ALMA MATER STUDIORUM · UNIVERSITÀ DI BOLOGNA

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Adiabatic extraction models for beam dynamics

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E cominciò: «Le cose tutte quante
hanno ordine tra loro»

Pd., I, 103–104

He had been gifted, too, with sterner powers.
Even while a child he laid his daring hand
On Science' golden key; and ere the tastes
Or sports of boyhood yet had passed away
Oft would he hold communion with the mind
Of Newton, and with awed enthusiasm learn
The eternal Laws which bind the Universe
And which the Stars obey.

William Rowan Hamilton
The Enthusiast, 1826

Abstract

Il presente lavoro prende le mosse da un problema di dinamica dei fasci relativo a un metodo di separazione di un fascio di particelle recentemente messo in funzione nell'sincrotrone PS al CERN. In questo sistema, variando adiabaticamente i parametri di un campo magnetico, nello spazio delle fasi si creano diverse isole di stabilità (risonanze) in cui le particelle vengono catturate.

Dopo una parte introduttiva in cui si ricava, a partire dalle equazioni di Maxwell, l'hamiltoniana di una particella sottoposta ai campi magnetici che si usano negli acceleratori, e una presentazione generale della teoria dell'invarianza adiabatica, si procede analizzando la dinamica di tali sistemi.

Inizialmente si prende in considerazione l'hamiltoniana mediata sulle variabili veloci, considerando perturbazioni (*kick*) dei termini dipolare e quadrupolare. In ognuno dei due casi, si arriva a determinare la probabilità che una particella sia catturata nella risonanza.

Successivamente, attraverso un approccio perturbativo, utilizzando le variabili di azione ed angolo, si calcola la forza della risonanza 4:1 per un *kick* quadrupolare.

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Introduction

Particle accelerators are the main research tool in high energy physics. Great discoveries at CERN have been often awarded with Nobel prizes, and giant-size experiments at LHC are widely recognized as the farthest frontiers of science.

While many are fascinated by mysteries of elusive particles such as Higgs boson, most people ignore that there exist a specific branch of applied physics which focuses on the study of the dynamics of particle beams inside accelerators. Knowledge of beam dynamics is fundamental in order to design systems of magnets useful to guide particles in synchrotrons. Different types of magnets have different effects on the particle beam, as their can be used to focus or to bend charged particles.

Beams are often modeled as classical dynamic systems, and the main theoretical background of their study is hamiltonian mechanics. Versatility of hamiltonian approach to such problems is well shown by the fact that in our work we will apply to accelerator physics some result obtained for celestial mechanics, as the same equations appear in such different fields.

PS and Multi-turn extraction

In this work, we focus our study on the problem to optimize the injection between the PS and the SPS synchrotrons at CERN. PS (*Proton Synchrotron*) is the oldest major ring accelerator at CERN, which in 1959, when it was put into service, was the highest energy accelerator in the world, with its 28 GeV. Fifty years later it has of course been outgunned by newer machines, but it still fulfill important tasks in CERN accelerators chain architecture (see fig. 1). Indeed, PS is now used to preaccelerate beams and to feed them into other accelerators, such as SPS (*Super Proton Synchrotron*).

PS has a lenght of 628 m, while SPS is 11 times longer. Since

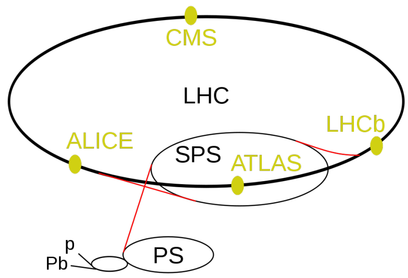


FIGURE 1 – Scheme of the accelerators chain at CERN. (Wikimedia Commons)

the 1970s in order to fill in the fastest way the larger accelerator, PS beams were split in five beamlets which were later put into SPS, thus filling 10/11 of SPS using two PS laps.

The method historically used to split beams, which we will call *Continous Tranfer* (CT), is very simple: an electrostatic kicker split a fraction of particles from the beam, that was separated by means of a septum. Then, the beam underwent a $\pi/2$ rotation and the proceeding were iterated four times, in order to produce five beamlets as requested.

Unfortunately, this system has its own flaws. First of all, it has a limited efficiency since part of the beam is lost at the septum. Furthermore, the beamlets injected in SPS have different optical parameters and do not optimally match, and, worse, the high radiation levels on the electrostatic foil made maintenance difficult to perform.

In 2002, Cappi and Giovannozzi [6] proposed a new method for splitting: *Multiturn Extraction* (MTE), which does not rely on any mechanical action. Their idea is the following: non-linear magnets (sextupole and octupole elements) generate a stable nonlinear resonance in the transverse phase space. Adiabatically varying the tune of the machine will cause some fractions of the beam to cross separatrices and to be trapped into different stable islands, thus being *captured into resonance* and be driven towards their final destination.

A computer simulation [6] of the beam evolution in MTE is shown in fig. (2), and in [5] preliminary results of the application of this system are also shown.

MTE method has been implemented at PS since September 2015,

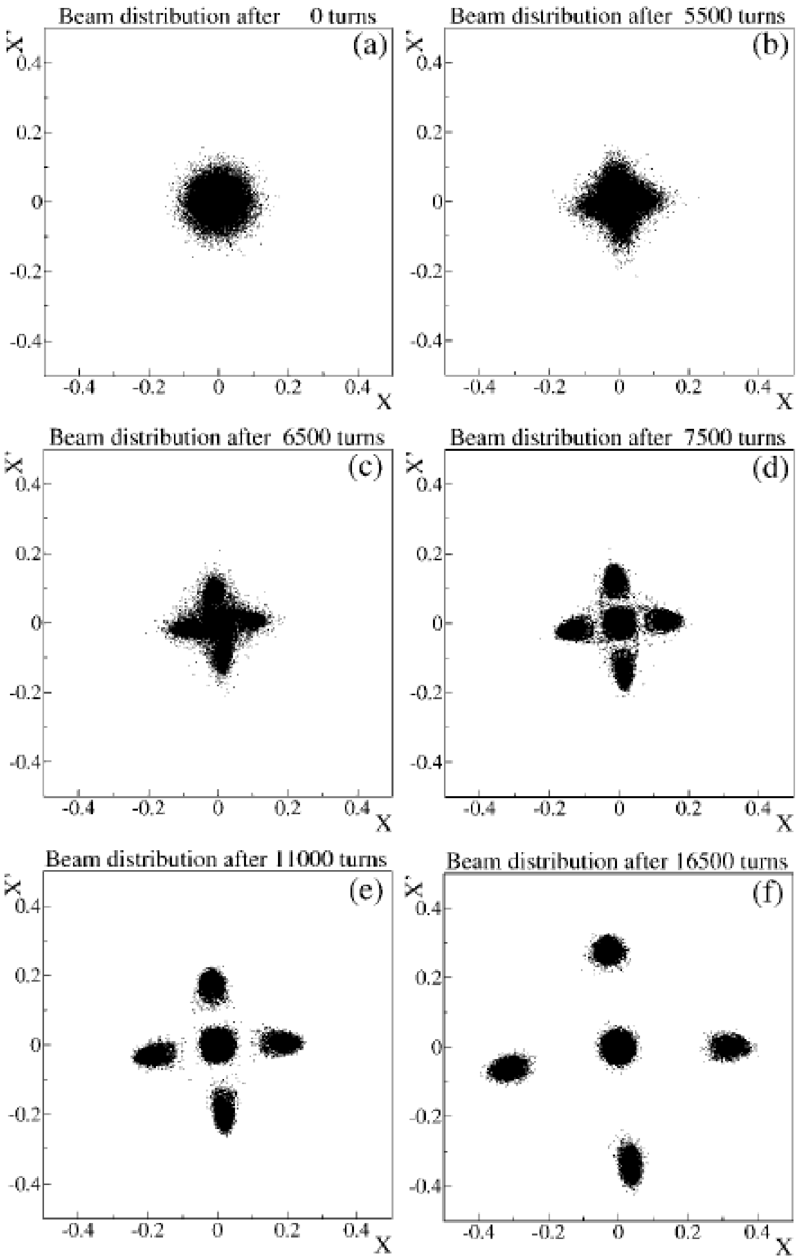


FIGURE 2 – Computer simulation of the phase space evolution of a gaussian particle distribution during MTE. The separation process is clearly visible: while the magnetic field tune is adiabatically varied, five different stability islands appear. [6].

however some theoretical problems are still open for the application of the neoadiabatic theory to particle accelerators [11]. A wide study of dynamical properties is needed to determine how the magnetic field parameters have to be tuned in order to optimize the probability of capture into resonance. Now, a full dissertation on this subject would involve 4D KAM theory and the neoadiabatic theory for Hamiltonian systems, whose geometrical tools are too advanced for a bachelor thesis. We will focus on a simplified, 2D version of the problem which is accessible at our level of knowledge.

Therefore, the main topic of our work is to show the possibility of adiabatic extraction of the beam using a quadrupolar kick.

Structure of the work

The work is divided into four chapters:

In the first chapter, it is given an overview on hamiltonian dynamics tractation of magnetic fields. Magnetic field hamiltonian is deduced, and it is presented multipole expansion of magnetic fields, explaining the roles of different terms in accelerator physics.

In the second chapter, we focus on adiabatic invariance theory, proving the adiabatic invariance of action-angle variables. Some examples are also shown.

In the third chapter, we follow Nejštadt's approach [10, 11] to derive an averaged hamiltonian for magnetic fields in presence of dipolar and quadrupolar kick and to compute capture probabilities.

In the fourth chapter, we finally apply action-angle variables to a adiabatically forced non-linear hamiltonian to compute the strenght of $4-$order resonance using Fourier expansion.

Chapter 1

Magnetic field

Particle accelerators work using magnetic fields lattice (i.e. ad ordered structures of dipole, quadrupole and multipole magnets) in order to properly confine charged particles into the beam pipe. In our work, we will need some introductory notions from classical electromagnetism in order to explain how hamiltonians which we will investigate in Chapter 3 arise.

1.1 Hamiltonian of electromagnetic field

From classical electrodynamics (Maxwell equations) it is possible to write the hamiltonian of a particle subject to a magnetic field $\mathbf{B}$ and an electric one $\mathbf{E}$. [8]

Now, the kinetic energy of a relativistic free particle of mass m reads

$$T = -mc^2\sqrt{1 - \frac{v^2}{c^2}}$$

where v is the particle velocity.

Electric field contributes to the potential energy of the system with a term $e\Phi$, being e the electric charge and Φ the scalar potential ($\mathbf{E} = -\boldsymbol{\nabla}\Phi$)

Magnetic field potential energy is

$$V_{\mathbf{B}} = -\frac{e}{c}\mathbf{v} \cdot \mathbf{A}$$

where $\mathbf{A}$ is vector potential ($\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$)

Now we can write the lagrangian $\mathcal{L} = T - V$

$$\mathcal{L} = -mc^2\sqrt{1-\frac{v^2}{c^2}} + \frac{e}{c}\mathbf{v}\cdot\mathbf{A} - e\Phi$$

We have a generalized potential which depends on velocity $\mathbf{v}$. Thus, in order to write our hamiltonian we have to use the general formula $\mathcal{H} = \mathbf{p}\cdot\mathbf{v} - \mathcal{L}$

where $\mathbf{p}$ is the generalized momentum

$$\mathbf{p} = \frac{\partial\mathcal{L}}{\partial v} = \frac{m\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{e}{c}\mathbf{A}$$

and we finally obtain

$$\mathcal{H} = \sqrt{\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + m^2c^4} + e\Phi$$

We will thereafter neglect electric field potential Φ because we study the transverse dynamics (called *betatronic motion*) of the beam whereas the accelerating system (based on RF cavities) defines the longitudinal ones.

Furthermore, we take into account only transverse magnetic field so that $A_x = A_z = 0$.

