ALMA MATER STUDIORUM · UNIVERSITÀ DI BOLOGNA

FACOLTÁ DI SCIENZE MATEMATICHE, FISICHE E NATURALI
Corso di Laurea Magistrale in Matematica
Curriculum generale-applicativo

FX MODELLING UNDER COLLATERALIZATION

Tesi di Laurea Magistrale in equazioni differenziali stocastiche

Relatore:
Chiar.mo Prof.
Andrea Pascucci

Presentata da:
Chiara Summonte

Sessione II

Anno Accademico 2015-2016

Alla mia famiglia

Contents

List of Figures

Introduction

The financial crisis begun in the second half of 2007 has triggered, among many consequences, a deep evolution phase of the classical framework adopted for trading derivatives. In particular, credit and liquidity issues were found to have macroscopical impacts on the prices of financial instruments, both plain vanillas and exotics.

Today, terminated or not the crisis, the market has learnt the lesson and persistently shows such effects. These are clearly visible in the market quotes of plain vanilla interest rate derivatives, such as Deposits, Forward Rate Agreements (FRA), Swaps (IRS) and options (Caps, Floors and Swaptions). Since August 2007 the primary interest rates of the interbank market, e.g. Libor , Euribor, Eonia, and Federal Funds rate, display large basis spreads that have raised up to 200 basis points. Recently, the market has also included the effect of collateral agreements widely diffused among derivatives counterparties in the interbank market.

In this context, our aim is to describe how to coherently price derivatives with flows and/or collateral posting in different currencies in presence of market dislocations and relying on funding strategies based on FX swaps. We extend the usual arbitrage-free pricing framework to accommodate collateral accounts by means of a more general definition of dividend and gain processes and we give clear definitions of the relevant pricing measures. We finally apply these results to derive pricing formulae for derivative contracts under different collateralization agreements.

The structure of this thesis is the following.

In Chapter 1 we start by reviewing the market practice for interest rate yield curves construction and pricing interest rate derivatives, both in the traditional (old style) single-curve version, and in the modern multiple-curve version triggered by the credit crunch crisis. Then we quickly describe two one-factor short rate models, such as the Vasicek and the Hull-White models, with an example of calibration to market data for the Vasicek model.

In Chapter 2 we derive the classical Black-Scholes-Merton pricing formulas using replication arguments, PDE and Feynman-Kac. Afterwards we generalize this formula by considering more general cases such as perfect collateral for derivative and for both derivative and hedge, derivatives on a dividend paying asset subject to repo funding, multiple currencies, etc.

In Chapter 3, that constitutes the central contribution of this work, we start from some formulas obtained in chapter 2 and we derive generic pricing formulae for different combinations of cash flow and collateral currencies. Then we apply the results to the pricing of FX swaps and CCS, and we discuss curve bootstrapping. Finally we investigate some approximations usually done in the practice when evaluating CCS.

Introduzione

La crisi finanziaria iniziata nella seconda metà del 2007 ha innescato, tra le molte conseguenze, una fase di profonda evoluzione del quadro classico adottato per il trading di derivati. In particolare, si é riscontrato che i problemi di credito e liquidità hanno effetti macroscopici sui prezzi degli strumenti finanziari, sia plain vanilla sia esotici.

Oggi, terminata o no la crisi, il mercato ha imparato la lezione e mostra in modo persistente tali effetti. Questi effetti sono chiaramente visibili nelle quotazioni di mercato dei derivati plain vanilla su tassi di interesse, come ad esempio Depositi, Forward Rate Agreement (FRA), swap (IRS) e opzioni (cap, floor e swaption). Dal mese di agosto 2007, i tassi di interesse primari del mercato interbancario, per esempio Libor, Euribor, Eonia, e il tasso dei fondi federali, mostrano un ampio basis spread arrivato fino a 200 punti base. Recentemente, il mercato ha anche incluso l'effetto dei collateral agreements ampiamente utilizzati nel mercato interbancario.

In questo contesto, il nostro scopo è quello di descrivere come prezzare coerentemente derivati con flussi di moneta e/o collaterale postato in diverse valute, in presenza di dislocazioni di mercato e contando su strategie di finanziamento basate su FX swap. Estendiamo il framework usuale di arbitrage-free pricing per accogliere collateral accounts attraverso una più generale definizione dei processi di dividendi e di guadagno e diamo chiare definizioni delle misure di pricing piú rilevanti. Applichiamo infine questi risultati per ricavare formule di pricing per contratti di derivati sotto diversi accordi di collateralizzazione.

La struttura di questa tesi è la seguente.
Nel capitolo 1 rivediamo la prassi di mercato per la costruzione delle curve dei rendimenti dei tassi di interesse e la determinazione dei prezzi dei derivati su tassi di interesse, sia nella versione tradizionale "single-curve" sia nella versione moderna "multiple curve" innescata dalla crisi del credito. In seguito descriviamo rapidamente due modelli per i tassi short, come il modello di Vasicek e quello di Hull-White, con un esempio di calibrazione ai dati di mercato per il modello di Vasicek.
Nel capitolo 2 deriviamo la formule classica di pricing di Black-Scholes-Merton utilizzando argomenti di replicazione, PDE e Feynman-Kac. Successivamente generalizziamo questa formula considerando i casi più generali come perfect collateral sia per i derivati sia per la copertura, derivati su asset che pagano dividendi soggetti a finanziamenti di tipo repo, valute multiple, ecc.
Nel capitolo 3, che costituisce il contributo centrale di questo lavoro, partiamo da alcune formule ottenuto nel capitolo 2 e deriviamo formule generiche di pricing per combinazioni diverse di cash flow e valute per il collaterale. Applichiamo poi i risultati al pricing dei FX swap e CCS, e discutiamo di bootstrapping. Infine analizziamo alcune approssimazioni applicate usualmente nella valutazione dei CCS.

Chapter 1

Interest rate derivatives

1.1 Classical vs modern pricing framework

A practice widely spread after the credit crunch is the one for which derivative contracts are traded along with insurances to protect from default events. An amount of cash or high quality assets is usually posted on a prefixed schedule to the counterparty to match the marked-to-market value of the position. The assets used as insurance are known as **collaterals** or **margins**. How to manage the collateral account during the life of the contract (margining procedure) and what happens on default of one of the counterparties is regulated by a bilateral agreement documented by ISDA, known as Credit Support Annex (CSA). In particular, the agreement regulates the possibility of re-hypothecating the collateral assets, namely to use them for funding purposes.

Trade between two counterparties under collateral	**Counterparty A**	**Counterparty B**
Trade value V(t) Collateral account C(t) Collateral interest R(t)	Positive (receiver) Receives collateral Pays interest	Negative (payer) Posts collateral Receives interest
Trade value V(t) Collateral account C(t) Collateral interest R(t)	Negative (receiver) Posts collateral Receives interest	Positive (payer) Receives collateral Pays interest

Figure 1.1: Collateral scheme

In banking, collateral has two meanings:

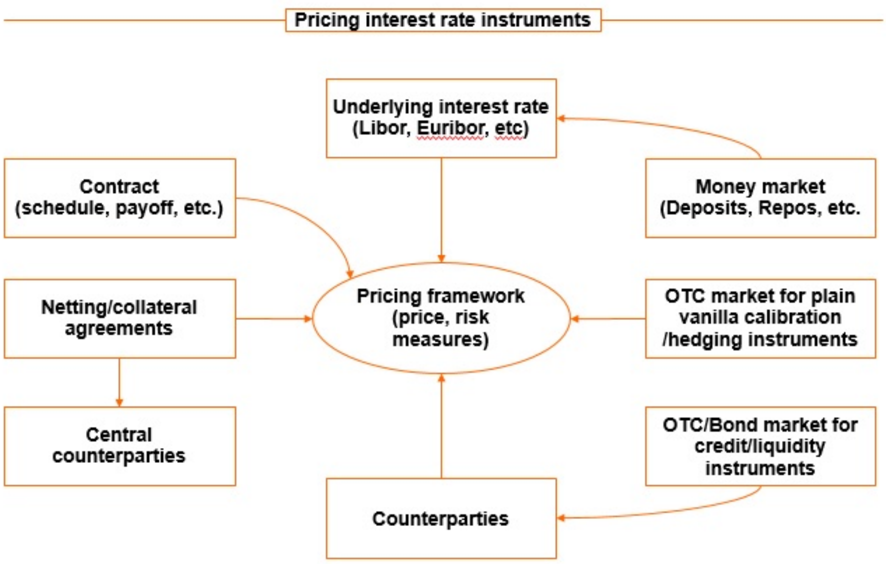
- **asset-based lending**: the traditional secured lending, with unilateral obligations, secured in the form of property, surety, guarantee or other;
- **capital market collateralization**: used to secure trade transactions, with bilateral obligations, secured by more liquid assets such as cash or securities, also known as margin.

As a consequence, the classical theoretical framework adopted to price derivatives has become obsolete. Well-known relations described on standard textbooks and holding since decades had to be abandoned in one day.

The modern theoretical framework, still under active research and development, includes a larger set of market information and of relevant risk factors, credit and funding risk in particular, and requires to review "from scratch" the no-arbitrage models used on the market for derivatives' pricing and risk analysis.

In order to understand and to price interest rate linked instruments we must know their characteristics.

- The underlying: the interest rates, such as ECB rates, Libor, Euribor, Eonia, etc.
- The money market where such interest rates are traded and the basic lending/borrowing contracts, such as Deposits.
- The contract mechanics: the schedule with all the contract relevant dates, the payoff, and any other condition affecting the price.
- The counterparties: leading to credit/default issues, and to the corresponding credit/debt market (CDS and Bonds).
- The collateral: leading to liquidity/funding issues and to Central Counterparties.
- The OTC market where the basic plain vanilla derivatives, are traded used for yield curve and volatility construction, for calibration and for hedging purposes.
- The pricing model, to calculate prices and risk measures (sensitivities, VaR, etc.)



Hence, the results of the financial crisis are several changes in the market, which can be summarized in these stylized facts:

1. Banks are not credit risk free and are not too big to fail, credit and liquidity risk in market benchmark interest rates (Ibor), tenor dependency.
2. Explosion of spot/forward market Ibor/OIS and Ibor/Ibor tenor basis.
3. Explosion of single and cross currency basis swap rates.
4. Break of the classic-no-arbitrage relationships between market FRA rates and forward rates implicit in market Deposits.

5. Shift from unsecured to secured money market funding, diffusion of collateralization, CSA chaos, new ISDA Standard CSA.

6. Trades migration to Central Counterparties (CCPs).

7. Shift towards CSA discounting for collateralized trades, changes in market quotations for OTC derivatives, multiplication of interest rate yield curves used for pricing interest rate derivatives.

8. Reactions of regulators.

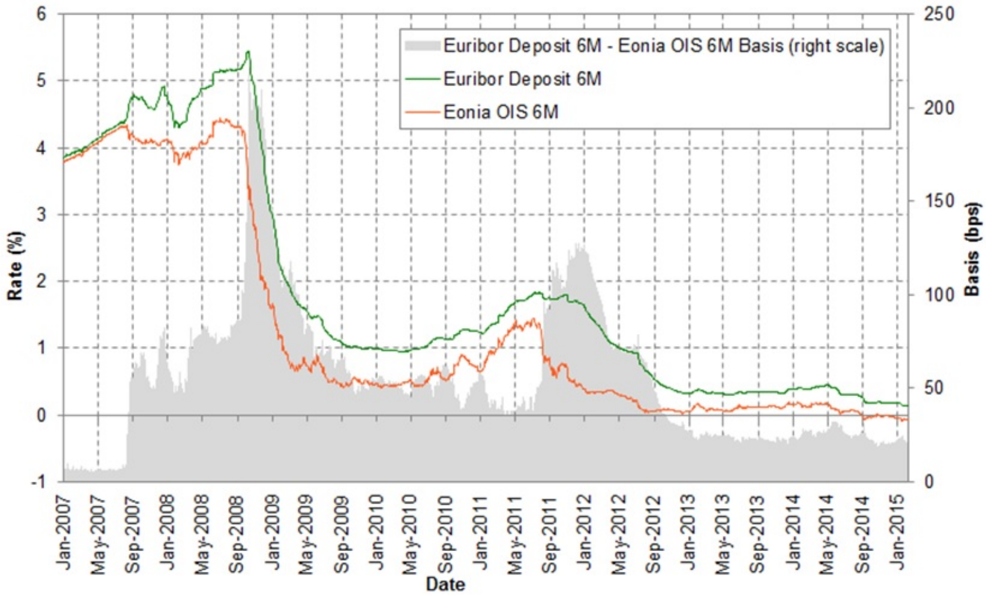


Figure 1.2: Euribor6M Depo vs Eur OIS 6M (spot) rates. Quotations Jan. 2007−Jan. 2015 (source: Bloomberg)

We will discuss now some important interest rate financial instruments. In particular, we will focus on Deposits, Futures and Swaps, in order to construct a discount curve that will help us to calibrate Vasicek and Hull-White short rate models.

We will adopt a systematic description approach based on the following scheme.

- Instrument description: payoff, pictures, discussion, etc.
- Instrument pricing: derivation of the relevant pricing formulas.
- Market data: possible quotations of the financial instrument available on the market.
- Discussion: classical vs modern pricing, etc.

Interest rate instruments depend, in general, on two distinct interest rates.

1. The underlying rate of the instrument, and related quantities, indexed with "x", such as $L_x(T_1, T_2)$.

2. The discount rate associated to the instrument, and related quantities, indexed with "d", such as $P_d(T_1, T_2)$. Notice that the discount rate depends on the funding sources.

1.2 Forward rates

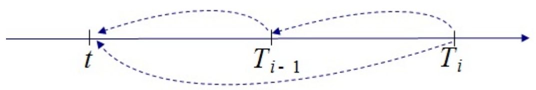
Forward rates $F(t;T_{i-1},T_i)$ are interest rates observed at a generic time instant t, resetting at future time T_{i-1} and spanning the future time interval $[T_{i-1};T_i]$ (called rate tenor), with $t < T_{i-1} < T_i$. Forward rates can be expressed in terms of Zero Coupon Bonds by recurring to a simple no-arbitrage argument. If we define the forward Zero Coupon Bond observed at time t as

$$P(t;T_{i-1},T_i) := \mathbb{E}_t^{\mathbb{Q}}[P(T_{i-1};T_i)]$$

we may write the following no arbitrage relation for deterministic $N(T_i)$

$$N(t) = P(t;T_i)N(T_i) = P(t;T_{i-1})P(t;T_{i-1},T_i)N(T_i)$$

The financial meaning of expression above is that, given a deterministic amount of money $N(T_i)$ at time T_i, its equivalent amount of money, or value, at time $t < T_i$ must be unique, both if we discount directly in one single step from T_i to t, using the discount factor $P(t;T_i)$, and if we discount in two steps, first from T_i to T_{i-1}, using the forward discount factor $P(t;T_{i-1},T_i)$ and then from T_{i-1} to t, using $P(t;T_{i-1})$.



At this point we may define the simple compounded forward rate $F(t;T_{i-1},T_i)$ associated to $P(t;T_{i-1},T_i)$ as

$$P(t;T_{i-1},T_i) = \frac{P(t;T_i)}{P(t;T_{i-1})} := \frac{1}{1+F(t;T_{i-1},T_i)\tau(T_{i-1},T_i)}.$$

By inverting we obtain the familiar no arbitrage expression

$$F_i(t) := F(t;T_{i-1},T_i) = \frac{1}{\tau(T_{i-1},T_i)}\left[\frac{1}{P(t;T_{i-1},T_i)} - 1\right] = \frac{P(t;T_{i-1}) - P(t;T_i)}{\tau(T_{i-1},T_i)P(t;T_i)}$$

We notice that, for $t \to T_{i-1}^-$ forward rates converge to spot Libor rates.

Theorem 1.1. *Forward rates are martingales under their "natural" T_i-forward measure $\mathbb{Q}^{T_i}$:*

$$F_i(t) = \mathbb{E}_t^{\mathbb{Q}^{T_i}}[F_i(u)] \quad \forall t < u < T_{i-1} < T_i$$

Proof. The quantity

$$\Pi(t) := P(t;T_i)F_i(t)\tau(T_{i-1},T_i) = P(t;T_{i-1}) - P(t;T_i)$$

is the time-t price of a tradable asset, since it is a combination of two (tradable) Zero Coupon Bonds. Hence, under the $\mathbb{Q}^{T_i}$-forward measure,

$$\frac{\Pi(t)}{P(t;T_i)} = \frac{P(t;T_i)F_i(t)}{P(t;T_i)} = F_i(t) = \mathbb{E}_t^{\mathbb{Q}^{T_i}}\left[\frac{\Pi(u)}{P(u;T_i)}\right]$$
$$= \mathbb{E}_t^{\mathbb{Q}^{T_i}}\left[\frac{F_i(u)P(u;T_i)}{P(u;T_i)}\right] = \mathbb{E}_t^{\mathbb{Q}^{T_i}}[F_i(u)]$$

In particular, setting $u = T_{i-1}$ we obtain

$$F_i(t) = \mathbb{E}_t^{\mathbb{Q}^{T_i}}[L(T_{i-1},T_i)]$$

□

Notice that (risky) Libor rates $L_x(T_{i-1}, T_i)$ are not martingales under the T_i-forward measure $\mathbb{Q}^{T_i}$,

$$F_i(t) = \mathbb{E}_t^{\mathbb{Q}^{T_i}}[L(T_{i-1}, T_i)] \neq \mathbb{E}_t^{\mathbb{Q}^{T_i}}[L_x(T_{i-1}, T_i)] := F_{x,i}(t)$$

since the underlying Libor rate is, in general, different from the funding rate associated to the probability measure.
We define

$$F_{x,i}(t) := \mathbb{E}_t^{\mathbb{Q}^{T_i}}[L_x(T_{i-1}, T_i)]$$

the risky forward rate.
When the funding rate and the underlying rate are the same, or in case of vanishing interest rate basis, we obtain the classical (pre-credit crunch) single curve limit

$$F_{x,i}(t) = \mathbb{E}_t^{\mathbb{Q}^{T_i}}[L_x(T_{i-1}, T_i)] \longrightarrow \mathbb{E}_t^{\mathbb{Q}^{T_i}}[L(T_{i-1}, T_i)] = F_i(t)$$

Properties of the risky forward rate:

1. at fixing date T_{i-1} it coincides with the Libor rate

$$F_{x,i}(T_{i-1}) = L_x(T_{i-1}, T_i)$$

2. It is a martingale under the T_i-forward discounting measure associated to the numeraire $P_d(t; T_i)$:

$$F_{x,i}(t) = \mathbb{E}_t^{\mathbb{Q}^{T_i}}[L_x(T_{i-1}, T_i)] = \mathbb{E}_t^{\mathbb{Q}^{T_i}}[F_{x,i}(T_{i-1})]$$

3. FRA contracts are quoted on the market in terms of their forward rates, thus it is "what you read on the screen". A forward rate term structure can be stripped from FRA quotations.

4. The risky forward rate is the basic building block of the new theoretical interest rate framework.

Instantaneous forward rate:
Instantaneous forward rates $f(t;T)$ are abstract forward rates observed at time t and spanning an infinitesimal future time interval $[T; T+dt]$ (with infinitesimal rate tenor). They are thus obtained through the limit

$$\begin{aligned} f(t;T) &:= \lim_{T'\to T^+} F(t;T,T') = -\lim_{T'\to T^+} \frac{1}{P(t;T')} \frac{P(t;T') - P(t;T)}{\tau(T,T')} \\ &= -\frac{1}{P(t;T)} \frac{\partial P(t;T)}{\partial T} = -\frac{\partial lnP(t;T)}{\partial T} \end{aligned}$$

Integrating the equation above we can express the Zero Coupon Bond as an integral of instantaneous forward rates as

$$P(t;T) = exp\left[-\int_t^T f(t;u)du\right]$$

Instantaneous forward rates can also be calculated as expectations of future short rates under the T-forward measure $\mathbb{Q}^T$. In fact, setting $u = T$ in the martingality relation for the forward rate we obtain

$$f(t;T) = \mathbb{E}_t^{\mathbb{Q}^T}[r(T)]$$

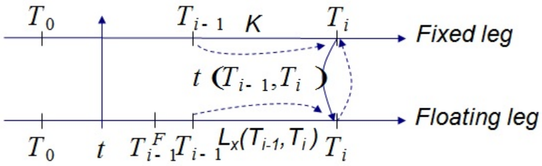
1.3 Forward Rate Agreements

Forward Rate Agreements (FRA) are standard OTC contracts with two legs starting at time T_0. The floating leg pays the interest accrued with a (risky) Libor $L_x(T_{i-1}, T_i)$ reset at time T^F_{i-1}, and spanning the time interval $[T_{i-1}; T_i]$. The fixed leg pays the interest accrued with a fixed rate K over the same time interval $[T_{i-1}; T_i]$. There are two types of FRA:

1. **Standard (or textbook) FRA**:
   the payoff off the standard FRA at payment date T_i is given by

$$\boldsymbol{FRA}_{Std}(T_i; \boldsymbol{T}, K, w) = Nw[L_x(T_{i-1}, T_i) - K]\tau_x(T_{i-1}, T_i)$$

   where $w = \pm 1$ for a payer/receiver FRA (referred to the fixed leg), respectively, and for simplicity we have assumed that both rates are annual, simply compounded, and share the same year fraction and day count convention. The price of the standard FRA at time $t < T_{i-1}$ is given, under the



   payment T_i-forward measure, by

$$\begin{aligned}\boldsymbol{FRA}_{Std}(t; \boldsymbol{T}, K, w) &= P_d(t; T_i)\mathbb{E}^{\mathbb{Q}^{T_i}}_t [\boldsymbol{FRA}_{Std}(T_i; \boldsymbol{T}, K, w)] \\ &= NwP_d(t; T_i) \left\{\mathbb{E}^{\mathbb{Q}^{T_i}}_t [L_x(T_{i-1}, T_i)] - K\right\} \tau_x(T_{i-1}, T_i) \\ &= NwP_d(t; T_i)F_{x,i}(t) - K\tau_x(T_{i-1}, T_i)\end{aligned}$$

   The FRA rate at time t is defined as the fixed rate K that makes null the FRA present value,

$$R^{FRA}_{x,Std}(t; \boldsymbol{T}) = F_{x,i}(t)$$

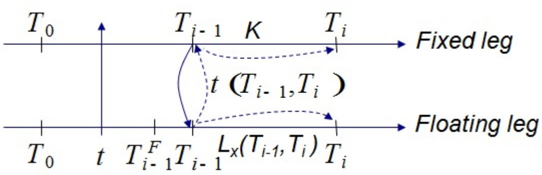
   Obviously, the FRA rate collapses on the Deposit rate for $T_1 \to T_0^+$

$$\lim_{T_1 \to T_0^+} R^{FRA}_{Std}(t; \boldsymbol{T}) = R^{Depo}(t; \boldsymbol{T})$$

2. **Market FRA**:
   the payoff of the market FRA at payment date T_{i-1} (not T_i) is given by

$$\begin{aligned}\boldsymbol{FRA}_{Mkt}(T_{i-1}; \boldsymbol{T}, K, w) &= N\frac{w[L_x(T_{i-1}, T_i) - K]\tau_x(T_{i-1}, T_i)}{1 + L_x(T_{i-1}, T_i)\tau_x(T_{i-1}, T_i)} \\ &= \frac{\boldsymbol{FRA}_{Std}(T_{i-1}; \boldsymbol{T}, K, w)}{1 + L_x(T_{i-1}, T_i)\tau_x(T_{i-1}, T_i)}\end{aligned}$$

   Notice that the payment is anticipated at date T_{i-1}, discounted from T_i to T_{i-1} using the Libor



rate itself.

The price of the market FRA at time $t < T_i$ is given, under the payment T_{i-1} forward measure, by

$$\begin{aligned}\boldsymbol{FRA}_{Mkt}(t;\boldsymbol{T},K,w) &= P_d(t;T_{i-1})\mathbb{E}_t^{\mathbb{Q}^{T_{i-1}}}[\boldsymbol{FRA}_{Mkt}(T_{i-1};\boldsymbol{T},K,w)] \\ &= NwP_d(t;T_{i-1})\mathbb{E}_t^{\mathbb{Q}^{T_{i-1}}}\left\{\frac{[L_x(T_{i-1},T_i)-K]\tau_x(T_{i-1},T_i)}{1+L_x(T_{i-1},T_i)\tau_x(T_{i-1},T_i)}\right\} \\ &= NwP_d(t;T_{i-1})\left\{1-[1+K\tau_x(T_{i-1},T_i)]\mathbb{E}_t^{\mathbb{Q}^{T_{i-1}}}\left[\frac{1}{1+L_x(T_{i-1},T_i)\tau_x(T_{i-1},T_i)}\right]\right\}\end{aligned}$$

Notice that, in this case, the price depends on the expectation of the forward discount factor

$$P_x(T_{i-1},T_i) := \frac{1}{1+L_x(T_{i-1},T_i)\tau_x(T_{i-1},T_i)}$$

under the payment T_{i-1} forward measure.

Switching from T_{i-1} to T_i forward measure we obtain

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}^{T_{i-1}}}\left[\frac{1}{1+L_x(T_{i-1},T_i)\tau_x(T_{i-1},T_i)}\right] &= \frac{P_d(t;T_{i-1})}{P_d(t,T_i)}\mathbb{E}_t^{\mathbb{Q}^{T_i}}\left[\frac{1}{P_d(T_{i-1},T_i)}\frac{1}{1+L_x(T_{i-1},T_i)\tau_x(T_{i-1},T_i)}\right] = \\ &= \frac{1}{1+\tau_d(T_{i-1},T_i)F_{d,i}(t)}\mathbb{E}_t^{\mathbb{Q}^{T_i}}\left[\frac{1+L_d(T_{i-1},T_i)\tau_d(T_{i-1},T_i)}{1+L_x(T_{i-1},T_i)\tau_x(T_{i-1},T_i)}\right]\end{aligned}$$

$$\boldsymbol{FRA}_{Mkt}(t;\boldsymbol{T},K,w) = NwP_d(t;T_{i-1})\left\{1-\frac{1+K\tau_x(T_{i-1},T_i)}{1+F_{d,i}(t)\tau_d(T_{i-1},T_i)}\right\}\mathbb{E}_t^{\mathbb{Q}^{T_i}}\left[\frac{1+L_d(T_{i-1},T_i)\tau_d(T_{i-1},T_i)}{1+L_x(T_{i-1},T_i)\tau_x(T_{i-1},T_i)}\right]$$

$$R_{x,Mkt}^{FRA}(t;\boldsymbol{T}) = \frac{1}{\tau_x(T_{i-1},T_i)}\left\{\frac{1+F_{d,i}(t)\tau_d(T_{i-1},T_i)}{\mathbb{E}_t^{\mathbb{Q}^{T_i}}\left[\frac{1+L_d(T_{i-1},T_i)\tau_d(T_{i-1},T_i)}{1+L_x(T_{i-1},T_i)\tau_x(T_{i-1},T_i)}\right]}\right\}$$

Thus the price of the market FRA depends on the model chosen for the joint distribution of the two Libor rates $L_d(T_{i-1},T_i)$ and $L_x(T_{i-1},T_i)$ under the forward measure $\mathbb{Q}^{T_i}$.
Assuming some model for the dynamics of $L_d(T_{i-1},T_i)$ and $L_x(T_{i-1},T_i)$ under the forward measure $\mathbb{Q}_d^{T_i}$ we obtain

$$\mathbb{E}_t^{\mathbb{Q}^{T_i}}\left[\frac{1+L_d(T_{i-1},T_i)\tau_d(T_{i-1},T_i)}{1+L_x(T_{i-1},T_i)\tau_x(T_{i-1},T_i)}\right] = \frac{1+F_{d,i}(t)\tau_d(T_{i-1},T_i)}{1+F_{x,i}(t)\tau_x(T_{i-1},T_i)}e^{C_x^{FRA}(t;T_{i-1})}$$

$$\boldsymbol{FRA}_{Mkt}(t;\boldsymbol{T},K,w) = NwP_d(t;T_{i-1})\left[1-\frac{1+K\tau_x(T_{i-1},T_i)}{1+F_{x,i}(t)\tau_x(T_{i-1},T_i)}e^{C_x^{FRA}(t;T_{i-1})}\right]$$

where $C_x(t;T_{i-1})$ is a convexity adjustment, whose detailed expression depends on the chosen model.

A possible choice is that of Mercurio (2010), in which the two FRA rates are modeled as shifted lognormal martingales under the forward measure $\mathbb{Q}^{T_i}$,

$$\begin{aligned}&\frac{dF_{d,i}(t)}{F_{d,i}(t)+\frac{1}{\tau_d(T_{i-1},T_i)}} = \sigma_{d,i}dW_d^{Q^{T_i}}(t) \\ &\frac{dF_{x,i}(t)}{F_{x,i}(t)+\frac{1}{\tau_x(T_{i-1},T_i)}} = \sigma_{x,i}dW_x^{Q^{T_i}}(t) \\ &dW_d^{Q^{T_i}}(t)dW_x^{Q^{T_i}}(t) = \rho_{d,x,i}dt \\ &C(t;T_{i-1}) = [\sigma_{x,i}^2 - \sigma_{x,i}\sigma_{d,i}\rho_{d,x,i}]\tau(t,T_{i-1})\end{aligned}$$

The size of the convexity adjustment results to be below 1 bp, even for long maturities, for typical post credit crunch market situations[1].

[1]see Mercurio 2010

```
16:14 30JAN15              ICAP LONDON                UK69580         ICAPSHORT2
Contact Reuters EXEU         EURO Short Swaps / FRAs           +44 (0)20 7532 3530
              1M Swaps              IMM Dated                           3m FRAs
  2x1       0.029/-0.021     1y  MAR/MAR 0.078-0.028           1x4   0.087-0.037
  3x1       0.023/-0.027     1y  JUN/JUN 0.075-0.025           2x5   0.086-0.036
  4x1       0.018/-0.032     1y  SEP/SEP 0.078-0.028           3x6   0.082-0.032
  5x1       0.013/-0.037     1y  DEC/DEC 0.088-0.038           4x7   0.081-0.031
  6x1       0.010/-0.040     2y  MAR/MAR 0.090-0.040           5x8   0.078-0.028
  7x1       0.008/-0.042     2y  JUN/JUN 0.099-0.049           6x9   0.075-0.025
  8x1       0.005/-0.045     3y  MAR/MAR 0.128-0.078
  9x1       0.004/-0.046                                             6m FRAs
 10x1       0.002/-0.048                IMM Fras              1x7   0.173-0.123
 11x1       0.001/-0.049                                      2x8   0.171-0.121
 12x1      -0.001/-0.051     1x7          0.174-0.124         3x9   0.168-0.118
                             2x8          0.170-0.120         4x10  0.166-0.116
 1y /3      0.078/ 0.028     3x9          0.167-0.117         5x11  0.164-0.114
 15m/3      0.077/ 0.027     4x10         0.165-0.115         6x12  0.163-0.113
 18m/3      0.078/ 0.028                                      12x18 0.169-0.119
 21m/3      0.082/ 0.032                                      18x24 0.199-0.149

 1y /6      0.162/ 0.112
 15m/6      0.150/ 0.100                                             12m FRA
 18m/6      0.165/ 0.115                                      12x24 0.298-0.248
 21m/6      0.159/ 0.109                ICAP OIS Fix Menu <ICAPOISFIX01>
ICAP Global Index <ICAP>                  Forthcoming changes <ICAPCHANG
```

Figure 1.3: Forward Rate Agreement: market quotes

In the classical, single-curve limit, with vanishing interest rate basis, we have

$$\boldsymbol{FRA}_{Std}(t;\boldsymbol{T},K,w) \to NwP(t;T_i)[F_i(t)-K]\tau(T_{i-1},T_i)$$

$$\boldsymbol{FRA}_{Mkt}(t;\boldsymbol{T},K,w) \to NwP(t;T_{i-1})\left[1-\frac{1+\tau(T_{i-1},T_i)K}{1+\tau(T_{i-1},T_i)F_i(t)}\right] =$$

$$= NwP(t;T_i)[F_i(t)-K]\tau(T_{i-1},T_i)$$

$$= \boldsymbol{FRA}_{Std}(t;\boldsymbol{T},K,w)$$

$$R^{FRA}_{Mkt}(t;\boldsymbol{T}) \to R^{FRA}_{Std}(t;\boldsymbol{T}) = F_i(t) = \frac{1}{\tau(T_{i-1},T_i)}\left[\frac{P(t;T_{i-1})}{P(t;T_i)}-1\right]$$

	Forward Rate Agreement pricing formulas
Classical (single-curve)	$\boldsymbol{FRA}_{Std}(t;T_{i-1},T_i,K,w) = NwP(t;T_i)[F_i(t)-K]\tau(T_{i-1},T_i)$ $R^{FRA}_{Std}(t;\boldsymbol{T}) = F_i(t) = \mathbb{E}_t^{Q^{T_i}}[L(T_{i-1},T_i)]$ $\boldsymbol{FRA}_{Mkt}(t;T_{i-1},T_i,K,w) = \boldsymbol{FRA}_{Std}(t;T_{i-1},T_i,K,w)$ $R^{FRA}_{Mkt}(t;\boldsymbol{T}) = R^{FRA}_{Std}(t;\boldsymbol{T})$
Modern (multi-curve)	$\boldsymbol{FRA}_{Std}(t;T_{i-1},T_i,K,w) = NwP_d(t;T_i)[F_{x,i}(t)-K]\tau_x(T_{i-1},T_i)$ $R^{FRA}_{x,Std}(t;\boldsymbol{T}) = F_{x,i}(t) := \mathbb{E}_t^{Q^{T_i}}[L_x(T_{i-1},T_i)]$ $\boldsymbol{FRA}_{Mkt}(t;\boldsymbol{T},K,w) = NwP_d(t;T_{i-1})\left[1-\frac{1+K\tau_x(T_{i-1},T_i)}{1+F_{x,i}(t)\tau_x(T_{i-1},T_i)}e^{C_x^{FRA}(t;T_{i-1})}\right]$ $R^{FRA}_{x,Mkt}(t;\boldsymbol{T}) = \frac{1}{\tau_x(T_{i-1},T_i)}\left\{[1+\tau_x(T_{i-1},T_i)F_{x,i}(t)]e^{C_x^{FRA}(t;T_{i-1})}-1\right\}$

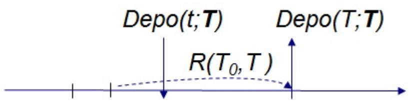
1.4 Deposits

Interest Rate Certificates of Deposits are standard OTC zero coupon contracts such that:

- at start date T_0, counterparty A, called the lender, pays a nominal amount N to counterparty B, called the borrower,
- at maturity date T the borrower pays back to the lender the nominal amount N plus the interest accrued over the period $[T_0, T]$ (called rate tenor) at the annual simply compounded interest rate $R(T_0, T)$, fixed at time $T_0^F < T_0$, where $[T_0^F, T_0]$ is the settlement period, usually equal to two working days in the EUR market.

