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GEOMETRIC PROPERTIES OF 2-DIMENSIONAL MINIMAL SURFACES IN A SUB-RIEMANNIAN MANIFOLD WHICH MODELS THE VISUAL CORTEX

Tesi di Laurea in Analisi Geometrica

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Abstract

In this paper we study the notion of degree for submanifolds embedded in an equiregular sub-Riemannian manifold and we provide the definition of their associated area functional. In this setting we prove that the Hausdorff dimension of a submanifold coincides with its degree, as stated by Gromov in [**19**]. Using these general definitions we compute the first variation for surfaces embedded in low dimensional manifolds and we obtain the partial differential equation associated to minimal surfaces. These minimal surfaces have several applications in the neurogeometry of the visual cortex.

Sommario

In questa tesi studiamo la nozione di grado per sottovarietà immerse in una varietà sub-riemanniana e forniamo la definizione del funzionale dell'area ad esse associato. In questo ambiente proviamo che la dimensione di Hausdorff di una sottovarietà coincide effettivamente con il suo grado, come affermò Gromov nel suo lavoro [19]. Utilizzando queste definizioni generali calcoliamo la variazione prima dell'area per sottovarietà immerse in varietà di dimensione bassa e otteniamo l'equazione alle derivate parziali associata alle superfici minime. Queste superfici minime hanno diverse applicazioni nella neurogeometria della corteccia celebrale.

Introduction

In the last years variational problems in sub-Riemannian geometry have become the object of many studies. In particular, Pauls in [30], Hurtado, Ritoré and Rosales in [23], Capogna, Citti and Manfredini in [4] , Cheng Hwang and Yang in [6] deal with minimal surfaces in Heisenberg group $\mathbb{H}^1$. Moreover, in [1,5,15–17] these problems in a more general setting, as contact sub-Riemannian manifolds or Caront groups, are studied

Our aim in this master thesis is to study the area functional for a smooth submanifold embedded in a sub-Riemannian manifold. First of all, we recall that a sub-Riemannian manifold N is a smooth manifold endowed with a distribution $\mathcal{H}$ which is a subbundle of the tangent bundle and a horizontal metric h defined only on the distribution. In the present work a crucial assumption is that distribution $\mathcal{H}$ verifies the celebrated *Hörmander rank condition* at each point p in N. Let $X_1, \cdots, X_k$ be a local frame for $\mathcal{H}$ where $k = \dim(\mathcal{H})$, we say that $\mathcal{H}$ verifies the Hörmander rank condition if the vector fields $X_1, \cdots, X_k$ and all their commutators of any order generate all the tangent space. This condition has been deeply studied after the first studies by Hörmander in [21], Rothschild and Stein in [34], Nagel, Stein and Wainger in [29] and Montgomery in [28]. Under this condition, Chow's Theorem implies that any couple of points can be connected by *horizontal curves* (see [7]). Thus, it is possible to define the Carnot-Carathéodory distance on N as the infimum of length of horizontal curves joining two given points. Iterated Lie brackets of horizontal vector fields generate a flag of subbundles

$$\mathcal{H} \subset \mathcal{H}^2 \subset \cdots \subset \mathcal{H}^r \subset \cdots \subset \mathcal{H}^s = TN, \tag{1}$$

where

$$\mathcal{H}^{r+1} = \mathcal{H}^r + [\mathcal{H}, \mathcal{H}^r], \qquad [\mathcal{H}, \mathcal{H}^k] = \{[X, Y] : X \in \mathcal{H}, Y \in \mathcal{H}^k\}.$$

Moreover, the integer list $(n_1(p), \cdots, n_r(p))$ where $n_i(p) = \dim(\mathcal{H}^i)$ is called the *growth vector* of $\mathcal{H}$ at p.

Our goal is to give a suitable definition of the area for a submanifold embedded in N. Franchi, Serapioni and Serra Cassano deeply studied regular surfaces in sub-Riemannian structures (see [12–14]). In the present work, we follow Magnani-Vittone [27] and Le Donne-Magnani [24] approach, consisting in considering submanifold, regular in Euclidean sense, and its associated degree. They show that the area of a submanifold in the Engel's group in [24] and in stratified

Carnot groups in [27] and [26] is strictly connected to this notion of degree of a submanifold. Here, we consider only *equiregular* sub-Riemannian manifolds where the growth vector is pointwise constant in N. We define the *degree* on an adapted basis $(X_1, \cdots, X_n)$ to flag (1) that exists by the Hörmander rank condition. We say that X_i has degree r if X_i lies in $\mathcal{H}^r$ but X_i does not belong to $\mathcal{H}^{r-1}$. Then, taking a m-vector $X^J = X_{j_1} \wedge \cdots \wedge X_{j_m}$ we set the degree $d(X^J)$ of X^J as the sum of degrees of each vector in the wedge product

$$d(X^J) = d(X_{j_1}) + \cdots + d(X_{j_m}).$$

Essentially, the degree is a pointwise property of the tangent space at p , indeed the m-vector $\tau_\Sigma(p)$ which represents the tangent space of the submanifold is a linear combination of m-vectors as X^J

$$\tau_\Sigma(p) = \sum_J \tau_J X^J|_p,$$

thanks to the linearity of the wedge product. Automatically, the degree of $\tau_\Sigma(p)$ is the maximum integer $d(X^J)$ such that τ_J is different from zero. This definition of degree is equivalent to Gromov's definition of degree (see [19]). Basically the degree measures the intersection between each layer of the flag and the tangent space therefore it is strictly connected to the geometrical structure submanifold inherited by the ambient sub-Riemannian structure.

When the ambient space is a Riemannian manifold equipped with a metric g it is clear how we define the area of a submanifold, $area(\Sigma, g)$, using the Riemannian area element depending on the metric g. When we consider a sub-Riemannian manifold there is a lack of a metric on the tangent bundle, since there exists only a horizontal metric h on the subbundle given by the distribution. In order to give the definition of area we extend the horizontal metric h to a Riemannian metric g such that g makes $H_i = \mathcal{H}^{i+1}/\mathcal{H}^i$ spaces orthogonal and $g|_\mathcal{H} = h$. Then it is natural to construct a sequence of metrics g_r defined on the basis $(X_1, \cdots, X_n)$ as

$$g_r(X_i, X_j) = \left(r^{\frac{d(X_i)+d(X_j)-2}{2}} \right)^{-1} g(X_i, X_j) \quad i, j = 1, \cdots, n. \tag{2}$$

Clearly, the restriction of g_r to the distribution $\mathcal{H}$ is equal to the horizontal metric h, $g_r|_\mathcal{H} = h$ and when we let r tend to zero the metric blows up out of $\mathcal{H}$. Thus, the metric g_r in the limit provides a good representation of h and shows that only horizontal curves are allowed. Indeed, a curve not tangent to the distribution at each point has infinite length

Since we have a sequence that converges to the sub-Riemannian metric, we define the sub-Riemannian area for a m-dimensional submanifold Σ of degree d embedded in a sub-Riemannian structure by

$$A(\Sigma) = \lim_{r \to 0} r^{\frac{d-m}{2}} area(\Sigma, g_r). \tag{3}$$

Moreover, Gromov in his book [19] claimed that the Hausdorff dimension of a submanifold M embedded in an equiregular sub-Riemannian manifold N is its degree but he did not prove it. Ghezzi and Jean in [18] provided a proof of this assertion only for a a *strongly equiregular* submanifold (see definition 2.10).

In the present work we provide a different proof for Ghezzi and Jean's result and then we give a proof Gromov's affirmation. Basically, the intersection between the tangent space of M and each layer of the stratification generates a flag and the strongly equiregular assumption assures that the dimension of each space of the flag is constant pointwise in M. We consider privileged coordinates adapted to the flag given by the exponential map in a neighborhood of a point p. Thanks to the Ball-Box Theorem, balls are equivalent to boxes (see [28, Theorem 2.4.2]). Thus we cover the intersection between boxes and submanifold M, which in privileged coordinates is

$$\mathrm{Box}^w(r') \cap \{x \in \mathbb{R}^n \ : \ x_{m+1} = \cdots = x_n = 0\}$$

with boxes of size $1/k$, thus we obtain that the Hausdorff dimension is equal to d. Then we realize that the degree of vector fields is lower semicontinuos, therefore, if we fix the degree of a submanifold M, a simple vector fields of m-vector tangent to M can not switch its degree in a neighborhood of a point p. Hence, the flag used in the previous proof has locally constant dimension, then we can apply the precedent argument to a neighborhood of a point p.

In the Euclidean space a standard definition of the mean curvature for a submanifold is obtained by the first variation of the area functional. Nowadays, a central problem in Geometric Analysis is to provide a good definition of the mean curvature in different settings, as in sub-Riemannian geometry, by computing the first variation of the area functional. Our principal motivation to minimize the area functional came from the neurogeometry of the brain where we learnt that the order of the mean curvature operator could be greater that two.

A mathematical model of simple cells S of the visual cortex V1 using the sub-Riemannian geometry of the roto-translational Lie group was proposed by Citti and Sarti (see [8], [9]). In their work, the perceptual completion is obtained through minimal surfaces and therefore they studied the regularity and foliation properties of minimal surfaces in S= $E(2)$. Their techniques have several applications in image completion. In [11] it was conjectured that endstopping cells E are sensible to curvature and a sub-Riemannian structure modelling their structure was proposed in [31]. In this work we shall consider an extension of the results of [9] to this family of cells. The space S will be identified with $\mathbb{R}^2 \times S^1$. We shall consider the distribution generated by the vector fields

$$X = \cos(\theta)\partial_x + \sin(\theta)\partial_y \quad \text{and} \quad Y = \partial_\theta,$$

and we shall compute the first variation of the area for a θ-graph in S to obtain the minimal surface equation

$$X\left(\frac{X(\theta)}{\sqrt{1+X(\theta)^2}}\right)=0.$$

This equation along characteristic curves is equivalent to $\theta''=0$, where the derivative $'$ is taken with respect to the arc-length parameter of the curve, see [15], [16]. Adding an additional variable, the curvature, to the three-dimensional space S we obtain the space E= $\mathbb{R}^2\times S^1\times\mathbb{R}$, where we consider the distribution generated by

$$X_1=\cos(\theta)\partial_x+\sin(\theta)\partial_y+k\partial_\theta \quad \text{and} \quad X_2=\partial_k,$$

(see also [31]). In this setting we are interested in (θ,κ)-graphs which are 2-dimensional surfaces. Notice that a (θ,k)-graph has a foliation property if and only if the equation $X_1(\theta)=\kappa$ holds. Moreover, this condition implies that the degree of the surface is four. Thus applying definition (3) we obtain

$$A(\Sigma)=\int_\Omega\sqrt{1+X_1(\kappa)^2}\,dx\,dy.$$

Critical points of this area functional satisfy the following minimal PDE equation

$$X_4(X_1(\theta))+X_1(X_4(\theta)))+X_1(g)X_4(\theta)+X_1^2(g)=0, \tag{4}$$

where we set

$$g=\frac{X_1(\kappa)}{\sqrt{1+X_1(\kappa)^2}}.$$

Notice that equation (4) is a third-order partial differential equation which we attempt to read along characteristic curves as we do in S. However, there is $X_4(\theta)$ term which corresponds to the derivative in the direction perpendicular to the tangent direction of characteristic curves projected on the retinal plane. Therefore, we consider different horizontal metrics h_1,h_2,h_3 that imply different minimal PDEs, but we have not succeeded in reading these equations along characteristic curves.

In Chapter 1 we provide the definition of a sub-Riemannian manifold, of a distribution, with its natural Carnot-Carathéodory distance, and some basic notions about the geodesic equation. Then we compare the involutive condition that implies the Frobenius Theorem with the Hörmander rank condition that implies the Chow Theorem. Furthermore, we define the exponential map, Lie derivative, regular surfaces and we report the Rothschild-Stein's theorem that assures that the tangent space to a sub-Riemannian structure is a Carnot group. In the last section we supply some example of sub-Riemannian manifold as the rototraslation group S and 2-jet space E, then we show that tangent space to S and E are respectively the Heisenberg group and the Engel's group.

Chapter 2 focuses on the degree and the definition of area functional for an embedded submanifold in an equiregular sub-Riemannian manifold. In this chapter we find condition under which the area functional is independent of the extension of the horizontal metric h up to a positive constant. Moreover, we show that this general definition of area corresponds to the area of hypersurfaces in the Heisenberg group. Then we study the geometry of surfaces and the area functional in S and E. In conclusion we prove the Gromov's conjecture about the Hausdorff dimension of a submanifold.

Finally, in Chapter 3 we compute the first variation of the area functional for a surface in S and E and we obtain the PDEs associated to the minimal surfaces. In E we notice that only variations preserving degree four are allowed, otherwise the area functional changes. Then we study general variations preserving the degree d of a submanifold in a general equiregular sub-Riemannian manifold and we obtain a PDE system of equations where the coefficients of the vector field X inducing the variation are involved. This system restricts the range of permitted variations.

Contents

CHAPTER 1

Introduction to sub-Riemannian geometry

1. Definition of sub-Riemannian structure

Following Montgomery, we give the following definition.

DEFINITION 1.1. A sub-Riemannian geometry on a smooth manifold N consists of a distribution $\mathcal{H} \subset TN$, which is a vector subbundle of the tangent bundle of N, together with a fiber inner-product h on this subbundle.

We will call $\mathcal{H}$ the *horizontal distribution* and the inner product h will be referred to as the *horizontal metric*. A vector field is horizontal if it is everywhere tangent to $\mathcal{H}$. A C^1 curve in N is said to be horizontal if the tangent vector is horizontal at every point. Let $\gamma : [a, b] \rightarrow N$ be a smooth horizontal curve, we define the length of γ by

$$l(\gamma) = \int_a^b \sqrt{h(\dot\gamma(t), \dot\gamma(t))}\, dt. \tag{5}$$

Notice that we define the length only for horizontal curves, where the inner product exists. We use the length to define the distance between two points p and q in N, as in Riemannian geometry:

$$d(p, q) = \inf\{l(\gamma) \;:\; \gamma \quad \text{is horizontal curve such that} \quad \gamma(a) = p, \gamma(b) = q\}. \tag{6}$$

If there is not a horizontal curve which joints p and q, we set that the distance is infinite. This is the well-known *Carnot-Carathéodory distance*, for brevity *C-C distance*.

Now, it is natural to whether there is a condition that assures that the distance between each points p and q in N is always finite. In other words, given every pair of points p and q in N, we want to know under which condition there exists a horizontal curve γ such that $\gamma(a) = p$ and $\gamma(b) = q$. In order to answer this question we have to introduce the Hörmander rank condition. Given a distribution $\mathcal{H}$ of dimension k with inner product h, we can consider an orthonormal local frame $X_1, \cdots, X_k$. On the other hand, we can give $X_1, \cdots, X_k$ and say that $\mathcal{H}$ is the distribution generated by these vector fields and set a horizontal inner product such that $X_1, \cdots, X_k$ is an orthonormal basis. We prefer the first approach, because in visual cortex cases we know the vector fields which generate the distribution but we have doubts about the choice of horizontal metric h.
Below, we recall some well-known definitions and theorems.

DEFINITION 1.2. Let $\varphi : M \to N$ be smooth function. The vector fields X on M and Y on N are called φ-related if $d\varphi \circ X = Y \circ \varphi$.

DEFINITION 1.3. Let X, Y be first-order regular differential operators (i.e. vector fields). Their *commutator* is defined by

$$[X, Y] = XY - YX$$

and it is also a first-order differential operator. We define the *Lie algebra generated by* $X_1, \cdots, X_k$ and denote it by

$$\mathcal{L}(X_1, \cdots, X_k)$$

the linear span of the operators $X_1, \cdots, X_k$ and their commutators of any order. We set that a commutator has degree r,

$$d(X) = r \quad \text{if} \quad X = [\cdots[X_{i_1}, X_{i_2}], \cdots, X_{i_r}] = Ad(X_{i_1}, \cdots, X_{i_r})$$

with $i_1, \cdots, i_r \in \{1, \cdots, k\}$.

In order to understand how the Hörmander rank condition is connected to the connectivity it is useful to remind what is an involutive distribution.

DEFINITION 1.4. A smooth distribution $\mathcal{H}$ is called *involutive* if $[X, Y] \in \mathcal{H}$ whenever X and Y are smooth vector fields lying in $\mathcal{H}$. In other words, let $X_1, \cdots, X_k$ be a local frame of $\mathcal{H}$, then

$$\mathcal{L}(X_1, \cdots, X_k) = \text{span}\{X_1, \cdots, X_k\}.$$

DEFINITION 1.5. Let (M, φ) be a submanifold of N. We say that M is an *integral manifold* of a distribution $\mathcal{H}$ on N if

$$d\varphi(T_pM) = \mathcal{H}_{|\varphi(p)} \qquad \text{for each } p \in M.$$

THEOREM 1.1 (Frobenius). *Let $\mathcal{D}$ be a k-dimensional smooth distribution on N. Then, $\mathcal{D}$ is involutive if and only if there exists an integral manifold of $\mathcal{D}$ passing through every point of N.*

PROOF. Here, we prove only that the existence of an integral manifold of $\mathcal{D}$ implies that $\mathcal{D}$ is involutive. We have to prove that $[X, Y] \in \mathcal{D}$ whenever X and Y are smooth vector fields lying in $\mathcal{D}$. By hypothesis, let (M, φ) be an integral manifold of $\mathcal{D}$ through $p = \varphi(m)$, therefore

$$d\varphi : T_mM \to \mathcal{D}_{|\varphi(m)}$$

is a isomorphism. Then, there exist vector fields $\bar{X}$ and $\bar{Y}$ such that

$$d\varphi(\bar{X}_{|m}) = X_{|\varphi(m)}, \qquad d\varphi(\bar{Y}_{|m}) = Y_{|\varphi(m)}.$$

Moreover, $\bar{X}$ and $\bar{Y}$ are smooth and φ-related. By [**36**, Proposition 1.55], which assures that if $\bar{X}$ and $\bar{Y}$ are φ-related then $[\bar{X}, \bar{Y}]$ and $[X, Y]$ are φ-related, we have $[X, Y] = d\varphi([\bar{X}, \bar{Y}]) \in \mathcal{D}$. The proof of other implication is done by induction on the dimension of the distribution, for further details see [**36**, Theorem 1.60]. □

When a k-dimensional distribution $\mathcal{H}$ on N is involutive and we consider a point p, by Frobenius Theorem, we have that there exists an integral manifold passing through p. This integral manifold of dimension k, called the leaf through p, is generated by the set of horizontal paths through the fixed point p. Therefore, if q does not lie in the leaf of p we can not connect p and q by a horizontal curve. Thus, the distance between p and q would be infinite. Opposite to involutive distributions we have the ones that verify Hörmander rank condition, also known as bracket-generating distributions.

Definition 1.6. We say that a distribution $\mathcal{H}$ on a n-dimensional manifold N verifies the *Hörmander rank condition* if any local frame $\{X_1, \cdots, X_k\}$ for $\mathcal{H}$ satisfies

$$\dim(\mathcal{L}(X_1, \cdots, X_k))(p) = n \qquad \forall\, p \,\in\, N.$$

In other words, the Lie algebra generated by $X_1, \cdots, X_k$ is all the tangent bundle. Let s be the smallest natural number such that $X_1, \cdots, X_k$ and their commutators of degree smaller that or equal to s span the all tangent space. We will call s the *step* at a point p and the local basis $X_1, \cdots, X_k, X_{k+1}, \cdots, X_n$ made out of commutators of $X_1, \cdots, X_k$ is chosen such that, for every point, the *local homogeneous dimension*

$$Q = \sum_{j=1}^{n} d(X_j) \tag{7}$$

is minimal.

Theorem 1.2 (Chow). *If a distribution $\mathcal{H} \subset TN$ verifies Hörmander rank condition, then the set of points that can be connected to p in N by a horizontal path is the connected component of N containing p.*

We suggest the reader to see [**28**, 2.2] for the heuristic Hermman's proof of Chow Theorem or [**28**, 2.4] for a standard proof.

Example 1.1. Let us show two examples of sub-Riemannian manifolds that do not verify the Hörmander rank condition at each point. Let $\mathbb{R}^2$ be the plane and let

$$f(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ e^{-\frac{1}{x^2}} & \text{if} \quad x \neq 0 \end{cases}$$

be a C^∞ function such that $f^{(n)}(0) = 0$ for all $n = 1, 2, 3, \cdots$. We consider the distribution generated by the vector fields

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 \\ f(x) \end{pmatrix}.$$

If we consider a point $p = (x, y)$ such that $x \neq 0$, X_1 and X_2 generate all the tangent space. Let $p = (x, y)$ be a point in $\mathbb{R}^2$ such that $x = 0$, we have $X_2(p) = 0$.

Moreover, all brackets of length n are equal to zero at $p = (0, y)$, the proof follows from induction. At first for $n = 1$, we have

$$[X_1, X_2](0, y) = \left(\lim_{h \to 0} \frac{e^{-\frac{1}{h^2}} - 0}{h} \right) \partial_y = \left(\lim_{x \to \infty} x \, e^{-x^2} \right) \partial_y = 0.$$

Now, we assume that

$$X_{n+2} := [X_1, \cdots, [X_1, X_2], \cdots] = f^{(n)}(0) \partial_y = 0.$$

Moreover, since it holds

$$f^{(n)}(x) = \frac{P_{2n-2}(x) \, e^{-\frac{1}{x^2}}}{x^{3n}},$$

we have

$$\begin{aligned} X_{n+3}(p) = [X_1, X_{n+2}](p) &= \lim_{h \to 0} \frac{\frac{P_{2n-2}(h) \, e^{-\frac{1}{h^2}}}{h^{3n}} - 0}{h} \partial_y|_p \\ &= \lim_{x \to \infty} P_k(x) e^{-x^2} \, \partial_y|_p = 0. \end{aligned}$$

Hence, this distribution does not verify the Hörmander rank condition. However, it is possible to possible to connect all possible points with integral curves. Indeed, the only problem could be at point $(0, y)$, but it is possible to leave this point by the vector field X_1 and there we can reach every point.

Thus, Hörmander rank condition assures that the Carnot-Carathéodory distance is finite. Furthermore, in definition (6) there is an infimum over all possible horizontal path with fixed endponits. A natural question may be when this infimum is reached. To answer this question we have to minimize the length functional $l(\gamma)$ defined in (5) over all possible horizontal curves γ with fixed endponits. As in Riemannian geometry minimizing the length functional is equal to minimizing the energy functional

$$E(\gamma) = \int_\gamma \frac{1}{2} \|\dot\gamma\|^2, \qquad \text{where} \quad \|\dot\gamma\|^2 = h(\dot\gamma, \dot\gamma). \tag{8}$$

Indeed, thanks to Cauchy-Schwarz inequality, we have

$$l(\gamma) = \int_\gamma \|\dot\gamma\| \cdot 1 \leqslant \sqrt{\int_\gamma \|\dot\gamma\|^2} \, \sqrt{b - a} = \sqrt{2 \, E(\gamma)} \, \sqrt{b - a}$$

with equality if and only if $\|\dot\gamma\| = c$. We denote $\bar\alpha$ the curve α covered at constant speed $\|\dot{\bar\alpha}\| = c$. Therefore, if γ minimizes E and η is another horizontal path connecting p and q, it follows

$$l(\gamma) \leqslant \sqrt{2 \, E(\gamma)} \, \sqrt{b - a} \leqslant \sqrt{2 \, E(\bar\eta)} \, \sqrt{b - a} = l(\bar\eta) = l(\eta),$$

then γ minimizes the length functional l. On the other hand, if γ minimizes l and η is another horizontal path connecting p and q, we have

$$\sqrt{2\ E(\bar{\gamma})}\ \sqrt{b-a} = l(\bar{\gamma}) = l(\gamma) \leqslant l(\eta) \leqslant \sqrt{2\ E(\eta)}\ \sqrt{b-a},$$

then γ minimizes the energy functional E.

Definition 1.7. An absolutely continuous horizontal path that realizes the distance between two points is called a minimizing *geodesic* or simply a geodesic.

In a sub-Riemannian setting there is a lack of a covariant two-tensor like the Riemannian metric in Riemannian geometry. However, it is possible to define a contravariant symmetric two-tensor, a section of $TN \otimes TN$. This tensor is called the *cometric* and has rank k, the dimension of the distribution. In [28, 1.5] Montgomery shows that from this cometric acting on covectors it is possible to define the *sub-Riemannian Hamiltonian* or kinetic energy

$$H : T^*N \to \mathbb{R}, \qquad H(q,p) = \frac{1}{2}((p,p))_q,$$

where $q \in N$, $p \in T_q^*N$ and $((\cdot,\cdot))$ is the cometric, such that $\frac{1}{2}\|\dot{\gamma}\|^2 = H(q,p)$. Here, $q = \gamma(t)$ and p such that $\dot{\gamma}(t) = \beta_{\gamma(t)}(p)$. Minimizing the energy functional E we obtain the Hamiltonian differential equations

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i} \tag{9}$$

These Hamiltonian differential equations are called the normal geodetic equations. The following theorem holds

Theorem 1.3. *Let $(\gamma(t), p(t))$ be a solution to Hamilton's differential equations on T^*N for the sub-Riemannian Hamiltonian H and let $\gamma(t)$ be its projection to N. Then every sufficiently short arc of γ is a minimizing sub-Riemannian geodesic. Moreover γ is the unique minimizing geodesic joining its endpoints.*

Definition 1.8. The projection γ of the previous theorem is called a *normal geodesics*.

Remark 1.1. There are sub-Riemannian manifolds that admit minimizing geodesics that do not solve the Hamilton's differential equations. These geodesics are called *singular geodesics*. Montgomery in his book [28] deeply studied this topic.

It is also worth mentioning another important theorem which shows that the topology induced by the Carnot-Carathéodory distance has the same topology of the smooth manifold N.

Theorem 1.4. *If the distribution $\mathcal{H}$ on N satisfies the Hörmander rank condition, then the topology on N induced by the Carnot-Carathéodory distance is the usual manifold topology.*

The proof follow from the well-known Ball-Box Theorem 2.2 (see [**28**, 2.4 and 2.5]).

REMARK 1.2. In the present work we follow Montgomery's approach. Some authors include the Hörmander rank condition in the definition of sub-Riemannian manifold.

1.1. Lie group and Carnot group. Let $(G, \cdot)$ be a Lie group (see [**36**, Definition 3.1]) and let $\mathfrak{g}$ be its Lie algebra. We consider $V \subset \mathfrak{g}$ a linear subspace of the Lie algebra. We can see the Lie Algebra as the space of all left invariant vector fields, i.e. l_g-related to themselves where the left translation by g in G is

$$l_g(h) = g \cdot h.$$

In this way, V is a left invariant distribution and the Hörmander rank condition corresponds to the fact that V Lie-generates $\mathfrak{g}$. If we set an inner product h on V we obtain a sub-Riemannian metric. Therefore (G, V, h) has a sub-Riemannian structure.

DEFINITION 1.9. We say that G is a *graded nilpotent Lie group* if the Lie algebra $\mathfrak{g}$ has the form

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_s$$

where $[V_i, V_j] = V_{i+j}$ and $V_r = 0$ if $r > s$. Therefore, all iterated brackets of length $r > s$ are zero. If we define an inner product h on V_1 and we suppose that V_1 Lie-generates $\mathfrak{g}$, then we obtain a sub-Riemannian manifold (G, V_1, h). We will call this structure a *Carnot group*.

