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Aspects of supergeometry in locally covariant quantum field theory

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Abstract

In questa tesi vengono presentati i più recenti risultati relativi all'estensione della *teoria dei campi localmente covariante* a geometrie che permettano di descrivere teorie di campo supersimmetriche. In particolare, si mostra come la definizione assiomatica possa essere generalizzata, mettendo in evidenza le problematiche rilevanti e le tecniche utilizzate in letteratura per giungere ad una loro risoluzione. Dopo un'introduzione alle strutture matematiche di base, varietà Lorentziane e operatori Green-iperbolici, viene definita l'algebra delle osservabili per la teoria quantistica del campo scalare. Quindi, costruendo un funtore dalla categoria degli spazio-tempo globalmente iperbolici alla categoria delle $*$-algebre, lo stesso schema viene proposto per le teorie di campo bosoniche, purché definite da un operatore Green-iperbolico su uno spazio-tempo globalmente iperbolico. Si procede con lo studio delle supervarietà e alla definizione delle geometrie di *background* per le super teorie di campo: le strutture di super-Cartan. Associando canonicamente ad ognuna di esse uno *spazio-tempo ridotto*, si introduce la categoria $\mathfrak{ghsCart}$ delle strutture di super-Cartan il cui spazio-tempo ridotto è globalmente iperbolico. Quindi, si mostra, in breve, come è possibile costruire un funtore da una sottocategoria di $\mathfrak{ghsCart}$ alla categoria delle super $*$-algebre e si conclude presentando l'applicazione dei risultati esposti al caso delle strutture di super-Cartan in dimensione $2|2$.

Contents

Introduction

Quantum field theory played a prominent role in the development of physical models throughout all the 20$^{\text{th}}$ century and continues nowadays. For this reason, growing effort has been devoted to the research for its mathematical rigorous formulation. A remarkable success in this direction has been achieved with the Haag-Kastler algebraic formulation of quantum field theory ([HK64]), which gives a formal description of the quantum fields in the Heisenberg picture encoding at same time the properties of causality – as intended on the Minkowski spacetime – and Lorentz covariance. Even though a theory developed on Minkowski spacetime is enough for all the applications in particle physics, a generalization to curved backgrounds is required in order to understand the physical problems coming from condensed matter physics and cosmology. This generalization is exactly what has been proposed by Brunetti, Fredenhagen and Verch in [BFV03] with the *general covariant locality principle*, formulated using the language of category theory, which makes possible a coherent description of the physics coherently on all the spacetimes, encoding in the algebra of fields causality and local Lorentz covariance. In particular, once the playground is defined restricting the class of all *Lorentzian manifolds* to those objects that are called *globally hyperbolic spacetimes* (organized in the category $\mathfrak{ghs}$), a quantum field theory is constructed as a functor from $\mathfrak{ghs}$ to the category of $*$-algebras, using the properties of the field operator defining the equations of motion for the classical theory.

It is important to notice that the introduction of a rather abstract mathematical language has remarkable practical effects in the development of technical results. First, the locally covariant approach is essential for the construction of interacting models on non-flat geometries, proving rigorous results on their renormalization and regularization properties. On the other hand, a deep understanding of the axiomatic scheme turns out to be fundamental for the extension of those results to different branch of physics, playing in different contexts, such as the *supersymmetric extensions of the Standard Model* ([DRW81]) or the *superspace formulation of supergravity* ([WZ77]). These are only two examples of supersymmetric models which can be described defining fields playing over exotic objects that mathematicians call *supermanifolds*. Hence, *supergeometry* – the branch of mathematics studying supermanifolds and supermanifolds transformations – has not to be intended as an exercise of pure mathematics, because it can be applied to the research of solution for some of the open problems of modern physics. Models based on super field theories are currently used to solve some theoretical pathologies showed by the standard model, in particular the well known *hierarchy problem* (see the introduction of [Ait05] for a brief presentation on the subject). Moreover, in cosmology, the so called *superparticles* seem to be good candidates for explaining *dark matter*. Together with the production of new physics, one of the sources of interest in super-quantum field theories is due to some regularity properties showed by interacting supersymmetric models, known as *non-renormalization theorems*.

The hope that a rigorous formulation of supersymmetric quantum field theories can shed light on these results, explains the last attempts to study super-QFT from the viewpoint of locally covariant quantum field theory ([HHS16], [Dap+15]). These works develop quantum field theory adopting as background geometry the so called *super-Cartan structures*, introduced in the superspace formulation of supergravity, with the respective category denoted by $\mathfrak{sCart}$, whose objects can be canonically associated to an ordinary oriented and time-oriented Lorentzian manifold. Then, techniques coming form ordinary quantum field theory can be borrowed in order to reproduce analogous constructions and results. Unfortunately, already at the level of non-interacting models the local covariance paradigm and the category theory language reveal a problematic behaviour when dealing with supersymmetry transformations. Indeed, using the ordinary tools of category theory, what comes out is a synthetic description of both bosonic and fermionic field theories, with the super-QFT functor failing to mix up the odd and the even components. The solution proposed by ([HHS16]) is that of clarifying the behaviour of free super-QFTs introducing advanced tools in category theory (in particular *enriched categories* and *enriched functors*) to obtain a sensible description of supersymmetry transformations, setting the basis for the perturbative analysis and the study of the renormalization properties.

The main purpose of this thesis is to present an overview of locally covariant quantum field theory devoting the last part to an exposition of the recent results in the description of super quantum field theories. Moreover, we propose a study of those super-QFT defined over $2|2$-dimensional super-Cartan structures, for which we give a definition of a suitable full subcategory of $\mathfrak{ghsCart}$, characterizing, in this new formalism, *enriched morphisms* and proving some properties of a super differential operator defined for its objects. A remarkable reason to study $2|2$-models in the recent version of LCQFT is that of setting the basis for an advanced analysis on the effect that the *enriched morphism*, i.e. supersymmetry transformations, could have on the degree of freedom showing up in the definition of super-Wick polynomials (see [Pin09] for an analogous idea applied to four dimensional conformal field theory).

Let us outline the content of the thesis. We begin with an introductory chapter exposing the fundamental mathematical structures for the development of locally covariant quantum field theory. First of all, we set the playground introducing the concept of *Lorentzian manifold*, on which we have to build up a causal structure, that in contrast to what happens on flat geometries is not naturally defined and has to be given as a datum, leading to the concept of spacetime (Definition 1.1.7). Yet, this latter notion includes a too wide class of manifolds in which unwanted frameworks can still be found (see Example 1.1.23). Thus, searching for additional requirements on the causal structure of the spacetimes, the class of globally hyperbolic spacetimes is defined and, if the idea of their definition moves from technical hypothesis on the topological properties of the domain, an useful characterization allows to understand them as oriented and time oriented Lorentzian manifolds having a *Cauchy surface* (Theorem 1.1.22). The importance of this statement relies especially in the fact that a Cauchy surface is needed to define sensible initial value problems for the *partial linear differential operators* encoding the dynamics of the classical fields. Indeed, in the second part of the chapter we present some results on differential operators, studying in particular the class of the so called *Green-hyperbolic operators* (Definition 1.2.9). For those operators we are able to give a description of the space of solutions, setting the bases for the construction of the algebra of the observables related to both the classical and the quantum theory. Then, we conclude analyzing briefly *normally hyperbolic operators*, which can be proven to be Green-hyperbolic using existence and uniqueness of solutions to the respective Cauchy initial value problem (Theorem 1.2.20).

In Chapter 2 we use the instruments developed to construct locally covariant quantum field theories on the line of the axioms proposed in [BFV03]. Before proceeding, we get some motivation for the axioms and the abstract constructions from the study of the Klein-Gordon (KG) theory: We set the framework for a sensible definition of the KG operator and, in Section 2.1.1, we use the results on Green-hyperbolic operators to build up a symplectic space for the classical observables in three different forms. Once equivalence of those three constructions has been proven (Diagram 2.1.1), we proceed describing two different schemes of quantization. The first consists of the standard association of a tensor algebra to a given vector space, on which commutator is implemented by equivalence relations defined using the symplectic form. The second approach we show is based on a more recent technique called *quantization by deformation*, concerning which the algebra of classical observable is quantized as a space of linear functionals where the pointwise product (characterizing the classical observables) is *deformed*, in a precise and rigorous sense, giving a product admitting canonical commutation relations. These $*$-algebras describe the quantum theory encoding coherently the property of the underlying spacetime and hence will be the prototype for the general construction of bosonic field theories in Section 2.3. Then, in Section 2.2, we show how spinor fields can be treated on curved Lorentzian manifolds. That exposition turns out to be useful, not only because it provides an other instructive example, but rather because it presents a simplified approach to *Cartan description* of Lorentzian manifolds, which, based on the notion of *Cartan structure* (Definition 2.2.4), provides an useful framework that can be easily extended in order to describe supergeometric backgrounds for quantum field theories. Finally, using the language of categories and functors, we describe the axiomatic formulation of quantum field theories and we give a precise definition of *general covariance*, showing how the results achieved for the KG field can be obtained for all Green-hyperbolic operators and *coherently* for all globally hyperbolic spacetimes.

Finally, Chapter 3 is devoted to the presentation of the most recent result on the extension of the *general covariance scheme* to field theories defined on supermanifolds. At this stage, what has been shown in Chapter 2 reveals useful as inspiration. On one hand, with a self-contained exposition on supergeometry we give the minimal instruments in order to understand the notion of supermanifold and to generalize Cartan structures to *super-Cartan structures* (Definition 3.2.2): a series of fundamental definitions and propositions is given, from the concept of super vector space to more advanced notions such as those of Berezin integration and super differential forms over supermanifolds. Thus, given a set of theoretical data needed to fix the amount of supersymmetry and the ordinary geometry to work on, the basic playground is analyzed, constructing the category $\mathfrak{ghsCart}$, consisting of those super-Cartan structures to which it is possible to associate *canonically* an ordinary globally hyperbolic spacetime, from which the notion of causality is borrowed for the treatment of super fields. On the other hand, following the ideas proposed in [HHS16], we sketch the attempt to reproduce the axiomatic framework of Section 2.3 defining a functor from a full subcategory of $\mathfrak{ghsCart}$ to the category of super $*$-algebras. Yet, in this construction *supersymmetry transformations* fails to be coherently encoded and hence advanced techniques in category theory should be used in order to solve those problematic behaviours.

In this part, since the subject could become too technical for the purposes of this thesis, we preferred to give only an heuristic idea of what has been done and to leave enough space for the presentation of a concrete model: In fact, only the formula defining *enriched morphisms* is presented and briefly discussed in order to be used in the following application. Then, we recollect and apply all the results showing some novel calculations for super field theory models on super-Cartan supermanifolds of dimension $2|2$, defining the category $2|2$-$\mathfrak{sLoc}$ imposing restrictive conditions called *supergravity supertorsion constraints* (see [WZ77] and [How79]), giving a classification of *first order supersymmetry transformations* and characterizing explicitly their action on the superalgebra of super fields.

Chapter 1

Basics on Lorentzian geometry and Green-hyperbolic operators

We begin this thesis with an introductory chapter exposing the fundamental mathematical structures for the development of locally covariant quantum field theory. After a brief overview of some remarkable results on *Lorentzian manifolds* and their properties, we give the definition of *globally hyperbolic spacetime*, i.e. an oriented and time-oriented Lorentzian manifold with a *Cauchy surface.* These additional requirements are fundamental in order to define a causal structure on the given manifold and to set differential equations, such as the equations of motion for the fields, with the respective initial values problem. Moreover, this special subclass of spacetimes turns out to be a satisfactory trade off between the regularity of the Minkowski spacetime and the generality we need for a formalism being successful in the treatment of physically interesting models (e.g. DeSitter spacetimes, Friedmann-Robertson-Walker spacetimes). In fact, defining the fields as sections of a suitable vector bundle and introducing the dynamics of the fields via linear differential operators, we find that a certain regularity on the structure of the framework manifold and the nature of those operators are crucial for the well-posedness of *Cauchy problems.* These operators are called *normally hyperbolic operators* and their importance does not lie in the fact that they allow to find explicit solution for the *Cauchy problems* but rather is due to the possibility of constructing out implicitly *Green's operators*, using existence and uniqueness of solutions. In conclusion, great attention will be paid to explain how operators admitting *advanced and retarded Green's operators* characterize the space of sections over which they act in order to establish the preliminary result for a rigorous treatment of the quantization scheme of the algebra of field, presented in Chapter 2.

1.1 Lorentzian geometry

In this section we present some notions of Lorentzian geometry. We want to provide the reader with a brief introduction to this subject, hence definitions and basic results on differential geometry will be understood.

From Lorentzian manifolds to globally hyperbolic spacetimes

Definition 1.1.1. We call *Lorentzian manifold* a pair (M, g) where M is a d-dimensional manifold and g is a metric on M with signature $(+, -, \dots, -)$.

Even though Lorentzian manifolds themselves are of great mathematical interest, this definition is too general for our purposes. Indeed, in order to build up a suitable framework for many physical models, we have to enrich their structure with further requirements. In particular, a causal structure is of great importance for physically motivated studies on Lorentzian manifolds. We recall here how it can be achieved on the manifold called Minkowski spacetime.

Example 1.1.2. Let's fix M as $\mathbb{R}^4$ and g as the Minkowski metric η. As well known, points in $\mathbb{R}^4$ label physical events. Given two distinct points $p, q \in \mathbb{R}^4$, we present a geometric method to say whether p and q are causally connected and then whether p lies in the future of q (or viceversa). First we recall that given a vector space endowed with a constant Lorentzian metric, i.e. a metric with signature as in Definition 1.1.1, vectors $v \in V$ can be classified using the quantity *length* $\|v\|_\eta := \eta(v, v)$: we define the class of *causal* and *spacelike* vectors for which the length $\|v\|_\eta$ is respectively non negative and negative. Then causal vectors can be divided in two subclass, that of *lightlike* vectors with null norm and that of *timelike* vectors with strictly positive norm. Then, in $\mathbb{R}^4$ we fix the canonical base $(\mathbf{e}_t, \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ and the standard global coordinates (t, x, y, z) in order to construct two distinguished regions: given a point q we take the vector v_{pq} connecting it to p and we look at the time-component and at $\eta(v_{pq}, v_{pq})$, defining the two sets:

$$J^+_{\mathbb{R}^4}(p) = \left\{ q \in \mathbb{R}^4,\ \|v_{pq}\|_\eta \geq 0 \quad \text{and} \quad \eta(\mathbf{e}_t, v_{pq}) > 0 \right\} \cup \{p\}$$
$$J^-_{\mathbb{R}^4}(p) = \left\{ q \in \mathbb{R}^4,\ \|v_{pq}\|_\eta \geq 0 \quad \text{and} \quad \eta(\mathbf{e}_t, v_{pq}) < 0 \right\} \cup \{p\}$$

In other words, with respect to p, we have divided $\mathbb{R}^4$ in three regions, namely, besides $J^+_{\mathbb{R}^4}(p)$ and $J^-_{\mathbb{R}^4}(p)$, there exists also the collection of points p for which v_{pq} is spacelike, or, equivalently, for which p and q are causally separated. Notice that the point p itself plays a distinguished and separate role and it is here conventionally assumed to include it in both $J^+_{\mathbb{R}^4}(p)$ and $J^-_{\mathbb{R}^4}(p)$. The regions $J^+_{\mathbb{R}^4}(p)$ and $J^-_{\mathbb{R}^4}(p)$ are called the *future (resp. past) light cone*, stemming from p.

Looking at this example, we propose a classification of vectors belonging to the tangent space T_pM

Definition 1.1.3. Consider a Lorentzian manifold (M, g). For each point $p \in M$ and each tangent vector $v \in \mathrm{T}_pM$ we say that v is

- *timelike* if $g_p(v, v) > 0$,
- *lightlike* if $g_p(v, v) = 0$,
- *causal* if $g_p(v, v) \geq 0$,
- *spacelike* if $g_p(v, v) < 0$.

Let's notice that this classification depends on the given metric, for this reason these definitions should make explicit reference to that metric (namely g-*timelike* rather than *timelike*). Anyway, often there is no risk of misunderstanding and we will omit this further specification, unless necessary. Given two points p, q in Minkowski spacetime, in order to state if they are causally connected we used the fact that $\mathbb{R}^4$, as a vector space, is isomorphic to its tangent space and has a global system of coordinates. Additional structures are needed to achieve analogous results for a generic Lorentzian manifold.

Definition 1.1.4. We say that a Lorentzian manifold (M, g) is *time orientable* if there exists a vector field $T \in \Gamma(M, \mathrm{T}M)$ over M such that $T(p)$ is time-like for each $p \in M$. We call *time orientation* of the Lorentzian manifold (M, g) a choice of such a vector field over M and we denote the chosen vector field with $\mathfrak{t}$.

A time orientation on a Lorentzian manifold gives a rule to classify curves connecting two points.

Definition 1.1.5. Consider a Lorentzian manifold (M, g) and a smooth curve $\gamma : I \to M$, where $I \subseteq \mathbb{R}$ is an interval, with its tangent vector field at point p, $X_{\gamma,p}$. We say that γ is *timelike*, *lightlike*, *causal* or *spacelike* if $X_{\gamma,p}$ is respectively *timelike*, *lightlike*, *causal* or *spacelike*, for each p along γ. Given a time oriented Lorentzian manifold $(M, g, \mathfrak{o}, \mathfrak{t})$ and a causal curve γ we say that this curve is: *future directed* if $g_p(X_{\gamma,p}, \mathfrak{t}(p)) > 0$ for each p along γ; *past directed* if $g_p(X_{\gamma,p}, \mathfrak{t}(p)) < 0$ for each p along γ.

Also here explicit reference to the time orientation chosen should be done; analogously, we will keep it understood when not necessary. Moreover, concerning curves in M, when a Lorentzian manifold is endowed with a time orientation, also the notion of inexstensibility can be slightly generalized to *future or past inexstendibility*.

Definition 1.1.6. Consider a future (past) directed causal curve $\gamma(s)$, denoting with s a parametrisation, then p is a future (past) endpoint if, for every neighbourhood U of p, there exist an s_0 such that $\gamma(s) \in U, \forall s > s_0$.

Now we are ready to define *spacetimes*, as Lorentzian manifolds regular enough to avoid some pathological situations from a physical point of view (such as disconnected backgrounds), and endowed with additional data, starting from which we can construct a causal structure on it.

