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Corso di Laurea in Matematica

La trasformata di Fourier nella valutazione di opzioni

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Introduction

The aim of this thesis is to provide a systematic analysis of the conditions required for the *existence* of Fourier transform valuation formulas in a general framework: i.e. when the underlying variable can depend on the path of the price process and the payoff function can be discontinuous. For example when considering a one-touch option on a Lévy-driven asset, both assumptions fail: the payoff function is clearly discontinuous, while a priori not much is known about the existence of a density for the distribution of the supremum of a Lévy process. The key idea in Fourier transform methods for option pricing lies in the separation of the *underlying process* and the *payoff function*. In this paper there are conditions on the moment generating function of the underlying random variable and the Fourier transform of the payoff function such that Fourier based valuation formulas hold true.
An interesting interplay between the continuity conditions imposed on the payoff function and the random variable arises naturally. The results of our analysis can be briefly summarized as follows: for general continuous payoff functions or for variables, whose distribution has a Lebesgue density, the valuation formulas using Fourier transforms are valid as Lebesgue integrals. When the payoff function is discontinuous and the random variable might not possess a Lebesgue density then we get pointwise convergence of the valuation formulas under additional assumptions, that are typically satisfied. The valuation formulas allow to compute prices of European options very fast, hence they allow the efficient calibration of the model to market data for a large variety of driving processes, such as Lévy processes. Indeed, for

lévy and affine processes the moment generating function is usually known explicitly, hence these models are tailor-made for Fourier transform pricing formulas. This thesis is organized as follows:
in Chapter 1 we present valuation formulas in the single asset case.
In Chapter 2 we review examples of commonly used payoff functions in dimension one.
In Chapter 3 we review example of characteristic function.
Finally, in Chapter4 we provide numerical examples for the valuation of options and the difference between this model and Black-Sholes model.

Introduzione in Italiano

Lo scopo di questa tesi è di fornire una analisi sistematica delle condizioni necessarie per l'esistenza delle formule di valutazione che impiegano la trasformata di Fourier in un quadro generale: vale a dire quando la variabile sottostante può dipendere dal percorso del processo del prezzo del sottostante e la funzione di payoff può essere discontinua. Per esempio, quando consideriamo una opzione one-touch, entrambe le ipotesi falliscono: la funzione di payoff è chiaramente discontinua, mentre a priori non è molto noto circa l'esistenza di una densità per la distribuzione del massimo di un processo di Lévy. L'idea chiave dei metodi con trasformata di fourier per prezzare le opzioni si trova nella separazione del processo sottostante e della funzione di payoff. Il risultato di questa analisi può essere brevemente riassunto come segue: in generale per funzioni payoff continue o per le variabili, la cui distribuzione ha un densità di Lebesgue le formule di valutazione che utilizzano le trasformata di Fourier sono un integrale di Lebesgue. Quando, la funzione di payoff è discontinua e la variabile casuale puó non avere una densità di Lebesgue, ci serviamo di una convergenza puntuale delle formule di valutazione, in presenza di ulteriori ipotesi, che in genere sono soddisfatte. Questa tesi è organizzata come segue:

Nel Capitolo 1 si presentano le formule nel caso di un singolo sottostante.
Nel Capitolo 2 abbiamo esempi di comuni funzioni di payoff.
Nel Capitolo 3 abbiamo esempi di funzioni caratteristiche.
Infine, nel Capitolo 4 forniamo esempi numerici: per la valutazione delle opzioni e per la differenza tra questo modello e il modello di Black-Sholes.

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Chapter 1

Option valuation: single asset

In this paper I will analize the work of Eberlein, Glau and Papapantoleon on the valuation of option with Fourier transform methods.

1.1 Underlying process

We model the price process of a financial asset as an *exponential Lévy process* $S = (S_t)_{0 \le t \le T}$, i.e. a stochastic process with representation

$$S_t = S_0 \mathrm{e}^{H_t} \qquad 0 \le t \le T \tag{1.1}$$

(shortly: $S = S_0 \mathrm{e}^H$), where $H = (H_t)_{0 \le t \le T}$ is a Lévy process with $H_0 = 0$.
Throughout this work, we assume that P is an (equivalent) martingale measure for the asset S ; moreover, for simplicity we assume that the dividend yield are zero.
By no-arbitrage theory the price of an option on S is calculated as its discounted expected payoff.
We will analyze and prove valuation formulas for options on an asset $S = S_0 \mathrm{e}^H$ with a payoff at maturity T that may depend on the whole path of S up to time T.
In order to incorporate both plain vanilla options and exotic options in a single framework we separate the *payoff function* from the *underlying process*, where:

1. the *underlying process* can be the log-asset price process or the supremum/infimum of the log-asset price process. This process will always be denoted by X i.e. $X = H$ or $X = \overline{H}$ or $X = \underline{H}$, where $\overline{H}$ or $\underline{H}$ are the supremum/infimum of the log-asset price process.

2. the *payoff function* is an arbitrary function $f : \mathbb{R} \to \mathbb{R}_+ \cup \{0\}$, for example $f(x) = (\mathrm{e}^x - K)^+$ or $f(x) = 1_{\{\mathrm{e}^x > B\}}$, for $K, B \in \mathbb{R}_+ \cup \{0\}$.

Clearly, we regard options as dependent on the *underlying process* X, i.e. on (some functional of) the logarithm of the asset price process S. The main advantage is that the characteristic function of X is easier to handle than that of (some functional of) S; for example, for a Lévy process $H = X$ is already known in advance.
Moreover, we consider exactly those options where we can incorporate the path-dependence of the option payoff into the underlying process X. European vanilla options are a trivial example, as there is no path-dependence; a non-trivial, example are options on the supremum. Other examples are the geometric Asian option and forward-start options.
In addition, we will assume that the initial value of the underlying process X is zero; this is the case in all natural examples in mathematical finance. The initial value S_0 of the asset price process S plays a particular role, because it is convenient to consider the option price as a function of it, or more specifically as a function of s=log S_0.
Hence, we express a general payoff as

$$\Phi\big(S_0\mathrm{e}^{H_t},\, 0 \leq t \leq T\big) = f(X_T + s)\,, \tag{1.2}$$

where f is a payoff function and X is the underlying process, i.e. an adapted process, possibly depending on the full history of H, with

$$X_t := \Psi(H_s,\, 0 \leq s \leq t) \quad \text{for } t \in [0,T],$$

and Ψ a measurable functional. Therefore, the time-0 price of the option is provided by the (discounted) expected payoff, i.e.

$$\mathbb{V}_f(X; s) = E\big[\Phi\big(S_t,\, 0 \leq t \leq T\big)\big] = E\big[f(X_T + s)\big]\,. \tag{1.3}$$

Note that we consider 'European style' options, in the sense that the holder or writer does not have the right to exercise or terminate the option before maturity. In case the interest rate r is non-zero the option price is given by

$$\mathbb{V}_f(X;s) = \mathrm{e}^{-rT} E\big[f(X_T + s)\big] \tag{1.4}$$

1.2 Option valuation

1.2.1 Option with continuous payoff function

The first result focuses on options with continuous payoff functions, such as European plain vanilla options, but also lookback options.
Let P_{X_T} denote the law and φ_{X_T} the (extended) characteristic function of the random variable X_T; that is

$$\varphi_{X_T}(\xi) = \mathrm{e}^{-t\psi(\xi)} \tag{1.5}$$

we allow $\xi \in \mathbb{C}$ whenever the integral defining $\varphi_{X_T}(\xi)$ converges .
The characteristic function is the Fourier transform of the law:

$$\varphi_{X_T}(\xi) = \int_R \mathrm{e}^{i\xi x} P_{X_T}(dx)$$

For any payoff function f let f_R denote the *dampened* payoff function, defined via

$$f_R(x) = \mathrm{e}^{-Rx} f(x) \tag{1.6}$$

for some $R \in \mathbb{R}$. Let $\widehat{f_R}$ denote the (extended) Fourier transform of a function f_R.