1.2. Exponential mapping. Another tool we need in this work is the exponential mapping induced by vector fields. As we need local results, we work in an open set (U, ψ) of N. Therefore, we enunciate the results for an open coordinate set Ω in $\mathbb{R}^n$, then we compose with diffeomorphism ψ^{-1} to see it in the manifold. Let $\Omega \subset \mathbb{R}^n$ be an open set and X be a smooth vector field on Ω. Fixed x in Ω, X induces a local one parameter group of transformations on Ω, $\{\sigma_X(t, x) = \sigma(t, x)\}_t$ which is the unique solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial\ \sigma(t, x)}{\partial t} = X_{|\sigma(t,x)} \\ \sigma(0, x) = x. \end{cases} \tag{10}$$

This unique solution always exists for $|t|$ sufficiently small. Moreover, if $X = X(u_1, \cdots, u_l)$ depends in smooth way on parameters $(u_1, \cdots, u_l)$ in an open set $U \subset \mathbb{R}^l$ and we consider compact sets $L \subset U$ and $K \subset \Omega$, there exists a constant ϵ_0 such that

$$\sigma : L \times] - \epsilon_0, \epsilon_0[\times K \to \Omega \tag{11}$$

is a smooth function. Thanks to the uniqueness of (10), there holds

$$\sigma(s,\sigma(t,x)) = \sigma(s+t,x) \quad \text{when} \quad x \in K,\ |s+t| < \epsilon_0, \tag{12}$$

$$\sigma_{\lambda X}(t,x) = \sigma_X(\lambda t,x) \quad \text{when} \quad x \in K,\ |\lambda t| < \epsilon_0. \tag{13}$$

Now, by equation (12) the function $x \to \sigma(-t,x)$ is a C^∞ inverse of $x \to \sigma(t,x)$. Therefore, $x \to \sigma(t,x)$ is a diffeomorphism on a compact set of Ω, for $|t|$ sufficiently small. In this sense we construct a parameter group of diffeomorphisms.

Definition 1.10. We define the *exponential mapping* by

$$\exp(X)(x) = \sigma_X(1,x)$$

whenever the right hand side is defined.

For all t sufficiently small, $\sigma_X(t,x) = \sigma_{tX}(1,x) = \exp(tX)(x)$ is always well-defined. Now, let $X_1,\cdots,X_l$ be smooth vector fields on Ω and $(u_1,\cdots,u_l)$ be parameters in $\mathbb{R}^k$. Then, if

$$|u| = \sqrt{\sum_{i=1}^{l} u_i^2} \tag{14}$$

is sufficiently small, $|u| < \epsilon_0$, we have that the function

$$(u_1,\cdots,u_l,x) \to \exp\left(\sum_{i=1}^{l} u_i\ X_i\right)(x) \tag{15}$$

is well-defined and smooth. For further details see [**29**, Appendix]. Notice that if X a vector field in $\mathcal{H}$ on N

$$\gamma(t) = \exp\left(tX\right)(p) = \sigma_X(t,p)$$

is a horizontal curve. Here, we denote in the same way the exponential mapping defined on $\Omega \subset \mathbb{R}^n$ and its image through ϕ^{-1}, where $\phi : U \to \Omega$ is a diffeomorphism, U is an open set in N and $p = \phi(x)$.

Since the exponential mapping is a local diffeomorphism between the tangent space and the manifold, it makes sense the following definition

Definition 1.11. Given $X_1,\cdots,X_k$ local frame of $\mathcal{H}$ around p and commutators $X_{k+1},\cdots,X_n$ that minimize the local homogeneous dimension Q. The *canonical coordinates* of q around p are the coefficients $(u_1,\cdots,u_n)$ such that

$$q = \exp\left(\sum_{i=1}^{n} u_i\ X_i\right)(p)$$

Let h be the metric making $X_1, \cdots, X_k$ orthonormal, we can extend the norm (14), with $l = k$, to a homogeneous norm on the whole space

$$\|u\| = \left(\sum_{i=1}^{n} |u_i|^{Q/d(X_i)} \right)^{1/Q}. \tag{16}$$

2. Differential Operators

The notion of exponential mapping allow us to define the Lie derivative in the direction X.

Definition 1.12. Let X be a fixed vector field. We call *Lie derivative* of f in the direction of the vector X on the tangent space to N at a point p the derivative with respect to t of the function $f(\exp(tX)(p))$ at $t = 0$.

Obviously, if f is C^1 the Lie derivative

$$\frac{d}{dt}\Big|_{t=0} f(\exp(tX)(p)) = df_p \left(\frac{d}{dt}\Big|_{t=0} \exp(tX)(p) \right) = df_p(X_p) = X_p(f)$$

is equal to the directional derivative Xf, but the Lie derivative can exist even if the directional derivative does not.

Definition 1.13. Let $U \subset N$ be an open set. Let $X_1, \cdots, X_k$ be a family of smooth vector fields defined on U and $f : U \to \mathbb{R}^m$. If the Lie derivatives $X_j f^i$ exist at p in U, for $j = 1, \cdots, k$ and $i = 1, \cdots, m$, we define the horizontal Jacobian of f at p as the matrix:

$$\mathcal{J}_{\mathcal{H}} f(p) = \begin{pmatrix} X_1 f^1(p) & \cdots & X_k f^1(p) \\ \vdots & \ddots & \vdots \\ X_1 f^m(p) & \cdots & X_k f^m(p) \end{pmatrix} = \begin{pmatrix} \nabla_{\mathcal{H}} f^1 \\ \vdots \\ \nabla_{\mathcal{H}} f^m \end{pmatrix}.$$

A function f is of class $C^1_{\mathcal{H}}$ if every element of $\mathcal{J}_{\mathcal{H}} f$ is continuous with respect to the Carnot-Carathéodory distance (6). A function f is $C^2_{\mathcal{H}}$ if every element of $\mathcal{J}_{\mathcal{H}} f$ is of class $C^1_{\mathcal{H}}$. The space $C^k_{\mathcal{H}}$ is defined by induction.

Remark 1.3. Let $f : U \to \mathbb{R}^m$ be a C^1_E function. Then, $J_{\mathcal{H}} f$ is the $(1,0)$ version of the restriction of the differential df to $\mathcal{H}$, $df|_{\mathcal{H}}$, when the inner product h is the one which makes $X_1, \ldots, X_k$ an orthonormal basis. Indeed, we have

$$df|_{\mathcal{H}}(X_j) = \begin{pmatrix} X_j f^1 \\ \vdots \\ X_j f^m \end{pmatrix} = \begin{pmatrix} h(\mathrm{grad}(f^1), X_j) \\ \vdots \\ h(\mathrm{grad}(f^m), X_j) \end{pmatrix},$$

where $\mathrm{grad}(f^i) = \sum_{l=1}^{k} a_i^l \, X_l$ and we have

$$X_j f^i = h\left(\sum_{l=1}^{k} a_i^l \, X_l, X_j \right) = a_i^j.$$

Therefore, $\mathrm{grad} f^i = \nabla_{\mathcal{H}} f^i$.

REMARK 1.4. Obviously, a function of class $C^1_{\mathcal{H}}$ need not to be of class C^1_E. If $X_1, \cdots, X_k$ satisfy the Hörmander rank condition with step s, then a function f of class $C^s_{\mathcal{H}}$ belongs to C^1_E. Hence, let $X_1, \cdots, X_k$ be vector fields that satisfy the Hörmander rank condition, then

$$f \in C^\infty_{\mathcal{H}} \quad \text{if and only if} \quad f \in C^\infty_E.$$

DEFINITION 1.14. Let U be an open set in N. A function $f : U \to \mathbb{R}^l$ is differentiable at a point $p \in U \subset N$ in the intrinsic sense if

$$f^i(\exp(\sum_{j=1}^n u_j X_j)(p)) - f^i(p) = \sum_{j=1}^k u_j X_j f^i(p) + o(\|u\|) \qquad i = 1, \cdots, l,$$

or in other words

$$f(\exp(\sum_{j=1}^n u_j X_j))(p) - f(p) = \mathcal{J}_{\mathcal{H}} f(p)(u_1, \cdots, u_k)^T + o(\|u\|)$$

THEOREM 1.5. *Let $U \subset N$ be a connected open set and suppose that $f : U \to \mathbb{R}^l$ is differentiable in the intrinsic sense at every point of $p \in U$. Fix p in U, let*

$$q = exp(\sum_{j=1}^n u_j X_j)(p)$$

be a point next to p and ξ in $\mathbb{R}^l$. Then there exists z in U such that

$$\langle f(q) - f(p), \xi\rangle = \langle J_H f(z)(u_1, \cdots, u_k)^T, \xi\rangle.$$

PROOF.

$$F : [0,1] \to \mathbb{R}, \qquad F(t) = \langle f(\exp(\sum_{j=1}^n t u_j X_j)(x)), \xi\rangle.$$

F is continuous in $[0,1]$. For every $t \in]0,1[$,

$$
\begin{aligned}
F'(t) &= \frac{d}{dt}\left(\sum_{i=1}^{l} f^i(\exp(\sum_{j=1}^{n} te_jX_j)(p))\ \xi_i\right)\\
&= \sum_{i=1}^{l} \lim_{t'\to t}\frac{1}{t'}\Big(f^i(\exp(\sum_{j=1}^{n} t'u_jX_j)\exp(\sum_{j=1}^{n} tu_jX_j)(p))+\\
&- f^i(\exp(\sum_{j=1}^{n} tu_jX_j)(p))\Big)\ \xi_i\\
&= \sum_{i=1}^{l}\sum_{j=1}^{k} u_jX_jf^i(\exp(\sum_{j=1}^{n} tu_jX_j)(p))\ \xi_i\\
&= \sum_{i=1}^{l}(\mathcal{J}_H f(\exp(\sum_{j=1}^{n} te_jX_j)(p))(u_1,\cdots,u_k)^t)_i\ \xi_i.
\end{aligned}
$$

By the mean value theorem, there exists $\tau \in [0,1]$ such that $F(1)-F(0)=F'(\tau)$. Hence, for $z = \exp\left(\sum_{j=1}^{n} \tau u_jX_j\right)(p)$ there follows

$$\langle f(q)-f(p),\xi\rangle = F(1)-F(0) = \langle \mathcal{J}_H f(z)(u_1,\cdots,u_m)^t,\xi\rangle.$$

□

Franchi, Serapioni and Serra Cassano gave a definition of a regular hypersurfaces in a Carnot group in [**13**, Definition 1.6]. A natural generalization of this definition is the following definition of a regular submanifold in a sub-Riemannian manifold.

DEFINITION 1.15. A *regular submanifold* of dimension m is a subset of N that can be locally represented as the zero-set of a function f in $C^1_{\mathcal{H}}(N,\mathbb{R}^l)$, where $l=n-m$ such the rank of $\mathcal{J}_{\mathcal{H}}f$ is equal to l.

However, in the present study we consider general smooth submanifolds in N and then we study the relation between their tangent space and the distribution by the notion of degree. Magnani, Vittone and Le Donne further expanded this approach in [**24**, **26**, **27**].

Here, we report an important result, the Rothschild-Stein's Theorem of lifting and approximation [**34**, Theorem 5] that shows how it is possible to approximate general free up vector fields (i.e. the vector field of the distribution and their commutators up to step s are linear independents see [**34**]) with polynomial vector field generating a free Lie Group. The proof of this theorem is connected to the Mitchell's Theorem which shows that the tangent space to a sub-Riemannian manifold is its nilpotentization, which is a Carnot group.

Theorem 1.6 (Rothschild-Stein's theorem of lifting and approximation). *Let $\mathcal{H}$ be a distribution generated by $X_1, \cdots, X_k$ vector fields on N and let p be a point in N such that $\mathcal{H}$*

(i) verifies the Hörmander rank condition
(ii) is free up to step s at p.

Choose $X_{k+1}, \cdots, X_n$ commutators such that $X_1, \cdots, X_n$ span the all tangent bundle, there they determine a canonical coordinates $(u_1, \cdots, u_n)$ around p. Let G be the Carnot Group of step s with k generators and $\mathfrak{g}$ its Lie algebra. Then there are $Y_1, \cdots, Y_n$ vector field in $\mathfrak{g}$ and neighborhoods U of p in N and Ω of 0 in G with the following properties. We consider

$$\Theta : U \times U \to \Omega, \qquad \Theta(\xi, \eta) = exp\left(\sum_{i=1}^{n} u_i Y_i\right)(0),$$

where $\eta = exp\left(\sum_{i=1}^{n} u_i Y_i\right)(\xi)$. Therefore, if we fix ξ in U the mapping

$$\eta \to \Theta_\xi(\eta) = \Theta(\xi, \eta) = (u_1, \cdots, u_n)$$

is a coordinate chart for U centered at ξ. In this chart

$$X_i = Y_i + R_i \qquad i = 1, \cdots, k$$

where R_i is a differential operator of local degree $\leqslant 0$.

3. Examples and applications

3.1. Heisenberg group. Let $(\mathbb{H}^3, *)$ be a simply connected Lie group whose Lie algebra is

$$\mathfrak{h} = \mathfrak{h}^1 \oplus \mathfrak{h}^2$$

where $\mathfrak{h}^1 = \text{span}\{X, Y\}$ and $\mathfrak{h}^2 = \text{span}\{Z\}$ with X, Y, Z satisfying the following bracket relations

$$[X, Y] = 2Z \quad [X, Z] = 0 \quad [Y, Z] = 0. \tag{17}$$

We can identify $\mathfrak{h}$ with $\mathbb{R}^3$, since the exponential mapping

$$\exp : \mathfrak{h} \longrightarrow \mathbb{H}^3$$

is a global diffeomorphism. Indeed we can introduce global coordinates on $\mathbb{H}^3$ by

$$\begin{array}{cccc} \varphi : & \mathbb{R}^3 & \longrightarrow & \mathbb{H}^3 \\ & (x, y, t) & \mapsto & \exp(xX + yY + tZ) \end{array}$$

Therefore, we can identify $\mathbb{H}^3$ with $\mathbb{R}^3$ and on $\mathbb{R}^3$ a pair of vector fields that satisfy (17) are

$$X = \partial_x - y\partial_t, \qquad Y = \partial_y + x\partial_t.$$

These vector fields lie on the kernel of the contact form

$$w = dt - (xdy - ydx).$$

The group operation on $\mathbb{R}^3$ with these two vector fields and $Z = \partial_t$ is

$$(x, y, t)\cdot (x', y', t') = (x + x', y + y', t + t' + xy' - yx').$$

Moreover, we have that $(\mathbb{H}^3, \mathfrak{h}^1, h)$ is a Carnot group, where h is a arbitrary metric on the distribution $\mathfrak{h}^1$. Notice that the coefficient 2 in front of Z in (17) does not affect the structure of Carnot group and would change only the group operation providing a group isomorphic to the one just now presented . Therefore, we should define the Heisenberg group as the only one Carnot group of dimension three such that Lie algebra $\mathfrak{h} = \mathfrak{h}^1 \oplus \mathfrak{h}^2$ satisfies the following conditions

$$\operatorname{rank}(\mathfrak{h}^1) = 2 \quad \text{and} \quad \operatorname{rank}(\mathfrak{h}^2) = 1.$$

Montgomery in [**28**] shows the connection between the Dido problem and the geodesics in Heisenberg group. We suggest the reader this lecture.

3.2. Rototraslation Group. Citti and Sarti in [**9**] proposed a model of low-level vision to mathematically model the functional structures of the primary visual cortex, they based their model on the previous works [**20**] by Hoffman and [**32**] by Petitot-Tondut where differential geometry models the visual cortex. To understand this model we make a brief exposition of the functional architecture of the visual cortex, see for further details [**22**, Chapter 4]. The acquisition of the visual system starts in the retina, that after projects the information to the lateral geniculate nucleus and from there to the primary visual cortex V1. We can identify the retinal structure with a plane $\mathbb{R}^2$. The primary visual cortex V1 processes the orientation via the *simple cells* and other features by *complex cells* (estimation of motion direction, detection of angles, curvature). The receptive field of a cell is the domain of the retinal plane to which the cell is connected with neural synapses of the retinal-geniculate-cortical path. When the receptive field of a cell is stimulated by a visual signal, the cell reacts generating spikes. On the receptive field there area "on", if the spikes respond to positive signal, and "off" area, if the spikes respond to negative signal. This behavior can be mathematically modeled by a function Ψ_0 defined on the retinal plane. The retinotopic structure is a logarithmic conformal mapping between the retina and V1, that Citti and Sarti ignored in their study. Cortical cells are organized in columns corresponding to parameters as orientation, curvature, ocular dominance and color by the *hypercolumnar structure*. This structure, for simple cells, means that over each point of the retina there is a set of cells (hypercolumn) which are sensitive to all possible orientations. The *non-maxima suppression* selects the orientation of maximum output of the hypercolumn in response to a visual stimulus and suppresses all the others. There is also the *connectivity structure* that connects cells with the same orientation belonging to different hypercolumns.

Citti and Sarti consider a gray level image I on the retina as a real stimulus and they assume that over each point (x, y) in the retina plane the cells in the hypercolumn can code the direction of the level line of I. They assume that the cell in the hypercolumn, which gives a maximal response, is sensible to the direction of the level line of I in a point (x, y). The gradient $\nabla I = (I_x, I_y)$ is perpendicular to the level lines, then a tangent vector to a level line is $(-I_y, I_x)$ or $(I_y, -I_x)$. In order to save the information of direction they consider the angle $\theta(x, y) = -\arctan(I_x, I_y), \quad \theta \in [0, \pi]$. This process associates to each retinal point (x, y) a point (x, y, θ) in the three-dimensional space $\mathbb{R}^2 \times S^1$. Therefore, each level line $\gamma(t) = (x(t), y(t))$ is lifted to a curve $\tilde{\gamma}(t) = (x(t), y(t), \theta(x(t), y(t)))$ in $\mathbb{R}^2 \times S^1$. Notice that the vector field on the retinal plane tangent to the level lines of I, γ, at the point (x, y) is

$$X_\theta = \cos(\theta(x, y))\partial_x + \sin(\theta(x, y))\partial_y. \tag{18}$$

A tangent vector on $\mathbb{R}^2 \times S^1$ to the lifted curve $\tilde{\gamma}$ is a linear combination of the vector field

$$X = \cos(\theta)\partial_x + \sin(\theta)\partial_y, \qquad Y = \partial_\theta.$$

We define the distribution $\mathcal{H} = \text{span}\{X, Y\}$, which is the kernel of the one-form

$$\omega = \sin(\theta)dx - \cos(\theta)dy.$$

The horizontal inner product h is the one which makes X and Y an orthonormal basis. Therefore, $(S := \mathbb{R}^2 \times S^1, \mathcal{H}, h)$ is a contact sub-Riemannian manifold, which verifies the Hörmander rank condition. Indeed, we have

$$T = [X, Y] = \sin(\theta)\partial_x - \cos(\theta)\partial_y$$

and the rank of

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ 0 & 0 & 1 \\ \sin(\theta) & -\cos(\theta) & 0 \end{pmatrix}$$

is three, since the matrix is invertible. One can think that each level line is lifted to space S separately, but the mechanism of non-maxima suppression is applied to the whole image producing a regular surface, see [**9**, 1.4.2] for the simple cell activity and [**9**, 1.6.1] in order to understand the non-maxima suppression. Therefore a image lifted to S is a regular surface in particular a θ-graph,

$$G_\theta = \{(x, y, \theta) \in S \ : \ \theta = \theta(x, y)\}.$$

In [**9**] they apply the iteration of the joint work of sub-Riemannian diffusion and non-maxima suppression to G_θ in order to provide a succession of surfaces that converges to a minimal surface in the rototraslation space. This minimal surface would be the surface we elaborate in the visual cortex and allows to propagate existing information and to complete the boundaries. Thus, we understand the reason why it is useful to study the PDE for minimal surfaces in this sub-Riemannian setting, a topic deeply studied by Galli in [**15**] and Galli and Ritoré in [**16**].

If we consider a curve in the retinal plane $\gamma(t) = (x(t), y(t))$ with tangent vector X_θ defined in (18), it follows that γ is parametrized by the arc. Indeed,

$$|\dot\gamma(t)|^2 = \dot x(t)^2 + \dot y(t)^2 = \cos(\theta)^2 + \sin(\theta)^2 = 1.$$

When we lift the curve γ to S, we obtain $(\tilde\gamma)(t) = (x(t), y(t), \theta(x(t), y(t)))$. The length of $\tilde\gamma$ is

$$l(\tilde\gamma) = \int_a^b \sqrt{h(\tilde\gamma(t), \tilde\gamma(t))}\, dt = \int_a^b \sqrt{1 + \dot\theta(t)^2}\, dt. \tag{19}$$

Notice that $\dot\theta(t) = k$, where k is the curvature of the curve γ in the retinal plane. Let us remind that the elastica functional for a curve in the plane is

$$\mathcal{E}(\gamma) = \int_\gamma \sqrt{1 + k^2},$$

therefore, it follows that the length of a lifted curve $\tilde\gamma$ is equal to the elastica functional of γ on the plane.

Now, we show an application of the Rothschild-Stein's Theorem at this simple case. We expand $\cos(\theta)$ and $\sin(\theta)$ at the first order at the point (x_0, y_0, θ_0)

$$\begin{aligned} X =& \underbrace{(\cos(\theta_0) - \sin(\theta_0)(\theta - \theta_0))\ \partial_x + (\sin(\theta_0) + \cos(\theta_0)(\theta - \theta_0))\ \partial_y}_{\tilde X} \\ &+ \underbrace{o(|\theta - \theta_0|)\ \partial_x + o(|\theta - \theta_0|)\ \partial_y}_{R}, \\ Y =& \partial_\theta. \end{aligned}$$

Now, we have

$$\begin{aligned} \tilde X =& (\cos(\theta_0) - \sin(\theta_0)(\theta - \theta_0))\ \partial_x + (\sin(\theta_0) + \cos(\theta_0)(\theta - \theta_0))\ \partial_y, \\ Y =& \partial_\theta, \\ [\tilde X, Y] =& -\sin(\theta_0)\partial_x + \cos(\theta_0)\partial_y. \end{aligned}$$

Let us consider the following transformation, that is essentially the exponential mapping in S,

$$\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} = \begin{pmatrix} \cos(\theta_0)\ x + \sin(\theta_0) y \\ -\sin(\theta_0)\ x + \cos(\theta_0) y \\ \theta - \theta_0 \end{pmatrix} \tag{20}$$

and

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos(\theta_0)\frac{\partial}{\partial x'} - \sin(\theta_0)\frac{\partial}{\partial y'} \\ \frac{\partial}{\partial y} &= \sin(\theta_0)\frac{\partial}{\partial x'} + \cos(\theta_0)\frac{\partial}{\partial y'}. \end{aligned}$$

In these new coordinates

$$\begin{aligned} \tilde{X} =& \partial x' + \theta' \, \partial_{y'}, \\ Y =& \partial_{\theta'}, \\ [\tilde{X}, Y] =& -\partial_{y'}. \end{aligned}$$

This is the Heisenberg algebra.

3.3. Engel group. The *Engel group* $\mathbb{E}$ is a simply connected Carnot group whose Lie algebra is

$$\mathfrak{e} = V_1 \oplus V_2 \oplus V_3$$

where

$$\operatorname{rank}(V_1) = 2, \quad \operatorname{rank}(V_2) = 1 \quad \text{and} \quad \operatorname{rank}(V_3) = 1.$$

Since the exponential is a global diffeomorphism

$$\exp : \mathfrak{e} \longrightarrow \mathbb{E}$$

we can represent the Engel group by $\mathbb{R}^4$ where

$$\begin{aligned} &X_1 = \partial_{x_1} - x_3\partial_{x_2} - x_4\partial_{x_3} \quad \text{and} \quad X_2 = \partial_{x_4} \quad \text{that generate} \quad V_1, \\ &X_3 = [X_1, X_2] = \partial_{x_3} \quad \text{that generates} \quad V_2, \\ &X_4 = [X_1, [X_1, X_2]] = \partial_{x_2} \quad \text{that generates} \quad V_3. \end{aligned}$$

This representation will be useful in the following section to show that this group is the tangent space to a four dimensional Engel sub-Riemannian manifold. Whereas Le Donne and Magnani present a more standard representation in [24] where

$$\begin{aligned} &X_1 = \partial_{x_1}, \quad X_2 = \partial_{x_2} + x_1\partial_{x_3} + \frac{x_1^2}{2}\partial_{x_4}, \\ &X_3 = [X_1, X_2] = \partial_{x_3} + x_1\partial_{x_4}, \\ &X_4 = [X_1, [X_1, X_2]] = \partial_{x_4}. \end{aligned}$$

3.4. Curvature and orientation. Let $E = \mathbb{R}^2 \times S^1 \times \mathbb{R}$ and let

$$X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y + k\partial_\theta, \qquad X_2 = \partial_k \tag{21}$$

be vector fields on E, we set $\mathcal{H} = \operatorname{span}\{X_1, X_2\}$. To define a sub-Riemannian manifolds we need an inner product on the distribution $\mathcal{H}$. In the present work we will use two different metrics on the distribution $\mathcal{H}$: h_1, the one which makes X_1 and X_2 orthonormal, and h_2, the one induced by the Euclidean metric

$$h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad h_2 = \begin{pmatrix} 1 + k^2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{22}$$

Therefore, $(E, \mathcal{H}, h_1)$ and $(E, \mathcal{H}, h_2)$ are sub-Riemannian manifolds, we will specify the metric we use. These vector fields satisfy the Hörmander rank condition. Indeed, we have

$$X_3 = [X_1, X_2] = -\partial_\theta \tag{23}$$

$$X_4 = [X_1, [X_1, X_2]] = -\sin(\theta)\partial_x + \cos(\theta)\partial_y. \tag{24}$$

We are interested in studying minimal surface in this setting, in particular (θ,κ)-graphs. Therefore, we need the implicit function theorem to provide a manageable parametrization. It would be a problem if we considered the definition of regular surface adopted in [13]. Indeed we should take into account

$$\mathcal{J}_H f = \begin{pmatrix} X_1 f^1 & X_2 f^1 \\ X_1 f^2 & X_2 f^2 \end{pmatrix}.$$

Magnani in [26] deal with implicit function theorem in stratified groups with respect to the intrinsic notion differentiability. However, we consider smooth submanifolds. Therefore our implicit function theorem is standard.

