu\nu}] =& i(\eta_{\rho\mu}K_{\nu} - \eta_{\rho\nu}K_{\mu}) \\ [P_{\rho}, L_{\mu\nu}] =& i(\eta_{\rho\mu}P_{\nu} - \eta_{\rho\nu}P_{\mu}) \\ [L_{\mu\nu}, L_{\rho\sigma}] =& i(\eta_{\rho\nu}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\rho\mu}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}). \end{aligned} \tag{1.6}$$

If we define the generators

$$\begin{aligned} J_{\mu\nu} =& L_{\mu\nu} \\ J_{-1,\mu} =& \frac{1}{2}(P_\mu - K_\mu) \\ J_{-1,0} =& D \\ J_{0,\mu} =& \frac{1}{2}(P_\mu + K_\mu) \end{aligned}, \tag{1.7}$$

and imposing that $J_{ab} = -J_{ba}$, and one finds that the Lie algebra takes the form

$$[J_{ab}, J_{cd}] = i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}) \tag{1.8}$$

where the index span from -1 to the dimension of the space and η has an extra minus sign. This means that there is an isomorphism between the conformal group in d dimension and the $SO(d+1,1)$ group. Similarly, in a Minkowskian spacetime of d dimension, whose Lorentz group is $SO(d-1,1)$, the conformal group is isomorphic to $SO(d,2)$.

It is interesting for our purpose to consider the general field transformation for the dilatations

$$\Phi'(x') = \alpha^{-2\Delta}\Phi(x) = \Lambda(x)^{-\Delta}\Phi(x). \tag{1.9}$$

More in general all conformal transformation can be written in the form

$$\Phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta}{4}} \Phi(x) \tag{1.10}$$

where Δ is the scaling dimension which depends on the kind of field.

Now to the Weyl transformation. Weyl transformations, or rescaling, are not a coordinate transformation but simply a local rescaling of the metric

$$g'_{\mu\nu}(x) = \Lambda(x)g_{\mu\nu}(x) \tag{1.11}$$

notice that on the left hand side of the equation we do not have x'. We can then say that a conformal transformation leaves the metric unchanged up to a Weyl rescaling. To have Weyl invariance though we have to introduce transformations that act also on the fields and follow the rule

$$\Phi'(x) = \Lambda(x)^{-\Delta}\Phi(x). \tag{1.12}$$

1.2 Trace of the energy-momentum tensor

Now we would like to address the relationship between conformal and Weyl invariance and the consequences on the energy momentum tensor . To do so we recall that for a model Φ coupled to gravity we can define the energy momentum tensor as

$$T_{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}}S[\Phi, g]. \tag{1.13}$$

This definition gives directly the improved Belifante energy momentum tensor and as a concept it is obtained remembering that $T_{\mu\nu}$ is the source of the gravitational field. Now if we consider an action coupled to gravity, where gravity is only a background, and we assume that such action is invariant under local Weyl rescaling, then we must have that

$$0 = \delta S = \delta_\Phi S + \delta_g S = \int d^4x\sqrt{g}\Big[\Big(\frac{\delta\mathcal{L}}{\delta(\partial\Phi)}\delta(\partial\Phi) + \frac{\delta\mathcal{L}}{\delta\Phi}\delta\Phi\Big) - \frac{1}{2}T_{\mu\nu}\delta g^{\mu\nu}\Big] \tag{1.14}$$

where in this equation δ_Φ is the variation with respect to the field and δ_g with respect to the metric. We see that the first term on the right hand side is proportional to the Euler Lagrange equation, so that on-shell it vanishes, and we obtain

$$\int d^4x\sqrt{g}T^\mu{}_\mu\sigma(x) = 0 \tag{1.15}$$

where we have defined an infinitesimal parameter $\sigma(x) = \ln\Lambda(x)$ so that

$$\delta g^{\mu\nu} = -\sigma(x)g^{\mu\nu}. \tag{1.16}$$

We see that as a consequence of the invariance of the action the energy momentum tensor must be traceless on-shell. Now assuming that we have an action invariant under local Weyl rescaling we ask ourselves if it is also invariant under conformal transformations.

To answer this question let us consider an infinitesimal coordinate transformation

$$x'^{\mu} = x^{\mu} - \epsilon^{\mu}(x) \tag{1.17}$$

so that the metric varies as

$$\delta g_{\mu\nu}(x) = g'_{\mu\nu}(x) - g_{\mu\nu}(x) = \nabla_{\mu}\epsilon_{\nu}(x) + \nabla_{\nu}\epsilon_{\mu}(x). \tag{1.18}$$

To have a true symmetry (as opposite to a background symmetry where the background is also transformed) one may try to compensate the variation of the metric with an infinitesimal Weyl transformation, as in (1.16), and require that

$$\delta g_{\mu\nu} = \nabla_{\mu}\epsilon_{\nu} + \nabla_{\nu}\epsilon_{\mu} + \sigma g_{\mu\nu} = 0. \tag{1.19}$$

Taking the trace of this equation one finds

$$\sigma = -\frac{2}{d}\nabla_{\mu}\epsilon^{\mu} \tag{1.20}$$

and thus obtains the conformal Killing equation

$$\nabla_{\mu}\epsilon_{\nu} + \nabla_{\nu}\epsilon_{\mu} - \frac{2}{d}g_{\mu\nu}\nabla_{\alpha}\epsilon^{\alpha} = 0. \tag{1.21}$$

Solutions of this equation are the conformal Killing vectors ϵ^{μ} that generate the conformal transformations. In flat space they are precisely the vectors that produce the finite conformal transformations reported in eq. (1.2).

To summarize if a model in curved space is invariant under local Weyl rescalings of the background metric, its energy momentum tensor is traceless and as a consequences the model is also invariant under conformal transformations. For this reason in the following we will only check local Weyl invariance in the models we are going to discuss.

1.3 Scalar Field

In this section we would like to introduce a scalar field model in 4-dimensional euclidean curved space which is also invariant under Weyl transformation. The classical action is similar to the one for a massless scalar field minimally coupled to gravity, but to guarantee the invariance under Weyl transformations one needs to add an improvement term proportional to R (our conventions can be found in appendix A.3)

$$S = \int d^4x \sqrt{g} \frac{1}{2} \Big(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{6} R \phi^2 \Big) . \tag{1.22}$$

Varying the action one can obtain the equation of motion. Using covariant derivatives, the variation reads

$$\delta S = \int d^4x \sqrt{g} (\nabla^\mu \phi \nabla_\mu \delta\phi - \frac{1}{6} R \phi \delta\phi) \tag{1.23}$$

that after an integration by part becomes

$$\delta S = \int d^4x \sqrt{g} (-\Box\phi - \frac{1}{6} R\phi)\delta\phi \tag{1.24}$$

so that we can read off the equation of motion

$$\Big(\Box + \frac{1}{6} R\Big)\phi = 0. \tag{1.25}$$

In the above relation we have defined the covariant laplacian $\Box = \nabla_\mu \nabla^\mu$ that acts on a scalar.

At this point, recalling the definition of the energy-momentum tensor $T_{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}}$, we would like to calculate its explicit expression (see appendix A.3 for the variation of the scalar curvature R)

$$\begin{aligned} \delta_g S = &\int d^4x \sqrt{g} \frac{1}{2} \Big(-\frac{1}{2} g_{\mu\nu} \nabla^\rho \phi \nabla_\rho \phi \delta g^{\mu\nu} + \nabla_\mu \phi \nabla_\nu \phi \delta g^{\mu\nu} + \frac{1}{12} g_{\mu\nu} R \phi^2 \delta g^{\mu\nu} + \\ &- \frac{1}{6} R_{\mu\nu} \phi^2 \delta g^{\mu\nu} - \frac{1}{6} \phi^2 \nabla_\mu \nabla_\nu \delta g^{\mu\nu} + \frac{1}{6} \phi^2 g_{\mu\nu} \Box \delta g^{\mu\nu} \Big) = \\ = &\int d^4x \sqrt{g} \frac{1}{2} \Big(-\frac{1}{2} g_{\mu\nu} \nabla^\rho \phi \nabla_\rho \phi \delta g^{\mu\nu} + \nabla_\mu \phi \nabla_\nu \phi \delta g^{\mu\nu} + \frac{1}{12} g_{\mu\nu} R \phi^2 \delta g^{\mu\nu} + \\ &- \frac{1}{6} R_{\mu\nu} \phi^2 \delta g^{\mu\nu} - \frac{1}{3} \nabla_\mu \phi \nabla_\nu \phi \delta g^{\mu\nu} - \frac{1}{3} \phi \nabla_\mu \nabla_\nu \phi \delta g^{\mu\nu} + \\ &+ \frac{1}{3} \nabla_\rho \phi \nabla^\rho \phi g_{\mu\nu} \delta g^{\mu\nu} + \frac{1}{3} \phi \Box \phi g_{\mu\nu} \delta g^{\mu\nu} \Big) \end{aligned} \tag{1.26}$$

that reads

$$
\begin{aligned}
T_{\mu\nu} =& \frac{1}{6}\nabla^{\rho}\phi\nabla_{\rho}\phi g_{\mu\nu} - \frac{2}{3}\nabla_{\mu}\phi\nabla_{\nu}\phi - \frac{1}{3}\phi\Box\phi g_{\mu\nu} + \\
&+ \frac{1}{3}\phi\nabla_{\mu}\nabla_{\nu}\phi - \frac{1}{12}g_{\mu\nu}R\phi^{2} + \frac{1}{6}R_{\mu\nu}\phi^{2}.
\end{aligned}
\tag{1.27}
$$

Now we can show that this tensor is indeed traceless

$$
\begin{aligned}
T_{\mu\nu}g^{\mu\nu} =& \frac{4}{6}\nabla^{\rho}\phi\nabla_{\rho}\phi - \frac{2}{3}\nabla_{\mu}\phi\nabla_{\nu}\phi - \frac{4}{3}\phi\Box\phi + \\
&+ \frac{1}{3}\phi\Box\phi - \frac{1}{3}R\phi^{2} + \frac{1}{6}R\phi^{2} = \\
=& -\phi\Box\phi - \frac{1}{6}R\phi^{2} = \phi(-\Box\phi - \frac{1}{6}R\phi) = 0;
\end{aligned}
\tag{1.28}
$$

in the last line we applied the equation of motion. As already mentioned, this is a consequence of Weyl invariance.

Now, we would like to show explicitly the invariance of the action under local Weyl rescalings. We define $\sigma(x) = \ln\Lambda(x)$ so that we can write the infinitesimal transformations

$$\delta g_{\mu\nu} = \sigma(x)g_{\mu\nu} \tag{1.29}$$

$$\delta\phi = -\frac{1}{2}\sigma(x)\phi, \tag{1.30}$$

after some calculation is also possible to show that

$$\delta R = -\sigma R + 3\nabla^{\mu}\nabla_{\mu}\sigma = -\sigma R + 3(\partial_{\mu}\partial^{\mu}\sigma - \Gamma^{\mu\nu}{}_{\nu}\partial_{\mu}\sigma). \tag{1.31}$$

Now we can proceed to calculate the variation of the action

$$
\begin{aligned}
\delta\mathcal{L} =& \sigma\sqrt{g}\Big(g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi - \frac{1}{6}R\phi^{2}\Big) + \\
&+ \frac{1}{2}\sqrt{g}\Big(-\sigma g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi - g^{\mu\nu}\phi\partial_{\mu}\phi\partial_{\nu}\sigma - \sigma g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi + \frac{1}{6}\sigma R\phi^{2} + \\
&- \frac{1}{2}\phi^{2}\partial_{\mu}\partial^{\mu}\sigma + \frac{1}{2}\phi^{2}\Gamma^{\mu\nu}{}_{\nu}\partial_{\mu}\sigma + \frac{1}{6}\sigma R\phi^{2}\Big) = \\
=& -\frac{1}{2}\sqrt{g}\Big(g^{\mu\nu}\phi\partial_{\mu}\phi\partial_{\nu}\sigma + \frac{1}{2}\phi^{2}\partial_{\mu}\partial^{\mu}\sigma - \frac{1}{2}\phi^{2}\Gamma^{\mu\nu}{}_{\nu}\partial_{\mu}\sigma\Big)
\end{aligned}
\tag{1.32}
$$

where we have applied the above mentioned rules taking into account that $g^{\mu\nu}$ has an infinitesimal transformation with the opposite sign. Since we are interested in the variation of the action we may add for free a total derivative $\frac{1}{4}\partial_\mu(\sqrt{g}g^{\mu\nu}\phi^2\partial_\nu\sigma)$ and obtain

$$\begin{aligned}\delta\mathcal{L} =&\frac{1}{2}\sqrt{g}\Big(-g^{\mu\nu}\phi\partial_\mu\phi\partial_\nu\sigma+\frac{1}{4}g^{\rho\sigma}\partial_\mu g_{\rho\sigma}g^{\mu\nu}\phi^2\partial_\nu\sigma+\frac{1}{2}\partial_\mu g^{\mu\nu}\phi^2\partial_\nu\sigma+\\ &+g^{\mu\nu}\phi\partial_\mu\phi\partial_\nu\sigma+\frac{1}{2}\phi^2\partial_\mu\partial^\mu\sigma-\frac{1}{2}\phi^2\partial_\mu\partial^\mu\sigma+\frac{1}{2}\phi^2\Gamma^{\mu\nu}{}_\nu\partial_\mu\sigma\Big)=\\ =&\frac{1}{2}\sqrt{g}\Big(\frac{1}{4}g^{\rho\sigma}\partial_\mu g_{\rho\sigma}g^{\mu\nu}\phi^2\partial_\nu\sigma+\frac{1}{2}\partial_\mu g^{\mu\nu}\phi^2\partial_\nu\sigma+\frac{1}{2}\phi^2\Gamma^{\mu\nu}{}_\nu\partial_\mu\sigma\Big),\end{aligned} \tag{1.33}$$

