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# Alternative derivative expansion in Functional RG and application

**Tesi di Laurea Magistrale in Teoria di Campo**

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# Sommario

Vogliamo dare una breve introduzione sul metodo del gruppo di Rinormalizzazione Funzionale nella teoria quantistica dei campi, che é intrinsecamente un metodo non perturbativo, in termini dell'equazione di Polchinski per l'azione 'Wilsoniana' e dell'equazione di Wetterich per il generatore dei vertici propri. Nell'ultimo caso mostriamo una semplice applicazione per una teoria con un singolo campo scalare reale nelle approsimazioni LPA e LPA'. Nel primo caso invece, mostriamo una versione "Hamiltoniana" dell'equazione di Polchinski che consiste nel fare una trasformazione di Legendre per il flusso della corrispondente Lagrangiana effettiva, sostituendo le derivate dei campi di ordine qualsisasi con i relativi momenti. Questo approccio é utile per studiare nuovi troncamenti nelle espansioni derivative. Applichiamo poi questa formulazione ad una teoria con un singolo campo scalare reale e, come nuovo risultato, deriviamo l'equazione di flusso per una teoria con $N$ campi scalari reali con una simmetria interna $O(N)$. All'interno di questo nuovo approccio analizziamo numericamente le soluzioni invarianti di scala per $N = 1$ e $d = 3$ (ovvero il modello critico di Ising), al primo ordine dell'espansione derivative e con un numero infinito di costanti di accoppiamento, codificate da due funzioni $V(\phi)$ e $Z(\phi)$, ottenendo cosí una stima per la dimensione anomala con un'accuratezza del 10% (confrontata con i risultati del Monte Carlo).

# Abstract

We give a brief review of the Functional Renormalization method in quantum field theory, which is intrinsically non perturbative, in terms of both the Polchinski equation for the Wilsonian action and the Wetterich equation for the generator of the proper verteces. For the latter case we show a simple application for a theory with one real scalar field within the LPA and LPA' approximations. For the first case, instead, we give a covariant "Hamiltonian" version of the Polchinski equation which consists in doing a Legendre transform of the flow for the corresponding effective Lagrangian replacing arbitrary high order derivative of fields with momenta fields. This approach is suitable for studying new truncations in the derivative expansion. We apply this formulation for a theory with one real scalar field and, as a novel result, derive the flow equations for a theory with $N$ real scalar fields with the $O(N)$ internal symmetry. Within this new approach we analyze numerically the scaling solutions for $N = 1$ in $d = 3$ (critical Ising model), at the leading order in the derivative expansion with an infinite number of couplings, encoded in two functions $V(\phi)$ and $Z(\phi)$, obtaining an estimate for the quantum anomalous dimension with a 10% accuracy (confronting with Monte Carlo results).

# Contents

# Introduction

Quantum field theory and Statistical field theory are the two main stones of the modern physics developed in the XX century. The Standard Model of fundamental interactions, for example, is formulated entirely within the framework of QFT. One of the extraordinary results of the QFT is that the coupling constants for the interactions are not constants at all but they are running constants depending on the energy scale at with we are studying the physical system, measuring different observables. Renormalization is one of the central tool that allows to deal with this concept and coherently derive measurable quantities, relating phenomena at different scales of observations and being intrinsically associated to the related concept of effective theories. The one who gave the strongest impulse towards the modern paradigm of renormalization was K. Wilson leading to the so called Wilson's renormalization group [35]. The basic idea is to study the correlation functions (or their generators) after integrating quantum or thermal fluctuations not all at once but with some coarse-graining procedure. In the case one wants to employ this in a continuous way a popular approach is based on integrating out fluctuations momentum shell by momentum shell. The continuous procedures generate the so called *exact renormalization group equations*, which can be used to describe, at least in principle, all kinds of non-perturbative quantum/statistical field theories, *i.e.* where there are no small parameters one can fruitfully expand in. This approach has the merit to give tools to studying both the universal (critical properties) and non universal of a given theory given its content in term of degrees of freedom and symmetries and can deal with infinitely many couplings.

There is a clear need for such approaches within the archetypal example of low energy QCD, but perhaps more importantly in the need to better understand the possibilities to access to several features of the standard model, which are just started to be investigated pertubatively in a systematic way at the level of effective theories with the inclusion of 6 dimensional operators. At the level of fundamental physics they may be also useful go get more insights in the SSB sector up to the Planck scale. In fact, it is well known that General Relativity cannot admit within perturbation theory a coherent formulation in terms of a quantum field, but this methods open the road to non perturbative studies within the paradigm of asymptotic safety introduced by S. Weinberg [28]. Such scenario could give possible UV complete models for fundamental physics including gravity without invoking as the only viable solution models like string theory whose aim is also to give a more unified picture.

Every exact RG equation has to be truncated, choosing some approximation, into a solvable set of equations that encode the wanted pieces of information.

Since any truncation induces errors that can be very hard to estimate, keep under control and reduce, it is important to have a rich pool of approximation schemes available. The goal of this thesis is precisely to show an alternative covariant Hamiltonian formulation of the Polchinski equation that can be useful for studying new truncation methods in the derivative expansion.

The plain of the work is as follow. In Chapter 1 we give a brief review of the Functional Renormalization method in terms of the Wetterich equation for the generator of the proper verteces and discuss some of the approximation schemes generally involved in calculations. In Chapter 2 we show one example, using the Wetterich equation to find a simple flow equation for a theory of one real $Z_2$ symmetric scalar field theory in the local potential approximation (LPA) and successively with the introduction of an approximate estimate for the anomalous dimension (LPA'). Moreover we analyze this simple theory in $d = 3$ with various techniques: spike-plot method, shooting from large field value and polynomial analysis near the origin to show how to extract the leading critical exponent $\nu$ of the critical Ising model. In Chapter 3 we review a new covariant Hamiltonian version of the Polchinski equation obtained from a generalized Legendre transform of the corresponding effective Lagrangian and introducing generalized covariant momenta fields as recently proposed in [36]. This method is giving a more powerful way to systematically generate different families of schemes for the derivative expansions. As a novel result we extend this formulation for the case of the $O(N)$ system with a Wilsonian action truncated at the leading order of the derivative expansion *i.e.* $\mathcal{O}(\partial^2)$ and some resummations of it, which can be easily extended to higher orders. We also make some consideration for the large $N$ limit case. In Chapter 4 we analyze numerically the fixed point equations for the case of $N = 1$ and $d = 3$, that corresponds to the continuum QFT description of the critical Ising model. We use the shooting method from large field value and compare it with the polynomial analysis near the origin and the spike-plot method. Finally we arrive at a numerical estimate for the anomalous dimension $\eta_\phi$ to be compared with the result obtained by Monte Carlo simulation.

# Chapter 1

# The Functional RG method

Modern physics is based upon two main theories: *Quantum Field Theory* and *Statistical Field Theory*. These two theories are intimately bound together, for example they have been molded by the concept of the renormalization group and the description of one theory is tied to the other via the Wick rotation. The renormalization group deals with the physics of scales. A central theme is the understanding of the macroscopic physics at long distances (low energy) in term of the fundamental microscopic interactions. Bridging this gap from micro to macro scales requires a thorough understanding of quantum or statistical fluctuations on all the scales in between. All particle physics is described by gauge theories and these theories, during the transition from micro to macro scales, turn from weak to strong coupling. In the regime of weak coupling we can use analytical perturbative methods and in the regime of strong coupling we need other techniques like lattice gauge theories. *Functional methods* begin to bridge the gap since they are not restricted to weak couplings and can still largely be treated analytically. This is the great advantage of functional methods. In particular *Functional Renormalization Group* combines this functional methods with the renormalization group idea of treating the fluctuation not all at once but successively from scale to scale. In other words it means that the correlations functions are not studied after having averaged over all fluctuations but it is considered only the change of the correlation functions induced by integration of fluctuations over a momentum shell. From the mathematical viewpoint this allows to transform the functional-integral structure of standard field theory into a functional differential structure.

The central tool of the FRG is given by a flow equation, a functional differential equation that describes the evolution of the correlations functions or their generating functional under the influence of fluctuations at different momentum scales. This equation connects the microscopic correlation functions in a perturbative domain in an *exact* manner with the desired full correlation functions after having integrated out all the fluctuations. Hence solving the flow equation is equivalent to solve the full theory.

## 1.1 Basics of Euclidean QFT

Now we are going to give a brief introduction to functional method, a necessary tool for the further developments we want to build [21]. From now on we will assume to be in the framework of euclidean field theory, it means that we will deal with fields $\phi_A$ on an Euclidean spacetime $\mathcal{M}$. Throughout all this thesis, the natural system of units will be used: $c = \hbar = 1$. As in QFTs, that are used to describe particle physics, the spacetime is Minkowskian, we will also assume that a Wick rotation to imaginary time can always be done.

In field theory all physical informations are stored in objects called *n-point correlation functions*

$$G^{(n)}_{A_1,\dots,A_n} = \left\langle \phi^{A_1} \dots \phi^{A_n} \right\rangle \tag{1.1}$$

where the labels $A_i$ are written in hyper condensed form (deWitt condensed form) and are of the form $A_i = (a_i, x_i)$ where $x_i$ are coordinates on $\mathcal{M}$ and $a_i$ contains information about the geometric nature of the field. As usual Einstein summation convention will be adopted when repeated indexes appear. For example if we have two fields $\phi$ and $\psi$ with the same type of indexes their inner product is

$$\phi^A \psi_A = \int dx \sum_a \phi^a(x) \psi_a(x) \tag{1.2}$$

We will concentrate on cases in which $\phi$ is a map $\phi : \mathcal{M} \to \mathcal{N}$ with $\mathcal{N}$ a Riemann manifold, thus, in this case, $a_i$ are indexes in some coordinate basis of $\mathcal{N}$.

In order to compute the expectation value of a general field configuration $\mathcal{O}[\phi]$ we need a measure

$$\mathcal{D}\phi$$

on the space of all possible fields and a probability density

$$\mathcal{P}[\phi]$$

for $\phi$. Now we can write the mean value as

$$\langle \mathcal{O}[\phi] \rangle = \frac{1}{Z} \int \mathcal{D}\phi \, \mathcal{P}[\phi] \, \mathcal{O}[\phi] \tag{1.3}$$

where $Z$ is a normalization factor for our probability such that $\langle 1 \rangle = 1$. Our correlation functions is obtained as

$$\left\langle \phi^{A_1} \dots \phi^{A_n} \right\rangle = \frac{1}{Z} \int \mathcal{D}\phi \, \mathcal{P}[\phi] \, \phi^{A_1} \dots \phi^{A_n} \tag{1.4}$$

In classical field theory the configuration of $\phi$ is assumed to be known once enough boundary conditions are specified and the equation of motion are solved. If we call this field configuration $\phi^A_{\text{classical}}$ then it is obvious that the probability density must be a delta functional

$$\mathcal{P}_{\text{classical}}[\phi] = \delta[\phi - \phi_{\text{classical}}] \tag{1.5}$$

with respect to the measure $\mathcal{D}\phi$. In the general quantum case the field configurations are weighted with an exponential of the action $S[\phi]$

$$\mathcal{P}_{\text{quantum}}[\phi] = e^{-S[\phi]} \tag{1.6}$$

An elegant and systematic way to compute the correlation functions is obtained introducing source current $J_A$ coupled to our fields $\phi^A$ and define a generating functional as follows [25]:

$$Z[J] = \int \mathcal{D}\phi \, e^{-S[\phi]+J_A\phi^A} = Z \left\langle e^{J_A\phi^A} \right\rangle \tag{1.7}$$

where second equality is obtained defining the probability density as

$$\mathcal{P}[\phi] = \frac{e^{-S[\phi]}}{Z} \tag{1.8}$$

where

$$Z = \int \mathcal{D}\phi \, e^{-S[\phi]} \tag{1.9}$$

is also called *partition function*. It is clear that $Z[J]$ is the generating functional of all the correlation functions

$$G^{(n)}_{A_1,\ldots,A_n} = \left\langle \phi^{A_1} \ldots \phi^{A_n} \right\rangle = \frac{1}{Z} \left. \frac{\delta^n Z[J]}{\delta J_{A_n} \ldots \delta J_{A_1}} \right|_{J=0} \tag{1.10}$$

in fact

$$\frac{\delta Z[J]}{\delta J_A} = \int \mathcal{D}\phi \, e^{-S[\phi]+J_B\phi^B} \phi_A \tag{1.11}$$

More interesting from the physical point of view are the connected correlation functions that are obtained from another generating functional defined as follows

$$Z[J] \equiv e^{W[J]} \tag{1.12}$$

taking $J$ functional derivatives

$$G^{(n)}_{c,\,A_1,\ldots,A_n} = \left. \frac{\delta^n W[J]}{\delta J_{A_n} \ldots \delta J_{A_1}} \right|_{J=0} \tag{1.13}$$

**Example**

Consider a free action with only the kinetic term

$$S_0[\phi] = \frac{1}{2}\phi^A \mathcal{K}_{AB} \phi^B \tag{1.14}$$

The $J$-dependent partition function is

$$\begin{aligned} Z_0[J] &= \int \mathcal{D}\phi \, e^{-\frac{1}{2}\phi^A\mathcal{K}_{AB}\phi^B + J_A\phi^A} && (1.15) \\ &= \int \mathcal{D}\tilde{\phi} \, e^{-\frac{1}{2}\tilde{\phi}^A\mathcal{K}_{AB}\tilde{\phi}^B} \, e^{\frac{1}{2}J_A\mathcal{K}^{AB}J_B} && (1.16) \\ &= \mathcal{C} \, e^{\frac{1}{2}J_A\mathcal{K}^{AB}J_B} && (1.17) \end{aligned}$$

where $\tilde{\phi}^A = \phi^A - \mathcal{K}^{AB}J_B$ and $\mathcal{K}^{AB}$ is the inverse of the kinetic operator *i.e.* $\mathcal{K}_{AB}\mathcal{K}^{BC} = \delta^C_A$.

We can see that in the free case only the two points correlation function is different from zero

$$G_0^{AB} = \lim_{J\to 0} \frac{1}{Z_0[J]} \frac{\delta^2 Z_0[J]}{\delta J_A \delta J_B} = \mathcal{K}^{AB} \tag{1.18}$$

and it represents the propagation of field $\phi$ from state $A$ to state $B$. If one looks at the associated functional

$$W_0[J] = \frac{1}{2} J_A \mathcal{K}^{AB} J_B \tag{1.19}$$

one easily realizes that it is the generator of the (only) connected function of the system.

Now we want to introduce some interaction encoded in a potential $V[\phi]$

$$S[\phi] = S_0[\phi] + V[\phi] \tag{1.20}$$

it's not difficul to believe that the $J$-dependent partition function is

$$Z[J] = \int \mathcal{D}\phi\, e^{-S_0[\phi]-V[\phi]+J_A\phi^A} = e^{-V[\frac{\delta}{\delta J}]}\, Z_0[J] \equiv e^{W[J]} \tag{1.21}$$

This is the starting point for all the *perturbative expansion*: if we expand the prefactor $e^{-V}$ we obtain a combinations of free propagators and vertexes that depend on the type of potential. It is possible to show that the $n$-point correlation functions generated by $W[J]$ are those of $Z[J]$ provided one removes all the diagrams that are disconnected.

Now we want to introduce a third functional, probably the most important one in quantum field theory.
First of all we define a new field called *classical field*

$$\phi_{c\ell}^A \equiv \left\langle \phi^A \right\rangle_J = \frac{\delta W[J]}{\delta J_A} \tag{1.22}$$

in fact

$$\left\langle \phi^A \right\rangle_J = \frac{\int \mathcal{D}\phi e^{-S[\phi]+J\cdot\phi}\, \phi^A}{\int \mathcal{D}\phi e^{-S[\phi]+J\cdot\phi}} = \frac{1}{Z[J]} \frac{\delta Z[J]}{\delta J_A} = \frac{\delta}{\delta J_A} \ln Z[J] = \frac{\delta W[J]}{\delta J_A}$$

that is the normalized vacuum expectation value of the self-interacting local quantum field operator $\phi^A$ in the presence of an external classical source $J$. Thus $\phi_{c\ell}^A$ is a quantum object in spite of the name. Note that in the limit of vanishing external source, if there is translation invariance in the theory, we have

$$\phi_{c\ell}^A\,|_{J=0} = \text{constant}$$

with constant equal to zero if and only if the spontaneous symmetry breaking does not occur.
We can now suitably define the Legendre functional transformation:

$$\Gamma[\phi_{c\ell}] \equiv \phi_{c\ell}^A\, J_A - W[J] \tag{1.23}$$

where we assume that the functional $\phi^A_{c\ell} = \phi^A_{c\ell}[J]$ is invertible so that $J = J[\phi_{c\ell}]$ and $\Gamma$ becomes a functional of $\phi^A_{c\ell}$. The Legendre transformed functional $\Gamma$ is called **Effective Action** and it is the generator of the so called *one particle irreducible correlation functions* (1PI) or *proper verteces*. The Effective Action has a very important interpretation: it is a classical action that can reproduce the quantum correlations, in other words it encodes all the quantum behavior of the system. Property of the effective action is to be convex *i.e.* the matrix

$$\frac{\delta^2 \Gamma}{\delta\phi^A\,\delta\phi^B}$$

must have positive semidefinite eigenvalues. As we can see from the eq.(1.23)

$$\frac{\delta\Gamma}{J_A} = 0 \quad \Rightarrow \quad \phi^A_{c\ell} = \frac{\delta W[J]}{\delta J_A} \tag{1.24}$$

the effective action is an extreme in the function $J$.

We define *proper verteces* the non-local coefficients in the expansion of $\Gamma[\phi_{c\ell}]$ in powers of $\phi_{c\ell}$. In fact we can write

$$\Gamma[\phi_{c\ell}] = \sum_{n=1}^{\infty} \frac{1}{n!} \left[\prod_{j=1}^{n} \int d^D x_j\, \phi_{c\ell}(x_j)\right] \Gamma^{(n)}(x_1.\cdots, x_n) \tag{1.25}$$

where

$$\Gamma^{(n)}(x_1.\cdots, x_n) = \left.\frac{\delta\Gamma[\phi_{c\ell}]}{\delta\phi_{c\ell}(x_n)\cdots\delta\phi_{c\ell}(x_1)}\right|_{\phi_{c\ell}=0} \tag{1.26}$$

In momentum space they read

$$\Gamma^{(n)}_{x_1,\cdots,x_n} = \int \frac{d^D p_1}{(2\pi)^D} \cdots \int \frac{d^D p_n}{(2\pi)^D}\, e^{-ip_1\cdot x_1 \ldots -ip_n\cdot x_n} \widetilde{\Gamma}^{(n)}_{p_1,\ldots,p_n} (2\pi)^D \delta(p_1 + \ldots + p_n) \tag{1.27}$$

where we have emphasized the translational invariance. For example the two point proper vertex reads

$$\Gamma^{(2)}(x-y) = \int \frac{d^D p}{(2\pi)^D}\, e^{-ip\cdot(x-y)}\, \widetilde{\Gamma}^{(2)}(k) \tag{1.28}$$

## 1.2 Wilson Approach

One of the extraordinary result of the technique of perturbative renormalization is that, in nature, the couplings manifest themselves through scale dependence [24][14]. In QFT this aspect emerges when we need to renormalize the theory, so in the development of the theory, we want to add this feature from the very beginning defining scale dependent functional so that this characteristic is built-in in the formalism. The first who tried to implement this idea in field theory was Wilson [33][34].

Suppose to deal with a scalar field in an euclidean manifold $\mathbb{R}^D$, and suppose to expand the field in the momentum space

$$\phi(x) = \int \frac{d^D q}{(2\pi)^D}\, \phi_q \, e^{-iq\cdot x} \tag{1.29}$$

The natural functional measure for our field integration is in this case

$$\int \mathcal{D}\phi = \prod_{q\in\mathbb{R}^D} d\phi_q \tag{1.30}$$

so the partition function is

$$Z = \prod_{q\in\mathbb{R}^D} \int d\phi_q e^{-S[\phi]} \tag{1.31}$$

We know that this integral is ill defined because it is divergent, for this reason we have to introduce a cutoff $\Lambda$ and a certain action $S_\Lambda[\phi]$ to regularize the integral and deal with finite correlations. The modifications of both the measure and the action must be such as to reproduce the same partition function.

$$Z = \prod_{q\leq\Lambda} \int d\phi_q e^{-S_\Lambda[\phi]} \tag{1.32}$$

We can give a physical interpretation to the new action and think at $S_\Lambda[\phi]$ as an UV action that contains all the information of the theory at energies greater than $\Lambda$. Following the same trick we can introduce a scale $k$ and define a new action $S_k[\phi]$ in this way

$$\begin{aligned} Z &= \prod_{\substack{q\in\mathbb{R}^D \\ |q|\leq k}} \int d\phi_q \underbrace{\prod_{\substack{q\in\mathbb{R}^D \\ k\leq|q|\leq\Lambda}} \int d\phi_q \, e^{-S_\Lambda[\phi]}}_{\equiv e^{-S_k[\phi]}} \\ &= \prod_{\substack{q\in\mathbb{R}^D \\ |q|\leq k}} \int d\phi_q \, e^{-S_k[\phi]} \end{aligned} \tag{1.33}$$

where

$$e^{-S_k[\phi]} = \prod_{\substack{q\in\mathbb{R}^D \\ k\leq|q|\leq\Lambda}} \int d\phi_q \, e^{-S_\Lambda[\phi]} \tag{1.34}$$

We thus interpret $S_k[\phi]$ as the result that comes integrating all modes with $k \leq |q| \leq \Lambda$. Moving $k$ towards zero means that we are moving us in the direction of a theory in which all scales contribute to our effective theory. This new action is called **Wilson effective action** and describes the physics at the associated scale $k$, in other words only modes with $|q| \simeq k$ are active at that scale. We can understand this even from another point of view using a blocking procedure.

This procedure was first used by Kadanoff in the study of scaling behavior of spin chain systems [8]. The idea was to divide the spin chain into blocks and perform a local average in order to obtain an "effective spin" for each block. This new effective-spin-chain has to be rescaled at the original lattice scale in order to compare the two systems. In this way we can construct an effective Hamiltonian for the system made of block spins. To be more precise lets start from the initial partition function

$$Z = \sum_{\sigma_i} e^{-\beta\mathcal{H}[\sigma_i]} \tag{1.35}$$

and taking into account the procedure of "decimation" we can write this relation

$$\sum_{\sigma_A}\prod_A \delta\big(\sigma_A - \frac{1}{\alpha^D}\sum_{i\in A}\sigma_i\big) = 1 \tag{1.36}$$

because in the sum $\sum_{\sigma_A}$ only one particular configuration of spin $\{\sigma_A\}$ correspond to that of block spins. So we get the result

$$Z = \sum_{\sigma_i} e^{-\beta\mathcal{H}[\sigma_i]} = \sum_{\sigma_A} e^{-\beta\mathcal{H}_{\text{eff}}[\sigma_A]} \tag{1.37}$$

where

$$e^{-\beta\mathcal{H}_{\text{eff}}[\sigma_A]} = \sum_{\sigma_i}\prod_A \delta\big(\sigma_A - \frac{1}{\alpha^D}\sum_{i\in A}\sigma_i\big)e^{-\beta\mathcal{H}[\sigma_i]} \tag{1.38}$$

This procedure can be iterated an infinite number of times, obtaining from each step a new effective Hamiltonian at larger scale. Therefore this technique is useful in the study of systems near phase transition where the correlation length $\xi \to \infty$ tends to infinite and collective behaviors emerge.

Wilson extended this idea for system with an infinite number of degree of freedom *i.e.* for fields. We can define in a similar way a *"blocked"* field as the convolution product of the scalar field $\phi$ with a "smearing function" $\rho_k(x)$ [10]

$$\phi_k(x) = \int dy\, \rho_k(x-y)\phi(y) \tag{1.39}$$

that has to provide an averaging of our field over a region of typical size $k^{-D}$.
We define also a coarse grained functional

$$e^{-S_k[\Phi]} = \int_\Lambda \mathcal{D}\phi\, \delta(\Phi - \phi_k)e^{-S[\phi]} \tag{1.40}$$

that is the continuum case of eq.(1.38) where we have stressed the cutoff $\Lambda$, necessary for the convergence of the integral (in a certain sense we can say that $\Lambda$ can be related to $a^{-1}$ with $a$ be the size of the spin lattice). If we choose for the smearing function, the step function

$$\widetilde{\rho}_k(q) = \theta(k-q) \tag{1.41}$$

we have a clear separation between slow modes

$$\Phi(x) \equiv \phi_<(x) = \int d^Dy\, \phi(y) \int \frac{d^Dq}{(2\pi)^D}\, e^{-q\cdot(x-y)}\, \theta(k-|q|) \tag{1.42}$$

and fast modes

$$\phi_>(x) = \phi(x) - \phi_<(x) \tag{1.43}$$

therefore we can write

$$e^{-S_k[\Phi]} = \int_\Lambda \mathcal{D}\phi_> \int_\Lambda \mathcal{D}\phi_< \, \delta(\Phi - \phi_<)\, e^{-S[\phi_< + \phi_>]} = \int \mathcal{D}\phi_> \, e^{-S[\Phi + \phi_>]} \tag{1.44}$$

that morally is the same as eq.(1.34) and tells us that after integrating out fast modes we obtain an effective action for slow modes. It's important to note that, as always happens in coarse–graining procedures, there is a hidden scheme dependence in the method. Therefore we have been capable to build a theory for $\phi_k(x)$ and construct an effective theory that describes effects of energies of order $k$ or less.

The equation describing the evolution of $S_k[\Phi]$ was derived by Polchinscki [23] and it reads

$$\frac{\partial S_k[\Phi]}{\partial k} = \frac{1}{2}\int \frac{d^D q}{(2\pi)^D} \frac{\partial R_k(q)}{\partial k} \left( \frac{\delta^2 S_k}{\delta\widetilde{\Phi}(q)\delta\widetilde{\Phi}(-q)} - \frac{\delta S_k}{\delta\widetilde{\Phi}(q)} \frac{\delta S_k}{\delta\widetilde{\Phi}(-q)} \right) \tag{1.45}$$

where $R_k(q)$ is a certain cutoff function (we will discuss exhaustively this equation in the third chapter).

## 1.3 Wetterich's non perturbative FRG equation

Wetterich in 1993 formulated an alternative approach to functional renormalization based on a scale dependent effective action $\Gamma_k$ rather than a scale dependent action $S_k$ [32].

In terms of $\Gamma$ we are looking for an interpolating action $\Gamma_k$, which is called **Effective Average Action**, with a momentum-shell parameter $k$ such that it satisfies

$$\begin{cases} \Gamma_{k\to\Lambda} = S_{\text{bare}} \\ \Gamma_{k\to 0} = \Gamma \end{cases} \tag{1.46}$$

The bare action is the microscopic classical action to be quantized and $\Gamma$ is the full quantum effective action that includes all quantum fluctuations for all momenta. $\Lambda$ is an ultraviolet cutoff which represents the physical energy scale beyond which QFT loses its validity. The theory is said UV complete if $\Lambda$ can be sent to $\infty$. $\Gamma_k$ is an effective action for average fields, this average is taken over a volume $\approx k^{-\mathrm{D}}$ so only the degree of freedom with momenta greater than the coarse-graining scale $k$ are effectively integrated out. We want to obtain a differential equation that describes the flow of $\Gamma_k$ compared to $k$. These are exactly the same goals of the Wilson approach.

The key idea is to modify the partition function adding an *infrared* cutoff term, depending on the coarse-graining scale $k$, that has the property to kill the propagation of "slow" modes and keep unaltered "fast" modes. So we have

$$Z_k[J] = \int \mathcal{D}\phi \, e^{-S[\phi] + J\cdot\phi - \Delta S_k[\phi]} \tag{1.47}$$

where the IR cutoff term has to satisfy the condition

$$\Delta S_{k=0}[\phi] = 0 \quad \Rightarrow \quad Z_{k=0}[J] = Z[J] \tag{1.48}$$

moreover, since it modifies the propagation of $\phi^A$ modes, it must be quadratic in the fields

$$\Delta S_k[\phi] = \frac{1}{2}\,\phi^A R^k_{AB}\phi^B \tag{1.49}$$

Let $\mathcal{K}_{AB}$ be the kinetic operator of the action and $\psi_i^B$ its eigen-fields

$$\mathcal{K}^A_B\,\psi_i^B = \lambda_i^2\psi_i^A \tag{1.50}$$

now we can separate "fast" modes ($\lambda_i^2 \gtrsim k^2$) from "slow" modes ($\lambda_i^2 \lesssim k^2$). We ask the kernel $R^k_{AB}$ to be a function of $\mathcal{K}$ such that moving to the eigen-fields we have

$$\mathcal{K}^A_B + R^{k,\,A}_B[\mathcal{K}] \to \lambda_i^2 + R_k[\lambda_i^2] \tag{1.51}$$

The conditions that $R_k[\lambda^2]$ has to satisfy are:

- $\lim_{k\to 0} R_k[\lambda^2] = 0$ equivalent to the condition $Z_{k=0}[J] = Z[J]$;
- $R_k[\lambda^2] > 0$ for $\lambda^2 < k^2$ which implements the IR regularization *i.e.* the regulator screens the IR modes in a mass-like fashion;
- $R_k[\lambda^2] \simeq 0$ for $\lambda^2 > k^2$ which implies that rapid modes are unaffected by the coarse-graining and are integrated out;
- $\lim_{k\to\infty} R_k[\lambda^2] = \infty$ so that no modes are propagating and quantum fluctuations are exponentially suppressed in the path integral (the stationary point of the classical action $S$ becomes dominant).

As before we first define the generator of connected Green functions

$$W_k[J] = \ln Z_k[J] \tag{1.52}$$

and the classical field which will be now $k$-dependent

$$\phi^A_{cl,k} \equiv \frac{\delta W_k[J]}{\delta J_A} = \frac{1}{Z_k[J]}\frac{\delta Z_k[J]}{\delta J_A} = \left\langle \phi^A \right\rangle_{k,J} \tag{1.53}$$

It is important to notice that if we keep $J$ fixed then $\phi_{c\ell}$ will depend on the scale $k$ and *viceversa*, if we keep $\phi_{c\ell}$ fixed then $J$ will depend on the scale $k$. In this case the **effective average action** takes the following form

$$\Gamma_k[\phi_{c\ell}] \equiv \underbrace{J_A\phi^A_{c\ell} - W_k[J]}_{\equiv \hat{\Gamma}_k[\phi_{c\ell}]} - \Delta S_k[\phi_{c\ell}] \tag{1.54}$$

where again $J$ has to be inverted as a functional of $\phi_{c\ell}$. Another important observation is that, because of the last term, eq.(1.54) is not a Legendre transformation so the effective average action is still not convex, convexity is restored in the limit $k \to 0$.

Now we want to derive the equations that governs the behavior of the functionals that we have introduced before, respect the sliding scale $k$ [32][1][4].

$$\left.\frac{\partial Z_k[J]}{\partial k}\right|_{\text{J fix}} = \int \mathcal{D}\phi\, e^{-S[\phi]+J\cdot\phi-\Delta S_k[\phi]} \left(-\frac{\partial}{\partial k}\Delta S_k[\phi]\right) \tag{1.55}$$

$$= \int \mathcal{D}\phi\, e^{-S[\phi]+J\cdot\phi-\Delta S_k[\phi]} \left(-\frac{1}{2}\phi^A \partial_k R^k_{AB}\phi^B\right) \tag{1.56}$$

$$= -\frac{1}{2}\frac{\delta}{\delta J_A}\left(\partial_k R^k_{AB}\right)\frac{\delta}{\delta J_B}\; Z_k[J] \tag{1.57}$$

$$\left.\frac{\partial W_k[J]}{\partial k}\right|_{\text{J fix}} = \frac{1}{Z_k[J]}\partial_k Z_k[J] \tag{1.58}$$

$$= -\frac{1}{2}\partial_k R^k_{AB}\left(\frac{\delta W_k}{\delta J_A}\frac{\delta W_k}{\delta J_B} + \frac{\delta^2 W_k}{\delta J_A \delta J_B}\right) \tag{1.59}$$

This is a functional differential equation for $W_k[J]$ and is related to the Polchinski equation [23] which has been presented in eq.(1.45).

$$\frac{\delta\hat{\Gamma}_k[\phi_{c\ell}]}{\delta\phi^A_{c\ell}} = \frac{\delta J_B}{\delta\phi^A_{c\ell}}\phi^B_{c\ell} + J_A - \frac{\delta W_k}{\delta J_B}\frac{\delta J_B}{\delta\phi^A_{c\ell}} = J_A \tag{1.60}$$

$$= \frac{\delta\Gamma_k}{\delta\phi^A_{c\ell}} + \frac{\delta\Delta S_k}{\delta\phi^A_{c\ell}} \tag{1.61}$$

$$= \frac{\delta\Gamma_k}{\delta\phi^A_{c\ell}} + R^k_{AB}\phi^B_{c\ell} \tag{1.62}$$

Now we want to compute the k-derivative at $J$ fixed of the eq. (1.54), to do this we have to take into account that [4]

$$\left.\frac{\partial}{\partial k}\right|_{J\text{fix}} = \left.\frac{\partial}{\partial k}\right|_{\phi\text{fix}} + \left.\frac{\partial\phi^A}{\partial k}\right|_{J\text{fix}}\frac{\delta}{\delta\phi^A} \tag{1.63}$$

so we get

$$\left.\frac{\partial\Gamma_k}{\partial k}\right|_J = \left.\frac{\partial\Gamma_k}{\partial k}\right|_\phi + \left.\frac{\partial\phi^A}{\partial k}\right|_J\frac{\delta\Gamma_k}{\delta\phi^A} = J_A\left.\frac{\partial\phi^A}{\partial k}\right|_J - \left.\frac{\partial W_k}{\partial k}\right|_J - \left.\frac{\partial\Delta S_k}{\partial k}\right|_J \tag{1.64}$$

$$\left.\frac{\partial\Gamma_k}{\partial k}\right|_\phi = \left(J_A - \frac{\delta\Gamma_k}{\delta\phi^A}\right)\left.\frac{\partial\phi^A}{\partial k}\right|_J - \left.\frac{\partial W_k}{\partial k}\right|_J - \left.\frac{\partial\Delta S_k}{\partial k}\right|_J \tag{1.65}$$

$$= R^k_{AB}\phi^B\left.\frac{\partial\phi^A}{\partial k}\right|_J + \frac{1}{2}\partial_k R^k_{AB}\left(\frac{\delta W_k}{\delta J_A}\frac{\delta W_k}{\delta J_B} + \frac{\delta^2 W_k}{\delta J_A\delta J_B}\right) \tag{1.66}$$

$$-\frac{1}{2}\left.\frac{\partial R^k_{AB}}{\partial k}\right|_J\phi^A\phi^B - R^k_{AB}\phi^A\left.\frac{\partial\phi^B}{\partial k}\right|_J \tag{1.67}$$

where we have used the eq.(1.59) and eq.(1.62). Now, by definition of $\phi^A_{c\ell}$ we obtain

$$\left.\frac{\partial\Gamma_k}{\partial k}\right|_\phi = \frac{1}{2}\partial_k R^k_{AB}\frac{\delta^2 W_k}{\delta J_A\delta J_B} \tag{1.68}$$

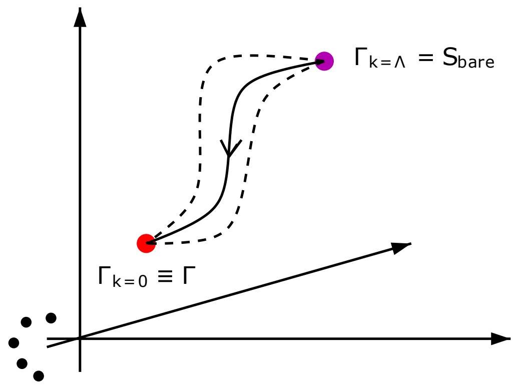



**Figure 1.1:** Sketch of the RG flow in the theory space. Each axis correspond to a different operator which spans the effective action.

The last step is to find a relation between the second derivative of $W_k$ and $\Gamma_k$.

$$G^{(2)}_{k,AB} \equiv \frac{\delta^2 W_k}{\delta J_A \delta J_B} = \frac{\delta \phi^A}{\delta J_B} = \left(\frac{\delta J_B}{\delta \phi^A}\right)^{-1} = \left(\frac{\delta^2 \hat{\Gamma}_k}{\delta \phi^A \delta \phi^B}\right)^{-1} = \left(\Gamma^{(2)}_{k,AB} + R_{k,AB}\right)^{-1} \tag{1.69}$$

We can see that the propagator is modified by the presence of the cutoff term $R_k$. If we define a new variable, the "time" of the renormalization flow

$$t = \ln \frac{k}{\Lambda} \Rightarrow \partial_t = k\partial_k \tag{1.70}$$

we conclude that the **exact renormalization group equation (ERGE)** is

$$\boxed{\begin{aligned}\dot{\Gamma}_k[\phi_{c\ell}] &= \frac{1}{2}\text{Tr}\left(G_k[\phi_{c\ell}]\dot{R}_k\right) \\ &= \frac{1}{2}\text{Tr}\left[\left(\Gamma^{(2)}_{k,AB} + R_{k,AB}\right)^{-1}\dot{R}_k\right]\end{aligned}} \tag{1.71}$$

Now I want to spend a few words on this equation enunciating its properties [5]:

- The flow equation is a functional differential equation for $\Gamma_k[\phi_{c\ell}]$ and not an integral-differential equation as we shall see in the next section. This equation is not approximate so in principle the results are exact.

- The solution of the ERGE is an RG trajectory in the *theory space i.e.* in the space of all the action functional spanned by all possible invariant operators of fields. Start and end of the trajectory are the bare action and the full action respectively. Note that there is a built-in dependence of the ERGE on the choice of the regulator so even the trajectory will depend on the particular shape of $R_k$ (see fig.1.1).

- The ERGE can be interpreted as a 1-loop equation, where the modified propagator performs a loop with a single insertion of the derivative of the

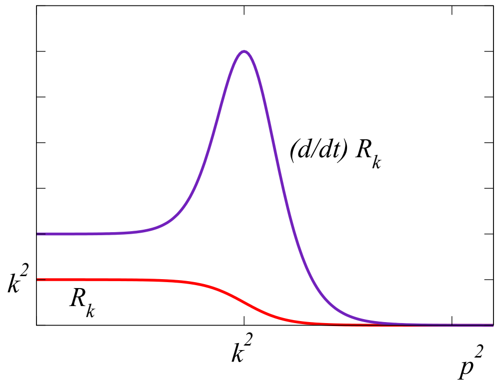
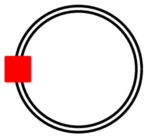



**Figure 1.2:** Sketch of a regulator function $R_k(p^2)$ and its derivative $\dot{R}_k$.We can see that the derivative implements the Wilsonian idea of integration out fluctuations within a momentum shell near $p^2 \simeq k^2$. Moreover we see that the regulator provides for an IR regulator for all modes with $p^2 \lesssim k^2$

cutoff term. The one loop structure derives from the choice of $\Delta S_k$ to be quadratic in the fields [12].

$$\dot{\Gamma}_k[\phi] = \frac{1}{2} \quad \dot{R}_k \tag{1.72}$$



- The purpose of the regulator is twofold: by construction it is an IR regulator as we can see in the occurrence of $R_k$ in the denominator of the ERGE and moreover, $\dot{R}_k$ acts as an UV regulator thanks to the conditions $\lim_{q^2/k^2 \to 0} R_k(q^2) > 0$, $\lim_{k^2/q^2 \to 0} R_k(q^2) = 0$ and thanks to the fact that its predominant support lies on a smeared momentum shell near $p^2 \sim k^2$. A typical shape of the regulator and of its derivative is given in fig.1.2. The peaked structure of $\dot{R}_k$ implements the Wilsonian idea of integrating over momentum shells and implies that the flow is localized in momentum space.

### 1.3.1 Alternative form of the ERGE

From the eq.(1.47) and the definition of $\hat{\Gamma}_k$ (1.54) we have

$$e^{-\hat{\Gamma}_k[\phi_{c\ell}]} = \int \mathcal{D}\phi \, e^{-S[\phi]-\Delta S_k[\phi]+\int J\cdot(\phi-\phi_{c\ell})} \tag{1.73}$$

that is an integro-differential equation for $\hat{\Gamma}_k[\phi_{c\ell}]$ because

$$J_A = \frac{\delta \hat{\Gamma}_k[\phi_{c\ell}]}{\delta \phi^A_{c\ell}} \tag{1.74}$$

We want to obtain a similar equation for $\Gamma_k[\phi_{c\ell}]$. First of all we expand the eq.(1.73)

$$e^{-\Gamma_k[\phi_{c\ell}]-\Delta S_k[\phi_{c\ell}]} = \int \mathcal{D}\phi \, e^{-S[\phi]-\Delta S_k[\phi]+\int \frac{\delta(\Gamma_k+\Delta S_k)[\phi_{c\ell}]}{\delta\phi_{c\ell}}\cdot(\phi-\phi_{c\ell})} \tag{1.75}$$

$$e^{-\Gamma_k[\phi_{c\ell}]} = \int \mathcal{D}\phi \, e^{-S[\phi]-\Delta S_k[\phi]+\Delta S_k[\phi_{c\ell}]+\int \frac{\delta(\Gamma_k+\Delta S_k)[\phi_{c\ell}]}{\delta\phi_{c\ell}}\cdot(\phi-\phi_{c\ell})} \tag{1.76}$$

Now we introduce the fluctuation field $\chi \equiv \phi - \phi_{c\ell}$ and taking into account that the IR cutoff term is quadratic in the fields

$$\begin{aligned} \Delta S_k[\phi] &= \Delta S_k[\phi_{c\ell}] + \Delta S_k[\chi] + \chi^A R^k_{AB}\phi^B_{c\ell} && (1.77) \\ &= \Delta S_k[\phi_{c\ell}] + \Delta S_k[\chi] + \chi^A \left.\frac{\delta \Delta S_k[\phi]}{\delta\phi^A}\right|_{\phi_{c\ell}} && (1.78) \end{aligned}$$

and that the path integral is invariant under translation, we finally obtain the following integro-differential equation for $\Gamma_k[\phi_{c\ell}]$

$$\begin{aligned} e^{-\Gamma_k[\phi_{c\ell}]} &= \int \mathcal{D}\chi \, e^{-S[\phi_{c\ell}+\chi]-\Delta S_k[\chi]+\int \chi \frac{\delta\Gamma_k}{\delta\phi_{c\ell}}} && (1.79) \\ &= \int \mathcal{D}\phi \, e^{-S[\phi]-\Delta S_k[\phi-\phi_{c\ell}]+\int(\phi-\phi_{c\ell})\frac{\delta\Gamma_k}{\delta\phi_{c\ell}}} && (1.80) \end{aligned}$$

Now suppose to start from a general theory which has an effective action defined as in eq.(1.80) where $\phi_{c\ell}$ is an unknown field configuration. Taking the functional derivative respect $\phi_{c\ell}$ we get

$$\left(R^k_{AB} + \frac{\delta^2\Gamma_k}{\delta\phi^A_{c\ell}\delta\phi^B_{c\ell}}\right)\int \mathcal{D}\phi\mathcal{P}[\phi](\phi-\phi_{c\ell})^B = 0 \tag{1.81}$$

and provided that $\Gamma^{(2)}_k + R_k$ has a null kernel *i.e.* invertible, we can say that

$$\phi^A_{c\ell} = \langle\phi^A\rangle = \int \mathcal{D}\phi\mathcal{P}[\phi]\phi^A \tag{1.82}$$

where

$$\mathcal{P}[\phi] = \frac{e^{-S[\phi]-\Delta S_k[\phi-\phi_{c\ell}]+\int(\phi-\phi_{c\ell})\frac{\delta\Gamma_k}{\delta\phi_{c\ell}}}}{\int\mathcal{D}\phi\, e^{-S[\phi]-\Delta S_k[\phi-\phi_{c\ell}]+\int(\phi-\phi_{c\ell})\frac{\delta\Gamma_k}{\delta\phi_{c\ell}}}} \tag{1.83}$$

so we obtained that the classical field is exactly the average of the field $\phi$. Taking again another derivative we get

$$\delta^A_B = \int \mathcal{D}\phi\,\phi^A \frac{\delta}{\delta\phi^B_{c\ell}}\mathcal{P}[\phi] \tag{1.84}$$

if we expand

$$\begin{aligned} &\delta^A_B \int \mathcal{D}\phi e^{[\dots]} + \phi^A_{c\ell}\int \mathcal{D}\phi\, e^{[\dots]}\left\{(\phi-\phi_{c\ell})^C\left(R^k_{CB} + \frac{\delta^2\Gamma_k}{\delta\phi^B_{c\ell}\delta\phi^C_{c\ell}}\right) - \frac{\delta\Gamma_k}{\delta\phi^B_{c\ell}}\right\} = \\ &\quad = \int \mathcal{D}\phi\,\phi^A\, e^{[\dots]}\left\{(\phi-\phi_{c\ell})^C\left(R^k_{CB} + \frac{\delta^2\Gamma_k}{\delta\phi^B_{c\ell}\delta\phi^C_{c\ell}}\right) - \frac{\delta\Gamma_k}{\delta\phi^B_{c\ell}}\right\} \end{aligned}$$

$$\delta^A_B = \int \mathcal{D}\phi \mathcal{P}[\phi] \left\{ (\phi^A \phi^C - \phi^A_{c\ell}\phi^C) \left( R^k_{CB} + \frac{\delta^2 \Gamma_k}{\delta\phi^B_{c\ell}\delta\phi^C_{c\ell}} \right) \right\} \tag{1.85}$$

$$= \left( \left\langle \phi^A \phi^C \right\rangle - \phi^A_{c\ell}\phi^C_{c\ell} \right) \left( R^k_{CB} + \frac{\delta^2 \Gamma_k}{\delta\phi^B_{c\ell}\delta\phi^C_{c\ell}} \right) \tag{1.86}$$

$$= \left( \left\langle \phi^A \phi^C \right\rangle - \left\langle \phi^A \right\rangle \left\langle \phi^C \right\rangle \right) \left( R^k_{CB} + \frac{\delta^2 \Gamma_k}{\delta\phi^B_{c\ell}\delta\phi^C_{c\ell}} \right) \tag{1.87}$$

$$= G^{AC}_k \left( R^k_{CB} + \frac{\delta^2 \Gamma_k}{\delta\phi^B_{c\ell}\delta\phi^C_{c\ell}} \right) \tag{1.88}$$

again we have that the full connected two-points propagator is

$$G_k = \left( R_k + \Gamma^{(2)}_k \right)^{-1} \tag{1.89}$$

We are ready to deduce the ERGE: taking the $k$-derivative of the eq.(1.79) at $\phi_{c\ell}$ constant we get

$$\partial_k \Gamma_k[\phi_{c\ell}] = \frac{1}{2} \partial_k R^k_{AB} \int \mathcal{D}\chi \mathcal{P}[\chi] \chi^A \chi^B - \partial_k \frac{\delta \Gamma_k}{\delta \phi^A_{c\ell}} \int \mathcal{D}\chi \mathcal{P}[\chi] \chi^A \tag{1.90}$$

but $\left\langle \chi^A \right\rangle = 0$ and $\left\langle \chi^A \chi^B \right\rangle = G^{AB}_k$. The final result is exactly the same as before

$$\partial_k \Gamma_k[\phi_{c\ell}] = \frac{1}{2} \mathrm{Tr} \left[ G_k \partial_k R_k \right] \tag{1.91}$$

### 1.3.2 Truncation methods

The Wetterich's equation cannot be solved exactly for an arbitrary $\Gamma_k$ therefore some approximations on the effective action have to be taken. There are two main truncation methods used in the literature: the *derivative expansion* and the *vertex expansion*. The last one was introduced and extensively investigated by Tim R. Morris [17] and it is widely used in condensed matter physics and in low energy QCD physics.

It is very important to emphasize that, because these methods are not based on expansions in some small coupling parameters, they are essentially non perturbative. The consequence of making these approximation is to transform the ERGE into a system of differential equations sometimes much more easy to solve.

The *derivative expansion* consists in expanding the effective action in powers of derivative of the fields. This methods is often applied to problems where one is interested in low momenta or when the local dynamics is known to dominate. This is the most used approximation technique and its convergence properties have been largely discussed (see for example [19]). We will use this approximation in a rather different way in the third chapter.

The *vertex expansion* consists in expanding the effective action in powers of the field.

$$\Gamma_k[\phi_{c\ell}] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^D x_1 \dots \int d^D x_n \, \Gamma^{(n)}_k(x_1, \dots, x_n) \phi_{c\ell}(x_1) \dots \phi_{c\ell}(x_n) \tag{1.92}$$

Upon inserting this expansion into the ERGE we obtain an infinite tower of functional equations that describe the flow of the $n$-points function.

Lets show the first three terms

$$\begin{aligned}\dot{\Gamma}_k^{(1)}[\phi_{c\ell}] = \frac{\delta}{\delta\phi_{c\ell}}\dot{\Gamma}_k[\phi_{c\ell}] &= -\frac{1}{2}\mathrm{Tr}\left\{\dot{R}_k\frac{1}{\Gamma_k^{(2)}+R_k}\frac{\delta\Gamma_k^{(2)}}{\delta\phi_{c\ell}}\frac{1}{\Gamma_k^{(2)}+R_k}\right\}\\ &= -\frac{1}{2}\mathrm{Tr}\left\{\dot{R}_k\frac{1}{\Gamma_k^{(2)}+R_k}\Gamma_k^{(3)}\frac{1}{\Gamma_k^{(2)}+R_k}\right\}\end{aligned} \tag{1.93}$$

$$\begin{aligned}\dot{\Gamma}_k^{(2)}[\phi_{c\ell}] = \frac{\delta^2}{\delta\phi_{c\ell}\delta\phi_{c\ell}}\dot{\Gamma}_k[\phi_{c\ell}] =& 2\times\frac{1}{2}\mathrm{Tr}\left\{\dot{R}_k\frac{1}{\Gamma_k^{(2)}+R_k}\Gamma_k^{(3)}\frac{1}{\Gamma_k^{(2)}+R_k}\Gamma_k^{(3)}\frac{1}{\Gamma_k^{(2)}+R_k}\right\}\\ &-\frac{1}{2}\mathrm{Tr}\left\{\dot{R}_k\frac{1}{\Gamma_k^{(2)}+R_k}\Gamma_k^{(4)}\frac{1}{\Gamma_k^{(2)}+R_k}\right\}\end{aligned} \tag{1.94}$$

$$\begin{aligned}\dot{\Gamma}_k^{(3)}[\phi_{c\ell}] =& -\frac{1}{2}\mathrm{Tr}\left\{\dot{R}_k\frac{1}{\Gamma_k^{(2)}+R_k}\Gamma_k^{(5)}\frac{1}{\Gamma_k^{(2)}+R_k}\right\}\\ &+6\times\frac{1}{2}\mathrm{Tr}\left\{\dot{R}_k\frac{1}{\Gamma_k^{(2)}+R_k}\Gamma_k^{(4)}\frac{1}{\Gamma_k^{(2)}+R_k}\Gamma_k^{(3)}\frac{1}{\Gamma_k^{(2)}+R_k}\right\}\\ &-6\times\frac{1}{2}\mathrm{Tr}\left\{\dot{R}_k\frac{1}{\Gamma_k^{(2)}+R_k}\Gamma_k^{(3)}\frac{1}{\Gamma_k^{(2)}+R_k}\Gamma_k^{(3)}\frac{1}{\Gamma_k^{(2)}+R_k}\Gamma_k^{(3)}\frac{1}{\Gamma_k^{(2)}+R_k}\right\}\end{aligned} \tag{1.95}$$

What we can see is that

$$\dot{\Gamma}_k^{(n)}[\phi_{c\ell}] = \mathcal{F}_n\left[\phi_{c\ell},\Gamma_k^{(2)},\ldots,\Gamma_k^{(n+2)}\right] \tag{1.96}$$

so we have a *hierarchy of the flow equations* and this is the problem. How we can do meaningful calculations? We need to truncate the effective action and restrict it to correlations on $n_{\max}$ fields but, in doing so, we no longer have a closed system of equations. First we need to write down a most general ansatz for the effective action that must contain all invariants that are compatible with the symmetries of the theory. Than one truncates by reducing higher $n$-point functions to contact terms or to a simplified momentum dependence or neglecting even higher correlations outright. For practical applications this is obviously the most problematic part, and it requires a lot of physical insight to make the correct physical choices.