Theorem 1.7 (Implicit Function Theorem). *Let $\Omega \subset E$ be an open set and let $f : \Omega \to \mathbb{R}^2$ be a continuous and $C^2_H(\Omega, \mathbb{R}^2)$ function. If*

$$\Sigma = \{\xi = (x, y, \theta, k) \in \Omega \ : \ f(x, y, \theta, k) = 0\}$$

and suppose that

$$\det \begin{pmatrix} X_2 f^1 & X_3 f^1 \\ X_2 f^2 & X_3 f^2 \end{pmatrix} (\bar{\xi}) \neq 0. \tag{25}$$

Then there exist neighborhoods $I, J \subset \mathbb{R}^2$ such that

$$\Sigma \cap (I \times J) = \{(x, y, u_1(x, y), u_2(x, y)) \ : \ (x, y) \in I\}.$$

Proof. First of all, notice that f in $C^2_H(\Omega, \mathbb{R}^2)$ implies $C^1_E(\Omega, \mathbb{R}^2)$ by Remark 1.4. Let $(\mathbb{R}^3 \times]0, 2\pi[, \mathbb{1} \times e^{i\theta} = \psi_1)$ and $(\mathbb{R}^3 \times]-\pi, \pi[, \mathbb{1} \times e^{i(\pi+\theta)} = \psi_2)$ be the two cards of E. If we want to express a point p of E we can use the coordinates

$$\psi_i^{-1} : U_i \to W_i, \quad \psi_i^{-1}(p) = (x, y, \theta, k) \quad i = 1, 2$$

where we have set

$$U_1 = \mathbb{R}^3 \times S^1_{/(1,0)}, \quad W_1 = \mathbb{R}^3 \times]0, 2\pi[, \quad U_2 = \mathbb{R}^3 \times S^1_{/(-1,0)} \quad W_2 = \mathbb{R}^3 \times] -\pi, \pi[.$$

Now we have $X_2 = \partial_k, \quad X_3 = -\partial_\theta$ and we can define the transformation

$$G(x, y, \theta, k) = (x, y, -\theta, k).$$

With this choice it follows

$$\partial_k (f \circ G) = X_2 f \qquad \partial_\theta (f \circ G) = dG(\partial_\theta) = X_3 f.$$

Therefore the condition (25) is equivalent to

$$\det \begin{pmatrix} \partial_\theta (f\circ G)^1 & \partial_k (f\circ G)^1 \\ \partial_\theta (f\circ G)^2 & \partial_k (f\circ G)^2 \end{pmatrix} (\bar{x},\bar{y},-\bar{\theta},\bar{k}) \neq 0. \tag{26}$$

Now thanks to the classic implicit function theorem, there exist neighborhoods I of $(\bar{x},\bar{y})$ and $\tilde{u_1} : I \to \mathbb{R}$ and $u_2 : I \to \mathbb{R}$ such that

$$(f\circ G)(x,y,\tilde{u_1}(x,y),u_2(x,y)) = 0.$$

Therefore we set $u_1 = -\tilde{u_1}$ it follows

$$f(x,y,u_1(x,y),u_2(x,y)) = 0$$

and the proof is complete. □

Now, we show that the Carnot group that approximates the structure of E is the Engel group. As we did in (3.2), we want to deduce approximate vector fields for this structure. Then, we expand at the first order $\sin(\theta)$ and $\cos(\theta)$ around θ_0

$$\begin{aligned} X_1 =& \underbrace{(\cos(\theta_0) - \sin(\theta_0)(\theta-\theta_0))\ \partial_x + (\sin(\theta_0)+\cos(\theta_0)(\theta-\theta_0))\ \partial_y + k\ \partial_\theta}_{Y_1} \\ &+ \underbrace{o(|\theta-\theta_0|)\ \partial_x + o(|\theta-\theta_0|)\ \partial_y}_{R}, \\ X_2 =& \partial_k. \end{aligned}$$

Therefore, we have

$$\begin{aligned} [X_1,X_2] &= -\partial_\theta \\ [X_1,[X_1,X_2]] &= \underbrace{\sin(\theta_0)\partial_x - \cos(\theta_0)\partial_y}_{Y_4} + o(1)\partial_x + o(1)\partial_y. \end{aligned}$$

If we cut at the first order, we obtain the following structure

$$\begin{aligned} Y_1 &= \cos(\theta_0) - \sin(\theta_0)(\theta-\theta_0)\ \partial_x + \sin(\theta_0) + \cos(\theta_0)(\theta-\theta_0)\ \partial_y + k\ \partial_\theta, \\ Y_2 &= \partial_k, \\ Y_3 &= -\partial_\theta, \\ Y_4 &= -\sin(\theta_0)\partial_x + \cos(\theta_0)\partial_y. \end{aligned}$$

Under the following transformations

$$\begin{pmatrix} x' \\ y' \\ \theta' \\ k' \end{pmatrix} = \begin{pmatrix} \cos(\theta_0)\ x + \sin(\theta_0) y \\ -\sin(\theta_0)\ x + \cos(\theta_0) y \\ \theta - \theta_0 \\ k \end{pmatrix}. \tag{27}$$

Renaming $(x', y', \theta', k') = (x, y, \theta, k)$, we obtain

$$\begin{aligned} Y_1 &= \partial_x + \theta\partial_y + k\partial_\theta, \qquad Y_2 = \partial_k, \\ Y_3 &= -\partial_\theta, \\ Y_4 &= \partial_y. \end{aligned}$$

This algebra generates a four dimensional Carnot group which is known as the Engel group.

In the sub-Riemannian manifold E we have already presented we have

$$X_1 = X + k\,Y,$$

where X and Y are the vector field of S. We can lift a horizontal curve $\gamma = (x(t), y(t), \theta(t))$ in S to a curve $\bar{\gamma} = (x(t), y(t), \theta(t), \dot{\theta}(t))$ in E. Therefore we have

$$\begin{aligned} \dot{\bar{\gamma}}(t) &= (\dot{x}(t), \dot{y}(t), \dot{\theta}(t), \ddot{\theta}(t)) = X_1 + \ddot{\theta}(t)X_2, \\ |\dot{\bar{\gamma}}(t)|^2_{h_1} &= 1 + \ddot{\theta}(t)^2, \\ |\dot{\bar{\gamma}}(t)|^2_{h_2} &= 1 + \dot{\theta(t)}^2 + \ddot{\theta}(t)^2, \end{aligned}$$

and the length of the curve $\bar{\gamma}$ is

$$l_{h_i}(\bar{\gamma}) = \int_a^b |\dot{\bar{\gamma}}(t)|_{h_i}\, dt.$$

After all, to know the curvature $K_s(\gamma)$ of the horizontal curve γ in S we have to derive the orthonormal tangent vector

$$E_1(t) = \frac{X(t) + \dot{\theta}(t)Y}{\sqrt{1 + \dot{\theta}(t)^2}}.$$

Hence, it follows

$$\frac{dE_1}{dt}(t) = \frac{\ddot{\theta}(t)}{1 + \dot{\theta}(t)^2}\, \frac{Y - \dot{\theta}(t)X}{\sqrt{1 + \dot{\theta}(t)^2}} + \frac{\dot{\theta}(t)}{\sqrt{1 + \dot{\theta}(t)^2}}\, T.$$

We have only the metric h_1 on the distribution, therefore for us the orthonormal to E_1 is

$$N = \frac{Y - \dot{\theta}(t)X}{\sqrt{1 + \dot{\theta}(t)^2}}$$

and the curvature for the horizontal curve is

$$K_S(\gamma) = \frac{\ddot{\theta}(t)}{1 + \dot{\theta}(t)^2}. \tag{28}$$

In this setting the elastica functional on a horizontal curve in S is not equal to the length of the lifted curve in E. Therefore, in order to deduce the previous property we substitute the vector field X_1 and X_2 by

$$Z_1 = \cos(\varphi)X + \sin(\varphi)Y$$
$$Z_2 = \partial_\varphi$$

where we have set $k = \tan(\varphi)$. In this way, if we set $\mathcal{H} = \mathrm{span}\{Z_1, Z_2\}$ and we choose as horizontal metric h_3 the one making Z_1 and Z_2 orthonormal, we have a sub-Riemannian manifold

$$(\tilde{E} = \mathbb{R}^2 \times S^1 \times S^1, \mathcal{H}, h_3)$$

where $\mathcal{H}$ is bracket-generating distribution. Indeed, there holds

$$Z_3 = [Z_1, Z_2] = \sin(\varphi)X - \cos(\varphi)Y$$
$$Z_4 = [Z_1, Z_3] = -\sin(\theta)\partial_x + \cos(\theta)\partial_y$$

and $Z_1, \cdots, Z_4$ are linear independents

$$\det \begin{pmatrix} \cos(\varphi)\cos(\theta) & \cos(\varphi)\sin(\theta) & \sin(\varphi) & 0 \\ 0 & 0 & 0 & 1 \\ \sin(\varphi)\cos(\theta) & \sin(\varphi)\sin(\theta) & -\cos(\varphi) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \end{pmatrix} \neq 0.$$

Here, if we consider a lifted curve γ form $\mathbb{R}^2$ to S φ is the real number in $]-\pi, \pi[$ such that

$$\dot{\gamma}(t) = \cos(\varphi)X + \sin(\varphi)Y.$$

Therefore, the tangent vector to a lifted curve $\bar{\gamma}(t) = (\gamma(t), \varphi(t))$ is

$$|\dot{\bar{\gamma}}(t)|^2 = 1 + \dot{\varphi}(t)^2$$

and its length

$$l(\bar{\gamma}(t)) = \int_a^b \sqrt{1 + \dot{\varphi}(t)^2}.$$

Now, we set $\varphi(t) = \arctan(\dot{\theta}(t))$, thus

$$\dot{\varphi}(t) = \frac{\ddot{\theta}(t)}{1 + \dot{\theta}(t)^2},$$

that is equal to (28). Hence, the length of a lifted curve $\bar{\gamma} = (\gamma, \varphi)$ in $\tilde{E}$ is equal to the elastica functional in S evaluate in γ

$$\mathcal{E}(\gamma) = \int_a^b \left(1 + \left(\frac{\ddot{\theta}(t)}{1 + \dot{\theta}(t)^2}\right)^2\right)^{\frac{1}{2}} dt.$$

REMARK 1.5. Notice that locally $k = \tan(\varphi)$ is a change of coordinates. Therefore we can see $\tilde{E}$ locally as E with an other metric h_3. Indeed, under this change of coordinates

$$X_1 = \frac{1}{\cos(\varphi)}(\cos(\varphi)X + \sin(\varphi)Y),$$
$$X_2 = \frac{1}{\cos(\varphi)^2}\ \partial_\varphi.$$

Therefore, h_3 on the distribution generated by X_1, X_2 should be

$$h_3 = \begin{pmatrix} \cos(\varphi)^2 & 0 \\ 0 & \cos(\varphi)^4 \end{pmatrix}.$$

In these coordinates the commutators change but they are proportional to Z_3 and Z_4 where the factors depend only on φ.

CHAPTER 2

Area in a sub-Riemannian manifold

Our purpose in this chapter is to give a general definition of the area in a sub-Riemannian manifold for a submanifold of arbitrary dimension using the notion of degree studied by Magnani-Vittone in [**27**] and Le Donne-Magnani in [**24**]. In this chapter we prove basic properties of the degree and the area, then we show that our definition of area for a hypersurface in the Heisenberg group coincides with the one used in [**33**] and we provide a suitable definition of area for a surface in S and E. Finally, in the last section we compare the Hausdorff dimension of a submanifold to the notion of degree.

Let N be a smooth manifold and n be the dimension of N. Let $\mathcal{H}$ be a distribution on N and U be an open subset of N. Locally $\{X_1, \cdots, X_k\}$ span $\mathcal{H}$ on the open set U. The distribution $\mathcal{H}$ is a subbundle of constant dimension k of the tangent space TU, see [**28**]. Moreover, h is a metric defined only on the subbundle $\mathcal{H}$. Therefore, $(N, \mathcal{H}, h)$ has a structure of sub-Riemannian manifold and furthermore we suppose that $X_1, \cdots, X_k$ verify the Hörmander rank condition. The Lie brackets of vector fields in $\mathcal{H}$ generate a flag of subbundles

$$\mathcal{H} \subset \mathcal{H}^2 \subset \cdots \subset \mathcal{H}^r \subset \cdots \subset TN, \tag{29}$$

with

$$\mathcal{H}^2 = \mathcal{H} + [\mathcal{H}, \mathcal{H}], \qquad \mathcal{H}^{r+1} = \mathcal{H}^r + [\mathcal{H}, \mathcal{H}^r],$$

where

$$[\mathcal{H}, \mathcal{H}^k] = \{[X, Y] \ : \ X \in \mathcal{H}, \ Y \in \mathcal{H}^k\}.$$

The fact that $X_1, \cdots, X_k$ verify Hörmander rank condition is equivalent, at least in the case of N compact, to the assumption that there is an s such that $\mathcal{H}^s = TN$. Henceforth, we will do this assumption. Here, we follow Montgomery's book [**28**, 2.3]. The flag of subbundles at a point p is a flag of subspaces of T_pN

$$\mathcal{H}_p \subset \mathcal{H}_p^2 \subset \cdots \subset \mathcal{H}_p^s = T_pN \tag{30}$$

and we set $n_i(p) = \dim \mathcal{H}_p^i$. The integer list $(n_1(p), \cdots, n_s(p))$ of dimensions is called the *growth vector* of $\mathcal{H}$ at p. Moreover, the smallest s such that $\mathcal{H}_p^s = T_pN$ is called the step of the distribution $\mathcal{H}$ at the point p.

Definition 2.1. A distribution $\mathcal{H}$ on a manifold N is *regular* at a point p in N if the growth vector is constant in a neighborhood of p.

EXAMPLE 2.1. In order to show an example of sub-Riemannian manifold with not regular points, we consider the Grŭsin plane G_2. Let us consider the plane $\mathbb{R}^2$ with coordinates x, y and the sub-Riemannian metric which makes orthonormal the vector fields

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 \\ x \end{pmatrix}.$$

These vector fields span all the tangent space, except along the line $x = 0$. There, if we add the Lie bracket

$$[X_1, X_2] = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

the distribution verifies the Hörmander rank condition and thus the hypothesis of the Chow Theorem. Outside the line $x = 0$, the sub-Riemannian metric is

$$ds = dx^2 + \frac{1}{x^2} dy^2,$$

that it is essentially a Riemannian metric. Now, if we consider a point p in $\{(x, y) \in \mathbb{R}^2 \ : \ x \neq 0\}$ then we can find a sufficiently small neighborhood where the dimension of the growth vector $n_1(p)$ is constantly equal to two. Hence, p is a regular point. If we suppose that $p \in \{(x, y) \in \mathbb{R}^2 \ : \ x = 0\}$ the growth vector will be $(n_1(p), n_2(p)) = (1, 1)$. Then an open neighborhood of p has to intersect the set where $x \neq 0$, and so the growth vector is equal to $n_1(p) = 2$. Therefore, for each neighborhood of a point in the line $x = 0$, we have that the growth is not constant. Hence, this is not a regular point. We suggest the reader to see [**2**, page 31] for an example of singular point where the dimension of the first layer is constant. There the sub-Riemaniann structure is $\mathbb{R}^3$ equipped of the distribution

$$\mathcal{H} = \text{span}\{X_1 = \partial_x + \frac{1}{2} y^2 \partial_z, \quad X_2 = \partial_y\}.$$

Each point in the surface $\pi = \{y = 0\}$ is a singular point. Indeed, the growth vector for p in π is $n(p) = (2, 2, 3)$ and for p not in π is $n(p) = (2, 3)$, since

$$[X_1, X_2] = -y\partial_z \quad [X_2, [X_1, X_2]] = -\partial_z.$$

We set $H_i = \mathcal{H}^i / \mathcal{H}^{i-1}$ and define

$$\text{Gr}(\mathcal{H}) = \mathcal{H} \oplus \mathcal{H}^2/\mathcal{H} \oplus \cdots \oplus \mathcal{H}^s/\mathcal{H}^{s-1} = H_1 \oplus \cdots \oplus H_s, \tag{31}$$

which is the graded bundle corresponding to the flag of bundles $\mathcal{H}^i$. When p is a regular point for the distribution, we call $\text{Gr}(\mathcal{H}))(p)$ the *nilpotentization* of $\mathcal{H}$ at p.

DEFINITION 2.2. Given a set of vector fields $X_1, \cdots, X_k \in TN$ and a multi-index

$$J = (j_1, \cdots, j_l) \in \{1, \cdots, k\}^l,$$

we set

$$X_J = [\cdots [X_{j_1}, X_{j_2}], \cdots, X_{j_l}].$$

We say that X_J is the commutator of length l of $X_1, \cdots, X_k$ if $X_J \in \mathcal{H}^l$ and $X_J \notin \mathcal{H}^{l-1}$.

Let U be a neighborhood of a regular point p in N, the tangent bundle TU can be written as

$$TU = H_1 \oplus \cdots \oplus H_s,$$

where

$$\begin{aligned}
H_1 &= \mathrm{span}\{X_1, \cdots, X_k\},\\
H_2 &= \mathrm{span}\{[X_i, X_j] \quad i \neq j \quad i, j = 1, \cdots, k \ : \ [X_i, X_j] \notin H_1\}\\
&= \mathrm{span}\{X_{k+1}, \cdots, X_{n_2}\},\\
&\ \vdots\\
H_l &= \mathrm{span}\{\ X_J \quad \text{a commutator of length } l \ : \ J \in \{1, \cdots, k\}^l\}\\
&= \mathrm{span}\{X_{n_{l-1}+1}, \cdots, X_{n_l}\},\\
&\ \vdots\\
H_s &= \mathrm{span}\{\ X_J \quad \text{a commutator of length } s \ : \ J \in \{1, \cdots, k\}^s\}\\
&= \mathrm{span}\{X_{n_{s-1}+1}, \cdots, X_n\}.
\end{aligned}$$

Where s is the step defined in 1.6. A frame

$$(X_1, \cdots, X_k, X_{n_1+1} \cdots, X_{n_2}, \cdots, X_{n_{s-1}+1}, \cdots, X_n)$$

is an adapted basis to the flag (30) at each point.

DEFINITION 2.3. An order basis $(v_1, \cdots, v_n)$ is said to be adapted to a flag

$$V_1 \subset V_2 \subset \cdots \subset V_s$$

if the first $d_i = \dim(V_i)$ vectors form a basis V_i.

For each X_i in the adapted basis we can assign its length, which is also called *weight*. The assignment $i \mapsto w_i$ is called the *weighting* associated to the growth vector.

Moreover, we assume that the dimension dim $H_i(p)$ is constant in p in N for each $i = 1, \cdots, s$, which is known as the *equiregularity assumption*, for further details see [**19**, page 95].

1. Degree of a submanifold in a sub-Riemannian manifold

DEFINITION 2.4. Let $(X_1, X_2, \cdots, X_n)$ be an adapted basis to the flag of TN. The degree $d(j)$ of X_j is the unique integer r such that $X_j \in H_r$. Let

$$X^J := X_{j_1} \wedge \cdots \wedge X_{j_m} \qquad m \leqslant n$$

be a simple m-vector of $\Lambda_m(N)$, where $J = (j_1, j_2, \cdots, j_m)$ and $1 \leqslant j_1 < j_2 < \cdots < j_m \leqslant n$. The *degree* of X^J is the integer $d(J)$ defined by the sum $d(j_1)+\cdots+d(j_m)$.

Let

$$\tau = \sum_J \tau_J X^J$$

be a m-vector in $\Lambda_m(N)$ represented with respect to the fixed adapted basis $(X_1, \cdots, X_n)$ where τ_J are functions. The *degree* of τ is defined as the integer

$$d(\tau) = \max\{d(X^J) \in \mathbb{N} \ : \ \tau_J \neq 0\}.$$

Let M be a manifold such that $\dim(M) = m < n = \dim(N)$ and let $\Phi : M \to N$ be an embedding, which is defined to be an injective immersion which is an homeomorphism onto its image. We have that $d\Phi(T_pM) \subset T_{\Phi(p)}N$ for each p in M and that the dimension $\dim(d\Phi(T_pM)) = m$, due to the differential $d\Phi$ being injective. We set $\Sigma = \Phi(M)$. Therefore, the degree of a point p in M is defined as

$$d_\Sigma(p) = d(\tau_\Sigma(p)),$$

where

$$\tau_\Sigma(p) \in \{v_1 \wedge \cdots \wedge v_m \ : \ \mathcal{B} = (v_1, \cdots, v_m) \quad \text{a basis of} \quad d\Phi(T_pM)\}.$$

Obviously, the degree is independent of the choice of the basis of the tangent subspace. Indeed, if we consider another basis $\mathcal{B}' = (v_1', \cdots, v_m')$ of $d\Phi(T_pM)$, there holds

$$v_1 \wedge \cdots \wedge v_m = \det(M_{\mathcal{B},\mathcal{B}'}) \ v_1' \wedge \cdots \wedge v_m'.$$

Since $\det(M_{\mathcal{B},\mathcal{B}'}) \neq 0$, the degree is well-defined.

Definition 2.5. The projection of τ onto m-vectors of degree r is defined by

$$(\tau)_r = \sum_{d(J)=r} \tau_J X^J.$$

The *degree* $d(\Sigma)$ of a submanifold Σ is the integer

$$\max_{p \in \Sigma} d_\Sigma(p).$$

2. Equivalence between our degree and Gromov's degree

Mikhael Gromov gave a definition of degree in [19, 0.6.B], we want to show that his definition is equivalent to ours given in Section 1 of this Chapter.

Definition 2.6 (Gromov's degree). Let M be a submanifold of an equiregular sub-Riemannian manifold N equipped with its flag of subbundles

$$\mathcal{H}^1 \subset \mathcal{H}^2 \subset \cdots \subset \mathcal{H}^s = TN.$$

We set $\tilde{\mathcal{H}}_p^j = T_pM \cap \mathcal{H}_p^j$, then it follows

$$\tilde{\mathcal{H}}_p^1 \subset \tilde{\mathcal{H}}_p^2 \subset \cdots \subset \tilde{\mathcal{H}}_p^s = T_pM.$$

We denote $\tilde{m}_j = \dim(\tilde{\mathcal{H}}_p^j / \tilde{\mathcal{H}}_p^{j-1})$, then we set

$$\tilde{D}_H(p) = \sum_{j=1}^{s} j \ \tilde{m}_j.$$

This definition of degree is much more geometrical than the one we gave before, because it expresses the intersection between the tangent space of the submanifold and the flag of subbudles. The intersection of the tangent space T_pM and the flag of subbundles will be constant for k_1 subbudles and it will change at $k_1 + 1$, then it will be constant until the subbudle k_2 and then it will change at $k_2 + 1$ and then again constant until subbudle k_3. Iterating this process, we obtain the finite sequence $k_1, \cdots, k_r \ \in \ \{1, \cdots, s-1\}$. Therefore we have

$$\begin{aligned} M_1 &= T_pM \cap \mathcal{H}^1 = \cdots = T_pM \cap \mathcal{H}^{k_1} \subsetneq \\ M_2 &= T_pM \cap \mathcal{H}^{k_1+1} = \cdots = T_pM \cap \mathcal{H}^{k_2} \subsetneq \\ &\vdots \\ M_r &= T_pM \cap \mathcal{H}^{k_r+1} = \cdots = T_pM \cap \mathcal{H}^{s}, \end{aligned}$$

where we set $L_i = \dim(M_i) \quad i = 1, \cdots, r$. Obviously,

$$M_1 \subset M_2 \subset \cdots \subset M_r = T_pM, \qquad L_1 < L_2 < \cdots < L_r = m.$$

Now, we can choose $v_1, \cdots, v_{L_1}, v_{L_1+1}, \cdots, v_{L_2}, \cdots, v_{L_{r-1}+1}, \cdots, v_{L_r}$ a basis of the tangent space T_pM such that

$$\begin{aligned} v_1, \cdots, v_{L_1} \ &\in \ M_1 \\ v_{L_1+1}, \cdots, v_{L_2} \ &\in M_2 \setminus M_1 \\ &\vdots \\ v_{L_{i-1}+1}, \cdots, v_{L_i} \ &\in M_i \setminus M_{i-1} \\ &\vdots \\ v_{L_{r-1}+1}, \cdots, v_{L_r} \ &\in M_r \setminus M_{r-1}. \end{aligned}$$

If v is a vector in $\{v_{L_{i-1}+1}, \cdots, v_{L_i}\}$ then v belongs to $\mathcal{H}^{k_{i-1}+1} \setminus \mathcal{H}^{k_{i-1}}$. Therefore, the degree $d(v)$ of v is equal to $k_{i-1} + 1$. Thanks to our definition of degree (2.4), it follows that the degree of the m-vector is equal to

$$
\begin{aligned}
&d(v_1 \wedge \cdots \wedge v_{L_1} \wedge v_{L_1+1} \wedge \cdots \wedge v_{L_2} \wedge \cdots \wedge v_{L_{r-1}+1} \wedge \cdots \wedge v_{L_r}) = \\
&L_1 + (L_2 - L_1)(k_1 + 1) + \cdots + (L_i - L_{i-1})(k_{i-1} + 1) + \\
&+ \cdots + (L_r - L_{r-1})(k_{r-1} + 1).
\end{aligned}
$$

Now, let us compute Gromov's degree. In order to determine it we recall that $\tilde{m}_j = \dim(\tilde{\mathcal{H}}_p^j / \tilde{\mathcal{H}}_p^{j-1})$, where $\tilde{\mathcal{H}}_p^j = T_pM \cap \mathcal{H}_p^j$. Thus, we have

$$
\begin{aligned}
&\tilde{m}_1 = L_1,\ \tilde{m}_2 = 0, \cdots, \tilde{m}_{k_1} = 0, \\
&\tilde{m}_{k_1+1} = L_2 - L_1,\ \tilde{m}_{k_1+2} = 0, \cdots, \tilde{m}_{k_2} = 0, \\
&\qquad \vdots \\
&\tilde{m}_{k_{i-1}+1} = L_i - L_{i-1},\ \tilde{m}_{k_{i-1}+2} = 0, \cdots, \tilde{m}_{k_i} = 0, \\
&\qquad \vdots \\
&\tilde{m}_{k_{r-1}+1} = L_r - L_{r-1},\ \tilde{m}_{k_{r-1}+2} = 0, \cdots, \tilde{m}_{k_r} = 0.
\end{aligned}
$$

Therefore, the Gromov's degree at the point p is

$$
\begin{aligned}
\tilde{D}_H(p) =& L_1 + (L_2 - L_1)(k_1 + 1) + \cdots + (L_i - L_{i-1})(k_{i-1} + 1) \\
&+ \cdots + (L_r - L_{r-1})(k_{r-1} + 1).
\end{aligned}
$$

Hence, the two definitions of degree are equivalent.

Remark 2.1. In [25] Magnani writes about horizontal and non-horizontal points and also about the degree of a submanifold in a Carnot group. He sets that p in Σ is a non-horizontal point when $T_p\Sigma$ and $\mathcal{H}_p$ are transversal and p is horizontal when these subspaces are not transversal. In other words, a point p is horizontal if

$$\mathcal{H}_p - \dim(T_p\Sigma \cap \mathcal{H}_p) < k.$$

Overall, the notion of degree is more sophisticated than the one of horizontal point, because the degree detects all possible intersections between the tangent space and each layer of the distribution, not only between the tangent space and the distribution.