The payoff at maturity, from the point of view of the lender (the receiver of the nominal amount plus interests), is given by

$$\boldsymbol{Depo}(T; \boldsymbol{T}) = N[1 + R(T_0, T)\tau(T_0, T)]$$



The price at time t, such that $T_0 \leq t \leq T$, when the deposit rate $R(T_0, T)$ is already fixed, is given by:

- the future cash flows (the nominal amount and the interest amount), which, in this case, are deterministic and happen at the same cash flow date T,
- each discounted at the pricing date $t \leq T$

In formulas:

$$\boldsymbol{Depo}(T; \boldsymbol{T}) = NP(t; T)[1 + R(T_0, T)\tau(T_0, T)]$$

where $P(t; T)$ is a discount factor taking into account the time value of money. In case of Deposits, the discount factor is typically consistent with the deposit rate $R(t, T)$ quoted on the market at time t, such that

$$P(t; T) = \frac{1}{1 + R(t, T)\tau(t, T)}, \quad T_0 \leq t \leq T$$

This price is the same for all counterparties.

1.5 Futures

Interest rate Futures are the exchange-traded contracts on the 3 month interest rate (for example LIBOR futures are on the 3 month LIBOR rate). They are similar to FRAs, except that their terms (such as maturity dates) are regulated by the exchange.The Futures' payoff at the last settlement date T_{i-1}, is given by

$$\boldsymbol{Futures}(T_{i-1},\boldsymbol{T}) = N[1 - L_x(T_{i-1},T_i)]$$

This payoff is a classical example of "mixing apples and oranges" because, clearly, on the r.h.s. 1 is adimensional while the Libor rate $L_x(T_{i-1},T_i)$ has dimension $t-1$ and they cannot be directly summed together without an year fraction $t(T_{i-1},T_i)$. Thus we must look at it as a mere rule for computing the amount of currency to be margined everyday.

The Futures' price at time $t < T_{i-1}$ is given by

$$\begin{aligned}\boldsymbol{Futures}(t,\boldsymbol{T}) &= \mathbb{E}_t^{\mathbb{Q}}[D_d(t;t)\boldsymbol{Futures}(T_{i-1};\boldsymbol{T})] \\ &= N\Big\{1 - \mathbb{E}_t^{\mathbb{Q}}[L_x(T_{i-1},T_i)]\Big\} := N[1 - R_x^{Fut}(t;\boldsymbol{T})]\end{aligned}$$

under the risk neutral measure $\mathbb{Q}$ associated to the funding bank account $B(t)$. Notice that the Futures' daily margination mechanism implies that the payoff is regulated everyday, thus generating the unitary discount factor $D(t;t) = 1$ appearing in the first line above. The daily margination amount is calculated as $D = 1.000.000 \cdot \frac{(P_{today} - P_{yesterday})}{4}$.

Hence, in order to price Futures we have to compute the Futures' rate

$$R^{Fut}(t;T_{i-1},T_i) := \mathbb{E}_t^{\mathbb{Q}}[L_x(T_{i-1},T_i)] = \mathbb{E}_t^{\mathbb{Q}}[F_{x,i}(T_{i-1})]$$

Since the forward rate $F_{x,i}(t)$ is not a martingale under the risk neutral measure $\mathbb{Q}$, such computation requires the adoption of a model for the dynamics of $F_{x,i}(t)$. In general, we obtain that the Futures' rate is given by the corresponding (risky) forward rate corrected with a convexity adjustment

$$R_x^{Fut}(t;\boldsymbol{T}) := \mathbb{E}_t^{\mathbb{Q}}[L_x(T_{i-1},T_i)] = \mathbb{E}_t^{\mathbb{Q}^{T_i}}[L_x(T_{i-1},T_i)] + C_x^{Fut}(t,T_{i-1}) = F_{x,i}(t) + C_x^{Fut}(t,T_{i-1})$$

The expression of the convexity adjustment will depend on the particular model adopted and will contain, in general, the model's volatilities and correlations. For instance, under the multiple curve Libor Market Model of Mercurio (2009), the convexity adjustment takes the form

$$C_x^{Fut}(t,T_{i-1}) \cong F_{x,i}(t) exp\Big[\int_t^{T_{i-1}} \mu_{x,i}(u)du - 1\Big]$$

where

$$\begin{aligned}\int_t^{T_{i-1}} \mu_{x,i}(u)du &\cong \sigma_{x,i}\sum_{j=1}^{i} \frac{\tau_{d,j}\sigma_{d,j}\rho_{d,j}^{x,d}F_{d,j}(t)}{1+\tau_{d,j}F_{d,j}(t)}(T_{j-1}-t) \\ \frac{dF_{x,i}(t)}{F_{x,i}} &= \mu_{x,i}(t)dt + \sigma_{x,i}dW_x^{Q^{T_i}}(t) \\ \frac{dF_{d,i}(t)}{F_{d,i}} &= \mu_{d,i}(t)dt + \sigma_{d,i}dW_d^{Q^{T_i}}(t) \\ F_{d,j}(t) : &= \mathbb{E}_t^{\mathbb{Q}^{T_j}}[L_d(T_{j-1},T_j)] = \frac{1}{\tau_{d,j}}\Big[\frac{P_d(t;T_{j-1})}{P_d(t;T_j)} - 1\Big] \\ \sigma_{x,i}, \sigma_{d,j}, \rho_{i,j}^{x,d} &= \text{instantaneous (deterministic) volatilities} \\ &\quad \text{and correlation of } F_{x,i}(t), F_{d,j}(t) \text{respectively.}\end{aligned}$$

	Futures pricing formulas
Classical (single-curve)	$\boldsymbol{Futures}(t, \boldsymbol{T}) = N[1 - R^{Fut}(t; \boldsymbol{T})]$ $R^{Fut}(t; \boldsymbol{T}) := \mathbb{E}_t^{\mathbb{Q}}[L(T_{i-1}, T_i)] = F_i(t) + C^{Fut}(t, T_{i-1})$
Modern (multi-curve)	$\boldsymbol{Futures}(t, \boldsymbol{T}) = N[1 - R_x^{Fut}(t; \boldsymbol{T})]$ $R_x^{Fut}(t; \boldsymbol{T}) := \mathbb{E}_t^{\mathbb{Q}}[L_x(T_{i-1}, T_i)] = F_{x,i}(t) + C_x^{Fut}(t, T_{i-1})$

0#FEI: LIFFE EURIBOR LIF/IEU EUR

Mth	Last	Net.Ch	Bid	Ask	Bid/Asksize	Settle	Open	High	Low	Volume	Op.Int	Time
FEB5	b 99.940	-0.005	99.940	c99.945	100x668	99.945	99.945	99.950	99.940	3302	21425	17:08
MAR5	99.930	-0.015	c99.930	c99.935	11186x5606	99.945	99.935	99.945	99.930	74491	423669	17:17
APR5	s 99.945	-0.010	i99.935	99.945	3x356	99.955	99.950	99.950	99.945	500	757	17:09
MAY5	s 99.950	-0.005	99.935	i99.950	200x152	99.955	99.955	99.960	99.950	501	0	17:14
JUN5	s↓99.940	-0.025	c99.940	99.945	32256x939	99.965	99.960	99.965	99.940	67266	325564	17:17
JUL5					0x0	99.965				0	0	:
SEP5	99.950	-0.020	c99.950	c99.955	8894x5456	99.970	99.970	99.975	99.950	75434	355202	17:17
DEC5	s 99.950	-0.020	c99.950	c99.955	5332x11524	99.970	99.970	99.980	99.950	36325	267475	17:17
MAR6	99.945	-0.020	99.945	c99.950	828x30480	99.965	99.965	99.970	99.945	38881	228914	17:17
JUN6	↓99.930	-0.020	c99.930	c99.935	2215x2606	99.950	99.950	99.960	99.930	36979	207720	17:17
SEP6	99.915	-0.015	c99.910	c99.915	6333x980	99.930	99.930	99.940	99.905	29657	182033	17:17
DEC6	s 99.890	-0.015	c99.885	c99.890	8126x359	99.905	99.905	99.910	99.885	20033	189236	17:17
MAR7	s 99.865	-0.010	c99.860	99.865	5542x33	99.875	99.875	99.885	99.860	28156	143624	17:17
JUN7	s 99.835	-0.005	c99.830	c99.835	705x109	99.840	99.840	99.850	99.825	21880	150198	17:17
SEP7	s↓99.800	-0.005	99.800	c99.805	4x1167	99.805	99.800	99.810	99.790	18388	131362	17:17
DEC7	s 99.765	0	c99.760	c99.765	255x69	99.765	99.765	99.770	99.750	16558	136632	17:17
MAR8	b 99.730	+0.010	c99.725	c99.730	10x138	99.720	99.710	99.735	99.705	9886	62439	17:17
JUN8	99.690	+0.020	c99.685	c99.690	44x14	99.670	99.660	99.695	99.655	8515	31941	17:17
SEP8	s 99.650	+0.025	c99.645	c99.655	151x203	99.625	99.615	99.655	99.610	6342	17807	17:16
DEC8	b 99.610	+0.030	99.605	i99.610	18x22	99.580	99.580	99.610	99.565	5728	17878	17:16
MAR9	99.560	+0.030	i99.555	c99.565	18x7	99.530	99.530	99.560	99.515	406	6531	17:15
JUN9	b 99.510	+0.035	i99.505	i99.520	18x54	99.475	99.470	99.510	99.470	677	2990	17:15
SEP9	b↑99.455	+0.035	i99.445	i99.465	18x37	99.420	99.455	99.455	99.455	96	716	17:15
DEC9	99.385	+0.025	i99.385	i99.400	18x30	99.360	99.385	99.385	99.380	56	1013	17:15
MAR0	99.330	+0.020	i99.325	99.340	18x30	99.310	99.330	99.330	99.330	5	293	17:15
JUN0	s 99.275	+0.015	i99.270	99.290	18x30	99.260	99.275	99.275	99.275	5	252	17:15
SEP0			i99.215	i99.245	18x30	99.210				0	50	:
DEC0			i99.150	99.195	18x20	99.155				0	85	:

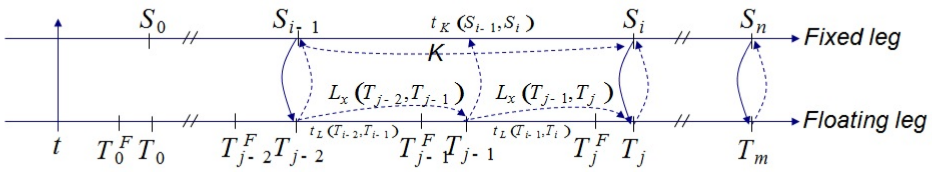
Figure 1.4: Futures: market quotes

1.6 Swap

Interest rate swaps are OTC contracts in which two counterparties agree to exchange two streams of cash flows, typically tied to a fixed rate K against floating rate. These payment streams are called fixed and floating leg of the swap, respectively, and they are characterized by two schedules $\boldsymbol{S}$, $\boldsymbol{T}$ and coupon payoffs.

$$
\begin{aligned}
&\boldsymbol{S} = \{S_0, ..., S_n\}, \text{fixed leg schedule} \\
&\boldsymbol{T} = \{T_0, ..., T_m\}, \text{floating leg schedule} \\
&S_0 = T_0, \quad S_n = T_m \\
&\boldsymbol{Swaplet}_{fix}(S_i; S_{i-1}, S_i, K) = NK\tau_K(S_{i-1}, S_i) \\
&\boldsymbol{Swaplet}_{float}(T_j; T_{j-1}, T_i) = NL_x(T_{j-1}, T_j)\tau_L(T_{j-1}, T_j)
\end{aligned}
$$

where τ_K and τ_L are the year fractions with the fixed and floating rate conventions.



The fixed vs floating interest rate swap coupon payoffs are

$$
\boldsymbol{Swaplet}_{fix}(S_i; S_{i-1}, S_i, K) = NK\tau_K(S_{i-1}, S_i), \quad i = 1, ..., n
$$

$$
\boldsymbol{Swaplet}_{float}(T_j; T_{j-1}, T_j) = NL_x(T_{j-1}, T_j)\tau_x(T_{j-1}, T_j), \quad j = 1, ..., m
$$

The coupon prices at time $t < max(S_i, T_j)$ are given by

$$
\begin{aligned}
\boldsymbol{Swaplet}_{fix}(t; S_{i-1}, S_i, K) &= P_d(t; S_i)\mathbb{E}_t^{\mathbb{Q}^{S_i}}[\boldsymbol{Swaplet}_{fix}(S_i; S_{i-1}, S_i, K)] \\
&= NP_d(t; S_i)K\tau_K(S_{i-1}, S_i) \\
\boldsymbol{Swaplet}_{float}(t; T_{j-1}, T_j) &= P_d(t; T_j)\mathbb{E}_t^{\mathbb{Q}^{T_j}}[\boldsymbol{Swaplet}_{float}(T_j; T_{j-1}, T_j)] \\
&= NP_d(t; T_j)F_{x,j}(t)\tau_x(T_{j-1}, T_j)
\end{aligned}
$$

The price of the fixed and floating swap legs is given, at time $t < T_0$, by

$$
\begin{aligned}
\boldsymbol{Swap}_{fix}(t; \boldsymbol{S}, K) &= \sum_{i=1}^{n} \boldsymbol{Swaplet}_{fix}(t; S_{i-1}, S_i, K) = KA_d(t, \boldsymbol{S}) \\
\boldsymbol{Swap}_{float}(t; \boldsymbol{T}) &= \sum_{j=1}^{m} \boldsymbol{Swaplet}_{float}(t; T_{j-1}, T_j) \\
&= \sum_{j=1}^{m} P_d(t; T_j)F_{x,j}(t)\tau_x(T_{j-1}, T_j)
\end{aligned}
$$

where the **swap annuity** $A_d(t, \boldsymbol{S})$ is defined as

$$
A_d(t, \boldsymbol{S}) = \sum_{i=1}^{n} P_d(t; S_i)\tau_K(S_{i-1}, S_i)
$$

The index $''d''$ reminds that the annuity is linked to the discount rate.

The total swap price is given, at time $t < T_0$, by

$$\begin{aligned}\boldsymbol{Swap}(t, \boldsymbol{T}, \boldsymbol{S}, K, \omega) &= \omega[\boldsymbol{Swap}_{float}(t; \boldsymbol{T}) - \boldsymbol{Swap}_{fix}(t; \boldsymbol{S}, K)] \\ &= N\omega\left[\sum_{j=1}^{m} P_d(t; T_j) F_{x,j}(t) \tau_x(T_{j-1}, T_j) - K A_d(t, \boldsymbol{S})\right]\end{aligned}$$

where $\omega = \pm 1$ for a payer/receiver swap (referred to the fixed leg).
The swap rate at time t is

$$\begin{aligned}R_x^{Swap}(t; \boldsymbol{T}, \boldsymbol{S}) &= \frac{\boldsymbol{Swap}_{float}(t; \boldsymbol{T})}{N\omega A_d(t, \boldsymbol{S})} \\ &= \frac{\sum_{j=1}^{m} P_d(t; T_j) F_{x,j}(t) \tau_x(T_{j-1}, T_j)}{A_d(t, \boldsymbol{S})}\end{aligned}$$

Hence the swap price can be written in terms of the swap rate as

$$\boldsymbol{Swap}(t, \boldsymbol{T}, \boldsymbol{S}, K, \omega) = N\omega[R_x^{Swap}(t; \boldsymbol{T}, \boldsymbol{S}) - K] A_d(t, \boldsymbol{S})$$

In the classical, single-curve limit, with vanishing interest rate basis, we have

$$\begin{aligned}\boldsymbol{Swap}_{fix}(t; \boldsymbol{S}, K) &\to NKA(t, \boldsymbol{S}) \\ \boldsymbol{Swap}_{float}(t; \boldsymbol{T}) &\to N \sum_{j=1}^{m} P(t; T_j) F(t) \tau_L(T_{j-1}, T_j) \\ &\cong N \sum_{j=1}^{m} [P(t, T_{j-1} - P(t, T_j)] = N[P(t, T_0) - P(t, T_m)]\end{aligned}$$

where we have used, in the last line, the single-curve expression of the forward rate and the telescopic property of the summation. The latter does hold exactly only if the floating leg schedule is regular (the periods do concatenate exactly with no gaps or overlappings). In practice the error is very small (of the order of 0.1 basis points).

The swap price and swap rate are given by

$$\boldsymbol{Swap}(t, \boldsymbol{T}, \boldsymbol{S}, K, \omega) \cong N\omega[P(t, T_0) - P(t, T_m) - KA(t, \boldsymbol{S})]$$

$$R^{Swap}(t, \boldsymbol{T}, \boldsymbol{S}) \cong \frac{P(t, T_0) - P(t, T_m)}{A(t, \boldsymbol{S})}$$

	Swap pricing formulas
Classical (single-curve)	$\boldsymbol{Swap}(t, \boldsymbol{T}, \boldsymbol{S}, K, \omega) = N\omega[R^{Swap}(t; \boldsymbol{T}, \boldsymbol{S}) - K] A_d(t, \boldsymbol{S})$ $R^{Swap}(t; \boldsymbol{T}, \boldsymbol{S}) \cong \frac{P(t,T_0) - P(t,T_m)}{A(t,\boldsymbol{S})}$
Modern (multi-curve)	$\boldsymbol{Swap}(t, \boldsymbol{T}, \boldsymbol{S}, K, \omega) = N\omega[R_x^{Swap}(t; \boldsymbol{T}, \boldsymbol{S}) - K] A_d(t, \boldsymbol{S})$ $R_x^{Swap}(t; \boldsymbol{T}, \boldsymbol{S}) = \frac{\sum_{j=1}^{m} P_d(t;T_j) F_{x,j}(t) \tau_x(T_{j-1},T_j)}{A_d(t,\boldsymbol{S})}$

12:59 30DEC11 ICAP UK69580 ICAPEURO

Euribor vs 6 mth 3/6 basis

Spot Starting Date

1 Yr	1.442-1.402	16Yrs	2.717-2.677	1 Yr	+33.5		
2 Yrs	1.330-1.290	17Yrs	2.727-2.687	2 Yrs	+26.4		
3 Yrs	1.400-1.360	18Yrs	2.729-2.689	3 Yrs	+22.4		
4 Yrs	1.565-1.525	19Yrs	2.725-2.685	4 Yrs	+19.8		
5 Yrs	1.756-1.716	20Yrs	2.717-2.677	5 Yrs	+17.9		
6 Yrs	1.941-1.901			6 Yrs	+16.3		
7 Yrs	2.096-2.056	21Yrs	2.707-2.667	7 Yrs	+15.0		
8 Yrs	2.220-2.180	22Yrs	2.696-2.656	8 Yrs	+13.9		
9 Yrs	2.324-2.284	23Yrs	2.683-2.643	9 Yrs	+12.9		
10Yrs	2.414-2.374	24Yrs	2.669-2.629	10Yrs	+12.1		
		25Yrs	2.654-2.614				
11Yrs	2.495-2.455						
12Yrs	2.566-2.526	26Yrs	2.639-2.599			10X12	0.172/0.132
13Yrs	2.624-2.584	27Yrs	2.624-2.584			10X15	0.304/0.264
14Yrs	2.667-2.627	28Yrs	2.610-2.570			10X20	0.323/0.283
15Yrs	2.698-2.658	29Yrs	2.597-2.557			10X25	0.260/0.220
		30Yrs	2.587-2.547			10X30	0.193/0.153
		35Yrs	2.571-2.531			10X35	0.177/0.137
		40Yrs	2.578-2.538			10X40	0.184/0.144
		50Yrs	2.590-2.550			10X50	0.196/0.156
		60Yrs	2.596-2.556			10X60	0.202/0.162

ICAP Euro IRS pages close @ 13:00 GMT. Pls refer to LIFFE/EUREX for futures lvls

12:57 30DEC11 ICAP UK69580 ICAPEURO2

For Further Details Please Call David Shepherd on +44 (0)207 532 3530

	Eonia IRS ACT/360	Euro Swap vs 3M Euribor 30/360	Euro Swap vs 1M Euribor 30/360
1YR	0.412-0.342	1.112-1.062	0.824-0.754
2YR	0.487-0.417	1.071-1.021	0.824-0.754
3YR	0.668-0.598	1.181-1.131	0.969-0.899
4YR	0.902-0.832	1.372-1.322	1.189-1.119
5YR	1.143-1.073	1.582-1.532	1.422-1.352
6YR	1.370-1.300	1.783-1.733	1.640-1.570
7YR	1.560-1.490	1.951-1.901	1.821-1.751
8YR	1.713-1.643	2.086-2.036	1.965-1.895
9YR	1.843-1.773	2.200-2.150	2.086-2.016
10YR	1.956-1.886	2.298-2.248	2.190-2.120
11YR	2.057-1.987	2.386-2.336	2.283-2.213
12YR	2.146-2.076	2.465-2.415	2.366-2.296
15YR	2.318-2.248	2.610-2.560	2.520-2.450
20YR	2.384-2.314	2.646-2.596	2.564-2.494
25YR	2.349-2.279	2.592-2.542	2.514-2.444
30YR	2.302-2.232	2.531-2.481	2.458-2.388

ICAP Euro IRS pages close @ 13:00 GMT. Pls refer to LIFFE/EUREX for futures lvls

12:57 30DEC11 ICAP UK69580 ICAP14

Euro Forwards

Please call +44 (0)20 7532 4120 for further details

Maturity

		1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
	1Y	1.198	1.359	1.587	1.817	2.025	2.190	2.318	2.423	2.512	2.591
	2Y	1.521	1.785	2.029	2.239	2.399	2.517	2.611	2.691	2.763	2.825
F	3Y	2.055	2.292	2.490	2.632	2.731	2.809	2.876	2.937	2.989	3.029
o	4Y	2.534	2.715	2.833	2.910	2.971	3.025	3.076	3.120	3.152	3.168
r	5Y	2.901	2.988	3.042	3.087	3.131	3.175	3.214	3.240	3.249	3.246
w	6Y	3.079	3.116	3.153	3.193	3.235	3.272	3.294	3.298	3.291	3.271
a	7Y	3.153	3.192	3.233	3.277	3.314	3.334	3.334	3.321	3.296	3.263
r	8Y	3.232	3.275	3.321	3.357	3.374	3.367	3.348	3.316	3.277	3.231
d	9Y	3.320	3.368	3.401	3.412	3.397	3.370	3.330	3.283	3.231	3.176
	10Y	3.417	3.444	3.444	3.417	3.381	3.332	3.277	3.218	3.158	3.099
	11Y	3.472	3.459	3.418	3.371	3.313	3.251	3.186	3.121	3.058	3.001
	12Y	3.445	3.389	3.336	3.270	3.203	3.133	3.064	2.999	2.941	2.889
	13Y	3.331	3.278	3.208	3.137	3.065	2.994	2.928	2.869	2.818	2.769
	14Y	3.223	3.143	3.068	2.993	2.920	2.853	2.795	2.746	2.698	2.653
	15Y	3.060	2.988	2.912	2.839	2.773	2.717	2.670	2.624	2.581	2.541
	20Y	2.416	2.389	2.350	2.314	2.280	2.249	2.223	2.202	2.187	2.183
	25Y	2.089	2.069	2.063	2.060	2.076	2.103	2.134	2.167	2.200	2.231
	30Y	2.245	2.290	2.331	2.369	2.403	2.433	2.458	2.480	2.499	2.514

ICAP Euro IRS pages close @ 13:00 GMT. Pls refer to LIFFE/EUREX for futures lvls

12:57 30DEC11 ICAP UK69580 ICAP15

Euro Forwards

Please call +44 (0)20 7532 4120 for further details

Maturity

		11Y	12Y	13Y	14Y	15Y	20Y	25Y	30Y
	1Y	2.660	2.716	2.755	2.783	2.798	2.767	2.683	2.613
	2Y	2.873	2.905	2.926	2.933	2.932	2.852	2.746	2.675
F	3Y	3.052	3.064	3.064	3.055	3.039	2.918	2.793	2.725
O	4Y	3.172	3.164	3.148	3.125	3.097	2.947	2.813	2.753
R	5Y	3.232	3.210	3.180	3.147	3.111	2.943	2.808	2.759
W	6Y	3.243	3.209	3.170	3.130	3.091	2.912	2.786	2.748
A	7Y	3.223	3.179	3.135	3.092	3.052	2.867	2.756	2.729
R	8Y	3.182	3.133	3.087	3.043	3.000	2.815	2.724	2.706
D	9Y	3.122	3.072	3.025	2.980	2.936	2.757	2.688	2.679
	10Y	3.045	2.996	2.948	2.903	2.859	2.695	2.649	2.648
	11Y	2.950	2.901	2.855	2.810	2.770	2.630	2.605	2.611
	12Y	2.839	2.793	2.749	2.708	2.671	2.563	2.558	2.571
	13Y	2.723	2.680	2.640	2.604	2.572	2.501	2.513	2.531
	14Y	2.611	2.572	2.537	2.507	2.481	2.447	2.474	2.496
	15Y	2.504	2.470	2.442	2.418	2.402	2.402	2.440	2.465
	20Y	2.188	2.199	2.214	2.231	2.249	2.331	2.381	2.416
	25Y	2.260	2.287	2.311	2.333	2.352	2.413	2.454	2.478
	30Y	2.525	2.534	2.542	2.548	2.554	2.578	2.591	2.594

ICAP Euro IRS pages close @ 13:00 GMT. Pls refer to LIFFE/EUREX for futures lvls

Figure 1.5: Swap: market quotes

1.7 One-factor short rate models

The theory of interest-rate modeling was originally based on the assumption of specific one-dimensional dynamics for the instantaneous spot rate process r. Modeling directly such dynamics is very convenient since all fundamental quantities (rates and bonds) are readily defined, by no-arbitrage arguments, as the expectation of a functional of the process r. Indeed, the existence of a risk-neutral measure implies that the arbitrage-free price at time t of a contingent claim with payoff H_T at time T is given by

$$H_t = \mathbb{E}_t[D(t,T)H_T] = \mathbb{E}_t\left[e^{-\int_t^T r(s)ds} H_T\right] \tag{1.7.1}$$

with $\mathbb{E}_t$ denoting the time t-conditional expectation under that measure. In particular, the zero-coupon-bond price at time t for the maturity T is characterized by a unit amount of currency available at time T, so that $H_T = 1$ and we obtain

$$P(t,T) = \mathbb{E}_t\left[e^{-\int_t^T r(s)ds}\right] \tag{1.7.2}$$

From this last expression it is clear that whenever we can characterize the distribution of $e^{-\int_t^T r(s)ds}$ in terms of a chosen dynamics for r, conditional on the information available at time t, we are able to compute bond prices P. From bond prices all kind of rates are available, so that indeed the whole zero-coupon curve is characterized in terms of distributional properties of r. The pioneering approach proposed by Vasicek (1977) was based on defining the instantaneous-spot-rate dynamics under the real-world measure. His derivation of an arbitrage-free price for any interest-rate derivative followed from using the basic Black and Scholes (1973) arguments, while taking into account the non-tradable feature of interest rates.

The construction of a suitable locally-riskless portfolio, as in Black and Scholes (1973), leads to the existence of a stochastic process that only depends on the current time and instantaneous spot rate and not on the maturities of the claims constituting the portfolio. Such process, which is commonly referred to as market price of risk, defines a Girsanov change of measure from the real-world measure to the risk-neutral one also in case of more general dynamics than Vasicek's. Precisely, let us assume that the instantaneous spot rate evolves under the real-world measure $\mathbb{Q}_0$ according to

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW^0(t)$$

where μ and σ are well-behaved functions and W^0 is a $\mathbb{Q}_0$-Brownian motion. It is possible to show[2] the existence of a stochastic process λ such that if

$$dP(t,T) = \mu^T(t, r(t))dt + \sigma^T(t, r(t))dW^0(t) \tag{1.7.3}$$

then

$$\frac{\mu^T(t, r(t)) - r(t)P(t,T)}{\sigma^T(t, r(t))} = \lambda(t)$$

for each maturity T, with λ that may depend on r but not on T. Moreover, there exists a measure $\mathbb{Q}$ that is equivalent to $\mathbb{Q}_0$ and is defined by the Radon-Nikodym derivative

$$\left.\frac{dQ}{dQ_0}\right|_{\mathcal{F}_t} = exp\left(-\frac{1}{2}\int_0^t \lambda^2(s)ds - \int_0^t \lambda(s)dW^0(s)\right)$$

where $W(t) = W^0(t) + \int_0^t \lambda(s)ds$ is a Brownian motion under $\mathbb{Q}$. The equation (1.7.3) expresses the bond-price dynamics in terms of the short rate r. It expresses how the bond price P evolves over time. Now recall that r is the instantaneous-return rate of a risk-free investment, so that the difference $\mu - r$ represents a difference in returns. It tells us how much better we are doing with respect to the risk-free case, i.e. with respect to putting our money in a riskless bank account. When we divide this quantity by σ_T, we are dividing by the amount of risk we are subject to, as measured by the bond-price volatility σ_T. This is why λ is referred to as "market price of risk". An alternative term could be "excess return with respect to a risk-free investment per unit of risk". The crucial observation is that in order to specify completely the model, we have to provide λ. In effect, the market price of risk λ connects the real-world measure to the risk-neutral measure as the main ingredient in the mathematical object $\frac{d\mathbb{Q}}{d\mathbb{Q}^0}$ expressing the connection between these two "worlds". The way of moving from one world to the other is characterized by our choice of λ. However, if we are just concerned with the pricing of (interestrate) derivatives, we can directly model the rate dynamics under the measure $\mathbb{Q}$, so that λ will be implicit in our dynamics. We put ourselves in the world $\mathbb{Q}$ and we do not bother about the way of moving to the world $\mathbb{Q}_0$. Then we would be in troubles only if we needed to move under the objective measure, but for pricing derivatives, the objective measure is not necessary, so that we can safely ignore it. Indeed, the value of the model parameters under the risk-neutral measure $\mathbb{Q}$ is what really matters in the pricing procedure, given also that the zero-coupon bonds are themselves derivatives under the above framework. All the models we consider in this chapter are presented under the risk-neutral measure, even when their original formulation was under the measure $\mathbb{Q}_0$. We will hint at the relationship between the two measures only occasionally, and will explore the interaction of the dynamics under the two different measures in the Vasicek case as an illustration.