In a circular accelerator, the best reference system for our particle is not the cartesian one: we use the transverse displacements x and y and the coordinate s which is the arc length of the reference trajectory, whose curvature we call r (fig. 1.1)

In this system, the hamiltonian reads

$$\mathcal{H} = c\sqrt{m^2c^2 + \left(p_s - \frac{e}{c}A_s\right)^2\left(1-\frac{x}{r}\right)^{-2} + p_x^2 + p_y^2} \tag{1.1}$$

This hamiltonian has time t as indipendent coordinate. It would be better to use the spacial coordinate s as the indipendent one. Now, being $-\mathcal{H}$ conjugate with t, s is conjugate to p_s, so we can use as new hamiltonian

$$\mathcal{K} = -p_s c$$

Solving (1.1) for p_s we have

$$\mathcal{K} = -\frac{e}{c}A_s - c\left(1+\frac{x}{r}\right)\sqrt{\mathcal{H}^2 - m^2c^2 - p_x^2 - p_y^2}$$

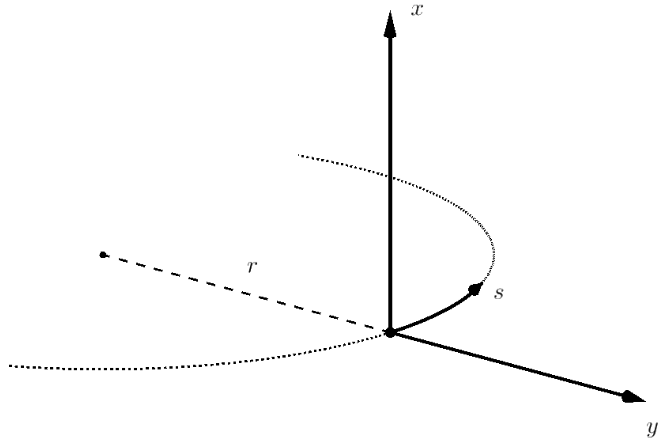


FIGURE 1.1 – Picture of the reference system (x, y, s) we use. The dotted line is the particle trajectory, whose curvature radius is r. The s coordinate is measured along the trajectory while x and y are orthogonal to it.

and, defining $p_c = \sqrt{\mathcal{H}^2/c^2 - m^2c^2}$ we obtain

$$\mathcal{K} = -\frac{e}{c}A_s - c\left(1 + \frac{x}{r}\right)\sqrt{p_c^2 - p_x^2 - p_y^2}$$

It is possible to introduce a scaling in the hamiltonian.

We impose $p'_x = p_x/p_c$, $p'_z = p_z/p_c$, $\mathcal{K}' = \mathcal{K}/p_c$ obtaining

$$\mathcal{K}' = -\frac{e}{cp_c}A_s - \left(1 + \frac{x}{r}\right)\sqrt{1 - p'^2_x - p'^2_y}$$

We now neglect the curvature effects, assuming $r \to \infty$. This assumption is not irrealistic if we consider long accelerators.

Therefore we have, renaming $\mathcal{K}'$ as $\mathcal{H}$:

$$\mathcal{H} = -\frac{e}{cp_c}A_s - \sqrt{1 - p'^2_x - p'^2_y}$$

which, for small values of p'_x and p'_y reduces to

$$\mathcal{H} = \frac{p'^2_x}{2} + \frac{p'^2_y}{2} - \frac{e}{cp_c}A_s \tag{1.2}$$

Therefore, we have a hamiltonian composed by a kinetic term $p^2/2$ and a potential one dependent on A_s. [12] We will now investigate how it is possible to write A_s for magnetic fields we consider in accelerator physics.

1.2 Multipole expansion of magnetic field

A magnetostatic field $\mathbf{B}$ in absence of currents obeys the equations:

$$\begin{cases} \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \\ \boldsymbol{\nabla} \times \mathbf{B} = 0 \end{cases} \tag{1.3}$$

the first equation being Gauß law for magnetostatics, and the second one being Ampère's law when $\mathbf{J} = 0$

Now, we immediately notice that these equations for $\mathbf{B}$ are exactly the same for electrostatic field in absence of charges, so the same methods can be exploited. Moreover, $\mathbf{B}$ is irrotational, so we can define a magnetostatic potential A_s such as

$$\mathbf{B} = -\boldsymbol{\nabla} A_s \tag{1.4}$$

and, combining (1.4) and the first of (1.3), we obtain Laplace equation

$$\nabla^2 A_s = 0 \tag{1.5}$$

As a consequence $A_s(x, y)$ is a so-called *harmonic function* (i.e. a real or imaginary part of a analytic function in the complex plane)

We now consider Lorentz force

$$\mathbf{F} = \frac{e}{c}\mathbf{v} \times \mathbf{B} \tag{1.6}$$

where e is the charge and $\mathbf{v}$ the particle velocity.

On a circular trajectory, considering only transversal fields, we must take into account also centrifugal force, and, at equilibrium conditions, we obtain

$$\frac{mv^2}{r} = \frac{e}{c}vB \tag{1.7}$$

where m is particle mass and r the trajectory radius. In the transversal field case, the vector product $\mathbf{v} \times \mathbf{B}$ reduces to the usual product vB.

It is straightforward to solve equation (1.7) for the radius r, obtaining

$$r = \frac{mvB}{ce} = \frac{pc}{eB} \tag{1.8}$$

being $p = mv$ the particle momentum.

We now define $\rho = Br = pc/e$ as *beam rigidity*, which is useful to normalize magnetic multipole terms.

Our Laplace equation can be solved making the following *ansatz* (*multipole expansion*):

$$A_s(x, y) = -\rho \sum_n \frac{1}{n!} A_n (x + \mathrm{i}y)^n \tag{1.9}$$

It is straightforward to proof that this *ansatz* really solves Laplace equation (1.5). We have:

$$\begin{aligned}
\nabla^2 A_s(x, y) &= \frac{\partial^2 A_n}{\partial x^2} + \frac{\partial^2 A_s}{\partial y^2} \\
&= -\rho \sum_n \frac{1}{n!} A_s \big[n(n-1)(x + \mathrm{i}y)^{n-2} - n(n-1)(x + \mathrm{i}y)^{n-2} \big] \\
&= 0
\end{aligned}$$

Now, the *ansatz* has a real (*normal*) and an imaginary (*skew*) part which are two independent solutions of the Laplace equation.

From Newton's binomial theorem,

$$(x + \mathrm{i}y)^n = \sum_k \binom{n}{k} \mathrm{i}^{n-k} x^{n-k} y^k \tag{1.10}$$

When $n - k$ is even, the k-th term of this expansion is real, while when it is odd we have a imaginary part term.

Taking the real part of (1.10) we obtain

$$\mathrm{Re}\{(x + \mathrm{i}y)^n\} = \sum_{m=0}^{n/2} (-1)^m \binom{n}{2m} x^{n-2m} y^{2m}$$

while for the imaginary one

$$\mathrm{Im}\{(x + \mathrm{i}y)^n\} = \sum_{m=0}^{(n+1)/2} (-1)^m \binom{n}{2m+1} x^{n-2m-1} y^{2m+1}$$

Thus, recalling the expression of A_s (1.9), we can write (remembering the definition of binomial coefficient)

$$\operatorname{Re} A_s = -\rho \sum_{m=0}^{n/2} A_n \frac{(-1)^m}{(2m)!(n-2m)!} x^{n-2m} y^{2m}$$

$$\operatorname{Im} A_s = -\rho \sum_{m=0}^{(n+1)/2} A_n \frac{(-1)^m}{(2m+1)!(n-2m-1)!} x^{n-2m-1} y^{2m+1}$$

In order to retrieve the expression of magnetic field $\mathbf{B} = (B_x, B_y, 0)$ we have to derive A_s, being

$$\mathbf{B} = \left(\frac{\partial A_s}{\partial y}, -\frac{\partial A_s}{\partial x}, 0 \right)$$

We will have two different fields if we use the real or the imaginary part. We will define $\mathbf{B}^{\mathrm{N}}$ for the real (normal) part and $\mathbf{B}^{\mathrm{S}}$ for the imaginary (skew) one.

Now,

$$\frac{\partial \operatorname{Re} A_s}{\partial x} = -\rho \sum_{m=0}^{n/2} A_n \frac{(-1)^m}{(2m)!(n-2m-1)!} x^{n-2m-1} y^{2m} = \frac{\partial \operatorname{Im} A_s}{\partial y}$$

and

$$\frac{\partial \operatorname{Re} A_s}{\partial y} = -\rho \sum_{m=0}^{n/2} A_n \frac{(-1)^m}{(2m-1)!(n-2m)!} x^{n-2m} y^{2m-1} = \frac{\partial \operatorname{Im} A_s}{\partial x}$$

so $\mathbf{B}^{\mathrm{S}}$ is obtained rotating $\mathbf{B}^{\mathrm{N}}$ of $\pi/2$.

We now analyze the first terms of the expansion. Specific magnets that generate a magnetic field with only one multipole expansion term are very useful for beam control in particle accelerators, because the magnitude of any term in the expansion can be set independently, such obtaining every desired magnetic field.

Dipolar field. For $n = 1$, we have

$$A_s^{(1)} = A_1(x + \mathrm{i}y)$$

so, giving different names to real and imaginary part coefficients,

$$\operatorname{Re} A_s^{(1)} = -\rho A_{10} x \qquad \operatorname{Im} A_s^{(1)} = -\rho A_{01} y$$

and the total **B** field reads

$$\mathbf{B}_1 = \rho \begin{pmatrix} A_{10} \\ A_{01} \\ 0 \end{pmatrix}$$

Thus, we have a uniform magnetic field, called *dipole field*, which is the one generated by a single bar magnet. Moreover, A_{10} and A_{01} represent the horizontal and vertical deflection curvature. Thus, the effect of dipolar field is beam deflection.

Quadrupolar field. The $n = 2$ term of the expansion (1.9) reads

$$A_s^{(2)} = -\rho \frac{A_2}{2} (x + \mathrm{i}y)^2$$

so

$$\operatorname{Re} A_s^{(2)} = -\rho \frac{A_2}{2} (x^2 - y^2) \qquad \operatorname{Im} A_s^{(2)} = -\rho A_2 xy$$

We rewrite the potential in the form

$$A_s^{(2)} = -\rho A_{20} \frac{1}{2} (x^2 - y^2) - \rho A_{11} xy$$

and we see that equipotential lines are equilateral hyperbola, and the imaginary part is rotated by $\pi/2$ respect to the real one.

The magnetic field is obtained performing the derivatives. We have:

$$\mathbf{B}_2^{\mathrm{S}} = \rho \frac{1}{2} A_{20} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \qquad \mathbf{B}_2^{\mathrm{N}} = \rho A_{11} \begin{pmatrix} y \\ x \\ 0 \end{pmatrix}$$

This field is called *quadrupole*. A quadrupolar field is the one generated by four bar magnets orthogonally arranged, with two north poles and two south poles facing each other, and it is useful to *focus* beams.

Sextupolar field. For $n = 3$ we have the so-called *sextupole* magnets whose effect in particle accelerators is remove *cromaticity*, i.e. to prevent particles with different energy from acquiring different linear dynamics.

We have:

$$A_s^{(3)} = -\rho A_{30}\frac{1}{6}(x^3 - 3xy^2) - \rho\frac{1}{6}A_{21}(3x^2y - y^2)$$

and

$$\mathbf{B}_3^{\mathrm{S}} = \rho A_{21}\begin{pmatrix} xy \\ x^2 - y^2 \\ 0 \end{pmatrix} \qquad \mathbf{B}_3^{\mathrm{N}} = -\rho A_{30}\begin{pmatrix} x^2 - y^2 \\ xy \\ 0 \end{pmatrix}$$

Octupolar field. Finally, for $n = 4$ we have

$$A_s^{(4)} = -\rho A_{40}\frac{1}{24}(x^4 - 6x^2y^2 + y^4) - \rho\frac{1}{24}A_{31}(x^3y - xy^3)$$

and

$$\mathbf{B}_4^{\mathrm{S}} = \rho A_{31}\begin{pmatrix} 3x^2y - y^3 \\ x^3 - 3xy^2 \\ 0 \end{pmatrix} \qquad \mathbf{B}_4^{\mathrm{N}} = \rho A_{40}\begin{pmatrix} x^3 - 3xy^2 \\ 3x^2y - y^3 \\ 0 \end{pmatrix}$$

Now, if we take into account only the unidimensional case, we use only the real or the imaginary part of A_s. Let us restrict to real values, including only horizontal effects.