Definition 1.1.7. We call *spacetime* a quadruple $(M, g, \mathfrak{o}, \mathfrak{t})$ where (M, g) is a d-dimensional smooth, connected, orientable, time-orientable Lorentzian manifold, $\mathfrak{o}$ is choice for the orientation and $\mathfrak{t}$ is a choice for time-orientation. Sometimes the quadruple defining a spacetime will be denoted with the shortand notation $\mathbf{M}$.

We have to inform the reader that the definition of spacetime is not the same throughout the literature. Sometimes, in physical focused papers, it is restricted to four dimensional manifolds, but this is not our case; firstly because no further hypothesis on dimension are needed to achieve the principal results of this section, secondly, because we will work in this thesis on different models with geometrical backgrounds of various dimensions.

Remark 1.1.8. If the choice of a time orientable manifold has been extensively justified, connectedness and orientability need a brief explanation. The former, when dealing with manifolds, entails path connectedness, that plays an important role in the definition of a causal structure for the spacetime. Indeed, as we will explore in the following, a causal relation between two points of the manifolds is defined looking at the properties of the curve connecting them. This practically means that causal structure of different connected components are not comparable and then, asking for connection, we can endow the manifold with a global causal structures. Orientability is crucial in all the applications of the Stoke's theorem, using integration.

On a spacetime, given a point $p \in M$, or a generic subset Ω, thanks to the previous definition, we can start defining causal structure labelling some particular subset of the manifold. These definitions, as we will see, are very helpful in the characterisation of the causal structure of $(M, g, \mathfrak{o}, \mathfrak{t})$.

Definition 1.1.9. Consider a spacetime $(M, g, \mathfrak{o}, \mathfrak{t})$ and point $p \in M$. We define:

- the *chronological future of the point p in M*, denoted by $I^+_M(p)$, as the set of those point q in $M \setminus \{p\}$, such that there exists a future directed timelike curve from p to q;
- the *causal future of the point p in M*, denoted by $J^+_M(p)$, as the set of those point q in M (including p by convention) such that there exists a future directed causal curve starting from p to q;

Moreover, the *chronological and causal past of the point p in M* ($I^-_M(p)$,$J^-_M(p)$) can be defined with obvious suitable changes. Using these notions for points $p \in M$, we can define the analogous concepts for subsets $\Omega \subseteq M$ taking the union over the points in Ω, e.g. we define the *chronological future of the subset Ω in M* as $I^+_M(\Omega) = \bigcup_{p\in\Omega} I^+_M(p)$, and we denote the unions $I^+_M(p) \cup I^-_M(p)$ and $J^+_M(p) \cup J^-_M(p)$ with $I_M(p)$ and respectively with $J_M(p)$. Finally we define the *Cauchy development of Ω in M* as the subset $D_M(\Omega)$ of the points $q \in M$ such that every inextensible causal curve in M passing through q meets Ω.

Looking now at the proprieties of the chronological future and past of a subset S of a spacetime we can underline some interesting topological proprieties: future, past, timelike and spacelike compactness.

Definition 1.1.10. Let $(M, g, \mathfrak{o}, \mathfrak{t})$ be a spacetime and let U be a subset of M. We say that U is *future compact*, resp. *past compact* if $S \cap J^+_M(p)$, resp. $S \cap J^-_M(p)$ is a compact set for every $p \in M$. We call S *timelike compact* if $S \cap J_M(p)$ is compact for each $p \in M$. Finally we will define as *spacelike compact*, those subset S such that exist a compact $K \subseteq M$ for which the inclusion $S \subseteq J_M(K)$ holds.

Chronological future and past allow us to establish when there is causal connection between two different subsets. This notion is crucial in general relativity and we will pay great attention when implementing it in the construction of quantum and classical field theories.

Definition 1.1.11. Let $(M, g, \mathfrak{o}, \mathfrak{t})$ be a spacetime. We say that two subsets U_1 and U_2 of M are *causally separated* if the intersection $J_M(U_1) \cap U_2$ is the empty set.

Remark 1.1.12. Sometimes we may say that U_1 and U_2 are causally separated (or equivalently that U_1 is causally separated from U_2), meaning that there is no point of U_1 from which start a causal curve going to a point of U_2. These conditions are equivalent: a proof can be found in [Ben11, Remark 1.2.8]

An open connected subset Ω of a spacetime is a spacetime in its own right. For example take the d-dimensional spacetime $(M, g, \mathfrak{o}, \mathfrak{t})$ and let Ω be a connected open subset of M. Then undoubtedly Ω can be seen as a d-dimensional submanifold of M and hence a d-dimensional manifold in its own right. Moreover it becomes a spacetime when suitable concepts of restriction for metric, orientation and time orientation are provided $g|_\Omega$, $\mathfrak{o}|_\Omega$ and $\mathfrak{t}|_\Omega$. Denoting this spacetime with $(\Omega, g|_\Omega, \mathfrak{o}|_\Omega, \mathfrak{t}|_\Omega)$, we can define the domain of dependence in Ω for all its points, $J_\Omega(p)$. This set could not coincide with the restriction to the subset of the domain of dependence of the point p in the whole spacetime. This bring us to the definition of *causal compatibility*.

Definition 1.1.13. Let $(M, g, \mathfrak{o}, \mathfrak{t})$ be a spacetime and let S be an open subset of M. Then S is *causally compatible* if $J^\pm_S(p) = J^\pm_M(p) \cap S$ for each $p \in S$.

We underline the fact that each causal curve that is contained in $S \subseteq M$ can be directly seen also as a causal curve contained in M, hence it always holds the inclusion $J_S(p) \subseteq J_M(p) \cap S$ for each $p \in S$ and then the real condition of causal compatibility is the other inclusion.

When a causal structure is defined, also notions such as convexity, of great importance when working with analytical techniques, need to be generalised. In the context of Lorentzian geometry a meaningful notion is that of causal convexity.

Definition 1.1.14. Let $(M, g, \mathfrak{o}, \mathfrak{t})$ be a spacetime and let S be a subset of M. We say that S is *causally convex* if each causal curve in M that with endpoints in S is entirely contained in S.

Remark 1.1.15. It can be trivially proven that causal convexity of subset $U \subseteq M$ implies causal compatibility (see [Ben11, Remark 1.2.10]).

Now that all these notions have been introduced, the meaning of map between spacetimes preserving the causal structure can be fixed.

Remark 1.1.16. When dealing with smooth maps between manifolds we can select a class of functions preserving some additional structures. For example, given two Riemmanian manifolds (M, g) and (N, h), we may pick out from the class of smooth maps the class of *isometric embeddings*, i.e. embeddings $\chi : M \leftarrow N$ such that $\chi^* h = g$. Under this condition we can identify the manifold M and its Riemannian structure with its image $\chi(M) \subseteq N$, submanifold of N, endowed with the induced metric $h|_{\chi(M)}$. When dealing with spacetimes, preservation of the additional causal structure plays a key role. Then we ask for maps ψ

that, given two spacetimes $(M, g, \mathfrak{o}, \mathfrak{t})$ and $(N, h, \mathfrak{u}, \mathfrak{l})$, is an isometric embedding of M in N, orientation and time-orientation are preserved when pushed forward and $\psi(M)$ is causally compatible in N. Formally, we ask for the further conditions $\psi_*\mathfrak{o} = \mathfrak{u}|_{\psi(M)}$ and $\psi_*\mathfrak{t} = \mathfrak{l}|_{\psi(M)}$. Concluding this remark, let's notice that once one asks for open embeddings, i.e. maps with the image that is an open subset of the target manifold, thanks to Remark 1.1.15, the hypothesis of causal compatibility can be achieved asking the image to be causally convex.

Even if we restricted the whole class of Lorentzian manifolds to those for which is possible to define an orientation and a time orientation, we are still working with a too large set. Indeed, generic spacetime can show in some cases properties which are pathological from a physical point of view: the classical example is the famous Gödel spacetime, whose causal structure admits closed causal curves.

Example 1.1.17. [1] Let's fix $M = \mathbb{R}^4$ and define the metric, heuristically, by the line element, using global coordinates (t, x, y, z) reads:

$$ds^2 = -\left(dt + e^{2ky}dx\right)^2 + dy^2 + \frac{e^{4ky}}{2}dx^2 + dz^2,$$

where $k \in \mathbb{R}$ (constant). Changing coordinates by:

$$e^{2ky} = \cosh(2kr) + \sinh(2kr)\cos\varphi, \quad \sqrt{2}kxe^{2ky} = \sinh(2kr)\sin\varphi,$$
$$\frac{kt}{\sqrt{2}} = \frac{kt'}{\sqrt{2}} - \frac{\varphi}{2} + \arctan\left(e^{-2kr}\tan\frac{\varphi}{2}\right),$$

where $|k(t-t')| < \frac{\pi}{\sqrt{2}}$, $r \in [0, \infty)$ and $\varphi \in [0, 2\pi)$, then

$$ds^2 = -dt'^2 + dr^2 + dz^2 - \frac{\sqrt{8}}{k}\sinh^2(kr)d\varphi dt' + \frac{1}{k^2}\left(\sinh^2(kr) - \sinh^4(kr)\right)d\varphi^2.$$

If one takes the any curve of the form $\gamma(\varphi) = (\tilde{t}', \tilde{z}, \tilde{r}, \varphi)$, where $\tilde{t}', \tilde{z}$ are arbitrary constants, $\tilde{r} \geq (1/k)\ln(1+\sqrt{2})$ and $\varphi \in I \subseteq [0, 2\pi)$, is a closed causal curve.

As a first attempt to avoid this pathological situations, one could think to exclude from the class of spacetimes those admitting closed causal curves. In this case the spacetime is said to satisfy the *causality condition*. However, even if a spacetime satisfies the latter condition, it can show unpleasant properties; this is clear when one has to work with spacetimes for which the metric could be slightly perturbed, indeed, in this situation,

[1] We show this example as proposed by [BDH13]

"almost closed" causal curves become closed (see [Wal10, pg. 197] for an example). With this in mind, we can now define the so called *strong causality condition*

Definition 1.1.18. A spacetime $(M, g, \mathfrak{o}, \mathfrak{t})$ is said to satisfy the strong causality condition if for each point $p \in M$ and for each open neighbourhood U of p in M there exists an open neighbourhood $V \subseteq U$ of p in M such that each future directed (or equivalently past directed) causal curve which starts and ends in V must be entirely contained in U.

However, the discussion on causality is far more deep and many shades of causality can be proposed and their relations investigated. Authors like Bernal, Sanchez and Minguzzi wrote thorough papers on these arguments, like [MS08] and [BS07]. In the latter was showed how causality and strong causality can be equivalent when an additional hypothesis holds.

Proposition 1.1.19. *Given a spacetime $(M, g, \mathfrak{t}, \mathfrak{o})$, assume that for all $p, q \in M$ $J^+(p) \cap J^-(q)$ is compact, then the following two conditions are equivalent:*

i) (M, g) is causal, i.e., there are no closed causal curves.

ii) (M, g) is strongly causal, i.e., for any $p \in M$, given any neighbourhood U of p there exists a neighbourhood $V \subset U$, $p \in V$, such that any future-directed (and hence also any past-directed) causal curve $\gamma : [a, b] \to M$ with endpoints at V is entirely contained in U.

Now, we recall that the dynamics for field theories shall be discussed in terms of an initial value problem for differential equations ruled by suitable operators. Indeed, what we need is a framework where Cauchy problems are well posed. Going on towards this aim we will firstly define Cauchy surfaces (together with a preliminary useful notion), as special subset of our spacetimes, that seem to be the candidates for defining initial data of a Cauchy problem.

Definition 1.1.20. Let $(M, g, \mathfrak{o}, \mathfrak{t})$ be a spacetime. $U \subseteq M$ is said to be *achronal (acausal)* in M if each timelike (causal) curve in M meets it at most once. We say that Σ is a *Cauchy surface of* M if it is an achronal subset of M and $D_M(\Sigma) = M$

Now, all the ingredient are ready for the definition of the class of 'good enough' spacetimes: Globally Hyperbolic Spacetimes (GHST).

Definition 1.1.21. A spacetime $(M, g, \mathfrak{o}, \mathfrak{t})$ is said globally hyperbolic if the following conditions hold:

i) $\boldsymbol{M}$ fulfils the *strong causality condition*;

ii) $J^+(p) \cap J^-(q)$ is a compact subset of M for all pairs of points (p, q) in M.

Throughout the literature many different definitions of GHST are presented, hence we give here a characterization theorem for GHST (as reported essentially in [BGP07]).

Theorem 1.1.22. *Let $\boldsymbol{M}$ be a spacetime. Then the following conditions are equivalent:*

i) $\boldsymbol{M}$ is globally hyperbolic;

ii) $\mathbf{M}$ *has a Cauchy surface* Σ *;*

iii) there exists an isometric embedding from the Lorentzian manifold (M, g) to $(\mathbb{R} \times \Sigma, -\beta \mathrm{d}t^2 + g_t)$, where Σ is a $(d-1)$-dimensional manifold and β is a smooth strictly positive function, g_t is a family of Riemannian metric on $\{t\} \times \Sigma$ varying smoothly with t. Moreover we have that for each $t \in \mathbb{R}$, $\{t\} \times \Sigma$ is the image through the isometric embedding of a smooth spacelike Cauchy surface of $\mathbf{M}$.

Globally hyperpolic spacetimes seem to solve the problem of finding a physically sensible class of spacetimes that is not too much restrictive. Moreover, they are a safe harbour also from a mathematical point of view. Once fields are interpreted as solution of equations of motion ruled by suitable operators, existence and global well-definiteness can be achieved in this framework. Unfortunately, there are some cases in which physically interesting manifolds are not part of this class. The most famous example is given by the so called *anti-de Sitter* spacetime. As for the Gödel spacetime, the latter admits closed causal curves.

Example 1.1.23. We briefly present here a description of the AdS_n spacetime[2]. It can be defined as a locus of points of $\mathbb{R}^{n+1}$ using the set of coordinates $(u, v, x_1, \ldots, x_{n-1})$

$$M = \left\{(u, v, x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n-1}^2 - u^2 - v^2 = -R^2\right\} \tag{1.1.1}$$

From a topological point of view, this manifold is homeomorphic to $S^1 \times \mathbb{R}^{n-1}$ and, when the ambient space $\mathbb{R}^{n+1}$ is endowed with the metric $g = diag(1, 1, -1, \ldots, -1)$, the induced metric $h := g|_M$ has Lorentzian signature. Since also orientability and time-orientability

[2] As reported in [BD15]

holds, we can choose a non degenearate volume form and a timelike vector field and build up the anti-de Sitter spacetime over the base manifold M. Moreover, when $n > 2$, (M, h) is a maximally symmetric solution to the Einstein's equation with a negative cosmological constant Λ. In other words it is a manifold of constant curvature $R = \frac{2n}{n-2}\Lambda$. The curve $\gamma : \mathbb{R} \to \mathbb{R}^{n+1}$

$$\gamma(s) = (R\sin(s), R\cos(s), 0, \dots, 0) \tag{1.1.2}$$

is closed, a circle in particular, whose tangent vector is everywhere time-like with respect to the induced metric.

Treating this particular situation special results shall be proven, because the general framework we are going to present is heavily based on the property of global hyperbolicity. Now, we want to know how key proprieties such as global hypebolicity persist when we take subsets of a given spacetime.

Proposition 1.1.24. *Let $(M, g, \mathfrak{o}, \mathfrak{t})$ be a globally hyperbolic spacetime and let S be a subset of M. If S is causally convex, then it is also globally hyperbolic.*

Proof. We follow here the proof by [Ben11]. Since $(M, g, \mathfrak{o}, \mathfrak{t})$ is a globally hyperbolic spacetime, then S trivially satisfy the strong causality condition. Then we have to show that with the induced spacetime structure $(S, g|_S, \mathfrak{o}|_S.\mathfrak{t}|_S)$, the set

$$J_S^+(p) \cap J_S^-(q) := C_S(p, q) \tag{1.1.3}$$

is compact for each p and q in S. We know that for each couple of point in M, $C_M(p, q)$ is compact and we shall use the fact that causal convexity entails causal compatibility to prove the compactness of $C_S(p, q)$. We fix p, $q \in S$. Since causal compatibility holds, we have that $J_S^{\pm}(r) = J_M^{\pm}(r) \cap S$, for each $r \in S$. It follows that

$$J_S^+(p) \cap J_S^-(q) = J_M^+(p) \cap J_M^-(q) \cap S.$$

Since M is globally hyperbolic, we deduce that $J_M^+(p) \cap J_M^-(q)$ is compact with respect to the topology of M. But $C_M(p, q)$ is also fully contained in S. Taking an arbitrary point r in $J_M^+(p) \cap J_M^-(q)$ we can always find a future directed causal curve in M from p to r and a past directed causal curve in M from q to r. Joining, after a suitable chancing in the parametrization of the latter, the two curves can be joined to obtain a future directed causal curve γ in M from p to q, that is entirely contained in S because its endpoints $p, q \in S$ and S is causally convex. Then the inclusion $C_M(p, q)(q) \subseteq S$ actually holds together with

$$C_S(p,q) = C_M(p,q) \cap S = C_M(p,q)$$

This chain of equalities complete the proof. □

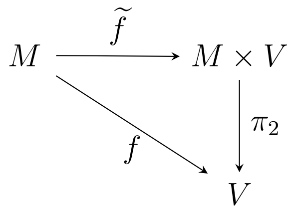
The real importance of this proposition will be clear in the following: when dealing with GHST we have sufficient conditions to embed it, in the sense showed in Remark 1.1.16, and understand its image as globally hyperbolic subspacetime of the target spacetime and this will be crucial when understanding quantum field theories in the framework of category theory.

1.2 Green-hyperbolic operators

Once properties of the framework have been fixed, we can proceed understanding which operators and which equations are suitable to regulate the dynamics of our models. Furthermore, for this class of operators, we shall prove the well-posedness of initial value problems (uniqueness and existence of solutions). Throughout this Section we will set all the mathematical details and we will show how the space of solutions of a given operator can be enriched with a nice structure.
Often those operators are called wave operators, or wave-like operator, but actually these labels describe a wider class of equations than those which usually people refer to with the same name. We start defining the space of sections of a vector bundle.

Definition 1.2.1. Let $\pi : E \to M$ be a vector bundle over M. A *smooth section of* E is a smooth function s from M to E such that $\pi \circ s = \mathrm{id}_M$. The *space of (smooth) sections* $\Gamma(M,E)$ consist of all the smooth sections of E and the *space of smooth sections with compact support* $\Gamma_c(M,E)$ is the set of all the compactly supported smooth sections of E.