We introduce in particular the classical short-rate model: the Vasicek model (1977), which is an endogenous term-structure model, meaning that the current term structure of rates is an output rather than an input of the model. The Vasicek model will be defined, under the risk-neutral measure $\mathbb{Q}$, by the dynamics

$$dr(t) = k[\theta - r(t)]dt + \sigma dW(t), \quad r(0) = r_0$$

This dynamics has some peculiarities that make the model attractive. The equation is linear and can be solved explicitly, the distribution of the short rate is Gaussian, and both the expressions and the distributions of several useful quantities related to the interest-rate world are easily obtainable. Besides, the endogenous nature of the model is now clear. Since the bond price $P(t,T) = \mathbb{E}_t\left\{e^{-\int_t^T r(s)ds}\right\}$ can be

[2] See for instance Björk (1997).

computed as a simple expression depending on k, θ, σ and $r(t)$, once the function $T \mapsto P(t, T; k, \theta, \sigma, r(t))$ is known, we know the whole interest-rate curve at time t. This means that, if $t = 0$ is the initial time, the initial interest rate curve is an output of the model, depending on the parameters k, θ, σ in the dynamics (and on the initial condition r_0).

A classical problem with the above problems is their endogenous nature. If we have the initial zero-coupon bond curve $T \mapsto P^M(0, T)$ from the market, and we wish our model to incorporate this curve, we need forcing the model parameters to produce a model curve as close as possible to the market curve. For example, in the Vasicek case, we need to run an optimization to find the values of k, θ and σ such that the model initial curve $T \to P(0, T; k, \theta, \sigma, r(0))$ is as close as possible to the market curve $T \mapsto P^M(0, T)$. Although the values $P^M(0, T)$ are actually observed only at a finite number of maturities $P^M(0, T_i)$, three parameters are not enough to reproduce satisfactorily a given term structure. Moreover, some shapes of the zero-coupon curve $T \mapsto L^M(0, T)$ (like an inverted shape) can never be obtained with the Vasicek model, no matter the values of the parameters in the dynamics that are chosen. The point of this digression is making clear that these kind of models are quite hopeless: they cannot reproduce satisfactorily the initial yield curve, and so speaking of volatility structures and realism in other respects becomes partly pointless.
To improve this situation, exogenous term structure models are usually considered. Such models are built by suitably modifying the above endogenous models. The basic strategy that is used to transform an endogenous model into an exogenous model is the inclusion of "time-varying" parameters.
Typically, in the Vasicek case, one does the following:

$$dr(t) = k[\theta - r(t)]dt + \sigma dW(t) \longmapsto dr(t) = k[\theta(t) - r(t)]dt + \sigma dW(t).$$

Now the function of time $\theta(t)$ can be defined in terms of the market curve $T \to L^M(0, T)$ in such a way that the model reproduces exactly the curve itself at time 0. We will consider the the Hull and White (1990) extended Vasicek model, and throughout all the section, we will assume that the term structure of discount factors that is currently observed in the market is given by the sufficiently-smooth function $t \mapsto P^M(0, t)$. We then denote by $f^M(0, t)$ the market instantaneous forward rates at time 0 for a maturity t as associated with the bond prices $\{P^M(0, t) : t \geq 0\}$, i.e.

$$f^M(0, t) = -\frac{\partial ln P^M(0, t)}{\partial t}$$

1.7.1 The Vasicek model

The simplest term structure model of any practical significance is *Vasicek model.* Under the risk-neutral measure its dynamics is given by:

$$dr(t) = k[\theta - r(t)]dt + \sigma dW(t), \quad r(0) = r_0 \tag{1.7.4}$$

where r_0, k, σ are positive constants.
Integrating equation (1.7.4), we obtain, for each $s \leq t$

$$r(t) = r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right) + \sigma\int_s^t e^{-k(t-u)}dW(u) \tag{1.7.5}$$

so that $r(t)$ conditional on $\mathcal{F}_s$ is normally distributed with mean and variance given respectively by

$$\mathbb{E}[r(t)|\mathcal{F}_s] = r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right) Var[r(t)|\mathcal{F}_s] = \frac{\sigma^2}{2k}\left[1 - e^{-2k(t-s)}\right] \tag{1.7.6}$$

This implies that, for each time t, the rate $r(t)$ can be negative with positive probability. The possibility of negative rates is indeed a major drawback of the Vasicek model. However, the analytical tractability that is implied by a Gaussian density is hardly achieved when assuming other distributions for the process r. As a consequence of (1.7.6), the short rate r is mean reverting, since the expected rate tends, for t going to infinity, to the value θ. The fact that θ can be regarded as a long term average rate could be also inferred from the dynamics (1.7.4) itself. Notice, indeed, that the drift of the process r is positive whenever the short rate is below θ and negative otherwise, so that r is pushed, at every time, to be closer on average to the level θ. The price of a pure-discount bond can be derived by computing the expectation (1.7.2). We obtain

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)} \tag{1.7.7}$$

where

$$A(t,T) = exp\left\{\left(\theta - \frac{\sigma^2}{2k^2}\right)[B(t,T) - T + t] - \frac{\sigma^2}{4k}B(t,T)^2\right\}$$
$$B(t,T) = \frac{1}{k}\left[e^{-k(T-t)}\right]$$

If we fix a maturity T, the change of numeraire[3] toolkit imply that under the T-forward measure $\mathbb{Q}^T$

$$dr(t) = [k\theta - B(t,T)\sigma^2 - kr(t)]dt + \sigma dW^T(t) \tag{1.7.8}$$

where the $\mathbb{Q}^T$-Brownian motion W^T is defined by

$$dW^T(t) = dW(t) + \sigma B(t,T)dt$$

so that, for $s \leq t \leq T$,

$$r(t) = r(s)e^{-k(t-s)} + M^T(s,t) + \sigma\int_s^t e^{-k(t-u)}dW^T(u)$$

with

$$M^T(s,t) = \left(\theta - \frac{\sigma^2}{k^2}\right)\left(1 - e^{-k(t-s)}\right) + \frac{\sigma^2}{2k^2}\left[e^{-k(T-t)} - e^{-k(T+t-2s)}\right]$$

Therefore, under $\mathbb{Q}^T$, the transition distribution of $r(t)$ conditional on $\mathcal{F}_s$ is still normal with mean and variance given by

$$\mathbb{E}^T\{r(t)|\mathcal{F}_s\} = r(s)e^{-k(t-s)} + M^T(s,t)$$
$$Var^T\{r(t)|\mathcal{F}_s\} = \frac{\sigma^2}{2k}\left[1 - e^{-2k(t-s)}\right].$$

[3] see Appendix A.4

The price at time t of a European option with strike X, maturity T and written on a pure discount bond maturing at time S has been derived by Jamshidian (1989). Using the known distribution of $r(t)$ under $\mathbb{Q}^T$, the calculation of the expectation (1.7.1), where $H_T = (P(T,S) - X)^+$, yields

$$\boldsymbol{ZBO}(t,T,S,X) = w[P(t,S)\Phi(wh) - XP(t,T)\Phi(w(h-\sigma_p))]$$

where $w = 1$ for a call and $w = -1$ for a put, $\Phi(.)$ denotes the standard normal cumulative distribution function, and

$$\sigma_p = \sigma\sqrt{\frac{1-e^{-2k(T-t)}}{2k}}B(T,S)$$
$$h = \frac{1}{\sigma_p}ln\frac{P(t,S)}{P(t,T)X} + \frac{\sigma_p}{2}$$

We can also consider the objective measure dynamics of the Vasicek model as a proceee of the form

$$dr(t) = [k\theta - (k+\lambda\sigma)r(t)]dt + \sigma dW^0(t), \quad r(0) = r_0 \tag{1.7.9}$$

where λ is a new parameter, contributing to the market price of risk. Compare this $\mathbb{Q}_0$ dynamics to the $\mathbb{Q}$-dynamics (1.7.4). Notice that for $\lambda = 0$ the two dynamics coincide, i.e. there is no difference between the risk neutral world and the objective world. More generally, the above $\mathbb{Q}_0$-dynamics is expressed again as a linear Gaussian stochastic differential equation, although it depends on the new parameter λ. This is a tacit assumption on the form of the market price of risk process. Indeed, requiring that the dynamics be of the same nature under the two measures, imposes a Girsanov change of measure of the following kind to go from (1.7.4) to (1.7.9):

$$\left.\frac{d\mathbb{Q}}{d\mathbb{Q}_0}\right|_{\mathcal{F}_t} = exp\left(-\frac{1}{2}\int_0^t \lambda^2 r(s)^2 ds + \int_0^t \lambda r(s) dW^0(s)\right).$$

In other terms, we are assuming that the market price of risk process $\lambda(t)$ has the functional form

$$\lambda(t) = \lambda r(t)$$

in the short rate. Of course, in general there is no reason why this should be the case. However, under this choice we obtain a short rate process that is tractable under both measures.

1.7.2 The Hull-White model

The need for an exact fit to the currently-observed yield curve, led Hull and White to the introduction of a time-varying parameter in the Vasicek model. Notice indeed that matching the model and the market term structures of rates at the current time is equivalent to solving a system with an infinite number of equations, one for each possible maturity. Such a system can be solved in general only after introducing an infinite number of parameters, or equivalently a deterministic function of time.
In this section we stick to the extension where only one parameter, corresponding to the Vasicek θ, is chosen to be a deterministic function of time.

The model we analyze implies a normal distribution for the short-rate process at each time. Moreover, it is quite analytically tractable in that zero-coupon bonds and options on them can be explicitly priced. The Gaussian distribution of continuously-compounded rates then allows for the derivation of analytical formulas and the construction of efficient numerical procedures for pricing a large variety of derivative securities.

Hull and White (1990) assumed that the instantaneous short-rate process evolves under the risk-neutral measure according to

$$dr(t) = [\theta(t) - a(t)r(t)]dt + \sigma(t)dW(t) \tag{1.7.10}$$

where θ, a, σ are deterministic functions of time. Here we concentrate on the following extension of the Vasicek model being analyzed by Hull and White (1994)

$$dr(t) = [\theta(t) - ar(t)]dt + \sigma dW(t) \tag{1.7.11}$$

where a and σ are now positive constants and θ is chosen so as to exactly fit the term structure of interest rates being currently observed in the market. It can be shown that, denoting by $f^M(0,T)$ the market instantaneous forward rate at time 0 for the maturity T, i.e.,

$$f^M(0,T) = -\frac{\partial lnP^M(0,T)}{\partial T}$$

with $P^M(0,T)$ the market discount factor for the maturity T, we must have

$$\theta(t) = \frac{\partial f^M(0,t)}{\partial T} + af^M(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}) \tag{1.7.12}$$

where $\frac{\partial f^M}{\partial T}$ denotes partial derivative of f^M with respect to its second argument.
Equation (1.7.11) can be easily integrated so as to yield

$$r(t) = r(s)e^{-a(t-s)} + \int_s^t e^{-a(t-u)}\theta(u)du + \sigma\int_s^t e^{-a(t-u)}dW(u) \tag{1.7.13}$$

$$= r(s)e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)} + \sigma\int_s^t e^{-a(t-u)}dW(u) \tag{1.7.14}$$

where

$$\alpha(t) = f^M(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2 \tag{1.7.15}$$

Therefore, $r(t)$ conditional on $\mathcal{F}_s$ is normally distributed with mean and variance given respectively by

$$\mathbb{E}[r(t)|\mathcal{F}_s] = r(s)e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)}$$

$$Var[r(t)|\mathcal{F}_s] = \frac{\sigma^2}{2a}\left[1 - e^{-2a(t-s)}\right]$$

Notice that defining the process x by

$$dx(t) = -ax(t)dt + \sigma dW(t), \quad x(0) = 0 \tag{1.7.16}$$

we immediately have that, for each $s < t$

$$x(t) = x(s)e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)} dW(u)$$

so that we can write $r(t) = x(t) + \alpha(t)$ for each t.
For model (1.7.11), the risk-neutral probability of negative rates at time t is explicitly given by

$$\mathbb{Q}\{r(t) < 0\} = \Phi\left(-\frac{\alpha(t)}{\sqrt{\frac{\alpha^2}{2a}[1 - e^{-2at}]}}\right).$$

However such probability is almost negligible in practice.

Bond and Option Pricing

The price at time t of a pure discount bond paying off 1 at time T is given by the expectation (1.7.2). Such expectation is relatively easy to compute under the dynamics (1.7.11). Notice indeed that, due to the Gaussian distribution of $r(T)$ conditional on $\mathcal{F}_t$, $t \leq T$, $\int_t^T r(u)du$ is itself normally distributed. Precisely we can show that

$$\int_t^T r(u)du|\mathcal{F}_t \sim \mathcal{N}\left(B(t,T)[r(t) - \alpha(t)] + ln\frac{P^M(0,t)}{P^M(0,T)} + \frac{1}{2}[V(0,T) - V(0,t)], V(t,T)\right)$$

where

$$B(t,T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right]$$
$$V(t,T) = \frac{\sigma^2}{a^2}\left[T - t + \frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\right]$$

so that we obtain

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)} \tag{1.7.17}$$

where

$$A(t,T) = \frac{P^M(0,T)}{P^M(0,t)} exp\left\{B(t,T)f^M(0,t) - \frac{\sigma^2}{4a}(1 - e^{-2a})B(t,T)^2\right\}$$

Similarly, the price $\boldsymbol{ZBC}(t,T,S,X)$ at time t of a European call option with strike X, maturity T and written on a pure discount bond maturing at time S is given by the expectation

$$\boldsymbol{ZBC}(t,T,S,X) = \mathbb{E}\left(e^{-\int_t^T r(s)ds}(P(T,S) - X)^+|\mathcal{F}_t\right)$$

or, equivalently, by

$$\boldsymbol{ZBC}(t,T,S,X) = P(t,T)\mathbb{E}^T((P(T,S) - X)^+|\mathcal{F}_t).$$

To compute the latter expectation, we need to know the distribution of the process r under the T-forward measure Q^T. Since the process x corresponds to the Vasicek's r with $\theta = 0$, we can use formula (1.7.8) to get

$$dx(t) = [-B(t,T)\sigma^2 - ax(t)]dt + \sigma dW^T(t)$$

where the $\mathbb{Q}^T$-Brownian mption W^T is defined by $dW^T(t) = dW(t) + \sigma B(t,T)dt$, so that, for $s \leq t \leq T$,

$$x(t) = x(s)e^{-a(t-s)} - M^T(s,t) + \sigma \int_s^t e^{-a(t-u)} dW^T(u)$$

with

$$M^T(s,t) = \frac{\sigma^2}{a^2}\left[1 - e^{-a(t-s)}\right] - \frac{\sigma^2}{2a^2}\left[e^{-a(T-t)} - e^{-a(T+t-2s)}\right]$$

It's easy to find that the distribution of the short rate $r(t)$ conditional on $\mathcal{F}_s$ is, under the measure $\mathbb{Q}^T$, still Gaussian with mean and variance given respectively by

$$\mathbb{E}^T\{r(t)|\mathcal{F}_s\} = x(s)e^{-a(t-s)} - M^T(s,t) + \alpha(t),$$
$$Var^T\{r(t)|\mathcal{F}_s\} = \frac{\sigma^2}{2a}\left[1 - e^{-2a(t-s)}\right]$$

As a consequence, the European call-option price is

$$\boldsymbol{ZBC}(t,T,S,X) = P(t,S)\Phi(h) - XP(t,T)\Phi(h - \sigma_p)$$

where

$$\sigma_p = \sigma\sqrt{\frac{1 - e^{-2a(T-t)}}{2a}}B(T,S)$$
$$h = \frac{1}{\sigma_p}ln\frac{P(t,S)}{P(t,T)X} + \frac{\sigma_p}{2}$$

Analogously, the price $\boldsymbol{ZBP}(t,T,S,X)$ at time t of a European put option with strike X, maturity T and written on a pure discount bond maturing at time S is given by

$$\boldsymbol{ZBP}(t,T,S,X) = XP(t,T)\Phi(-h + \sigma_p) - P(t,S)\Phi(-h).$$

Through these formulas we can also price caps and floors since they can be viewed as portfolios of zero-bond options. To this end, we denote by $D = \{d_1, d_2, ..., d_n\}$ the set of the cap/floor payment dates and by $\mathcal{T} = \{t_0, t_1, ..., t_n\}$ the set of the corresponding times, meaning that t_i is the difference in years between d_i and the settlement date t, and where t_0 is the first reset time. Moreover, we denote by τ_i the year fraction from d_{i-1} to d_i, $i = 1, ..., n$. So the price at time $t < t_0$ of the cap with cap rate (strike) X, nominal value N and set of times $\mathcal{T}$ is given by

$$\boldsymbol{Cap}(t,\mathcal{T},N,X) = N\sum_{i=1}^{n}(1 + X\tau_i)\boldsymbol{ZBP}\left(t, t_{i-1}, t_i, \frac{1}{1 + X\tau_i}\right)$$

or, more explicitly,

$$\boldsymbol{Cap}(t,\mathcal{T},N,X) = N\sum_{i=1}^{n}[P(t,t_{i-1})\Phi(-h_i + \sigma_p^i) - (1 + X\tau_i)P(t,t_i)\Phi(-h_i)]$$

where

$$\sigma_p^i = \sigma\sqrt{\frac{1 - e^{-2a(t_{i-1}-t)}}{2a}}B(t_{i-1}, t_i),$$
$$h_i = \frac{1}{\sigma_p^i}ln\frac{P(t,t_i)(1 + X\tau_i)}{P(t,t_{i-1})} + \frac{\sigma_p^i}{2}$$

Analogously, the price of the corresponding floor is

$$\boldsymbol{Flr}(t,\mathcal{T},N,X) = N\sum_{i=1}^{n}[(1 + X\tau_i)P(t,t_i)\Phi(h_i) - P(t,t_{i-1})\Phi(h_i - \sigma_p^i)]$$

We are also able to explicitly price European options on coupon-bearing bonds. To this end, consider a European option with strike X and maturity T, written on a bond paying n coupons after the option maturity. Denote by T_i,$T_i > T$, and c_i the payment time and value of the i-th cash flow after T. Let

$\mathcal{T} := \{T_1, ..., T_n\}$ and $c := \{c_1, ..., c_n\}$. Denote by r^* the value of the spot rate at time T for which the coupon-bearing bond price equals the strike and by X_i the time-T value of a pure-discount bond maturing at T_i when the spot rate is r^*. Then the option price at time $t < T$ is

$$\boldsymbol{CBO}(t, T, \mathcal{T}, c, X) = \sum_{i=1}^{n} c_i \boldsymbol{ZBO}(t, T, T_i, X_i) \tag{1.7.18}$$

Given the analytical formula (1.7.18), also European swaptions can be analytically priced, since a European swaption can be viewed as an option on a coupon-bearing bond. Indeed, consider a payer swaption with strike rate X, maturity T and nominal value N, which gives the holder the right to enter at time $t_0 = T$ an interest rate swap with payment times $T = \{t1, ..., tn\}$, $t_1 > T$, where he pays at the fixed rate X and receives LIBOR set "in arrears". We denote by τ_i the year fraction from t_{i-1} to t_i, $i = 1, ..., n$ and set $c_i := X\tau_i$ for $i = 1, ..., n-1$ and $c_n := 1 + X\tau_i$. Denoting by r^* the value of the spot rate at time T for which

$$\sum_{i=1}^{n} c_i A(T, t_i) e^{-B(T,t_i)r^*} = 1$$

and setting $X_i := A(T, t_i) exp(_B(T, t_i) r^*)$, the swaption price at time $t < T$ is then given by

$$\boldsymbol{PS}(t, T, \mathcal{T}, N, X) = N \sum_{i=1}^{n} c_i \boldsymbol{ZBP}(t, T, t_i, X_i)$$

Analogously, the price of the corresponding receiver swaption is

$$\boldsymbol{RS}(t, T, \mathcal{T}, N, X) = N \sum_{i=1}^{n} c_i \boldsymbol{ZBC}(t, T, t_i, X_i)$$

1.7.3 An example of calibration

We present here an example of calibration to real-market data of the Vasicek one-factor model we have reviewed in the previous section. To this end, we use these market-priced instruments: From these

Deposit Rates

	Instrument	Start Date	End Date	Rate
	USD O/N	08/01/2007	09/01/2007	5,300%
1	USD 1W	10/01/2007	17/01/2007	5,303%
2	USD 2W	10/01/2007	24/01/2007	5,310%
1	USD 1M	10/01/2007	12/02/2007	5,320%
2	USD 2M	10/01/2007	12/03/2007	5,346%
3	USD 3M	10/01/2007	10/04/2007	5,360%

Futures

	Instrument	Price	Conv. Adj	All-in Rate
mar-07	EDH07	94,665	0,01	5,335%
jun-07	EDM07	94,790	0,10	5,209%
sep-07	EDU07	94,955	0,30	5,042%
dec-07	EDZ07	95,090	0,50	4,905%
mar-08	EDH08	95,170	0,80	4,822%
jun-08	EDM08	95,210	1,15	4,779%
sep-08	EDU08	95,235	1,55	4,750%
dec-08	EDZ08	95,230	2,00	4,750%

Swap Rates

	Instrument	Start Date	End Date	Rate
2	USD SWAP 02Y	10/01/2007	10/01/2009	5,124%
3	USD SWAP 03Y	10/01/2007	10/01/2010	5,048%
4	USD SWAP 04Y	10/01/2007	10/01/2011	5,034%
5	USD SWAP 05Y	10/01/2007	10/01/2012	5,038%
7	USD SWAP 07Y	10/01/2007	10/01/2014	5,065%
10	USD SWAP 10Y	10/01/2007	10/01/2017	5,115%
12	USD SWAP 12Y	10/01/2007	10/01/2019	5,150%
15	USD SWAP 15Y	10/01/2007	10/01/2022	5,193%
20	USD SWAP 20Y	10/01/2007	10/01/2027	5,229%
25	USD SWAP 25Y	10/01/2007	10/01/2032	5,239%
30	USD SWAP 30Y	10/01/2007	10/01/2037	5,238%

data we want to construct the zero-coupon bond curve defined as

$$C_x^P(t_0) := \{T \to P(t_0, T), T \geq t_0\}$$

In particular the yield curve bootstrapping formulas are just the pricing formulas discussed in chapter 1, applied to our plain vanilla instruments quoted on the market and selected as bootstrapping instruments.

Instrument	Quotation	Pricing formula
Deposits	Spot rate	$L(t, T_i)$
Futures	Futures price	$\boldsymbol{Futures}(t; \boldsymbol{T}) = N\{1 - [F_i(t) + C^{Fut}(t, T_{i-1}]\}$
Swaps	Swap rate	$R^{Swap}(t; \boldsymbol{T}, \boldsymbol{S}) = \frac{\sum_{j=1}^{m} P_d(t,T_j)F_j(t)\tau(T_{j-1},T_j)}{A_d(t,\boldsymbol{S})}$

We suppose that:

- $T = [T_0, T_1, ..., T_n]$ be the time grid of the market data selected as bootstrapping instruments (pillar);
- $R^{mkt}(T_0, T_i)$ the market rate quoted for the bootstrapping instrument associated to pillar i;

- We have already bootstrapped the yield curve until pillar $i-1$ and we want to compute the curve at pillar i.

Then, the bootstrapping algorithm proceeds as follows, for each typology of bootstrapping instruments:

- **Deposits**: $P(t_0, T_i) = \frac{1}{1+L(t_0,T_i)\tau(t_0,T_i)}$
- **Futures**: $P(t_0, T_i) = \frac{P(t_0,t_{i-1})}{1+\left[1-\frac{\boldsymbol{Futures(t;T)}}{100}-C^{Fut}(t,T_{i-1})\right]\tau(T_{i-1},T_i)}$
- **Swap**: $P(t_0, T_i) = \frac{P_d(t_0,T_i)P(t_0,T_{i-1})}{R_i^{Swap}(t_0)A_{d,i}(t_0)-R_{i-1}^{Swap}(t_0)A_{d,i-1}(t_0)+P_d(t_0,T_i))}$

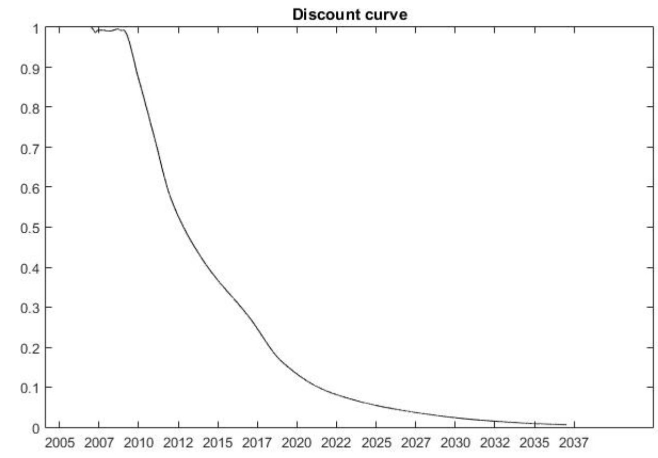
We determine the discount factors for the standard maturities in three steps:

1. Build the short end (approximately, the first 3 months) of the curve using deposit rates. This step involve some interpolation.
2. Build the intermediate (somewhere between 3 months and 5 years) part of the curve using the Eurodollar futures. The starting date for the first future has its discount rate set by interpolation from the already built short end of the curve. With the addition of each consecutive future contract to the curve the discount factor for its starting date is either (a) interpolated from the existing curve if it starts earlier than the end date of the last contract, or (b) extrapolated from the end date of the previous future.
3. Build the long end of the curve using swap rates as par coupon rates. Observe first that for a swap of maturity T_{mat} we can calculate the discount factor $P(0;T_{mat})$ in terms of the discount factors to the earlier coupon dates:

$$P(0,T_{mat}) = \frac{1 - S(T_{mat})\sum_{j=1}^{n-1}\alpha_j P(0,T_j)}{1+\alpha_n S(T_{mat})}$$

   We begin by interpolating the discount factors for coupon dates that fall within the previously built segment of the curve, and continue by inductively applying the above formula. The problem is that we do not have market data for swaps with maturities falling on all standard dates and interpolation is again necessary to deal with the intermediate dates.

We obtain: Then, implementing the equation (1.7.7) for the price of a zero-coupon bond, our aim is to



minimize the difference from this analytical formula, which depends on the parameters of the model, and the discount curve constructed from the market, i.e we have to find this minimum:

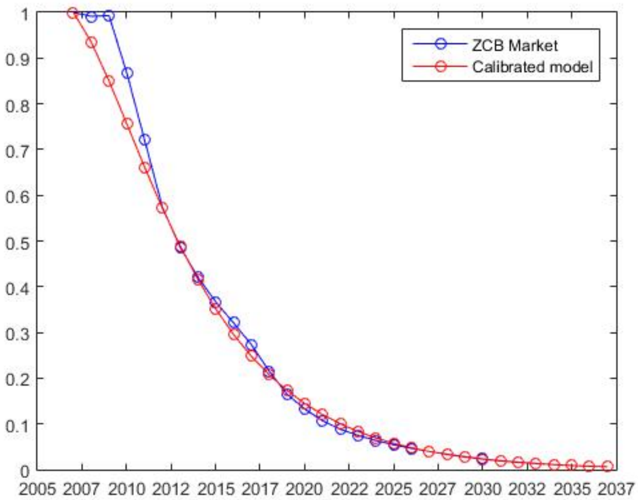
$$min \sum_{i=1}^{n} (P(0,T_i) - P^{mkt}(0,T_i))^2$$

In Matlab we have used the function *lsqnonlin.m* in this way

```
funz=@(x) ZCB_Mark./ZCB(t,TT1,x(1),x(2),x(3),r(1))-1;
x0=[0.2,0.05,0.03];
xmin=lsqnonlin(funz,x0);


k=xmin(1);
teta=xmin(2);
sigma=xmin(3);
```

Our results are summarized in this graph:

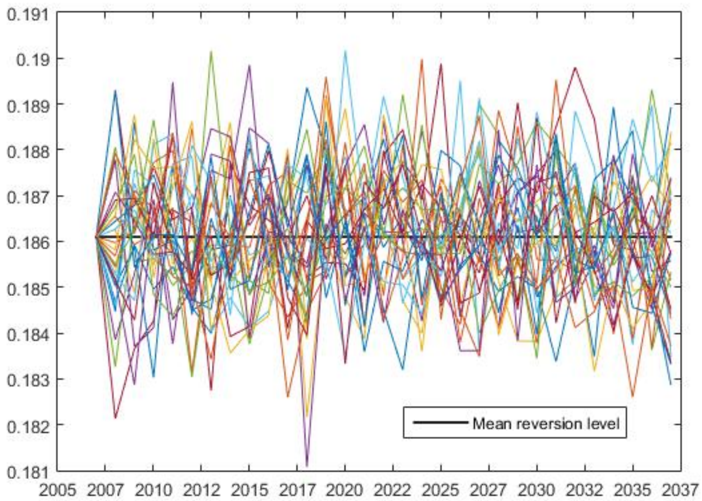


$k = 0.2728$

$\theta = 0.1861$

$\sigma = 0.001$

With these parameters, the trajectories created by the Vasicek model are the following



Chapter 2

Funding, collateral, funding value adjustment

2.1 Black-Scoles-Merton from a modern perspective

We consider a generic derivative Π depending on a single generic underlying asset A, with payoff $\Pi(T)$ at time T and price $\Pi(t)$ at time $t < T$.

We assume a market M that trades three financial instruments:

1. the asset A, with no dividends

2. the derivative Π

3. the funding account B_f (cash) for funding unsecured at rate r_f

We stress the following assumptions:

- Single asset A
- No collateral
- No counterparty risk
- No dividends
- Generic funding for asset A, no repo
- Deterministic interest rates (see the derivation)
- All the classical Black-Scholes-Merton assumptions

We will derive the classical Black-Scholes-Merton pricing formulas using replication arguments, PDE and Feynman-Kac. In particular, we will be able to understand when and where funding enters into the derivation and into the final result.

Dynamics under real measure P:

$$
\begin{aligned}
dA(t) &= \mu(t, A)dt + \sigma(t, A)dW^P(t) \\
dB_f(t) &= r_f(t)B_f(t)dt
\end{aligned}
$$

$$
\begin{aligned}
d\Pi(t) &= \frac{\partial \Pi}{\partial t}dt + \frac{\partial \Pi}{\partial A}dA(t) + \frac{1}{2}\frac{\partial^2 \Pi}{\partial^2 A}dA^2(t) \\
&= \widehat{\mathfrak{L}}_\mu \Pi(t)dt + \sigma(t,A)\frac{\partial \Pi}{\partial A}dW^P(t) \\
\widehat{\mathfrak{L}}_\mu &= \frac{\partial \Pi}{\partial t} + \mu(t,A)\frac{\partial \Pi}{\partial A} + \frac{1}{2}\sigma^2(t,A)\frac{\partial^2 \Pi}{\partial^2 A}
\end{aligned}
$$

following notation and basic assumptions previously established and dropping obvious indexes.

We now construct a replication strategy Θ of the derivative Π, by setting up a replication portfolio V such that

$$V(t) = \Pi(t), \quad \forall t \leq T$$

by combining appropriate amounts of the available assets

$$\boldsymbol{X}(t) := \begin{bmatrix} A(t) \\ B_f(t) \end{bmatrix}$$

$$\boldsymbol{\Theta}(t) := \begin{bmatrix} \Delta(t) \\ \Psi_f(t) \end{bmatrix}$$

$$V(t) = \boldsymbol{\Theta}(t)' \cdot \boldsymbol{X}(t) = \Delta(t)A(t) + \Psi_f(t)B_f(t)$$

where:

- $\boldsymbol{X}$ is the vector of the price processes of the assets,
- $\boldsymbol{\Theta}$ is the vector of the portfolio positions, or number of units, in each asset,
- V is the (scalar) value of the replication portfolio,
- $\boldsymbol{\Theta}(t)'$ denotes vector transposition.

The replication strategy is described also by its (vector) dividend processes $D(t)$, and gain processes $G(t)$ (namely profit and losses achieved by holding securities), such that

$$
\begin{aligned}
&\boldsymbol{D(t)} = \boldsymbol{0} \\
&\boldsymbol{G}(t) := \boldsymbol{X}(t) + \boldsymbol{D}(t) = \boldsymbol{X}(t)
\end{aligned}
$$

The gain processes of the assets, in SDE form, are given directly by the dynamics chosen before, as

$$d\boldsymbol{G}(t) = d\boldsymbol{X}(t) = \begin{bmatrix} dA(t) \\ dB_f(t) \end{bmatrix} = \begin{bmatrix} \mu(t,A)dt + \sigma(t,A)dW^P(t) \\ r_f(t)B_f(t)dt \end{bmatrix}$$

The gain process of the replication portfolio is given, in SDE form, by

$$
\begin{aligned}
dG(t) &:= \boldsymbol{\Theta}(t)' \cdot d\boldsymbol{G}(t) \\
&= \Delta(t)dA(t) + \Psi_f(t)dB_f(t) \\
&= [\mu(t,A)\Delta(t) + r_f(t)\Psi_f(t)B_f(t)]dt + \Delta(t)\sigma(t,A)dW^P(t)
\end{aligned}
$$

We now impose replication condition, and we obtain

$$\Pi(t) = V(t) = \Delta(t)A(t) + \Psi_f(t)B_f(t) \quad \forall t \leq T$$

$$\Rightarrow \Psi_f(t)B_f(t) = \Pi(t) - \Delta(t)A(t)$$

consistently with the fact that the funding account B_f is used to finance the borrowing of $\Delta(t)$ units of the underlying $A(t)$ at the funding rate $r_f(t)$.