Equation (1.9) now reads:

$$A_s(x) = -\rho\sum_n \frac{1}{n!}A_n x^n$$

and actually reduces to a Taylor expansion, whose coefficients A_n are

$$A_n = \frac{1}{\rho}\frac{\mathrm{d}^n}{\mathrm{d}x^n}A_s(x)\bigg|_{x=0} = \frac{1}{\rho}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}B(x)\bigg|_{x=0}$$

because A_s is a primitive of B.

If we use potentials up to the octupole term we have

$$A_s = -\rho k_1 x + k_2 x^2 + k_3 x^3 + k_4 x^4$$

and, substituting this expression in (1.2) we obtain

$$\mathcal{H} = \frac{p^2}{2} + k_1 x + k_2 x^2 + k_3 x^3 + k_4 x^4$$

which is the hamiltonian we are going to study in the next chapters. [12]

Chapter 2

Adiabatic invariants

At 1911 Solvay Conference in Brussels, Albert Einstein replied to Hendrick Lorentz stating that, for a pendulum whose lenght was periodically changed, after a period

> «If the length of a pendulum is changed infinitely slowly, its energy remains equal to $h\nu$ if it was originally $h\nu$.»[1]

This is the first appearance of *adiabatic invariance*, a concept arising when studying dynamical systems which depends upon parameters whose change in time is much *slowlier* than system motion. A classic example is a pendulum whose lenght is slowly changed: it is possible to show that the oscillation amplitude is function only of lenght, and that the quotient between energy and frequency is almost conserved, while energy and frequency can substantially change. This quotient is said to be an *adiabatic invariant* of the system.

[1] «M. Lorentz se rappelle une conversation qu il eut avec M. Einstein il y a déjà quelque temps, et dans laquelle il fut question d'un pendule simple qu'on raccourcirait en tenant le fil entre deux doigts, qu'on glisse vers le bas. Supposons qu'au commencement le pendule ait exactement un élément d'énergie tel qu'il correspond à la fréquence de ses oscillations, il semble alors qu'à la fin de l'expérience son énergie sera moindre que l'élément qui correspond à la nouvelle fréquence.

M. Einstein. — Si l'on modifie la longueur du pendule de manière infiniment lente, l'énergie de l'oscillation reste égale à $h\nu$, si elle était primitivement égale à $h\nu$: elle varie proportionnellement à la fréquence.»

(*La Théorie du Rayonnement et les Quanta*, report of the 1911 Solvay Conference, Gauthier-Villars, 1912, p. 450.)

The word *adiabatic*, whose introduction is due to Paul Ehrenfest[2], directly comes from thermodynamics. In that theory, adiabatic transformations do not dissipate heat. In this case, if, e.g., we slowly double the lenght of the pendulum and then we shorten it towards the original value, the initial and final energy are the same, and no energy is dissipated as heat, because the action (energy over frequency) remains almost unchanged during the whole process.

2.1 Definitions

Let $\mathcal{H}(p, q, \lambda)$ be a single-degree of freedom hamiltonian. For technical reason, connected with the fact that parameter variations should be made unaware of the system status, we suppose $\mathcal{H}$ is $\mathcal{C}^2$ with respect to λ. [1]

We request $\lambda(t)$ to be a parameter *slowly* dependent upon time, where slowness property is mathematically stated as follows: there exist ε such as

$$\frac{1}{n!}\left|\frac{\mathrm{d}^n\lambda}{\mathrm{d}t^n}\right| \leq \varepsilon^n$$

However, we will define $\lambda = \varepsilon t$. This assumption trivially satisfies the slowness condition.

We have

Definition 1. *A function $I(p, q, \lambda)$ is called* adiabatic invariant *of the system $\mathcal{H}$ if, for any $\varepsilon > 0$ there exist an $\varepsilon_0 > 0$, so that, if $\varepsilon < \varepsilon_0$, $0 < t < 1/\varepsilon$*

$$|I(p(t), q(t), \varepsilon t) - I(p(0), q(0), 0)| < c\varepsilon^\alpha \quad (\alpha > 0)$$

This definition means that, for an adiabatic invariant I, we have, for a time $t < 1/\varepsilon$, $\alpha = 1$ that $I(t) - I(0) = \mathcal{O}(\varepsilon)$..

Trivially, integrals of motion are adiabatic invariants, but we are interested in non-trivial examples. We will show that every unidimensional system admits an adiabatic invariant. The proof is rather constructive: we actually show that action variable is adiabatically invariant.

We define

[2] EHRENFEST P., *Over adiabatische veranderingen van een stelsel in verband met e theorie de quanta*, *Verslagen Kon. Akad. Amsterdam* **25** (1916) pp. 412—433

Definition 2 (Action-angle variables)**.** *In a unidimensional system with hamiltonian $\mathcal{H}(p,q)$, being $\gamma(E)$ a closed level curve of energy E in the phase space, in each region delimitated by separatrices, we define as* action variable $J(E)$ *the integral*

$$J = \frac{1}{2\pi} \oint_{\mathcal{H}=E} p\mathrm{d}q$$

and angle coordinate *the expression*

$$\theta = \frac{\partial}{\partial J} \int_{\mathcal{H}=E} p\mathrm{d}q$$

The change of coordinates $(p,q) \to (J,\theta)$ is canonical.

We also define the frequency ω *as*

$$\omega = \frac{\partial \mathcal{H}}{\partial J}$$

2.2 An example: the harmonic oscillator

Let we consider a unitary mass pendulum whose frequency ω is slowly variated according to the expression $\omega = \omega_0(1+\lambda)$, where $\lambda = \varepsilon t$.

Before taking into account the perturbation, we know that the hamiltonian reads

$$\mathcal{H} = \frac{p^2}{2} + \omega^2 \frac{q^2}{2} \tag{2.1}$$

and, fixed the energy E, the motion lies on an ellipse whose semiaxes a and b are

$$a = \sqrt{2E}$$
$$b = \sqrt{2E/\omega}$$

Thus, the action variable $J = (2\pi)^{-1} \int \mathrm{d}q\, p$ is obtained dividing for 2π the area of this ellipse which is πab. We have

$$J = \frac{ab}{2} = \frac{E}{\omega} = \frac{1}{2\omega}\left(p^2 + \omega^2 q^2\right)$$

Introducing now our perturbation (we choose $\omega_0 = 1$), we have

$$J = \frac{q^2(1+\varepsilon t)}{2} + \frac{p^2}{2(1+\varepsilon t)}$$

Now, we use Taylor expansion to rewrite

$$\frac{1}{1+\varepsilon t} = 1 - \varepsilon t + \mathcal{O}(\varepsilon)$$

and we have

$$J = \frac{q^2}{2} + \frac{p^2}{2} + \varepsilon\left(\frac{q^2 t}{2} - \frac{p^2 t}{2}\right) + \mathcal{O}(\varepsilon^2)$$

In order to prove that J is adiabatically invariant, we have to show that, over a period,

$$\Delta J = \mathcal{O}(\varepsilon^2)$$

We need the ε^2 order because, if ΔJ were only $\mathcal{O}(\varepsilon)$, for a time $1/\varepsilon$ we would have $\Delta J = \mathcal{O}(1)$.

Now, if T is the period of oscillation,

$$\Delta J = J(T) - J(0) = \varepsilon\left(\frac{q^2}{2} - \frac{p^2}{2}\right)T + \mathcal{O}(\varepsilon^2)$$

Rearranging eq. 2.1 we obtain

$$\frac{p^2}{2} = \mathcal{H}(t) - \frac{q^2}{2} - \varepsilon t \frac{q^2}{2}$$

and, substituting,

$$\Delta J = \varepsilon(\mathcal{H}(T) - \mathcal{H}(0)) - \varepsilon^2 T^2 \frac{q^2}{2} + \mathcal{O}(\varepsilon^2)$$

On a period,

$$\mathcal{H}(T) - \mathcal{H}(0) = \varepsilon T \frac{q^2}{2}$$

Therefore

$$\Delta J = \mathcal{O}(\varepsilon^2)$$

so J is adiabatically invariant.

2.3 The adiabatic theorem

We are now ready to state the main theorem of this chapter.

Theorem 1. *In a unidimensional system with hamiltonian $\mathcal{H}(p, q, \lambda)$, if $\omega(J, \lambda) \neq 0$, the action variable $J(E, \lambda)$ is adiabatically invariant.*

Proof. Time-dependent canonical changes of variables cause the hamiltonian to mutate according to the law

$$\tilde{\mathcal{H}} = \mathcal{H} + \frac{\partial F}{\partial t} \tag{2.2}$$

where F is the transforamation generatrix

$$F(q, E, \lambda) = \int_{\mathcal{H}=E} \mathrm{d}q\, p$$

Now, applying chain rule, $\lambda = \varepsilon t$, so

$$\tilde{\mathcal{H}} = \mathcal{H} + \varepsilon \frac{\partial F}{\partial \lambda}$$

We procede evaluating the term $\frac{\partial F}{\partial \lambda}$.

$$\left.\frac{\partial F}{\partial \lambda}\right|_{q,J} = \int \mathrm{d}q \left(\left.\frac{\partial p}{\partial \lambda}\right|_{q,E} + \left.\frac{\partial p}{\partial \lambda}\right|_{q,\lambda} \left.\frac{\partial \mathcal{H}}{\partial \lambda}\right|_{J} \right) \tag{2.3}$$

Now, differentiating the equation $\mathcal{H}(q, p(q, E, \lambda), \lambda) = E$ we obtain, remembering that q does not depend upon λ,

$$\left.\frac{\partial \mathcal{H}}{\partial p}\right|_{q,\lambda} \left.\frac{\partial p}{\partial \lambda}\right|_{q,E} + \frac{\partial \mathcal{H}}{\partial \lambda} = 0$$

and, solving the equation for $\frac{\partial p}{\partial \lambda}$,

$$\left.\frac{\partial p}{\partial \lambda}\right|_{q,E} = -\left.\frac{\partial \mathcal{H}}{\partial \lambda}\right|_{p,q} \left(\left.\frac{\partial \mathcal{H}}{\partial p}\right|_{q,\lambda} \right)^{-1}$$

From the definition of angle variable, it is straightforward to get

$$\frac{\partial \theta}{\partial q} = \frac{\partial p}{\partial J} = \frac{\partial p}{\partial \mathcal{H}} \frac{\partial \mathcal{H}}{\partial J} = \left(\frac{\partial \mathcal{H}}{\partial p} \right)^{-1} \omega$$

thus, integrating $\frac{\partial p}{\partial \lambda}$ we obtain

$$\int \mathrm{d}q \left.\frac{\partial p}{\partial \lambda}\right|_{q,E} = -\frac{1}{\omega}\int \mathrm{d}\theta \, \frac{\partial \mathcal{H}}{\partial \lambda}$$

In order to evaluate the second term of the integral (2.3). We procede in the same way. This time the equation to differentiate will be

$$\mathcal{H}(J,\lambda) = E$$

from which we get

$$\left.\frac{\partial \mathcal{H}}{\partial J}\right|_{\lambda} \left.\frac{\partial J}{\partial \lambda}\right|_{E} + \frac{\partial \mathcal{H}}{\partial \lambda}_{J} = 0$$

We integrate and obtain

$$\int \mathrm{d}\theta \, \left.\frac{\partial J}{\partial \lambda}\right|_{E} = -\int \mathrm{d}q \, \left.\frac{\partial p}{\partial E}\right|_{q,\lambda} \left.\frac{\partial H}{\partial \lambda}\right|_{J}$$