We finally remember again that these approximations are not expansions in some small parameters although, of course, the assumption is that higher order operators will be irrelevant and suppressed due to the existence of a large scale.

## 1.4 Asymptotic Safety

The effective average action $\Gamma_k[\phi_{c\ell}]$ can be parametrized with a basis of operators $\mathcal{O}_{i,k}[\phi_{c\ell}]$ that are compatible with the symmetries of the system. These operators can be interpreted as coordinates in the space of all allowed field theories.

$$\Gamma_k[\phi_{c\ell}] = \sum_i g_{i,k}\mathcal{O}_{i,k}[\phi_{c\ell}] \tag{1.97}$$

where $g_{i,k}$ are the couplings and form the dual space of the operator space. We can keep the basis fixed or not exactly as in quantum mechanics we have the Heisenberg or Schroedinger representations. For simplicity we shall take the basis fixed ($k$-independent) so when we consider the derivative respect the RG time we have

$$\dot{\Gamma}_k[\phi_{c\ell}] = \sum_i \dot{g}_{i,k}\mathcal{O}_i[\phi_{c\ell}] = \sum_i \beta_i\mathcal{O}_i[\phi_{c\ell}] \tag{1.98}$$

where $\beta_{i,k}$ are called **beta functions** for the couplings $g_{i,k}$. From the ERGE we have that $\dot{\Gamma}_k[\phi_{c\ell}]$ is a function of $\Gamma_k[\phi_{c\ell}]$ so the beta function has this natural parametrization

$$\beta_i = \beta_i(g,k) \tag{1.99}$$

If the operators $\mathcal{O}_{i,k}[\phi_{c\ell}]$ have a canonical dimension cm$^{d_i}$ than the corresponding couplings $g_i$ have the canonical dimension cm$^{-d_i}$ so the naive scaling is $g_{i,k} \sim k^{d_i}$. However we are looking for dimensionless couplings, because from the point of view of experimental physics it is clear that we always measure quantities compared to some reference scale, so we define

$$\tilde{g}_{i,k} \equiv g_{i,k}\, k^{-d_i} \tag{1.100}$$

that corresponds to the Kadanoff rescaling after blocking. The relative beta functions are

$$\tilde{\beta}_i = -d_i\,\tilde{g}_{i,k} + k^{-d_i}\beta_i \tag{1.101}$$

and because of the dimensionless of $\tilde{\beta}_i$, they must be functions only of $\tilde{g}$

$$\tilde{\beta}_i = \tilde{\beta}_i(\tilde{g}) \tag{1.102}$$

Having chosen the basis of operators fixed with the scale, the knowledge of the RG flow for the effective average action is equivalent to the knowledge of the flow for the beta functions. The theory space is infinite dimensional so even the dual space of couplings is infinite dimensional but a problem rises because we are not capable to do infinite experiments to measure all the couplings. For this reason we want a theory to be predictive and so we have to consistently constrain the initial condition $S_\Lambda$ to some finite dimensional subset of the coupling space (the RG flow doesn't change the number of parameters).

We define a **fixed point** for the beta functions as the set of dimensionless parameters $\tilde{g}_i^*$ such that

$$\tilde{\beta}_i(\tilde{g}^*) = 0 \tag{1.103}$$

so if we start with a theory at a fixed point than the RG flow doesn't modify the theory that will remain there at every scale. The study of the behavior of the flow near a given fixed point is usually done defining the **stability matrix**

$$\mathcal{M}_{ij} \equiv \frac{\partial \tilde{\beta}_i}{\partial \tilde{g}_j} \tag{1.104}$$

This matrix at the FP can be diagonalized

$$\mathcal{M}_{ij}\Big|_{\tilde{g}^*} = \mathrm{diag}(\lambda_{(1)}, \lambda_{(2)}, \dots) \tag{1.105}$$

in order to obtain a set of eigenvectors $\{v^{(a)}\}$ and eigenvalues$\{\lambda_{(a)}\}$ that can be separated into two classes

- $\lambda_{(a)} > 0$ means that the FP is *repulsive* in the corresponding direction;
- $\lambda_{(a)} < 0$ means that the FP is *attractive* in the corresponding direction.

The flow near a fixed point along the $i^{th}$ direction can be expressed as

$$\tilde{g}_i(t) = \tilde{g}_i^* + \delta\tilde{g}_i(t) \tag{1.106}$$

where $\delta\tilde{g}_i$ is a small fluctuation around the fixed point. The flow equation can now be linearized

$$\begin{aligned} \tilde{\beta}_i(g) = \tilde{\beta}_i(g^*) + \frac{\partial \tilde{\beta}_i}{\partial \tilde{g}_j}\bigg|_{g^*} \delta\tilde{g}_j = \mathcal{M}_{ij}\Big|_{\tilde{g}^*} \delta\tilde{g}_j \\ = \partial_t(\tilde{g}_i^* + \delta\tilde{g}_i) = \partial_t(\delta\tilde{g}_i) \end{aligned} \tag{1.107}$$

$$\implies \partial_t(\delta\tilde{g}_i) = \mathcal{M}_{ij}\Big|_{\tilde{g}^*} \delta\tilde{g}_j \tag{1.108}$$

After solving the eigenvalue problem

$$\mathcal{M}_{ij}\Big|_{\tilde{g}^*} v_j^{(a)} = \lambda_{(a)} v_i^{(a)} \tag{1.109}$$

we can expand the fluctuation in term of the eigenvectors

$$\delta\tilde{g}_i(t) = \sum_{(a)} c_{(a)}(t)\, v_i^{(a)} \tag{1.110}$$

and substituting in eq.(1.108) we get

$$\sum_{(a)} \dot{c}_{(a)}\, v_i^{(a)} = \mathcal{M}_{ij}\Big|_{\tilde{g}^*} \sum_{(a)} c_{(a)}\, v_j^{(a)} = \sum_{(a)} c_{(a)}\, \lambda_{(a)} v_i^{(a)} \tag{1.111}$$

$$\implies \dot{c}_{(a)} = \lambda_{(a)}\, c_{(a)} \tag{1.112}$$

that has a power law solution

$$c_{(a)}(t) = c_{(a)}(0)\, e^{\lambda_{(a)} t} = c_{(a)}(0) \left(\frac{k}{k_0}\right)^{\lambda_{(a)}} \tag{1.113}$$

Therefore the behavior of the couplings near a FP in the linearized problem is

$$\tilde{g}_i(t) = \tilde{g}_i^* + \sum_{(a)} c_{(a)}(0) \left(\frac{k}{k_0}\right)^{\lambda_{(a)}} v_i^{(a)} \tag{1.114}$$

The **critical exponents** of the model are defined as

$$\nu_{(a)} \equiv -\lambda_{(a)} \tag{1.115}$$

and can be separated into three classes

- *relevant* if $\nu_{(a)} > 0$;
- *marginal* if $\nu_{(a)} = 0$;
- *irrelevant* if $\nu_{(a)} < 0$.

An attractive FP has a very important physical meaning because it may represent the ending of the limit $\Lambda \to \infty$ (interpreted as an extension of $k$ integration from $\Lambda$ to $\infty$), in fact along the attractive direction $\tilde{g}_i \xrightarrow[k\to\infty]{} \tilde{g}_i^*$. Therefore, in this case, we can consistently take the UV limit and the theory is said to be **asymptotically safe** and **renormalizable** [28]. *Asymptotic freedom* is a special case of asymptotic safety.

In such a case the fixed point is characterized by $\tilde{g}_i^* = 0$, the fixed point is said Gaussian. QCD, for example, is a one coupling $g_s$ asymptotically free theory with fixed point $g_s^* = 0$.

However there is another important thing necessary for the theory to be predictive: the number of attractive directions must be finite. The UV limit is also called "continuum limit" because it corresponds in lattice theory to the limit $a \to 0$ with $a$ the size of the lattice. In cutoff-regulated theories and theories with a sliding scale, if we want our theory to be a low energy manifestation of a more fundamental action with the same degrees of freedom, it is necessary that a FP, with the mentioned properties, exists. Otherwise no meaningful UV-limit is possible.

# Chapter 2

# The scalar model

In this chapter we will apply the exact renormalization group equation for the effective average action in a simple but nontrivial example: the scalar model in $D$ dimensions with an arbitrary local potential and an anomalous dimension encoded in the presence of a wave function renormalization.

## 2.1 Local potential approximation (LPA)

We start our study of the scalar model from an effective average action that we truncate in a local potential form [17] [15] *i.e.*

$$\Gamma_k[\phi] = \int d^D x \left( \frac{1}{2} Z_\phi \, \partial_\mu \phi \, \partial_\mu \phi + V[\phi] \right) \tag{2.1}$$

Remember that we are in euclidean spacetime and that $V[\phi]$ can be an arbitrary functional of the real scalar field $\phi$. We have also omitted the subscript $c\ell$ in the field for sake of simplicity. The scalar field has the canonical engineering dimension $[\phi] = \mathrm{cm}^{-\frac{D}{2}+1}$ and $[V] = \mathrm{cm}^{-D}$.

Now we have to do some general steps before using the ERGE. As it is usual in renormalization procedures we define the renormalized field

$$\phi_R \equiv \sqrt{Z_\phi}\phi \tag{2.2}$$

and because in the asymptotic safety scenario we want to study the flow of dimensionless couplings, we need to deal with dimensionless objects.

$$\begin{aligned} \phi_* &\equiv k^{-D/2+1} \sqrt{Z_\phi}\, \phi & (2.3) \\ V_*[\phi_*] &\equiv k^{-D}\, V\left[ \frac{\phi_*}{k^{-D/2+1}\sqrt{Z_\phi}} \right] & (2.4) \end{aligned}$$

The coefficients in a power law expansion of the potential $V_*$ are called dimensionless renormalized couplings. From now on, $k$-derivatives are always performed at fixed $\phi$ so when applied to the dimensionless field we obtain

$$k\partial_k \phi_* = \partial_t \phi_* = -\left( \frac{D}{2} - 1 + \frac{\eta_\phi}{2} \right) \phi_* \tag{2.5}$$

where we have defined the anomalous dimension

$$\eta_\phi = -\frac{\dot{Z}_\phi}{Z_\phi} \tag{2.6}$$

The reason of the name is that from the eq.(2.6) we can see immediately that

$$\phi_* \sim k^{D/2-1+\eta_\phi/2} \tag{2.7}$$

so the anomalous dimension changes the scaling one naively expects for the field $\phi_*$.

Taking the total derivative with respect to $t$ of the eq.(2.4) we have to stress that there are two contributes, one given by the built-in dependence on $k$ through its expansion coefficients and another given by the argument $\phi_*$

$$\dot{V}_*[\phi_*] + V_*'[\phi_*]\left(-\phi_*\left[\frac{D}{2} - 1 + \frac{\eta_\phi}{2}\right]\right) = -DV_*[\phi_*] + k^{-D}\dot{V}[\phi] \tag{2.8}$$

but we are interested only in the variation of dimensionless renormalized couplings so

$$\dot{V}_*[\phi_*] = -DV_*[\phi_*] + V_*'[\phi_*]\,\phi_*\left[\frac{D}{2} - 1 + \frac{\eta_\phi}{2}\right] + k^{-D}\dot{V}[\phi] \tag{2.9}$$

From the ERGE we have

$$\partial_t\Gamma_k\big|_{\phi\,\mathrm{const}} = \int d^Dx\,\dot{V}[\phi] = \frac{1}{2}\mathrm{Tr}\left\{\dot{R}_K\,[\,R_k - Z_\phi\,\Box + V''\,]^{-1}\right\} \tag{2.10}$$

Now, because $R_k$ is an IR regulator, it must have the following form

$$R_k = Z_\phi\,\mathcal{R}_k \tag{2.11}$$

so we get

$$\int d^Dx\,\dot{V}[\phi] = \frac{1}{2}\mathrm{Tr}\left\{(\dot{R}_K - \eta_\phi R_k)\left[R_k - \Box + \frac{V''}{Z_\phi}\right]^{-1}\right\} \tag{2.12}$$

and moving to momentum space

$$\dot{V}[\phi] = \frac{1}{2}\int \frac{d^Dp}{(2\pi)^D}\big(\dot{R}_K(p^2) - \eta_\phi R_k(p^2)\big)\left[R_k(p^2) + p^2 + \frac{V''[\phi]}{Z_\phi}\right]^{-1} \tag{2.13}$$

The “optimized” cut-off kernel is [11]

$$\mathcal{R}_k(p^2) = (k^2 - p^2)\theta(k^2 - p^2) \tag{2.14}$$

$$\partial_k R_k(p^2) = 2k^2\theta(k^2 - p^2) \tag{2.15}$$

where the assumption of regularity in $k^2 = p^2$ is taken. From the relation

$$\frac{1}{R_k(p^2) + p^2 + \frac{V''}{Z_\phi}} = \frac{\theta(k^2 - p^2)}{k^2 + \frac{V''}{Z_\phi}} + \frac{\theta(p^2 - k^2)}{p^2 + \frac{V''}{Z_\phi}} \tag{2.16}$$

we have

$$\dot{V}[\phi] = \frac{1}{2}\int \frac{d^D p}{(2\pi)^D}\theta(k^2 - p^2)\frac{k^2(2-\eta_\phi) + \eta_\phi p^2}{k^2 + \frac{V''}{Z_\phi}} \tag{2.17}$$

and passing to spherical coordinates

$$\dot{V}[\phi] = \frac{\Omega_D}{2(2\pi)^D}\int_0^\infty dp\, p^{D-1}\,\theta(k^2 - p^2)\,\frac{k^2(2-\eta_\phi) + \eta_\phi p^2}{k^2 + \frac{V''}{Z_\phi}} \tag{2.18}$$

$$= \frac{\Omega_D k^{D+2}}{D(2\pi)^D}\frac{1 - \frac{\eta_\phi}{D+2}}{k^2 + \frac{V''}{Z_\phi}} \tag{2.19}$$

where $\Omega_D = 2\pi^{\frac{D}{2}}/\Gamma[\frac{D}{2}]$. Moreover in term of the dimensionless potential, taking into account that $V_*''[\phi_*] = k^{-D}V''[\phi]\left(\frac{d\phi}{d\phi_*}\right)^2 = \frac{1}{k^2 Z_\phi}V''[\phi]$ and substituting eq.(2.19) in eq.(2.9) we have the final result

$$\dot{V}_*[\phi_*] = -D\,V_*[\phi_*] + \frac{D-2+\eta_\phi}{2}\,\phi_*\,V_*'[\phi_*] + C_D\,\frac{1 - \frac{\eta_\phi}{D+2}}{1 + V_*''[\phi_*]} \tag{2.20}$$

where $C_D^{-1} = 2^{D-1}\pi^{D/2}\,D\,\Gamma[D/2]$. This is the **ERGE for the scalar model** in the local potential approximation. It is important to note that this equation is non linear due to the presence of the last term where we have the second derivative in the denominator.

## 2.2 Scalar anomalous dimension (LPA')

When we have calculated the flow equation for the effective average action we have taken $\phi_{c\ell}$=const, so the kinetic term does not play any role and it is not possible to evaluate the anomalous dimension. Therefore we want to obtain an equation that shows how $Z_\phi$ changes under the flow of the renormalization 'time'. The key idea is that we have to look at the flow of the two-point function rather than at the flow of the effective action itself. In momentum space the two-point function, for the choice made in eq.(2.1), is

$$\Gamma_k^{(2)}(p) = Z_\phi p^2 + V''[\phi] \tag{2.21}$$

so in the flow of $\dot{\Gamma}_K^{(2)}(p)$ the coefficient $\propto p^2$ will be our beta-function $\dot{Z}_\phi$. An important observation to do is that to evaluate the anomalous dimension we have to take a particular field configuration which we choose to be constant.

The $n$-point vertex is [24]

$$\Gamma^{(n)}_{k;x_1,\dots,x_n} = \frac{\delta^n\Gamma_k}{\delta\phi(x_1)\dots\delta\phi(x_n)} \tag{2.22}$$

and in the momentum space

$$\widetilde{\Gamma}^{(n)}_{k;p_1,\dots,p_n} = \Gamma^{(n)}_{k;p_1,\dots,p_n}\delta_{p_1+\dots+p_n} \tag{2.23}$$

where we have factorized the momentum conservation at each vertex due to translation invariance. For this reason $\Gamma_k^{(n)}$ is a function of $n-1$ momenta, for example

$$\Gamma^{(2)}_{k;x_1,x_2} = \Gamma^{(2)}_k(x_1 - x_2) \implies \Gamma^{(2)}_{k;p,-p} \equiv \Gamma^{(2)}_{k;p} \tag{2.24}$$

The exact renormalization group equation in momentum space is

$$\dot{\Gamma}_k[\phi] = \frac{1}{2}\int_p G_{k;p}\dot{R}_{k;p} \tag{2.25}$$

$$G_{k;p} = \left(\Gamma^{(2)}_{k;p} + R_{k;p}\right)^{-1} \tag{2.26}$$

where $R_{k;p} = R_k(p^2)$. Now we have to take derivative of the eq.(2.25) with respect to $\phi$

$$\dot{\Gamma}^{(1)}_{k;x}[\phi] = \frac{1}{2}\mathrm{Tr}\left(\dot{R}_k \frac{\delta}{\delta\phi_x}\frac{1}{\Gamma^{(2)} + R_k}\right) \tag{2.27}$$

Form the formal equality

$$\partial_x \mathcal{O}^{-1} = -\mathcal{O}^{-1}\left[\partial_x \mathcal{O}\right]\mathcal{O}^{-1} \tag{2.28}$$

we get

$$\dot{\Gamma}^{(1)}_{k;x}[\phi] = -\frac{1}{2}\mathrm{Tr}\left(\dot{R}_k G_k \Gamma^{(3)}_k G_k\right) = -\frac{1}{2}\left(\dot{R}_{k;z_1,z_2} G_{k;z_1 y_1}\Gamma^{(3)}_{k;xy_1y_2} G_{k;y_2z_2}\right) \tag{2.29}$$

Taking another derivative and using again this formal equality

$$\partial_x\partial_y \mathcal{O}^{-1} = -\mathcal{O}^{-1}\left[\partial_x\partial_y \mathcal{O}\right]\mathcal{O}^{-1} + \mathcal{O}^{-1}\left[\partial_x \mathcal{O}\right]\mathcal{O}^{-1}\left[\partial_y \mathcal{O}\right]\mathcal{O}^{-1} + \mathcal{O}^{-1}\left[\partial_y \mathcal{O}\right]\mathcal{O}^{-1}\left[\partial_x \mathcal{O}\right]\mathcal{O}^{-1} \tag{2.30}$$

we get

$$\begin{aligned}\dot{\Gamma}^{(2)}_{k;x,y}[\phi] = &-\frac{1}{2}\dot{R}_{k;z_4,z_3} G_{k;z_3z_1}\Gamma^{(4)}_{k;xyz_1z_2} G_{k;z_2z_4} \\ &+ \dot{R}_{k;y_2y_1} G_{k;y_1z_1}\Gamma^{(3)}_{k;xz_1z_2} G_{k;z_2z_3}\Gamma^{(3)}_{k;yz_3z_4} G_{k;Z_4y_2}\end{aligned} \tag{2.31}$$

and after taking the Fourier transformation

$$\begin{aligned}\dot{\Gamma}^{(2)}_k[p] = &-\frac{1}{2}\int \frac{d^Dq}{(2\pi)^D}\dot{R}_k(q)\widetilde{G}_k(q)\widetilde{G}_k(q)\Gamma^{(4)}_k(p,q,-p,-q) \\ &+ \int \frac{d^Dq}{(2\pi)^D}\dot{R}_k(q)\widetilde{G}_k(q)\widetilde{G}_k(q)\widetilde{G}_k(q+p)\Gamma^{(3)}_k(p,q,-p-q)\Gamma^{(3)}_k(-p,-q,p+q)\end{aligned} \tag{2.32}$$

*Proof.*

$$\int d^Dy\, \mathcal{O}_{x_1,y}\,[\mathcal{O}^{-1}]_{y,x_2} = \delta_{x_1,x_2}$$

now let's take the functional derivative respect $\phi_x$

$$\int_y \frac{\delta \mathcal{O}_{x_1,y}}{\delta\phi_x}\,[\mathcal{O}^{-1}]_{y,x_2} + \mathcal{O}_{x_1,y}\frac{\delta[\mathcal{O}^{-1}]_{y,x_2}}{\delta\phi_x} = 0$$

moltiply by $\mathcal{O}_{x_2,x_4}$ and integrate in $x_2$

$$\int_{x_2}\int_y \frac{\delta \mathcal{O}_{x_1,y}}{\delta\phi_x}[\mathcal{O}^{-1}]_{y,x_2}\mathcal{O}_{x_2,x_4} + \mathcal{O}_{x_1,y}\frac{\delta[\mathcal{O}^{-1}]_{y,x_2}}{\delta\phi_x}\mathcal{O}_{x_2,x_4} = 0$$

$$\frac{\delta \mathcal{O}_{x_1,x_4}}{\delta\phi_x} + \int_{x_2}\int_y \mathcal{O}_{x_1,y}\frac{\delta[\mathcal{O}^{-1}]_{y,x_2}}{\delta\phi_x}\mathcal{O}_{x_2,x_4} = 0$$

now we have to multiply by $[\mathcal{O}^{-1}]_{x_4,z_2}\,[\mathcal{O}^{-1}]_{z_1,x_1}$ and integrate in $x_1, x_4$

$$\int_{x_1}\int_{x_4}[\mathcal{O}^{-1}]_{z_1,x_1}\frac{\delta \mathcal{O}_{x_1,x_4}}{\delta\phi_x}[\mathcal{O}^{-1}]_{x_4,z_2} + \int_{x_1}\int_{x_4}\int_{x_2}\int_y [\mathcal{O}^{-1}]_{z_1,x_1}\mathcal{O}_{x_1,y}\frac{\delta[\mathcal{O}^{-1}]_{y,x_2}}{\delta\phi_x}\mathcal{O}_{x_2,x_4}[\mathcal{O}^{-1}]_{x_4,z_2} = 0$$

$$\Longrightarrow \int_{x_1}\int_{x_2}[\mathcal{O}^{-1}]_{z_1,x_1}\frac{\delta \mathcal{O}_{x_1,x_2}}{\delta\phi_x}[\mathcal{O}^{-1}]_{x_2,z_2} + \frac{\delta[\mathcal{O}^{-1}]_{z_1,z_2}}{\delta\phi_x} = 0$$

that is excatly the eq.(2.30).
Let's start from the first term of (2.32)

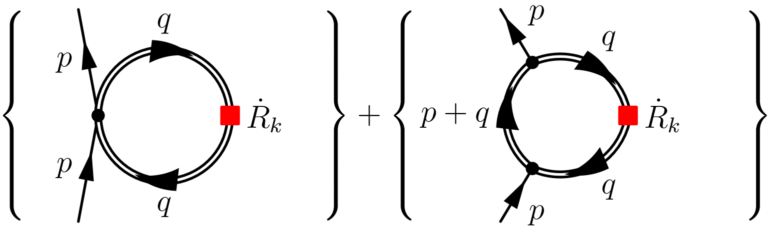
$$\begin{aligned}
&-\frac{1}{2}\int d^D z_1 \dots \int d^D z_4 \int_{p_1}\dots\int_{p_7} e^{-ip_1\cdot(z_3-z_1)}\widetilde{G}_k(p_1)e^{-ip_2\cdot(z_2-z_4)}\widetilde{G}_k(p_2)e^{-ip_7\cdot(z_4-z_3)}\dot{R}_k(p_7)\times\\
&\times e^{-ip_3\cdot x-ip_4\cdot y-ip_5\cdot z_1-ip_6\cdot z_2}\widetilde{\Gamma}^{(4)}_k(p_3,\dots,p_6)(2\pi)^D\delta(p_3+\dots+p_6) =\\
&= -\frac{1}{2}\int_{p_1}\dots\int_{p_7}(2\pi)^{4D}\delta(-p_1+p_5)\delta(p_6+p_2)\delta(p_1-p_7)\delta(p_7-p_2)e^{-ip_3\cdot x-ip_4\cdot y}\widetilde{G}_k(p_1)\widetilde{G}_k(p_2)\dot{R}_k(p_7)\times\\
&\times \widetilde{\Gamma}^{(4)}_k(p_3,\dots,p_6)(2\pi)^D\delta(p_3+\dots+p_6) =\\
&= -\frac{1}{2}\int_p\int_q e^{-ip\cdot(x-y)}\widetilde{G}^2_k(q)\dot{R}_k(q)\widetilde{\Gamma}^{(4)}_k(p,-p,q,-q)
\end{aligned}$$

The second term is

$$\begin{aligned}
&\int d^D x_1 \dots \int d^D x_6 \int_{p_1}\dots\int_{p_4}\int_{q_1}\dots\int_{q_6} e^{-ip_1\cdot(x_1-x_2)}\widetilde{G}_k(p_1)e^{-ip_2\cdot(x_3-x_4)}\widetilde{G}_k(p_2)e^{-ip_3\cdot(x_5-x_6)}\widetilde{G}_k(p_3)\times\\
&\times e^{-ip_4\cdot(x_6-x_1)}\dot{R}_k(p_4)e^{-iq_1\cdot x-iq_2\cdot x_2-iq_3\cdot x_3}\widetilde{\Gamma}^{(3)}_k(q_1,q_2,q_3)(2\pi)^D\delta(q_1+q_2+q_3)\times\\
&\times e^{-iq_6\cdot y-iq_5\cdot x_5-iq_4\cdot x_4}\widetilde{\Gamma}^{(3)}_k(q_4,q_5,q_6)(2\pi)^D\delta(q_4+q_5+q_6) =\\
&= \int_{p_1}\dots\int_{p_4}\int_{q_1}\dots\int_{q_6}(2\pi)^{6D}\delta(p_1-p_4)\delta(-p_1+q_2)\delta(p_2+q_3)\delta(-p_2+q_4)\delta(p_3+q_5)\delta(-p_3+p_4)\times\\
&\times (2\pi)^D\delta(q_1+q_2+q_3)(2\pi)^D\delta(q_4+q_5+q_6)e^{-iq_1\cdot x-iq_6\cdot y}\widetilde{G}_k(p_1)\widetilde{G}_k(p_2)\widetilde{G}_k(p_3)\dot{R}_k(p_4)\widetilde{\Gamma}^{(3)}_k(q_1,q_2,q_3)\times\\
&\times \widetilde{\Gamma}^{(3)}_k(q_4,q_5,q_6) =\\
&= \int_q\int_\ell e^{-iq\cdot(x-y)}\widetilde{G}^2_k(\ell)\widetilde{G}_k(\ell+q)\dot{R}_k(\ell)\widetilde{\Gamma}^{(3)}_k(q,\ell,-q-\ell)\widetilde{\Gamma}^{(3)}_k(-q,-\ell,\ell+q)
\end{aligned}$$

□

this equation can be expressed in term of Feynman diagrams

$\dot{\Gamma}^{(2)}_k(p) = -\frac{1}{2}$  (2.33)


In our LPA approach, all correlation functions for $n > 2$ points depends only on the potential $V$, therefore

$$\Gamma^{(n)}_{k;p_1,\dots,p_n} = V^{(n)}[\phi] \qquad n > 2 \tag{2.34}$$

and the flow equation for the two point function becomes

$$\dot{Z}_\phi p^2 + \dot{V}''[\phi] = -\frac{1}{2}V^{(4)}[\phi]\int_q \dot{R}_k(q)\widetilde{G}_k^2(q) + \big(V^{(3)}[\phi]\big)^2\int_q \dot{R}_k(q)\widetilde{G}_k^2(q)\widetilde{G}_k(q+p) \tag{2.35}$$

If we take $p_\mu = 0$

$$\dot{V}''[\phi] = -\frac{1}{2}V^{(4)}[\phi]\int_q \dot{R}_k(q)\widetilde{G}_k^2(q) + \big(V^{(3)}[\phi]\big)^2\int_q \dot{R}_k(q)\widetilde{G}_k^3(q) \tag{2.36}$$

we have an expression for $\dot{V}''[\phi]$ that we have to subtract from eq.(2.35) to obtain

$$\dot{Z}_\phi p^2 = \big(V^{(3)}[\phi]\big)^2\int_q \dot{R}_k(q)\widetilde{G}_k^2(q)\left(\widetilde{G}_k(q+p) - \widetilde{G}_k(q)\right) \tag{2.37}$$

where of course this equality is not consistent, we have to project the right hand side on the sector quadratic in $p$ to obtain $Z_\phi$. It's important to note that from the eq.(2.13) we have

$$\begin{aligned}\dot{V}'[\phi] &= -\frac{1}{2}V^{(3)}[\phi]\int_q \widetilde{G}_k^2(q)\dot{R}_k\\ \dot{V}''[\phi] &= -\frac{1}{2}V^{(4)}[\phi]\int_q \widetilde{G}_k^2(q)\dot{R}_k + \big(V^{(3)}[\phi]\big)^2\int_q \widetilde{G}_k^3(q)\dot{R}_k\end{aligned} \tag{2.38}$$

that is consistent with the last derivation. Now taking into account that $\frac{\partial}{\partial p^2}p^2 = \frac{1}{2D}\frac{\partial^2}{\partial p_\mu \partial p^\mu}p^2 = 1$ we have

$$\begin{aligned}\dot{Z}_\phi &= \frac{\partial}{\partial p^2}\bigg|_0\left(\dot{Z}_\phi p^2\right) = \frac{1}{2D}\frac{\partial^2}{\partial p^\mu \partial p_\mu}\bigg|_0\left(\dot{Z}_\phi p^2\right)\\ &= \big(V'''[\phi]\big)^2\int_q \dot{R}_k(q)\widetilde{G}_k^2(q)\frac{1}{2D}\frac{\partial^2}{\partial p_\mu \partial p^\mu}\bigg|_0 \widetilde{G}_k(q+p)\\ &= \big(V'''[\phi]\big)^2\int_q \dot{R}_k(q)\widetilde{G}_k^2(q)\frac{1}{2D}\frac{\partial^2}{\partial q_\mu \partial q^\mu}\widetilde{G}_k(q)\\ &= \big(V'''[\phi]\big)^2\int_q \dot{R}_k(q^2)\widetilde{G}_k^2(q^2)\left(\widetilde{G}_k'(q^2) + \frac{q^2}{D/2}\widetilde{G}_k''(q^2)\right)\end{aligned} \tag{2.39}$$

where the second equality is necessary to commute the derivative with the integration. As before we take the following IR regulator

$$R_k(q^2) = Z_\phi(k^2 - q^2)\theta(k^2 - q^2) \tag{2.40}$$

so we get

$$\dot{R}_k(q^2) = Z_\phi\big[-\eta_\phi(k^2 - q^2) + 2k^2\big]\theta(k^2 - q^2) + Z_\phi(k^2 - q^2)2k^2\delta(k^2 - q^2)$$

$$\widetilde{G}_k(q^2) = \frac{1}{\Gamma^{(2)}_{k;q} + R_{k;q}} = \theta(k^2 - q^2)\frac{1}{Z_\phi k^2 + V''[\phi]} + \theta(q^2 - k^2)\frac{1}{Z_\phi q^2 + V''}$$

$$\frac{\partial}{\partial q^\mu}\widetilde{G}_k(q^2) = -\theta(q^2-k^2)\frac{Z_\phi 2q_\mu}{(Z_\phi q^2+V'')^2} - 2q_\mu \frac{\delta(k^2-q^2)}{Z_\phi k^2+V''} + 2q_\mu \frac{\delta(q^2-k^2)}{Z_\phi q^2+V''}$$
$$= -\theta(q^2-k^2)\frac{Z_\phi 2q_\mu}{(Z_\phi q^2+V'')^2}$$
$$\frac{\partial^2}{\partial q^\mu \partial q^\nu}\widetilde{G}_k(q^2) = -4q_\mu q_\nu Z_\phi \delta(q^2-k^2)\frac{1}{(Z_\phi q^2+V'')^2} - \theta(q^2-k^2)\frac{\partial}{\partial q^\nu}\frac{Z_\phi 2q_\mu}{(Z_\phi q^2+V'')^2}$$

Taking into account the assumption of regularity at $k^2 = q^2$, the flow equation for the anomalous dimension is

$$\dot{Z}_\phi = -\big(V'''[\phi]\big)^2 \int_q Z_\phi\big[-\eta_\phi(k^2-q^2)+2k^2\big]\frac{[\theta(k^2-q^2)]^3}{\big(Z_\phi k^2+V''[\phi]\big)^2}\frac{1}{2D}\frac{4q^2 Z_\phi \delta(q^2-k^2)}{(Z_\phi q^2+V'')^2} \tag{2.41}$$

but this equation would be wrong because in the ERGE for the effective action only two theta function (or their derivatives) are present, one from the propagator and the other from the cutoff. This tells us that the integral of interest is actually

$$\begin{aligned}\dot{Z}_\phi &= -\big(V'''[\phi]\big)^2 Z_\phi^2 \frac{4(k^2)^2}{D}\frac{1}{(Z_\phi k^2+V''[\phi])^4}\int \frac{d^D q}{(2\pi)^D}\delta(q^2-k^2)\theta(k^2-q^2)\\ &= -\big(V'''[\phi]\big)^2 Z_\phi^2 \frac{4(k^2)^2}{D}\frac{1}{(Z_\phi k^2+V''[\phi])^4}\frac{\Omega_D}{(2\pi)^D}\int_0^\infty dr r^{D-1}\delta(r-k^2)\theta(k^2-r)\\ &= -\big(V'''[\phi]\big)^2 Z_\phi^2 \frac{4(k^2)^2}{D}\frac{1}{(Z_\phi k^2+V''[\phi])^4}\frac{\Omega_D}{(2\pi)^D}\frac{k^{D-2}}{4}\end{aligned} \tag{2.42}$$

(remember that $\int dx\delta(x)\theta(x) = 1/2$).

Now using $V_*[\phi_*]$ and noting that $V_*'''[\phi_*] = V'''[\phi]k^{D/2-3}\frac{1}{Z_\phi^{3/2}}$ we get the final result

$$\eta_\phi = -\frac{\dot{Z}_\phi}{Z_\phi} = C_D \frac{\big(V_*'''[\phi_*]\big)^2}{\big(1+V_*''[\phi_*]\big)^4} \tag{2.43}$$

Due to the approximation in the definition of anomalous dimension we adopted, it is always necessary to specify the field configuration one uses to calculate it. One usually works with the ground state $\phi = 0$ so, in this case, $\eta_\phi = 0$ because $V_*'''[\phi_* = 0] = 0$.

We can conclude that the complete flow of the renormalized dimensionless potential is

$$\dot{V}_*[\phi_*] = -D\, V_*[\phi_*] + \frac{D-2+\eta_\phi}{2}\phi_*\, V_*'[\phi_*] + C_D \frac{1-\frac{\eta_\phi}{D+2}}{1+V_*''[\phi_*]} \tag{2.44}$$
$$\eta_\phi = C_D \frac{\big(V_*'''[\phi_*]\big)^2}{\big(1+V_*''[\phi_*]\big)^4} \tag{2.45}$$
$$C_D^{-1} = 2^{D-1}\pi^{D/2}\, D\, \Gamma[D/2] \tag{2.46}$$

## 2.3 Example: study of the FP equation

As we said before in section 1.4 a fixed point is defined as that point where all the beta functions are zero *i.e.* the effective average action doesn't flow and remains the same at different scales $k$. In our LPA' scalar example we are looking for a dimensionless potential $v(\varphi)$ (the field $\varphi$ being constant can be treated as a real number) that satisfies the following non-linear second order differential equation

$$0 = -D\,v(\varphi) + \frac{D-2+\eta_\phi}{2}\,\varphi\,v'(\varphi) + C_D\,\frac{1-\frac{\eta_\phi}{D+2}}{1+v''(\varphi)} \tag{2.47}$$

In the case of a conventional $\mathbb{Z}_2$ symmetry, the potential is even and a function of $\varphi^2$ so setting $\rho = \varphi^2$ the equation can be written as

$$0 = -D\,v(\rho) + (D-2+\eta_\phi)\,\rho\,v'(\rho) + C_D\,\frac{1-\frac{\eta_\phi}{D+2}}{1+4\rho v''(\rho)+2v'(\rho)} \tag{2.48}$$

### 2.3.1 Numerical solution for $\eta_\phi = 0$ and $D$ generic

We start our analysis in the LPA approximation neglecting the flow equation for the wave function renormalization. The equation is

$$-\,D\,v(\varphi) + \frac{D-2}{2}\,\varphi\,v'(\varphi) + \frac{C_D}{1+v''(\varphi)} = 0 \tag{2.49}$$

with a condition due to the $\mathbb{Z}_2$ symmetry

$$v'(0) = 0 \implies v(0) = \frac{C_D/D}{1+v''(0)} \tag{2.50}$$

Therefore starting from a non-linear second order differential equation, due to the symmetry condition there is one parameter left, $v''(0) \equiv \sigma$ and we want to study the problem in the space of this parameter. We want to follow the strategy developed in [7], [17] which consists in solving the differential equation with a numerical shooting method varying the initial condition in the space of parameters that in this simple case is one dimensional.

Trying to numerically solve the non-linear differential equation imposing the two Cauchy initial conditions (2.50), one typically encounters a singularity at some value of $\varphi_{\text{critic}}$ where the algorithm stops. Such a value increases in a steep way close to the initial condition which correspond to a global solution, even if the numerical errors mask partially this behavior. Hence the strategy is to plot $\varphi_{\text{critic}}(\sigma)$ for different values of dimension. This is very useful to gain a first understanding of the positions of the possible FPs (for application of this strategy for a more complicated system such as a multi-meson Yukawa interactions see [30]).

It is called **critical dimension** that dimension for which the operator $\varphi^{2n}$ has the engineering dimension of $\text{cm}^{-D}$ so its coupling is dimensionless. It happens for

$$d_c(n \geq 2) = \frac{2n}{n-1} = 4, 3, \frac{8}{3}, \frac{5}{2}, \frac{12}{5}, \ldots \tag{2.51}$$

A theory perturbatively renormalizable by power counting cannot have couplings with a dimension of the inverse of a mass so the term $\varphi^{2n}$ with zero dimension is the last relevant term that can be present in such a theory. For example, in $D = 4$ the last term is $\varphi^4$ because $\varphi \sim \text{cm}^{-1}$, in $D = 3$ the last term is $\varphi^6$ because $\varphi \sim \text{cm}^{-1/2}$, in $D = 8/3$ the last term is $\varphi^8$ because $\varphi \sim \mathrm{cm}^{-1/3}$ and so on. Hence, what we expect is that below the threshold of the critical dimensions, new operators become relevant and new universality classes appear below these dimensions.

In fig. 2.1 we show the results of this analysis for various dimensions: $D = 4, 3.5, 3, 2.8, \frac{8}{3}, \frac{8}{3} - \frac{1}{10}, \frac{5}{2}, \frac{12}{5}$. For $D = 4$ we see a single spike for $\sigma = 0$ which corresponds to the Gaussian solution. For $4 > D \geq 3$ we have crossed the threshold below which the operator $\varphi^6$ becomes relevant so another spike with $\sigma \neq 0$ appears. For $3 > D \geq \frac{8}{3}$ we have crossed the threshold below which the operator $\varphi^8$ becomes relevant so a second spike with $\sigma \neq 0$ appears. Similar observations can be gathered for the other dimensions, every time ones crosses a critical dimension one more spike appears. For $\frac{5}{2} > D \geq \frac{12}{5}$ we have crossed the threshold below which the operator $\varphi^{12}$ becomes relevant so in total there are five spikes (two are very close to zero). In some cases these new spikes are too close to $\sigma = 0$ that a zoom near this region is needed, for example see fig.2.2 where we have zoomed the plot for $D = \frac{8}{3} - \frac{1}{10}$ and $D = \frac{12}{5}$.

### 2.3.2 Numerical solution for $\eta_\phi = 0$ and $D = 3$: asymptotic analysis

Now we want to construct the numerical solution for $v(\varphi)$ in the special case of three dimensions and with zero anomalous dimension in a domain that covers the asymptotic region. This might be call global scaling solution and its knowledge will be important for the study of the quality of polynomial expansion presented in the next section. The latter approach is very useful especially in the case of the LPA' which gives us access to a self-consistent computation of the anomalous dimension.

The fixed point equation in this case is

$$- 3\, v(\varphi) + \frac{1}{2}\, \varphi\, v'(\varphi) + \frac{1}{6\pi^2}\, \frac{1}{1 + v''(\varphi)} \tag{2.52}$$

In the asymptotic region we can neglect the non linear term so we have

$$- 3\, v(\varphi) + \frac{1}{2}\, \varphi\, v'(\varphi) = 0 \implies v(\varphi) = a\, \varphi^6 \tag{2.53}$$

this is the leading term of the potential so we can write the asymptotic behavior as

$$v_{\mathrm{as}}(\varphi) = a\, \varphi^6 + \varepsilon(\varphi) \tag{2.54}$$

where $a$ is a parameter that we will have to choose in a consistently way. Taking into account a polynomial ansatz for $\varepsilon(\varphi)$

$$\varepsilon(\varphi) = \sum_{k=-N}^{2} \lambda_{2k} \varphi^{2k} \tag{2.55}$$

and solving analytically the eq.(2.52) we get the solution up to the order $\varphi^{-24}$

$$\begin{aligned} v_{\text{as}}(\varphi) = \; & a\,\varphi^6 + \frac{1}{900\,a\,\pi^2\,\varphi^4} - \frac{1}{37800\,a^2\,\pi^2\,\varphi^8} + \frac{1}{1458000\,a^3\,\pi^2\,\varphi^{12}} \\ & - \frac{1}{2430000\,a^3\,\pi^4\,\varphi^{14}} - \frac{1}{53460000\,a^4\,\pi^2\,\varphi^{16}} + \frac{1}{19136250\,a^4\,\pi^4\,\varphi^{18}} \\ & + \frac{1}{1895400000\,a^5\,\pi^2\,\varphi^{20}} - \frac{131}{32148900000\,a^5\,\pi^4\,\varphi^{22}} \\ & + \frac{250\,a - 3\,\pi^4}{196830000000\,a^6\,\pi^6\,\varphi^{24}} + O\left(\frac{1}{\varphi^{26}}\right) \end{aligned} \tag{2.56}$$

Once the asymptotic expansion is determined we proceed with a shooting method *i.e.* with a numerical integration from the asymptotic region toward the origin. The properties of the solutions which reach the origin depend on the free parameter $a$ in the asymptotic expansion. In principle there is also a second parameter (as expected in the Cauchy problem) which is associated to a negligible contribution characterized by an essential singularity at $\varphi \to \infty$ like $e^{-b\varphi^2}$. By requiring the solution to be $\mathbb{Z}_2$ symmetric (with $v'(0) = 0$) one can uniquely fix the latter parameter to its fixed point value $a^*$ [15].

First of all we have to choose a point $\varphi_{\text{max}}$ from which starting the numerical integration towards the origin $\phi = 0$ and secondly, we have to choose a range of the asymptotic parameter $a$ such that the numerical integration itself can reach the origin because, in general, this does not happen for all values of $a$. For this particular case we have chosen $\varphi_{\text{max}} = 3$ and $a$ in the range $[10^{-3}, 10^{0.8}]$. In fig.2.3a we have shown a parametric plot where on the $x$ and $y$-axis there are $v(0)$ and $v'(0)$ respectively. We are interested on the right intersection of the $x$-axis that corresponds to a non trivial solution of the fixed point equation, in other words it corresponds to a potential with the right shape for a Wilson Fisher fixed point. Taking the corresponding value of $a^*$ (that in our case is $a^* = 3.50759$) we have solved numerically again the differential equation for that particular value and plotted the corresponding global scaling solution in fig.2.3b.

In fig.2.4a it is shown instead, the numerical solution obtained from the previous analysis of spike plots: the value of $\sigma = v''(0)$ corresponding to a non Gaussian solution in $D = 3$ is $\sigma^* = -0.1860664$. It is important to note that, whereas the numerical integration starting from the origin breaks down for a value of $\varphi \sim 0.426$, with the asymptotic behavior method we can construct a global scaling solution for all values of the scalar field. In fig.2.4b the two solutions, the one obtained from the spike plot method and the other one obtained from the asymptotic method, are shown in a zoomed area near the origin: we can see that there is a perfect overlap of the two solutions.

### 2.3.3 Polynomial analysis for $\eta_\phi = 0$ and $D = 3$

In this section we are going to discuss the use of polynomial parametrization and consequent truncations of the function $v(\varphi)$. First of all we will present the results obtained within the LPA which can be directly compared to the analysis in the previous section, secondly we will push forward the analysis to a self consistent inclusion of the wave function renormalization of the field.

**Expansion around $\phi = 0$**

In the symmetric regime, the physically meaningful parametrization of the scalar potential is a Taylor expansion around vanishing field

$$v(\rho) = \sum_{n=0}^{N} \lambda_n(t)\, \rho^n \tag{2.57}$$

where as usual $\rho = \varphi^2$.

To study the FP equation we have to find all the beta functions in this truncations $\{\beta_0,\, \beta_1,\, \beta_2,\, \ldots, \beta_N\}$ and set all of these equal to zero. The FP equation presents itself in the form of

$$\dot{v}(\rho) = \mathscr{F}\left[v(\rho)\right] \tag{2.58}$$

where $\mathscr{F}\left[v\right]$ is the right hand side of the eq.(2.48) with $\eta_\phi = 0$

$$-\,D\, v(\rho) + (D-2)\,\rho\, v'(\rho) + C_D\, \frac{1}{1 + 4\rho v''(\rho) + 2v'(\rho)} = \mathscr{F}\left[v(\rho)\right] \tag{2.59}$$

therefore our beta functions can be written as

$$\beta_n = \dot{\lambda}_n = \frac{1}{n!}\, \frac{d^n}{d\rho^n} \mathscr{F}\left[v(\rho)\right] \tag{2.60}$$

and the fixed point equation for the potential becomes now a system of $N$ equations in $N$ variables:

$$\begin{cases} \beta_0(\lambda_0, \lambda_1, \ldots, \lambda_N) = 0 \\ \beta_1(\lambda_0, \lambda_1, \ldots, \lambda_N) = 0 \\ \vdots \\ \beta_N(\lambda_0, \lambda_1, \ldots, \lambda_N) = 0 \end{cases} \tag{2.61}$$

We have studied the polynomial solution of the FP equation for a potential expanded in powers of $\rho = \phi^2$ till the order $\rho^8$. We found the following expression

$$\begin{aligned} v(\varphi) = &-93.0524\rho^8 - 39.5308\rho^7 - 4.836\rho^6 + 3.36645\rho^5 + 2.80432\rho^4 \\ &+ 1.38007\rho^3 + 0.607516\rho^2 - 0.0928136\rho^1 + 0.00691201 \end{aligned}$$

and with this solution, we have also studied the stability matrix at the corresponding fixed point

$$\mathcal{M}_{ij}\Big|_{\lambda_*} = \frac{\partial \beta_i}{\partial \lambda_j}\Big|_{\lambda_*}. \tag{2.62}$$

Among all the eigenvalues, there is one that is non trivial and negative ($\lambda_*^{(-)} = -1.54051$), corresponding to the attractive direction in the couplings space. From this eigenvalue we can obtain the $\nu$ critical exponent of the Ising model in $D = 3$ defined as

$$\nu \equiv -\frac{1}{\lambda_*^{(-)}}$$

that in our case is $\nu = 0.649136$ rather close to that obtain from the Monte-Carlo simulation $\nu \sim 0.62998$.

**Expansion around $\varphi = \kappa$**

In the regime of spontaneous symmetry breaking (SSB), the potential $v(\rho)$ develops a non trivial minimum at $\kappa = \varphi_0^2$ which becomes the preferred reference point for a different Taylor expansion

$$v(\rho) = \lambda_0(t) + \sum_{n=2}^{N} \lambda_n(t)\ (\rho - \kappa(t))^n \tag{2.63}$$

The sum starts from $n = 2$ because we want the potential to satisfy the condition

$$\left.\frac{\partial v}{\partial \rho}\right|_{\rho=\kappa(t)} = 0 \tag{2.64}$$

In this case to gain the beta functions we have to be a bit more careful because we have to consider even the flow of $\kappa(t)$.

The fixed point equation now reads

$$\dot{v}(\rho) = \dot{\lambda}_0 + \sum_{n=2}^{N} \dot{\lambda}_n\ (\rho - \kappa)^n - \sum_{n=2}^{N} \lambda_n\, n\ (\rho - \kappa)^{n-1}\,\dot{\kappa} = \mathscr{F}\left[v(\rho)\right] \tag{2.65}$$

where $\mathscr{F}$ is expanded in $\rho$ around $\kappa$ up to order $N$

$$\mathscr{F}\left[v(\rho)\right] = \sum_{m=0}^{N} \frac{1}{m!} \left.\frac{d^m \mathscr{F}(\rho)}{d\rho^m}\right|_{\rho=\kappa} (\rho - \kappa)^m \tag{2.66}$$

The condition of $\kappa$ as a minimum must be true even when the renormalization ''time'' flows, therefore

$$0 = \frac{d}{dt}\left(\frac{\partial v}{\partial \rho}\Big|_{\kappa(t)}\right) = \frac{d}{d\rho}\Big|_{\kappa}\frac{dv}{dt} + \frac{d^2 v}{d\rho^2}\Big|_{\kappa}\dot{\kappa} \tag{2.67}$$

from which we can obtain the beta function for the non trivial minimum

$$\Longrightarrow\ \dot{\kappa} = -\frac{\left.\frac{d}{d\rho}\right|_{\kappa}\dot{v}}{\left.\frac{d^2 v}{d\rho^2}\right|_{\kappa}} = -\frac{\mathscr{F}'(\rho = \kappa)}{v''(\rho = \kappa)} = -\frac{\mathscr{F}'(\kappa)}{2\lambda_2} \tag{2.68}$$

Substituting eq.(2.68) in eq.(2.65) we have

$$\dot{\lambda}_0 + \sum_{n=2}^{N} \dot{\lambda}_n\ (\rho - \kappa)^n = \mathscr{F}(\kappa) + \sum_{n=2}^{N} (\rho - \kappa)^n \left(\frac{1}{n!}\mathscr{F}^{(n)}(\kappa) - \frac{1}{2}(n+1)\frac{\lambda_{n+1}}{\lambda_2}\mathscr{F}'(\kappa)\right) \tag{2.69}$$

Therefore our beta functions are

$$\begin{cases} \beta_0 = \dot{\lambda}_0 = \mathscr{F}(\kappa) \\ \beta_n = \dot{\lambda}_n = \frac{1}{n!}\mathscr{F}^{(n)}(\kappa) - \frac{1}{2}(n+1)\frac{\lambda_{n+1}}{\lambda_2}\mathscr{F}'(\kappa) \\ \dot{\kappa} = -\frac{\mathscr{F}'(\kappa)}{2\lambda_2} \end{cases} \tag{2.70}$$

and solving the kernel of this system in the variables $\{\kappa, \lambda_0, \lambda_2, \lambda_3, \ldots, \lambda_N\}$ we obtain the fixed point solution for our potential.