The gain process of the replication portfolio becomes

$$dG(t) := \mu(t, A)\Delta(t)dt + d\Gamma(t, A) + \Delta(t)\sigma(t, A)dW^P(t)$$
$$d\Gamma(t, A) = [-r_f(t)\Delta(t)A(t) + r_f(t)\Pi(t)]dt$$

the cash amount $\Gamma(t)$ contained in the replication portfolio is split between:

- the derivative $\Pi(t)$, growing at the funding rate $r_f(t)$,
- the amount $\Delta(t)A(t)$, borrowed at the funding rate $r_f(t)$ to finance the purchase of $\Delta(t)$ units of the underlying asset $A(t)$.

We now impose the self-financing condition. The replication strategy is said self-financing if its dividend process (in/out cash flows generated by the strategy) is null,

$$D(t) = G(t) - V(t) = 0$$

We have just seen that this latter condition is already satisfied. Combining the conditions above, we have

$$dG(t) = dV(t) = d\Pi(t)$$

Introducing in this latter equation the expressions of $dG(t)$ and $d\Pi(t)$ obtained before, and rearranging terms we obtain the SDE

$$\left[\frac{\partial \Pi}{\partial t} + \mu(t, A)\left(\frac{\partial \Pi}{\partial A} - \Delta(t)\right) + \frac{1}{2}\sigma^2(t, A)\frac{\partial^2 \Pi}{\partial^2 A}\right] dt + \sigma(t, A)\left(\frac{\partial \Pi}{\partial A} - \Delta(t)\right) dW^P(t) = d\Gamma(t, A)$$

We finally impose the risk neutral condition $\Delta(t) = \frac{\partial \Pi}{\partial A}$, such that the stochastic (risky) term with $dW^P(t)$ disappears, and we obtain a **Black-Scholes** PDE equation for the derivative's price $\Pi(t)$

$$\widehat{\mathfrak{L}}_{r_f}\Pi(t) = r_f(t)\Pi(t)$$
$$\widehat{\mathfrak{L}}_{r_f} := \frac{\partial}{\partial t} + r_f(t)A(t)\frac{\partial}{\partial A} + \frac{1}{2}\sigma^2(t, A)\frac{\partial^2}{\partial^2 A}$$

Using the Feynman-Kac theorem we may switch from the PDE representation to the SDE representation given by

$$\Pi(t) = \mathbb{E}_t^{\mathbb{Q}}[D_f(t; T)\Pi(T)]$$
$$D_f(t; T) := exp\left[-\int_t^T r_f(u)du\right]$$
$$dA(t) = r_f(t)A(t)dt + \sigma(t, A)dW^{\mathbb{Q}}(t)$$

under the risk neutral funding probability measure $\mathbb{Q}$ associated, in this case, to the funding account $B_f(t)$. We conclude that we discount at the **funding rate**.

Remark 2.1. Replication at work
This proof makes clear how the replication and funding mechanism works in practice. The market risk generated by the derivatives' position is hedged using the risky asset A. The replication strategy is constructed with a combination of the instruments available on the market: the (single) asset A and the funding account B_f. The former allows to include into the replication strategy the appropriate amount of risk to hedge the risk generated by the derivative. The latter describe the amount of cash that we must borrow or lend on the market at the funding rate r_f to finance the hedging. The cash is split between the amount $\Delta(t)A(t)$, borrowed to finance the purchase of $\Delta(t)$ units of the risky asset $A(t)$, and the amount Π.

Remark 2.2. Probability measure
The probability measure $\mathbb{Q}$ introduced via Feynman-Kac is associated to the risk neutral drift r_f appearing in the SDE dynamics of the asset A. In the classical financial world $\mathbb{Q}$ was traditionally associated to a Libor Bank account, reflecting the average funding rate on the interbank money market, considered a good proxy of a risk free rate. Nowadays, in the modern financial world, there are no risk free rates, and $\mathbb{Q}$ must be interpreted simply as the risk neutral measure associated to the funding account B_f. We call it the funding measure.

Remark 2.3. Static hedge
In particular, self-financing implies the absence of strategy dividends and

$$\begin{aligned}
&dV(t) = dG(t)\\
&\Rightarrow d[\boldsymbol{\Theta}(t)' \cdot \boldsymbol{X}(t)] = \boldsymbol{\Theta}(t)' \cdot d\boldsymbol{G}(t) = \boldsymbol{\Theta}(t)' \cdot d\boldsymbol{X}(t)\\
&d[\Delta(t)A(t) + \Psi_f(t)B_f(t)] = \Delta(t)dA(t) + \Psi_f(t)dB_f(t)\\
&\Rightarrow d\boldsymbol{\Theta}(t) = \boldsymbol{0}\\
&\Rightarrow \boldsymbol{\Theta}(t) = constant
\end{aligned}$$

This is the well known feature of the classical Black-Scholes derivation:

- the position $\Delta(t)A(t)$ in the risky asset S is self-financing in its own, because its variation $d[\Delta(t)A(t)]$ is funded by the risky asset variation alone, $\Delta(t)dA(t)$.
- the position is static, $\Delta(t) = constant$.

We stress that this is a consequence of the absence of dividends. In general this equality does not hold

$$d[\boldsymbol{\Theta}(t)' \cdot \boldsymbol{X}(t)] \neq \boldsymbol{\Theta}(t)' \cdot d\boldsymbol{X}(t)$$

Remark 2.4. Zero Coupon Bond
We can define unsecured Zero Coupon Bonds, such that

$$P_f(T;T) = 1$$

$$P_f(t;T) = \mathbb{E}_t^{\mathbb{Q}}[D_f(t;T)]$$

Remark 2.5. We can switch from risk neutral funding measure $\mathbb{Q}$, associated to numeraire $B_f(t)$, to

$T-$forward measure $\mathbb{Q}^T$, associated to numeraire $P_f(t,T)$, using the Radon-Nikodym derivative

$$
\begin{aligned}
M_f(t;T) &:= \frac{P_f(t;T)}{B_f(t)} \\
\Pi(t;T) &= \mathbb{E}_t^{\mathbb{Q}}[D_f(t,T)\Pi(T)] \\
&= M_f(t;T)\mathbb{E}_t^{\mathbb{Q}^T}\left[\frac{D_f(t,T)}{M_f(T;T)}\Pi(T)\right] \\
&= \frac{P_f(t;T)}{B_f(t)}\mathbb{E}_t^{\mathbb{Q}^T}\left[\frac{B_f(t)}{P_f(T;T)}\Pi(T)\right] \\
&= P_f(t;T)\mathbb{E}_t^{\mathbb{Q}^T}[\Pi(T)]
\end{aligned}
$$

2.2 Multiple funding sources

We assume that derivatives' counterparties may finance their derivatives' activity by borrowing and lending funds on the market through a variety of market operations, such as trading Deposits, Repos (Repurchase Agreements), Bonds, etc. at their corresponding funding rates. We also assume that derivatives' counterparties eventually reduce the counterparty risk through the adoption of bilateral collateral agreements (CSA) or trade migration to Central Counterparties (CCPs). In particular, we will identify three sources of funding associated with derivatives.

- **Money market**
  Money market funding is the traditional unsecured funding source for banks and financial institutions. Borrowing and lending is based on the trading of Certificates of Deposit (Depo). A Depo is an unsecured cash zero coupon loan. In case of default of the borrower during the Depo life, the lender suffers a loss. In banks and financial institutions, a derivative trading desk may borrow and lend unsecured funds through a treasury desk.

- **Repo market**
  Another common funding source is repo funding. In this case, borrowing and lending is based on trading Repurchase Agreement contracts (Repo). A Repo is a secured cash zero coupon loan such that, at time t, counterparty B, the borrower, sales an asset A to counterparty L, the lender, receiving upfront the corresponding asset value $A(t)$, under the agreement to buy it back and pay an interest $R_R(t,T)$, called repo rate, at maturity $T > t$. Thus, the borrower pays, at maturity T the amount

  $$\Pi_R(T, A) = A(t)[1 + R_R(t,T)\tau(t,T)]$$

  The repo is secured by the asset A itself, used as collateral. In case of default of the borrower during the repo life, the lender keeps the asset A. Thus, a repo is equivalent to a combination between a spot sale (the initial legal transfer of the asset to the lender in exchange for transfer of money to the borrower), and a forward contract (repayment of the loan to the lender and return of the collateral of the borrower at maturity). Possible coupons and or dividends generated by the asset A during the repo life are transferred by the lender to the borrower. Looking at the forward contract component of the repo, the repo price $\Pi_R(t)$ is such that $\Pi_R(t) = 0$ if the contract is traded at par at time t.
  In banks and financial institutions, a derivative trading desk may borrow and lend secured repo funds through a repo desk.

- **Collateral**
  Real collateral agreements are regulated mostly under the Credit Support Annex (CSA) of the ISDA

Standard Master Agreement. For pricing purposes it is useful to introduce an abstract "perfect" collateral, with the following properties.

- Zero initial margin or initial deposit
- Zero threshold
- Zero minimum transfer amount
- Fully symmetric
- Cash collateral only
- Continuous margination
- Instantaneous settlement
- Instantaneous margination rate $r_c^\beta(t)$ in currency β
- In case of default of one counterparty: neither close out amounts nor legal risk to the closing of the deal or availability of the collateral

As a consequence we have that, in general, the collateral value perfectly matches the derivative's value,

$$\Pi^\alpha(t, A) = x_{\alpha\beta}(t) C^\beta(t), \quad \forall t \leq T.$$

In banks and financial institutions, the collateral associated with a derivative trading desk is operated by a collateral desk.

Multiple funding accounts

Following the previous discussion, we assume that the amount of cash borrowed or lent by a counterparty in the market M from multiple funding sources is associated to multiple funding accounts B_x, where index x denotes the specific source of funding, with value $B_x^\alpha(t)$ and (symmetric) funding rate $r_x^\alpha(t)$ in currency α at time t, such that

$$dB_x^\alpha(t) = r_x^\alpha(t) B_x^\alpha(t) dt, \quad B_x^\alpha(0) = 1$$

$$B_x^\alpha(t) = exp\left[\int_0^t r_x^\alpha(u) du\right]$$

$$D_x^\alpha(t, T) := \frac{B_x^\alpha(t)}{B_x^\alpha(T)} = exp\left[-\int_t^T r_x^\alpha(u) du\right]$$

Remark 2.6. 1. The collection of funding accounts $B_x(t)$ is assumed locally market risk free, since their dynamics do not contain stochastic terms, and thus the value of the account at time $t + dt$ depends only on the value of the account and of the rate at previous time t.

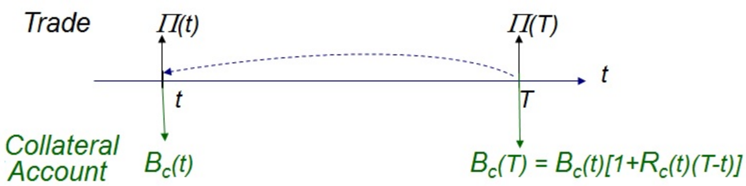
2. The collection of funding accounts $B_x(t)$ is assumed credit risk free, since the default of the borrowing counterparty is not included into their dynamics

3. The funding rates $r_x(t)$ may be, in general, stochastic.

2.3 Collateral: discrete margination

Example 2.1. A trivial (but intuitive) example
Let's suppose that the trade consists of single cash flow, such that we receive/pay an amount $\Pi(T)$ at

maturity T, corresponding to a present value $\Pi(t)$ at time $t < T$.
Let's suppose also that the trade is under perfect collateral, with two margination dates, at t and T: at time t we post the amount $B_c(t)$ into the collateral account, where it grows at the collateral rate $R_c(t)$ up to maturity T. By no arbitrage and self-financing, we must have



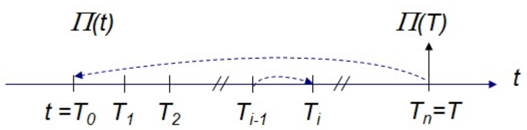
$$B_c(T) = B_c(t)[1 + R_c(t)(T - t)] = \Pi(T)$$

$$\Pi(t) = P_d(t, T)\Pi(T) = B_c(t)$$
$$\Rightarrow P_d(t, T) = \frac{1}{[1 + R_c(t)(T - t)]}$$

Thus no arbitrage requires discounting at the **collateral rate**.

In case of multiple discrete marginations, at each margination date T_i the counterparties must regulate the margin over the last time interval $\Delta T_i = [T_{i-1}, T_i]$ by exchanging the amount

$$\begin{aligned}\mathcal{M}(T_i) &= \Pi(T_i) - \Pi(T_{i-1}) - B_c(T_{i-1})R_c(T_{i-1})\Delta T_i \\ &= \Pi(T_i) - \Pi(T_{i-1})[1 + R_c(T_{i-1})\Delta T_i]\end{aligned}$$



The counterparty whose NPV has increased/decreased must receive/post the amount $M(T_i)$. At the same time interval the bank account $B(t)$ and discount factor $D(t, T)$ evolve with the (simple compounded) rate R as

$$B(T_i) = B(T_{i-1})[1 + R(T_{i-1})\Delta T_i]$$
$$D(t; T_i) = \frac{D(t; T_{i-1})}{1 + R(T_{i-1})\Delta T_i}, \quad D(t; t) = 1$$

The value at time t of all the future margination amounts is given by the discounted sum

$$\begin{aligned}\mathcal{M}(t; \boldsymbol{T}) &= \sum_{i=1}^{n} \mathbb{E}_t^Q[D(t; T_i)\mathcal{M}(T_i)] \\ &= \sum_{i=1}^{n} \mathbb{E}_t^Q\{D(t; T_i)[\Pi(T_i) - \Pi(T_{i-1})(1 + R_c(T_{i-1})\Delta T_i)]\} \\ &= \sum_{i=1}^{n} \mathbb{E}_t^Q\left[D(t; T_i)\Pi(T_i) - D(t; T_i)\frac{D_c(t; T_{i-1})}{D_c(t; T_i)}\Pi(T_{i-1})\right]\end{aligned}$$

where $D_c(t; T_i)$ is the discount factor associated to the collateral rate such that

$$B_c(T_i) = B_c(T_{i-1})[1 + R_c(T_{i-1})\Delta T_i]$$

$$D_c(t; T_i) = \frac{D_c(t; T_{i-1})}{1 + R_c(T_{i-1})\Delta T_i}$$

$$D_c(t;t) = 1$$

By no-arbitrage, the total value of the margination amount must be null,

$$\mathcal{M}(t;\boldsymbol{T}) = \sum_{i=1}^{n} \mathbb{E}_t^Q \left[D(t;T_i)\Pi(T_i) - D(t;T_i)\frac{D_c(t;T_{i-1})}{D_c(t;T_i)}\Pi(T_{i-1}) \right] = 0$$

This may be true if and only if $D_d(t;T_i) = D_c(t;T_i)$. Hence

$$\begin{aligned}\mathcal{M}(t;\boldsymbol{T}) &= \sum_{i=1}^{n} \mathbb{E}_t^Q [D_c(t;T_i)\Pi(T_i) - D_c(t;T_{i-1})\Pi(T_{i-1})] \\ &= \mathbb{E}_t^Q [D_c(t;T_n)\Pi(T_n) - D_c(t;T_0)\Pi(T_0)] \\ &= \mathbb{E}_t^Q [D_c(t;T)\Pi(T)] - \Pi(t) = 0\end{aligned}$$

We conclude that no arbitrage implies discounting at the **collateral rate**

$$\Pi(t) = \mathbb{E}_t^Q [D_c(t;T)\Pi(T)]$$

$$D(t;T_i) = D_c(t;T_i)$$

$$R(t;T_i) = R_c(t;T_i)$$

In the limit of continuous margination $\Delta T_i \to dt$ we have

$$\begin{aligned}D(t;T_i) \to D(t;T) &= exp\left[-\int_t^T r(u)du \right] \\ D_c(t;T_i) \to D_c(t;T) &= exp\left[-\int_t^T r_c(u)du \right] \\ \mathcal{M}(t;\boldsymbol{T}) \to \mathcal{M}(t;T) &= \mathbb{E}_t^Q [D_c(t;T)\Pi(T)] - \Pi(t) = 0 \\ \Pi(t) &= \mathbb{E}_t^Q [D_c(t;T)\Pi(T)]\end{aligned}$$

where $r(t)$ and $r_c(t)$ are the short rate and the collateral short rate, respectively.

In the limit of no margination $\Delta T_i \to T - t$ we have the same equations as above, but we make funding not at the collateral rate but at a generic funding spread $s_f(t)$ over the risk free rate

$$D(t;T_i) \longrightarrow exp\left\{ -\int_t^T [r(u) + s_f(u)]du \right\} := D(t;T)D_f(t;T)$$

$$\mathcal{M}(t;\boldsymbol{T}) \longrightarrow \mathcal{M}_f(t;\boldsymbol{T}) = \mathbb{E}_t^Q [D(t;T)D_f(t;T)\Pi(T)] - \Pi(t) = 0$$

$$\Pi(t) \longrightarrow \Pi_f(t) = \mathbb{E}_t^Q [D(t;T)D_f(t;T)\Pi(T)]$$

$$|\Pi_f(t)| \le |\Pi(t)|$$

Hence we discount at the **funding rate**.

If $s_f(t)$ is deterministic we obtain

$$\Pi_f(t) = P_f(t;T)\mathbb{E}_t^Q [D(t;T)\Pi(T)] = P_f(t;T)\Pi(t)$$

$$P_f(t;T) = D_f(t;T) = exp\left[-\int_t^T s_f(u)du \right]$$

$$|\Pi_f(t)| \le |\Pi(t)|$$

2.4 Perfect collateral

We now consider the case of derivative under perfect collateral.
Our economy admits, in this case, four financial instruments:

- the asset A, with no dividends
- the derivative Π under collateral C
- the funding account B_f for funding unsecured at rate r_f
- the collateral account B_c for funding secured by collateral at rate r_c

We hold all the assumptions of the previous case but the perfect collateral, such that

$$\Pi(t) = C(t), \quad \forall t \leq T$$

Dynamics under real measure P:

$$\begin{aligned} dA(t) &= \mu(t, A)dt + \sigma(t, A)dW^P(t) \\ dB_f(t) &= r_f(t)B_f(t)dt \\ dB_c(t) &= r_c(t)B_c(t)dt \end{aligned}$$

$$\begin{aligned} d\Pi(t) &= \widehat{\mathfrak{L}}_\mu \Pi(t)dt + \sigma(t, A)\frac{\partial \Pi}{\partial A}dW^P(t) \\ \widehat{\mathfrak{L}}_\mu &= \frac{\partial \Pi}{\partial t} + \mu(t, A)\frac{\partial}{\partial A} + \frac{1}{2}\sigma^2(t, A)\frac{\partial^2}{\partial^2 A} \end{aligned}$$

We now construct a replication strategy Θ of the derivative Π, by setting up a replication portfolio V such that

$$V(t, \boldsymbol{\Theta}, \boldsymbol{X}) = \Pi(t), \quad \forall t \leq T$$

by combining appropriate amounts of the available assets

$$\boldsymbol{X}(t) := \begin{bmatrix} A(t) \\ B_f(t) \\ B_c(t) \end{bmatrix}$$

$$\boldsymbol{\Theta}(t) := \begin{bmatrix} \Delta(t) \\ \Psi_f(t) \\ \Psi_c(t) \end{bmatrix}$$

$$V(t, \boldsymbol{\Theta}, \boldsymbol{X}) = \boldsymbol{\Theta}(t)' \cdot \boldsymbol{X}(t) = \Delta(t)A(t) + \Psi_f(t)B_f(t) + \Psi_c(t)B_c(t)$$

Dividend and gain processes of the replication strategy

$$\begin{aligned} \boldsymbol{D}(t) &= \boldsymbol{0} \\ \boldsymbol{G}(t) &:= \boldsymbol{X}(t) + \boldsymbol{D}(t) = \boldsymbol{X}(t) \end{aligned}$$

The gain processes of the assets, in SDE form, are given by

$$d\boldsymbol{X}(t) = \begin{bmatrix} dA(t) \\ dB_f(t) \\ dB_c(t) \end{bmatrix} = \begin{bmatrix} \mu(t,A)dt + \sigma(t,A)dW^P(t) \\ r_f(t)B_f(t)dt \\ r_c(t)B_c(t)dt \end{bmatrix}$$

The gain process of the replication portfolio is given, in SDE form, by

$$\begin{aligned} dG(t) &:= \boldsymbol{\Theta}(t)' \cdot d\boldsymbol{G}(t) \\ &= \Delta(t)dA(t) + \Psi_f(t)dB_f(t) + \Psi_c(t)dB_c(t) \\ &= [\mu(t,A)\Delta(t) + r_f(t)\Psi_f(t)B_f(t) + r_c(t)\Psi_c(t)B_c(t)]dt + \Delta(t)\sigma(t,A)dW^P(t) \end{aligned}$$

We now impose the perfect collateral and replication conditions,

$$\Pi(t) = C(t) = \Psi_c(t)B_c(t)$$

$$V(t,\boldsymbol{\Theta},\boldsymbol{X}) = \Pi(t), \quad \forall t \leq T$$

and we obtain

$$\begin{aligned} V(t,\boldsymbol{\Theta},\boldsymbol{X}) &:= \Delta(t)A(t) + \Psi_f(t)B_f(t) + \Psi_c(t)B_c(t) \\ &= \Delta(t)A(t) + \Psi_f(t)B_f(t) + \Pi(t) \\ &\Rightarrow \Psi_f(t)B_f(t) = -\Delta(t)A(t) \end{aligned}$$

Notice that:

- the funding account B_f is used to finance, at the funding rate $r_f(t)$, the borrowing of Δ units of the asset A;
- the collateral account is used to finance, at the collateral rate r_c, the derivative Π;
- no cash is left out the replication because of the perfect collateral (see the partial collateral case later).

The gain process of the replication portfolio becomes

$$\begin{aligned} dG(t,\boldsymbol{\Theta},\boldsymbol{X}) &= \mu(t,A)\Delta(t)dt + d\Gamma(t,A) + \Delta(t)\sigma(t,A)dW^P(t) \\ d\Gamma(t,A) :&= [-r_f(t)\Delta(t)A(t) + r_c(t)\Pi(t)]dt \end{aligned}$$

Notice that the cash amount $\Gamma(t,A)$ contained in the replication portfolio is split between:

- the derivative amount, equal to the cash in the collateral account B_c, growing at the collateral rate r_c,
- the amount $\Delta(t)A(t)$, borrowed at the funding rate $r_f(t)$ to finance the purchase of $\Delta(t)$ units of the asset $A(t)$.

We now impose the self-financing condition on the replication strategy

$$D(t,\boldsymbol{\Theta},\boldsymbol{X}) = G(t,\boldsymbol{\Theta},\boldsymbol{X}) - V(t,\boldsymbol{\Theta},\boldsymbol{X}) = 0$$

We have just seen that this latter condition is already satisfied. Combining the conditions above, we have

$$dG(t,\boldsymbol{\Theta},\boldsymbol{X}) = dV(t,\boldsymbol{\Theta},\boldsymbol{X}) = d\Pi(t)$$

Introducing in this latter equation the expressions of $dG(t, \boldsymbol{\Theta}, \boldsymbol{X})$ and $d\Pi(t)$ obtained before, and rearranging terms we obtain the SDE

$$\left[\frac{\partial \Pi}{\partial t} + \mu(t,A)\left(\frac{\partial \Pi}{\partial A} - \Delta(t)\right) + \frac{1}{2}\sigma^2(t,A)\frac{\partial^2 \Pi}{\partial^2 A}\right] dt + \sigma(t,A)\left(\frac{\partial \Pi}{\partial A} - \Delta(t)\right) dW^P(t) = d\Gamma(t,A)$$

We finally impose the risk neutral condition $\Delta(t) = \frac{\partial \Pi}{\partial A}$, such that the stochastic (risky) term with $dW^P(t)$ disappears, and we obtain a **generalised Black-Scholes** PDE equation for the derivative's price $\Pi(t, A)$

$$\begin{aligned}\widehat{\mathfrak{L}}_{r_f}\Pi(t) &= r_c(t)\Pi(t)\\ \widehat{\mathfrak{L}}_{r_f} &:= \frac{\partial}{\partial t} + r_f(t)A(t)\frac{\partial}{\partial A} + \frac{1}{2}\sigma^2(t,A)\frac{\partial^2}{\partial^2 A}\end{aligned}$$

Using the Feynman-Kac theorem we may switch from the PDE representation to the SDE representation given by

$$\Pi(t) = \mathbb{E}_t^{\mathbb{Q}}[D_c(t;T)\Pi(T)] \tag{2.4.1}$$

$$D_c(t;T) := exp\left[-\int_t^T r_c(u)du\right] \tag{2.4.2}$$

$$dA(t) = r_f(t)A(t)dt + \sigma(t,A)dW^{\mathbb{Q}}(t) \tag{2.4.3}$$

We conclude that we discount at the **collateral rate**.

Remark 2.7.

1. **Funding measure**:
   the probability measure $\mathbb{Q}$ introduced via Feynman-Kac is associated with the risk neutral drift r_f appearing in the SDE dynamics of the asset A. It is the same measure of the uncollateralised case, but now the numeraire is the collateral account B_c.

2. **Borrowing/lending return**:
   we notice that, over the time interval $[t, t + dt)$, the derivative $\Pi(t)$ and the cash flow $r_c(t)\Pi(t)dt$ generated by the (perfect) collateral margination, are equivalent to a derivative $\Pi(t)$ without collateral but with a continuous dividend yield $r_c(t)\Pi(t)dt$, such that $r_f - r_c$ the rate is the actual (instantaneous) borrowing/lending cost/return including the collateral.

3. **Zero Coupon Bonds**:
   we can define secured (collateralized) Zero Coupon Bonds, such that

$$P_c(T;T) = 1$$

$$P_c(t;T) = \mathbb{E}_t^Q[D_c(t;T)]$$

4. **T-Forward measure**:
   we can switch from the funding measure associated with numeraire $B_c(t)$ to T-forward measure $Q \subset T$ associated with secured (collateralised) numeraire $P_c(t;T)$ using the corresponding Radon-

Nikodym derivative as follows

$$\begin{aligned} M_c(t,T) &= \frac{P_c(t;T)}{B_c(t)} \\ \Pi_c(t) &= \mathbb{E}_t^Q[D_c(t;T)\Pi(T)] \\ &= M(t,T)\mathbb{E}_t^{Q_c^T}\left[\frac{D_c(t;T)}{M(T,T)}\Pi(T)\right] \\ &= P_c(t;T)\mathbb{E}_t^{Q_c^T}\left[\frac{1}{P_c(T,T)}\Pi(T)\right] \\ &= P_c(t;T)\mathbb{E}_t^{Q_c^T}[\Pi(T)] \end{aligned}$$

We remember that any positive martingale process (not necessary the ratio between two numeraires) can be used as Radon-Nikodym derivative in a measure change.

5. **(Im)perfect collateral**:
   Since real CSAs are far from being perfect, a more general proof is required to take into account imperfect collateral, such that $B_c(t) \neq \Pi(t)$, also in terms of different currencies.

6. **Equity asset**:
   The lending/borrowing of some assets is often realised through repo contracts and funded at the repo rate. Furthermore, assets may pay, in general, dividends. Typical example is equity asset. Thus a more general proof is necessary to deal with these special case.

7. **Funding Valuation Adjustment (FVA)**:
   Comparing the collateralised vs uncollateralised prices we can define a Funding Value Adjustment (FVA) such that, in additive form,

$$\begin{aligned} \Pi_f(t) &= \mathbb{E}_t^Q[D_f(t;T)\Pi(T)] \\ &= \mathbb{E}_t^Q\left[\frac{D_f(t;T)}{D_c(t;T)}D_c(t;T)\Pi(T)\right] \\ &= \mathbb{E}_t^Q[D_c(t;T)\Pi(T)] + FVA_{f,c}(t) \\ &= \Pi_c(t) + FVA_{f,c}(t) \\ FVA_{f,c} &:= \Pi_f(t) - \Pi_c(t) \end{aligned}$$

The calculation of the FVA above depends on the correlated dynamics of the funding and collateral rates, and is thus model dependent. If the variance of the ratio between the funding and the collateral (stochastic) discounts is negligible with respect to the variance of the remaining discounted payoff [1],

$$Var\left[\frac{D_f(t;T)}{D_c(t;T)}\right] \ll Var[D_c(t;T)\Pi(T)]$$

we may write

$$\begin{aligned} \Pi_f(t) &= \mathbb{E}_t^Q\left[\frac{D_f(t;T)}{D_c(t;T)}D_c(t;T)\Pi(T)\right] \\ &\cong \mathbb{E}_t^Q\left[\frac{D_f(t;T)}{D_c(t;T)}\right]\mathbb{E}_t^Q[D_c(t;T)\Pi(T)] \\ &= \mathbb{E}_t^Q\left[\frac{D_f(t;T)}{D_c(t;T)}\right]\Pi_c(t) \\ FVA_{f,c}(t) &\cong \left\{\mathbb{E}_t^Q\left[\frac{D_f(t;T)}{D_c(t;T)}\right] - 1\right\}\Pi_c(t) \end{aligned}$$

[1] This condition is less restrictive than the deterministic basis limit used below.

In the limit of deterministic basis we obtain the simple expression

$$\mathbb{E}_t^Q\left[\frac{D_f(t;T)}{D_c(t;T)}\right] \cong \frac{\mathbb{E}_t^Q[D_f(t;T)]}{\mathbb{E}_t^Q[D_c(t;T)]} = \frac{P_f(t;T)}{P_c(t;T)} = e^{-\int_t^T s_{f,c}(u)du}$$

$$FVA_{f,c} \cong \left[e^{-\int_t^T s_{f,c}(u)du} - 1\right]\Pi_c(t) \cong -\left[\int_t^T s_{f,c}(u)du\right]\Pi_c(t)$$

$$s_{f,c}(t) := r_f(t) - r_c(t)$$

2.5 Perfect collateral for derivative and hedge

We now consider the special case of perfect collateral, in which also the asset A is perfectly collateralized, with its distinct collateral rate.
In this case our economy admits, four financial instruments:

- the asset A under collateral C_A
- the derivative Π under collateral C_Π
- the derivatives' collateral account $B_{c\Pi}$ for funding Π secured by collateral at rate $r_{c\Pi}$
- the asset's collateral account B_{cA} for funding A secured by collateral at rate r_{cA}

The proof follows the previous case of perfect collateral, with the substitutions

$$\begin{aligned} &B_f \to B_{cA}, \quad r_f \to r_{cA} \\ &B_c \to B_{c\Pi}, \quad r_c \to r_{c\Pi} \\ &C \to C_\Pi \end{aligned}$$

We obtain the PDE

$$\begin{aligned} &\widehat{\mathfrak{L}}_{r_cA}\Pi(t) = r_{c\Pi}(t)\Pi(t) \\ &\widehat{\mathfrak{L}}_{r_cA} := \frac{\partial}{\partial t} + r_{cA}(t)A(t)\frac{\partial}{\partial A} + \frac{1}{2}\sigma^2(t,A)\frac{\partial^2}{\partial^2 A} \end{aligned}$$

Using the Feynman-Kac

$$\begin{aligned} &\Pi(t) = \mathbb{E}_t^{\mathbb{Q}}[D_{c\Pi}(t;T)\Pi(T)] \\ &D_{c\Pi}(t;T) := exp\left[-\int_t^T r_{c\Pi}(u)du\right] \\ &dA(t) = r_{cA}(t)A(t)dt + \sigma(t,A)dW^{\mathbb{Q}}(t) \end{aligned}$$

We conclude that, in case of perfect collateral for both the derivative and the hedge, the stochastic process for the asset A is driven by the collateral rate r_{cA} associated with the asset A.