Now we compare this expressione of $\frac{\partial J}{\partial \lambda}$ with the one obtained differentiating the definition of J

$$\left.\frac{\partial J}{\partial \lambda}\right|_{E} = \frac{1}{2\pi}\oint_{\mathcal{H}=E} \mathrm{d}q \, \left.\frac{\partial p}{\partial \lambda}\right|_{q,E}$$

The expression of $\frac{\partial F}{\partial \lambda}$ now reads

$$\frac{\partial F}{\partial \lambda} = -\frac{1}{\omega(E,\lambda)}\int \mathrm{d}\theta \left.\frac{\partial \mathcal{H}}{\partial \lambda}\right|_{p,q} - \int \mathrm{d}\theta \, \left.\frac{\partial J}{\partial \lambda}\right|_{E}$$

Rewriting $\frac{\partial J}{\partial \lambda}$ as follows:

$$\frac{\partial J}{\partial \lambda} = \frac{\partial J}{\partial \mathcal{H}}\frac{\partial \mathcal{H}}{\partial \lambda}$$

integrating and remembering the definition of ω we finally obtain

$$\frac{\partial J}{\partial \lambda} = -\frac{1}{2\pi\omega}\int_0^{2\pi} \mathrm{d}\theta \, \frac{\partial \mathcal{H}}{\partial \lambda} = -\frac{1}{\omega}\left\langle \frac{\partial \mathcal{H}}{\partial \lambda} \right\rangle$$

Thus, we have

$$\frac{\partial F}{\partial \lambda} = -\frac{1}{\omega}\int \mathrm{d}\theta \left(\frac{\partial \mathcal{H}}{\partial \lambda} - \left\langle \frac{\partial \mathcal{H}}{\partial \lambda} \right\rangle \right) \tag{2.4}$$

This integral, extended to a period, gives result zero, proving that it is possible to choose a generatrix F such as $\frac{\partial F}{\partial \lambda}$ has zero average over a period.

Being $\left\langle \frac{\partial F}{\partial \lambda} \right\rangle = 0$, we have $\left\langle \tilde{\mathcal{H}} \right\rangle = \langle \mathcal{H} \rangle$.

Now, in order to prove the adiabatic invariance of J we can consider its time derivative $\dot{J}$.

We have

$$\dot{J} = \varepsilon \frac{\partial F}{\partial \lambda}$$

so

$$\dot{J} = \frac{\varepsilon}{\omega} \left(\frac{\partial \mathcal{H}}{\partial \lambda} - \left\langle \frac{\partial \mathcal{H}}{\partial \lambda} \right\rangle \right)$$

Now, we write ΔJ for a period T:

$$\Delta J = J(T) - J(0) = \int_0^T \mathrm{d}t \, \dot{J}$$

and we integrate over θ, performing the substitution

$$\mathrm{d}t = \frac{\mathrm{d}\theta}{\omega} + \mathcal{O}(\varepsilon)$$

which comes from the equation of motion for θ:

$$\dot{\theta} = \omega + \mathcal{O}(\varepsilon)$$

obtaining

$$\Delta J = \varepsilon \int_0^{2\pi} \frac{1}{\omega^2} \left(\frac{\partial \mathcal{H}}{\partial \lambda} - \left\langle \frac{\partial \mathcal{H}}{\partial \lambda} \right\rangle \right) + \mathcal{O}(\varepsilon^2)$$

Now, ω depends on the energy E and on the paramater λ. On a period, the variation of these quantities is by definition $\mathcal{O}(\varepsilon)$, so we can extract ω^2 from the integral.

Having shown that eq. 2.4 is zero, ΔJ is therefore $\mathcal{O}(\varepsilon^2)$, this proving our initial statement. [1–4]

□

It should be pointed out that this proof is valid only in the unidimensional case. If we study adiabatic invariants in higher dimensions, we will stumble upon the fact that, unless frequencies are commensurable, trajectories are not closed, but they densely envelope invariant tori. Thus, we are not able to compute averages and we cannot build an adiabatic invariant for every system. [9, p. 234]

Chapter 3

Averaged hamiltonian

We consider a simplied model for the betatronic dynamics in a particle accelerator using a one dimensional hamiltonian (flat beam) with average non linear effects (sextupole and octupole terms).

Let we start from a magnetic field hamiltonian in multipole expansion as derived in Chapter 1, including terms up to the octupole (Q^4) one. We will analyze two different ways of slowly varying parameters of the magnetic field: in the first section we will use a dipolar kick, i.e. the dipole term will be perturbated, while in the second one we will use a quadrupolar kick, varying the Q^2 term.

3.1 Dipolar kick

For a dipolar kick, our hamiltonian reads

$$\mathcal{H} = \frac{P^2}{2} + \omega_0 \frac{Q^2}{2} - k_3 \frac{Q^3}{3} + k_4 \frac{Q^4}{4} + Q\varepsilon \cos(\omega t)$$

where the term $\varepsilon \cos(\omega t)$ is a perturbation of the dipolar term.

We will now proceed imposing a canonical change of variables

$$\begin{cases} Q = \sqrt{2\rho/\omega_0} \sin\phi \\ P = \sqrt{2\rho\omega_0} \cos\phi \end{cases} \tag{3.1}$$

which causes $\mathcal{H}$ to become

$$\mathcal{H} = \omega_0 \rho - \frac{k_3}{3\omega_0^{3/2}} (2\rho)^{3/2} \sin^3\phi + \frac{k_4}{\omega_0^2} \rho^2 \sin^4\phi + \varepsilon \sqrt{\frac{2\rho}{\omega_0}} \sin\phi \cos(\omega t) \tag{3.2}$$

3.1.1 Average dynamics

We want to average the system near the $1:1$ resonance γ, so we apply another canonical transformation $(\rho, \phi) \to (\tilde{\rho}, \gamma)$

$$\gamma = \phi - \omega t \qquad \Rightarrow \qquad \phi = \gamma + \omega t$$

while the radius rho remains unvaried $\tilde{\rho} = \rho$.

Now, this transformation is time-dependent, so the new hamiltonian cannot be rewritten trivially substituting the expression of ϕ. We need a generatrix function

$$F(\tilde{\rho}, \phi) = \tilde{\rho}\gamma = \tilde{\rho}(\phi - \omega t) = \rho(\phi - \omega t)$$

and the new hamiltonian $\mathcal{H}'$ becomes

$$\mathcal{H}' = \mathcal{H} + \frac{\partial F}{\partial t} = \mathcal{H} - \rho\omega$$

i.e.

$$\begin{aligned}\mathcal{H}' = \rho(\omega_0 - \omega) &- \frac{k_3}{3\omega_0^{3/2}}(2\rho)^{3/2}\sin^3(\gamma + \omega t) \\ &+ \frac{k_4}{\omega_0^2}\rho^2 \sin^4(\gamma + \omega t) + \varepsilon\sqrt{\frac{2\rho}{\omega_0}}\sin(\gamma + \omega t)\cos(\omega t)\end{aligned}$$

We are now ready to average the hamiltonian near γ over the *fast* dynamics given by ωt.

We remember that, over a period, $\langle \sin^3 x \rangle = 0$

In fact,

$$\begin{aligned}\langle \sin^3 x \rangle &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}x\, \sin^3 x = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}t \left(\frac{e^{it} - e^{-it}}{2i}\right)^3 = \\ &= -\frac{1}{16i\pi}\int_0^{2\pi} \mathrm{d}t\big(e^{3it} - 3e^{it} + 3e^{-it} - e^{-3it}\big) = 0\end{aligned}$$

and the third-order term cancel.

We will now calculate $\langle \sin^4 x \rangle$.

$$\begin{aligned}\langle \sin^4 x \rangle &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}x\, \sin^4 x = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}t \left(\frac{e^{it} - e^{-it}}{2i}\right)^4 = \\ &= \frac{1}{32\pi}\int_0^{2\pi} \mathrm{d}t\big(e^{4it} - 4e^{2it} + 6 - 4e^{-2it} - e^{-4it}\big) = \\ &= \frac{1}{32\pi} \cdot 6 \cdot 2\pi = \frac{3}{8}\end{aligned}$$

Finally, we have to compute $\langle \sin(\gamma + \omega t)\cos(\omega t)\rangle$:

$$\begin{aligned}\langle \sin(\gamma + \omega t)\cos(\omega t)\rangle &= \langle[\sin\gamma\cos(\omega t) + \cos\gamma\sin(\omega t)]\cos(\omega t)\rangle \\ &= \sin\gamma\left\langle\cos^2(\omega t)\right\rangle + \cos\gamma\,\langle\sin(\omega t)\rangle\,\langle\cos(\omega t)\rangle\end{aligned}$$

The second term is trivially zero, being $\langle\sin(\omega t)\rangle = 0$, while for the first we need $\langle\cos^2 x\rangle$

$$\begin{aligned}\left\langle\cos^2 x\right\rangle &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}x\,\cos^2 x = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}t\left(\frac{e^{it}+e^{-it}}{2}\right)^2 = \\ &= \frac{1}{8\pi}\int_0^{2\pi}\mathrm{d}t\big(e^{2it} + 2 + e^{2it}\big) = \\ &= \frac{1}{8\pi}\cdot 2\cdot 2\pi = \frac{1}{2}\end{aligned}$$

Substituting all these results, the averaged hamiltonian $\mathcal{H}$ reads

$$\mathcal{H} = \rho(\omega_0 - \omega) + \frac{3k_4}{8\omega_0^2}\rho^2 + \frac{\varepsilon}{2}\sqrt{\frac{2\rho}{\omega_0}}\sin\gamma$$

We are now ready for another coordinate change. We set

$$\begin{cases} x = \sqrt{2\rho}\sin\gamma \\ y = \sqrt{2\rho}\cos\gamma \end{cases} \tag{3.3}$$

resulting in the hamiltonian

$$\mathcal{H} = (\omega_0 - \omega)\left(\frac{x^2+y^2}{2}\right) + \frac{3k_4}{8\omega_0^2}\left(\frac{x^2+y^2}{2}\right)^2 + \frac{\varepsilon}{2\sqrt{\omega_0}}x$$

Finally, rescaling and letting

$$\begin{cases} \lambda = -\dfrac{16}{3}\dfrac{\omega_0^2}{k_4}(\omega_0 - \omega) \\ \mu = \dfrac{16}{3}\dfrac{\omega_0^{3/2}}{k_4}\varepsilon \end{cases}$$

our hamiltonian reads

$$\mathcal{H} = (x^2+y^2)^2 - \lambda(x^2+y^2) + \mu x \tag{3.4}$$

This specific hamiltonian has been widely studied, and has appearead in many different fields, including celestial mechanics. [10, 11]

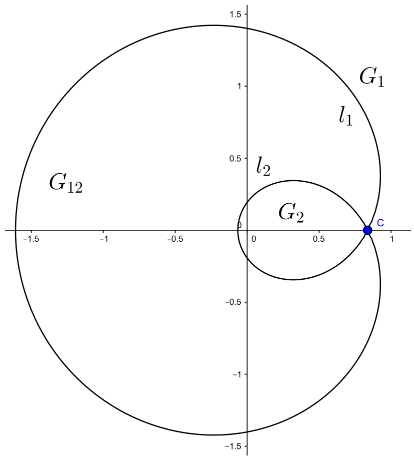


FIGURE 3.1 – Separatrices in phase space (x, y) for hamiltonian (3.4) with $\lambda = 2$, $\mu = 1$. The topology of the phase space here portraited, with the two separatrices l_1 and l_2 dividing the space into regions G_1, G_2 and G_{12} is obtained when $\lambda > \frac{3}{2}\mu^{2/3}$. The saddle point C is also shown.