We have studied this system of equations for a potential expanded in powers of $\big(\rho - \kappa(t)\big)^n$ till the order $n = 8$. The FP solution that we have obtained is

$$v(\rho) = 236.194\,(\rho - 0.0612931)^8 - 5.15436\,(\rho - 0.0612931)^7 \tag{2.71}$$
$$- 17.7191\,(\rho - 0.0612931)^6 - 0.896651\,(\rho - 0.0612931)^5 \tag{2.72}$$
$$+ 3.38833\,(\rho - 0.0612931)^4 + 2.17835\,(\rho - 0.0612931)^3 \tag{2.73}$$
$$+ 0.933984\,(\rho - 0.0612931)^2 + 0.00386081 \tag{2.74}$$

and again, as before, we have studied the stability matrix for this solution and founded a non trivial negative eigenvalue ($\lambda_*^{(-)} = -1.53969$) corresponding to the critical exponent $\nu = 0.649481$ that is slightly different from that one obtained with a polynomial expansion around the origin.

In fig.2.5 we have plotted the polynomial solutions for the FP equation: on the left panel there is the one obtained from an expansion around the origin $\varphi = 0$ and on the right panel there is the one obtained from an expansion around the non trivial minimum $\varphi = \kappa$. In fig.2.6 we have shown a comparison of these solutions with the global scaling solution obtained from the asymptotic method. What one can deduce is that the polynomial expansion around a non trivial minimum is a better approximation for the FP potential than the polynomial expansion around null field, in other words it has a greater convergence ray. But even in this better case there is a deviation from the global scaling solution from a certain field value onwards.

### 2.3.4 Polynomial analysis for $\eta_\phi \neq 0$ and $D = 3$

The polynomial expansion presented in the previews section, as we said, gives us access to a self-consistent computation of the anomalous dimensions in some approximation. It can be computed using an *iterative method* or a *direct method*. We will use in both cases the expansion around the $\kappa$ minimum because it is a better approximation.

#### Iterative method

In this case we start from the fixed point equation without the anomalous dimension.

$$0 = -D\,v(\rho) + (D-2)\,\rho\,v'(\rho) + C_D\,\frac{1}{1 + 4\rho v''(\rho) + 2v'(\rho)} \tag{2.75}$$

As we stressed in section 2.2 the expression for $\eta_\phi$ must be evaluated for a certain field configuration and we choose the non trivial minimum. Now we have a non-zero anomalous dimension $\eta_\phi(\kappa_0)$ and we repeat the polynomial analysis of the FP equation but now with the insertion of $\eta_\phi(\kappa_0)$.

$$0 = -D\,v(\rho) + (D - 2 + \eta_\phi(\kappa_0))\,\rho\,v'(\rho) + C_D\,\frac{1 - \frac{\eta_\phi(\kappa_0)}{D+2}}{1 + 4\rho v''(\rho) + 2v'(\rho)} \tag{2.76}$$

This second step will provide us with a new value for the non trivial minimum, say $\kappa_1$ and we repeat the study of the FP equation with $\eta_\phi(k_1)$. Proceeding iteratively we will arrive at a convergent value for the anomalous dimension

$$\eta_\phi^* = \lim_{j\to\infty} \eta_\phi(\kappa_j) \tag{2.77}$$

where $j$ denotes the $j$-th step in this iterative method.

### Non iterative method

In this case we start directly from the eq.(2.76) where the anomalous dimension is expressed by the eq.(2.45)

$$\eta_\phi = C_D \left. \frac{(v'''(\varphi))^2}{(1+v''(\varphi))^4} \right|_{\varphi_0} = C_D\, 16\, \rho \frac{(2\rho v'''(\rho) + 3v''(\rho))^2}{(1+4\rho v''(\rho) + 2v'(\rho))^4} \tag{2.78}$$

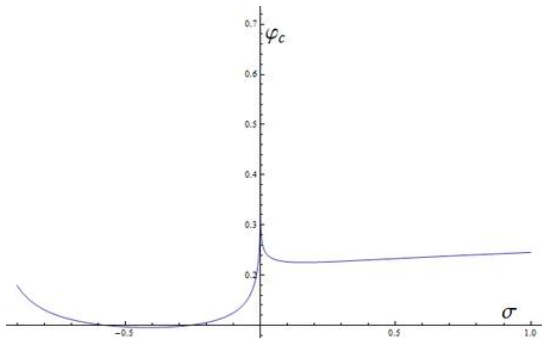
Choosing $\varphi_0^2 = \kappa$ and taking into account the expansion (2.63) we have

$$\eta_\phi = \frac{1}{6\pi^2} \frac{16\cdot 36\,\kappa\,(\lambda_2 + 2\,\kappa\,\lambda_3)^2}{(1+8\,\kappa\,\lambda_2)^4} \tag{2.79}$$

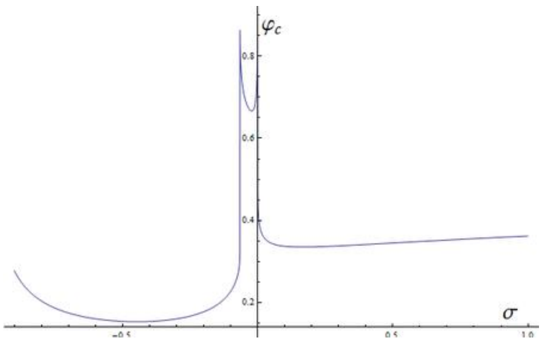
The beta functions are exactly the same as in eq.(2.70) but with a more complicated function $\mathscr{F}(\rho)$.

In both cases the value for the anomalous dimension at the FP solution, with a truncation up to the order $(\rho-\kappa)^6$ is $\eta_\phi = 0.111948$ that is three times the right value, but with this approximation we cannot do better.

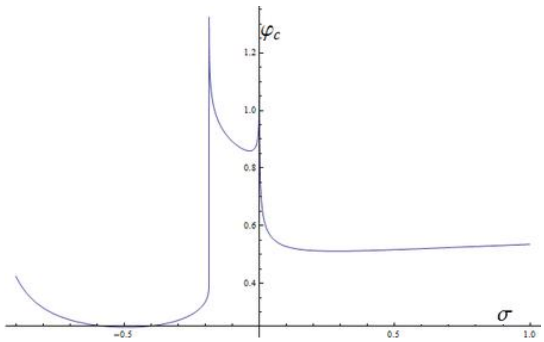
In fig.2.7 we have shown the solution of the potential at the FP with anomalous dimension and a comparison with the solution in absence of the anomalous dimension, what we can see is a slightly chance of the minimum position and of the value $v(0)$.



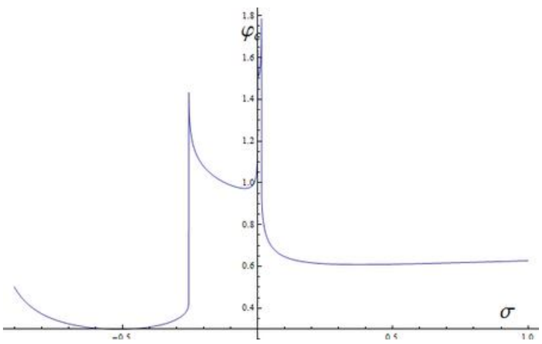

**(a)** $D = 4$



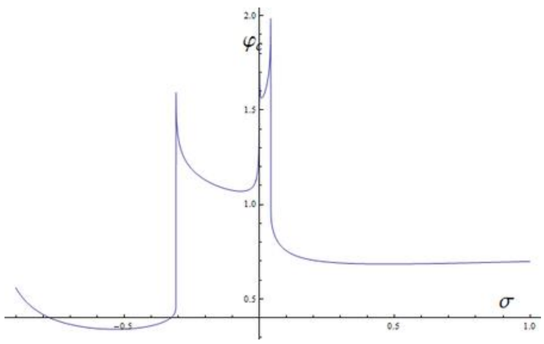

**(b)** $D = 3.5$



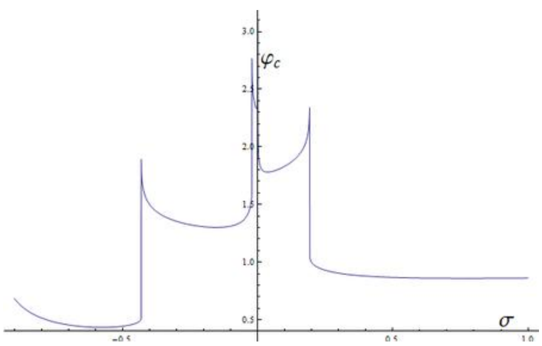

**(c)** $D = 3$



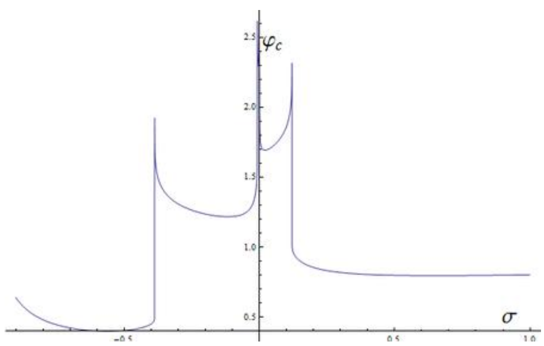

**(d)** $D = 2.8$




**(e)** $D = \frac{8}{3}$

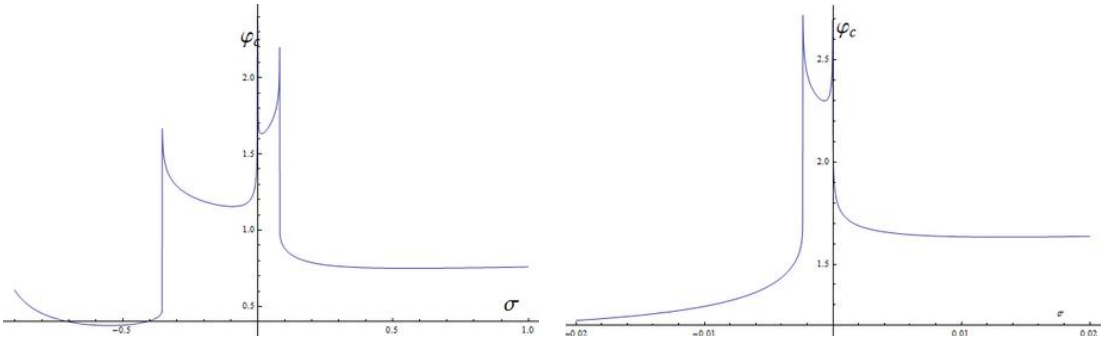



**(f)** $D = 2.42$



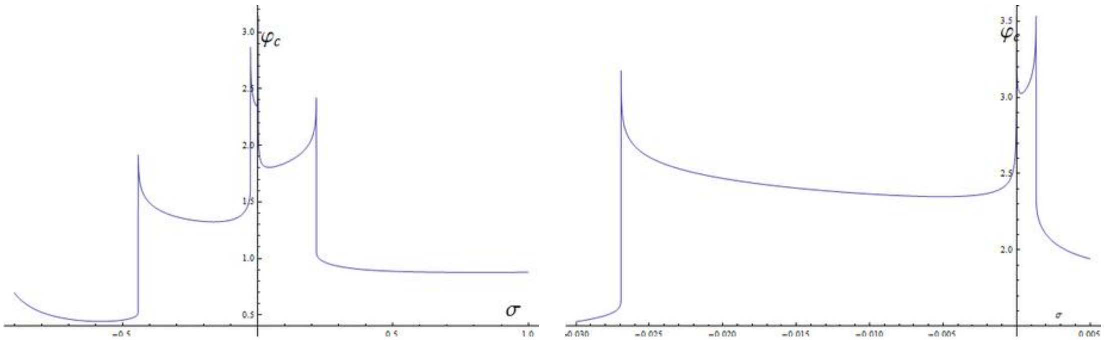

**(g)** $D = \frac{5}{2}$

**Figure 2.1:** For different values of dimension we have plotted $\phi_{\text{critic}}$ (the point at which the numerical integration from the origin breaks down) as a function of the parameter $\sigma = v''(0)$.




**(a)** Spike plot of $\phi_{\text{critic}}(\sigma)$ at dimension $D = \frac{8}{3} - \frac{1}{10}$

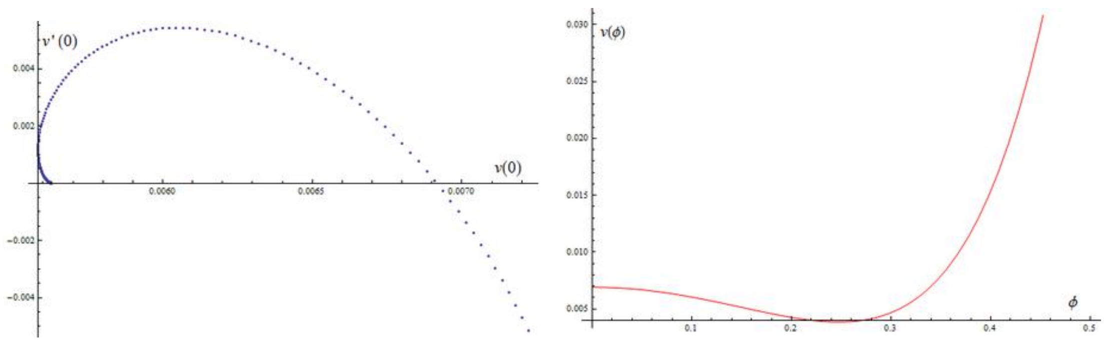
**(b)** Zoomed area around the origin $\sigma = 0$ for the spike plot on the left.




**(c)** Spike plot of $\phi_{\text{critic}}(\sigma)$ at dimension $D = \frac{12}{5}$

**(d)** Zoomed area around the origin $\sigma = 0$ for the spike plot on the left.

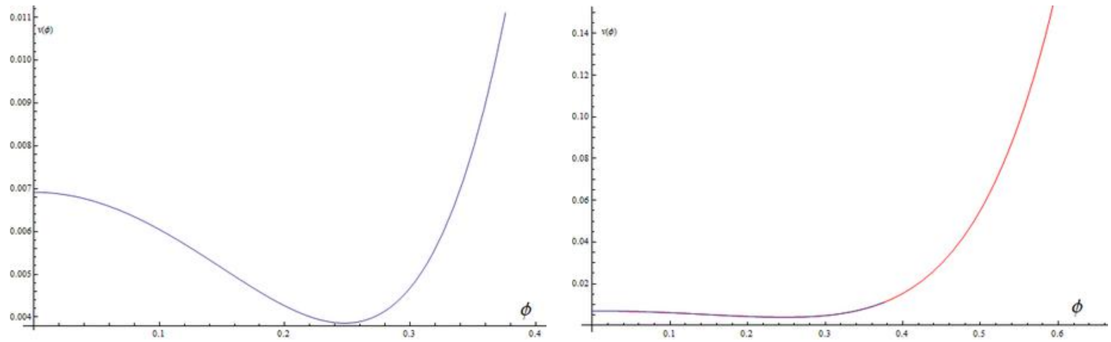
**Figure 2.2**




**(a)** Parametric plot of $v'(0)$ as a function of $v(0)$ for $10^{-3} < a < 10^{0.8}$

**(b)** The potential $v(\varphi)$ for the global scaling solution corresponding to that value of $a^*$ for which $v'(0) = 0$

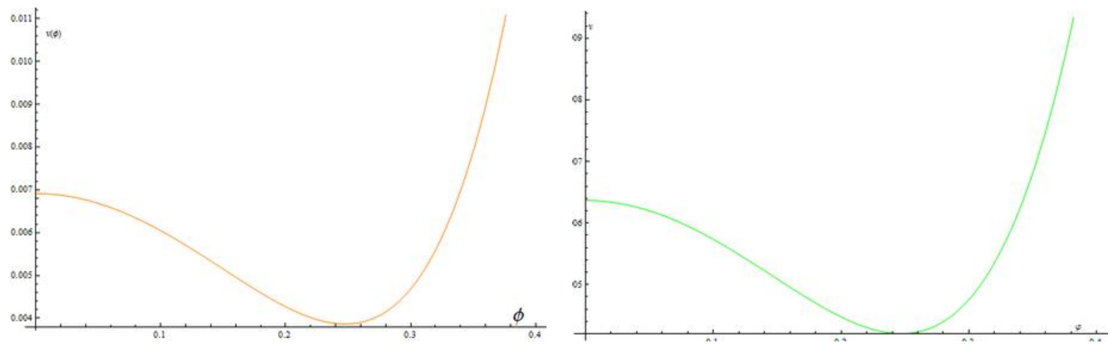
**Figure 2.3**



**(a)** Numerical solution of the fixed point equation in LPA for the potential $v(\varphi)$ obtained with the spike-plot-method.

**(b)** Comparison between the two potentials obtained from the spike-plot method and from the asymptotic method in LPA.

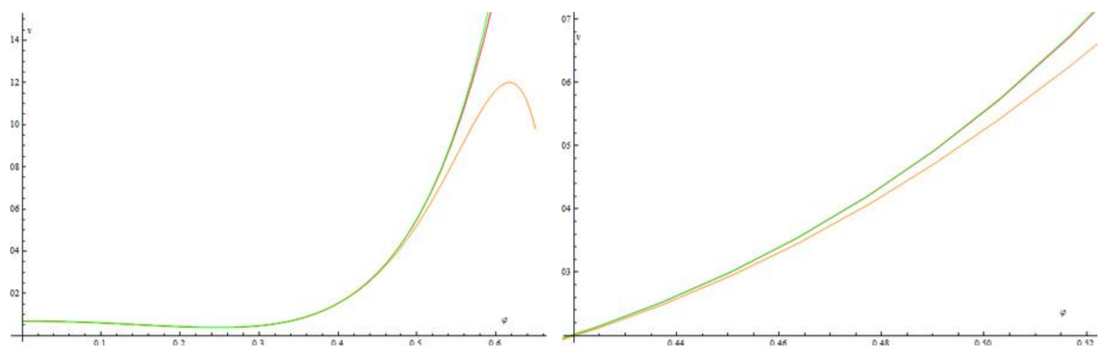
**Figure 2.4**



**(a)** Potential $v(\varphi)$ in the LPA with a polynomial expansion around the origin up to the order $\rho^8$.

**(b)** Potential $v(\varphi)$ in the LPA with a polynomial expansion around a non trivial minimum up to the order $(\rho - \kappa)^8$.

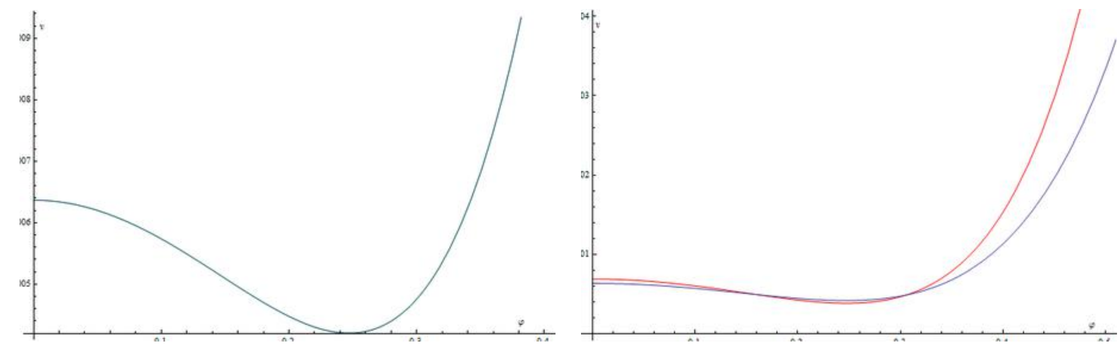
**Figure 2.5**



**(a)** Comparison between the solutions of the FP equation in LPA obtain from the expansion around the origin (orange line), the expansion around a minimum $\kappa \neq 0$ (green line) and the global scaling solution (red line).

**(b)** Zoomed area (of the figure on the left) near the region of deviation from the global scaling solution.

**Figure 2.6**



**(a)** FP solution for the potential $v(\varphi)$ in LPA' using polynomial expansion around the non trivial minimum up to the order $(\rho-\kappa)^6$ (iterative and direct method gives the same result).

**(b)** Comparison between the global scaling solution with $\eta_\phi = 0$ and the polynomial expansion for $\eta_\phi \neq 0$.

**Figure 2.7**

# Chapter 3

# A covariant "Hamiltonian" approach for the FRG equation

In this chapter we are going to follow a different approach for the study of the Functional Renormalization Group equation in the framework of a theory with one real scalar field.

The starting point is the Polchinski equation that is the flow equation for a Wilsonian effective action depending on the effective scale $\Lambda$. From this equation we can obtain the flow for the corresponding effective Lagrangian and for a covariant effective Hamiltonian doing the Legendre transform of the previous one as proposed in [36]. In a leading order of the derivative expansion for this Hamiltonian, we will obtain the flow equation for an arbitrary $\phi$-dependent potential $V(\phi)$ and for the wave function renormalization $Z(\phi)$.

Finally, we shall obtain a novel result by applying this approach to the $O(N)$ scalar model with $N$ real scalar fields $\varphi^a$, $a = 1, 2, \ldots, N$, to obtain for the first time the flow equation for the arbitrary potential $V(\rho)$ and the two wave function renormalization $Z(\rho)$ and $Y(\rho)$ in this formulation.

## 3.1 Polchinski's non perturbative FRG equation

As we have already learned in the previous chapter, the basic idea behind the continuous RG is the following: rather than integrated over all momentum modes $p$ in one go, one first integrates out modes between a cutoff scale $\Lambda_0$ and a very much lower energy scale $\Lambda$ [16]. The remaining integral from $\Lambda$ to zero may again be expressed as a partition function, but the bare action $S_{\Lambda_0}$ is replaced by a complicated effective action $S_\Lambda$ and the overall cutoff $\Lambda_0$ by the effective cutoff $\Lambda$. We can regard the cutoff $\Lambda$ as an infrared cutoff for the modes $q > \Lambda$ *i.e* for the modes that have already been integrated out or as an ultraviolet cutoff for the modes $q < \Lambda$. The fundamental aspect of this idea is that all the Green functions must be the same so that all the physics must be the same under this procedure of integrating out modes.

We can introduce the effective cutoff by modifying the euclidean propagator as follow

$$\frac{1}{q^2} = \frac{C_{UV}(q,\Lambda)}{q^2} + \frac{C_{IR}(q,\Lambda)}{q^2} \tag{3.1}$$

where $C_{UV}(q,\Lambda)$ acts as an UV cutoff *i.e.* $\lim_{q\to\infty} C_{UV}=0$ and $\lim_{q\to 0} C_{UV}=1$ forbidding the propagation of fast modes whereas $C_{IR}(q,\Lambda)$ acts as an IR cutoff *i.e.* $\lim_{q\to 0} C_{IR}=0$ and $\lim_{q\to\infty} C_{IR}=1$ forbidding the propagation of slow modes [23].

The partition function for a theory with one scalar field is expressed by the functional integral

$$Z[J]=\int \mathcal{D}\phi\, e^{-\frac{1}{2}\phi\cdot q^2\cdot\phi - S_{\Lambda_0}[\phi]+J\cdot\phi} \tag{3.2}$$

where $-\frac{1}{2}\phi\cdot q^2\cdot\phi = -\frac{1}{2}\int_q \phi(q)q^2\phi(-q) = -\frac{1}{2}\int dx\partial_\mu\phi\partial_\mu\phi$ and $J\cdot\phi=\int dx J(x)\phi(x)$.

Now we want to show that the functional integral can be split into two integrals, one for the fast modes that we call $\phi_>$ and one for the slow modes that we call $\phi_<$, so that the partition function can be written as [18]

$$Z[J]=\int\mathcal{D}\phi_>\int\mathcal{D}\phi_<\, e^{-\frac{1}{2}\phi_>\cdot\Delta_{IR}^{-1}\cdot\phi_> - \frac{1}{2}\phi_<\cdot\Delta_{UV}^{-1}\cdot\phi_< - S_{\Lambda_0}[\phi_>+\phi_<]+J\cdot(\phi_>+\phi_<)} \tag{3.3}$$

*Proof.* Let's split the propagator for $\phi$ as $D=D_1+D_2$ and the kinetic term as $K_1=D_1^{-1}$ , $K_2=D_2^{-1}$ , $K=D^{-1}=\frac{1}{D_1+D_2}$. We want to find the modes propagated by $K_1$ and $K_2$.

$$\frac{D_1}{D_1+D_2}=1-D_2D^{-1} \implies D^{-1}=D_1^{-1}-D_1^{-1}D^{-1}D_2=K_1-K_1KD_2$$
$$\frac{D_2}{D_1+D_2}=1-D_1D^{-1} \implies D^{-1}=D_2^{-1}-D_2^{-1}D^{-1}D_1=K_2-K_2KD_1$$

the kinetic term becomes

$$\frac{1}{2}\phi\cdot D^{-1}\cdot\phi=\frac{1}{4}\phi K_1(1-KD_2)\phi+\frac{1}{4}\phi K_2(1-KD_1)\phi$$

and if we define $\phi_1\equiv(1-KD_2)\phi$, $\phi_2\equiv(1-KD_1)\phi$ we have

$$\begin{aligned}\frac{1}{2}\phi\cdot D^{-1}\cdot\phi =&\frac{1}{4}\phi_1K_1\phi_1+\frac{1}{4}\phi_2K_2\phi_2+\frac{1}{4}\phi_2K_1\phi_1+\frac{1}{4}\phi_1K_2\phi_2\\ =&\frac{1}{4}\phi_1K_1\phi_1+\frac{1}{4}\phi_2K_2\phi_2+\frac{1}{4}\{\phi(1-KD_1)K_1(1-KD_2)\phi+\phi(1-KD_2)K_2(1-KD_1)\phi\}\\ =&\frac{1}{4}\phi_1K_1\phi_1+\frac{1}{4}\phi_2K_2\phi_2+\frac{1}{4}\{\phi(K_1+K_2)\phi-\phi K\phi-\phi(K_1KD_2+K_2KD_1)\phi\}\\ =&\frac{1}{4}\phi_1K_1\phi_1+\frac{1}{4}\phi_2K_2\phi_2+\frac{1}{4}\phi K\phi\end{aligned}$$

and this implies that

$$\phi K\phi=\phi_1K_1\phi_1+\phi_2K_2\phi_2$$

Turning back to the original notation we have

$$\begin{aligned}\phi_> \equiv&(1-q^2\Delta_{UV})\phi=(1-C_{UV})\phi=C_{IR}\phi=q^2\Delta_{IR}\phi\\ \phi_< \equiv&(1-q^2\Delta_{IR})\phi=(1-C_{IR})\phi=C_{UV}\phi=q^2\Delta_{UV}\phi\end{aligned}$$

□

Now consider only the integration over fast modes

$$\begin{aligned}Z[J]&=\int\mathcal{D}\phi_<\ e^{-\frac{1}{2}\phi_<\cdot\Delta_{UV}^{-1}\cdot\phi_<}\int\mathcal{D}\phi_>\ e^{-\frac{1}{2}\phi_>\cdot\Delta_{IR}^{-1}\cdot\phi_>-S_{\Lambda_0}[\phi_>+\phi_<]+J\cdot(\phi_>+\phi_<)}\\ Z[J]&=\int\mathcal{D}\phi_<\ e^{-\frac{1}{2}\phi_<\cdot\Delta_{UV}^{-1}\cdot\phi_<}Z_\Lambda[J,\phi_<]\end{aligned} \tag{3.4}$$

and doing the following shift $\phi'_> \equiv \phi_> - \Delta_{IR} \cdot J$ we trivially get

$$\begin{aligned} Z_\Lambda[J,\phi_<] =& e^{\frac{1}{2}J\cdot\Delta_{IR}\cdot J + J\cdot\phi_<} \int \mathcal{D}\phi'_> \, e^{-\frac{1}{2}\phi'_>\cdot\Delta_{IR}^{-1}\cdot\phi'_> - S_{\Lambda_0}[\phi'_> + \Delta_{IR}\cdot J + \phi_<]} \\ \equiv& e^{\frac{1}{2}J\cdot\Delta_{IR}\cdot J + J\cdot\phi_<} \, e^{-S_\Lambda[\Delta_{IR}\cdot J + \phi_<]} \end{aligned} \tag{3.5}$$

for some functional $S_\Lambda$. It is straightforward to note that $Z_\Lambda$ and $S_\Lambda$ do not depend on both $J$ and $\phi_<$ independently but on the sum $\Phi = \phi_< + \Delta_{IR} \cdot J$. The partition function will be:

$$Z[J] = e^{\frac{1}{2}J\cdot\Delta_{IR}\cdot J} \int \mathcal{D}\phi_< e^{-\frac{1}{2}\phi_<\cdot\Delta_{UV}^{-1}\cdot\phi_< - S_\Lambda[\Phi] + J\cdot\phi_<} \tag{3.6}$$

so we see that $S_\Lambda$ is nothing but the interaction part of the Wilsonian effective action $S_\Lambda^{tot}$.

To obtain the RG equation for the effective action $S_\Lambda$ we have to differentiate $Z_\Lambda$ with respect $\Lambda$. From the eq.(3.4) and eq.(3.5), taking into account that the dependence on $\Lambda$ is present only through $\Delta_{IR}^{-1}$ [18] , we obtain

$$\boxed{\left.\frac{\partial S_\Lambda[\Phi]}{\partial\Lambda}\right|_\Phi = \frac{1}{2}\frac{\delta S_\Lambda}{\delta\Phi}\cdot\frac{\partial\Delta_{UV}}{\partial\Lambda}\frac{\delta S_\Lambda}{\delta\Phi} - \frac{1}{2}\frac{\delta}{\delta\Phi}\cdot\frac{\partial\Delta_{UV}}{\partial\Lambda}\frac{\delta S_\Lambda}{\delta\Phi}} \tag{3.7}$$

*Proof.*

$$\begin{aligned} \frac{\partial}{\partial\Lambda}Z_\Lambda[J,\phi_<] =& \int \mathcal{D}\phi_> \, (-)\frac{1}{2}\phi_> \cdot \frac{\Delta_{IR}^{-1}}{\partial\Lambda} \cdot \phi_> \, e^{-\frac{1}{2}\phi_>\cdot\Delta_{IR}^{-1}\cdot\phi_> - S_{\Lambda_0}[\phi_> + \phi_<] + J\cdot(\phi_> + \phi_<} \\ =& -\frac{1}{2}\left(\frac{\delta}{\delta J} - \phi_<\right)\frac{\partial\Delta_{IR}^{-1}}{\partial\Delta}\left(\frac{\delta}{\delta J} - \phi_<\right) Z_\Lambda[J,\phi_<] \end{aligned}$$

From the definition of the effective action $Z_\Lambda[\Phi] = e^{\frac{1}{2}J\cdot\Delta_{IR}\cdot J + J\cdot\phi_< - S_\Lambda[\Phi]}$ we have

$$\frac{\partial}{\partial\Lambda}Z_\Lambda = Z_\Lambda\left(\frac{1}{2}J\cdot\frac{\partial\Delta_{IR}}{\partial\Lambda}\cdot J - \left.\frac{\partial S_\Lambda}{\partial\Lambda}\right|_\Phi - \frac{\delta S_\Lambda}{\delta\Phi}\cdot\frac{\partial\Delta_{IR}}{\partial\Lambda}\cdot J\right)$$

$$\begin{aligned} \left(\frac{\delta}{\delta J} - \phi_<\right) Z_\Lambda =& Z_\Lambda\left(\Delta_{IR}\cdot J - \frac{\delta S_\Lambda}{\delta\Phi}\cdot\Delta_{IR}\right) \\ \left(\frac{\delta}{\delta J} - \phi_<\right)^2 Z_\Lambda =& -\phi_< Z_\Lambda\left(\Delta_{IR}\cdot J - \frac{\delta S_\Lambda}{\delta\Phi}\cdot\Delta_{IR}\right) + Z_\Lambda\left(\Delta_{IR} - \frac{\delta^2 S_\Lambda}{\delta\Phi^2}\cdot\Delta_{IR}^2\right) \\ &+ Z_\Lambda\left(\Delta_{IR}\cdot J - \frac{\delta S_\Lambda}{\delta\Phi}\cdot\Delta_{IR}\right)\left(\Delta_{IR}\cdot J + \phi_< - \frac{\delta S_\Lambda}{\delta\Phi}\cdot\Delta_{IR}\right) \end{aligned}$$

therefore we obtain the final equality

$$\begin{aligned} & Z_\Lambda\left(\frac{1}{2}J\cdot\frac{\partial\Delta_{IR}}{\partial\Lambda}\cdot J - \left.\frac{\partial S_\Lambda}{\partial\Lambda}\right|_\Phi - \frac{\delta S_\Lambda}{\delta\Phi}\cdot\frac{\partial\Delta_{IR}}{\partial\Lambda}\cdot J\right) = \\ & \frac{1}{2}\frac{1}{\Delta_{IR}^2}\frac{\partial\Delta_{IR}}{\partial\Lambda}Z_\Lambda\left(\Delta_{IR}^2\frac{\delta S_\Lambda}{\delta\Phi}\frac{\delta S_\Lambda}{\delta\Phi} - \frac{\delta^2 S_\Lambda}{\delta\Phi^2}\Delta_{IR}^2 + \Delta_{IR} + \Delta_{IR}^2 J^2 - 2J\Delta_{IR}^2\frac{\delta S_\Lambda}{\delta\Phi}\right) \end{aligned}$$

that implies

$$-\left.\frac{\partial S_\Lambda}{\partial\Lambda}\right|_\Phi = \frac{1}{2}\frac{\partial\Delta_{IR}}{\partial\Lambda}\left(\frac{\delta S_\Lambda}{\delta\Phi}\right)^2 - \frac{1}{2}\frac{\partial\Delta_{IR}}{\partial\Lambda}\frac{\delta^2 S_\Lambda}{\delta\Phi^2} + \frac{1}{2}\frac{\partial}{\partial\Lambda}\ln\Delta_{IR}$$

Taking into account that $\Delta_{IR} + \Delta_{UV} = 1/q^2$ so $\partial_\Lambda\Delta_{IR} + \partial_\Lambda\Delta_{UV} = 0$ and dropping the constant term field-independent we gain the final result (3.7) □

## 3.2 Momenta fields and the derivative expansion

In this section we would like to develop a new method, as showed in [36], to construct approximate solutions of functional renormalization group equations. We already know that the two main approximation strategies to truncate the FRG equations into a solvable set of equations are the vertex expansion and the derivative expansion. The first is an expansion in field variables while retaining the full momentum dependence whereas the second is an expansion in powers of momenta while retaining the full field dependence. This last approximation makes use of local actions with definite given powers of field derivative and it relies on the assumption that the system have one mass scale $m$ such that high powers of $\frac{\partial^2}{m^2}$ play a progressive less important role.

Apart convergence the problem of the DE is essentially a combinatoric computational difficulty in obtaining the flow equations for a hight order of the DE. The standard way to deal with such a problem is to compute the flow of a full function of $p^2$. Moreover, in a DE setup one have to keep $\phi$ generic and constant in order to describe infinitely many vertices. The traditional way to take into account these two aspects simultaneously is by means of an Hamiltonian formalism.

We restrict ourselves to the case of a $\mathbb{Z}_2$ real scalar field theory. First of all we want to find the RG equation for the effective Lagrangian starting from the Polchinski's equation [23] and taking a truncation that is essentially an arbitrary high order of the DE. At a later time we want to translate this equation into a RG equation for the Hamiltonian density replacing the arbitrary-order derivative

$$\frac{d^n}{dx^{\mu_1}\dots dx^{\mu_n}}\phi(x) \to \pi^{\mu_1\cdots\mu_n}$$

with a symmetric tensor field of arbitrary high rank order. This is the crucial point that makes this approach different from the DE: for example two terms like $-\phi\partial^2\phi$ and $\partial\phi\partial\phi$ get translated into two different tensorial objects $\phi\pi^{\mu\nu}\delta_{\mu\nu}$ and $\pi^\mu\pi_\mu$. We are interested in an expansion of the Hamiltonian in momenta field of increasing rank order. Certainly related to this work is the analysis of the FRG equations of the effective Hamiltonian outlined in [29] but with an important difference in the present formulation: the Legendre transform is taken after each RG steps thus obtaining a derivative-free effective Hamiltonian *i.e.* not depending on derivatives of $\phi$ and $\pi$.

### 3.2.1 Lagrangian flow equation

Let's start from the Polchinski's equation for the effective action

$$\dot{S}[\phi] = \frac{1}{2}\int dx \int dy \left( \frac{\delta S[\phi]}{\delta\phi(x)}\dot{C}(x-y)\frac{\delta S[\phi]}{\delta\phi(y)} - \frac{\delta}{\delta\phi(x)}\dot{C}(x-y)\frac{\delta S[\phi]}{\delta\phi(y)} \right) \tag{3.8}$$

where $\dot{S} = -\Lambda\partial_\Lambda S$ *i.e.* $t = \ln\frac{\Lambda_0}{\Lambda}$ and $C_\Lambda$ is the regularized UV propagator that in the momentum space can be expressed as $C_\Lambda(q^2) = \Lambda^{\eta-2}\frac{K(q^2/\Lambda^2)}{q^2/\Lambda^2}$ with $K$ an UV cutoff function.

*It is important to recall that the classic ERG procedure consists of two steps:*

*a coarse-graining followed by a rescaling.* At the moment we are working with dimensionfull quantities (with also the possibility for the scalar field to have a non trivial anomalous dimension) so only the first step has been taken into account. Only at the end of our treatment we will make the last step rescaling all quantities to dimensionless ones.

We are interested in a Lagrangian density depending on generically high derivatives of the scalar field $\phi$. Using multi-indices $M \equiv (\mu_1, \ldots, \mu_m)$ with $m \in \mathbb{N}$ we denote

$$\phi_M(x) = \phi_{\mu_1,\ldots,\mu_m}(x) = \frac{d^m}{dx^{\mu_1} \ldots dx^{\mu_m}} \phi(x) = \frac{d^M}{dx^M} \phi(x) \tag{3.9}$$

and our Lagrangian will be

$$\mathcal{L} = \mathcal{L}\left(x, \phi_M(x)\right). \tag{3.10}$$

Now we are ready to write the Polchinski equation as a partial differential equation for $\mathcal{L}$ taking into account of our truncation. The first functional variation of the effective action $S[\phi] = \int dx \mathcal{L}\big(x, \phi_M(x)\big)$ is

$$\begin{aligned} \delta S[\phi] = \int dx \Bigg[ \frac{\partial \mathcal{L}}{\partial \phi(x)} \delta\phi(x) + \frac{\partial \mathcal{L}}{\partial \phi_\mu(x)} \delta\phi_\mu(x) + \frac{\partial \mathcal{L}}{\partial \phi_{\mu\nu}(x)} \delta\phi_{\mu\nu}(x) + \ldots \\ \cdots + \frac{\partial \mathcal{L}}{\partial \phi_{\mu_1 \ldots \mu_m}} \delta\phi_{\mu_1 \ldots \mu_m}(x) \Bigg] = \int dx \frac{\partial \mathcal{L}}{\partial \phi_M(x)} \delta\phi_M(x) \end{aligned} \tag{3.11}$$

where the sum over the multi-indices $M$ is understood. Because

$$\delta\phi_M(x) = \delta \frac{d^M}{dx^M} \phi(x) = \frac{d^M}{dx^M} \delta\phi(x) \tag{3.12}$$

integrating by parts we obtain

$$\begin{aligned} \delta S[\phi] = \int dx \delta\phi(x) \Bigg[ \frac{\partial \mathcal{L}}{\partial \phi(x)} - \frac{\partial}{\partial x^\mu} \frac{\delta \mathcal{L}}{\partial \phi_\mu} + \frac{\partial^2}{\partial x^\mu \partial x^\nu} \frac{\partial \mathcal{L}}{\partial \phi_{\mu\nu}} + \ldots \\ \cdots + (-1)^m \frac{d^m}{dx^{\mu_1} \ldots dx^{\mu_m}} \frac{\partial \mathcal{L}}{\partial \phi_{\mu_1 \ldots \mu_m}} \Bigg] = \int dx \delta\phi(x) (-1)^M \frac{d^M}{dx^M} \frac{\partial \mathcal{L}}{\partial \phi_M(x)} \end{aligned} \tag{3.13}$$

and, for the definition of functional derivative, we gain

$$\begin{aligned} \frac{\delta S}{\delta \phi(x)} =& \frac{\partial \mathcal{L}}{\partial \phi(x)} - \frac{d}{dx^\mu} \frac{\partial \mathcal{L}}{\partial \phi_\mu} + \frac{d^2}{dx^\mu dx^\nu} \frac{\partial \mathcal{L}}{\partial \phi_{\mu\nu}} + \cdots + (-1)^m \frac{d^m}{dx^{\mu_1} \ldots dx^{\mu_m}} \frac{\partial \mathcal{L}}{\partial \phi_{\mu_1 \ldots \mu_m}} \\ =& (-1)^M \frac{d^M}{dx^M} \frac{\partial \mathcal{L}}{\partial \phi_M(x)} \end{aligned} \tag{3.14}$$

To obtain the second order functional derivative it is useful to rewrite the first derivative as an integral of some Lagrangian density

$$\frac{\delta S}{\delta \phi(x)} = (-1)^M \int dy \delta(y - x) \frac{d^M}{dy^M} \frac{\partial \mathcal{L}}{\partial \phi_M(y)} = \int dy \frac{\partial \mathcal{L}}{\partial \phi_M(y)} \frac{d^M}{dy^M} \delta(y - x) \tag{3.15}$$

$$
\begin{aligned}
\delta\frac{\delta S}{\delta\phi(x)} &= \int dy \frac{\delta^2\mathcal{L}}{\partial\phi_M(y)\partial\phi_N(y)}\delta\phi_N(y)\frac{d^M}{dy^M}\delta(y-x) \\
&= \int dy \frac{\delta^2\mathcal{L}}{\partial\phi_M(y)\partial\phi_N(y)}\left(\frac{d^N}{dy^N}\delta\phi(y)\right)\frac{d^M}{dy^M}\delta(y-x) \\
&= \int dy(-1)^N\delta\phi(y)\frac{d^N}{dy^N}\left[\frac{\partial^2\mathcal{L}}{\partial\phi_M(y)\partial\phi_N(y)}\delta_M(y-x)\right]
\end{aligned}
\tag{3.16}
$$

so the second functional derivative is

$$
\frac{\delta^2 S[\phi]}{\delta\phi(x)\delta\phi(y)} = (-1)^N\frac{d^N}{dy^N}\left[\frac{\partial^2\mathcal{L}}{\partial\phi_M(y)\partial\phi_N(y)}\frac{d^M}{dy^M}\delta(y-x)\right]
\tag{3.17}
$$

The Polchinski equation for the present truncation will be

$$
\begin{aligned}
\dot{S} =& \frac{1}{2}\int_{x,y}(-1)^M\left[\frac{d^M}{dx^M}\frac{\partial\mathcal{L}}{\partial\phi_M(x)}\right]\dot{C}(x-y)(-1)^N\left[\frac{d^N}{dy^N}\frac{\partial\mathcal{L}}{\partial\phi_N(y)}\right] \\
&-\frac{1}{2}\int_{x,y}\dot{C}(x-y)(-1)^N\frac{d^N}{dy^N}\left[\frac{\partial^2\mathcal{L}}{\partial\phi_M(y)\partial\phi_N(y)}\frac{d^M}{dy^M}\delta(y-x)\right] \\
=& \frac{1}{2}\int_{x,y}\frac{\partial\mathcal{L}}{\partial\phi_M(x)}\left[\frac{d^M}{dx^M}\frac{d^N}{dy^N}\dot{C}(x-y)\right]\frac{\partial\mathcal{L}}{\partial\phi_N(y)} \\
&-\frac{1}{2}\int_{x,y}\left[\frac{d^N}{dy^N}\dot{C}(x-y)\right]\frac{\partial^2\mathcal{L}}{\partial\phi_M(y)\partial\phi_N(y)}\frac{d^M}{dy^M}\delta(y-x)
\end{aligned}
\tag{3.18}
$$

Assuming that the regularized propagator is an even function of the position in space $C(x) = C(-x)$ than

$$
\begin{aligned}
\dot{S} =& (-1)^N\frac{1}{2}\int_{x,y}\frac{\partial\mathcal{L}}{\partial\phi_M(x)}\left[\frac{d^M}{dx^M}\frac{d^N}{dx^N}\dot{C}(x-y)\right]\frac{\partial\mathcal{L}}{\partial\phi_N(y)} \\
&-\frac{1}{2}\int_{x,y}\left[(-1)^N\frac{d^N}{dx^N}\dot{C}(x-y)\right]\frac{\partial^2\mathcal{L}}{\partial\phi_M(y)\partial\phi_N(y)}(-1)^M\frac{d^M}{dx^M}\delta(y-x) \\
=& (-1)^N\frac{1}{2}\int_{x,y}\frac{\partial\mathcal{L}}{\partial\phi_M(x)}\frac{\partial\mathcal{L}}{\partial\phi_N(y)}\left[\frac{d^{M+N}}{dx^M dx^N}\dot{C}(x-y)\right] \\
&-\frac{1}{2}\int_{x,y}\left[(-1)^N\frac{d^{M+N}}{dx^M dx^N}\dot{C}(x-y)\right]\frac{\partial^2\mathcal{L}}{\partial\phi_M(y)\partial\phi_N(y)}\delta(y-x)
\end{aligned}
$$

As a consequence, one can recast the RG equation for the effective action in the following form:

$$
\int_x \dot{\mathcal{L}}(x) = \frac{(-1)^N}{2}\left\{\int_{x,y}\frac{\partial\mathcal{L}}{\partial\phi_M(x)}\frac{\partial\mathcal{L}}{\partial\phi_N(y)}\dot{C}_{MN}(x-y) - \dot{C}_{MN}(0)\int_x\frac{\partial^2\mathcal{L}}{\partial\phi_M(x)\partial\phi_N(x)}\right\}
\tag{3.19}
$$

where we have defined

$$
\dot{C}_{MN}(x-y) \equiv \frac{d^{M+N}}{dx^M dx^N}\dot{C}(x-y)
\tag{3.20}
$$

The first term on the r.h.s is a non-local term and to project this onto a local form we have to expand it about the point $x$

$$
\frac{\partial\mathcal{L}}{\partial\phi_N}(y) = \frac{\partial\mathcal{L}}{\partial\phi_N}(x) + \sum_{L\neq 0}\frac{1}{L!}(y-x)^L\frac{d^L}{dx^L}\frac{\partial\mathcal{L}}{\partial\phi_N}(x)
\tag{3.21}
$$

where if $L$ is a multi-indices $L = (\mu_1, \ldots, \mu_\ell)$ then $L! = \ell!$ and $z^L = z^{\mu_1} \cdot \ldots \cdot z^{\mu_\ell}$. Because we want to restrict ourselves to the pointlike interaction limit and so neglect the explicit $x$-dependence in the Lagrangian, the total $x^\lambda$-derivative will be

$$\frac{d}{dx^\mu} = \frac{d\phi_M(x)}{dx^\mu}\frac{\partial}{\partial\phi_M(x)} = \phi_{M\mu}(x)\frac{\partial}{\partial\phi_M(x)} \tag{3.22}$$

the second derivative is

$$\begin{aligned}\frac{d^2}{dx^\mu dx^\nu} &= \phi_{M\mu}\frac{\partial}{\partial\phi_M}\left(\phi_{N\nu}\frac{\partial}{\partial\phi_N}\right) = \phi_{M\mu}\left(\delta_{N\nu,M}\frac{\partial}{\partial\phi_N} + \phi_{N\nu}\frac{\partial^2}{\partial\phi_M\partial\phi_N}\right)\\ &= \phi_{N\mu\nu}\frac{\partial}{\partial\phi_N} + \phi_{M\mu}\phi_{N\nu}\frac{\partial^2}{\partial\phi_M\partial\phi_N}\end{aligned}$$

the third derivative is

$$\begin{aligned}\frac{d^3}{dx^{\mu_1}dx^{\mu_2}dx^{\mu_3}} =&\phi_{M\mu_1\mu_2\mu_3}\frac{\partial}{\partial\phi_M}+\\ &+(\phi_{M_1\mu_1}\phi_{M_2\mu_2\mu_3} + \phi_{M_1\mu_2}\phi_{M_2\mu_3\mu_1} + \phi_{M_1\mu_3}\phi_{M_2\mu_1\mu_2})\frac{\partial^2}{\partial\phi_{M_1}\partial\phi_{M_2}}+\\ &+\phi_{M_1\mu_1}\phi_{M_2\mu_2}\phi_{M_3\mu_3}\frac{\partial^3}{\partial\phi_{M_1}\partial\phi_{M_2}\partial\phi_{M_3}}\end{aligned}$$

the forth derivative is

$$\begin{aligned}&\frac{d^4}{dx^{\mu_1}\ldots dx^{\mu_4}} = \phi_{M\mu_1\mu_2\mu_3\mu_4}\frac{\partial}{\partial\phi_M}+\\ &+(\phi_{M_1\mu_1\mu_2}\phi_{M_2\mu_3\mu_4} + \phi_{M_1\mu_1\mu_3}\phi_{M_2\mu_2\mu_4} + \phi_{M_1\mu_1\mu_4}\phi_{M_2\mu_2\mu_3}\\ &+\phi_{M_1\mu_1}\phi_{M_2\mu_2\mu_3\mu_4} + \phi_{M_1\mu_2}\phi_{M_2\mu_1\mu_3\mu_4} + \phi_{M_1\mu_3}\phi_{M_2\mu_1\mu_2\mu_4} + \phi_{M_1\mu_4}\phi_{M_2\mu_1\mu_2\mu_3})\frac{\partial^2}{\partial\phi_{M_1}\partial\phi_{M_2}}+\\ &+(\phi_{M_1\mu_1\mu_2}\phi_{M_2\mu_3}\phi_{M_3\mu_4} + \phi_{M_1\mu_1\mu_3}\phi_{M_2\mu_2}\phi_{M_3\mu_4} + \phi_{M_1\mu_1\mu_4}\phi_{M_2\mu_2}\phi_{M_3\mu_3}\\ &+\phi_{M_1\mu_2\mu_3}\phi_{M_2\mu_1}\phi_{M_3\mu_4} + \phi_{M_1\mu_2\mu_4}\phi_{M_2\mu_1}\phi_{M_3\mu_3} + \phi_{M_1\mu_3\mu_4}\phi_{M_2\mu_1}\phi_{M_3\mu_2})\frac{\partial^3}{\partial\phi_{M_1}\partial\phi_{M_2}\partial\phi_{M_3}}\\ &+\phi_{M_1\mu_1}\phi_{M_2\mu_2}\phi_{M_3\mu_3}\phi_{M_4\mu_4}\frac{\partial^4}{\partial\phi_{M_1}\partial\phi_{M_2}\partial\phi_{M_3}\partial\phi_{M_4}}\end{aligned}$$

and so on iterating this procedure.
It is straightforward to derive the $L-$th derivative that can be written as

$$\frac{d^L}{dx^L} = \sum_{i=1}^{L}\phi_{(M_1}\phi_{M_2}\ldots\phi_{M_i)}\frac{\partial^i}{\partial\phi_{M_1}\partial\phi_{M_2}\ldots\partial\phi_{M_i}} \tag{3.23}$$

where $\phi_{(M_1}\phi_{M_2}\ldots\phi_{M_i)}$ denotes a sum over all possible ways of ditributing the indices inside $L$ on the $i-$entries $\phi_{M_1}\phi_{M_2}\ldots\phi_{M_i}$ under the rules that there must be at least one index out of $L$ per entry, that ordering inside each entry does not matter and that permutations of $M_1\ldots M_i$ do not matter. So the equation (3.19) becomes

$$
\begin{aligned}
\dot{\mathcal{L}}(x) &= -\frac{(-1)^N}{2}\dot{C}_{MN}(0)\frac{\partial^2\mathcal{L}}{\partial\phi_M\partial\phi_N}(x) \\
&+ \frac{(-1)^N}{2}\frac{\partial\mathcal{L}}{\partial\phi_M}(x)\int_y \dot{C}_{MN}(x-y)\left[\frac{\partial\mathcal{L}}{\partial\phi_N}(x) + \frac{(y-x)^L}{L!}\frac{d^L}{dx^L}\frac{\partial\mathcal{L}}{\partial\phi_N}(x)\right] \\
&= -\frac{(-1)^N}{2}\dot{C}_{MN}(0)\frac{\partial^2\mathcal{L}}{\partial\phi_M\partial\phi_N}(x) + \frac{(-1)^N}{2}\frac{\partial\mathcal{L}}{\partial\phi_M}(x)\frac{\partial\mathcal{L}}{\partial\phi_N}(x)\int_y \dot{C}_{MN}(x-y) \\
&+ \frac{(-1)^N}{2}\frac{\partial\mathcal{L}}{\partial\phi_M}(x)\frac{1}{L!}\sum_{i=1}^{L}\phi_{(M_1}\ldots\phi_{M_i)}\frac{\partial^{i+1}\mathcal{L}}{\partial\phi_{M_1}\ldots\partial\phi_{M_i}\partial\phi_N}(x)\int_y \dot{C}_{MN}(x-y)(y-x)^L
\end{aligned}
\tag{3.24}
$$

where obviously the sum over $L$ is understood. Assuming that the regularized propagator is an even function of the space position $C(z) = C(-z)$ we have

$$
\begin{aligned}
\int_y (y-x)^L \frac{d^{M+N}}{dx^M dx^N}\dot{C}(x-y) &= (-1)^{M+N}\int_y (y-x)^L \frac{d^{M+N}}{dy^M dy^N}\dot{C}(y-x) \\
&= (-1)^{M+N}\int_z z^L \frac{d^{M+N}}{dz^M dz^N}\dot{C}(z) \equiv (-1)^{M+N} J_{L,MN}
\end{aligned}
\tag{3.25}
$$

Whenever $M + N > L$, integrating by parts and assuming that the regulator is such that the boundary terms vanish, one gets $J_{L,MN} = 0$.
Finally to sum up, the projection of the Polchinski equation on the ansatz of a local effective Lagrangian, depending on arbitrary high order field derivatives, gives

$$
\begin{aligned}
\dot{\mathcal{L}} = &-\frac{(-1)^N}{2}\dot{C}_{MN}(0)\frac{\partial^2\mathcal{L}}{\partial\phi_M\partial\phi_N} + \frac{1}{2}\left(\frac{\partial\mathcal{L}}{\partial\phi}\right)^2\dot{\widetilde{C}}(0) \\
&+ \frac{(-1)^M}{2}\frac{\partial\mathcal{L}}{\partial\phi_M}\frac{J_{L,MN}}{L!}\sum_{i=1}^{L}\phi_{(M_1}\ldots\phi_{M_i)}\frac{\partial^{i+1}\mathcal{L}}{\partial\phi_{M_1}\ldots\partial\phi_{M_i}\partial\phi_N}
\end{aligned}
\tag{3.26}
$$

with $L \neq 0$ whereas the sum over $M, N, M_1, \ldots, M_i$ includes the empty index. We have introduced $\widetilde{C}(p)$ as the Fourier transform of $C(z)$.