2.6 Perfect collateral, dividends, repo

We now consider the case of derivatives on a dividend paying asset A subject to repo funding. In practice, the asset A, instead of being traded directly, funded through an unsecured funding account B_f at the funding rate r_f, is traded indirectly through repo contracts, secured by the asset A and funded at the repo rate r_R.
Hence, our economy admits, in this case, four financial instruments:

- the asset A, and its dividends at rate r_D
- the derivative Π under collateral C
- the collateral account B_c for funding secured by collateral at rate r_c
- the repo contract $\Pi_R(A)$ for funding secured by asset A at repo rate r_R

Dynamics under real measure P:

$$\begin{aligned} dA(t) &= \mu(t, A)dt + \sigma(t, A)dW^P(t) \\ dB_c(t) &= r_c(t)B_c(t)dt \\ d\Pi(t) &= \widehat{\mathfrak{L}}_\mu \Pi(t)dt + \sigma(t, A)\frac{\partial \Pi}{\partial A}dW^P(t) \\ \widehat{\mathfrak{L}}_\mu &= \frac{\partial}{\partial t} + \mu(t, A)\frac{\partial}{\partial A} + \frac{1}{2}\sigma^2(t, A)\frac{\partial^2}{\partial^2 A} \end{aligned}$$

Regarding the repo contract dynamics, we must remember that, for the repo holder:

- there is a continuous positive cash flow of dividends $+r_D(t)A(t)dt$
- there is a continuous negative cash flow of repo interests $-r_R(t)A(t)dt$,
- the price is linearly dependent on the asset A

Thus, using Ito's Lemma, we obtain

$$\begin{aligned} d\Pi_R(t) &= \frac{\partial \Pi_R}{\partial t}dt + \frac{\partial \Pi_R}{\partial A}dA(t) + \frac{1}{2}\frac{\partial^2 \Pi_R}{\partial^2 A}dA^2(t) + r_D(t)A(t)dt - r_R(t)A(t)dt \\ &= dA(t) + [r_D(t) - r_R(t)]A(t)dt \end{aligned}$$

(We have considered unit nominal).

The replication strategy of the derivative Π is obtained by combining appropriate amounts $[\Delta,\ \Psi]$ of the available assets $[\Pi_R,\ B_c]$

$$\boldsymbol{X}(t) := \begin{bmatrix} 0 \\ B_c(t) \end{bmatrix}$$
$$\boldsymbol{\Theta}(t) := \begin{bmatrix} \Delta(t) \\ \Psi_c(t) \end{bmatrix}$$

where we have taken into account that the repo contract is always traded at par on the market, such that $\Pi_R(t) = 0$ for time t trading.

The value of the replication portfolio V is thus simply given by

$$V(t, \boldsymbol{\Theta}, \boldsymbol{X}) = \boldsymbol{\Theta}(t)' \cdot \boldsymbol{X}(t) = \Psi_c(t) B_c(t)$$

(remember there is a zero repo value on the r.h.s.)

The gain processes of the assets, in SDE form, are given by

$$d\boldsymbol{G}(t) = \begin{bmatrix} d\Pi_R(t) \\ dB_c(t) \end{bmatrix} = \begin{bmatrix} \{\mu(t, A) + [r_D(t) - r_R(t)]A(t)\}dt + \sigma(t, A)dW^P(t) \\ r_c(t)B_c(t)dt \end{bmatrix}$$

The gain process of the replication portfolio is given, in SDE form, by

$$\begin{aligned} dG(t) &:= \boldsymbol{\Theta}(t)' \cdot d\boldsymbol{G}(t) \\ &= \{[\mu(t, A) + (r_D(t) - r_R(t))A(t)]\Delta(t) + r_c(t)\Psi_c(t)B_c(t)\}dt + \Delta(t)\sigma(t, A)dW^P(t) \end{aligned}$$

The dividend processes of the assets may be obtained by difference

$$\begin{aligned} d\boldsymbol{D}(t) &= d\boldsymbol{G}(t) - d\boldsymbol{X}(t) \\ &= \begin{bmatrix} dA(t) + [r_D(t) - r_R(t)]A(t)dt \\ dB_c(t) \end{bmatrix} - \begin{bmatrix} 0 \\ dB_c(t) \end{bmatrix} \\ &= \begin{bmatrix} dA(t) + [r_D(t) - r_R(t)]A(t)dt \\ 0 \end{bmatrix} \\ \boldsymbol{D}(0) &= \boldsymbol{0} \end{aligned}$$

This is consistent with the presence of dividends assumed at the beginning. Notice that in this case we have

$$d[\boldsymbol{\Theta}(t)' \cdot \boldsymbol{X}(t)] \neq \boldsymbol{\Theta}(t)' \cdot d\boldsymbol{X}(t)$$

We now impose the perfect collateral and replication conditions,

$$\Psi_c(t)B_c(t) = C(t) = \Pi(t) = V(t, \boldsymbol{\Theta}, \boldsymbol{X}), \quad \forall t \leq T$$

The gain process of the replication portfolio becomes

$$\begin{aligned} dG(t, \boldsymbol{\Theta}, \boldsymbol{X}) &:= \Delta(t)\{dA(t) + [r_D(t) - r_R(t)]A(t)dt\} + dB_c(t) \\ &= \mu(t, A)\Delta(t)dt + d\Gamma(t, A) + \Delta(t)\sigma(t, A)dW^P(t) \\ d\Gamma(t, A) &= \{[r_D(t) - r_R(t)]\Delta(t)A(t) + r_c(t)\Pi(t)\}\, dt \end{aligned}$$

We observe at this stage that the cash amount $\Gamma(t, A)$ contained in the replication portfolio is split between:

- the cash in the collateral account B_c, growing at the collateral rate r_c,
- the cash generated by the dividends paid by the asset A at the dividend rate r_D,
- the amount $\Delta(t)A(t)$, borrowed at the repo rate $r_R(t)$ to finance the purchase of $\Delta(t)$ units of the asset $A(t)$, secured by the asset itself.

We now impose the self-financing condition: the dividend process of the replication strategy must be null,

$$D(t, \boldsymbol{\Theta}, \boldsymbol{X}) = G(t, \boldsymbol{\Theta}, \boldsymbol{X}) - V(t, \boldsymbol{\Theta}, \boldsymbol{X}) = 0$$

$$\Rightarrow dG(t, \mathbf{\Theta}, \boldsymbol{X}) = dV(t, \mathbf{\Theta}, \boldsymbol{X}) = d\Pi(t).$$

Introducing the gain process of the strategy on the l.h.s, the derivative process on the r.h.s of the previous equation, and rearranging terms we obtain the SDE

$$\left\{\frac{\partial \Pi}{\partial t} + \mu(t, A)\left[\frac{\partial \Pi}{\partial A} - \Delta(t)\right] + \frac{1}{2}\sigma^2(t, A)\frac{\partial^2 \Pi}{\partial^2 A}\right] dt + \sigma(t, A)\left[\frac{\partial \Pi}{\partial A} - \Delta(t)\right] dW^P(t) = d\Gamma(t, A)$$

We finally impose the risk neutral condition $\Delta(t) = \frac{\partial \Pi}{\partial A}$, such that the stochastic (risky) term with $dW^P(t)$ disappears, and we obtain a **generalised Black-Scholes** equation for the derivative's price $\Pi(t)$

$$\widehat{\mathfrak{L}}_{r_R - r_D}\Pi(t) = r_c(t)\Pi(t)$$
$$\widehat{\mathfrak{L}}_{r_R - r_D} := \frac{\partial}{\partial t} + (r_R(t) - r_D(t))A(t)\frac{\partial}{\partial A} + \frac{1}{2}\sigma^2(t, A)\frac{\partial^2}{\partial^2 A}$$

Using the Feynman-Kac theorem we may switch from the PDE representation to the SDE representation given by

$$\Pi(t) = \mathbb{E}_t^{\mathbb{Q}}[D_c(t; T)\Pi(T)]$$
$$D_c(t; T) := exp\left[-\int_t^T r_c(u)du\right]$$
$$dA(t) = [r_R(t) - r_D(t)]A(t)dt + \sigma(t, A)dW^{\mathbb{Q}}(t)$$

We conclude that we discount at the **collateral rate**.

Remark 2.8. 1. **Funding measure**:
the probability measure $\mathbb{Q}$ introduced via Feynman-Kac is associated with the risk neutral drift r_R appearing in the SDE dynamics of the asset A. It is the same measure of the collateralised case, with numeraire the collateral account B_c.

2. **Risk neutral drift**:
The repo rate r_R is the correct rate to be used in the risk neutral dynamics of assets subject to repo. We may think that the repo (short) rate $r_R(t)$ is associated with a repo account B_R, such that
$$dB_R(t) = r_R(t)B_R(t)dt$$
The repo rate, being associated with a secured transaction, may be considered a good proxy of a risk free rate. In practice, overnight repos are close to unsecured overnight rates.

3. **Funding Valuation Adjustment (FVA)**:
The price $\Pi(t)$ is different from the no repo case, because of the different risk neutral drift in the SDE dynamics of the asset A. According to the consideration above, the difference is small. We conclude that FVA is negligible.

2.7 Partial collateral

We now relax the hypothesis of perfect collateral and consider the more general case of partial collateral $C(t) \neq \Pi(t)$, in the same currency of the derivative.

Our economy admits, in this case, four financial instruments:

- the asset A
- the derivative Π under partial collateral C
- the funding account B_f for funding unsecured at rate r_f
- the collateral account B_c for funding secured by collateral at rate r_c

In general, we may assume the following dynamics under the real measure P:

$$\begin{aligned}
dA(t) &= \mu(t, A)dt + \sigma(t, A)dW^P(t) \\
dB_f(t) &= r_f(t)B_f(t)dt \\
dB_c(t) &= r_c(t)B_c(t)dt \\
d\Pi(t) &= \widehat{\mathfrak{L}}_\mu \Pi(t)dt + \sigma(t, A)\frac{\partial \Pi}{\partial A}dW^P(t) \\
\widehat{\mathfrak{L}}_\mu &= \frac{\partial}{\partial t} + \mu(t, A)\frac{\partial}{\partial A} + \frac{1}{2}\sigma^2(t, A)\frac{\partial^2}{\partial^2 A}
\end{aligned}$$

The replication strategy of the derivative Π is obtained by combining appropriate amounts $\mathbf{\Theta}$ of the available assets

$$\boldsymbol{X}(t) := \begin{bmatrix} A(t) \\ B_f(t) \\ B_c(t) \end{bmatrix}$$

$$\mathbf{\Theta}(t) := \begin{bmatrix} \Delta(t) \\ \Psi_f(t) \\ \Psi_c(t) \end{bmatrix}$$

The value of the replication portfolio V is thus simply given by

$$V(t, \mathbf{\Theta}, \boldsymbol{X}) = \mathbf{\Theta}(t)' \cdot \boldsymbol{X}(t) = \Delta(t)A(t) + \Psi_f(t)B_f(t) + \Psi_c(t)B_c(t)$$

The gain and dividend processes of the assets, in SDE form, are given directly by the dynamics discussed before, as

$$\begin{aligned}
d\boldsymbol{G}(t) :&= \begin{bmatrix} dA(t) \\ dB_f(t) \\ dB_c(t) \end{bmatrix} = \begin{bmatrix} \mu(t, A)dt + \sigma(t, A)dW^P(t) \\ r_f(t)B_f(t)dt \\ r_c(t)B_c(t)dt \end{bmatrix} \\
d\boldsymbol{D}(t) &= \mathbf{0}
\end{aligned}$$

The gain process of the replication portfolio is given, in SDE form, by

$$\begin{aligned}
dG(t) &:= \mathbf{\Theta}(t)' \cdot d\boldsymbol{G}(t) \\
&= \{\mu(t, A)\Delta(t) + r_f(t)\Psi_f(t)B_f(t) + r_c(t)\Psi_c(t)B_c(t)\}dt + \Delta(t)\sigma(t, A)dW^P(t)
\end{aligned}$$

We now impose the perfect collateral and replication conditions,

$$V(t, \mathbf{\Theta}, \boldsymbol{X}) = \Pi(t) = \Delta(t)A(t) + \Psi_f(t)B_f(t) + \Psi_c(t)B_c(t), \quad \forall t \leq T$$

$$\Rightarrow \Psi_f(t)B_f(t) = \Pi(t) - \Psi_c(t)B_c(t) = \Pi(t) - \Delta(t)A(t) - C(t)$$

consistently with the fact that the funding account B_f is used to finance the borrowing of $\Delta(t)$ units of the asset $A(t)$ at the funding rate $r_f(t)$.

The gain process of the replication portfolio becomes

$$\begin{aligned} dG(t, \boldsymbol{\Theta}, \boldsymbol{X}) &= \mu(t, A)\Delta(t)dt + d\Gamma(t, A) + \Delta(t)\sigma(t, A)dW^P(t) \\ d\Gamma(t, A) &= \{-r_f(t)\Delta(t)A(t) + r_f(t)\Pi(t) - [r_f(t) - r_c(t)]C(t)\}dt \\ &= \{-r_f(t)\Delta(t)A(t) + r_c(t)C(t) - r_f(t)[\Pi(t) - C(t)]\}dt \end{aligned}$$

We observe that the cash amount $\Gamma(t, A)$ in the replication portfolio is split between:

- the collateral C, growing at the collateral rate r_c,
- the amount $\Delta(t)A(t)$, borrowed at the funding rate $r_f(t)$ to finance the purchase of $\Delta(t)$ units of the asset $A(t)$,
- the off-collateral amount $\Pi(t) - C(t)$, borrowed/lent at the funding rate $r_f(t)$

We now impose the self-financing condition: the dividend process of the replication strategy must be null,

$$\begin{aligned} D(t, \boldsymbol{\Theta}, \boldsymbol{X}) &= G(t, \boldsymbol{\Theta}, \boldsymbol{X}) - V(t, \boldsymbol{\Theta}, \boldsymbol{X}) = 0 \\ \Rightarrow dG(t, \boldsymbol{\Theta}, \boldsymbol{X}) &= dV(t, \boldsymbol{\Theta}, \boldsymbol{X}) = d\Pi(t). \end{aligned}$$

Introducing, in the previous equation, the gain process of the replication portfolio on the l.h.s, the derivative process on the r.h.s, and rearranging terms, we have

$$\left\{\frac{\partial \Pi}{\partial t} + \mu(t, A)\left[\frac{\partial \Pi}{\partial A} - \Delta(t)\right] + \frac{1}{2}\sigma^2(t, A)\frac{\partial^2 \Pi}{\partial^2 A}\right] dt + \sigma(t, A)\left[\frac{\partial \Pi}{\partial A} - \Delta(t)\right] dW^P(t) = d\Gamma(t, A)$$

Finally, we impose the risk neutral condition $\Delta(t) = \frac{\partial \Pi}{\partial A}$ and we obtain a **generalised Black-Scholes** equation for the derivative's price $\Pi(t)$

$$\begin{aligned} \widehat{\mathfrak{L}}_{r_f}\Pi(t) &= r_f(t)\Pi(t) - [r_f(t) - r_c(t)]C(t) \\ \widehat{\mathfrak{L}}_{r_f} &:= \frac{\partial}{\partial t} + (r_f(t))A(t)\frac{\partial}{\partial A} + \frac{1}{2}\sigma^2(t, A)\frac{\partial^2}{\partial^2 A} \end{aligned}$$

Using the Feynman-Kac theorem we obtain the SDE representation

$$\begin{aligned} \Pi(t) &= \mathbb{E}_t^{\mathbb{Q}}\left[D_c(t; T)\Pi(T) + \int_t^T D_c(t; u)[r_f(u) - r_c(u)][\Pi(u) - C(u)]du\right] \\ D_c(t; T) &:= exp\left[-\int_t^T r_c(u)du\right] \\ dA(t) &= r_f(t)A(t)dt + \sigma(t, A)dW^{\mathbb{Q}}(t) \end{aligned}$$

Remark 2.9. Funding Value Adjustment (FVA)
We may rewrite the previous formula as follows

$$\begin{aligned} \Pi_{f,c}(t) &= \mathbb{E}_t^Q\left[D_c(t; T)\Pi(T) + \int_t^T D_c(t; u)[r_f(u) - r_c(u)][\Pi_{f,c}(u) - C(u)]du\right] \\ &= \Pi_c(t) + FVA_{f,c}(t) \\ FVA_{f,c} : &= \Pi_{f,c}(t) - \Pi_c(t) \\ &= \mathbb{E}_t^Q\left[\int_t^T D_c(t; u)[r_f(u) - r_c(u)][\Pi_{f,c}(u) - C(u)]du\right] \end{aligned}$$

In this case, the FVA amounts to the expected difference between the trade and collateral values, weighted with the difference between funding and collateral rates, integrated over the residual life of the trade.

2.8 Perfect collateral, stochastic rates

We consider now the case of stochastic funding rates with perfect collateral. Stochastic funding means that funding rates are stochastic, there are more risk factors to hedge.

Our economy admits, in this case, six financial instruments:

- the asset A,
- the uncollateralized asset A_f, e.g. a zero coupon bond $P_f(t, s)$
- the collateralized asset A_c, e.g. a zero coupon bond $P_c(t, s)$
- the derivative Π under perfect collateral C
- the funding account B_f for funding unsecured at rate r_f
- the collateral account B_c for funding secured by collateral at rate r_c

Notice that:

- Non tradable assets, such as interest rates, inflation, fx, etc. can't appear directly as hedging instruments, but through some corresponding tradable financial instruments. In particular, funding rates r_x may enter in the form of Zero Coupon Bonds, $A_x(t) = P_x(t, s)$.
- Additional financial instruments associated with (stochastic) funding rates are required to hedge the additional risk factors.

In general, we may assume the following dynamics under the real measure P:

$$\begin{aligned}
dA(t) &= \mu_A(t, A)dt + \boldsymbol{\sigma}_A(t, A)d\boldsymbol{W}^P(t)\\
dA_x(t) &= \mu_x(t, A_x)dt + \boldsymbol{\sigma}_x(t, A_x)d\boldsymbol{W}^P(t)\\
dB_x(t) &= r_x(t)B_x(t)dt\\
d\Pi(t) &= \widehat{\mathfrak{L}}_\mu \Pi(t)dt + \left[\frac{\partial \Pi}{\partial A}\boldsymbol{\sigma}_A(t, A) + \frac{\partial \Pi}{\partial A_f}\boldsymbol{\sigma}_f(t, A_f) + \frac{\partial \Pi}{\partial A_c}\boldsymbol{\sigma}_c(t, A_c)\right] \cdot d\boldsymbol{W}^P(t)\\
\widehat{\mathfrak{L}}_\mu &:= \frac{\partial}{\partial t} + \mu_A(t, A)\frac{\partial}{\partial A} + \mu_f(t, A_f)\frac{\partial}{\partial A_f} + \mu_c(t, A_c)\frac{\partial}{\partial A_c} + \frac{1}{2}\boldsymbol{\Sigma^2(t, A, A_f, A_c)}(t) \cdot \boldsymbol{\frac{\partial^2}{\partial^2 A}}\\
\boldsymbol{\Sigma^2(t, A, A_f, A_c)} \cdot \boldsymbol{\frac{\partial^2}{\partial^2 A}} &:= \boldsymbol{\sigma}_A^2(t, A)\frac{\partial^2}{\partial^2 A} + \boldsymbol{\sigma}_f^2(t, A_f)\frac{\partial^2}{\partial^2 A_f} + \boldsymbol{\sigma}_c^2(t, A_c)\frac{\partial^2}{\partial^2 A_c} +\\
&\quad + \boldsymbol{\sigma}_A(t, A) \cdot \left[\boldsymbol{\sigma}_f^2(t, A_f)\frac{\partial^2}{\partial A \partial A_f} + \boldsymbol{\sigma}_c^2(t, A_c)\frac{\partial^2}{\partial A \partial A_c}\right] + \boldsymbol{\sigma}_f(t, A_f) \cdot \boldsymbol{\sigma}_c(t, A_c)\frac{\partial^2}{\partial A_f \partial A_c}\\
dW_i^P(t)dW_j^P(t) &= \delta_{i,j}dt\\
x &:= \{f, c\}
\end{aligned}$$

The replication strategy of the derivative Π is obtained by combining appropriate amounts $\boldsymbol{\Theta}$ of the available assets

$$\boldsymbol{X}(t) := \begin{bmatrix} A(t) \\ A_f(t) \\ A_c(t) \\ B_f(t) \\ B_c(t) \end{bmatrix}$$

$$\boldsymbol{\Theta}(t) := \begin{bmatrix} \Delta_A(t) \\ \Delta_f(t) \\ \Delta_c(t) \\ \Psi_f(t) \\ \Psi_c(t) \end{bmatrix}$$

The value of the replication portfolio V is thus simply given by

$$V(t, \boldsymbol{\Theta}, \boldsymbol{X}) = \boldsymbol{\Theta}(t)' \cdot \boldsymbol{X}(t) = \Delta_A(t)A(t) + \Delta_f(t)A_f(t) + \Delta_c(t)A_c(t) + \Psi_f(t)B_f(t) + \Psi_c(t)B_c(t)$$

The gain and dividend processes of the assets, in SDE form, are given directly by the dynamics discussed before, as

$$\begin{aligned} d\boldsymbol{G}(t) : &= \begin{bmatrix} dA(t) \\ dA_f(t) \\ dA_c(t) \\ dB_f(t) \\ dB_c(t) \end{bmatrix} = \begin{bmatrix} \mu_A(t, A)dt + \boldsymbol{\sigma}_A(t, A) \cdot d\boldsymbol{W}^P(t) \\ \mu_f(t, A_f)dt + \boldsymbol{\sigma}_f(t, A_f) \cdot d\boldsymbol{W}^P(t) \\ \mu_c(t, A_c)dt + \boldsymbol{\sigma}_c(t, A_c) \cdot d\boldsymbol{W}^P(t) \\ r_f(t)B_f(t)dt \\ r_c(t)B_c(t)dt \end{bmatrix} \\ d\boldsymbol{D}(t) &= \boldsymbol{0} \end{aligned}$$

The gain process of the replication portfolio is given, in SDE form, by

$$\begin{aligned} dG(t) &:= \boldsymbol{\Theta}(t)' \cdot d\boldsymbol{G}(t) \\ &= \{\mu_A(t, A)\Delta_A(t) + \mu_f(t, A_f)\Delta_f(t) + \mu_c(t, A_c)\Delta_c(t) + r_f(t)\Psi_f(t)B_f(t) + r_c(t)\Psi_c(t)B_c(t)\}dt + \\ &+ [\Delta_A(t)\boldsymbol{\sigma}_A(t, A) + \Delta_f(t)\boldsymbol{\sigma}_f(t, A_f) + \Delta_c(t)\boldsymbol{\sigma}_c(t, A_c)]d\boldsymbol{W}^P(t) \end{aligned}$$

We now impose the perfect collateral and replication conditions,

$$\begin{aligned} V(t, \boldsymbol{\Theta}, \boldsymbol{X}) &= \Pi(t) \\ &= \Delta_A(t)A(t) + \Delta_f(t)A_f(t) + \Delta_c(t)A_c(t) + \Psi_f(t)B_f(t) + \Psi_c(t)B_c(t) \\ &= \Delta_A(t)A(t) + \Delta_f(t)A_f(t) + \Psi_f(t)B_f(t) + \Pi(t) \end{aligned}$$

where we have used the generalised perfect collateral condition

$$\Pi(t) = C(t) = \Psi_c(t)B_c(t) + \Delta(t)A_c(t)$$

Thus we obtain the generalised unsecured funding condition

$$\Psi_f(t)B_f(t) = -\Delta_A(t)A(t) - \Delta_f(t)A_f(t)$$

consistently with the fact that the funding account B_f is used to finance the borrowing of $\Delta_A(t)$ units of the asset $A(t)$ and $\Delta_f(t)$ units of the asset $A_f(t)$ at the funding rate $r_f(t)$.

The gain process of the replication portfolio becomes

$$
\begin{aligned}
dG(t, \boldsymbol{\Theta}, \boldsymbol{X}) = d\Pi(t) &= \widehat{\mathfrak{L}}_{\mu}\Pi(t)dt = \\
&= [\mu_A(t,A)\Delta_A(t) + \mu_f(t,A_f)\Delta_f(t) + \mu_c(t,A_c)\Delta_c(t)]dt + \\
&+ d\Gamma(t, A, A_f, A_c) + \\
&+ [\Delta_A(t)\boldsymbol{\sigma}_A(t,A) + \Delta_f(t)\boldsymbol{\sigma}_f(t,A_f) + \Delta_c(t)\boldsymbol{\sigma}_c(t,A_c)] \cdot d\boldsymbol{W}^P(t) \\
d\Gamma(t,A) &= -r_f(t)[\Delta_A(t)A(t) + \Delta_f(t)A_f(t)] + r_c(t)[\Pi(t) - \Delta_c(t)A_c(t)]dt
\end{aligned}
$$

We observe that the cash amount $\Gamma(t, A)$ in the replication portfolio is split between:

- the collateral amount $C(t) = \Pi - \Delta_c(t)A_c(t)$, growing at the collateral rate r_c,
- the amount $\Delta(t)A(t)$, borrowed at the funding rate $r_f(t)$ to finance the purchase of $\Delta(t)$ units of the asset $A(t)$, and $\Delta_f(t)$ units of the asset $A_f(t)$.

We now impose the self-financing condition: the dividend process of the replication strategy must be null,

$$
\begin{aligned}
D(t, \boldsymbol{\Theta}, \boldsymbol{X}) = G(t, \boldsymbol{\Theta}, \boldsymbol{X}) - V(t, \boldsymbol{\Theta}, \boldsymbol{X}) = 0 \\
\Rightarrow dG(t, \boldsymbol{\Theta}, \boldsymbol{X}) = dV(t, \boldsymbol{\Theta}, \boldsymbol{X}) = d\Pi(t).
\end{aligned}
$$

Introducing, in the previous equation, the gain process of the replication portfolio on the l.h.s, the derivative process on the r.h.s, and rearranging terms, we have

$$
\begin{aligned}
&\left\{\frac{\partial \Pi}{\partial t} + \mu_A(t,A)\left[\frac{\partial \Pi}{\partial A} - \Delta_A(t)\right] + \mu_f(t,A_f)\left[\frac{\partial \Pi}{\partial A_f} - \Delta_f(t)\right] + \mu_c(t,A_c)\left[\frac{\partial \Pi}{\partial A_c} - \Delta_c(t)\right] + \right. \\
&\left.\frac{1}{2}\boldsymbol{\Sigma}^2(t,A,A_f,A_c)\frac{\partial^2 \Pi}{\partial^2 A}\right\}dt + \left\{\left[\frac{\partial \Pi}{\partial A} - \Delta_A(t)\right]\boldsymbol{\sigma}_A(t,A) + \left[\frac{\partial \Pi}{\partial A_f} - \Delta_f(t)\right]\boldsymbol{\sigma}_f(t,A_f) + \right. \\
&\left. + \left[\frac{\partial \Pi}{\partial A_c} - \Delta_c(t)\right]\boldsymbol{\sigma}_c(t,A_c)\right\} \cdot d\boldsymbol{W}^P(t) = d\Gamma(t,A,A_f,A_c)
\end{aligned}
$$

Finally, imposing the **generalised** risk neutral condition

$$
\Delta_A(t) = \frac{\partial \Pi}{\partial A} \quad \Delta_f(t) = \frac{\partial \Pi}{\partial A_f} \quad \Delta_c(t) = \frac{\partial \Pi}{\partial A_c}
$$

we obtain a **generalised Black-Scholes** PDE equation for the derivative's price $\Pi(t)$

$$
\begin{aligned}
&\widehat{\mathfrak{L}}_{f,c}\Pi(t) = r_c(t)\Pi(t) \\
&\widehat{\mathfrak{L}}_{f,c} := \frac{\partial}{\partial t} + r_f(t)\left[A(t)\frac{\partial}{\partial A} + A_f(t)\frac{\partial}{\partial A_f}\right] + r_c(t)A_c(t)\frac{\partial}{\partial A_c}\frac{1}{2}\boldsymbol{\Sigma}^2(t,A,A_f,A_c) \cdot \frac{\boldsymbol{\partial^2}}{\boldsymbol{\partial^2 A}}
\end{aligned}
$$

Using the Feynman-Kac theorem we obtain the SDE representation

$$
\begin{aligned}
&\Pi(t) = \mathbb{E}_t^{\mathbb{Q}}\left[D_c(t;T)\Pi(T)\right] \\
&D_c(t;T) := exp\left[-\int_t^T r_c(u)du\right] \\
&dA(t) = r_f(t)A(t)dt + \boldsymbol{\sigma}_A(t,A) \cdot d\boldsymbol{W}^{\mathbb{Q}}(t) \\
&dA_x(t) = r_x(t)A_x(t)dt + \boldsymbol{\sigma}_x(t,A_x) \cdot d\boldsymbol{W}^{\mathbb{Q}}(t)
\end{aligned}
$$

We conclude that we discount at the **collateral rate**.

Remark 2.10. IR dynamics
In the risk neutral measure the dynamics of the stochastic rates r_x is expressed via the functional dependence on the instruments A_x. We could proceed the other way around, postulating from the very beginning, stochastic process of the rates in the risk-neutral measure.
This freedom allows to select their dynamics with appropriate characteristics that can be calibrated to the corresponding yield curves. For example, we could choose the Hull-White dynamics such that, under the risk neutral measure $\mathbb{Q}$,

$$dr_x(t) = [k_x(t) - a_x(t)r_x(t)]dt + \boldsymbol{\sigma}_{r_x}(t) \cdot d\boldsymbol{W}^Q(t)$$

Remark 2.11. IR instruments
Next, we can specify the interest rate instruments A_x used in the replication. One possible choice are unsecured and (perfectly) collateralised zero coupon bonds, $A_x(t) = P_x(t, T')$, with some maturity $T' > T$.
The parameters $k_x(t)$ in the Hull-White dynamics above can be calibrated to the market zero coupon bond curves. Using Ito's lemma we obtain the price dynamics of the zero coupon bonds as

$$dP_x(t, T') = r_x(t)P_x(t, T')dt - P_x(t, T')\boldsymbol{\sigma}_{P_x}(t, T') \cdot d\boldsymbol{W}^Q(t)$$

$$\boldsymbol{\sigma}_{P_x}(t, T') = \boldsymbol{\sigma}_{r_x}(t) \int_t^{T'} e^{-\int_t^u a_x(v)dv} du$$

$$x = f, c$$

Notice that the zero coupon bonds dynamics inherit appropriate drifts corresponding to their underlying (short) rates r_x.

2.9 General case

We now consider the more general case of multiple assets A, multiple stochastic funding rates r_x, dividends and partial collateral.

Our economy admits, in this case, multiple financial instruments: the vector of assets $A_x = [A_f, A_c, A_R]$ according to the funding of their associated hedging strategies (unsecured, collateral, repo) the derivative Π on assets A_x under partial CSA the vector of funding accounts $B_x = [B_f, B_c, B_R]$ with funding rates $r_x = [r_f, r_c, r_R]$ The generalised funding conditions become

$$\Delta_f(t)A_f(t) + \Psi_f(t)B_f(t) = \Pi(t) - C_\Pi(t)$$

$$\Delta_c(t)A_c(t) + \Psi_c(t)B_c(t) = C_\Pi(t)$$

$$\Delta_R(t)A_R(t) + \Psi_R(t)B_R(t) = 0$$

We obtain a generalised Black-Scholes PDE equation for the derivative's price $\Pi(t)$

$$\widehat{\mathcal{L}}_r\Pi(t) = r_f(t)\Pi(t) - [r_f(t) - r_c(t)]C_\Pi(t)$$

$$\widehat{\mathcal{L}}_r = \frac{\partial}{\partial t} + \sum_x [r_x(t) - r_D(t)]\boldsymbol{A}_x(t) \cdot \frac{\partial}{\partial \boldsymbol{A}_x} + \frac{1}{2}\sum_{x,y} \boldsymbol{\sigma}_x(t, \boldsymbol{A}_x) \cdot \boldsymbol{\sigma}_y(t, \boldsymbol{A}_y) \cdot \frac{\partial^2}{\partial \boldsymbol{A}_x \partial \boldsymbol{A}_y}$$

and, using Feynman-Kac, the SDE representation

$$\Pi(t) = \mathbb{E}_t^{\mathbb{Q}}\left[D_c(t;T)\Pi(T) - \int_t^T D_c(u,T)[r_f(u) - r_c(u)][\Pi(u) - C_\Pi(u)]du\right]$$

$$D_x(t;T) := exp\left[-\int_t^T r_x(u)du\right]$$
$$dA_x(t) = [r_x(t) - r_D(t)]A_x(t)dt + \boldsymbol{\sigma}_x(t, \boldsymbol{A}_x) \cdot d\boldsymbol{W}^Q(t)$$

2.10 Multiple currency

We consider now the case of different currency for the derivative and its funding.

Our economy admits, in this case, five financial instruments:

- the asset A^α, in currency α, with no dividends
- the derivative Π^α under partial collateral C^β in currency β
- the funding account B_f^α for funding unsecured in currency α at rate r_f^α
- the funding account B_f^β for funding unsecured in currency β at rate r_f^β
- the collateral account B_c^β for funding secured by collateral C^β at rate r_c^β

We also assume that the derivative Π is under perfect collateral, such that

$$\Pi^\alpha(t) = x^{\alpha\beta}(t)C^\beta(t), \quad \forall t \leq T$$

where $x^{\alpha\beta}$ is the spot exchange rate expressing the value in currency α of one unit of currency β.