Definition 1.1. For extended Fourier transform we consider

$$\widehat{f_R}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{i\xi x} f_R(x) dx \tag{1.7}$$

we allow $\xi \in \mathbb{C}$ whenever the integral defining above converges.

In order to derive a valuation formula for an option with an arbitrary *continuous* payoff function f, we will impose the following conditions.

(C1) Assume that $f_R, \widehat{f_R} \in L^1(\mathbb{R})$.

(C2) Assume that $E\big[S_T^R\big]$ is finite.

Theorem 1.2.1. *If the asset price process is modeled as an exponential Lévy process and conditions (C1)–(C2) are in force, then the time-0 price function is given by*

$$E\big[f(X_T+s)\big] = \frac{\mathrm{e}^{Rs}}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-i\xi s - T\psi(-(\xi+iR))}\widehat{f}(iR+\xi)d\xi \tag{1.8}$$

Proof. Using (1.3) and (1.6) we have

$$E\big[f(X_T+s)\big] = \int_\Omega f(X_T+s)dP = \mathrm{e}^{Rs}\int_R \mathrm{e}^{Rx} f_R(x+s)P_{X_T}(dx) \tag{1.9}$$

By assumption (C1), $f_R \in L^1(\mathbb{R})$, and the Fourier transform

$$\widehat{f_R}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{i\xi x} f(x)dx,$$

is well defined for every $\xi \in \mathbb{R}$.

Now for (C1) $\widehat{f_R} \in L^1(\mathbb{R})$ so, using the Inversion Theorem (cf. [Theorem A.37.]Pascucci07), $\widehat{f_R}$ can be inverted and f_R can be represented, for *all* $x \in \mathbb{R}$, as

$$f_R(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-ix\xi}\widehat{f_R}(u)d\xi. \tag{1.10}$$

Now, returning to the valuation problem (1.9) we get that

$$\begin{aligned}
E\big[f(X_T+s)\big] &= \mathrm{e}^{Rs}\int_{\mathbb{R}} \mathrm{e}^{Rx}\left(\frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-i(x+s)\xi}\widehat{f_R}(\xi)d\xi\right)P_{X_T}(\mathrm{d}x) \\
&= \frac{\mathrm{e}^{Rs}}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-i\xi s}\left(\int_{\mathbb{R}} \mathrm{e}^{-i(\xi+iR)x}P_{X_T}(\mathrm{d}x)\right)\widehat{f_R}(\xi)d\xi \\
&= \frac{\mathrm{e}^{Rs}}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-i\xi s}\varphi_{X_T}(-(\xi+iR)\widehat{f}(\xi+iR)d\xi \\
\text{Now for(1.5)} &= \frac{\mathrm{e}^{Rs}}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-i\xi s - T\psi(-(\xi+iR))}\widehat{f}(iR+\xi)d\xi
\end{aligned} \tag{1.11}$$

where for the second equality we have applied Fubini's theorem; moreover, for the Third equality we have

$$\widehat{f_R}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{i\xi x} f_R(x)dx = \int_{\mathbb{R}} \mathrm{e}^{i\xi x}\mathrm{e}^{-Rx} f(x)dx$$
$$= \int_{\mathbb{R}} \mathrm{e}^{i\xi x}\mathrm{e}^{-Rx} f(x)dx = \widehat{f}(\xi + iR)$$

Finally, for the application of Fubini's theorem we use again assumptions (C1) and (C2): indeed the summability is guaranteed by

$$\int_{\mathbb{R}} \mathrm{e}^{Rx} \int_{\mathbb{R}} |\mathrm{e}^{-i\xi(x+s)}\widehat{f_R}(\xi)| d\xi P_{X_T}(\mathrm{d}x) = ||\widehat{f_R}||_{L^1} E\big[\mathrm{e}^{RX_T}\big]$$

$\square$

Remark 1 Theorem 1.2.1 can be straightforwardly generalized to the multi-dimensional case.

Remark 2 Assumption (C1) implies that f is a continuous function. Theorem 1.2.3 below provides a pricing formula for discontinuous payoffs. Moreover (C2) is an integrability condition equivalent to

$$E\big[S_T^R\big] = \mathrm{e}^{Rs} E\big[\mathrm{e}^{RX_T}\big] = \mathrm{e}^{Rs} \int_{\mathbb{R}} \mathrm{e}^{Rx} P_{X_T}(dx) < \infty$$

that is, the measure $\mathrm{e}^{Rx} P_{X_T}(dx)$ is finite.

Remark 3 If we apply the Substitution of the variable ξ we can find that:

$$E\big[f(X_T + s)\big] = \frac{\mathrm{e}^{Rs}}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{ius - T\psi(-(-u+iR))} \widehat{f}(iR - u)du$$

where $u = -\xi$. This result derive since:

- The integral is on the whole real axis.
- $E\big[f(X_T + s)\big]$ is always positive.

We could also replace assumption (C1) with the following condition

$$\text{(C1}'\text{): } f_R \in L^1(\mathbb{R}) \quad \text{and} \quad \widehat{\mathrm{e}^{Rx} P_{X_T}} \in L^1(\mathbb{R}).$$

Proof. Using (1.3) and (1.6) we have

$$E\big[f(X_T+s)\big] = \int_\Omega f(X_T+s)dP = \mathrm{e}^{Rs}\int_R \mathrm{e}^{Rx} f_R(x+s)P_{X_T}(dx)$$

$$\widehat{\mathrm{e}^{Rx}P_{X_T}}(dx) = \int_\mathbb{R} \mathrm{e}^{iux}\mathrm{e}^{Rx}P_{X_T}(dx) = \int_\mathbb{R} \mathrm{e}^{i(u-iR)x}P_{X_T}(dx) = \varphi_{X_T}(u-iR)$$

Now we apply the inversion formula

$$\mathrm{e}^{Rx}P_{X_T}(dx) = \frac{1}{2\pi}\int_\mathbb{R} \mathrm{e}^{-ixu}\varphi_{X_T}(u-iR)\mathrm{d}u.$$

Now returning to the evaluation problem we get that

$$\begin{aligned}
E\big[f(X_T+s)\big] &= \mathrm{e}^{Rs}\int_\mathbb{R}\left(\frac{1}{2\pi}\int_\mathbb{R}\mathrm{e}^{-ixu}\varphi_{X_T}(u-iR)\mathrm{d}u\right)f_R(x+s)dx\\
&= \frac{\mathrm{e}^{Rs}}{2\pi}\int_\mathbb{R}\varphi_{X_T}(u-iR)\left(\int_\mathbb{R}\mathrm{e}^{-i(y-s)u}f_R(y)dy\right)\mathrm{d}u\\
&= \frac{\mathrm{e}^{Rs}}{2\pi}\int_\mathbb{R}\mathrm{e}^{ius}\left(\int_\mathbb{R}\mathrm{e}^{i(-u)y}f_R(y)dy\right)\varphi_{X_T}(u-iR)\mathrm{d}u\\
&= \frac{\mathrm{e}^{Rs}}{2\pi}\int_\mathbb{R}\mathrm{e}^{ius}\varphi_{X_T}(u-iR)\widehat{f}(iR-u)\mathrm{d}u.\\
&= \frac{\mathrm{e}^{Rs}}{2\pi}\int_\mathbb{R}\mathrm{e}^{ius-T\psi(u-iR))}\widehat{f}(iR-u)du
\end{aligned}$$

where for the second equality we have applied Fubini's theorem and we have changed the variable (x+s=y $\Longrightarrow dx = dy$).