Remark 1.2.2. Let's notice that one can use trivial vector bundles to give a different characterization of vector valued functions on a manifold. Suppose we are given a function $f : M \to V$, taking into account the trivial vector bundle $(M \times V, \pi_1, M, V)$ and denoting with $\pi_{1,2}$ the projection with respect to the first and second coordinate, we can define a section $\widetilde{f} : M \to M \times V$ such that the following diagram commutes



This shows that $\widetilde{f}$ is a section of the trivial bundle $M \times V$ a that it's possible to find a bijection between $C^\infty(M, V)$ and $\Gamma(M, M \times V)$.

Then we shall proceed enriching the structure of vector bundles with a non degenerate bilinear form.

Definition 1.2.3. Let (E, π, M, V) be a real vector bundle. A *Bosonic (Fermionic) non-degenerate bilinear form* on E is a smooth map, denoted by $\langle \cdot, \cdot \rangle_E$, acting on the vector bundle $E \otimes E$, being fiberwise symmetric (antisymmetric) and non degenerate, i.e.:

(i) for each $p \in M$, $\langle \cdot, \cdot \rangle_E$ is a symmetric (antisymmetric) bilinear form;

(ii) for each $p \in M$, taking $v \in E_p$ is such that $\langle v, w \rangle_E = 0$ for each $w \in E_p$, then $v = 0$.

The explicit reference to the vector bundle E will be removed when clear from the context. Since we consider spacetimes for which an orientation is given, we can pick out the metric induced volume form to define integration for compactly supported functions defined on the base manifold M. Via integration, inner products, like those of Definition 1.2.3, induce a non degenerate pairing between smooth sections and compactly supported smooth sections of a given vector bundle E.

Definition 1.2.4. Let (E, π, M, V) be a real vector bundle. We define the *non degenerate pairing induced by the non degenerate bilinear form* $\langle \cdot, \cdot \rangle_E$ *as the map*

$$(\cdot, \cdot)_E : \Gamma_0(M, E) \times \Gamma(M, E) \to \mathbb{R} \qquad (\sigma, \tau) \mapsto \int_M \langle \sigma, \tau \rangle_E \,\mathrm{dvol}_{\boldsymbol{M}} \,. \tag{1.2.1}$$

Notice that, for (1.2.1) *to be meaningful, we could consider both* $\tau, \sigma \in \Gamma(M, E)$ *provided that* $\mathrm{supp}(\tau) \cap \mathrm{supp}(\sigma)$ *is compact*

Now we are ready to present some results on linear partial differential operators acting on sections of a given vector bundle.

Definition 1.2.5. Let $(E, \pi, M, V), (F, \rho, M, W)$ be two vector bundles over the same d-dimensional manifold M. A *linear partial differential operator of order at most* k is a linear map $P : \Gamma(M, E) \to \Gamma(M, F)$ that can be written locally (in charts) as

$$P = \sum_{i=0}^{k} \sum_{j_1,\ldots,j_i=1}^{d} A_{j_1,\ldots,j_i} \partial_{j_1} \cdots \partial_{j_i} \tag{1.2.2}$$

where $\{A_{j_1,\ldots,j_i}\}$ are smooth $\mathrm{Hom}(V, W)$-valued functions and $\{\partial_i\}$ denotes the partial derivatives

We shall briefly note that partial differential operators don't change supports of sections, in particular they induce linear map $P : \Gamma_c(M, E) \to \Gamma_c(M, E)$. Before introducing further objects for the development of field theories, we give here the definition of *principal symbol* useful in many concrete applications. It can be used to provide characterization of certain classes of partial differential operator because it is an useful and simple notion that can be associated also to differential operators defined by complicated expressions.

Definition 1.2.6. Given a spacetime $(M, g, \mathfrak{o}, \mathfrak{t})$ and a vector bundle E over M, let be P be a differential operator defined on the space of section section $\Gamma(M, E)$ by the Equation (1.2.2), then the *principal symbol* at the point p is the polynomial in the variable $\xi \in \mathbb{R}^d$

$$\sigma_P(p, \xi) = \sum_{j_1,\ldots,j_k=1}^{d} A_{j_1,\ldots,j_i}(p) \xi^{j_1} \ldots \xi^{j_k} \ . \tag{1.2.3}$$

Remark 1.2.7. Sometimes the defintion of linear partial differential operator is given using the multi-index notation (α), in this case the Example 1.2.3 takes the form

$$\sigma_P(p, \xi) = \sum_{|\alpha|=k} A_\alpha(p) \xi^\alpha.$$

Not all operators in the LPDOs' class are of physical interest or allow a mathematical and rigorous formulation of quantum field theories on GHST. Pairings induced by non degenerate inner products will be useful to define the formal adjoint operator of a given LPDO and, in the following, to endow our space of solution with appropriate and physically meaningful symplectic forms (for Bosonic bilinear forms) and inner products (for Fermionic ones).

Definition 1.2.8. Given a linear partial differential operator $P : \Gamma(M, E) \to \Gamma(M, F)$, its *formal adjoint* $P^* : \Gamma(M, F) \to \Gamma(M, E)$ is a linear partial differential operator such that, taking the non-degenerate pairings of Eq. (1.2.1) for E and respectively F, $(P^* f, g)_E = (f, Pg)_F$ for each $f \in \Gamma(M, F)$ and $g \in \Gamma(M, E)$, satisfying $\operatorname{supp}(f) \cap \operatorname{supp}(g)$ compact. A linear partial differential operator $P : \Gamma(M, E) \to \Gamma(M, E)$ is *formally self-adjoint* if $P^* = P$.

Now we have all the basic tools to define and to deal with Green's operators: according to our aims, they have a central role in the study of the structure of the space of solution, when the equation of motion is ruled by a particular class of operators called Green-hyperbolic operators (GHOs).

Definition 1.2.9. Let consider a spacetime $(M, g, \mathfrak{o}, \mathfrak{t})$, a vector bundle E over M and a linear partial differential operator $P : \Gamma(M, E) \to \Gamma(M, E)$. A linear map $G^\pm : \Gamma_c(M, E) \to \Gamma(M, E)$ is an *retarded/advanced Green's operator* for P if for each $f \in \Gamma_c(M, E)$:

(i) $PG^\pm f = f$;

(ii) $G^\pm P f = f$;

(iii) $\operatorname{supp}(G^\pm f) \subseteq J^\pm_M(\operatorname{supp}(f))$.

A linear partial differential operator P admitting both advanced and retarder Green's operator is called *Green-hyperbolic.*

The study of GHOs is the main subject of this section and will be the starting point for the construction of classical and quantum field theories. The first question one usually wants to answer is whether, given a GHO, its Green's operators are unique. Still in the recent past, authors believed that proving uniqueness required green hyperbolicity of the adjoint operator (see [BDH13]), but as showed by [Bär14] this condition is not necessary. We recall these results, starting from the proof that Green's operators admits unique extensions to wider space of sections.

Theorem 1.2.10. *There exist unique linear continuous extensions*

$$\overline{G}^+ : \Gamma_{pc}(M, E) \to \Gamma_{pc}(M, E) \quad \textit{and} \quad \overline{G}^- : \Gamma_{fc}(M, E) \to \Gamma_{fc}(M, E)$$

of G^+ and G^- respectively, such that

(i) $\overline{G}^+ P f = f$ for all $f \in \Gamma_{pc}(M, E)$;

(ii) $P\overline{G}^+ f = f$ *for all* $f \in \Gamma_{pc}(M, E)$*;*

(iii) $\text{supp}(\overline{G}^+ f) \subset J^+_M(\text{supp} f)$ *for all* $f \in \Gamma_{pc}(M, E)$*;*

and similarly for $\overline{G}^-$*.*

Proof. Reporting the proof proposed by [Bär14], we only show how $\overline{G}^+$ is defined, the steps for $\overline{G}^-$ being analogous. Given $f \in \Gamma_{pc}(M.E)$ and a point $x \in M$ we define $(\overline{G}_+ f)(x)$ as follows: Since $J^-(x) \cap \text{supp} f$ is compact we can choose a cutoff function such that $\chi f \in \Gamma_c(M, \mathbb{R})$ with $\chi \equiv 1$ on a neighbourhood of $J^-(x) \cap \text{supp} f$. Now we put

$$(\overline{G}^+ f)(x) := (G^+(\chi f))(x). \tag{1.2.4}$$

This is clearly an extension of G^+: when $f \in \Gamma_c(M, E)$ then the choice $\chi \equiv 1$ on the whole manifold gives what we need. First we show that this definition is independent of the choice of χ. Namely, let χ' be another function such that $\chi' f$ has compact support, we show that for each $x \in M$, $(G^+((\chi - \chi')f))(x) = 0$. In order to achieve this result, we use the third property of Definition 1.2.9 for which

$$x \in \text{supp}(G_+((\chi - \chi')f)) \subset J^+(\text{supp}((\chi - \chi')f))$$

The latter means that there exists a causal curve from the $\text{supp}((\chi - \chi')f)$ to x and $\text{supp}((\chi - \chi')f) \cap J^-(x)$ is nonempty. On the other hand,

$$\text{supp}((\chi - \chi')f) \cap J^-_M(x) = \text{supp}(\chi - \chi') \cap \text{supp} f \cap J^-_M(x).$$

But on $\text{supp} f \cap J^-_M(x)$, $\chi = \chi' \equiv 1$ and then the intersection on the right end side is the empty set. This lead to a contradiction with the previous sentence. Now, we have to prove that the target space of the map defined by (1.2.4) is $\Gamma_{sc}(M, E)$. Since the support of $\overline{G}^+ f$ depends on G^+, we shall prove only that is a smooth function. Given $x \in M$ the smoothness is trivial for each point in $I^-_M(x)$, indeed, a unique χ can be fixed and we have $\overline{G}^+ f = G^+(\chi f)$ that is a smooth function because it lies on the codomain of G^+. Noticing each point $x' \in M$ is in a set of the type $I^-_M(y)$ for some y, we can conclude that $\overline{G}^+ f$ is smooth on the whole M.

Linearity of the operator $\overline{G}^+$ is a matter of choosing a suitable χ. Let $f_1, f_2 \in \Gamma_{pc}(M, E)$, let's take a smooth function χ such that χf_1, χf_2 have compact support and $\chi \equiv 1$ on neighbourhoods of both $\mathrm{supp}(f_1) \cap J^-(x)$ and $\mathrm{supp}(f_2) \cap J^-(x)$. Then also $\chi(f_1 + f_2) \in \Gamma_c(M, E)$ and $\chi \equiv 1$ on neighbourhoods of both $\mathrm{supp}(f_1 + f_2) \cap J^-(x)$. Then all the following equivalences are well defined

$$\begin{aligned}(\overline{G}_+(f_1 + f_2))(x) &= (G_+(\chi f_1 + \chi f_2))(x) \\ &= (G_+(\chi f_1)(x) + (G_+(\chi f_2))(x) \\ &= (\overline{G}_+ f_1)(x) + (\overline{G}_+ f_2))(x).\end{aligned}$$

We shall proceed with the prove of the three properties. i.) Let $x \in M$ and χ as before. In particular, we may choose $\chi \equiv 1$ also on a neighbourhood of x, showing (ii).

$$(P\overline{G}_+ f)(x) = (PG_+(\chi f))(x) = (\chi f)(x) = f(x).$$

ii.) Proving (i) is a bit more complicated. Let's start computing

$$\begin{aligned}(\overline{G}_+ Pf)(x) &= (G_+(\chi \cdot Pf))(x) \\ &= (G_+ P(\chi f))(x) + (G_+([\chi, P]f))(x) \\ &= f(x) + (G_+([\chi, P]f))(x).\end{aligned}$$

Using the same trick as before, we simply show that $x \notin \mathrm{supp}(G_+([\chi, P]f))$. The coefficients of the differential operator $[\chi, P]$ vanish where χ is locally constant, in particular on $\mathrm{supp} f \cap J^-(x)$ where $\chi \equiv 1$ Now we find

$$\begin{aligned}\mathrm{supp}(G_+([\chi, P]f)) &\subset J^+(\mathrm{supp}([\chi, P]f)) \\ &\subset J^+(\mathrm{supp} f \setminus J^-(x)) \\ &\subset J^+(\mathrm{supp} f) \setminus \{x\}\end{aligned}$$

and therefore $x \notin \mathrm{supp}(G_+([\chi, P]f))$.

iii.) For the last we see that given $f \in \Gamma_{pc}(M, E)$

$$\mathrm{supp}(\overline{G}_+ f) \subset \bigcup_\chi \mathrm{supp}(G_+(\chi f)) \subset \bigcup_\chi J^+(\mathrm{supp}(\chi f)) \subset J^+(\mathrm{supp} f).$$

Here the union is taken over all χ such that $\chi f \in \Gamma_c(M, \mathbb{R})$. □

Now we are going to prove that operations such as composition or direct sum preserve properties of GHOs. The importance of these results relies in the fact that we are able to prove Green-hyperbolicity for a wide class of interesting operators (normally hyperbolic operators), and we can understand physically interesting operators out of this class as composition or direct sum of normally hyperbolic operator. The direct sum and composition of two Green-hyperbolic operators is again Green hyperbolic.

Proposition 1.2.11. *Let* $P, P' : \Gamma(M,E) \to \Gamma(M,E)$ *and* $Q : \Gamma(M,F) \to \Gamma(M,F)$ *be Green hyperbolic. Then the operators*

i) $P \oplus Q : \Gamma(M, E \oplus F) \to \Gamma(M, E \oplus F)$

ii) $P \circ P' : \Gamma(M,E) \to \Gamma(M,E)$

are also Green hyperbolic.

Proof. Proving the first is highly trivial. Denoting with $G^{\pm}_P$ and $G^{\pm}_Q$ are the Green's operators for P and Q respectively, then the direct sum $G^{\pm}_P \oplus G^{\pm}_Q$ defines Green's operator for $P \oplus Q$. When dealing with the proof of the second statement, one should think simply to compose Green's operator in reverse order, but unfortunately this composition doesn't make sense: this problem is solved using extensions $\overline{G}^{\pm}$. The Green's operator $G^{+} : \Gamma_c(M,E) \to \Gamma(M,E)$ is defined as follows:

$$G^{+}_{P \circ P'} := \iota_{pc} \circ \overline{G}^{\pm}_{P'} \circ \overline{G}^{\pm}_{P} \circ \iota_c$$

where we used the inclusions $\iota_c : \Gamma_c(M,E) \hookrightarrow \Gamma_{pc}(M,E)$ and $\iota_{pc} : \Gamma_{pc}(M,E) \hookrightarrow \Gamma(M,E)$. □

Now, we shall present two results that seem to invert the previous proposition. As we will show, when Green-hyperbolicity can't be proven directly for a linear partial differential operator, could be useful relating them to operators for which this property is already known: tools like these are typically used when dealing with *Dirac type operators*.

Proposition 1.2.12. *Let* $P : \Gamma(M,E) \to \Gamma(M,E)$ *be a differential operator and let* E *carry a non-degenerate bilinear form. If* P^2 *is Green-hyperbolic, so is* P*. Let* $P^* : \Gamma(M,E) \to \Gamma(M,E)$ *be the formally adjoint operator. If* P^*P *and* PP^* *are Green hyperbolic, then* P *and* P^* *are Green-hyperbolic too.*

As the last of this group of results, we present one that allows to compute Green's operators of the formal adjoint.

Proposition 1.2.13. *Let consider a globally hyperbolic spacetime* $\mathbf{M}$ *and a vector bundle* E *defined over* M. *Taking* $P : \Gamma(M, E) \to \Gamma(M, E)$ *Green-hyperbolic operator, with Green-hyperbolic formal adjoint* $P^* : \Gamma(M, E) \to \Gamma(M, E)$, *denoting the retarded and advanced Green's operators respectively* $G^{\pm}$ *and* $G^{\star\pm}$. *Then, for all* $\sigma, \tau \in \Gamma_c(M, E)$, *it holds that*

$$(G^{*\mp}\tau, \sigma)_E = (\tau, G^{\pm}\sigma)_E$$

i.e. $(G^{\pm})^* = (G^*)^{\mp}$.

Proof. Let's prove this statement for G^+. By Definition 1.2.9 of Green's operator we know that $PG^+ = \mathrm{id}_{\Gamma_c(M,E)}$ and then

$$(G^{*-}\tau, \sigma)_E = (G^{*-}\tau, PG^+\sigma)_E = (P^*G^{*-}\tau, G^+\sigma)_E = (\tau, G^+\sigma)_E$$

holds true for arbitrary $\sigma \in \Gamma_c(M, E)$ and $\tau \in \Gamma_c(M, E)$, because the set

$$\mathrm{supp}\left(G^{*-}\tau\right) \cap \mathrm{supp}\left(G^+\sigma\right)$$

is compact and the chain of equalities is well-defined at every step. □

We now introduce the causal propagator G of an operator P, built up from its Green's operators.

Definition 1.2.14. Let E be a vector bundle over a spacetime $(M, g, \mathfrak{o}, \mathfrak{t})$ and P be a Green-hyperbolic operator on $\Gamma(M, E)$. Then $G = G^+ - G^-$ is the *causal propagator* for P defined by $G^{\pm}$.

The properties of G can be obtained directly from those of Definition 1.2.9. For $\tau \in \Gamma_c(M, E)$:

(i) $PG\tau = 0$;

(ii) $GP\tau = 0$;

(iii) $\mathrm{supp}(Gf) \subseteq J_M(\mathrm{supp}(f))$.

Starting from the latter, we are ready to prove the central theorem of this section in order to understand the theory of solution of the equation ruled by a given Green-hyperbolic operator P.

Theorem 1.2.15. *Let $(M, g, \mathfrak{o}, \mathfrak{t})$ be a spacetime and E a vector bundle defined over M. Let's take an operator $P : \Gamma(M, E) \to \Gamma(M, E)$ being Green-hyperbolic. Denoting with G its causal propagator, then*

$$\{0\} \to \Gamma_c(M, E) \xrightarrow{P} \Gamma_c(M, E) \xrightarrow{G} \Gamma_{sc}(M, E) \xrightarrow{P} \Gamma_{sc}(M, E) \to \{0\} \tag{1.2.5}$$

is an exact sequence, i.e. each kernel of the arrows depicted above coincides with the image of the previous map.

Proof. We prove the theorem going step by step from left to right.