now we remember that the covariant derivative of the metric vanishes so that we can use

$$\begin{aligned}\partial_\rho g_{\mu\nu} =&\Gamma^\sigma{}_{\rho\mu}g_{\sigma\nu}+\Gamma^\sigma{}_{\rho\nu}g_{\mu\sigma}\\ \partial_\mu g^{\mu\nu}=&-\Gamma^\mu{}_{\mu\rho}g^{\rho\nu}-\Gamma^\nu{}_{\mu\rho}g^{\mu\rho}\end{aligned} \tag{1.34}$$

and, after a substitution in the equation we can say that

$$\delta\mathcal{L}=\frac{1}{2}\sqrt{g}\Big(\frac{1}{2}\Gamma^{\mu\nu}{}_\mu\phi^2\partial_\nu\sigma-\frac{1}{2}\Gamma^\mu{}_\mu{}^\nu\phi^2\partial_\nu\sigma-\frac{1}{2}\Gamma^{\nu\mu}{}_\mu\phi^2\partial_\nu\sigma+\frac{1}{2}\phi^2\Gamma^{\mu\nu}{}_\nu\partial_\mu\sigma\Big)=0 \tag{1.35}$$

that proves the invariance of the action. This calculation is a bit complex, but quite explicit. Thus, we would like to present the same calculation done in a more covariant way

$$\begin{aligned}\delta\mathcal{L}=&\sigma\sqrt{g}\Big(\nabla_\mu\phi\nabla^\mu\phi-\frac{1}{6}R\phi^2\Big)+\\ &+\frac{1}{2}\sqrt{g}\Big(-\sigma\nabla_\mu\phi\nabla^\mu\phi-\phi\nabla_\mu\phi\nabla^\mu\sigma-\sigma\nabla_\mu\phi\nabla^\mu\phi+\\ &+\frac{1}{6}\sigma R\phi^2-\frac{1}{2}\phi^2\Box\sigma+\frac{1}{6}\sigma R\phi^2\Big)=\\ =&-\frac{1}{2}\sqrt{g}\Big(\phi\nabla_\mu\phi\nabla^\mu\sigma+\frac{1}{2}\phi^2\Box\sigma\Big)=\\ =&-\frac{1}{2}\sqrt{g}\Big(\frac{1}{2}\nabla_\mu\phi^2\nabla^\mu\sigma+\frac{1}{2}\phi^2\Box\sigma\Big)=\\ =&-\frac{1}{4}\sqrt{g}\nabla_\mu\Big(\frac{1}{2}\phi^2\nabla^\mu\sigma+\frac{1}{2}\phi^2\nabla^\mu\sigma\Big)\end{aligned} \tag{1.36}$$

which is a total derivative that can be dropped upon integration, so that $\delta S=0$. This proves Weyl invariance.

1.4 Vector field

Now we are going to do the same for the vector field $A_\mu(x)$. We will introduce an action and prove that it is classically invariant under local Weyl rescaling. The action we consider is the free Maxwell action minimally coupled to gravity

$$S = -\int d^4x \frac{1}{4}\sqrt{g} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \tag{1.37}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. First we would like to obtain the equations of motion. To do so we vary the action

$$\begin{aligned} \delta S &= \int d^4x \sqrt{g}\Big(-\frac{1}{4} F^{\mu\nu} \delta F_{\mu\nu}\Big) = \int d^4x \sqrt{g}\Big(-\frac{1}{4} F^{\mu\nu} \nabla_\mu \delta A_\nu\Big) = \\ &= \int d^4x \sqrt{g}\Big(\frac{1}{4} \nabla_\mu F^{\mu\nu}\Big) \delta A_\nu, \end{aligned} \tag{1.38}$$

in the calculation we used the property $[\delta, \nabla] = 0$ and also the antisymmetric nature of $F^{\mu\nu}$. From the last line we can read off the equations of motion

$$\nabla_\nu F^{\mu\nu} = 0. \tag{1.39}$$

We also want to calculate the energy momentum tensor, so starting again from the action we recall that varying only the metric one finds

$$\delta S = \int d^4x \sqrt{g}\Big(-\frac{1}{2} T_{\mu\nu} \delta g^{\mu\nu}\Big) = \int d^4x \sqrt{g}\Big(\frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu}\Big). \tag{1.40}$$

Thus we consider

$$\begin{aligned} \delta S &= \int d^4x \Big(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} (\delta\sqrt{g}) - \sqrt{g}\frac{1}{2} F_{\mu\rho} F_\nu{}^\rho \delta g^{\mu\nu}\Big) = \\ &= \int d^4x \sqrt{g} \frac{1}{2}\Big(F^{\mu\rho} F^\nu{}_\rho - \frac{1}{4} g^{\mu\nu} F_{\sigma\rho} F^{\sigma\rho}\Big) \delta g_{\mu\nu}, \end{aligned} \tag{1.41}$$

from the last line we can read off

$$T_{\mu\nu} = F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\sigma\rho} F^{\sigma\rho}. \tag{1.42}$$

This tensor is conserved and has a null trace

$$T^{\mu}{}_{\mu} = g^{\mu\nu} F_{\mu\rho} F_{\nu}{}^{\rho} - \frac{1}{4} 4 F_{\sigma\rho} F^{\sigma\rho} = 0. \tag{1.43}$$

Now we want to prove the invariance of the action under a Weyl rescaling. The transformation we need are

$$\begin{aligned} \delta g_{\mu\nu} =& \sigma g_{\mu\nu} \\ \delta g^{\mu\nu} =& - \sigma g^{\mu\nu} \\ \delta A_{\mu} =& 0 \end{aligned} \tag{1.44}$$

where once again $\sigma = \ln(\Lambda)$. Now remembering $\delta\sqrt{g} = \frac{1}{2}\sigma\sqrt{g} g^{\mu\nu} g_{\mu\nu}$ we can calculate

$$\begin{aligned} \delta\mathcal{L} =& -\frac{1}{8}\sigma\sqrt{g} g^{\tau\omega} g_{\tau\omega} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + \frac{1}{2}\sigma\sqrt{g} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} = \\ =& -\frac{1}{2}\sigma\sqrt{g} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + \frac{1}{2}\sigma\sqrt{g} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} = 0 \end{aligned} \tag{1.45}$$

which shows the invariance.

1.5 Dirac field

Before introducing this model we have to digress a little on how to couple a fermion to gravity. Introducing the veilbein $e^{a}{}_{\mu}$ by setting

$$g_{\mu\nu} = \eta_{ab}\, e_{\mu}{}^{a}\, e_{\nu}{}^{b} \tag{1.46}$$

one gains a new gauge symmetry: the local Lorentz transformations of tangent space. The covariant derivative needs a corresponding connection (the spin connection) so that for a tensor or spinor field V of tangent space one has

$$\nabla_{\mu} V = \partial_{\mu} V + \frac{1}{2}\omega_{\mu ab} M^{ab} V \tag{1.47}$$

where M^{ab} are the generators of the Lorentz group in the representation of the field V and satisfy the algebra normalized as

$$[M^{ab}, M^{cd}] = \eta^{bc}M^{ad} - \eta^{ac}M^{bd} - \eta^{bd}M^{ac} + \eta^{ad}M^{bc}. \tag{1.48}$$

The spin connection without torsion is defined by requiring the vielbein to be covariantly constant

$$\nabla_\mu e_\nu{}^a \equiv \partial_\mu e_\nu{}^a - \Gamma^\rho{}_{\mu\nu} e_\rho{}^a + \omega_\mu{}^a{}_b e_\nu{}^b = 0 \tag{1.49}$$

which can be solved for $\omega_\mu{}^{ab}$ by

$$\omega_\mu{}^{ab} = e^{\nu b}(\Gamma^\rho{}_{\mu\nu} e_\rho{}^a - \partial_\mu e_\nu{}^a) \tag{1.50}$$

or equivalenty by

$$\begin{aligned}\omega_\mu{}^{ab} &= \frac{1}{2} e^{\rho a} e^{\nu b} e_{\mu c}(\partial_\nu e_\rho{}^c - \partial_\rho e_\nu{}^c) \\ &\quad + \frac{1}{2} e^{\nu a}(\partial_\mu e_\nu{}^b - \partial_\nu e_\mu{}^b) - \frac{1}{2} e^{\nu b}(\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a) \ .\end{aligned} \tag{1.51}$$

This last expression shows manifestly the antisymmetry under exchange of the indices a and b. The spin connections transforms as a gauge field for local Lorentz transformations

$$e_\mu{}^a \to e'_\mu{}^a = \Lambda^a{}_b e_\mu{}^b \tag{1.52}$$

$$\omega_\mu{}^{ab} \to \omega'_\mu{}^{ab} = \Lambda^{ac}\partial_\mu \Lambda^b{}_c + \Lambda^a{}_c \Lambda^b{}_d \, \omega_\mu{}^{cd} \ . \tag{1.53}$$

In addition, the curvatures corresponding to the different connections (for the spin connection we define it as $[\nabla_\mu, \nabla_\nu] = \frac{1}{2} R_{\mu\nu ab}(\omega) M^{ab}$) are related since

$$[\nabla_\mu, \nabla_\nu] e_\rho{}^a = 0 \quad \to \quad R_{\mu\nu\rho a}(\Gamma) = R_{\mu\nu\rho a}(\omega) \tag{1.54}$$

where of course indices are made flat or curved by using the vierbien $e^a{}_\mu$ and its inverse.

Now, thanks to this geometrical set-up, we can introduce the action for a spin $\frac{1}{2}$ Dirac field

$$S = -\int d^4xe\overline{\psi}\gamma^\mu\nabla_\mu\psi\,; \tag{1.55}$$

where the covariant derivative is defined as

$$\nabla_\mu = \partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^a\gamma^b\ . \tag{1.56}$$

It is also important to notice that the gamma matrices have now a curved space variant defined as

$$\gamma^\mu = e^\mu{}_a\gamma^a\ . \tag{1.57}$$

Now we want to obtain the equations of motion so that we vary the action

$$\begin{aligned}\delta S &= -\int d^4xe\delta\overline{\psi}\gamma^\mu\nabla_\mu\psi + e\overline{\psi}\gamma^\mu\nabla_\mu\delta\psi = \\ &= -\int d^4xe\Big(\delta\overline{\psi}(\gamma^\mu\nabla_\mu\psi) - (\nabla_\mu\overline{\psi}\gamma^\mu)\delta\psi\Big)\end{aligned} \tag{1.58}$$

and from the least action principle we find the Dirac equation

$$\gamma^\mu\nabla_\mu\psi = 0 \tag{1.59}$$

and its complex conjugate. In this situation it is natural to define the energy momentum tensor for the Dirac field as

$$T^\mu{}_a = \frac{1}{e}\frac{\delta S}{\delta e_\mu{}^a}. \tag{1.60}$$

The reason for this is that when varying the action now we want to consider $\delta e_\mu{}^a$ instead of $\delta g^{\mu\nu}$. This definition is consistent with the previous one, in fact

$$T_\mu{}^a = -\frac{1}{e}\frac{\delta S}{\delta e^\mu{}_a} = -\frac{1}{g}\frac{\delta S}{\delta g^{\alpha\beta}}\frac{\delta g^{\alpha\beta}}{\delta e^\mu{}_a} = -\frac{2}{g}\frac{\delta S}{\delta g^{\alpha\beta}}e^{\beta a}\delta^\alpha{}_\mu = T_{\alpha\beta}e^{\beta a}\delta^\alpha{}_\mu. \tag{1.61}$$

As a consequence we obtain the same result that we got before, that is the energy momentum tensor is covariantly conserved, symmetric and traceless. To see this we may again compute

$$0 = \delta S = \delta_\Phi S + \delta_e S = \int d^4xe\Big[\Big(\frac{\delta\mathcal{L}}{\delta(\partial\Phi)}\delta(\partial\Phi) + \frac{\delta\mathcal{L}}{\delta\Phi}\delta\Phi\Big) + T^\mu{}_a\delta e_\mu{}^a\Big], \tag{1.62}$$

so that on-shell only the last term survives (here we have collectively denoted by Φ the fields ψ and $\overline{\psi}$, and used right derivatives). If we consider at this point a reparametrization

$$\delta e_\mu{}^a = \nabla_\mu \epsilon^a \tag{1.63}$$

the variation becomes

$$\int d^4x e T^\mu{}_a \nabla_\mu \epsilon^a = -\int d^4x e (\nabla_\mu T^\mu{}_a) \epsilon^a = 0 \tag{1.64}$$

proving that the tensor is indeed covariantly conserved

$$\nabla_\mu T^\mu{}_a = 0. \tag{1.65}$$

To prove that the tensor is symmetric we have to consider a Lorentz transformation

$$\delta e_\mu{}^a = \omega^a{}_b e^b{}_\mu, \tag{1.66}$$

recalling that, for a Lorentz transformation, the infinitesimal parameter ω_{ab} is antisymmetric we get

$$\int d^4x e T^\mu{}_a \omega^a{}_b e^b{}_\mu = \int d^4x e T^{ba} \omega_{ab} = 0. \tag{1.67}$$

This imply that as we stated the energy momentum tensor is symmetric. Lastly considering a Weyl rescaling

$$\delta e_\mu{}^a = \frac{1}{2}\sigma(x) e_\mu{}^a \tag{1.68}$$

we get

$$\int d^4x \frac{1}{2} e T^\mu{}_a \sigma(x) e_\mu{}^a = \int d^4x \frac{1}{2} e T^\mu{}_\mu \sigma(x) = 0 \tag{1.69}$$

that prove the tensor is traceless.