3. Semicontinuity of the degree

Here our aim is to prove that the degree of a vector field on a sub-Riemaniann manifold $(N, \mathcal{H})$, defined in 2.4, is lower semicontinuous at a regular point p in N. Let $U \subset N$ be an open neighborhood of p and let

$$v(q) = \sum_{i=1}^{s} \sum_{j=n_{i-1}}^{n_i} c_{ij}(q)(X_j)_q \tag{32}$$

be a smooth vector field on U, where $(X_1, \cdots, X_n)$ is an adapted basis to the flag of the tangent space generated by the bracket-generating distribution

$$\mathcal{H} = \text{span}\{X_1, \cdots, X_k\}.$$

In (32) we adopt the convention that $n_0 = 1$. Let d be an integer number such that $c_{dk}(p) \neq 0$ where k is a integer in $\{n_{d-1}, \cdots, n_d\}$ and $c_{ij}(p) = 0$ when $i = d+1, \cdots, s$. Therefore, the degree $d(v(p))$ of v at p is equal to d. Since coefficients are continuous, there exists $U' \subset U$ neighborhood such that $c_{dk}(q) \neq 0$ for each q in U'. Therefore for each q in U' the degree of $v(q)$ is greater than or equal to the degree of $v(p)$,

$$d(v(q)) \geqslant d(v(p)) = d.$$

The degree at q could be strictly greater than d if there is a coefficient $c_{ij}(p)$ with $i = d+1, \cdots, s$ that is equal to zero at p but over U' is different to zero. Hence, we have

$$\liminf_{q \to p} d(v(q)) \geqslant d(v(p)).$$

4. Sub-Riemaniann area of a submanifold

We extend the metric h defined on $\mathcal{H}$ to a Riemaniann metric g such that $g_{|\mathcal{H}} = h$ and the spaces $H_i(p)$ are orthogonal for each p in N. Now, let $r > 0$ be a real number and we will consider the Riemannian metrics g_r such that

$$g_r(X_i, X_j) = \left(r^{\frac{d(X_i)+d(X_j)-2}{2}} \right)^{-1} g(X_i, X_j) \quad i, j = 1, \cdots, n, \tag{33}$$

where X_i and X_j belongs to the adapted basis $(X_1, \cdots, X_n)$. We will consider the m-vector fields

$$\tilde{X}^J = \left(r^{\frac{d(X_{j_1})-1}{2}} X_{j_1} \right) \wedge \cdots \wedge \left(r^{\frac{d(X_{j_m})-1}{2}} X_{j_m} \right),$$
$$J = (j_1, j_2, \cdots, j_m), \quad 1 \leqslant j_1 < \cdots < j_m \leqslant n.$$

Let $(U, \varphi = y^1, \cdots, y^m)$ be local coordinates in $p \in M$ and $\left(\left.\frac{\partial}{\partial y^1}\right|_p, \cdots, \left.\frac{\partial}{\partial y^m}\right|_p \right)$ is a basis of T_pM. Furthermore,

$$\mathcal{B} = \left(d\Phi\left(\left.\frac{\partial}{\partial y^1}\right|_p \right), \cdots, d\Phi\left(\left.\frac{\partial}{\partial y^m}\right|_p \right) \right)$$

is a basis of $d\Phi(T_pM)$ and we can express

$$d\Phi\left(\left.\frac{\partial}{\partial y^1}\right|_p \right) \wedge \cdots \wedge d\Phi\left(\left.\frac{\partial}{\partial y^m}\right|_p \right) = \sum_J \tilde{\tau}_J \, \tilde{X}^J$$

with respect to the basis $\{\tilde{X}^J \ : J = (j_1, \cdots, j_m)\}$. We can take into account the Jacobian matrix (g_{ij}) which is equal to

$$\begin{pmatrix} g_r\left(d\Phi\left(\left.\frac{\partial}{\partial y^1}\right|_p\right), d\Phi\left(\left.\frac{\partial}{\partial y^1}\right|_p\right)\right) & \cdots & g_r\left(d\Phi\left(\left.\frac{\partial}{\partial y^1}\right|_p\right), d\Phi\left(\left.\frac{\partial}{\partial y^m}\right|_p\right)\right) \\ \vdots & \ddots & \vdots \\ g_r\left(d\Phi\left(\left.\frac{\partial}{\partial y^m}\right|_p\right), d\Phi\left(\left.\frac{\partial}{\partial y^1}\right|_p\right)\right) & \cdots & g_r\left(d\Phi\left(\left.\frac{\partial}{\partial y^m}\right|_p\right), d\Phi\left(\left.\frac{\partial}{\partial y^m}\right|_p\right)\right) \end{pmatrix}.$$

Now, we will define a metric on the m-vectors using the metric g_r. Let $e_1, \cdots, e_m$ and $e'_1, \cdots, e'_m$ be vectors in $d\Phi(T_pM)$. Thanks to the metric we can define the one-forms

$$w_i(v) = g_r(v, e_i) \quad \forall \ v \ \in \ d\Phi(T_pM), \quad i = 1, \cdots, m.$$

We set
(34)

$$\begin{aligned} g_r(e_1 \wedge \cdots \wedge e_m, e'_1 \wedge \cdots \wedge e'_m) &= (w_1 \wedge \cdots \wedge w_m)(e'_1, \cdots, e'_m) \\ &= \sum_{\sigma \in \mathbf{S}_m} \operatorname{sgn}(\sigma)(w_1 \otimes \cdots \otimes w_m)(e'_{\sigma(1)}, \cdots, e'_{\sigma(m)}) \\ &= \sum_{\sigma \in \mathbf{S}_m} \operatorname{sgn}(\sigma) g_r(e'_{\sigma(1)}, e_1) \cdots g_r(e'_{\sigma(m)}, e_m). \end{aligned}$$

Therefore, by (34) and the Leibniz formula for the determinant there follows

$$\left| d\Phi\left(\left.\frac{\partial}{\partial y^1}\right|_p\right) \wedge \cdots \wedge d\Phi\left(\left.\frac{\partial}{\partial y^m}\right|_p\right) \right|_{g_r}^2 = \det(g_{ij})(p).$$

Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ be an atlas of the manifold M and let $\{\Psi_\alpha\}_{\alpha \in I}$ be a partition of unity subordinated to the cover $\{U_\alpha\}$ such that the compact supports of Ψ_α are completely contained in U_α. Therefore the Riemannian area is

$$\begin{aligned} area(\Phi(M), g_r) = &\sum_{\alpha \in I} \int_{\varphi(U_\alpha)} \left| d(\Phi \circ \Psi_\alpha)\left(\left.\frac{\partial}{\partial y^1}\right|_{\varphi_\alpha^{-1}(\xi)}\right) \wedge \cdots \right. \\ &\left. \cdots \wedge d(\Phi \circ \Psi_\alpha)\left(\left.\frac{\partial}{\partial y^m}\right|_{\varphi_\alpha^{-1}(\xi)}\right) \right|_{g_r} \circ \varphi_\alpha^{-1}(\xi) \ d\xi, \end{aligned}$$

where $\xi \in \mathbb{R}^m$ and $d\xi = d\xi^1 \cdots d\xi^m$. Notice that here we consider the Lebesgue measure on the chart, but the same argument holds if M is equipped with a different measure μ. When we consider the measure μ we will compute the integral respect $d\mu$ instead of $d\xi$.

If we set

$$\Phi(M)_i = \{p \in M \ : \ d_{\Phi(M)}(p) = i\}$$

the submanifold $\Phi(M)$ can be written as

$$\Phi(M) = \Phi(M)_m \cup \cdots \cup \Phi(M)_d,$$

where $d = d(\Phi(M))$ is the degree of the submanifold. Finally, we define the sub-Riemannian area of $\Phi(M)$ as

$$A(\Phi(M), g) = \lim_{r\to 0} \sum_{i=m}^{d} r^{\frac{i-m}{2}}\ area(\Phi(M)_i, g_r). \tag{35}$$

Remark 2.2. Let (M, μ) a manifold of degree d embedded in N. Notice that only coefficients of m-vector fields of degree d survive because when r tends to zero the metric factor depending on the degree neutralize the factor $r^{\frac{d-k}{2}}$ that multiplies the Riemannian area. Therefore, we have

$$\begin{aligned} A(\Phi(M), g) = &\sum_{\alpha\in I} \int_{\varphi(U_\alpha)} \Bigg| \Bigg(d(\Phi\circ\Psi_\alpha) \left(\frac{\partial}{\partial y^1}\Big|_{\varphi_\alpha^{-1}(\xi)} \right) \wedge \cdots \\ &\cdots \wedge d(\Phi\circ\Psi_\alpha) \left(\frac{\partial}{\partial y^m}\Big|_{\varphi_\alpha^{-1}(\xi)} \right) \Bigg)_d \Bigg|_g \circ \varphi_\alpha^{-1}(\xi)\ d\mu, \end{aligned}$$

where $|\cdot|_g$ denotes the norm on the m-vector induced by g and $(\cdot)_d$ denotes the projection on the m-vector of degree d.

5. An interesting case of sub-Riemanian area

In general the sub-Riemanian area is dependent of the metric extension of the horizontal metric h.

Example 2.2. Let $\mathbb{H}^1 \otimes \mathbb{H}^1$ be the direct product of Heisenberg space where we consider real coordinates (x, y, z, x', y', z') and the Lie algebra is generated by

$$\begin{aligned} &X = \partial_x - y\partial_z, \quad Y = \partial_y + x\partial_z, \quad Z = \partial_z, \\ &X' = \partial_{x'} - y'\partial_{z'}, \quad Y' = \partial_{y'} + x\partial_{z'}, \quad Z' = \partial'_z, \end{aligned}$$

and the only commutator relations not null are

$$[X, Y] = 2Z \quad [X', Y'] = 2Z'.$$

Therefore we have 4-dimensional distribution $\mathcal{H}$ generated by X, Y, X', Y' and let h the horizontal metric making X, Y, X', Y' an orthonormal basis. Let Ω be a bounded open set of $\mathbb{R}^2$, there we consider the surface Σ parametrized by

$$\begin{aligned} \Phi : \quad \Omega \quad &\longrightarrow \quad \mathbb{H}^1 \otimes \mathbb{H}^1 \\ (s,t) \quad &\rightarrow \quad (s, 0, u(s,t), 0, t, u(s,t)) \end{aligned}$$

where u is a smooth function such that $u_t(s,t) \equiv 0$. Therefore, it follows

$$\begin{aligned} \Phi_s =& (1, 0, u_s, 0, 0, u_s) = X + u_s\ Z + u_s\ Z', \\ \Phi_t =& (0, 0, 0, 0, 1, 0) = Y'. \end{aligned}$$

and we have
$$\Phi_s \wedge \Phi_t = X \wedge Y' + u_s(Z \wedge Y' + Z' \wedge Y').$$
When $u_s(s,t)$ is different from zero the degree is three. Now, we consider the following metrics
$$g_{\lambda,\nu} = \begin{pmatrix} I_4 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \nu \end{pmatrix}$$
on the basis (X, Y, X', Y', Z, Z'). These metrics extend the horizontal metric and make H_1 and H_2 spaces orthogonal. The sub-Riemannian area of Σ depends on the metric elected, indeed
$$A(\Sigma, g_{\lambda,\nu}) = \int_\Omega u_s(\lambda + \nu)\ dxdy.$$
Therefore, the PDE for the minimal surfaces obtain by the first variation of the area functional depends on the metric extension of h.

However, let $d = d(\Sigma)$ be the degree of the submanifold of dimension m. We assume that m-vector
$$v := d\Phi\left(\left.\frac{\partial}{\partial y^1}\right|_p\right) \wedge \cdots \wedge d\Phi\left(\left.\frac{\partial}{\partial y^m}\right|_p\right)$$
is expressed as the m-vector of an adapted basis, we suppose that the terms of degree d are a wedge product of $m-1$ vectors of the first layer H_1 and one vector of H_{d-m+1}. Moreover, we suppose that the H_{d-m+1} has dimension one. We will call all these assumptions the *HC hypothesis*. In the definition of this condition we have been inspired by the fact that in the Heisenberg group, in the visual cortex S and E the required assumptions are satisfied.

Here, we assume the HC hypothesis. Therefore, let $X_{i_1}, \cdots, X_{i_{m-1}}$ be elements of the basis in the first layer and X_m be the only vector of the basis in H_{d-m+1} and the terms of degree d of v are
$$\sum_{i=1}^{l} a_i X_{i_1} \wedge \cdots \wedge X_{i_{m-1}} \wedge X_m$$
If we apply the metric g_r extended to the m-vectors
$$\begin{aligned} &g_r(X_{i_1} \wedge \cdots \wedge X_{i_{m-1}} \wedge X_{i_m}, X_{i_1} \wedge \cdots \wedge X_{i_{m-1}} \wedge X_{i_m}) \\ &= \sum_{\sigma \in \mathbf{S}_m} \operatorname{sgn}(\sigma) g_r(X_{\sigma(i_1)}, X_{i_1}) \cdots g_r(X_{\sigma(i_{m-1})}, X_{i_{m-1}}) g_r(X_{\sigma(m)}, X_m) \end{aligned} \tag{36}$$
Since the metric makes the layer orthogonal the permutation σ must fix the last index m, i.e. $\sigma(m) = m$. Let g and $\bar{g}$ be two metrics such that $g_{|H} = h$, $\bar{g}_{|H} = h$, then there exists λ a positive real number such that
$$g_r(X_m, X_m) = \lambda \bar{g}_r(X_m, X_m) = \frac{\lambda}{r^{d-m}} \bar{g}(X_m, X_m).$$

Going back to (36), we have

$$(36) = \frac{\lambda}{r^{d-m}}\bar{g}(X_m, X_m) \sum_{\tau \in \mathbf{S}_{m-1}} \operatorname{sgn}(\tau) h(X_{\tau(i_1)}, X_{i_1}) \cdots h(X_{\tau(i_{m-1})}, X_{i_{m-1}})$$
$$= \lambda \bar{g}_r(X_{i_1} \wedge \cdots \wedge X_{i_{m-1}} \wedge X_{i_m}, X_{i_1} \wedge \cdots \wedge X_{i_{m-1}} \wedge X_{i_m}).$$

Hence, we have

$$A(\Phi(M), g) = \lim_{r\to 0} \quad r^{\frac{d-m}{2}} A(\Phi(M), g_r)$$
$$= \lim_{r\to 0} \quad r^{\frac{d-m}{2}} \int_M \left(\sum_{i=1}^{k} a_i^2 |X_{i_1} \wedge \cdots \wedge X_{i_{m-1}} \wedge X_m|_{g_r} \right)^{\frac{1}{2}}$$
$$= \int_M \left(\sum_{i=1}^{k} a_i^2 \; g(X_m, X_m) \sum_{\tau \in \mathbf{S}_{m-1}} \operatorname{sgn}(\tau) h(X_{\tau(i_1)}, X_{i_1}) \cdots h(X_{\tau(i_{m-1})}, X_{i_{m-1}}) \right)^{\frac{1}{2}}$$
$$= \int_M \left(\sum_{i=1}^{k} a_i^2 \; \lambda \bar{g}(X_m, X_m) \sum_{\tau \in \mathbf{S}_{m-1}} \operatorname{sgn}(\tau) h(X_{\tau(i_1)}, X_{i_1}) \cdots h(X_{\tau(i_{m-1})}, X_{i_{m-1}}) \right)^{\frac{1}{2}}$$
$$= \lambda A(\Phi(M), \bar{g}).$$

In conclusion, when the HC hypothesis holds the sub-Riemannian area definition is independent of the extension of the metric h up to a positive constant

$$A(\Phi(M)) = \lim_{r\to 0} \quad \sum_{i=m}^{d} \lambda_i \, r^{\frac{i-m}{2}} A(\Phi(M)_i, g_r). \tag{37}$$

6. Area of a hypersurface in the Heisenberg group

The Heisenberg group $\mathbb{H}^n$ is the Lie group $(\mathbb{R}^{2n+1}, *)$ where the product is defined, for any pair of points $(z,t) = (z_1, \cdots, z_n, t)$, $(z', t') = (z'_1, \cdots, z'_n, t')$ in $\mathbb{R}^{2n+1} = \mathbb{C}^{2n} \times \mathbb{R}$, by

$$(z,t) * (z', t') = \left(z + z', t + t' + \sum_{i=1}^{n} Im(z_i \bar{z}_i') \right).$$

A basis of left invariant vector fields is given by $\{X_1, \cdots, X_n, Y_1, \cdots, Y_n, T\}$, where

$$X_i = \frac{\partial}{\partial_{x_i}} + y_i \frac{\partial}{\partial_t}, \quad Y_i = \frac{\partial}{\partial_{y_i}} - x_i \frac{\partial}{\partial_t} \quad i = 1, \cdots, n, \quad T = \frac{\partial}{\partial_t}.$$

The only non-trivial relation is $[X_i, Y_i] = -2T$. The horizontal distribution at p in $\mathbb{H}^n$ is defined by $\mathcal{H}_p = span\{(X_i)_p, \ (Y_i)_p, \ i = 1, \cdots, n\}$. Let $\Omega \subset \mathbb{R}^{2n+1}$ be an open set and let Σ be a C^1 hypersurface given by the zero set level of the function

$$f(x_1, y_1, \cdots, x_n, y_n, t) = u(x_1, y_1, \cdots, x_n, y_n) - t = 0.$$

Let Σ_0 be the set of characteristic points, where the tangent hyperplane coincides with the horizontal distribution. It is given by the condition

$$\nabla_H f = (u_{x_1} - y_1, u_{y_1} + x_1, \cdots, u_{x_n} - y_n, u_{y_n} + x_n) = 0.$$

We can see the hypersurface as the embedding

$$\Phi : \mathbb{R}^{2n} \to \mathbb{H}^n, \qquad \Phi(x_1, y_1, \cdots, x_n, y_n) = (x_1, y_1, \cdots, x_n, y_n, u(x_1, y_1, \cdots, x_n, y_n)).$$

We set a metric h on the subbundle $\mathcal{H}$ such that X_i, Y_i orthonormal for each $i = 1, \cdots, n$ and we extend the metric h to g that makes X_i, Y_i, T orthonormal, then we have by (33)

$$g_r = g|_{\mathcal{H}} + \frac{1}{r} g|_T.$$

Now, we have $Z_i = \sum_{j=1}^{2n+1} B_i^j(\xi) e_j$, where we set

$$Z_1 = X_1,\ Z_2 = Y_1, \cdots, Z_{2n-1} = X_n,\ Z_{2n} = Y_n,\ Z_{2n+1} = T$$

and

$$B(\xi) = \begin{pmatrix} & & & y_1 \\ & & & -x_1 \\ & I_{2n} & & \vdots \\ & & & y_n \\ & & & -x_n \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

We write the canonical basis respect to $(Z_1, \cdots, Z_{2n+1})$ using the inverse matrix $B(\xi)$

$$e_i = \sum_{j=1}^{2n+1} (B(\xi)^{-1})_i^j Z_j, \tag{38}$$

where

$$B(\xi)^{-1} = \begin{pmatrix} & & & -y_1 \\ & & & x_1 \\ & I_{2n} & & \vdots \\ & & & -y_n \\ & & & x_n \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

We consider $\Phi_{x_i} = \sum_{j=1}^{2n+1} \Phi_{x_i}^j e_j$ and by equation (38) we obtain

$$\Phi_{x_i} = \sum_{j=1}^{2n+1} \Phi_{x_i}^j \sum_{k=1}^{2n+1} (B(\xi)^{-1})_j^k Z_k = X_i + (u_{x_i} - y_i)T,$$
$$\Phi_{y_i} = Y_i + (u_{y_i} + x_i)T.$$

In this case we have

$$\tilde{Z}^{J_{2n+1}} = Z_1 \wedge \cdots \wedge Z_{2n},$$
$$\tilde{Z}^{J_{2i-1}} = (r^{\frac{1}{2}})\ Z_1 \wedge \cdots \wedge \hat{Z}_{2i+1} \wedge \cdots \wedge Z_{2n+1},$$
$$\tilde{Z}^{J_{2i}} = (r^{\frac{1}{2}})\ Z_1 \wedge \cdots \wedge \hat{Z}_{2i} \wedge \cdots \wedge Z_{2n+1}.$$

Then, we make the wedge product

$$\begin{aligned} v :=& \Phi_{x_i} \wedge \Phi_{y_1} \wedge \cdots \wedge \Phi_{x_n} \wedge \Phi_{y_n} \\ =& X_1 \wedge \cdots \wedge Y_n \\ &+ \sum_{i=1}^{n} (u_{x_i} - y_i) X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge Y_n \wedge T \\ &+ \sum_{i=1}^{n} (u_{y_i} + x_i) X_1 \wedge \cdots \wedge \hat{Y}_i \wedge \cdots \wedge Y_n \wedge T \\ =& \tilde{Z}^{J_{2n+1}} + \sum_{i=1}^{n} (u_{x_i} - y_i)\ r^{-\frac{1}{2}} \tilde{Z}^{J_{2i-1}} + \sum_{i=1}^{n} (u_{y_i} + x_i)\ r^{-\frac{1}{2}} \tilde{Z}^{J_{2i}} \end{aligned}$$

Therefore,

$$g_r(v, v) = 1 + \sum_{i=1}^{n} (u_{x_i} - y_i)^2\ r^{-1} + \sum_{i=1}^{n} (u_{y_i} + x_i)^2\ r^{-1}$$

and the Riemaniann area is

$$area(\Sigma \setminus \Sigma_0, g_r) = \int_{\Omega} \left(1 + \sum_{i=1}^{n} (u_{x_i} - y_i)^2\ r^{-1} + \sum_{i=1}^{n} (u_{y_i} + x_i)^2\ r^{-1} \right)^{\frac{1}{2}} d\lambda$$

where $d\lambda = dx_1 dy_1 \cdots dx_n dy_n$ is the Lebesgue measure.

Notice that in $\Sigma \setminus \Sigma_0$ the degree is equal to $2n + 1$ and the dimension of the hypersurface is equal to $2n$ thus, thanks to the definition (35) of sub-Riemannian area, we have

$$A(\Sigma \setminus \Sigma_0) = \lim_{r\to 0} \int_\Omega r^{\frac{2n+1-2n}{2}} \left(1 + \sum_{i=1}^{n} (u_{x_i} - y_i)^2 \ r^{-1} + \sum_{i=1}^{n} (u_{y_i} + x_i)^2 \ r^{-1}\right)^{\frac{1}{2}} d\lambda$$

$$= \lim_{r\to 0} \int_\Omega \left(r + \sum_{i=1}^{n} (u_{x_i} - y_i)^2 \ + \sum_{i=1}^{n} (u_{y_i} + x_i)^2\right)^{\frac{1}{2}} d\lambda$$

$$= \int_\Omega \left(\sum_{i=1}^{n} (u_{x_i} - y_i)^2 \ + \sum_{i=1}^{n} (u_{y_i} + x_i)^2\right)^{\frac{1}{2}} d\lambda$$

7. Area of a surface in a sub-Riemannian manifold

We are interested in surfaces in a sub-Riemaniann manifold N, therefore in this section we will show all the terms defined in section 4 for a surface. Obviously, $(N, \mathcal{H}, h)$ satisfies Hörmander rank condition and locally

$$\mathcal{H} = \text{span}\{X_1, \cdots, X_k\}.$$

By Hörmander rank condition we consider a local frame $X_1, \cdots, X_n$ where X_{k+1}, $\cdots$, X_n are commutators of $X_1, \cdots, X_k$. As we show in section 4 the tangent space is

$$TN = H_1 \oplus \cdots \oplus H_s$$

Let Ω be a subset of $\mathbb{R}^2$, we take into account the surface Σ parametrized by

$$\Phi : \Omega \to N.$$

Let g be the metric that extends h and makes the layers orthogonal. Furthermore, let g_r be the metric defined in (33), we will consider the 2-vector fields

$$\tilde{X}_i \wedge \tilde{X}_j = \left(r^{\frac{d(X_i)-1}{2}} X_i\right) \wedge \left(r^{\frac{d(X_j)-1}{2}} X_j\right) \qquad i, j = 1, \cdots, n \quad i \neq j$$

and we express

$$\Phi_x \wedge \Phi_y = \sum_{1\leqslant i<j\leqslant n} \tilde{\tau_{ij}} \ \tilde{X}_i \wedge \tilde{X}_j$$

with respect to the basis $\{\tilde{X}_i \wedge \tilde{X}_j \ : \quad 1 \leqslant i < j \leqslant n\}$. Let

$$(g_{ij}) = \begin{pmatrix} g_r(\Phi_x, \Phi_x) & g_r(\Phi_x, \Phi_y) \\ g_r(\Phi_x, \Phi_y) & g_r(\Phi_y, \Phi_y) \end{pmatrix}.$$

be the metric matrix of g_r which induces on the 2-vector the new metric

$$g_r(X \wedge Y, Z \wedge T) = g_r(X, Z) g_r(Y, T) - g_r(X, T) g_r(Y, Z).$$

Thanks to this definition, we have

$$g_r(\Phi_x \wedge \Phi_y, \Phi_x \wedge \Phi_y) = g_r(\Phi_x, \Phi_x) g_r(\Phi_y, \Phi_y) - g_r(\Phi_x, \Phi_y) g_r(\Phi_x, \Phi_y) = \det(g_{ij}).$$

Hence,

$$A(\Sigma, g_r) = \int_\Omega \sqrt{g_r(\Phi_x \wedge \Phi_y, \Phi_x \wedge \Phi_y)}\, dxdy.$$

Setting $\Sigma_i = \{p \in \Sigma \ : \ d_\Sigma(p) = i\}$ the surface Σ can be written as

$$\Sigma = \Sigma_2 \cup \cdots \cup \Sigma_d, \qquad \text{where } d = d(\Sigma) \quad \text{is the degree of the surface.}$$

Finally, the sub-Riemannian area of Σ is

$$A(\Sigma, g) = \lim_{r \to 0} \sum_{i=2}^{d} \int_{\Phi^{-1}(\Sigma_i)} r^{\frac{i-2}{2}} \sqrt{g_r(\Phi_x \wedge \Phi_y, \Phi_x \wedge \Phi_y)}\, dxdy. \tag{39}$$

8. Sub-Riemannian area of a surface in S

Let $H_1 = \mathcal{H}$ be the subbundle of the tangent bundle TS generated by

$$X = \cos(\theta)\partial_x + \sin(\theta)\partial_y \quad \text{and} \quad Y = \partial_\theta.$$

The tangent space can be written as

$$TS = H_1 \oplus H_2$$

where

$$H_2 = \operatorname{span}\{T = \sin(\theta)\partial_x - \cos(\theta)\partial_y\}.$$

We shall equip S with a sub-Riemannian metric h defined on its horizontal distribution, such that X, Y are an orthonormal basis for this metric. Thus, we extend the metric h with a Riemaniann metric g such that $g_{|H_0} = h$ and H_2 is orthogonal to H_1. By definition (33), it follows

$$g_r = g_{|H_1} + \frac{1}{r} g_{|H_2}.$$

Now, let Ω be an open set of $\mathbb{R}^2$. We consider a f-graph parametrized by

$$\begin{array}{cccc} \Phi : & \Omega & \longrightarrow & S \\ & (x, y) & \mapsto & (x, y, f(x, y)) \end{array}$$

and its tangent vectors

$$\begin{aligned} \Phi_x &= \partial_x + f_x\partial_\theta = \cos(\theta)X + \sin(\theta)T + f_xY \\ \Phi_y &= \partial_y + f_y\partial_\theta = \sin(\theta)X - \cos(\theta)T + f_yY. \end{aligned}$$

Therefore, the wedge product of these tangent vectors is

$$\Phi_x \wedge \Phi_y = -X \wedge T - T(f)X \wedge Y + X(f)T \wedge Y.$$

The degree of this surface is three and it can not change due to the coefficient of $X \wedge T$ never vanishes , hence the sub-Riemaniann area is

$$A(\Phi(\Omega)) = \int_\Omega \left(1 + X(f)^2\right)^{\frac{1}{2}}\, dxdy.$$

9. Geometry of surfaces in 4-dimensional sub-Riemannian manifolds

Our aim in this section is to introduce a suitable area functional for 2 - dimensional surfaces embedded in two different 4-dimensional sub-Riemannian manifolds E and $\tilde{E}$ defined in the first chapter section 3.4. We are interested in surfaces that are union of lifted curves of the level set of an image on the retinal plane to the space E or $\tilde{E}$. Therefore, we have to study the geometry of these particular surface, their foliation properties and their degree, so that we can provide a suitable area functional. Let us remind that

$$(E = \mathbb{R}^2 \times S^1 \times \mathbb{R}, \mathcal{H}_E, h_i), \qquad (\tilde{E} = \mathbb{R}^2 \times S^1 \times S^1, \mathcal{H}_{\tilde{E}}, h_3)$$

where we have

$$\mathcal{H}_E = \mathrm{span}\{X_1, X_2\}, \qquad \mathcal{H}_{\tilde{E}} = \mathrm{span}\{Z_1, Z_2\}.$$

The tangent bundles can be written as

$$TE = H_1 \oplus H_2 \oplus H_3, \qquad T\tilde{E} = H_1 \oplus H_2 \oplus H_3$$

where $H_2 = \mathrm{span}\{X_3\}$ (respectively $H_2 = \mathrm{span}\{Z_3\}$) and $H_3 = \mathrm{span}\{X_4\}$ (respectively $H_2 = \mathrm{span}\{Z_3\}$).