Now we want to simplify the last equation taking a Lagrangian which is an arbitrary function of $\phi$ and $\phi_\mu$ so only the first order derivative of the scalar field is presented in this truncation. With this ansatz for the Lagrangian

$$
\mathcal{L} = \mathcal{L}(\phi, \phi_\mu) \tag{3.27}
$$

the flow equation becomes

$$\begin{aligned}\dot{\mathcal{L}} =& \frac{1}{2}\dot{\tilde{C}}(0)\left(\frac{\partial\mathcal{L}}{\partial\phi}\right)^2 - \frac{1}{2}\dot{C}(0)\frac{\partial^2\mathcal{L}}{\partial\phi^2} + \frac{1}{2}\dot{C}_{\mu\nu}(0)\frac{\partial^2\mathcal{L}}{\partial\phi_\mu\partial\phi_\nu} \\ &+ \frac{1}{2}\sum_{\ell=1}^{\infty}\frac{1}{\ell!}J_{\lambda_1\ldots\lambda_\ell,\mu}\,\phi_{\lambda_1}\ldots\phi_{\lambda_\ell}\left(\frac{\partial\mathcal{L}}{\partial\phi}\frac{\partial^{\ell+1}\mathcal{L}}{\partial\phi^\ell\partial\phi_\mu} - \frac{\partial\mathcal{L}}{\partial\phi_\mu}\frac{\partial^{\ell+1}\mathcal{L}}{\partial\phi^{\ell+1}}\right) \\ &- \frac{1}{2}\sum_{\ell=2}^{\infty}\frac{1}{\ell!}J_{\lambda_1\ldots\lambda_\ell,\mu\nu}\,\phi_{\lambda_1}\ldots\phi_{\lambda_\ell}\frac{\partial\mathcal{L}}{\partial\phi_\mu}\frac{\partial^{\ell+1}\mathcal{L}}{\partial\phi^\ell\partial\phi_\nu} \\ &+ \frac{1}{2}\sum_{\ell=1}^{\infty}\frac{1}{\ell!}J_{\lambda_1\ldots\lambda_\ell,0}\,\phi_{\lambda_1}\ldots\phi_{\lambda_\ell}\frac{\partial\mathcal{L}}{\partial\phi}\frac{\partial^{\ell+1}\mathcal{L}}{\partial\phi^{\ell+1}}\end{aligned} \tag{3.28}$$

where the second and third terms come from the first term of eq.(3.26) for $M = N = 0$ and $M = N = 1$ (the term from $M = 0, N = 1$ and $M = 1, N = 0$ canceled each other because of the factor $(-1)^N$). The forth term comes from the third term of eq.(3.26) for $M = 1, N = 0$ and $M = 0, N = 1$. The fifth term comes form the third term of eq.(3.26) for $M = N = 1$ and the sixth term comes form the third term of eq.(3.26) for $M = N = 0$.
(We have to take into account that in the third term of eq.(3.26), for each $L \neq 0$, in the sum over $i$ only the term with $i = L$ survives because all other terms are out of our truncation).

If we want further to project the flow on the sector quadratic in $\phi_\mu$ and neglect higher powers, the following terms survive

$$\begin{aligned}\dot{\mathcal{L}} =& \frac{1}{2}\dot{\tilde{C}}(0)\left(\frac{\partial\mathcal{L}}{\partial\phi}\right)^2 - \frac{1}{2}\dot{C}(0)\frac{\partial^2\mathcal{L}}{\partial\phi^2} + \frac{1}{2}\dot{C}_{\mu\nu}(0)\frac{\partial^2\mathcal{L}}{\partial\phi_\mu\partial\phi_\nu} \\ &+ \frac{1}{2}J_{\lambda,\mu}\,\phi_\lambda\left(\frac{\partial\mathcal{L}}{\partial\phi}\frac{\partial^2\mathcal{L}}{\partial\phi\partial\phi_\mu} - \frac{\partial\mathcal{L}}{\partial\phi_\mu}\frac{\partial^2\mathcal{L}}{\partial\phi^2}\right) \\ &+ \frac{1}{4}J_{\mu\nu,0}\,\phi_\mu\phi_\nu\frac{\partial\mathcal{L}}{\partial\phi}\frac{\partial^3\mathcal{L}}{\partial\phi^3}\end{aligned} \tag{3.29}$$

where the last term can be rewritten as

$$-\frac{1}{4}J_{\mu\nu,0}\phi_\mu\phi_\nu\left(\frac{\partial^2\mathcal{L}}{\partial\phi^2}\right)^2$$

because they differ by a total derivative, always taking into account our projection ($\phi_M$ absent for $N > 1$). Infact:

*Proof.*

$$\phi_\mu\phi_\nu\frac{\partial\mathcal{L}}{\partial\phi}\frac{\partial^3\mathcal{L}}{\partial\phi^3} = \phi_\mu\phi_\nu\frac{\partial}{\partial\phi}\left(\frac{\partial\mathcal{L}}{\partial\phi}\frac{\partial^2\mathcal{L}}{\partial\phi^2}\right) - \phi_\mu\phi_\nu\left(\frac{\partial^2\mathcal{L}}{\partial\phi^2}\right)^2$$

but the total $x^\mu$ derivative, because of our ansatz is

$$\frac{d}{dx^\mu} = \phi_{M\mu}\frac{\partial}{\partial\phi_M} \simeq \phi_\mu\frac{\partial}{\partial\phi}$$

therefore

$$\begin{aligned}
\phi_\mu\phi_\nu \frac{\partial\mathcal{L}}{\partial\phi}\frac{\partial^3\mathcal{L}}{\partial\phi^3} =&\phi_\mu \frac{d}{dx^\nu}\left(\frac{\partial\mathcal{L}}{\partial\phi}\frac{\partial^2\mathcal{L}}{\partial\phi^2}\right) - \phi_\mu\phi_\nu\left(\frac{\partial^2\mathcal{L}}{\partial\phi^2}\right)^2 \\
=&\frac{d}{dx^\nu}\left(\phi_\mu\frac{\partial\mathcal{L}}{\partial\phi}\frac{\partial^2\mathcal{L}}{\partial\phi^2}\right) - \phi_{\mu\nu}\frac{\partial\mathcal{L}}{\partial\phi}\frac{\partial^2\mathcal{L}}{\partial\phi^2} - \phi_\mu\phi_\nu\left(\frac{\partial^2\mathcal{L}}{\partial\phi^2}\right)^2 \\
\simeq& - \phi_\mu\phi_\nu\left(\frac{\partial^2\mathcal{L}}{\partial\phi^2}\right)^2
\end{aligned}$$

□

Later on we will follow these conventions for the regulator dependent terms as in [2]

$$\dot{\tilde{C}}(0) = -2\Lambda^{\eta-2}K_0 \tag{3.30}$$

$$\dot{C}_{\mu\nu}(0) = 2\delta^\nu_\mu\Lambda^{d+\eta}\frac{I_1}{d} \tag{3.31}$$

$$J_{\lambda,\mu} = 2\delta^\lambda_\mu\Lambda^{\eta-2}K_0 \tag{3.32}$$

$$J_{\mu\nu,0} = -4\delta^\mu_\nu\Lambda^{\eta-4}K_1 \tag{3.33}$$

$$\dot{C}(0) = -2\Lambda^{d-2+\eta}I_0 \tag{3.34}$$

where $K_0, I_0, K_1, I_1$ are all positives real constant parameters. We emphasize again that the presence of the anomalous dimension is because we consider from the beginning the possibility for the scalar field $\phi$ to have a non trivial anomalous dimension.

### 3.2.2 Covariant "Hamiltonian" flow equation

Now we want to look for an Hamiltonian translation of the flow equation (3.29). In the traditional Hamiltonian formalism we deal with the field $\phi(x)$ and its momenta $\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(x)}$ defined as the partial derivative of the Lagrangian with respect to time-derivative of the field itself. Here, for the actual truncation $\mathcal{L} = \mathcal{L}(\phi, \phi_M)$ we can define a generalized covariant momenta as follow

$$\pi_M(x) \equiv i\frac{\partial\mathcal{L}}{\partial\phi_M(x)} \tag{3.35}$$

and a generalized Hamiltonian in the euclidean space-time

$$\mathcal{H}(\phi, \pi_M) = i\pi_M\phi_M + \mathcal{L}(\phi, \phi_M) \tag{3.36}$$

where all the field-derivatives have to be inverted and expressed in term of the momenta. We again drop the explicit $x$-dependence in the Lagrangian and also in the Hamiltonian.

To translate the flow equation for $\mathcal{L}$ into flow equation for $\mathcal{H}$ we need some preliminaries formulas

$$\frac{\partial\mathcal{L}}{\partial\phi} = \frac{\partial\mathcal{H}}{\partial\phi} \quad , \quad \frac{\partial^2\mathcal{L}}{\partial\phi\partial\phi} = \frac{\partial^2\mathcal{H}}{\partial\phi\partial\phi} \tag{3.37}$$

$$\frac{\partial \mathcal{H}}{\partial \pi_M} = i\phi_M + i\pi_N \frac{\partial \phi_N}{\partial \pi_M} + \frac{\partial \mathcal{L}}{\partial \phi_N}\frac{\partial \phi_N}{\partial \pi_M} = i\phi_M$$

therefore

$$\phi_M = -i\frac{\partial \mathcal{H}}{\partial \pi_M} \tag{3.38}$$

$$\frac{\partial^2 \mathcal{L}}{\partial \phi_M \partial \phi_N} = \frac{\partial}{\partial \phi_M}(-i\pi_N) = -i\left(\frac{\partial \phi_M}{\partial \pi_N}\right)^{-1} = \left(\frac{\partial^2 \mathcal{H}}{\partial \pi_N \partial \pi_M}\right)^{-1} \tag{3.39}$$

Because the variable $\phi$ and $\phi_M$ are independent

$$0 = \frac{d\phi_M}{d\phi} = \frac{\partial \phi_M}{\partial \phi} + \frac{\partial \pi_N}{\partial \phi}\frac{\partial \phi_M}{\partial \pi_N} = -i\frac{\partial^2 \mathcal{H}}{\partial \phi \partial \pi_M} + i\frac{\partial^2 \mathcal{L}}{\partial \phi \partial \phi_N}(-i)\frac{\partial^2 \mathcal{H}}{\partial \pi_N \partial \pi_M}$$

and multiplying by $\left(\frac{\partial^2 \mathcal{H}}{\partial \pi_M \partial \pi_L}\right)^{-1}$ we have

$$\frac{\partial^2 \mathcal{L}}{\partial \phi \partial \phi_M} = i\frac{\partial^2 \mathcal{H}}{\partial \phi \partial \pi_N}\left(\frac{\partial^2 \mathcal{H}}{\partial \pi_N \partial \pi_M}\right)^{-1} \tag{3.40}$$

Neglecting the explicit spacetime, *i.e.* momentum, dependence of the couplings, then $\mathcal{H}$ depends on the position $x$ only through the fields $\phi(x)$ and $\pi_M(x)$ and this is a good feature because we can study the flow of the Hamiltonian by setting both fields to constant values. Under this approximation, the RG equation of $\mathcal{H}$ is encoded in a partial differential equation for a function of infinitely many fields which are symmetric tensors of arbitrary rank order.

Now we could consider a further approximation, for example neglecting the dynamics of momenta with rank bigger than one, so taking under consideration an arbitrary function $\mathcal{H} = \mathcal{H}(\phi, \pi_\mu)$. This is exactly the case of the eq.(3.29) that becomes

$$\begin{aligned} \dot{\mathcal{H}} =& -\Lambda^{\eta-2}K_0\left(\frac{\partial \mathcal{H}}{\partial \phi}\right)^2 + \Lambda^{d-2+\eta}I_0\frac{\partial^2 \mathcal{H}}{\partial \phi^2} + \Lambda^{d+\eta}\frac{I_1}{d}\left(\frac{\partial^2 \mathcal{H}}{\partial \pi_\mu \partial \pi_\mu}\right)^{-1} \\ &+ \Lambda^{\eta-2}K_0\frac{\partial \mathcal{H}}{\partial \pi_\mu}\left[\frac{\partial \mathcal{H}}{\partial \phi}\frac{\partial^2 \mathcal{H}}{\partial \phi \partial \pi_\nu}\left(\frac{\partial^2 \mathcal{H}}{\partial \pi_\mu \partial \pi_\nu}\right)^{-1} + \pi_\mu\frac{\partial^2 \mathcal{H}}{\partial \phi^2}\right] \\ &+ \Lambda^{\eta-4}K_1\frac{\partial \mathcal{H}}{\partial \pi_\mu}\frac{\partial \mathcal{H}}{\partial \pi_\mu}\frac{\partial \mathcal{H}}{\partial \phi}\frac{\partial^3 \mathcal{H}}{\partial \phi^3} \end{aligned} \tag{3.41}$$

where the conventions (3.34) are taken. Our Hamiltonian, because of the Lorentz symmetry, depends only on two scalar variables: $\phi$ and $\sigma \equiv \frac{\pi_\mu \pi_\mu}{2}$.
Our task is now to compute the previous equation in term of these new scalar variables. To invert the second derivative in the third and forth term we introduce two projectors

$$P^\perp_{\mu\nu} \equiv \delta_{\mu\nu} - \frac{\pi_\mu \pi_\nu}{\pi^2} \tag{3.42}$$

$$P^L_{\mu\nu} \equiv \frac{\pi_\mu \pi_\nu}{\pi^2} \tag{3.43}$$

which have the properties: $P^\perp \pi = 0, P^L \pi = \pi, (P^\perp)^2 = P^\perp, (P^L)^2 = P^L, P^L P^\perp = P^\perp P^L = 0, P^\perp + P^L = 1$

$$\frac{\partial \mathcal{H}}{\partial \pi_\mu} = \frac{\partial \mathcal{H}}{\partial \sigma}\frac{\partial \sigma}{\partial \pi_\mu} = \mathcal{H}'_\sigma \pi_\mu \tag{3.44}$$

$$\frac{\partial^2\mathcal{H}}{\partial\pi_\mu\partial\pi_\nu} = \frac{\partial^2\mathcal{H}}{\partial\sigma^2}\frac{\partial\sigma}{\partial\pi_\mu}\frac{\partial\sigma}{\partial\pi_\nu} + \frac{\partial\mathcal{H}}{\partial\sigma}\frac{\partial^2\sigma}{\partial\pi_\mu\partial\pi_\nu} = P^{\perp}_{\mu\nu}\mathcal{H}'_\sigma + P^{L}_{\mu\nu}(\mathcal{H}'_\sigma + 2\sigma\mathcal{H}''_\sigma) \tag{3.45}$$

and the inverse can be written as

$$\left(\frac{\partial^2\mathcal{H}}{\partial\pi_\mu\partial\pi_\nu}\right)^{-1} = \frac{P^{\perp}_{\mu\nu}}{\mathcal{H}'_\sigma} + \frac{P^{L}_{\mu\nu}}{\mathcal{H}'_\sigma + 2\sigma\mathcal{H}''_\sigma} \tag{3.46}$$

where with the notation $\mathcal{H}'_\sigma$ we mean the first derivative of the Hamiltonian with respect the variable $\sigma$ and so on for the others.
The last second derivative we need is

$$\frac{\partial^2\mathcal{H}}{\partial\phi\partial\pi_\mu} = \pi_\mu\frac{\partial^2\mathcal{H}}{\partial\sigma\partial\phi} = \pi_\mu\mathcal{H}''_{\sigma\phi} \tag{3.47}$$

and the eq.(3.41), multiplying by $\Lambda^{-d}$, can be rewritten as

$$\begin{aligned}\Lambda^{-d}\dot{\mathcal{H}} = &-\Lambda^{\eta-d-2}K_0(\mathcal{H}'_\phi)^2 + \Lambda^{\eta-2}I_0\mathcal{H}''_\phi + \Lambda^{\eta}\frac{I_1}{d}\left(\frac{d-1}{\mathcal{H}'_\sigma} + \frac{1}{\mathcal{H}'_\sigma + 2\sigma\mathcal{H}''_\sigma}\right)\\ &+ \Lambda^{\eta-d-2}K_0\left[2\sigma\frac{\mathcal{H}'_\phi\mathcal{H}'_\sigma\mathcal{H}''_{\phi\sigma}}{\mathcal{H}'_\sigma + 2\sigma\mathcal{H}''_\sigma} + 2\sigma\mathcal{H}'_\sigma\mathcal{H}''_\phi\right]\\ &+ \Lambda^{\eta-d-4}K_1 2\sigma(\mathcal{H}'_\sigma)^2\mathcal{H}'_\phi\mathcal{H}'''_\phi\end{aligned} \tag{3.48}$$

Suppose now to rescale $\mathcal{H}$, $\phi$, $\sigma$ in this way: $\mathcal{H} = a\bar{\mathcal{H}}$, $\phi = b\bar{\phi}$, $\sigma = c\bar{\sigma}$. Then we have

$$\begin{aligned}\Lambda^{-d}a\dot{\bar{\mathcal{H}}} =&\Lambda^{\eta-2}\frac{aI_0}{b^2}\bar{\mathcal{H}}''_{\bar{\phi}} + \Lambda^{\eta}\frac{acI_1}{a^2 d}\left(\frac{d-1}{\bar{\mathcal{H}}'_{\bar{\sigma}}} + \frac{1}{\bar{\mathcal{H}}'_{\bar{\sigma}} + 2\bar{\sigma}\bar{\mathcal{H}}''_{\bar{\sigma}}}\right)\\ &+ \Lambda^{\eta-d-2}\frac{a^2K_0}{b^2}\left[-(\bar{\mathcal{H}}'_{\bar{\phi}})^2 + 2\bar{\sigma}\bar{\mathcal{H}}'_{\bar{\sigma}}\bar{\mathcal{H}}''_{\bar{\phi}} + 2\bar{\sigma}\frac{\bar{\mathcal{H}}'_{\bar{\phi}}\bar{\mathcal{H}}'_{\bar{\sigma}}\bar{\mathcal{H}}''_{\bar{\phi}\bar{\sigma}}}{\bar{\mathcal{H}}'_{\bar{\sigma}} + 2\bar{\sigma}\bar{\mathcal{H}}''_{\bar{\sigma}}}\right]\\ &+ \Lambda^{\eta-d-4}\frac{K_1a^4}{cb^4}2\bar{\sigma}(\bar{\mathcal{H}}'_{\bar{\sigma}})^2\bar{\mathcal{H}}'_{\bar{\phi}}\bar{\mathcal{H}}'''_{\bar{\phi}}\end{aligned} \tag{3.49}$$

and it is easy to see that we can absorb the three regulator dependent parameters $K_0, I_0, I_1$ but not the parameter $K_1$ in the last term. In details, the appropriate choice for the parameters is:

$$a = \frac{I_0}{K_0}, \qquad b^2 = I_0, \qquad c = \frac{I_0^2}{K_0^2 I_1}. \tag{3.50}$$

After this rescaling and dropping the bar for the sake of brevity we obtain

$$\begin{aligned}\Lambda^{-d}\dot{\mathcal{H}} =&\Lambda^{\eta-2}\mathcal{H}''_\phi + \Lambda^{\eta}\frac{1}{d}\left(\frac{d-1}{\mathcal{H}'_\sigma} + \frac{1}{\mathcal{H}'_\sigma + 2\sigma\mathcal{H}''_\sigma}\right)\\ &+ \Lambda^{\eta-d-2}\left[-(\mathcal{H}'_\phi)^2 + 2\sigma\mathcal{H}'_\sigma\mathcal{H}''_\phi + 2\sigma\frac{\mathcal{H}'_\phi\mathcal{H}'_\sigma\mathcal{H}''_{\phi\sigma}}{\mathcal{H}'_\sigma + 2\sigma\mathcal{H}''_\sigma}\right]\\ &+ \Lambda^{\eta-d-4}\frac{K_1I_1}{K_0I_0}2\sigma(\mathcal{H}'_\sigma)^2\mathcal{H}'_\phi\mathcal{H}'''_\phi\end{aligned} \tag{3.51}$$

where as usual the last term can be rewritten, apart from a boundary term, as

$$-\Lambda^{\eta-d-4}\frac{K_1I_1}{K_0I_0}2\sigma(\mathcal{H}'_\sigma)^2(\mathcal{H}''_\phi)^2 \tag{3.52}$$

where the pre-factor $\Lambda^{\eta-d-4}$ is the correct one because of the full quantum dimension of the fields so that the term has zero dimension as $\Lambda^{-d}\dot{\mathcal{H}}$.

### The second step in the RG procedure: rescaling

Before employing further approximations, we have to write down the right flow equation for the dimensionless fields. At criticality the fields $\phi$ and $\sigma$ scale with $\Lambda$ according to their full quantum dimension that we call $D_\phi$ and $D_\sigma$ respectively. The full dimensionality of $\pi_\mu$ is such that the term $\pi_\mu \phi_\mu$ in the definition of the Hamiltonian should have the same dimension as $\mathcal{H}$, in other words it should scale with $\Lambda^d$. For this reason $D_\pi = d - D_\phi - 1$ so, knowing that $D_\phi = (d-2+\eta)/2$ we get $D_\pi = (d-\eta)/2$. The rescaling we have to do is:

$$\mathcal{H} = \Lambda^d \tilde{\mathcal{H}}, \qquad \phi = \Lambda^{D_\phi}\tilde{\phi}, \qquad \sigma = \Lambda^{2D_\pi}\tilde{\sigma} \tag{3.53}$$

where with tilde we denote dimensionless Hamiltonian and fields. Now we have to start from this equality

$$\tilde{H}(\tilde{\phi},\tilde{\sigma}) = \Lambda^{-d}\mathcal{H}(\phi,\sigma) \tag{3.54}$$

and apply a total derivative with respect to the RG time on both side of this equality.

$$-\Lambda\frac{d}{d\Lambda}\tilde{\mathcal{H}}(\tilde{\phi},\tilde{\sigma}) = -\Lambda\partial_\Lambda\Big|_{\tilde{\phi},\tilde{\sigma}}\tilde{\mathcal{H}} - \Lambda\frac{\partial\tilde{\phi}}{\partial\Lambda}\tilde{\mathcal{H}}'_{\tilde{\phi}} - \Lambda\frac{\partial\tilde{\sigma}}{\partial\Lambda}\tilde{\mathcal{H}}'_{\tilde{\sigma}} = d\Lambda^{-d}\mathcal{H} - \Lambda^{-d}\Lambda\frac{d}{d\Lambda}\mathcal{H} \tag{3.55}$$

$$\implies -\Lambda\partial_\Lambda\Big|_{\tilde{\phi},\tilde{\sigma}}\tilde{\mathcal{H}} = \Lambda\frac{\partial\tilde{\phi}}{\partial\Lambda}\tilde{\mathcal{H}}'_{\tilde{\phi}} + \Lambda\frac{\partial\tilde{\sigma}}{\partial\Lambda}\tilde{\mathcal{H}}'_{\tilde{\sigma}} + d\tilde{\mathcal{H}} + \Lambda^{-d}\dot{\mathcal{H}} \tag{3.56}$$

Recalling that $\tilde{\phi} = \Lambda^{-D_\phi}\phi = \Lambda^{-(d-2+\eta)/2}\phi$, $\tilde{\sigma} = \Lambda^{-2D_\pi}\sigma = \Lambda^{-(d-\eta)}\sigma$ we gain

$$\partial_t\Big|_{\tilde{\phi},\tilde{\sigma}}\tilde{\mathcal{H}} = -\frac{d-2+\eta}{2}\tilde{\phi}\tilde{\mathcal{H}}'_{\tilde{\phi}} - (d-\eta)\tilde{\sigma}\tilde{\mathcal{H}}'_{\tilde{\sigma}} + d\tilde{\mathcal{H}} + \Lambda^{-d}\dot{\mathcal{H}} \tag{3.57}$$

where the last term is exactly the eq.(3.51) that must be expressed as a function of the new variables. To sum up the final result is

$$\begin{aligned}\partial_t\Big|_{\tilde{\phi},\tilde{\sigma}}\tilde{\mathcal{H}} =& -\frac{d-2+\eta}{2}\tilde{\phi}\tilde{\mathcal{H}}'_{\tilde{\phi}} - (d-\eta)\tilde{\sigma}\tilde{\mathcal{H}}'_{\tilde{\sigma}} + d\tilde{\mathcal{H}} \\ &+ \tilde{\mathcal{H}}''_{\tilde{\phi}} + \frac{1}{d}\left(\frac{d-1}{\tilde{\mathcal{H}}'_{\tilde{\sigma}}} + \frac{1}{\tilde{\mathcal{H}}'_{\tilde{\sigma}} + 2\tilde{\sigma}\tilde{\mathcal{H}}''_{\tilde{\sigma}}}\right) \\ &- (\tilde{\mathcal{H}}'_{\tilde{\phi}})^2 + 2\tilde{\sigma}\tilde{\mathcal{H}}'_{\tilde{\sigma}}\tilde{\mathcal{H}}''_{\tilde{\phi}} + 2\tilde{\sigma}\frac{\tilde{\mathcal{H}}'_{\tilde{\phi}}\tilde{\mathcal{H}}'_{\tilde{\sigma}}\tilde{\mathcal{H}}''_{\tilde{\phi}\tilde{\sigma}}}{\tilde{\mathcal{H}}'_{\tilde{\sigma}} + 2\tilde{\sigma}\tilde{\mathcal{H}}''_{\tilde{\sigma}}} \\ &+ \frac{K_1 I_1}{K_0 I_0}2\tilde{\sigma}(\tilde{\mathcal{H}}'_{\tilde{\sigma}})^2\tilde{\mathcal{H}}'_{\tilde{\phi}}\tilde{\mathcal{H}}'''_{\tilde{\phi}}\end{aligned} \tag{3.58}$$

The left hand side of this equation is the time derivative acting only on the intrinsic time dependence therefore only on the couplings. Finally it is important to note that if one choose a family of cutoff functions such that $K_1 = 0$ then the equation will become regulator independent.

### 3.2.3 RG equations for $\tilde{V}(\tilde{\phi})$ and $\tilde{Z}(\tilde{\phi})$

Now we want to do a further approximation and project the previous equation on the sector quadratic in the first derivative of the scalar field. The most general Lagrangian will be:

$$\mathcal{L} = V(\phi) + \frac{1}{2}Z(\phi)\partial_\mu\phi\partial_\mu\phi \tag{3.59}$$

(from now on we will deal with dimensionless quantities so we drop the tilde for sake of brevity). With this ansatz the momenta field will be

$$\pi_\mu = i\frac{\partial\mathcal{L}}{\partial\phi_\mu} = iZ(\phi)\phi_\mu \tag{3.60}$$

and so it is trivial to invert it and express the first derivative of the field in term of the momenta

$$\phi_\mu = -i\frac{\pi_\mu}{Z(\phi)} \tag{3.61}$$

Using the Legendre transform we get

$$\mathcal{H} = i\pi_\mu\phi_\mu + \mathcal{L} = V(\phi) + \frac{\sigma}{Z(\phi)} \tag{3.62}$$

As we can see $\mathcal{H}''_\sigma = 0$ so the eq.(3.58) simplifies and becomes (dropping the tildes)

$$\begin{aligned}\dot{\mathcal{H}} =& -\frac{d-2+\eta}{2}\phi\mathcal{H}'_\phi - (d-\eta)\sigma\mathcal{H}'_\sigma + d\mathcal{H} \\ &+ \mathcal{H}''_\phi + \frac{1}{\mathcal{H}'_\sigma} - (\mathcal{H}'_\phi)^2 + 2\sigma\mathcal{H}'_\sigma\mathcal{H}''_\phi + 2\sigma\mathcal{H}'_\phi\mathcal{H}''_{\phi\sigma} \\ &+ \frac{K_1 I_1}{K_0 I_0}2\sigma(\mathcal{H}'_\sigma)^2\mathcal{H}'_\phi\mathcal{H}'''_\phi\end{aligned} \tag{3.63}$$

As we have said before the left hand side is the intrinsic time derivative so it is

$$\dot{\mathcal{H}} = \dot{V} - \frac{\dot{Z}}{Z^2}\sigma \tag{3.64}$$

whereas in the right hand side we have to replace our Hamiltonian $\mathcal{H}(\phi,\sigma)$:

$$\begin{aligned}\mathcal{H} =& V(\phi) + \frac{\sigma}{Z(\phi)} \\ \mathcal{H}'_\phi =& V' - \frac{Z'}{Z^2}\sigma \\ \mathcal{H}''_\phi =& V'' - \frac{Z''Z - 2(Z')^2}{Z^3} \\ \mathcal{H}'_\sigma =& \frac{1}{Z} \\ \mathcal{H}''_{\phi\sigma} =& -\frac{Z'}{Z^2}\end{aligned}$$

Projecting the r.h.s, one more time, on the sector linear in the momenta field $\sigma$ we gain two flow equations for the dimensionless potential and renormalization function

$$\dot{V} = dV - \frac{1}{2}(d-2+\eta)\phi V' - (V')^2 + V'' + Z \tag{3.65}$$

$$\dot{Z} = -\eta Z - \frac{1}{2}(d-2+\eta)\phi Z' - 2ZV'' + Z'' - 2\frac{(Z')^2}{Z} - 2bV'V''' \tag{3.66}$$

$$\doteq -\eta Z - \frac{1}{2}(d-2+\eta)\phi Z' - 2ZV'' + Z'' - 2\frac{(Z')^2}{Z} + 2b(V'')^2 \tag{3.67}$$

where as usual with the notation $\doteq$ we mean that the two equations are the same apart from an overall boundary term and the coefficient $b$ is the regulator dependent pre-factor present in the last term of the eq.(3.63) *i.e.* $b = \frac{K_1 I_1}{K_0 I_0}$.

These equations differ from the ones obtained by a first order of the DE using the Polchinski flow equation in a Lagrangian approach [2]. The equation for $\dot{V}$ is the same apart for the fact that in the DE there is a regulator dependent coefficient multiplying $Z$ that cannot be removed by rescaling. Here, this coefficient can safely set equal to one. The equation for $\dot{Z}$ is rather different: the first four terms and the last one are morally the same but others are different. We rewrite here the equations derived in [2]

$$\dot{V} = dV - \frac{1}{2}(d-2+\eta)\phi V' - (V')^2 + V'' + 2\frac{I_1 K_0}{I_0} Z$$

$$\doteq -\eta Z - \frac{1}{2}(d-2+\eta)\phi Z' - 4ZV'' + Z'' - 2Z'V' + \frac{K_1}{K_0^2}(V'')^2 - \frac{\eta}{2}$$

In the next section, we shall analyze the scaling solutions of equations (3.65) and (3.67) with $b = 0$ and $b \neq 0$, which describe for $d = 3$ the critical Ising model. Once we will get the scaling solutions for the potential $V_*(\phi)$ and the wave function renormalization $Z_*(\phi)$, *i.e.* the solutions at the fixed point, we can linearize the system and study small variations from the fixed point solutions. Setting

$$V(\phi) = V_*(\phi) + v(\phi) \tag{3.68}$$

$$Z(\phi) = Z_*(\phi) + z(\phi) \tag{3.69}$$

where $v(\phi)$ and $z(\phi)$ are the small variations, we have to substitute them in the previous equations and take only the first order in $v$ and $z$. Therefore we will arrive at the following equations

$$\dot{v} = dv - \frac{d-2+\eta}{2}\phi v' - 2V_*'v' + v'' + z \tag{3.70}$$

$$\dot{z} = -\eta z - \frac{d-2+\eta}{2}\phi z' - 2Z_* v'' - 2Z_* V_*'' z + z'' + 2\frac{(Z_*')^2}{Z_*^2} z - 4\frac{Z_*'}{Z_*} z' + 4bV_*''v'' \tag{3.71}$$

that is linear because it can be rewrite in a matrix-like form as

$$\begin{pmatrix} \dot{v} \\ \dot{z} \end{pmatrix} = \mathcal{M}_\phi \begin{pmatrix} v \\ z \end{pmatrix} \tag{3.72}$$

where $\mathcal{M}_\phi$ is a matrix whose entries are differential operators acting on the small function variations $v$ and $z$. Studying the eigenvalue problem of this system we can obtain, for example, the value of $\nu$ which is one of the critical exponents in the Ising model.

## 3.3 O(N) model

In this section we want to apply the strategy explained previously for the case of $N$ real scalar fields with an internal $O(N)$ symmetry. As we have done before we will start from the Polchinski equation to derive a renormalization group equation for a Lagrangian that will depend on arbitrary high order derivatives of the $N$ fields. Subsequently we will translate this equation into a RG equation for the covariant Hamiltonian density replacing the arbitrary-order derivative

$$\frac{d^n}{dx^{\mu_1}\dots dx^{\mu_n}}\varphi^a(x) \to \pi^a_{\mu_1\dots\mu_n} \tag{3.73}$$

with a symmetric tensor field that will contain, in this case, an internal index. At the end, we will project this equation (depending on momenta field with arbitrary high rank order) on the sector quadratic in momenta itself so dropping all that with rank bigger than two.

### 3.3.1 Lagrangian flow equation

Now we are ready to apply the previous ideas for a scalar theory with $O(N)$ symmetry. The extension of the Polchinski equation (3.8) to the $O(N)$ model is straightforward

$$\dot{S}[\varphi] = \frac{1}{2}\int_x\int_y\left(\frac{\delta S[\varphi]}{\delta\varphi^a(x)}\dot{C}_{ab}(x-y)\frac{\delta S[\varphi]}{\delta\varphi^b(y)} - \frac{\delta}{\delta\varphi^a(x)}\dot{C}_{ab}(x-y)\frac{\delta S[\varphi]}{\delta\varphi^b(y)}\right) \tag{3.74}$$

but for the $O(N)$ symmetry the regularized propagator is diagonal

$$\dot{C}_{ab}(x-y) = \delta_{ab}\dot{C}(x-y) \tag{3.75}$$

We are now interested in local truncations which correspond to a Lagrangian density depending on generically high derivatives of the fields, in other words our effective action is

$$S_\Lambda[\varphi] = \int d^dx\mathcal{L}\left(x, \varphi^a_M(x)\right) \tag{3.76}$$

Following exactly the same calculations of the previous section we have

$$\frac{\delta S[\varphi]}{\delta\varphi^a(x)} = (-1)^M\frac{d^M}{dx^M}\frac{\partial\mathcal{L}}{\partial\varphi^a_M(x)} \tag{3.77}$$

$$\frac{\delta^2 S[\varphi]}{\delta\varphi^a(x)\delta\varphi^b(y)} = (-1)^N\frac{d^N}{dy^N}\left[\frac{\partial^2\mathcal{L}}{\partial\varphi^a_M(y)\partial\varphi^b_N(y)}\frac{d^M}{dy^M}\delta(y-x)\right]. \tag{3.78}$$

Substituting these in the former equation and doing some integrations by part we have

$$\begin{aligned}\dot{S} =& (-1)^N\frac{1}{2}\int_x\int_y\frac{\partial\mathcal{L}}{\partial\varphi^a_M(x)}\frac{\partial\mathcal{L}}{\partial\varphi^a_N(y)}\frac{d^{M+N}}{dx^Mdx^N}\dot{C}(x-y)\\ &-(-1)^N\frac{1}{2}\int_x\frac{\partial^2\mathcal{L}}{\partial\varphi^a_M(x)\partial\varphi^a_N(x)}\frac{d^{M+N}}{dz^Mdz^N}\dot{C}(z)\Big|_{z=0}.\end{aligned} \tag{3.79}$$

The first term is non local so we take a Taylor expansion around the $x$ point

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \varphi_N^a}(y) =&\frac{\partial \mathcal{L}}{\partial \varphi_N^a}(x) + \sum_{L\neq 0} \frac{1}{L!}\,(y-x)^L \frac{d^L}{dx^L}\frac{\partial \mathcal{L}}{\partial \varphi_N^a}(x)\\ =&\frac{\partial \mathcal{L}}{\partial \varphi_N^a}(x) + \sum_{L\neq 0} \frac{1}{L!}\,(y-x)^L \sum_{i=1}^{L} \varphi^{a_1}_{(M_1}\cdots\varphi^{a_i}_{M_i)\,L}\,\frac{\partial^{i+1}\mathcal{L}}{\partial\varphi^{a_1}_{M_1}\ldots\partial\varphi^{a_i}_{M_i}\partial\varphi^a_N}(x)\end{aligned} \tag{3.80}$$

and therefore the flow equation for the Lagrangian takes the form:

$$\begin{aligned}\dot{\mathcal{L}} =& -\frac{(-1)^N}{2}\dot{C}_{MN}(0)\frac{\partial^2\mathcal{L}}{\partial\varphi^a_M\partial\varphi^a_N} + \frac{1}{2}\frac{\partial \mathcal{L}}{\partial\varphi^a}\frac{\partial \mathcal{L}}{\partial\varphi^a}\dot{\tilde{C}}(0)\\ &+\frac{(-1)^M}{2}\frac{\partial \mathcal{L}}{\partial\varphi^a_M}\frac{J_{L,MN}}{L!}\sum_{i=1}^{L}\varphi^{a_1}_{(M_1}\cdots\varphi^{a_i}_{M_i)\,L}\frac{\partial^{i+1}\mathcal{L}}{\partial\varphi^{a_1}_{M_1}\ldots\partial\varphi^{a_i}_{M_i}\partial\varphi^a_N}.\end{aligned} \tag{3.81}$$

This is the straightforward generalization to $O(N)$ case of eq.(3.26) with $L \neq 0$ (sum over $L$ is implicit here) whereas the sum over $M, N, M_1, \ldots, M_i$ includes the empty index. Now we want to simplify the last equation taking a Lagrangian which is an arbitrary function of $\varphi^a$ and $\varphi^a_\mu$ so only the first order derivative is present in this truncation.
With this ansatz for the Lagrangian

$$\mathcal{L} = \mathcal{L}(\varphi^a, \varphi^a_\mu) \tag{3.82}$$

the flow equation becomes

$$\begin{aligned}\dot{\mathcal{L}} =&\frac{1}{2}\dot{\tilde{C}}(0)\frac{\partial\mathcal{L}}{\partial\varphi^a}\frac{\partial\mathcal{L}}{\partial\varphi^a} - \frac{1}{2}\dot{C}(0)\frac{\partial^2\mathcal{L}}{\partial\varphi^a\partial\varphi^a} + \frac{1}{2}\dot{C}_{\mu\nu}(0)\frac{\partial^2\mathcal{L}}{\partial\varphi^a_\mu\partial\varphi^a_\nu}\\ &+\frac{1}{2}\sum_{\ell=1}^{\infty}\frac{1}{\ell!}J_{\lambda_1\ldots\lambda_\ell,\mu}\,\varphi^{a_1}_{\lambda_1}\cdots\varphi^{a_\ell}_{\lambda_\ell}\left(\frac{\partial\mathcal{L}}{\partial\varphi^b}\frac{\partial^{\ell+1}\mathcal{L}}{\partial\varphi^{a_1}\ldots\partial\varphi^{a_\ell}\partial\varphi^b_\mu} - \frac{\partial\mathcal{L}}{\partial\varphi^b_\mu}\frac{\partial^{\ell+1}\mathcal{L}}{\partial\varphi^{a_1}\ldots\partial\varphi^{a_\ell}\partial\varphi^b}\right)\\ &-\frac{1}{2}\sum_{\ell=2}^{\infty}\frac{1}{\ell!}J_{\lambda_1\ldots\lambda_\ell,\mu\nu}\,\varphi^{a_1}_{\lambda_1}\cdots\varphi^{a_\ell}_{\lambda_\ell}\frac{\partial\mathcal{L}}{\partial\varphi^b_\mu}\frac{\partial^{\ell+1}\mathcal{L}}{\partial\varphi^{a_1}\ldots\partial\varphi^{a_\ell}\partial\varphi^b_\nu}\\ &+\frac{1}{2}\sum_{\ell=1}^{\infty}\frac{1}{\ell!}J_{\lambda_1\ldots\lambda_\ell,0}\,\varphi^{a_1}_{\lambda_1}\cdots\varphi^{a_\ell}_{\lambda_\ell}\frac{\partial\mathcal{L}}{\partial\varphi^b}\frac{\partial^{\ell+1}\mathcal{L}}{\partial\varphi^{a_1}\ldots\partial\varphi^{a_\ell}\partial\varphi^b}\end{aligned} \tag{3.83}$$

where the second and third term come from the first one of eq.(3.81) for $M = N = 0$ and $M = N = 1$ respectively (because of the factor $(-1)^N$ the cases $M = 0, N = 1$ and $M = 1, N = 1$ have opposite sign). The two terms in the second line come from the sum in the eq. (3.81) for the cases $N = 1, M = 0$ and $N = 0, M = 1$. The terms in the third and forth lines come from the former sum corresponding to the cases $N = 1, M = 1$ and $M = N = 0$ respectively. (Remember that in eq.(3.81), for each $L \neq 0$, in the sum over $i$ only the term with $i = L$ survives because all other terms are out of our truncation *i.e.* have derivatives of the field greater than one). Note that we still have a sum of infinitely many terms but this is no longer the case if one further projects the flow on the sector quadratic in $\varphi^a_\mu$ which select the following terms

$$\begin{aligned}\dot{\mathcal{L}} =&\frac{1}{2}\dot{\tilde{C}}(0)\frac{\partial\mathcal{L}}{\partial\varphi^a}\frac{\partial\mathcal{L}}{\partial\varphi^a} - \frac{1}{2}\dot{C}(0)\frac{\partial^2\mathcal{L}}{\partial\varphi^a\partial\varphi^a} + \frac{1}{2}\dot{C}_{\mu\nu}(0)\frac{\partial^2\mathcal{L}}{\partial\varphi^a_\mu\partial\varphi^a_\nu}\\ &+\frac{1}{2}J_{\nu,\mu}\,\varphi^a_\nu\left(\frac{\partial\mathcal{L}}{\partial\varphi^b}\frac{\partial^2\mathcal{L}}{\partial\varphi^a\partial\varphi^b_\mu} - \frac{\partial\mathcal{L}}{\partial\varphi^b_\mu}\frac{\partial^2\mathcal{L}}{\partial\varphi^a\partial\varphi^b}\right)\\ &+\frac{1}{4}J_{\mu\nu,0}\,\varphi^a_\mu\varphi^c_\nu\frac{\partial\mathcal{L}}{\partial\varphi^b}\frac{\partial^3\mathcal{L}}{\partial\varphi^a\partial\varphi^c\partial\varphi^b}\end{aligned} \tag{3.84}$$

Again the last term can be expressed, apart from a boundary term, as

$$-\frac{1}{4}J_{\mu\nu,0}\varphi^a_\mu\varphi^c_\nu\frac{\partial^2\mathcal{L}}{\partial\varphi^a\partial\varphi^b}\frac{\partial^2\mathcal{L}}{\partial\varphi^b\partial\varphi^c} \tag{3.85}$$

### 3.3.2 Hamiltonian flow equation

As we did in the previous section, we look for an Hamiltonian translation of this discussion. First of all we define the momenta fields for the $O(N)$ case that have an internal ''color" index

$$\pi^a_M(x) \equiv i\frac{\partial\mathcal{L}}{\partial\varphi^a_M(x)} \tag{3.86}$$

and an Hamiltonian density

$$\mathcal{H}(\varphi^a,\pi^a_M) = i\pi^a_M\varphi^a_M + \mathcal{L}(\varphi^a,\varphi^a_M) \tag{3.87}$$

that depends on the $N$ fields and on arbitrary high order rank momenta. Moreover we need the following relations

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\varphi^a} =&\frac{\partial\mathcal{H}}{\partial\varphi^a}\\ \varphi^a_M =& -i\frac{\partial\mathcal{H}}{\partial\pi^a_M}\\ \frac{\partial^2\mathcal{L}}{\partial\varphi^a_M\partial\varphi^b_N} =&\left(\frac{\partial^2\mathcal{H}}{\partial\pi^b_N\partial\pi^a_M}\right)^{-1}\\ \frac{\partial^2\mathcal{L}}{\partial\varphi^a\partial\varphi^b_M} =&i\frac{\partial^2\mathcal{H}}{\partial\varphi^a\partial\pi^c_N}\left(\frac{\partial^2\mathcal{H}}{\partial\pi^c_N\partial\pi^b_M}\right)^{-1}\end{aligned} \tag{3.88}$$

where the last one comes from the independence of $\varphi^a_M$ and $\varphi^b$ *i.e.* $\frac{d\varphi^a_M}{d\varphi^b} = \frac{\partial\varphi^a_M}{\partial\varphi^b} + \frac{\partial\pi^c_N}{\partial\varphi^b}\frac{\partial\phi^a_M}{\partial\pi^c_N} = 0$.