3.1.2 Study of phase space topology

For λ larger or equal than a critical value λ^*, the phase space topology is the one in fig. 3.1. There are two equilibrium points and a saddle point C, whose coordinate are $(x_C, 0)$, where x_C is the largest root of the equilibrium equation

$$\frac{\partial \mathcal{H}}{\partial x} = 4x^3 - 2\lambda x + \mu = 0$$

This equation has three real solutions if

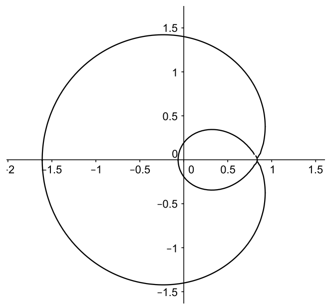
$$\lambda > \frac{3}{2}\mu^{2/3}$$

thus obtaining an expression for critical value λ^*.

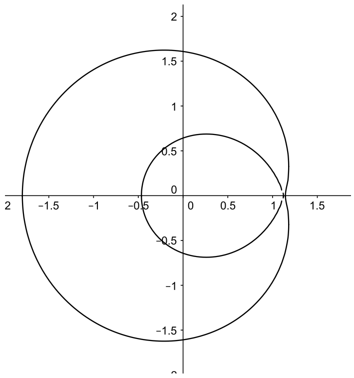
An expression of x_C is retrievable using the formulae for third grade equations. It reads

$$x_C = \frac{\sqrt{6\lambda}}{3} \cos\left(\frac{\pi}{6} + \frac{1}{3}\arcsin\left(\frac{3\sqrt{6}}{4}\frac{\mu}{\lambda^{3/2}}\right)\right)$$

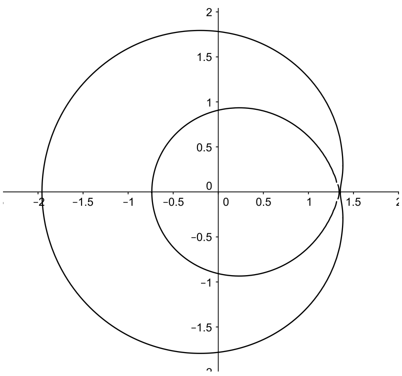
We show in figures (3.2 — 3.3) the geometry of separatrices in the phase spaces for different values of λ and μ. We call G_2 the



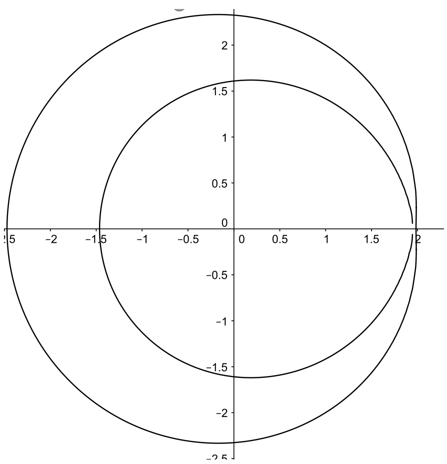
(A) $\lambda = 2,\ \mu = 1$



(B) $\lambda = 3,\ \mu = 1$

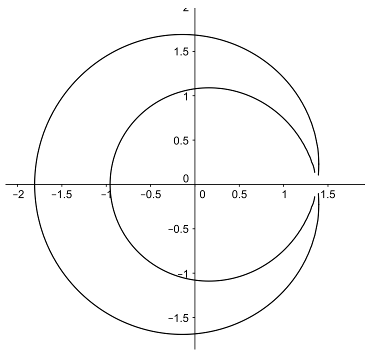


(C) $\lambda = 4,\ \mu = 1$

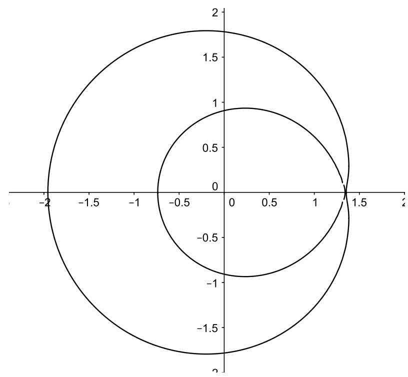


(D) $\lambda = 8,\ \mu = 1$

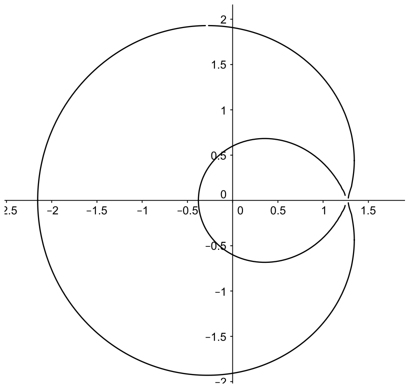
FIGURE 3.2 – Portraits of the separatrices in the phase space (x, y) of hamiltonian (1.1) for different values of λ, having fixed $\mu = 1$. As λ increases, both the inner (G_2) and the outer (G_{12}) regions of the phase space inflates, but G_2 gets bigger faster, eventually covering G_{12} for $\lambda \gg \mu$.



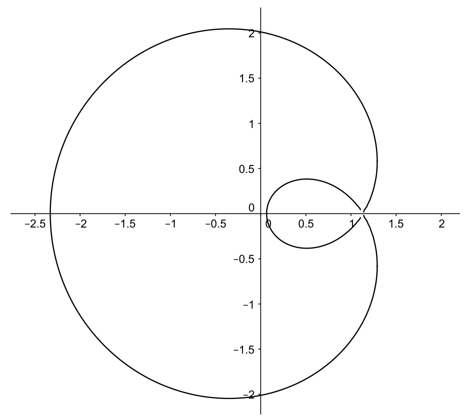
(A) $\lambda = 4$, $\mu = 0.5$



(B) $\lambda = 4$, $\mu = 1$



(C) $\lambda = 4$, $\mu = 2$



(D) $\lambda = 4$, $\mu = 3.25$

FIGURE 3.3 – Portraits of the separatrices in the phase space (x, y) of hamiltonian (3.4) for different values of μ, having fixed $\lambda = 4$. As μ increases, the outer separatrix slowly inflates, while the inner one gets much smaller, eventually (fig. 3.3d) excluding the origin $(0, 0)$ from the inner region.

region delimitated by the smaller closed curve, G_1 the region outside separatrices and G_{12} the remaining inner part of the phase space. A point in G_2 will orbit not far from the origin, while a point in G_{12} will be transported away. Going back to our initial idea, a particle which happens to be in G_{12} will form one of the beamlet to be bent, while particles in G_1 will stay in the main beam.

We also call l_1 the outer, bigger separatrix curve and l_2 the inner, smaller one.

Fixed λ, if a point is initially in G_1, as the separatrix inflates when λ increases, the point will, at a certain $\tilde{\lambda}$, be trapped into G_{12}, and be *captured into resonance*, or eventually cross the smaller separatrix, passing into resonance without being captured.

In an averaged approach, where the phase space we study, resulting from a deformation of the original hamiltonian phase space, is omeomorphic to the original one, we cannot exactly know which one of the two scenarios will happen. Although, we can consider the capture into resonance as a random event, depending on initial conditions. Therefore, it is possible to compute the *probability* of capture into G_{12}.

We now define $\mathcal{H}_C$ to be the hamiltonian at point C

$$\mathcal{H}_C = \mathcal{H}(x_C, 0, \lambda)$$

and we introduce a new hamiltonian $\mathcal{K}$ as

$$\mathcal{K} = \mathcal{H} - \mathcal{H}_C$$

This way, in the regions of the phase space (see fig. 3.1) G_1 and G_2 we have $\mathcal{K} > 0$; in G_{12} $\mathcal{K} < 0$ and on the separatrices $\mathcal{K} = 0$.

3.1.3 Capture probability

In [11], Nejštadt proves why capture probability $\mathcal{P}$ can be computed using the formula

$$\mathcal{P} = \frac{I_1 - I_2}{I_1} \tag{3.5}$$

where

$$I_1 = -\oint_{l_1} \mathrm{d}t \, \frac{\partial \mathcal{H}}{\partial \lambda} \qquad I_2 = -\oint_{l_2} \mathrm{d}t \, \frac{\partial \mathcal{H}}{\partial \lambda}$$

The integrals have to be computed at $\lambda = \tilde{\lambda}$ where the point crosses the separatrix.

Without giving a formal proof of this statement, we should note that, according to Liouville's theorem, phase space volume is conserved. Now, on a variation of λ, a phase volume V_1 will enter G_{12} from G_1, and a phase volume V_2 will leave G_{12} entering G_2. So, the total G_{12} volume gain is $(V_1 - V_2)/V_2$, and the two integrals I_1 and I_2 are the flow of phase volume across l_1 and l_2. This justifies Nejštadt's formula.

Now, I_1 and I_2 are quite hard to compute, but an exact method is possible to perform. First of all, we need to move the origin on the saddle point $(x_C, 0)$ performing the transformation $x \to x + x_C$, obtaining the hamiltonian

$$\mathcal{K} = \mathcal{H} - \mathcal{H}_C = (x^2+y^2)^2 + 4x_C x(x^2+y^2) - (\tilde{\lambda} - 2x_C^2)(x^2+y^2) + 4x_C^2 x^2$$

We go back to (ρ, γ) coordinates and we get

$$\mathcal{K} = 4\rho^2 + 8x_C\sqrt{2\rho}\rho\sin\gamma - 2(\tilde{\lambda} - 2x_C^2)\rho + 8x_C^2\rho\sin^2\gamma$$

On the separatrices $\mathcal{K} = 0$, so we have the equation (dividing for ρ and factorizing)

$$2\left(\sqrt{\rho} + \sqrt{2}x_C\sin\gamma\right)^2 = \left(\tilde{\lambda} - 2x_C^2\right)$$

which solved gives

$$\rho(\gamma) = \frac{1}{2}\left(\sqrt{\tilde{\lambda} - 2x_C^2}\right) \pm 2x_C\sin\gamma)^2 \tag{3.6}$$

We rewrite the differential $\mathrm{d}t$ using

$$\mathrm{d}t = \frac{\mathrm{d}\gamma}{\dot{\gamma}} = \frac{\mathrm{d}\gamma}{\partial\mathcal{K}/\partial\rho} = \frac{\sqrt{2}\mathrm{d}\gamma}{2\sqrt{\rho}\sqrt{\tilde{\lambda} - 2x_C^2}}$$

Now,

$$\frac{\partial\mathcal{H}}{\partial\lambda} = -\rho$$

so, performing the substitutions, our integral reads

$$\oint_{l_i} \mathrm{d}\gamma \, \frac{\sqrt{2}\rho}{2\sqrt{\tilde{\lambda} - 2x_C^2}} = \oint_{l_i} \mathrm{d}\gamma \, \frac{\sqrt{2\rho}}{2\sqrt{\tilde{\lambda} - 2x_C^2}}$$

From (3.6) we get the expression of $\sqrt{\rho}$ and the integral finally becomes

$$\oint_{l_i} \mathrm{d}\gamma \, \frac{\sqrt{\tilde{\lambda} - 2x_C^2 \pm 2x_c \sin\gamma}}{2\sqrt{\tilde{\lambda} - 2x_C^2}} = \frac{1}{2}\oint_{l_i} \mathrm{d}\gamma \pm \frac{x_C}{\sqrt{\lambda - 2x_C^2}} \oint_{l_i} \mathrm{d}\gamma \, \sin\gamma \tag{3.7}$$

We observe that the integrand function only depends on the angle γ, so in order to get the exact result we have to determine the dominion of γ spanning the separatrix curves. Having translated the origin in the hyperbolic point C, we obtain (see fig. 3.1) that for l_1 γ must extend from the angle $+\Theta/2$ to $2\pi - \Theta/2$, while for l_2 we will have $\gamma \in [\pi - \Theta/2, \pi + \Theta/2]$, where Θ is the angle between the tangent lines to the sepatrices in C, which can be computed by Morse lemma.