And

$$\int_\mathbb{R}\mathrm{e}^{i(-u)y}f_R(y)dy = \int_\mathbb{R}\mathrm{e}^{i(-u)y}\mathrm{e}^(Ry)f(y)dy = \int_\mathbb{R}\mathrm{e}^{i(-u+i\mathbb{R})y}f(y)dy = \widehat{f}(iR-u)$$

Finally, the application of Fubini's theorem is justified since

$$\begin{aligned}
\int_\mathbb{R}\int_\mathbb{R}|\mathrm{e}^{-iux}||\varphi_{X_T}(u-iR)|\mathrm{d}u|f_R(x+s)|dx &\le \int_\mathbb{R}\left(\int_\mathbb{R}|\varphi_{X_T}(u-iR)|\mathrm{d}u\right)|f_R(x+s)|dx\\
&\le KK' < \infty,
\end{aligned}$$

where we have used Assumption (C1') $\square$

Apart from $\widehat{f_R} \in L^1(\mathbb{R})$, the prerequisites of Theorem 1.2.1 are quite easy to check in specific cases. In general, it is also an interesting question to know when the Fourier transform of an integrable function is integrable. The problem is well understood for smooth (C^2 or C^∞) functions, but the functions we are dealing with are typically not smooth. Hence, we will provide below an easy-to-check condition for a non-smooth function to have an integrable Fourier transform.
Let us consider the Sobolev space $W_2^1(\mathbb{R})$, with

$$W_2^1(\mathbb{R}) = \Big\{ g \in L^2(\mathbb{R}) \;\Big|\; \partial g \text{ exists and } \partial g \in L^2(\mathbb{R}) \Big\},$$

where ∂g denotes the *weak* derivative of a function g; .
Let $g \in W_2^1(\mathbb{R})$, then we get that

$$\widehat{\partial g}(u) = -iu\widehat{g}(u) \tag{1.12}$$

and $\widehat{g}, \widehat{\partial g} \in L^2(\mathbb{R})$.

Lemma 1.2.2. *Let $f_R \in W_2^1(\mathbb{R})$, then $\widehat{f_R} \in L^1(\mathbb{R})$.*

Proof. Using the above results, we have that

$$\infty > \int_{\mathbb{R}} \left(\big|\widehat{f}_R(u)\big|^2 + \big|\widehat{\partial f}_R(u)\big|^2 \right) \mathrm{d}u = \int_{\mathbb{R}} \big|\widehat{f}_R(u)\big|^2 \big(1 + |u|^2\big) \mathrm{d}u. \tag{1.13}$$

Now, by the Hölder inequality and (1.13), we get that

$$\begin{aligned} \int_{\mathbb{R}} \big|\widehat{f}_R(u)\big| \mathrm{d}u &= \int_{\mathbb{R}} \big|\widehat{f}_R(u)\big| \frac{1+|u|}{1+|u|} \mathrm{d}u \\ &\leq \left(\int_{\mathbb{R}} \big|\widehat{f}_R(u)\big|^2 (1+|u|)^2 \mathrm{d}u \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \frac{1}{(1+|u|)^2} \mathrm{d}u \right)^{\frac{1}{2}} < \infty \end{aligned}$$

and the result is proved. □

Example 1 (Call option): For a Call option we have

$$C(T, S_0, K) = \frac{K^{1-R} S_0^R}{2\pi} \int_{\mathbb{R}} \frac{\mathrm{e}^{-i\xi log \frac{S_0}{K} - T\psi(-(\xi + iR))}}{(i\xi - R)(1 + i\xi - R)}$$

Proof. By (1.8) and Section 2.1 □

1.2.2 Option with discontinuous payoff function

Next, we deal with the valuation formula for options whose payoff function can be *discontinuous*, while at the same time the measure P_{X_T} does *not* necessarily possess a Lebesgue density. Such a situation arises typically when pricing one-touch options in purely discontinuous Lévy models. Hence, we need to impose different conditions, and we derive the valuation formula as a pointwise limit.

In this subsections we will make use of the following notation; we define the function $\bar{f}_R$ and the measure ϱ as follows

$$\bar{f}_R(x) := f_R(-x) \quad \text{and} \quad \varrho(\mathrm{d}x) := \mathrm{e}^{Rx} P_{X_T}(\mathrm{d}x).$$

Moreover $\varrho(\mathbb{R}) = \int \varrho(\mathrm{d}x)$, while $\bar{f}_R * \varrho$ denotes the convolution of the function $\bar{f}_R$ with the measure ϱ. In this case we will use the following assumptions.

(D1) Assume that $f_R \in L^1(\mathbb{R})$.

(D2) Assume that $E\big[S_T^R\big]$ exists ($\Longleftrightarrow \varrho(\mathbb{R}) < \infty$).

(D3) Assume that the map $x \mapsto E\big[f(X_T + x)\big]$ is continuous at $-s$ and has bounded variation in a neighborhood of $-s$.

Theorem 1.2.3. *Let the asset price process be modeled as an exponential Lévy process and conditions (D1)–(D2) be in force. The time-0 price function is given by*

$$E\big[f(X_T + s)\big] = \frac{\mathrm{e}^{Rs}}{2\pi} \lim_{A\to\infty} \int_{-A}^{A} \mathrm{e}^{-i\xi s - T\psi(-(\xi+iR))} \widehat{f}(iR+\xi)d\xi \tag{1.14}$$

For the proof we use the following theorem:

Theorem 1.2.4. *(Jordan) If $f \in L^1(\mathbb{R})$ is of Bounded Variation in the interval $[a,b]$ then $\forall x \in]a,b[$*

$$\frac{1}{2}\big(f(x^+) + f(x^-)\big) = \frac{1}{2\pi} \lim_{A\to\infty} \int_{-A}^{A} \mathrm{e}^{-ixy} \widehat{f}(y)dy$$

Proof. Starting from (1.9), we can represent the option price function as a convolution of $\bar{f}_R$ and ϱ as follows

$$
\begin{aligned}
E\big[f(X_T+s)= & \, \mathrm{e}^{Rs}\int_{\mathbb{R}} \mathrm{e}^{Rx} f_R(x+s) P_{X_T}(\mathrm{d}x) \\
= & \, \mathrm{e}^{Rs}\int_{\mathbb{R}} \bar{f}_R(-s-x)\varrho(\mathrm{d}x) = \mathrm{e}^{Rs}\bar{f}_R * \varrho(-s).
\end{aligned} \tag{1.15}
$$

Using that $f_R \in L^1(\mathbb{R})$, hence also $\bar{f}_S \in L^1(\mathbb{R})$, and $\varrho(\mathbb{R}) < \infty$ we get that $\bar{f}_R * \varrho \in L^1(\mathbb{R})$, since

$$
\|\bar{f}_R * \varrho\|_{L^1(\mathbb{R})} \leq \varrho(\mathbb{R})\, \|\bar{f}_R\|_{L^1(\mathbb{R})} < \infty; \tag{1.16}
$$

compare with Young's inequality, (cf. [IV.1.6]Katznelson04).
Therefore, the Fourier transform of the convolution is well defined and we can deduce that, for all $u \in \mathbb{R}$,

$$
\widehat{\bar{f}_R * \varrho}(u) = \widehat{\bar{f}}_s(u) \cdot \widehat{\varrho}(u);
$$

By (1.16) we can apply the inversion theorem for the Fourier transform, (cf.Teorema(Jordan) 2-6 B. Pini) and get

$$
\frac{1}{2}\big(\bar{f}_R * \varrho(-s^+) + \bar{f}_R * \varrho(-s^-)\big) = \frac{1}{2\pi}\lim_{A\to\infty}\int_{-A}^{A} \mathrm{e}^{-i\xi s}\widehat{\varrho}(-\xi)\widehat{\bar{f}_R}(-\xi)d\xi, \tag{1.17}
$$

if there exists a neighborhood of $-s$ where $-s \mapsto \bar{f}_R * \varrho(-s)$ is of bounded variation.

We proceed as follows: first we show that the function $s \mapsto \bar{f}_R * \varrho(-s)$ has bounded variation; then we show that this map is also continuous, which yields that the left hand side of (1.17) equals $\bar{f}_R * \varrho(-s)$.

For that purpose, we re-write (1.15) as

$$
\bar{f}_R * \varrho(-s) = \mathrm{e}^{-Rs} E\big[f(X_T+s)\big], ;
$$

then, $\bar{f}_R * \varrho$ is of bounded variation on a compact interval $[a, b]$ if and only if $E\big[f(X_T+s)\big] \in BV([a,b])$; this holds because the map $s \mapsto \mathrm{e}^{-Rs}$ is of bounded variation on any bounded interval on $\mathbb{R}$, and the fact that the space

$BV([a,b])$ forms an algebra.
Moreover, $-s$ is a continuity point of $\bar{f}_R * \varrho$ if and only if $E\big[f(X_T + \cdot)\big]$ is continuous at $-s$.