9.1. Degree of a surface. Let Σ be a 2-dimensional submanifold in E. The degree of a 2-vector

$$\tau = \sum_{1 \leqslant i < j \leqslant 4} \tau_{ij} X_i \wedge X_j \in \Lambda_2(E)$$

is given by

$$d(\tau) = \max\{d_i + d_j \mid \tau_{ij} \neq 0\}$$

where d_i is the degree of X_i, hence $d_1 = d_2 = 1, \quad d_3 = 2$ and $d_4 = 3$. Then define the pointwise degree at x of a 2-dimensional submanifold Σ in E by

$$d_\Sigma(x) = d(\tau_\Sigma(x))$$

where $\tau_\Sigma(x)$ is a 2-vector of Σ at $x \in \Sigma$, in other words

$$\tau_\Sigma(x) \in \{v_1 \wedge v_2 \ : \ \forall \ (v_1, v_2) \text{ basis of } T_x\Sigma\}.$$

Let (e_1, e_2) and (v_1, v_2) be two basis of $T_x\Sigma$ then there exists $\lambda \neq 0$ such that

$$e_1 \wedge e_2 = \lambda(v_1 \wedge v_2).$$

Hence, the definition of degree is independent of the choice of the basis. The degree $d(\Sigma)$ of Σ is the integer

$$\max_{x \in \Sigma} d_\Sigma(x).$$

In E, we have $X_i = \sum_{j=1}^{4} A_i^j(\xi) e_j$, where

$$A(\xi) = \begin{pmatrix} \cos(\theta) & \sin(\theta) & k & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \end{pmatrix}.$$

We can write the canonical basis respect to $X_1, \cdots, X_4$

$$e_i = \sum_{j=1}^{4} (A(\xi)^{-1})_i^j X_j, \tag{40}$$

where

$$A(\xi)^{-1} = \begin{pmatrix} \cos(\theta) & 0 & k\cos(\theta) & -\sin(\theta) \\ \sin(\theta) & 0 & k\sin(\theta) & \cos(\theta) \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

We set $(u_1, u_2) = (x, y)$ and $\Phi = (\Phi^1, \Phi^2, \Phi^3, \Phi^4)$. We consider $\Phi_{u_i} = \sum_{j=1}^{4} \Phi_{u_i}^j e_j$ and by equation (40) we obtain

$$\begin{aligned} \Phi_{u_i} =& \sum_{j=1}^{4} \Phi_{u_i}^j \sum_{k=1}^{4} (A(\xi)^{-1})_j^k X_k \\ =& \Phi_{u_i}^1 (\cos(\theta) X_1 + k\cos(\theta) X_3 - \sin(\theta) X_4) \\ &+ \Phi_{u_i}^2 (\sin(\theta) X_1 + k\sin(\theta) X_3 + \cos(\theta) X_4) \\ &- \Phi_{u_i}^3 X_3 + \Phi_{u_i}^4 X_2 \\ =& (\cos(\Phi^3)\Phi_{u_i}^1 + \sin(\Phi^3)\Phi_{u_i}^2) X_1 + \Phi_{u_i}^4 X_2 + (\Phi^4(\cos(\Phi^3)\Phi_{u_i}^1 + \sin(\Phi^3)\Phi_{u_i}^2) \\ &- \Phi_{u_i}^3) X_3 + (-\sin(\Phi^3)\Phi_{u_i}^1 + \cos(\Phi^3)\Phi_{u_i}^2) X_4. \end{aligned}$$

Computing the wedge product, it follows

$$\begin{aligned} \Phi_x \wedge \Phi_y =& (\cos(\Phi^3)\Phi_u^{14} + \sin(\Phi^3)\Phi_u^{24}) X_1 \wedge X_2 \\ &- (\cos(\Phi^3)\Phi_u^{13} + \sin(\Phi^3)\Phi_u^{23}) X_1 \wedge X_3 \\ &+ \Phi_u^{12} X_1 \wedge X_4 \\ &+ (\Phi_u^{34} - \Phi^4(\cos(\Phi^3)\Phi_u^{14} + \sin(\Phi^3)\Phi_u^{24})) X_2 \wedge X_3 \\ &+ (\sin(\Phi^3)\Phi_u^{14} - \cos(\Phi^3)\Phi_u^{24}) X_2 \wedge X_4 \\ &+ (\Phi^4\Phi_u^{12} - \sin(\Phi^3)\Phi_u^{13} + \cos(\Phi^3)\Phi_u^{23}) X_3 \wedge X_4, \end{aligned} \tag{41}$$

where we set

$$\Phi_u^{ij} = \det \begin{pmatrix} \Phi_x^i & \Phi_y^i \\ \Phi_x^j & \Phi_y^j \end{pmatrix}.$$

According to the notion of pointwise degree, we have that

(42)
$$d_\Sigma(\Phi(u)) = \begin{cases} 5 & \text{if } c_{34}(u) \neq 0 \\ 4 & \text{if } |c_{14}(u)| + |c_{24}(u)| > 0 \quad \text{and} \quad c_{34}(u) = 0 \\ 3 & \text{if } |c_{13}(u)| + |c_{23}(u)| > 0 \quad \text{and} \quad c_{34}(u) = c_{14}(u) = c_{24}(u) = 0 \\ 2 & \text{if } c_{34}(u) = c_{14}(u) = c_{24}(u) = c_{13}(u) = c_{23}(u) = 0 \end{cases}$$

where we set
$$\Phi_{u_1} \wedge \Phi_{u_2} = \sum_{1 \leqslant i < j \leqslant 4} c_{ij}(u) X_i \wedge X_j.$$
Notice that the degree of Σ can never be equal to 2. Indeed, if d_Σ was equal to 2 the submanifold Σ would be a integrable manifold for the distribution $\mathcal{H}$, then $\mathcal{H}$ would be involutive by Frobenius Theorem. However, the distribution $\mathcal{H}$ is bracket-generating and not involutive.

Now, we want to study the degree for a submanifold Σ in $\tilde{E}$. Evidently, each Z_i has the same degree of X_i and let $\Phi = (\Phi^1, \Phi^2, \Phi^3, \Phi^4)$ be a parametrization of Σ. We set $(x, y) = (u_1, u_2)$ and we consider
$$\Phi_{u_i} = \sum_{j=1}^{4} \Phi^j_{u_i} e_j$$
As we did before, we can express the canonical basis respect to $Z_1, \cdots, Z_4$

$$e_i = \sum_{j=1}^{4} (A(\xi)^{-1})^j_i Z_j, \tag{43}$$

where
$$A(\xi) = \begin{pmatrix} \cos(\varphi)\cos(\theta) & \cos(\varphi)\sin(\theta) & \sin(\varphi) & 0 \\ 0 & 0 & 0 & 1 \\ \sin(\varphi)\cos(\theta) & \sin(\varphi)\sin(\theta) & -\cos(\varphi) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \end{pmatrix},$$
$$A(\xi)^{-1} = \begin{pmatrix} \cos(\varphi)\cos(\theta) & 0 & \sin(\varphi)\cos(\theta) & -\sin(\theta) \\ \cos(\varphi)\sin(\theta) & 0 & \sin(\varphi)\sin(\theta) & \cos(\theta) \\ \sin(\varphi) & 0 & -\cos(\varphi) & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$
Therefore, we have
$$\begin{aligned} \Phi_{u_i} &= \sum_{j=1}^{4} \Phi^j_{u_i} e_j \\ &= \left(\cos(\Phi^4)\cos(\Phi^3)\ \Phi^1_{u_i} + \cos(\Phi^4)\sin(\Phi^3)\ \Phi^2_{u_i} + \sin(\Phi^4)\ \Phi^3_{u_i}\right)\ Z_1 \\ &\quad + \Phi^4_{u_i}\ Z_2 + \left(\sin(\Phi^4)\cos(\Phi^3)\Phi^1_{u_i} + \sin(\Phi^4)\sin(\Phi^3)\Phi^2_{u_i} - \cos(\Phi^4)\Phi^3_{u_i}\right)\ Z_3 \\ &\quad + \left(-\sin(\Phi^3)\Phi^1_{u_i} + \cos(\Phi^3)\Phi^2_{u_i}\right) Z_4. \end{aligned}$$

Now, we make the wedge product

(44)

$$\begin{aligned}\Phi_x \wedge \Phi_y = & \left(\cos(\Phi^4)\cos(\Phi^3)\Phi_u^{14} + \cos(\Phi^4)\sin(\Phi^3)\Phi_u^{24} + \sin(\Phi^4)\Phi_u^{34}\right) Z_1 \wedge Z_2 \\ & - \left(\cos(\Phi^3)\Phi_u^{13} + \sin(\Phi^3)\Phi_u^{23}\right) Z_1 \wedge Z_3 \\ & + \left(\cos(\Phi^4)\Phi_u^{12} + \sin(\Phi^4)\sin(\Phi^3)\Phi_u^{13} - \sin(\Phi^4)\cos(\Phi^3)\Phi_u^{23}\right) Z_1 \wedge Z_4 \\ & - \left(\sin(\Phi^4)\cos(\Phi^3)\Phi_u^{14} + \sin(\Phi^4)\sin(\Phi^3)\Phi_u^{24} - \cos(\Phi^4)\Phi_u^{34}\right) Z_2 \wedge Z_3 \\ & + \left(\sin(\Phi^3)\Phi_u^{14} - \cos(\Phi^3)\Phi_u^{24}\right) Z_2 \wedge Z_4 \\ & + \left(\sin(\Phi^4)\Phi_u^{12} - \cos(\Phi^4)\sin(\Phi^3)\Phi_u^{13} + \cos(\Phi^4)\cos(\Phi^3)\Phi_u^{23}\right) Z_3 \wedge Z_4.\end{aligned}$$

The argument (42) holds also in this case.

9.2. Structure of surfaces. Let Σ be a surface of E. It is useful to define the set of *characteristic points* as

$$\Sigma_0 = \{p \in \Sigma \ : \ T_p\Sigma = \mathcal{H}_p\},$$

where $\mathcal{H}_p$ is the horizontal plane generated by the vectors $X_1(p)$ and $X_2(p)$. Now we consider the surface $\Sigma = \{(x, y, \theta(x, y), k(x, y))\}$ and we are interested in the intersection between the tangent space of the surface $T_p\Sigma$ and the horizontal plane $\mathcal{H}_p$. With this parametrization the tangent vectors are

$$\Phi_x = (1, 0, \theta_x, k_x) \qquad \Phi_y = (0, 1, \theta_y, k_y)$$

and the horizontal plane is spanned by

$$X_1 = cos(\theta)\partial_x + sen(\theta)\partial_y + k\partial_\theta, \qquad X_2 = \partial_k.$$

In order to know $T_p\Sigma \cap \mathcal{H}_p$ it is necessary to take in account the rank of the matrix

$$B = \begin{pmatrix} 1 & 0 & \theta_x & k_x \\ 0 & 1 & \theta_y & k_y \\ \cos(\theta) & \sin(\theta) & k & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{45}$$

Obvious rank$(B) \geqslant 3$, indeed

$$\det \begin{pmatrix} 1 & 0 & k_x \\ 0 & 1 & k_y \\ 0 & 0 & 1 \end{pmatrix} \neq 0.$$

The surface would have characteristic points if the vectors Φ_x and Φ_y were a linear combination of X_1 and X_2, i.e. rank$(B) = 2$. Hence, this surface Σ does not have characteristic points, i.e. $\Sigma_0 = \emptyset$. Moreover,

$$\text{rank}(B) = 3 \quad \Leftrightarrow \quad \det \begin{pmatrix} \cos(\theta) & \sin(\theta) & k \\ 1 & 0 & \theta_x \\ 0 & 1 & \theta_y \end{pmatrix} = 0 \tag{46}$$

$$\Leftrightarrow \quad \boxed{k - \theta_x \cos(\theta) - \theta_y \sin(\theta) = 0.} \tag{47}$$

Let us apply the same argument to a surface in the sub-Riemannian manifold $\tilde{E}$. Here, we consider the surface $\Sigma = \{(x, y, \theta(x, y), \varphi(x, y))\}$ and its tangent vectors

$$\Phi_x = (1, 0, \theta_x, \varphi_x), \qquad \Phi_y = (0, 1, \theta_y, \varphi_y).$$

Now, in order to know $T_p\Sigma \cap \mathcal{H}_p$ it is necessary to take in account the rank of the matrix

$$B = \begin{pmatrix} 1 & 0 & \theta_x & \varphi_x \\ 0 & 1 & \theta_y & \varphi_y \\ \cos(\varphi)\cos(\theta) & \cos(\varphi)\sin(\theta) & \sin(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{48}$$

Clearly, the rank$(B) \geqslant 3$, indeed

$$\det \begin{pmatrix} 1 & 0 & \varphi_x \\ 0 & 1 & \varphi_y \\ 0 & 0 & 1 \end{pmatrix} \neq 0.$$

Therefore, the surface does not have characteristic points and has a foliation property if and only if

$$\text{rank}(B) = 3 \quad \Leftrightarrow \quad \det \begin{pmatrix} 1 & 0 & \theta_x \\ 0 & 1 & \theta_y \\ \cos(\varphi)\cos(\theta) & \cos(\varphi)\sin(\theta) & \sin(\varphi) \end{pmatrix} = 0 \tag{49}$$

$$\Leftrightarrow \quad \boxed{\sin(\varphi) - \cos(\varphi)\,(\theta_x \cos(\theta) - \theta_y \sin(\theta)) = 0.} \tag{50}$$

9.3. Sub-Riemannian area of a surface. Let Σ be the surface parametrized by $\Phi(x, y) = (x, y, \theta(x, y), k(x, y))$. Thanks to (41), we have

(51)

$$\begin{aligned} \Phi_x \wedge \Phi_y = &(\cos(\theta)k_y - \sin(\theta)k_x)X_1 \wedge X_2 - (\cos(\theta)\theta_y - \sin(\theta)\theta_x)X_1 \wedge X_3 + \\ &+ X_1 \wedge X_4 + (\theta_x k_y - \theta_y k_x - k(\cos(\theta)k_y - \sin(\theta)k_x))X_2 \wedge X_3 \\ &+ (\sin(\theta)k_y + \cos(\theta)k_x)X_2 \wedge X_4 + (k - \sin(\theta)\theta_y - \cos(\theta)\theta_x)X_3 \wedge X_4. \end{aligned}$$

By the foliation condition (47),

$$k - \theta_x \cos(\theta) - \theta_y \sin(\theta) = 0 \quad \Leftrightarrow \quad X_1(\theta) = k$$

we have that $d(\Sigma) \leqslant 4$. Moreover, the coefficient of $X_1 \wedge X_4$ never vanishes, therefore $d(\Sigma) = 4$. Let Ω a subset $\mathbb{R}^2$, let $\Phi : \Omega \to E$ be a parametrization of Σ.

Now we use the tools developed in this chapter to define the area of Σ. We set the metric g such that

$$g(X_i, X_i) = 1 \quad i = 1, \cdots, 4 \quad \text{and} \quad g(X_i, X_j) = 0 \quad i \neq j.$$

Therefore, here we equip the distribution $\mathcal{H}$ on E with the metric h_1, later we will consider also the metric h_2. Now, let $r > 0$ be a real number and we will consider the Riemannian metrics

$$g_r = g_{|H_1} + \frac{1}{r} g_{|H_2} + \frac{1}{r^2} g_{|H_3}.$$

An orthonormal basis for g_r is

$$\tilde{X}_1 = X_1, \quad \tilde{X}_2 = X_2, \quad \tilde{X}_3 = \sqrt{r} X_3, \quad \tilde{X}_4 = r X_4$$

Taking into account the surface

$$\Sigma = \{(x, y, \theta, k) \ : \ \theta(x, y) - \theta = 0, \quad k - k(x, y) = 0)\}$$

and setting

$$\tilde{X}_i \wedge \tilde{X}_j = \left(r^{\frac{d(X_i)-1}{2}} X_i \right) \wedge \left(r^{\frac{d(X_j)-1}{2}} X_j \right)$$

we express the 2-vector (51) respect to basis $(\tilde{X}_1, \cdots, \tilde{X}_4)$

(52)

$$\begin{aligned} \Phi_x \wedge \Phi_y =& (\cos(\theta) k_y - \sin(\theta) k_x) \tilde{X}_1 \wedge \tilde{X}_2 - r^{-\frac{1}{2}} (\cos(\theta)\theta_y - \sin(\theta)\theta_x) \tilde{X}_1 \wedge \tilde{X}_3 \\ & + r^{-1} \tilde{X}_1 \wedge \tilde{X}_4 + r^{-\frac{1}{2}} (\theta_x k_y - \theta_y k_x - k(\cos(\theta) k_y - \sin(\theta) k_x)) \tilde{X}_2 \wedge \tilde{X}_3 \\ & + r^{-1} (\sin(\theta) k_y + \cos(\theta) k_x) \tilde{X}_2 \wedge \tilde{X}_4 \\ & + r^{-\frac{3}{2}} (k - \sin(\theta)\theta_y - \cos(\theta)\theta_x) \tilde{X}_3 \wedge \tilde{X}_4. \end{aligned}$$

Then we take into account the Jacobian matrix

$$(g_{ij}) = \begin{pmatrix} g_r(\Phi_x, \Phi_x) & g_r(\Phi_x, \Phi_y) \\ g_r(\Phi_x, \Phi_y) & g_r(\Phi_y, \Phi_y) \end{pmatrix}.$$

In this case the metric on the 2-vector induced by the metric g_r is

$$g_r(X \wedge Y, Z \wedge T) = g_r(X, Z) g_r(Y, T) - g_r(X, T) g_r(Y, Z).$$

We have taken into account the definition (34), where there are only even and odd permutations. Thanks to this definition, we have

$$g_r(\Phi_x \wedge \Phi_y, \Phi_x \wedge \Phi_y) = g_r(\Phi_x, \Phi_x) g_r(\Phi_y, \Phi_y) - g_r(\Phi_x, \Phi_y) g_r(\Phi_x, \Phi_y) = \det(g_{ij}).$$

Hence, it follows

$$|\Phi_x \wedge \Phi_y|^2 = g_r(\Phi_x \wedge \Phi_y, \Phi_x \wedge \Phi_y) = \sum c_{ij}^2 \ r^{-d(X_i \wedge X_j)+2}.$$

The area element is

$$d\Sigma_r = \left(\sum c_{ij}^2 \ r^{-d(X_i \wedge X_j)+2} \right)^{\frac{1}{2}} \ dxdy,$$

therefore, the area of the surface Σ with the metric g_r is

$$A(\Sigma, g_r) = \int_\Omega \left(\sum c_{ij}^2 \; r^{-d(X_i \wedge X_j)+2}\right)^{\frac{1}{2}} \; dxdy.$$

Now by the definition of the area of a surface in the a sub-Riemannian manifold (39), it follows

$$\begin{aligned} A(\Sigma) &= \lim_{r \to 0} \quad r^{\frac{d(\Sigma)-2}{2}} A(\Sigma, g_r) \\ &= \lim_{r \to 0} \quad r^{\frac{d(\Sigma)-2}{2}} \int_\Omega r^{-\frac{d(\Sigma)+2}{2}} \left(\sum c_{ij}^2 \; r^{-d(X_i \wedge X_j)+2+d(\Sigma)-2}\right)^{\frac{1}{2}} \; dxdy \\ &= \lim_{r \to 0} \quad \int_\Omega \left(\sum c_{ij}^2 \; r^{-d(X_i \wedge X_j)+d(\Sigma)}\right)^{\frac{1}{2}} \; dxdy. \end{aligned}$$

If $|r| < 1$ and the functions c_{ij} are in $L^2(\Omega)$ it is possible to apply dominated convergence theorem

$$\begin{aligned} A(\Sigma) &= \int_\Omega \lim_{r \to 0} \quad \left(\sum c_{ij}^2 \; r^{-d(X_i \wedge X_j)+d(\Sigma)}\right)^{\frac{1}{2}} \; dxdy \\ &= \int_\Omega \left(c_{14}^2 + c_{24}^2\right)^{\frac{1}{2}} \; dxdy = \int_\Omega (1 + (X_1 k)^2)^{\frac{1}{2}} \; dxdy \\ &= \int_\Omega (1 + (X_1^2 \theta)^2)^{\frac{1}{2}} \; dxdy. \end{aligned}$$

Let us apply the same arguments to a surface Σ in $\tilde{E}$, parametrized by

$$\Phi(x, y) = (x, y, \theta(x, y), \varphi(x, y)).$$

By equation (44), it follows

$$\begin{aligned} \Phi_x \wedge \Phi_y = &\left(\cos(\varphi)\cos(\theta)\varphi_y - \cos(\varphi)\sin(\theta)\varphi_x + \sin(\varphi)\Phi_u^{34}\right) Z_1 \wedge Z_2 \\ &- \left(\cos(\theta)\theta_y - \sin(\theta)\theta_x\right) Z_1 \wedge Z_3 \\ &+ \left(\cos(\varphi) + \sin(\varphi)\sin(\theta)\theta_y + \sin(\varphi)\cos(\theta)\theta_x\right) Z_1 \wedge Z_4 \\ &- \left(\sin(\varphi)\cos(\theta)\varphi_y - \sin(\varphi)\sin(\theta)\varphi_x - \cos(\varphi)\Phi_u^{34}\right) Z_2 \wedge Z_3 \\ &+ \left(\sin(\theta)\varphi_y + \cos(\theta)\varphi_x\right) Z_2 \wedge Z_4 \\ &+ \left(\sin(\varphi) - \cos(\varphi)\sin(\theta)\theta_y - \cos(\varphi)\cos(\theta)\theta_x\right) Z_3 \wedge Z_4 \end{aligned} \tag{53}$$

By the foliation property (50)

$$\sin(\varphi) - \cos(\varphi)\left(\theta_x \cos(\theta) - \theta_y \sin(\theta)\right) = 0 \quad \Leftrightarrow \quad \tan(\varphi) = X(\theta)$$

the degree of Σ, $d(\Sigma) \leqslant 4$. Moreover, taking into account the coefficient of $Z_1 \wedge Z_4$ we notice that it does not vanish everywhere. Indeed, we have

$$\cos(\varphi) + \sin(\varphi) X(\theta) = \frac{\cos(\varphi)^2 + \sin(\varphi)^2}{\cos(\varphi)} = \frac{1}{\cos(\varphi)} \neq 0.$$

Hence, assuming the foliation condition the surface Σ has degree four, $d(\Sigma) = 4$. In conclusion, the area of a surface $\Sigma = \Phi(\Omega)$ in $\tilde{E}$ is

(54)
$$A(\Sigma) = \int_\Omega \left(\frac{1}{\cos(\varphi)^2} + (\cos(\theta)\varphi_x + \sin(\theta)\varphi_y)^2 \right)^{\frac{1}{2}} \, dxdy = \int_\Omega \frac{(1 + Z_1(\varphi)^2)^{\frac{1}{2}}}{\cos(\varphi)} \, dxdy.$$

Indeed, we consider only terms that have degree four which survive when we let the metric g_r blows up.

9.4. Sub-Riemannian area of a surface in E with an arbitrary horizontal metric. Now, we want to deduce the sub-Riemannian area formula for a (θ,κ)-graph embedded in E equipped with an arbitrary metric h on the horizontal subbudle

$$(h_{ij})(p) = \begin{pmatrix} h(X_1, X_1)(p) & h(X_1, X_2)(p) \\ h(X_2, X_1)(p) & h(X_2, X_2)(p) \end{pmatrix}, \tag{55}$$

where $p = (x, y, \theta(x, y), \kappa(x, y))$. We will use the same notation of the previous section, therefore we have

$$TE = H_1 \oplus H_2 \oplus H_3.$$

We extend the metric h to a metric g such that

$$\begin{aligned} &g(v, w) = 0 \quad v \in H_1, \quad w \in H_2 \oplus H_3, \\ &g(X_i, X_j) = 0 \quad 3 \leqslant i < j \leqslant 4, \quad g(X_3, X_3) = g(X_4, X_4) = 1. \end{aligned}$$

In this case the HC hypothesis holds, then, by the section 5, we know that the definition of sub-Riemannian area is independent of the metric extension up to a constant, see equation (37). Here, we consider the equation (52), the only different from the previous case are the follows terms

$$\begin{aligned} g_r(\tilde{X}_1 \wedge \tilde{X}_4, \tilde{X}_1 \wedge \tilde{X}_4) &= g(X_1, X_1) g(X_4, X_4) - g(X_1, X_4)^2 = h(X_1, X_1) \\ g_r(\tilde{X}_2 \wedge \tilde{X}_4, \tilde{X}_2 \wedge \tilde{X}_4) &= g(X_2, X_2) g(X_4, X_4) - g(X_2, X_4)^2 = h(X_2, X_2) \end{aligned}$$

Therefore, we have

$$A(\Sigma) = \lambda \int_\Omega \left(h(X_1, X_1) + h(X_2, X_2) \; X_1(k))^2 \right)^{\frac{1}{2}} \, dxdy. \tag{56}$$

Hence, when we consider the horizontal metric h_2, induced by the euclidean metric, the area functional will be

$$A(\Sigma) = \int_\Omega (1 + k^2 + X_1(k)^2)^{\frac{1}{2}} \, dxdy. \tag{57}$$

10. Hausdorff dimension

Our goal in this section is to prove that Hausdorff dimension of a submanifold embedded in an equiregular sub-Riemannian manifold is equal to its degree. This result was speculated by Gromov in [**19**, O.6.B], but it hasn't been proved yet. Ghezzi and Jean in [**18**] proved the result only for a strongly equiregular submanifold, where intersection of tangent space and each layer H_i has constant dimension for each point of the submanifold. First of all we undertake an alternative way to show Ghezzi and Jean's result. In Section 10.4 we remove strongly equiregular hypothesis and will prove Gromov's conjecture.