In a straightforward manner, neglecting the dynamics of momenta with rank bigger than two and taking moreover the quadratic sector of the Lagrangian flow equation, eq.(3.84) becomes

$$\dot{\mathcal{H}} = \Lambda^{d-2+\eta} I_0 \frac{\partial^2 \mathcal{H}}{\partial \varphi^a \partial \varphi^a} + \Lambda^{d+\eta} \frac{I_1}{d} \left( \frac{\partial^2 \mathcal{H}}{\partial \pi^a_\mu \partial \pi^a_\mu} \right)^{-1}$$
$$+ \Lambda^{\eta-2} K_0 \frac{\partial \mathcal{H}}{\partial \pi^a_\mu} \left[ -\frac{\partial \mathcal{H}}{\partial \varphi^a} \frac{\partial \mathcal{H}}{\partial \varphi^a} + \frac{\partial \mathcal{H}}{\partial \varphi^b} \frac{\partial^2 \mathcal{H}}{\partial \varphi^a \partial \pi^c_\nu} \left( \frac{\partial^2 \mathcal{H}}{\partial \pi^c_\nu \partial \pi^b_\mu} \right)^{-1} + \pi^b_\mu \frac{\partial^2 \mathcal{H}}{\partial \varphi^a \partial \varphi^b} \right]$$
$$+ \Lambda^{\eta-4} K_1 \frac{\partial \mathcal{H}}{\partial \pi^a_\mu} \frac{\partial \mathcal{H}}{\partial \pi^c_\mu} \frac{\partial \mathcal{H}}{\partial \varphi^b} \frac{\partial^3 \mathcal{H}}{\partial \varphi^a \partial \varphi^b \partial \varphi^c} \tag{3.89}$$

where the last term can be expressed, apart for a boundary term, as

$$- \Lambda^{\eta-4} K_1 \frac{\partial \mathcal{H}}{\partial \pi^a_\mu} \frac{\partial \mathcal{H}}{\partial \pi^c_\mu} \frac{\partial^2 \mathcal{H}}{\partial \varphi^a \partial \varphi^b} \frac{\partial^2 \mathcal{H}}{\partial \varphi^b \partial \varphi^c} \tag{3.90}$$

Because of the $O(N)$ symmetry the most general quadratic Lagrangian we can take is [20]

$$\mathcal{L} = V(\rho) + \frac{1}{2} Z(\rho) \partial_\mu \varphi^a \partial_\mu \varphi^a + \frac{1}{2} Y(\rho) \varphi^a \partial_\mu \varphi^a \varphi^b \partial_\mu \varphi^b \tag{3.91}$$

(where $\rho = \frac{\varphi^a \varphi^a}{2}$) that correspond to the $\mathcal{O}(\partial^2)$ of the derivative expansion. Note that there is one more renormalization function $Y(\rho)$ in spite of the case of one single scalar field.

With $N$ scalar fields we can define two projectors

$$P^{ab}_L = \frac{\varphi^a \varphi^b}{\varphi^2} \quad \text{and} \quad P^{ab}_\perp = \delta^{ab} - \frac{\varphi^a \varphi^b}{\varphi^2} \tag{3.92}$$

which have the properties: $P_L \varphi = \varphi$, $P_\perp \varphi = 0$, $P_L^2 = P_L$, $P_\perp^2 = P_\perp$, $P_L P_\perp = P_\perp P_L = 0$, $P_L + P_\perp = 1$. With this truncation the momenta fields will be

$$\frac{\partial \mathcal{L}}{\partial \varphi^a_\mu} = (Z P_\perp + X P_L)^{ab} \varphi^b_\mu \equiv -i \pi^a_\mu \tag{3.93}$$

where we have defined $X(\rho) = Z(\rho) + 2\rho Y(\rho)$. Thanks to the properties of projectors we can readily invert the last relation and express the derivative field as a function of momenta

$$\varphi^a_\mu = -i \pi^b_\mu \left( \frac{P_\perp}{Z} + \frac{P_L}{X} \right)^{ba} \tag{3.94}$$

Substituting this inverse relation into the definition of the Hamiltonian we get

$$\mathcal{H} = V(\rho) + \frac{\sigma_\perp}{Z(\rho)} + \frac{\sigma_L}{X(\rho)} = \mathcal{H}(\rho, \sigma_L, \sigma_\perp) \tag{3.95}$$

where $\sigma_L = \frac{1}{2} P^{ab}_L \pi^a_\mu \pi^b_\mu$ and $\sigma_\perp = \frac{1}{2} P^{ab}_\perp \pi^a_\mu \pi^b_\mu = \sigma - \sigma_L$ with $\sigma = \frac{\pi^a_\mu \pi^a_\mu}{2}$.

*Proof.*

$$\partial_\mu \varphi^a \partial_\mu \varphi^a = \partial_\mu \varphi^a \delta^{ab} \partial_\mu \varphi^b = \varphi^a_\mu (P_L + P_\perp)^{ab} \varphi^b_\mu$$

$$\varphi^a \partial_\mu \varphi^a \varphi^b \partial_\mu \varphi^b = \frac{\varphi^a \varphi^b}{\varphi^2} \varphi^a_\mu \varphi^b_\mu \varphi^2 = P^{ab}_L 2\rho \varphi^a_\mu \varphi^b_\mu$$

$$
\begin{aligned}
\mathcal{H} =& i\pi^a_\mu \varphi^a_\mu + \mathcal{L} \\
=& i\pi^a_\mu(-i)\pi^b_\mu \left(\frac{P_L^{ab}}{X} + \frac{P_\perp^{ab}}{Z}\right) + V + \frac{1}{2}Z\varphi^a_\mu (P_L + P_\perp)^{ab}\varphi^b_\mu + \frac{1}{2}Y P_L^{ab} 2\rho \varphi^a_\mu \varphi^b_\mu \\
=& \pi^a_\mu \pi^b_\mu \left(\frac{P_L^{ab}}{X} + \frac{P_\perp^{ab}}{Z}\right) + V - \frac{1}{2}Z(P_L + P_\perp)^{ab} \left(\frac{P_\perp}{Z} + \frac{P_L}{X}\right)^{ac} \pi^c_\mu \left(\frac{P_\perp}{Z} + \frac{P_L}{X}\right)^{bd} \pi^d_\mu \\
& - \rho Y P_L^{ab} \left(\frac{P_\perp}{Z} + \frac{P_L}{X}\right)^{ac} \pi^c_\mu \left(\frac{P_\perp}{Z} + \frac{P_L}{X}\right)^{bd} \pi^d_\mu \\
=& V + \sigma_L \left(\frac{2}{X} - \frac{Z}{X^2} - \frac{2\rho Y}{X^2}\right) + \sigma_\perp \left(\frac{2}{Z} - \frac{1}{Z}\right) \\
=& V + \frac{\sigma_\perp}{Z} + \frac{\sigma_L}{X}
\end{aligned}
$$

□

Now we want to express the flow equation for $\mathcal{H}$ in terms of the new scalar variables $\rho, \sigma_L, \sigma_\perp$. Let's start from the first and second derivative of the Hamiltonian with respect the momenta field:

$$
\frac{\partial \mathcal{H}}{\partial \pi^a_\mu} = \left.\frac{\partial \mathcal{H}}{\partial \sigma_\perp}\right|_{\sigma_L} \frac{\partial \sigma_\perp}{\partial \pi^a_\mu} + \left.\frac{\partial \mathcal{H}}{\partial \sigma_L}\right|_{\sigma_\perp} \frac{\partial \sigma_L}{\partial \pi^a_\mu} = \left(\mathcal{H}'_\perp P_\perp + \mathcal{H}'_L P_L\right)^{ab} \pi^b_\mu \tag{3.96}
$$

$$
\frac{\partial^2 \mathcal{H}}{\partial \pi^a_\mu \partial \pi^b_\nu} = \left.\frac{\partial \mathcal{H}}{\partial \sigma_\perp}\right|_{\sigma_L} \frac{\partial^2 \sigma_\perp}{\partial \pi^a_\mu \partial \pi^b_\nu} + \left.\frac{\partial \mathcal{H}}{\partial \sigma_L}\right|_{\sigma_\perp} \frac{\partial^2 \sigma_L}{\partial \pi^a_\mu \partial \pi^b_\nu} = \delta^{\mu\nu} \left(\mathcal{H}'_\perp P_\perp + \mathcal{H}'_L P_L\right)^{ab} \tag{3.97}
$$

Using the projectors it is immediate to write down the inverse of the second derivative

$$
\left(\frac{\partial^2 \mathcal{H}}{\partial \pi^a_\mu \partial \pi^b_\nu}\right)^{-1} = \delta^{\mu\nu} \left(\frac{P_\perp}{\mathcal{H}'_\perp} + \frac{P_L}{\mathcal{H}'_L}\right)^{ab} \tag{3.98}
$$

Now let's consider derivatives involving scalar fields

$$
\begin{aligned}
\frac{\partial \mathcal{H}}{\partial \varphi^a} &= \left.\frac{\partial \mathcal{H}}{\partial \rho}\right|_{\sigma_L, \sigma_\perp} \frac{\partial \rho}{\partial \varphi^a} + \left.\frac{\partial \mathcal{H}}{\partial \sigma_L}\right|_{\rho, \sigma_\perp} \frac{\partial \sigma_L}{\partial \varphi^a} + \left.\frac{\partial \mathcal{H}}{\partial \sigma_\perp}\right|_{\rho, \sigma_L} \frac{\partial \sigma_L}{\partial \varphi^a} \\
&= \frac{\partial \mathcal{H}}{\partial \rho}\varphi^a + \frac{\partial \sigma_L}{\partial \varphi^a}\left(\frac{\partial \mathcal{H}}{\partial \sigma_L} - \frac{\partial \mathcal{H}}{\partial \sigma_\perp}\right) \\
&= \mathcal{H}'_\rho \varphi^a + (\mathcal{H}'_L - \mathcal{H}'_\perp)\left(\pi^a_\mu \frac{(\varphi \cdot \pi_\mu)}{2\rho} - \sigma_L \frac{\varphi^a}{\rho}\right)
\end{aligned} \tag{3.99}
$$

$$
\begin{aligned}
\frac{\partial^2 \mathcal{H}}{\partial \varphi^b \partial \pi^a_\mu} =& \frac{\partial \mathcal{H}}{\partial \sigma_\perp}\frac{\partial^2 \sigma_\perp}{\partial \varphi^b \partial \pi^a_\mu} + \frac{\partial \mathcal{H}}{\partial \sigma_L}\frac{\partial^2 \sigma_L}{\partial \varphi^b \partial \pi^a_\mu} + \frac{\partial \sigma_\perp}{\partial \pi^a_\mu}\left(\frac{\partial^2 \mathcal{H}}{\partial \rho \partial \sigma_\perp}\varphi^b\right) + \frac{\partial \sigma_L}{\partial \pi^a_\mu}\left(\frac{\partial^2 \mathcal{H}}{\partial \rho \partial \sigma_L}\varphi^b\right) \\
=& (\mathcal{H}'_L - \mathcal{H}'_\perp)\left[\frac{(\varphi \cdot \pi_\mu)}{2\rho}(P_\perp - P_L)^{ab} + \frac{\varphi^a \pi^b_\mu}{2\rho}\right] \\
&+ \varphi^b \pi^c_\mu \left(P_\perp \mathcal{H}''_{\rho\perp} + P_L \mathcal{H}''_{\rho L}\right)^{ac}
\end{aligned} \tag{3.100}
$$

$$\begin{aligned}\frac{\partial^2\mathcal{H}}{\partial\varphi^a\partial\varphi^b} =& \mathcal{H}'_\rho\delta^{ab} + \varphi^a\left(\mathcal{H}''_\rho\varphi^b + \mathcal{H}''_{\rho\perp}\frac{\partial\sigma_\perp}{\partial\varphi^b} + \mathcal{H}''_{\rho L}\frac{\partial\sigma_L}{\partial\varphi^b}\right)\\ &+ \mathcal{H}''_{\rho L}\varphi^b\frac{\partial\sigma_L}{\partial\varphi^a} + \mathcal{H}'_L\frac{\partial^2\sigma_L}{\partial\varphi^a\partial\varphi^b} + \mathcal{H}''_{\rho\perp}\varphi^b\frac{\partial\sigma_\perp}{\partial\varphi^a} + \mathcal{H}'_\perp\frac{\partial^2\sigma_\perp}{\partial\varphi^a\partial\varphi^b}\\ =& \delta^{ab}\mathcal{H}'_\rho + \mathcal{H}''_\rho 2\rho P_L^{ab}\\ &+ \left(\mathcal{H}''_{\rho L} - \mathcal{H}''_{\rho\perp}\right)\left[\frac{\varphi^a\pi^b_\mu}{2\rho}(\varphi\cdot\pi_\mu) + \frac{\varphi^b\pi^a_\mu}{2\rho}(\varphi\cdot\pi_\mu) - 4\sigma_L P_L^{ab}\right]\\ &+ (\mathcal{H}'_L - \mathcal{H}'_\perp)\left[\frac{\pi^a_\mu\pi^b_\mu}{2\rho} - \frac{\varphi^b\pi^a_\mu + \varphi^a\pi^b_\mu}{2\rho^2}(\varphi\cdot\pi_\mu) - \frac{\sigma_L}{\rho}(P_\perp^{ab} - 3P_L^{ab})\right]\end{aligned} \tag{3.101}$$

$$\begin{aligned}\frac{\partial^3\mathcal{H}}{\partial\varphi^a\partial\varphi^b\partial\varphi^c} =& \frac{\partial}{\partial\varphi^c}\left(\mathcal{H}'_\rho\delta^{ab} + \mathcal{H}''_\rho\varphi^a\varphi^b + \dots\right)\\ =& \mathcal{H}''_\rho\varphi^c\delta^{ab} + \mathcal{H}''_\rho\varphi^b\delta^{ac} + \mathcal{H}''_\rho\varphi^a\delta^{bc} + \mathcal{H}'''_\rho\varphi^a\varphi^b\varphi^c + \dots\end{aligned} \tag{3.102}$$

where we have left out all those terms that have quadratic momenta fields (the reason will be clear later on).

Now we have to substitute these derivatives into the flow equation for $\mathcal{H}$ and truncate all to the quadratic sector on the momenta $\pi^a_\mu$. Let's start from the first term of the eq.(3.89)

$$\frac{\partial\mathcal{H}}{\partial\varphi^a}\frac{\partial\mathcal{H}}{\partial\varphi^a} = \left\{\mathcal{H}'_\rho\varphi^a + (\mathcal{H}'_L - \mathcal{H}'_\perp)\left[\pi^a_\mu\frac{(\varphi\cdot\pi_\mu)}{2\rho} - \sigma_L\frac{\varphi^a}{\rho}\right]\right\}^2 \doteq 2\rho(\mathcal{H}'_\rho)^2 \tag{3.103}$$

where the symbol $\doteq$ stands for up to term more than quadratic in the momenta. In the second and third term we have to trace over the inner-space indices

$$\frac{\partial^2\mathcal{H}}{\partial\varphi^a\partial\varphi^a} = N\mathcal{H}'_\rho + 2\rho\mathcal{H}''_\rho + (\mathcal{H}'_L - \mathcal{H}'_\perp)\left(\frac{\sigma_L + \sigma_\perp}{\rho} - N\frac{\sigma_L}{\rho}\right) \tag{3.104}$$

$$\left(\frac{\partial^2\mathcal{H}}{\partial\pi^a_\mu\partial\pi^a_\mu}\right)^{-1} = d\left(\frac{N-1}{\mathcal{H}'_\perp} + \frac{1}{\mathcal{H}'_L}\right) \tag{3.105}$$

The forth term is

$$\begin{aligned}&\frac{\partial\mathcal{H}}{\partial\pi^a_\mu}\frac{\partial\mathcal{H}}{\partial\varphi^b}\frac{\partial^2\mathcal{H}}{\partial\varphi^a\partial\pi^c_\nu}\left(\frac{\partial^2\mathcal{H}}{\partial\pi^c_\nu\partial\pi^b_\mu}\right)^{-1} = \left(\mathcal{H}'_\perp P_\perp^{ab}\pi^b_\mu + \mathcal{H}'_L P_L^{ab}\pi^b_\mu\right)\cdot\\ &\cdot\left[\mathcal{H}'_\rho\varphi^b + (\mathcal{H}'_L - \mathcal{H}'_\perp)\left(\pi^b_\mu\frac{(\varphi\cdot\pi_\mu)}{2\rho} - \sigma_L\frac{\varphi^b}{\rho}\right)\right]\cdot\\ &\cdot\left[(\mathcal{H}'_L - \mathcal{H}'_\perp)\left(\frac{(\varphi\cdot\pi_\nu)}{2\rho}(P_\perp - P_L)^{ac} + \frac{\varphi^c\pi^a_\nu}{2\rho}\right) + \varphi^a\pi^d_\nu\left(P_\perp\mathcal{H}''_{\rho\perp} + P_L\mathcal{H}''_{\rho L}\right)^{dc}\right]\cdot\\ &\cdot\delta^{\mu\nu}\left(\frac{P_\perp}{\mathcal{H}'_\perp} + \frac{P_L}{\mathcal{H}'_L}\right)^{cb} \doteq\\ &\doteq (\mathcal{H}'_L - \mathcal{H}'_\perp)\,2\sigma_\perp\frac{\mathcal{H}'_\perp\mathcal{H}'_\rho}{\mathcal{H}'_L} + 4\rho\sigma_L\mathcal{H}'_L\mathcal{H}'_\rho\mathcal{H}''_{\rho L}\end{aligned} \tag{3.106}$$

The fifth term is

$$\begin{aligned}
&\frac{\partial \mathcal{H}}{\partial \pi^a_\mu}\pi^b_\mu \frac{\partial^2 \mathcal{H}}{\partial \varphi^a \partial \varphi^b} = \left(\mathcal{H}'_\perp P_\perp + \mathcal{H}'_L P_L\right)^{ac} \pi^c_\mu \pi^b_\mu \cdot \\
&\cdot \Bigg[\delta^{ab}\mathcal{H}'_\rho + \mathcal{H}''_\rho 2\rho P^{ab}_L + \left(\mathcal{H}''_{\rho L} - \mathcal{H}''_{\rho\perp}\right)\left(\frac{\varphi^a \pi^b_\mu}{2\rho}(\varphi\cdot\pi_\mu) + \frac{\varphi^b\pi^a_\mu}{2\rho}(\varphi\cdot\pi_\mu) - 4\sigma_L P^{ab}_L\right) \\
&+ \left(\mathcal{H}'_L - \mathcal{H}'_\perp\right)\left(\frac{\pi^a_\mu\pi^b_\mu}{2\rho} - \frac{\varphi^b\pi^a_\mu + \varphi^a\pi^b_\mu}{2\rho^2}(\varphi\cdot\pi_\mu) - \frac{\sigma_L}{\rho}(P^{ab}_\perp - 3P^{ab}_L)\right)\Bigg] \doteq \\
&\doteq 2\sigma_L\left(\mathcal{H}'_\rho\mathcal{H}'_L + 2\rho\mathcal{H}''_\rho\mathcal{H}'_L\right) + 2\sigma_\perp\mathcal{H}'_\rho\mathcal{H}'_\perp
\end{aligned} \tag{3.107}$$

Finally the last term is

$$\begin{aligned}
&\frac{\partial \mathcal{H}}{\partial \pi^a_\mu}\frac{\partial \mathcal{H}}{\partial \pi^c_\mu}\frac{\partial \mathcal{H}}{\partial \varphi^b}\frac{\partial^3 \mathcal{H}}{\partial\varphi^a\partial\varphi^b\partial\varphi^c} = \\
&= \left(\mathcal{H}'_\perp P_\perp + \mathcal{H}'_L P_L\right)^{ad}\pi^d_\mu \left(\mathcal{H}'_\perp P_\perp + \mathcal{H}'_L P_L\right)^{ce}\pi^e_\mu \cdot \frac{\partial \mathcal{H}}{\partial \varphi^b}\frac{\partial^3 \mathcal{H}}{\partial\varphi^a\partial\varphi^b\partial\varphi^c} \\
&\doteq 12\rho\sigma_L\mathcal{H}'_\rho(\mathcal{H}'_L)^2\mathcal{H}''_\rho + 8\rho^2\sigma_L\mathcal{H}'_\rho(\mathcal{H}'_L)^2\mathcal{H}'''_\rho + 4\rho\mathcal{H}'_\rho\mathcal{H}''_\rho\sigma_\perp(\mathcal{H}'_\perp)^2
\end{aligned} \tag{3.108}$$

and the corresponding one apart from a boundary term is

$$\begin{aligned}
&-\frac{\partial \mathcal{H}}{\partial \pi^a_\mu}\frac{\partial \mathcal{H}}{\partial \pi^c_\mu}\frac{\partial^2 \mathcal{H}}{\partial\varphi^a\partial\varphi^b}\frac{\partial^2 \mathcal{H}}{\partial\varphi^b\partial\varphi^c} = \\
&= -\left(\mathcal{H}'_\perp P_\perp + \mathcal{H}'_L P_L\right)^{ad}\pi^d_\mu \left(\mathcal{H}'_\perp P_\perp + \mathcal{H}'_L P_L\right)^{ce}\pi^e_\mu \cdot \\
&\quad \cdot (\mathcal{H}'_\rho\delta^{ab} + \mathcal{H}''_\rho 2\rho P^{ab}_L + \dots)(\mathcal{H}'_\rho\delta^{bc} + \mathcal{H}''_\rho 2\rho P^{bc}_L + \dots) \\
&\doteq -(\mathcal{H}'_\rho)^2\left(2\sigma_\perp(\mathcal{H}'_\perp)^2 + 2\sigma_L(\mathcal{H}'_L)^2\right) - 8\rho\sigma_L(\mathcal{H}'_L)^2\mathcal{H}''_\rho(\mathcal{H}'_\rho + \rho\mathcal{H}''_\rho)
\end{aligned} \tag{3.109}$$

We can finally collect all these intermediate results and rewrite down the flow equation for the Hamiltonian density within the truncation for a quadratic functional of the momenta:

$$\begin{aligned}
\Lambda^{-d}\dot{\mathcal{H}} =& \Lambda^{-2+\eta}I_0\left\{N\mathcal{H}'_\rho + 2\rho\mathcal{H}''_\rho + (\mathcal{H}'_L - \mathcal{H}'_\perp)\left(\frac{\sigma_L+\sigma_\perp}{\rho} - N\frac{\sigma_L}{\rho}\right)\right\} \\
&+ \Lambda^\eta I_1\left(\frac{N-1}{\mathcal{H}'_\perp} + \frac{1}{\mathcal{H}'_L}\right) \\
&+ \Lambda^{-d+\eta-2}K_0\Bigg\{-2\rho(\mathcal{H}'_\rho)^2 + (\mathcal{H}'_L - \mathcal{H}'_\perp)\,2\sigma_\perp\frac{\mathcal{H}'_\perp\mathcal{H}'_\rho}{\mathcal{H}'_L} + 4\rho\sigma_L\mathcal{H}'_\rho\mathcal{H}''_{\rho L} \\
&\qquad + 2\sigma_L\left(\mathcal{H}'_\rho\mathcal{H}'_L + 2\rho\mathcal{H}''_\rho\mathcal{H}'_L\right) + 2\sigma_\perp\mathcal{H}'_\rho\mathcal{H}'_\perp\Bigg\} \\
&+ \Lambda^{-d+\eta-4}K_1\left\{12\rho\sigma_L\mathcal{H}'_\rho(\mathcal{H}'_L)^2\mathcal{H}''_\rho + 8\rho^2\sigma_L\mathcal{H}'_\rho(\mathcal{H}'_L)^2\mathcal{H}'''_\rho + 4\rho\mathcal{H}'_\rho\mathcal{H}''_\rho\sigma_\perp(\mathcal{H}'_\perp)^2\right\}
\end{aligned} \tag{3.110}$$

where the last term can be also written as

$$-\Lambda^{-d+\eta-4}K_1\left\{(\mathcal{H}'_\rho)^2\left[2\sigma_\perp(\mathcal{H}'_\perp)^2 + 2\sigma_L(\mathcal{H}'_L)^2\right] + 8\rho\sigma_L(\mathcal{H}'_L)^2\mathcal{H}''_\rho(\mathcal{H}'_\rho + \rho\mathcal{H}''_\rho)\right\} \tag{3.111}$$

Suppose now to rescale $\mathcal{H}$, $\phi$, $\sigma$ in this way: $\mathcal{H} \to a\mathcal{H}$, $\phi \to b\phi$, $\sigma \to c\sigma$. Then we have

$$\Lambda^{-d} a\dot{\mathcal{H}} = \Lambda^{-2+\eta}\frac{I_0 a}{b}\left\{N\mathcal{H}'_\rho + 2\rho\mathcal{H}''_\rho + (\mathcal{H}'_L - \mathcal{H}'_\perp)\left(\frac{\sigma_L+\sigma_\perp}{\rho} - N\frac{\sigma_L}{\rho}\right)\right\}$$
$$+\Lambda^{\eta}\frac{I_1 c}{a}\left(\frac{N-1}{\mathcal{H}'_\perp} + \frac{1}{\mathcal{H}'_L}\right)$$
$$+\Lambda^{-d+\eta-2}\frac{K_0 a^2}{b}\left\{-2\rho(\mathcal{H}'_\rho)^2 + (\mathcal{H}'_L - \mathcal{H}'_\perp)\,2\sigma_\perp\frac{\mathcal{H}'_\perp\mathcal{H}'_\rho}{\mathcal{H}'_L} + 4\rho\sigma_L\mathcal{H}'_\rho\mathcal{H}''_{\rho L}\right.$$
$$\left.+2\sigma_L\left(\mathcal{H}'_\rho\mathcal{H}'_L + 2\rho\mathcal{H}''_\rho\mathcal{H}'_L\right) + 2\sigma_\perp\mathcal{H}'_\rho\mathcal{H}'_\perp\right\}$$
$$+\Lambda^{-d+\eta-4}\frac{K_1 a^4}{b^2 c}\left\{12\rho\sigma_L\mathcal{H}'_\rho(\mathcal{H}'_L)^2\mathcal{H}''_\rho + 8\rho^2\sigma_L\mathcal{H}'_\rho(\mathcal{H}'_L)^2\mathcal{H}'''_\rho + 4\rho\mathcal{H}'_\rho\mathcal{H}''_\rho\sigma_\perp(\mathcal{H}'_\perp)^2\right\} \tag{3.112}$$

and it is easy to see that we can absorb the three regulator dependent parameters $K_0, I_0, I_1$ but not the parameter $K_1$ in the last term. In details, the appropriate choice for these parameters is:

$$a = \frac{I_0}{K_0}, \qquad b = I_0, \qquad c = \frac{I_0^2}{K_0^2 I_1}. \tag{3.113}$$

whereas the last term will remain regulator dependent

$$\Lambda^{-d+\eta-4}\frac{K_1 I_1}{K_0 I_0}\left\{12\rho\sigma_L\mathcal{H}'_\rho(\mathcal{H}'_L)^2\mathcal{H}''_\rho + 8\rho^2\sigma_L\mathcal{H}'_\rho(\mathcal{H}'_L)^2\mathcal{H}'''_\rho + 4\rho\mathcal{H}'_\rho\mathcal{H}''_\rho\sigma_\perp(\mathcal{H}'_\perp)^2\right\}$$
$$\doteq -\Lambda^{-d+\eta-4}\frac{K_1 I_1}{K_0 I_0}\left\{(\mathcal{H}'_\rho)^2\left[2\sigma_\perp(\mathcal{H}'_\perp)^2 + 2\sigma_L(\mathcal{H}'_L)^2\right] + 8\rho\sigma_L(\mathcal{H}'_L)^2\mathcal{H}''_\rho(\mathcal{H}'_\rho + \rho\mathcal{H}''_\rho)\right\} \tag{3.114}$$

### 3.3.3 Rescaling and RG equations for $\tilde{V}(\tilde{\rho})$, $\tilde{Z}(\tilde{\rho})$ and $\tilde{Y}(\tilde{\rho})$

At criticality the fields $\rho$ and $\sigma = \sigma_L + \sigma_\perp$ scale with $\Lambda$ according to their full quantum dimension that we call $D_\rho$ and $D_\sigma$ respectively. The full dimensionality of $\pi^a_\mu$ is such that the term $\pi^a_\mu\phi^a_\mu$ in the definition of the Hamiltonian should have the same dimension as $\mathcal{H}$, in other words it should scale with $\Lambda^d$. For this reason $D_\pi = d - D_\phi - 1$ so, knowing that $D_\phi = (d-2+\eta)/2$ we get $D_\pi = (d-\eta)/2$. The rescaling we have to do is:

$$\mathcal{H} = \Lambda^d\tilde{\mathcal{H}}, \qquad \rho = \Lambda^{2D_\phi}\tilde{\rho}, \qquad \sigma = \Lambda^{2D_\pi}\tilde{\sigma} \tag{3.115}$$

where with tilde we denote dimensionless Hamiltonian and fields. Now we have to start from this equality

$$\tilde{H}(\tilde{\rho}, \tilde{\sigma}_L, \tilde{\sigma}_\perp) = \Lambda^{-d}\mathcal{H}(\rho, \sigma_L, \sigma_\perp) \tag{3.116}$$

and apply a total derivative with respect the RG time on both side of this equality. It is not hard to believe that the implicit time derivative of the Hamiltonian is

$$\partial_t\big|_{\tilde{\rho},\tilde{\sigma}_L,\tilde{\sigma}_\perp}\tilde{\mathcal{H}} = -(d-2+\eta)\tilde{\rho}\tilde{\mathcal{H}}'_{\tilde{\rho}} - (d-\eta)(\tilde{\sigma}_L\tilde{\mathcal{H}}'_{\tilde{\sigma}_L} + \tilde{\sigma}_\perp\tilde{\mathcal{H}}'_{\tilde{\sigma}_\perp}) + d\tilde{\mathcal{H}} + \Lambda^{-d}\dot{\mathcal{H}} \tag{3.117}$$

where the last term is exactly the eq.(3.112) that must be expressed as a function of the new dimensionless variables. To sum up the final result will be (dropping the tilde for sake of brevity):

$$\begin{aligned}
\partial_t\big|_{\phi,\sigma_L,\sigma_\perp}\mathcal{H} =& -(d-2+\eta)\rho\mathcal{H}'_\rho-(d-\eta)(\sigma_L\mathcal{H}'_L+\sigma_\perp\mathcal{H}'_\perp)+d\mathcal{H}\\
&+N\mathcal{H}'_\rho+2\rho\mathcal{H}''_\rho+(\mathcal{H}'_L-\mathcal{H}'_\perp)\left(\frac{\sigma_L+\sigma_\perp}{\rho}-N\frac{\sigma_L}{\rho}\right)+\frac{N-1}{\mathcal{H}'_\perp}+\frac{1}{\mathcal{H}'_L}\\
&-2\rho(\mathcal{H}'_\rho)^2+(\mathcal{H}'_L-\mathcal{H}'_\perp)\,2\sigma_\perp\frac{\mathcal{H}'_\perp\mathcal{H}'_\rho}{\mathcal{H}'_L}+4\rho\sigma_L\mathcal{H}'_\rho\mathcal{H}''_{\rho L}\\
&\qquad+2\sigma_L\left(\mathcal{H}'_\rho\mathcal{H}'_L+2\rho\mathcal{H}''_\rho\mathcal{H}'_L\right)+2\sigma_\perp\mathcal{H}'_\rho\mathcal{H}'_\perp\\
&+\frac{K_1I_1}{K_0I_0}\left\{12\rho\sigma_L\mathcal{H}'_\rho(\mathcal{H}'_L)^2\mathcal{H}''_\rho+8\rho^2\sigma_L\mathcal{H}'_\rho(\mathcal{H}'_L)^2\mathcal{H}'''_\rho+4\rho\mathcal{H}'_\rho\mathcal{H}''_\rho\sigma_\perp(\mathcal{H}'_\perp)^2\right\}
\end{aligned}\tag{3.118}$$

where the last term can be expressed also as

$$-\frac{K_1I_1}{K_0I_0}\left\{(\mathcal{H}'_\rho)^2\left[2\sigma_\perp(\mathcal{H}'_\perp)^2+2\sigma_L(\mathcal{H}'_L)^2\right]+8\rho\sigma_L(\mathcal{H}'_L)^2\mathcal{H}''_\rho(\mathcal{H}'_\rho+\rho\mathcal{H}''_\rho)\right\}\tag{3.119}$$

Finally it is important to note that if one choose a family of cutoff functions such that $K_1=0$ then the equation will become regulator independent.

The left hand side of this equation is the RG ‘’time” derivative acting only on the intrinsic RG time dependence which reads:

$$\dot{\mathcal{H}}=\dot V-\frac{\dot Z}{Z^2}\sigma_\perp-\frac{\dot X}{X^2}\sigma_L\tag{3.120}$$

where $\dot X=\dot Z+2\rho\dot Y$. Projecting the right hand side of eq.(3.118) on the ansatz

$$\mathcal{H}(\rho,\sigma_\perp,\sigma_L)=V(\rho)+\frac{\sigma_\perp}{Z(\rho)}+\frac{\sigma_L}{X(\rho)}\tag{3.121}$$

and using these relations

$$\begin{aligned}
\mathcal{H}'_\rho&=V'-\frac{Z'}{Z^2}\sigma_\perp-\frac{X'}{X^2}\sigma_L\\
\mathcal{H}''_\rho&=V''+\sigma_\perp\left(-\frac{Z''}{Z^2}-2\frac{(Z')^2}{Z^3}\right)+\sigma_L\left(-\frac{X''}{X^2}-2\frac{(X')^2}{X^3}\right)\\
\mathcal{H}'''_\rho&=V'''+\sigma_L\left(-\frac{X'''}{X^2}-6\frac{(X')^3}{X^4}+6\frac{X'X''}{X^3}\right)+\sigma_\perp\left(-\frac{Z^3}{Z^2}-6\frac{(Z')^3}{Z^4}+6\frac{Z'Z''}{Z^3}\right)\\
\mathcal{H}'_\perp&=Z^{-1}\\
\mathcal{H}'_L&=X^{-1}\\
\mathcal{H}''_{L\rho}&=-\frac{X'}{X^2}
\end{aligned}\tag{3.122}$$

we obtain the final result

$$\dot V=dV-\rho(d+\eta-2)V'+2\rho V''+NV'-2\rho(V')^2+2\rho Y+NZ\tag{3.123}$$

$$\begin{aligned}\dot{Z} =& - 2V'(Z - 2\rho Y) + \rho Z'(2 - d - \eta) - 4\rho V'Z' - \eta Z + NZ' \\ &- 4\rho\frac{(Z')^2}{Z} + \frac{2YZ}{Z + 2\rho Y} + 2\rho Z'' - 4b\rho V'V''\end{aligned} \tag{3.124}$$

$$\begin{aligned}\dot{Y} =& - 2V''(Z - 2\rho Y) - 8\frac{\rho V''YZ}{Z + 2\rho Y} - 2N\frac{Y^2}{Z} + \rho Y'(2 - d - \eta) + Y(2 - d - 2\eta) \\ &- 4YV' + 2V'Z' + NY' + 2\rho Y'' - 8\rho Y'\frac{Z' + \rho Y'}{Z + 2\rho Y} + 4Y'\frac{Z - 2\rho Y}{Z + 2\rho Y} \\ &+ 4YZ'\frac{\rho Z' - 2Z}{Z(Z + 2\rho Y)} - 4Y^2\frac{Z - \rho Y}{Z(Z + 2\rho Y)} - 4bV'(V'' + \rho V''')\end{aligned} \tag{3.125}$$

where the coefficient $b$ is the regulator dependent coefficient that comes from the last term of the RG equation for the Hamiltonian *i.e.* $b = \frac{K_1 I_1}{K_0 I_0}$. Instead if we take under our consideration the flow different from this one apart from the boundary term, we gain in this case

$$\dot{V} = dV - \rho(d + \eta - 2)V' + 2\rho V'' + NV' - 2\rho(V')^2 + 2\rho Y + NZ \tag{3.126}$$

$$\begin{aligned}\dot{Z} =& - 2V'(Z - 2\rho Y) + \rho Z'(2 - d - \eta) - 4\rho V'Z' - \eta Z + NZ' \\ &- 4\rho\frac{(Z')^2}{Z} + \frac{2YZ}{Z + 2\rho Y} + 2\rho Z'' + 2b(V')^2\end{aligned} \tag{3.127}$$

$$\begin{aligned}\dot{Y} =& - 2V''(Z - 2\rho Y) - 8\frac{\rho V''YZ}{Z + 2\rho Y} - 2N\frac{Y^2}{Z} + \rho Y'(2 - d - \eta) + Y(2 - d - 2\eta) \\ &- 4YV' + 2V'Z' + NY' + 2\rho Y'' - 8\rho Y'\frac{Z' + \rho Y'}{Z + 2\rho Y} + 4Y'\frac{Z - 2\rho Y}{Z + 2\rho Y} \\ &+ 4YZ'\frac{\rho Z' - 2Z}{Z(Z + 2\rho Y)} - 4Y^2\frac{Z - \rho Y}{Z(Z + 2\rho Y)} + 4bV''(V' + \rho V'')\end{aligned} \tag{3.128}$$

### 3.3.4 Large $N$ limit

We are interested now in making some consideration for the large $N$ limit case, when essentially only the transverse modes are involved in the dynamic of the system. In $d = 3$ this problem has been first studied, with a power-law cutoff and for the the average effective action (Wetterich's equations), in [20]. From these earlier studies we know that the contribution of $Y_\Lambda$ to the running of $V_\Lambda$ and $Z_\Lambda$ is of the order $1/N$, so that it can be neglected in the large $N$ limit. Moreover, in the same limit, the only known solutions of the fixed point equations for $V$ and $Z$ have a field-independent $Z$ with anomalous dimension $\eta = 0$. No other solutions, with a non vanishing $\eta$ at $N = \infty$, are known up to now.

For this reason it is interesting to study the $N = \infty$ limit of the Wilsonian action, which is different from the generator of the average proper vertices, within our new formalism and see if we can obtain different results. In order to perform the limit it is convenient to do the following rescaling

$$\rho = N\tilde{\rho}, \qquad V = N\tilde{V}, \qquad Y = \frac{1}{N}\tilde{Y}. \tag{3.129}$$

and substituting them into the flow equation and dropping all sub-leading terms in the large $N$ limit we obtain

$$\dot{\tilde{V}} = d\tilde{V} + \tilde{Z} + \tilde{V}' + (2-d-\eta)\tilde{\rho}\tilde{V}' - 2\tilde{\rho}(\tilde{V}')^2 \tag{3.130}$$

$$\dot{\tilde{Z}} = \tilde{\rho}\tilde{Z}'(2-d-\eta) - 4\tilde{\rho}\tilde{V}'\tilde{Z}' - \eta\tilde{Z} + \tilde{Z}' - 2\tilde{V}'(\tilde{Z} - 2\tilde{\rho}\tilde{Y}) - 4b\tilde{\rho}\tilde{V}'\tilde{V}'' \tag{3.131}$$

$$\begin{aligned}\dot{\tilde{Y}} =& \tilde{\rho}\tilde{Y}'(2-d-\eta) + \tilde{Y}(2-d-2\eta) - 2\frac{\tilde{Y}^2}{\tilde{Z}} + \tilde{Y}' + 2\tilde{V}'\tilde{Z}' - 4\tilde{Y}\tilde{V}' \\ & - 2\tilde{V}''(\tilde{Z} - 2\tilde{\rho}\tilde{Y}) - 8\frac{\tilde{\rho}\tilde{V}''\tilde{Y}\tilde{Z}}{\tilde{Z} + 2\tilde{\rho}\tilde{Y}} - 4b\tilde{V}'(\tilde{V}'' + \tilde{\rho}\tilde{V}''')\end{aligned} \tag{3.132}$$

A first important observation is that the equations for $\tilde{V}$ and $\tilde{Z}$ do not decouple from that one for $\tilde{Y}$.

### Some comments

Suppose now to be in $d = 3$ and in the special case of $b = 0$. We want to ask if there is a quadratic solution for the potential, therefore we make this ansatz

$$\tilde{V}(\rho) = v_0 + v_1\rho \tag{3.133}$$

Substituting this into the eq.(3.130), at the fixed point we have

$$0 = 3(v_0 + v_1\rho) + \tilde{Z} + v_1 - (1+\eta)\rho v_1 - 2\rho v_1^2$$

therefore there are two possible solutions if $Z$=const

$$\begin{cases} v_1 = 0, \vee\, v_1 = 1 - \frac{\eta}{2} \\ v_0 = -\frac{Z+v_1}{3} \end{cases}$$

the eq.(3.131) at the fixed point gives

$$0 = -\eta\tilde{Z} - 2v_1(\tilde{Z} - 2\tilde{\rho}\tilde{Y})$$

that implies

$$\begin{cases} v_1 = 0 \implies \tilde{Z} = 0 \vee \eta = 0 \\ v_1 \neq 0 \implies \tilde{Z} = \tilde{\rho}\tilde{Y}\frac{4v_1}{\eta + 2v_1} \implies \tilde{Y} = \frac{a}{\tilde{\rho}} \end{cases}$$

If we take $\tilde{Y} = a/\rho$ the eq.(3.132) at the fixed point gives

$$0 = (1+\eta)\frac{a}{\tilde{\rho}} - \frac{a}{\tilde{\rho}}(1+2\eta) - \frac{2a^2}{\tilde{\rho}^2\tilde{Z}} - \frac{a}{\tilde{\rho}^2} - 4v_1\frac{a}{\tilde{\rho}}$$

that implies

$$a = -\frac{\tilde{Z}}{2} \qquad \wedge \qquad \begin{cases} a = 0 \\ v_1 = -\frac{\eta}{4} \end{cases}$$

instead, if we take $v_1 = 0$ the fixed point equation for $\tilde{Y}$ becomes

$$\tilde{Y}(2\tilde{Y} + \tilde{Z}) = \tilde{Y}'\tilde{Z}(1-\tilde{\rho}) \qquad \implies \qquad \tilde{Y} = -\frac{\alpha\tilde{Z}}{2(\alpha - \tilde{Z} + \tilde{\rho}\tilde{Z})}$$

where $\alpha$ is a constant of integration.

To sum up we have three possible solutions if we make the assumption of a quadratic potential:

- $\tilde{V} = (1 - \frac{\eta}{2})(\rho - \frac{1}{3})$ and $\tilde{Z} = \tilde{Y} = 0$ that is non physical

- $\eta = 0$, $\tilde{V} = v_0 = -\frac{\tilde{Z}}{3}$, $\tilde{Y} = -\frac{\alpha\tilde{Z}}{2(\alpha - \tilde{Z} + \tilde{\rho}\tilde{Z})}$ that may be possible

- $\eta = 4$, $\tilde{V} = \frac{2a+1}{3} - \rho$, $\tilde{Z} = -2a$ and $\tilde{Y} = \frac{a}{\tilde{\rho}}$ that is again non physical.

# Chapter 4

# Numerical analysis for $N = 1$ in $d = 3$

In this chapter we want to give a detailed description of a numerical analysis that we have done for the case of one real scalar field in $d = 3$. The model corresponds to the continuum QFT description of the critical Ising model for which some accurate results for the critical exponents are known from numerical Monte Carlo analysis [6], many loop perturbative computations, from conformal bootstrap approach [27] and also from many functional RG analysis (see for example [3][22][13]). We chose the method of shooting from large field value [20]. To this end we have to find the leading asymptotic solutions for the fixed point equations, that will be parametrized by $\eta$ itself and other parameters that govern the asymptotic behavior of solutions, and make some checks also from the origin. Successively we will follow this strategy: because of the $\mathbb{Z}_2$ symmetry we know from the principle that we must find an even solution with zero first derivative at the origin; for this reason, if we have two extra parameters $A, B$ parametrizing the asymptotic solutions, we will than plot a discrete set of point in the $(V'(0), A)$-plain and $(Z'(0), B)$-plain for different value of $\eta$. If there are solutions for some value of the anomalous dimension, they should correspond to a couple of value $(A_{\rm fp}, B_{\rm fp})$ for which the first derivatives at the origin can set to zero at arbitrary high precision.

We have studied eqs. (3.65),(3.66) and (3.67) that are two coupled second order non linear differential equations. At a first time we have set the parameter $b = 0$, in this way we have two equations that do not depend on the particular choice of the cutoff function, even if a choice is actually made: all those regulator functions $K(q^2/\Lambda^2)$ such that $K_1 = 0$ are possible choices. In other words we have to impose the condition

$$J_{\mu\nu,0} = 0 \implies \int d^3z z^2 \dot{C}(z^2) = 0 \tag{4.1}$$

In this case we will see that the only solutions obtained from the numerical analysis have a negative wave function renormalization corresponding to a ghost field. We will conclude that for $b = 0$ there isn't any interesting physical solutions. Nevertheless we want to start from this case because it is easier than the case with $b \neq 0$. We shall then analyze the case where the cutoff is more generic leading to $b > 0$. In this case we found that physical scaling solutions with positive $Z(\phi)$ exist if the parameter $b$ lies in the range $0 < b \leq b_{\rm max} \sim 3$ and that the variation of the anomalous dimension in this range presents a minimum: this property is in agreement with the principle of minimum sensitivity.

## 4.1 Asymptotic expansions for $b = 0$

To find the asymptotic expansion for $V$ and $Z$ we have to find first the leading term as some power in the field $\phi$ and than take corrections to this one adding powers in the field that are sub-leading. At each steps we have to set the coefficient of the higher power equal to zero, this implies an algebraic equation for the coefficient of the $i$-th correction.

Let's take a first look at the fixed point equations

$$0 =dV - \frac{1}{2}(d-2+\eta)\phi V' - (V')^2 + V'' + Z \tag{4.2}$$

$$0 =-\eta Z - \frac{1}{2}(d-2+\eta)\phi Z' - 2ZV'' + Z'' - 2\frac{(Z')^2}{Z} \tag{4.3}$$

for large field value we can safely say that $V \gg Z$ and taking the ansatz $V_{as} = A_0\phi^{\alpha_0}$ also the second derivative can be neglected. We obtain the equation

$$0 = dA_0\phi^{\alpha_0} - \frac{d-2+\eta}{2}\alpha_0 A_0\phi^{\alpha_0} - A_0^2\alpha_0^2\phi^{2\alpha_0-2}$$

To avoid the square of the first derivative having a power bigger than $V$ we must impose the condition

$$\alpha_0 = 2\alpha_0 - 2 \implies \alpha_0 = 2$$

and the relative coefficient is a second order algebraic equation that has to be solved in $A_0$

$$dA_0 - \frac{d-2+\eta}{2}\alpha_0 A_0 - \alpha_0^2 A_0^2 = 0 \implies A_0 = \frac{1}{2}\left(1 - \frac{\eta}{2}\right)$$

Therefore the leading behavior of the potential is

$$V_{as}(\phi) = \frac{1}{2}\left(1 - \frac{\eta}{2}\right)\phi^2 + \dots \tag{4.4}$$

Consider now the fixed point equation for $Y = 1/Z$

$$0 = -\eta Y + \frac{d-2+\eta}{2}\phi Y' - 2V''Y - Y'' \tag{4.5}$$

and taking into account the leading term for $V_{as}(\phi)$ and the ansatz $Y_{as}(\phi) = B\phi^\beta$ we have

$$0 = -\eta B\phi^\beta + \frac{d-2+\eta}{2}B\beta\phi^\beta - 2\left(1 - \frac{\eta}{2}\right)B\phi^\beta - B\beta(\beta-1)\phi^{\beta-2}$$

The last term is sub-leading so we remain with the condition that the coefficient of $\phi^\beta$ must be zero and this fixes the leading power behavior for $Y_{as}(\phi)$

$$0 = -\eta B + \frac{d-2+\eta}{2}B\beta - 2\left(1 - \frac{\eta}{2}\right)B \implies \beta = \frac{4}{d-2+\eta}$$

therefore we have

$$\begin{aligned} Y_{as}(\phi) &= B\phi^{\frac{4}{d-2+\eta}} + \dots \\ Z_{as}(\phi) &= \frac{1}{B}\phi^{-\frac{4}{d-2+\eta}} + \dots \end{aligned} \tag{4.6}$$

### 4.1.1 First two corrections for $V_{as}(\phi)$

Now consider the first correction for the potential

$$V_{as} = \frac{1}{2}\left(1 - \frac{\eta}{2}\right)\phi^2 + A\phi^\alpha$$

the leading terms in eq.(4.2) will be those of power $\phi^\alpha$ because $\alpha < 2 \implies 2\alpha - 2 < \alpha$

$$0 = \phi^\alpha \left[dA - A\frac{d-2+\eta}{2} - 2A\alpha\left(1 - \frac{\eta}{2}\right)\right] - A^2\alpha^2\phi^{2\alpha-2} + A\alpha(\alpha-1)\phi^{\alpha-2} + 1 - \frac{\eta}{2}$$

and setting the coefficient of $\phi^\alpha$ equal to zero we find $\alpha = \frac{2d}{d+2-\eta}$ thus we have

$$V_{as}(\phi) = \frac{1}{2}\left(1 - \frac{\eta}{2}\right)\phi^2 + A\phi^{\frac{2d}{d+2-\eta}} + \dots \tag{4.7}$$

where $A$ is the free parameter that governs the behavior of $V$ for large field values. Setting $d = 3$ and $\eta = 0.03$ we see that $\alpha \simeq 1.28$ and $\beta \simeq 3.08$ so till we reach the power $\phi^{-\beta}$ in the expansion for the potential we can neglect the presence of $Z$ in eq.(4.2).

Now let's take a bit of attention for the second correction

$$V_{as} = A_0\phi^2 + A\phi^\alpha + A_1\phi^{\alpha_1}$$

Substituting this into the previous equation, the survived terms are

$$\begin{aligned}0 =& da_1\phi^{\alpha_1} - \frac{d-2+\eta}{2}a_1\alpha_1\phi^{\alpha_1} - A^2\alpha^2\phi^{2\alpha-2} - \alpha_1^2 a_1^2\phi^{2\alpha_1-2} - 4A_0a_1\alpha_1\phi^{\alpha_1} - 2A\alpha\alpha_1\phi^{\alpha+\alpha_1-2}\\ &+ 1 - \frac{\eta}{2} + A\alpha(\alpha-1)\phi^{\alpha-2} + a_1\alpha_1(\alpha_1-1)\phi^{\alpha_1-2}\end{aligned}$$

and the maximum exponent must be set equal to $\alpha_1$. Therefore

$$\alpha_1 = \max\{2\alpha-2, \alpha-2, 0\} = 2\alpha - 2 = 2\frac{d-2+\eta}{d+2-\eta}$$

and setting equal to zero the relative coefficient we can solve the algebraic equation for $A_1$ as a function of $d$ and $\eta$

$$A_1 = \frac{4A^2d^2}{(2-\eta)(d+2-\eta)^2}$$

This is the philosophy to find all powers and coefficients for the asymptotic expansion. Iterating this procedure we arrive to the final form

$$V_{as}(\phi) = \frac{1}{2}\left(1 - \frac{\eta}{2}\right)\phi^2 + A\phi^{\frac{2d}{d+2-\eta}} + \sum_{i=1}^{13} A_i(A,d,\eta)\phi^{\alpha_i(A,d,\eta)} + \sum_{i=14}^{18} A_i(A,B,d,\eta)\phi^{\alpha_i(A,B,d,\eta)} \tag{4.8}$$

where we have emphasized that till the 13-th term there isn't any contribution from $Z_{as}(\phi)$ so powers and coefficients do not depend on the free parameter $B$. See the full expansion in the Appendix 4.4.1.