We have the following dynamics under the real measure P^α:

$$\begin{aligned}
dA^\alpha(t) &= \mu(t, A^\alpha)dt + \sigma(t, A^\alpha)dW^{P,\alpha}(t) \\
dB_f^\alpha(t) &= r_f^\alpha(t)B_f^\alpha(t)dt \\
dB_f^\beta(t) &= r_f^\beta(t)B_f^\beta(t)dt \\
dB_c^\beta(t) &= r_c^\beta(t)B_c^\beta(t)dt \\
d\Pi^\alpha(t) &= \left[\frac{\partial \Pi^\alpha}{\partial t} + \mu(t, A^\alpha)\frac{\partial \Pi}{\partial A} + \frac{1}{2}\sigma^2(t, A^\alpha)\frac{\partial^2 \Pi^\alpha}{\partial^2 A}\right]dt + \sigma^2(t, A^\alpha)\frac{\partial \Pi}{\partial A}dW^{P,\alpha}(t)
\end{aligned}$$

We now construct the replication strategy of the derivative Π^α by setting up a replication portfolio V^α such that

$$V^\alpha(t, \boldsymbol{\Theta}, \boldsymbol{X}) = \boldsymbol{\Pi}^\alpha(t), \quad \forall t \leq T$$

by combining appropriate amounts $\boldsymbol{\Theta}$ of the available assets $\boldsymbol{X}$

$$\boldsymbol{X}(t) := \begin{bmatrix} A^\alpha(t) \\ B_f^\alpha(t) \\ B_f^\beta(t) \\ B_c^\beta(t) \end{bmatrix}$$

$$\boldsymbol{\Theta}(t) := \begin{bmatrix} \Delta(t) \\ \Psi_f^\alpha(t) \\ \Psi_f^\beta(t)x^{\alpha\beta}(t) \\ \Psi_c^\beta(t)x^{\alpha\beta}(t) \end{bmatrix}$$

The value of the replication portfolio $V^\alpha(t, \boldsymbol{\Theta}, \boldsymbol{X})$ is given by

$$V(t, \boldsymbol{\Theta}, \boldsymbol{X}) = \boldsymbol{\Theta}(t)' \cdot \boldsymbol{X}(t) = \Delta(t)A^\alpha(t) + \Psi_f^\alpha(t)B_f^\alpha(t) + \Psi_f^\beta(t)x^{\alpha\beta}(t)B_f^\beta(t) + \Psi_c^\beta(t)x^{\alpha\beta}(t)B_c(t)$$

The gain and dividend processes of the assets, in SDE form, are given directly by the dynamics discussed before, as

$$\begin{aligned} d\boldsymbol{G}(t) :&= \begin{bmatrix} dA^\alpha(t) \\ dB_f^\alpha(t) \\ dB_f^\beta(t) \\ dB_c^\beta(t) \end{bmatrix} = \begin{bmatrix} \mu(t, A^\alpha)dt + \sigma(t, A^\alpha)dW^{P,\alpha}(t) \\ r_f^\alpha(t)B_f^\alpha(t)dt \\ r_f^\beta(t)B_f^\beta(t)dt \\ r_c^\beta(t)B_c^\beta(t)dt \end{bmatrix} \\ d\boldsymbol{D}(t) &= \boldsymbol{0} \\ d\boldsymbol{G}(t) &= d\boldsymbol{X}(t) \end{aligned}$$

We now impose the perfect collateral and replication conditions, and we obtain

$$\begin{aligned} V^\alpha(t, \boldsymbol{\Theta}, \boldsymbol{X}) &= \Delta(t)A^\alpha(t) + \Psi_f^\alpha(t)B_f^\alpha(t) + \Psi_f^\beta(t)x^{\alpha\beta}(t)B_f^\beta(t) + \Psi_c^\beta(t)x^{\alpha\beta}(t)B_c^\beta(t) \\ &= \Delta(t)A^\alpha(t) + \Psi_f^\alpha(t)B_f^\alpha(t) + \Psi_f^\beta(t)x^{\alpha\beta}(t)B_f^\beta(t) + \Pi(t) \\ V^\alpha(t, \boldsymbol{\Theta}, \boldsymbol{X}) &= \Pi^\alpha, \quad \forall t \leq T \\ \Rightarrow \Psi_f^\alpha(t)B_f^\alpha(t) &= -\Delta(t)A^\alpha(t) - \Psi_f^\beta(t)x^{\alpha\beta}(t)B_f^\beta(t) \end{aligned}$$

consistently with the fact that the funding account B_f^α is used to finance, in currency α the borrowing of $\Delta(t)$ units of the asset $A^\alpha(t)$ and $\Psi_f(t)x^{\alpha\beta}(t)$ units of cash $B_f^\beta(t)$ in currency β, at the funding rate $r_f^\alpha(t)$.

The gain process of the replication portfolio, in SDE form, is given by

$$\begin{aligned} dG^\alpha(t, \boldsymbol{\Theta}, \boldsymbol{X}) &= dX^\alpha(t, \boldsymbol{\Theta}, \boldsymbol{X}) = \boldsymbol{\Theta}(t)' \cdot d\boldsymbol{G}(t) \\ &= \mu(t, A^\alpha)\Delta(t)dt + d\Gamma^\alpha(t, A^\alpha) + \Delta(t)\sigma(t, A^\alpha)dW^{P,\alpha}(t) \\ d\Gamma^\alpha(t, A^\alpha) :&= \left\{ -r_f^\alpha(t)\Delta(t)A^\alpha(t) + \left[r_f^\beta(t) - r_f^\alpha(t)\right] \Psi_f^\beta(t)x^{\alpha\beta}(t)B_f^\beta(t) + r_c^\beta(t)\Pi^\alpha(t)\right\}dt \\ &= \left\{ -r_f^\alpha(t)\Delta(t)A^\alpha(t) + \left[r_c^\beta(t) + r_f^\beta(t) - r_f^\alpha(t)\right] \Pi^\alpha(t)\right\}dt \end{aligned}$$

where we have chosen $\Psi_f^\beta x^{\alpha\beta}B_f^\beta(t) = \Pi^\alpha(t)$, such that $\Psi_f^\beta x^{\alpha\beta}B_f^\beta(t)$ units of cash $B_f^\beta(t)$ are used to fund the derivative $\Pi^\alpha(t)$.

We observe that the cash amount $\Gamma^\alpha(t)$ in the replication portfolio is split between:

- the amount $\Delta(t)A^\alpha(t)$ in currency α, borrowed at the funding rate $r_f^\alpha(t)$ to finance the purchase of $\Delta(t)$ units of the asset $A^\alpha(t)$,
- the amount $B_c^\beta(t)$ in the collateral account (currency β), growing at the cross currency collateral rate $r_c^\beta(t) + r_f^\beta(t) - r_f^\alpha$

We now impose the self-financing condition: the dividend process of the replication strategy must be null,

$$D^\alpha(t, \boldsymbol{\Theta}, \boldsymbol{X}) = G^\alpha(t, \boldsymbol{\Theta}, \boldsymbol{X}) - V^\alpha(t, \boldsymbol{\Theta}, \boldsymbol{X}) = 0$$

$$\Rightarrow dG^\alpha(t, \boldsymbol{\Theta}, \boldsymbol{X}) = dV^\alpha(t, \boldsymbol{\Theta}, \boldsymbol{X}) = d\Pi^\alpha(t).$$

Introducing, in the previous equation, the gain process of the replication portfolio on the l.h.s, the derivative process on the r.h.s, and rearranging terms, we have

$$\left\{\frac{\partial \Pi^\alpha}{\partial t} + \mu(t, A^\alpha)\left[\frac{\partial \Pi^\alpha}{\partial A} - \Delta(t)\right] + \frac{1}{2}\sigma^2(t, A^\alpha)\frac{\partial^2 \Pi^\alpha}{\partial^2 A}\right] dt +$$
$$+\sigma(t, A^\alpha)\left[\frac{\partial \Pi^\alpha}{\partial A} - \Delta(t)\right] dW^{P,\alpha}(t) = d\Gamma^\alpha(t, A^\alpha)$$

Finally, we impose the risk neutral condition $\Delta(t) = \frac{\partial \Pi}{\partial A}$ and we obtain a **generalised Black-Scholes** equation for the derivative's price $\Pi^\alpha(t)$

$$\widehat{\mathfrak{L}}_{r_f^\alpha}\Pi^\alpha(t) = \left[r_c^\beta(t) + r_f^\beta(t) - r_f^\alpha(t)\right]\Pi^\alpha(t)$$
$$\widehat{\mathfrak{L}}_{r_f^\alpha} := \frac{\partial}{\partial t} + r_f^\alpha(t)A^\alpha(t)\frac{\partial}{\partial A} + \frac{1}{2}\sigma^2(t, A^\alpha)\frac{\partial^2}{\partial^2 A}$$

Using the Feynman-Kac theorem we obtain the SDE representation

$$\Pi^\alpha(t) = \mathbb{E}_t^{\mathbb{Q}^\alpha}\left[D_{c,f}^{\alpha,\beta}(t;T)\Pi^\alpha(T)\right]$$
$$D_{c,f}^{\alpha,\beta}(t;T) := exp\left[-\int_t^T [r_c^\beta(u) + r_f^\beta(u) - r_f(u)^\alpha]du\right] = D_c^\beta(t;T)\frac{D_f^\beta(t;T)}{D_f^\alpha(t;T)}$$
$$dA^\alpha(t) = r_f^\alpha(t)A^\alpha(t)dt + \sigma_A(t, A^\alpha)dW^{\mathbb{Q}^\alpha}(t)$$

Remark 2.12. Discounting

$$D_{c,f}^{\alpha,\beta}(t;T) := exp\left[-\int_t^T \left[r_c^\beta(u) + r_f(u)^\beta(u) - r_f^\alpha\right] du\right]$$

contains the **cross currency basis**.

Remark 2.13. Funding Value Adjustment (FVA)
Comparing the single *vs* double currency collateralised prices we can define a Funding Value Adjustment (FVA) such that, in additive form,

$$\Pi_f(t) = \mathbb{E}_t^{Q^\alpha}\left[\frac{D_c^\beta(t;T)D_f^\beta(t;T)}{D_c^\alpha(t;T)D_f^\alpha(t;T)}\right]\mathbb{E}_t^{Q^\alpha}\left[D_c^\alpha(t;T)\Pi^\alpha(T)\right]$$
$$= \mathbb{E}_t^{Q^\alpha}\left[\frac{D_c^\beta(t;T)D_f^\beta(t;T)}{D_c^\alpha(t;T)D_f^\alpha(t;T)}\right]\Pi_c^\alpha(t)$$
$$FVA_{f,c}^{\alpha,\beta}(t) \cong \left\{\mathbb{E}_t^{Q^\alpha}\left[\frac{D_c^\beta(t;T)D_f^\beta(t;T)}{D_c^\alpha(t;T)D_f^\alpha(t;T)}\right] - 1\right\}\Pi_c^\alpha(t)$$

In the limit of deterministic basis we obtain the simple expression

$$\mathbb{E}_t^{Q^\alpha}\left[\frac{D_c^\beta(t;T)D_f^\beta(t;T)}{D_c^\alpha(t;T)D_f^\alpha(t;T)}\right] \cong e^{-\int_t^T s_{f,c}^{\alpha,\beta}(u)du}$$
$$FVA_{f,c}^{\alpha,\beta} \cong \left[e^{-\int_t^T s_{f,c}^{\alpha,\beta}(u)du} - 1\right]\Pi_c^\alpha(t)$$
$$\cong -\left[\int_t^T s_{f,c}^{\alpha,\beta}(u)du\right]\Pi_c^\alpha(t)$$
$$s_{f,c}^{\alpha,\beta}(t) := r_c^\beta(t) + r_f^\beta(t) - r_f^\alpha(t)$$

Remark 2.14. Special cases
All the cases analysed before may be recovered as special cases of the last formula.

- perfect collateral: set $x^{\alpha\beta}(t)B_c^\beta(t) \to \Pi^\alpha(t, A) \quad \forall t$
- single currency: set $\alpha \to \beta, x^{\alpha\beta}(t) \to 1 \quad \forall t$
- no collateral: set $B_c(t) \to 0 \quad \forall t$

Chapter 3

FX modelling in collateralized markets

The **foreign exchange market** (forex, FX, or currency market) is a global decentralized market for the trading of currencies. This includes all aspects of buying, selling and exchanging currencies at current or determined prices. In terms of volume of trading, it is by far the largest market in the world. The main participants in this market are the larger international banks. Financial centres around the world function as anchors of trading between a wide range of multiple types of buyers and sellers around the clock, with the exception of weekends. The foreign exchange market does not determine the relative values of different currencies, but sets the current market price of the value of one currency as demanded against another.

The foreign exchange market works through financial institutions, and it operates on several levels. Behind the scenes banks turn to a smaller number of financial firms known as "dealers", who are actively involved in large quantities of foreign exchange trading. Most foreign exchange dealers are banks, so this behind-the-scenes market is sometimes called the "interbank market", although a few insurance companies and other kinds of financial firms are involved. Trades between foreign exchange dealers can be very large, involving hundreds of millions of dollars. Because of the sovereignty issue when involving two currencies, forex has little (if any) supervisory entity regulating its actions.

An investor, funding derivative contracts and hedging instruments along with their collateral accounts, requires liquidity in one or more currencies. Cash in foreign currencies is usually obtained by trading FX spot and swap contracts. Thus, market dislocations may produce additional costs in funding and hedging activities and, during turbolent periods, can also lead to severe liquidity shortages.
We notice that these funding costs depend on the particular funding strategy adopted by the investor. Indeed, there are different ways to raise money in a foreign currency. The actual funding policy adopted by an institution is a collection of different strategies, driven not only by financial factors. Thus, to introduce an arbitrage-free pricing framework we need to select a particular funding policy. Here we explicitly assume that a domestic investor can fund in foreign currencies only by means of FX swaps. Thus, prices of derivative contracts with cash flows or collateral accounts expressed in foreign currencies should include funding costs originating from the FX swap market.
In general, the FX market does not quote instruments sufficient to fix all the degrees of freedom of dynamical models describing the relevant financial risks. Moreover, the market of cross-currency products is essentially USD based, so that we need to perform triangulations to connect currencies for which no

quotes are available. This section sets within this context and aims to shed some light both from a theoretical and a market practice point of view.

3.1 Funding strategies in domestic and foreign currencies

If some cash flow is expressed in a different currency, we should describe how the investor can obtain cash in such currencies to fulfill the contractual agreements. In the following it is crucial to assume that the investor can fund without restrictions in one particular currency by accessing a risk-free bank account, and we call such currency the domestic currency. All the other currencies are called foreign currencies. The problem, discussed in the introduction, of the limited access to on-shore liquidity channels for off-shore institutions, create market segmentation between currencies. Hence, what we discuss here may lead to asymmetrical evaluation of financial contracts.

If we want to fund in foreign currencies we have to trade market instruments paying cash flows in such currencies, and, if required, to remunerate their collateral accounts in the proper currency. It's important to describe the market strategy used to implement funding in foreign currencies, since the collateralization procedures required by the strategy will affect the pricing formulae.

Two instruments commonly used to implement funding strategies in foreign currencies are the **FX spot** and **forward** contracts. Morevoer, the FX money market quotes also combinations of a long (short) FX spot contract and of a short (long) FX forward contract, usually named **FX swap**.
Here we focus on FX swap contracts, in which the investor borrows cash from the counterparty in foreign currency while lending domestic currency to the same party. At inception one unit of domestic currency is exchanged against the equivalent amount of foreign currency, while at maturity one unit of domestic currency is exchanged back against a given quantity of foreign currency that was determined at inception by market bid-ask dynamics. Let χ_t be the FX market rate converting one unit of foreign currency into a quantity of domestic currency as seen at time t and let e_t be the domestic collateral accrual rate for FX products. A FX swap contract started at t and collateralized at e will exchange, at its maturity T, one unit of domestic currency against $\frac{1}{X_t(T;e)}$ units of foreign currency. The quantity $X_t(T;e)$ correspond to the market quote for the given FX swap. In the following we denote

$$D(t,T,x) := \exp\left\{-\int_t^T x_u du\right\}$$

where x_t is a generic rate.
Then, we use FX swap contracts to build funding strategies in a foreign currency. In particular, if we assume that the margining procedure occurs on a continuous time basis and it is able to remove all credit risk (perfect collateralization), we can use the pricing formula (2.4.1) to get

$$V_t^{FXswap} := \mathbb{E}_t\left[\left(\frac{\chi_T}{X_t(T;e)} - 1\right) D(t,T;e)\right] \tag{3.1.1}$$

The FX forward rate is determined to sell the FX swap contract at par, so that, putting (3.1.1) equal to zero, we get

$$X_t(T;e) = \frac{\mathbb{E}_t[\chi_T D(t,T;e)]}{\mathbb{E}_t[D(t,T;e)]} = \mathbb{E}_t^{T;e}[\chi_T]. \tag{3.1.2}$$

Notice that FX forward rates depend on collateral rates, and, as a consequence, FX forward rates observed in instruments with different collateralization are different.

Remark 3.1. The reference leg of a FX swap contract can be expressed in foreign currency, namely we can consider a FX swap contract where at inception one unit of foreign currency is exchanged against the equivalent amount of domestic currency, while at maturity one unit of foreign currency is exchanged back against a given quantity of domestic currency. If we still assume domestic collateralization at overnight rate e_t, we can apply equation (2.4.1) to obtain

$$\widetilde{V}_t^{FXswap} := \mathbb{E}_t\left[\left(\chi_T - \widetilde{X}_t(T;e)\right) D(t,T;e)\right]$$

where $\widetilde{X}_t(T;e)$ is the par rate of the contract, so that

$$\widetilde{X}_t(T;e) = \frac{\mathbb{E}_t[\chi_T D(t,T;e)]}{\mathbb{E}_t[D(t,T;e)]} = \mathbb{E}_t^{T;e}[\chi_T] = X_t(T;e)$$

Thus, we get that this forward rate is exactly the same as the one given in equation (3.1.2).

3.1.1 Collateralized foreign measure

Using the FX forward rate we can define the collateralized foreign measure $\mathbb{Q}^b$ by means of the following Radon-Nikodym derivative

$$Z_t^f(e) := \left.\frac{d\mathbb{Q}^b}{d\mathbb{Q}}\right|_t := \frac{\chi_t}{\chi_0} D(0,t;e - b^f(e)) \tag{3.1.3}$$

where we define the basis rate $b_t^f(e)$

$$b_t^f(e)dt := e_t dt - \mathbb{E}_t\left[\frac{d\chi_t}{\chi_t}\right] \tag{3.1.4}$$

We can notice that $Z_t^f(e)$ is a $\mathbb{Q}$ martingale normalized so that $Z_0^f(e) = 1$. Hence, if we use the above measure in the definition of FX forward rate, we get

$$\mathbb{E}_t^b\left[D(t,T;b^f(e))\right] = \frac{1}{\chi_t}\mathbb{E}_t\left[\chi_T D(t,T;e)\right] \tag{3.1.5}$$

The equation (3.1.5) establish that the result, expressed in foreign currency units, is the same as discounting the flux by means of the basis collateral rate $b^f(e)$ under an appropriate measure $\mathbb{Q}^b$.
Now, defining the effective foreign funding curve as

$$P_t^f(T;e) := \mathbb{E}_t^b\left[D(t,T;b^f(e))\right], \tag{3.1.6}$$

we have

$$P_t^f(T;e) = \frac{X_t(T;e)}{\chi_t} P_t(T;e) \tag{3.1.7}$$

By means of the following Radon-Nikodym derivative

$$Z_t^f(e) := \left.\frac{d\mathbb{Q}^{T;b}}{d\mathbb{Q}^b}\right|_t := \frac{\mathbb{E}_t^b[D(0,T;b^f(e))]}{P_0^f(T;e)} = \frac{D(0,t;b^f(e))P_t^f(T;e)}{P_0^f(T;e)} \tag{3.1.8}$$

we have also the definition of the collateralized foreign T-forward measure $\mathbb{Q}^{T;b}$, that will serve us later.

3.2 Derivation of pricing formulae

At this point we are able to price contracts with cash flows and/or collateral accounts expressed in foreign currencies inclusive of funding costs originating from dislocations in the FX market. In the following we consider three cases:

1. domestic coupon contracts with collateral posted in a foreign currency;
2. contracts with cash flows denominated in a foreign currency but domestic collateral;
3. contracts with foreign cash flows and collateral.

3.2.1 Pricing domestic contracts collateralized in foreign currency

We consider the case of a derivative collateralized with assets in foreign currency remunerated at c_t^f rate. We have to evaluate the cost of carry of the collateral account in foreign currency. The collateral taker must remunerate the collateral assets posted in foreign currency at the contractual rate c_t^f, while he is funding in the domestic currency with the risk-free bank account.

We give a description of the problem by detailing the funding strategy followed by the collateral taker to remunerate the account. At each collateralization time t we have to remunerate the collateral account, so that at $t + \Delta t$ we must have

$$C_t^f(1 + c_t^f \Delta t)$$

in the collateral account, where c_t^f is the derivative collateral rate. In order to obtain such foreign cash, we enter at time t into a FX swap with notional

$$X_t(t + \Delta t; e) C_t^f (1 + c_t^f \Delta t).$$

On the other hand the FX swap require to pay back the notional in domestic currency at $t + \Delta t$, which we can fulfill by entering at time t into a risk-free zero-coupon bond with notional

$$P_t(t + \Delta t; r) X_t(t + \Delta t; e) C_t^f (1 + c_t^f \Delta t)$$

where $P_t(T; r) := \mathbb{E}_t[D(t, T; r)]$ is the price of the risk-free zero-coupon bond. Thus, the dividend to be payed at each margining date is equal to

$$\chi_t C_t^f - P_t(t + \Delta t; r) X_t(t + \Delta t; e) C_t^f (1 + c_t^f \Delta t)$$

We can solve the FX forward rate in term of the basis curve by using equation (3.1.7):

$$P_t^f(T; e) = \frac{X_t(T; e)}{\chi_t} P_t(T; e) \quad \Rightarrow \quad X_t(t + \Delta t; e) = \chi_t \frac{P_t^f(t + \Delta t; e)}{P_t(t + \Delta t; e)}$$

and we obtain

$$\chi_t C_t^f \left(1 - P_t(t + \Delta t; r) \frac{P_t^f(t + \Delta t; e)}{P_t(t + \Delta t; e)} (1 + c_t^f \Delta t) \right)$$

In the limit of small time intervals Δt we get a continuous dividend equal to

$$\chi_t C_t^f (r_t - c_t^f + b_t^f(e) - e_t) \Delta t$$

where $b_t^f(e)$ is given by equation (3.1.4).

If we assume perfect collateralization, namely $V_t \doteq \chi_t C_t^f$, we can substitute the collateral costs in (2.4.1) with the above expression to obtain the following proposition. The discount rate for a derivative perfectly collateralized in foreign currency with CSA accrual rate given by c_t^f, and funded by means of FX swaps collateralized at the overnight rate e_t, is given by $c_t^f - b_t^f(e) + e_t$, so that in case of perfect collateralization we get

$$V_t = \int_t^T \mathbb{E}_t \left[D(t,u;r) \left(d\pi_u + V_u (r_u - c_u^f + b_u^f(e) - e_u) du \right) \right] \tag{3.2.1}$$

$$= \int_t^T \mathbb{E}_t \left[D(t,u;c^f - b^f(e) + e) d\pi_u \right]. \tag{3.2.2}$$

3.2.2 Pricing foreign contracts collateralized in domestic or foreign currency

Contracts expressed in foreign currencies can be priced by using the collateralized foreign measure $\mathbb{Q}^b$ introduced by equation (3.1.3). We consider a foreign derivative collateralized in domestic currency accruing at c_t rate, and funded by means of FX swaps whose collateral account is remunerated at domestic overnight rate e_t. The contractual coupons can be converted at each payment dates at the spot FX rate, and we obtain from equation (2.4.1)

$$V_t = \int_t^T \mathbb{E}_t \left[D(t,u;c) \chi_u d\pi_u^f \right]$$

We change measure to the collateralized foreign measure $\mathbb{Q}^b$, and we get

$$V_t^f := \frac{V_t}{\chi_t} = \int_t^T \mathbb{E}_t^b \left[D(t,u;c + b^f(e) - e) d\pi_u^f \right] \tag{3.2.3}$$

which can be simplified for contracts with collateral rate equal to the overnight rate in the following form

$$V_t^f \doteq \int_t^T \mathbb{E}_t^b \left[D(t,u;b^f(e)) d\pi_u^f \right] = \int_t^T P^f(t,u;e) \mathbb{E}_t^{u;b} \left[d\pi_u^f \right], \quad c_t \doteq e_t$$

where last expectation on the right-hand side is computed under the basis forward measure $\mathbb{Q}^{T;b}$.

As last case we consider a foreign coupon derivative collateralized in another foreign currency accruing at $c_t^{f'}$ rate, and funded by means of FX swaps whose collateral account is remunerated at domestic overnight rate e_t. The contractual coupons can be converted at each payment dates at the spot FX rate, and we obtain from equation (3.2.2)

$$V_t = \int_t^T \mathbb{E}_t \left[D(t,u;c^{f'} - b^{f'}(e) + e) \chi_u d\pi_u^f \right]$$

We change measure to the collateralized foreign measure $\mathbb{Q}^b$, and we get

$$V_t^f = \int_t^T \mathbb{E}_t^b \left[D(t,u;c^{f'} - b^{f'}(e) + b^f(e)) d\pi_u^f \right] \tag{3.2.4}$$

which can be simplified for contracts collateralized in the same foreign currency of the cash flows in the following form.

$$V_t^f \doteq \int_t^T \mathbb{E}_t^b \left[D(t,u;c^f) d\pi_u^f \right], \quad f \doteq f'.$$

In the following Table we summarize the results.

π_t	C_t		Pricing formula
d	d		$V_t = \int_t^T \mathbb{E}_t[D(t,u;c)d\pi_u]$
d	f		$V_t = \int_t^T \mathbb{E}_t[D(t,u;c^f - b^f(e) + e)d\pi_u]$
f	d		$V_t = \int_t^T \mathbb{E}_t^b[D(t,u;c + b^f(e) - e)d\pi_u^f]$
f	$f^{'}$		$V_t^f = \int_t^T \mathbb{E}_t^b[D(t,u;c^{f'} - b^{f'}(e) + b^f(e))d\pi_u^f]$

Table 3.1: Pricing formulae for derivative contracts with domestic (d) or foreign (f or $f^{'}$) contractual coupons π_t and/or collateral accounts C_t. Cash flows in the foreign currencies are always funded by means of FX swaps with domestic collateralization with accrual rate equal to the overnight rate e_t.

We have seen how the choice of the collateral and cash-flow currencies modifies the pricing equation. In the following section we apply these results to understand the impact of a change of collateralization currency in derivative pricing. In particular, we will focus on FX swap and CCS pricing.

3.3 Pricing FX market instruments

In this section we want to apply the theory just developed to the pricing of FX market products, with a special focus on FX swaps and CCS. Alike the single-currency interest rate markets, we find quotes only for a given set of instrument typologies and for standardized expiry/maturity dates, whereas traders need to price and hedge off-market products, with customized features. The most straightforward way to achieve this task is to bootstrap a set of convenient discounting and forwarding curves, able to take into account collateral posting in foreign currencies. In terms of availability and liquidity, the instruments that can be used to calibrate those curves are **FX swaps** for short to mid maturities and **CCS** with notional resetting for mid to long maturities.

3.3.1 Pricing FX swaps in the market practice

Even if prices of derivative contracts collateralized in different currencies are different, liquid market instruments such as FX swaps are quoted without mentioning the currency used for collateralization, since uncertainties are usually hidden in the bid-ask spread quoted market.
We now investigate the consequence of this approximation.

We consider a domestic investor funding in foreign currencies by means of FX swap contracts collateralized in domestic currency, as we did in the previous section. In this setting, we wish to price a FX swap collateralized in foreign currency and remunerated at the foreign overnight rate e_t^f. We can apply equation (3.2.2) to obtain

$$V_t^{FXswap/f} := \mathbb{E}_t\left[\left(\frac{\chi_T}{X_t(T;e^f,e)} - 1\right) D(t,T;e^f - b^f(e) + e)\right]$$

where we name $X_t(T;e^f,e)$ the par rate of the contract. We can solve for the par rate to get

$$X_t(T;e^f,e) = \frac{\mathbb{E}_t\left[\chi_T D(t,T;e^f - b^f(e) + e)\right]}{\mathbb{E}_t\left[D(t,T;e^f - b^f(e) + e)\right]} = X_t(T;e)\left(1 + \gamma_t^\chi(e^f,e)\right)$$

where the convexity $\gamma_t^\chi(e^f,e)$ of the FX forward rate due to a change in collateral currency is defined as

$$\gamma_t^\chi(e^f,e) := \frac{Cov_t^{T;e}\left[\chi_T, D(t,T;e^f - b^f(e))\right]}{X_t(T;e)\mathbb{E}_t^{T;e}[D(t,T;e^f - b^f(e))]}$$

We can observe that, if the above covariance is null, e.g. this occurs when the spread between the basis rate $b^f(e)$ and the foreign overnight rate e^f is a deterministic function of time, then the convexity is zero, and, in turn, the par rate is equal to the forward rate given in equation (3.1.2). In general, we cannot estimate the convexity $\gamma_t^{\chi}(e^f, e)$ from market data, since V_t^{FXswap} and $V_t^{FXswap/f}$ share the same quote on the FX market, and there are not other liquid quotes with this information.

The approach of quoting FX par swap rates regardless of the chosen collateral currency also impacts FX swap triangulations. Let us consider three currencies $\{x, y, z\}$ and the FX swaps between such currencies. Market practice is to quote par swap rates such that

$$\frac{X_t^{x\to z}(T)}{X_t^{y\to z}(T)} \approx X_t^{x\to y}(T).$$

By means of the theoretical framework we developed, we want to delimit the validity of such a relationship. Let us focus on the FX forward rates from currencies x, and y, to currency z. We first assume that the collateral accounts are in currency z, and remunerated at the overnight rate e_t^z. We name such rates $X_t^{x\to z}(T; e^z)$ and $X_t^{y\to z}(T; e^z)$. If we calculate the ratio between them, we get

$$\frac{X_t^{x\to z}(T; e^z)}{X_t^{y\to z}(T; e^z)} = \frac{\mathbb{E}_t[\chi_T^{x\to z} D(t, T; e^z)]}{\mathbb{E}_t[\chi_T^{y\to z} D(t, T; e^z)]} = \chi_t^{x\to y} \frac{P_t^x(T; e^z)}{P_t^y(T; e^z)}$$

where the last step comes from the triangulation rules of spot FX market, namely

$$\chi_t^{x\to z} = \chi_t^{x\to y}\chi_t^{y\to z}$$

On the other hand, by equation (3.2.3) the FX forward rate from currency x to currency y with collateralization in currency z at e_t^z overnight rate is given by

$$X_t^{x\to y}(T; e^z) = \frac{\mathbb{E}_t^{b^y}[\chi_T^{x\to y} D(t, u; b^y(e^z))]}{\mathbb{E}_t^{b^y}[D(t, u; b^y(e^z))]} = \frac{\mathbb{E}_t[\chi_T^{x\to z} D(t, u; e^z)]}{\chi_t^{y\to z}\mathbb{E}_t^{b^y}[D(t, u; b^y(e^z))]} = \chi_t^{x\to y} \frac{P_t^x(T; e^z)}{P_t^y(T; e^z)}$$

in which the second equality comes from the equation (3.1.5): $\mathbb{E}_t^b[D(t, T; b^f(e))] = \frac{1}{\chi_t}\mathbb{E}_t[\chi_T D(t, T; e)]$. Thus, the triangulation rule for FX forward rates holds only if all the rates share the same collateralization currency

$$X_t^{x\to z}(T; e^z) = X_t^{x\to y}(T; e^z) X_t^{y\to z}(T; e^z) \tag{3.3.1}$$

The same relationship does not hold if we change the collateralization currency. For instance, we consider the FX forward rates from currency x to currency y with collateralization in currency y at e_t^y overnight rate. We get from equation (3.2.4)

$$\begin{aligned}
X_t^{x\to y}(T; e^y) &= \frac{\mathbb{E}_t^{b^y}[\chi_T^{x\to y} D(t, u; e^y)]}{\mathbb{E}_t^{b^y}[D(t, u; e^y)]} = \frac{\mathbb{E}_t[\chi_T^{x\to z} D(t, u; e^y - b^y(e^z) + e^z)]}{\mathbb{E}_t[\chi_T^{y\to z} D(t, u; e^y - b^y(e^z) + e^z)]} \\
&= \frac{X_t^{x\to z}(T; e^z)\left(1 + \gamma_t^{\chi^{x\to z}}(e^y, e^z)\right)}{X_t^{y\to z}(T; e^z)\left(1 + \gamma_t^{\chi^{y\to z}}(e^y, e^z)\right)} \\
&= X_t^{x\to y}(T; e^z)\left(\frac{1 + \gamma_t^{\chi^{x\to z}}(e^y, e^z)}{1 + \gamma_t^{\chi^{y\to z}}(e^y, e^z)}\right)
\end{aligned}$$

If the market approximation

$$\gamma_t^{\chi}(e^y, e^z) \approx 0 \tag{3.3.2}$$

is assumed, we obtain that the triangulation rule for FX forward rates is holding again.