Let us now prove explicitly the invariance of the action under Weyl transformations. The Weyl rescaling that we need are

$$\delta e_\mu{}^a = \frac{1}{2}\sigma(x) e_\mu{}^a \tag{1.70}$$

$$\delta e^\mu{}_a = -\frac{1}{2}\sigma(x) e^\mu{}_a \tag{1.71}$$

$$\delta e = e e^{\mu}{}_{a} \delta e_{\mu}{}^{a} = 2\sigma(x) e \tag{1.72}$$

$$\delta\omega_{\mu ab} = \frac{1}{2}(e_{\mu a} e^{\nu}{}_{b} - e_{\mu b} e^{\nu}{}_{a})\partial_{\nu}\sigma(x) \tag{1.73}$$

$$\delta\psi = -\frac{3}{4}\sigma(x)\psi \tag{1.74}$$

$$\delta\overline{\psi} = -\frac{3}{4}\sigma(x)\overline{\psi}. \tag{1.75}$$

Now we can calculate the variation of the action, remembering while doing so that the vielbein is a local object and as such it depends on the coordinates,

$$\begin{aligned} \delta\mathcal{L} =& \delta(-e\overline{\psi}\gamma^{\mu}\nabla_{\mu}\psi) = -2\sigma e\overline{\psi}\gamma^{\mu}\nabla_{\mu}\psi + \frac{3}{4}\sigma e\overline{\psi}\gamma^{\mu}\nabla_{\mu}\psi + \frac{1}{2}\sigma e\overline{\psi}\gamma^{\mu}\nabla_{\mu}\psi + \\ &- \frac{1}{8}e\overline{\psi}\gamma^{\mu}(e_{\mu a}e^{\nu}{}_{b} - e_{\mu b}e^{\nu}{}_{a})\gamma^{a}\gamma^{b}\psi\partial_{\nu}\sigma + \frac{3}{4}\sigma e\overline{\psi}\gamma^{\mu}\nabla_{\mu}\psi + \frac{3}{4}e\overline{\psi}\gamma^{\mu}\psi\partial_{\mu}\sigma = \\ =& - \frac{1}{8}e\overline{\psi}\gamma^{\mu}(e_{\mu a}e^{\nu}{}_{b} - e_{\mu b}e^{\nu}{}_{a})\gamma^{a}\gamma^{b}\psi\partial_{\nu}\sigma + \frac{3}{4}e\overline{\psi}\gamma^{\mu}\psi\partial_{\mu}\sigma, \end{aligned} \tag{1.76}$$

remembering eq. (1.57) we obtain

$$\begin{aligned} \delta\mathcal{L} =& - \frac{1}{8}e\overline{\psi}(\eta_{ca}e^{\mu}{}_{b} - \eta_{cb}e^{\mu}{}_{a})\gamma^{c}\gamma^{a}\gamma^{b}\psi\partial_{\mu}\sigma + \frac{3}{4}e\overline{\psi}\gamma^{\mu}\psi\partial_{\mu}\sigma = \\ =& - \frac{1}{8}e\overline{\psi}[4\gamma^{\mu} - \eta_{cb}e^{\mu}{}_{a}(-\gamma^{c}\gamma^{b}\gamma^{a} + 2\eta^{ab}\gamma^{c})]\psi\partial_{\mu}\sigma + \frac{3}{4}e\overline{\psi}\gamma^{\mu}\psi\partial_{\mu}\sigma = \\ =& - \frac{1}{8}e\overline{\psi}(4\gamma^{\mu} + 4\gamma^{\mu} - 2\gamma^{\mu})]\psi\partial_{\mu}\sigma + \frac{3}{4}e\overline{\psi}\gamma^{\mu}\psi\partial_{\mu}\sigma = 0, \end{aligned} \tag{1.77}$$

where we used the relations $\eta_{ab} = e_{\mu a}e^{\mu}{}_{b}$ and $\gamma^{a}\gamma_{a} = 4$. This concludes the proof of the invariance of this classical Dirac model under local Weyl rescaling.

Chapter 2

Quantum anomaly, Fujikawa method and consistent regulator

2.1 Fujikawa method to evaluate quantum anomaly

The origin of anomalies in (perturbative) quantum field theory can be traced back to the fact that in the computation of loop corrections, one has to specify a regularization scheme. The latter, in general, does not preserve all of the symmetries of the classical action. After renormalizing, one can eliminate the regulating parameter (like the momentum cut-off Λ, the ϵ parameter of dimensional regularization, or the mass M of Pauli-Villars fields) and it may happen that some (necessarily finite) non-symmetrical terms survive, causing the breaking of those symmetries not preserved by the regularization. Still, it may happen that these terms can be cancelled by adding local counterterms to the effective action, whose variation cancel the anomaly. If this is not the case, one has a true anomaly. In the language of generating functionals, it means that the effective action Γ does not satisfy the corresponding Ward identity. The piece which breaks the Ward identity is identified as the consistent anomaly, where "consistent" refers to the fact that the anomaly is obtained from the variation of the effective action, and thus satisfies certain integrability conditions [4]. In the following we will apply the method of Fujikawa [5, 6] for computing the anomalies, improved by the scheme of ref. [7] to identify a consistent regulator. The latter scheme makes the anomaly calculation equivalent to a Feynman graph calculation regulated à la Pauli-Villars, which is necessarily consistent.

In Fujikawa's method, one recognizes the anomaly as arising from the non-invariance under a symmetry transformation of the measure $D\phi$ of the path integral

$$Z = \int D\phi \; e^{-S[\phi]} \tag{2.1}$$

here written in euclidean time. To review Fujikawa's method, let us consider an infinitesimal symmetry transformation of the form $\delta_\alpha \phi^i = \alpha f^i(\phi, \partial_\mu \phi)$, with infinitesimal constant parameter α, that leaves the action invariant, i.e. $\delta_\alpha S[\phi] = 0$. Promoting the parameter α to be an arbitrary function $\alpha(x)$, one identifies the Noether current J_μ associated to the symmetry by calculating

$$\delta_{\alpha(x)} S[\phi] = \int d^4x \; J^\mu \partial_\mu \alpha(x) \tag{2.2}$$

and recognizing from this expression the form of J^μ. Terms proportional to an undifferentiated α cannot be present, as for constant parameter one has a symmetry. On-shell $\delta S[\phi] = 0$ for arbitrary variations (least action principle), and after performing an integration by parts in (2.2) one deduces that the Noether current J^μ is classically conserved

$$\partial_\mu J^\mu = 0 \; . \tag{2.3}$$

The quantum theory is defined by the path integral of eq. (2.1). Under a dummy change of integration variables the path integral is left invariant

$$\int D\phi' \; e^{-S[\phi']} = \int D\phi \; e^{-S[\phi]} \; . \tag{2.4}$$

Let us apply this property to an infinitesimal change of integration variables $\phi^i \to \phi'^i = \phi^i + \delta_{\alpha(x)} \phi^i$, where $\delta_{\alpha(x)} \phi^i$ is given by an infinitesimal symmetry transformation with the parameter α replaced by the arbitrary function $\alpha(x)$. In relating the path integral written in terms of ϕ'^i to the one written in terms of ϕ^i (in a condensed notation we include the space-time dependence into the index i), one may use that

$$S[\phi'] = S[\phi] + \delta_{\alpha(x)} S[\phi] \tag{2.5}$$

and taking into account the path integral jacobian $\mathcal{J}$

$$\mathcal{J} = \text{Det}\ \frac{\partial \phi'^i}{\partial \phi^j} = 1 + \text{Tr}\ \frac{\partial \delta_{\alpha(x)} \phi^i}{\partial \phi^j} \equiv 1 + \text{Tr}\ \Delta \tag{2.6}$$

one finds from (2.4)

$$\langle \text{Tr}\ \Delta - \delta_{\alpha(x)} S[\phi] \rangle = 0 \ . \tag{2.7}$$

After an integration by parts this is re-written as

$$\int d^4x\, \alpha(x) \partial_\mu \langle J^\mu \rangle = -\ \text{Tr}\ \Delta \tag{2.8}$$

which shows that the Noether current is not conserved at the quantum level if the path integral measure carries a nontrivial jacobian

$$\partial_\mu \langle J^\mu \rangle \neq 0 \ . \tag{2.9}$$

Here above, quantum expectation values have been indicated by $\langle \cdots \rangle$ and defined as normalized averages within the path integral. We have assumed that the jacobian is independent of the quantum fields, so that it can be pulled out of the expectation value. Of course, one must remember to check that the candidate anomaly computed from (2.8) cannot be canceled by the variation of a local counterterm.

To proceed further, one must define carefully the formal expressions appearing in the above reasonings. Ideally, one would like to fully specify the path integration measure, so that the evaluation of the jacobian would be a well-defined task. In practice, one is able to compute gaussian path integrals only, and resort to perturbative methods for more complicated cases. Nevertheless, one can still obtain the one-loop anomalies by regulating the trace in (2.8), as shown by Fujikawa [5, 6]. Employing a negative-definite operator $\mathcal{R}$ the candidate anomaly is regulated as

$$\mathcal{A} = \lim_{M \to \infty} \text{Tr}\ \Delta\, e^{\frac{\mathcal{R}}{M^2}} \ . \tag{2.10}$$

This functional trace is written in a more explicit notation (for a single scalar field) as

$$\text{Tr}\ \Delta = \int d^4x \int d^4y\, \Delta(x,y) \delta^4(x-y) \ , \qquad \Delta(x,y) = \frac{\delta(\delta_{\alpha(x)} \phi(x))}{\delta \phi(y)} \tag{2.11}$$

and regulated by the differential operator $\mathcal{R}(x)$ acting on the x coordinates as

$$\lim_{M\to\infty} \mathrm{Tr}\ \Delta\, e^{\frac{\mathcal{R}}{M^2}} = \lim_{M\to\infty} \int d^4x \int d^4y\, \Delta(x,y)\, e^{\frac{\mathcal{R}(x)}{M^2}}\, \delta^4(x-y)\ . \tag{2.12}$$

For an arbitrary regulator $\mathcal{R}$, it is not obvious which kind of anomaly one is going to get. In the next section we will provide a method to obtain consistent regulator.

2.2 Consistent regulator via Pauli Villars

A well-defined algorithm for determining those regulators $\mathcal{R}$ which produce consistent anomalies has been established in [7] (see also [8]). The basic idea is to first use a Pauli-Villars (PV) regularization [9], compute the anomalies due to the non-invariance of the PV mass term, and then read off the regulators and jacobians to be used in the Fujikawa's scheme in order to reproduce the anomalies. Since the PV method yields consistent anomalies, being a Feynman graph calculation, one obtains "consistent" regulators.

In more details the PV method for computing one-loop anomalies goes as follows. Let us denote by ϕ a collection of quantum fields with quadratic action

$$\mathcal{L}_\phi = \frac{1}{2}\phi^T T \mathcal{O} \phi \tag{2.13}$$

invariant under a linear symmetry of the form

$$\delta\phi = K\phi\ . \tag{2.14}$$

The case of linear symmetries is enough for the present purposes. The one-loop of this theory is regulated by subtracting a loop of a massive PV fields χ with action

$$\mathcal{L}_\chi = \frac{1}{2}\chi^T T \mathcal{O} \chi + \frac{1}{2} M \chi^T T \chi \tag{2.15}$$

where the last term describes the mass of the PV fields[1]. The invariance of the original

[1]More generally, one should employ a set of PV fields with mass M_i and statistic c_i to be able to regulate and cancel all possible one-loop divergences [9], but for the sake of the present exposition it is enough to consider only one copy of the PV fields. Also, the mass M in the PV lagrangian generically carries an appropriate positive power, according to the mass dimension of the differential operator $\mathcal{O}$.

action is extended to an invariance of the massless part of the PV action by defining

$$\delta\chi = K\chi \tag{2.16}$$

so that only the PV mass term may break the symmetry (if one can find a symmetrical mass term, then the symmetry will be anomaly free). One refers to $T\mathcal{O}$ as the kinetic matrix and to T as the mass matrix. They both depend on eventual background fields, which may get transformed under the symmetry variation as well. The anomalous response of the path integral under a symmetry variation is now due to the mass term only, since the measure of the PV fields χ can be defined in such a way that their jacobian cancels the jacobian of the original fields ϕ, as argued in [7]. Under the symmetry transformation (2.16) the mass term lagrangian of the PV fields varies as

$$\delta\mathcal{L}_\chi = \frac{1}{2}M\chi^T(TK + K^TT + \delta T)\chi\ . \tag{2.17}$$

Using this into the variation of the PV-regulated path integral one computes the anomaly in the Noether current as

$$\begin{aligned} \int d^4x\, \alpha(x)\partial_\mu\langle J^\mu\rangle &= -\lim_{M\to\infty} \mathrm{Tr}\Big[\frac{1}{2}M\Big(TK + K^TT + \delta T\Big)\Big(TM + T\mathcal{O}\Big)^{-1}\Big] \\ &= -\lim_{M\to\infty} \mathrm{Tr}\Big[\Big(K + \frac{1}{2}T^{-1}\delta T\Big)\Big(1 + \frac{\mathcal{O}}{M}\Big)^{-1}\Big] \end{aligned} \tag{2.18}$$

where we replaced K^TT by TK, since both T and $T\mathcal{O}$ are symmetric, and used the χ-propagator from (2.15) to close the χ-loop (recall its relative minus sign with respect to the ϕ-loop). The limit $M \to \infty$ indicates that the PV fields are removed by making them infinitely massive, so that in (2.18) only a mass independent term survives, which gives the anomaly[2].

At this stage one may notice that the expansion of $(1 + \frac{\mathcal{O}}{M})^{-1}$ leads to Feynman graphs, just as the expansion of $e^{-\frac{\mathcal{O}}{M}}$ whenever $\mathcal{O}$ is a positive definite operator. Hence one may cast the anomaly calculation as a typical calculation of a Fujikawa's jacobian

[2]Eventual diverging term are removed by using a set of PV-fields entering the loop with suitable coefficients c_i, instead of a single PV field, as reminded in the previous footnote. It is not necessary to explicitate this procedure further.

as in (2.10) by identifying

$$\Delta = K + \frac{1}{2}T^{-1}\delta T \;, \qquad \mathcal{R} = -\mathcal{O} \;. \tag{2.19}$$

This freedom in regulating path integral jacobians by using suitable functions of the regulator $\mathcal{R}$ was already noticed in [5, 6], and used in [7] to make the above connection.