Indeed, we have

$$\tan(\Theta/2) = \sqrt{\frac{6x_C^2 - \tilde{\lambda}}{\tilde{\lambda} - 2x_C^2}}$$

and, using a well-known trigonometric relation

$$\cos\Theta = \frac{1 - \tan^2(\Theta/2)}{1 + \tan^2(\Theta/2)} = \frac{\tilde{\lambda}}{2x_C^2} - 2$$

so that

$$\Theta = \arccos\left(\frac{\tilde{\lambda}}{2x_C^2} - 2\right)$$

Coming back to the integral (3.7) we should notice that the two dominions of integration are symmetric, and that the second term integrand, $\sin\gamma$, is an odd function. This means that that term is zero, and we finally have

$$I_1 = \frac{1}{2}\int_{\Theta/2}^{2\pi - \Theta/2} \mathrm{d}\gamma = \pi - \Theta/2$$

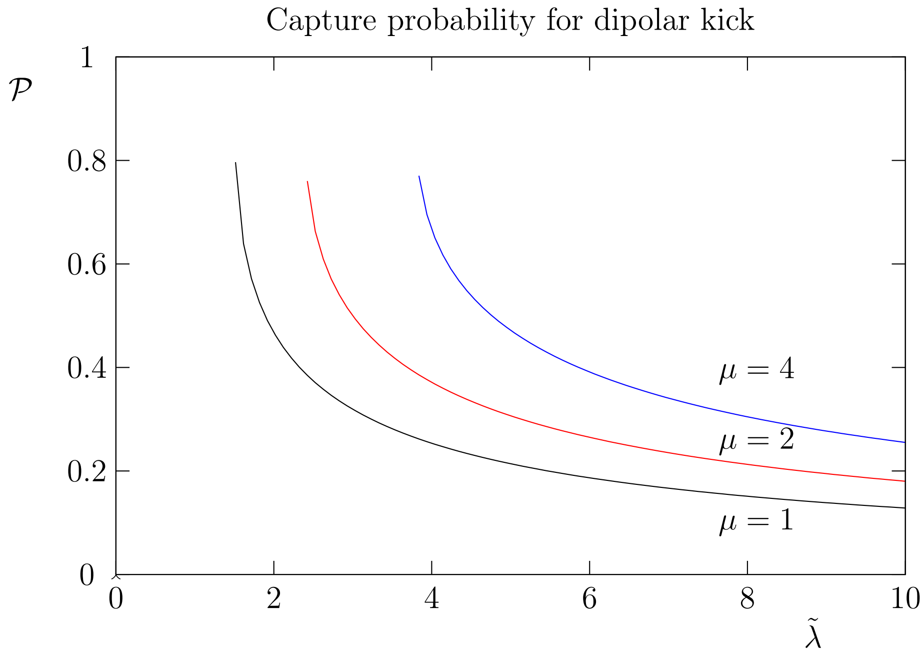


FIGURE 3.4 – Plot of capture into resonance probability $\mathcal{P}$ (from eq. 3.8) for hamiltonian (3.4) as function of $\tilde{\lambda}$ for different values of μ. Probability is asintotically 1 for $\tilde{\lambda} \to \tilde{\lambda}^* = \frac{3}{2}\mu^{2/3}$ and tends to 0 as $\tilde{\lambda} \to \infty$. On larger values of μ, the minimal $\tilde{\lambda}$ necessary for capture to occur increases, but probability of capture for larger $\tilde{\lambda}$ is better.

$$I_2 = \frac{1}{2}\int_{\pi-\Theta/2}^{\pi+\Theta/2} \mathrm{d}\gamma = \Theta/2$$

We therefore have a simple formula for probability $\mathcal{P}$. From (3.5) we obtain

$$\mathcal{P}(\tilde{\lambda}) = \frac{\pi - \Theta}{\pi - \Theta/2} \tag{3.8}$$

We show in fig. (3.4) $\mathcal{P}$ plotted over $\tilde{\lambda}$ for different values of μ.

3.2 Quadrupolar kick

Introducing a quadrupolar magnetic field kick in our base hamiltonian, we obtain

$$\mathcal{H} = \frac{P^2}{2} + \omega_0 \frac{Q^2}{2} - k_3 \frac{Q^3}{3} + k_4 \frac{Q^4}{4} + \frac{Q^2}{2} \varepsilon \cos(\omega t)$$

We proceed as we did in the precedent section, following step-by-step Nejštadt's approach in [10].

Applying the same canonical change of variable $(P, Q) \to (\rho, \phi)$ we used in the first section (eq. 3.1) we obtain a new hamiltonian which reads

$$\mathcal{H} = \omega_0 \rho - \frac{k_3}{3\omega_0^{3/2}} (2\rho)^{3/2} \sin^3 \phi + \frac{k_4}{\omega_0^2} \rho^2 \sin^4 \phi + \frac{\varepsilon \rho}{\omega_0} \sin^2 \phi \cos(\omega t)$$

3.2.1 Average dynamics

Now we have to average our system near the resonance. It is straightforward to show that using $1:1$ resonance as in the dipolar case would give a null result. Thus, the right substitution to perform is

$$\gamma = 2\phi - \omega t \qquad \phi = \frac{1}{2}(\gamma + \omega t)$$

This is a canonical transformation $(\rho, \phi) \to (\tilde{\rho}, \gamma)$ whose generatrix function reads

$$F(\tilde{\rho}, \phi) = \tilde{\rho}\gamma = \tilde{\rho}(2\phi - \omega t) = \rho(2\phi - \omega t)$$

and

$$\frac{\partial F}{\partial t} = \rho\omega$$

so the new hamiltonian $\mathcal{H}'$ is

$$\begin{aligned} \mathcal{H}' = \rho(\omega_0 - \omega) - \frac{k_3}{3\omega_0^{3/2}} (2\rho)^{3/2} \sin^3 \left(\frac{\gamma - \omega t}{2} \right) + \\ + \frac{k_4}{\omega_0^2} \rho^2 \sin^4 \left(\frac{\gamma - \omega t}{2} \right) + \\ + \frac{\rho \varepsilon}{\omega_0} \sin^2 \left(\frac{\gamma - \omega t}{2} \right) \cos(\omega t) \end{aligned}$$

Now, we have to average this integral over a period $4\pi/\omega$.
We have, reusing results from the precedent section

$$\left\langle \sin^3\left(\frac{\gamma-\omega t}{2}\right)\right\rangle = 0$$
$$\left\langle \sin^4\left(\frac{\gamma-\omega t}{2}\right)\right\rangle = \frac{3}{8}$$

We have now to compute

$$\left\langle \sin^2\left(\frac{\gamma-\omega t}{2}\right)\cos\left(\omega t\right)\right\rangle$$

.

Let $\omega t = \theta$.

$$\begin{aligned}\left\langle \sin^2\left(\frac{\gamma-\theta}{2}\right)\cos\theta\right\rangle &= \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta\left(\frac{e^{\frac{i\theta+i\gamma}{2}}-e^{\frac{-i\theta-i\gamma}{2}}}{2i}\right)^2\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)\\ &= -\frac{1}{16\pi}\int_0^{2\pi}\mathrm{d}\theta\big(e^{i\theta}e^{i\gamma}+e^{-i\theta}e^{-i\gamma}-2\big)\left(e^{i\theta}+e^{-i\theta}\right)\\ &= -\frac{1}{8\pi}\frac{e^{i\gamma}+e^{-i\gamma}}{2}\int_0^{2\pi}\mathrm{d}\theta = -\frac{\cos\gamma}{4}\end{aligned}$$

and the averaged hamiltonian finally reads

$$\langle\mathcal{H}\rangle = \rho(\omega_0-\omega)+\frac{3k_4}{8\omega_0^2}\rho^2-\frac{\varepsilon}{4\omega_0}\rho\cos\gamma \tag{3.9}$$

Finally, we write this hamiltonian using the (x, y) coordinates introduced in eq. (3.3) and, after a rescaling, we have

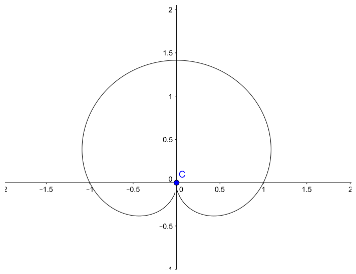
$$\mathcal{H} = (x^2+y^2)^2-\lambda(x^2+y^2)-\mu y\sqrt{x^2+y^2} \tag{3.10}$$

where we have defined

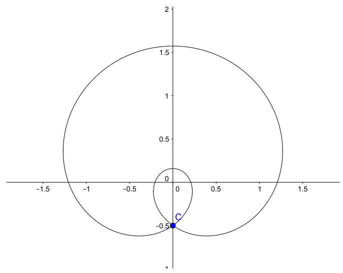
$$\begin{cases}\lambda = -\dfrac{16}{3}\dfrac{\omega_0^2}{k_4}(\omega_0-\omega)\\ \mu = \dfrac{8}{3\sqrt{2}}\dfrac{\omega_0}{k_4}\varepsilon\end{cases}$$

3.2.2 Study of phase space topology

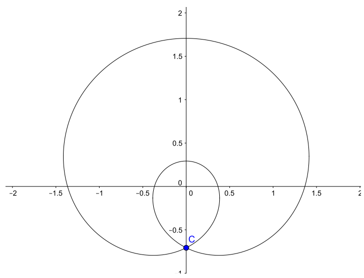
In figures (3.5 — 3.6) we show the geometry of separatrices in the phase spaces for different values of λ and μ.



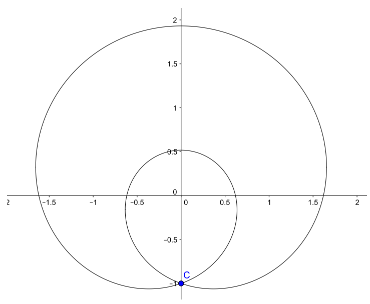
(A) $\lambda = 1$, $\mu = 1$



(B) $\lambda = 1.5$, $\mu = 1$

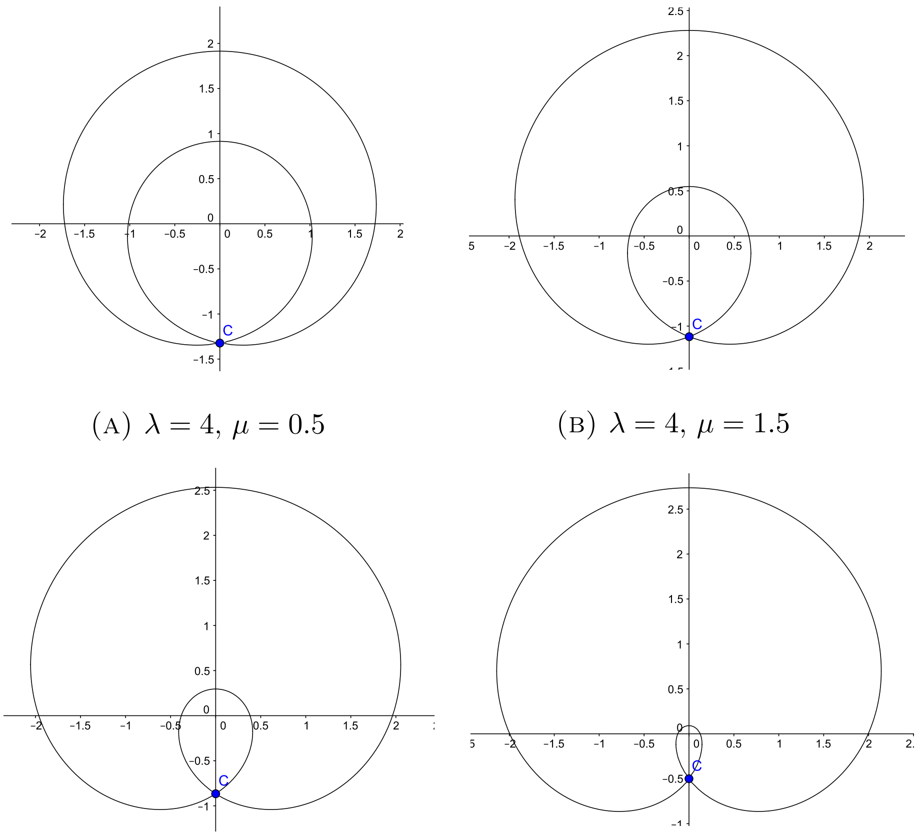


(C) $\lambda = 2$, $\mu = 1$



(D) $\lambda = 3$, $\mu = 1$

FIGURE 3.5 – Portraits of separatrices in the phase space (x, y) of hamiltonian (3.10) for different values of λ, having fixed $\mu = 1$. In the first picture, we have the critical case $\lambda = \lambda^*$. As λ increases, both the inner and the outer separatrices inflate.



