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Bianchi type II cosmology in Hořava Lifshitz gravity

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Sommario

In questo lavoro abbiamo esaminato uno spazio tempo di tipo Bianchi II nel contesto di una formulazione proiettabile della gravità Horava Lifshitz; le risultanti equazioni di campo sono state analizzate qualitativamente nel regime infrarosso $\lambda = 1$. Sono state trovate le soluzioni analitiche per un modello semplificato in cui vengono considerati solo i termini di curvatura più alta proporzionali al cubo del tensore di Ricci spaziale. La dinamica risultante è ancora descritta da una transizione fra due epoche di Kasner, ma si è trovata una legge di trasformazione fra gli indici di Kasner in disaccordo rispetto a quella prevista in relatività generale.

Abstract

In this work a Bianchi type II space-time within the framework of projectable Horava Lifshitz gravity was investigated; the resulting field equations in the infrared limit $\lambda = 1$ were analyzed qualitatively. We have found the analytical solutions for a toy model in which only the higher curvature terms cubic in the spatial Ricci tensor are considered. The resulting behavior is still described by a transition among two Kasner epochs, but we have found a different transformation law of the Kasner exponents with respect to the one of Einstein's general relativity.

Contents

Introduction

In 2009 a new class of renormalizable quantum gravity theories, inspired by the Lifshitz scalar theory of critical phenomena, was proposed by the Czech physicist Petr Horava. The improved UV behavior of these theories is obtained at expense of the Lorentz invariance, so it is a crucial point that in the low energy limit they must be able to recover it, flowing towards general relativity. These theories, usually referred to as *Horava-Lifshitz* gravities, assume that the space-time is foliated into space like hypersurfaces, with the spatial and the time coordinates which scale anisotropically at high energies; this fundamental decomposition is the responsible for the Lorentz symmetry breaking. Beyond renormalizability, there are also cosmological implications of the anisotropic scaling which deserve attention, and could be used to discriminate among the many formulations of such gravity theories.
The task of this thesis is to study a class of homogeneous cosmological models, the Bianchi type II universes, in the infrared limit of a projectable version of Horava-Lifshitz gravity; the work is structured as follows,

- In chapter one we briefly review some standard results of Einstein's general relativity; in the first two sections is presented the geometrical background which is required in section three, where the explicit solution of the field equations for the Bianchi type II vacuum space-time are obtained.

- Chapter two is intended as a brief survey about Horava's proposal and the further formulations of the theory, here are also reported some cosmological features of Horava-Lifshitz gravities.

- Chapter three is devoted to the analysis of a Bianchi type II spatial metric within space-time anisotropic scaling, the field equations are obtained and qualitatively studied; then the results are compared with those of general relativity.

Chapter 1

Bianchi type II space-time

In this chapter we are going to analyze and discuss some classical results of general relativity in the contest of the cosmological solutions called *homogeneous spaces*; among these we study with a particular attention the Bianchi type II anisotropic spacetime.

The structure of the chapter is the following; we start by briefly introducing Einstein equations, then we drift our attention to the study of large-scale solutions of these focusing on the class of the homogeneous Bianchi cosmologies.

Finally we work out explicitly the solutions for the simplest one with spatial curvature, the Bianchi type-II.

1.1 General relativity

Nowadays our understanding of gravitation is built on Einsteins's general theory of relativity, which is basically a geometrical description of the dynamical interactions between the space-time and his energy-matter contents.

Under "geometrical" we mean that it relates locally the mathematical proprieties of space-time, encoded in the metric tensor $g_{\mu\nu}$, with the stress-energy tensor $T_{\mu\nu}$, which accounts the density and flux of energy-momentum; it is also dynamical in the sense that it describes the evolution of these interactions in a predictable way.

The core of General Relativity is the set of differential equations [1, 2]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{1.1}$$

Together with the so called first and second Bianchi's Identities:

$$R_{\mu\nu\lambda\eta} + R_{\mu\eta\nu\lambda} + R_{\mu\lambda\eta\nu} = 0, \tag{1.2}$$

$$R_{\mu\nu\lambda\eta;\sigma} + R_{\mu\nu\sigma\lambda;\eta} + R_{\mu\nu\eta\sigma;\lambda} = 0, \tag{1.3}$$

where we use the notation $A_{\mu;\nu}$ to represent the covariant derivative of A_μ with respect to x_ν.

We call the system (1.1) *Einstein's field equations* (EFE); these holds in any reference frame because they are tensor equalities, so we say that they are covariant.

This is probably the main feature of General relativity and it is encoded in one of its first principles , the general principle of relativity; the other pillars of the theory are the equivalence principle, which states that these equations must produce locally the same physics for all the observers, and the corrispondence principle, which state that classical Newtonian gravitation must be recovered as asymptotic limit.

The main characters acting the system (1.1) are the Ricci tensor $R_{\mu\nu}$, the 4-D metric $g_{\mu\nu}$, the cosmological constant Λ and the stress-energy tensor $T_{\mu\nu}$; in order to fully appreciate the matter-geometry coupling that derives from the EFE we have to focus on the formers, and this could be done only in the framework of differential geometry.

1.1.1 Riemannian geometry

Differential geometry could be considered as the study of the most general mathematical proprieties of a given differentiable manifold; e.g. a set endowed with smooth functions that relate this to another set, or subset, in $\Re^n$.

Even if a satisfactory survey of this branch of mathematics could be done without, it is very useful introduce an object called metric tensor, which permits to define distances and shapes at least locally anywhere on the manifold. To be clearer the manifold which we refer is the universe, and the representation of the metric that we use is a 4x4 matrix which, by contraction with two vectors that define two points in the universe, describes the line element, e.g. the distance, between these points.

In GR the metric tensor is a dynamical entity which plays the role of field in the EFE; the source of this dynamic is the energy-momentum content of a given portion of the manifold and, as in classical mechanics, the sources are related to the second order derivatives of the fields; indeed the Ricci tensor

$R_{\mu\nu}$ is linked in non-trivial way to the second derivative of the metric tensor.

As we said, GR is a covariant theory of gravitation and the physics laws must be the same for all the observers, no matter in which reference frame they find themselves; on the other side, in order to describe physics at different points of the manifold or processes involving some large portion of space, we need a way to link reference frames related to distant observers.

This is achieved defining a rule for the transport of a vector along a manifold, obviusly this rule is by no means unique and in general we have to define vectors, curves and family of curves called congruences which estabilish paths on which parallel transport of vectors is well defined; mathematically speaking we need an *affine connection* which links vectors at different points.

More technically, a vector at the point x in the manifold is a function which relates this point to some coordinates in $\mathbb{R}^n$, the set of such functions at x defines the tangent space of this point; finally the affine connection is an object defined in the set of all tangent spaces at each point, called tangent bundle, which permits to compare vectors from different tangent spaces.

Covariant Derivative

We are able to describe the parallel transport by means of the *covariant derivative*; once that a vector basis $\mathbf{e}_i$ is given, we say that a vector $\mathbf{u}$ is parallelly transported along the curve γ defined by the vector $\mathbf{v}$ if holds[3]

$$\nabla_{\mathbf{v}}\mathbf{u} = 0. \tag{1.4}$$

We can give the following coordinate dependent representation of covariant derivative of a vector A_i given the coordinates x_j,

$$\nabla_j A_i = \frac{\partial A_i}{\partial x^j} - \Gamma^k_{ij} A_k, \tag{1.5}$$

in the last equation the terms Γ^k_{ij} depicts the contribute of the connection, acting on the components of vector A providing for keeping parallel during the transport.

Levi-Civita affine connection

The fundamental theorem of Riemmanian geometry states that on a Riemmanian or pseudo Riemmanian manifold there is an unique torsion-free connection which preserve the metric tensor by parallel transport, and this connection is the Levi-Civita connection of the metric[4].

For our purposes, the torsion could be simply considered as the anti-symmetric part of the connection, so the Levi-Civita connection must be symmetric.

These conditions could be written:

$$\Gamma^{\eta}_{\mu\nu} = \Gamma^{\eta}_{\nu\mu}, \tag{1.6}$$
$$\nabla_{\eta} g_{\mu\nu} = 0, \tag{1.7}$$

since we have a metric tensor defined on our manifold we can use it to express the Levi-Civita connection:

$$\Gamma^{\eta}_{\mu\nu} = \frac{1}{2} g^{\eta\sigma} (g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma} + g_{\nu\lambda} C^{\lambda}_{\sigma\mu} + g_{\mu\lambda} C^{\lambda}_{\sigma\nu}) + \frac{1}{2} C^{\eta}_{\mu\nu}, \tag{1.8}$$

the coefficients $C^{\eta}_{\mu\nu}$ depends on the choice of the basis vector e_i; in general they are defined by:

$$[e_{\mu}, e_{\nu}] = C^{\eta}_{\mu\nu} e_{\eta}, \tag{1.9}$$

where the $C^{\eta}_{\mu\nu}$ are also called structure functions.

Coordinates for which the $C^{\eta}_{\mu\nu}$ terms vanish are called holonomic; in this case the Levi-Civita affine connection components are called Christoffel symbols.

Riemann curvature tensor

The curvature of a Riemannian manifold is expressed by the so called Riemann tensor, that is, the only object linear in the second order derivative of the metric that transform in tensorial way.

In terms of the Christoffel symbols, once that an holonomic vetor basis is chosen, Riemann tensor is given by:

$$R^{\lambda}_{\eta\mu\nu} = \partial_{\nu} \Gamma^{\lambda}_{\eta\mu} - \partial_{\mu} \Gamma^{\lambda}_{\eta\nu} + \Gamma^{\rho}_{\eta\mu} \Gamma^{\lambda}_{\rho\nu} - \Gamma^{\rho}_{\eta\nu} \Gamma^{\lambda}_{\rho\mu}. \tag{1.10}$$

In a general coordinate basis it is enough to get a representation of the covariant derivative; then the Riemann tensor is defined by:

$$[\nabla_{\mu}, \nabla_{\nu}] A^{\lambda} = R^{\lambda}_{\sigma\mu\nu} A^{\sigma} - T^{\sigma}_{\mu\nu} \nabla_{\sigma} A^{\lambda}, \tag{1.11}$$

where $T^{\sigma}_{\mu\nu}$ is the torsion tensor.

In the last formula we can appreciate the interpretation of Riemann tensor as the measure of the manifold's curvature; indeed it measures the difference in the parallel transport of a vector along two different directions.

One can easily see from eq.(1.10) that in a torsion-free manifold the following proprieties for the curvature tensor holds:

$$R_{\lambda\eta\mu\nu} = -R_{\eta\lambda\mu\nu} = R_{\eta\lambda\nu\mu}, \tag{1.12}$$
$$R_{\lambda\eta\mu\nu} = R_{\mu\nu\lambda\eta}, \tag{1.13}$$
$$\tag{1.14}$$

these constraints, together with the first Bianchi identity (1.2), fix the number N of indipendent components of the Riemann tensor in a torsion free manifold of dimension n:

$$N = \frac{1}{12}n^2(n^2 - 1). \tag{1.15}$$

Riemann contractions

Riemann tensor doesn't enter in the EFE (1.1) which are tensorial equalities of rank 2; so we have to define the following contractions of $R_{\lambda\eta\mu\nu}$ called Ricci's tensor and scalar:

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}, \tag{1.16}$$
$$R = g^{\lambda\eta}R_{\lambda\eta}. \tag{1.17}$$

Note that Ricci tensor is defined in such a way that is symmetric, just like the metric and the stress-energy tensor.

Another useful object which could be defined from the Riemann tensor is the Kretschmann scalar, defined by:

$$K = R^{\lambda\sigma\mu\nu}R_{\lambda\sigma\mu\nu}, \tag{1.18}$$

which is helpful when one find divergences of the Ricci scalar that are due to unhappies choices of the reference frame and dont represent a singularity of the geometry, in this case the Kretschmann scalar is finite.

1.1.2 Killing equations

Since the system (1.1) involves only symmetric tensors on a four-dimensional space-time, we are left with the problem to solve 10 partial higly non-linear differential equations.

This task could be more easy if we postulate further symmetries for the solutions; mathematically speaking this is achieved by finding transformations which leaves the metric unchanged, and this naturally leads to the concept of killing equation[3].

Lie derivative

To be sure that those symmetries pertain to the structure of the manifold, and are not a merely effect of a particulare choice of reference frame, the transformation laws must be tensorial.

Let us wrote explicitly an infinitesimal transormation of the coordinates at point x:

$$x_{\mu}^{'} = x_{\mu} + \epsilon a_{\mu}, \tag{1.19}$$

with ϵ infinitesimal parameter; the direction and the magnitude of the transformation are given by the vector field:

$$X = a^{\mu}\partial_{\mu}. \tag{1.20}$$

In order to study how this coordinate's change affects the tensorial objects defined on the manifold it is very useful to introduce the concept of Lie derivation[4].

To start consider a vector field X, this induces a 1-parameter group of automorphisms on the set of tensors defined on the manifold; let's call such transformations φ_t, roughly speaking this implies that, if K is a tensor, $\varphi_t K = K^{'}$ is still a tensor of the same species, so φ_t preserve the algebra of tensor fields.

Then we can define the *Lie derivative* of a tensor T with respect to the vector field X at point p by:

$$\pounds_X T_p = \lim_{t \to 0} \frac{1}{t}[T_p - (\varphi_t T)_p]. \tag{1.21}$$

More explicitly we can wrote down the action of the Lie derivative on functions and vector fields:

$$\pounds_X f \quad = Xf, \tag{1.22}$$

$$\pounds_X Y \quad = [X, Y], \tag{1.23}$$

where we use brackets[ ] to indicate the commutator among the two fields.

$\pounds$ is a well defined derivation because it is linear and satisfy Leibniz rule; moreover Lie derivation has the following properties inherited from those of commutators (which are Lie product's):

$$\pounds_{[X,Y]} = [\pounds_X, \pounds_Y], \tag{1.24}$$

$$[\pounds_X, [\pounds_Y, \pounds_Z]] + [\pounds_Z, [\pounds_X, \pounds_Y]] + [\pounds_Y, [\pounds_Z, \pounds_X] = 0. \tag{1.25}$$

Since, as was just told, $\pounds$ is a good derivation, we can write down his action on tensors of general rank keeping in mind that those are built up by tensor product; derivation is linear with respect to that, and if we are working with an affine connection given on the manifold, in a general coordinate system, holds:

$$(\pounds_X T)^{\lambda\sigma}_{\mu\nu} = T^{\lambda\sigma}_{\mu\nu;\eta} a^\eta - T^{\eta\sigma}_{\mu\nu} a^\lambda_{;\eta} - T^{\lambda\eta}_{\mu\nu} a^\sigma_{;\eta} + T^{\lambda\sigma}_{\eta\nu} a^\eta_{;\mu} + T^{\lambda\sigma}_{\mu\eta} a^\eta_{;\nu}. \tag{1.26}$$

Note that Lie derivative is defined even without connection or metric; actually it is a relation among tangent spaces of the manifold.

Killing vectors

Consider a manifold endowed with a metric tensor g; if exist some vector field ξ such that:

$$\pounds_\xi g = 0, \tag{1.27}$$

then we call it Killing vector field (KVF), if there are many of them, say ξ_1, ξ_2, it could be proved that:

$$(\pounds_{[\xi_1,\xi_2]} g) = 0. \tag{1.28}$$

Equation (1.27) in terms of coordinates, if a symmetric connection is given, looks like:

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0. \tag{1.29}$$

Since the left hand side of the latter is symmetric in both indexes μ, ν there are utmost $\frac{n(n+1)}{2}$ indipendent first-order differential equations, called Killing equations.

We have seen that the Lie product, or commutator, of two KVF's is still a killing vector field, so they form an algebra; of particulary interest are those KVF's algebras which are simultaneously Lie algebras, since we know that them are completely determined by a set of structure constants.

KVF's and Lie groups

Suppose to have found a set of KVF's which realize the following algebra:

$$[\xi_i, \xi_j] = C^k_{ij}\xi_k, \tag{1.30}$$

$$C^l_{im}C^m_{jk} + C^l_{km}C^m_{ij} + C^l_{jm}C^m_{ki} = 0. \tag{1.31}$$

Then we have determined an isometry Lie group G on the manifold; reversing the point of view one can also think that a given set of structure constants and vector fields satisfing the above equations generates a manifold.

We call a manifold with an isometry group G maximally symmetric if it has a maximal number of Killing vector fields; it could be proved that there are $\frac{n(n+1)}{2}$ of them and among these $\frac{n(n-1)}{2}$ constitute a Lie subgroup of rotations which is a subspace of the manifold, moreover maximally symmetric spaces have constant curvature.
There are also manifolds which instead contain a maximally symmetric *subspace*; these are caracterized by a Lie subalgebra of Killing vector fields with less than $\frac{n(n+1)}{2}$ generators.