(i) the first step means injectivity of P. Let's take $f \in \Gamma_c(M, E)$, we have that

$$Pf = 0 \quad \Rightarrow \quad G^+Pf = 0 \quad \Rightarrow \quad f = 0.$$

(ii) Let's show now that Im $P|_{\Gamma_c(M,E)}$ = Ker $G|_{\Gamma_c(M,E)}$. We take $f \in \Gamma_c(M, E) \cap$ Im $P|_{\Gamma_c(M,E)}$, i.e. $f = Pu$ with $u \in \Gamma_c(M, E)$, for the second of the properties listed above

$$Gf = GPu = 0.$$

Now, taking h s.t. $Gh = 0$ and splitting the causal propagator, we have $G^+h = G^-h = h_0$. Let's notice that supp$h_0 \subseteq J^+(\text{supp}h) \cap J^-(\text{supp}h)$, and for this reason is compact. Then, we can write

$$h = PG^+h = Ph_0.$$

(iii) In order to prove that Ker$P|_{\Gamma_{sc}(M,E)}$ = Im$G|_{\Gamma_c(M,E)}$, we take u such that $Pu = 0$ and its support is spacelike-compact and consider $u_+ = \chi_+u$ and $u_- = \chi_-u$, where $\{\chi_+, \chi_-\}$ is a partition of unity subordinate to the open cover $\{I^+_M(\Sigma_-), I^-_M(\Sigma_+)\}$ and Σ_+, Σ_- are disjoint spacelike Cauchy surfaces, with Σ_+ lying in the future of Σ_-. Per linearity $Pu_+ + Pu_- = Pu = 0$. Together with the support properties of u (spacelike-compact support) and χ_+, χ_- (past-compact, respectively future-compact, support), this identity entails that $Pu_+ = -Pu_-$ has compact support, hence we can define $f = \pm Pu_\pm$. We show that $u = GPu_+$. Indeed, using the extension of Green's hyperbolc operators in order to define the linear operator $\overline{G} := \overline{G}^+ - \overline{G}^-$ we get

$$\overline{G}f = \overline{G}^+f - \overline{G}^-f = u_+ + u_- = u.$$

But, because $f \in \Gamma_c(M,E)$, the extended propagator match the causal propagator and we have $u = \bar{G}f = Gf$. Let's prove the other inclusion, taking a section of the form Gv in $\Gamma_{sc}(M,E)$, easily $PGv = 0$, then $Gv \in \text{Ker } G|_{\Gamma_{sc}(M,E)}$.

(iv) Surjectivity of the operator P acting on $\Gamma_{sc}(M,E)$ comes from the fact that given $u \in \Gamma_{sc}(M,E)$ and a partition of unity defined like at the previous point, we take the section $v = \overline{G}^+ u_+ + \overline{G}^- u_-$ and we get

$$Pv = P\overline{G}^+ u_+ + P\overline{G}^- u_- = u_+ + u_- = u$$

and the latter point concludes the proof.

□

Using the extensions of Green's operator we can prove a similar sequence involving different spaces of sections. We do not give explicit proof of this proposition here, because it's essentially a reproduction of ideas and techniques used in the latter.

Proposition 1.2.16. *Let $(M, g, \mathfrak{o}, \mathfrak{t})$ be a spacetime and E a vector bundle defined over M. Let's take an operator $P : \Gamma(M,E) \to \Gamma(M,E)$ being Green-hyperbolic. Denoting with $\overline{G}$ its extended causal propagator, then*

$$\overline{G} = \overline{G}^+ - \overline{G}^- \tag{1.2.6}$$

Then

$$\{0\} \to \Gamma_{tc}(M,E) \xrightarrow{P} \Gamma_{tc}(M,E) \xrightarrow{\overline{G}} \Gamma(M,E) \xrightarrow{P} \Gamma(M,E) \to \{0\} \tag{1.2.7}$$

is an exact sequence.

Now, we can introduce the class of normally hyperbolic operators. As anticipated above this is a set of LPDOs distinguished from a physical point of view: for those operators we are able to prove Green-hyperbolicity, using the fact that a Cauchy problem ruled by the operator admits unique and global solutions. Concluding this chapter we will make precise what we've just said.

Definition 1.2.17. Let (E, π, M, V) be a real vector bundle over a d-dimensional Lorentzian manifold (M, g). We say that a partial differential operator $P : \Gamma(M,E) \to \Gamma(M,E)$ of second order is *normally hyperbolic* if it can be locally written

$$P = -\sum_{i,j=1}^{d} g^{ij}\partial_i\partial_j + \sum_{i=1}^{d} A_i\partial_i + A \,. \tag{1.2.8}$$

Given a section J of a vector bundle E, called *source*, and a differential operator P taking values on $\Gamma(M, E)$, we say that a partial differential equation $P\sigma = J$ is a *wave equation* if P is normally hyperbolic.

Remark 1.2.18. Formal adjoint operators of normally hyperbolic operator are normally hyperbolic. Then all results we prove for NHOs hold true for the formal adjoint too.

Theorem 1.2.19. *Let $(M, g, \mathfrak{o}, \mathfrak{t})$ be a globally hyperbolic spacetime. Consider $\Sigma \subseteq M$ being a Cauchy surface for $\mathbf{M}$, whit a normal future-pointing vector denoted by $\mathfrak{n}$. Then, take a vector bundle (E, π, M, V), endowed with a connection ∇, a normally hyperbolic operator P acting on sections of $\Gamma(M, E)$. For each initial data $u_0, u_1 \in \Gamma_c(\Sigma, E)$ and each source $f \in \Gamma_c(M, E)$, the Cauchy problem*

$$Pu = f \text{ on } M, \quad u = u_0,\ \nabla_{\mathfrak{n}} u = u_1 \quad \text{ on } \Sigma, \tag{1.2.9}$$

has a unique solution $u \in \Gamma(M, E)$.

Moreover, the support of u satisfies the following property

$$\mathrm{supp}(u) \subseteq J_M(\mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup \mathrm{supp}(f)) \ , \tag{1.2.10}$$

and the map $\Gamma_c(\Sigma, E) \times \Gamma_c(\Sigma, E) \times \Gamma_c(M, E) \to \Gamma(M, E), (u_0, , u_1, f) \mapsto u$, taking a set of initial data and source and assigning the unique solution of respective Cauchy problem, is linear and continuous.

We skip the technicalities of the proof here, but we suggest the interested reader to refer to [Bär14] for a complete exposition on Cauchy problems for normally hyperbolic operators, with thorough proofs and useful insights. What is of central importance is the fact that from existence and uniqueness of a global solution we can build up Green's operators for a NHO.

Theorem 1.2.20. *Let's take a spacetime $(M, g, \mathfrak{o}, \mathfrak{t})$ and E, vector bundle over M. Given a normally hyperbolic operator P acting on section of $\Gamma(M, E)$, then P is Green-hyperbolic.*

Proof. We have to explicitly construct Green's operator satisfying conditions of definition 1.2.9 (we show here only the construction for $G^{\pm}$). Given a compactly supported section f we solve the Cauchy problem:

$$Pu = f \text{ on } M, \quad u = 0 \text{ on } \Sigma_-, \tag{1.2.11}$$

where Σ_- is a Cauchy surface s.t supp$f \subseteq J(\Sigma_-)$ and supp$f \cap \Sigma_- = \emptyset$, and we denote its unique solution with u_f. We claim that a Green's operator G^+ for P is the map

$$f \mapsto G^+ f \ := \ u_f \tag{1.2.12}$$

Let's check first whether supp$(G^+ f) \subseteq J^+(\text{supp} f)$, and whether this map does not depend on the choice of the Cauchy surface. We define the manifold $\widetilde{M} = M \setminus J^+(\text{supp} f)$ and, noticing that Σ^- is fully included in $\widetilde{M}$, we solve the Cauchy problem 1.2.11 in $\widetilde{M}$, where $f|_{\widetilde{M}}$ is the null function. The solution of a Cauchy problem with null initial data is $u = 0$, but by construction also u_f solves it. We conclude that $u_f|_{\tilde{M}} = 0$ and then supp$G^+ f =$ supp$u_f \subseteq J^+(\text{supp} f)$. Let's take now another Cauchy surface Σ'_-, still disjoint from supp$f$ and such that supp$f \subseteq J(\Sigma'_-)$, the map (1.2.12) defines a function u'_f that coincide with u_f on Σ_- and then is also solution of the Cauchy problem with initial data on Σ_-, but by uniqueness they are the same function on whole M. Concerning the first property of the definition of Green's operator, this map trivially satisfy

$$PG^+ f = Pu_f = f$$

The second property follows since, for any space-like Cauchy surface Σ disjoint from the causal future of the support of Pf, u_{Pf} is a solution of the Cauchy problem $Pu_{Pf} = Pf$ with vanishing initial data on Σ. Consequently $u_{Pf} - f$ is a solution of a Cauchy problem with vanishing initial data on Σ and vanishing source, meaning that $u_{Pf} - f = 0$. Hence the map $f \mapsto u_f$ is a Green's operator for P. □

Chapter 2

Locally covariant quantum field theory

In this chapter we use all the tools developed to build up quantum field theories in a mathematical rigorous framework. In Section 2.1, in order to get motivations for the introduction of the axioms and the formal language of the last section, we present a scheme for the quantization of the Klein-Gordon theory. We prove that the KG operator is normally hyperbolic, hence, using Theorem 1.2.15, we associate a *symplectic space* to the operator with three different, but equivalent, methods, constructing the space of linear observables for the classical field theory. We proceed then to algebraic quantization, which analogously can be achieved working via two different paths. The first consist in the definition of the tensor algebra out of a symplectic space, imposing the canonical commutation relations by quotient of a suitable ideal. The second scheme is know as *quantization by deformation* and provides an algebra of observable working on linear functionals and *deforming*, in a precise and rigorous sense, the pointwise product in a product admitting canonical commutators. These $*$-algebras describe the quantum theory encoding coherently the property of the underlying spacetime: *locality* and *causality* are correctly implemented, together with the less-known *time slice axiom*. Then, in Section 2.2, we show how spinor fields can be treated on curved Lorentzian manifold. This part is useful because it comprehends a simple introduction on *Cartan geometry*, which allows to describe Lorentzian manifolds in term of *Cartan structure* and provide a framework which can be extended for the description of field theories on *supermanifolds* (or, more precisely, *super-Cartan structures*). Finally, using the language of categories and functors, we describe the axiomatic formulation of quantum field theories and we give a precise definition of *general covariance*, showing how the results achieved for the KG field can be obtained for all Green-hyperbolic operators and *coeherently* for all globally hyperbolic spacetimes.

2.1 A first example: the Klein-Gordon field

We start our discussion with the simplest example of a classical field: the free scalar field. Despite this field is the least relevant from a physical point of view, it is an important prototype for a deep understanding of quantum field theories in curved backgrounds. Furthermore, the generalisation of the results of this chapter to the case of super free scalar fields is the final aim of this theory.

In order to write a generalized version of the well known Klein-Gordon equation, we can invoke two heuristic rules:

a) the principle of general covariance, for which equations of physics are derived from those on Minkowski spacetime substituting any mathematical object derived from the metric η with the analogous object built up on the general curved metric g;

b) the requirement that in the limit $g \to \eta$, equations reduce to that valid in special relativity.

Naturally, these two rules, the first in particular, are not well defined and, in general, when used alone, they hardly lead to the correct physical model. Let's see how they work when applied to the case of our concern: the massive scalar field. The equation of motion in $(\mathbb{R}^d, \eta)$, in the general global coordinates $\{x^i\}_{i=1,\dots,d}$ already used in Example 1.1.2, using the Einstein convention looks like:

$$(\eta^{ij}\partial_i\partial_j + m^2)\phi = 0 \tag{2.1.1}$$

Hence, with the substitution $\eta \to g$, we get:

$$(g^{ij}\nabla_i\nabla_j + m^2)\phi = 0 \tag{2.1.2}$$

where ∇_i is the covariant derivative associated to the Levi-Civita connection. Unfortunately, there are many other generalisations of (2.1.1) compatible with this procedure, namely all the equations of the form:

$$\left(\Box_\nabla + f(R) + m^2\right)\phi = 0 \tag{2.1.3}$$

where f is an analytic function and $\Box_\nabla$ is the short notation for the D'Alembert operator associated to the Levi-Civita connection. This regularity condition for f is reasonable and turns out to be fundamental when studying renormalization properties of the quantum scalar field (see [HW01]). A thorough discussion of all the possible forms for the function f is not of our concern. We take $f = \xi R$ where ξ is constant. We label the field operator with:

$$K^g_{\xi,m} = (\Box_\nabla + \xi R + m^2) \tag{2.1.4}$$

where we choose the notation in order to make clear the dependence on the metric and we call *minimal coupling* the case $\xi = 0$ and *conformal coupling* that for which $\xi = \frac{d-2}{4(d-1)}$. Let's notice that when dealing with 2D models conformal and minimal coupling coincide.

Remark 2.1.1. Let's show how solutions of the conformal coupled scalar field operator, with $m = 0$, behave under conformal transformations of the spacetime. Given two d-dimensional Lorentzian manifold (M, g) and $(\widetilde{M}, \widetilde{g})$ and a diffeomorphism $\chi : M \to \widetilde{M}$ such that $\chi^* \widetilde{g} = S^2 g$, denoting with $\widetilde{R}$ and $\widetilde{\nabla}$ the quantities related to the metric of the target space, we can relate solutions of the transformed equation to the untransformed field. Indeed, writing the never vanishing factor $S^2 = e^{2\varphi}$, for the sake of simplicity, and recalling the formulas

$$\Box_{\tilde{\nabla}} \phi = e^{-2\varphi} \left(\Box_\nabla \phi + (d-2) g^{\mu\nu} \partial_\nu \varphi \partial_\mu \phi\right) \tag{2.1.5}$$

$$\widetilde{R} = e^{-2\varphi} \left[R + \frac{4(d-1)}{(d-2)} e^{-(d-2)\varphi/2} \Box \left(e^{(d-2)\varphi/2}\right)\right] \tag{2.1.6}$$

we get

$$\left(\Box_{\tilde{\nabla}} + \xi_d \widetilde{R}\right) \phi =$$
$$e^{-2\varphi} \left[\underbrace{\Box_\nabla \phi + (d-2) g^{ij} \partial_i \varphi \partial_j \phi + e^{-(d-2)\varphi/2} \phi \Box \left(e^{(d-2)\varphi/2}\right)}_{= e^{-2\alpha\phi} \Box_\nabla (e^{2\alpha\varphi}) \phi} + \frac{d-2}{4(d-1)} R \phi \right],$$

with the equivalence in the underbrace justified by the calculation

$$\begin{aligned} e^{-2\alpha\varphi} \Box_\nabla \left(e^{2\alpha\varphi}\right) \phi =& e^{-2\alpha\varphi} \nabla_\mu \nabla^\mu \left(e^{2\alpha\varphi}\right) \phi \\ =& e^{-2\alpha\varphi} \left\{ \nabla^\mu \left[\partial_\mu \left(e^{2\alpha\varphi} \phi\right)\right] + \nabla^\mu \left[\left(e^{2\alpha\varphi}\right) \partial_\mu \phi\right] \right\} \\ =& e^{-2\alpha\varphi} \left[\Box_\nabla \left(e^{2\alpha\varphi}\right) \phi + 4\alpha e^{2\alpha\varphi} \partial^\mu \varphi \partial_\mu \phi + e^{2\alpha\varphi} \Box_n a \phi\right] \end{aligned}$$

where $\alpha = -\frac{d-2}{4}$. Then, we conclude that

$$\left(\Box_{\tilde{\nabla}} + \xi_d \widetilde{R}\right) \widetilde{\phi} = S^{\left(\frac{d-6}{2}\right)} \left(\left(\Box_{\nabla} + \xi_d R\right)\right) S^{\left(-\frac{d-2}{2}\right)} \widetilde{\phi} \tag{2.1.7}$$

the field $\widetilde{\phi} = S^{\left(\frac{d-2}{2}\right)} \phi$ solving the transformed equation.

Now we have defined an equation of motion and before proceeding we remind the reader that equations like that of the massive scalar field could come both from heuristic arguments and from Lagrangian densities generalized to curved backgrounds. Here we are ready to understand the scalar field operator using the formalism developed in Chapter 1. Given a manifold M, we define K as an operator acting on the space of section $\Gamma(M, M \times \mathbb{R})$, where the vector bundle is taken to be $(M \times \mathbb{R}, \mathrm{pr}_M, M, \mathbb{R})$. As we showed in Remark 1.2.2, the sections of this vector bundle are in bijective correspondence with the space of real smooth functions $C^\infty(M)$. Moreover, we can define a bilinear form integrating the point-wise product of sections

$$(\cdot, \cdot)_{\mathbb{R}} : C^\infty(M) \times C^\infty(M) \to \mathbb{R} \qquad (f, g) \mapsto \int_M f \cdot g \;\, \mathrm{dvol}_{\boldsymbol{M}} \,, \tag{2.1.8}$$

where f and g are taken to be smooth real sections with supports intersecting on a compact set.

Remark 2.1.2. Here we introduce quickly another formalism which is often convenient to work with, for the development of quantum field theory. Using the definition of differential forms over a manifold, with notions and notations used in [Tay96, ch. 9,10], we can define the *Hodge-d'Alembert operator*

$$\Box_{H,k} : \Lambda^k(M) \to \Lambda^k(M) \qquad \Box_{H,k} := d\delta + \delta d \tag{2.1.9}$$

where $\Lambda^k(M)$ is the vector space of the k-forms on M. The operator (2.1.9) shows up in many general contexts, but for our aims is enough to notice that, since $\Lambda^0(M) = C^\infty(M)$,

$$\Box_{H,0} := \delta d \tag{2.1.10}$$

is an operator acting on the space of the smooth functions on M. The second term of the sum disappeared in (2.1.10) because δ is the null operator when acting on $\Lambda^0(M)$. Now we can see how the operator (2.1.10) is related to the operator $\Box_\nabla$ used before. Taking

$f \in C^\infty(M)$, using the multi-index notation, we get

$$\begin{aligned}\Box_{H,0} u =& \mathrm{d}\delta u + \delta\,\mathrm{d}u = \delta\,\mathrm{d}u = \delta\,\partial_i u\,\mathrm{d}x^i = \star\mathrm{d}\star(\partial_i u\,\mathrm{d}x^i)\\ =& \frac{1}{(n-1)!}\star\mathrm{d}\left(\varepsilon_{j,j_1,\dots,j_{n-1}}\sqrt{|g|}g^{jl}\partial_l u\,\mathrm{d}x^{j_1}\wedge\dots\mathrm{d}x^{j_{n-1}}\right)\\ =& \frac{1}{(n-1)!}\star\left(\varepsilon_{j,j_1,\dots,j_{n-1}}\partial_i\left(\sqrt{|g|}g^{jl}\partial_l u\right)\,\mathrm{d}x^i\wedge\mathrm{d}x^{j_1}\wedge\dots\mathrm{d}x^{j_{n-1}}\right)\\ =& \frac{1}{(n-1)!}\frac{1}{\sqrt{|g|}}\varepsilon^{i,j_1,\dots,j_{n-1}}\varepsilon_{j,j_1,\dots,j_{n-1}}\partial_i\left(\sqrt{|g|}g^{jl}\partial_l u\right)\,\mathrm{d}x^i\wedge\mathrm{d}x^{j_1}\wedge\dots\mathrm{d}x^{j_{n-1}}\\ =& -\frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}\,g^{il}\partial_l u)\end{aligned} \tag{2.1.11}$$

proving with this chain of equivalence that $\Box_{H,0} = -\Box_\nabla$

In the following, the group of results at the end of the previous chapter will be crucial in the construction of quantum field theories. Hence, we show here that $K^g_{\xi,m}$ is normally hyperbolic and hence Green-hyperbolic.