For many cases the regulator $\mathcal{O}$ is enough, while in other cases (typically when $\mathcal{O}$ is a first order differential operator) one has to improve it. A way of doing this is achieved by inserting the identity $1 = (1 - \frac{\mathcal{O}}{M})(1 - \frac{\mathcal{O}}{M})^{-1}$ into (2.18), so that the functional trace becomes

$$\mathrm{Tr}\left[\left(K + \frac{1}{2}T^{-1}\delta T\right)\left(1 - \frac{\mathcal{O}}{M}\right)\left(1 - \frac{\mathcal{O}^2}{M^2}\right)^{-1}\right] , \tag{2.20}$$

which can be simplified using the invariance of the kinetical part of the action

$$\chi^T (T\mathcal{O}K + \frac{1}{2}\delta(T\mathcal{O}))\chi = 0 \tag{2.21}$$

obtaining

$$\Delta = K + \frac{1}{2}T^{-1}\delta T + \frac{1}{2}\delta\mathcal{O}M^{-1} \;, \qquad \mathcal{R} = \mathcal{O}^2 \tag{2.22}$$

valid if $\mathcal{O}^2$ becomes a positive definite operator. To calculate the anomaly we then have to calculate

$$\mathcal{A} = \lim_{M\to\infty} \mathrm{Tr}\; \Delta\, e^{\frac{\mathcal{R}}{M^2}} \;. \tag{2.23}$$

To do so in the next section we are going to introduce the heat kernel.

2.3 Heat kernel

The heat kernel is the solution of the heat equation

$$-\frac{\partial}{\partial\beta}K = \widehat{H}K, \tag{2.24}$$

where $\widehat{H}$ is a second order elliptic differential operator. This equation in quantum physics is obtained via the Wick rotation of the Schroedinger equation, where $\widehat{H}$ is the hamiltonian of the system. The heat kernel is especially useful for one-loop calculations in QFT, in particular for Laplace type operator. The solution of the above equation, the heat kernel itself, can be written as

$$K(x, y, \beta, \widehat{H}) = \langle y|e^{-\beta\widehat{H}}|x\rangle. \tag{2.25}$$

satisfying the boundary condition

$$K(x, y, 0, \widehat{H}) = \delta(x - y). \tag{2.26}$$

This is a formal solution because as the differential operator becomes more complicated is not always possible to find an exact solution, and one as to turn to perturbative methods. It is possible to show that on manifolds without boundaries one can expand the heat kernel as

$$K(\beta, x, y, \widetilde{H}) = K(\beta, x, y, \widetilde{H}_0)(1 + \beta a_2(x, y) + \beta^2 a_4(x, y) + ...) \tag{2.27}$$

where $\widetilde{H}_0$ is the free hamiltonian operator. The coefficients $a_{2n}(x, y)$ in the limit $x \to y$ are regular, they are called heat kernel coefficient and are known for all the most common operator of quantum physics. For all the calculation involved in this thesis we will have the following situation

$$\lim_{M\to\infty} \mathrm{Tr}\ \Delta\, e^{\frac{\mathcal{R}}{M^2}} = \lim_{M\to\infty} \int d^4x \int d^4y\, \Delta(x, y)\, \langle y|e^{\frac{\mathcal{R}(x)}{M^2}}|x\rangle\, \delta^4(x - y)\ . \tag{2.28}$$

where M^2 plays the role of β so that the heat kernel for us will appear in the form

$$\langle x|e^{\frac{\mathcal{R}(x)}{M^2}}|x\rangle. \tag{2.29}$$

In the calculations below we will only need the coefficient $a_4(x, x)$ at coinciding points, that we are going to denote by $a_4(x)$. For a second order elliptic differential operator in a curved space without additional structures (i.e. no gauge fields or scalar potentials) one can check that, by dimensional analysis, it is quadratic in the curvature so that it takes the form

$$a_4(x) = aR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + bR_{\mu\nu}R^{\mu\nu} + cR^2 + d\Box R + e\sqrt{g}\epsilon_{\mu\nu\rho\sigma}R^{\mu\nu\alpha\beta}R^{\rho\sigma}{}_{\alpha\beta} \tag{2.30}$$

where the constants a, b, c, d and e depend on the specific operator. Of course, the term proportional to e may be present only for chiral operators (i.e. not invariant under parity). For the explicit calculation of the various coefficients for various differential operators we refer to [10].

As we shall see, for conformal fields, the coefficient $a_4(x)$ will identify the trace anomaly. In general, this anomaly must satisfy certain integrability conditions that arise from the fact that the anomaly can be seen as arising from the variation of a functional, the effective action. These consistency conditions have been worked out in [20], and imply that only a certain combinations of the curvatures can appear in a_4. These combinations are the topological Euler density E_4

$$E_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \tag{2.31}$$

the square of the Weyl tensor C^2

$$C^2 \equiv C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2 \tag{2.32}$$

the topological Pontryagin density P

$$P = \sqrt{g}\epsilon_{\mu\nu\rho\sigma}R^{\mu\nu\alpha\beta}R^{\rho\sigma}{}_{\alpha\beta} \tag{2.33}$$

and the term $\Box R$. This last term can be removed by a local counterterm, and thus is not considered as a true anomaly. Thus only three coefficients characterize the trace anomaly of conformal fields, so that up to the $\Box R$ terms one expects for them an expression of the form

$$a_4(x) = \alpha E_4 + \beta C^2 + \gamma P. \tag{2.34}$$

However the topological Pontryagin density P has never been observed to arise in conformal field theories until the recent claims made in [2, 3].

2.4 Scalar field trace anomaly

In this section we would like to calculate the trace anomaly for a scalar field using the methods introduced so far. We cast the model in euclidian space, so that after a Wick rotation we have the euclidean action

$$S = \int d^4x \sqrt{g}\frac{1}{2}\Big(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{6}R\phi^2\Big)\,. \tag{2.35}$$

We want to write the lagrangian in the form $\mathcal{L} = \phi T\mathcal{O}\phi$ to apply the method we have just explained. After a partial integration we get

$$S = -\int d^4x \sqrt{g}\frac{1}{2}\phi\Big(\Box + \frac{1}{6}R\Big)\phi\,. \tag{2.36}$$

where $\Box$ is the scalar laplacian

$$\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu = g^{\mu\nu}\nabla_\mu\partial_\nu = \frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}g^{\mu\nu}\partial_\nu. \tag{2.37}$$

This covariant calculation can be check by standard algebra, without using the concept of covarinat derivativest

$$\begin{aligned}\mathcal{L} =& \sqrt{g}\frac{1}{2}\Big(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{6}R\phi^2\Big) = \\ =& \frac{1}{2}\sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{12}\sqrt{g}R\phi^2 - \frac{1}{2}\partial_\mu(\sqrt{g}g^{\mu\nu}\phi\partial_\nu\phi) = \\ =& -\frac{1}{4}\sqrt{g}g^{\alpha\beta}\partial_\mu g_{\alpha\beta}g^{\mu\nu}\phi\partial_\nu\phi - \frac{1}{2}\sqrt{g}\partial_\mu g^{\mu\nu}\phi\partial_\nu\phi - \frac{1}{2}\sqrt{g}\phi g^{\mu\nu}\partial_\mu\partial_\nu\phi - \frac{1}{12}\sqrt{g}R\phi^2,\end{aligned} \tag{2.38}$$

where we added a total derivative. Now remembering that the covariant derivative of the metric is zero

$$\begin{aligned}\mathcal{L} =& -\frac{1}{4}\sqrt{g}g^{\alpha\beta}({\Gamma^\rho}_{\mu\alpha}g_{\rho\beta} + {\Gamma^\rho}_{\mu\beta}g_{\alpha\rho})g^{\mu\nu}\phi\partial_\nu\phi + \frac{1}{2}\sqrt{g}({\Gamma^\mu}_{\mu\rho}g^{\rho\nu} + {\Gamma^\nu}_{\mu\rho}g^{\mu\rho})\phi\partial_\nu\phi + \\ & -\frac{1}{2}\sqrt{g}\phi g^{\mu\nu}\partial_\mu\partial_\nu\phi - \frac{1}{12}\sqrt{g}R\phi^2 = \\ =& \frac{1}{2}\phi[-\sqrt{g}(g^{\mu\nu}\partial_\mu\partial_\nu - {\Gamma^{\nu\mu}}_\mu\partial_\nu + \frac{1}{6}R)]\phi = \\ =& \frac{1}{2}\phi[-\sqrt{g}(\nabla^\nu\nabla_\nu + \frac{1}{6}R)]\phi,\end{aligned} \tag{2.39}$$

in the last line we reintroduced the covariant derivative.

To calculate the anomaly at this point we introduce a lagrangian for a Pauli Villars field χ with a covariant mass term

$$\mathcal{L} = -\frac{1}{2}\sqrt{g}\chi\Big(\Box + \frac{1}{6}R\Big)\chi + \frac{1}{2}\sqrt{g}M\chi^2, \tag{2.40}$$

so that using the definitions previously described we recognize

$$T\mathcal{O} = -\sqrt{g}(\Box + \frac{1}{6}R) \tag{2.41}$$

$$T = \sqrt{g} \tag{2.42}$$

$$\mathcal{O} = -(\Box + \frac{1}{6}R) \tag{2.43}$$

$$T^{-1} = \frac{1}{\sqrt{g}}. \tag{2.44}$$

Now we have to recall

$$\delta g_{\mu\nu} = \sigma(x)g_{\mu\nu} \tag{2.45}$$

$$\delta\phi = -\frac{1}{2}\sigma(x)\phi \tag{2.46}$$

to calculate K and δT. We can immediately recognize

$$K = -\frac{1}{2}\sigma \tag{2.47}$$

and with some calculation we get

$$\delta T = \frac{1}{2}\sqrt{g}g^{\mu\nu}\sigma g_{\mu\nu} = 2\sigma\sqrt{g}. \tag{2.48}$$

Now since $\mathcal{O}$ is a second order differential operator we can stop here, and say that the anomaly is equal to

$$\begin{aligned}\mathcal{A} &= \mathrm{Tr}\,\Delta e^{-\frac{\mathcal{O}}{M^2}} = \\ &= \mathrm{Tr}\Big[(K + \frac{1}{2}T^{-1}\delta T)e^{-\frac{\mathcal{O}}{M^2}}\Big] = \mathrm{Tr}\Big[(-\frac{1}{2}\sigma + \frac{1}{2}\frac{1}{\sqrt{g}}2\sigma\sqrt{g}\Big)e^{\frac{\Box+\frac{1}{6}R}{M^2}}\Big] = \\ &= \mathrm{Tr}\Big(\frac{1}{2}\sigma e^{\frac{\Box+\frac{1}{6}R}{M^2}}\Big).\end{aligned} \tag{2.49}$$

To explicitly calculate the functional trace we use an heat kernel approach considering the expansion

$$\mathcal{A} = \frac{M^4}{(4\pi)^2}\int d^4x\sqrt{g}\sigma(x)\sum\frac{a_{2n(x)}}{M^{2n}} \tag{2.50}$$

where we recognize the Seeley-DeWitt coefficients at coinciding points [11, 12]. The negative powers of M disappear as it goes to infinity, the positive one instead can be eliminated considering a set of Pauli Villars field with appropriate masses. The result we are interested in, as a consequence, has only one term left which after calculation gives rise to

$$\mathcal{A} = \frac{1}{5760\pi^2}\int d^4x\sqrt{g}\sigma(x)(R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - R_{\mu\nu}R^{\mu\nu} - \Box R) \tag{2.51}$$

in agreements with the results found in the literature.

We want now to find the value of the trace of the energy-momentum tensor; to do so recalling equation (2.7) we can write

$$\int d^4x\sqrt{g}\sigma(x)T^{\mu}{}_{\mu} = -\frac{M^4}{(4\pi)^2}\int d^4x\sqrt{g}\sigma(x)\sum\frac{a_{2n(x)}}{M^{2n}}. \tag{2.52}$$

The energy-momentum tensor trace as a consequence is equal to

$$T^{\mu}{}_{\mu} = -\frac{a_4}{(4\pi)^2} = -\frac{1}{5760\pi^2}(R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - R_{\mu\nu}R^{\mu\nu} + \Box R). \tag{2.53}$$

so that, up the $\Box R$ term

$$T^{\mu}{}_{\mu}(x) = \frac{1}{11520\pi^2}(E_4 - 3C^2). \tag{2.54}$$

Obviously, the Pontryagin term cannot arise as the theory is not chiral.

2.5 Vector field trace anomaly

Next step we will calculate the trace anomaly also for a massless vector field. The action has already been presented earlier in Minkowski space, but here we wish to proceed in euclidean space so that after a Wick rotation we have

$$S = \int d^4x\sqrt{g}\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}\,. \tag{2.55}$$

Now to compute the trace anomaly we have to quantize the system. However the model enjoys the standard gauge invariance, $\delta A_\mu(x) = \partial_\mu \Lambda(x)$, and one must introduce a gauge fixing procedure to proceed with the quantization. We choose as gauge condition the covariant Feynman gauge, which gives rise to Faddeev-Popov ghosts as well. As the model is coupled to gravity, the latter cannot be neglected. These modifications will alter the form of the energy momentum tensor (in particular there will appear also the contribution of the ghosts fields). However one may prove by using BRST methods that these modifications do not modify the result for gauge invariant observables, and in particular the expectation value of the energy-momentum tensor and its trace.