1.1.3 Isometries and relativistic cosmology

Einstein's equations make it possible to study the properties of the universe as a whole in terms of its geometrical features.
The branch of physics that emerges from this framework is called relativistic cosmology[5], and could be interpreted as the classification and description of the possible universes allowed by general relativity; this is a quite different approach compared to that of the standard observational cosmology, which instead moves from the structure of the universe at present times in order to describe its past and its future.

In the following we are going to list some manifolds whose metrics possesses a group of isometry and which are of particular interest in the contest of cosmological solutions of the Einstein's field equations.

Minkowsky space

The Minkowsky space is characterized by the following line element in cartesian coordinates:

$$ds^2 = dt^2 - (dx^2 + dy^2 + dz^2), \tag{1.32}$$

so that the metric is diagonal with the components:

$$g_{\mu\nu} = (1, -1 - 1 - 1).$$

The Minkowsky space is obviusly a solution of Einstein field equations in vacuum since it is the proper manifold to describe special relativity, and by corrispondence principle GR restore special relativity in low-gravitational limits.

This model has a maximally symmetric group of isometry, the killing vector fields in 3+1 dimensions are:

- 4 space-time translations, with generators $\partial_t, \partial_x, \partial_y, \partial_z$,
- 3 spatial rotations given by $x\partial_y - y\partial_x$, $y\partial_z - z\partial_y$, $z\partial_x - x\partial_z$,
- 3 space-time boost generators $x\partial_t + t\partial_x$, $y\partial_t + t\partial_y$, $z\partial_t + t\partial_z$.

One can easily check that among these $\frac{n(n-1)}{2}$ form a Lie algebra of rotations and n are translation-like; Minkowsky spaces have constant curvature $= 0$.

De Sitter space

De Sitter space of dimension n could be defined as an hyperboloid on a Minkowsky space of dimension $n+1$, in the four-dimensional case we have[6]

$$ds^2 = d\xi_0^2 - (d\xi_1^2 + d\xi_2^2 + d\xi_3^2 + d\xi_4^2), \tag{1.33}$$

$$\xi_0^2 - (\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2) = -R^2 \tag{1.34}$$

De Sitter space is maximally symmetric and it is a vacuum solution of the EFE with a positive cosmological constant, so it represents a universe with positive constant curvature.

Several choices of coordinates could be made in order to satisfy the above equations that correspond to different parametrizations of the hyperboloid. To begin with let us consider the following parametrization of the coordinates :

$$\begin{aligned}
\xi_0 &= R\sinh\frac{t}{R} + e^{\frac{t}{R}}\frac{x^2+y^2+z^2}{2R} \\
\xi_1 &= e^{\frac{t}{R}}x, \\
\xi_2 &= e^{\frac{t}{R}}y, \\
\xi_3 &= e^{\frac{t}{R}}z, \\
\xi_4 &= R\cosh\frac{t}{R} - e^{\frac{t}{R}}\frac{x^2+y^2+z^2}{2R},
\end{aligned}$$

in terms of the new variables t, x, y, z the line element becomes:

$$ds^2 = dt^2 - \exp\frac{2t}{R}\left(dx^2 + dy^2 + dz^2\right). \tag{1.35}$$

Another admitted parametrization is given by:

$$\begin{aligned}
\xi_0^{'} &= R\sinh\frac{t}{R}, \\
\xi_1^{'} &= R\cosh\frac{t}{R}\sin\chi\sin\theta\cos\phi, \\
\xi_2^{'} &= R\cosh\frac{t}{R}\sin\chi\sin\theta\sin\phi, \\
\xi_3^{'} &= R\cosh\frac{t}{R}\sin\chi\cos\theta, \\
\xi_4^{'} &= R\cosh\frac{t}{R}\cos\chi,
\end{aligned}$$

where χ runs from 0 to π, the corresponding line element is:

$$ds^2 = dt^2 - R^2\cosh^2(\frac{t}{R})(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2(\theta)d\phi^2). \tag{1.36}$$

We can also have:

$$\begin{aligned}
\xi_0^{''} &= R\sinh\frac{t}{R}\cosh\chi, \\
\xi_1^{''} &= R\sinh\frac{t}{R}\sinh\chi\sin\theta\cos\phi, \\
\xi_2^{''} &= R\sinh\frac{t}{R}\sinh\chi\sin\theta\sin\phi, \\
\xi_3^{''} &= R\sinh\frac{t}{R}\sinh\chi\cos\theta, \\
\xi_4^{''} &= R\cosh\frac{t}{R},
\end{aligned}$$

with the line element

$$ds^2 = dt^2 - R^2\sinh^2(\frac{t}{R})(d\chi^2 + \sinh^2(d\theta^2 + \sin^2(\theta)d\phi^2). \tag{1.37}$$

We have seen that different parameterizations of the De Sitter hyperboloid give raise to a variety of cosmological models with different line elements, those belong to a class of cosmologies called Friedmann Lemaître Robertson Walker models(FLRW), and satisfy the so called *Cosmological principle.*

Friedmann spaces

Let us suppose that our universe is such that on large scales it looks almost the same in every spatial direction and also that its subparts shares an identical evolution in time; these hypoteses are known as the *homogeneity* and *isotropy* of space time and summarize the cosmological principle.
From these assumptions derives the following functional dependence of the line element:

$$ds^2 = N(t)dt^2 - a(t)^2 dl^2 = N(t)dt^2 + g_{ij}d\omega^i d\omega^j, \tag{1.38}$$

where dl^2 is the general spatial line element given by contraction of the spatial metric g_{ij} with the related basis $\{\omega_i\}$, $N(t)$ is a function of time called shift wich could be removed with a gauge transformation (rescaling the time coordinate to fix $N(t) = 1$) , finally $a(t)$ is the *scale factor*, which encodes the fact that the spatial line element evolves isotropically in time; the requirement of homogeneity constrains the components of the metric to be only functions of time.

Now we are going to show that if we impose that the spatial sections at fixed time are three-dimensional euclidean spaces we can obtain two maximally symmetric models; infact we can easily calculate the Christoffel symbols for eq (1.38), obtaining the following non-zero components:

$$\Gamma^x_{tx} = \Gamma^y_{ty} = \Gamma^z_{tz} = \frac{\dot{a}}{a},$$

$$\Gamma^t_{xx} = \Gamma^t_{yy} = \Gamma^t_{zz} = \dot{a}a.$$

So that the resulting Killing equations from eq.(1.29) becomes $\dot{a}$ dependents; with the choice $\dot{a} = 0$ we obtain the same KVF of the Minkowsky space, so that Friedmann spaces with constant scale factor and Euclidean flat spatial sections are Minkowsky spaces.

If we look for models with $\dot{a} \neq 0$ integrating Killing equations we are left with a dynamical equation for the scale factor:

$$\ddot{a}a - \dot{a}^2 = 0,$$

with solution

$$a(t) = e^{\frac{t}{C}}, \tag{1.39}$$

where C is a positive integration constant.
We know yet that this is a maximally symmetric model, since we meet it in eq.(1.35); Killing vectors in these spaces are 4 translations, 3 spatial rotations and 3 so called *conformal transformations*.

Since we have constrained the spatial sections to be the those of the euclidean three-dimensional space we have obtained cosmological models which are flat, but there are also Friedmann models which arent and whose spatial sections are very different from the latters.
We have yet obtained them as parametrizations of the De Sitter space; eq.(1.36) describe a closed universe with spherical spatial sections, eq.(1.37) defines instead an open universe with an hyperbolic three-dimensional geometry.

All the models we have seen until now are maximally symmetric, in the next section we introduce such cosmologies which do not possess a complete set of Killing vector fields, these are homogeneous just like the Friedmann models but on the other hand are non isotropic; this lack is manifest in the spatial metric of the models since appears in it more than one scale factor.

1.2 Homogeneous cosmologies

The cosmological principle describes universes whose content is uniformly distribuited among the space, so that it must look the same in each point of it; even if we experience daily deviations from this assumption, just looking the sun or ourselves, from a large scale point of view this is close to be true on the basis of the observation, actually the standard model of cosmology ΛCDM agrees with a flat FRLW space-time geometry with positive cosmological constant.

By the way, from the standpoint of relativistic cosmology, the fundamental true are the EFE so that each cosmological solution of these has the same dignity; moreover if we can deduce properties of the general solutions of the EFE we could be more predictable about the consequences, no matters in which cosmological models we are working.[5]
In this framework collocates themself the study of homogeneous cosmologies.

1.2.1 Mathematical apparatus

In Homogeneous spaces we deal with models of universe which possess a set of *spatial* Killing vectors that form a Lie algebra; since we are interested in 3+1 dimension manifolds, if we are able to list all the three-dimensional Lie algebras we can easily perform the classification of the homogeneous spaces.

In the following paraghraps we set up some techniques that will make us capable for writting down the EFE for homogeneous spaces as a set of ordinary differential equations, due to the fact that the metric coefficients turns out to be functions of cosmic time only.

Synchronous reference frame

To begin with, let us suppose that exist a t variable, which we call cosmic time, and in terms of this cosmic time the line element has the form:

$$ds^2 = dt^2 - g(t)_{ij} dx^i dx^j, \tag{1.40}$$

where $\{ij\}$ runs from 1 to 3.
We call the coordinates in which this metric holds synchronous reference frame.
In this reference frame the trajectories of an observer at rest are four-dimensional geodesics.

The advantage of working in terms of cosmic time is that the formulae for the Ricci tensor result really simplified; in fact, defining the *extrinsic curvature*:

$$K_{ij} = \partial_t g_{ij}, \tag{1.41}$$

we can write the non-vanishing Christoffel symbols as:

$$\Gamma^0_{ij} = \frac{1}{2} K_{ij}, \tag{1.42}$$

$$\Gamma^i_{0j} = \frac{1}{2} K^i_j, \tag{1.43}$$

$$\Gamma^i_{jk} = \lambda^i_{jk}, \tag{1.44}$$

where the λ are the Christoffel symbols built from the spatial metric g_{ij}.

Using these expressions in the definition of the Ricci tensor we obtain [1]:

$$R_{00} = -\frac{1}{2} \dot{K}^i_i - \frac{1}{4} K^i_j K^j_i, \tag{1.45}$$

$$R_{0j} = \frac{1}{2} \left[K^i_{j;i} - K^i_{i;j} \right], \tag{1.46}$$

$$R_{ij} = \frac{1}{2} \dot{K}_{ij} + \frac{1}{4} \left[K_{ij} K^l_l - 2 K^l_i K_{lj} \right] + P_{ij}, \tag{1.47}$$

here P_{ij} is the three-dimensional Ricci tensor obtained using the spatial metric g_{ij}.

Maurer Cartan forms

We have seen that when we work with a representation of the metric such that its components are functions of time only, it is possible to express the four-dimensional Ricci tensor in terms of the tensors K_{ij} and P_{ij}, now we want to show how such coordinate basis could be found.

To start suppose that the spatial sections of the manifold possess a maximally symmetric group of isometries; since the space has dimension 3 we have 6 Killing vectors ξ_i; by means of the algebra of these Killing vectors we can define a basis e_i such that:

$$[e_i, \xi_j] = 0. \tag{1.48}$$

Actually we impose that at point p the two fields coincides:

$$e_i(p) = \xi_i(p),$$

then, iterating Killing transformation on the e_i (via exponentation, since we are working with Lie algebras of KVF) we obtain a set of vector fields which satisfy:

$$[e_i, e_j] = D^k_{ij} e_k.$$

Now, expanding the $\{e\}$ vectors in the basis of the KVF:

$$e_i = \alpha^j_i \xi_j,$$

with the α coefficients constrained to be such that:

$$\alpha^i_j(p) = \delta^i_j.$$

In terms of this expansion eq.(1.48) becomes:

$$[\alpha^k_i \xi_k, \xi_j] = 0 = \alpha^k_i C^l_{kj} \xi_l - \xi_j(\alpha^k_i \xi_k),$$

the above relation, calculated at point p give us:

$$\xi_i \alpha^k_j = C^k_{ji}. \tag{1.49}$$

Now consider the Lie braket among two e_i's:

$$\begin{aligned}[e_i, e_j] = [\alpha^l_i \xi_l, \alpha^m_j \xi_m] &= \alpha^l_i \alpha^m_j C^n_{lm} \xi_n + \alpha^l_i (\xi_l \alpha^m_j) \xi_m - \alpha^m_j (\xi_m \alpha^l_i) \xi_l \\ &= D^k_{ij} e_k = D^k_{ij} a^l_k \xi_l,\end{aligned}$$

evaluating the above equality at the point p gives us:

$$D^k_{ij}\delta^l_k\xi_l = \delta^l_i\delta^m_j C^n_{lm}\xi_n - \delta^m_j C^l_{im}\xi_l + \delta^l_i C^m_{jl}\xi_m,$$

$$D^k_{ij}\xi_k = C^k_{ji}\xi_k, \tag{1.50}$$

so that the Killing vector fields and the vectors $\{e_i\}$ share the same Lie algebra, since they got the same set of structure constants.
Taking the duals ω^i of the vectors $\{e_i\}$ we came to the so called *Maurer-Cartan* 1-forms of the basis $\{e_i\}$[4].

To summarize we were able to construct, starting from the algebra of KVF, first the basis of vectors $\{e_i\}$ that have the same algebra, then the Maurer-Cartan 1-forms ω which are invariant under the action of the Lie group since $\pounds_\xi\omega = 0$.
In this new basis holds $\omega(e) = const$ and the metric tensor that emegerges is invariant on the spatial sections defined by the spatial Killing vector fields.
Finally we want mention that it could be proved that for the differential of ω holds the *called Maurer-Cartan structural equation*:

$$d\omega(\xi_1, \xi_2) = -\frac{1}{2}\omega[\xi_1, \xi_2] \tag{1.51}$$

So that even $d\omega$ share the same structure of the Lie algebra.

1.2.2 Bianchi classification

We have seen that with the request of homogeneity it is possible to construct a spatial line element in which the metric coefficients are spatially constant, since we were able to choose a basis that share the same Lie algebra of the Killing vector fields, so to get a metric tensor which is invariant under the transformation of the isotropy group.

As was mentioned before, we could be able to classify all the homogeneous cosmologies if we can list all the possible 3-dimensional Lie algebras.
This task was achieved by Bianchi in [7], a complete derivation with the associated basis vector for each model could be found in [8].
Let us consider the following decomposition of the structure constants of the Lie algebra:

$$C^k_{ij} = \varepsilon_{ijl}C^{lk}, \tag{1.52}$$

the new tensor C^{ij} could be decomposed in its symmetric and antisymmetric parts[9]:

$$C^{ij} = n^{ij} + \varepsilon^{ijk}a_k. \tag{1.53}$$

In terms of this new rank 2 tensor it is possible to write the Jacobi identity in the following way:

$$\varepsilon_{ijk}C^{ij}C^{kl} = 0, \tag{1.54}$$

combining the last two equation we get:

$$\begin{aligned}0 =&\varepsilon_{ijk}(\varepsilon^{ijm}a_m + n^{ij})(\varepsilon^{klm}a_m + n^{kl})\\ =&\varepsilon_{ijk}(\varepsilon^{ijm}a_m\varepsilon^{klm}a_m + \varepsilon^{ijm}a_m n^{kl} + n^{ij}\varepsilon^{klm}a_m + n^{ij}n^{kl})\\ =&\delta_k^m a_m n^{kl}.\end{aligned}$$

We have rewrited the Jacobi identity for the structure constants in terms of the system:

$$n^{ij}a_j = 0. \tag{1.55}$$

By a coordinate transformation the tensor n^{ij} is put in diagonal form, moreover we can choose without loss of generality that the vector a_j lies along his principal direction[1]; we are left with

$$\begin{aligned}a_j &= (a, 0, 0),\\ n^{11}a &= 0,\end{aligned}$$

now, listing all the possible choices of n_{ii} and a which realize the Jacobi identity we obtain the celebrated Bianchi classification:

Type	a	n^{11}	n^{22}	n^{33}
I	0	0	0	0
II	0	1	0	0
VII$_0$	0	1	1	0
VI$_0$	0	1	-1	0
IX	0	1	1	1
VIII	0	1	1	-1
V	1	0	0	0
IV	1	0	0	1
VII$_a$	a	0	1	1
III	1	0	1	-1
VI$_a$	a	0	1	-1

Here the nomenclature whit Roman numbers is due to Bianchi and is matter of convention, we also mention a useful and modern schematization given by Ellis and Mac Callum which could be found in [10].