(A) $\lambda = 4,\ \mu = 0.5$

(B) $\lambda = 4,\ \mu = 1.5$

(C) $\lambda = 4,\ \mu = 2.5$

(D) $\lambda = 4,\ \mu = 3.5$

FIGURE 3.6 – Portratits of separatrices in the phase space (x, y) of hamiltonian (3.10) for different values of μ, having fixed $\lambda = 4$. As μ increases, the outer separatrix slowly inflates, while the inner one decreases its size reaching eventually the critical condition for $\mu = \lambda$, but always keeping the origin $(0, 0)$ inside.

For $\lambda > \lambda^* = \mu$, we have two elliptical points and a hyperbolic one. The saddle point C has coordinates $(0, y_C)$ where y_C is the smaller root of the equation

$$\left.\frac{\partial \mathcal{H}}{\partial y}\right|_{x=0} = 4y^3 - 2\lambda|y| - 2\mu y = 0$$

We search a solution on the negative semiaxis, so $|y| = -y$.
Thus

$$y_C = -\sqrt{\frac{\lambda - \mu}{2}}$$

and, in order to obtain a real value, we retrieve $\lambda > \mu$.

We now want to obtain an explicit expression of the separatrices. On that curves, the condition $\mathcal{H} = \mathcal{H}_C$ is satisfied.

We have

$$\mathcal{H}_C = y_C^4 - \lambda y_C^2 - \mu y_C|y_C| = y_C^4 - \lambda y_C^2 + \mu y_C^2 = -\left(\frac{\lambda - \mu}{2}\right)^2$$

so, the equation $\mathcal{H} - \mathcal{H}_C = 0$, reduces, going back to (ρ, γ) coordinates:

$$4\rho^2 - 2\rho(\mu\cos\gamma + \lambda) + \left(\frac{\lambda - \mu}{2}\right)^2$$

and solving this second-grade equation we get

$$\rho(\gamma) = \frac{\mu\cos\gamma + \lambda \pm \sqrt{(\mu\cos\gamma + \lambda)^2 - (\lambda - \mu)^2}}{4}$$

3.2.3 Capture probability

The capture probability reads, being l_i a separatrix (see fig. 3.7), as in (3.5)

We have

$$I_i = \left.\int_{l_i} \mathrm{d}t\, \rho\right|_{\lambda = \tilde{\lambda}}$$

We rewrite this integral as follows

$$\int_{l_i} \mathrm{d}t\, \rho = \int \mathrm{d}\gamma \frac{\rho(\gamma)}{\dot{\gamma}}$$

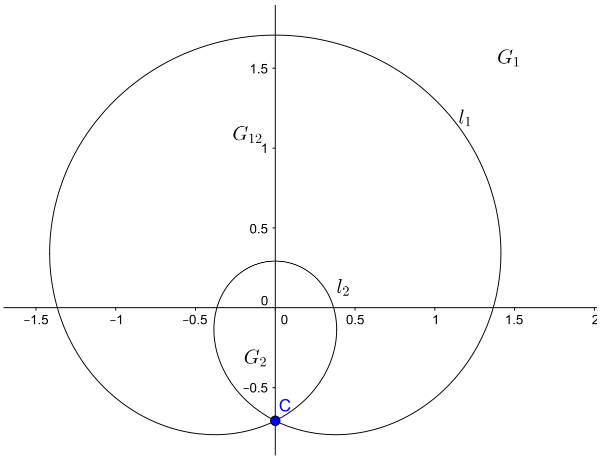


FIGURE 3.7 – Separatrices in phase space (x, y) for hamiltonian (3.10) with $\lambda = 2$, $\mu = 1$. The topology of the phase space here portraited, with the two separatrices l_1 and l_2 dividing the space into regions G_1, G_2 and G_{12} is obtained when $\lambda > \mu$. The saddle point C is also shown.

where $\dot{\gamma}$ is obtained from Hamilton equation:

$$\begin{aligned}\dot{\gamma} = \frac{\partial \mathcal{H}}{\partial \rho} &= 8\rho - 2\lambda - 2\mu\cos\gamma \\ &= 2(\mu\cos\gamma + \lambda) \pm 2\sqrt{(\mu\cos\gamma + \lambda)^2 - (\lambda - \mu)^2} - 2\lambda - 2\mu\cos\gamma \\ &= \pm 2\sqrt{(\mu\cos\gamma + \lambda)^2 - (\lambda - \mu)^2}\end{aligned}$$

and we find

$$\begin{aligned}I_i &= \int_{l_i} \mathrm{d}\gamma \, \frac{\mu\cos\gamma + \tilde{\lambda} \pm \sqrt{(\mu\cos\gamma + \tilde{\lambda})^2 - (\tilde{\lambda} - \mu)^2}}{\pm 8\sqrt{(\mu\cos\gamma + \tilde{\lambda})^2 - (\tilde{\lambda} - \mu)^2}} \\ &= \int \mathrm{d}\gamma \left(\frac{1}{8} \pm \frac{\mu\cos\gamma + \tilde{\lambda}}{8\sqrt{(\mu\cos\gamma + \tilde{\lambda})^2 - (\tilde{\lambda} - \mu)^2}} \right)\end{aligned}$$

As we did not perform any translation of the origin, this integrals have to be computed for $\gamma \in [0, 2\pi]$.

We define

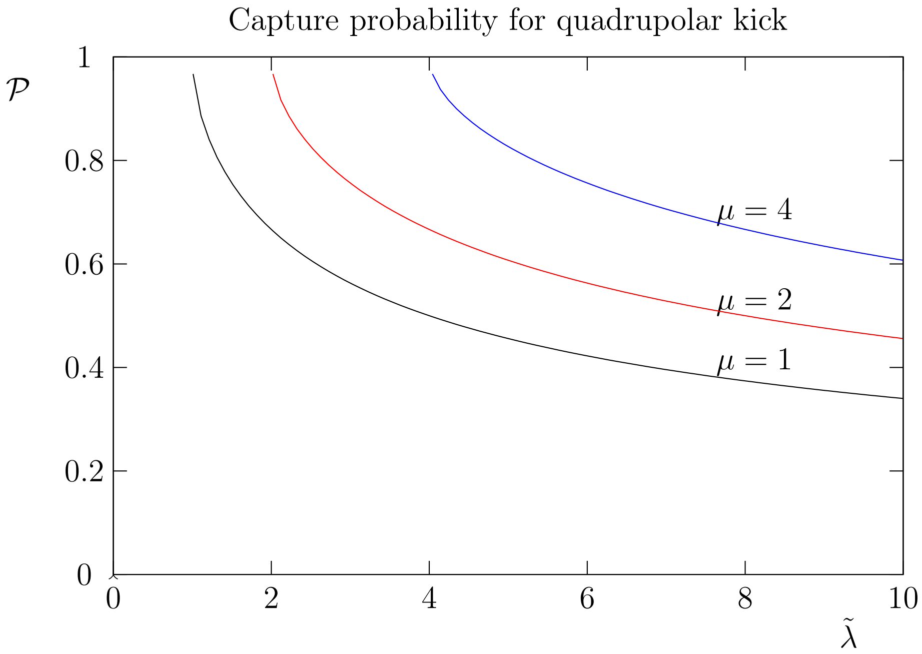


FIGURE 3.8 – Plot of capture into resonance probability $\mathcal{P}$ (from eq. 3.12) for hamiltonian (3.10) as function of $\tilde{\lambda}$ for different values of μ. Probability is asintotically 1 for $\tilde{\lambda} \to \tilde{\lambda}^* = \mu$ and tends to 0 as $\tilde{\lambda} \to \infty$. For larger values of μ, the minimal $\tilde{\lambda}$ necessary for capture to occur increases, but probability of capture for larger $\tilde{\lambda}$ is better.

$$\Xi = \int_0^{2\pi} \mathrm{d}\gamma \frac{\mu\cos\gamma + \tilde{\lambda}}{\sqrt{(\mu\cos\gamma + \tilde{\lambda})^2 - (\tilde{\lambda} - \mu)^2}} = 4 \arcsin\sqrt{\frac{\mu}{\tilde{\lambda}}} \tag{3.11}$$

The computation of Ξ, which is performed reconducting the integral to notable results tabled in [7], is shown in Appendix A.

Thus, we finally obtain

$$\mathcal{P} = \frac{I_2 - I_1}{I_1} = \frac{2\Xi}{2\pi + \Xi} \tag{3.12}$$

Fig. (3.8) shows how the probability $\mathcal{P}$ changes varying λ for different values of μ.

Chapter 4

Perturbation approach

Let we take into account the hamiltonian we studied in the second section of the latter chapter.

$$\mathcal{H} = \frac{P^2}{2} + \omega_0^2 \frac{Q^2}{2} + k_3 \frac{Q^3}{3} + k_4 \frac{Q^4}{4} + \varepsilon \frac{Q^2}{2} \cos(\omega t)$$

We rewrite it using (ρ, ϕ) coordinates as in equation (3.2), and, for $\varepsilon = 0$ we obtain

$$\mathcal{H} = \omega_0 \rho + \frac{2^{3/2} k_3}{3 \omega_0^{3/2}} \rho^{3/2} \sin^3 \phi + \frac{k_4}{\omega_0^{3/2}} \rho^2 \sin^4 \phi$$

Being $\varepsilon = 0$, we are able to perform a canonical transformation $(\rho, \phi) \to (J, \theta)$ where J and θ are *action-angle variables*, which causes the new hamiltonian to read

$$\mathcal{H}(J) = \omega_0 J + \omega_2 \frac{J^2}{2}$$

In order to find the expression of J and θ we consider the Lie trasformation $e^{D_{F(J,\theta)}}$ where we define

$$F(J, \theta) = J^{3/2} f_3(\theta) + J^2 f_4(\theta)$$

where $f_3(\theta)$ and $f_4(\theta)$ have to be found from the equation

$$e^{D_{F(J,\theta)}} \mathcal{H}(\rho, \phi) = \mathcal{H}(J)$$

We recall that

$$D_F = \{\cdot, F\}$$

so

$$e^{D_F}\mathcal{H} = \mathcal{H} + \{\mathcal{H},F\} + \frac{1}{2}\{\{\mathcal{H},F\},F\} + \frac{1}{6}\{\{\{\mathcal{H},F\},F\},F\} + \cdots$$

In order to investigate 4−order resonance, we can restrict to terms of J up to J^2, so we will need only to explicitly compute the first two terms of the exponential series.