Proposition 2.1.3. *Given a globally hyperbolic spacetime $(M, g, \mathfrak{o}, \mathfrak{t})$ and the vector bundle $(M\times\mathbb{R}, \mathrm{pr}_M, M, \mathbb{R})$, the linear partial differential operator, acting on sections $\Gamma(M, M\times\mathbb{R})$, defined by*

$$K^g_{\xi,m} = (\Box_\nabla + \xi R + m^2) \tag{2.1.12}$$

where $\Box_\nabla$ is the d'Alembert operator associated to the Levi-Civita connection is normally hyperbolic

Proof. Locally we can rewrite equation (2.1.12) as

$$\begin{aligned}K^g_{\xi,m}\phi &= \left(\nabla^i\nabla_i + \xi R + m^2\right)\phi\\ &= \left(g^{ij}\partial_i\partial_j + g^{ij}\Gamma^k_{ij}\partial_k + \xi R + m^2\right)\phi\end{aligned}$$

that perfectly fits the Definition 1.2.17 of normally hyperbolic operator. □

We know from the previous chapter that normal hyperbolicity of the operator entails Green-hyperbolicity, using the existence and uniqueness of a proper Cauchy problem. In some cases, Green's operators can be explicitly computed and this is the case for the Klein-Gordon operator in flat spacetimes.

Example 2.1.4. Let's write the operator of Definition 2.1.4 for the flat spacetime using the system of global coordinates and the greek index notation $\{x^\mu\}_{\mu=0,..,3}$

$$K^\eta_{\xi,m} = \eta^{\mu\nu}\partial_\mu\partial_\nu\phi + m^2\phi = 0 \tag{2.1.13}$$

We know that the Green's operators are given by

$$G^\pm : C_c^\infty(M) \to C_{sc}^\infty(M) \qquad f \mapsto G^\pm f := \lim_{\epsilon\to 0}\int_{\mathbb{R}^4} dy\, f(y) G^\pm_\epsilon(x,y) \tag{2.1.14}$$

where, as in Definition 1.2.9, the retarded Green's operator is G^+ and the advanced is the other one. The function $G^\pm_\epsilon$ is defined via integration over the "momentum space":

$$G^\pm_\epsilon(x,y) = \int_{\mathbb{R}^4} \mathrm{dvol}_{\mathbb{R}^4} \frac{e^{-i\eta(p,(x-y))}}{m^2 + \mathbf{p}\cdot\mathbf{p} - (p_0 \pm i\epsilon)^2} \tag{2.1.15}$$

The support of the distributions $G^\pm$ imply the inclusion $G^\pm f \subseteq J^\pm_{\mathbb{R}^4}(\mathrm{supp} f)$, and hence we recovered the construction used in books like [PS95]

Once the theory of scalar field has been put in a rigorous framework, we can study the possible methods to achieve quantization in generic curved backgrounds. We will try to give a useful introduction to some of this techniques, following the scheme of [Hac10].

2.1.1 The space of classical observables

Given an equation of motion, a physical system is described by *the space of observables* associated to that equation, usually described in mathematical terms as a commuting algebra $\mathcal{A}$ endowed with a Poisson structure, i.e. a binary operation $\{\cdot,\cdot\} : \mathcal{A}\times\mathcal{A} \to \mathcal{A}$ satisfying the following three conditions:

Skew symmetry $\{a,b\} = -\{b,a\}$;

Jacobi identity $\{a,\{b,c\}\} + \{b,\{c,a\}\} + \{c,\{a,b\}\} = 0$;

Leibniz's Rule $\{ab,c\} = a\{b,c\} + b\{a,c\}$;

for $a,b,c \in \mathcal{A}$. The scope of this paragraph is to explain how such an association can be concretely obtained in a framework compatible with general relativity. This means, mathematically speaking, that the brackets defining the Poisson structure shall be coherent with the principle of general covariance. The existence of such covariant Poisson bracket has been proven for the first time by [Pei52].

We start our discussion with one of the key results of all the chapter that is a characterisation of the space of the compactly supported smooth functions on a GHST. This proposition is the starting point for the definition of a *space of observables* and the implementation of the 'quantization machinery'. From now on we will simplify the notation, removing superscripts and subscripts from the symbol of the scalar field operator. References to the mass term and to the coupling function are not needed in general. One can use Theorem 1.2.15 in order to build up a symplectic space structure from $C^\infty_c(M)$: due to the exactness of the sequence of (1.2.5) the quotient space

$$\mathcal{E}^{K}(M) := C^\infty_c(M)/K(C^\infty_c(M)) \tag{2.1.16}$$

is well defined. Moreover we can find an isomorphism to the space of space-like compact solution of equation (2.1.3). First we notice that the space $\mathrm{Sol}^K_{sc}(M)$ of solutions with spacelike-compact support of the equation $Ku = 0$ on M, coincides with the kernel of K restricted to spacelike-like compact smooth functions, which is a trivial fact. Then one can apply the sequence in order to construct explicitly the isomorphism.

Proposition 2.1.5. *Consider the space of functions $C^\infty(M)$ over a globally hyperbolic spacetime $(M, g, \mathfrak{o}, \mathfrak{t})$. Let $K : C^\infty(M) \to C^\infty(M)$ be the scalar operator with Green's operator $G^{K\pm}$. Then the space $\mathrm{Sol}^K_{sc}(M)$ of solutions with spacelike-compact support defined by*

$$\mathrm{Sol}^K_{sc}(M) := \left\{\psi \in C^\infty_{sc}(M) : K\psi = 0\right\}$$

isomorphic to the quotient space $\mathcal{E} := C^\infty_c(M)/K(C^\infty_c(M))$ via the map

$$\mathcal{G}^K : C^\infty(M)/K(C^\infty(M)) \to \mathrm{Sol}^K_{sc}(M) \quad [f] \mapsto G^K f \tag{2.1.17}$$

Proof. For this proposition we give a schematic proof:

i) $\mathcal{G}^K$ is a well defined map. Given two elements of a class $[f]$, f and $f' = f + Kh$. We calculate
$$\mathcal{G}^K([f']) = Gf + GKh = Gf = \mathcal{G}^K([f])\,.$$

ii) $\mathcal{G}^K$ is an injective map. $\mathcal{G}^K([f]) = 0 \Rightarrow Gf = 0$. That is $f \in \mathrm{Ker}G^K$, but $\mathrm{Ker}G^K = \mathrm{Im}K|_{C^\infty_c(M)}$, hence $f = Kh \Rightarrow [f] = [0]$

iii) $\mathcal{G}^K$ is an surjective map. Given $\psi \in \mathrm{Sol}^K_{sc}$, $\psi \in \mathrm{Ker}K|_{C^\infty_{sc}(M)} = \mathrm{Im}G^K|_{C^\infty_c(M)}$, hence $\psi = G^K f$ with $f \in C^\infty_c(M)$

□

Besides its mathematical characterisation, also the physical interpretation of $\mathrm{Sol}^K_{sc}(M)$ is noteworthy. As we will see in the following, it plays the role of the *classical phase space* of the theory. This fact becomes clear once additional structure is provided for the space of solution, in fact this structure is a skew-symmetric, non degenerate bilinear form that can be defined directly on $\mathrm{Sol}^K_{sc}(M)$ or at the level of the space $\mathcal{E}^K(M)$. The connection between these two structures will be proven in the following. Here we first endow $\mathcal{E}^K(M)$ with a symplectic form defined using the pairing between smooth sections and the causal propagator associated to K.

Proposition 2.1.6. *Let $(M, g, \mathfrak{o}, \mathfrak{t})$ be a globally hyperbolic spacetime. Let $K : C^\infty(M) \to C^\infty(M)$ be the Klein-Gordon operator and denote with G^K the associated advanced-minus-retarded operator. Then the map presented below defines a non-degenerate bilinear form:*

$$\tau^K_M : \mathcal{E}^K(M) \times \mathcal{E}^K(M) \to \mathbb{R}, \qquad ([f],[h]) \mapsto \tau^K_M([f],[h]) := (f, G^K h)_{\mathbb{R}} \tag{2.1.18}$$

where $f \in [f]$ and $h \in [h]$ are two arbitrary representatives. Furthermore, τ^K_M is skew-symmetric, and then defines a symplectic form on $\mathcal{E}^K(M)$.

Proof. The definition of τ^K_M given by (2.1.18) doesn't depend on the representatives. Taking indeed $f' = f + Ku$, $h' = h + Kv$

$$\begin{aligned}\tau^K_M([f'],[h']) =& (f + Ku, G^K(h + Kv))_{\mathbb{R}} \\ =& (f, G^K h)_{\mathbb{R}} + (Ku, G^K h)_{\mathbb{R}} = \tau^K_M([f],[h])\end{aligned}$$

where we used that $G^K Ku = 0$ for $u \in C^\infty_c(M)$ and secondly that $G^{K*} = -G^K$. Bilinearity is clear by definition and so we can concentrate on non-degeneracy. We take $[f] \in \mathcal{E}^K(M)$ such that

$$\tau^K_M([f'],[f]) = 0 \quad \forall [f'] \in \mathcal{E}^K$$

this entails that $(f', G^K f)_{\mathbb{R}} = 0$, $\forall f' \in C^\infty_c(M)$ and that $G^K f = 0$, for the non-degeneracy of the pairing $(\cdot,\cdot)_{\mathbb{R}}$. Hence $f \in \mathrm{Ker}G^K$ that coincides with $\mathrm{Im}K|_{C^\infty_c(M)}$, and so is another way to label the class $[0]$ generated by all the elements of the for Ku with $u \in C^\infty_c(M)$. □

The symplectic space $(\mathcal{E}^K(M), \tau^K_M)$ induces a symplectic structure on $\mathrm{Sol}^K_{sc}(M)$ via the standard push-forward of tensors associated to the map $\mathcal{G}^K$. Indeed, it can be easily proven that

$$\begin{aligned}\mathcal{G}^K_*\tau^K_M := \tau^K_M \circ \left((\mathcal{G}^K)^{-1} \otimes (\mathcal{G}^K)^{-1}\right) : \mathrm{Sol}^K_{sc}(M) \times \mathrm{Sol}^K_{sc}(M) \to \mathbb{R}\\ \mathcal{G}^K_*\tau^K_M(\phi_1, \phi_2) = \tau^K_M\left((\mathcal{G}^K)^{-1}(\phi_1), (\mathcal{G}^K)^{-1}(\phi_2)\right)\end{aligned}$$

is a non degenerate, antisymmetric bilinear form on the space of spacelike compatcly supported solutions, once two elements of $\mathrm{Sol}^K_{sc}(M)$ are interpreted as images of two elements in $\mathcal{E}^K(M)$. Unfortunately, despite the induced structure is enough to proceed with theoretical presentation of the classical and quantum scalar field theory, we shall find a more practical version of the symplectic form. This will turn out to be useful when dealing with explicit computations and in order to give a physically meaningful interpretation. Adapting ideas and methods of [HS13] to our context, let us show how an alternative but equivalent structure can be defined on $\mathrm{Sol}^K_{sc}(M)$: for each element ϕ we know that there exist a compact set C such that $\mathrm{supp}(\phi) \subseteq J^{\pm}(C)$ and we can rewrite this function as the sum $\phi = \phi^+ + \phi^-$. This splitting is not unique but for our purpose we have just to notice that given another splitting $\phi^{+'} + \phi^{-'}$, the function $\rho := \phi^{+'} - \phi^+ = \phi^- - \phi^{-'}$ is a compactly supported function and $\mathrm{supp}(\rho) \subseteq C$. Given such a decomposition, we can define the map

$$\sigma^K : Sol^K_{sc}(M) \times Sol^K_{sc}(M) \to \mathbb{R} \qquad \sigma^K(\phi_1, \phi_2) = (K\phi_1^+, \phi_2)_{\mathbb{R}} \tag{2.1.19}$$

The integral defining this bilinear form exists once ϕ_1 and ϕ_2 are solutions: from $K\phi_1 = 0$, as stressed in the proof of Theorem 1.2.15, we deduce that $K\phi_1^+ = -K\phi_1^-$ is a compactly supported function, hence integration is sensible in (2.1.19). Moreover, due to the consideration above, we can prove the independence on the splitting. Given another splitting for ϕ_1,

$$\begin{aligned}(K\phi_1^{+'}, \phi_2)_{\mathbb{R}} =& (K\phi_1^+, \phi_2)_{\mathbb{R}} + (K\rho, \phi_2)_{\mathbb{R}}\\ =& (K\phi_1^+, \phi_2)_{\mathbb{R}} + (\rho, K\phi_2)_{\mathbb{R}} = (K\phi_1^+, \phi_2)_{\mathbb{R}},\end{aligned} \tag{2.1.20}$$

where the last equivalence is due to self-adjointness of K and to the fact that $K\phi_2 = 0$. After seeing this proof, one should be tempted to integrate by parts directly in the definition (2.1.19) and then claim that the map is trivial, but actually, since ϕ_1^+ and ϕ_2 do not have compact overlapping supports integration by parts is not allowed. The properties needed to proceed with the construction of the symplectic space should be checked: here we sketch the proof.

Proposition 2.1.7. *The map defined by* (2.1.19) *has the following properties*

i) The map σ^K is antisymmetric

ii) For all $\phi_1, \phi_2 \in \mathrm{Sol}^K_{sc}(M)$, there exist $[u], [v] \in \mathcal{E}^K$

$$\sigma^K(\phi_1, \phi_2) = \tau^K_M\big([u], [v]\big) \ . \tag{2.1.21}$$

Proof. Starting from the point i), we take arbitrary $\phi_1, \phi_2 \in \mathrm{Sol}^K_{sc}(M)$ and we consider the splittings $\phi_i = \phi_i^+ + \phi_i^-$, $i = 1, 2$. Recalling that given a a past compact set A^+ and a future compact set A^-, their intersection is a compact set and that, as noticed before, from $K\phi_i = 0$ follows that $K\phi_i^+ = -K\phi_i^-$. Then

$$\sigma^K(\phi_1, \phi_2) = (K\phi_1^+, \phi_2)_{\mathbb{R}} = (K\phi_1^+, \phi_2^+)_{\mathbb{R}} + (K\phi_1^+, \phi_2^-)_{\mathbb{R}}$$

and the splitting due to the linearity of the pairing makes sense because both terms are well-defined. Indeed $\mathrm{supp}(\phi_i^+) \cap \mathrm{supp}(\phi_j^-)$ is compactly supported for each couple (i, j) and the term $(K\phi_1^+, \phi_2^+)_{\mathbb{R}}$ can be rewritten as $-(K\phi_1^-, \phi_2^+)_{\mathbb{R}}$, that falls in the case just examined. Hence, we can proceed with

$$\begin{aligned}\sigma^K(\phi_1, \phi_2) &= -(K\phi_1^-, \phi_2^+)_{\mathbb{R}} + (\phi_1^+, K\phi_2^-)_{\mathbb{R}} = -(\phi_1^-, K\phi_2^+)_{\mathbb{R}} - (\phi_1^+, K\phi_2^+)_{\mathbb{R}} \\ &= -(\phi_1, K\phi_2^+)_{\mathbb{R}} = -(K\phi_2^+, \phi_1)_{\mathbb{R}} = -\sigma^K(\phi_2, \phi_1),\end{aligned} \tag{2.1.22}$$

For the proof of the point ii) we recall that using the isomorphism $\mathcal{G}^K$ of Proposition 2.1.5, we can interpret a solution ϕ of $K\phi = 0$ as the image of a function $u \in C_c^\infty(M)$

$$\phi_{[u]} = \mathcal{G}^K([u]) := G^K u$$

Then, given two elements in $\mathrm{Sol}^K_{sc}(M)$ we denote them with $\phi_{[u]}, \phi_{[v]}$ and we can use the convenient decomposition $G^K u = (G^K)^+ u - (G^K)^- u$ given by the Green's operator in order to find

$$\sigma^K(\phi_{[u]}, \phi_{[v]}) = (K(G^K)^+ u, G^K v)_{\mathbb{R}} = (u, G^K v)_{\mathbb{R}} = \tau^K_M\left([u], [v]\right) \ , \tag{2.1.23}$$

where we used the definition of Green's operator and of the symplectic form on the space $\mathcal{E}^K(M)$ □

Since we introduced this symplectic structure on the space of space-compact solutions in order to compare our formalism with the standard one of quantum field theories in flat backgrounds, underlining the relationship between $(\mathrm{Sol}^K_{sc}(M), \sigma^K)$ and the space of function of (2.1.16) turns out to be useful. Hence, we can summarize our results with:

$$(Sol^K_{sc}(M), \sigma^K) \overset{\mathrm{Symp}}{\simeq} (\mathcal{E}^K(M), \tau^K_M)$$

where the superscript "Symp" indicates that the isomorphism defined preserves the symplectic structures. The symplectic form σ^K defined in (2.1.19) is enough for a full theoretical comprehension of the phase space, but, in order to underline the connection with the ordinary quantization, we show how that form can be written as an integration of *a current* over any Cauchy surface. Using the formalism introduced in Remark 2.1.2, we present here a technical Lemma, the proof of which can be found in [Tay96, Ch. 10].

Lemma 2.1.8. *Given a smooth compact manifold M with boundary ∂M and two differential forms $u, v \in \Lambda^0(M)$ we get*

$$(\Box_{H,0}u, v)_{\mathbb{R}} - (u, \Box_{H,0}v)_{\mathbb{R}} = \int_{\partial M} \mathrm{dvol}_{\partial M} \left(\langle dv, \mathfrak{n}\rangle u - \langle du, \mathfrak{n}\rangle v\right) \tag{2.1.24}$$

where $\langle \cdot, \cdot \rangle$ is the dual pairing between differential forms and vector fields.