Bianchi I, the Kasner solution

In 1922 Kasner has found the following solution for the EFE in vacuum[11]:

$$ds^2 = dt^2 - t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{3p_3}dz^2, \tag{1.56}$$

$$p_1 + p_2 + p_3 = 1, \tag{1.57}$$
$$p_1^2 + p_2^2 + p_3^2 = 1, \tag{1.58}$$

which corresponds to the type-I of the Bianchi classification; since all the components of n^{ij} and the vector a_j vanish there are no structure constant different from 0.

Since we have three Kasner indexes p_1, p_2, p_3 togheter with the two constraints of the above system of equations, we actually can express them in terms of just one parameter u , suppose without loss of generality that $p_1 \leq p_2 \leq p_3$, then[12]:

$$\begin{aligned} p_1 &= \frac{-u}{1+u+u^2}, \\ p_2 &= \frac{1+u}{1+u+u^2}, \\ p_3 &= \frac{u(1+u)}{1+u+u^2}. \end{aligned}$$

This parametriztion holds for $u > 1$, anyway it's easy to see that if $u < 1$:

$$\begin{aligned} p_1\left(\frac{1}{u}\right) &= p_1(u), \\ p_2\left(\frac{1}{u}\right) &= p_3(u), \\ p_3\left(\frac{1}{u}\right) &= p_2(u). \end{aligned}$$

Looking for explicit values of those exponents one find that except for the solutions with one index $p_i = 1$ and the others vanishing, which could be proved corresponds to the Minkowsky space solution, the Kasner indexes must be distribuited in the following way:

$$-\frac{1}{3} \leq p_1 \leq 0, \tag{1.59}$$

$$0 \leq p_2 \leq \frac{2}{3}, \tag{1.60}$$

$$\frac{2}{3} \leq p_3 \leq 1. \tag{1.61}$$

Thus we finally conclude that in the Bianchi-I cosmology, which corresponds to a flat but anisotropic universe, there are three different scale factors referred to the spatial axes with two of them that increase with time and one which conversely decreases.
Note also that the spatial volume of a Bianchi-I spacetime grows with time due to equations (1.57)-(1.58), thus in the limit $t \to 0$ we have a Big-bang like singularity.

BKL oscillatory approach

The behavior of the universe near the big bang singularity beyond the hypotesis of homogeneity and isotropy has attracted the attemption of general relativists since the sixties; if the presence of such singularity is a general peculiarity of the Einstein's equations or an incident due to the high symmetry accounted in the Friedmann models was for a long time matter of controversy.
In [13] is stated that the occurrence of the singularity is a general conseguence of the Einstein field equation under certain assumptions, in [12, 14] was showed that the physics near the singular point could be generally described in terms of suitable generalizations of the Kasner solution.
In [1] the so called *oscillatory approach to the initial singularity* is explained via qualitative analysis of the set of Einstein's differential equations for homogeneous spaces; actually one founds that the Kasner solution is still a good approssimation when the effects of the spatial curvature can be neglected; these effects anyway grow during the evolution and finally ruin the validity of the Kasner description. Surprisingly we see that the character of the spatial curvature perturbations is periodic, in the sense that first it is growing and after starts to decrease, and when it becomes again negligible the universe is still described by a Kasner solution but with a different set of Kasner exponents.
If we start with a set of Kasner exponents p_1, p_2, p_3; then they trasform under the effect of the perturbation due to the spatial curvature in the following way:

$$
\begin{aligned}
p_l^{'} &= \frac{\mid p_1 \mid}{1 - \mid 2p_1 \mid}, \\
p_m^{'} &= -\frac{2 \mid p_1 \mid - p_2}{1 - \mid 2p_1 \mid}, \\
p_n^{'} &= \frac{p_3 - 2 \mid p_1 \mid}{1 - \mid 2p_1 \mid}.
\end{aligned}
$$

In terms of the u parameter the transition among the Kasner regimes is obtained by simply shifting the value of u to $u-1$; the set of exponents in the new regime becomes:

$$\begin{aligned} p_1^{'} &= p_2(u-1), \\ p_2^{'} &= p_1(u-1), \\ p_3^{'} &= p_3(u-1). \end{aligned}$$

The interval of time in which the universe is unaffected by the spatial curvature perturbation is called Kasner epoch, the lenght of this Kasner epoch has stochastic character.
BKL showed that the universe, while approaching the singularities, passes trought a divergent number of such transitions of Kasner epochs, and the overrall behavior is that the past singularity is never reached.

In the next section we are going to study in detail how the spatial curvature can produce such transitions between Kasner epochs in the simplest case of lowest degree of anisotropy; this will furnish the sufficient background to figure out the Bianchi IX dynamics in terms of a chaotic collection of Bianchi type II transitions.

1.3 Bianchi type-II cosmological model

In this section we are going to solve analitically the EFE for a Bianchi type II cosmology; in literature there are well-know solutions obtained via hamiltonian methods, see for example [16], [17] and [18], however in the present work we want to be close to the BKL *Lagrangian* description of the oscillatory approach, and since we were not able to find this kind of solutions in some preceeding work we are going to calculate them by ourselves.

1.3.1 Bianchi II EFE in vacuum

To begin with let us calculate explicitly the EFE in vacuum with the metric induced by the B-II Lie algebra. Using the following parametrization from [5] we have the KVF's:

$$\begin{aligned} \xi_1 =& \partial_y, \\ \xi_2 =& \partial_z, \\ \xi_3 =& \partial_x + z\partial_y. \end{aligned}$$

The related vector basis that commute with the above is:

$$\begin{aligned} e_1 =& \partial_y, \\ e_2 =& x\partial_y + \partial_z, \\ e_3 =& \partial_x, \end{aligned}$$

from which we obtain, taking the duals:

$$\begin{aligned} \omega^1 =& (dy - xdz), \\ \omega^2 =& dz, \\ \omega^3 =& dx, \end{aligned}$$

and the Maurer-Cartant structural equations look:

$$\begin{aligned} d\omega^1 &= \omega^2 \wedge \omega^3, \\ d\omega^2 &= 0, \\ d\omega^3 &= 0. \end{aligned}$$

Using these variables we can write the spatial B-II line element as:

$$ds^2_{B-II} = B(t)^2(\omega^1)^2 + A(t)^2(\omega^3)^2 + C(t)(\omega^3)^2, \tag{1.62}$$

where $A(t), B(t), C(t)$ are the scale factors in this basis.

The extrinsic curvature

The extrinsic curvature has the same structure for all the Homogeneous cosmologies, since as told we could always find a coordinate basis for which the metric results diagonal and function of time only.
Taking the time derivative of the spatial metric tensor we obtain:

$$K = \begin{pmatrix} 2A\dot{A} & 0 & 0 \\ 0 & 2B\dot{B} & 0 \\ 0 & 0 & 2C\dot{C} \end{pmatrix}, \tag{1.63}$$

then we have:

$$\begin{aligned} \partial_t K^i_i &= 2\partial_t \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) = 2 \left[\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} - \left(\frac{\dot{A}^2}{A} + \frac{\dot{B}^2}{B} + \frac{\dot{C}^2}{C} \right) \right], \\ K^i_j K^j_i &= 4 \left[\frac{\dot{A}^2}{A} + \frac{\dot{B}^2}{B} + \frac{\dot{C}^2}{C} \right]. \end{aligned}$$

We are working with the vacuum equations for a diagonal metric so the EFE reduces to $R_{ii} = 0$; here in the spatials equations of 4-D Ricci tensor we have:

$$\partial_t\left(\sqrt{g}K_1^1\right) = \left(\dot{A}BC + A\dot{B}C + AB\dot{C}\right)\frac{2\dot{A}}{A} + ABC\left(\frac{\ddot{A}}{A^2} - \frac{\dot{A}^2}{A^2}\right) = \frac{2\partial_t\left(\dot{A}BC\right)}{ABC},$$

$$\partial_t\left(\sqrt{g}K_2^2\right) = \left(\dot{A}BC + B\dot{A}C + AB\dot{C}\right)\frac{2\dot{B}}{B} + ABC\left(\frac{\ddot{B}}{B^2} - \frac{\dot{B}^2}{B^2}\right) = \frac{2\partial_t\left(A\dot{B}C\right)}{ABC},$$

$$\partial_t\left(\sqrt{g}K_3^3\right) = \left(\dot{A}BC + B\dot{A}C + AB\dot{C}\right)\frac{2\dot{C}}{C} + ABC\left(\frac{\ddot{C}}{C^2} - \frac{\dot{C}^2}{C^2}\right) = \frac{2\partial_t\left(AB\dot{C}\right)}{ABC}.$$

The spatial 3D Ricci tensor

The other main ingredient which appears in the EFE's in synchronous time is the spatial Ricci tensor P_{ij}, built by using the spatial metric g_{ij}.

We know that the components of the spatial metric expressed in terms of the basis $\{e_i\}$ are constant on spatial hypersurfaces, so the Levi-Civita connection is a function of the structure constants only; after some algebra it could be shown that the spatial Ricci tensor assumes the form:

$$P_b^a = \frac{1}{2g}[2C^{bd}C_{ad} + C^{bd}C_{da} + C^{db}C_{ad} - C_d^d(C_a^b + C_a^b) + \delta_a^b((C_d^d)^2 - 2C^{df}C_{df})],$$

and, since we have just one constant structure different from 0 in the Bianchi type II algebra, which is $C_{23}^1 = 1$, we have $C^{ab} \neq 0$ only if $a = b = 1$; in this case P_{ab} becomes:

$$P_j^i = \frac{1}{2A^2B^2C^2}[-(g_{11})^2\delta_j^i + 2(g_{11})^2\delta_j^1\delta_1^i], \tag{1.64}$$

or:

$$P = \frac{1}{2A^2B^2C^2}\begin{pmatrix} A^4 & 0 & 0 \\ 0 & -A^4 & 0 \\ 0 & 0 & -A^4 \end{pmatrix}. \tag{1.65}$$

EFE in synchronous time

Summing up we are left with the following formulae for the components of the Ricci tensor:

$$
\begin{aligned}
R_0^0 &= -\frac{1}{2}\partial_t K_\alpha^\alpha - \frac{1}{4}K_\alpha^\beta K_\beta^\alpha, \\
R_\alpha^0 &= -\frac{1}{2}K_\beta^\gamma (C_{\gamma\alpha}^\beta - \delta_\alpha^\beta C_{\eta\gamma}^\eta), \\
R_\beta^\alpha &= -\frac{1}{2\sqrt{g}}(\dot{\sqrt{g}K_\alpha^\beta}) - P_\beta^\alpha.
\end{aligned}
$$

The set of vacuum Einstein field equations for this model becomes:

$$
\begin{aligned}
R_0^0 =& -\frac{1}{2}\left\{\left[\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} - \left(\frac{\dot{A}^2}{A} + \frac{\dot{B}^2}{B} + \frac{\dot{C}^2}{C}\right)\right]\right\} - \frac{1}{4}\left\{\left[\frac{\dot{A}^2}{A} + \frac{\dot{B}^2}{B} + \frac{\dot{C}^2}{C}\right]\right\} = 0, \\
R_1^1 =& \frac{1}{2ABC}\left[2\frac{\partial_t(\dot{A}BC)}{ABC}\right] - \frac{A^4}{2A^2B^2C^2}.
\end{aligned}
$$

Switching the role of A with B and C in the first term in right side of the last equation and taking the correct sign for the P components, we obtain the others two field equations:

$$
\begin{aligned}
R_2^2 &= \frac{1}{2ABC}\left[2\frac{\partial_t(A\dot{B}C)}{ABC}\right] + \frac{A^4}{2A^2B^2C^2} = 0, \\
R_3^3 &= \frac{1}{2ABC}\left[2\frac{\partial_t(AB\dot{C})}{ABC}\right] + \frac{A^4}{2A^2B^2C^2} = 0.
\end{aligned}
$$

Thus we are left with the following system of ordinary differential equations:

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} = 0, \tag{1.66}$$

$$\frac{\partial_t\left(\dot{A}BC\right)}{ABC} = \frac{-A^4}{2A^2B^2C^2}, \tag{1.67}$$

$$\frac{\partial_t\left(A\dot{B}C\right)}{ABC} = \frac{A^4}{2A^2B^2C^2}, \tag{1.68}$$

$$\frac{\partial_t\left(AB\dot{C}\right)}{ABC} = \frac{A^4}{2A^2B^2C^2}. \tag{1.69}$$

EFE from ADM - Einstein Hilbert action

The EFE could be derived also by varying the following action, called the Einstein Hilbert action [1]:

$$S_{EH} = \int d^4x\sqrt{-g}R. \tag{1.70}$$

The above functional of the metric could be reformulated in a suitable way if we suppose that the space-time is foliated into spacelike surfaces Σ_t which depends on the time coordinate, or, in other words, if we choose to use coordinates in which the time components of the metric $g_{t\mu}$ do not have dependence on spatial coordinates.
The resulting procedure is called *ADM formalism*[19] from the creators R.Arnowitt, S. Deser, C.W.Misner. To start consider a metric of a following type:

$$ds^2 = N(t)dt^2 - g_{ij}(N^i dt + dx^i)(N^j dt + dx^j), \tag{1.71}$$

where the function $N(t)$ is the lapse, N^i is the shift and g_{ij} is the spatial metric.
Using this decomposition, we came to an action of the form:

$$S_{ADM} = \int dtd^3x\sqrt{g}N(t)\left[(K^{ij}K_{ij} - K^2) + P\right], \tag{1.72}$$

with $K_{ij} = \frac{1}{2N}(\partial_t g_{ij} - N_{i;j} - N_{j;i})$, and P is the three-dimensional Ricci scalar.

In homogeneous cosmologies this is a suitable formalism, indeed we use coordinate basis in which the space-time foliation is supposed, but with the constraint on the shift $N^i = 0$.
The ADM action for the B-II metric (1.62) is:

$$S_{ADM} = \int dtd^3x\sqrt{g}N(t)\left[\frac{1}{N^2}\left(\frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + \frac{\dot{C}^2}{C^2}\right) - \frac{1}{N^2}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)^2 - \frac{A^2}{2B^2C^2}\right], \tag{1.73}$$

then, expanding the second term we get

$$= \int dtd^3x\sqrt{g}\left[-\frac{2}{N}\left(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC}\right) - N\frac{A^2}{2B^2C^2}\right],$$

and finally, varying respect to N

$$+\frac{2}{N^2}(\dot{A}\dot{B}C + \dot{A}B\dot{C} + A\dot{B}\dot{C}) - \frac{A^3}{2BC} = 0.$$

This is the Friedmann-type equation, now we can fix the gauge $N(t) = 1$ to obtain:

$$S_{ADM} = \int dtd^3x - \left[2(\dot{A}\dot{B}C + \dot{A}B\dot{C} + A\dot{B}\dot{C}) + \frac{A^3}{2BC}\right], \tag{1.74}$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} = +\frac{A^2}{4B^2C^2}. \tag{1.75}$$

Now varying with respect to A, B, C the action gives:

$$\left(\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC}\right) = \frac{3}{4}\frac{A^2}{B^2C^2}, \tag{1.76}$$

$$\left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB}\right) = -\frac{1}{4}\frac{A^2}{B^2C^2}, \tag{1.77}$$

$$\left(\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC}\right) = -\frac{1}{4}\frac{A^2}{B^2C^2}, \tag{1.78}$$

from which it follows

$$2\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} = -\frac{1}{2}\frac{A^2}{B^2C^2},$$

$$2\frac{\ddot{A}}{A} + \frac{3}{4}\frac{A^2}{B^2C^2} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} - \frac{\dot{B}\dot{C}}{BC} = -\frac{1}{2}\frac{A^2}{B^2C^2},$$

which becomes, using the Friedmann equation:

$$\frac{\ddot{A}}{A} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} = -\frac{A^2}{2B^2C^2}$$

and finally the left-hand side of this equation could be written:

$$\frac{\partial_t(\dot{A}BC)}{ABC} = \frac{-A^4}{2A^2B^2C^2}$$

which is the same of eq. (1.67); proceeding in a similar way we can obtain also eq. (1.68) and (1.69).
To recover the familiar form of the R^0_0 equation we have to sum up eq. (1.76), (1.77), (1.78) to get:

$$2\left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}\right) + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} = \frac{1}{4}\frac{A^2}{B^2C^2}$$

And using (1.75) we obtain eq. (1.66).