### 4.1.2 First two corrections for $Y_{as}(\phi)$

Now consider the first correction for the inverse of the wave function renormalization

$$Y_{as} = B\phi^{\frac{4}{d-2+\eta}} + B_1\phi^{\beta_1}$$

and substitute this into eq.(4.5) that gives

$$\begin{aligned} 0 = &- \eta B_1\phi^{\beta_1} + \frac{d-2+\eta}{2}\beta_1 B_1\phi^{\beta_1} - \beta(\beta-1)B\phi^{\beta-2} - \beta_1(\beta_1-1)B_1\phi^{\beta_1-2} \\ &- 2B\phi^{\beta}\left[A\alpha(\alpha-1)\phi^{\alpha-2} + \sum_{i=1}^{i=n} A_i\alpha_i(\alpha_i-1)\phi^{\alpha_i-2}\right] \\ &- 2B_1\phi^{\beta_1}\left[2A_0 + A\alpha(\alpha-1)\phi^{\alpha-2} + \sum_{i=1}^{i=n} A_i\alpha_i(\alpha_i-1)\phi^{\alpha_i-2}\right] \end{aligned}$$

It is immediate to see that the biggest exponent apart from $\beta_1$ is $\beta+\alpha-2$ therefore we have found the exponent of the first correction

$$\beta_1 = \beta + \alpha - 2 = \frac{4}{d-2+\eta} + \frac{2d}{d+2-\eta} - 2$$

and setting equal to zero the corresponding coefficient we can find the correct value of $B_1$

$$-\eta B_1 + \frac{d+2-\eta}{2}B_1\beta_1 - 4A_0B_1 - 2BA\alpha(\alpha-1) = 0 \implies B_1 = \frac{4ABd}{(d+2-\eta)(\eta-2)}$$

In the same way we can find the second exponent and coefficient. For the ansatz

$$Y_{as} = B\phi^{\frac{4}{d-2+\eta}} + B_1\phi^{\beta_1} + B_2\phi^{\beta_2}$$

eq.(4.5) becomes

$$\begin{aligned} 0 = &- \eta B_2\phi^{\beta_2} + \frac{d-2+\eta}{2}\beta_2 B_2\phi^{\beta_2} - \beta(\beta-1)B\phi^{\beta-2} - \beta_1(\beta_1-1)B_1\phi^{\beta_1-2} \\ &- \beta_2(\beta_2-1)B_2\phi^{\beta_2-2} - 2B\phi^{\beta}\sum_{i=1}^{i=n} A_i\alpha_i(\alpha_i-1)\phi^{\alpha_i-2} \\ &- 2B_1\phi^{\beta_1}\left[A\alpha(\alpha-1)\phi^{\alpha-2} + \sum_{i=1}^{i=n} A_i\alpha_i(\alpha_i-1)\phi^{\alpha_i-2}\right] \\ &- 2B_2\phi^{\beta_2}\left[2A_0 + A\alpha(\alpha-1)\phi^{\alpha-2} + \sum_{i=1}^{i=n} A_i\alpha_i(\alpha_i-1)\phi^{\alpha_i-2}\right] \end{aligned}$$

where the exponent of the second correction will be

$$\beta_2 = \max\{\beta-2, \beta+\alpha_1-2, \beta_1+\alpha-2\} = \beta_1+\alpha-2 = \frac{4}{d-2+\eta} + \frac{4d}{d+2-\eta} - 4$$

and setting equal to zero the corresponding coefficient

$$-\eta B_2 + \frac{d-2+\eta}{2}B_2\beta_2 - 4B_2A_0 - 2B_1A\alpha(\alpha-1) - 2B_1A_1\alpha_1(\alpha_1-1) = 0$$

we find

$$B_2 = \frac{32 d^2 A^2 B}{(d+2-\eta)^3 (2-\eta)}$$

Iterating this procedure we have found the following form

$$Y_{as}(\phi) = B\phi^{\beta(d,\eta)} + \sum_{i=1}^{21} B_i(A, B, d, \eta)\phi^{\beta_i(d,\eta)} \tag{4.9}$$

where all the coefficients till the 15-th are linear in $B$ whereas the following terms became non linear in $B$ because of the contribution due to $A_j$ and $\alpha_j$ with $j \geq 14$. Again see the Appendix 4.4.1 for the full expansion.

### 4.1.3 Asymptotic expansion for $Z_{as}(\phi)$

The shooting method from large field value needs some expansions, up to $n$ terms, for the potential $V(\phi)$ and $Y(\phi)$ but the series of $Y(\phi)$ and $Z(\phi)$ are the same only in the limit $n \to \infty$. For this reason it is important to solve the two coupled differential equations for both $(V, Y)$ and $(V, Z)$ as a check for the numerical integration itself.

If the expansion for $Y_{as}$ is

$$Y_{as}(\phi) = B\phi^{\beta} + \sum_{i=1}^{n} B_i \phi^{\beta_i} \tag{4.10}$$

then that for $Z_{as}$ will be

$$Z_{as}(\phi) = \frac{1}{B}\phi^{-\beta}\left(1 + \sum_{i=1}^{n} \frac{B_i}{B}\phi^{\beta_i - \beta}\right)^{-1} \tag{4.11}$$

thus expanding in series power $(1+x)^{-1}$ we have

$$\begin{aligned} Z_{as}(\phi)B\phi^{\beta} =& 1 - \sum_{i=1}^{n} \frac{B_i}{B}\phi^{\beta_i - \beta} + \sum_{i,j=1}^{n} \frac{B_i B_j}{B^2}\phi^{\beta_i + \beta_j - 2\beta} - \sum_{i,j,k=1}^{n} \frac{B_i B_j B_k}{B^3}\phi^{\beta_i+\beta_j+\beta_k - 3\beta} + \\ & \cdots + (-1)^m \sum_{i_1,\ldots i_m=1}^{n} \frac{B_{i_1}\ldots B_{i_m}}{B^m}\phi^{\beta_{i_1}+\cdots+\beta_{i_m} - m\beta} + \ldots \end{aligned} \tag{4.12}$$

Taking under consideration the previous asymptotic expansion for $Y_{as}(\phi)$ it is not difficult to believe that the first fifth terms are

$$\begin{aligned} Z_{as}(\phi) =& \frac{1}{B}\phi^{-\beta} - \frac{B_1}{B^2}\phi^{\beta_1 - 2\beta} + \left(\frac{B_1^2}{B^3} - \frac{B_2}{B^2}\right)\phi^{\beta_2 - 2\beta} - \frac{B_3}{B^2}\phi^{\beta_3 - 2\beta} \\ & + \left(-\frac{B_4}{B^2} + 2\frac{B_1 B_2}{B^3} - \frac{B_1^3}{B^4}\right)\phi^{\beta_4 - 2\beta} + \ldots \end{aligned} \tag{4.13}$$

See Appendix 4.4.1 for the successive terms till the 14-th term.

## 4.2 Numerical analysis for $b = 0$

We are now ready to integrate our two fixed point equations for $V(\phi)$ and $Y(\phi)$ that we rewrite here

$$0 = dV - \frac{d-2+\eta}{2}\phi V' - (V')^2 + V'' + \frac{1}{Y} \tag{4.14a}$$

$$0 = -\eta Y + \frac{d-2+\eta}{2}\phi Y' - 2V''Y - Y'' \tag{4.14b}$$

This is a system of two second order non-linear differential equations therefore we have to give four Cauchy initial conditions to numerically solve it. This condition are given by our asymptotic expansions and they reads

$$\begin{cases} V(\phi_{\max}) = V_{as}(\phi_{\max}, A, B, d, \eta) \\ V'(\phi_{\max}) = V'_{as}(\phi_{\max}, A, B, d, \eta) \\ Y(\phi_{\max}) = Y_{as}(\phi_{\max}, A, B, d, \eta) \\ Y'(\phi_{\max}) = Y'_{as}(\phi_{\max}, A, B, d, \eta) \end{cases}$$

where $\phi_{\max}$ is the starting point of the shooting. We have therefore three parameters $A, B, \eta$ ($d$ has been fixed to 3) and in the space of all their possible values we must try to integrate the system from $\phi_{\max}$ to the origin and find the domain inside which we can reach $\phi = 0$. The full asymptotic expansion for $V$ and $Z$ or $Y$ contains actually also other two parameters, according to the fact of dealing with a system of two second order differential equations. However it is suspected that these are associated to corrections characterized by an essential singularity at $\phi \to \infty$ which are sub-leading compared to any power like correction.

Before going on we have to take into account some issues that are important to make a consistently numerical integration. These problems are essentially about the asymptotic expansions. First of all it is not difficult to see that there is a maximum value of $\eta$ beyond which our asymptotic expansions lose their validity because the powers of $\phi$ are not ordered any more: this value is for $d = 3$ exactly 0.2. Secondly the value of $\phi_{\max}$ is not arbitrary but it is actually a function of $A$ and $B$: we have to choose it in a range of $\phi$ where the first derivatives of $V_{as}$ and $Y_{as}$ change very slowly and where the convergence for the asymptotic expansion is quite good. However the value of $\phi_{\max}$ cannot be to high because the numerical integration could become less reliable due to numerical errors.

For this last reason it is important to plot the asymptotic expansions at various order for different value of the parameters $(A, B)$ and look at the change in the convergence. For example, taking the values of $A = \{-2, -3, -5, -7\}$ and $B = \{-0.002, -0.02, -0.2, -2\}$ and plotting $V_{as}(\phi)$ for different maximum order, say from the the 10-th to the 18-th order, it is easily to see that only the parameter $A$ heavily affects the convergence of $V_{as}$: near $A = -2$ one can safely set $\phi_{\max} \simeq 5.5$ instead near $A = -7$ one have to increase the starting point of integration at $\phi_{\max} \simeq 12$. Taking the same values of $A$ and $B$ and plotting $Y_{as}(\phi)$ for different maximum order, say from the 16-th to the 21-th order, it easily to see that varying the parameter $B$ there is simply a scaling of $Y_{as}$ so, even in this case, only the parameter $A$ heavily affect the convergence of $Y_{as}$: near $A = -2$ one can safely set

$\phi_{\max} \simeq 4$ but near $A = -7$ one have to increase the starting point of integration at $\phi_{\max} \simeq 13$. Same considerations can be deduced if one consider $B > 0$.

Since the interesting solutions are in the negative sector of $A$ and not smaller than $-5$ we can set $\phi_{\max} = 8$. Then we start by fixing an arbitrary initial $\eta$, for instance $\eta = 0.036$ which correspond to the best numerical estimates for the critica Ising model, and plot $V'(0)$ and $Z'(0)$ as a functions of $A$, for several values of $B$. In fig.4.1 we have plotted $V'(0)$ (blue dots) and $Y'(0)$ (purple dots) in the range $-6 < A < -3$ for $B = \{-0.05, -0.01, -0.005, -0.001\}$ looking from left to right of the panel. For $A > -3$ the integration fails in reaching the origin. We can see that there is one zero for $V'(0)$ for all range of $B$ investigated but two zeros for $Y'(0)$ appear only if the value of $B$ is sufficiently small $B \lesssim -0.0045$. We call with $A^*_{\mathrm{V}}$ the zero for $V'(0)$, with $A^*_{\mathrm{Y}}$ the right zero for $Y'(0)$ and with $\Delta A^* = A^*_{\mathrm{V}} - A^*_{\mathrm{Y}}$ the difference between them. Reducing $B$ we have found that there is a value $B^*$ at with $\lim_{B\to B^*} \Delta A^* = 0$ and we call $A^* = \lim_{B\to B^*} A^*_{\mathrm{V}}$. In other words, fixing $\eta = 0.036$, we have found a couple of value $(A^*, B^*)$ corresponding to a global solution which reaches the origin with a zero first derivative, as required by the $\mathbb{Z}_2$ symmetry.

In fig.4.3 we have plotted the solution founded for $\eta = 0.036$ and $A = A^*, B = B^*$. In the upper left panel there is the numerical solution found with the method of shooting from large field values (blue line) compared with the asymptotic expansion (red line). In the upper right panel there is the numerical solution compared with the polynomial solution (green line). In the lower left panel there is the numerical solution for the wave function renormalization (purple line) compared with the asymptotic expansion (red line) and again in the lower right panel there is the comparison with the polynomial solution (black line). We have already explained how the polynomial analysis works in Chapter 2.3.3 so we won't say anything more about it.

Successively we have repeated all this procedure for different values of $\eta$ and we have collected a set of points in the space of the parameters $(\eta, A, B)$ each of them correspond to a global scaling solution for our two differential equations eq.(4.14). In fig.4.2 we have plotted the final result that corresponds to a family of FP global solution for the two equations under consideration. On the left side there is $A^*$ as a function of $\eta$ whereas on the right side there is $B^*$ as a function of $\eta$.

### 4.2.1 Final comments

- We started our analysis from two coupled differential equations obtained setting to zero the regulator-dependent parameter $b = \frac{K_1 I_1}{K_0 I_0}$. Thanks to this trick our equations do not contain any terms depending on the cutoff function but implicitly we have made a choice because we have restricted ourselves to a family of cutoff functions such that $K_1 = 0$. Therefore, strictly speaking, it is not true that our equations are universal. On the other hand this is expected since Wilsonian action depends on the coarse-graining scheme.

- The global scaling solution for the potential presents a non trivial minimum so it has the right shape for a Wilson-Fisher Fixed Point. But the solution for the wave function renormalization poses a problem because it is negative so it should represent propagation of a ghost field, being negative the kinetic term in the Lagrangian. The reason is that the only solutions that we have found are for negative values of $B$. We have analyzed even the case for $B > 0$ but no points $(A^*, B^*)$ with $V'(0) = 0 = Y'(0)$ has been found.

- The line of FPs solution in the parameter space seems pretty flat and suggests that there isn't any privilege value for the anomalous dimension, in particular there isn't any special behavior around $\eta$ expected (0.03612).

These results show that the numerical construction from the asymptotic region is not able to lead to physical solutions for the special case of $b = 0$. For this reason a more systematic analysis from the origin should be done.

### LPA case

We want to give finally the numerical result for the very simple case of the local potential approximation where the wave function renormalization is set to a constant and only the flow equation for the potential is considered. Therefore the equation that we have to study is

$$\dot{V} = dV - \frac{d-2+\eta}{2}\phi V' - (V')^2 + V'' + C \tag{4.15}$$

where $\eta = 0$ since we are in LPA and $C$ a constant that we firstly set to zero.

We have used the method of shooting from large field value therefore the asymptotic expansion for the potential is exactly that one obtained before but truncated up to the 13-th order correction. We have chosen $\phi_{\max} = 10$ as the starting point for the numerical integration and, as usual, in fig.4.4 (left panel) we have plotted the first derivative of the potential at the origin as a function of the parameter $A$. We can see two zeros for $V'(0)$: one for $A = A_1^* = 0$ and another for some value $A = A_2^* < 0$. The solution for the potential at $A_1^*$ corresponds to a free theory because it is exactly the quadratic leading term of the asymptotic expansion, whereas the numerical solution for $A_2^*$ is a non trivial solution corresponding to a Wilson Fisher fixed point. We have plotted the last one on the right panel of fig.4.4.

If we add a constant $C \neq 0$ in the equation, there is a shift of the two zeros $A_1^*$ and $A_2^*$, positive if $C < 0$ and negative if $C > 0$. The solutions are again of the two types described before but simply shifted of some constant.

## 4.3 Asymptotic expansions for $b \neq 0$

In the previous section we have studied numerically the fixed point equations for $V(\phi)$ and $Y(\phi)$ and we have gathered that no *physical* solutions exist in all range of parameter space. For this reason we want now to relax the assumption that we have made *i.e.* we want to take under consideration the possibility for the regulator dependent cutoff $K_1$ to be non zero. In this case, only the flow equation for the wave function renormalization change and acquires one more term that can be written in two different and equivalent ways (equivalent because they differs apart for a boundary term but within the truncation of a quadratic Lagrangian in the first derivative of the scalar field). The equations under consideration are:

$$\dot{V} = dV - \frac{d-2+\eta}{2}\phi V' - (V')^2 + V'' + Y^{-1} \tag{4.16a}$$

$$\dot{Y} = \eta Y - \frac{d-2+\eta}{2}\phi Y' + 2YV'' + Y'' + 2bV'V'''Y^2 \tag{4.16b}$$

$$= \eta Y - \frac{d-2+\eta}{2}\phi Y' + 2YV'' + Y'' - 2b(V'')^2 Y^2 \tag{4.16c}$$

We have studied both the eqs. (4.16b) and (4.16c) but in this thesis we want to show only the analysis made for the second equation because it is that one which has a physical solution.

We want to start finding the asymptotic expansion for the fixed point solutions. Let's start from the leading terms for the potential $V_{as} = A_0\phi^{\alpha_0} + A\phi^{\alpha}$ then we have

$$0 = \phi^{\alpha_0}\left(dA_0 - \frac{d-2+\eta}{2}A_0\alpha_0\right) + \phi^{\alpha}\left(dA - \frac{d-2+\eta}{2}A\alpha\right) - A_0^2\alpha_0^2\phi^{2\alpha_0-2} - 2A_0A\alpha_0\alpha\phi^{\alpha_0+\alpha-2} - A^2\alpha^2\phi^{2\alpha-2} + A_0\alpha_0(\alpha_0-1)\phi^{\alpha_0-2} + A\alpha(\alpha-1)\phi^{\alpha-2}$$

again the condition $\alpha_0 = 2\alpha_0 - 2$ fixes the first power $\alpha_0 = 2$, setting to zero the relative coefficient implies $A_0 = \frac{2-\eta}{4}$ and setting to zero the coefficient of $\phi^{\alpha}$ fixes the second power $\alpha = \frac{2d}{d+2-\eta}$. Therefore we have the leading behavior for the potential

$$V_{as}(\phi) = \left(1 - \frac{\eta}{2}\right)\frac{\phi^2}{2} + A\phi^{\frac{2d}{d-\eta+2}} + \dots \tag{4.17}$$

To obtain the leading term for $Y_{as}$ we have to substitute these two terms into the eq.(4.16c) giving

$$0 = \eta B\phi^{\beta} - \frac{d-2+\eta}{2}\beta B\phi^{\beta} + 2B\phi^{\beta}\left(2A_0 + A\alpha(\alpha-1)\phi^{\alpha-2}\right) + B\beta(\beta-1)\phi^{\beta-2} - 2bB^2\phi^{2\beta}\left(2A_0 + A\alpha(\alpha-1)\phi^{\alpha-2}\right)^2$$

and because of the last term $Y^2(V'')^2$ with $2\beta$ as the higher power we have to set $\beta = 2\beta$ so $\beta = 0$. Setting to zero the relative coefficient

$$\eta B + 4BA_0 - 8bB^2A_0^2 = 0 \implies B = \frac{4}{b(2-\eta)^2}$$

we can see two important differences with the case discussed in the previous section. First of all the free parameter $B$ in the leading term of the asymptotic behavior of $Y_{as}$ has been fixed by the leading power condition so we still have again two parameters but the second one is the regulator coefficient $b$; secondly the function $Y$ asymptotically converges to a constant and does not diverge to infinity as some power in the field.

The leading behavior for $Y$ will be simply a constant

$$Y_{as}(\phi) = \frac{4}{b(2-\eta)^2} + \dots \tag{4.18}$$

### 4.3.1 First correction for $Y_{as}(\phi)$

Now consider the first correction for the inverse of the wave function renormalization

$$Y_{as}(\phi) = B + B_1\phi^{\beta_1} \tag{4.19}$$

then the FP equation for $Y$ imposes

$$\begin{aligned}
0 =&\eta B_1\phi^{\beta_1} - \frac{d-2+\eta}{2}\beta_1 B_1\phi^{\beta_1} \\
&+ 2BA\alpha(\alpha-1)\phi^{\alpha-2} + 2B_1\phi^{\beta_1}\left(2A_0 + A\alpha(\alpha-1)\phi^{\alpha-2}\right) \\
&+ B_1\beta_1(\beta_1-1)\phi^{\beta_1-2} \\
&- 2bB^2\left(4A_0A\alpha(\alpha-1)\phi^{\alpha-2} + A^2\alpha^2(\alpha-1)^2\phi^{2\alpha-4}\right) \\
&- 2b\left(2BB_1\phi^{\beta_1} + B_1^2\phi^{2\beta_1}\right)\left(2A_0 + A\alpha(\alpha-1)\phi^{\alpha-2}\right)^2
\end{aligned}$$

where the leading power fixes the value of the first correction exponent $\beta_1 = \alpha - 2$ and setting to zero the relative coefficient

$$\eta B_1 - \frac{d-2+\eta}{2}\beta_1 B_1 + 2BA\alpha(\alpha-1) + 4B_1A_0 - 8bB^2A_0A\alpha(\alpha-1) - 16bBB_1A_0^2 = 0$$

fixes the values of the first correction coefficient. The expansion up to the first order is

$$Y_{as}(\phi) = \frac{4}{b(2-\eta)^2} + \frac{16Ad(2+\eta)(d-2+\eta)}{b(2+d-\eta)(\eta-2)^3(8+\eta(d-6)+\eta^2)}\phi^{\frac{2\eta-4}{d+2-\eta}} + \dots \tag{4.20}$$

In general, taking the following ansatz for the asymptotic expansions

$$\begin{aligned}
V_{as}(\phi) =& \sum_{j=1} A_j\phi^{\alpha_j} \\
Y_{as}(\phi) =& B + \sum_{i=1} B_i\phi^{\beta_i}
\end{aligned}$$

the FP equation for $Y$ becomes

$$\begin{aligned}
0 =&\eta\left(B + B_i\phi^{\beta_i}\right) - \frac{d-2+\eta}{2}B_i\beta_i\phi^{\beta_i} \\
&+ 2\left(B + B_i\phi^{\beta_i}\right)A_j\alpha_j(\alpha_j-1)\phi^{\alpha_j-2} + B_j\beta_j(\beta_j-1)\phi^{\beta_j-2} \\
&- 2bA_iA_j\alpha_i(\alpha_i-1)\alpha_j(\alpha_j-1)\phi^{\alpha_i+\alpha_j-4}\left(B^2 + 2BB_k\phi^{\beta_k} + B_kB_\ell\phi^{\beta_k+\beta_\ell}\right)
\end{aligned}$$

and iterating this procedure we have arrived till the 13-th correction in the expansion for $Y_{as}(\phi)$ (See Appendix 4.4.1 for the full expansion).

### 4.3.2 First two corrections for $V_{as}(\phi)$

Now we want to show explicitly the first two corrections to the leading behavior for the potential because in this case $Z_{as}$ contributes to $V_{as}$ from the second correction. Let's take

$$V_{as} = A_0\phi^2 + A\phi^\alpha + A_1\phi^{\alpha_1} + A_2\phi^{\alpha_2}$$

and $Z_{as} = Z_0$ because asymptotically it should be a constant so the FP point equation for $V$ implies

$$
\begin{aligned}
0 =& d(A_0\phi^2 + A\phi^{\alpha} + A_1\phi^{\alpha_1} + A_2\phi^{\alpha_2}) \\
&- \frac{d-2+\eta}{2}(2A_0\phi^2 + A\alpha\phi^{\alpha} + A_1\alpha_1\phi^{\alpha_1} + A_2\alpha_2\phi^{\alpha_2}) \\
&- \left(2A_0\phi + A\alpha\phi^{\alpha-1} + A_1\alpha_1\phi^{\alpha_1-1} + A_2\alpha_2\phi^{\alpha_2-1}\right)^2 \\
&+ 2A_0 + A\alpha(\alpha-1)\phi^{\alpha-2} + A_1\alpha_1(\alpha_1-1)\phi^{\alpha_1-2} + A_2\alpha_2(\alpha_2-1)\phi^{\alpha_2-2} + Z_0 \\
=& d(A_1\phi^{\alpha_1} + A_2\phi^{\alpha_2}) - \frac{d-2+\eta}{2}(A_1\alpha_1\phi^{\alpha_1} + A_2\alpha_2\phi^{\alpha_2}) \\
&- 4A_0A_1\alpha_1\phi^{\alpha_1} - 4A_0A_2\alpha_2\phi^{\alpha_2} \\
&- 2AA_1\alpha\alpha_1\phi^{\alpha+\alpha_1-2} - 2AA_2\alpha\alpha_2\phi^{\alpha+\alpha_2-2} - 2A_1A_2\alpha_1\alpha_2\phi^{\alpha_1+\alpha_2-2} \\
&- A^2\alpha^2\phi^{2\alpha-2} - A_1^2\alpha_1^2\phi^{2\alpha_1-2} - A_2^2\alpha_2^2\phi^{2\alpha_2-2} \\
&+ 2A_0 + A\alpha(\alpha-1)\phi^{\alpha-2} + A_1\alpha_1(\alpha_1-1)\phi^{\alpha_1-2} + A_2\alpha_2(\alpha_2-1)\phi^{\alpha_2-2} + Z_0
\end{aligned}
$$

the highest power of this expression $2\alpha - 2$ fixes the value of $\alpha_1$ and setting to zero the corresponding coefficient gives

$$A_1 = -\frac{4A^2d^2}{(\eta-2)(d-\eta+2)^2}$$

The survived terms are

$$
\begin{aligned}
0 =& dA_2\phi^{\alpha_2} - \frac{d-2+\eta}{2}A_2\alpha_2\phi^{\alpha_2} - 4A_0A_2\alpha_2\phi^{\alpha_2} \\
&- 2AA_1\alpha\alpha_1\phi^{\alpha+\alpha_1-2} - 2AA_2\alpha\alpha_2\phi^{\alpha+\alpha_2-2} - 2A_1A_2\alpha_1\alpha_2\phi^{\alpha_1+\alpha_2-2} \\
&- A_1^2\alpha_1^2\phi^{2\alpha_1-2} - A_2^2\alpha_2^2\phi^{2\alpha_2-2} \\
&+ 2A_0 + A\alpha(\alpha-1)\phi^{\alpha-2} + A_1\alpha_1(\alpha_1-1)\phi^{\alpha_1-2} + A_2\alpha_2(\alpha_2-1)\phi^{\alpha_2-2} + Z_0
\end{aligned}
$$

where the highest power is zero because $\alpha + \alpha_1 - 2, 2\alpha_1 - 2, \alpha - 2, \alpha_1 - 2$ are all negative, therefore to the second correction contributes the leading term of $Z_{as}$

$$dA_2 + 2A_0 + Z_0 = 0 \implies A_2 = -\frac{(\eta-2)(b(\eta-2)-2)}{4d}$$

Iterating this procedure in a straightforward way we have found an expansion for $V_{as}(\phi)$ up to 12-th correction. (See Appendix 4.4.1 for the full expansion).

### 4.3.3 Asymptotic expansion for $Z_{as}(\phi)$

As we said before it is important to solve numerically the two coupled differential equations for both $(V, Y)$ and $(V, Z)$. Our expansion for $Y_{as}$ is

$$Y_{as}(\phi) = B + \sum_{i=1}^{n} B_i\phi^{\beta_i}$$

then that for $Z_{as}$ will be

$$Z_{as}(\phi) = \frac{1}{B}\left(1 + \sum_{i=1}^{n}\frac{B_i}{B}\phi^{\beta_i}\right)^{-1}$$

thus expanding in series power $(1+x)^{-1}$ we have

$$Z_{as}(\phi) = 1 - \sum_{i=1}^{n} \frac{B_i}{B}\phi^{\beta_i} + \sum_{i,j=1}^{n} \frac{B_i B_j}{B^2}\phi^{\beta_i+\beta_j} - \sum_{i,j,k=1}^{n} \frac{B_i B_j B_k}{B^3}\phi^{\beta_i+\beta_j+\beta_k} + \dots$$

Taking under consideration the previous asymptotic expansion for $Y_{as}(\phi)$ it is not difficult to obtain the first fifth terms

$$Z_{as}(\phi) = \frac{1}{B} - \frac{B_1}{B^2}\phi^{\beta_1} + \left(\frac{B_1^2}{B^3} - \frac{B_2}{B^2}\right)\phi^{\beta_2} - \frac{B_3}{B^2}\phi^{\beta_3} + \left(-\frac{B_4}{B^2} + 2\frac{B_1 B_2}{B^3} - \frac{B_1^3}{B^4}\right)\phi^{\beta_4} + \dots$$

See Appendix 4.4.1 for the full expansion till the 14-th term.

## 4.4 Numerical analysis for $b \neq 0$

Differently from the previous analysis, in this case we have studied the FP equations for $U \equiv V'$ and $Z$ that reads

$$0 = dU - \frac{d-2+\eta}{2}U - \frac{d-2+\eta}{2}\phi U' - 2UU' + U'' + Z' \tag{4.21a}$$

$$0 = -\eta Z^2 - \frac{d-2+\eta}{2}\phi Z Z' - 2Z^2 U' + Z Z'' - 2(Z')^2 + 2bZ(U')^2 \tag{4.21b}$$

but the same results can be obtained from integration in $V$ and $Y$. This choice is also useful because in this way we can overcome the problem of throwing away the constant in the Polchinski equation: in fact adding some constant from the very beginning of Polchinski' s flow equation affects only the equation for $V$ but we can alway shift the potential of some constant.

The Cauchy initial conditions that we have to impose are as usual four

$$\begin{cases} U(\phi_{\max}) = V'_{as}(\phi_{\max}, A, b, d, \eta) \\ U'(\phi_{\max}) = V''_{as}(\phi_{\max}, A, b, d, \eta) \\ Z(\phi_{\max}) = Z_{as}(\phi_{\max}, A, b, d, \eta) \\ Z'(\phi_{\max}) = Z'_{as}(\phi_{\max}, A, b, d, \eta) \end{cases}$$

with $A, b, \eta$ our three free parameters (again we neglect the other two parameters associated to an essential singularity at $\phi \to \infty$) and we shall restrict to the three dimensional case $d = 3$. As we have previously discussed, the starting point for the numerical shooting method $\phi_{\max}$ is, in principle, a function of all the three parameters and we have to chose it sufficiently large (but not to much) such that the convergence of the asymptotic expansion is pretty good. For this reason, as an example, we have plotted in figure 4.5 the asymptotic expansions of $Y_{as}(\phi)$ for the last ten orders. We have fixed $\eta = 0.041$ and varied the other two parameter: $A = \{-4, -3, -2, -1\}$ and $b = \{0.01, 0.5, 2.2, 3.5\}$. It is like putting the cart before the horse, but we have chosen these values because it is in this domain that we have seen a family of interesting *physical* solutions. What we can see from these figures is that only the parameter $A$ has a strong influence on the convergence of the asymptotic expansions, in particular the greater its absolute value is, the worst the convergence. Moreover the expansion of $V_{as}$ is much better then that of $Y_{as}$. Same considerations can be done for different value of $\eta$. Because the interesting solutions are in the negative sector of $A$ and not smaller than -2 we can safely set $\phi_{\max} = 20$.

Now the philosophy in searching for a solution is exactly the same as before but with a difference: before we fixed $\eta$ and tried to find a couple of values $(A^*, B^*)$ at which $V'(0)$ and $Z'(0)$ could set to zero with arbitrary high precision; here we fixe $b$ and try to find a couple of value $(A^*, \eta^*)$ at which $V'(0)$ and $Z'(0)$ manage to set to zero with arbitrary high precision. The reason for this choice can be seen in fig.4.6 where we have plotted $(U(0), A)$ (blue dots) and $(Z'(0), A)$ (purple dots) for $b = 2$ and different values of $\eta$. The parameter $\eta$ has a great influence on $Z'(0)$ in fact for $\eta = 0.025$ there is only two zeros, one for $A^*_{\rm Z} = 0$ and another for $A^*_{\rm Z} \simeq -0.8$ but growing $\eta$ appears two more zeros, for example for $\eta = 0.04$ there is one zero at $A^*_{\rm Z} \simeq -1.55$ and another at $A^*_{\rm Z} \simeq -1.3$. It is indeed the one on the right that has the property $\lim_{\eta\to\eta^*} A^*_{\rm Z} = A^*_{\rm U}$ for some value of $\eta^*$, as can be seen in the fig. 4.7, because there is a crossing of the two plots *i.e.* for $\eta = 0.040$ $A^*_{\rm Z} < A^*_{\rm U}$ and for $\eta = 0.046$ $A^*_{\rm Z} > A^*_{\rm U}$. This is the method that we have used to find the solution for our two differential equations.

Successively we have repeated this procedure for different values of $b$ and we have found a range $0 < b \lesssim 3$ within it there is a family $\mathscr{F}^*$ of FPs $(\eta^*, b^*)$ each of them is a possible *physical* solution. For $3 \lesssim b$ we have found no solutions in fact, as we can see in fig.4.8, fixing a value of $\eta \in \mathscr{F}^*$ and growing $b$ there is some value $b_{\max}$ such that for $b > b_{\max}$ only one zero is present in the plot of $(Z'(0), A)$ and correspond to $A = 0$.

## 4.4.1 Final comments

- The family of fixed point solutions $\mathscr{F}^*$ has been plotted in fig.4.10. The potential, for each point, has the the right shape for a Wilson-Fisher FP *i.e.* it has a non trivial minimum for a value of the scalar field different from zero and so it can give rise to a SSB phase in the IR depending of the bare action chosen as a starting point of the flow. The solution for the wave function renormalization is *positive* and so it represents the propagation of a physical scalar field because now the kinetic term in the Lagrangian has the right sign.

- In fig.4.9 we have plotted the solution for one particular point, the one corresponding to $b = 2.3$. In the upper panel there is on the left side the first derivative of the potential $U(\phi)$ compared with the asymptotic expansion $V'_{as}(\phi)$ and on the right side the wave function renormalization $Z(\phi)$ compared with the asymptotic expansion $Z_{as}(\phi)$. As suspected the convergence to the asymptotic behavior $V$ is faster than those of $Z$. In the bottom panel there is a zoom area near the origin where we can see the non trivial minimum at $\phi \simeq 1.12$.

- As a check for our numerical results, we have tried to analyzed the two fixed point equations with a polynomial approach exactly as we did in the previous case with $b = 0$ or in the chapter 2.3.3 but unfortunately we didn't find any solution. Probably the polynomial solution has a zero convergence radius. For this reason we followed a more functional strategy which shows us in a graphical way the existence of our solutions [30]. Suppose to fix $b$ (for example $b = 2.3$) and also $\eta$ and numerical integrate our equations from the origin to $\phi = 3$, for example. Two Cauchy conditions are imposed by the $\mathbb{Z}_2$ symmetry ($U(0) = 0 = Z'(0)$) whereas the left two are undetermined. If we choose $U'(0) = U'_{\text{numeric}}(0)$ and $Z(0) = Z_{\text{numeric}}(0)$ (at the fixed value of $b$) then we expect that our numerical integration can reach $\phi = 3$. In fact it is so, but if we slightly vary these two values the integration stops at some $\phi_{\text{critic}} < 3$. Therefore the strategy is to plot $\phi_{\text{critic}}$ as a function of the two left parameters and see where there is a steep increase of it. The point $(U'_{\text{numeric}}(0), Z_{\text{numeric}}(0))$ should correspond to a spike in the 3D plot and this is exactly what we found in fig.4.11 (bottom panel), in particular in fig.4.11a and fig.4.11b we have plotted the sections corresponding to $U'(0) = U'_{\text{numeric}}(0)$ and $Z(0) = Z_{\text{numeric}}(0)$ respectively.

- The family $\mathscr{F}^*$ has a minimum (at $\eta = 0.04126$) and this is a nice result because it is in agreement with the principle of *minimum sensitivity*, in other words we can choose the value of $b = b_0^*$ corresponding to that minimum and in this way, having a small variation of $\eta$ near $b_0^*$, we can evaluate the anomalous dimension with a good approximation. For example if we choose the range $2.0 < b < 2.6$ then the variation of the anomalous dimension is about 3% therefore we can finally say that our result is $\eta = 0.0413 \pm 0.0012$.

  We have found solutions even for small values of $b$, the last point we have analyzed is for $b = 0.005$ but we suppose that we could go further on as long as $b \neq 0$ , in fact for $b = 0$ there is a singularity in the asymptotic expansion and the numerical integration will fail of course.

- As a final comment that shows one more time the consistency of this numerical analysis is that for all $(b, \eta)$ there is also a trivial Gaussian solution correspond to $A = 0$. In this case the numerical solutions coincide exactly with the asymptotic expansions that reduce to their leading behavior, quadratic for the potential and constant for the wave function renormalization:

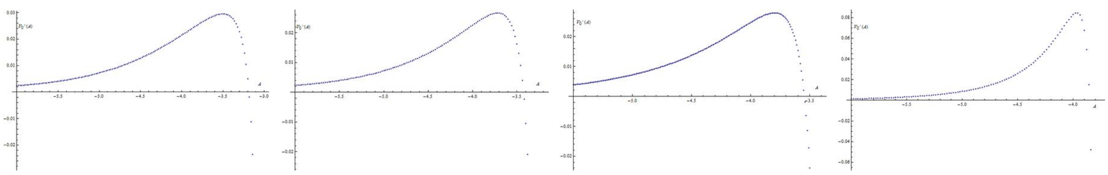
$$V(\phi) = \frac{1}{4}(2-\eta)\phi^2 - \frac{(\eta-2)(b(\eta-2)-2)}{4d} \tag{4.22}$$

$$Z(\phi) = \frac{1}{4}b(2-\eta)^2 \tag{4.23}$$

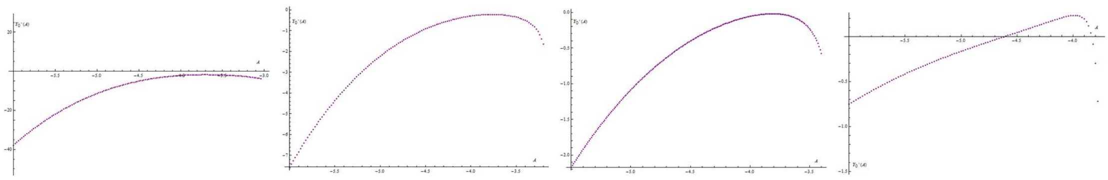
  but these are actually analytic solutions for the two fixed point equations.

  Now we understand why in the previous section we studied the simple LPA case. The shape of the function $V'\big|_0(A)$ in the LPA for $b = 0$ is essentially the same as in this case with $b \neq 0$ even if the asymptotic solutions for the potential are quite different already from the 4-th term. In fact there is a trivial Gaussian solution for the potential corresponding to $A_1^* = 0$ and a Wilson Fisher solution corresponding to some $A_2^* < 0$. Moreover the presence of the function $Z(\phi)$ in the fixed point equation for $V(\phi)$ ensures the stability of the FP parameter $A_1^*$ at the zero value even if the wave function renormalization tends asymptotically to a constant different from zero *i.e.* $\lim_{\phi\to\infty} Z(\phi) = \text{const} \neq 0$.

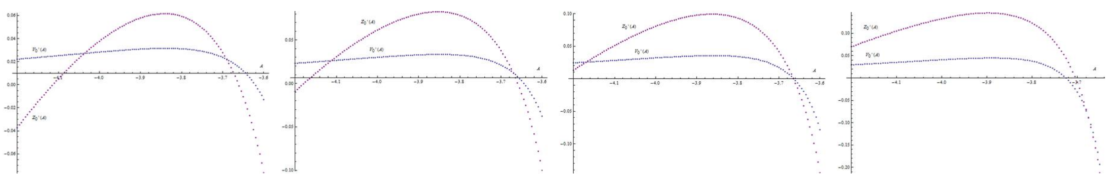
All our numerical analysis in finding the family of fixed points $\mathscr{F}^*$ has been done with a double machine precision because we have seen that our two equations are very sensible to the initial conditions. The reason is essentially because the convergence of $Z_{as}(\phi)$ is not so good even at $\phi_{\max} = 20$ (for example the relative difference between the 7-th order and the 13-th order is about 0.017%). To overcome this problem we can increase the starting point of integration or increase the order of the asymptotic expansion. Letting the last way for a future study we have tried to increase $\phi_{\max}$ for the fixed value of $b = 2.3$ and what we have seen (see fig.4.12) is that the value of $\eta^*$ decreases and seems to converge to a value of $\eta \simeq 0.04078$ (about 0.8% of the initial value) which is though different from the expected value of 0.0361(2).



**(a)** $B = -0.05$ **(b)** $B = -0.01$ **(c)** $B = -0.005$ **(d)** $B = -0.001$

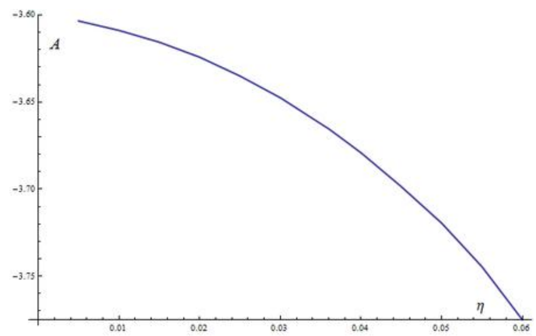


**(e)** $B = -0.05$ **(f)** $B = -0.01$ **(g)** $B = -0.005$ **(h)** $B = -0.001$

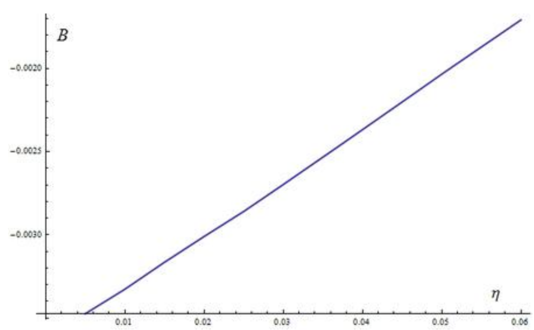


**(i)** $B = -0.0032$ **(j)** $B = -0.0028$ **(k)** $B = -0.0025$ **(l)** $B = -0.0018$

**Figure 4.1:** $(V'(0), A)$ (upper panel) and $(Y'(0), A)$ (middle panel) for different value of $B$ at fixed $\eta = 0.036$. In the lower panel there is an overlap of the two functions to show graphically the existence of a $B^*$-value at with $V'(0)$ and $Y'(0)$ are zero for the same value of $A^*$.




**(a)**




**(b)**

**Figure 4.2:** On the left the parameter $A^*$ corresponding to the zeros for $V'(0)$ and $Z'(0)$ as a function of $\eta$ and on the right the parameter $B^*$ with the same property as a function of $\eta$.

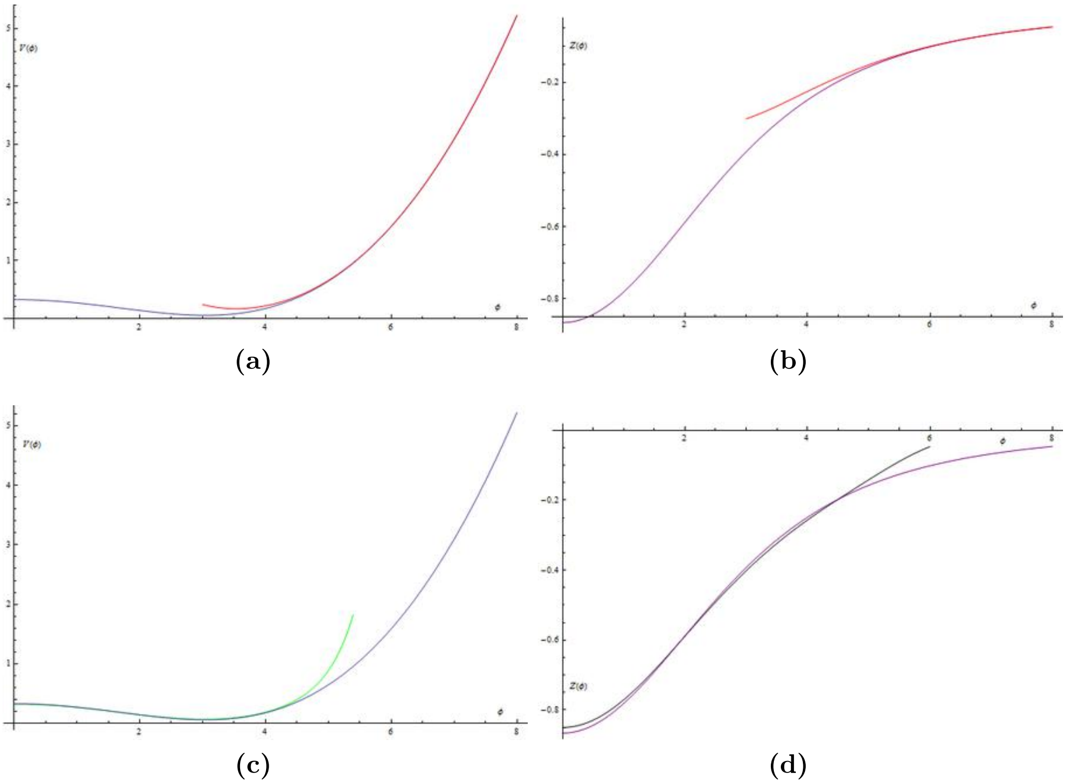



**Figure 4.3:** Upper panel: the function $V(\phi)$ and $Z(\phi)$ for $\eta = 0.036$, at the values of $A = A^*, B = B^*$ corresponding to the zeros for $V'(0)$ and $Z'(0)$ (the red lines are the asymptotic behaviors). These solutions are obtained by numerical shooting integration from $\phi = 8$ to $\phi = 0$.
Bottom panel: the functions $V(\phi)$ and $Z(\phi)$ are compared with the polynomial solutions founded for an order of $\phi^{14}$ (green line for the potential and black line for the wave function renormalization).

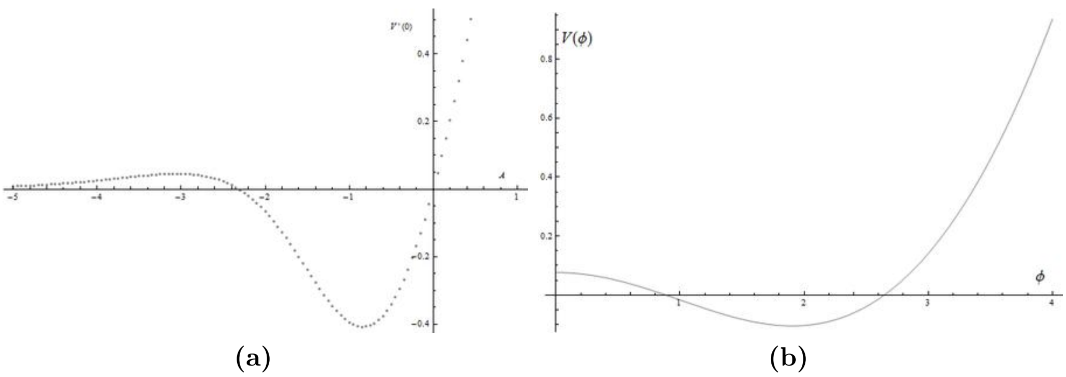
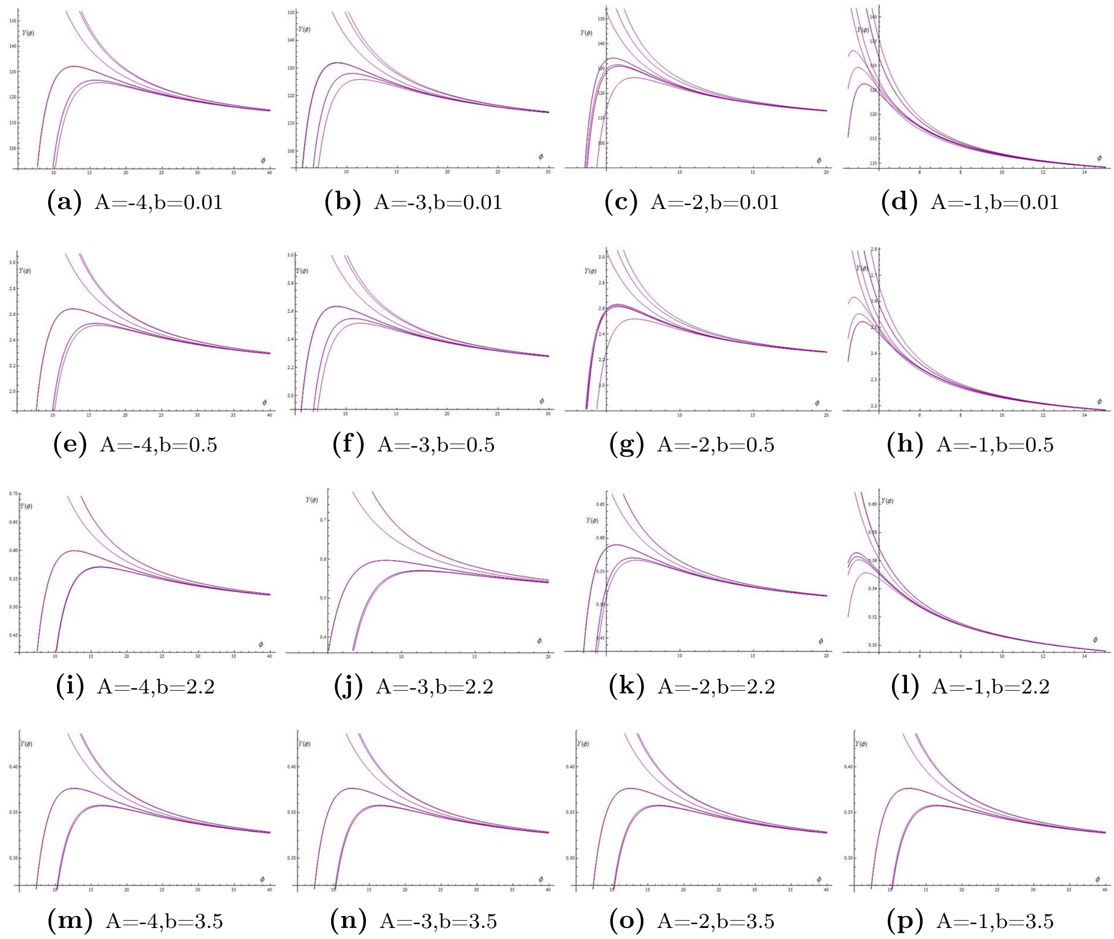



**Figure 4.4:** On the left panel the first derivative of the potential in the origin $V'(0)$ as a function of the asymptotic parameter $A$ in the LPA case for $b = 0$. On the right panel the global scaling solution corresponding to the left zero for $V'(0)$.



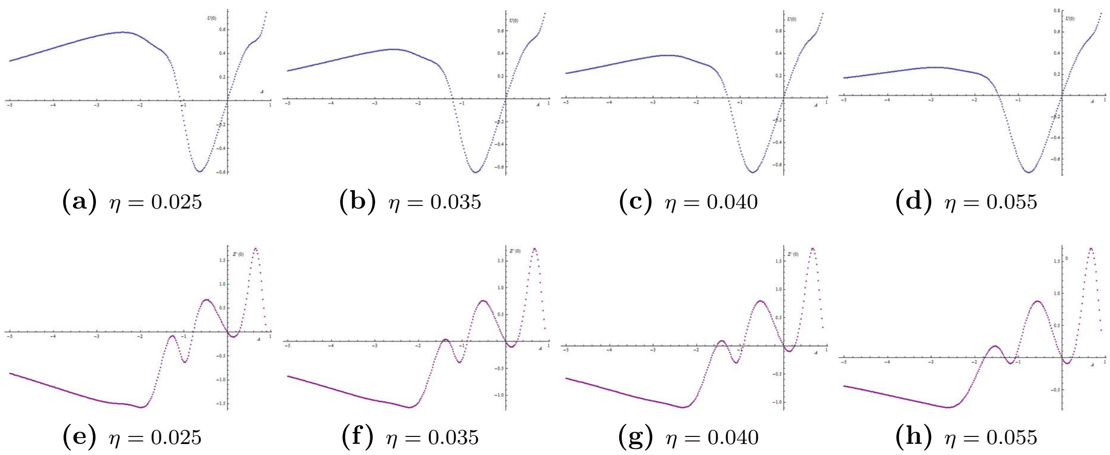
(a) A=-4,b=0.01 (b) A=-3,b=0.01 (c) A=-2,b=0.01 (d) A=-1,b=0.01

(e) A=-4,b=0.5 (f) A=-3,b=0.5 (g) A=-2,b=0.5 (h) A=-1,b=0.5

(i) A=-4,b=2.2 (j) A=-3,b=2.2 (k) A=-2,b=2.2 (l) A=-1,b=2.2

(m) A=-4,b=3.5 (n) A=-3,b=3.5 (o) A=-2,b=3.5 (p) A=-1,b=3.5

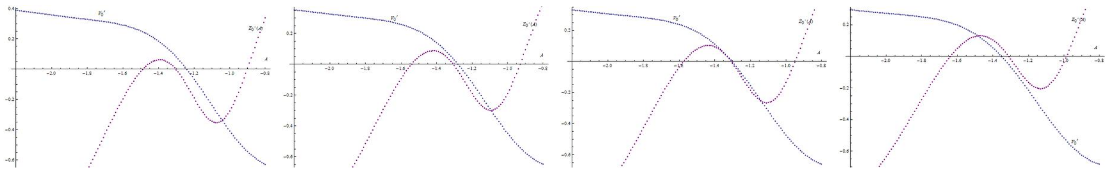
**Figure 4.5:** We have plotted in a matrix-like form for different values of $A = \{-4, -3, -2, -1\}$ and $b = \{0.01, 0.5, 2.2, 3.5\}$ the asymptotic expansions of $Y_{as}(\phi)$ from the 6-th order to the 14-th order. The convergence becomes worst for decreasing value of $A$ and it is essentially unaffected by variation of $b$. Since the interesting values of $A$ are not smaller than $-2$ we can safely set the starting point of numerical integration at $\phi_{\max} = 20$.