In addition, we have that

$$\widehat{\bar{f}_R}(-\xi) = \int_{\mathbb{R}} \mathrm{e}^{-i\xi x}\mathrm{e}^{Rx} f-(x)\mathrm{d}x = \widehat{f}(iR+\xi) \tag{1.18}$$

and

$$\widehat{\varrho}(-\xi) = \int_{\mathbb{R}} \mathrm{e}^{-i\xi x}\mathrm{e}^{Rx} P_{X_T}(\mathrm{d}x) = \varphi_{X_T}(-\xi - iR) = e^{-T\psi(-(\xi+iR))} \tag{1.19}$$

Hence, (1.17) together with (1.18), (1.19) and the considerations regarding the continuity and bounded variation properties of the value function yield the required result. □

Example 2 (digital option) The payoff of a digital call option with barrier $B \in \mathbb{R}_+$ is $1_{\{\mathrm{e}^x > B\}}$ so

$$C(T, S_0, K) = \frac{B^{-R}S_0^R}{2pi} \lim_{A\to\infty} \int_{-A}^{A} -\frac{\mathrm{e}^{-i\xi log\frac{S_0}{B} - T\psi(-(\xi+iR))}}{i(\xi+iR)} d\xi \tag{1.20}$$

Proof. Just use Theorem 1.2.4 with fourier transform of the payoff evaluate in section 2.2 □

Chapter 2

Example of payoff functions

Here we list some representative examples of payoff functions used in finance, together with their Fourier transforms and comment on whether they satisfy some of the required assumptions for option pricing.

2.1 Call and Put Option

The payoff of the standard call option with strike $K \in \mathbb{R}_+$ is $f(x) = (\mathrm{e}^x - K)^+$. Let $z \in \mathbb{C}$ with $\Im z \in (1, \infty)$, then the Fourier transform of the payoff function of the call option is

$$\widehat{f}(z) = \int_{\mathbb{R}} \mathrm{e}^{izx}(\mathrm{e}^x - K)^+ \mathrm{d}x = \int_{-\infty}^{lnK} 0(\mathrm{e}^{izx})\mathrm{d}x + \int_{lnK}^{\infty} (\mathrm{e}^x - K)\mathrm{e}^{izx}\mathrm{d}x$$

$$= \int_{\ln K}^{\infty} \mathrm{e}^{(1+iz)x}\mathrm{d}x - K\int_{\ln K}^{\infty} \mathrm{e}^{izx}\mathrm{d}x$$

Now

$$\int_{\ln K}^{\infty} \mathrm{e}^{(1+iz)x}\mathrm{d}x = \frac{1}{1+iz}\int_{\ln K}^{\infty} \mathrm{e}^{(1+iz)x}(1+iz)\mathrm{d}x = \frac{1}{1+iz}\Big[\mathrm{e}^{(iz+1)x}\Big]_{lnK}^{\infty}$$

Now we use $\Im z \in (1, \infty)$ so

$$= 1 + iz\left[0 - \mathrm{e}^{(iz+1)lnK}\right] = -\frac{K^{iz+1}}{iz+1}$$

and

$$-K\int_{\ln K}^{\infty} \mathrm{e}^{izx}\mathrm{d}x = -\frac{K}{iz}\int_{\ln K}^{\infty} iz(\mathrm{e}^{izx})\mathrm{d}x = -\frac{K}{iz}\left[\mathrm{e}^{izx}\right]_{lnK}^{\infty}$$

Now we use $\Im z \in (1, \infty)$ so

$$= -\frac{K}{iz}\left[0 - K^{iz}\right] = K^{iz}\frac{K}{iz}$$

Finally

$$\widehat{f}(z) = -\frac{K^{iz+1}}{iz+1} + K^{iz}\frac{K}{iz} = \frac{K^{1+iz}}{iz(1+iz)}. \tag{2.1}$$

Now, regarding the dampened payoff function of the call option, we easily get for $R \in (1, \infty)$ that $f_R \in L^1_{\mathrm{bc}}(\mathbb{R}) \cap L^2(\mathbb{R})$ (where $L^1_{\mathrm{bc}}(\mathbb{R})$ is the space of bounded and continuous function in L^1). The weak derivative of f_R is

$$\partial f_R(x) = \begin{cases} 0, & \text{if } x < \ln K, \\ \mathrm{e}^{-Rx}(\mathrm{e}^x - R\mathrm{e}^x + RK), & \text{if } x > \ln K. \end{cases} \tag{2.2}$$

Again, we have that $\partial f_R \in L^2(\mathbb{R})$. Therefore, $f_R \in W^1_2(\mathbb{R})$ and using Lemma 1.2.2 we can conclude that $\widehat{f_R} \in L^1(\mathbb{R})$. Summarizing, condition (C1) of Theorem 1.2.1 is fulfilled for the payoff function of the call option.

Similarly, for a put option, where $f(x) = (K - \mathrm{e}^x)^+$, we have that

$$\widehat{f}(z) = \frac{K^{1+iz}}{iz(1+iz)}, \qquad \Im z \in (-\infty, 0). \tag{2.3}$$

Analogously to the case of the call option, we can conclude for the dampened payoff function of the put option that $f_R \in L^1_{\mathrm{bc}}(\mathbb{R})$ and $f_R \in W^1_2(\mathbb{R})$ for $R < 0$, yielding $\widehat{f_R} \in L^1(\mathbb{R})$. Hence, condition (C1) Theorem 1.2.1 is also fulfilled for the payoff function of the put option.

2.2 Digital Option

The payoff of a digital call option with barrier $B \in \mathbb{R}_+$ is $1_{\{e^x > B\}}$. Let $z \in \mathbb{C}$ with $\Im z \in (0, \infty)$, then the Fourier transform of the payoff function of the digital call option is

$$\widehat{f}(z) = \int_{\mathbb{R}} e^{izx} 1_{\{e^x > B\}} dx = \int_{-\infty}^{lnB} 0 e^{izx} dx + \int_{lnB}^{\infty} e^{izx} dx$$

$$= \frac{1}{iz} \int_{lnB}^{\infty} ize^{izx} dx = \frac{1}{iz} \left[e^{izx} \right]_{lnB}^{\infty}$$

Now we use $\Im z \in (0, \infty)$ so

$$= \frac{1}{iz} \left[0 - B^{iz} \right] = -\frac{B^{iz}}{iz} \tag{2.4}$$

Similarly, for a digital put option, where $f(x) = 1_{\{e^x < B\}}$, we have that

$$\widehat{f}(z) = \frac{B^{iz}}{iz}, \qquad \Im z \in (-\infty, 0). \tag{2.5}$$

For the dampened payoff function of the digital call and put option, we can easily check that $f_R \in L^1(\mathbb{R})$ for $R \in (0, \infty)$ and $R \in (-\infty, 0)$.