EFE in logarithmic time

We can simplify the above equations using the logarithmic time τ defined by:

$$dt = ABCd\tau \tag{1.79}$$

such that the scale factors are represented by

$$
\begin{aligned}
A(t) &= e^{\alpha(\tau)}, \\
B(t) &= e^{\beta(\tau)}, \\
C(t) &= e^{\gamma(\tau)},
\end{aligned}
$$

Note that the derivation with respect to this new time coordinate implies the appearance of a factor $\frac{1}{ABC}$; the left sides of the equations (1.67), (1.68) and (1.69) contain terms like:

$$
\begin{aligned}
\partial_t(\dot{A}BC) =& \frac{1}{ABC}\partial_\tau \left[\left(\frac{1}{ABC}\partial_\tau A\right) BC\right] \\
=& \frac{1}{ABC}\partial_\tau \left[\frac{1}{ABC}\dot{\alpha}ABC\right] \\
=& \frac{1}{ABC}\ddot{\alpha}.
\end{aligned}
$$

Subtituting this in eq. 1.67 we get:

$$
\frac{\ddot{\alpha}}{A^2B^2C^2} = -\frac{1}{2A^2B^2C^2}A^4,
$$

where the dot is used now to indicate a derivation with respect to τ; proceeding in a similar way for all the variables we obtain:

$$
2\ddot{\alpha} = -e^{4\alpha}, \tag{1.80}
$$

$$
2\ddot{\beta} = e^{4\alpha}, \tag{1.81}
$$

$$
2\ddot{\gamma} = e^{4\alpha}, \tag{1.82}
$$

together with:

$$
\frac{1}{2}(\ddot{\alpha} + \ddot{\beta} + \ddot{\gamma}) = \dot{\alpha}\dot{\beta} + \dot{\alpha}\dot{\gamma} + \dot{\beta}\dot{\gamma} = \frac{1}{4}e^{4\alpha}. \tag{1.83}
$$

1.3.2 Solving EFE in log time

The EFE in logarithmic time in the new variables α, β, γ are decoupled and, moreover, do not cointain first order derivatives with respect to τ; our task is now to solve analitically the new, simpler, system of equations.

Equation for α

To begin with, consider eq. (1.80):

$$\begin{aligned} 2\ddot{\alpha} &= -e^{4\alpha}, \\ 2\ddot{\alpha}\dot{\alpha} + e^{4\alpha}\dot{\alpha} &= 0, \\ \partial_\tau \left[\dot{\alpha}^2 + \frac{1}{4}e^{4\alpha}\right] &= 0, \end{aligned}$$

from which we get:

$$\dot{\alpha}^2 + \frac{1}{4}e^{4\alpha} = H^2, \tag{1.84}$$

where H^2 is a positive integration constant.
To proceed further we define:

$$x^2 = H^2 - \frac{1}{4}e^{4\alpha},$$

so that:

$$\begin{aligned} x &= \pm\sqrt{H^2 - \frac{1}{4}e^{4\alpha}}, \\ 2xdx &= -\, e^{4\alpha}d\alpha \\ d\alpha &= -\frac{2xdx}{e^{-\alpha}} = -\frac{2xdx}{4(H^2 - x^2)} \end{aligned}$$

In light of this new variable x we can rewrite eq.(1.75)

$$\begin{aligned} \left(\frac{d\alpha}{d\tau}\right)^2 &= H^2 - \frac{1}{4}e^{4\alpha}, \\ \left(\frac{-2xdx}{d\tau 4(H^2 - x^2)}\right)^2 &= x^2, \\ d\tau &= \pm\frac{dx}{2H^2(1 - \frac{x^2}{H^2})}. \end{aligned}$$

Now, making once again a coordinate change:

$$y = \frac{x}{H},$$

we are left with a simpler equation

$$\pm d\tau = \frac{dy}{2H(1 - y^2)},$$

which is integrable and gives:

$$\pm \int d\tau = \int \frac{dy}{2H(1-y^2)},$$
$$\pm(\tau + k_1) = \frac{1}{2H} \int dy \frac{1}{1-y^2},$$
$$= \frac{1}{4H} \log \frac{1+y}{1-y} + k_2.$$

Then, defining $k = k_1 - k_2$ and exponentiating:

$$e^{\pm 4H(\tau+k)} = \frac{1+y}{1-y},$$

we came to an equation for y:

$$y(1 + e^{\pm 4H(\tau+k)}) + 1 - e^{\pm 4H(\tau+k)} = 0,$$
$$y = \frac{e^{\pm 4H(\tau+k)} - 1}{(1 + e^{\pm 4H(\tau+k)})},$$

we are now closer to the end; indeed substituting the explicit value of y:

$$\pm \frac{1}{H} \sqrt{H^2 - \frac{1}{4} e^{4\alpha}} = \frac{e^{\pm 4H(\tau+k)} - 1}{(1 + e^{\pm 4H(\tau+k)})}$$

and squaring both sides

$$\frac{1}{H^2} \left[H^2 - \frac{1}{4} e^{4\alpha} \right] = \left[\frac{e^{\pm 4H(\tau+k)} - 1}{(1 + e^{\pm 4H(\tau+k)})} \right]^2,$$

we arrive to an equation where the exponential of α can be isolated; using some algebra we obtain

$$\frac{1}{H^2} \left[H^2 - \frac{1}{4} e^{4\alpha} \right] = \left[\frac{e^{\pm 4H(\tau+k)} - 1}{(1 + e^{\pm 4H(\tau+k)})} \right]^2,$$
$$\frac{1}{4H^2} e^{4\alpha} = 1 - \left[\frac{e^{\pm 4H(\tau+k)} - 1}{(1 + e^{\pm 4H(\tau+k)})} \right]^2,$$
$$\frac{1}{4H^2} e^{4\alpha} = \frac{(1 + e^{\pm 4H(\tau+k)})^2 - [e^{\pm 4H(\tau+k)} - 1]^2}{(1 + e^{\pm 4H(\tau+k)})^2}$$

$$e^{4\alpha} = \frac{16H^2 e^{\pm 4H(\tau+k)}}{(1 + e^{\pm 4H(\tau+k)})^2},$$

And finally taking the logarithm:

$$\alpha(\tau) = \frac{1}{4} log \frac{16H^2 e^{\pm 4H(\tau+k)}}{(1 + e^{\pm 4H(\tau+k)})^2}. \tag{1.85}$$

Equations for β and γ

Now that we have the explicit form of α it becomes easier to write down the τ-dependent scale factors; to proceed it is enough to observe that

$$\ddot{\alpha} + \ddot{\beta} = 0 = \ddot{\alpha} + \ddot{\gamma}$$

by integrating twice the first of the above equations we obtain:

$$\beta(\tau) = \omega\tau + \sigma - \alpha(\tau) \tag{1.86}$$

where ω and σ are integration constants.

Proceeding in a similar way for γ, we obtain:

$$\gamma(\tau) = \Omega\tau + \Sigma - \alpha(\tau). \tag{1.87}$$

Hypergeometric function

Now that we have found the solutions of the system (1.80), (1.81), (1.82) we have to perform the inverse transformation from eq.(1.79) for come back to the scale factors in terms of the cosmic time t; the problem reduces to solving the following integral:

$$\int_0^\infty dt = \int_{-\infty}^\infty d\tau e^{\alpha+\beta+\gamma} \tag{1.88}$$

whose solution is given in terms of a *Hypergeometric function* ${}_2F_1$, then t is given by:

$$t = -\frac{e^{\sigma+\Sigma+\tau(\omega+\Omega)}{}_2F_1[-\frac{1}{2}, \frac{-H+\omega+\Omega}{4H}, \frac{3H+\omega+\Omega}{4H}, -e^{4H(k+\tau)}]}{2(H-\omega-\Omega)\sqrt{1+e^{4H(k+\tau)}}[\frac{H^2 e^{4H(k+\tau}}{(1+e^{4H(\tau+k)})^2}]^{\frac{1}{4}}} \tag{1.89}$$

1.3.3 Asymptotic solutions

Examining the solutions (1.85), (1.86), (1.87) we see that we deal with two sets of solutions which depend on six integration constants $H, k, \omega, \Omega, \sigma, \Sigma$, which should satisfy a constraint, imposed by the Friedmann equation.

To analyze the asymptotic behavior of the system we have to choose one of the two sets of solutions, since they depends on the sign of the exponential which appears in α, let us select the one with plus sign, so that we are working with:

$$\alpha(\tau) = \frac{1}{4} log \frac{16H^2 e^{4H(\tau+k)}}{(1+e^{4H(\tau+k)})^2},$$
$$\beta(\tau) = \omega\tau + \sigma - \alpha(\tau),$$
$$\gamma(\tau) = \Omega\tau + \Sigma - \alpha(\tau).$$

Now consider the two asymptotic limits of the above system of equation; when $\tau \to +\infty$:

$$\alpha(\tau_{\to+\infty}) \approx C_\alpha - H\tau, \tag{1.90}$$

$$\beta(\tau_{\to+\infty}) \approx C_\beta + (\omega + H)\tau, \tag{1.91}$$

$$\gamma(\tau_{\to+\infty}) \approx C_\gamma + (\Omega + H)\tau, \tag{1.92}$$

with $C_{\alpha,\beta,\gamma}$ constants.
In the opposite limit $\tau \to -\infty$ instead:

$$\alpha(\tau_{\to-\infty}) \approx C_\alpha^{'} + H\tau, \tag{1.93}$$

$$\beta(\tau_{\to-\infty}) \approx C_\beta^{'} + (\omega - H)\tau, \tag{1.94}$$

$$\gamma(\tau_{\to-\infty}) \approx C_\gamma^{'} + (\Omega - H)\tau. \tag{1.95}$$

The first thing to note here is that in α the linear term in τ changes sign, passing from $-H$ to H; the τ linear terms in γ and β otherwise has a sign which depends on the magnitude of the constants ω, Ω and H.

The distant future

Using the above asymptotic form of α, β, γ it is possible to express again the scale factors in terms of the cosmic time t; consider eq.(1.79):

$$\begin{aligned} dt &= ABC d\tau \\ &\sim e^{(H+\omega+\Omega)\tau} d\tau \end{aligned}$$

By integration we obtain:

$$\tau \sim \frac{\log[(H + \omega + \Omega)\tau]}{H + \omega + \Omega}.$$

Now, using this expression for τ in the asymptotic solution we get the scale factors in terms of the cosmic time t:

$$A(t_{\to+\infty}) = e^{\alpha(\tau)} \sim e^{\frac{-H \log t}{H+\omega+\Omega}} \sim t^{\frac{-H}{H+\omega+\Omega}}, \tag{1.96}$$

$$B(t_{\to+\infty}) = e^{\beta(\tau)} \sim e^{\frac{(\omega+H) \log t}{H+\omega+\Omega}} \sim t^{\frac{\omega+H}{H+\omega+\Omega}}, \tag{1.97}$$

$$C(t_{\to+\infty}) = e^{\gamma(\tau)} \sim e^{\frac{(\omega+H) \log t}{H+\omega+\Omega}} \sim t^{\frac{\Omega+H}{H+\omega+\Omega}}. \tag{1.98}$$

Consider now the asymptotic form of the Friedmann equation (1.83) in the distant future:

$$(-H)(\omega + H) + (-H)(\Omega + H) + (\omega + H)(\Omega + H) = \frac{1}{4} e^{-4H\tau}.$$

The right hand side of this equation, due to its τ dependence, vanishes in this limit so that we are left with:

$$H^2 = \omega\Omega \tag{1.99}$$

Remarkably the exponents of the scale factors in this limit satisfies:

$$\frac{-H + H + \omega + H + \Omega}{H + \omega + \Omega} = 1$$

$$\frac{[(-H)^2 + (\omega + H)^2 + (\Omega + H)^2}{(H + \omega + \Omega)^2} = 1 \tag{1.100}$$

The first equality is trivial, the second instead is proved by use of the Friedmann equation, indeed:

$$\frac{H^2 + \omega^2 + H^2 + 2\omega H + \Omega^2 + H^2 + 2\Omega H}{(H + \omega + \Omega)^2} =$$
$$= \frac{3H^2 + \Omega^2 + \omega^2 + 2H(\omega + \Omega)}{(H + \omega + \Omega)^2}$$
$$= \frac{(H^2 + \Omega^2 + \omega^2 + 2H\omega + 2H\Omega + 2\Omega\omega}{(H + \omega + \Omega)^2} = 1.$$

What we have found is that, in the far future limit, the Bianchi type II metric becomes a Kasner solution, because it satisfies the Kasner condition on the time exponents.

The distant past

We now proceed to analyze what happens in the opposite limit $\tau \to -\infty$ for our class of solutions; in this case eq.(1.79) gives, using equations (1.93), (1.94) and (1.95):

$$\tau = \frac{\log[(\omega + \Omega - H)t}{\omega + \Omega - H}.$$

So that the scale factors for $\tau \to -\infty$, $t \to 0$ become:

$$A(t_{\to 0}) = t^{\frac{H}{\omega+\Omega-H}}, \tag{1.101}$$

$$B(t_{\to 0}) = t^{\frac{\omega-H}{\omega+\Omega-H}}, \tag{1.102}$$

$$C(t_{\to 0}) = t^{\frac{\Omega-H}{\omega+\Omega-H}}. \tag{1.103}$$

The first Kasner condition is trivially satisfied, while for the second one we have:

$$\frac{3H^2 + \omega^2 + \Omega^2 - 2H(\omega + \Omega)}{(\omega + \Omega - H)^2}$$
$$= \frac{H^2 + \omega^2 + \Omega^2 - 2H\omega - 2H\Omega + 2\omega\Omega}{(\omega + \Omega - H)^2} = 1$$

where we use again the Friedmann equations, which reads:

$$(H)(\omega - H) + (H)(\Omega - H) + (\omega - H)(\Omega - H) = e^{H\tau}$$

when τ goes to $-\infty$ the right-hand side of the equation disappears and we are left with:

$$-H^2 + \omega\Omega = 0$$

which is identical to eq.(1.99).

So we have found that in the distant past limit the Bianchi type II universe behaves like a Kasner space-time.

1.3.4 Miscellanea on B-II spacetime

We have seen that the B-II geometry has the following dynamics :

- The $A(t)$ scale factor in asymptotic regimes behaves like a power of the t coordinate with an exponent that switches the sign during the evolution.
- In both asymptotic regimes we found an universe which looks flat but anisotropic, e.g. Kasner spaces, but with a different anisotropy manifested in the reshuffling of the Kasner exponents.
- Starting from t close to 0 we have the A scale factor which grows since has a positive exponent, and one among B or C which is decreasing, nominally B if $\omega \leq \Omega$.
- In some intermediate regime, A stops growing reaching a maximum and start decreasing; in the same way B arrives to its minimum and then starts to increase.
- When both A and B have completed their transitions, reaching the regime $t \to \infty$, the universe find itself in another Kasner epoch.

We have performed the qualitative analysis of the dynamic in B-II space-times in terms of the cosmic time, but the solution founded in terms of the logarithmic time is exact and to recover the complete dynamics with the inverse transformation needs only the evaluation of the Hypergeometric ${}_2F_1$. In the above analysis we have considered just one set of solutions, the one with the $+$ sign in the argument of the exponent which appears in α, but the other set of solution has qualitatively the same behavior just switching the role of the initial and final states, as we could expect since we are working with the classical general relativity which is symmetric with respect to the inversion of time.

Chapter 2

Horava Lifshitz gravity

It is a know fact that quantum gravity is non-renormalizable; among the attempts made to overcome this issue collocates Horava's proposal of Lifshitz type gravity [20], which make it renormalizable at the expense of the Lorentz invariance.

The structure of the chapter is the following; in the first section we introduce Horava's original formulation and the related conceptual features, in section two we drift our attention to further developements of the theory, listing some results and remained open questions; finally, in the last section, we focus on the cosmological implications of this new space-time standpoint.

2.1 Horava-Lifshitz background

If Lorentz symmetry holds beyond ultra-high-energy is matter of debate at least since the nineties [22] [23]; even if there is no observed experimental deviation from Lorentz invariance there are many challenging theoretical frameworks in which it is just a low-energy emergent symmetry of space-time, Horava gravity is one of such theories.

2.1.1 Abandoning Lorentz invariance

Treating Einstein's general relativity from the standpoint of quantum field theory, one is faced with the problem of dealing with a dimensionful gravitational coupling constant $[G_N] = -2$ in mass units, which hampers the perturbative renormalizability of the theory.
Since the graviton propagator, the Green function in momentum-space, scales

with the four-momentum k_μ as

$$G(\omega, k) = \frac{1}{k^2},$$

where k is

$$k = \sqrt{\omega^2 - \vec{k}^2},$$

when we calculate Feynman diagrams at increasing loop orders it is necessary to introduce counterterms with increasing degree in curvature in order to keep them finite.
Such unhealty behavior could be overcomed if we modify the propagator in such a way that it brokes Lorentz-invariance:

$$G(\omega, k) = \frac{1}{k^2 - G_N k^4},$$

and this is obtained adding new high-order curvature terms in the Lagrangian, which become responsible for the Lorentz symmetry breaking; since the new propagator at high momentum is dominated by the k^4 term, the ultraviolet divergences are under control.
By the way, such propagators are not the end of the story, indeed they have two poles:

$$\frac{1}{k^2 - G_N k^4} = \frac{1}{k^2} - \frac{1}{k^2 - \frac{1}{G_N}},$$

and the second corresponds to ghost excitations because it corresponds to a negative kinetic energy, and would imply a violation of unitarity; the family of theories with this kind of regularization are called *higher derivative* gravitational theories.