(a) $\eta = 0.025$ (b) $\eta = 0.035$ (c) $\eta = 0.040$ (d) $\eta = 0.055$

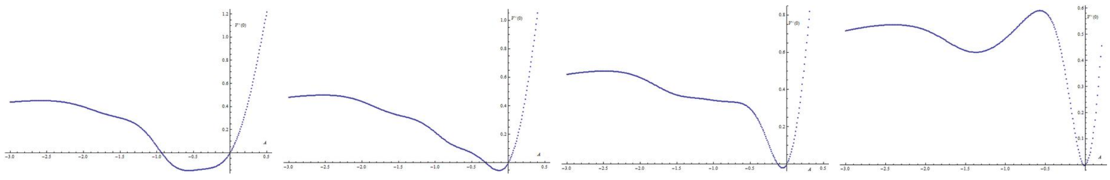
(e) $\eta = 0.025$ (f) $\eta = 0.035$ (g) $\eta = 0.040$ (h) $\eta = 0.055$

**Figure 4.6:** $V'(0)$ in the upper panel and $Z'(0)$ in the bottom panel at fixed $b = 2$ for different values of $\eta = \{0.025, 0.035, 0.04, 0.055\}$

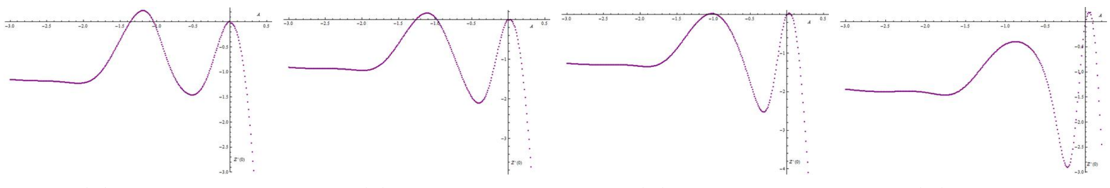


**(a)** $\eta = 0.037$ **(b)** $\eta = 0.040$ **(c)** $\eta = 0.042$ **(d)** $\eta = 0.046$

**Figure 4.7:** $V'(0)$ and $Z'(0)$ at fixed $b = 2$ for different values of $\eta = \{0.037, 0.040, 0.042, 0.046\}$



**(a)** b=3.1 **(b)** b=3.3 **(c)** b=3.5 **(d)** b=3.8



**(e)** b=3.1 **(f)** b=3.3 **(g)** b=3.5 **(h)** b=3.8

**Figure 4.8:** $V'(0)$ in the upper panel and $Z'(0)$ in the bottom panel at fixed $\eta = 0.048$ for different values of $b = \{3.1, 3.3, 3.5, 3.8\}$

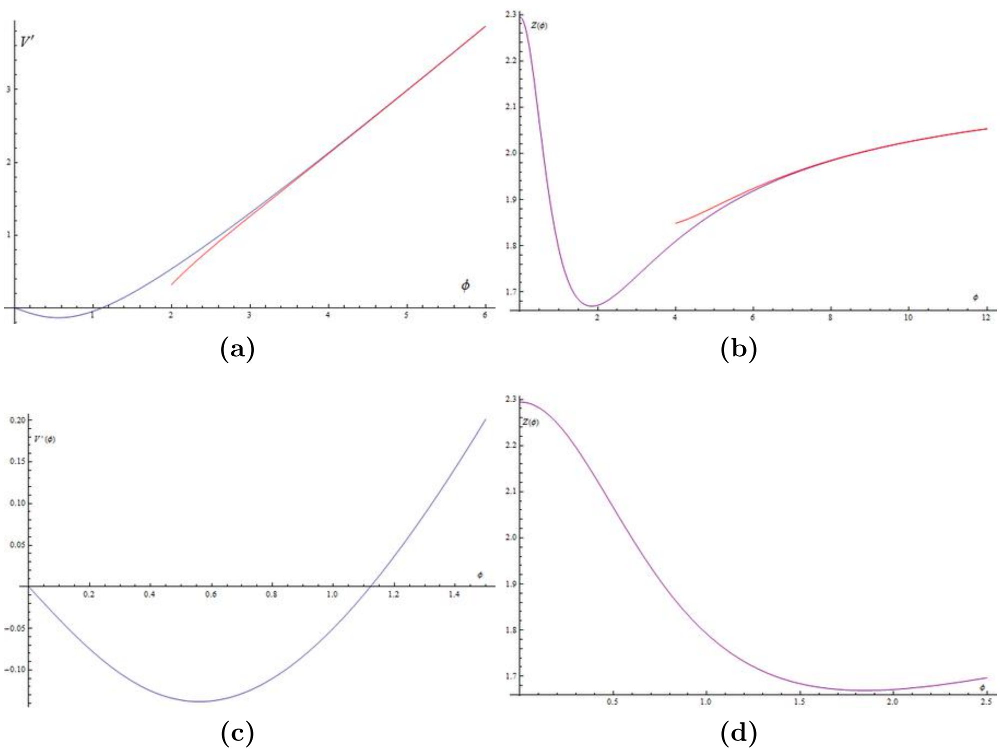
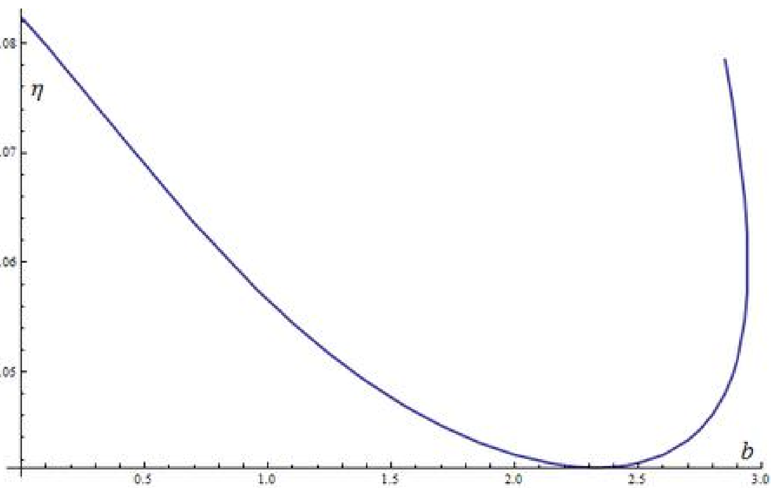
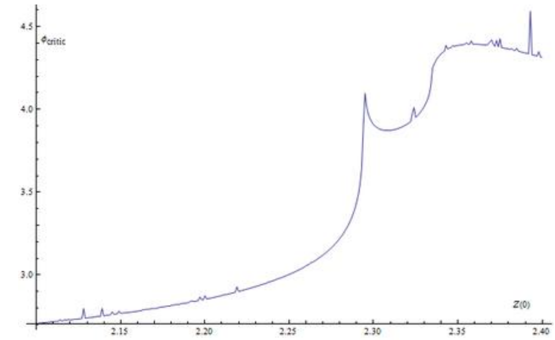



**Figure 4.9:** Upper panel: the function $U(\phi) = V'(\phi)$ and $Z(\phi)$ for $b = 2.3$, at the values of $A = A^*, \eta = \eta^* = 0.04127$ corresponding to the zeros for $V'(0)$ and $Z'(0)$ (red lines are the asymptotic behaviors). These solutions are obtained by numerical shooting integration from $\phi = 20$ to $\phi = 0$.
Bottom panel: the functions $U(\phi)$ and $Z(\phi)$ near the origin.

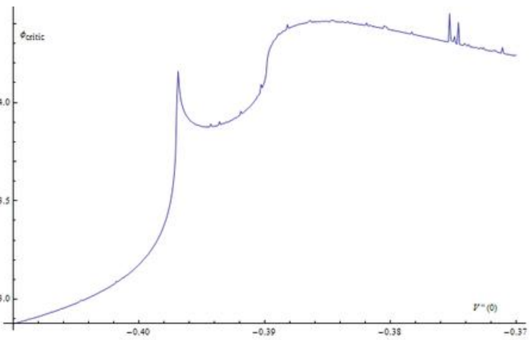



**(a)**

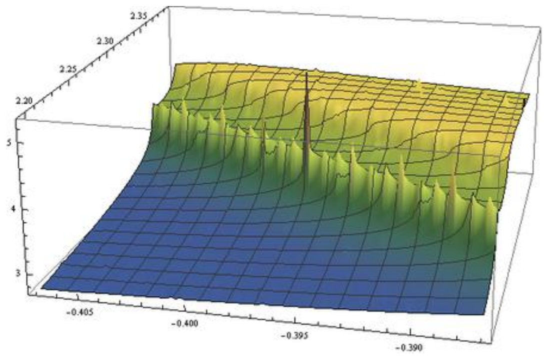
**Figure 4.10:** The anomalous dimension $\eta^*$ as a function of the regulator-dependent coefficient $b$. Each point corresponds to a Wilson Fisher FP solution for the Polchinski flow equations.



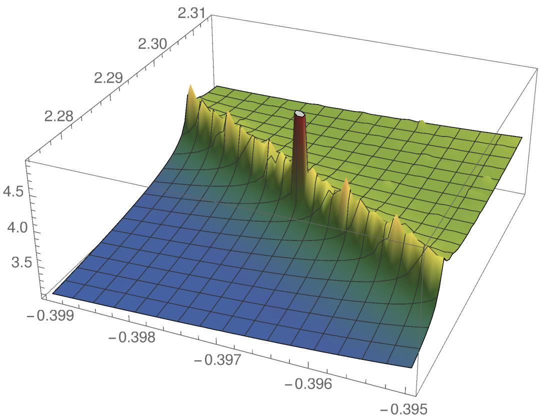

**(a)** Spike plot of $\phi_{\text{critic}}$ as a function of $Z(0)$ for the value of $V''(0)$ corresponding to the global solution.




**(b)** Spike plot of $\phi_{\text{critic}}$ as a function of $V''(0)$ for the value of $Z(0)$ corresponding to the global solution.

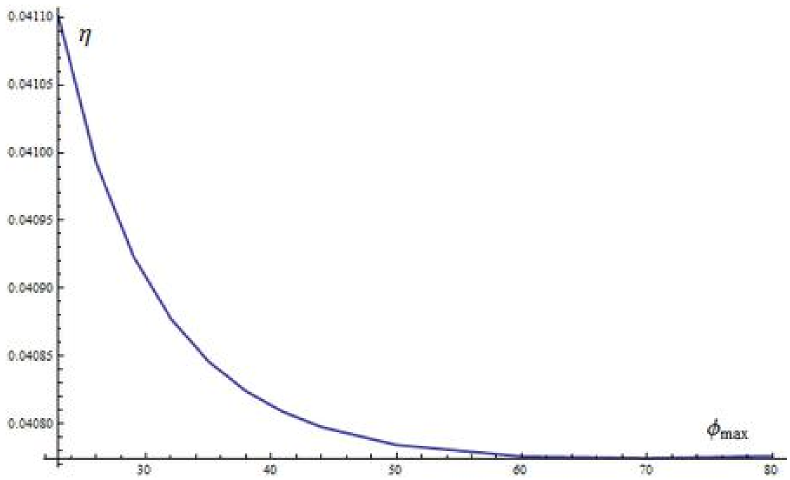



**(c)** Spike plot of $\phi_{\text{critic}}$ as a function of $\big(Z(0), V''(0)\big)$ in a neighborhood of the values corresponding to the global solution.




**(d)** Zoom area near the global solution.

**Figure 4.11:** From the numerical solution founded with the shooting method for $b = 2.3$ and $\eta = 0.04127(4)$ we have obtained that $Z(0) = 2.294(2)$ and $V''(0) = -0.3969(7)$. Fixing $(b, \eta)$ we can do a numerical integration from the origin and plot $\phi_{\text{critic}}$ as a function of the parameters $\big(Z(0), V''(0)\big)$: near the values corresponding to the global solution $\phi_{\text{critic}}$ increases in a steep way and we can see a spike.




**(a)**

**Figure 4.12:** Variation of the anomalous dimension $\eta^*$ as a function of $\phi_{\max}$, the starting point of numerical integration, for the fixed value $b = 2.3$.

# Conclusion

In this thesis we have presented a covariant "Hamiltonian" version of the Polchinski equation for the Wilson action, as proposed in [36]. We have applied this formulation for a theory with only one real scalar field and also for a theory with $N$ real scalar fields with the $O(N)$ internal symmetry. In both cases we have finally truncated the RG flow equation of the Hamiltonian at the quadratic order in the covariant momentum fields that, in some sense, corresponds to the leading order $\mathcal{O}(\partial^2)$ in the derivative expansion, with a full "Hamiltonian flow" which can include also a resummed family of contributions. For the former case without resummation, we have obtained the flow equations in the $N = 1$ case for an arbitrary potential $V(\phi)$ and for the wave function renormalization $Z(\phi)$ whereas in the $O(N)$ for an arbitrary potential $V(\phi)$ and for the two wave functions $Z(\phi)$ and $Y(\phi)$. These coupled equations are all of the second order but highly non linear.

In the latter case we have observed that, despite the usually partial decoupling present in the truncated 1PI vertex generator in the large $N$ limit, the flow equations of the Wilsonian action for $V(\phi)$ and $Z(\phi)$ do not decouple from that one for $Y(\phi)$. In this limit we have found a special analytic scaling solutions, with a quadratic potential which may be interesting. A more general analysis could be carried on in this direction.

In the case of $N = 1$ real scalar field we have analyzed numerically, using a numerical shooting method from large field value, the scaling solutions in $d = 3$ which corresponds to the critical Ising model. In the equation for $\dot{Z}$ there is a regulator dependent term with a non negative coefficient $b$.

We have analyzed firstly the case $b = 0$ and we have found a family of Wilson Fisher fixed points but with a negative solution for the wave function renormalization $Z(\phi)$ (see fig.4.3), which are therefore unphysical since the scalar field would be a ghost. Moreover we have found no special behavior, for values of the anomalous dimension $\eta$ close to the correct 0.036, in the parameter space $(\eta, A, B)$ (see fig.4.2) where $A, B$ are the parameters that govern the asymptotic expansion for the scaling solutions (neglecting the parameters which parametrize deformations from it with an essential singular behavior at $\phi = \infty$). Moreover we have done also a polynomial analysis around the origin that confirms the existence of these spurious unphysical solutions (see again fig.4.3), at least up to some order of the polynomial truncation $\sim \phi^{14}$. The polynomial analysis around the origin could be probably improved on performing a conformal mapping from a wedge region which avoid the closest singularity on a complex $\phi$ onto a disk. In the new variables the polynomial truncations should present better convergence properties. In conclusion a more systematic numerical analysis from the origin should be done.

Secondly we have analyzed the case of $b > 0$ taking under consideration a more general cutoff. We have found a family of Wilson Fisher fixed points in the range of $0 < b \leq b_{\max} \sim 3$ with a positive $Z(\phi)$ so describing in this case the propagation of a physical scalar field (see fig.4.12a). Moreover in the parameter space $(\eta, b)$ we have found a minimum at $\eta = 0.04126$ for a certain value $b_0$ and this is in agreement with the principle of minimum sensitivity, in other words we can choose a particular cutoff regulator such that the variation for the anomalous dimension is minimized. In this case with $b > 0$ the starting point for the shooting method that we have chosen is $\phi_{\max} = 20$. Since the convergence of $V_{as}(\phi)$ and $Z_{as}(\phi)$ is faster if we increase $\phi_{\max}$ we expect that $\mu$ is a function of $\phi_{\max}$ itself. For this reason we have taken the particular value $b = 2.3$ and analyzed the variation of $\mu$ increasing $\phi_{\max}$ (see fig.4.12). We have seen that the anomalous dimension from the value of $\mu \simeq 0.04110$ converges to the value of $\simeq 0.04077$ and this convergence is important because tells us that the numerical problem of having $\mu = \mu(\phi_{\max})$ is under control. Despite the previous case, the polynomial analysis around the origin made for $b > 0$ seems to have a zero convergence radius but a more systematic analysis has to be done especially for high order polynomials. In this case clearly the conformal mapping techniques mentioned above can help to find a polynomial representation around the origin. Waiting to perform such an analysis we have used an alternative way: a 3D-spike plot method based upon integrating the two fixed point equations from the origin, on varying the two left initial conditions (the other two are fixed by the $\mathbb{Z}_2$ symmetry condition) near some fixed point solution previously founded. The existence of a spike for the right values of the two parameters confirms the existence of the global solution obtained from the large field method (see fig.4.11).

Obvious lines of developments which may follow our work also include:

1. study the $O(N)$ model described by the $V, Z, Y$ truncation;

2. perform an analysis in fractional dimensions to see the appearing of multi-critical scaling solutions;

3. one should also perform, as illustrated, a linear analysis around the fixed point to extract the critical exponent $\nu$;

4. investigate the resummed covariant Hamiltonian equation at all orders in the momenta $\pi_\mu \pi^\mu$ which may lead to better numerical results;

5. derive the Lagrangian equation at quartic order in $\phi_\mu$, which would add other three terms to the flow equation, depending only on $K_1$, *i.e.* to a specific class of coarse-graining procedures and the corresponding covariant Hamiltonian formulation.

In the steps 4 and 5 it is expected to obtain results with a better accuracy for estimate of the anomalous dimension.

# Asymptotic expansion for $V_{as}(\phi), Y_{as}(\phi), Z_{as}(\phi)$ in the case of $b = 0$

$$\mathbf{V_{as}}(\phi) = \mathbf{A_0}\phi^{\mathbf{2}} + \mathbf{A}\phi^{\alpha} + \sum_{\mathbf{i=1}}^{\mathbf{18}} \mathbf{A_i}\phi^{\alpha_{\mathbf{i}}} \tag{24}$$

We have stopped the $V_{as}(\phi)$ expansion up to the 18-th term which correspond to that one involving the second term in the expansion for $Z_{as}(\phi)$ *i.e* $Z_1\phi^{\zeta_1}$.

$$A_0 = \frac{2-\eta}{4} \tag{25}$$

$$A_1 = -\frac{4A^2d^2}{(\eta-2)(d-\eta+2)^2} \tag{26}$$

$$A_2 = \frac{\eta-2}{2d} \tag{27}$$

$$A_3 = \frac{16A^3d^3(d+\eta-2)}{(\eta-2)^2(d-\eta+2)^4} \tag{28}$$

$$A_4 = -\frac{2Ad(d+\eta-2)}{(d-\eta+2)^3} \tag{29}$$

$$A_5 = -\frac{64A^4d^4(3d+5(\eta-2))(d+\eta-2)}{3(\eta-2)^3(d-\eta+2)^6} \tag{30}$$

$$A_6 = \frac{8A^2d^2\left(d^2+2d(\eta-2)-5(\eta-2)^2\right)(d+\eta-2)}{(\eta-2)(d-2\eta+4)(d-\eta+2)^5} \tag{31}$$

$$A_7 = \frac{128A^5d^5\left(6d^2+23d(\eta-2)+21(\eta-2)^2\right)(d+\eta-2)}{3(\eta-2)^4(d-\eta+2)^8} \tag{32}$$

$$A_8 = -\frac{32A^3d^3\left(d^4+4d^3(\eta-2)-15d^2(\eta-2)^2-22d(\eta-2)^3+44(\eta-2)^4\right)(d+\eta-2)}{(\eta-2)^2(d-3\eta+6)(d-2\eta+4)(d-\eta+2)^7} \tag{33}$$

$$A_9 = -\frac{1024A^6d^6\left(5d^3+32d^2(\eta-2)+65d(\eta-2)^2+42(\eta-2)^3\right)(d+\eta-2)}{5(\eta-2)^5(d-\eta+2)^{10}} \tag{34}$$

$$A_{10} = -\frac{2Ad(\eta-2)(d-3\eta+6)(d+\eta-2)}{(d-\eta+2)^6} \tag{35}$$

$$A_{11} = \big\{128A^4d^4\big(3d^6+17d^5(\eta-2)-100d^4(\eta-2)^2-266d^3(\eta-2)^3+859d^2(\eta-2)^4 \tag{36}$$
$$+777d(\eta-2)^5-1674(\eta-2)^6\big)(d+\eta-2)\big\}$$
$$/\big\{3(\eta-2)^3(d-4\eta+8)(d-3\eta+6)(d-2\eta+4)(d-\eta+2)^9\big\} \tag{37}$$

$$A_{12} = \big\{2048A^7d^7\big(90d^4+837d^3(\eta-2)+2780d^2(\eta-2)^2+3917d(\eta-2)^3$$
$$+1980(\eta-2)^4\big)(d+\eta-2)\big\}/\big\{45(\eta-2)^6(d-\eta+2)^{12}\big\} \tag{38}$$

$$A_{13} = \frac{16A^2d^2\left(2d^4-7d^3(\eta-2)-30d^2(\eta-2)^2+95d(\eta-2)^3-64(\eta-2)^4\right)(d+\eta-2)}{(2d-3\eta+6)(d-2\eta+4)(d-\eta+2)^8} \tag{39}$$

$$A_{14} = -\frac{d+\eta-2}{B\left(d^2+d\eta-2\eta+4\right)} \tag{40}$$

$$A_{15} = -\big\{256A^5d^5\big(6d^8+41d^7(\eta-2)-384d^6(\eta-2)^2-1244d^5(\eta-2)^3+6980d^4(\eta-2)^4$$
$$+9435d^3(\eta-2)^5-38906d^2(\eta-2)^6-19512d(\eta-2)^7+55584(\eta-2)^8\big)(d+\eta-2)\big\}$$
$$/\big\{3(\eta-2)^4(d-5\eta+10)(d-4\eta+8)(d-3\eta+6)(d-2\eta+4)(d-\eta+2)^{11}\big\} \tag{41}$$

$$A_{16} = -\big\{16384A^8d^8\big(315d^5+3933d^4(\eta-2)+18718d^3(\eta-2)^2+42538d^2(\eta-2)^3$$
$$+46263d(\eta-2)^4+19305(\eta-2)^5\big)(d+\eta-2)\big\}/\big\{315(\eta-2)^7(d-\eta+2)^{14}\big\} \tag{42}$$

$$A_{17} = -\big\{96A^3d^3\big(2d^7-9d^6(\eta-2)-85d^5(\eta-2)^2+415d^4(\eta-2)^3-81d^3(\eta-2)^4$$
$$-1894d^2(\eta-2)^5+3032d(\eta-2)^6-1404(\eta-2)^7\big)(d+\eta-2)\big\}$$
$$/\big\{(\eta-2)(d-3\eta+6)(2d-3\eta+6)(d-2\eta+4)^2(d-\eta+2)^{10}\big\} \tag{43}$$

$$A_{18} = \frac{4Ad(d+2)(d+\eta-2)^2}{B(\eta-2)(d-\eta+2)^2(d+\eta)\left(d^2+d\eta-2\eta+4\right)} \tag{44}$$

$$\alpha = \frac{2d}{d-\eta+2} \tag{45}$$

$$\alpha_1 = \frac{4d}{d-\eta+2} - 2 \tag{46}$$

$$\alpha_2 = 0 \tag{47}$$

$$\alpha_3 = \frac{6d}{d-\eta+2} - 4 \tag{48}$$

$$\alpha_4 = \frac{2(\eta-2)}{d-\eta+2} \tag{49}$$

$$\alpha_5 = \frac{2(d+3\eta-6)}{d-\eta+2} \tag{50}$$

$$\alpha_6 = \frac{4(\eta-2)}{d-\eta+2} \tag{51}$$

$$\alpha_7 = \frac{2(d+4\eta-8)}{d-\eta+2} \tag{52}$$

$$\alpha_8 = \frac{6(\eta-2)}{d-\eta+2} \tag{53}$$

$$\alpha_9 = \frac{2(d+5(\eta-2))}{d-\eta+2} \tag{54}$$

$$\alpha_{10} = \frac{2(\eta-2)}{d-\eta+2} - 2 \tag{55}$$

$$\alpha_{11} = \frac{8(\eta-2)}{d-\eta+2} \tag{56}$$

$$\alpha_{12} = \frac{2(d+6(\eta-2))}{d-\eta+2} \tag{57}$$

$$\alpha_{13} = \frac{4(\eta-2)}{d-\eta+2} - 2 \tag{58}$$

$$\alpha_{14} = -\beta = -\frac{4}{d+\eta-2} \tag{59}$$

$$\alpha_{15} = \frac{10(\eta-2)}{d-\eta+2} \tag{60}$$

$$\alpha_{16} = \frac{2(d+7(\eta-2))}{d-\eta+2} \tag{61}$$

$$\alpha_{17} = \frac{6(\eta-2)}{d-\eta+2} - 2 \tag{62}$$

$$\alpha_{18} = \beta_1 - 2\beta = \frac{2d}{d-\eta+2} - \frac{4}{d+\eta-2} - 2 \tag{63}$$

$$\mathbf{Y_{as}}(\phi) = \mathbf{B}\phi^{\beta} + \sum_{\mathbf{i=1}}^{\mathbf{23}} \mathbf{B_i}\phi^{\beta_i} \tag{64}$$

We have stopped the $Y_{as}(\phi)$ expansion up to the 21-th term which correspond to that one involving the last term in the expansion for $V_{as}(\phi)$ *i.e* $A_{18}\phi^{\alpha_{18}}$.

$$B_1 = \frac{4ABd}{(\eta-2)(d-\eta+2)} \tag{65}$$

$$B_2 = -\frac{32A^2Bd^2}{(\eta-2)(d-\eta+2)^3} \tag{66}$$

$$B_3 = \frac{4B(d+\eta-6)}{(d+\eta-2)^3} \tag{67}$$

$$B_4 = \frac{64A^3Bd^3(d+5(\eta-2))}{(\eta-2)^2(d-\eta+2)^5} \tag{68}$$

$$\begin{aligned} B_5 =& \big\{8ABd\big(d^5\eta - 4d^4\big(\eta^2-2\big) - 2d^3(\eta-2)((\eta-16)\eta+8) + 4d^2(\eta-2)^2(\eta(3\eta-11)-8) \\ &+ d(\eta-2)^3(\eta(9\eta-62)+144) + 4(\eta-2)^4(3\eta-10)\big)\big\} \\ &/\big\{(\eta-2)(d-2\eta+4)(d-\eta+2)^4(d+\eta-2)^3\big\} \end{aligned} \tag{69}$$

$$B_6 = -\frac{512A^4Bd^4(d+7(\eta-2))(d+3\eta-6)}{3(\eta-2)^3(d-\eta+2)^7} \tag{70}$$

$$\begin{aligned} B_7 =& 6 - \big\{128A^2Bd^2\big(d^6(\eta-1) + d^5((19-9\eta)\eta-6) + 2d^4(\eta-2)(9(\eta-1)\eta-2) \\ &+ 2d^3(\eta-2)^2(\eta(17\eta-91)+66) - d^2(\eta-2)^3(31(\eta-3)\eta-74) \\ &- d(\eta-2)^4(\eta(41\eta-227)+382) - 4(\eta-2)^5(\eta(\eta+2)-14)\big)\big\} \\ &/\big\{(\eta-2)(d-3\eta+6)(d-2\eta+4)(d-\eta+2)^6(d+\eta-2)^3\big\} \end{aligned} \tag{71}$$

$$B_8 = \frac{512A^5Bd^5(3d+7(\eta-2))(d+9(\eta-2))(d+4\eta-8)}{3(\eta-2)^4(d-\eta+2)^9} \tag{72}$$

$$B_9 = -\frac{4B(d+\eta-6)(d+\eta-4)(3d+3\eta-10)}{(d+\eta-2)^6} \tag{73}$$

$$\begin{aligned} B_{10} =& 6\big\{128A^3Bd^3(d^8(3\eta-4) - 4d^7(\eta(7\eta-17)+8) - d^6(\eta-2)(\eta(11\eta-54)+8) \\ &+ 2d^5(\eta-2)^2(\eta(491\eta-1466)+984) - d^4(\eta-2)^3(\eta(1219\eta-1986)+296) \\ &- 4d^3(\eta-2)^4(\eta(794\eta-3703)+3248) + d^2(\eta-2)^5(\eta(1595\eta-4606)-2744) \\ &+ 2d(\eta-2)^6(\eta(1543\eta-7702)+11096) + 16(\eta-2)^7(\eta(31\eta-55)-74))\big\} \\ &/\big\{(\eta-2)^2(d-4\eta+8)(d-3\eta+6)(d-2\eta+4)(d-\eta+2)^8(d+\eta-2)^3\big\} \end{aligned} \tag{74}$$

$$B_{11} = -\frac{8192A^6Bd^6(d+5(\eta-2))(d+11(\eta-2))(d+2\eta-4)(d+3\eta-6)}{5(\eta-2)^5(d-\eta+2)^{11}} \tag{75}$$

$$\begin{aligned} B_{12} =& -\big\{8ABd\big(6d^{11}\eta + d^{10}(\eta(30-49\eta)+48) + 2d^9(\eta(\eta(46\eta+75)-378)+184) \\ &+ 4d^8(\eta(\eta(\eta(39\eta-545)+1428)-892)-224) - 8d^7(\eta-2)(\eta(\eta(\eta(61\eta-360)-8) \\ &+ 1484)-1184) - 2d^6(\eta-2)^2(\eta(\eta(3\eta(35\eta-874)+13364)-18296)+4928) \\ &+ 4d^5(\eta-2)^3(7\eta(\eta(\eta(31\eta-293)+652)+356)-7408) \\ &+ 8d^4(\eta-2)^4(\eta(\eta(5\eta(7\eta-142)+4806)-11628)+7072) \\ &- 2d^3(\eta-2)^5(\eta(\eta(\eta(271\eta-3530)+14884)-18792)-5632) \\ &- d^2(\eta-2)^6(\eta(\eta(\eta(245\eta-2706)+16684)-55576)+70720) \\ &- 2d(\eta-2)^7(\eta(\eta(\eta(32\eta+321)-4680)+16708)-19152) \\ &- 12(\eta-3)(\eta-2)^8(5\eta-14)((\eta-6)\eta+16)\big)\big\} \\ &/\big\{(\eta-2)(2d-3\eta+6)(d-2\eta+4)(d-\eta+2)^7(d+\eta-2)^6\big\} \end{aligned} \tag{76}$$

$$B_{13} = -\big\{2048A^4Bd^4\big(d^{10}(2\eta-3)+d^9\big(-21\eta^2+56\eta-32\big)-2d^8\big(19\eta^3-84\eta^2+94\eta-4\big)$$
$$+2d^7(\eta-2)^2\big(835\eta^2-2614\eta+1954\big)-6d^6(\eta-2)^3\big(329\eta^2-865\eta+676\big)$$
$$-4d^5(\eta-2)^4\big(4937\eta^2-17387\eta+14171\big)+2d^4(\eta-2)^5\big(11073\eta^2-27062\eta+12658\big)$$
$$+2d^3(\eta-2)^6\big(28485\eta^2-127354\eta+123622\big)-d^2(\eta-2)^7\big(21972\eta^2-68793\eta-5234\big)$$
$$-3d(\eta-2)^8\big(17053\eta^2-79828\eta+104236\big)-12(\eta-2)^9\big(871\eta^2-2434\eta+544\big)\big)\big\}$$
$$/\big\{3(\eta-2)^3(d-5\eta+10)(d-4\eta+8)(d-3\eta+6)(d-2\eta+4)(d-\eta+2)^{10}(d+\eta-2)^3\big\} \tag{77}$$

$$B_{14} = 6\big\{8192A^7Bd^7\big(30d^5+809d^4(\eta-2)+7489d^3(\eta-2)^2+30689d^2(\eta-2)^3$$
$$+56829d(\eta-2)^4+38610(\eta-2)^5\big)\big\}/\big\{45(\eta-2)^6(d-\eta+2)^{13}\big\} \tag{78}$$

$$B_{15} = \big\{128A^2Bd^2\big(6d^{13}(\eta-1)+d^{12}\big(-102\eta^2+229\eta-94\big)$$
$$+d^{11}\big(637\eta^3-2252\eta^2+2274\eta-540\big)$$
$$+d^{10}\big(-1448\eta^4+5397\eta^3-4268\eta^2-2972\eta+2944\big)$$
$$+d^9\big(-827\eta^5+18600\eta^4-92008\eta^3+182592\eta^2-154384\eta+43328\big)$$
$$+2d^8(\eta-2)^2\big(3647\eta^4-28069\eta^3+59598\eta^2-37436\eta-280\big)$$
$$-2d^7(\eta-2)^3\big(1591\eta^4+3090\eta^3-68106\eta^2+144944\eta-78600\big)$$
$$-2d^6(\eta-2)^4\big(6752\eta^4-69641\eta^3+230436\eta^2-257204\eta+67888\big)$$
$$+2d^5(\eta-2)^5\big(3384\eta^4-18503\eta^3-19966\eta^2+209292\eta-202872\big)$$
$$+d^4(\eta-2)^6\big(12246\eta^4-140447\eta^3+598974\eta^2-1043588\eta+526792\big)$$
$$-d^3(\eta-2)^7\big(2495\eta^4-31386\eta^3+97054\eta^2+6080\eta-257528\big)$$
$$-d^2(\eta-2)^8\big(4152\eta^4-47577\eta^3+234620\eta^2-567924\eta+544352\big)$$
$$-d(\eta-2)^9\big(1419\eta^4-2518\eta^3-39716\eta^2+167384\eta-187680\big)$$
$$-6(\eta-2)^{10}\big(141\eta^4-1298\eta^3+4880\eta^2-9032\eta+6864\big)\big)\big\}$$
$$/\big\{(\eta-2)(d-3\eta+6)(2d-3\eta+6)(d-2\eta+4)^2(d-\eta+2)^9(d+\eta-2)^6\big\} \tag{79}$$

$$B_{16} = \frac{8(d+\eta+2)}{(d+\eta-2)(d+\eta)\big(d^2+d\eta-2\eta+4\big)} \tag{80}$$

$$B_{17} = \big\{1024A^5Bd^5\big(3d^{12}(5\eta-8)-8d^{11}\big(23\eta^2-66\eta+43\big)$$
$$-2d^{10}\big(227\eta^3-950\eta^2+984\eta+16\big)+2d^9(\eta-2)^2\big(12797\eta^2-41744\eta+33120\big)$$
$$-36d^8(\eta-2)^3\big(1248\eta^2-4315\eta+4168\big)-18d^7(\eta-2)^4\big(33743\eta^2-116960\eta$$
$$+97400\big)+2d^6(\eta-2)^5\big(561839\eta^2-1770692\eta+1422128\big)+2d^5(\eta-2)^6\big(2250831\eta^2$$
$$-8669872\eta+7836128\big)-d^4(\eta-2)^7\big(5753615\eta^2-17904442\eta+12319504\big)$$
$$-2d^3(\eta-2)^8\big(6105729\eta^2-26803992\eta+27485308\big)+8d^2(\eta-2)^9\big(573661\eta^2$$
$$-2011669\eta+853426\big)+48d(\eta-2)^{10}\big(222745\eta^2-997720\eta+1215148\big)$$
$$+1152(\eta-2)^{11}\big(2158\eta^2-6967\eta+4042\big)\big)\big\}/\big\{3(\eta-2)^4(d-6\eta+12)\cdot$$
$$\cdot(d-5\eta+10)(d-4\eta+8)(d-3\eta+6)(d-2\eta+4)(d-\eta+2)^{12}(d+\eta-2)^3\big\} \tag{81}$$

$$B_{18} = \frac{16B(d+\eta-6)(d+\eta-4)(d+\eta-3)(3d+3\eta-10)(5d+5\eta-14)}{3(d+\eta-2)^9} \tag{82}$$

$$B_{19} = -\big\{131072A^8Bd^8\big(45d^6+1494d^5(\eta-2)+17839d^4(\eta-2)^2+100964d^3(\eta-2)^3$$
$$+291259d^2(\eta-2)^4+411750d(\eta-2)^5+225225(\eta-2)^6\big)\big\}$$
$$/\big\{315(\eta-2)^7(d-\eta+2)^{15}\big\} \tag{83}$$

$$
\begin{aligned}
B_{20} =& - \big\{128A^3Bd^3\big(12d^{16}(3\eta-4) - 8d^{15}\big(95\eta^2 - 253\eta + 148\big)\\
&+ d^{14}\big(5595\eta^3 - 23688\eta^2 + 32380\eta - 14384\big) + d^{13}\big(-3046\eta^4 + 1066\eta^3 + 31216\eta^2\\
&- 47992\eta + 11072\big) + d^{12}\big(-160013\eta^5 + 1413768\eta^4 - 4865536\eta^3 + 8113392\eta^2 - 6502576\eta\\
&+ 1976000\big) + 2d^{11}(\eta-2)^2\big(375680\eta^4 - 2257697\eta^3 + 4891828\eta^2 - 4519476\eta + 1510592\big)\\
&- 2d^{10}(\eta-2)^3\big(389927\eta^4 - 1010082\eta^3 - 1016572\eta^2 + 4189800\eta - 2484928\big)\\
&- 4d^9(\eta-2)^4\big(587953\eta^4 - 5652599\eta^3 + 16705576\eta^2 - 19253964\eta + 7404704\big)\\
&+ 2d^8(\eta-2)^5\big(2522577\eta^4 - 16982006\eta^3 + 33509204\eta^2 - 19306952\eta - 1181856\big)\\
&+ 4d^7(\eta-2)^6\big(585868\eta^4 - 9011479\eta^3 + 39997956\eta^2 - 63700076\eta + 32180000\big)\\
&- d^6(\eta-2)^7\big(9429049\eta^4 - 83097418\eta^3 + 244302500\eta^2 - 257663416\eta + 69210784\big)\\
&- 2d^5(\eta-2)^8\big(530051\eta^4 - 11507921\eta^3 + 73568760\eta^2 - 173719732\eta + 124478176\big)\\
&+ d^4(\eta-2)^9\big(7661863\eta^4 - 77899386\eta^3 + 292034348\eta^2 - 455102648\eta + 218332448\big)\\
&+ 2d^3(\eta-2)^{10}\big(620380\eta^4 - 4807909\eta^3 + 22007860\eta^2 - 64637764\eta + 78059328\big)\\
&- 4d^2(\eta-2)^{11}\big(495645\eta^4 - 5656948\eta^3 + 25985752\eta^2 - 56295024\eta + 47817936\big)\\
&- 8d(\eta-2)^{12}\big(130536\eta^4 - 692511\eta^3 + 354532\eta^2 + 3415508\eta - 5040992\big)\\
&- 96(\eta-2)^{13}\big(5058\eta^4 - 42237\eta^3 + 137800\eta^2 - 214132\eta + 136432\big)\big)\big\}\\
&/\big\{(\eta-2)^2(2d-5\eta+10)(d-4\eta+8)(d-3\eta+6)(2d-3\eta+6)(d-2\eta+4)^2\cdot\\
&\cdot(d-\eta+2)^{11}(d+\eta-2)^6\big\} \qquad (84)
\end{aligned}
$$

$$
\begin{aligned}
B_{21} =& \big\{16Ad\big(d^5 + 6d^4 + d^3\big(-6\eta^2 + 32\eta - 48\big) - 4d^2\big(2\eta^3 - 9\eta^2 + 2\eta + 24\big)\\
&+ d\big(-3\eta^4 + 80\eta^2 - 160\eta + 48\big) - 2(\eta-2)^2\big(5\eta^2 - 4\big)\big)\big\}/\big\{(d-\eta+2)^3\cdot\\
&(d+\eta-2)(d+\eta)\big(d^2 + d\eta - 2\eta + 4\big)\big(d^2 - d\eta + 4d - 2\eta^2 + 6\eta - 4\big)\big\} \qquad (85)
\end{aligned}
$$

$$\beta = \frac{4}{d+\eta-2} \qquad (86)$$

$$\beta_1 = \frac{2d}{d-\eta+2} + \frac{4}{d+\eta-2} - 2 \qquad (87)$$

$$\beta_2 = 4\left(\frac{d}{d-\eta+2} + \frac{1}{d+\eta-2} - 1\right) \qquad (88)$$

$$\beta_3 = \frac{4}{d+\eta-2} - 2 \qquad (89)$$

$$\beta_4 = \frac{6d}{d-\eta+2} + \frac{4}{d+\eta-2} - 6 \qquad (90)$$

$$\beta_5 = \frac{2d}{d-\eta+2} + \frac{4}{d+\eta-2} - 4 \qquad (91)$$

$$\beta_6 = \frac{8d}{d-\eta+2} + \frac{4}{d+\eta-2} - 8 \qquad (92)$$

$$\beta_7 = \frac{4d}{d-\eta+2} + \frac{4}{d+\eta-2} - 6 \qquad (93)$$

$$\beta_8 = \frac{10d}{d-\eta+2} + \frac{4}{d+\eta-2} - 10 \tag{94}$$

$$\beta_9 = \frac{4}{d+\eta-2} - 4 \tag{95}$$

$$\beta_{10} = \frac{6d}{d-\eta+2} + \frac{4}{d+\eta-2} - 8 \tag{96}$$

$$\beta_{11} = 4\left(\frac{3d}{d-\eta+2} + \frac{1}{d+\eta-2} - 3\right) \tag{97}$$

$$\beta_{12} = \frac{2d}{d-\eta+2} + \frac{4}{d+\eta-2} - 6 \tag{98}$$

$$\beta_{13} = \frac{8d}{d-\eta+2} + \frac{4}{d+\eta-2} - 10 \tag{99}$$

$$\beta_{14} = \frac{14d}{d-\eta+2} + \frac{4}{d+\eta-2} - 14 \tag{100}$$

$$\beta_{15} = 4\left(\frac{d}{d-\eta+2} + \frac{1}{d+\eta-2} - 2\right) \tag{101}$$

$$\beta_{16} = \alpha_{14} + \beta - 2 = -2 \tag{102}$$

$$\beta_{17} = \frac{10d}{d-\eta+2} + 4\left(\frac{1}{d+\eta-2} - 3\right) \tag{103}$$

$$\beta_{18} = \frac{4}{d+\eta-2} - 6 \tag{104}$$

$$\beta_{19} = 4\left(\frac{4d}{d-\eta+2} + \frac{1}{d+\eta-2} - 4\right) \tag{105}$$

$$\beta_{20} = \frac{6d}{d-\eta+2} + \frac{4}{d+\eta-2} - 10 \tag{106}$$

$$\beta_{21} = \alpha_{18} + \beta - 2 = \frac{2d}{d-\eta+2} - 4 \tag{107}$$

$$Z_{as}(\phi) = Z_0\phi^{\zeta_0} + \sum_{i=1}^{14} Z_i\phi^{\zeta_i} \tag{108}$$

$$Z_0 = \frac{1}{B} \tag{109}$$

$$Z_1 = -\frac{B_1}{B^2} \tag{110}$$

$$Z_2 = \frac{B_1^2}{B^3} - \frac{B_2}{B^2} \tag{111}$$

$$Z_3 = -\frac{B_3}{B^2} \tag{112}$$

$$Z_4 = -\frac{B_1^3}{B^4} + \frac{2B_1B_2}{B^3} - \frac{B_4}{B^2} \tag{113}$$

$$Z_5 = \frac{2(B_1B_3)}{B^3} - \frac{B_5}{B^2} \tag{114}$$

$$Z_6 = \frac{B_1^4}{B^5} - \frac{3\left(B_1^2B_2\right)}{B^4} + \frac{2B_1B_4}{B^3} + \frac{B_2^2}{B^3} - \frac{B_6}{B^2} \tag{115}$$

$$Z_7 = -\frac{3\left(B_1^2B_3\right)}{B^4} + \frac{2B_1B_5}{B^3} + \frac{2B_2B_3}{B^3} - \frac{B_7}{B^2} \tag{116}$$

$$Z_8 = -\frac{B_1^5}{B^6} + \frac{4B_1^3B_2}{B^5} - \frac{3\left(B_1^2B_4\right)}{B^4} - \frac{3\left(B_1B_2^2\right)}{B^4} + \frac{2B_1B_6}{B^3} + \frac{2B_2B_4}{B^3} - \frac{B_8}{B^2} \tag{117}$$

$$Z_9 = \frac{B_3^2}{B^3} - \frac{B_9}{B^2} \tag{118}$$

$$Z_{10} = \frac{4B_1^3B_3}{B^5} - \frac{3\left(B_1^2B_5\right)}{B^4} - \frac{6(B_1B_2B_3)}{B^4} + \frac{2B_1B_7}{B^3} + \frac{2B_2B_5}{B^3} + \frac{2B_3B_4}{B^3} - \frac{B_{10}}{B^2} \tag{119}$$

$$\begin{aligned} Z_{11} =& \frac{B_1^6}{B^7} - \frac{5\left(B_1^4B_2\right)}{B^6} + \frac{4B_1^3B_4}{B^5} + \frac{6B_1^2B_2^2}{B^5} - \frac{3\left(B_1^2B_6\right)}{B^4} - \frac{6(B_1B_2B_4)}{B^4} - \frac{B_2^3}{B^4} \\ &+ \frac{2B_1B_8}{B^3} + \frac{2B_2B_6}{B^3} + \frac{B_4^2}{B^3} - \frac{B_{11}}{B^2} \end{aligned} \tag{120}$$

$$Z_{12} = -\frac{3\left(B_1B_3^2\right)}{B^4} + \frac{2B_1B_9}{B^3} + \frac{2B_3B_5}{B^3} - \frac{B_{12}}{B^2} \tag{121}$$

$$\begin{aligned} Z_{13} =& -\frac{5\left(B_1^4B_3\right)}{B^6} + \frac{4B_1^3B_5}{B^5} + \frac{12B_1^2B_2B_3}{B^5} - \frac{3\left(B_1^2B_7\right)}{B^4} - \frac{6(B_1B_2B_5)}{B^4} - \frac{6(B_1B_3B_4)}{B^4} \\ &- \frac{3\left(B_2^2B_3\right)}{B^4} + \frac{2B_1B_{10}}{B^3} + \frac{2B_2B_7}{B^3} + \frac{2B_3B_6}{B^3} + \frac{2B_4B_5}{B^3} - \frac{B_{13}}{B^2} \end{aligned} \tag{122}$$

$$Z_{14} = -\frac{B_1^7}{B^8} + \frac{6B_1^5B_2}{B^7} - \frac{5\left(B_1^4B_4\right)}{B^6} - \frac{10\left(B_1^3B_2^2\right)}{B^6} + \frac{4B_1^3B_6}{B^5} + \frac{12B_1^2B_2B_4}{B^5} + \frac{4B_1B_2^3}{B^5}$$
$$- \frac{3\left(B_1^2B_8\right)}{B^4} - \frac{6(B_1B_2B_6)}{B^4} - \frac{3\left(B_1B_4^2\right)}{B^4} - \frac{3\left(B_2^2B_4\right)}{B^4} + \frac{2B_1B_{11}}{B^3}$$
$$+ \frac{2B_2B_8}{B^3} + \frac{2B_4B_6}{B^3} - \frac{B_{14}}{B^2} \tag{123}$$

$$\zeta_0 = -\beta \tag{124}$$

$$\zeta_1 = \beta_1 - 2\beta \tag{125}$$

$$\zeta_2 = \beta_2 - 2\beta \tag{126}$$

$$\zeta_3 = \beta_3 - 2\beta \tag{127}$$

$$\zeta_4 = \beta_4 - 2\beta \tag{128}$$

$$\zeta_5 = \beta_5 - 2\beta \tag{129}$$

$$\zeta_6 = \beta_6 - 2\beta \tag{130}$$

$$\zeta_7 = \beta_7 - 2\beta \tag{131}$$

$$\zeta_8 = \beta_8 - 2\beta \tag{132}$$

$$\zeta_9 = \beta_9 - 2\beta \tag{133}$$

$$\zeta_{10} = \beta_{10} - 2\beta \tag{134}$$

$$\zeta_{11} = \beta_{11} - 2\beta \tag{135}$$

$$\zeta_{12} = \beta_{12} - 2\beta \tag{136}$$

$$\zeta_{13} = \beta_{13} - 2\beta \tag{137}$$

$$\zeta_{14} = \beta_{14} - 2\beta \tag{138}$$

# Asymptotic expansion for $V_{as}(\phi), Y_{as}(\phi), Z_{as}(\phi)$ in the case of $b \neq 0$

$$\mathbf{V_{as}}(\phi) = \mathbf{A_0}\phi^{\mathbf{2}} + \mathbf{A}\phi^{\alpha} + \sum_{\mathbf{i=1}}^{\mathbf{12}} \mathbf{A_i}\phi^{\alpha_\mathbf{i}} \tag{139}$$

We have stopped the $V_{as}(\phi)$ expansion up to the 12-th term which correspond to that one involving the 6-th term in the expansion for $Z_{as}(\phi)$ *i.e* $Z_6\phi^{\zeta_6}$.