10.1. Hausdorff measure. Here we report some relevant definitions and theorems about the Hausdorff theory of measure, for a complete dissertation see [**3**].

DEFINITION 2.7. Let (X, d) be a metric space and α be a real number. Let $\{S_i\}_{i\in I}$ be a countable covering of X, i.e. a collection of sets such that $X \subset \bigcup S_i$, we define its α-weight $w_\alpha(\{S_i\})$ by the formula

$$w_\alpha(\{S_i\}) = \sum_{i\in I} (\text{diam } S_i)^\alpha.$$

For an $\varepsilon > 0$ define $H^\alpha_\varepsilon(X)$ by

$$H^\alpha_\varepsilon(X) = \inf\{w_\alpha(\{S_i\}) : \text{diam } S_i < \varepsilon \text{ for all } i\}$$

where the infimum is taken over all countable coverings of X by sets of diameter smaller that ε. If there is not a covering set the infimum is $+\infty$. Now, the α-*dimensional Hausdorff measure* of X is defined by the formula

$$H^\alpha(X) = C(\alpha) \lim_{\varepsilon\to 0} H^\alpha_\varepsilon(X)$$

where $C(\alpha)$ is a positive normalization constant. We set $H^\alpha(\emptyset) = 0$. Since we estimate a infimum over a smaller set we have $H^\alpha_{\varepsilon_1}(X) \geqslant H^\alpha_{\varepsilon_2}(X)$ when $\varepsilon_1 < \varepsilon_2$. Therefore, $H^\alpha(X)$ can be a nonnegative real number or $+\infty$.

THEOREM 2.1. *For a metric space X there exists a* $\alpha_0 \in [0, +\infty]$ *such that* $H^\alpha(X) = 0$ *for all* $\alpha > \alpha_0$ *and* $H^\alpha(X) = +\infty$ *for all* $\alpha < \alpha_0$ *.*

DEFINITION 2.8. The value α_0 from Theorem 2.1 is called the *Hausdorff dimension* of X and denoted by dim_H(X)

In order to define the α-dimensional spherical Hausdorff measure $\mathcal{S}^\alpha(X)$ we provide the same definition of the α-dimensional Hausdorff measure, but all possible countable coverings are $\{B(x_i, r_i)\}_{i\in I}$ instead of general sets S_i. Here $B(x_i, r_i)$

denotes a ball respect to the metric d centered in x_i. Moreover, we will consider a subset of a metric space that is a metric space with the restricted metric.

10.2. Ball-Box Theorem. Let us remind that a ball of radius ε in sub-Riemannian manifold N is defined as

$$B(p,\varepsilon) = \{q \in N \ : \ d_c(p,q) < \varepsilon\},$$

where d_c denotes the Carnot-Carathéodory distance. In order to understand Ball-Box Theorem we have to remind what are adapted coordinates and boxes.

DEFINITION 2.9. Coordinates $y_1, \cdots, y_n$ centered in p are said to be *linearly adapted* to the distribution $\mathcal{H}$ at p if $\mathcal{H}^i(p)$ is annihilated by the differentials $dy_{n_i+1}, \cdots, dy_n$ at p, where $n_i = n_i(p)$ are components of growth vector at p. We define the w-weighted box at p of size ε as

$$\mathrm{Box}^w(0,\varepsilon) = \{y \in \mathbb{R}^n \ : \ |y_i| < \varepsilon^{w_i}, \ i = 1, \cdots, n\}.$$

The Ball-Box Theorem claims that the image through the coordinates of C-C balls are uniformly equivalent to the boxes.

THEOREM 2.2 (Ball-Box Theorem). *There exist linearly adapted coordinates $\varphi = (y_1, ..., y_n)$ and positive constants $c < C$ and $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$,*

$$Box^w(c\varepsilon) \ \subset \ \varphi(B(p,\varepsilon)) \ \subset \ Box^w(C\varepsilon).$$

We suggest the reader to see [28, 2.4] for a detailed proof of this theorem.

10.3. Hausdorff dimension of a strongly equiregular submanifold of degree d. Since in this subsection we consider strongly equiregular submanifolds giving the definition may be appropriate, for further details see [18].

DEFINITION 2.10. Let M be a submanifold of dimension m of equiregular sub-Riemannian mainfold N and p in M. We consider the flag at p

$$\{0\} \subset (\mathcal{H}_p \cap T_pM) \subset (\mathcal{H}_p^2 \cap T_pM) \subset \cdots \subset (\mathcal{H}_p^s \cap T_pM) = T_pM. \tag{58}$$

Let us remind that we set, in section 1,

$$\tilde{\mathcal{H}}_p^j = T_pM \cap \mathcal{H}_p^j, \qquad \tilde{m}_j = \mathrm{rank}(\tilde{\mathcal{H}}_p^j/\tilde{\mathcal{H}}_p^{j-1}).$$

A submanifold is said *strongly equiregular* if $\tilde{\mathcal{H}}_p^j = T_pM \cap \mathcal{H}_p^j$ has constant dimension for each p in M and $i = 1, \cdots, s$.

The Hausdorff dimension at a point p in N has a local nature, therefore it is sufficient to work in a small neighborhood of p, $B(p,r) \cap N$. Bellaïche in [2, Chapter 4] show how to construct polynomial privileged coordinates. However, we use the exponential map, as Ghezzi and Jean did in [18], in order to provide privileged coordinates such that we have $M \cap B(p,r) = \{x_{m+1} = \cdots = x_n = 0\}$.

Definition 2.11. A system of privileged coordinates is a system of local coordinates $z_1, ..., z_n$ centered at p such that:

(i) $z_1, ..., z_n$ are linearly adapted at p
(ii) The order of z_j at p is exactly w_j.

Now, we show how it is possible to create privileged coordinates using the exponential map. Thanks to the strongly equiregular hypothesis, given $r > 0$ small enough, we can find m vector fields $Y_1, \cdots, Y_m$ adapted to the flag (58) at each q in $B(p,r) \cap M$. Moreover reducing r if necessary, we can find $Y_{m+1}, \cdots, Y_n$ vector fields such that $Y_1, \cdots, Y_n$ is an adapted basis to flag (29) at each q in $B(p,r) \cap N$. We define the local diffeomorphism $\Phi_p : \mathbb{R}^n \longrightarrow N$ by

$$\Phi_p(x) = \exp\left(\sum_{i=m+1}^{n} x_i \; Y_i\right) \circ \exp\left(\sum_{i=1}^{m} x_i \; Y_i\right)(p) \tag{59}$$

The inverse $\varphi_p = \Phi_p^{-1}$ provides a system of coordinates which are privileged. Therefore, in these coordinates φ_p the set $M \cap B(p,r)$ coincides with

$$\Big\{\exp\left(\sum_{i=1}^{m} x_i \; Y_i\right)(p) \;:\; x_1, \cdots, x_m \;\in\; \Omega\Big\} \subset \Big\{\Phi_p(x) \;:\; x_{m+1} = \cdots = x_n = 0\Big\}, \tag{60}$$

where Ω is an open set of $\mathbb{R}^m$.

From now on we will work in these coordinates $\varphi = (x_1, \cdots, x_n)$ centered in p. By the Ball-Box Theorem we have that a ball is uniformly equivalent to a box in $\mathbb{R}^n$. Therefore, instead of $\varphi_p(M \cap B(p,r))$ we consider

$$S := \text{Box}^w(r') \cap \{x \in \mathbb{R}^n \;:\; x_{m+1} = \cdots = x_n = 0\},$$

where $r' < r$. Now, if we want to know the Hausdorff dimension we have to cover this set with boxes of size $\varepsilon = \frac{1}{k}$ and we have to know how many boxes we need.

$$\begin{aligned} H^l_{1/k}(S) &= \sum_{boxes} \left(\frac{c}{k}\right)^l = ([r'k]+1)^{\tilde{m}_1} \; ([r'k]+1)^{2\tilde{m}_2} \cdots \; ([r'k]+1)^{s\tilde{m}_s} \left(\frac{c}{k}\right)^l \\ &= \left(\prod_{i=1}^{s} ([r'k]+1)^{i\tilde{m}_i}\right) \left(\frac{c}{k}\right)^l = ([r'k]+1)^{\sum_{i=1}^{s} i\tilde{m}_i} \left(\frac{c}{k}\right)^l \approx r'^d \; c^l \; \frac{k^d}{k^l}. \end{aligned} \tag{61}$$

Therefore, we have

$$\lim_{k\to\infty} H^l_{1/k}(S) = \begin{cases} \infty & l < d \\ 0 & l > d \end{cases} \tag{62}$$

Now, thanks to Theorem 2.1 and Definition 2.8 it follows that the Hausdorff dimension of a strongly equiregular submanifold in sub-Riemanian manifold is its degree.

10.4. Hausdorff dimension of equiregular submanifold of degree d. The strongly equiregularity is a global property of N and the Hausdorff dimension at a point p has a local nature. Therefore, to prove that the Hausdorff dimension at p of a submanifold of degree d is equal to d we need only strongly regularity.

DEFINITION 2.12. A submanifold M is said *strongly regular* at p if N is an equiregular manifold and there exists an open neighborhood U of p in N such that $\tilde{\mathcal{H}}_q^j = T_qM \cap \mathcal{H}_q^j$ has constant dimension for each q in $M \cap U$ and $i = 1, \cdots, s$.

PROPOSITION 2.1. *Let M be a submanifold of a sub-Riemannian manifold N with constant degree d. Then, a regular point p in M is strongly regular at p.*

PROOF. Let $(v_1(q), \cdots, v_m(q))$ be a basis of the tangent space to M for each q in M and we consider a smooth m-tensor on M

$$v = v_1 \wedge \cdots \wedge v_m.$$

The degree of v, which is the sum of the degree of the simple vector v_i

$$d(v(q)) = \sum_{i=1}^{m} d(v_i(q)),$$

must be equal to d, for each q in M. By the semicontinuity of the degree, for each $i = 1, \cdots, m$ there exists an open neighborhood U_i of p such that

$$d(v_i(q)) \geqslant d(v_i(p)) \qquad q \in U_i.$$

Now, we consider the open set

$$U = U_1 \cap \cdots \cap U_m.$$

Let q be a point in U, we claim that

$$d(v_i(q)) = d(v_i(p)).$$

Indeed, suppose that there exists a $j \in \{1, \cdots, m\}$ such that $d(v_j(q)) > d(v_j(p))$ then it follows

$$d = d(v_j(p)) + \sum_{\substack{i=1,\\ i\neq j}}^{m} d(v_i(p)) \leqslant d(v_j(p)) + \sum_{\substack{i=1,\\ i\neq j}}^{m} d(v_i(q)) < \sum_{i=1}^{m} d(v_i(q)) = d$$

which is impossible. Thus, we have $d(v_i(q)) = d(v_i(p))$ for each $i = 1, \cdots, m$ and q in U. This means that if l tangent vectors to M at p lie in the j layer then l tangent vectors to M at q in U lie in the j layer. In other words, we have

$$\operatorname{rank}(\tilde{\mathcal{H}}_q^j) = \operatorname{rank}(T_qM \cap \mathcal{H}_q^j) = \operatorname{rank}(T_pM \cap \mathcal{H}_p^j) = \operatorname{rank}(\tilde{\mathcal{H}}_p^j).$$

Hence, M is strongly regular at p. □

Now, if we consider a $B(p, r) \subset U$, where U is the one used in the previous proof, we can consider $Y_1, \cdots, Y_m$ vector fields adapted to the flag (58) at each point q in $B(p, r) \cap N$ as we did in section 10.4 and we can repeat the proof we did for the strongly equiregular submanifold. Overall, the degree of a smooth submanifold in a equiregular sub-Riemaniann manifold N represents its Hausdorff dimension.

It would be really interesting to try to prove that the measure of area for a submanifold of degree d given (35) is absolutely continuous respect to d-Hausdorff measure of the submanifold. In [**27**] Vittone and Magnani dealt with this problem for a submanifold in a Carnot group and Franchi, Serapioni and Serra Cassano in [**13**] faced this problem for an hypersurface in an Heisenberg group.

CHAPTER 3

First variation of area functional

First of all in this chapter we compute the first variation of the area functional for a θ-graph in S and we obtain the well known second-order partial differential equation (65) (for further details see [**15**]). Then we make the first variation of a (θ,κ)-graph embedded in E endowed with the horizontal metric making X_1 and X_2 orthonormal and the one induced by the Euclidean metric. Moreover, we compute the first variation of a (θ,κ)-graph in $\tilde{E}$. Basically, as the minimal equation has a local nature working in $\tilde{E}$ with $Z_1, \cdots, Z_4$ is the same as working in E with $X_1, \cdots, X_4$ with a change of coordinates and a new horizontal metric. Thus, we obtain three different minimal equations which are third-order partial differential equations. The high order is due to the fact that only variations preserving the degree are allowed otherwise the area functional could change expression. In section 4 we study general variations, induced by a vector field X, that preserve the degree d of submanifold in a general equiregular sub-Riemannian manifold. Hence, we obtain a PDE system of equations where the coefficients of the vector field X inducing the variation are involved . This system restricts the range of permitted variations. At the end of this chapter we verify that this PDE system is only an equation for (θ,κ)-graphs and it is the same condition determined in section 2 by the implicit function theorem.

Remark 3.1. In the sequel we will use the vector fields $X_1, \cdots, X_4$ to derive functions which depend only on the variable x and y, above all because we are interested in (θ, k)-graphs. Let $f : \Omega \to \mathbb{R}$ be a function from an open set $\Omega \subset \mathbb{R}^2$, then we consider the derivate at the point $p = (x, y, k, \theta)$

$$\begin{aligned} X_1(f(x,y)) &= \cos(\theta)\partial_x(f(x,y)) + \sin(\theta)\partial_y(f(x,y)) + k\partial_\theta(f(x,y)) \\ &= \cos(\theta)\partial_x(f(x,y)) + \sin(\theta)\partial_y(f(x,y)). \end{aligned}$$

This derivative acts on the functions as if the vector field X_1 were projected onto the retinal plane, we will denote this projection by $\overline{X_1}$. Therefore when the vector field X_1 acts on a function f depending on x, y, we will use the notation $\overline{X_1}(f)$ instead of $X_1(f)$, in order to remember that the partial derivate ∂_θ respect to θ vanishes. When X_2 and X_3 act on a function depending only on x, y they vanish

$$X_2(f(x,y)) = 0, \qquad X_3(f(x,y)) = 0.$$

Whereas the projection of $\overline{X_4}$ is equal to X_4. We will apply the same notation to $Z_1, \cdots, Z_4$ vector fields on $\tilde{E}$. They will be projected to $\overline{Z_1}, \cdots, \overline{Z_4}$ vector fields on the retinal plane.

1. Minimal surface equation in S

We consider the group of rigid motions of the Euclidean plane, i.e. $E(2)$. The underlying manifold is $\mathbb{R}^2 \times S^1$ where the horizontal distribution $\mathcal{H}$ is generated by the vector fields

$$X = \cos(\theta)\partial_x + \sin(\theta)\partial_y, \quad Y = \partial_\theta,$$

the Reeb vector field is

$$T = \sin(\theta)\partial_x - \cos(\theta)\partial_y$$

and the contact for is $\omega = \sin(\theta)dx - \cos(\theta)dy$. We shall consider $E(2)$ with a sub-Riemannian metric h defined on its horizontal distribution, such that X, Y are an orthonormal basis for this metric. A surface Σ in $E(2)$ is the zero level set of a function $u(x, y, \theta)$. We are interested in surfaces which have the following zero set level

$$u(x, y, \theta) = f(x, y) - \theta.$$

We gave a definition of area of a f-graph in section 8 of chapter 2 and we showed that

$$A(\Sigma) = \int_\Omega \left(1 + X(f)^2\right)^{\frac{1}{2}} \, dxdy.$$

Here, we want to show that we obtain the same functional $A(\Sigma)$ if we consider the following definition of area

Definition 3.1. Let Σ be a surface in $E(2)$. We define the area of Σ by

$$\mathcal{A}(\Sigma) = \int_\Sigma |N_h| d\Sigma,$$

where N_h is the horizontal projection of the Riemannian normal onto the horizontal distribution and $d\Sigma$ is the Riemannian area measure.

In this case the unit normal vector of Σ is

$$\begin{aligned} N =& \frac{(Xu)X + (Yu)Y + (Tu)T}{\sqrt{(Xu)^2 + (Yu)^2 + (Tu)^2}} \\ =& \frac{X(f)X - Y + T(f)T}{\sqrt{f_x^2 + f_y^2 + 1}} \end{aligned}$$

and the projection of N onto the horizontal plane is

$$N_h = \frac{X(f)X - Y}{\sqrt{f_x^2 + f_y^2 + 1}}.$$

Now we are interested in calculating the Riemannian area measure $d\Sigma$. In order to develop this computation we can consider the graph of function f

$$G_f = \{(x, y, f(x, y)) \; : \; (x, y) \in \Omega\}.$$

Let us consider $U = \Omega\times]0, \pi[$, we define the diffeomorphism

$$\varphi : U \longrightarrow \mathbb{R}^3 \qquad \varphi(x, y, t) = (x, y, t - f(x, y))$$

and its inverse $\varphi^{-1}(x, y, z) = (x, y, z + f(x, y))$. Therefore, we have

$$d\varphi^{-1}\Big(\frac{\partial}{\partial_x}\Big) = (1, 0, f_x), \qquad d\varphi^{-1}\Big(\frac{\partial}{\partial_y}\Big) = (0, 1, f_y).$$

Taking into account this matrix

$$(g_{ij}) = \begin{pmatrix} 1 + (f_x)^2 & f_x f_y \\ f_x f_y & 1 + (f_y)^2 \end{pmatrix},$$

it is possible to compute the Riemannian area measure

$$d\Sigma = \sqrt{det(g_{ij})}\; dxdy = \big(1 + |\nabla f|^2\big)^{\frac{1}{2}}\; dxdy. \tag{63}$$

Finally, by definition (3.1) the area of a surface G_f is

$$\mathcal{A}(G_f) = \int_\Omega |N_h| d\Sigma = \int_\Omega \frac{\sqrt{1 + (X(f))^2}}{\sqrt{1 + |\nabla f|^2}} \sqrt{1 + |\nabla f|^2}\; dxdy \tag{64}$$

which is the same functional $A(G_f)$ deduced in section 8.

Now let v be a function in $C_0^\infty(\Omega)$. We will compute the first variation of area formula with respect to a variation $\theta + tv$ to obtain

$$\frac{d}{dt}\Big|_{t=0} A(G_{\theta+tv}) = \int_\Omega \frac{X(\theta)}{\sqrt{1 + X(\theta)^2}} \frac{d}{dt}\Big|_{t=0} X_{\theta+tv}(\theta + tv).$$

Now we use

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} X_{\theta+tv}(\theta + tv) &= \frac{d}{dt}\Big|_{t=0} \cos(\theta + tv)(\theta + tv)_x + \sin(\theta + tv)(\theta + tv)_y \\ &= -v \sin(\theta)\theta_x + \cos(\theta)v_x + v \cos(\theta)\theta_y + \sin(\theta)v_y \\ &= -v\, T(\theta) + X(v), \end{aligned}$$

and the divergence theorem

$$\begin{aligned} \int_\Omega wX(v) &= \int_\Omega w \,\langle(\cos(\theta), \sin(\theta)), \nabla v\rangle \\ &= -\int_\Omega v \,\big((w \cos(\theta))_x + (w \sin(\theta))_y\big) \\ &= -\int_\Omega \big(vX(w) - vwT(\theta)\big), \end{aligned}$$

so that

$$\int_\Omega w\left(X(v) - v\,T(\theta)\right) = -\int_\Omega vX(w).$$

This implies

$$\frac{d}{dt}\bigg|_{t=0} A(G_{\theta+tv}) = -\int_\Omega v\,X\left(\frac{X(\theta)}{\sqrt{1+X(\theta)^2}}\right).$$

Hence the Euler-Lagrange equation for a minimal surface is

$$X\left(\frac{X(\theta)}{\sqrt{1+X(\theta)^2}}\right) = 0. \tag{65}$$

As the unit normal to the characteristic curves in the surface is given by

$$Z = \frac{X + X(\theta)Y}{\sqrt{1+X(\theta)^2}},$$

and the function θ only depends on (x,y), the minimal surface equation is equivalent to $\theta'' = 0$, where $'$ is the arc-length parameter of the characteristic curves.

2. First variation of the area with fixed degree in E

In $(E, \mathcal{H}, h_1)$ the sub-Riemannian area element for (θ,κ)-graphs of degree four, satisfying $\kappa = X_1(\theta)$, is given by

$$A(G_{\theta,\kappa}) = \int_\Omega \sqrt{1 + X_1(\kappa)^2}\; dxdy.$$

The characteristic direction is given by

$$Z = \frac{X_1 + X_1(\kappa)X_2}{\sqrt{1+X_1(\kappa)^2}},$$

where $X_2 = \partial_k$. Therefore, when we consider a function $f(x,y)$ defined on an open set $\Omega \subset \mathbb{R}^2$ we have

$$Z(f) = \frac{\overline{X_1}(f)}{\sqrt{1+\overline{X_1}(\kappa)^2}}.$$

Now we want to induce variations that preserve the degree of the surface, which is four. A general variation of a (θ,κ)-graph by a function (v,w) would be $(\theta + tv, \kappa + sw)$. We seek a condition assuring that the $(\theta + tv, \kappa + sw)$-graph still has degree four. Thus, we set

$$f_{(t,s)}(x,y) = (x, y, \theta + tv, \kappa + sw)$$

and, by condition (47), the degree of the surface $G_{f_{(t,s)}}$ is four if and only if

$$G(t,s,x,y) = \cos(\theta+tv)(\theta+tv)_x + \sin(\theta+tv)(\theta+tv)_y - (\kappa+sw) = 0.$$

We wish to find a function $s : (-\varepsilon, \varepsilon) \to \mathbb{R}$ such that $G(t, s(t), x, y) = 0$, therefore we have to apply the implicit function theorem. We have

$$\left.\frac{\partial G}{\partial t}\right|_{t=0} = vX_4(\theta), + X_1(v)$$
$$\left.\frac{\partial G}{\partial s}\right|_{s=0} = -w.$$

Hence, if the function $w(x, y) \neq 0$ for each (x, y) in Ω, then there exists $s(t)$ such that

$$G(t, s(t), x, y) = 0$$

and

$$s'(0) = -\frac{\frac{\partial G}{\partial t}}{\frac{\partial G}{\partial s}} = \frac{vX_4(\theta) + X_1(v)}{w}. \tag{66}$$

Therefore,

$$\left.\frac{d}{dt}\right|_{t=0} (\theta + tv, \kappa + s(t)w) = (v, s'(0)w) = (v, vX_4(\theta) + X_1(v)). \tag{67}$$

Since we search for critical points of the area functional, we derive with respect to t a variation

$$\left.\frac{d}{dt}\right|_{t=0} A(\theta + tv, \kappa + s(t)w) = \int_\Omega \frac{X_1(\kappa)}{\sqrt{1 + X_1(\kappa)^2}} \left.\frac{d}{dt}\right|_{t=0} (X_1)_{(\theta+tv,\kappa+s(t)w)}(\kappa + s(t)w).$$

We have

$$\begin{aligned}
&\left.\frac{d}{dt}\right|_{t=0} (X_1)_{(\theta+tv,\kappa+s(t)w)}(\kappa + s(t)w) = \\
&= \left.\frac{d}{dt}\right|_{t=0} \cos(\theta + tv)(\kappa + s(t)w)_x + \sin(\theta + tv)(\kappa + s(t)w)_y \\
&= v\ X_4(\kappa + s(0)w) + X_1(s'(0)w) = v\ X_4(\kappa) + X_1(vX_4(\theta) + X_1(v)) \\
&= v\ X_4(\kappa) + X_1(v)X_4(\theta) + v\ X_1(X_4(\theta)) + X_1^2(v).
\end{aligned}$$

Now we have to use integration by parts for the second and the fourth terms

$$\mathrm{II} = \int_\Omega \frac{X_1(\kappa)}{\sqrt{1 + X_1(\kappa)^2}}\ X_1(v)X_4(\theta)\ dx\ dy \tag{68}$$

$$\mathrm{IV} = \int_\Omega \frac{X_1(\kappa)}{\sqrt{1 + X_1(\kappa)^2}}\ X_1^2(v)\ dx\ dy \tag{69}$$

Now it would be useful to enunciate a integration by parts theorem for this setting

LEMMA 3.1. *Let Ω be a open set in $\mathbb{R}^2$. Let $f, g : \Omega \to \mathbb{R}$ be functions in $C_0^1(\Omega)$. Let*

$$X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y + k\partial_\theta, \qquad X_4 = -\sin(\theta)\partial_x + \cos(\theta)\partial_y$$

be the vector fields defined in (21) and (24). Then, there holds

$$\int_\Omega g\ X_1 f\ dxdy = -\int_\Omega f\ (g\ X_4\theta + X_1 g)dxdy.$$

PROOF. Thanks to the formula of integration by parts in $\Omega \subset \mathbb{R}^2$, where $\nu = (\nu_x, \nu_y)$ is the outward unit surface normal to $\partial\Omega$, it follows

$$\int_\Omega g\cos(\theta)\partial_x f + g\sin(\theta)\partial_y f\ dxdy = \int_{\partial\Omega} g\ f(\cos(\theta)\nu_x + \sin(\theta)\nu_y)\ d\sigma$$
$$-\int_\Omega f\ \partial_x(g\cos(\theta)) + f\ \partial_y(g\sin(\theta))\ dxdy$$
$$= \int_{\partial\Omega} g\ f\ (\cos(\theta), \sin(\theta)) \cdot \nu\ d\sigma - \int_\Omega f\ (g\ X_4\theta + X_1 g)dxdy.$$

□

First of all, we define

$$g := \frac{X_1(\kappa)}{(1 + (X_1(\kappa))^2)^{\frac{1}{2}}}. \tag{70}$$

Thus, if we use this Lemma 3.1 in (68) it follows

$$\int_\Omega g\ X_4(\theta)X_1(v)\ dx\ dy = -\int_\Omega v\ \left(g\ (X_4\theta)^2 + X_1(g)X_4(\theta) + g\ X_1(X_4(\theta))\right) dxdy.$$

On the other hand, Lemma 3.1 applies twice to (69), it gives

$$\int_\Omega g\ X_1^2(v)\ dx\ dy = -\int_\Omega X_1(v)\ (gX_4(\theta) + X_1(g))\ dxdy$$
$$= \int_\Omega v\ \left((g\ X_4(\theta) + X_1(g))\ X_4(\theta) + X_1(g\ X_4(\theta) + X_1(g)\right)\ dxdy.$$

Hence, we have

$$\int_\Omega v\left(X_4(\kappa)g + X_1(X_4(\theta))g + X_1(g)X_4(\theta) + X_1^2(g)\right)\ dxdy = 0. \tag{71}$$

Therefore, the minimal surface equation is

$$X_4(X_1(\theta))g + X_1(X_4(\theta))g + X_1(g)X_4(\theta) + X_1^2(g) = 0. \tag{72}$$