In order to proceed, we shall rewrite (2.1.24) using functions and derivatives. Recalling the relation (2.1.11) of Remark 2.1.2, taking two smooth functions ϕ_1, ϕ_2 it becomes:

$$(\Box_\nabla \phi_1, \phi_2)_{\mathbb{R}} - (\phi_1, \Box_\nabla \phi_2) = \int_{\partial M} \mathrm{dvol}_{\partial M} \left(-\phi_1 \nabla_{\mathfrak{n}} \phi_2 + \phi_2 \nabla_{\mathfrak{n}} \phi_1\right)$$

Now, in order to apply Lemma 2.1.8, we choose a Cauchy surface Σ of the GHST, $(M, g, \mathfrak{o}, \mathfrak{t})$, and we split the manifold M in two regions, $M^\pm = J^\pm(\Sigma)$. This splitting is such that $M = M^+ \cup M^-$ and $M^+ \cap M^- = \Sigma$ and induces a decomposition of the bilinear form σ^K, when acting on two smooth space-compactly supported solution ϕ_1, ϕ_2:

$$\sigma^K(\phi_1, \phi_2) = -(K\phi_1^-, \phi_2^+)^+_{\mathbb{R}} + (\phi_1^+, K\phi_2^-)^-_{\mathbb{R}}. \tag{2.1.25}$$

where for each f, g we denote

$$(f,g)^{\pm}_{\mathbb{R}} = \int_{M^{\pm}} \mathrm{dvol}_M f \cdot g$$

Let us notice that with this splitting we decomposed the bilinear form in two terms which are integrals over compact submanifold. Indeed $\mathrm{supp}(\phi_1^{\pm}) \subseteq J^{\pm}(C)$ for some compact set C and $J^{\pm}(C) \cap M^{\mp}$ is trivially compact.

Proposition 2.1.9. *Given a globally hyperbolic spacetime $(M, g, \mathfrak{o}, \mathfrak{t})$ and the scalar field operator K the symplectic form of* (2.1.19) *can be written as*

$$\sigma^K(\phi_1, \phi_2) = \int_{\Sigma} \mathrm{dvol}_{\Sigma} \; (\phi_1 \nabla_{\mathfrak{n}} \phi_2 - \phi_1 \nabla_{\mathfrak{n}} \phi_2) \tag{2.1.26}$$

where Σ is an arbitrary Cauchy surface of $(M, g, \mathfrak{o}, \mathfrak{t})$.

Proof. We start from the splitting of (2.1.25)

$$\sigma^K(\phi_1, \phi_2) = -(K\phi_1^-, \phi_2)^+_{\mathbb{R}} + (\phi_1^+, K\phi_2)^-_{\mathbb{R}}.$$

As stressed before, both integrals of the last term are performed over a compact region of the spacetime, respectively

$$\mathrm{supp}(\phi_1^{\mp}) \cap \mathrm{supp}(\phi_2) \cap J^{\pm}(\Sigma) = C^{\pm},$$

and moreover the only region of the boundary $\partial C^{\pm}$ where integrands are non vanishing, repsectively, $\partial C^{\pm} \cap \Sigma$. Hence, applying Lemma 2.1.8 in a suitable version, denoting with $\mathfrak{n}$ the future pointing normal vector to Σ and $\tilde{\mathfrak{n}} = -\mathfrak{n}$, we get:

$$\begin{aligned}\sigma^K(\phi_1, \phi_2) &= \int_{\Sigma} \mathrm{dvol}_{\Sigma} \, (\phi_1^- \nabla_{\mathfrak{n}} \phi_2 - \phi_2 \nabla_{\mathfrak{n}} \phi_1^- - \phi_1^+ \nabla_{\tilde{\mathfrak{n}}} \phi_2 + \phi_2 \nabla_{\tilde{\mathfrak{n}}} \phi_1^+) \\ &= \int_{\Sigma} \mathrm{dvol}_{\Sigma} \; (\phi_1 \nabla_{\mathfrak{n}} \phi_2 - \phi_2 \nabla_{\mathfrak{n}} \phi_1) \end{aligned} \tag{2.1.27}$$

□

Let's notice that, since the decomposition depends only on the chosen Cauchy surface, independence from the splitting of the form σ^K entails independence from the choice of the Cauchy surface. Writing σ^K as in (2.1.26), allows to recollect the standard symplectic form and a physically meaningful interpretation of our formalism. First we notice that for the continuity of the map $\mathcal{G}^K$ and for the result of Proposition 2.1.7, the symplectic form τ_M^K defines a bidistribution in $D'(M \times M)$, i.e. the dual space of compactly supported functions defined on the product manifold $M \times M$. We denote this bidistribution with the symbol Δ, and we define the integral kernel with $\Delta(x, y)$. This fact can be transposed to the symplectic form σ^K with an heuristic interpretation of the association $[u] \to \phi_{[u]}$ as distribution, defined via the so called "unsmeared field" $\phi(x)$ (for a rigorous treatment of this interpretation we recommend the reading of [Hac10, pag. 53]). Taking this into account we can write

$$\sigma^K(\phi(x), \phi(y)) = \Delta(x, y).$$

Furthermore, we know that $\tau_M^K(f, g)$ vanishes when the supports of the functions g and f are causally separated, this entails, looking at the level of integral kernel, that $\Delta(x, y)$ is zero when the points x, y are causally separated. For this property, throughout the literature, the commutator is said to satisfy the causality condition. Concluding, in order to underline the connection with the standard formalism of QFT we shall understand σ^K as the equal time commutator. We present here without proof a proposition by Dimock [Dim80], as exposed by [Hac10], that states some properties of the causal propagator and its distributional integral kernel.

Proposition 2.1.10. *Given a globally hyperbolic spacetime $(M, g, \mathfrak{o}, \mathfrak{t})$ and a Green-hyperbolic operator K, with its causal propagator G^K. Given any Cauchy surface Σ, for all $f \in C_c^\infty(\Sigma)$ the following statement holds true:*

$$\nabla_{\mathfrak{n}} G^K f|_\Sigma = f, \qquad G^K f|_\Sigma = 0 \tag{2.1.28}$$

that in terms of integral kernels can be read as

$$\nabla_{\mathfrak{n}} \Delta(x, y)|_{\Sigma\times\Sigma} = \delta_\Sigma(x, y) \qquad \Delta(x, y)|_\Sigma = 0 \tag{2.1.29}$$

Recalling now the theorem Theorem 1.1.22, we see that any Cauchy surface can be seen as a 'time constant sub manifold'. Indeed given a Cauchy surface Σ the existence of the diffeomorphism claimed at point iii) of Theorem 1.1.22 and the topological decomposition of M as $\mathbb{R} \times \Sigma$, with the metric $-\beta dt^2 + g_t$, allow us to fix $t = t_0$ and to interpret Σ as a time constant submanifold because it is diffeomorphic to $t_0 \times \Sigma$. Moreover, we know for the same argument and for what has been stressed at the end of Proposition 2.1.9 that σ^K

is constant when moving on the family of Cauchy surfaces. Hence we can write

$$\begin{aligned} \sigma^K\left(\nabla_{\mathfrak{n}}\phi(x), \phi(y)\right) =& \nabla_{\mathfrak{n}}\Delta(x,y)|_{\Sigma\times\Sigma} = \delta_\Sigma(x,y) \\ \sigma^K\left(\phi(x), \phi(y)\right) =& \Delta(x,y) \end{aligned} \tag{2.1.30}$$

We can now conclude this subsection seeing what happens in the usual example of the flat spacetime.

Remark 2.1.11. What we have just shown can be used to shed light on the connection between our formalism and the ordinary development of QFT in Minkowski spacetime. We build up the GHST taking the Lorentzian manifold $(\mathbb{R}^4, \eta)$ and defining the volume form induced by η and a timelike vector field using the system of global coordinates (x^0, x^1, x^2, x^3), hence we get the 4-tuple $(\mathbb{R}^4, \eta, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \mathbf{e_0})$. The family of Cauchy surfaces is easily defined as $\Sigma_t = \{x \in \mathbb{R}^4 | x^0 = t\}$. In this framework the time-orientation vector $\mathbf{e_0}$ is also the unit constant vector field normal to Σ_t, hence $\nabla_{\mathfrak{n}} = \partial_0$. Then $\nabla_{\mathfrak{n}}\Phi(x) = \partial_0\Phi(x) = \dot{\Phi}(x) = \Pi(x)$ and equations (2.1.30) looks like

$$\begin{aligned} \sigma^K\left(\Pi(x), \phi(y)\right) &= \nabla_{\mathfrak{n}}\Delta(x,y)|_{\Sigma\times\Sigma} = \delta_\Sigma(x-y) = \delta^{(3)}(\mathbf{x}-\mathbf{y}) \\ \sigma^K\left(\phi(x), \phi(y)\right) &= \Delta(x-y) \end{aligned}$$

where the function $\Delta(x-y)$ is the well known Pauli-Jordan distribution.

Functionals as observables

Here we show how the classical theory of the scalar field can be set in a more general framework. The space of classical observables of the theory will be defined as the space of functionals on the *space of field configurations*, $C^\infty(M)$ and the equivalence with the previous construction will be proven. As a first step, in this space we can define the notion of *linear observable.*

Definition 2.1.12. Given a GHST, $(M, g, \mathfrak{o}, \mathfrak{t})$, and the normally hyperbolic operator K, we define the set of *linear observables* for the scalar field theory ruled by K as the set of functionals generated this way by $f \in C_c^\infty(M)$:

$$F_f : C^\infty(M) \to \mathbb{R} \qquad \phi \mapsto F_f(\phi) = (f, \phi)_{\mathbb{R}}. \tag{2.1.31}$$

If we denote with $\mathcal{L}(M)$ the space of functionals defined by (2.1.31) we can define the map

$$\mathcal{F} : C_c^\infty(M) \to \mathcal{L}(M) \quad f \mapsto \mathcal{F}(f) := F_f(\phi). \tag{2.1.32}$$

This definition already shows some pathologies: first, let us notice that some ambiguities could arise when we work on the so called *space of on shell field configuration*, i.e. the space $\mathrm{Sol}^K(M) = \mathrm{Ker}K|_{C^\infty(M)}$. Indeed there exist two different elements $f \in C_c^\infty(M)$, such that $F_h(\phi) = F_f(\phi)$ for each $\phi \in \mathrm{Sol}^K(M)$. This is due to the fact that we are working with the kernel of the operator K, indeed given $f \in C_c^\infty(M)$ and defining $\tilde{f} = f + Kh$ for some $h \in C_c^\infty(M)$, we get the chain of equalities

$$F_{\tilde{f}}(\phi) = (\tilde{f}, \phi)_{\mathbb{R}} = (f + Kh, \phi)_{\mathbb{R}} = F_f(\phi) + F_{Kh}(\phi)$$

but, using the self-adjointness of K, the last term becomes

$$F_{Kh}(\phi) = (Kh, \phi)_{\mathbb{R}} = \int_M \mathrm{dvol}_M\,(Kh)\,\phi = \int_M \mathrm{dvol}_M\,h(K\phi) = 0$$

Hence we found that the kernel of the linear map $f \mapsto F_f$ is nonempty and the latter is not injective. Yet, due to the linearity, we have simply to figure out which is the space of null observables and then, understanding its degeneracy, we can quotient out redundancies.

Proposition 2.1.13. *The space of linear null observables*

$$\mathcal{N}^K(M) = \{F_f(\phi) = 0, \forall \phi \in \mathrm{Sol}^K(M)\} \tag{2.1.33}$$

coincides with the image of $\mathrm{Im}\left(K|_{C_c^\infty(M)}\right)$ *via the isomorphism* $\mathcal{F}$ *defined in* (2.1.32)

Proof. The statement can be rephrased as

$$\mathcal{F}\left(\mathrm{Im}\left(K|_{C_c^\infty(M)}\right)\right) = \mathcal{N}^K(M).$$

We prove first the inclusion $\mathcal{F}\left(\mathrm{Im}\left(K|_{C_c^\infty(M)}\right)\right) \subseteq \mathcal{N}^K(M)$: Given an element in $C_c^\infty(M)$ of the form Kf, for all ϕ we write

$$\mathcal{F}(Kf)(\phi) = F_{Kf}(\phi) = (Kf, \phi)_{\mathbb{R}} = (f, K\phi)_{\mathbb{R}} = 0$$

then $\mathcal{F}(Kf)(\cdot) \in \mathcal{N}^K$, and the inclusion above holds true.

Now we prove the inclusion $\mathcal{F}\left(\mathrm{Im}\left(K|_{C_c^\infty(M)}\right)\right) \supseteq \mathcal{N}^K(M)$: Given F_f such that $F_f(\phi) = 0$, $\forall \phi \in \mathrm{Sol}^K$, this can be written in terms of pairing

$$F_f(\phi) = (f, \phi)_{\mathbb{R}} = 0, \quad \forall \phi \mathrm{Sol}^K_{sc}(M)\,.$$

Recalling now that $\mathrm{Sol}^K = \mathrm{Ker}(K|_{C^\infty(M)})$ and invoking the extended exact sequence (1.2.7) of Theorem 1.2.16 for K we have

$$F_f(\phi) = (f, \phi)_{\mathbb{R}} = 0 = (f, \overline{G}^K h)_{\mathbb{R}} = 0, \quad \forall h \in C^\infty_{tc}(M)$$

Then, we can use non degeneracy of the pairing $(\cdot,\cdot)$ and that $\overline{G}^* = -\overline{G}$ in order to notice that $\overline{G}^K f = G^K f = 0$, and so that $f \in \mathrm{Ker}\left(G^K|_{C_c^\infty(M)}\right) = \mathrm{Im}\left(K|_{C_c^\infty(M)}\right)$. Concluding that $f = Ku$ for some $u \in C_c^\infty(M)$. □

Then we are ready to define

$$\mathcal{O}^K(M) := \mathcal{L}(M)/\mathcal{N}(M)$$

and denoting with a slight abuse of notation the map $\mathcal{F} : \mathcal{E}^K(M) \to \mathcal{O}^K(M)$ such that

$$\mathcal{F}([f]) := F_{[f]},$$

we find an isomorphism, through which we can even push-forward the symplectic structure. Indeed, we easily define:

$$\begin{aligned} \mathcal{F}_*\tau_M^K : \mathcal{O}^K(M) \times \mathcal{O}^K(M) &\to \mathbb{R} \\ \left(F_{[f]}, F_{[h]}\right) &\mapsto \mathcal{F}_*\tau_M^K\left(F_{[f]}, F_{[h]}\right) := \tau_M^K([f],[h]) \end{aligned} \tag{2.1.34}$$

Hence we gave a third way of understanding the space of observables associated to the field theory ruled by the operator K, and we showed how these three formulation are equivalent. We can summarize the situation with the following diagram

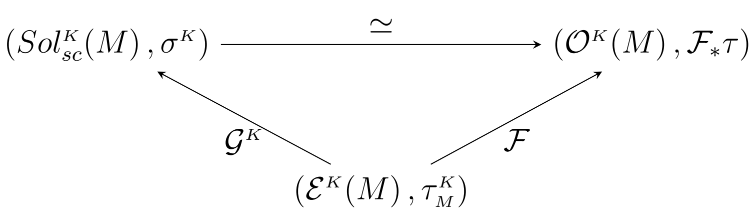


Diagram 2.1.1

and, with this diagram in mind we can activate the quantization machinery.

2.1.2 Algebraic quantization

Even thought the construction of classical field theories showed strong analogies with the methods used when dealing with flat spacetimes, quantization shows remarkable differences. Indeed, it is a well know fact that in generally curved spacetimes a sensible notion of ‘particle’ fail to be found. At the level of mathematical formalisms this means the impossibility to pick out one preferred state from the class of suitable states for the theory. The algebraic quantum field theory gave in the past a good recipe to avoid this problem, that is defining an algebra of fields, working independently from the choice of a unique state or a class of Hilbert spaces unitarily related. In this framework, as a first step, we try to associate to our symplectic space of classical fields a $*$-*algebra* of *quantized fields*. After the characterisation of the space of classical observable for the scalar field theory, we have to find a suitable method to proceed in this direction. We recall that we have got three different, but related, schemes for the construction of a space of observables, endowed with a symplectic structure. Since we know, as summarized in Diagram 2.1.1, that there is not any theoretical difference between this three formulations, each of them can be used depending on the practical advantages we need.

As for the classical field theory, also the construction of a theory for a quantized field can be achieved in many different ways: we are going to show here two of these techniques. Before proceeding, it's important to underline that the $*$-algebra of fields has to be intimately related with the spacetime being the background for the theory. Hence, more often in the literature authors refer to a net of algebras associated to a spacetime and to all its causally compatible subregions, implementing two conditions which encode topological and causal properties of the background. These are:

i) **Isotony**, meaning that the algebra has to be coherent with the operation of inclusion between two regions of the spacetime, i.e. the algebra associated to a region O is a subalgebra for the algebra associated to each opens set O' such that $O \subseteq O'$ and O is causally compatible with O'.

ii) **Causality**, meaning that elements of subalgebras associated to causally separated regions of the spacetime are asked to commute. In physical terms, this means that observables defined on causally non-related regions should be simultaneously measurable, i.e. the measurement of one them should not affect the other.

iii) **Time slice property**, meaning that if a causally convex open set $O \subseteq M$ contains a Cauchy surface for $\mathbf{M}$, then the algebra associated to O has to be isomorphic to the algebra associated to M. This practically means that once you characterize the space of observable associated to an arbitrary small open region around a Cauchy surface Σ, then you know the space of observable of the whole spacetime.