3.3.2 Cross-Currency swaps

FX swaps are quoted with sufficient liquidity only for short maturities, i.e. up to two-four years depending on the considered currency pair. For longer maturities, market participant exchange amounts of currency by means of Cross-Currency swaps (CCS). A CCS is a foreign exchange derivative between two institutions to exchange the principal and/or interest payments of a loan in one currency for equivalent amounts, in net present value terms, in another currency. A currency swap should be distinguished from interest rate swap, for in currency swap, both principal and interest of loan is exchanged from one party to another party for mutual benefits.

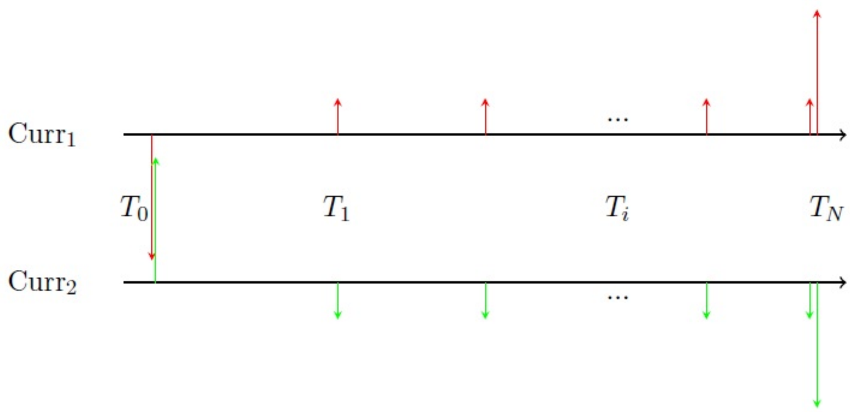


Figure 3.1: A cash flow description of a constant-notional CCS

In the simplest case, depicted in Figure 3.1, each of the two parties lends to the other an amount of money at the swap start date T_0, receives for it (floating) rate interests at dates $T_1, ..., T_i, ..., T_N$, and gets the notional back at maturity T_N. These CCS are called **constant-notional** because the principal amount used to value interests is established once for all at inception. For most currency pairs, standardized CCS are structured such that, at inception, notionals are equivalent (using FX rate for T_0) and the deal is entered at-par.

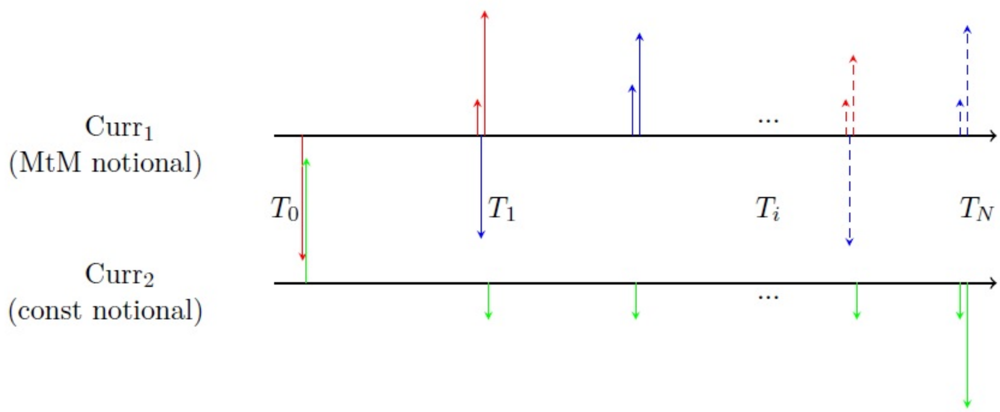


Figure 3.2: A cash flow description of a marked-to-market CCS

The FX market mainly quotes marked-to-market (MtM) CCS, which are built by appending a series of at-par single period CCS one after the other. The resulting contract is shown in Figure 3.2, and it behaves like a pair of rolling loans, where notionals are exchanged at each interest rate payment date, hence reducing the counterparty risk and the FX risk of the structure. Legs with a MtM notional are

dubbed renotioning or resetting legs, to distinguish them from constant-notional ones.
Most of quoted and liquid CCS have the following features:

- the major currency has a renotioning leg, while a minor currency has a constant-notional leg;
- the major currency has interests indexed to flat Libor rates, while a minor currency has interests based on Libor rates plus a spread;
- the spread over a minor currency floaters is chosen such that the CCS is at-par (equilibrium spreads are CCS market quotes);
- payments occur quarterly.

Now we want to value CCS net present value (NPV) and, in order to do that, we take the point of view of a domestic investor pricing FX swaps, constant-notional CCS and marked-to-market CCS with collateral posted in domestic currency and remunerated at the same rate e; therefore all foreign flows will be priced by the formula in the third row of Table 3.1. The extension to different funding strategies is straightforward by means of the results of Section (3.2).

1. **Constant-Notional CCS**

Let's consider a constant-notional CCS, where interests are indexed to domestic and foreign Libor rates $L_t^x(T)$, t being the fixing date, T the maturity and $x \in \{d, f\}$. We assume that the domestic market quotes single-currency interest rate swaps with a floating leg indexed to the same Libor rates the CCS domestic leg is indexed to and where the standard collateralization is based on the same collateral rate e. Hence, we define the domestic Libor forward rate $F_t(T_i; e)$ as the forward for the Libor rate $L_{T_{i-1}}(T_i)$ when the collateral is posted in natural currency, i.e.

$$F_t(T_i; e) := \frac{\mathbb{E}_t[D(t, T_i; e) L_{T_{i-1}}(T_i)]}{P_t(T_i; e)} = \mathbb{E}_t^{T_i;e}[L_{T_{i-1}}(T_i)]. \tag{3.3.3}$$

For sake of generality we allow CCS interests to be equal to Libor rates plus a spread, which we name s for the domestic leg and s^f for the foreign leg. According to first and third rows of Table 3.1, we compute the CCS price, expressed in domestic units, as

$$V_t^{CCS} := V_t^{CCS/d} - V_t^{CCS/f} \tag{3.3.4}$$

where the net present values of the domestic and foreign legs are given by

$$V_t^{CCS/d} := N\left(-P_t(T_0; e) + \sum_{i=1}^{N} \tau_i (F_t(T_i; e) + s) P_t(T_i; e) + P_t(T_N; e)\right) \tag{3.3.5}$$

$$V_t^{CCS/f} := \chi_t N^f \left(-P_t^f(T_0; e) + \sum_{i=1}^{N} \tau_i (F_t^f(T_i; e) + s^f) P_t^f(T_i; e) + P_t^f(T_N; e)\right) \tag{3.3.6}$$

and where N and N^f stand respectively for the notionals of the domestic and foreign legs, τ_i is the year fraction calculated between T_{i-1} and T_i. Most importantly, in valuing the net present value of the foreign leg we introduced the foreign basis forward Libor rates $F_t^f(T_i; e)$ observed under domestic collateralization as

$$F_t^f(T_i; e) := \frac{\mathbb{E}_t^b\left[D(t, T_i; b^f(e)) L_{T_{i-1}}^f(T_i)\right]}{P_t^f(T_i; e)} = \mathbb{E}_t^{T_i;b}\left[L_{T_{i-1}}^{T_i;b}(T_i)\right] \tag{3.3.7}$$

2. **Marked-to-market contributions**

In this case the resetting of the notional creates an asymmetry between the two legs, so that different pricing formulae will be needed according to the leg on which the marking-to-market operates.
If the renotioning leg is the domestic one, we get

$$V_t^{CCS} := V_t^{MtMCCS/d} - V_t^{CCS/f} \tag{3.3.8}$$

where the net present value of the MtM leg is given by

$$\begin{aligned} V_t^{MtMCCS/d} = N^f \sum_{i=1}^{N} P_t(T_i;e)\mathbb{E}_t^{T_i;e}\left[\chi_{T_{i-1}}(1+\tau_i(L_{T_{i-1}}(T_i)+s))\right] \\ -N^f \sum_{i=1}^{N} P_t(T_i;e)\mathbb{E}_t^{T_{i-1};e}[\chi_{T_{i-1}}] \end{aligned} \tag{3.3.9}$$

while the constant-notional leg is defined as in Equation (3.3.6).
In the other case, where the foreign leg has a renotioning feature, we have

$$V_t^{CCS} := V_t^{CCS/d} - V_t^{MtMCCS/f} \tag{3.3.10}$$

where the net present value of the MtM leg reads

$$\begin{aligned} V_t^{MtMCCS/f} = \chi_t N \sum_{i=1}^{N} P_t^f(T_i;e)\mathbb{E}_t^{T_i;b}\left[\frac{1}{\chi_{T_{i-1}}}\left(1+\tau_i(L^f_{T_{i-1}}(T_i)+s^f)\right)\right] \\ -\chi_t N \sum_{i=1}^{N} P_t^f(T_{i-1};e)\mathbb{E}_t^{T_{i-1};b}[\frac{1}{\chi_{T_{i-1}}}] \end{aligned} \tag{3.3.11}$$

while the constant-notional leg is defined as in Equation (3.3.5).

These two pricing formulae share the same structure. The contribution of the first summation represents the redemption of the lending which occurrs at the end of each period plus the payment of matured interests, while the second summation corresponds to the lending of an amount of currency at each coupon period start date, such as to be at par with the constant-notional of the other leg. Let us now separately analyze the structure of the MtM contributions.

Domestic Marked-to-Market Leg

We start by analyzing the domestic MtM leg. The second summation term in equation (3.3.9), where the exchange rate read at T_{i-1} is immediately paid, corresponds to the flow of a FX swap, and it is simply given by

$$\sum_{i=1}^{N} P_t(T_{i-1};e)\mathbb{E}_t^{T_{i-1};e}[\chi_{T_{i-1}}] = \chi_t \sum_{i=1}^{N} P_t^f(T_{i-1};e).$$

On the other hand the first summation, over interests and notional repayments, involves two terms linked to the correlation structure of discounting, forwarding and exchange rates. In particular we need to evaluate a FX forward with delayed payment, namely

$$\mathbb{E}_t^{T_i;e}[\chi_{T_{i-1}}] = \mathbb{E}_t^{T_i;e}[X_{T_{i-1}}(T_{i-1};e)] \tag{3.3.12}$$

and a floating domestic payment with stochastic notional

$$\mathbb{E}_t^{T_i;e}[\chi_{T_{i-1}}L_{T_{i-1}}(T_i)] = \mathbb{E}_t^{T_i;e}[X_{T_{i-1}}(T_{i-1};e)F_{T_{i-1}}(T_i;e)] \tag{3.3.13}$$

We wrote these contributions in terms of the forward exchange rate $X_t(T_{i-1}; e)$ and of the forward Libor rate $F_t(T_i; e)$, which are martingales under the terminal measures $\mathbb{Q}^{T_{i-1};e}$ and $\mathbb{Q}^{T_i;e}$, respectively. These two measures are linked by standard Radon-Nikodym derivative

$$Z_t(T_{i-1}, T_i; e) := \left.\frac{d\mathbb{Q}^{T_i;e}}{\mathbb{Q}^{T_{i-1};e}}\right|_t = \frac{P_t(T_i; e)}{P_t(T_{i-1}; e)} \frac{P_0(T_{i-1}; e)}{P_0(T_i; e)} \tag{3.3.14}$$

As a consequence, the first term must be valued taking into account the correlation between the forward exchange rate and the collateral curve (pure change of measure effect), while the second also need to incorporate the correlation with forward Libors (change of measure plus covariation). Any estimate of these contributions requires defining joint distribution with covariation effects. We notice that the FX delayed payment terms would be present even in the case of a fixed rate MtM CCS.

Foreign Marked-to-Market Leg

We focus on the foreign MtM leg, whose tractation is totally analogous to the one carried on for the domestic case. The term bound to the lending of a marked-to-market foreign notional is trivial, because it states that we are lending one unit of domestic currency at each date $T_0, ..., T_{N-1}$ and we get, by means of Eq. (3.1.5),

$$\chi_t \sum_{i=1}^{N} P_t^f(T_{i-1}; e)\mathbb{E}_t^{T_{i-1};b}\left[\frac{1}{\chi_{T_{i-1}}}\right] = \sum_{i=1}^{N} P_t(T_{i-1}; e)$$

Analogously to the domestic MtM case we then write the terms related to the payment of interests and notional redemptions relying on forward exchange rates and forward Libor rates. We have a FX forward with delayed payment

$$\mathbb{E}_t^{T_i;b}\left[\frac{1}{\chi_{T_{i-1}}}\right] = \mathbb{E}_t^{T_i;b}\left[\frac{1}{X_{T_{i-1}}(T_{i-1}; e)}\right] \tag{3.3.15}$$

and a floating foreign payment with stochastic notional

$$\mathbb{E}_t^{T_i;b}\left[\frac{L_{T_{i-1}}^f(T_i)}{\chi_{T_{i-1}}}\right] = \mathbb{E}_t^{T_i;b}\left[\frac{F_{T_{i-1}}^f(T_i; e)}{X_{T_{i-1}}(T_{i-1})}\right] \tag{3.3.16}$$

In this case the exchange rate to be used for notional purposes is the reverse rate $\frac{1}{\chi_t}$ converting domestic units into foreign one. Both expectations involve the basis forward measure $\mathbb{Q}^{T_i;b}$, under which the basis forward Libor rate $F_t^f(T_i; e)$ is a martingale. The forward exchange rate $X_t(T_{i-1}; e)$, as seen, is a martingale under the terminal measure $\mathbb{Q}^{T_{i-1};e}$, which is connected to $\mathbb{Q}^{T_i;b}$ by means of the Radon-Nikodym derivative

$$Z_t^f(T_{i-1}, T_i; e) := \left.\frac{d\mathbb{Q}^{T_i;b}}{d\mathbb{Q}^{T_{i-1};e}}\right|_t = \frac{X_t(T_{i-1}; e)P_t^f(T_i; e)}{P_t^f(T_{i-1}; e)} \frac{P_0(T_{i-1}; e)}{X_0(T_{i-1}; e)P_0^f(T_i; e)} \tag{3.3.17}$$

Once again, the first expectation involves a change of measure contribution, and depends on the correlation between $X_t(T_{i-1}; e)$ and $Z_t^{i-1,i}(e; b)$, while the second also depends on the interplay with basis forward Libor rates.

At the end of this section we tie together all the analysis brought on so far and quickly sketch a hypothetical bootstrapping procedure to infer basis discount factors, basis forward Libor rates, as well as correlation terms. We assume here that the market quotes, for a set of maturities, FX swaps, constant-notional CCS, and MtM CCS both with fixed and floating rate interests.

1^{st} *step*: we can apply Equation (3.1.7)

$$P_t^f(T;e) = \frac{X_t(T;e)}{\chi_t} P_t(T;e)$$

to a set of domestic-foreign FX swap with increasing maturities T_1, T_2,..., T_N, such as to derive the basis discount factors $P_t^f(T_i;e)$. These basis zero-coupon bonds constitute the fundamental pillars of a basis discounting curve used to discount foreign flows collateralized in domestic currency.

2^{nd} *step*: we consider constant-notional CCS and apply formulae of Section (3.3.2) (Point 1) to deduce basis forward Libor rates $F_t^f(T_i;e)$ for a set of maturities T_i hence building a basis forwarding curve. Marked-to-market CCS with domestic renotioning leg and fixed interests let us infer forward FX rates with deferred payments (Eq. (3.3.12)), which incorporate the correlation between domestic discounts and FX forwards, while CCS with domestic renotioning leg and floating interests are used to deduce the terms involving correlations between FX forwards and domestic Libor forwards (Eq. (3.3.13)). Analogously, Marked-to-market CCS with foreign renotioning leg and fixed interests let us value inverse forward FX rates with deferred payments (Eq. (3.3.16)), which incorporate the correlation between basis discounts and FX forwards, while CCS with foreign renotioning leg and floating interests are helpful to estimate correlations between FX forwards and foreign basis Libor forwards (Eq. (3.3.13)).

3.4 Effective discounting curve approach

The procedure described above constitute only a theoretical case study; in this section we discuss a practical curve bootstrapping procedure by approximating the pricing formula of MtM CCS.
We have already said before that the only FX quotes which are actively traded are FX swaps with short to mid maturities together with MtM CCS with flat floating interests and renotioning for the major currency leg versus floating interests plus spread and constant notional on the minor currency leg. The absence of a set of quotes allowing a sequential bootstrap of the foreign basis discounting curve, of the foreign basis forwarding curve and of correlations driving MtM corrections, forces market players need to find some approximations. A very common approach to take into account the information embedded in market quotes, and to quickly price CCS, consists in avoiding direct modelling of the dependencies among all the components of the swap price formula, and valuing net present values by means of an effective discounting curve approach.
The most natural way for an investor to achieve this result is to give relevance to its own domestic currency and price the CCS by means of four curves:

1. a domestic discounting curve, which is the same curve linked to the domestic collateral rate;

2. a domestic forwarding curve, which is the same curve obtained from single-currency standard floaters quoted in the domestic money market;

3. a foreign currency forwarding curve, which is the same curve obtained from single-currency standard floaters quoted in the foreign currency money market;

4. an implied foreign currency discounting curve, bootstrapped such as market CCS are repriced at par.

By means of this procedure we choose to use unadjusted foreign forwards, as if they were paid and collateralized in their own currency, and incorporate all the corrections discussed above into an implied foreign currency discounting curve. In this way we deduce an implied foreign currency curve that does not correspond to the basis foreign currency curve given in equation (1.7.8), and used in the previous section to present the theoretical curve bootstrapping procedure, unless some approximations hold.

3.4.1 Bootstrapping curves in the FX market

The short end of the implied curve could be straightforwardly stripped by FX swaps by means of Eq. (3.1.7). Thus, if we call T_c the longest maturity for which we can find on the market liquid quotes of FX swaps, we can write

$$P_t^{f,impl}(T;e) := \frac{X_t(T;e)}{\chi_t} P_t(T;e), \quad T \leq T_c$$

where $X_t(T;e)$ is given by the market.
Then, in order to cover the mid-long part of FX curves ($T > T_c$), we need to develop simplified pricing formulae for CCS to be used for bootstrapping purposes. The effective curve approach consists in disregarding all of the contributions that would require a dynamical model and in re-writing in a simple way the net present values given by equations (3.3.6), (3.3.9), (3.3.11) in terms of implied basis foreign zero-coupon bonds $P_t^{f,impl}(T;e)$. These latters will be calibrated such as to ensure that a set of relevant market instruments is priced at-par. Equation (3.3.5), wich only involves domestic flows and domestic collateral, is unchanged. Let us begin by analyzing the MtM domestic leg of a CCS. Its NPV can be cast

in the form

$$V_t^{MtMCCS/d} = \sum_{i=1}^{N} N_{i-1}^{impl}(-P_t(T_{i-1};e) + P_t(T_i;e)(1+\tau_i(F_t(T_i;e)+s))) \tag{3.4.1}$$

by introducing the coupon-dependent notionals

$$N_t^{impl} := N^f X_t^{impl}(T_i;e)$$

based on the implied forward exchange rates

$$X_t^{impl}(T_i;e) := \frac{\chi_t P_t^{f,impl}(T_i;e)}{P_t(T_i;e)}$$

As for the foreign leg, in presence of constant notional (Eq. (3.3.6)), we set

$$V_t^{CCS/f} = \chi_t N^f \left(-P_t^{f,impl}(T_0;e) + \sum_{i=1}^{N} \tau_i(\hat{F}_t^f(T_i;e^f) + s^f) P_t^{f,impl}(T_i;e) + P_t^{f,impl}(T_N;e) \right) \tag{3.4.2}$$

while if the leg is marked-to-market (Eq. (3.3.11)), we write

$$V_t^{MtMCCS/f} = \chi_t \sum_{i=1}^{N} N_{i-1}^{f,impl} \left(-P_t^{f,impl}(T_{i-1};e) + P_t^{f,impl}(T_i;e) \left(1 + \tau_i(\hat{F}_t^f(T_i;e^f) + s^f) \right) \right)$$

where we defined the maturity-dependent notionals

$$N_i^{f,impl} := N \frac{1}{X_t^{impl}(T_i;e)}$$

and, in the formulae related to foreign leg, we replaced basis foreign forward Libor rates with the foreign forward Libor rates $\hat{F}_t^f(T_i;e^f)$ which are the rates bootstrapped by a foreign investor by means of its own money market quotes with the analogous of Equation (3.3.3) discussed before. We use these rates because we are able to bootstrap them from market quotes.

The rationale behind this definition of implied curve is removing from the equation all the terms depending on dynamical parameters which cannot be bootstrapped by independent market quotes. In the following sections we highlight the terms we have approximated to understand the hypothesis under which the implied curve $P_t^{f,impl}(T;e)$ can be identified with the basis curve $P_t^f(T;e)$.

Approximating a MtM Domestic Leg

We consider Eq. (3.3.9) and, neglecting correlations between FX spot rate and interest rate risk factors, get

$$\mathbb{E}_t^{T_i}[\chi_{T_{i-1}}] \approx X_t(T_{i-1};e), \quad \mathbb{E}_t^{T_i}[\chi_{T_{i-1}} L_{T_{i-1}}(T_i)] \approx X_t(T_{i-1};e) F_t(T_i;e).$$

The assumption

$$X_t^{impl}(T_i;e) \approx X_t(T_i;e) \tag{3.4.3}$$

leads to Eq. (3.4.1) and to the identification $P_t^f(T_i;e) \approx P_t^{f,impl}(T_i;e)$.

Approximating a Constant-Notional Foreign Leg

Let us focus on Eq. (3.3.6). Since the foreign basis forward Libor rates $F_t^f(T_i;e)$ in such equation cannot be deduced from independent market quotes, a possible solution is to replace the forward Libor rate by means of the foreign forward Libor rates as given by the single-currency foreign money market, namely

$$F_t^f(T_i;e) \approx \hat{F}_t^f(T_i;e^f). \tag{3.4.4}$$

This choice naturally suggests to identify $P_t^f(T_i;e) \approx P_t^{f,impl}(T_i;e)$ and to price this leg by means of Eq. (3.4.2).

Approximating a MtM Foreign Leg Finally, we consider Eq. (3.3.9), disregard correlations between FX spot rate and interest rate risk factors to approximate

$$\mathbb{E}_t^{T_i;b}\left[\frac{1}{\chi_{T_{i-1}}}\right] \approx \frac{1}{X_t(T_{i-1};e)}, \quad \mathbb{E}_t^{T_i;b}\left[\frac{L_{T_{i-1}}(T_i)}{\chi_{T_{i-1}}}\right] \approx \frac{F_t^f(T_i;e)}{X_t(T_{i-1};e)}$$

and get a simple formula for the net present value of the foreign leg of a MTM CCS as

$$V_t^{MtMCCS/f} \approx \chi_t \sum_{i=1}^{N} N_{i-1}^f \left(-P_t^f(T_{i-1};e) + P_t^f(T_i;e)\left(1 + \tau_i(F_t^f(T_i;e) + s^f)\right)\right).$$

Under the approximations of Eqs. (3.4.3) and (3.4.4) once again we can inteerprete the implied discounts $P_t^{f,impl}(T_i;e)$ as proxy for the true basis discount $P_t^f(T_i;e)$.

Appendix A

Theoretical framework

A.1 Itô calculus: some hints.

Definition A.1. We denote by $\mathbb{L}^2_{loc}$ the family of processes $(u_t)_{t\in[0,T]}$ that are progressively measurable with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and such that

$$\int_0^T u_t^2 dt < \infty \quad a.s \tag{A.1.1}$$

It is interesting to note that the space $\mathbb{L}^2_{loc}$ is invariant with respect to changes of equivalent probability measures (see def. (A.5)): if (A.1.1) holds and $\mathbb{Q} \approx \mathbb{P}$ then we have of course

$$\int_0^T u_t^2 dt < \infty \quad \mathbb{Q} - a.s.$$

Definition A.2. An Itô process is a stochastic process X of the form

$$X_t = X_0 + \int_0^t u_s ds + \int_0^t \sigma_s dW_s, \quad t \in [0,T] \tag{A.1.2}$$

where X_0 is a $\mathcal{F}_0$-measurable random variable, $\mu \in \mathbb{L}^1_{loc}$ and $\sigma \in \mathbb{L}^2_{loc}$

Formula (A.1.2) is usually written in the "differential form"

$$dX_t = \mu_t dt + \sigma_t dW_t \tag{A.1.3}$$

Corollary A.1. *If X is the Itô process in* (A.1.2)*, then its quadratic variation process is given by*

$$\langle X \rangle_t = \int_0^t \sigma_s^2 ds$$

or, in differential terms,

$$d \langle X \rangle_t = \sigma_t^2 dt$$

The differential representation of an Itô process is unique, that is the drift and diffusion coefficients are determined uniquely[1]

Theorem A.1. *Itô formula for Brownian motion*
Let $f \in C^2(\mathbb{R})$ and let W be a real Brownian motion. Then $f(W)$ is an Itô process and we have

$$df(W_t) = f^{'}(W_t)dW_t + \frac{1}{2}f^{''}(W_t)dt^2$$

[1]A proof can be found in [I] p. 169
[2]A complete proof can be found in [I] pp. 170-172

Theorem A.2. *Itô formula-general version*
Let X be the Itô process in (A.1.3) *and $f = f(t, x) \in C^{1,2}(\mathbb{R}^2)$. Then the stochastic process*

$$Y_t = f(t, X_t)$$

is an Itô process and we have

$$df(t, X_t) = \partial_t f(t, X_t)dt + \partial_x f(t, X_t)dX_t + \frac{1}{2}\partial_{xx} f(t, X_t)d\langle X\rangle_t . \tag{A.1.4}$$

Remark A.1. Since, by Corollary (A.1), we have

$$d\langle X\rangle_t = \sigma_t^2 dt$$

formula (A.1.4) can be written more explicitly as follows

$$df = \left(\partial_t f + \mu_t \partial_x f + \frac{1}{2}\sigma_t^2 \partial_{xx} f\right) dt + \sigma_t \partial_x f dW_t, \tag{A.1.5}$$

where $f = f(t, X_t)$.

Definition A.3. An N-dimensional Itô process is a stochastic process of the form

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s, \quad t \in [0, T] \tag{A.1.6}$$

where X_0 is $\mathcal{F}_0$-measurable, W is a d-dimensional Brownian motion, $\mu \in \mathbb{L}^1_{loc}$ is a $(N \times 1)$-vector and $\sigma \in \mathbb{L}^2_{loc}$ is a $(N \times d)$-matrix.

Formula (A.1.6) can be written in the equivalent differential form

$$dX_t = \mu_t dt + \sigma_t dW_t$$

or, more explicitly

$$dX_t^i = \mu_t^i dt + \sum_{j=1}^{d} \sigma_t^{ij} dW_t^j, \quad i = 1, ..., N$$

Lemma A.1.1. *Consider an Itô process X of the form* (A.1.6) *and set*

$$C = \sigma\sigma^*.$$

Then we have

$$\left\langle X^i, X^j \right\rangle_t = \int_0^t C_s^{ij} ds, \quad t \geq 0$$

or, in differential notation,

$$d\langle X\rangle_t = C_t dt$$

In practice, given two Itô processes X, Y in $\mathbb{R}^N$, the computation of $\langle X, Y\rangle_t$ can be handled by applying the following "rule":

$$d\left\langle X^i, Y^j \right\rangle_t = dX_t^i dY_t^j$$

where the product on the right-hand side of the previous equality can be computed using the following formal rules:

$$dtdt = dtdW^i = dW^i dt = 0, \quad dW^i dW^j = \delta_{ij} dt,$$

and δ_{ij} denotes Kronecker's delta

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Theorem A.3. *Let X be an Itô process of the form* (A.1.6) *and $f = f(t,x) \in C^{1,2}(\mathbb{R} \times \mathbb{R}^N)$. Then*

$$df = \partial_t f dt + \nabla f \cdot dX_t + \frac{1}{2} \sum_{i,j=1}^{N} \partial_{x_i x_j} f d \left\langle X^i, X^j \right\rangle_t, \tag{A.1.7}$$

with $f = f(t, X_t)$ and $\nabla f = (\partial_{x_1} f, ..., \partial_{x_N} f)$.
In compact form, if we put $C = \sigma\sigma^*$ and recall Lemma (A.1.1)*, then the formula* (A.1.7) *becomes*

$$\begin{aligned} df &= \left(\frac{1}{2} \sum_{i,j=1}^{N} C_t^{ij} \partial_{x_i x_j} f + \mu_t \cdot \nabla f + \partial_t f \right) dt + \nabla f \cdot \sigma_t dW_t \\ &= \left(\frac{1}{2} \sum_{i,j=1}^{N} C_t^{ij} \partial_{x_i x_j} f + \sum_{i=1}^{N} \mu_t^i \partial_{x_i} f + \partial_t f \right) dt + \sum_{i=1}^{N} \sum_{h=1}^{d} \partial_{x_i} f \sigma_t^{ih} dW_t^h. \end{aligned}$$

A.2 Bank account and risk neutral measure

In order to understand the modern interest rate market after the credit crunch, we must set up a framework with solid theoretical basis able to explain the observed data, or, in other words, to price plain vanilla derivatives according to the available market quotations.

The (spot) instantaneous rate, abbreviated into short rate, is an abstract rate spanning an infinitesimal time interval (with infinitesimal rate tenor). In fact, setting $T_1 = t$, $T_2 = T$ and taking the limit $t^+ \leftarrow T$ we obtain

$$R_k(t,T) \xrightarrow[t^+ \leftarrow T]{} R_1(t,T) \xrightarrow[t^+ \leftarrow T]{} R_\infty(t,T) \xrightarrow[t^+ \leftarrow T]{} L_k(t,T) \xrightarrow[t^+ \leftarrow T]{} r(t)$$

From any equation above, we may obtain, setting $T = t + dt$,

$$N(t + dt) = N(t)[1 + r(t)dt]$$

where $N(t)$ is the nominal al time t and dt is an infinitesimal time interval. Thus the dynamics of $N(t)$ is given by

$$dN(t) := N(t + dt) - N(t) = N(t)r(t)dt$$

and can be easily integrated over a finite time interval $[T_1, T_2]$ to obtain

$$N(T_2) = N(T_1) exp \int_{T_1}^{T_2} r(t)dt.$$

Thus, the short rate is a continuously compounded annual rate.

The bank account, or money market account, is, in financial mathematics, an ideal financial instrument representing the behaviour of an abstract loan that rewards its holder with the risk free rate. Denoting with $B(t)$ the bank account value at time t, it must evolve in time according to the dynamics

$$\begin{cases} dB(t) = r(t)B(t)dt, \\ B(0) = 1. \end{cases}$$

This simple differential problem can be integrated to obtain the solution

$$B(T) = B(0) exp \int_0^T r(t)dt$$

The factor $B(0) = 1$ on the right hand side (r.h.s) is tipically omitted, in this case one should remember that the dimensionality is correct (currency on both sides).

Being the short rate a stochastic process, also the bank account is a stochastic process. From a financial point of view, the bank account is such that one unit of currency invested at time $t = 0$ accrues, over an infinitesimal time interval dt, at the (stochastic) short rate $r(t)$.
Thus the bank account is particularly suitable as reference asset, or numeraire, since it allows to put into relation amounts of currencies observed at different times.
The value (price) of any contract Π can be expressed in units of $B(t)$ and $B(T)$ through the risk neutral pricing formula

$$\frac{\Pi(t)}{B(t)} = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{\Pi(T)}{B(T)}\right]$$

$$\Pi(t) = \mathbb{E}_t^{\mathbb{Q}}[D(t,T)\Pi(T)]$$

$$D(t,T) := \frac{B(t)}{B(T)} = exp\left[-\int_t^T r(u)du\right]$$

where $\mathbb{Q}$ denotes the risk neutral measure associated to the numeraire $B(t)$ and $\mathbb{E}_t^{\mathbb{Q}}[.]$ denotes the expectation at time $t < T$ under measure $\mathbb{Q}$.
We remark that the stochastic discount factor is adimensional, being the ratio between two bank account values, and it depends on the short rate over the time interval $[t, T]$.