2.3 Asset-or-Nothing Digital Option

A variant of the digital option is the so-called *asset-or-nothing digital*, where the option holder receives one unit of the *asset*, instead of *currency*, depending on whether the underlying reaches some barrier or not. The payoff of the asset-or-nothing digital call option with barrier $B \in \mathbb{R}_+$ is $f(x) = \mathrm{e}^x 1_{\{\mathrm{e}^x > B\}}$, and the Fourier transform, for $z \in \mathbb{C}$ with $\Im z \in (1, \infty)$, is

$$\widehat{f}(z) = \int_{\mathbb{R}} \mathrm{e}^{izx} 1_{\{\mathrm{e}^x > B\}} \mathrm{d}x = \int_{-\infty}^{lnB} 0 \mathrm{e}^{izx} \mathrm{e}^x \mathrm{d}x + \int_{lnB}^{\infty} \mathrm{e}^{izx} \mathrm{e}^x dx$$

$$= \frac{1}{1+iz} \int_{lnB}^{\infty} (1+iz) \mathrm{e}^{(1+iz)x} \mathrm{d}x = \frac{1}{1+iz} \left[\mathrm{e}^{(1+iz)x} \right]_{lnB}^{\infty}$$

Now we use $\Im z \in (1, \infty)$ so

$$= \frac{1}{1+iz} \left[0 - B^{1+iz} \right] = -\frac{B^{1+iz}}{1+iz} \tag{2.6}$$

Similarly, for a asset-or-nothing digital put option, where $f(x) = \mathrm{e}^x 1_{\{\mathrm{e}^x < B\}}$, we have that

$$\widehat{f}(z) = \frac{B^{1+iz}}{1+iz}, \qquad \Im z \in (-\infty, 0). \tag{2.7}$$

2.4 Double Digital Option

The payoff of the double digital call option with barriers $\underline{B}, \overline{B} > 0$ is $1_{\{\underline{B}<\mathrm{e}^x<\overline{B}\}}$. Let $z \in \mathbb{C}\setminus\{0\}$, then the Fourier transform of the payoff function is

$$\widehat{f}(z) = \int_{ln\underline{B}}^{ln\overline{B}} \mathrm{e}^{izx}\mathrm{d}x = \frac{1}{iz}\left[\mathrm{e}^{izx}\right]_{ln\underline{B}}^{ln\overline{B}} = \frac{1}{iz}\left(\overline{B}^{iz} - \underline{B}^{iz}\right) \tag{2.8}$$

The dampened payoff function of the double digital option satisfies $g \in L^1(\mathbb{R})$ for all $R \in \mathbb{R}$.

Moreover, we can decompose the value function of the double digital option as

$$E\big[f(X_T + s)\big] = E\big[f_1(X_T + s)\big] - E\big[f_2(X_T + s)\big]$$

where $f_1(x) = 1_{\{\mathrm{e}^x<\overline{B}\}}$ and $f_2(x) = 1_{\{\underline{B}\leq\mathrm{e}^x\}}$.

2.5 Self-Quanto Option

The payoff of a self-quanto call option with strike $K \in \mathbb{R}_+$ is $f(x) = \mathrm{e}^x(\mathrm{e}^x - K)^+$. Let $z \in \mathbb{C}$ with $\Im z \in (2, \infty)$, then the Fourier transform of the payoff function of the self-quanto call option is

$$\widehat{f}(z) = \int_{lnK}^{\infty} \mathrm{e}^{izx}\mathrm{e}^x(\mathrm{e}^x - K)\mathrm{d}x = \int_{lnK}^{\infty} \mathrm{e}^{(2+iz)x}\mathrm{d}x + \int_{lnK}^{\infty} -K\mathrm{e}^{(iz+1)x}\mathrm{d}x$$

Now if $\Im z \in (2, \infty)$ the first integral is

$$\frac{1}{2+iz}K^{iz+2}$$

and the second integral is

$$\frac{1}{1+iz}K^{iz+1}$$

so

$$\widehat{f}(z) = \frac{K^{2+iz}}{(1+iz)(2+iz)}. \tag{2.9}$$

Similarly, for a self-quanto put option, where $f(x) = \mathrm{e}^x(K - \mathrm{e}^x)^+$, we get

$$\widehat{f}(z) = \frac{K^{2+iz}}{(1+iz)(2+iz)}, \qquad \Im z \in (-\infty, 1).$$

Analogously to the case of the call and put option, we can conclude for the dampened payoff function of the self-quanto option that $f_R \in L^1_{\mathrm{bc}}(\mathbb{R}) \cap W^1_2(\mathbb{R})$ for $R \in (2, \infty)$ and $R \in (-\infty, 1)$ respectively; hence $\widehat{f_R} \in L^1(\mathbb{R})$ in both cases. Summarizing, condition (C1) of Theorem 1.2.1 is fulfilled for the payoff function of the self-quanto option.

2.6 Power Option

The payoff of a power call option with strike $K \in \mathbb{R}_+$ and power 2 is $f(x) = [(\mathrm{e}^x - K)^+]^2$. Let $z \in \mathbb{C}$ with $\Im z \in (2, \infty)$, then the Fourier transform of the payoff function of the power call option is

$$\widehat{f}(z) = \int_{lnK}^{\infty} \mathrm{e}^{izx}(\mathrm{e}^x + K)^2 \mathrm{d}x = \int_{lnK}^{\infty} \mathrm{e}^{(iz+2)x} + K^2\mathrm{e}^{izx} - 2K\mathrm{e}^{(iz+1)x}\mathrm{d}x$$

$$= \frac{1}{iz+2}\Big[\mathrm{e}^{(iz+2)x}\Big]_{lnK}^{\infty} + \frac{K^2}{iz}\Big[\mathrm{e}^{izx}\Big]_{lnK}^{\infty} - \frac{2K}{iz+1}\Big[\mathrm{e}^{(iz+1)x}\Big]_{lnK}^{\infty}$$

Now we use $\Im z \in (2, \infty)$ so

$$= \frac{1}{iz+2}K^{iz+2} + \frac{K^2}{iz}K^{iz} - \frac{2K}{iz+1}K^{iz+1} = \frac{2K^{2+iz}}{iz(1+iz)(2+iz)} \tag{2.10}$$

Similarly, for a power put option, where $f(x) = [(K - \mathrm{e}^x)^+]^2$, we get

$$\widehat{f}(z) = -\frac{2K^{2+iz}}{iz(1+iz)(2+iz)}, \qquad \Im z \in (-\infty, 0). \tag{2.11}$$

Once again, we can easily conclude for the dampened payoff function of the power option that $f_R \in L^1_{\mathrm{bc}}(\mathbb{R}) \cap W^1_2(\mathbb{R})$ for $R \in (2, \infty)$ and $R \in (-\infty, 0)$ respectively; hence $\widehat{f}_R \in L^1(\mathbb{R})$ in both cases. Summarizing, condition (C1) of Theorem 1.2.1 is fulfilled for the payoff function of the power call and put option.

Chapter 3

Example of characteristic function

In probability theory and statistics, the characteristic function of any random variable completely defines its probability distribution. Thus it provides the basis of an alternative route to analytical results compared with working directly with probability density functions or cumulative distribution functions.

In addition to univariate distributions, characteristic functions can be defined for vector- or matrix-valued random variables, and can even be extended to more generic cases.

The characteristic function always exists when treated as a function of a real-valued argument, unlike the moment-generating function. There are relations between the behavior of the characteristic function of a distribution and properties of the distribution, such as the existence of moments and the existence of a density function. The characteristic function provides an alternative way for describing a random variable. Similarly to the cumulative distribution function

$$F_X(x) = \mathrm{E}\ \mathbf{1}_{\{X \leq x\}}$$

which completely determines behavior and properties of the probability distribution of the random variable X, the characteristic function

$$\varphi_X(t) = \mathrm{E}[\mathrm{e}^{itX}]$$

also completely determines behavior and properties of the probability distribution of the random variable X. The two approaches are equivalent in the sense that knowledge of one of the functions can always be used in order to find the other one, yet they both provide different insight for understanding the features of our random variable. However, in particular cases, there can be differences in whether these functions can be represented as expressions involving simple standard functions.
If a random variable admits a density function, then the characteristic function is its dual, in the sense that each of them is a Fourier transform of the other. If a random variable has a moment-generating function, then the characteristic function can be extended to the complex domain so that

$$\varphi_X(-it) = M_X(t).$$

Note however that the characteristic function of a distribution always exists, even when the probability density function or moment-generating function does not.

Definition 3.1. For a scalar random variable X the characteristic function is defined as the expected value of e^{itX}, where i is the imaginary unit, and $t \in \mathbb{R}$ is the argument of the characteristic function:

$$\varphi_X : \mathbb{R} \to \mathbb{C}; \quad \varphi_X(t) = \mathrm{E}\left[\mathrm{e}^{itX}\right] = \int_{-\infty}^{\infty} \mathrm{e}^{itx}\, dF_X(x) \qquad \left(= \int_{-\infty}^{\infty} \mathrm{e}^{itx} f_X(x)\, dx\right)$$

Here F_X is the cumulative distribution function of X, and the integral is of the Riemann-Stieltjes kind. If random variable X has a probability density function f_X, then the characteristic function is its Fourier transform, and the last formula in parentheses is valid.