Thus, we add to the classical action the gauge fixing term

$$\mathcal{L}_{gf} = \frac{1}{2}\sqrt{g}\,(\nabla^\mu A_\mu)^2 \tag{2.56}$$

and the corresponding Faddeev-Popov ghost term

$$\mathcal{L}_{FP} = \sqrt{g}\,\nabla^\mu b\,\partial_\mu c \tag{2.57}$$

where b and c are the scalar Faddeev-Popov ghosts of anticommuting character. To have a rigid scale invariance the ghosts must scale as the scalar fields of the previous section,

$$\delta g_{\mu\nu} = \sigma g_{\mu\nu} \tag{2.58}$$

$$\delta b = -\frac{1}{2}\sigma b \tag{2.59}$$

$$\delta c = -\frac{1}{2}\sigma c \tag{2.60}$$

but the full Weyl invariance is broken in the gauge fixing sector.

Let us now rewrite the kinetic terms of the various fields to expose the kinetic differential operators and check that the gauge fixing procedure has produced invertible operators. For the gauge field A_μ we have

$$\mathcal{L}_A = \mathcal{L} + \mathcal{L}_{gf} = \sqrt{g}\Big(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}(\nabla^\mu A_\mu)^2\Big) \tag{2.61}$$

and up to total derivatives (that integrate to zero in the action) we manipulate it and rewrite it as

$$
\begin{aligned}
\mathcal{L}_A &= \sqrt{g}\left(\frac{1}{4}(\nabla^\mu A^\nu - \nabla^\nu A^\mu)(\nabla_\mu A_\nu - \nabla_\nu A_\mu) + \frac{1}{2}\nabla^\mu A_\mu \nabla^\nu A_\nu\right) \\
&= \sqrt{g}\left(\frac{1}{2}\nabla^\mu A^\nu \nabla_\mu A_\nu - \frac{1}{2}\nabla^\nu A^\mu \nabla_\mu A_\nu + \frac{1}{2}\nabla^\mu A_\mu \nabla^\nu A_\nu\right) \\
&= -\sqrt{g}\left(\frac{1}{2}A^\nu \nabla^\mu \nabla_\mu A_\nu - \frac{1}{2}A^\mu \nabla^\nu \nabla_\mu A_\nu + \frac{1}{2}A_\mu \nabla^\mu \nabla^\nu A_\nu\right) \\
&= -\sqrt{g}\left(\frac{1}{2}A^\mu \Box_1 A_\mu - \frac{1}{2}A^\mu \nabla_\nu \nabla_\mu A^\nu + \frac{1}{2}A^\mu \nabla_\mu \nabla_\nu A^\nu\right) \\
&= -\sqrt{g}\left(\frac{1}{2}A^\mu \Box_1 A_\mu + \frac{1}{2}A^\mu [\nabla_\mu, \nabla_\nu] A^\nu\right) \\
&= -\sqrt{g}\left(\frac{1}{2}A^\mu \Box_1 A_\mu + \frac{1}{2}A^\mu R_{\mu\nu}{}^\nu{}_\lambda A^\lambda\right) \\
&= -\sqrt{g}\left(\frac{1}{2}A^\mu \Box_1 A_\mu + \frac{1}{2}A^\mu R_{\mu\nu} A^\nu\right) \\
&= \sqrt{g}\,\frac{1}{2}A^\mu(-g_{\mu\nu}\Box_1 - R_{\mu\nu})A^\nu
\end{aligned}
\tag{2.62}
$$

where $\Box_1 = \nabla^\mu \nabla_\mu$ is the covariant laplacian acting on vectors. It is obviously invertible (at least for small curvatures, as it approaches the flat laplacian).

Similarly, for the ghost fields a partial integration puts the action in the form

$$
\mathcal{L}_{FP} = \sqrt{g}\, b(-\nabla^\mu \partial_\mu)c = \sqrt{g}\, b(-\Box)c \tag{2.63}
$$

where $\Box = \nabla^\mu \partial_\mu$ is the covariant laplacian acting on scalars, which appeared already in the previous section. There are two ghosts, so they contribuite to the anomaly as two scalar fields, but notice that no improvement terms is present, so it will be different from the one already obtained.

Now to find the regulator we consider mass terms of the form

$$
\mathcal{L}_{A,M} = \frac{1}{2}M\sqrt{g}g^{\mu\nu}A_\mu A_\nu \tag{2.64}
$$

and

$$
\mathcal{L}_{PV,M} = M\sqrt{g}bc \tag{2.65}
$$

to be used for the corresponding PV fields.

Then in the A_μ sector we can recognize the following quantities

$$T\mathcal{O} = -\sqrt{g}(g^{\mu\nu}\Box_1 + R^{\mu\nu}) \tag{2.66}$$

and

$$T = \sqrt{g}g^{\mu\nu} \tag{2.67}$$

Now we need to calculate T^{-1} and δT to identify $\mathcal{O}$ and Δ as in (2.19). We do not need additional terms since $\mathcal{O}$ is a second order operator. We find

$$T^{-1} = \frac{1}{\sqrt{g}} g_{\mu\nu} \tag{2.68}$$

and recalling once again that

$$\begin{aligned} \delta g_{\mu\nu} &= \sigma g_{\mu\nu} \\ \delta A_\mu &= 0 \end{aligned} \tag{2.69}$$

we calculate

$$\delta T = 2\sigma\sqrt{g}g^{\mu\nu} - \sigma\sqrt{g}g^{\mu\nu} = \sigma\sqrt{g}g^{\mu\nu} \tag{2.70}$$

$$K = 0 \tag{2.71}$$

so that

$$\begin{aligned} \Delta &= \sigma\delta_\mu{}^\nu \\ \mathcal{O} &= -\delta_\mu{}^\nu\Box_1 - R_\mu{}^\nu. \end{aligned} \tag{2.72}$$

Thus the contribution to the trace anomaly is

$$\mathcal{A}_A = \lim_{M\to\infty} \mathrm{Tr}(\sigma\delta_\mu{}^\nu e^{\frac{\delta_\mu{}^\nu\Box_1 + R_\mu{}^\nu}{M^2}}). \tag{2.73}$$

where only the M^0 survives in the limit (after renomalization). It produces a Seeley-DeWitt coefficient that can be extracted from the literature [11, 12].

However, this will not give the final result, yet, as we have to add the ghosts contribution. As we have mentioned before, the b, c ghost sector produces the contribution of two scalar without the improvement term, with a global minus sing as it correspond to anticommuting fields. Thus proceeding as before one finds at the end the contribution

$$\mathcal{A}_{bc} = -2 \lim_{M\to\infty} \mathrm{Tr}(\sigma e^{\frac{\Box}{M^2}}). \tag{2.74}$$

where only the M^0 survives in the limit (after renomalization). The corresponding Seeley-DeWitt coefficient can again be extracted from the literature [11, 12].

Adding up the two contributions, one finds the complete trace anomaly for the spin 1 field that reads

$$\mathcal{A} = \frac{1}{5760\pi^2} \int d^4x \sqrt{g}\sigma(x)(-13R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} + 88R_{\mu\nu}R^{\mu\nu} - 25R^2 - 18\Box R). \tag{2.75}$$

To evaluate the energy momentum tensor as we did before we use the eqation

$$\int d^4x \sqrt{g}\sigma(x)T^\mu{}_\mu = -\mathcal{A}, \tag{2.76}$$

so that the result is

$$T^\mu{}_\mu = \frac{1}{5760\pi^2}(13R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 88R_{\mu\nu}R^{\mu\nu} + 25R^2 + 18\Box R). \tag{2.77}$$

Disregarding the term $\Box R$ since it can be eliminated via a counter-term we rewrite the last expression using E_4 and C^2

$$T^\mu{}_\mu(x) = \frac{1}{11520\pi^2}(62E_4 - 36C^2). \tag{2.78}$$

Once again since this is not a chiral theory the Pontryagin term cannot arise.

Chapter 3

Trace anomaly for fermions

3.1 Trace anomaly for a Dirac model

The lagrangian of a massless Dirac fermion ψ has been discussed in section 1.5 and can be written as

$$\mathcal{L} = -e\,\overline{\psi}\gamma^\mu \nabla_\mu \psi \tag{3.1}$$

where we recall that the covariant derivative reads

$$\nabla_\mu = \partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^a\gamma^b \tag{3.2}$$

and that γ^μ contains the inverse of the vielbein $e_\mu{}^a$, i.e. $\gamma^\mu = e^\mu{}_a\gamma^a$.

We want to evaluate the Weyl anomaly and to do so we are going to use the Fujikawa method with consistent regulator as described previously. Thus we need to add a mass term for the PV part and manipulate the lagrangian to put in the form

$$\mathcal{L} = \frac{1}{2}\chi^T T\mathcal{O}\chi + \frac{1}{2}M\chi^T T\chi \tag{3.3}$$

where the operator $T\mathcal{O}$ has to be the same for both the fermion and the PV part. So we manipulate la lagrangian to cast it in the required form

$$
\begin{aligned}
\mathcal{L} =& - e\,\overline{\psi}\gamma^\mu \nabla_\mu \psi = -\frac{1}{2} e\overline{\psi}\gamma^\mu\nabla_\mu\psi - \frac{1}{2}e\overline{\psi}\gamma^\mu\partial_\mu\psi - \frac{1}{8}e\overline{\psi}\omega_{\mu ab}\gamma^c\gamma^a\gamma^b e^\mu{}_c\psi \\
=& -\frac{1}{2}e\overline{\psi}\gamma^\mu\nabla_\mu\psi + \frac{1}{2}\partial_\mu(e\overline{\psi}e^\mu{}_a)\gamma^a\psi + \\
&-\frac{1}{8}e\overline{\psi}\omega_{\mu ab}(\gamma^a\gamma^b\gamma^c + 2\eta^{ca}\gamma^b - 2\eta^{cb}\gamma^a)e^\mu{}_c\psi = \\
=& -\frac{1}{2}e\overline{\psi}\gamma^\mu\nabla_\mu\psi + \frac{1}{2}e\partial_\mu\overline{\psi}\gamma^\mu\psi + \frac{1}{2}e\overline{\psi}e^\nu{}_a\partial_\mu e_\nu{}^a\gamma^\mu\psi + \frac{1}{2}e\overline{\psi}\partial_\mu e^\mu{}_a\gamma^a\psi + \\
&-\frac{1}{8}e\overline{\psi}\omega_{\mu ab}\gamma^a\gamma^b\gamma^\mu\psi - \frac{1}{4}e\overline{\psi}(\omega_{\mu cb}\gamma^b - \omega_{\mu ac}\gamma^a)e^{\mu c}\psi = \\
=& -\frac{1}{2}e\overline{\psi}\gamma^\mu\nabla_\mu\psi + \frac{1}{2}e\overline{\psi}(\overleftarrow{\partial_\mu} - \frac{1}{4}\omega_{\mu ab}\gamma^a\gamma^b)\gamma^\mu\psi + \\
&+\frac{1}{2}e\overline{\psi}e^\nu{}_a(\Gamma_{\mu\nu}{}^\kappa e_\kappa{}^a - \omega_\mu{}^a{}_b e_\nu{}^b)\gamma^\mu\psi - \frac{1}{2}e\overline{\psi}(\Gamma_{\kappa\mu}{}^\mu e^\kappa{}_a + \omega_{\mu a}{}^b e^\mu{}_b)\gamma^a\psi \\
&-\frac{1}{2}e\overline{\psi}\omega_{\mu ab}\gamma^b e^{\mu a}\psi = \\
=& -\frac{1}{2}e\overline{\psi}\gamma^\mu\nabla_\mu\psi - \frac{1}{2}e\psi^T(\gamma^\mu)^T(\partial_\mu - \frac{1}{4}\omega_{\mu ab}(\gamma^a\gamma^b)^T)\overline{\psi}^T = \\
=&\frac{1}{2}\psi_c{}^T(-eC^T\gamma^\mu\nabla_\mu)\psi + \frac{1}{2}\psi^T(-e(\gamma^\mu)^T\widetilde{\nabla}_\mu C)\psi_c
\end{aligned}
\tag{3.4}
$$

where we have found convenient to use the charge conjugated field

$$
\psi_c = C^{-1}\overline{\psi}^T \tag{3.5}
$$

and the operator

$$
\widetilde{\nabla}_\mu = \partial_\mu - \frac{1}{4}\omega_{\mu ab}(\gamma^a\gamma^b)^T. \tag{3.6}
$$

In order to reach the final line we added the total derivative term $\frac{1}{2}\partial_\mu(e\overline{\psi}\gamma^\mu\psi)$ and then moved γ_μ from the right to the left of the ω term. Then we remembered the property we discussed in chapter 2, and in particular the fact that $\omega_{\mu ab}$ is antisymmetric in its lower roman indices and that the covariant derivative of the vielbein is zero. Then in the final line we introduced ψ_c. Thus we may rewrite the lagrangian in the form

$$
\mathcal{L} = \frac{1}{2}\begin{pmatrix}\psi^T & \psi_c{}^T\end{pmatrix}\begin{pmatrix}0 & -e(\gamma^\mu)^T\widetilde{\nabla}_\mu C \\ -eC^T\gamma^\mu\nabla_\mu & 0\end{pmatrix}\begin{pmatrix}\psi \\ \psi_c\end{pmatrix}. \tag{3.7}
$$

Now we add the Dirac mass term which will be the one producing the anomaly

$$\begin{aligned}\Delta\mathcal{L}_M =& e\frac{M}{2}(\overline{\psi}\psi + \overline{\psi}\psi) = e\frac{M}{2}(-\psi^T C C^{-1}\overline{\psi}^T + \overline{\psi}(C^{-1})^T C^T \psi) = \\ =& e\frac{M}{2}(-\psi^T C \psi_c + {\psi_c}^T C^T \psi)\end{aligned} \tag{3.8}$$

that written as a matrix reads

$$\Delta\mathcal{L}_M = \frac{M}{2}\begin{pmatrix}\psi^T & {\psi_c}^T\end{pmatrix}\begin{pmatrix}0 & -eC \\ eC^T & 0\end{pmatrix}\begin{pmatrix}\psi \\ \psi_c\end{pmatrix}. \tag{3.9}$$