In [20] Horava tries to overcome these problems postulating that space and time, in their quantum nature, are deeply different and have to be treated separately.
This idea is inherited from the physics of condensed matter systems in which the action shows *anisotropic scaling* between space and time; e.g. the theory of a Lifshitz scalar in $D + 1$ dimensions with an action of the form:

$$S = \int d^D x dt \left[(\dot{\Phi})^2 - (\triangle\ \Phi)^z \right], \tag{2.1}$$

with anisotropic scaling of degree z.
Considering the case $z = 2$, if we add to the action a term

$$-c^2 \int d^D x dt \partial_i \Phi \partial^i \Phi, \tag{2.2}$$

we obtain a theory that, at long distances, when spatial second order terms become negligible in the infrared, recovers Lorentz invariance.
Horava theory is based on the assumption that the action for the gravitational interaction has such kind of anisotropic scaling between space and time, with the Lorentz symmetry which is hopefully recovered in the infrared, where space-time behaves in agreement with Einstein's general relativity.

2.1.2 Horava's construction

The starting point of Horava construction is the following line element:

$$ds^2 = -N^2c^2dt^2 + g_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right), \tag{2.3}$$

in which N is the shift, N^i the lapse and g_{ij} is the spatial metric, exactly as in the ADM decomposition (1.71).
In general relativity we can represent the line element in this way since we have freedom to foliate space-time in terms of spacelike surfaces Σ_t, anyway, in Horava gravity, this decomposition is not just a choice of coordinate parametrization permitted by the Gauge symmetry but rather the postulate that space-time has *a priori* such preferred foliation.

To implement the concept of anisotropic scaling let us postulate that space and time have the following dimensions:

$$[dx] = [k]^{-1},$$
$$[dt] = [k]^{-z},$$

where k is an object with the dimensions of momentum; in condensed matter physics the dynamical critical exponent z measures the degree of anisotropy between space and time.
In this way we have implicitly introduced a scale $Z = \frac{[dx]^z}{[dt]}$; two kinds of units are used to express it[24]:

- the so called *theoretician's units* which set $Z = 1$,

- in terms of the momentum ζ we have $Z = \zeta^{-z+1}c$, from which we see that one cannot simultaneously set both $c = 1$ and $Z = 1$ without loose the ability of perform power counting analysis

Using the first choice of units we postulate the classical scaling dimension of the fields to be [20]:

$$[N^i] = \frac{[dx]}{[dt]} = [k^{z-1}],$$
$$[g_{ij}] = [N] = 1,$$

The kinetic term

In order to construct a kinetic term that contains only first time derivatives of the spatial metric and moreover is invariant under the foliation-preserving diffeomorphisms, e.g. such diffeomorphisms which conserve the foliation of space time in spatial hypersurfaces Σ_t, we are forced to consider as a building block the extrinsic curvature, or the second fundamental form [25]

$$K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right) \tag{2.4}$$

so that the general kinetic term for the Horava Lifshitz action is:

$$S_{kin} = \frac{1}{g_K}\int d^3x d^t N\sqrt{g}\left[K_{ij}K^{ij} - \lambda K^2\right] \tag{2.5}$$

where g_K is the coupling constant for the kinetic term of the action; in theoretician's units we have:

$$[g_K] = [k]^{d-z}.$$

The only difference between the action (2.5) and the general ADM kinetic term is that if one wants the theory to be invariant under all space-time diffeomorphism and not just those foliation-preserving one is forced to take $\lambda = 1$.

The potential term

The potential term in usual ADM action is simply P, the spatial Ricci scalar built from the three-dimensional metric g_{ij}; however, since we have postulated the anisotropic scaling among space and time, in Horava gravity there are many more terms with increasing order of spatial derivatives.
Horava considered the 3+1 dimensional case with dynamical critical exponent $z = 3$ just because with this choice the coupling constant of the kinetic term becomes dimensionless; the general potential term with this scaling is:

$$S_{V_{HL}} = \int d^3x dt\sqrt{g}NV(g_{ij}), \tag{2.6}$$

with

$$[S_{V(g_{ij})}] = [k]^6.$$

The list of independent operators which are compatible with the dimensional constraint is very large, and contains terms cubic and quadratic in curvature tensor P_{ij}; those of dimension $[k]^6$ are:

$$P^3,\ PP^i_jP^j_i\ ,\ P^i_jP^j_kP^k_i\ ,\ P\nabla^2 P\ ,\ \nabla_i P_{jk}\nabla^i P^{jk},$$

while the lower dimension terms are:

$$P^2,\ P^i_jP^j_i,\ P,$$

and it is easy to see that in this general case the number of terms and so the number of coupling constants becomes huge.

In order to keep the number of terms in the potential under control Horava proposed two additional symmetries for the action, the *projectability* and the *detailed balance* conditions; the first encoded in the request for the lapse N to be a function of time only, while the second inspired by the theory of critical phenomena, where is supposed that the potential is quadratic with respect to a tensor E_{ij} which is itself derivable from the variation of some action W. These two symmetries set the potential to be:

$$S_{V_{HL}} = \frac{g_K^2}{8}\int d^3x dt\sqrt{g}NE^{ij}G_{ijkl}E^{kl} \tag{2.7}$$

$$\sqrt{g}E^{ij} = \frac{\delta W(g_{ij})}{\delta g_{ij}} \tag{2.8}$$

where G_{ijkl} is the inverse of the generalized DeWitt supermetric[26]; since we are working in 3+1 dimensions the only tensor of third order in spatial derivatives which satisfy the above conditions is the *Cotton* tensor C_{ij}, defined to be the traceless part of the spatial Ricci tensor, in analogy with the Weyl tensor for the four-dimensional case:

$$C_{ij} = \epsilon^{ikl}\nabla_k\left((P^j_l - \frac{1}{4}P\delta^j_l\right),$$

which is obtained by varying the following spatial action [25]

$$W = \int d^3x\epsilon^{ijk}\left(\Gamma^m_{il}\partial_j\Gamma^l_{km} + \frac{2}{3}\Gamma^n_{il}\Gamma^l_{jm}\Gamma^m_{kn}\right). \tag{2.9}$$

Summing up we are left with the potential

$$\begin{aligned} S_{V_{HL}} &= \int dtd^3x\sqrt{g}N\left[\frac{-g_K^2}{2\omega^4}C^{ij}C_{ij}\right] \\ &= \int dtd^3x\sqrt{g}N\left[\frac{-g_K^2}{2\omega^4}\left(\nabla_i P_{jk}\nabla^i P^{jk} - \nabla_i P_{jk}\nabla^j P^{ik} - \frac{1}{8}\nabla_i P\nabla^i P\right)\right]. \end{aligned}$$

If we now add to the action W the folling deformation:

$$\mu\int d^3x\sqrt{g}\left(R - 2\Lambda_W\right), \tag{2.10}$$

the potential term in the action $S_{V_{HL}}$ becomes:

$$\begin{aligned} V_{HL} =& \frac{-g_K^2}{2\omega^4}C^{ij}C_{ij} + \frac{g_k^2\mu}{2\omega^2}\epsilon^{ijk}P_{il}\nabla_j P_k^l - \frac{g_K^2\mu^2}{8}P^{ij}P_{ij} \\ &+ \frac{g_K^2\mu^2}{8(1-3\lambda)}\left(\frac{1-4\lambda}{4}P^2 + \Lambda_W P - 2\Lambda_W^2\right), \end{aligned}$$

at long distances this potential is dominated by the last two terms, the cosmological constant and the spatial curvature, and the theory flows in the infrared to $z = 1$ so that Lorentz invariance is accidentally restored.

2.2 Beyond Horava's original proposal

2.2.1 Beyond detailed balance

The detailed balance condition is a simplifying ansatz known in physics at least since the 1872, when Boltzmann used it in order to prove the *H-theorem*[27]; fundamentally it is a hypotesis about the equilibrium condition among systems.

Suppose that we have a set of states $\{n\}$ and the density matrix ρ, defining as usual the conditional probability:

$$\rho_{nm} =< n|\rho|m >,$$

the evolution in time of the probability distribution of a state is usually written in terms of a master equation:

$$\partial_t\rho(n) = -\sum_m \phi_{mn}\rho(n) + \sum_m \phi_{nm}\rho(m).$$

With this equation we are stating that the variation on the probability distribution of the state n is given by the difference between the incoming and

outgoing fluxes ϕ, with the fluxes defined to be the sums of the conditional probabilities of n and the rest of the states.
Looking for stationary distributions lead us to

$$\partial_t \rho(n) = 0 \tag{2.11}$$

$$\sum_m \phi_{mn}\rho(n) = \sum_m \phi_{nm}\rho(m). \tag{2.12}$$

Askind for detailed balance is a stronger condition with respect to the latter, and is realized by requiring that:

$$\phi_{mn}\rho(n) = \phi_{nm}\rho(m), \tag{2.13}$$

which is pratically a condition on the reversability of each process.

The detailed balance condition in Horava's original proposal is encoded in equations (2.8) and (2.9), here we constraint the potential to be derivable from an action which is function of the spatial metric only, in some sense eq. (2.9) is the analogue of eq. (2.13).

Troubles with detailed balance

In [20] Horava writes, in regard of detailed balance, that *The reason for this restriction is purely pragmatic, to limit the proliferation of independent couplings.*
The absence of physical reasons justifying detailed balance, together with the occurrence of some inconsistencies with respect to observational data, has naturally led to an extension of the theory without this assumption but which keep on track the projectability condition.

Let us list some of the inconsistencies of the detailed balance version of the theory:

- The cosmological constant appears in the potential with the wrong sign to be compatible with cosmological observation[21]
- The deformation (2.10) generates a fifth-order operator which violates parity [28][29]
- Using detailed balance without projectability seriously prejudices the power counting renormalizability of the theory according to [30]
- In [31] [32] is stated that the theory without matter does not have a general perturbative IR limit, and so general relativity is not restored at low energies and the scalar graviton of the theory shows *strong coupling.*

Projectable and parity invariant Horava-Lifshitz gravity

Imposing the projectability condition $N = N(t)$ without detailed balance and suppressing parity violating terms leave us more freedom in the choice of the coupling parameters, actually one is left with the following linear combination of terms[24]:

$$\begin{aligned} V_{HL} =& g_0\zeta^6 + g_1\zeta^4 P + g_2\zeta^2 P^2 + g_3\zeta^2 P^i_j P^j_i + \\ &+ g_4 P^3 + g_5 P^i_j P^j_i P + g_6 P^i_j P^j_k P^k_i + \\ &+ g_7 P\nabla^2 P + g_8 \nabla_i P_{jk} \nabla^i P^{jk}, \end{aligned} \tag{2.14}$$

where we have introduced the factors ζ's to keep the coupling constants dimensionless.
In this version of the theory we don't have constraints on the sign of the cosmological constant, moreover the power counting renormalizability seems ensured [33].

Note that if in the original theory the detailed balance hypotesys was asked in order to simplify the form of the potential, the projectable version without detailed balance still remains quite simple and reduces to the original form of the potential [20] with suitable choices of the coupling constants (except for the parity violating operators which were suppressed for convenience).
Finally we want mention that the projectability condition could always be imposed in general relativity by the gauge symmetry, and that there are many solutions of physical interest in which it must hold globally, like in the Schwarzschild space-time with Painleve-Gullstrand coordinates or in general FLRW and Bianchi cosmologies[21].

2.2.2 Beyond projectability

The request that the lapse N is function of time only is sufficient to set a well-behaved potential term for the action functional, because we have now the ability to keep under control the couplings g_i in order to resemble general relativity at long distances; by the way, we still have a potentially dangerous Lorentz violating term in the kinetic part of the action if $\lambda \neq 1$, which correspond to an extra scalar mode graviton.

Another feature of the projectable version of the theory is that, varying with respect to the lapse $N(t)$ the action, we obtain the Hamiltonian constraint

$$H = \int d^3x\sqrt{g}\,[\mathcal{T}(K) + V_{HL}] = 0,$$

this spatially integrated Hamiltonian constraint is the best we can obtain without relaxing the projectability condition and is a common feature of all the projectable versions of the theory independently from the form of the potential; this is another difference with respect to general relativity case in which N is a function of both space and time coordinates and whose variation leads to a local super-Hamiltonian constraint[24].

In order to avoid pathological behaviors of the scalar graviton arising from $\lambda \neq 1$ several extension of Horava model were considered in which the projectability condition is relaxed; in these models one is forced to impose further symmetries since, otherwise, the number of terms in the potential increase of a magnitude order with respect to the projectable version, and terms containing the spatial derivative of the lapse are introduced.

Among the explored non-projectable extensions of the theory we mention [34]:

- *Healthy extensions* theories, in which the preferred foliation of space time is promoted to be dynamical [35]
- The attempt for imposing *further symmetries* as done for example in [36] and [37]
- The work [38] which is set in the constest of the *analogous spacetimes* programme.

We must remark howewer that no one of the above strategies was yet fully satisfying, moreover the capability of performing power counting analysis in the non-projectable case is complicated because of the appearence of singularities in the propagators which are not local in space, as stated in [33], and whose implications are yet to explore.

2.3 Horava Lifshitz cosmology

There are a number of interesting cosmological implications of Horava Lifshitz gravity both in IR and UV limits; since in general the results depend on the specific version of the theory we have to distinguish among those statement which are due to the main principle of the theory, the anisotropic scaling, and those related to the particulars simplifying assumptions invoked for the potential.

We have already seen in the preceeding chapter that it is a common strategy, in the search for standard cosmological solutions of the Einstein's field equations, to consider some additional symmetries for the space-time manifold. Since in the majorities of these solutions we use metrics in which the time coordinate is indipendent from the spatial ones the projectability condition, which we stress could be always implemented thanks to the gauge symmetry in Einstein's general relativity, once promoted to a fundamental principle shows interesting features.

2.3.1 Projectable Horava Lifshitz cosmology

Dark matter in HL gravities

Since in the projectable version of Horava gravity we have a spatially integrated Hamiltonian constraint, instead of a local super Hamiltonian as in general relativity, in FLRW models the 00 component of the Einstein equations becomes invalid. The conseguence is that we are forced to use the second Friedmann equation, which contains second order time derivatives of the scale factor, to impose the initial conditions.
The first integral of this second Friedmann equation gives us something which looks like the super Hamiltonian constraint plus other terms which follow from the integration, in flat FLRW background the equation explicitly looks:

$$\frac{3(3\lambda-1)}{2}\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G_N\left(\rho + \frac{C_0}{a^3}\right),$$

where C_0 is the dark matter integration constant; actually it is not a real matter contribute, in the sense that it does not come from adding a matter sector in the action, but emerges as a result of the projectability assumption, and behaves like pressure-less dust at least at low energies [39].

Bouncing universes

Considering non-flat FRLW backgrounds with potential (2.14) the second Friedmann equation becomes [40]:

$$-\frac{3\lambda-1}{2}\left[2\partial_t\left(\frac{\dot{a}}{a}\right)+3\left(\frac{\dot{a}}{a}\right)^2\right] = 8\pi G_N P - \frac{\alpha_3\kappa^3}{a^6} - \frac{\alpha_2\kappa^2}{a^4} + \frac{\kappa}{a^2} - \Lambda, \quad (2.15)$$

where the constants α_2, α_3 are linear combinations of the coupling parameters related to the sixth order and fourth order in spatial derivative terms which enters in the potential.

The above equation admits the following first integral:

$$\frac{3(3\lambda-1)}{2}\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G_N\left(\rho+\frac{C_0}{a^3}\right)-\frac{\alpha_3\kappa^3}{a^6}-\frac{3\alpha_2\kappa^2}{a^4}-\frac{3\kappa}{a^2}+\Lambda, \qquad (2.16)$$

which reduces to the one for the flat FLRW background when the spatial curvature vanishes.
This first integral could be rewritten like an energy conservation equation for a;

$$\frac{\dot{a}^2}{2}+\frac{2}{(3\lambda-1)}V(a)=0,$$

note that the qualitative behavior of the system is complitely determined from the potential $V(a)$, whose sign depends on the value of λ; if $\lambda \geq \frac{1}{3}$ the potential $V(a)$ must be $\leqslant 0$.
When $V(a)=0$ the scale factor changes behavior, reaching a maximum or a minimum; the overall qualitative analysis shows that for suitable choices of the coupling constant exist bouncing solutions for the Friedmann curved spaces, so there is in HL gravity a mechanism due to which cosmological singularities can be avoided.
Let us consider the following examples:

- If there is just one value a_0 of the scale factor in which the potential vanish such that for $a < a_0$, $V(a) > 0$ and $V(a) < 0$ for $a > a_0$ then the universe is contracting at the beginning of the evolution and after a bounce at $a = a_0$ starts to expand.