To build up an algebra with these properties we first present here a procedure that makes possible to associate a $*$-algebra to a symplectic space, and then we force suitable causality conditions, quotienting by a suitable ideal. Hence, given a symplectic space (V, τ) one can start introducing an algebra A consisting of the vector space $\mathcal{T}^{\mathbb{C}}V = \bigoplus_{k\in\mathbb{N}_0} V_{\mathbb{C}}^{\otimes k}$, i.e. the direct sum of all the tensor powers of the complexification $V_{\mathbb{C}}$ of the vector space V, where we have set $V_{\mathbb{C}}^{\otimes 0} = \mathbb{C}$. Therefore, we can interpret elements of $\mathcal{T}^{\mathbb{C}}V$ as sequences of the form $\{\mathbf{v_k} \in V_{\mathbb{C}}^{\otimes k}\}_{k\in\mathbb{N}_0}$ with finite non-zero terms. Each term $\mathbf{v}_k$ can be written as a linear combination, with complex coefficients, of terms of the form $v_1 \otimes \cdots \otimes v_k$ for $v_1, \ldots, v_k \in V$. A product on this set is given by the binary operation $\cdot : \mathcal{T}^{\mathbb{C}}V \times \mathcal{T}^{\mathbb{C}}V \to \mathcal{T}^{\mathbb{C}}V$ given by

$$\{\mathbf{u}_k\} \cdot \{\mathbf{v}_k\} = \{\mathbf{w}_k\}, \qquad \mathbf{w}_k = \sum_{i+j=k} u_i \otimes v_j. \tag{2.1.35}$$

With respect to this product the identity $\mathbb{1}$ is given by the sequence $v_0 = 1$, $v_k = 0\,\forall k$. An involution for this algebra can be defined on the elements of the form $v_1 \otimes \cdots \otimes v_k \in V^{\otimes k}$ and then extended to all the elements of the direct sum:

$$(v_1 \otimes \cdots \otimes v_k)^* = (\overline{v}_k \otimes \cdots \otimes \overline{v}_1)$$

As evident this construction can be used over each symplectic space, but from now on, in order to compare results with ordinary physically focused literature, we will use $(\mathcal{E}^K(M), \tau_M^K)$ as a prototype and we will denote the elements of the algebra with $\Phi_{[\mathbf{f}]_k}$. Then, we can notice that elements in $\mathcal{E}^K(M)$ of the form $[f]$ have a faithful correspondence in $\mathscr{A}(M) = \mathcal{T}^{\mathbb{C}}\mathcal{E}^K(M)$, given by $\{[\mathbf{f}]_\mathbf{k}\}$ such that $[\mathbf{f}]_1 = [f]$ and the other terms are vanishing, i.e. $\{0.[f], 0, \ldots, 0\}$. The algebra $\mathscr{A}(M)$ already encodes the property of isotony quoted above, indeed for each region $O \subseteq O'$, the inclusion $C_c^\infty(O) \hookrightarrow C_c^\infty(O')$ in the sense that we can trivially understand $f \in C_c^\infty(O)$ as an element in $C_c^\infty(O')$ extending it by zero. This entails $\mathcal{E}^K(O) \subseteq \mathcal{E}^K(O')$. Conversely, the causality condition has to be introduced 'by hand'. We take the ideal $\mathcal{J}_\tau^{Bos}$ generated by the elements of the form

$$\Phi_{[f]} \cdot \Phi_{[h]} - \Phi_{[h]} \cdot \Phi_{[f]} - i\tau([f], [h])\mathbb{1},$$

for all $[f],[h] \in V$. Then we can ultimate the construction of the algebra defining the quotient

$$\mathscr{A}^{K}(M) = \mathcal{T}^{\mathbb{C}}\mathcal{E}^{K}(M)\,/\mathcal{J}_{\tau}^{Bos}. \tag{2.1.36}$$

This algebra satisfies the causality condition by definition and what has yet to be proven is the compatibility of this construction with the time slice axiom. We first present the situation of the classical theory and then we deduce what happens at the level of the algebra. Hence given a causally connected open subset O containing a Cauchy surface Σ, we first prove that the embedding $\mathcal{E}^{K}(O) \hookrightarrow \mathcal{E}^{K}(M)$ given by the extension by zero $[f] \mapsto [\text{ext}_{M} f]$ is a symplectomorphism. This map is trivially well defined and linear, it preserves the symplectic form because

$$\tau_{O}^{K}([f],[h]) = \int_{O} \text{dvol}_{O} f G^{K} h = \int_{M} \text{dvol}_{M} \text{ext}_{M}(f) G^{K} \text{ext}_{M}(h) = \tau_{M}^{K}([\text{ext}_{M}(f)],[\text{ext}_{M}(f)]),$$

and is injective because functions of the form Ku are extended to $K\text{ext}_{M}u$. The only thing that shall be checked is the surjectivity.Taken a smooth section f with compact support in M, we take two Cauchy surfaces Σ^{+}, Σ^{-} contained in O, respectively in the future and past of Σ. We then define the open cover $\{I^{+}(\Sigma^{-}), I^{-}(\Sigma^{+})\}$ and we take a partition of unit subordined to it, $\chi^{+} + \chi^{-} = 1$, on M. We, hence define $f' = -K(\chi^{-}G^{K}f)$, that has support in O, because, $K(\chi^{+}G^{K} + \chi^{-}G^{K})f = 0$ entails that $f' = K(\chi^{+}G^{K})$. But, f' differs from f only for the image through K of a compactly supported section. Indeed, $f' = f - K(\chi^{-}G^{K+})f + \chi^{+}G^{K-}f$ (we left the check to the reader). And hence we found a function f' such that $[\text{ext}_{M}(f')] = [f]$. At the level of the algebra this symplectic isomorphism induces a $*$-isomorphism, that is defined associating the element $\Phi_{[\mathbf{f}]_{k}}$ to the element $\Phi_{[\mathbf{ext_{M}f}]_{k}}$. We do not prove this fact here because all this results can be seen easily with the formalism of category theory as exposed in the next section. We can recollect the claims of this subsection on the algebra $\mathscr{A}^{K}(\mathbf{M})$[1] in the following theorem

Theorem 2.1.14. *Given a globally hyperbolic spacetime* $(M, g, \mathfrak{o}, \mathfrak{t})$, *and for a given causally connected open subset* $U \subseteq M$, *let* $\mathscr{A}^{K}(U)$ *be the unital* $*$*-algebra of observables for the real scalar field introduced in* (2.1.36). *Then the following properties hold:*

Isotony *if* $O \subseteq O'$, $\mathscr{A}^{K}(\mathbf{O}) \subseteq \mathscr{A}^{K}(\mathbf{O}')$.

Causality *If* O, O' *are causally separated in* $\mathbf{M}$, *then elements in the algebra* $\mathscr{A}^{K}(\mathbf{O})$

[1] Since we understood that properties of the algebra depend on the causal structure via τ, we now label the algebra of the observable with the letter referring to the spacetime.

commutes with elements in $\mathscr{A}^K(\mathbf{O}')$. *Practically, if* $f, g \in C_c^\infty(M)$ *be such that* $\text{supp}(f) \cap J(\text{supp}(g)) = \emptyset$*, then* $\Phi_{[f]} \cdot \Phi_{[g]} = \Phi_{[g]} \cdot \Phi_{[f]}$.

Time-slice axiom *Let* $O \subseteq M$ *be a causally convex open neighbourhood of a spacelike Cauchy surface* Σ *for* $\mathbf{M}$. *Then the algebras* $\mathscr{A}^K(\mathbf{O})$ *and* $\mathscr{A}^K(\mathbf{M})$ *are* $*$*-ismorphic.*

Quantization by deformation

The formalism just presented is very abstract and certainly hard to manipulate when extending the algebra of fields to deal with models of interacting field theories. Here we briefly show how, with a different approach, it is possible to get the quantized version of the classical scalar field theory. This approach is called *deformation quantization* and consist of 'deforming' the commutative product of a Poisson algebra in order to define a new non-commuting algebra. Given an algebra $(A, \cdot_s)$ (where $\cdot_s$ is a commutative product) endowed with a Poisson structure $\{,\}$ one studies the class of deformed products $\star_\hbar$[2] such that for all pairs $a, b \in A$

$$\lim_{\hbar \to 0} a \star_\hbar b = a \cdot_s b, \quad \lim_{\hbar \to 0} \frac{1}{\hbar}[a, b]_{\star_\hbar} = \{a, b\} \tag{2.1.37}$$

Here, we briefly show how an algebra of quantum local observables can be constructed deforming the ordinary commuting product between classical observables. Deformation techniques can be naturally applied to algebraic quantum field theory dealing with the space of linear observables $\mathcal{L}$ defined in the previous paragraph. But, in order not to restrict our discussion to very limited cases, we present this methods applied to a more general context. In the literature, many authors refer to the *the space of observables* as the set of real valued functionals from the space of field configuration $C^\infty(M)$. Often, one picks out from this set a subset of functionals suitable for the treatment of the particular problem of interest. Namely, one can select all the regular polynomial functionals defined by

$$F(\phi) = \sum_{n=0}^{N} \int_{M^n} \mathrm{dx}_1 \ldots \mathrm{dx}_n \, f_n(x_1, \ldots, x_n)\phi(x_1)\ldots\phi(x_n)$$

where $f_n \in C_c^\infty(M^n)$. This set, denoted by $\mathcal{P}_{reg}$ will be the framework for our exposition. It is trivial to notice that $\mathcal{L} \subset \mathcal{P}_{reg}$. Now, without any intention to give a thorough exposition, we show how the space of local functionals and hence its subset $\mathcal{L}$ can be endowed with a

[2]Until now the quantity $\hbar$ has been set equal to one, but in this subsection it cannot be fixed because it is used as deforming parameter

non commuting deformed product using the Plank's constant as parameter. We recall that $\mathcal{P}_{reg}$ is naturally endowed with the pointwise commuting product

$$\mu : \mathcal{P}_{reg} \times \mathcal{P}_{reg} \to \mathcal{P}_{reg} \quad \mu\,(F, G)\,(\phi) = F(\phi) \cdot G(\phi),\ \forall \phi \in \mathrm{Sol}^{K} \tag{2.1.38}$$

Over this space we define the Poisson bracket using the notion of functional derivative. Given a funcrional F, its nth functional derivative is a compactly supported distributional density in n variables, symmetrical under permutations of arguments, defined by

$$\langle F^{(n)}(\phi), v^{\otimes n} \rangle = \frac{d^n}{d\lambda^n} F(\phi + \lambda v)\big|_{\lambda=0} \,. \tag{2.1.39}$$

For the sake of the synthesis we cannot go deeply in the well-posedness of this definition and we recommend to the reader a brief discussion on this tools applied to algebraic QFT in [BDF+09, Section 2]. Now, before proceeding we refer to the previous section for the interpretation of the symplectic form as a distribution in $\mathcal{D}'(M \times M)$, that can be represented by the integral kernel.

Now, considering the integral kernel $\Delta(x, y)$ associated to the symplectic form τ_M^K and recalling the interpretation proposed in the previous section of the latter as a distribution in $\mathcal{D}'(M \times M)$, we can define the Poisson bracket for this vector space as

$$\{F, G\}_{Poi}\,(\phi) = \left(F^{(1)}(\phi), \Delta G^{(1)}(\phi)\right)_{\mathbb{R}} \,, \tag{2.1.40}$$

that perfectly fits the symplectic form obtained via pull-back in (2.1.34), when applied to the space of linear functionals $\mathcal{L}$. Moreover it can be easily proven that if on entry of the bracket (2.1.40) is an element of the space $\mathcal{N}^K(M)$, the result is an element in this space, i.e. $\mathcal{N}^K(M)$ is an ideal for the bracket and so the quotient that defines $\mathcal{O}^K(M) = \mathcal{L}/\mathcal{N}^K(M)$ can be extended to all the space $\mathcal{P}_{reg}$, defining $\mathcal{P}^K_{reg} = \mathcal{P}_{reg}/\mathcal{N}^K(M)$. Now we define the product solving the problem (2.1.37)

$$(F \star G)(\phi) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left(\frac{1}{2}\right)^n \left(F^{(n)}(\phi), \Delta^{\otimes n} G^{(n)}(\phi)\right)_{\mathbb{R}} \,, \tag{2.1.41}$$

Let us notice that in the case of linear functional of the form $F_{[f]} \in \mathcal{O}^K(M)$ this product gives exactly

$$F_{[f]} \star_\hbar F_{[h]} := \mu\left(F_{[f]}, F_{[h]}\right) + i\hbar\tau_M^K([f],[h])\mathbb{1} \tag{2.1.42}$$

because the functional derivative of $F_{[h]}(\phi) = \int_M \mathrm{dvol}_M\, h \cdot \phi$ gives the function h interpreted as distribution acting on the space $C^\infty(M)$. Hence we can easily deduce that the commutator evaluated on two element of $\mathcal{O}^K(M)$ gives

$$\left[F_{[f]}, F_{[h]}\right]_\hbar := i\hbar\tau^K_M([f],[h])\mathbb{1}$$

defining exactly the same algebra of the previous section. Whereas in the extended algebra $\mathcal{P}^K$ we get:

$$[F,G]_\hbar = i\hbar\left(F^{(1)}(\phi), \Delta G^{(1)}(\phi)\right)_{\mathbb{R}} + \mathrm{O}(\hbar^2)$$

satisfying the deformation condition and quantizing this way an extended class of functionals.

Summarizing we have shown how this approach generates the same algebra that is given working with the machinery of algebraic quantization. Moreover, it is clear that this new request allow to comprehend the problem of quantizing the linear observables within a wider class of problems that is the quantization of more general structures. Starting from an algebra with a commutative product in a way such that the (2.1.37) holds true. As seen, this particular formulation is not fundamental in treating the free theory of scalar field but it gives a very enlightening insight on the procedure of quantization and turns out to be a very powerful tool to treat algebraic quantization of interacting field theories. Unfortunately a deeper discussion on the noteworthy applications of this methods to the treatment of interactions needs preliminary notions we haven't got yet. Hence, we recommend the reader the consultation of one or two enlightening references, such as [DF01], [BDF+09], [FR15].

2.2 Cartan structures and spinor fields

The generalisation of the free scalar field equation to curved background has been introduced using well known notions in ordinary differential geometry. For the spinor field, the analogous problem needs different instruments and a different description of the underlying background. In this section, we are going to give a new picture to describe Lorentzian manifolds not only with the aim of defining a suitable coupling between geometry (curvature) and spinor field, but also setting a good starting point for the presentation of *super Cartan structures* that will be the basic framework for a rigorous definition of *super-QFT*.

Non-coordinate bases for Lorentzian manifolds

At the beginning of our discussion, we give an equivalent formulation of geometry over n-dimensional Lorentzian manifolds replacing the information given by the metric in a set of n one-forms. This approach can be formulated in the very general context of *Cartan geometry* (see for example [Sha97])), but since the Cartan theory could be too complicate and technical for the scope of this thesis, we will use an adapted version of the subject, making explicit use of bases decomposition and coordinates. In doing this we will make use of Einstein convention for sums throughout all the section.

The tangent space at a point p of a given Lorentzian manifold M is generated by the so called *coordinate basis*, $\{\partial_i|_p\}_{i=1}^n$, i.e. vectors defined by the chosen system of charts $\{U_\alpha, \phi_\alpha\}$, defined by

$$\partial_i|_p(f) = \frac{\partial}{\partial x^i}\left(f \circ \phi_\alpha\right)|_p$$

with $p \in U_\alpha$ for a given α, with the basis of T_p^*M induced by duality denoted with $\{\mathrm{d}x^i\}_{i=1}^n$. Another set of vectors spanning T_pM can be obtained using the action of the general linear group $\mathrm{GL}(n, \mathbb{R})$. Denoting this set with $\{\hat{V}_i|_p\}_{i=1}^n$, the relation $\hat{V}_i|_p = \sum_i e^i_j \partial_i|p$ where e^i_j are n^2 real numbers defining an element of the general linear group. Moreover, when the manifold is endowed with a Lorentzian metric g, a subgroup of transformation can be extracted by asking for preservation of a certain orthonormality condition ruled by the flat metric η, i.e. (omitting the point p for simplicity and details on the running indexes, because obvious):

$$g(\hat{V}_j, \hat{V}_i) = g(V_j^k \partial_k, V_i^h \partial_h) = V_i^h V_j^k g_{kh} = \eta_{ij} \tag{2.2.1}$$

Looking at the last equation, we can see that the Lorentzian metric elements at a point g_{ij}, represented as a matrix, can be put in the diagonal form once a suitable basis is chosen. We choose a set of tangent vectors satisfying this condition and we denote them with the same symbol $\hat{V}_i$. This defines a *non-coordinate basis* and the dual basis of the space of one-forms $\{\hat{e}^i\}_{i=1}^n$, which allows to rewrite the metric as

$$g = \eta_{ij} \hat{e}^i \otimes \hat{e}^j. \tag{2.2.2}$$

We recall that transformation from the set of one-forms to the coordinate basis of the cotangent space can be defined by $\hat{e} = e^i_j \mathrm{d}x^j$ and satisfy the relation $V_l^j e^i_j = \delta^i_l$. Since there are many non-coordinate bases, a natural question is how many of these satisfy the constraint to (2.2.1), answering this question we set

$$\hat{e}'^{i}|_p = \Lambda^i_j(p)\hat{e}^j|_p$$

and we find the equation defining the *Lorentz group* $\mathrm{SO}(1, d-1)$,

$$\Lambda^k_i(p)\eta_{kh}\Lambda^h_j(p) = \eta_{ij}.$$

Remark 2.2.1. The transformations coefficients defined by e^i_j can be used for many different purposes. As first example we define the 'moving γ-matrices'. More precisely, given the *Clifford algebra* generated by the elements $\{\gamma^i\}$ fulfilling

$$\gamma^i\gamma^j + \gamma^j\gamma^i = \eta^{ij}$$

we define the elements $\widetilde{\gamma}^i = e^i_j\gamma^j$ and we find that an analogous relation, ruled by the coefficient of the metric at a point g^{ij}, holds true

$$\widetilde{\gamma}^i\widetilde{\gamma}^j + \widetilde{\gamma}^j\widetilde{\gamma}^i = g^{ij}$$

The local action of the Poincaré group

Summarising, at each point we have a class of non-coordinate bases of the tangent and cotangent space, each element of this class is linked to another element by a linear map - that we will call *local Lorentz transformation* $(\Lambda^i_j(p))$ - and to the coordinate basis with the linear transformation defined by the coefficients $e^i_j(p)$. Now, we can use this two facts to encode the action of the Lorentz group and of the spin group for defining an equation of motion for the spinor field. For an n dimensional manifold, the idea is that of defining an object "coupling" the set of n one-forms with the elements of the translations Lie algebra $(\mathfrak{t})$, using then the action of the adjoint action of the Poincaré group on these elements. This new object is called *vielbein* and is defined as follow.

Definition 2.2.2. Given a Lorentzian manifold (M, g), denoting by $\mathfrak{t}$ the translation algebra, the *vielbein* $\hat{e}$ is a one-form section with values in the translation algebra, i.e. an element of $\Omega^1(M, \mathfrak{t})$ satisfying a non degenerate condition in the sense that. For each point p, we take a chart (U, ϕ), where $p \in U$, that induces a system of local coordinates $\{x^i\}_{i=1}^n$, we can expand the vielbein in terms of the coordinate basis T^*_pM and an orthonormal basis of $\mathfrak{t}$:

$$\hat{e}|_p = \hat{e}^i|_p \otimes P_i = e^i_j(p)\mathrm{d}x^j|_p \otimes P_i \tag{2.2.3}$$

where the coefficients $e^i_j(p)$ define a non degenerate transformation to the non coordinate basis $\{\hat{e}^i\}_{i=1}^n$. We denote the inverse of this transformation with V^k_h, using which we can write the *associated frame of vector fields* $\hat{V}_h = V^k_h \partial_k$.