Using eq. (3.7) as the expression for $T\mathcal{O}$ we obtain

$$\mathcal{O} = \begin{pmatrix}-\gamma^\mu\nabla_\mu & 0 \\ 0 & C^{-1}(\gamma^\mu)^T\widetilde{\nabla}_\mu C\end{pmatrix}. \tag{3.10}$$

These expressions for the cases we have discussed can be rearranged by using

$$\begin{aligned}C^{-1}(\gamma^\mu)^T\widetilde{\nabla}_\mu C =& C^{-1}(\gamma^\mu)^T C\partial_\mu + \\ & -\frac{1}{4}\omega_{\mu ab}C^{-1}(\gamma^\mu)^T C C^{-1}(\gamma^b)^T C C^{-1}(\gamma^a)^T C = -\gamma^\mu\nabla_\mu.\end{aligned} \tag{3.11}$$

Now we need to calculate K, δT and $\delta\mathcal{O}$ to do so we recall the Weyl rescaling transformations

$$\delta {e_\mu}^a = \frac{1}{2}\sigma(x){e_\mu}^a \tag{3.12}$$

$$\delta {e^\mu}_a = -\frac{1}{2}\sigma(x){e^\mu}_a \tag{3.13}$$

$$\delta e = 2\sigma(x)e \tag{3.14}$$

$$\delta\omega_{\mu ab} = \frac{1}{2}(e_{\mu a}{e^\kappa}_b - e_{\mu b}{e^\kappa}_a)\partial_\kappa\sigma(x) \tag{3.15}$$

$$\delta\psi = -\frac{3}{4}\sigma(x)\psi \tag{3.16}$$

$$\delta\psi_c = -\frac{3}{4}\sigma(x)\psi_c \tag{3.17}$$

that in a straightforward way, when considering T, where the only varying part is e, leads to

$$\delta T = \begin{pmatrix} 0 & -2e\sigma(x)C \\ 2e\sigma(x)C^T & 0 \end{pmatrix}. \tag{3.18}$$

For the variation of $\mathcal{O}$ we need to compute

$$\begin{aligned}
\delta(\gamma^\mu \Delta_\mu) =& \gamma^a \delta e^\mu{}_a \partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^c \delta e^\mu{}_c \gamma^a\gamma^b + \frac{1}{4}\delta\omega_{\mu ab}\gamma^c e^\mu{}_c \gamma^a \gamma^b = \\
=& -\frac{1}{2}\sigma(x)\gamma^\mu\partial_\mu - \frac{1}{8}\sigma(x)\gamma^\mu\omega_{\mu ab}\gamma^a\gamma^b + \\
&+ \frac{1}{8}(e_{\mu a}e^\kappa{}_b - e_{\mu b}e^\kappa a)\partial_\kappa\sigma(x)\gamma^c e^\mu{}_c\gamma^a\gamma^b = \\
=& -\frac{1}{2}\sigma(x)\gamma^\mu\nabla_\mu + \frac{1}{8}\eta_{ac}\gamma^c\gamma^a\gamma^\kappa\partial_\kappa\sigma(x) + \\
&- \frac{1}{8}\eta_{cb}(-\gamma^\kappa\gamma^c\gamma^b + 2\eta^{\kappa c}\gamma^b)\partial_\kappa\sigma(x) \\
=& -\frac{1}{2}\sigma(x)\gamma^\mu\nabla_\mu + \frac{3}{4}\gamma^\mu\partial_\mu\sigma(x)
\end{aligned} \tag{3.19}$$

in this last calculation we apply the above transformation law then used the relation $\eta_{ab} = e^\mu{}_a e_{\mu b}$, move γ^κ and then use $\eta_{ab}\gamma^a\gamma^b = 4\mathbb{1}$. In the end we obtain

$$\delta\mathcal{O} = \begin{pmatrix} \frac{1}{2}\sigma\gamma^\mu\nabla_\mu - \frac{3}{4}\gamma^\mu\partial_\mu\sigma & 0 \\ 0 & \frac{1}{2}\sigma\gamma^\mu\nabla_\mu - \frac{3}{4}\gamma^\mu\partial_\mu\sigma. \end{pmatrix} \tag{3.20}$$

Next we will like to calculate $\mathcal{O}^2$ and the results is

$$\mathcal{O}^2 = \begin{pmatrix} \gamma^\mu\nabla_\mu\gamma^\nu\nabla_\nu & 0 \\ 0 & \gamma^\mu\nabla_\mu\gamma^\nu\nabla_\nu \end{pmatrix} \tag{3.21}$$

This is the regulator in both the Dirac and the Weyl case. Next we need K that we can simply read off from the relations (3.16) and (3.17)

$$K = \begin{pmatrix} -\frac{3}{4}\sigma & 0 \\ 0 & -\frac{3}{4}\sigma \end{pmatrix}. \tag{3.22}$$

The last matrix that we need to evaluate is T^{-1} that can be easily show to be

$$T^{-1} = \begin{pmatrix} 0 & \frac{(C^T)^{-1}}{e} \\ -\frac{C^{-1}}{e} & 0 \end{pmatrix}. \tag{3.23}$$

We recall now that we are trying to get to

$$\mathcal{A} = \mathrm{Tr}\,\Delta e^{\frac{\mathcal{O}^2}{M^2}} = \mathrm{Tr}\lim_{M\to\infty}(K + \frac{1}{2}T^{-1}\delta T + \frac{1}{2}\delta\mathcal{O}M^{-1})e^{\frac{\mathcal{O}^2}{M^2}} \tag{3.24}$$

and we have obtained

$$\begin{aligned}\Delta &= \begin{pmatrix} -\frac{3}{4}\sigma & 0 \\ 0 & -\frac{3}{4}\sigma \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & \frac{(C^T)^{-1}}{e} \\ -\frac{C^{-1}}{e} & 0 \end{pmatrix}\begin{pmatrix} 0 & -2e\sigma(x)C \\ 2e\sigma(x)C^T & 0 \end{pmatrix} + \\ &+ \frac{1}{2M}\begin{pmatrix} \frac{1}{2}\sigma\gamma^\mu\nabla_\mu - \frac{3}{4}\gamma^\mu\partial_\mu\sigma & 0 \\ 0 & \frac{1}{2}\sigma\gamma^\mu\nabla_\mu - \frac{3}{4}\gamma^\mu\partial_\mu\sigma \end{pmatrix} = \\ &= \begin{pmatrix} \frac{1}{4}\sigma + \frac{1}{4M}\sigma\gamma^\mu\nabla_\mu - \frac{3}{8M}\gamma^\mu\partial_\mu\sigma & 0 \\ 0 & \frac{1}{4}\sigma + \frac{1}{4M}\sigma\gamma^\mu\nabla_\mu - \frac{3}{8M}\gamma^\mu\partial_\mu\sigma \end{pmatrix},\end{aligned} \tag{3.25}$$

now since $\mathcal{O}^2$ is in diagonal form we recall the propriety

$$e^{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}} = \begin{pmatrix} e^A & 0 \\ 0 & e^B \end{pmatrix} \tag{3.26}$$

which leads to

$$\mathcal{A} = \mathrm{Tr}\begin{pmatrix} (\frac{1}{4}\sigma + \frac{1}{4M}\sigma\gamma^\mu\nabla_\mu - \frac{3}{8M}\gamma^\mu\partial_\mu\sigma)e^{\frac{(\gamma^\mu\nabla_\mu)^2}{M^2}} & 0 \\ 0 & (\frac{1}{4}\sigma + \frac{1}{4M}\sigma\gamma^\mu\nabla_\mu - \frac{3}{8M}\gamma^\mu\partial_\mu\sigma)e^{\frac{(\gamma^\mu\nabla_\mu)^2}{M^2}} \end{pmatrix} \tag{3.27}$$

so that we get

$$\mathcal{A} = \mathrm{Tr}\Big[(\frac{1}{2}\sigma + \frac{1}{2M}\sigma\gamma^\mu\nabla_\mu - \frac{3}{4M}\gamma^\mu\partial_\mu\sigma)e^{\frac{(\gamma^\mu\nabla_\mu)^2}{M^2}}\Big]. \tag{3.28}$$

At this point if we use the series expansion of the exponential each member of the series has an even number of gamma matrices; this implies that both the second and the third term of the above result will generate a series with an odd number of gamma matrices for which the Dirac trace is always null. Thanks to this consideration we can rewrite $\mathcal{A}$ as

$$\mathcal{A} = \mathrm{Tr}\Big[\frac{1}{2}\sigma e^{\frac{(\gamma^\mu \nabla_\mu)^2}{M^2}}\Big] \tag{3.29}$$

which is the expression of trace anomaly for a Dirac fermion already considered by Fujikawa. Using the heat kernel results that can be extracted from the literature we get

$$\mathcal{A} = \frac{1}{5760\pi^2}\int d^4x\sqrt{g}\sigma(x)(-\frac{7}{2}R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 4R_{\mu\nu}R^{\mu\nu} + \frac{5}{2}R^2 + 6\Box R). \tag{3.30}$$

We can obtain the trace of the energy momentum tensor from

$$\int d^4x\sqrt{g}\frac{1}{2}\sigma(x)T^\mu{}_\mu = \mathcal{A} \tag{3.31}$$

The sign differs from the one used so far since we are now dealing with a path integral defined for fermions and no longer for bosons, as a consequence the trace of the jacobian (2.6) has the opposite sign. The energy-momentum tensor trace as a consequence is equal to

$$T^\mu{}_\mu = -\frac{1}{5760\pi^2}(\frac{7}{2}R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} + 4R_{\mu\nu}R^{\mu\nu} - \frac{5}{2}R^2 - 6\Box R). \tag{3.32}$$

Now as explained in the second chapter we would like to check the consistency condition by rewriting the tensor as a combination of E_4, C^2 and P. Since this is not a chiral theory we certainly cannot obtain a contribution from P and we find the well-known result

$$T^\mu{}_\mu(x) = \frac{1}{11520\pi^2}(11E_4 - 18C^2)\ . \tag{3.33}$$

3.2 Trace anomaly for a Weyl model

Finally, let us consider the case of our interest, a left handed Weyl spinor λ defined by

$$\gamma^5\lambda = \lambda \tag{3.34}$$

with γ^5 the chirality matrix (see appendix for our conventions). The lagrangian for such a chiral spinor cannot contain a Dirac mass term, as the Lorentz scalar $\overline{\lambda}\lambda$ vanishes. Originally the neutrino was treated this way.

Let us review in some details the model in flat space to build the stage for later analysis. The standard lagrangian for the massless Weyl fermion λ in flat space reads

$$\mathcal{L} = -\overline{\lambda}\gamma^a\partial_a\lambda \tag{3.35}$$

with the spinor λ containing just two complex (Grassmann valued) independent functions instead of four, because of the chiral constraint (3.34). It is invariant under $U(1)$ phase transformation

$$\lambda \to \lambda' = e^{i\alpha}\lambda\ , \qquad \overline{\lambda} \to \overline{\lambda}' = e^{-i\alpha}\overline{\lambda} \tag{3.36}$$

that gives rise to a conserved fermion number. The related conserved current

$$j^a = i\overline{\lambda}\gamma^a\lambda \tag{3.37}$$

is called the chiral current, as the fermion is chiral. It is anomalous when the coupling to curved space is introduced. As anticipated, a Dirac mass term preserving the chiral $U(1)$ symmetry cannot be introduced, as the $\overline{\lambda}\lambda$ bilinear vanishes. However, one may add to (3.35) a Majorana mass term of the form

$$\Delta\mathcal{L}_M = \frac{M}{2}(\lambda^T C\lambda + h.c.) \tag{3.38}$$

with M a real mass parameter, C the charge conjugation matrix, and "h.c." denoting the hermitian conjugate. This mass term is real, Lorentz invariant, and nonvanishing for Grassmann valued spinors (C is antisymmetric). However it violates the $U(1)$ symmetry (3.36). The latter symmetry is sometimes needed in applications, so that a Majorana mass term is excluded in those cases (e.g. for the left handed neutrino, which has an hypercharge that couples to a gauge field). In other situations the Majorana mass is not excluded by symmetries (e.g. for the right handed neutrino, which has vanishing hypercharge) and it might be present. The possibility of using a Majorana mass will be useful for constructing a Pauli-Villars ultraviolet regulator for the massless chiral fermion. Let us explicitate the Majorana mass term as

$$\begin{aligned}\Delta\mathcal{L}_M &= \frac{M}{2}(\lambda^T C\lambda + h.c.) = \frac{M}{2}(\lambda^T C\lambda + \lambda^\dagger C^\dagger\lambda^*)\\ &= \frac{M}{2}(\lambda^T C\lambda + \overline{\lambda}\beta^{-1}C^\dagger(\beta^{-1})^T\overline{\lambda}^T) = \frac{M}{2}(\lambda^T C\lambda - \overline{\lambda}C^{-1}\overline{\lambda}^T).\end{aligned} \tag{3.39}$$

This last form is useful for considering λ and $\overline{\lambda}$ as independent fields when varying the action to find the field equations, as well as for performing the path integral quantization. To obtain it we have used that β is symmetric and anticommuting with C, and that $C^\dagger = C^{-1}$, properties which are certainly true in the explicit chiral representation reported in the appendix (the final result is probably representation independent, though we have not tried to prove it in detail). The equation of motion from $\mathcal{L}+\Delta\mathcal{L}_M$ are easily found to be given by the coupled equations

$$
\begin{aligned}
&\gamma^a\partial_a\lambda + MC^{-1}\overline{\lambda}^T = 0 \\
&\partial_a\overline{\lambda}\gamma^a + M\lambda^T C = 0 \; .
\end{aligned}
\tag{3.40}
$$