3.1 CGMY Model

Let $H = (H_t)_{0\leq t\leq T}$ be a CGMY Lévy process, another name for this process is (generalized) tempered stable process.
The characteristic function of H_t, $t \in [0, T]$, is

$$\varphi_{H_t}(u) = \exp\Big(tC\,\Gamma(-Y)\big[(M-iu)^Y + (G+iu)^Y - M^Y - G^Y\big]\Big) \tag{3.1}$$

for $Y \neq 0$ where the parameter space is $C, G, M > 0$ and $Y \in (-\infty, 2)$. and the moment generating function exists for $R \in \mathcal{I} = [-G, M]$. The sample paths of the CGMY process have unbounded variation if $Y \in [1, 2)$, bounded variation if $Y \in (0, 1)$, and are of compound Poisson type if $Y < 0$.

3.2 Normal distribution

For the standard normal random variable, the characteristic function is

$$\varphi(u) = \int_{-\infty}^{\infty} \mathrm{e}^{iux}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}x^2}dx = \mathrm{e}^{-\frac{1}{2}u^2}. \tag{3.2}$$

For a generic normal distribution with mean μ and variance σ^2, the characteristic function is

$$\varphi(u;\,\mu,\sigma^2) = \mathrm{E}[\mathrm{e}^{iu\mathcal{N}(\mu,\sigma^2)}] = \mathrm{e}^{i\mu u-\frac{1}{2}\sigma^2u^2}. \tag{3.3}$$

Chapter 4

Application

Assume we are interested in pricing a *European option* on the asset $S_T = S_0 \mathrm{e}^H$, e.g. a call, a put or a digital option. Then, it is sufficient to know the characteristic function of the random variable $X_T \equiv H_T$, and H_T must possess a moment generating function for $R \in \mathcal{I}$ with $\mathcal{I} \subseteq \mathbb{R}$.
Examples of options that can be treated include plain vanilla call and put options with payoff $(S_T - K)^+$ and $(K - S_T)^+$, digital cash-or-nothing and asset-or-nothing options, with payoffs $1_{\{S_T > B\}}$ and $S_T 1_{\{S_T > B\}}$, double digital options, with payoff $1_{\{\underline{B} < S_T < \overline{B}\}}$, self-quanto and power options. Below we describe some characteristic examples of models used in mathematical finance.

4.1 Numerical evaluation: CGMY model

As an illustration of the applicability of Fourier-based valuation formulas we present a numerical example on the pricing of a call option. As driving motion we consider CGMY model.

4.1.1 $C(K,t)$

From Theorem 1.2.1 (with no-zero interest rate r) and from (2.1) we obtain

$$C(t, S_0, K) = \frac{\mathrm{e}^{-rT} K^{1-R} S_0^R}{2\pi} \int_{\mathbb{R}} \frac{\mathrm{e}^{-i\xi log \frac{S_0}{K} - T\psi(-(\xi+iR))}}{(i\xi - R)(1 + i\xi - R)} \tag{4.1}$$

where ψ is the characteristic exponent function (3.1)

$$\psi(\xi) = \Big(- C\,\Gamma(-Y)\big[(M - i\xi)^Y + (G + i\xi)^Y - M^Y - G^Y\big]\Big)$$

The choice of parameters (CGMY) is based on *Carr, Peter, Geman, Hélyette, Madan, Dilip B., Yor, Marc (2002). "The fine structure of asset returns: an empirical investigation".*
The interest rate r=0.05.
For the *implementation* we use MATLAB

```
%%%%%%%%%%%%CGMY_call_option.m%%%%%%%%%%%%%

close all
clear all

% parametri
% Y in [-inf,2] (interessante in [1,2])
% scelta 1
Y = 1.50683;
C = 0.08;
G = 25.04;
M = 25.04;
% R in [-G,M]
R = 2;
r = 0.05;
S0 = 100;
% tempi
```

```
t = linspace(0,1,20);
% strike
k = linspace(85,135,20);
% integrazione numerica
    lt = length(t);
    lk = length(k);
    V = zeros(lt,lk);
    for i=1:lt
        for j=1:lk
            V(i,j) = CGMY_value1(t(i),k(j),Y,C,G,M,R,r,S0);
        end
    end

[K,T] = meshgrid(k,t);

surf(K,T,Q)
xlabel('k')
ylabel('t')
zlabel('V')
```

where the function value$(t(i), k(j), Y, C, G, M, R, r, S0)$ is

```
%%%%%%%%%%%%CGMY_value.m%%%%%%%%%%%%%
function [V] = CGMY_value1 (t,k,Y,C,G,M,R,r,S0)

        % integrazione numerica

        % estremi di integrazione
        a = -100000;
        b = 100000;

        h = exp(-r.*t)*k^(1-R).*S0^R./(2*pi);
```

```
        V = h*quad(@(u)CGMY_integrand1(u,t,k,C,G,M,Y,R,S0),a,b);
end
```

where CGMY_integrand1 is

```
%%%%%%%%CGMY_integrand1.m%%%%%%%%%%%%
function [y] = CGMY_integrand1 (u,t,k,C,G,M,Y,R,S0)

psi = CGMY_characteristic_exp1(+u-1i*R,C,G,M,Y);
y = exp(+1i.*u.*log(S0./k)-t.*psi)./((-1i.*u-R).*(1-1i.*u-R));
```

where characteristic_exp1 is

```
%%%%%%%%CGMY_charcteristic_exp1.m%%%%%%%%%%
function [psi] = CGMY_characteristic_exp (u,C,G,M,Y)

psi = -C*gamma(-Y).*((M-1i.*u).^Y+(G+1i.*u).^Y-M^Y-G^Y
```