Thus, applying Poisson bracket definition

$$\begin{aligned}\{\mathcal{H},F\} &= \frac{\partial\mathcal{H}}{\partial\rho}\frac{\partial F}{\partial\theta} - \frac{\partial\mathcal{H}}{\partial\theta}\frac{\partial F}{\partial\rho} \\ &= J^{3/2}\omega_0 f_3'(\theta) \\ &\quad + J^2\left(\omega_0 f_4'(\theta) + \frac{\sqrt{2}k_3 f_3'(\theta)\sin^3\theta}{\omega_0^{3/2}} - \frac{3\sqrt{2}k_3 f_3(\theta)\sin^2\theta\cos\theta}{\omega_0^{3/2}}\right) \\ &\qquad + \mathcal{O}(J^2)\end{aligned}$$

$$\{\{\mathcal{H},F\},F\} = J^2\left(-\frac{3\omega_0 f_3(\theta) f_3''(\theta)}{2} + \frac{3\omega_0 f_3'^2(\theta)}{2}\right) + \mathcal{O}(J^2)$$

We have the equation

$$e^{D_F}\mathcal{H}_0 = \mathcal{H}_0(J) = \omega_0 J + \frac{\omega_2}{2}J^2$$

and regrouping $J^{3/2}$ and J^2 terms we obtain

$$f_3'(\theta) = \frac{2^{3/2}k_3}{3\omega_0^{5/2}}\sin^3\theta$$

$$f_4'(\theta) = \frac{3}{2}\big(f_3(\theta)f_3''(\theta) - f_3'^2(\theta)\big) + \frac{\omega_2}{\omega_0} + \frac{k_4}{\omega_0^3}\sin^4\theta$$

Substituting and integrating, we get, for $f_3(\theta)$ and $f_4(\theta)$:

$$f_3(\theta) = \frac{2^{3/2}k_3}{3\omega_0^{5/2}}\int \mathrm{d}\theta\, \sin^3\theta = -\frac{2^{3/2}k_3}{3\omega_0^{5/2}}\cos\theta(2+\sin^2\theta)$$

$$f_4(\theta) = \cos^3\theta\left(\frac{2k_3^2\sin^3\theta}{\omega_0^5} + \frac{15k_3^2\sin\theta}{3\omega_0^5}\right) - \cos\theta\left(\frac{k_4\sin^3\theta}{4\omega_0^3} - \frac{\sqrt{2}k_3\sin^2\theta}{\omega_0^{5/2}}\right) - \frac{30k_3^2 - 3\omega_0^2k_4 - 4\omega_0^4\omega_2}{8\omega_0^5}\theta$$

We are now ready to use another time Poisson bracket formalism to get the expressions of ϕ and ρ in terms of θ and J, obtaining, for terms up to J^2

$$\phi = \theta + \frac{3}{2}J^{1/2}f_3(\theta) + J\left(2f_4(\theta) + \frac{3}{2}f_3(\theta)f_3'(\theta)\right)$$

$$\rho = J - J^{3/2}f_3'(\theta) - J^2\left(f_4(\theta) - \frac{3}{2}f_3(\theta)f_3''(\theta) + \frac{3}{2}f_3'^2(\theta)\right)$$

When we impone a $k-$order resonance, the condition reads

$$k\omega \pm \omega_r = 0$$

where

$$\omega = \frac{\partial\mathcal{H}}{\partial J}$$

is the proper frequency of the system.

This means that, when this equation is satisfied, if we rewrite the perturbation term $\frac{Q^2}{2}$ as a Fourier series, the only term which will not cancel when averaging will be the $\cos(k\theta)$ one. We can neglect the $\sin(k\theta)$ term because we chose a cosinusoidal perturbation, which applied to a sinuosoidal term cancel on satisfying the resonance condition.

Let we set $k = 4$. We have

$$Q = \sqrt{2\rho}\sin\phi$$

so

$$\frac{Q^2}{2} = \rho\sin^2\phi$$

.

Now, up to J^2 order we have, being $\sin^2\phi \sim \phi^2$

$$\frac{Q^2}{2} = J\theta^2 + J^{3/2}\big(3\theta f_3(\theta) - \theta^2 f_3'(\theta)\big) \\ + J^2\left(4\theta f_4(\theta) - \theta^2 f_4'(\theta) + \frac{3}{2}f_3(\theta)f_3''(\theta) - \frac{3}{2}f_3'^2(\theta) + \frac{9}{4}f_3^2(\theta)\right)$$

and we can compute the Fourier series term $a_4(J)$ using the well-known formula

$$a_4(J) = \frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta\, \frac{Q^2}{2}\cos(4\theta)$$

This integral is quite laborious to perform, so we computed it using a computer algebra system, which gave the result

$$a_4(J) = \frac{J}{4} + J^2\left(\frac{3}{8}\frac{\omega_2}{\omega_0} + \frac{475 - 96\pi^2}{2304}\frac{k_4}{\omega_0^3} - \frac{1151}{160}\frac{k_3^2}{\omega_0^5}\right)$$

Now, ω_2 is the coefficient of the term ρ^2 in the averaged hamiltonian (3.9) which reads

$$\omega_2 = \frac{3k_4}{8\omega_0^2}$$

Performing the substitution we obtain

$$a_4(J) = \frac{J}{4} + J^2\left[\left(\frac{9}{64} + \frac{475 - 96\pi^2}{2304}\right)\frac{k_4}{\omega_0^3} - \frac{1151}{160}\frac{k_3^2}{\omega_0^5}\right]$$

and, approximating the numerical constants,

$$a_4(J) = 0.25J + J^2\left(-0.064\frac{k_4}{\omega_0^3} - 7.194\frac{k_3^2}{\omega_0^5}\right) \tag{4.1}$$

This final result fully depends on magnetic field parameters k_3 and k_4, on proper frequency of the system ω_0 and on action J, which, from the point of view of beam dynamics has to be interpreted as *beam emittance*, the property of the beam which measures the average spread in the phase space of beam particles coordinates. [3]

Conclusion

We have shown that, in a simplified, unidimensional model, it is possibile to trap particles into resonance using an adiabatically varied magnetic field, performing a perturbation of the dipolar and the quadrupole term. Moreover, in figures (3.4, 3.8) we plot the capture probability over the averaged hamiltonian parameters. The dependencies are similar, but for comparable values of the parameters, the quadrupolar kick shows a better probability.

Finally, the strength of the $4:1$ resonance computed in (4.1) fixes a minimum value for the perturbation strength in order to achieve the adiabatic trapping.

A further study should focus on the comparation of these theoretical results with the real ones obtained using MTE at CERN, in order to understand how accurate are these models.

Appendix A

Computation of Ξ

The integral

$$\Xi = \int_0^{2\pi} \mathrm{d}\gamma \, \frac{\mu\cos\gamma + \lambda}{\sqrt{(\mu\cos\gamma+\lambda)^2 - (\lambda-\mu)^2}}$$

which appears in eq. (3.11) when computing the probability of capture for a quadropolar kick is quite hard to solve analitically. We have performed the integration reconducting Ξ to a sum of notable integrals whose value has been tabled by Gradštein and Ryžik in [7].

First of all, we notice that there is no difference integrating this function for $\gamma \in [-\pi, \pi]$, so our integral will read

$$\int_{-\pi}^{\pi} \mathrm{d}\gamma \, \frac{\mu\cos\gamma + \lambda}{\sqrt{(\mu\cos\gamma+\lambda)^2 - (\lambda-\mu)^2}}$$

Now, we perform the parametric substitution for goniometric functions.

We define

$$t = \tan\frac{\gamma}{2}$$

and we have

$$\cos\gamma = \frac{1-t^2}{1+t^2} \qquad \mathrm{d}\gamma = \frac{2}{1+t^2}\mathrm{d}t$$

For $\gamma = 0$, we have, as $\gamma \to \pm\pi$, $\tan(\pm\pi/2) \to \infty$, so our integral will rewrite, after some elementary algebrical manipulations

$$\Xi = \int_{-\infty}^{\infty} \mathrm{d}t \, \frac{\lambda(1+t^2) + \mu(1-t^2)}{\sqrt{\mu}(1+t^2)\sqrt{\lambda(1+t^2) - \mu t^2}}$$

Having defined $k = \mu/\lambda$, we can split this integral in three parts:

$$\Xi_1 = \frac{1}{k} \int \frac{\mathrm{d}t}{\sqrt{(1-k^2)t^2+1}}$$

$$\Xi_2 = k \int \frac{\mathrm{d}t}{(1+t^2)\sqrt{(1-k^2)t^2+1}}$$

$$\Xi_3 = k \int \mathrm{d}t \, \frac{t^2}{(1+t^2)\sqrt{(1-k^2)t^2+1}}$$

We of course have

$$\Xi = (\Xi_1 + \Xi_2 - \Xi_3)\Big|_{-\infty}^{\infty}$$

Let we take into account Ξ_1.
Being $c = (1-k^2)$, this integral is reconductible to the form

$$\int \mathrm{d}t \, \frac{1}{\sqrt{1+ct^2}}$$

When $c > 0$ (which is true, being $\lambda > \mu$), this integral reads [7, 2.27, p. 99]

$$\frac{1}{\sqrt{c}} \ln\left(\sqrt{ct^2+1} + \sqrt{c}t\right)$$

We notice, by the way, that this is an expression of hyperbolic arc sine function.
Substituting c, we obtain (setting the integration constant to 0)

$$\Xi_1 = \frac{1}{k\sqrt{1-k^2}} \ln\left(\sqrt{(1-k^2)t^2+1} + t\sqrt{1-k^2}\right)$$

Let us now move on Ξ_2.
We perform the substitution

$$s = 1+t^2 \qquad \mathrm{d}t = \frac{\mathrm{d}s}{2\sqrt{s-1}}$$

and we get

$$\Xi_2 = \frac{k}{2} \int \frac{\mathrm{d}s}{s\sqrt{s^2(1-k^2)+s(2k^2-1)-k^2}}$$

In [7, 2.266, p. 94] we find the value of integrals similar to

$$\int \frac{\mathrm{d}x}{x\sqrt{R(x)}}$$

where $R(x)$ is a second-grade polynomial. If, as in our case, $\Delta < 0$, we find that our integral is expressed in terms of arc sine function. Therefore we have, performing all the substitutions

$$\Xi_2 = k\frac{1}{k}\arcsin\left(\frac{kt}{\sqrt{1+t^2}}\right) = \arcsin\left(\frac{kt}{\sqrt{1+t^2}}\right)$$

Finally, also for Ξ_3 we make the substitution

$$s = 1 + t^2 \qquad \mathrm{d}t = \frac{\mathrm{d}s}{2\sqrt{s-1}}$$

obtaining

$$\Xi_3 = \frac{k}{2}\int \mathrm{d}s \, \frac{s-1}{s\sqrt{s-1}\sqrt{1+(1-k^2)(s-1)}}$$

This integral can be split into the sum of two integrals

$$\begin{aligned}\Xi_3 &= \Xi_{3a} - \Xi_{3b} \\ &= \frac{k}{2}\int \frac{\mathrm{d}s}{\sqrt{k^2(1-s)+s}\sqrt{s-1}} + \frac{k}{2}\int \frac{\mathrm{d}s}{s\sqrt{s-1}\sqrt{k^2(1-s)+s}}\end{aligned}$$

and, using the same formulae from [7], we have

$$\Xi_3 = -\arcsin\left(\frac{kt}{\sqrt{1+t^2}}\right) + \frac{k}{\sqrt{1-k^2}}\ln\left(\sqrt{(1-k^2)t^2+1} + t\sqrt{1-k^2}\right)$$

so

$$\Xi = 2\arcsin\left(\frac{kt}{\sqrt{1+t^2}}\right)\Bigg|_{-\infty}^{\infty} + \frac{\sqrt{1-k^2}}{k}\ln\left(\sqrt{(1-k^2)t^2+1} + t\sqrt{1-k^2}\right)\Bigg|_{-\infty}^{\infty}$$

Now, the first term is easy to evaluate as

$$4\arcsin k$$

while, for the second one we must be careful while taking the limits.

Let we go back to γ variable. We have

$$t = \tan(\gamma/2) = \frac{\sin(\gamma/2)}{\cos(\gamma/2)}$$

obtaining

$$\ln\left(\frac{\sqrt{2\cos\gamma - k^2\sin^2\gamma + 2} + \sqrt{1-k^2}\sin\gamma}{1+\cos\gamma}\right)\Bigg|_0^{2\pi} = 0$$

and we get

$$\Xi = 4\arcsin k = 4\arcsin\sqrt{\frac{\mu}{\lambda}}$$

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