One can transpose the second one and rewrite it by using the properties of the charge conjugation matrix as

$$
\gamma^a\partial_a(C^{-1}\overline{\lambda}^T) + M\lambda = 0 \; . \tag{3.41}
$$

In this form it may be inserted into the first equation, after applying to it the $\gamma^a\partial_a$ operator, to recognize the mass shell condition (Klein-Gordon equation)

$$
(-\partial^a\partial_a + M^2)\lambda = 0 \; . \tag{3.42}
$$

This shows that the parameter M is indeed a mass for the chiral spinor λ. The breaking of the chiral $U(1)$ fermion number symmetry is evident from the massive equations (3.40), that mix λ with its complex conjugate field contained in $\overline{\lambda}$. A phase transformation does not leave those equations invariant. Let us describe the basic dynamical variables as a column vector

$$
\chi = \begin{pmatrix} \lambda \\ \overline{\lambda}^T \end{pmatrix} \tag{3.43}
$$

so that the massive lagrangian

$$
\mathcal{L} = -\overline{\lambda}\partial\!\!\!/\lambda + \frac{M}{2}(\lambda^T C\lambda - \overline{\lambda}C^{-1}\overline{\lambda}^T) \tag{3.44}
$$

with $\partial\!\!\!/ = \gamma^a\partial_a$ can be written in the general form (after an integration by part)

$$
\mathcal{L} = \frac{1}{2}\chi^T T\mathcal{O}\chi + \frac{1}{2}M\chi^T T\chi \tag{3.45}
$$

with

$$T\mathcal{O} = \begin{pmatrix} 0 & -\partial\!\!\!/^T \\ -\partial\!\!\!/ & 0 \end{pmatrix} \ , \qquad T = \begin{pmatrix} C & 0 \\ 0 & -C^{-1} \end{pmatrix} \tag{3.46}$$

with $\partial\!\!\!/^T = \gamma^{aT}\partial_a$, so that

$$\mathcal{O} = \begin{pmatrix} 0 & C^{-1}\partial\!\!\!/^T \\ C\partial\!\!\!/ & 0 \end{pmatrix} \ , \qquad \mathcal{O}^2 = \begin{pmatrix} \partial\!\!\!/^2 & 0 \\ 0 & C\partial\!\!\!/^2 C^{-1} \end{pmatrix} . \tag{3.47}$$

Once covariantized the operator $\mathcal{O}^2$ will be used as regulator for our anomaly calculation.

Using the charge conjugated field $\lambda_c = C^{-1}\overline{\lambda}^T$ instead of $\overline{\lambda}^T$, things become perhaps more evident. The equations of motion can be written as

$$\begin{aligned} \partial\!\!\!/\lambda + M\lambda_c = 0 \\ \partial\!\!\!/\lambda_c + M\lambda = 0 \end{aligned} \tag{3.48}$$

and if one uses as independent variables

$$\tilde{\chi} = \begin{pmatrix} \lambda \\ \lambda_c \end{pmatrix} \tag{3.49}$$

the lagrangian takes the form

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\lambda_c^T C\partial\!\!\!/\lambda + \lambda^T C\partial\!\!\!/\lambda_c) + \frac{M}{2}(\lambda^T C\lambda + \lambda_c^T C\lambda_c) \\ &= \frac{1}{2}\tilde{\chi}^T\tilde{T}\tilde{\mathcal{O}}\tilde{\chi} + \frac{1}{2}M\tilde{\chi}^T\tilde{T}\tilde{\chi} \end{aligned} \tag{3.50}$$

then

$$\tilde{T}\tilde{\mathcal{O}} = \begin{pmatrix} 0 & C\partial\!\!\!/ \\ C\partial\!\!\!/ & 0 \end{pmatrix} \ , \qquad \tilde{T} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \ , \qquad \tilde{\mathcal{O}} = \begin{pmatrix} 0 & \partial\!\!\!/ \\ \partial\!\!\!/ & 0 \end{pmatrix} \tag{3.51}$$

and the consistent regulator takes the form

$$\tilde{\mathcal{O}}^2 = \begin{pmatrix} \partial\!\!\!/^2 & 0 \\ 0 & \partial\!\!\!/^2 \end{pmatrix} . \tag{3.52}$$

We also recall that λ_c has opposite chirality of λ, as from (3.34) one gets

$$\gamma_5 \lambda_c = -\lambda_c \; . \tag{3.53}$$

3.3 Anomaly calculation for a Weyl fermion

The lagrangian of a massless Weyl fermion λ in flat space written in eq. (3.35) can be immediately covariantized and reads just as the Dirac lagrangian

$$\mathcal{L} = -e\,\overline{\lambda}\gamma^\mu \nabla_\mu \lambda \tag{3.54}$$

but now one must remember the chiral relation

$$\gamma^5 \lambda = \lambda \; . \tag{3.55}$$

In the same way as the Dirac case we have

$$\nabla_\mu = \partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^a\gamma^b \tag{3.56}$$

$$\gamma^\mu = e^\mu{}_a \gamma^a \; . \tag{3.57}$$

We want to evaluate the Weyl anomaly, and to do so we are going to use the Fujikawa method with consistent regulator as explained in section 2 and used later on. To do so we need to add a mass term for the PV part and manipulate the lagrangian to obtain the form

$$\mathcal{L} = \frac{1}{2}\chi^T T \mathcal{O} \chi + \frac{1}{2} M \chi^T T \chi \tag{3.58}$$

where the operator $T\mathcal{O}$ has to be the same for both the fermion and the PV part. As this operator is exactly the same as the Dirac case, reading from (3.4) to (3.7) we find again

$$\mathcal{L} = \frac{1}{2}\begin{pmatrix} \lambda^T & \lambda_c{}^T \end{pmatrix} \begin{pmatrix} 0 & -e(\gamma^\mu)^T \widetilde{\nabla}_\mu C \\ -eC^T\gamma^\mu \nabla_\mu & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \lambda_c \end{pmatrix} . \tag{3.59}$$

We could also use a different approach and treat λ as a Dirac fermion while adding to it the projector $\frac{1+\gamma^5}{2}$. With this approach the calculation stays almost the same but we obtain [1]

$$\mathcal{L} = \frac{1}{2}\begin{pmatrix}\lambda^T & {\lambda_c}^T\end{pmatrix}\begin{pmatrix}0 & -e\left(\frac{1+\gamma^5}{2}\right)(\gamma^\mu)^T\widetilde{\nabla}_\mu C\\ -eC^T\gamma^\mu\nabla_\mu\left(\frac{1+\gamma^5}{2}\right) & 0\end{pmatrix}\begin{pmatrix}\lambda\\ \lambda_c\end{pmatrix}. \tag{3.60}$$

Now we want to calculate the mass part using the mass term described before, we get

$$\Delta\mathcal{L}_M = e\frac{M}{2}(\lambda^T C\lambda - \overline{\lambda}C^{-1}\overline{\lambda}^T) \tag{3.61}$$

which we can rewrite as

$$\Delta\mathcal{L}_M = \frac{M}{2}\begin{pmatrix}\lambda^T & {\lambda_c}^T\end{pmatrix}\begin{pmatrix}eC & 0\\ 0 & -eC^T\end{pmatrix}\begin{pmatrix}\lambda\\ \lambda_c\end{pmatrix} \tag{3.62}$$

where we have introduced λ_c as well. If we are thinking of using Dirac spinors and projecting them, we need to consider a mass term with projectors added to it

$$\Delta\mathcal{L}_M = e\frac{M}{2}(\lambda^T C\Big(\frac{1+\gamma^5}{2}\Big)\lambda - \overline{\lambda}\Big(\frac{1-\gamma^5}{2}\Big)C^{-1}\overline{\lambda}^T) \tag{3.63}$$

which we rewrite as

$$\Delta\mathcal{L}_M = \frac{M}{2}\begin{pmatrix}\lambda^T & {\lambda_c}^T\end{pmatrix}\begin{pmatrix}eC\left(\frac{1+\gamma^5}{2}\right) & 0\\ 0 & -eC^T\left(\frac{1-\gamma^5}{2}\right)\end{pmatrix}\begin{pmatrix}\lambda\\ \lambda_c\end{pmatrix}. \tag{3.64}$$

Using (3.59) and (3.60) as expression for $T\mathcal{O}$ for both the cases we obtain

$$\mathcal{O} = \begin{pmatrix}0 & -C^{-1}(\gamma^\mu)^T\widetilde{\nabla}_\mu C\\ \gamma^\mu\nabla_\mu & 0\end{pmatrix}. \tag{3.65}$$

This expression can be rearranged by using

$$\begin{aligned}C^{-1}(\gamma^\mu)^T\widetilde{\nabla}_\mu C =& C^{-1}(\gamma^\mu)^T C\partial_\mu +\\ &-\frac{1}{4}\omega_{\mu ab}C^{-1}(\gamma^\mu)^T CC^{-1}(\gamma^b)^T CC^{-1}(\gamma^a)^T C = -\gamma^\mu\nabla_\mu\end{aligned} \tag{3.66}$$

[1] We don't need a projector for the $\overline{\lambda}$ part because the operator $\gamma^\mu\nabla_\mu$ always changes chirality and the projector are of course idempotent.

so we find

$$\mathcal{O} = \begin{pmatrix} 0 & \gamma^\mu \nabla_\mu \\ \gamma^\mu \nabla_\mu & 0 \end{pmatrix} \tag{3.67}$$

as one could have expected from the analogous expression in flat space (3.52).

Now we need to calculate K, δT and $\delta \mathcal{O}$. The Weyl fermion transforms in the same way as the Dirac ones does, so we use the transformation from (3.12) to (3.17). From these transformations, when considering T where the only varying part is e, we obtain

$$\delta T = \begin{pmatrix} 2e\sigma(x)C & 0 \\ 0 & -2e\sigma(x)C^T \end{pmatrix} \tag{3.68}$$

or in the other case simply

$$\delta T = \begin{pmatrix} 2e\sigma(x)C\left(\frac{1+\gamma^5}{2}\right) & 0 \\ 0 & -2e\sigma(x)C^T\left(\frac{1-\gamma^5}{2}\right) \end{pmatrix}. \tag{3.69}$$

For the variation of $\mathcal{O}$ we remember the calculation (3.19) which is the same also for this case, so the results are

$$\delta \mathcal{O} = \begin{pmatrix} 0 & -\frac{1}{2}\sigma\gamma^\mu \nabla_\mu + \frac{3}{4}\gamma^\mu \partial_\mu \sigma \\ -\frac{1}{2}\sigma\gamma^\mu \nabla_\mu + \frac{3}{4}\gamma^\mu \partial_\mu \sigma & 0 \end{pmatrix}. \tag{3.70}$$

This matrix anyway will not play any role in the calculation of the anomaly since does not have any diagonal term and we are going to be interested in the trace. Next we will like to calculate $\mathcal{O}^2$ and the results is

$$\mathcal{O}^2 = \begin{pmatrix} \gamma^\mu \nabla_\mu \gamma^\nu \nabla_\nu & 0 \\ 0 & \gamma^\mu \nabla_\mu \gamma^\nu \nabla_\nu \end{pmatrix}. \tag{3.71}$$

This is the regulator in both the Dirac and the Weyl case. Next we need K that we can simply read off from relations (3.16) and (3.17)

$$K = \begin{pmatrix} -\frac{3}{4}\sigma & 0 \\ 0 & -\frac{3}{4}\sigma \end{pmatrix}. \tag{3.72}$$

the last matrix that we need to evaluate is T^{-1} that being in diagonal form can be easily show to be

$$T^{-1} = \begin{pmatrix} \frac{C^{-1}}{e} & 0 \\ 0 & -\frac{(C^T)^{-1}}{e} \end{pmatrix} . \tag{3.73}$$

We recall now that we are trying to get to

$$\mathcal{A} = \mathrm{Tr}\, \Delta e^{\frac{\mathcal{O}^2}{M^2}} = \mathrm{Tr} \lim_{M\to\infty} (K + \frac{1}{2} T^{-1}\delta T + \frac{1}{2}\delta\mathcal{O}M^{-1}) e^{\frac{\mathcal{O}^2}{M^2}} \tag{3.74}$$

and we have obtained

$$\begin{aligned} \Delta = & \begin{pmatrix} -\frac{3}{4}\sigma & 0 \\ 0 & -\frac{3}{4}\sigma \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \frac{C^{-1}}{e} & 0 \\ 0 & -\frac{(C^T)^{-1}}{e} \end{pmatrix} \begin{pmatrix} 2e\sigma C & 0 \\ 0 & -2e\sigma C^T \end{pmatrix} + \\ & + \frac{1}{2M} \begin{pmatrix} 0 & \frac{1}{2}\sigma\gamma^\mu\nabla_\mu - \frac{3}{4}\gamma^\mu\partial_\mu\sigma \\ -\frac{1}{2}\sigma\gamma^\mu\nabla_\mu + \frac{3}{4}\gamma^\mu\partial_\mu\sigma & 0 \end{pmatrix} = \\ = & \begin{pmatrix} \frac{1}{4}\sigma & \frac{1}{4M}\sigma\gamma^\mu\nabla_\mu - \frac{3}{8M}\gamma^\mu\partial_\mu\sigma \\ -\frac{1}{4M}\sigma\gamma^\mu\nabla_\mu + \frac{3}{8M}\gamma^\mu\partial_\mu\sigma & \frac{1}{4}\sigma \end{pmatrix} . \end{aligned} \tag{3.75}$$