- If there are two extremum points a_1, a_2 for which the potential vanish and such that in the interval $a_1 < a < a_2$ we have $V(a) < 0$, and $V(a) < 0$ outside, then the evolution becomes periodic.

- If $V(a_0) = 0$ is a local minimum or maximum then we have static solutions which are respectively stable and non-stable.

Cosmological implications of the anisotropy

In [41] is presented another intriguing feature of the Horava Lifshitz cosmology which is based on the principle of anisotropic scaling, whitin this framework indeed there is a mechanism thanks to which it is possible to generate cosmological perturbations without requiring inflation.
In standard relativistic cosmology we have the following dispersion relation

for the linearized Fourier modes of the cosmological perturbations in FLRW backgrounds:

$$\omega^2 = \frac{c_s^2 k^2}{a^2},$$

with c_s, k^2 the speed of sound and the comoving wawe number. We see that these modes are insensitive of the expansion rate of the universe if $\omega^2 >> \left(\frac{\dot{a}}{a}\right)^2 = H^2$; on the other hand, when H^2 becomes larger respect to ω^2 then the *Hubble friction* due to the expansion of the universe freezes the mode which remains almost constant. Cosmological perturbations are generated by the quantum fluctuations that follow the freeze-out, then the condition for such perturbations is:

$$\frac{d}{dt}\left(\frac{H^2}{\omega^2}\right) > 0.$$

In Horava gravities however, the dispersion relation is modified due to the anisotropic scaling, qualitatively we have:

$$\omega^2 \propto M^2 \left(\frac{k^2}{M^2 a^2}\right)^z,$$

from which we obtain that the necessary condition to get cosmological perturbation is:

$$\frac{d^2 a^z}{dt^2} > 0,$$

this condition is satisfied also for non accellerating universe, for example a power law expansion $a \propto t^p$ with $p > \frac{1}{3}$ suffices; the conseguence is that we don't need inflation in order to generate the spectrum of cosmological perturbation.

2.3.2 Horava-Lifshitz homogeneous cosmologies

The problem of singularities in quantum cosmology reflects our lack of knowledge about quantum gravity; Hawking and Penrose [13] have shown that general relativity predicts under certain assumptions the occurrence of a space-time singularity usually referred to as *Big bang*, unfortunatelly general relativity cannot describe the physics of the singularity itself.
The study of Friedmann universes, together with the developement of cosmological dark energy models, have generated scenarios in which other kind of *sudden future singularities* arise quite naturally [42]; for this reasons it is required that a satisfactory theory of quantum gravity have to be able of

provide those answers which general relativity cannot do about these singularities.
Since Horava gravity in curved FLRW background allows for bouncing solutions there is a mechanism in this framework thanks to which the cosmological singularities can be avoided, look for what happens beyond the assumption of spatial homogeneity and isotropy could show if such avoiding mechanisms are present even in less symmetric cases.

The ADM decomposition of space time is particularly suited for the homogeneous cosmologies, in these models indeed it is possible to construct from the algebra of Killing vector fields such coordinate basis for which the components of the spatial metric are function of time only; this, together with the projectability condition, reduces hugely the number of terms compatible with the anisotropic scaling in the Horava-Lifshitz potential part of the action, moreover, like in general relativity, the field equations still are a set of ordinary differential equations in time.

We know yet that in general relativity there exist a mechanism which avoid the occurrence of a singularity, that is, the BKL oscillatory approach [14] related to the study of Bianchi type IX spacetimes, here the stochastic character of the solutions of EFE inspired the name *mixmaster universe*[43]. It is interesting to know if such chaotical character is present even in a Bianchi IX spacetime with anisotropic scaling both in the IR and UV regimes; some analysis were performed in [44, 45] but looks like that the answer about the occurrence of chaoticity is sensible on the specific model of HL gravity used. In the contest of homogeneous Horava-like cosmologies we want also mention the work [46] in which the analysis of the Bianchi type III Horava Lifshitz spacetime was performed, here the explicit field equations are given for generic λ but they reduces to an Abel equation of the first kind whose general solution was not obtained.

The above mentioned analysis were performed via Hamiltonian methods, howewer in the next chapter we are going to study the proprieties of the B-II HL cosmological model with a different approach, more close to the one used in the BKL analysis; the main reason for such approach is that the mixmaster dynamics in general relativity is qualitatively described by a succession of Kasner transitions with stocastic character, the dynamics of each individual transition from one Kasner epoch to another is well described by the dynamic of the Bianchi II spacetime.
We hope that the knowledge of the Bianchi II space-time with anisotropic scaling will be useful to clarify at least qualitatively the physics of Bianchi

IX space-time in Horawa Lifshitz background.

Chapter 3

Bianchi type II in Horava Lifshitz gravity

It is supposed that Horava gravity recovers Einstein's general relativity in the low energy limit, where $\lambda = 1$ and the higher curvature terms in the potential become negligible; however, it is interesting to ask how smooth it is the transition from the quantum anisotropic regime in the UV and the general relativity fixed point in the IR.
Even if a satisfactory answer to this question can be given only with a complete analysis of the renormalization group equations of the theory, comparing known solutions of Einstein's general relativity with the analogues in Horava gravity, close to the IR, is a good starting point for contradict or reinforce the correspondence principle which is at the grounds of the proposal [20].
Homogeneous cosmologies were deeply studied in the contest of general relativity and provide a wide arena for testing gravity theories; with this motivation in this chapter we are going to study within the framework of Horava gravity the behavior of the Bianchi type II spacetime.
The ultimate reason for doing so is that it represents the first homogeneous model in the Bianchi classification with no trivial algebra of the spatial Killing vector fields; actually in this work we shall consider the simplest curved manifold which shows anisotropy both in the spatial sections and in the scaling between space and time.

The structure of the chapter is the following; in the first section we consider the potential and the kinetic term for the Horava-Lifshitz action using the B-II spatial metric, then we proceed in varying the obtained functional to derive the field equations for the scale factors; section two is devoted to the qualitative analysis of the system of differential equations.

Finally, in section three, we will compare the solutions obtained with the standard results of Einstein's general relativity for the Bianchi type II space-time.

3.1 The field equations

3.1.1 The kinetic term

To begin with we have to construct a suitable kinetic term for the action, since in Bianchi models we freely set the shift $N^i = 0$ the extrinsic curvature is simply given by:

$$K_{ij} = \frac{1}{2N}\partial_t g_{ij},$$

and the kinetic part of the action becomes:

$$\begin{aligned} S^{Kin}_{BII-HL} &= \int d^3x dt \sqrt{g} N(t) \left[K^{ij}K_{ij} - \lambda K^2\right] \\ &= \int d^3x dt \sqrt{g} N(t) \left[\frac{1}{N^2}\left(\frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + \frac{\dot{C}^2}{C^2}\right) - \frac{\lambda}{N^2}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)^2\right] \\ &= \int d^3x dt \sqrt{g} N \left[\left(\frac{1-\lambda}{N^2}\right)\left(\frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + \frac{\dot{C}^2}{C^2}\right) - \frac{2\lambda}{N^2}\left(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC}\right)\right], \end{aligned} \tag{3.1}$$

and is easy to verify that the above restores the kinetic term proper of the general relativity when $\lambda = 1$.

3.1.2 The potential term

We have seen in the previous chapter that anisotropic scaling allows terms up to sixth order in spatial derivatives inside the potential part of the action, so we have to calculate terms quadratic and cubic in the spatial Ricci tensor P^α_β. In chapter one we have found the explicit form of this tensor when the invariant coordinate basis is used:

$$P^a_b = \frac{1}{2g}[2C^{bd}C_{ad} + C^{bd}C_{da} + C^{db}C_{ad} - C^d_d(C^b_a + C^b_a) + \delta^b_a((C^d_d)^2 - 2C^{df}C_{df})].$$

For the B-II spatial metric we have

$$P^i_j = \frac{1}{2A^2B^2C^2}[-(g_{11})^2\delta^i_j + 2(g_{11})^2\delta^1_j\delta^i_1]. \tag{3.2}$$

from which we can calculate:

$$\begin{aligned}
P &= -\frac{A^4}{2A^2B^2C^2},\\
P^2 &= \frac{1}{4}\frac{A^8}{A^4B^4C^4},\\
P^i_jP^j_i &= \frac{3}{4}\frac{A^8}{A^4B^4C^4},\\
P^3 &= -\frac{1}{8}\frac{A^{12}}{A^6B^6C^6},\\
P^i_jP^j_kP^k_i &= -\frac{1}{8}\frac{A^{12}}{A^6B^6C^6},\\
P^i_jP^j_iP &= -\frac{3}{8}\frac{A^{12}}{A^6B^6C^6}.
\end{aligned}$$

In the potential there are also terms that contain the spatial covariant derivative of the spatial Ricci tensor, anyway it is straightforward to realize that those terms vanish since the spatial Ricci tensor components are functions of time only.
Finally we are left with the following potential term for the action:

$$V_{BII-HL} = c_1\frac{A^4}{A^2B^2C^2} + c_2\frac{A^8}{A^4B^4C^4} + c_3\frac{A^{12}}{A^6B^6C^6}, \tag{3.3}$$

with

$$\begin{aligned}
c_1 &= -\frac{1}{2}g_1\\
c_2 &= \frac{1}{4}g_2 - \frac{3}{4}g_3\\
c_3 &= -\frac{1}{8}g_4 - \frac{3}{8}g_5 - \frac{1}{8}g_6
\end{aligned}$$

3.1.3 The field equations

We are interested in the low-energy limit of the Horava Lifshitz gravity because we would like to establish if the overall behavior is different from the one expected in general relativity; consequently we will set $\lambda = 1$ in the kinetic term (3.1) and subtracting (3.3) we arrive to an action of the form:

$$S_{BII-HL} = \int dtd^3x\sqrt{g}N(t)\left[-\frac{2}{N^2}\left(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC}\right) + V_{BII-HL}\right] \tag{3.4}$$

We want to stress that choosing $\lambda = 1$ we are preventing us from the appearence of the extra scalar graviton, so it is a reasonable assumption if we want to recover a description close to the standard picture of general relativity.

After some algebra on eq.(3.4) we are left with the following Lagrangian:

$$\mathcal{L} = \left[\frac{-2}{N}\left(\dot{A}\dot{B}C + \dot{A}B\dot{C} + A\dot{B}\dot{C}\right) + N(t)\left(c_1\frac{A^3}{BC} + c_2\frac{A^5}{B^3C^3} + c_3\frac{A^7}{B^5C^5}\right)\right],$$

to begin with let us varying the above with respect to the lapse N

$$\frac{2}{N^2}\left(\dot{A}\dot{B}C + \dot{A}B\dot{C} + A\dot{B}\dot{C}\right) + \left(c_1\frac{A^3}{BC} + c_2\frac{A^5}{B^3C^3} + c_3\frac{A^7}{B^5C^5}\right) = 0,$$

which is the Friedmann-like equation, we can rewrite it choosing the gauge $N = 1$:

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} = -\frac{1}{2}\left(c_1\frac{A^2}{B^2C^2} + c_2\frac{A^4}{B^4C^4} + c_3\frac{A^6}{B^6C^6}\right). \tag{3.5}$$

to obtain the variation with respect to A in this gauge we have to calculate:

$$\partial_t\frac{\partial\mathcal{L}}{\partial\dot{A}} = -2\left(\ddot{B}C + B\ddot{C} + 2\dot{B}\dot{C}\right)$$
$$\frac{\partial\mathcal{L}}{\partial A} = \left(-2\dot{B}\dot{C} + 3c_1\frac{A^2}{BC} + 5c_2\frac{A^4}{B^3C^3} + 7c_3\frac{A^6}{B^5C^5}\right)$$

thus the field equation is

$$-2\left(\ddot{B}C + B\ddot{C} + 2\dot{B}\dot{C}\right) - \left(-2\dot{B}\dot{C} + 3c_1\frac{A^2}{BC} + 5c_2\frac{A^4}{B^3C^3} + 7c_3\frac{A^6}{B^5C^5}\right) = 0$$

and after some algebra it becomes

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} = -\frac{1}{2}\left(3c_1\frac{A^2}{B^2C^2} + 5c_2\frac{A^4}{B^4C^4} + 7c_3\frac{A^6}{B^6C^6}\right) \tag{3.6}$$

Now, proceeding in a similar way for the others scale factors we obtain:

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} = +\frac{1}{2}\left(c_1\frac{A^2}{B^2C^2} + 3c_2\frac{A^4}{B^4C^4} + 5c_3\frac{A^6}{B^6C^6}\right) \tag{3.7}$$
$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = +\frac{1}{2}\left(c_1\frac{A^2}{B^2C^2} + 3c_2\frac{A^4}{B^4C^4} + 5c_3\frac{A^6}{B^6C^6}\right) \tag{3.8}$$

It is useful to define the quantity $\tilde{V}$ by

$$\tilde{V} = \frac{A^2}{B^2C^2}$$

so that the dynamical equations can be written:

$$\begin{aligned}
\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} &= -\frac{1}{2}\left(+3c_1\tilde{V} + 5c_2\tilde{V}^2 + 7\tilde{V}^3\right), \\
\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} &= +\frac{1}{2}\left(c_1\tilde{V} + 3c_2\tilde{V}^2 + 5\tilde{V}^3\right), \\
\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} &= +\frac{1}{2}\left(c_1\tilde{V} + 3c_2\tilde{V}^2 + 5\tilde{V}^3\right), \\
\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} &= -\frac{1}{2}\left(c_1\tilde{V} + c_2\tilde{V}^2 + c_3\tilde{V}^3\right).
\end{aligned}$$

Field equations in logarithmic time

The left hand side of the dynamical equations share the same structure of the Bianchi II space-time in general relativity; for this reason it is useful to write them as functions of the logarithmic time τ defined in (1.79), as in the classical case, indeed, the second order time derivatives of the fields in terms of $\alpha(\tau), \beta(\tau), \gamma(\tau)$ result decoupled.
To begin with note that:

$$\begin{aligned}
\frac{\ddot{B}}{B} &= \frac{1}{B}\partial_t\left(\frac{\partial_\tau B}{ABC}\right) \\
&= \frac{1}{B}\frac{\partial_\tau}{ABC}\left(\dot{\beta}e^{-\alpha-\gamma}\right) \\
&= \frac{1}{AB^2C}\left(\ddot{\beta} - \dot{\beta}\left(\dot{\alpha} + \dot{\gamma}\right)\right)e^{-\alpha-\gamma} \\
&= \frac{1}{A^2B^2C^2}\left(\ddot{\beta} - \dot{\beta}\left(\dot{\alpha} + \dot{\gamma}\right)\right),
\end{aligned}$$

and also that holds

$$\frac{1}{A^2B^2C^2} = \frac{A^2}{A^4B^4C^4} = \frac{1}{A^4}\tilde{V}.$$

Using these relations in the field equations we are left with:

$$\begin{aligned}
\left[\ddot{\beta} + \ddot{\gamma} - \dot{\beta}\left(\dot{\alpha} + \dot{\gamma}\right) - \dot{\gamma}\left(\dot{\alpha} + \dot{\beta}\right) + \dot{\beta}\dot{\gamma}\right] &= -\frac{1}{2}A^4\left(3c_1 + 5c_2\tilde{V} + 7c_3\tilde{V}^2\right), \\
\left[\ddot{\alpha} + \ddot{\gamma} - \dot{\alpha}\left(\dot{\beta} + \dot{\gamma}\right) - \dot{\gamma}\left(\dot{\alpha} + \dot{\beta}\right) + \dot{\alpha}\dot{\gamma}\right] &= \frac{1}{2}A^4\left(c_1 + 3c_2\tilde{V} + 5c_3\tilde{V}^2\right), \\
\left[\ddot{\alpha} + \ddot{\beta} - \dot{\alpha}\left(\dot{\beta} + \dot{\gamma}\right) - \dot{\beta}\left(\dot{\alpha} + \dot{\gamma}\right) + \dot{\alpha}\dot{\gamma}\right] &= \frac{1}{2}A^4\left(c_1 + 3c_2\tilde{V} + 5c_3\tilde{V}^2\right),
\end{aligned}$$

together with the constraint

$$\left(\dot{\alpha}\dot{\beta}+\dot{\alpha}\dot{\gamma}+\dot{\beta}\dot{\gamma}\right)=-\frac{1}{2}A^4\left(c_1+c_2\tilde{V}+c_3\tilde{V}^2\right). \tag{3.9}$$

By use of this Friedmann-like equation in the rest of the system we obtain:

$$\begin{aligned}
\ddot{\beta}+\ddot{\gamma} &= -\frac{A^4}{2}\left(4c_1+6c_2\tilde{V}+8c_3\tilde{V}^2\right), \\
\ddot{\alpha}+\ddot{\beta} &= \frac{A^4}{2}\left(2c_2\tilde{V}+4c_3\tilde{V}^2\right), \\
\ddot{\alpha}+\ddot{\gamma} &= \frac{A^4}{2}\left(2c_2\tilde{V}+4c_3\tilde{V}^2\right),
\end{aligned}$$

finally, adding these last two equations and extracting the value $\ddot{\beta}+\ddot{\gamma}$ from the first we obtain the α field equation:

$$\ddot{\alpha}=\frac{A^4}{2}\left(2c_1+5c_2\tilde{V}+8c_3\tilde{V}^2\right). \tag{3.10}$$

Now it is straightforward to calculate the β and γ dynamical equations

$$\ddot{\beta}=-\frac{A^4}{2}\left(2c_1+3c_2\tilde{V}+4c_3\tilde{V}^2\right), \tag{3.11}$$

$$\ddot{\gamma}=\ddot{\beta}. \tag{3.12}$$

3.2 Qualitative analysis

Comparing equations (3.9)-(3.12) with (1.80)-(1.83) we see that when the terms proportional to $\tilde{V}$ becomes negligible, if the first coupling coefficient in the potential is set to $g_1=1$ the dynamical equations of general relativity are restored.