In case of a contract paying multiple coupons $\{\Pi(T_1), ..., \Pi(T_N)\}$ at multiple cash flow dates $\{T_1, ..., T_N\}$ we have

$$\Pi(t) = \sum_{i=1}^{N} \mathbb{E}_t^{\mathbb{Q}}[D(t,T)\Pi(T)]$$

Notice that the expectation is a linear operator. The expression above can be written in integral form

$$\Pi(t) = \int_t^T \mathbb{E}_t^{\mathbb{Q}}[D(t,u)d\pi(u)]$$

where $T = T_N$, by introducing the cumulative coupon process

$$\pi(t) := \sum_{i=1}^{N} \not\Vdash_{t>T_i} \Pi(T_i).$$

In fact

$$\Pi(t) = \int_t^T \mathbb{E}_t^{\mathbb{Q}}[D(t,u)d\pi(u)] = \int_t^T \mathbb{E}_t^{\mathbb{Q}}[D(t,u)\sum_{i=1}^{N}\delta(T_i - u)\Pi(u)du] = \sum_{i=1}^{N}\mathbb{E}_t^{\mathbb{Q}}[D(t,T)\Pi(T)]$$

A.3 Feynman-Kac theorem

The Feynman-Kac theorem, under certain assumptions, allows us to express the solution of a given partial differential equation (PDE) as the expected value of a function of a suitable diffusion process whose drift and diffusion coefficient are defined in terms of the PDE coefficients.

Theorem A.4 (Feynman-Kac). *Let A be a generic asset with price process $A(t)$ solution of the SDE*

$$
\begin{aligned}
&dA(t) = \mu(t,A)dt + \sigma(t,A)dW^{\mathbb{Q}}(t), \quad 0 \le t \le T,\\
&\mu, \sigma \in \mathfrak{L}^2[0,T],\\
&A(t=0) = A_0 \in \mathbb{R}^+,
\end{aligned}
$$

under some probability measure $\mathbb{Q}$. Let also Π be a derivative on A with price $\Pi(t, A(t)) = \Pi(t)$ at time t, solution of the parabolic PDE

$$\widehat{\mathfrak{L}}\Pi(t) = r(t)\Pi(t),$$

$$\widehat{\mathfrak{L}} := \frac{\partial}{\partial t} + \mu(t,A)\frac{\partial}{\partial A} + \frac{1}{2}\sigma^2(t,A)\frac{\partial^2}{\partial^2 A}$$

$$\Pi \in \mathfrak{C}^{1,2}[[0,T]\times\mathbb{R}], \quad \sigma(t,A)\Pi(t) \in \mathfrak{L}^2[0,T],$$

$$\Pi(T) \in \mathbb{R}^+, t \in [0,T] \subset \mathbb{R}^+$$

Then the derivatives' price $\Pi(t)$ admits the representation

$$\Pi(t) = \mathbb{E}_t^{\mathbb{Q}}[D(t,T)\Pi(T)],$$

$$D(t,T) = exp\left[-\int_t^T r(u)du\right]$$

where $\mathbb{Q}$ is the measure such that

$$dA(t) = \mu(t,A)dt + \sigma(t,A)dW^{\mathbb{Q}}(t)$$

Theorem A.5 (Generalised Feynman-Kac). *In case of the more general parabolic PDE*

$$\mathfrak{L}_\mu = r(t,A(t))\Pi(t) + \Phi(t,A(t)),$$

we have the more general expectation

$$\Pi(t) = \mathbb{E}_t^{\mathbb{Q}}\left[D(t,T,A)\Pi(T) + \int_t^T D(t,u,A(u))\phi(u,A(u))du\right],$$

$$D(t,T,A) = exp\left[-\int_t^T r(u,A(u))du\right]$$

where $\mathbb{Q}$ is the measure such that

$$dA(t) = \mu(t,A)dt + \sigma(t,A)dW^{\mathbb{Q}}(t)^3$$

This theorem is important because it establishes a link between the PDE's of traditional analysis and physics and diffusion processes in stochastic calculus. Solutions of PDE's can be interpreted as expectations of suitable transformations of solutions of stochastic differential equations and vice versa.

[3]See e.g Darrel Duffie (2001), Tomas Bjork (2009)

A.4 Zero Coupon Bond

The Zero Coupon Bond is the simplest interest rate derivative. It is a contract in which one party guarantees to the other party the payment of one unit of currency at maturity date T, with no other payments. The contract payoff at time T is thus denoted by $P(T;T) = 1$ and the contract value at time $t < T$ by $P(t;T)$. The dimension is currency (c) and the units are, e.g., Euro. Using the risk neutral pricing formula we have the pricing expression

$$P(t;T) = \mathbb{E}_t^{\mathbb{Q}}[D(t,T)P(T;T)] = \mathbb{E}_t^{\mathbb{Q}}[D(t,T)].$$

As for the bank account the dimensionality of the equation above is correct when one remembers that there is an hidden nominal amount $N = 1$ units of currency on the r.h.s.

Notice that the Zero Coupon Bond value, being the price of a contract between two counterparties, has to be exactly known at any time $t < T$, and thus it is a deterministic (not stochastic) quantity (being an expectation). This is the main difference with respect to the stochastic discount factor.

Both the stochastic discount factor $D(t,T)$ and the Zero Coupon Bond $P(t;T)$ "move" an amount of money backward in time. In financial terms we say that the amount of money is discounted from time T to time $t < T$, thus $D(t,T)$ and $P(t;T)$ are both called discount factors. The reciprocals $\frac{1}{D(t;T)}$ and $\frac{1}{P(t;T)}$ "move" an amount of money forward in time from t to $T > t$ and are called capitalization factors.

The main difference between the two types of discount/capitalization factors is that, in general, the Zero Coupon Bond is deterministic, while the stochastic discount factor is not. Thus, given a deterministic amount of money $N(T2)$ at time $T2$, we have

$$N(T_1) = P(T_1;T_2)N(T_2),$$

$$N'(T_1) = D(T_1;T_2)N(T_2) \neq N(T_1),$$

$$\mathbb{E}_{T_1}^{\mathbb{Q}}[N'(T_1)] = P(T_1;T_2)N(T_2) = N(T_1)$$

where $N(T_1)$ is deterministic and $N'(T_1)$ is, in general, stochastic.
In case of deterministic interest rates, we have $D(t,T) = P(t;T)$ and $N'(T_1) = N(T_1)$ are all deterministic quantities.

There exist a relationship between interest rates and Zero Coupon Bonds. Using the general expression

$$N(T_1) = P(T_1;T_2)N(T_2)$$

and the definitions of simple/discrete/continuous compounded rates give before, we obtain the following expressions

Interest rate	Expression in terms of Zero Coupon Bond
Simple compounding (Libor)	$L(T_1,T_2) = \frac{1}{\tau(T_1,T_2)}\left[\frac{1}{P(T_1;T_2)} - 1\right]$
Discrete compounding	$R_k(T_1,T_2) = \frac{k}{P(T_1;T_2)^{\frac{1}{k\tau(T_1,T_2)}}} - k$
Continuous compounding	$R_\infty(T_1,T_2) = -\frac{1}{\tau(T_1,T_2)} lnP(T_1;T_2)$

We may invert the preceding relations to express the Zero Coupon Bond in terms of the different interest rates

Interest rate	Expression in terms of Zero Coupon Bond
Simple compounding (Libor)	$P(T_1, T_2) = \frac{1}{1+L(T_1;T_2)\tau(T_1,T_2)}$
Discrete compounding	$P(T_1, T_2) = \frac{1}{\left[1+\frac{R_k(T_1,T_2)}{k}\right]^{k\tau(T_1,T_2)}}$
Continuous compounding	$P(T_1, T_2) = e^{-R_\infty(T_1,T_2)\tau(T_1,T_2)}$

The Zero Coupon Bond is particularly important in interest rate modeling because, similarly to the bank account, it can be used as reference asset (numeraire) to put into relation amounts of currencies observed at different times.

The value (price) of any asset π at any times t and $T > t$ can be expressed in units of $P(t, T)$ and $P(T, T)$, respectively, through the T-forward (Libor) pricing formula

$$\frac{\Pi(t)}{P(t;T)} = \mathbb{E}_t^{\mathbb{Q}^T}\left[\frac{\Pi(T)}{P(T;T)}\right] = \mathbb{E}_t^{\mathbb{Q}^T}[\Pi(T)]$$

$$\Pi(t) = P(t;T)\mathbb{E}_t^{\mathbb{Q}^T}[\Pi(T)]$$

where $\mathbb{Q}^{\,T}$ denotes the T-forward (Libor) measure associated to the numeraire $P(t;T)$.

A.5 Change of measure

Radon-Nikodym theorem

Definition A.4. Given any two measures $\mathbb{P}$, $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, we say that $\mathbb{Q}$ is $\mathbb{P}$-*absolutely continuous on* $\mathcal{F}$ if, for every $A \in \mathcal{F}$ such that $\mathbb{P}(A) = 0$, we have $\mathbb{Q}(A) = 0$. In this case we write $\mathbb{Q} \ll \mathbb{P}$ or $\mathbb{Q} \ll_{\mathcal{F}} \mathbb{P}$ if we want to highlight the σ-algebra that we are considering; indeed it is apparent that the notion of absolute continuity depends on the σ-algebra under consideration: if $\mathcal{G} \subseteq \mathcal{F}$ are σ-algebras, then $\mathbb{Q} \ll_{\mathcal{G}} \mathbb{P}$ does not necessarily imply that $\mathbb{Q} \ll_{\mathcal{F}} \mathbb{P}$.

Definition A.5. If $\mathbb{Q} \ll \mathbb{P}$ and $\mathbb{P} \ll \mathbb{Q}$, then we say that the measure $\mathbb{P}$ and $\mathbb{Q}$ are *equivalent* and we write $\mathbb{P} \approx \mathbb{Q}$. In case $\mathbb{P}$, $\mathbb{Q}$ are probability measures, $\mathbb{Q} \ll_{\mathcal{F}} \mathbb{P}$ implies that the $\mathbb{P}$-negligible events in $\mathcal{F}$ are also $\mathbb{Q}$-negligible, but the converse may not be true. Obviously, if $\mathbb{Q} \ll_{\mathcal{F}} \mathbb{P}$, then for every $A \in \mathcal{F}$ such that $P(A) = 1$ we have $\mathbb{Q} = 1$, i.e. the certain events for $\mathbb{P}$ are certain also for $\mathbb{Q}$, but the converse is not generally true.

Theorem A.6. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a finite-measure space. If* $\mathbb{Q}$ *is a finite measure on* $(\Omega, \mathcal{F})$ *and* $\mathbb{Q} \ll_{\mathcal{F}} \mathbb{P}$, *then there exists* $L : \Omega \to \mathbb{R}$, $L \geq 0$, *such that*

i) L *is* $\mathcal{F}$*-measurable;*

ii) L *is* $\mathbb{P}$*-integrable;*

iii) $\mathbb{Q}(A) = \int_A LdP$ *for every* $A \in \mathcal{F}$

Further, L *is* $\mathbb{P}$*-almost surely unique (i.e. if* L' *verifies the same properties of* L*, then* $\mathbb{P}(L = L') = 1$*). We say that* L *is the density of* $\mathbb{Q}$ *with respect to* $\mathbb{P}$ *on* $\mathcal{F}$ *or also the Radon-Nikodym derivative of* $\mathbb{Q}$ *with respect to* $\mathbb{P}$ *on* $\mathcal{F}$ *and we write without distinction* $L = \frac{d\mathbb{Q}}{d\mathbb{P}}$ *or* $d\mathbb{Q} = Ld\mathbb{P}$*. In order to emphasize the dependence on* $\mathcal{F}$*, we also write*

$$L = \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}} \tag{A.5.1}$$

Remark A.2. Let $\mathbb{P}$, $\mathbb{Q}$ be the probability measures on the space $(\Omega,\mathcal{F})$ with $\mathbb{Q} \ll \mathbb{P}$ and set $L = \frac{d\mathbb{Q}}{d\mathbb{P}}$. Using Dynkin's theorem, we can show that $X \in L^1(\Omega,\mathbb{Q})$ if and only if $XL \in L^1(\Omega,\mathbb{P})$ and in that case

$$\mathbb{E}^{\mathbb{Q}}[X] = \mathbb{E}^{\mathbb{P}}[XL] \tag{A.5.2}$$

where $\mathbb{E}^{\mathbb{P}}$ and $\mathbb{E}^{\mathbb{Q}}$ denote the expectations under the probability measures $\mathbb{P}$ and $\mathbb{Q}$ respectively. In other words

$$\int_\Omega X d\mathbb{Q} = \int_\Omega X \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right) d\mathbb{P}$$

and this justifies the notation (A.5.1).

On the probability space $(\Omega,\mathcal{F},\mathbb{P})$ we consider a sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$ and a probability measure $\mathbb{Q} \ll_{\mathcal{F}} \mathbb{P}$ (therefore also $\mathbb{Q} \ll_{\mathcal{G}} \mathbb{P}$). We denote by $L^{\mathcal{F}}$ (resp. $L^{\mathcal{G}}$) the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathcal{F}$ (resp. on $\mathcal{G}$). In general $L^{\mathcal{F}} \neq L^{\mathcal{G}}$ since $L^{\mathcal{F}}$ may not be $\mathcal{G}$-measurable. On the other hand, we have

$$L^{\mathcal{G}} = \mathbb{E}^{\mathbb{P}}[L^{\mathcal{F}}|\mathcal{G}]$$

Indeed $L^{\mathcal{G}}$ is integrable and $\mathcal{G}$-measurable and we have

$$\int_G L^{\mathcal{G}} dP = \mathbb{Q}(G) = \int_G L^{\mathcal{G}} dP, \quad G \in \mathcal{G}$$

since $\mathcal{G} \in \mathcal{F}$.
A result on the change of probability measure for conditional expectations, analogous to formula (A.5.2), is given by the following:

Theorem A.7 (Bayes' formula)**.** *Let $\mathbb{P}$, $\mathbb{P}$ be the probability measures on $(\Omega,\mathcal{F})$ with $\mathbb{Q} \ll_{\mathcal{F}} \mathbb{P}$. If $X \in L^1$ $(\Omega,\mathbb{Q})$, $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$ and we set $L = \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}}$, then we have*

$$\mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}] = \frac{\mathbb{E}^{\mathbb{P}}[XL|\mathcal{G}]}{\mathbb{E}^{\mathbb{P}}[L|\mathcal{G}]}$$

Proof. We put $V = \mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}]$ and $W = \mathbb{E}^{\mathbb{P}}[L|\mathcal{G}]$. We have to prove that

i) $\mathbb{Q}(W > 0) = 1$;

ii) $VW = \mathbb{E}^{\mathbb{P}}[XL|\mathcal{G}]$

Concerning i), since $\{W = 0\} \in \mathcal{G}$, we have

$$\mathbb{Q}(W = 0) = \int_{\{W=0\}} L dP = \int_{\{W=0\}} W dP = 0.$$

Concerning ii), VW is obviously $\mathcal{G}$-measurable and for every $G \in \mathcal{G}$ we have

$$\begin{aligned}\int_G VW dP &= \int_G \mathbb{E}^{\mathbb{P}}[VL|\mathcal{G}] dP = \int_G VL dP \\ &= \int_G \mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}] dQ = \int_G X dQ = \int_G XL dP\end{aligned}$$

□

Definition A.6. $\mathbb{Q}^{N_1}$, $\mathbb{Q}^{N_2}$ are *equivalent martingale measures* associated to the two numeraires N_1 and N_2 if

$$\frac{\Pi(t)}{N_1(t)} = \mathbb{E}_t^{\mathbb{Q}^{N_1}}\left[\frac{\Pi(T)}{N_1(T)}\right]$$
$$\frac{\Pi(t)}{N_2(t)} = \mathbb{E}_t^{\mathbb{Q}^{N_2}}\left[\frac{\Pi(T)}{N_2(T)}\right]$$

Comparing the two equations above we obtain

$$\Pi(t) = \mathbb{E}_t^{\mathbb{Q}^{N_1}}\left[\frac{N_1(t)}{N_1(T)}\Pi(T)\right] = \mathbb{E}_t^{\mathbb{Q}^{N_2}}\left[\frac{N_2(t)}{N_2(T)}\Pi(T)\right]$$

thus the change from measure $\mathbb{Q}^{N_1}$ to measure $\mathbb{Q}^{N_2}$ is given by

$$\mathbb{E}_t^{\mathbb{Q}^{N_2}}\left[\frac{\Pi(T)}{N_2(T)}\right] = \frac{N_1(t)}{N_2(t)}\mathbb{E}_t^{\mathbb{Q}^{N_1}}\left[\frac{\Pi(T)}{N_1(T)}\right]$$

Why changing the numeraire?

Let $X(t)$ the stochastic process underlying the payoff of the derivative Π:

- Suppose that $N_2(t)$ is a numeraire, a strictly positive tradable asset and $X(t)N_2(t)$ is the price of a tradable asset;
- In this case $\frac{[X(t)N_2(t)]}{N_2(t)}$ is a martingale under $\mathbb{Q}_2$, such that we may assume simple stochastic dynamics for it with simple distributions under $\mathbb{Q}_2$, e.g. lognormal martingale dynamics

$$\frac{dX(t)}{X(t)} = \sigma(t)dW^{\mathbb{Q}^{N_2}}(t)$$

$$lnX(T) \approx \mathcal{N}\left[lnX(t) - \frac{1}{2}\int_t^T \sigma^2(u)du; \int_t^T \sigma^2(u)du\right]$$

Now, if $\frac{\Pi(t)}{N_2(t)}$ is simple enough w.r.t. $\frac{\Pi(t)}{N_1(t)}$, we are able to compute its expectation under $\mathbb{Q}_2$.

$$\mathbb{E}_t^{\mathbb{Q}^{N_2}}\left[\frac{\Pi(T)}{N_2(T)}\right]$$

Example A.1 (Change between risk neutral and $T-$forward measures)**.** If we choose

$$\mathbb{Q}^{N_1} = \mathbb{Q}, \mathbb{Q}^{N_2} = \mathbb{Q}_T$$

$$N_1(t) = B(t), N_2(t) = P(t;T)$$

we obtain

$$\begin{aligned}\Pi(t) = \mathbb{E}_t^{\mathbb{Q}}[D(t,T)\Pi(T)] &= B(t)\mathbb{E}_t^{\mathbb{Q}}\left[\frac{\Pi(T)}{B(T)}\right]\\ &= B(t)\frac{P(t;T)}{B(t)}\mathbb{E}_t^{\mathbb{Q}^T}\left[\frac{\Pi(T)}{P(T;T)}\right]\\ &= P(t;T)\mathbb{E}_t^{\mathbb{Q}^T}[\Pi(T)]\end{aligned}$$

Example A.2 (Change between T_1 and T_2 forward measures)**.** If we choose

$$\mathbb{Q}^{N_1} = \mathbb{Q}^{T_1}, \mathbb{Q}^{N_2} = \mathbb{Q}^{T_2} \quad T_1 < T_2$$

$$N_1(t) = P(t;T_1), N_2(t) = P(t;T_2)$$

we obtain

$$\Pi(t) = \mathbb{E}_t^{\mathbb{Q}^{T_1}}\left[\frac{P(t;T_1)}{P(T';T_1)}\Pi(T')\right] = \mathbb{E}_t^{\mathbb{Q}^{T_2}}\left[\frac{P(t;T_2)}{P(T';T_2)}\Pi(T')\right]$$

$$t \leq T_1 \leq T_2 \leq T'$$

Example A.3 (Change between foreign and domestic risk neutral measures)**.** If we choose

$$\mathbb{Q}^{N_1} = \mathbb{Q}^f, \mathbb{Q}^{N_2} = \mathbb{Q}^d$$

$$N_1(t) = B_f(t), N_2(t) = B_d(t)$$

$$\Pi_d(t) = x_{fd}(t)\Pi_f(t)$$

where x_{fd} is the spot exchange rate from currency f to currency d, we obtain

$$\begin{aligned}\Pi_f(t) &= \mathbb{E}_t^{\mathbb{Q}^f}[D_f(t,T)\Pi_f(T)] \\ \Pi_d(t) &= \mathbb{E}_t^{\mathbb{Q}^d}[D_d(t,T)\Pi_d(T)] \\ &= x_{fd}(t)\mathbb{E}_t^{\mathbb{Q}^f}[D_f(t,T)\Pi_f(T)] \\ &= \mathbb{E}_t^{\mathbb{Q}^d}[D_d(t,T)x_{fd}(T)\Pi_f(T)]\end{aligned}$$

hence

$$\mathbb{E}_t^{\mathbb{Q}^f}[D_f(t,T)\Pi_f(T)] = \mathbb{E}_t^{\mathbb{Q}^d}\left[D_d(t,T)\frac{x_{fd}(T)}{x_{fd}(t)}\Pi_f(T)\right], \quad \forall t \leq T$$

Example A.4 (Change between risk neutral and forward swap measures)**.** If we choose

$$\mathbb{Q}^{N_1} = \mathbb{Q}, \mathbb{Q}^{N_2} = \mathbb{Q}_{\mathbb{S}}$$

$$N_1(t) = B(t), N_2(t) = A(t,\mathbb{S})$$

where $A(t,\mathbb{S}) = \sum_{i=1}^{n} P(t,S_i)\tau_k(S_{i-1},S_i)$, we obtain

$$\begin{aligned}\Pi(t) = \mathbb{E}_t^{\mathbb{Q}}[D(t,T)\Pi(T)] &= \mathbb{E}_t^{\mathbb{Q}}\left[\frac{B(t)}{B(T)}\Pi(T)\right] \\ &= A(t,\mathbb{S})\mathbb{E}_t^{\mathbb{Q}_{\mathbb{S}}}\left[\frac{\Pi(T)}{A(T,\mathbb{S})}\right], \quad \forall t \leq T \leq S_0\end{aligned}$$

.

The Girsanov theorem shows how a SDE changes due to changes in the underlying probability measure. It is based on the fact that the SDE drift depends on the particular probability measure $\mathbb{P}$ in our probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$, and that, if we change the probability measure in a "regular" way, the drift of the equation changes while the diffusion coefficient remains the same. The Girsanov theorem can be thus useful when we want to modify the drift coefficient of a SDE. Indeed, suppose that we are given two measures $\mathbb{P}^*$ and $\mathbb{P}$ on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t)$. Two such measures are said to be equivalent, written $\mathbb{P}^* \sim \mathbb{P}$, if they share the same sets of null probability (or of probability one, which is equivalent). Therefore two measures are equivalent when they agree on which events of $\mathcal{F}$ hold almost surely. Accordingly, a proposition holds almost surely under $\mathbb{P}$ if and only if it holds almost surely under $\mathbb{P}^*$. Similar definitions apply also for the measures restriction to $\mathcal{F}_t$, thus expressing equivalence of the two measures up to time t. When two measures are equivalent, it is possible to express the first in terms of the second through the Radon-Nikodym derivative. Indeed, there exists a martingale ρ_t on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ such that

$$\mathbb{P}^*(A) = \int_A \rho_t(w)d\mathbb{P}(w), \quad A \in \mathcal{F}_t$$

which can be written in a more concise form as

$$\left.\frac{d\mathbb{P}^*}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \rho_t$$

The process ρ_t is called the Radon-Nikodym derivative of $\mathbb{P}^*$ with respect to $\mathbb{P}$ restricted to $\mathcal{F}_t$. When in need of computing the expected value of an integrable random variable X, it may be useful to switch from one measure to another equivalent one. Indeed, it is possible to prove that the following equivalence holds:

$$\mathbb{E}^*[X] = \int_\Omega X(w)d\mathbb{P}^*(w) = \int_\Omega X(w)\frac{d\mathbb{P}^*}{d\mathbb{P}}(w)d\mathbb{P}(w) = \mathbb{E}\left[X\frac{d\mathbb{P}^*}{d\mathbb{P}}\right]$$

where $\mathbb{E}^*$ and $\mathbb{E}$ denote expected values with respect to the probability measures $\mathbb{P}^*$ and $\mathbb{P}$, respectively. More generally, when dealing with conditional expectations, we can prove that

$$\mathbb{E}^*[X|\mathcal{F}_t] = \frac{\mathbb{E}\left[X\frac{d\mathbb{P}^*}{d\mathbb{P}}|\mathcal{F}_t\right]}{\rho_t}$$

Theorem A.8. *Change of numeraire (Girsanov)*
Given two numeraires $N_1(t)$, $N_2(t)$, and a generic asset $X(t)$ following the stochastic diffusion processes under the martingale measure $\mathbb{Q}_1$ associated to N_1,

$$dX(t) = \mu_X^{Q_1}[t, X(t)] + \boldsymbol{\sigma}_X[t, X(t)]' \cdot d\boldsymbol{W}^{Q_1}(t)$$

$$dN_1(t) = \mu_{N_1}^{Q_1}[t, N_1(t)] + \boldsymbol{\sigma}_{N_1}[t, N_1(t)]' \cdot d\boldsymbol{W}^{Q_1}(t)$$

$$dN_2(t) = \mu_{N_2}^{Q_1}[t, N_2(t)] + \boldsymbol{\sigma}_{N_2}[t, N_2(t)]' \cdot d\boldsymbol{W}^{Q_1}(t)$$

$$dW_i^{Q_1}(t)dW_j^{Q_1}(t) = \rho_{ij}(t)dt, \quad i,j = 1, ..., F$$

where $\boldsymbol{W}(t)$ is a F-dimensional vector of correlated brownian motions under $\mathbb{Q}_1$ and the volatilities are F-dimensional vectors, with $1 \leq F \leq 3$. The dynamics of $X(t)$ under Q_2 is

$$dX(t) = \mu_X^{Q_2}[t, X(t)] + \boldsymbol{\sigma}_X[t, X(t)] \cdot d\boldsymbol{W}^{Q_2}(t)$$

$$\mu_X^{Q_2}[t, X(t)] = \mu_X^{Q_1}[t, X(t)] + \boldsymbol{\sigma}_X[t, X(t)] \cdot \boldsymbol{\rho}(t) \cdot \left[\frac{\boldsymbol{\sigma}_{N_2}[t, X(t)}{N_2(t)} - \frac{\boldsymbol{\sigma}_{N_1}[t, X(t)]}{N_1(t)}\right]'$$

$$d\boldsymbol{W}^{Q_2}(t) = d\boldsymbol{W}^{Q_1}(t) - \boldsymbol{\rho}(t) \cdot \left[\frac{\boldsymbol{\sigma}_{N_2}[t, X(t)}{N_2(t)} - \frac{\boldsymbol{\sigma}_{N_1}[t, X(t)]}{N_1(t)}\right]' dt$$

A.6 Replication

- **Market**:
  we assume a market M trading n assets $\boldsymbol{A}$ with price, cumulative dividend and cumulative gain processes

$$\boldsymbol{G}(t) = \boldsymbol{A}(t) + \boldsymbol{D}(t) = \begin{bmatrix} G_1(t) \\ \vdots \\ G_n(t) \end{bmatrix} = \begin{bmatrix} A_1(t) \\ \vdots \\ A_n(t) \end{bmatrix} + \begin{bmatrix} D_1(t) \\ \vdots \\ D_n(t) \end{bmatrix}$$

- **Asset price dynamics**:
  we assume Ito process under real measure P

$$d\boldsymbol{A}(t) = \mu(t, \boldsymbol{A})dt + \boldsymbol{\sigma}(t, \boldsymbol{A}) \cdot d\boldsymbol{W}^P(t), \quad \boldsymbol{A}(0) = \boldsymbol{A}_0$$

$$dW_i^P(t)dW_j^P(t) = \rho_{ij}dt, \quad \forall i,j = 1, ..., d$$

- **Cumulative dividend dynamics**:
  we assume continuous dividends proportional to asset $\boldsymbol{A}$ with instantaneous dividend rate $r_D(t)$

$$d\boldsymbol{D}(t) = \boldsymbol{A}(t)\boldsymbol{r}_D(t)dt, \quad \boldsymbol{D}(0) = \boldsymbol{0}$$

$$D(t) = \int_0^t \boldsymbol{A}(u)\boldsymbol{r}_D(u)du$$

- **Cumulative gain dynamics**:

$$d\boldsymbol{G}(t) = d\boldsymbol{A}(t) + d\boldsymbol{D}(t) = [\mu(t,\boldsymbol{A}) + \boldsymbol{r}_D(t)\boldsymbol{A}(t)]dt + \boldsymbol{\sigma}(t,\boldsymbol{A}) \cdot d\boldsymbol{W}^P(t)$$
$$\boldsymbol{G}(0) = \boldsymbol{A}(0) + \boldsymbol{D}(0) = \boldsymbol{G}_0$$

- **Derivative price dynamics**:
  the market trades also a derivative, with price at time t denoted by $\Pi(t)$, depending on the other assets A. The dynamics of the derivative price is obtained from Ito's lemma

$$\begin{aligned} d\Pi(t) &= \frac{\partial \Pi}{\partial t}dt + \frac{\partial \Pi'}{\partial \boldsymbol{A}} \cdot d\boldsymbol{A}(t) + \frac{1}{2}d\boldsymbol{A}(t)' \cdot \frac{\partial^2 \Pi}{\partial^2 \boldsymbol{A}} \cdot d\boldsymbol{A}(t) \\ &= \widehat{\mathfrak{L}}_\mu(t,\boldsymbol{A})\Pi(t)dt + \frac{\partial \Pi'}{\partial \boldsymbol{A}} \cdot \boldsymbol{\sigma}(t, bsA) \cdot d\boldsymbol{W}^P(t) \\ \widehat{\mathfrak{L}} &:= \frac{\partial}{\partial t} + \boldsymbol{\mu}(t,\boldsymbol{A})' \cdot \frac{\partial}{\partial \boldsymbol{A}} + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{f=1}^{d}\boldsymbol{\sigma}_{i,f}(t,\boldsymbol{A})\boldsymbol{\sigma}_{i,j}(t,\boldsymbol{A})\frac{\partial^2}{\partial A_i \partial A_j} \end{aligned}$$

- **Exchange rate**:
  since we want to deal with the multiple currency funding case, we introduce in M also a spot exchange rate at time t: $N^\alpha(t) = x^{\alpha,\beta}(t)N^\beta(t)$

- **Trading strategy**:
  a trading strategy is a (multidimensional) process $\boldsymbol{\Theta(t)}$ such that

$$\begin{aligned} V(t,\boldsymbol{\Theta},\boldsymbol{A}) &:= \boldsymbol{\Theta(t)} \cdot \boldsymbol{A}(t), \quad V(0,\boldsymbol{\Theta},\boldsymbol{A}) = \boldsymbol{\Theta}(0)' \cdot \boldsymbol{A}(0) := V_0(\boldsymbol{\Theta},\boldsymbol{A}) \\ G(t,\boldsymbol{\Theta},\boldsymbol{A}) &:= \int_0^t \boldsymbol{\Theta}(u)' \cdot d\boldsymbol{G}(u), \quad G(0,\boldsymbol{\Theta},\boldsymbol{A}) = 0 \\ D(t,\boldsymbol{\Theta},\boldsymbol{A}) &:= G(t,\boldsymbol{\Theta}) - [V(t,\boldsymbol{\Theta},\boldsymbol{A}) - V_0(\boldsymbol{\Theta},\boldsymbol{A})], \quad D(0,\boldsymbol{\Theta},\boldsymbol{A}) = 0 \end{aligned}$$

  are the price, cumulative gain and cumulative dividend processes, respectively. The components Θ of the trading strategy are interpreted as the number of units (or nominal) of the asset $\boldsymbol{A}$ held at time t.

- **Self financing**:
  a trading strategy $\boldsymbol{\Theta}$ is self-financing if its associated dividend process is null

$$D(t,\boldsymbol{\Theta},\boldsymbol{A}) = 0$$

$$\Rightarrow dG(t,\boldsymbol{\Theta},\boldsymbol{A}) = dV(t,\boldsymbol{\Theta},\boldsymbol{A})$$

  Intuitively, a self-financing strategy is such that it's cumulated gains are generated only by the changes in the asset prices, and no additional cash flows occur during it's life.

- **Replication**:
  a derivative price $\Pi(t)$ is replicated through a self-financing trading strategy $\Theta(t)$ if

$$\Pi(t, \boldsymbol{A}) = V(t, \boldsymbol{\Theta}, \boldsymbol{A}), \quad \forall t \in [0, T]$$

$$G_{\Pi}(t, \boldsymbol{A}) = G(t, \boldsymbol{\Theta}, \boldsymbol{A}), \quad \forall t \in [0, T]$$

  where $G_{\Pi}(t, A)$ is the cumulative gain process associated to derivative Π.

Bibliography

[I] Pascucci A., *PDE and Martingale Methods in Option Pricing*, Springer, Bocconi University Press, Milan, 2011

[II] Brigo D. and Mercurio F., *Interest Rate Models-Theory and Practice With Smile, Inflation and Credit*, Springer-Verlag, Berlin, 2006

[III] Moreni N. and Pallavicini A., *FX Modelling in Collateralized Markets: foreign measures, basis curves and pricing formulae*, http://papers.ssrn.com/sol3/results.cfm, 2015

[IV] Lesniewski A., *Interest Rate and Credit Models*, http://www.math.nyu.edu/ alberts/spring07/, 2007