Let $\gamma(s)$ be a characteristic curve in E such that

$$\gamma'(s) = Z_{\gamma(s)}.$$

The characteristic curve lives in (θ, κ)-graph, therefore we have

$$\gamma(s) = (x(s), y(s), \theta(x(s), y(s)), \kappa(x(s), y(s)).$$

Hence,

$$\kappa' = \frac{X_1(\kappa) + X_1(\kappa)X_2(\kappa)}{\sqrt{1 + X_1(\kappa)^2}} = \frac{X_1(\kappa)}{\sqrt{1 + X_1(\kappa)^2}} = g$$

$$\kappa'' = \frac{X_1\left(\frac{X_1(\kappa)}{\sqrt{1+X_1(\kappa)^2}}\right)}{\sqrt{1 + X_1(\kappa)^2}} = \frac{X_1(X_1(k))}{(1 + X_1(k)^2)^2}$$

$$\kappa''' = \frac{X_1\left(\dfrac{X_1\,(g)}{\sqrt{1 + X_1(\kappa)^2}}\right)}{\sqrt{1 + X_1(\kappa)^2}}.$$

We can express $X_1^2(g)$ in terms of the the derivatives of κ,

$$\frac{X_1^2(g)}{\sqrt{1 + X_1(\kappa)^2}} = \frac{X_1\left(\dfrac{X_1\,(g)}{\sqrt{1 + X_1(\kappa)^2}}\right)}{\sqrt{1 + X_1(\kappa)^2}} - \frac{1}{\sqrt{1 + X_1(\kappa)^2}} X_1\left(\frac{1}{\sqrt{1 + X_1(\kappa)^2}}\right) X_1(g).$$

In this case the equation (72) along the characteristic curves is

$$\frac{(X_4(X_1(\theta)) + X_1(X_4(\theta)))}{\sqrt{1 + X_1(\kappa)^2}}\kappa' + \kappa'' X_4(\theta) + \kappa''' - X_1\left(\frac{1}{\sqrt{1 + X_1(\kappa)^2}}\right)\kappa'' = 0. \tag{73}$$

Now,

$$X_1\left(\frac{1}{\sqrt{1 + X_1(\kappa)^2}}\right) = -\frac{X_1(\kappa)\ X_1(X_1(\kappa))}{(1 + X_1(\kappa)^2)^{\frac{3}{2}}}$$
$$= -\kappa'\ \frac{X_1(X_1(\kappa))}{(1 + X_1(\kappa)^2)} = -\kappa'\ \kappa''\ (1 + X_1(\kappa)^2)$$

Consequently, the equation (73) is equivalent to

$$\frac{(X_4(X_1(\theta)) + X_1(X_4(\theta)))}{\sqrt{1 + X_1(\kappa)^2}}\kappa' + \kappa'' X_4(\theta) + \kappa''' + \kappa'\ \kappa''(1 + X_1(\kappa)^2) = 0. \tag{74}$$

REMARK 3.2. The Lie bracket of the vector fields X_1, X_4 is the vector field

$$[X_1, X_4] = a\ \partial_x + b\ \partial_y + c\ \partial_\theta + d\ \partial_k.$$

If we want to know the effect of this vector field on a function $f(x, y)$, we only need to compute

$$a = [X_1, X_4](x) = X_1(-\sin(\theta)) = -\kappa\cos(\theta)$$
$$b = [X_1, X_4](y) = X_1(\cos(\theta)) = -\kappa\sin(\theta)$$

and it follows

$$[X_1, X_4](f(x, y)) = -\kappa\overline{X_1}(f(x, y)).$$

Now, we consider a (θ, k)-graph and we will show that

$$X_4(X_1(f(x, y))) - X_1(X_4(f(x, y)))$$

is different from $[X_1, X_4](f(x, y))$.
At first, we have

$$\begin{aligned}\overline{X_4}(\overline{X_1}f) =& X_4(\cos(\theta)\ f_x + \sin(\theta)f_y)\\ =& -\sin(\theta)(\cos(\theta)\ f_x + \sin(\theta)f_y)_x + \cos(\theta)(\cos(\theta)\ f_x + \sin(\theta)f_y)_y\\ =& -\sin(\theta)(-\sin(\theta)\theta_x\ f_x + \cos(\theta)\ f_{xx} + \cos(\theta)\theta_x f_y + \sin(\theta)f_{xy})\\ &+ \cos(\theta)(-\sin(\theta)\theta_y\ f_x + \cos(\theta)\ f_{xy} + \cos(\theta)\theta_y\ f_y + \sin(\theta)f_{yy}).\end{aligned} \tag{75}$$

On the other hand,

$$\begin{aligned}\overline{X_1}(\overline{X_4}f) =& \overline{X_1}(-\sin(\theta)\ f_x + \cos(\theta)f_y)\\ =& \cos(\theta)(-\sin(\theta)\ f_x + \cos(\theta)f_y)_x + \sin(\theta)(-\sin(\theta)\ f_x + \cos(\theta)f_y)_y\\ =& \cos(\theta)(-\cos(\theta)\theta_x\ f_x - \sin(\theta)\ f_{xx} - \sin(\theta)\theta_x f_y + \cos(\theta)f_{xy})\\ &+ \sin(\theta)(-\cos(\theta)\theta_y\ f_x - \sin(\theta)\ f_{xy} - \sin(\theta)\theta_y\ f_y + \cos(\theta)f_{yy}).\end{aligned} \tag{76}$$

Hence, subtracting (75) and (76), we have

$$\overline{X_4}(\overline{X_1}f) - \overline{X_1}(\overline{X_4}f) = \theta_x\ f_x + \theta_y\ f_y.$$

When the function f is $\theta(x, y)$, it follows

$$\overline{X_4}(\overline{X_1}\theta) - \overline{X_1}(\overline{X_4}\theta) = \theta_x^2 + \theta_y^2.$$

Now, we consider the sub-Riemannian manifold $(E, \mathcal{H}, h_2)$ where h_2 is the horizontal metric induced by Euclidean metric. We computed the area functional for a (θ,κ)-graph with this metric in (57) and we obtained

$$A(G_{(\theta,k)}) = \int_\Omega (1 + \kappa^2 + X_1(\kappa)^2)^{\frac{1}{2}}\ dxdy.$$

Here, the characteristic vector is

$$Z = \frac{X_1 + X_1(\kappa)X_2}{\sqrt{1 + \kappa^2 + X_1(\kappa)^2}}.$$

Since the degree of a (θ,κ)-graph is independent of the horizontal metric, a variation preserving the degree has to satisfy (66) and conditions deduced at the beginning of this section. Now, we compute the first variation

$$\frac{d}{dt}\Big|_{t=0} A(\theta+tv,\kappa+s(t)w) = \underbrace{\int_\Omega \frac{\kappa}{\sqrt{1+\kappa^2+X_1(\kappa)^2}}\frac{d}{dt}\Big|_{t=0}(\kappa+s(t)w)}_{\mathrm{I}} + \underbrace{\int_\Omega \frac{X_1(\kappa)}{\sqrt{1+\kappa^2+X_1(\kappa)^2}}\frac{d}{dt}\Big|_{t=0} X_{1(\theta+tv,\kappa+s(t)w)}(\kappa+s(t)w)}_{\mathrm{II}}.$$

Setting

$$g := \frac{X_1(\kappa)}{\sqrt{1+\kappa^2+X_1(\kappa)^2}}$$

by (71) we deduce that

$$\mathrm{II} = \int_\Omega v\left(X_4(\kappa)g + X_1(X_4(\theta))g + X_1(g)X_4(\theta) + X_1^2(g)\right)\,dx\,dy. \tag{77}$$

Therefore, we have only to develop

$$\mathrm{I} = \int_\Omega \frac{\kappa}{\sqrt{1+\kappa^2+X_1(\kappa)^2}}\frac{d}{dt}\Big|_{t=0}(\kappa+s(t)w).$$

Now, we have

$$\frac{d}{dt}\Big|_{t=0}(\kappa+s(t)w) = vX_4(\theta) + X_1(v)$$

and by lemma 3.1 it follows

$$\int_\Omega \frac{\kappa}{\sqrt{1+\kappa^2+X_1(\kappa)^2}}(X_1(v)+vX_4(\theta)) = -\int_\Omega vX_1\left(\frac{\kappa}{\sqrt{1+\kappa^2+X_1(\kappa)^2}}\right).$$

Hence, thanks to the arbitrary of v the minimal equation with h_2 as horizontal metric is

$$X_4(\kappa)g + X_1(X_4(\theta))g + X_1(g)X_4(\theta) + X_1^2(g) - X_1\left(\frac{\kappa}{\sqrt{1+\kappa^2+X_1(\kappa)^2}}\right) = 0. \tag{78}$$

Here, the following derivative rules hold

$$\kappa' = \frac{X_1(\kappa)}{\sqrt{1+\kappa^2+X_1(\kappa)^2}} = g$$

$$\kappa'' = \frac{X_1\left(\frac{X_1(\kappa)}{\sqrt{1+\kappa^2+X_1(\kappa)^2}}\right)}{\sqrt{1+\kappa^2+X_1(\kappa)^2}} = \frac{X_1^2(\kappa)(1+\kappa^2) - \kappa\ X_1(\kappa)^2}{(1+\kappa^2+X_1(\kappa)^2)^{\frac{3}{2}}}$$

$$\kappa''' = \frac{X_1\left(\frac{X_1\,(g)}{\sqrt{1+\kappa^2+X_1(\kappa)^2}}\right)}{\sqrt{1+\kappa^2+X_1(\kappa)^2}}.$$

We can express $X_1^2(g)$ in terms of the the derivatives of κ,

$$X_1^2(g) = X_1\left(\frac{X_1\,(g)}{\sqrt{1+\kappa^2+X_1(\kappa)^2}}\right) - X_1\left(\frac{1}{\sqrt{1+\kappa^2+X_1(\kappa)^2}}\right)X_1(g).$$

Putting these equations in (78), we have

$$\begin{aligned}
&X_4(\kappa)g + X_1(X_4(\theta))g + X_1(g)X_4(\theta)\\
&+ X_1\left(\frac{X_1\,(g)}{\sqrt{1+\kappa^2+X_1(\kappa)^2}}\right) - X_1\left(\frac{1}{\sqrt{1+\kappa^2+X_1(\kappa)^2}}\right)X_1(g)\\
&-\left(\frac{X_1(\kappa)}{\sqrt{1+\kappa^2+X_1(\kappa)^2}}\right) + \kappa X_1\left(\frac{1}{\sqrt{1+\kappa^2+X_1(\kappa)^2}}\right) = 0.
\end{aligned}$$

Hence, it follows

$$\begin{aligned}
&\frac{(X_4(\kappa)+X_1(X_4(\theta))-1)}{\sqrt{1+\kappa^2+X_1(\kappa)^2}}\ \kappa' + \kappa''\left(X_4(\theta) - X_1\left(\frac{1}{\sqrt{1+\kappa^2+X_1(\kappa)^2}}\right)\right)\\
&+\kappa''' + \kappa\left(\frac{1}{\sqrt{1+\kappa^2+X_1(\kappa)^2}}\right)' = 0.
\end{aligned}$$

3. First variation of the area in a different manifold

In $\tilde{E}$ the area functional for a (θ,φ)-graph satisfying the foliation condition is given by

$$A(\Sigma) = \int_\Omega \left(\frac{1+Z_1(\varphi)^2}{\cos(\varphi)^2}\right)^{\frac{1}{2}}\ dxdy. \tag{79}$$

Furthermore, the foliation condition implies that intersection between the distribution and the tangent plane has dimension one. The characteristic vector is an unitary vector that gives the direction of foliation curves and it is equal to

$$Z = \frac{Z_1 + Z_1(\varphi)Z_2}{\sqrt{1+Z_1(\varphi)^2}}.$$

Here, a variation which preserves the degree has to satisfy the following condition

$$G(t,s,x,y) = \cos(\theta+tv)(\theta+tv)_x + \sin(\theta+tv)(\theta+tv)_y - \tan(\varphi+sw) = 0.$$

Therefore, we have

$$\begin{aligned} \left.\frac{\partial G}{\partial t}\right|_{t=0} &= v\ Z_4(\theta) + X(v), \\ \left.\frac{\partial G}{\partial s}\right|_{s=0} &= -\left(1+\tan(\varphi)^2\right) w. \end{aligned}$$

If we consider $w(x,y) \neq 0$ we can apply the implicit function theorem to express s respect to t such that

$$G(t, s(t), x, y) = 0.$$

Furthermore, we have

$$s'(0) = -\frac{\dfrac{\partial G}{\partial t}}{\dfrac{\partial G}{\partial s}} = \frac{v\ Z_4(\theta) + X(v)}{(1+\tan(\varphi)^2)\ \ w} = \frac{\cos(\varphi)}{w}\left(v\cos(\varphi)\ Z_4(\theta) + Z_1(v)\right).$$

Hence, it follows

$$\left.\frac{d}{dt}\right|_{t=0}(\theta+tv, \varphi+s(t)w) = (v, s'(0)w) = \left(v, \cos(\varphi)(v\cos(\varphi)\ Z_4(\theta) + Z_1(v))\right). \tag{80}$$

Then, we can compute the first variation

$$\begin{aligned} &\left.\frac{d}{dt}\right|_{t=0} A(\theta+tv, \varphi+s(t)w) = \\ &\int_\Omega \frac{\sin(\varphi)}{\cos(\varphi)^2}\sqrt{1+Z_1(\varphi)^2}\ s'(0)w \\ &+ \int_\Omega \frac{1}{\cos(\varphi)}\ \frac{Z_1(\varphi)}{\sqrt{1+Z_1(\varphi)^2}}\ \left.\frac{d}{dt}\right|_{t=0} Z_{1(\theta+tv,\varphi+s(t)w)}(\varphi+s(t)w). \end{aligned} \tag{81}$$

Now, it is useful to develop

(82)
$$
\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} Z_{1(\theta+tv,\varphi+s(t)w)}(\varphi+s(t)w) = \\
&= \frac{d}{dt}\Big|_{t=0} \cos(\varphi+s(t)w)\left(\cos(\theta+tv)(\varphi+s(t)w)_x+\sin(\theta+tv)(\varphi+s(t)w)_y\right)\\
&= -\sin(\varphi)\ s'(0)w\ X(\varphi)+\cos(\varphi)\ v\ Z_4(\varphi+s(0)w)+Z_1(s'(0)w)\\
&= -\sin(\varphi)(v\cos(\varphi)\ Z_4(\theta)+Z_1(v))\ Z_1(\varphi)+v\ \ \cos(\varphi)Z_4(\varphi)\\
&\quad + Z_1\left(\cos(\varphi)(v\cos(\varphi)\ Z_4(\theta)+Z_1(v))\right)\\
&= -2\sin(\varphi)(v\cos(\varphi)\ Z_4(\theta)+Z_1(v))\ Z_1(\varphi)+v\ \ \cos(\varphi)Z_4(\varphi)\\
&\quad +\cos(\varphi)Z_1\left(v\cos(\varphi)\ Z_4(\theta)+Z_1(v)\right).
\end{aligned}
$$

Then if we set

$$h := \frac{Z_1(\varphi)}{\sqrt{1+Z_1(\varphi)^2}}$$

and we put (82) in (81) we have

$$
\begin{aligned}
(81) = &\int_\Omega \frac{\sin(\varphi)}{\cos(\varphi)}\sqrt{1+Z_1(\varphi)^2}\ (v\cos(\varphi)\ Z_4(\theta)+Z_1(v))\\
&-\int_\Omega 2\ \frac{h}{\cos(\varphi)}\sin(\varphi)(v\cos(\varphi)\ Z_4(\theta)+Z_1(v))\ Z_1(\varphi)+\int_\Omega v\ h\ Z_4(\varphi)\\
&+\underbrace{\int_\Omega h\ Z_1\left(v\cos(\varphi)\ Z_4(\theta)+Z_1(v)\right)}_{\mathrm{I}}.
\end{aligned}
\tag{83}
$$

In this setting there exists an integration by parts similar to Lemma 3.1

LEMMA 3.2. *Let Ω be an open set of $\mathbb{R}^2$ and let $f,g:\Omega\to\mathbb{R}$ be real functions in C^1_0. Let*

$$\overline{Z}_1=\cos(\varphi)X,\qquad \overline{Z}_3=\sin(\varphi)X\qquad Z_4=-\sin(\theta)\partial_x+\cos(\theta)\partial_y$$

be vector field on Ω. Then, there holds

$$\int_\Omega g\ \overline{Z}_1(f) = -\int_\Omega f\ \left(g\ \ \cos(\varphi)\ Z_4(\theta)-\overline{Z}_3(\varphi)g+\overline{Z}_1(g)\right).$$

Now, we apply this Lemma to I in (83)

$$\mathrm{I} = -\int_\Omega (v\cos(\varphi)\ Z_4(\theta)+Z_1(v))\left(h\ \ \cos(\varphi)\ Z_4(\theta)-Z_3(\varphi)\ h+Z_1(h)\right).$$

Therefore, (83) becomes

$$\begin{aligned}(81) = &\int_\Omega \frac{\sin(\varphi)}{\cos(\varphi)}\sqrt{1+Z_1(\varphi)^2}\;(v\cos(\varphi)\;Z_4(\theta)+Z_1(v))\\ &-\int_\Omega \frac{h}{\cos(\varphi)}\sin(\varphi)(v\cos(\varphi)\;Z_4(\theta)+Z_1(v))\;Z_1(\varphi)+\int_\Omega v\;h\;Z_4(\varphi)\\ &-\int_\Omega (v\cos(\varphi)\;Z_4(\theta)+Z_1(v))\,(h\;\cos(\varphi)\;Z_4(\theta)+Z_1(h)).\end{aligned}$$

If we add the first and the second term we obtain

$$\begin{aligned}(81) = &\underbrace{\int_\Omega \frac{\sin(\varphi)}{\cos(\varphi)}\frac{(v\cos(\varphi)\;Z_4(\theta)+Z_1(v))}{\sqrt{1+Z_1(\varphi)^2}}}_{\text{III}}+\int_\Omega v\;h\;Z_4(\varphi)\\ &\underbrace{-\int_\Omega (v\cos(\varphi)\;Z_4(\theta)+Z_1(v))\,(h\;\cos(\varphi)\;Z_4(\theta)+Z_1(h))}_{\text{II}}.\end{aligned}$$

Applying once more Lemma 3.2 to II and III, it follows

$$\begin{aligned}\text{II} &= -\int_\Omega v\;Z_3(\varphi)(h\;\cos(\varphi)\;Z_4(\theta)+Z_1(h))+\int_\Omega v\;Z_1(h\;\cos(\varphi)\;Z_4(\theta)+Z_1(h)),\\ \text{III} &= +\int_\Omega v\;Z_3(\varphi)\frac{\tan(\varphi)}{\sqrt{1+Z_1(\varphi)^2}}-\int_\Omega v\;Z_1\left(\frac{\tan(\varphi)}{\sqrt{1+Z_1(\varphi)^2}}\right).\end{aligned}$$

Therefore, it follows

$$\begin{aligned}(81) = &+\int_\Omega v\;Z_3(\varphi)\frac{\tan(\varphi)}{\sqrt{1+Z_1(\varphi)^2}}-\int_\Omega v\;Z_1\left(\frac{\tan(\varphi)}{\sqrt{1+Z_1(\varphi)^2}}\right)+\int_\Omega v\;h\;Z_4(\varphi)\\ &-\int_\Omega v\;Z_3(\varphi)(h\;\cos(\varphi)\;Z_4(\theta)+Z_1(h))+\int_\Omega v\;Z_1(h\;\cos(\varphi)\;Z_4(\theta)+Z_1(h)).\end{aligned}$$

By the arbitrariness of v we have that the minimal equation is

$$\begin{aligned}&Z_3(\varphi)\frac{\tan(\varphi)}{\sqrt{1+Z_1(\varphi)^2}}-Z_1\left(\frac{\tan(\varphi)}{\sqrt{1+Z_1(\varphi)^2}}\right)+h\;Z_4(\varphi)\\ &-Z_3(\varphi)(h\;\cos(\varphi)\;Z_4(\theta)+Z_1(h))+Z_1(h\;\cos(\varphi)\;Z_4(\theta)+Z_1(h))=0.\end{aligned}\tag{84}$$

There holds

$$\varphi' = \frac{Z_1(\varphi)}{\sqrt{1+Z_1(\varphi)^2}} = h,$$

$$\varphi'' = \frac{Z_1\left(\frac{Z_1(\varphi)}{\sqrt{1+Z_1(\varphi)^2}}\right)}{\sqrt{1+Z_1(\varphi)^2}} = \frac{Z_1^2(\varphi)}{(1+Z_1(\varphi)^2)^2},$$

$$\varphi''' = \frac{Z_1\left(\frac{Z_1(g)}{\sqrt{1+Z_1(\varphi)^2}}\right)}{\sqrt{1+Z_1(\varphi)^2}}.$$

Cleaning equation (84) up, we obtain

(85)

$$\begin{aligned}&-h + Z_1(\varphi)\tan(\varphi)\frac{Z_1^2(\varphi)}{(1+Z_1(\varphi)^2)^{\frac{3}{2}}} + h\ Z_4(\varphi) - Z_3(\varphi)(h\ \cos(\varphi)\ Z_4(\theta) + Z_1(h))\\ &+ Z_1(h)\ \cos(\varphi)\ Z_4(\theta) - h\sin(\varphi)Z_1(\varphi)Z_4(\theta) + h\cos(\varphi)Z_1(Z_4(\theta))\\ &+ Z_1\left(\frac{Z_1(h)^2}{\sqrt{1+Z_1(\varphi)}}\right) - Z_1\left(\frac{1}{\sqrt{1+Z_1(\varphi)^2}}\right)Z_1(h) = 0.\end{aligned}$$

Developing the equation we have

$$\begin{aligned}&h(-2\sin(\varphi)Z_1(\varphi)Z_4(\theta) - 1 + \ Z_4(\varphi) + \cos(\varphi)Z_1(Z_4(\theta)))\\ &+ Z_1(h)\ \cos(\varphi)\ Z_4(\theta)+\\ &+ Z_1\left(\frac{Z_1(h)}{\sqrt{1+Z_1(\varphi)^2}}\right) - Z_1(\varphi)Z_1(h)^2 = 0.\end{aligned} \tag{86}$$

Cleaning up, it follows

$$\begin{aligned}&\frac{\varphi'}{\sqrt{1+Z_1(\varphi)^2}}(-2\sin(\varphi)Z_1(\varphi)Z_4(\theta) - 1 + \ Z_4(\varphi) + \cos(\varphi)Z_1(Z_4(\theta)))\\ &+ \varphi''\ \cos(\varphi)\ Z_4(\theta) + \varphi''' - \varphi'\ Z_1(h)^2 = 0.\end{aligned} \tag{87}$$

4. First variation of the area of a submanifold

Let $(N, \mathcal{H}, h)$ be a sub-Riemannian manifold, where h is a metric on the horizontal space $\mathcal{H}$. As we did in 4 we extend the horizontal metric to a Riemannian metric which makes the $H_i(p)$ space orthogonal for each p in N. Now, we consider M a m-dimensional submanifold of degree d in N and our aim is to find a condition on the vector field X, that induces the first variation, in order to preserve the degree d of the submanifold in the deformation. Let p be a point in M, we consider $e_1, \cdots, e_m$ a basis of T_pM for the auxiliary Riemannian metric g. Let X be a vector field with compact support, we can extend $e_1, \cdots, e_m$ to $E_1, \cdots, E_m$

along the flow line of X passing through p. Let $I = (-\delta, \delta)$ be an open set of $\mathbb{R}$, we will denote by $\{\varphi_t\}_{t\in I}$ the one parametric group of local diffeomorphism of a neighborhood U of p in N associated to the vector field X, see [35]. The condition $deg(\varphi_t(M)) = d$ is equivalent to

(*i*) $g(E_1(t) \wedge \cdots \wedge E_m(t), v) = 0$ for each p in M, t in I and $v = v_1 \wedge \cdots \wedge v_m$ m-vector of degree greater than d at p,

(*ii*) there exists w m-vector of degree d at $\varphi_t(p)$ such that

$$g(E_1(t) \wedge \cdots \wedge E_m(t), w) \neq 0.$$

Now, we can take into account the derivative of condition (i) respect to vector X_p

$$X_p\ g(E_1 \wedge \cdots \wedge E_m, v) = 0. \tag{88}$$

Let us denote by ∇ the Levi-Civita connection, compatible with the metric g and symmetric. Thus, we have

$$g\left(\sum_{i=1}^{m} e_1 \wedge \cdots \wedge \nabla_{X_p} e_i \wedge \cdots \wedge e_m, v\right) + g\left(e_1 \wedge \cdots \wedge e_m, \nabla_{X_p} v\right) = 0. \tag{89}$$

Notice that $\nabla_{E_i} X = \nabla_X E_i$, since $[E_i, X] = 0$. Moreover, $\nabla_{e_i} X$ can be replaced by $(\nabla_{e_i} X)^\perp$, since $e_i \wedge e_i = 0$. Hence, we can write the previous equation as

$$g\left(\sum_{i=1}^{m} e_1 \wedge \cdots \wedge (\nabla_{e_i} X)^\perp \wedge \cdots \wedge e_m, v\right) + g\left(e_1 \wedge \cdots \wedge e_m, \nabla_{X_p} v\right) = 0. \tag{90}$$

Now, we consider a coordinate neighborhood $(U, x^1, \cdots, x^n)$ of p in N such that

$$U \cap M = \{x^{m+1} = \cdots = x^n = 0\}.$$

In this chart the $e_1, \cdots, e_m$ can be replaced by $\left.\frac{\partial}{\partial x^1}\right|_p, \cdots, \left.\frac{\partial}{\partial x^m}\right|_p$ and we can express the vector field X as

$$X = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x^i}.$$

In order to develop the equation (90) we have to compute the term

$$\nabla_{e_j} X = \nabla_{\frac{\partial}{\partial x^j}} \left(\sum_{i=1}^{n} f_i \frac{\partial}{\partial x^i}\right) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_j} \frac{\partial}{\partial x_i} + f_i \nabla_{\frac{\partial}{\partial x_j}} \left(\frac{\partial}{\partial x_i}\right). \tag{91}$$

Therefore, it follows

$$
\begin{aligned}
&g\left(\frac{\partial}{\partial x_1}\wedge\cdots\wedge\nabla_{e_j}X\wedge\cdots\wedge\frac{\partial}{\partial x_m},v\right)=\\
&\sum_{i=m+1}^{n}\frac{\partial f_i}{\partial x_j}\,g\left(\frac{\partial}{\partial x_1}\wedge\cdots\wedge\frac{\partial}{\partial x_i}\wedge\cdots\wedge\frac{\partial}{\partial x_m},v\right)\\
&+\sum_{i=1}^{n}f_i\,g\left(\frac{\partial}{\partial x_1}\wedge\cdots\wedge\nabla_{\frac{\partial}{\partial x_j}}\left(\frac{\partial}{\partial x_i}\right)\wedge\cdots\wedge\frac{\partial}{\partial x_m},v\right).
\end{aligned}\tag{92}
$$

Thus, we put this term in equation (90)

$$
\begin{aligned}
&\sum_{j=1}^{m}\sum_{i=m+1}^{n}\frac{\partial f_i}{\partial x_j}\,g\left(\frac{\partial}{\partial x_1}\wedge\cdots\wedge\frac{\partial}{\partial x_i}\wedge\cdots\wedge\frac{\partial}{\partial x_m},v\right)\\
&+\sum_{j=1}^{m}\sum_{i=1}^{n}f_i\,g\left(\frac{\partial}{\partial x_1}\wedge\cdots\wedge\nabla_{\frac{\partial}{\partial x_j}}\left(\frac{\partial}{\partial x_i}\right)\wedge\cdots\wedge\frac{\partial}{\partial x_m},v\right)\\
&+\sum_{j=1}^{m}\sum_{i=1}^{n}f_i\,g\left(\frac{\partial}{\partial x_1}\wedge\cdots\wedge\frac{\partial}{\partial x_m},v_1\wedge\cdots\wedge\nabla_{\frac{\partial}{\partial x_i}}v_j\wedge\cdots\wedge v_m\right)=0.
\end{aligned}\tag{93}
$$

Since v is an arbitrary m-vector with degree greater that d, we have a system of partial differential equations of first order. If we want to know how many restrictions are involved, we have to know the $\dim(\Lambda^r_m(U)_p)$, where $\Lambda^r_m(U)_p$ is the space of m-vectors which have degree r greater that d at p. In this case it is crucial the assumption that the distribution $\mathcal{H}$ is equiregular, i.e. the dimension of the layers $H_i(p)$ is constant for each p in N, therefore the dimension $\dim(\Lambda^r_m(U)_p)$ will be constant as well. Thus, the number of equations l is constant at each point. Let us remind that M has degree d and that the tangent space

$$TN = H_1\oplus\cdots\oplus H_s,$$

then the PDE system has l equations, where

$$l=\sum_{r=d+1}^{s}\dim(\Lambda^r_m(U)_p).$$

The set of vector fields on an open set, $\mathcal{X}(U)$, is an infinite-dimensional space. Equations (93) give conditions that reduce the entirety of vector fields admissible for the first variation. This set will be denoted as $\mathcal{A}(U)\subset\mathcal{X}(U)$.