$$A_0 = \frac{2-\eta}{4} \tag{140}$$

$$A_1 = -\frac{4A^2d^2}{(\eta-2)(d-\eta+2)^2} \tag{141}$$

$$A_2 = -\frac{(\eta-2)(b(\eta-2)-2)}{4d} \tag{142}$$

$$A_3 = \frac{16A^3d^3(d+\eta-2)}{(\eta-2)^2(d-\eta+2)^4} \tag{143}$$

$$A_4 = \frac{Ad(d+\eta-2)\left(b\left(\eta^2-4\right)(d-\eta+2)-2\left((d-6)\eta+\eta^2+8\right)\right)}{(d-\eta+2)^3\left((d-6)\eta+\eta^2+8\right)} \tag{144}$$

$$A_5 = -\frac{64A^4d^4(3d+5(\eta-2))(d+\eta-2)}{3(\eta-2)^3(d-\eta+2)^6} \tag{145}$$

$$\begin{aligned} A_6 = -&\Big\{4A^2d^2(d+\eta-2)\big(b(\eta-2)\big(d^4\left(\eta^2+2\eta-2\right)-4d^3(\eta-2)-2d^2(\eta-2)^2\left(2\eta^2+5\eta-2\right) \\ &+4d(\eta-2)^3(3\eta+7)+(\eta-2)^4\left(3\eta^2-4\eta-26\right)\big) \\ &-2\big(d^4(\eta-1)\eta+2d^3(\eta-2)^2(2\eta-1)-2d^2(\eta-2)^2(9\eta-10) \\ &-2d(\eta-2)^3\left(4\eta^2-13\eta-2\right)-5(\eta-2)^4\left(\eta^2-7\eta+12\right)\big)\big)\Big\} \\ &/\Big\{(\eta-2)(d-2\eta+4)(d-\eta+2)^5\left(d(\eta-1)+\eta^2-5\eta+6\right)\left((d-6)\eta+\eta^2+8\right)\Big\} \end{aligned} \tag{146}$$

$$A_7 = \frac{128A^5d^5\left(6d^2+23d(\eta-2)+21(\eta-2)^2\right)(d+\eta-2)}{3(\eta-2)^4(d-\eta+2)^8} \tag{147}$$

$$A_8 = 0 \tag{148}$$

$$\begin{aligned}
A_9 =& \big\{16A^3d^3(d+\eta-2)\big(-b(\eta-2)\big(-d^7\left(3\eta^3+4\eta^2-16\eta+8\right) \\
&+ d^6\left(-3\eta^4+\eta^3+52\eta^2-116\eta+64\right)+d^5(\eta-2)^2\left(36\eta^3+57\eta^2-134\eta+16\right) \\
&+ 2d^4(\eta-2)^3\left(15\eta^3-22\eta^2-244\eta+208\right)-d^3(\eta-2)^4\left(99\eta^3+142\eta^2-332\eta-344\right) \\
&- d^2(\eta-2)^5\left(87\eta^3-287\eta^2-1218\eta+1280\right)+d(\eta-2)^6\left(66\eta^3+89\eta^2-598\eta-1120\right) \\
&+ 2(\eta-2)^7\left(30\eta^3-119\eta^2-194\eta+832\right)\big)-2\big(d^7\eta\left(3\eta^2-7\eta+4\right) \\
&+ d^6\left(21\eta^4-113\eta^3+210\eta^2-152\eta+32\right)-12d^5(\eta-2)^2\left(11\eta^2-24\eta+12\right) \\
&- 2d^4(\eta-2)^3\left(81\eta^3-255\eta^2+108\eta+88\right)-d^3(\eta-2)^4\left(189\eta^3-1497\eta^2+2700\eta-1168\right) \\
&+ 3d^2(\eta-2)^5\left(51\eta^3-9\eta^2-724\eta+832\right)+22d(\eta-2)^6\left(15\eta^3-101\eta^2+188\eta-64\right) \\
&+ 44(\eta-2)^7\left(3\eta^3-29\eta^2+92\eta-96\right)\big)\big)\big\} \\
&/\big\{(\eta-2)^2(d-3\eta+6)(d-2\eta+4)(d-\eta+2)^7 \\
&\big(d(\eta-1)+\eta^2-5\eta+6\big)\left((d-6)\eta+\eta^2+8\right)\left(d(3\eta-4)+3\eta^2-14\eta+16\right)\big\}
\end{aligned} \tag{149}$$

$$A_{10} = -\frac{1024A^6d^6\left(5d^3+32d^2(\eta-2)+65d(\eta-2)^2+42(\eta-2)^3\right)(d+\eta-2)}{5(\eta-2)^5(d-\eta+2)^{10}} \tag{150}$$

$$\begin{aligned}
A_{11} =& \big\{Ad(\eta-2)(d+\eta-2)\big(b(\eta-2)(\eta+2)(d-3\eta+6)(d-\eta+2)\big(b\left(\eta^2-4\right)(d-\eta+2) \\
&+ 2\left(d^2-3d\eta+4d+2\eta-4\right)\big)\big)\big\} \\
&/\big\{\big(2(d-\eta+2)^6\left((d-6)\eta+\eta^2+8\right)\big)\left(d^2-d\eta-2\left(\eta^2-5\eta+6\right)\right)\big\} \\
&+ \big\{Ad(\eta-2)(d+\eta-2)\big(2(d-\eta+2)\big(b\left(\eta^2-4\right)(d-\eta+2) \\
&- 2\left((d-6)\eta+\eta^2+8\right)\big)-4(\eta-2)\big(b\left(\eta^2-4\right)(d-\eta+2)-2\left((d-6)\eta+\eta^2+8\right)\big)\big)\big\} \\
&/\big\{2(d-\eta+2)^6\left((d-6)\eta+\eta^2+8\right)\big\}
\end{aligned} \tag{151}$$

$$\begin{aligned}
A_{12} =& -\{64A^4d^4(d+\eta-2)\big(b(\eta-2)\big(3\big(6\eta^4+2\eta^3-51\eta^2+68\eta-24\big)d^{10} \\
&+ \big(30\eta^5-34\eta^4-569\eta^3+1894\eta^2-2104\eta+768\big)d^9 \\
&- 4(\eta-2)^2\big(117\eta^4+49\eta^3-791\eta^2+779\eta-136\big)d^8 \\
&- 2(\eta-2)^3\big(336\eta^4-252\eta^3-5595\eta^2+10606\eta-4720\big)d^7 \\
&+ 4(\eta-2)^4\big(795\eta^4+366\eta^3-3858\eta^2-547\eta+3508\big)d^6 \\
&+ 4(\eta-2)^5\big(1299\eta^4-2655\eta^3-16965\eta^2+35719\eta-14656\big)d^5 \\
&- 2(\eta-2)^6\big(2928\eta^4+4026\eta^3-7725\eta^2-69394\eta+74656\big)d^4 \\
&- 2(\eta-2)^7\big(6552\eta^4-19028\eta^3-59309\eta^2+148970\eta-43120\big)d^3 \\
&- (\eta-2)^8\big(654\eta^4-30898\eta^3+10895\eta^2+346160\eta-383352\big)d^2 \\
&+ 3(\eta-2)^9\big(2850\eta^4-9322\eta^3-20477\eta^2+66152\eta+3136\big)d \\
&+ 18(\eta-2)^{10}\big(210\eta^4-1340\eta^3+437\eta^2+10524\eta-16368\big)\big)+ \\
&- 2\big(3\eta\big(6\eta^3-23\eta^2+29\eta-12\big)d^{10} \\
&+ \big(174\eta^5-1165\eta^4+2955\eta^3-3470\eta^2+1800\eta-288\big)d^9 \\
&- 2(\eta-2)^2\big(42\eta^4+489\eta^3-2087\eta^2+2476\eta-888\big)d^8 \\
&- 4(\eta-2)^3\big(828\eta^4-3674\eta^3+4797\eta^2-1296\eta-676\big)d^7 \\
&- 2(\eta-2)^4\big(2202\eta^4-21241\eta^3+56013\eta^2-53844\eta+15816\big)d^6 \\
&+ 2(\eta-2)^5\big(6702\eta^4-20141\eta^3-16797\eta^2+82836\eta-52264\big)d^5 \\
&+ 4(\eta-2)^6\big(8136\eta^4-61888\eta^3+145509\eta^2-118032\eta+20292\big)d^4
\end{aligned}$$

$$+ 4(\eta-2)^7\big(1704\eta^4 - 43632\eta^3 + 204971\eta^2 - 324168\eta + 157900\big)d^3$$
$$- (\eta-2)^8\big(36462\eta^4 - 231371\eta^3 + 365893\eta^2 + 132196\eta - 411216\big)d^2$$
$$- 3(\eta-2)^9\big(11838\eta^4 - 116129\eta^3 + 403597\eta^2 - 572156\eta + 259584\big)d$$
$$- 1674(\eta-2)^{10}\big(6\eta^4 - 73\eta^3 + 329\eta^2 - 652\eta + 480\big)\big)\big)\big\}$$
$$/\big\{3(d-4\eta+8)(d-3\eta+6)(d-2\eta+4)(d-\eta+2)^9(\eta-2)^3\cdot$$
$$\cdot\big(\eta^2 - 5\eta + d(\eta-1) + 6\big)\big(\eta^2 + (d-6)\eta + 8\big)\big(2\eta^2 - 9\eta + d(2\eta-3) + 10\big)\cdot$$
$$\cdot\big(3\eta^2 - 14\eta + d(3\eta-4) + 16\big)\big\} \tag{152}$$

$$\alpha = \frac{2d}{d-\eta+2} \tag{153}$$

$$\alpha_1 = \frac{4d}{d-\eta+2} - 2 \tag{154}$$

$$\alpha_2 = \zeta_0 = 0 \tag{155}$$

$$\alpha_3 = \frac{6d}{d-\eta+2} - 4 \tag{156}$$

$$\alpha_4 = \zeta_1 = \frac{2(\eta-2)}{d-\eta+2} \tag{157}$$

$$\alpha_5 = \frac{8d}{d-\eta+2} - 6 \tag{158}$$

$$\alpha_6 = \zeta_2 = \frac{4(\eta-2)}{d-\eta+2} \tag{159}$$

$$\alpha_7 = \frac{10d}{d-\eta+2} - 8 \tag{160}$$

$$\alpha_8 = \zeta_3 = -2 \tag{161}$$

$$\alpha_9 = \frac{6(\eta-2)}{d-\eta+2} \tag{162}$$

$$\alpha_{10} = \frac{2(d+5(\eta-2))}{d-\eta+2} \tag{163}$$

$$\alpha_{11} = \zeta_5 = \frac{2(\eta-2)}{d-\eta+2} - 2 \tag{164}$$

$$\alpha_{12} = \zeta_6 = \frac{8(\eta-2)}{d-\eta+2} \tag{165}$$

$$\mathbf{Y_{as}}(\phi) = \mathbf{B} + \sum_{\mathbf{i=1}}^{\mathbf{13}} \mathbf{B_i}\phi^{\beta_\mathbf{i}} \tag{166}$$

$$B = \frac{4}{b(2-\eta)^2} \tag{167}$$

$$B_1 = \frac{16Ad(\eta+2)(d+\eta-2)}{b(\eta-2)^3(d-\eta+2)\left((d-6)\eta+\eta^2+8\right)} \tag{168}$$

$$\begin{aligned} B_2 =&\big\{64A^2d^2\big(d^4\left(\eta^2+2\eta-4\right)-2d^3\left(\eta^4-2\eta^3-6\eta^2+24\eta-24\right) \\ &-2d^2(\eta-2)^2\left(3\eta^3-\eta^2-24\eta+24\right)-2d(\eta-2)^3\left(3\eta^3-4\eta^2-26\eta+36\right) \\ &-(\eta-2)^4\left(2\eta^3-5\eta^2-14\eta+44\right)\big)\big\} \\ &/\big\{b(\eta-2)^4(d-\eta+2)^3\left(d(\eta-1)+\eta^2-5\eta+6\right)\left((d-6)\eta+\eta^2+8\right)^2\big\} \end{aligned} \tag{169}$$

$$B_3 = 0 \tag{170}$$

$$\begin{aligned} B_4 =&-\big\{256A^3d^3\big(d^7(\eta-2)^2(\eta+4)+d^6\left(-3\eta^5+16\eta^4+20\eta^3-212\eta^2+440\eta-320\right) \\ &+d^5\left(-30\eta^6+141\eta^5+72\eta^4-1896\eta^3+5168\eta^2-5936\eta+2624\right) \\ &-d^4(\eta-2)^2\left(105\eta^5-296\eta^4-952\eta^3+4892\eta^2-7224\eta+3520\right) \\ &-d^3(\eta-2)^3\left(180\eta^5-659\eta^4-1554\eta^3+9780\eta^2-14456\eta+7136\right) \\ &-d^2(\eta-2)^4\left(165\eta^5-792\eta^4-948\eta^3+10700\eta^2-18792\eta+10048\right) \\ &-d(\eta-2)^5\left(78\eta^5-475\eta^4-6\eta^3+5676\eta^2-13400\eta+8928\right) \\ &-(\eta-2)^6\left(15\eta^5-112\eta^4+128\eta^3+1092\eta^2-3752\eta+3520\right)\big)\big\} \\ &\big\{b(\eta-2)^4(d-\eta+2)^5\left(d(\eta-1)+\eta^2-5\eta+6\right)\left((d-6)\eta+\eta^2+8\right)^3\cdot \\ &\cdot\left(d(3\eta-4)+3\eta^2-14\eta+16\right)\big\} \end{aligned} \tag{171}$$

$$B_5 = \frac{16Ad(\eta+2)(d-3\eta+6)(d+\eta-2)\left(b\left(\eta^2-4\right)(d-\eta+2)+2\left(d^2-3d\eta+4d+2\eta-4\right)\right)}{b(\eta-2)^2(d-\eta+2)^4\left((d-6)\eta+\eta^2+8\right)\left(d^2-d\eta-2\left(\eta^2-5\eta+6\right)\right)} \tag{172}$$

$$\begin{aligned} B_6 =&\big\{1024A^4d^4\big(3d^{11}(\eta-2)^2\left(\eta^4-12\eta^2+34\eta-24\right) \\ &+d^{10}\left(-12\eta^8+121\eta^7-276\eta^6-1146\eta^5+8558\eta^4-23652\eta^3+35400\eta^2-28464\eta+9600\right) \\ &-d^9\big(216\eta^9-1933\eta^8+4552\eta^7+15972\eta^6-136074\eta^5 \\ &+425896\eta^4-765712\eta^3+835008\eta^2-518304\eta+141696\big) \\ &-d^8(\eta-2)^2\big(1548\eta^8-9935\eta^7+5748\eta^6+140518\eta^5-631666\eta^4 \\ &+1354604\eta^3-1655736\eta^2+1115088\eta-324480\big) \\ &-2d^7(\eta-2)^3\big(3024\eta^8-20495\eta^7+12386\eta^6+294824\eta^5-1303646\eta^4 \\ &+2699468\eta^3-3144312\eta^2+2000464\eta-544320\big) \\ &-2d^6(\eta-2)^4\big(7308\eta^8-55405\eta^7+53372\eta^6+766522\eta^5-3624294\eta^4 \\ &+7631348\eta^3-8823656\eta^2+5481584\eta-1442944\big) \\ &-2d^5(\eta-2)^5\big(11592\eta^8-100205\eta^7+154038\eta^6+1258888\eta^5-6886922\eta^4 \\ &+15463444\eta^3-18529256\eta^2+11693168\eta-3080128\big) \\ &-2d^4(\eta-2)^6\big(12348\eta^8-121903\eta^7+273580\eta^6+1278526\eta^5-8847570\eta^4 \\ &+22134252\eta^3-28621880\eta^2+19114320\eta-5234816\big) \\ &-d^3(\eta-2)^7\big(17568\eta^8-196975\eta^7+597578\eta^6+1478128\eta^5-14884286\eta^4 \end{aligned}$$

$$+ 43113164\eta^3 - 62383096\eta^2 + 45772752\eta - 13577536\big)$$
$$- d^2(\eta-2)^8\big(8028\eta^8 - 101405\eta^7 + 393012\eta^6 + 357298\eta^5 - 7714662\eta^4$$
$$+ 26783988\eta^3 - 44644456\eta^2 + 37238000\eta - 12417408\big)$$
$$- d(\eta-2)^9\big(2136\eta^8 - 30137\eta^7 + 143206\eta^6 - 45776\eta^5 - 2182594\eta^4$$
$$+ 9442436\eta^3 - 18513352\eta^2 + 18037296\eta - 7020480\big)$$
$$- 3(\eta-2)^{10}\big(84\eta^8 - 1313\eta^7 + 7420\eta^6 - 10070\eta^5 - 82542\eta^4$$
$$+ 473044\eta^3 - 1108168\eta^2 + 1283120\eta - 603776\big)\big)\big\}$$
$$/\big\{3b(\eta-2)^5(d-\eta+2)^7\left(d(\eta-1)+\eta^2-5\eta+6\right)^2\left((d-6)\eta+\eta^2+8\right)^4\cdot$$
$$\cdot\left(d(2\eta-3)+2\eta^2-9\eta+10\right)\left(d(3\eta-4)+3\eta^2-14\eta+16\right)\big\} \qquad (173)$$

$$B_7 = -\big\{64A^2d^2(d+\eta-2)\big(b(\eta-2)^2(\eta+2)\big(d^8\left(\eta^2+3\eta-6\right) - 3d^7\left(\eta^3-\eta^2-8\eta+4\right)$$
$$+ d^6\left(-25\eta^4 - 19\eta^3 + 286\eta^2 - 508\eta + 408\right)$$
$$+ d^5\left(49\eta^5 + 7\eta^4 - 344\eta^3 + 160\eta^2 - 48\eta + 528\right)$$
$$+ d^4(\eta-2)^2\left(93\eta^4 - 261\eta^3 - 1894\eta^2 + 3196\eta - 1080\right)$$
$$- d^3(\eta-2)^3\left(121\eta^4 + 141\eta^3 - 1750\eta^2 - 1612\eta + 1464\right)$$
$$- d^2(\eta-2)^4\left(147\eta^4 - 679\eta^3 - 1834\eta^2 + 9172\eta - 2184\right)$$
$$+ d(\eta-2)^5\left(75\eta^4 + 27\eta^3 - 1634\eta^2 + 860\eta + 6312\right)$$
$$+ 2(\eta-2)^6\left(39\eta^4 - 199\eta^3 - 106\eta^2 + 2072\eta - 3048\right)\big)$$
$$- 4\big(d^9\left(\eta^2+2\eta-4\right) + d^8\left(-2\eta^4 - 9\eta^3 + 8\eta^2 + 48\eta - 40\right)$$
$$+ d^7\left(20\eta^5 - 13\eta^4 - 130\eta^3 + 220\eta^2 - 264\eta + 272\right)$$
$$+ d^6\left(-50\eta^6 + 97\eta^5 + 214\eta^4 - 628\eta^3 + 1328\eta^2 - 2944\eta + 2272\right)$$
$$- d^5(\eta-2)^2\left(68\eta^5 - 493\eta^4 - 634\eta^3 + 3732\eta^2 - 2616\eta + 80\right)$$
$$+ d^4(\eta-2)^3\left(202\eta^5 - 341\eta^4 - 2588\eta^3 + 3484\eta^2 + 1416\eta - 336\right)$$
$$+ d^3(\eta-2)^4\left(156\eta^5 - 1447\eta^4 + 938\eta^3 + 11764\eta^2 - 15928\eta + 1072\right)$$
$$- d^2(\eta-2)^5\left(198\eta^5 - 375\eta^4 - 3568\eta^3 + 8380\eta^2 + 6152\eta - 12336\right)$$
$$- 2d(\eta-2)^6\left(86\eta^5 - 643\eta^4 + 720\eta^3 + 4250\eta^2 - 10488\eta + 4464\right)$$
$$- 2(\eta-2)^7\left(8\eta^5 - 147\eta^4 + 607\eta^3 - 106\eta^2 - 3392\eta + 4704\right)\big)\big)\big\}$$
$$/\big\{b(\eta-2)^3(d-2\eta+4)(d-\eta+2)^6\left(d^2 - 2d(\eta-1) - 3\eta^2 + 14\eta - 16\right)\cdot$$
$$\cdot\left(d(\eta-1)+\eta^2-5\eta+6\right)\left((d-6)\eta+\eta^2+8\right)^2\left(d^2 - d\eta - 2\left(\eta^2-5\eta+6\right)\right)\big\} \qquad (174)$$

$$B_8 = -\big\{2048A^5d^5\left(6(\eta-2)^2\big(3\eta^6 - 8\eta^5 - 32\eta^4 + 244\eta^3 - 628\eta^2 + 784\eta - 384\big)d^{14}\right.$$
$$+ \big(-90\eta^{10} + 1167\eta^9 - 4748\eta^8 - 3255\eta^7 + 107870\eta^6 - 501440\eta^5 + 1298480\eta^4$$
$$- 2134272\eta^3 + 2230272\eta^2 - 1362432\eta + 371712\big)d^{13}$$
$$- 2\big(1140\eta^{11} - 13124\eta^{10} + 50537\eta^9 + 36024\eta^8 - 1163696\eta^7 + 5633136\eta^6 - 15708208\eta^5$$
$$+ 29048912\eta^4 - 36517568\eta^3 + 30290880\eta^2 - 15059712\eta + 3416064\big)d^{12}$$
$$- 4(\eta-2)^2\big(5955\eta^{10} - 51798\eta^9 + 107863\eta^8 + 588161\eta^7 - 4703270\eta^6 + 15909908\eta^5$$
$$- 33051848\eta^4 + 45132288\eta^3 - 39914880\eta^2 + 20900352\eta - 4948992\big)d^{11}$$
$$- 2(\eta-2)^3\big(70080\eta^{10} - 614584\eta^9 + 1193366\eta^8 + 7696497\eta^7 - 57212574\eta^6 + 184262828\eta^5$$
$$- 363122408\eta^4 + 468042384\eta^3 - 389141920\eta^2 + 190884736\eta - 42153984\big)d^{10}$$
$$- (\eta-2)^4\big(528750\eta^{10} - 4920951\eta^9 + 10297728\eta^8 + 62107635\eta^7 - 474680326\eta^6$$
$$+ 1521805360\eta^5 - 2936969104\eta^4 + 3662826496\eta^3 - 2920374272\eta^2 + 1365180928\eta$$
$$- 286198784\big)d^9 - 2(\eta-2)^5\big(684540\eta^{10} - 6937497\eta^9 + 17001465\eta^8 + 82024785\eta^7$$

$$
\begin{aligned}
&- 697508538\eta^6 + 2310425136\eta^5 - 4501744768\eta^4 + 5579117712\eta^3 - 4366978720\eta^2 \\
&+ 1986212224\eta - 402809856\big)d^8 - 8(\eta-2)^6\big(315945\eta^{10} - 3526077\eta^9 + 10427319\eta^8 \\
&+ 35843748\eta^7 - 369180748\eta^6 + 1305084756\eta^5 - 2641084088\eta^4 + 3342404032\eta^3 \\
&- 2635437824\eta^2 + 1194036736\eta - 239083520\big)d^7 - 2(\eta-2)^7\big(1696320\eta^{10} - 20913297\eta^9 \\
&+ 74670114\eta^8 + 159310158\eta^7 - 2258277760\eta^6 + 8770800796\eta^5 - 18902896248\eta^4 \\
&+ 25063392880\eta^3 - 20434638944\eta^2 + 9460439680\eta - 1915020288\big)d^6 - (\eta-2)^8\big(3328470\eta^{10} \\
&- 45277101\eta^9 + 192926220\eta^8 + 186940941\eta^7 - 4943047242\eta^6 + 21678782112\eta^5 \\
&- 50828042064\eta^4 + 72203157248\eta^3 - 62367859712\eta^2 + 30270651904\eta - 6356632576\big)d^5 \\
&- 2(\eta-2)^9\big(1183500\eta^{10} - 17703654\eta^9 + 88664611\eta^8 - 6279924\eta^7 - 1891906800\eta^6 \\
&+ 9646241464\eta^5 - 25040424448\eta^4 + 38812564848\eta^3 - 36262979936\eta^2 + 18890163328\eta \\
&- 4222298112\big)d^4 - 4(\eta-2)^{10}\big(297435\eta^{10} - 4871488\eta^9 + 28243883\eta^8 - 28647785\eta^7 \\
&- 484835394\eta^6 + 2982562308\eta^5 - 8699658312\eta^4 + 14923099776\eta^3 - 15334032512\eta^2 \\
&+ 8744577536\eta - 2129386496\big)d^3 - 2(\eta-2)^{11}\big(200640\eta^{10} - 3581214\eta^9 + 23705402\eta^8 \\
&- 44725715\eta^7 - 306312910\eta^6 + 2411774452\eta^5 - 8012648952\eta^4 + 15372092464\eta^3 \\
&- 17590214112\eta^2 + 11158657664\eta - 3019505664\big)d^2 - (\eta-2)^{12}\big(81570\eta^{10} - 1579277\eta^9 \\
&+ 11790376\eta^8 - 31920183\eta^7 - 99831890\eta^6 + 1135028080\eta^5 - 4359384816\eta^4 \\
&+ 9427465216\eta^3 - 12118823936\eta^2 + 8658089472\eta - 2649375744\big)d - 6(\eta-2)^{13}\big(1260\eta^{10} \\
&- 26343\eta^9 + 219473\eta^8 - 765933\eta^7 - 812734\eta^6 + 19383136\eta^5 - 87798976\eta^4 + 215420944\eta^3 \\
&- 312826272\eta^2 + 253892992\eta - 89263104\big)\big)\big\}/\big\{3b(d-\eta+2)^9(\eta-2)^6\cdot \\
&\cdot\big(\eta^2 - 5\eta + d(\eta-1) + 6\big)^2\big(\eta^2 + (d-6)\eta + 8\big)^5\big(2\eta^2 - 9\eta + d(2\eta-3) + 10\big)\cdot \\
&\cdot\big(3\eta^2 - 14\eta + d(3\eta-4) + 16\big)\big(5\eta^2 - 22\eta + d(5\eta-8) + 24\big)\big\} \qquad (175)
\end{aligned}
$$

$$
B_9 = 0 \qquad (176)
$$

$$
\begin{aligned}
B_{10} = &- \big\{256A^3d^3(d+\eta-2)\big(2\big(3(\eta-2)^2(\eta+4)d^{15} - 3\big(3\eta^5 + 5\eta^4 - 56\eta^3 - 40\eta^2 + 328\eta - 256\big)d^{14} \\
&+ \big(93\eta^6 - 439\eta^5 - 940\eta^4 + 5680\eta^3 - 8408\eta^2 + 8000\eta - 5248\big)d^{13} \\
&+ \big(21\eta^7 + 1381\eta^6 - 3424\eta^5 - 2640\eta^4 + 23304\eta^3 - 78848\eta^2 + 135872\eta - 80896\big)d^{12} \\
&+\big( - 3633\eta^8 + 19022\eta^7 - 14080\eta^6 - 182000\eta^5 + 796856\eta^4 - 1503136\eta^3 + 1389376\eta^2 \\
&- 576448\eta + 90368\big)d^{11} + 2\big(3009\eta^9 - 10434\eta^8 - 10254\eta^7 + 76810\eta^6 - 89472\eta^5 \\
&+ 64760\eta^4 - 371280\eta^3 + 1207680\eta^2 - 1688896\eta + 871424\big)d^{10} + 2(\eta-2)^2\big(15315\eta^8 \\
&- 122773\eta^7 + 111680\eta^6 + 1154676\eta^5 - 4306304\eta^4 + 7097128\eta^3 - 6185872\eta^2 + 2260672\eta \\
&+ 11264\big)d^9 - 2(\eta-2)^3\big(17787\eta^8 + 8909\eta^7 - 594942\eta^6 + 902992\eta^5 + 2971176\eta^4 \\
&- 9513152\eta^3 + 10233632\eta^2 - 4852096\eta + 581632\big)d^8 - (\eta-2)^4\big(136962\eta^8 - 1250687\eta^7 \\
&+ 1588424\eta^6 + 13164508\eta^5 - 48178784\eta^4 + 63004784\eta^3 - 35736192\eta^2 + 5415872\eta + 3091200\big)d^7 \\
&+ (\eta-2)^5\big(33387\eta^8 + 733895\eta^7 - 6618610\eta^6 + 8275060\eta^5 + 48173088\eta^4 - 157777360\eta^3 \\
&+ 176265824\eta^2 - 83899072\eta + 15777792\big)d^6 + (\eta-2)^6\big(298905\eta^8 - 2559239\eta^7 + 2118516\eta^6 \\
&+ 36086612\eta^5 - 118610120\eta^4 + 98855808\eta^3 + 62180064\eta^2 - 112882688\eta + 35882496\big)d^5 \\
&+ (\eta-2)^7\big(152049\eta^8 - 2867745\eta^7 + 14760686\eta^6 - 7966736\eta^5 - 143437048\eta^4 + 442132080\eta^3 \\
&- 494152384\eta^2 + 197655168\eta - 12683264\big)d^4 - (\eta-2)^8\big(188157\eta^8 - 994208\eta^7 - 5058880\eta^6 \\
&+ 45643032\eta^5 - 85449208\eta^4 - 86244288\eta^3 + 448092864\eta^2 - 459290560\eta + 132708608\big)d^3 \\
&- 2(\eta-2)^9\big(114954\eta^8 - 1400369\eta^7 + 5063876\eta^6 + 4105142\eta^5 - 74403516\eta^4 + 194326400\eta^3 \\
&- 192112560\eta^2 + 31123360\eta + 34762240\big)d^2 - 4(\eta-2)^{10}\big(20955\eta^8 - 347387\eta^7 + 2094974\eta^6
\end{aligned}
$$

$$
\begin{aligned}
&- 4372320\eta^5 - 8530052\eta^4 + 63645964\eta^3 - 132479304\eta^2 + 119866912\eta - 37342464\big)d \\
&- 48(\eta-2)^{11}\big(186\eta^8 - 4297\eta^7 + 36260\eta^6 - 136060\eta^5 + 131618\eta^4 + 717178\eta^3 \\
&- 2761676\eta^2 + 3899456\eta - 2062080\big)\big) - b(\eta-2)\big(\big(3\eta^5 + 11\eta^4 - 36\eta^3 - 16\eta^2 \\
&+ 152\eta - 160\big)d^{14} + \big(-9\eta^6 + 23\eta^5 + 54\eta^4 - 244\eta^3 + 88\eta^2 + 576\eta - 448\big)d^{13} \\
&- \big(219\eta^7 + 277\eta^6 - 4070\eta^5 + 4700\eta^4 + 10016\eta^3 - 31216\eta^2 + 37184\eta - 19840\big)d^{12} \\
&+ \big(453\eta^8 + 29\eta^7 - 1314\eta^6 - 12296\eta^5 + 13120\eta^4 + 71920\eta^3 - 129248\eta^2 + 26752\eta \\
&+ 36096\big)d^{11} + 2\big(1521\eta^9 - 5366\eta^8 - 29893\eta^7 + 171496\eta^6 - 180960\eta^5 - 483704\eta^4 \\
&+ 1548800\eta^3 - 1815200\eta^2 + 1069632\eta - 293376\big)d^{10} - 2(\eta-2)^2\big(1977\eta^8 + 5861\eta^7 \\
&- 6272\eta^6 - 162748\eta^5 + 230288\eta^4 + 582160\eta^3 - 1512704\eta^2 + 1030336\eta - 100096\big)d^9 \\
&- 2(\eta-2)^3\big(9375\eta^8 - 27449\eta^7 - 235752\eta^6 + 735208\eta^5 + 362408\eta^4 - 3095376\eta^3 \\
&+ 3711648\eta^2 - 1619200\eta + 128768\big)d^8 + 2(\eta-2)^4\big(4509\eta^8 + 43613\eta^7 - 48954\eta^6 \\
&- 1331880\eta^5 + 2249504\eta^4 + 4333808\eta^3 - 12298720\eta^2 + 9310336\eta - 2011904\big)d^7 \\
&+ (\eta-2)^5\big(58503\eta^8 - 212615\eta^7 - 1393030\eta^6 + 5236908\eta^5 + 4090912\eta^4 - 23024336\eta^3 \\
&+ 13160288\eta^2 + 7448896\eta - 6622976\big)d^6 + (\eta-2)^6\big(11283\eta^8 - 316937\eta^7 + 151302\eta^6 \\
&+ 8593664\eta^5 - 18649984\eta^4 - 26237488\eta^3 + 90780832\eta^2 - 66535168\eta + 14042368\big)d^5 \\
&- (\eta-2)^7\big(81567\eta^8 - 292765\eta^7 - 1909936\eta^6 + 7318824\eta^5 + 9897672\eta^4 - 44741104\eta^3 \\
&- 7335776\eta^2 + 73806336\eta - 35301632\big)d^4 - (\eta-2)^8\big(56751\eta^8 - 547609\eta^7 + 15178\eta^6 \\
&+ 12425000\eta^5 - 30364288\eta^4 - 27726896\eta^3 + 146096480\eta^2 - 107116160\eta + 11085568\big)d^3 \\
&+ 2(\eta-2)^9\big(13986\eta^8 - 7965\eta^7 - 585903\eta^6 + 934338\eta^5 + 7490556\eta^4 - 18737968\eta^3 \\
&- 17217632\eta^2 + 72052768\eta - 36087808\big)d^2 + 4(\eta-2)^{10}\big(9990\eta^8 - 76566\eta^7 - 12343\eta^6 \\
&+ 1545566\eta^5 - 3941784\eta^4 - 2265968\eta^3 + 18665008\eta^2 - 16652256\eta - 2151168\big)d \\
&+ 24(\eta-2)^{11}\big(459\eta^8 - 4776\eta^7 + 10330\eta^6 + 62140\eta^5 - 354016\eta^4 + 432920\eta^3 \\
&+ 936032\eta^2 - 2946560\eta + 2205696\big)\big)\big)\big\}/\big\{b(d-3\eta+6)(d-2\eta+4)(d-\eta+2)^8\cdot \\
&\cdot(\eta-2)^3\big(d^2 + (4-3\eta)d - 4\eta^2 + 18\eta - 20\big)\big(d^2 - 2(\eta-1)d - 3\eta^2 + 14\eta - 16\big)\cdot \\
&\cdot\big(\eta^2 - 5\eta + d(\eta-1) + 6\big)\big(\eta^2 + (d-6)\eta + 8\big)^3\big(3\eta^2 - 14\eta + d(3\eta-4) + 16\big)\cdot \\
&\cdot\big(d^2 - \eta d - 2\big(\eta^2 - 5\eta + 6\big)\big)\big\}
\end{aligned}
\tag{177}
$$

$$
\begin{aligned}
B_{11} =&\big\{16384A^6d^6\big(5(\eta-2)^2\big(18\eta^{10} - 135\eta^9 + 195\eta^8 + 2084\eta^7 \\
&- 14688\eta^6 + 49584\eta^5 - 104992\eta^4 + 146136\eta^3 - 130304\eta^2 + 67456\eta - 15360\big)d^{19} \\
&+ \big(-540\eta^{14} + 9864\eta^{13} - 70584\eta^{12} + 194270\eta^{11} + 576959\eta^{10} - 7852575\eta^9 \\
&+ 38515162\eta^8 - 121100604\eta^7 + 271712168\eta^6 - 448919440\eta^5 + 546269920\eta^4 - 478288960\eta^3 \\
&+ 285735680\eta^2 - 104330240\eta + 17551360\big)d^{18} + \big(-18900\eta^{15} + 313092\eta^{14} \\
&- 2110239\eta^{13} + 5524481\eta^{12} + 17879918\eta^{11} - 235613022\eta^{10} + 1178042264\eta^9 \\
&- 3858889920\eta^8 + 9219265256\eta^7 - 16644527264\eta^6 + 22883417024\eta^5 - 23711884800\eta^4 \\
&+ 17996035200\eta^3 - 9460633600\eta^2 + 3081973760\eta - 468828160\big)d^{17} - (\eta-2)^2\big(284040\eta^{14} \\
&- 3868722\eta^{13} + 19696698\eta^{12} - 16201666\eta^{11} - 366422163\eta^{10} + 2572537287\eta^9 \\
&- 9883654234\eta^8 + 26209938236\eta^7 - 51201200424\eta^6 + 75114418928\eta^5 - 82374329120\eta^4 \\
&+ 65836041920\eta^3 - 36343038720\eta^2 + 12416102400\eta - 1980866560\big)d^{16} - 4(\eta-2)^3\big(623700\eta^{14} \\
&- 8452656\eta^{13} + 41596293\eta^{12} - 19202185\eta^{11} - 871965036\eta^{10} + 5695768936\eta^9 \\
&- 20892994500\eta^8 + 53072643632\eta^7 - 99360828392\eta^6 + 139742217360\eta^5 - 147005032800\eta^4 \\
&+ 112806563520\eta^3 - 59857182720\eta^2 + 19683133440\eta - 3027650560\big)d^{15} \\
&- 4(\eta-2)^4\big(3645000\eta^{14} - 50796414\eta^{13} + 253646154\eta^{12} - 98709930\eta^{11}
\end{aligned}
$$

$$
\begin{aligned}
&- 5561697549\eta^{10} + 35797123311\eta^{9} - 129093731726\eta^{8} + 320658081164\eta^{7} - 584252693432\eta^{6} \\
&+ 796879209744\eta^{5} - 811027077280\eta^{4} + 601207286080\eta^{3} - 307901646080\eta^{2} + 97673994240\eta \\
&- 14489036800\big)d^{14} - 4(\eta-2)^{5}\big(15218280\eta^{14} - 222235074\eta^{13} + 1163055177\eta^{12} \\
&- 626390751\eta^{11} - 25325980202\eta^{10} + 167129052666\eta^{9} - 606435798040\eta^{8} + 1500014934904\eta^{7} \\
&- 2700141925736\eta^{6} + 3615814802672\eta^{5} - 3595960710560\eta^{4} + 2595724293952\eta^{3} \\
&- 1291298020864\eta^{2} + 397204292608\eta - 57059993600\big)d^{13} - 4(\eta-2)^{6}\big(47420100\eta^{14} \\
&- 733503888\eta^{13} + 4106114868\eta^{12} - 3478943974\eta^{11} - 85253778733\eta^{10} + 595839146179\eta^{9} \\
&- 2217911020662\eta^{8} + 5553388714780\eta^{7} - 10022457366808\eta^{6} + 13350832227248\eta^{5} \\
&- 13123071387232\eta^{4} + 9314810237120\eta^{3} - 4538744792320\eta^{2} + 1363520224256\eta \\
&- 190905833472\big)d^{12} - 2(\eta-2)^{7}\big(226872360\eta^{14} - 3738797010\eta^{13} + 22650329277\eta^{12} \\
&- 28629766871\eta^{11} - 432262251846\eta^{10} + 3298683209628\eta^{9} - 12807415882384\eta^{8} \\
&+ 32942816212096\eta^{7} - 60453232360616\eta^{6} + 81192016903728\eta^{5} - 79873866612064\eta^{4} \\
&+ 56386225942848\eta^{3} - 27183693036032\eta^{2} + 8046889294848\eta - 1106728103936\big)d^{11} \\
&- 2(\eta-2)^{8}\big(424323900\eta^{14} - 7469866404\eta^{13} + 49254074892\eta^{12} - 86261346490\eta^{11} \\
&- 830748035309\eta^{10} + 7156115125569\eta^{9} - 29407728273326\eta^{8} + 78714803198308\eta^{7} \\
&- 148796117820408\eta^{6} + 204154924105840\eta^{5} - 203649624706208\eta^{4} + 144785724178880\eta^{3} \\
&- 69870766151936\eta^{2} + 20596142908416\eta - 2808779776000\big)d^{10} - 2(\eta-2)^{9}\big(627412500\eta^{14} \\
&- 11806690896\eta^{13} + 84881868621\eta^{12} - 193872945365\eta^{11} - 1203591269656\eta^{10} \\
&+ 12212330953634\eta^{9} - 53831454121068\eta^{8} + 151625891406712\eta^{7} - 298651382251528\eta^{6} \\
&+ 423735605386128\eta^{5} - 434100636830368\eta^{4} + 314878120233408\eta^{3} - 154057660608512\eta^{2} \\
&+ 45770867920896\eta - 6258118848512\big)d^{9} - 2(\eta-2)^{10}\big(737296560\eta^{14} - 14822923050\eta^{13} \\
&+ 116114538042\eta^{12} - 330420403654\eta^{11} - 1283487614577\eta^{10} + 16375661936049\eta^{9} \\
&- 78426936583678\eta^{8} + 234690464813188\eta^{7} - 486390250898488\eta^{6} + 721305200077200\eta^{5} \\
&- 767833754078048\eta^{4} + 575405513955648\eta^{3} - 289165297623296\eta^{2} + 87729990756352\eta \\
&- 12179092148224\big)d^{8} - 4(\eta-2)^{11}\big(344295900\eta^{14} - 7385296932\eta^{13} \\
&+ 62902201221\eta^{12} - 215492902501\eta^{11} - 471471044976\eta^{10} + 8572996046784\eta^{9} \\
&- 45212321002876\eta^{8} + 144930991369744\eta^{7} - 318611422705640\eta^{6} + 498299397633488\eta^{5} \\
&- 556787266338144\eta^{4} + 435981746150336\eta^{3} - 227849284858880\eta^{2} + 71524917760000\eta \\
&- 10218973337600\big)d^{7} - 4(\eta-2)^{12}\big(254152080\eta^{14} - 5806420506\eta^{13} + 53614593246\eta^{12} \\
&- 215432117194\eta^{11} - 187558247805\eta^{10} + 6922706181379\eta^{9} - 40834160649774\eta^{8} \\
&+ 141250766819452\eta^{7} - 331558812621496\eta^{6} + 550839114930128\eta^{5} - 651474321562784\eta^{4} \\
&+ 538187227898432\eta^{3} - 295722971589888\eta^{2} + 97233474783232\eta - 14488600385536\big)d^{6} \\
&- 4(\eta-2)^{13}\big(146512800\eta^{14} - 3557932686\eta^{13} + 35498611233\eta^{12} - 163952653515\eta^{11} \\
&+ 33723868074\eta^{10} + 4222044106746\eta^{9} - 28425431299808\eta^{8} + 106845263734712\eta^{7} \\
&- 269227888472872\eta^{6} + 477836778396720\eta^{5} - 602128208030624\eta^{4} + 528900285871168\eta^{3} \\
&- 308380454969856\eta^{2} + 107339416337408\eta - 16884317681664\big)d^{5} - 4(\eta-2)^{14}\big(64597500\eta^{14} \\
&- 1663892244\eta^{13} + 17876468304\eta^{12} - 93395759910\eta^{11} + 104833630171\eta^{10} \\
&+ 1879304461071\eta^{9} - 14877527865958\eta^{8} + 61205071143212\eta^{7} - 166296405425112\eta^{6} \\
&+ 316683328937840\eta^{5} - 427322137341024\eta^{4} + 401542558988736\eta^{3} - 250268793527552\eta^{2} \\
&+ 93035589628928\eta - 15609633458176\big)d^{4} - (\eta-2)^{15}\big(84173040\eta^{14} - 2294762418\eta^{13} \\
&+ 26457490335\eta^{12} - 154362906049\eta^{11} + 297903785570\eta^{10} + 2303542424284\eta^{9} \\
&- 22528764900408\eta^{8} + 102283959891104\eta^{7} - 300842349667640\eta^{6} + 616738132496592\eta^{5}
\end{aligned}
$$

$$- 894464615915296\eta^4 + 903273818400960\eta^3 - 605271148161536\eta^2 + 242021183499264\eta$$
$$- 43689575825408\big)d^3 - (\eta-2)^{16}\big(19097100\eta^{14} - 549877968\eta^{13} + 6780930192\eta^{12}$$
$$- 43717465054\eta^{11} + 116294662593\eta^{10} + 434719460247\eta^9 - 5785729506266\eta^8$$
$$+ 29299298634716\eta^7 - 93642779618984\eta^6 + 207188777133072\eta^5 - 323825446045408\eta^4$$
$$+ 352605081399872\eta^3 - 255135604574976\eta^2 + 110361365980160\eta - 21591994331136\big)d^2$$
$$- (\eta-2)^{17}\big(2695140\eta^{14} - 81793476\eta^{13} + 1075420503\eta^{12} - 7594679675\eta^{11}$$
$$+ 25324801284\eta^{10} + 38292423222\eta^9 - 894073315612\eta^8 + 5126099874792\eta^7$$
$$- 17876555760536\eta^6 + 42777393075184\eta^5 - 72183405054688\eta^4 + 84943402257984\eta^3$$
$$- 66583099165696\eta^2 + 31300539725824\eta - 6679882547200\big)d - 5(\eta-2)^{18}\big(35640\eta^{14}$$
$$- 1137762\eta^{13} + 15901914\eta^{12} - 122106394\eta^{11} + 484476249\eta^{10} + 14360963\eta^9$$
$$- 12389304946\eta^8 + 82281543404\eta^7 - 314677444744\eta^6 + 815953310384\eta^5 - 1488362674336\eta^4$$
$$+ 1895615890880\eta^3 - 1613311153920\eta^2 + 827272577024\eta - 193730101248\big)\big)\big\}$$
$$/\big\{5b(d-\eta+2)^{11}(\eta-2)^7\big(\eta^2-5\eta+d(\eta-1)+6\big)^3\big(\eta^2+(d-6)\eta+8\big)^6$$
$$\big(2\eta^2-9\eta+d(2\eta-3)+10\big)\big(3\eta^2-13\eta+d(3\eta-5)+14\big)\cdot$$
$$\cdot\big(3\eta^2-14\eta+d(3\eta-4)+16\big)^2\big(5\eta^2-22\eta+d(5\eta-8)+24\big)\big\} \quad (178)$$

$$B_{12} = -\big\{8Ad(\eta+2)(3d-5\eta+10)(d-3\eta+6)(d-2\eta+4)(d+\eta-2)\big(b^2\big(\eta^2-4\big)^2(d-\eta+2)^2$$
$$+ 8b\big(\eta^2-4\big)\big(d^3+d^2(5-3\eta)+2d\big(\eta^2-3\eta+2\big)-(\eta-2)^2\big)+4\big(2d^4+d^3(16-11\eta)$$
$$+ 2d^2\big(7\eta^2-19\eta+12\big)+d\eta\big(-3\eta^2+8\eta-4\big)+2(\eta-2)^2\big(\eta^2-5\eta+8\big)\big)\big)\big\}$$
$$/\big\{b(\eta-2)^2(d-\eta+2)^7\big(2d^2-d\eta-3\eta^2+14\eta-16\big)\big((d-6)\eta+\eta^2+8\big)\big(d^2-d\eta$$
$$- 2\big(\eta^2-5\eta+6\big)\big)\big\} \quad (179)$$

$$B_{13} = 0 \quad (180)$$

$$\beta_1 = \frac{2(\eta-2)}{d-\eta+2} \quad (181)$$

$$\beta_2 = \frac{4(\eta-2)}{d-\eta+2} \quad (182)$$

$$\beta_3 = -2 \quad (183)$$

$$\beta_4 = \frac{6(\eta-2)}{d-\eta+2} \quad (184)$$

$$\beta_5 = \frac{2(\eta-2)}{d-\eta+2} - 2 \quad (185)$$

$$\beta_6 = \frac{8(\eta-2)}{d-\eta+2} \quad (186)$$

$$\beta_7 = \frac{4(\eta-2)}{d-\eta+2} - 2 \quad (187)$$

$$\beta_8 = \frac{10(\eta - 2)}{d - \eta + 2} \tag{188}$$

$$\beta_9 = -4 \tag{189}$$

$$\beta_{10} = \frac{6(\eta - 2)}{d - \eta + 2} - 2 \tag{190}$$

$$\beta_{11} = \frac{12(\eta - 2)}{d - \eta + 2} \tag{191}$$

$$\beta_{12} = \frac{2(\eta - 2)}{d - \eta + 2} - 4 \tag{192}$$

$$\beta_{13} = \frac{8(\eta - 2)}{d - \eta + 2} - 2 \tag{193}$$

$$\mathbf{Z_{as}} = \frac{\mathbf{1}}{\mathbf{B}} + \sum_{\mathbf{i=1}}^{\mathbf{13}} \mathbf{Z_i} \phi^{\zeta_\mathbf{i}} \tag{194}$$

$$Z_1 = -\frac{B_1}{B^2} \tag{195}$$

$$Z_2 = \frac{{B_1}^2}{B^3} - \frac{B_2}{B^2} \tag{196}$$

$$Z_3 = -\frac{B_3}{B^2} \tag{197}$$

$$Z_4 = -\frac{B_1^3}{B^4} + 2\frac{B_1 B_2}{B^3} - \frac{B_4}{B^2} \tag{198}$$

$$Z_5 = 2\frac{B_1 B_3}{B^3} - \frac{B_5}{B^2} \tag{199}$$

$$Z_6 = \frac{B_1^4}{B^5} - 3\frac{B_1^2 B_2}{B^4} + 2\frac{B_1 B_4}{B^3} + \frac{B_2^2}{B^3} - \frac{B_6}{B^2} \tag{200}$$

$$Z_7 = -3\frac{B_1^2 B_3}{B^4} + 2\frac{B_1 B_5}{B^3} - \frac{B_7}{B^2} \tag{201}$$

$$Z_8 = -\frac{B_1^5}{B^6} + 4\frac{B_1^3 B_2}{B^5} - 3\frac{B_1^2 B_4}{B^4} - 3\frac{B_1 B_2^2}{B^4} + 2\frac{B_1 B_6}{B^3} + 2\frac{B_2 B_4}{B^3} - \frac{B_8}{B^2} \tag{202}$$

$$Z_9 = \frac{B_3^2}{B^3} \tag{203}$$

$$Z_{10} = 4\frac{B_1^3 B_3}{B^5} - 3\frac{B_1^2 B_5}{B^4} - \frac{B_1 B_2 B_3}{B^4} + 2\frac{B_1 B_7}{B^3} + 2\frac{B_2 B_5}{B^3} - \frac{B_{10}}{B^2} \tag{204}$$

$$\begin{aligned} Z_{11} = & \frac{B_1^6}{B^7} - 5\frac{B_1^4 B_2}{B^6} + 4\frac{B_1^3 B_4}{B^5} + 6\frac{B_1^2 B_2^2}{B^5} - 3\frac{B_1^2 B_6}{B^4} - 6\frac{B_1 B_2 B_4}{B^4} - \frac{B_2^3}{B^4} \\ & + 2\frac{B_1 B_8}{B^3} + 2\frac{B_2 B_6}{B^3} + \frac{B_4^2}{B^3} - \frac{B_{11}}{B^2} \end{aligned} \tag{205}$$

$$Z_{12} = -3\frac{B_1 B_3^2}{B^4} + 2\frac{B_9 B_1}{B^3} - \frac{B_{12}}{B^2} \tag{206}$$

$$Z_{13} = 4\frac{B_1^3 B_5}{B^5} - 3\frac{B_1^2 B_7}{B^4} - 6\frac{B_1 B_2 B_5}{B^4} + 2\frac{B_{10} B_1}{B^3} + 2\frac{B_2 B_7}{B^3} + 2\frac{B_4 B_5}{B^3} - \frac{B_{13}}{B^2} \tag{207}$$

$$\zeta_1 = \beta_1 \tag{208}$$

$$\zeta_2 = \beta_2 \tag{209}$$

$$\zeta_3 = \beta_3 \tag{210}$$

$$\zeta_4 = \beta_4 \tag{211}$$

$$\zeta_5 = \beta_5 \tag{212}$$

$$\zeta_6 = \beta_6 \tag{213}$$

$$\zeta_7 = \beta_7 \tag{214}$$

$$\zeta_8 = \beta_8 \tag{215}$$

$$\zeta_9 = \beta_9 \tag{216}$$

$$\zeta_{10} = \beta_{10} \tag{217}$$

$$\zeta_{11} = \beta_{11} \tag{218}$$

$$\zeta_{12} = \beta_{12} \tag{219}$$

$$\zeta_{13} = \beta_{13} \tag{220}$$

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