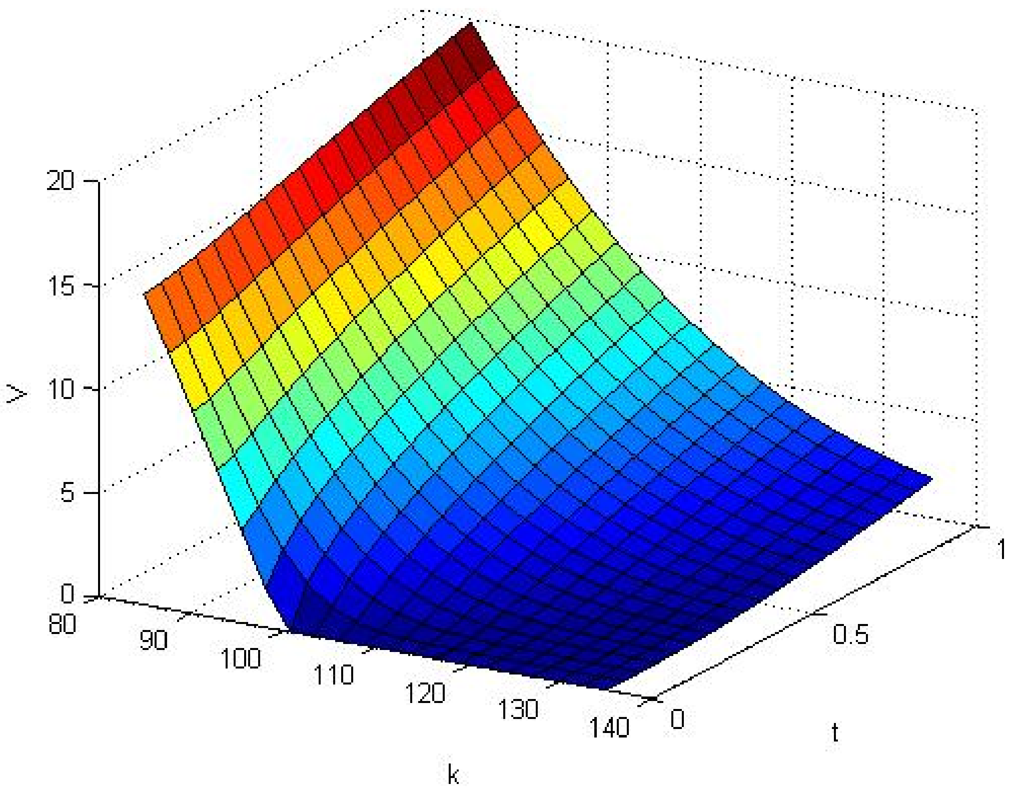

Figure 4.1: Call Price in the CGMY model

4.1.2 $C(K, R)$

With small changes in `CGMY_call_option1.m` we can see how, for fixed t, the price of an option changes with respect to the dampening coefficent R and the Strike K.

```
%%%%%%%%%%%%%CGMY_call_option_R.m%%%%%%%%%%%%%

close all
clear all


% parametri
% Y in [-inf,2] (interessante in [1,2])
```

```
Y = 1.50683;
C = 0.08;
G = 25.04;
M = 25.04;
% interest rate
r = 0.05;
%tempo
t = 0.5;
S0 = 100;

% coefficente di penalizzazione
R = linspace(1.1,25,20);
% strike
k = linspace(85,130,20);
% integrazione numerica

    lR = length(R);
    lk = length(k);
    V = zeros(lR,lk);
    for i=1:lR
        for j=1:lk
            V(i,j) = CGMY_value1(t,k(j),Y,C,G,M,R(i),r,S0);
        end
    end

[K,RR] = meshgrid(k,R);

surf(K,RR,V)
xlabel('k')
ylabel('R')
zlabel('V')
```

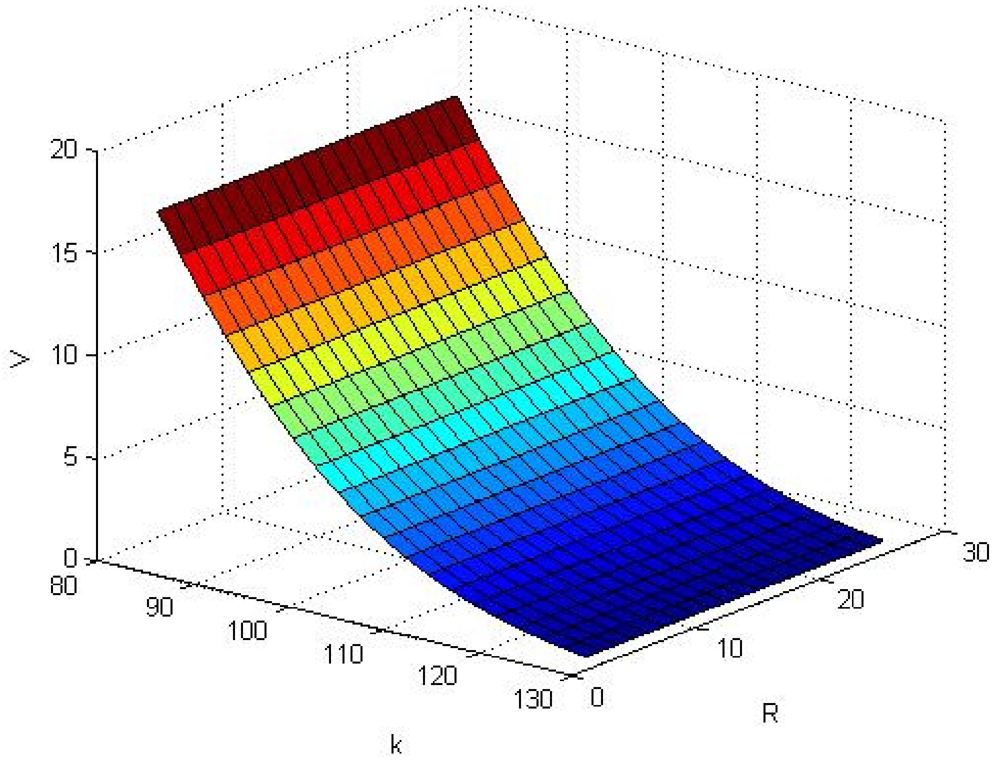

Figure 4.2: Call Price changes respect to R

4.2 Fourier transform valuation Vs Black-Sholes model

In this section we want to see the difference between the valuation of the price of a call option using Black-Sholes formula and the valuation using Fourier transform method. With a little modification of our implementation we can see this difference.
First of all we change the `characteristic_exp.m` and we put inside the characteristic function of a normal distribution:

$$\varphi(u;\, \mu, \sigma^2) = \mathrm{E}[\mathrm{e}^{iu\mathcal{N}(\mu,\sigma^2)}] = \mathrm{e}^{i\mu u - \frac{1}{2}\sigma^2 u^2}$$

In Black-Sholes model $S_T = S_0 \mathrm{e}^{X_T}$ where $X_T \sim \mathcal{N}((r - \frac{\sigma^2}{2})t, \sigma^2 t)$.
So the characteristic function becomes

$$\varphi_{H_T}(u) = \mathrm{e}^{-t\psi(u)}$$

Where

$$\psi(u) = -iu(r - \frac{\sigma^2}{2}) + \frac{1}{2}\sigma^2 u^2 \tag{4.2}$$

We choose $r = 0.05$ & $\sigma = 0.30$

```
%%%%%%%%Characteristic _exp1.m%%%%%%%%%
function [psi] = characteristic_exp (u,r,d)
psi = -1i*u.*(r-0.5*d^2)+0.5*d^2.*u.^2;
```

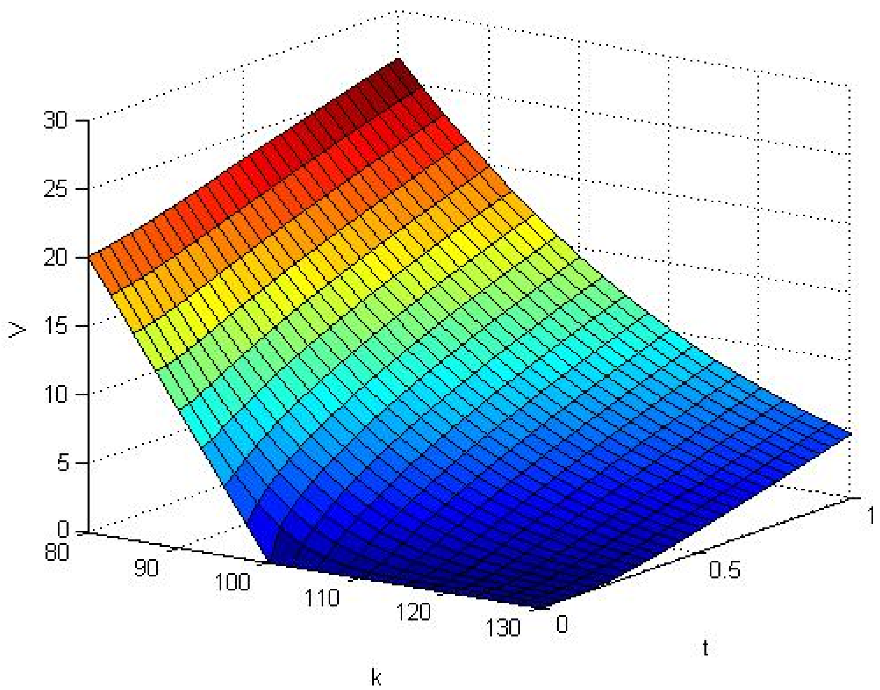

Figure 4.3: Call price with characteristic function of a normal distribution

With the same changes made in 4.1.2 we can see how, for fixed t, the price of an option changes with respect to the dampening coefficent R and the Strike K.