Then, we compute the dimension of $\Lambda^r_m(U)_p$ in order to know the number of the equations of the PDE system. Let $v=v_1\wedge\cdots\wedge v_m$ be an m-vector and let $r=d(v)$ be the fixed degree, we can choose k_1 vectors of the wedge product in the first layer H_1, k_i vectors in the i layer H_i and k_s vectors in H_s such that

$$k_1+\cdots+k_s=m.$$

Moreover, if the degree is $r = d(v)$, the coefficients k_i have to verify

$$1 \cdot k_1 + 2 \cdot k_2 + \cdots + s \cdot k_s = r.$$

Now, for each layer we can choose k_i vectors, disregarding their order, from a basis of H_i, i.e from $\dim(H_i)$ elements. The number that correspond to those choices is the binomial coefficient

$$\binom{\dim(H_i)}{k_i}.$$

Hence, we can compute the dimension of the space of m-vector of degree r as

$$\sum_{\substack{k_0+\cdots+k_s=m,\\ 0\cdot k_0+1\cdot k_1+\cdots+s\cdot k_s=r}} \left(\prod_{i=0}^{s} \binom{\dim(H_i)}{k_i} \right) = \dim(\Lambda_m^r(U)_p). \tag{94}$$

Let (M, μ) be an embedded submanifold of degree d of N with a Riemannian metric g.

$$\Phi : M \to N$$

Thanks to Definition 35 we have

$$A(M) = \int_M |(e_1 \wedge \cdots \wedge e_m)_d| \, d\mu$$

where $e_1, \cdots, e_m$ is an basis of T_pM.

Definition 3.2. Let X be a vector field with compact support in $\mathcal{A}(M)$, i.e that verifies the PDE system given by (93), and let $\{\varphi_t\}$ be the flow associated to the vector field X. We have that the first variation for the sub-Riemannian area of M is

$$\left.\frac{d}{dt}\right|_{t=0} A(\varphi_t(M)). \tag{95}$$

As we have done previously, we extend $e_1, \cdots, e_m$ to $E_1, \cdots, E_m$ along the flow line of X passing through p. Moreover, we have defined a metric g on the m-vectors of degree d, $\Lambda_m^d(U)$, thus there exists a orthonormal basis $\mathcal{D} = (w_1, \cdots, w_l)$, where l is the dimension of $\Lambda_m^d(U)$, which we calculated in (94), and w_i are m-vector in $\Lambda_m^d(U)$. Now, using the dominated convergence theorem, we put the derivative under the integral sign and it follows

$$\frac{d}{dt}\Big|_{t=0} A(\varphi_t(M)) =$$
$$\int_M \frac{d}{dt}\Big|_{t=0} |\,(E_1(t)\wedge\cdots\wedge E_m(t))_d\,|\; d\mu$$
$$= \int_M X_p \left(\sum_{w\in\mathcal{D}} g(E_1(t)\wedge\cdots\wedge E_m(t), w)^2\right)^{\frac{1}{2}} d\mu$$
$$= \int_M \frac{1}{|(e_1\wedge\cdots\wedge e_m)_d|} \sum_{w\in\mathcal{D}} X_p\; g(E_1(t)\wedge\cdots\wedge E_m(t), w)\; d\mu$$
$$= \int_M \frac{1}{|(e_1\wedge\cdots\wedge e_m)_d|} \sum_{w\in\mathcal{D}} g\left(\sum_{i=1}^m e_1\wedge\cdots\wedge(\nabla_{e_i}X)^\perp\wedge\cdots\wedge e_m, w\right) d\mu$$
$$+ \int_M \frac{1}{|(e_1\wedge\cdots\wedge e_m)_d|} \sum_{w\in\mathcal{D}} g\left(e_1\wedge\cdots\wedge e_m, \nabla_{X_p}w\right)\; d\mu.$$

5. PDE system restrictions for a graph

Let us recall the vector fields

$$X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y + k\partial_\theta, \qquad X_2 = \partial_k,$$

which generate the distribution $\mathcal{H}$, and their Lie bracket derivatives

$$X_3 = [X_1, X_2] = -\partial_\theta,$$
$$X_4 = [X_1,[X_1,X_2]] = -\sin(\theta)\partial_x + \cos(\theta)\partial_y.$$

As we did previously, we consider the metric g on the tangent bundle that makes X_1, X_2, X_3, X_4 orthonormal. We want to re-write the PDE system (93) for a (θ,k)-graph of degree four in E. The only 2-vector of degree greater than four is $X_3\wedge X_4$, therefore the PDE system is essentially an equation. The tangent vectors to the embedding Φ are

$$\Phi_x = \partial_x + \theta_x\partial_\theta + k_x\partial_k,$$
$$\Phi_y = \partial_y + \theta_y\partial_\theta + k_y\partial_k.$$

As we are working with the vector fields X_i, we express the vector field that induces the first variation respect to X_i

$$X = \sum_{i=1}^4 f_i X_i, \qquad \text{where} \quad f_i \in C_0^\infty(E)$$

We can consider as tangent vectors a linear combination of Φ_x and Φ_y, for example

$$
\begin{aligned}
e_1 &= \cos(\theta)\Phi_x + \sin(\theta)\Phi_y = X_1 + X_1(k)X_2 \\
e_2 &= -\sin(\theta)\Phi_x + \cos(\theta)\Phi_y = X_4 - X_4(\theta)X_3 + X_4(k)X_2.
\end{aligned}
$$

Instead of developing local equation (93) it would be better to consider equation (90). Therefore, we have

$$
\begin{aligned}
&\underbrace{g(\nabla_{e_1} X \wedge e_2,\ X_3 \wedge X_4)}_{A} \\
&+ \underbrace{g(e_1 \wedge \nabla_{e_2} X,\ X_3 \wedge X_4)}_{B} \\
&+ \underbrace{g(e_1 \wedge e_2,\ \nabla_X (X_3 \wedge X_4))}_{C} = 0.
\end{aligned} \tag{96}
$$

In order to develop this equation it would be useful to know

$$\nabla_{X_i} X_j \qquad i, j = 1, \cdots, 4,$$

which can be computed by the Koszul formula, see [**10**]. Let N be a Riemannian manifold of dimension n and let X, Y, Z be vector fields in $\mathfrak{X}(N)$, then there holds

$$
\begin{aligned}
2\ g(Z, \nabla_Y X) =& X\ g(Y, Z) + Y\ g(Z, X) - Z\ g(X, Y) \\
& - g([X, Z], Y) - g([Y, Z], X) - g([X, Y], Z).
\end{aligned} \tag{97}
$$

Therefore, if we write the Koszul formula for our vector fields $X_1, \cdots, X_4$, we have

$$
\begin{aligned}
2\ g(X_k, \nabla_{X_i} X_j) =& X_j\ g(X_i, X_k) + X_i\ g(X_k, X_j) - X_k\ g(X_j, X_i) \\
& - g([X_j, X_k], X_i) - g([X_i, X_k], X_j) - g([X_j, X_i], X_k),
\end{aligned} \tag{98}
$$

where $i, j, k = 1, \cdots, 4$. Later we should use all possible Lie brackets, thus here we provide a list of them

$$
\begin{aligned}
&[X_3, X_4] = X_1 + k\ X_3, \\
&[X_1, X_4] = -k\ X_1 - k^2\ X_3 \\
&[X_2, X_4] = 0.
\end{aligned}
$$

We start from $\nabla_{X_1} X_2$:

$$
\begin{aligned}
2\ g(X_1, \nabla_{X_1} X_2) &= -2\ g([X_2, X_1], X_1) = -2\ g(X_3, X_1) = 0 \\
2\ g(X_2, \nabla_{X_1} X_2) &= -\ g([X_1, X_2], X_2) -\ g([X_2, X_1], X_2) = 0 \\
2\ g(X_3, \nabla_{X_1} X_2) &= -\ \cancel{g([X_2, X_3], X_1)} -\ \cancel{g([X_1, X_3], X_2)} - g([X_2, X_1], X_3) = 1 \\
2\ g(X_4, \nabla_{X_1} X_2) &= -\ g([X_2, X_4], X_1) -\ g([X_1, X_4], X_2) - g([X_2, X_1], X_4) \\
&= g(k\ X_1 + k^2\ X_3, X_2) = 0.
\end{aligned}
$$

Therefore, it follows

$$\nabla_{X_1} X_2 = \frac{1}{2} X_3 = \frac{1}{2}[X_1, X_2].$$

Moreover, thanks to the symmetry of the Levi-Civita connection, we have

$$\nabla_{X_1}X_2 - \nabla_{X_2}X_1 = [X_1, X_2] \quad \Rightarrow \quad \nabla_{X_2}X_1 = -\frac{1}{2}X_3.$$

Now, we compute $\nabla_{X_1}X_3$:

$$\begin{aligned}
2\ g(X_1, \nabla_{X_1}X_3) &= -2\ g([X_3, X_1], X_1) = 2\ g(X_4, X_1) = 0\\
2\ g(X_2, \nabla_{X_1}X_3) &= -\cancel{g([X_3, X_2], X_1)} - g([X_1, X_2], X_3) - \cancel{g([X_3, X_1], X_2)} = -1\\
2\ g(X_3, \nabla_{X_1}X_3) &= -g([X_3, X_3], X_1) - g([X_1, X_3], X_3) - g([X_3, X_1], X_3) = 0\\
2\ g(X_4, \nabla_{X_1}X_3) &= -g([X_3, X_4], X_1) - g([X_1, X_4], X_3) - g([X_3, X_1], X_4)\\
&= -1 + k^2 + 1 = k^2.
\end{aligned}$$

Therefore, we have

$$\nabla_{X_1}X_3 = \frac{1}{2}(-X_2 + k^2\ X_4).$$

In addition, by the symmetry of Levi-Civita connection, it follows

$$\nabla_{X_1}X_3 - \nabla_{X_3}X_1 = [X_1, X_3] = X_4 \quad \Rightarrow \quad \nabla_{X_3}X_1 = -\frac{1}{2}X_2 + \left(\frac{k^2}{2} - 1\right) X_4.$$

For the term $\nabla_{X_1}X_4$ we have:

$$\begin{aligned}
2\ g(X_1, \nabla_{X_1}X_4) &= -2\ g([X_4, X_1], X_1) = -2\ k\\
2\ g(X_2, \nabla_{X_1}X_4) &= -g([X_4, X_2], X_1) - g([X_1, X_2], X_4) - g([X_4, X_1], X_2) = 0\\
2\ g(X_3, \nabla_{X_1}X_4) &= -g([X_4, X_3], X_1) - g([X_1, X_3], X_4) - g([X_4, X_1], X_3)\\
&= 1 - 1 - k^2 = -k^2\\
2\ g(X_4, \nabla_{X_1}X_4) &= -g([X_4, X_4], X_1) - g([X_1, X_4], X_4) - g([X_4, X_1], X_4)) = 0.
\end{aligned}$$

Hence, it follows

$$\nabla_{X_1}X_4 = -k\ X_1 - \frac{k^2}{2}X_3.$$

Furthermore, the symmetry of Levi-Civita connection allow us to determine $\nabla_{X_4}X_1$ using the Lie bracket

$$\nabla_{X_1}X_4 - \nabla_{X_4}X_1 = [X_1, X_4] = -k\ X_1 - k^2\ X_3 \quad \Rightarrow \quad \nabla_{X_4}X_1 = \frac{k^2}{2}X_3.$$

We miss to calculate $\nabla_{X_1}X_1$:

$$\begin{aligned}
2\ g(X_1, \nabla_{X_1}X_1) &= 0\\
2\ g(X_2, \nabla_{X_1}X_1) &= -2\ g([X_1, X_2], X_1) = 0\\
2\ g(X_3, \nabla_{X_1}X_1) &= -2\ g([X_1, X_3], X_1) = 0\\
2\ g(X_4, \nabla_{X_1}X_1) &= -2\ g([X_1, X_4], X_1) = 2\ k.
\end{aligned}$$

Therefore, we have

$$\nabla_{X_1} X_1 = k\ X_4.$$

Now, we compute $\nabla_{X_2} X_2$:

$$\begin{aligned}
&2\ g(X_1, \nabla_{X_2} X_2) = -2\ g([X_2, X_1], X_2) = 0\\
&2\ g(X_2, \nabla_{X_2} X_2) = 0\\
&2\ g(X_3, \nabla_{X_2} X_2) = -2\ g([X_2, X_3], X_2) = 0\\
&2\ g(X_4, \nabla_{X_2} X_2) = -2\ g([X_2, X_4], X_2) = 0.
\end{aligned}$$

Therefore, we have

$$\nabla_{X_2} X_2 = 0.$$

Now, we compute $\nabla_{X_3} X_3$:

$$\begin{aligned}
&2\ g(X_1, \nabla_{X_3} X_3) = -2\ g([X_3, X_1], X_3) = 0\\
&2\ g(X_2, \nabla_{X_3} X_3) = -2\ g([X_3, X_2], X_3) = 0\\
&2\ g(X_3, \nabla_{X_3} X_3) = 0\\
&2\ g(X_4, \nabla_{X_3} X_3) = -2\ g([X_3, X_4], X_3) = -2\ k.
\end{aligned}$$

Therefore, we have

$$\nabla_{X_3} X_3 = -k\ X_4.$$

Now, we compute $\nabla_{X_4} X_4$:

$$\begin{aligned}
&2\ g(X_1, \nabla_{X_4} X_4) = -2\ g([X_4, X_1], X_4) = 0\\
&2\ g(X_2, \nabla_{X_4} X_4) = -2\ g([X_4, X_2], X_4) = 0\\
&2\ g(X_3, \nabla_{X_4} X_4) = -2\ g([X_4, X_3], X_4) = 0\\
&2\ g(X_4, \nabla_{X_4} X_4) = -0.
\end{aligned}$$

Therefore, we have

$$\nabla_{X_4} X_4 = 0.$$

Now, it is the time of $\nabla_{X_2} X_3$:

$$\begin{aligned}
&2\ g(X_1, \nabla_{X_2} X_3) = -\cancel{g([X_3, X_1], X_2)} - g([X_2, X_1], X_3) - \cancel{g([X_3, X_2], X_1)} = 1\\
&2\ g(X_2, \nabla_{X_2} X_3) = -g([X_3, X_2], X_2) - g([X_2, X_2], X_3) - g([X_3, X_2], X_2) = 0\\
&2\ g(X_3, \nabla_{X_2} X_3) = -g([X_3, X_3], X_2) - g([X_2, X_3], X_3) - g([X_3, X_2], X_3) = 0\\
&2\ g(X_4, \nabla_{X_2} X_3) = -g([X_3, X_4], X_2) - g([X_2, X_4], X_3) - g([X_3, X_2], X_4) = 0.
\end{aligned}$$

Thus, we have

$$\nabla_{X_2} X_3 = \frac{1}{2} X_1$$

and obviously, since $[X_3, X_2] = 0$, it follows

$$\nabla_{X_3} X_2 = \frac{1}{2} X_1.$$

Now, we compute $\nabla_{X_2} X_4$:

$$\begin{aligned}
2\ g(X_1, \nabla_{X_2} X_4) &= -\cancel{g([X_4, X_1], X_2)} - \cancel{g([X_2, X_1], X_4)} - \cancel{g([X_4, X_2], X_1)} = 0 \\
2\ g(X_2, \nabla_{X_2} X_4) &= -2\ g([X_4, X_2], X_2) = 0 \\
2\ g(X_3, \nabla_{X_2} X_4) &= -\cancel{g([X_4, X_3], X_2)} - \cancel{g([X_2, X_3], X_4)} - \cancel{g([X_4, X_2], X_3)} = 0 \\
2\ g(X_4, \nabla_{X_2} X_4) &= -g([X_2, X_4], X_4) - g([X_4, X_2], X_4) = 0.
\end{aligned}$$

Therefore, we have

$$\nabla_{X_2} X_4 = 0.$$

Since $[X_2, X_4] = 0$, it follows

$$\nabla_{X_4} X_2 = 0.$$

We miss the last term $\nabla_{X_3} X_4$:

$$\begin{aligned}
2\ g(X_1, \nabla_{X_3} X_4) &= -g([X_4, X_1], X_3) - g([X_3, X_1], X_4) - g([X_4, X_3], X_1) \\
&= -k^2 + 1 + 1 = -k^2 + 2 \\
2\ g(X_2, \nabla_{X_3} X_4) &= -g([X_4, X_2], X_3) - g([X_3, X_2], X_4) - g([X_4, X_3], X_2) = 0 \\
2\ g(X_3, \nabla_{X_3} X_4) &= -2\ g([X_4, X_3], X_3) = 2\ k \\
2\ g(X_4, \nabla_{X_3} X_4) &= -g([X_3, X_4], X_4) - g([X_4, X_3], X_4) = 0.
\end{aligned}$$

At the end, we have

$$\nabla_{X_3} X_4 = \left(-\frac{k^2}{2} + 1 \right)\ X_1 + k X_3.$$

Since $[X_3, X_4] = X_1 + k\ X_3$, it follows

$$\nabla_{X_4} X_3 = -\frac{k^2}{2}\ X_1.$$

Let us compute separately each term (A,B,C) of the equation (96), using the covariant derivatives we have just evaluated. First of all we take into account

$$\mathrm{A} = g(\nabla_{e_1} X \wedge e_2,\ X_3 \wedge X_4). \tag{99}$$

It is necessary to develop the following term

$$
\begin{aligned}
\nabla_{e_1} X \wedge e_2 &= \left(\nabla_{X_1 + X_1(k) X_2} \sum_{i=1}^{4} f_i \; X_i \right) \wedge e_2 \\
&= \left(\nabla_{X_1} \left(\sum_{i=1}^{4} f_i \; X_i \right) + X(k) \nabla_{X_2} \left(\sum_{i=1}^{4} f_i \; X_i \right) \right) \wedge e_2 \\
&= \left(\sum_{i=1}^{4} (X_1(f_i) + X_1(k) \; X_2(f_i)) X_i \right) \wedge e_2 \\
&\quad + \left(\sum_{i=1}^{4} f_i (\nabla_{X_1} X_i + X_1(k) \nabla_{X_2} X_i) \right) \wedge e_2 \\
&= \sum_{i=1}^{4} (X_1(f_i) + X_1(k) \; X_2(f_i)) \;\; X_i \wedge e_2 \\
&\quad + f_1 \left(k \; X_4 - \frac{X_1(k)}{2} \; X_3 \right) \wedge e_2 + \frac{1}{2} f_2 \; X_3 \wedge e_2 \\
&\quad + f_3 \left(\frac{1}{2}(-X_2 + k^2 X_4) + \frac{X_1(k)}{2} X_1 \right) \wedge e_2 \\
&\quad + f_4 \left(-k \; X_1 - \frac{k^2}{2} X_3 \right) \wedge e_2 .
\end{aligned} \tag{100}
$$

Notice that

$$
\begin{aligned}
g(X_i \wedge e_2, X_3 \wedge X_4) &= g(X_i, X_3) g(e_2, X_4) - g(X_i, X_4) g(e_2, X_3) \\
&= \begin{cases} 0 & i = 1, 2 \\ 1 & i = 3 \\ X_4(\theta) & i = 4. \end{cases}
\end{aligned} \tag{101}
$$

Putting (100) in (99) and taking into account (101), we have

$$
\begin{aligned}
\text{A} =& X_1(f_3) + X_1(k) \; X_2(f_3) + X_4(\theta) \; (X_1(f_4) + X_1(k) \; X_2(f_4)) \\
&+ \left(-\frac{X_1(k)}{2} + k \; X_4(\theta) \right) f_1 + \frac{1}{2} f_2 + \left(\frac{k^2}{2} X_4(\theta) \right) \; f_3 - \frac{k^2}{2} f_4
\end{aligned} \tag{102}
$$

Now, we consider the second term of equation (96)

$$
B = g(e_1 \wedge \nabla_{e_2} X, \;\; X_3 \wedge X_4).
$$

Before we start computing $e_1 \wedge \nabla_{e_2} X$, it is convenient to notice that

$$
g(e_1 \wedge X_i, X_3 \wedge X_4) = g(e_1, X_3) g(X_i, X_4) - g(e_1, X_4) g(X_i, X_3) = 0,
$$

since $e_1 = X_1 + X_1(k)X_2$ does not have components in the direction X_3 and X_4. Therefore, it follows that

$$\mathrm{B} = 0.$$

The last term of equation (96) is

$$\mathrm{C} = g(e_1 \wedge e_2,\ \nabla_X (X_3 \wedge X_4)) = \underbrace{g(e_1 \wedge e_2,\ \nabla_X X_3 \wedge X_4)}_{\mathrm{I}} + \underbrace{g(e_1 \wedge e_2,\ X_3 \wedge \nabla_X X_4)}_{\mathrm{II}}$$

Now,

$$\mathrm{I} = g(e_1, \nabla_X X_3)\ g(e_2, X_4) - \cancel{g(e_1, X_4)\ g(e_2, \nabla_X X_3)} = g(e_1, \nabla_X X_3).$$

Furthermore, we have

$$\sum_{i=1}^{4} f_i\ \nabla_{X_i} X_3 = \frac{1}{2} f_1 \left(-X_2 + k^2 X_4\right) + \frac{1}{2} f_2\ X_1 - k\ f_3\ X_4 - f_4\ \frac{k^2}{2}\ X_1.$$

If we make the scalar product of this term and e_1, we have

$$\mathrm{I} = -\frac{X_1(k)}{2} f_1 + \frac{1}{2} f_2 - \frac{k^2}{2}\ f_4. \tag{103}$$

Now, we consider

$$\mathrm{II} = \cancel{g(e_1, X_3) g(e_2, \nabla_X X_4)} - g(e_1, \nabla_X X_4) g(e_2, X_3) = X_4(\theta)\ g(e_1, \nabla_X X_4).$$

In addition, we have

$$\sum_{i=1}^{4} f_i\ \nabla_{X_i} X_4 = f_1 \left(-k\ X_1 - \frac{k^2}{2} X_3\right) + \left(\left(-\frac{k^2}{2} + 1\right)\ X_1 + k X_3\right)\ f_3.$$

If we make the scalar product of this term and e_1, we have

$$\mathrm{II} = \left(-k\ f_1 + \left(-\frac{k^2}{2} + 1\right)\ f_3\right)\ X_4(\theta).$$

Then, if we add I and II it follows

$$\mathrm{C} = \mathrm{I}\ +\ \mathrm{II} = \left(-\frac{X_1(k)}{2} - k\ X_4(\theta)\right) f_1 + \frac{1}{2} f_2 + \left(1 - \frac{k^2}{2}\right)\ X_4(\theta)\ f_3 - \frac{k^2}{2}\ f_4$$

Going back to equation (96), we have

$$
\begin{aligned}
0 =& \mathrm{A} + \mathrm{B} + \mathrm{C} \\
=& X_1(f_3) + X_1(k)\ X_2(f_3) + X_4(\theta)\ (X_1(f_4) + X_1(k)\ X_2(f_4)) \\
&+ \left(-\frac{X_1(k)}{2} + k\ X_4(\theta)\right) f_1 + \frac{1}{2} f_2 + \left(\frac{k^2}{2} X_4(\theta)\right)\ f_3 - \frac{k^2}{2} f_4 \\
&+ \left(-\frac{X_1(k)}{2} - k\ X_4(\theta)\right) f_1 + \frac{1}{2} f_2 + \left(1 - \frac{k^2}{2}\right)\ X_4(\theta)\ f_3 - \frac{k^2}{2}\ f_4 \\
=& X_1(f_3) + X_1(k)\ X_2(f_3) + X_4(\theta)\ (X_1(f_4) + X_1(k)\ X_2(f_4)) \\
&- X_1(k) f_1 + f_2 + X_4(\theta)\ f_3 - k^2 f_4.
\end{aligned} \tag{104}
$$

At the end, the PDE equation is

$$
\begin{aligned}
&X_1(f_3) + X_1(k)\ X_2(f_3) + X_4(\theta)\ (X_1(f_4) + X_1(k)\ X_2(f_4)) \\
&- X_1(k) f_1 + f_2 + X_4(\theta)\ f_3 - k^2 f_4 = 0.
\end{aligned} \tag{105}
$$

Remark 3.3. There are two variations making the surface flow into itself. This phenomenon happens when the vector fields X, inducing the variation, is equal to tangent vectors e_1 or e_2. Indeed the coefficients $f_1 = 1$, $f_2 = X_1(k)$, $f_3 = 0$ and $f_4 = 0$ of

$$e_1 = X_1 + X_1(k) X_2$$

satisfy equation (105). Moreover, if we put the coefficients $f_1 = 0$, $f_2 = X_4(k)$, $f_3 = -X_4(\theta)$ and $f_4 = 1$ of

$$e_2 = X_4 - X_4(\theta) X_3 + X_4(k) X_2.$$

in equation (105) we obtain

$$-X_1(X_4(\theta)) + X_4(X_1(\theta)) - X_4(\theta)^2 - X_1(\theta)^2$$

which thanks to Remark 3.2 is equal to

$$\theta_x^2 + \theta_y^2 - X_4(\theta)^2 - X_1(\theta)^2 = 0.$$

When we induce a variation $(\theta + tv, k + tw)$ of a (θ,k)-graph in direction θ and k the vector field inducing the variation is

$$X = w(x, y) X_2 - v(x, y) X_3.$$

Therefore the PDE equation (105) corresponds to

$$-\overline{X_1}(v) + w - X_4(\theta) v = 0.$$

This is the condition deduced in section 2 by the implicit function theorem.

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