Now since $\mathcal{O}^2$ is in diagonal form, we recall the property

$$e^{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}} = \begin{pmatrix} e^A & 0 \\ 0 & e^B \end{pmatrix} \tag{3.76}$$

wich leads to

$$\mathcal{A} = \mathrm{Tr} \begin{pmatrix} \frac{1}{4}\sigma e^{\frac{(\gamma^\mu\nabla_\mu)^2}{M^2}} & (\frac{1}{4M}\sigma\gamma^\mu\nabla_\mu - \frac{3}{8M}\gamma^\mu\partial_\mu\sigma) e^{\frac{(\gamma^\mu\nabla_\mu)^2}{M^2}} \\ (-\frac{1}{4M}\sigma\gamma^\mu\nabla_\mu + \frac{3}{8M}\gamma^\mu\partial_\mu\sigma) e^{\frac{(\gamma^\mu\nabla_\mu)^2}{M^2}} & \frac{1}{4}\sigma e^{\frac{(\gamma^\mu\nabla_\mu)^2}{M^2}} \end{pmatrix} . \tag{3.77}$$

At this point we have three different way of addressing the Trace. If we consider this equation without any manipulation we could make the statement that the only difference from the Dirac case will emerge when we will take the trace of the gamma matrices; this trace will be half of the one in the Dirac space since the Weyl space has only two degree of freedom instead of the four of the Dirac one. So we expect the result to be that the anomaly for the Weyl case should be half of the Dirac one. The second possible approach is not to consider the trace over the Weyl space but to consider it over the Dirac one. To do so we add the chiral projector keeping in mind that the upper diagonal term emerges from λ while the bottom diagonal term emerges from λ_c. This implies that we must use different projector for each term. Furthermore we notice that the operator $(\gamma^\mu \nabla_\mu)^2$ does not change chirality so we can write

$$\begin{aligned} \mathcal{A} =& \operatorname{Tr}\Big[\frac{1}{4}\sigma e^{\frac{(\gamma^\mu\nabla_\mu)^2}{M^2}}\Big(\frac{1+\gamma^5}{2}\Big) + \frac{1}{4}\sigma e^{\frac{(\gamma^\mu\nabla_\mu)^2}{M^2}}\Big(\frac{1-\gamma^5}{2}\Big)\Big] = \\ =& \operatorname{Tr}\Big(\frac{1}{4}\sigma e^{\frac{(\gamma^\mu\nabla_\mu)^2}{M^2}}\Big) \end{aligned} \tag{3.78}$$

which is, as in the Majorana case, half of the Weyl anomaly for the Dirac case. The last approach involves considering the projector from the lagrangian and carry out the whole calculation from there as we have done in parallel so far. We stopped at the calculation of T^{-1}, the reason being that the matrix that we can read in (3.64) is not invertible. In these circumstances the right thing to do is to restrict the trace to the space in which T is invertible. This T acts upon vectors with sixteen components, the vectors that guarantee the existence of T^{-1} are

$$\begin{pmatrix} P_L X \\ P_R Y \end{pmatrix} = \begin{pmatrix} \left(\frac{1+\gamma^5}{2}\right) X \\ \left(\frac{1-\gamma^5}{2}\right) Y \end{pmatrix} \tag{3.79}$$

where X and Y are spinors with four components. Now if we move the projectors from the vector to matrix remembering their idempotency we obtain

$$\begin{aligned} \Delta =& \begin{pmatrix} -\frac{3}{4}\sigma P_L & 0 \\ 0 & -\frac{3}{4}\sigma P_R \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \frac{C^{-1}}{e} & 0 \\ 0 & -\frac{(C^T)^{-1}}{e} \end{pmatrix}\begin{pmatrix} 2e\sigma C P_L & 0 \\ 0 & -2e\sigma C^T P_R \end{pmatrix} + \\ &+ \frac{1}{2M}\begin{pmatrix} 0 & P_L\frac{1}{2}\sigma\gamma^\mu\nabla_\mu - \frac{3}{4}\gamma^\mu\partial_\mu\sigma P_R \\ -P_R\frac{1}{2}\sigma\gamma^\mu\nabla_\mu + \frac{3}{4}\gamma^\mu\partial_\mu\sigma P_L & 0 \end{pmatrix} = \\ =& \begin{pmatrix} \frac{1}{4}\sigma P_L & P_L\frac{1}{2}\sigma\gamma^\mu\nabla_\mu - \frac{3}{4}\gamma^\mu\partial_\mu\sigma P_R \\ -P_R\frac{1}{2}\sigma\gamma^\mu\nabla_\mu + \frac{3}{4}\gamma^\mu\partial_\mu\sigma P_L & \frac{1}{4}\sigma P_R \end{pmatrix}. \end{aligned} \tag{3.80}$$

In this final formula we used P_L and P_R to have a less heavy notation. At this point since the regulator is the same as the case we encountered before we see that the anomaly reads

$$\begin{aligned}\mathcal{A} &= \mathrm{Tr}\Big[\frac{1}{4}\sigma e^{\frac{(\gamma^\mu\nabla_\mu)^2}{M^2}}\Big(\frac{1+\gamma^5}{2}\Big) + \frac{1}{4}\sigma e^{\frac{(\gamma^\mu\nabla_\mu)^2}{M^2}}\Big(\frac{1-\gamma^5}{2}\Big)\Big] = \\ &= \mathrm{Tr}\Big(\frac{1}{4}\sigma e^{\frac{(\gamma^\mu\nabla_\mu)^2}{M^2}}\Big)\end{aligned} \tag{3.81}$$

which is the same result that we have obtained before. The explicit expression is of course

$$\mathcal{A} = \frac{1}{11520\pi^2}\int d^4x\sqrt{g}\sigma(x)(-\frac{7}{2}R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 4R_{\mu\nu}R^{\mu\nu} + \frac{5}{2}R^2 + 6\Box R). \tag{3.82}$$

And as a consequence the energy momentum tensor is

$$T^\mu{}_\mu = -\frac{1}{11520\pi^2}(\frac{7}{2}R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} + 4R_{\mu\nu}R^{\mu\nu} - \frac{5}{2}R^2 - 6\Box R) \tag{3.83}$$

$$T^\mu{}_\mu(x) = \frac{1}{11520\pi^2}(\frac{11}{2}E_4 - 9C^2)\ . \tag{3.84}$$

Conclusions

Is now time to sum up the results and achievements of this thesis. In the first chapter we have given an introduction to Weyl and conformal transformations and we have shown explicitly how they relate to one another. We have also presented explicit calculations to show the invariance of the most common models. In the second and third chapters we have given an introduction to the Fujikawa method for the calculation on quantum anomalies and we have explicated its connection to the Pauli Villars regularization method; again we have presented an explicit end extended calculation of the trace anomaly for the scalar model, the vector model and for the Dirac one. Lastly we have applied the same method to the trace anomaly of a Weyl fermion, a result which is not as much well established. We have found that the trace anomaly for a Weyl fermion is half the anomaly of a Dirac fermion.

Appendix A

Conventions

A.1 Gamma matrices

The Dirac matrices with flat indices γ^a satisfy the Clifford algebra

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} \tag{A.1}$$

where the Minkowski metric η_{ab} is mostly plus. Thus γ^0 is anti-hermitian and the γ^i's are hermitian (we split the index a into time and space components as $a = (0, i)$). These hermiticity properties are expressed compactly by the relation

$$\gamma^{a\dagger} = -\beta\gamma^a\beta \tag{A.2}$$

where $\beta = i\gamma^0$. The latter is used in the definition of the Dirac conjugate $\overline{\psi}$ of the spinor ψ, defined by

$$\overline{\psi} = \psi^\dagger\beta \tag{A.3}$$

so that the product $\overline{\psi}\psi$ is a Lorentz scalar. The chirality matrix

γ^5 is defined by

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{A.4}$$

and satisfies

$$\{\gamma^5, \gamma^a\} = 0\ , \quad (\gamma^5)^2 = 1\ , \quad \gamma^{5\dagger} = \gamma^5\ . \tag{A.5}$$

It allows to introduce the left and right chiral projectors

$$P_L = \frac{1+\gamma_5}{2} \ , \qquad P_R = \frac{1-\gamma_5}{2} \tag{A.6}$$

that split a Dirac spinor ψ into its left- and right-handed components (Weyl spinors)

$$\psi = \lambda + \rho \ , \qquad \begin{cases} \lambda = \frac{1+\gamma_5}{2}\psi \\ \rho = \frac{1-\gamma_5}{2}\psi \end{cases} \ . \tag{A.7}$$

The latter transform irreducibly under the transformations of the Lorentz group connected to the identity (the proper, orthochronous Lorentz group): λ is a left-handed Weyl spinor and ρ is a right-handed Weyl spinor.

A.2 Chiral representation of the gamma matrices

A useful representation of the gamma matrices is the chiral one, defined in terms of two by two blocks by

$$\gamma^0 = -i \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \ , \qquad \gamma^i = -i \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{A.8}$$

so that

$$\gamma^5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \ , \qquad \beta = i\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \ . \tag{A.9}$$

It is a convenient representation as the Lorentz generators M^{ab} in the spinorial representation[1] as well as the chirality matrix γ^5 take a block diagonal form. Indeed, the spinorial representation of the Lorentz generators $M^{ab} = \frac{1}{4}[\gamma^a, \gamma^b] = \frac{1}{2}\gamma^{ab}$ become

$$M^{0i} = \frac{1}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \ , \qquad M^{ij} = \frac{i}{2}\epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \ .$$

[1] The related Lie algebra has the form $[M^{ab}, M^{cd}] = \eta^{bc}M^{ad} \pm 3$ terms.

In this representation the Dirac field and its chiral parts take the form

$$\psi = \begin{pmatrix} l \\ r \end{pmatrix} , \qquad \lambda = \begin{pmatrix} l \\ 0 \end{pmatrix} , \qquad \rho = \begin{pmatrix} 0 \\ r \end{pmatrix} \tag{A.10}$$

where l and r are two dimensional spinors (Weyl spinors).

The charge conjugation matrix C can be defined by

$$C = \gamma^2 \beta = -i \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix} \tag{A.11}$$

and has the property of relating the gamma matrices to their transposed ones

$$C\gamma^a C^{-1} = -\gamma^{aT} . \tag{A.12}$$

It is used to define the charge-conjugated field

$$\psi_c = C^{-1}\overline{\psi}^T \tag{A.13}$$

in which particles and antiparticles are interchanged. Indeed, one may check that if a Dirac spinor ψ satisfies the standard Dirac equation coupled to a $U(1)$ gauge field

$$[\gamma^a(\partial_a - ieA_a) + m]\psi = 0 \tag{A.14}$$

then ψ_c satisfies

$$[\gamma^a(\partial_a + ieA_a) + m]\psi_c = 0 . \tag{A.15}$$

It is easy to check that in the chiral representation the charge conjugation matrix satisfies

$$C = -C^T = -C^{-1} = -C^\dagger = C^* . \tag{A.16}$$

Note that if the ψ is chiral, say $\gamma^5\psi = \psi$, then its charge conjugated field ψ_c has the opposite chirality $\gamma^5\psi_c = -\psi_c$. A Majorana spinor can be defined as a spinor that is equal to its charged conjugated one

$$\psi = \psi_c \tag{A.17}$$

so that particles and antiparticles coincide. This constraint is incompatible with the chiral constraints, so that Majorana-Weyl fermions do not exist in 4 dimensions. We recall that for a Weyl spinor λ the scalar $\overline{\lambda}\lambda$ vanishes, so a Dirac mass term is not allowed. On the other hand the term

$$\lambda^T C\lambda \tag{A.18}$$

is a Lorentz scalar, and since C is antisymmetric it is non-vanishing if the spinor is taken to be Grassman valued (anticommuting). Thus in flat spacetime a mass term of the form

$$M\lambda^T C\lambda + h.c. \tag{A.19}$$

with M real ($h.c.$ indicates the hermitian conjugate) is allowed: it is real, Lorentz invariant and non-vanishing. This is the so-called Majorana mass term. It violates the fermion number symmetry generated by the group $U(1)$ of phase transformations.

A.3 Metric and connections

We use a mostly plus metric $g_{\mu\nu}$ and the Levi-Civita connection $\Gamma^\rho{}_{\mu\nu}$ (Christoffel symbols) that makes the metric covariantly constant

$$\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma_{\rho\mu}{}^\sigma g_{\sigma\nu} - \Gamma_{\rho\nu}{}^\sigma g_{\mu\sigma} = 0 \tag{A.20}$$

and it follows that

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \ . \tag{A.21}$$

On vectors with upper indices the covariant derivative acts as

$$\nabla_\mu V^\rho = \partial_\mu V^\rho + \Gamma^\rho{}_{\mu\nu}\, V^\nu \ . \tag{A.22}$$

We use the following conventions for the curvature tensors

$$[\nabla_\mu, \nabla_\nu] V^\sigma = R_{\mu\nu}{}^\sigma{}_\rho V^\rho \ , \quad R_{\mu\nu} = R_{\mu\rho}{}^\rho{}_\nu \ , \qquad R = R^\mu{}_\mu \tag{A.23}$$

so that the scalar curvature of a sphere is negative. The Riemann tensor $R_{\mu\nu}{}^{\sigma}{}_{\rho}$ is manifestly antisymmetric in its first two indices. It also satisfies

$$R_{\mu\nu\sigma\rho} = -R_{\mu\nu\rho\sigma} \tag{A.24}$$

as a consequence of using the Levi-Civita connection (it follows from $[\nabla_\mu, \nabla_\nu]\, g_{\sigma\rho} = 0$).

We would also like to explicit here the variation of δR with respect to $\delta g^{\mu\nu}$

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + \nabla_\mu\nabla_\nu\delta g^{\mu\nu} - \Box\delta(\ln g) = R_{\mu\nu}\delta g^{\mu\nu} + \nabla_\mu\nabla_\nu\delta g^{\mu\nu} - g_{\mu\nu}\Box\delta g^{\mu\nu}. \tag{A.25}$$

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