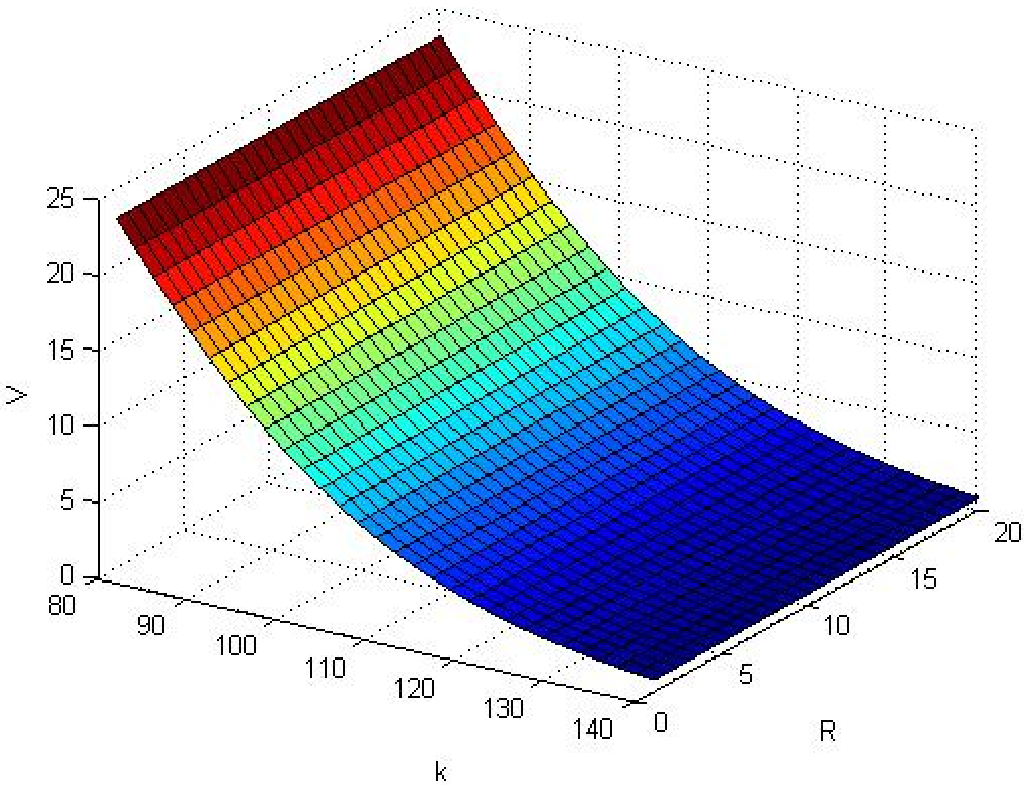

Figure 4.4: Call price with characteristic function of a normal distribution changes respect to R

Finally we want to see the Approximation error between `V` (Price of an option with Fourier transform method) and `price` (Price of an option with Black-Sholes formula) :

$$\frac{V - price}{price} \tag{4.3}$$

The value of a call option in terms of the Black-Scholes parameters is:

$$C(S,t) = SN(d_1) - Ke^{-r(t)}N(d_2)$$
$$d_1 = \frac{\ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})(t)}{\sigma\sqrt{t}}$$
$$d_2 = d_1 - \sigma\sqrt{t}.$$

where:

$N()$ is the cumulative distribution function of the standard normal distribution

t is the time to maturity

S is the spot price of the underlying asset

K is the strike price

r is the risk free rate (annual rate, expressed in terms of continuous compounding)

σ is the volatility in the log-returns of the underlying.

So we modify `Call_option.m` in following way:

```
%%%%%%%%%%call_option.m%%%%%%%%%%
close all
clear all

% parametri
r = 0.05;
```

```
d = 0.3;
% R coefficente di penalizzazione
R = 2;
S0 = 100;
% tempi
t = linspace(0.1,2,60);
% strike
k = linspace(70,110,51);
% integrazione numerica

lt = length(t);
lk = length(k);
V = zeros(lt,lk);
d1 = zeros(lt,lk);
d2 = zeros(lt,lk);
price = zeros(lt,lk);
for i=1:lt
for j=1:lk
V(i,j) = real(value(t(i),k(j),r,d,R,S0));
d1(i,j)=(log(S0./k(j))+(r+(d^2)/2).*(t(i)))/(d.*sqrt(t(i)));
d2(i,j)=d1(i,j)-d.*sqrt(t(i));
price(i,j)=S0.*normcdf(d1(i,j))+
                 -k(j).*exp(-r*(t(i))).*normcdf(d2(i,j));
end
end

[K,T] = meshgrid(k,t);

surf(K,T,(V-price)./price)
xlabel('k')
ylabel('t')
```

```
zlabel('(V-price)/price')
```

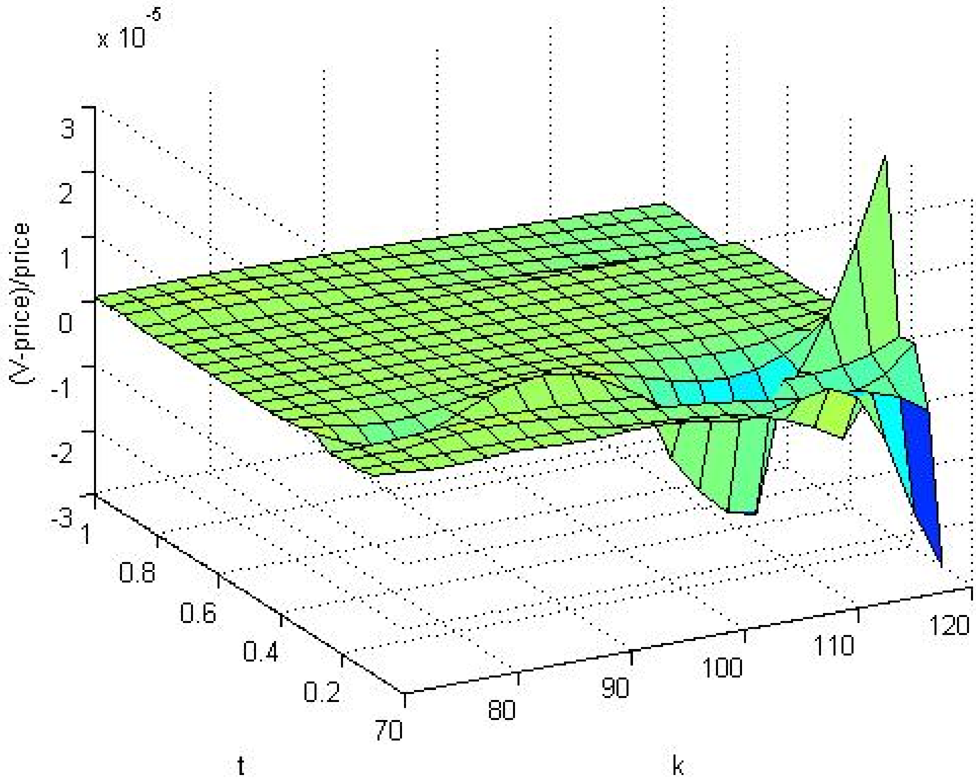

Figure 4.5: Approximation error

Bibliography

[1] Carr, Peter, Geman, Hélyette, Madan, Dilip B., Yor, Marc (2002). The fine structure of asset returns: an empirical investigation. *Journal of Business* 75 (2): 305-332

[2] Deitmar, A. (2004). A First Course in Harmonic Analysis (2nd ed.).*Springer.*

[3] Eberlein, Glau and Papapantoleon Analysis of fourier transform valuation formulas and applications *Applied Mathematical Finance* Januar (2010).

[4] Katznelson, Y.(2004) An Introduction to Harmonic Analysis (Third ed.) *Cambridge University Press.*

[5] Pascucci, A.(2007) Calcolo Stocasstico per la Finanza *Springer.*

[6] Pini, B.(1977) Terzo Corso di Analisi Matematica, Cap. 1- Operatori lineari negli spazi Ł^p *Cooperativa Libraria Universitaria Editrice Bologna.*

[7] Rudin, W.(1987) Real and Complex Analysis (Third ed.) *McGraw-Hill.*

[8] Sato, K. (1999) Lévy Processes and Infinitely Divisible Distributions. *Cambridge University Press.*

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