In order to estimate the effects of the $\tilde{V}$, $\tilde{V}^2$ terms on the overall dynamic we are going to perform two kind of analysis; firstly we shall do some perturbative considerations around the standard solution of general relativity obtained in chapter one, focusing on the asymptotic limits which are supposed to show Kasner-like behavior, secondly we will consider the case in which the new incoming terms in the potential are such that cannot be treated perturbatively.

3.2.1 Perturbative analysis

Supposing that the higher curvature terms due to the anisotropic scaling can be treated with standard perturbation techniques, the effective field equations become:

$$\ddot{\alpha}_{eff} = +\frac{1}{2}e^{4\alpha}\left(2c_1 + 5c_2 e^{2(3\alpha-\chi_1\tau-\chi_2)} + 8c_3 e^{4(3\alpha-\chi_1\tau-\chi_2)}\right), \tag{3.13}$$

$$\ddot{\beta}_{eff} = -\frac{1}{2}e^{4\alpha}\left(2c_1 + 3c_2 e^{2(3\alpha-\chi_1\tau-\chi_2)} + 4c_3 e^{4(3\alpha-\chi_1\tau-\chi_2)}\right), \tag{3.14}$$

$$\ddot{\gamma}_{eff} = -\frac{1}{2}e^{4\alpha}\left(2c_1 + 3c_2 e^{2(3\alpha-\chi_1\tau-\chi_2)} + 4c_3 e^{4(3\alpha-\chi_1\tau-\chi_2)}\right), \tag{3.15}$$

with the constraint

$$\left(\dot{\alpha}_{eff}\dot{\beta}_{eff} + \dot{\alpha}_{eff}\dot{\gamma}_{eff} + \dot{\beta}_{eff}\dot{\gamma}_{eff}\right) = -\frac{1}{2}e^{4\alpha}\left(c_1 + c_2 e^{2(3\alpha-\chi_1\tau-\chi_2)} + c_3 e^{4(3\alpha-\chi_1\tau-\chi_2)}\right) \tag{3.16}$$

where we have rewrited $\alpha - \beta - \gamma$ whit the use of equations (1.80)-(1.83) and defining

$$\chi_1 = (\omega + \Omega),$$
$$\chi_2 = (\sigma + \Sigma).$$

Asymptotic regimes

Since in the asymptotic regimes α, β, γ are linear functions of τ it is easy to solve the above set of differential equations for the effective scale factors, this will be enough to see if the transition between Kasner epochs is maintained in the presence of higher order curvature terms.
As in chapter one we choose the set of solution with the + sign in the exponential inside α; thus, when $\tau \to +\infty$, we know that $\alpha \to C_\alpha - H\tau$, so that:

$$\begin{aligned}\lim_{\tau\to+\infty} \ddot{\alpha}_{eff}(\tau) =& \frac{1}{2}e^{-4H\tau+C_\alpha}\left(2c_1 + 5c_2 e^{-2(3H\tau+\chi_1\tau+\chi_2+-3C_\alpha)} + 8c_3 e^{-4(3H\tau+\chi_1\tau+\chi_2+-3C_\alpha)}\right)\\ \approx& \frac{1}{2}\left(2c_1 e^{-4H\tau} + 5c_2 e^{-2\tau(5H+\chi_1)} + 8c_3 e^{-4\tau(4H+\chi_1)}\right).\end{aligned}$$

Then, by integrating twice the last line we obtain:

$$\lim_{\tau\to+\infty} \alpha_{eff}(\tau) \approx \frac{1}{2}e^{-4H\tau}\left[\frac{2c_1}{16H^2} + \frac{5c_2 e^{-2\tau(5H+\chi_1)}}{4(5H+\chi_1)^2} + \frac{8c_3 e^{-4\tau(4H+\chi_1)}}{16(4H+\chi_1)^2}\right] + G_\alpha\tau + \Xi,$$

where Ξ and G_α are integration constants.
If we want the effective scale factor and the classical solution obtained in

chapter 1 to be the same in this asymptotic limit for at least an interval $\Delta\tau$, then we are forced to set:

$$G_\alpha = -H, \quad \Xi = C_\alpha,$$

and we are finally left with:

$$\lim_{\tau\to+\infty} \alpha_{eff}(\tau) - \alpha(\tau) = = \frac{1}{2}e^{-4H\tau}\left[\frac{2c_1}{16H^2} + \frac{5c_2e^{-2\tau(5H+\chi_1)}}{4(5H+\chi_1)^2} + \frac{8c_3e^{-4\tau(4H+\chi_1)}}{16(4H+\chi_1)^2}\right]. \tag{3.17}$$

Analogously we have for the other scale factors:

$$\lim_{\tau\to+\infty} \beta_{eff}(\tau) - \beta(\tau) = = -\frac{1}{2}e^{-4H\tau}\left[\frac{2c_1}{16H^2} + \frac{3c_2e^{-2\tau(5H+\chi_1)}}{4(5H+\chi_1)^2} + \frac{4c_3e^{-4\tau(4H+\chi_1)}}{16(4H+\chi_1)^2}\right], \tag{3.18}$$

$$\lim_{\tau\to+\infty} \gamma_{eff}(\tau) - \gamma(\tau) = = \frac{1}{2}e^{-4H\tau}\left[\frac{2c_1}{16H^2} + \frac{3c_2e^{-2\tau(5H+\chi_1)}}{4(5H+\chi_1)^2} + \frac{4c_3e^{-4\tau(4H+\chi_1)}}{16(4H+\chi_1)^2}\right]. \tag{3.19}$$

Repeating the above analysis in the opposite asymptotic regime, so considering the limit $\tau \to -\infty$ of equations (3.13), (3.14), (3.15) we obtain:

$$\lim_{\tau\to-\infty} \alpha_{eff}(\tau) - \alpha(\tau) = = \frac{1}{2}e^{4H\tau}\left[\frac{2c_1}{16H^2} + \frac{5c_2e^{2\tau(5H-\chi_1)}}{4(5H-\chi_1)^2} + \frac{8c_3e^{4\tau(4H-\chi_1)}}{16(4H-\chi_1)^2}\right]. \tag{3.20}$$

$$\lim_{\tau\to-\infty} \beta_{eff}(\tau) - \beta(\tau) = = \frac{1}{2}e^{4H\tau}\left[\frac{2c_1}{16H^2} + \frac{3c_2e^{2\tau(5H-\chi_1)}}{4(5H-\chi_1)^2} + \frac{4c_3e^{4\tau(4H-\chi_1)}}{16(4H-\chi_1)^2}\right]. \tag{3.21}$$

$$\lim_{\tau\to-\infty} \gamma_{eff}(\tau) - \gamma(\tau) = = \frac{1}{2}e^{4H\tau}\left[\frac{2c_1}{16H^2} + \frac{3c_2e^{2\tau(5H-\chi_1)}}{4(5H-\chi_1)^2} + \frac{4c_3e^{4\tau(4H-\chi_1)}}{16(4H-\chi_1)^2}\right]. \tag{3.22}$$

These differences among the classical solutions and the effective ones are sensitive to the set of integration constants H, ω, Ω; indeed the classical behavior is recovered only if the linear coefficients of τ in the exponentials have a sign concordant with it, let us summarize what could happen:

- If $H < -\frac{\chi_1}{4}$ the classical behavior is ruined and α, β and γ grow exponentially with time in both the asymptotic regimes.

- If $H > \frac{\chi_1}{4}$ the classical behavior is recovered for the whole set of solutions because all the exponentials fall down rapidly to zero in both asymptotic regimes.

- If the integration constants are distribuited in a different way from the two just considered, so for $-\frac{\chi_1}{4} < H < \frac{\chi_1}{4}$, then the two asymptotic solutions aren't symmetric and we could have a Kasner-like behavior in the future but not in the past.

Miscellanea on the perturbative analysis

To perform the above calculation we simply put into the potential term the explicit form of α, β and γ which are solution in the standard general relativity case; unfortunately we have found that there are particular choices of the integration constants for which the difference between the classical solution and the effective one diverges, in this case the whole perturbative analysis lacks of validity and the obtained results are not significant.
Anyway we were able to found a subset of initial conditions for which the expected qualitative dynamic of general relativity is reproduced in the framework of Horava gravity, moreover we have seen that eventually responsible for such displacement from the classical B-II are the exponential terms that contain the combination $\alpha - \beta - \gamma$; for this reason in the next subsection we will consider a toy model to clarify the effect of such exponentials on the overall behavior when they are dominant.

3.2.2 Toy model

We have tried to solve the set of equations for our model with perturbative metods supposing that the effects of the higher curvature terms does not affect too much the classical solutions; this has led to inconsistencies because there are cases in which the effect of such perturbations looks dominant.
Motivated by such evidence in this subsection we will consider a toy model for which there is only one term dominant in the right hand side of equations (3.10)(3.11)(3.12) while the others are negligible; to begin with let us consider the following system of equations in which only the terms proportional to $\tilde{V}^2$

are considered:

$$\ddot{\alpha} = 4c_3 e^{4(2\alpha-\beta-\gamma}),$$
$$\ddot{\beta} = -2c_3 e^{4((2\alpha-\beta-\gamma}),$$
$$\ddot{\gamma} = -2c_3 e^{4((2\alpha-\beta-\gamma}).$$

Now let us consider the following linear combination:

$$2\alpha - \beta - \gamma = \varphi, \tag{3.23}$$
$$\ddot{\varphi} = 12c_3 e^{4\varphi} \tag{3.24}$$

the solution of the above differential equation is:

$$\varphi = \frac{1}{4}\log\left[\frac{-C_\varphi{}^2}{12c_3}\frac{e^{\pm\frac{C_\varphi}{\sqrt{2}}(\tau+q)}}{\left(1+e^{\pm\frac{C_\varphi}{\sqrt{2}}(\tau+q)}\right)^2}\right], \tag{3.25}$$

where C_φ and q are integration constants.
Using this into the field equation we obtain:

$$\begin{aligned}\ddot{\alpha} =& 4c_3 e^{4\varphi}\\ =& -\frac{C_\varphi{}^2}{3}\frac{e^{\pm\frac{c_\varphi}{\sqrt{2}}(\tau+q)}}{\left(1+e^{\pm\frac{c_\varphi}{\sqrt{2}}((\tau+q)}\right)^2}\\ =& \frac{1}{3}C_\varphi\sqrt{2}\frac{\partial}{\partial\tau}\left[\frac{1}{1+e^{\pm\frac{c_\varphi}{\sqrt{2}}((\tau+q)}}\right]\end{aligned}$$

so that

$$\dot{\alpha} = \frac{1}{3}C_\varphi\sqrt{2}\frac{1}{1+e^{\pm\frac{c_\varphi}{\sqrt{2}}((\tau+q)}} + G_{\alpha\tilde{V}^2}$$

with $G_{\alpha\tilde{V}^2}$ constant. Integrating once again we are left with:

$$\begin{aligned}\alpha =& \int d\tau\frac{1}{3}C_\varphi\sqrt{2}\frac{1}{1+e^{\pm\frac{c_\varphi}{\sqrt{2}}((\tau+q)}} + G_{\alpha\tilde{V}^2}\\ =& \int dy\frac{2}{3}\frac{1}{y(1+y)} + G_{\alpha\tilde{V}^2}\tau\\ =& \frac{2}{3}\log\left[\frac{e^{\pm\frac{c_\varphi}{\sqrt{2}}((\tau+q)}}{1+e^{\pm\frac{c_\varphi}{\sqrt{2}}((\tau+q)}}\right] + G_{\alpha\tilde{V}^2}\tau + \Upsilon\end{aligned}$$

with Υ an integration constant.
Proceeding in a similar way for the other two scale factors give us the following system of equation:

$$\alpha_{\tilde{V}^2} = \frac{2}{3}\log\left[\frac{e^{\pm\frac{c_\varphi}{\sqrt{2}}((\tau+q)}}{1+e^{\pm\frac{c_\varphi}{\sqrt{2}}((\tau+q)}}\right] + G_{\alpha\tilde{V}^2}\tau + \Upsilon_\alpha \tag{3.26}$$

$$\beta_{\tilde{V}^2} = -\frac{1}{3}\log\left[\frac{e^{\pm\frac{c_\varphi}{\sqrt{2}}((\tau+q)}}{1+e^{\pm\frac{c_\varphi}{\sqrt{2}}((\tau+q)}}\right] + G_{\beta\tilde{V}^2}\tau + \Upsilon_\beta \tag{3.27}$$

$$\gamma_{\tilde{V}^2} = -\frac{1}{3}\log\left[\frac{e^{\pm\frac{c_\varphi}{\sqrt{2}}((\tau+q)}}{1+e^{\pm\frac{c_\varphi}{\sqrt{2}}((\tau+q)}}\right] + G_{\gamma\tilde{V}^2}\tau + \Upsilon_\gamma. \tag{3.28}$$

Let us evaluate the asymptotic behavior of the logharitm inside the above equations; as in the classical case we choose the set of solution with the + sign, then when $\tau \to +\infty$ we have

$$\log\left[\frac{e^{\frac{C_\varphi}{\sqrt{2}}((\tau+q)}}{1+e^{\frac{C_\varphi}{\sqrt{2}}((\tau+q)}}\right] = -\log\left[1+e^{-\frac{C_\varphi}{\sqrt{2}}(\tau+q}\right] \approx 0,$$

and the asymptotic form of the scale factors is:

$$\alpha_{\tilde{V}^2} \approx G_{\alpha\tilde{V}^2}\tau, \tag{3.29}$$

$$\beta_{\tilde{V}^2} \approx G_{\beta\tilde{V}^2}\tau, \tag{3.30}$$

$$\gamma_{\tilde{V}^2} \approx G_{\gamma\tilde{V}^2}\tau. \tag{3.31}$$

With the Friedmann equation given by

$$G_{\alpha\tilde{V}^2}G_{\beta\tilde{V}^2} + G_{\alpha\tilde{V}^2}G_{\gamma\tilde{V}^2} + G_{\beta\tilde{V}^2}G_{\gamma\tilde{V}^2} = -\frac{1}{2}c_3e^{4\varphi} \to 0. \tag{3.32}$$

In the other regime, $\tau \to -\infty$, we have

$$\log\left[\frac{e^{\frac{C_\varphi}{\sqrt{2}}((\tau+q)}}{1+e^{\frac{C_\varphi}{\sqrt{2}}((\tau+q)}}\right] \approx \frac{C_\varphi}{\sqrt{2}}(\tau+q),$$

from which

$$\alpha_{\tilde{V}^2} \approx \left(\frac{\sqrt{2}}{3}C_\varphi + G_{\alpha\tilde{V}^2}\right)\tau, \tag{3.33}$$

$$\beta_{\tilde{V}^2} \approx \left(-\frac{1}{3\sqrt{2}}C_\varphi + G_{\beta\tilde{V}^2}\right)\tau, \tag{3.34}$$

$$\gamma_{\tilde{V}^2} \approx \left(-\frac{1}{3\sqrt{2}}C_\varphi + G_{\gamma\tilde{V}^2}\right)\tau. \tag{3.35}$$

With the constraint:

$$\begin{aligned} &G_{\alpha\tilde{V}^2}G_{\beta\tilde{V}^2}+G_{\alpha\tilde{V}^2}G_{\gamma\tilde{V}^2}+G_{\beta\tilde{V}^2}G_{\gamma\tilde{V}^2}\\ &-\frac{C_\varphi}{3\sqrt{2}}\left(-G_{\beta\tilde{V}^2}-G_{\gamma\tilde{V}^2}+2G_{\alpha\tilde{V}^2}\right)+\frac{1}{6}{C_\varphi}^2=0 \end{aligned} \tag{3.36}$$

Back to the syncronous time

Let us evaluate in the asymptotic regimes the inverse transformation (1.79); in the limit $\tau\to+\infty$ we have:

$$\begin{aligned} t&=\int d\tau e^{(G_{\alpha\tilde{V}^2}+G_{\beta\tilde{V}^2}+G_{\gamma\tilde{V}^2})\tau}\\ &=\frac{e^{(G_{\alpha\tilde{V}^2}+G_{\beta\tilde{V}^2}+G_{\gamma\tilde{V}^2})\tau}}{(G_{\alpha\tilde{V}^2}+G_{\beta\tilde{V}^2}+G_{\gamma\tilde{V}^2})}, \end{aligned}$$

from which

$$\tau\approx\log\frac{t}{(G_{\alpha\tilde{V}^2}+G_{\beta\tilde{V}^2}+G_{\gamma\tilde{V}^2})}, \tag{3.37}$$

so that the scale factors become:

$$A_{\tilde{V}^2}\approx t^{\frac{G_{\alpha\tilde{V}^2}}{G_{\alpha\tilde{V}^2}+G_{\beta\tilde{V}^2}+G_{\gamma\tilde{V}^2}}}, \tag{3.38}$$

$$B_{\tilde{V}^2}\approx t^{\frac{G_{\beta\tilde{V}^2}}{G_{\alpha\tilde{V}^2}+G_{\beta\tilde{V}^2}+G_{\gamma\tilde{V}^2}}}, \tag{3.39}$$

$$C_{\tilde{V}^2}\approx t^{\frac{G_{\gamma\tilde{V}^2}}{G_{\alpha\tilde{V}^2}+G_{\beta\tilde{V}^2}+G_{\gamma\tilde{V}^2}}}. \tag{3.40}$$

From which is easy to show that both the Kasner conditions for the exponents hold using (3.32).

In the limit $\tau\to-\infty$ we have:

$$t=\int d\tau e^{(G_{\alpha\tilde{V}^2}+G_{\beta\tilde{V}^2}+G_{\gamma\tilde{V}^2})\tau},$$

$$\tau\approx\log\frac{t}{(G_{\alpha\tilde{V}^2}+G_{\beta\tilde{V}^2}+G_{\gamma\tilde{V}^2})}, \tag{3.41}$$

with the scale factors given by

$$A_{\tilde{V}^2}\approx t^{\frac{G_{\alpha\tilde{V}^2}+C_\varphi\frac{2}{3\sqrt{2}}}{G_{\alpha\tilde{V}^2}+G_{\beta\tilde{V}^2}+G_{\gamma\tilde{V}^2}}}, \tag{3.42}$$

$$B_{\tilde{V}^2}\approx t^{\frac{G_{\beta\tilde{V}^2}-\frac{1}{3\sqrt{2}}C_\varphi}{G_{\alpha\tilde{V}^2}+G_{\beta\tilde{V}^2}+G_{\gamma\tilde{V}^2}}}, \tag{3.43}$$

$$C_{\tilde{V}^2}\approx t^{\frac{G_{\gamma\tilde{V}^2}-\frac{1}{3\sqrt{2}}C_\varphi}{G_{\alpha\tilde{V}^2}+G_{\beta\tilde{V}^2}+G_{\gamma\tilde{V}^2}}}. \tag{3.44}$$

The first Kasner condition is trivially satisfied, while for the second we have:

$$\frac{\left[\left(G_{\alpha\tilde{V}^2}+\frac{2C_\varphi}{3\sqrt{2}}\right)^2+\left(G_{\beta\tilde{V}^2}-\frac{C_\varphi}{3\sqrt{2}}\right)^2+\left(G_{\gamma\tilde{V}^2}-\frac{C_\varphi}{3\sqrt{2}}\right)^2\right]}{(G_{\alpha\tilde{V}^2}+G_{\beta\tilde{V}^2}+G_{\gamma\tilde{V}^2})^2}=$$
$$=\frac{\left[\frac{{C_\varphi}^2}{3}-\frac{2C_\varphi}{3\sqrt{2}}\left(-2G_{\alpha\tilde{V}^2}+G_{\beta\tilde{V}^2}+G_{\gamma\tilde{V}^2}\right)+{G_{\alpha\tilde{V}^2}}^2+{G_{\beta\tilde{V}^2}}^2+{G_{\gamma\tilde{V}^2}}^2\right]}{{G_{\alpha\tilde{V}^2}}^2+{G_{\beta\tilde{V}^2}}^2+{G_{\gamma\tilde{V}^2}}^2+2\left(G_{\alpha\tilde{V}^2}G_{\beta\tilde{V}^2}+G_{\alpha\tilde{V}^2}G_{\gamma\tilde{V}^2}+G_{\beta\tilde{V}^2}G_{\gamma\tilde{V}^2}\right)}=$$
$$=1$$

where we used equation (3.36) into the denominator.
It is possible to express the set of Kasner exponent in one regime in terms of the other one; let us define the exponents in the $\tau\to+\infty$ regime as:

$$p_1=\frac{G_{\alpha\tilde{V}^2}}{G_{\alpha\tilde{V}^2}+G_{\beta\tilde{V}^2}+G_{\gamma\tilde{V}^2}},$$
$$p_2=\frac{G_{\beta\tilde{V}^2}}{G_{\alpha\tilde{V}^2}+G_{\beta\tilde{V}^2}+G_{\gamma\tilde{V}^2}},$$
$$p_3=\frac{G_{\gamma\tilde{V}^2}}{G_{\alpha\tilde{V}^2}+G_{\beta\tilde{V}^2}+G_{\gamma\tilde{V}^2}},$$

then the set of exponents which corresponds to $\tau\to-\infty$ regime is:

$$p_1^{'}=p_1+2\zeta$$
$$p_2^{'}=p_2-\zeta$$
$$p_3^{'}=p_3-\zeta$$

where the factor ζ is given by:

$$\zeta=\frac{1}{3\sqrt{2}}\frac{C_\varphi}{G_{\alpha\tilde{V}^2}+G_{\beta\tilde{V}^2}+G_{\gamma\tilde{V}^2}}.$$

Since we have proved that in both regimes hold the Kasner conditions for the exponents it is straightforward to realize that:

$$p_1^{'\,2}+p_2^{'\,2}+p_3^{'\,2}=1$$
$$={p_1}^2+{p_2}^2+{p_3}^2+6\zeta^2+2\zeta\left(2p_1-p_2-p_3\right),$$

so that finally

$$\zeta=\frac{p_2+p_3-2p_1}{3}.$$

In conclusion we have obtained the following transformation rule for the Kasner exponents:

$$p_1^{'} = \frac{-p_1 + p_2 + p_3}{3}, \tag{3.45}$$

$$p_2^{'} = \frac{2p_2 - p_3 + 2p_1}{3}, \tag{3.46}$$

$$p_3^{'} = \frac{2p_3 - p_2 + 2p_1}{3}. \tag{3.47}$$

Using the clever BKL parametrization of the Kasner exponents in terms of the quantity u we can write:

$$p_1 = \frac{-u}{1 + u + u^2}$$

$$p_2 = \frac{1 + u}{1 + u + u^2}$$

$$p_3 = \frac{u\,(u + 1)}{1 + u + u^2},$$

From which we can read the transition law between the two Kasner epochs:

$$p_1^{'} = \frac{1}{3}\frac{u^2 + 3u + 1}{1 + u + u^2},$$

$$p_2^{'} = \frac{1}{3}\frac{+ - u^2 - u + 2}{1 + u + u^2},$$

$$p_3^{'} = \frac{1}{3}\frac{2u^2 - u - 1}{1 + u + u^2}.$$

Since u is ≥ 1, it is straightforward to realize that exponent referred to the A scale factor $p_1^{'}$ is now positive, and the one referred to B, $p_2^{'}$, is negative.

3.3 Discussions

In this chapter we have considered a manifold in which is implemented the anisotropic scaling $z = 3$ between space and time, and whose spatial sections geometry is encoded in the Bianchi type II algebra.
As explained in chapter two the conseguence of such anisotropic scaling are the presence of higher curvature terms up to sixth order spatial derivatives in the action functional, and the appearance of an extra scalar graviton which is the result of the reduced symmetry of the foliation preserving diffeomorphism; such graviton is the main source of troubles with Horava gravity theories because shows strong coupling even at low energies.

To avoid this kind of issues, in standard cosmological application of Horava gravity, one usually set the coupling parameter $\lambda = 1$; in this analysis we share this point of view and the only difference between our model and the one of general relativity relies in the presence of the Higher spatial curvature terms in the potential part of the action.

Implementing these assumptions we came to the set of differential equations (3.10) (3.11) and (3.12), and since we were not able to solve them analitically we have studied qualitatively the effects of the higher curvature terms in the asymptotic regimes of the model.
In general relativity indeed the Bianchi II dynamic in vacuum can be summarized in the statement that the spatial anisotropy cause a transition between two asymptotic Kasner epochs, so we were interested in verify if such transition is present even within the Lorentz symmetry breaking potential.

To begin with we moved from the hypotesis that the deformations of the potential do not affect too much the qualitative shape of the classical solutions, so we have solved the field equations with standard perturbative techniques inserting the asymptotic solutions which hold in general relativity in the explicit form of the potential.
We have found in equations (3.17)-(3.22) that there is a subset of initial conditions for which such perturbative analysis looks consistent and the behavior of the model is asymptotically the same of general relativity; on the other hand we have found that there are also cases for which the perturbative approach falls, since the solutions obtained strongly diverges from the classical ones.

If the perturbative analysis falls down we can argue that the potential terms due to the anisotropic scaling are not negligible with respect to those coming from the spatial anisotropy; with this motivation in mind we have considered the case in which the classical part of the potential is unimportant compared to the one coming from the higher curvature terms.
If this is the case, since we are interested in the asymptotic dynamic, we can simplify further our model focusing only on those terms which are cubic in the spatial Ricci tensor; the solutions for the resulting field equations are given in (3.26),(3.27) and (3.28).

Let us compare these solutions with those of general relativity:

$$
\begin{aligned}
\alpha_{HL}^{\tilde{V}^2} &\propto \log\left[\frac{e^{\pm\frac{c_\varphi}{\sqrt{2}}\tau}}{1+e^{\pm\frac{c_\varphi}{\sqrt{2}}\tau}}\right] + G_{\alpha\tilde{V}^2}\tau, \qquad && \alpha_{GR} \propto \log\frac{e^{\pm 4H\tau}}{(1+e^{\pm 4H\tau})^2};\\
\beta_{HL}^{\tilde{V}^2} &\propto -\log\left[\frac{e^{\pm\frac{c_\varphi}{\sqrt{2}}\tau}}{1+e^{\pm\frac{c_\varphi}{\sqrt{2}}\tau}}\right] + G_{\beta\tilde{V}^2}\tau, \qquad && \beta_{GR} \propto \omega\tau - \alpha_{GR}(\tau);\\
\gamma_{HL}^{\tilde{V}^2} &\propto -\log\left[\frac{e^{\pm\frac{c_\varphi}{\sqrt{2}}\tau}}{1+e^{\pm\frac{c_\varphi}{\sqrt{2}}\tau}}\right] + G_{\gamma\tilde{V}^2}\tau, \qquad && \gamma_{GR} \propto \Omega\tau - \alpha_{GR}(\tau);
\end{aligned}
$$

The main difference between the two set of solutions arises in the denominators inside the logarithms; indeed in the GR solutions the power two in the denominator ensures that in the asymptotic limits the α term switches sign, these logarithms are functions that reach an extremum and then change shape insensitive to the value of the integration constants.
On the other side, the logarithms inside the $\tilde{V}^2$ solutions do not have such switching character, these are in fact monotonic functions which tend to zero in one regime and become linear in τ in the other; in this sense looks that the whole dynamic is determined by the values of the integration constants $G_{\tilde{V}^2}$ and C_φ.

By the way this is not the end of the story; to fully understand the overall behavior in both cases we have to consider the Friedmann-like equations (1.83) and (3.9), keeping only the $\tilde{V}^2$ term in the latter we obtain

$$
\begin{aligned}
\alpha_{GR}\beta_{GR} + \alpha_{GR}\gamma_{GR} + \beta_{GR}\gamma_{GR} &= \frac{e^{4\alpha_{GR}}}{2},\\
\alpha_{\tilde{V}^2}\beta_{\tilde{V}^2} + \alpha_{\tilde{V}^2}\gamma_{\tilde{V}^2} + \beta_{\tilde{V}^2}\gamma_{\tilde{V}^2} &= -\frac{c_3}{2}e^{4(2\alpha_{\tilde{V}^2}-\beta_{\tilde{V}^2}-\gamma_{\tilde{V}^2})}.
\end{aligned}
$$

The curious fact is that the above Friedmann-like equations share the same behavior; indeed equations (3.23),(3.24) and (3.25) tell us that the combination $(2\alpha_{\tilde{V}^2} - \beta_{\tilde{V}^2} - \gamma_{\tilde{V}^2}) = \varphi$ satisfy a differential equation similar in structure to the one satisfied by α_{GR}, the conseguence is that in both cases in the asymptotic limits the right hand side of the Friedmann equations vanish.

These circumstances led us to prove that equations (3.38)-(3.40) and (3.42)-(3.44) for the scale factors imply that the asymptotic regimes of our toy model are Kasner's epochs. However the transition law from one set of Kasner exponents to the other is sensibly different compared to the one of general relativity, and it is showed in equations (3.45),(3.46),(3.47).

Conclusions

The privileged role of time in quantum gravity is one of the trickiest puzzle of the contemporary theoretical physics that inevitably collides with our faith in the principles of Einstein's general relativity.
It is our opinion that Horava's attempt [20] to describe phenomenologically such occurrence, with the postulate of anisotropic scaling inherited from the condensed matter physics, deserves particular attention at least for the following reasons: it shows an improved UV behavior which guarantees renormalizability of the gravitational interactions and it has a realistic chance of direct confrontation with experiments and observation already in the low-energy infrared regime[24].
In this work we have considered in Horava's scenario the simplest homogeneous cosmological model with spatial anisotropy, the Bianchi II space-time, due to the fact that in general relativity it is a key dowel to understand the behavior of the universe while approaching to the cosmological singularities, as was showed in BKL analysis [12, 14].
In chapter one we have reviewed a suitable mathematical formalism for deal with homogeneous cosmologies, we have also obtained the explicit solutions of the Einstein's field equations in vacuum B-II space-time, showing that the resulting dynamic is represented by a transition between two asymptotic Kasner epochs.
In chapter two we have presented Horava's original formulation, discussing the features which provide for renormalizability, and the related pathologies which affect the model due to its lack of symmetry with respect to general relativity. We have also stated that there are interesting cosmological implication in the projectable formulation of the theory; indeed the anisotropic scaling principle allows for the generation of scale-invariant cosmological perturbations without the need of accelerated expansion, mimics the effect of dark matter sources thanks to the higher curvature terms in the potential and admits a class of bouncing solutions in FLRW cosmological models.
Chapter three finally was devoted to the qualitative analysis of the system of differential equations obtained by varying Horava's action functional for

the Bianchi type II spatial metric. We have considered the evolution of the model at the $\lambda = 1$ fixed point in order to avoid effects from the extra scalar graviton which afflicts all the formulations of the theory, so that the only difference with respect to the action of general relativity relies in the sixth order derivatives of the spatial metric which enter the potential part of the action.

We have considered two different scenarios; firstly the one in which the quadratic and cubic combinations of the Ricci tensor are subdominant with respect to the classical potential of general relativity, secondly we have reversed their role considering as dominants the higher terms in spatial curvature; in both cases we have found that the asymptotic regimes of the model are of Kasner type. We have not considered the case in which the all terms of the potential are taken into account because we were not able to solve analitically the complete set of differential equations, however it is our opinion that taking the complete form of the potential into account will not affect the obtained qualitative picture, at least when we are close to the cosmological singularities; further numerical simulations are required to enforce this statement.

It is know that to the Bianchi type IX space-time is related the occurrence of chaos in general relativity, and that the dynamic of the model is well described by a succession of Kasner epochs whose individual length has a stochastic character [25]. If this is the case even in the contest of Horava Lifshitz gravity it is an open question [44, 45, 47], anyway, if the system could be described in terms of a chain of subsequent Bianchi II "bounces" like in general relativity, our analysis suggest that the overall dynamic toward the singularity is still described by a succession of Kasner epochs.

Maybe the understanding of the modified transition law among the Kasner epochs could clarify the issue of chaos in Horava-Lifshitz theories; we address this task to future works.

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