We notice that in the literature the definition of the vielbein can be different, it is presented as a non degenerate map from a $\mathrm{SO}(1, n\text{-}1)$-vector bundle to the tangent bundle of the manifold M. We are not going to show the equivalence of this two definitions, but we remark that in both cases the vielbein has the double aim: describing the geometry of the underlying Lorentzian manifold (M, g), done by the set of one forms $\{\hat{e}^i\}_{i=1}^n$ due to their relation with the metric, and encoding the local action of the Lorentz group, in a sense that we are going to make clear in the following remark.

Remark 2.2.3. Taking a Lie group G we define the adjoint action of the group on itself as a map $\psi : G \times G \to G$, $(g, h) \mapsto \psi(g, h) = ghg^{-1}$. Looking at this map on just one argument, we define the map $\psi_g = \psi(g, \cdot)$ and the map $\Psi : G \to \mathrm{Aut}(G)$, associating $g \mapsto \psi_g$. Fixing an element g, if we take the differential at the identity of ψ_g

$$\mathrm{d}\psi_g|e : \mathrm{T}_e G \to \mathrm{T}_e G \tag{2.2.4}$$

$$X \mapsto \mathrm{d}\psi_g|_e(X) \tag{2.2.5}$$

We recall that the tangent space at the identity is the Lie algebra $\mathfrak{g}$ associated to G and we denote, from now on, the map defined via 2.2.4 by $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$. This can be seen as an action $\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$ of the group on its Lie algebra, and taking the differential of Ad_g with the respect to the group element g, an action of the algebra on itself $\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is naturally defined. It is a well known fact that the maps from two elements can be explicitly computed $[\cdot, \cdot]_{\mathfrak{g}}$:

$$\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \qquad\qquad (x, y) \mapsto \mathrm{ad}_x(y) = [x, y]$$

and defines the so called adjoint representation of the algebra on itself. We sometimes will adopt the convention, as often in physics, for which the action of the Lie algebra element x is referred to as infinitesimal action of the Lie group element g associated to x, and denoted by δ_x

Let us take the Poincaré group $\mathrm{SO}(1,n\text{-}1)\ltimes\mathbb{R}^{1,n\text{-}1}$ as Lie group. We can use its infinitesimal action on the translation algebra $\mathfrak{t}$, seen as the normal subalgebra of the Poincaré algebra $\mathfrak{poinc}=\mathfrak{so}(1,n\text{-}1)\oplus\mathfrak{t}$, consisting of the elements of the form $0\oplus P$, where $P\in\mathfrak{t}$. Hence, denoting the generators of the algebra by $\{P_i, L_{jk}\}$, if we look at the infinitesimal transformations induced by the element $\varepsilon=\varepsilon_1^{ij}M_{ij}\oplus\varepsilon_2^k P_k$, we can calculate the action on $\mathfrak{t}$

$$\begin{aligned}\delta_\varepsilon(0\oplus P_h)=&\mathrm{ad}_\varepsilon(0\oplus P_h)\\ =&\left[\varepsilon_1^{ij}L_{ij}\oplus\varepsilon_2^k P_k, 0\oplus P_h\right]\\ =&\varepsilon_1^{ij}(0\oplus[L_{ij},P_h])\\ =&2(\varepsilon_1)^i_h(0\oplus P_i)\end{aligned}\tag{2.2.6}$$

that, since we are interested only in the action on $\mathfrak{t}$, can be written shortly $\delta_\varepsilon(P_h)=2\varepsilon^i_k P_i$. Going directly to finite transformation, via exponential map, the action of $\mathrm{SO}(1,n\text{-}1)\ltimes\mathbb{R}^{1,n\text{-}1}$ becomes

$$\mathrm{Ad}_{(\Lambda,a)}P_j=\Lambda^i_j P_i$$

where (Λ,a) is the element generated by $\varepsilon\in\mathfrak{poinc}$, and Λ^i_j are the matrix elements of the representation of the Lorentz group over $\mathfrak{t}$. We can use this fact to define an action of the Lorentz group on the vielbein, setting

$$\delta_\varepsilon(\hat{e})=[\varepsilon,\hat{e}]:=\varepsilon_1^{ij}\hat{e}^h\left[L_{ij},P_h\right]\qquad\qquad 2\varepsilon^i_h\hat{e}^h\otimes P_i\tag{2.2.7}$$

Together with the action on the algebra we want to operate on $\hat{e}$ using a notion of 'derivative', for this scope we have to choose a connection $\omega\in\Omega^1\left(M\otimes\mathfrak{so}(1,n\text{-}1)\right)$, defined on the generators by the antisymmetric coefficients by

$$\omega=\omega^{ij}\otimes L_{ij},\tag{2.2.8}$$

and acting on the vielbein this way:

$$\begin{aligned}\mathrm{d}_\omega\hat{e}:=&\mathrm{d}\hat{e}^k\otimes P_k+\omega^{ij}\wedge\hat{e}^k\otimes[L_{ij},P_k]\\ &=\mathrm{d}\hat{e}^k\otimes P_k+\omega^j_i\wedge\hat{e}^k\otimes P_k+\omega^i_j\wedge\hat{e}^j\otimes P_i\\ &=(\mathrm{d}\hat{e}^i+2\omega^i_j\wedge\hat{e}^j)\otimes P_i.\end{aligned}\tag{2.2.9}$$

The last equation defining a two form with values in $\mathfrak{t}$. Hence, we define an element of $\Omega^1(M \otimes \mathfrak{t})$: the torsion two-form $T = \mathrm{d}_\omega \hat{e} = T^i \otimes P_i$. Choosing a connection means imposing constraints on the torsion coefficients T^i and we want to motivate which constraint has to be applied recalling the relation with the geometry of the Lorentzian manifold (M, g) and its tangent bundle. We denote by ∇ a connection over $\mathrm{T}M$, hence the torsion associated to ∇ is defined by the formula:

$$\widetilde{T} : \mathrm{T}M \times \mathrm{T}M \to \mathrm{T}M(X, Y) \mapsto \nabla_X Y - \nabla_Y X + [X, Y]$$

With respect to the basis $\{\hat{V}_i\}_{i=1}^n$, we get the associated Christoffel symbols computing $\nabla_{\hat{V}_i}\hat{V}_j = \widetilde{\Gamma}^k_{ij}\hat{V}_k$. Moreover, we denote by the c^k_{ij} the structure coefficients of the vector fields algebra at a point p

$$[\hat{V}_j, \hat{V}_k] = c^i_{jk}\hat{V}_i$$

given by

$$\begin{aligned}[\hat{V}_j, \hat{V}_k] &= [V^l_j\partial_l, V^r_k\partial_r] = \hat{V}_j(V^r_k)\partial_r + V^l_jV^r_k\partial_l\partial_r - \hat{V}_k(V^l_j)\partial_l - V^r_kV^l_j\partial_r\partial_l \\ &= (\hat{V}_j(V^r_k) - \hat{V}_k(V^r_j))\partial_r = \underbrace{e^i_r(\hat{V}_j(V^r_k) - \hat{V}_k(V^h_j))}_{c^i_{jk}}\hat{V}_i\end{aligned}$$

or equivalently (noting that $e^i_h\hat{V}_j(V^h_k) = -V^h_k\hat{V}_j(e^i_h)$)

$$c^i_{jk} = V^h_j\hat{V}_k(e^i_h) - V^h_k\hat{V}_j(e^i_h) \tag{2.2.10}$$

Then

$$\begin{aligned}\widetilde{T}^i_{jk} &= \langle \hat{e}^i, \widetilde{T}(\hat{V}_j, \hat{V}_k)\rangle = \langle \hat{e}^i, \nabla_{\hat{V}_j}\hat{V}_k - \nabla_{\hat{V}_k}\hat{V}_j + [\hat{V}_j, \hat{V}_k]\rangle \\ \widetilde{T}^i_{jk} &= \widetilde{\Gamma}^i{}_{jk} - \widetilde{\Gamma}^i{}_{jk} + c^k_{ij}\end{aligned} \tag{2.2.11}$$

Now, we can make explicit the calculations of the coefficients T^i of $T = \mathrm{d}_\omega\hat{e}$ and compare the result with the formula (2.2.11).

$$\begin{aligned}T^i &= \mathrm{d}(e^i_j\mathrm{d}x^j) + 2(\omega^i_k)_j\hat{e}^j \wedge \hat{e}^k = \\ &= \partial_h\hat{e}^i_l\mathrm{d}x^h \wedge \mathrm{d}x^l + 2(\omega^i_k)_j\hat{e}^j \wedge \hat{e}^k \\ &= \left(V^l_k\hat{V}_j(e^i_l) + 2(\omega^i_k)_j\right)\hat{e}^j \wedge \hat{e}^k\end{aligned} \tag{2.2.12}$$

and, extracting coefficients by anti-symmetrization, we get

$$T^i_{jk} = \frac{1}{2}\left(V^l_k \hat{V}_j(e^i_l) - V^l_j \hat{V}_k(e^i_l)\right) + (\omega^i_k)_j - (\omega^i_j)_k$$

Hence, forcing the identification

$$2(\omega^i_j)_k \leftrightarrow \widetilde{\Gamma}^i_{jk},$$

we can identify also $T^i_{jk} \leftrightarrow -\frac{1}{2}\widetilde{T}^i_{jk}$, meaning that the two quantities are proportional up to constant. Now, we use this fact to induce a choice of the connection ruled by ω, seeing what happens when we take the connection ∇ to be the Levi-Civita connection over $\mathrm{T}M$. As well known, the latter is a torsion free connection and this entails $\widetilde{T}^i_{jk} = 0$ and then the proportionality relation allows to write the vanishing torsion constraint which uniquely determines ω

$$T_\omega = \mathrm{d}_\omega \hat{e} = 0$$

For a fixed ω, another object related to the curvature tensor of the manifold can be computed. It is an element of $\Omega^2(M, \mathfrak{so}(1, n\text{-}1))$ and is defined by

$$\begin{aligned} R_\omega = \mathrm{d}_\omega \omega &= \mathrm{d}\omega + \frac{1}{2}\,\omega^{ij} \wedge \omega^{kl} \otimes [M_{ij}, M_{kl}] \\ &= (\mathrm{d}\omega^{ij} + \omega^{ki} \wedge \omega^j_k) \otimes M_{ij}. \end{aligned} \tag{2.2.13}$$

Moreover, referring to [Nak03], we note that T_ω and R_ω satisfy two equations that are closely related to the ordinary *Bianchi identities* for the torsion and curvature tensor

$$\mathrm{d}T^i + \omega^i_k \wedge T^k = R^i_k \wedge \hat{e}^k \qquad \mathrm{d}R^i_j + \omega^i_l \wedge R^l_j - R^i_l \wedge \omega^l_j = 0$$

Transformation rule for ω This paragraph has the precise aim to present the most relevant fact of the vielbein formulation of Lorentzian geometry. We defined the torsion one-form associated to the connection ω, which takes values in the translation algebra $\mathfrak{t}$. This means that we can deduce a transformation rule for ω imposing it to be coherent with the torsion T_ω transformation rule.

From an infinitesimal transformation generated by $\varepsilon \in \mathfrak{so}(1, n\text{-}1)$, we know that, in analogy with (2.2.7)

$$\delta_\varepsilon(T_\omega) = 2\varepsilon^i_h {T_\omega}^h \otimes P_i \quad \text{and} \quad \widehat{\mathrm{Ad}}_{(\Lambda,a)}(T) = T^j \otimes \Lambda^i_j P_i,$$

which can be read on coefficients as

$$T'^j = \Lambda^j_i T^i \tag{2.2.14}$$

We calculate now $\mathrm{d}_\omega \hat{e}$, varying the vielbein coefficients ($\hat{e}^i \mapsto \Lambda^i_j \hat{e}^j$)

$$\begin{aligned} \mathrm{d}'_\omega \hat{e}' =& \left(\mathrm{d}(\Lambda^i_j \hat{e}^j) + \omega'^i_{\ k} \wedge \Lambda^k_l \hat{e}^r\right) \otimes P_i \\ =& \mathrm{d}\Lambda^i_j \wedge \hat{e}^j + \left(\Lambda^j_i \mathrm{d}\hat{e}^j + \omega'^i_{\ k} \wedge \Lambda^k_l \hat{e}^r\right) \otimes P_i \end{aligned} \tag{2.2.15}$$

and taking components

$$\Lambda^i_j \mathrm{d}\hat{e}^j + \left(\Lambda^h_r \omega'^i_{\ h} + \mathrm{d}\Lambda^i_r\right) \wedge \hat{e}^r. \tag{2.2.16}$$

Hence, if the set of one forms ω^i_j transform as

$$\omega^i_j \mapsto \Lambda^i_l \omega^l_k (\Lambda^{-1})^k_j - (\Lambda^{-1})^s_j \mathrm{d}\Lambda^i_s \tag{2.2.17}$$

then $\Lambda^h_r \omega'^i_{\ h} + \mathrm{d}\Lambda^i_r = \Lambda^h_r \omega^i_h$ and the equation (2.2.14) is fulfilled.

We conclude this subsection with the summary definition of *Cartan structure*, this notion will be the connection point with the supergeometric theory in the following.

Definition 2.2.4. Given a manifold M, we define as *Cartan structure* over M, a pair $(\hat{e}, \omega)$, where $\hat{e}$ is a vielbein respecting Definition 2.2.2 and ω is an element of $\Omega^1(M, \mathfrak{so}(1, n\text{-}1))$. We will call *Cartan manifold* the triple $(M, \hat{e}, \omega)$, to which torsion and curvature are assigned with the formulas

$$T_\omega := \mathsf{d}_\omega \hat{e} := \mathrm{d}\hat{e} + [\omega, \hat{e}]_\wedge \tag{2.2.18}$$

$$R_\omega := \mathsf{d}_\omega \omega := \mathrm{d}\omega + [\omega, \omega]_\wedge\,, \tag{2.2.19}$$

where, for $\rho \in \Omega^l(M, \mathfrak{g}), \sigma \in \Omega^k(M, \mathfrak{g})$ $[\rho, \sigma]_{\wedge, \mathfrak{g}} := \rho^i \wedge \sigma^j \otimes [X_i, X_j]_{\mathfrak{g}}$. We can associate a Lorentzian manifold (M, g) to each Cartan manifold, with g induced by the local frame of $\Omega^1(M)$ following equation (2.2.2).

An equation of motion for spinor fields

As anticipated, the main reason for a vielbein based description of the geometry of a Lorentzian manifold is that of encoding and making explicit the local action of the Lorentz group. This allow to proceed directly towards the study of spinor fields on curved backgrounds. We start recalling that, on Minkowski spacetime, a spinor field is defined as an element of a vector space where acts a representation of the double covering of the Poincaré group (the spin group Spin$(1, n\text{-}1)$). On non-flat globally hyperbolic spacetimes global transformation composing the Poincaré group are no longer valid symmetries for the physical system and the only tool we can use is the local action we described above. In the following, we will use the vielbein and the spin connection to define a representation of the spin group and write down an equation of motion, invariant under the local action of the Lorentz group.

Example 2.2.5. We recall that in ordinary approaches to quantum field theory (see [PS95, ch. 3.3, pg. 40]), a spinor is an element of $\mathbb{C}^4$ and the Spin$(1,3)$-group action on $\mathbb{C}^4$ is represented by the group of 4×4 matrices defined via exponential map

$$\mathsf{S}(\Lambda) = \exp\left(\frac{1}{2}\varepsilon_{\rho\sigma} S^{\rho\sigma}\right)$$

where γ^μ are elements in End$(\mathbb{C}^4)$, satisfying the Clifford algebra ruled by the Minkowskian metric and the set of matrices $S^{\rho\sigma} = \frac{1}{4}\left(\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho\right)$ represent the Lie algebra of the spin group (or equivalently of the Lorentz group). The transformation rule for spinors is

$$\psi^a(x) \mapsto \mathsf{S}(\Lambda)^a_b \psi^b(R(\Lambda)^{-1}x)$$

Where with $R(\Lambda)$ we denoted the representation over $\mathbb{R}^{1,3}$ of the Lorentz group. Then the equation of motion is written using global coordinates $\{x^\mu\}_{i=0}^3$

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

and invariance under the Lorentz group action is ensured by the well known property (sometimes called "covariance of gamma matrices")

$$\mathsf{S}(\Lambda)^{-1}\gamma^\mu\mathsf{S}(\Lambda) = \gamma^\nu R(\Lambda)^\mu_\nu \tag{2.2.20}$$

Even though the instruments are different, the guidelines for the definition of a spinor field equation are analogous. We take a representation of the group Spin$(1, n\text{-}1)$ on a vector space V and we define spinors as section in $\Gamma(M, V)$. This representation has to satisfy a relation like (2.2.20), hence denoting it by $\rho^V : \mathrm{SO}_0(1, n-1) \to \mathrm{End}(V)$, we write for

$\Lambda \in \mathrm{SO}_0(1, n-1)$

$$\rho^V(\Lambda)^{-1}\gamma^\mu\rho^v(\Lambda) = \gamma^\nu R(\Lambda)^\mu_\nu$$

where now $R(\Lambda)$ is a n-dimensional representation. Then, recalling that a notion of covariant derivative can be defined using the spin connection $\omega \in \Omega^1(M, \mathfrak{so}(1, n\text{-}1))$, we extend the representation ρ^V of the group to a representation of ω over V

$$\rho : \Omega^1(M, \mathfrak{so}(1, n\text{-}1)) \to \Omega^1(M, \mathrm{End}(V))$$

using which we can introduce the dynamics with the operator

$$\begin{aligned} \mathsf{d}_{\rho,\omega} : &\Gamma(M, V) \to \Omega^1(M, V) \\ &\psi \to \mathsf{d}_{\rho,\omega}\psi := \mathrm{d}\psi + \rho(\omega)\psi. \end{aligned} \tag{2.2.21}$$

Using the local frame, the derivative operator $\mathsf{d}_{\rho,\omega}$ can be expanded, calculating the symbols

$$\nabla_i^{\rho,\omega}\psi := \langle \hat{V}_i, \mathsf{d}_{\rho,\omega}\psi \rangle \tag{2.2.22}$$

with which we define the operator

$$\begin{aligned} \not{D}_{\rho,\omega} : &\Gamma(M, V) \to \Gamma(M, V) \\ &\psi \mapsto \not{D}_{\rho,\omega}\psi := \gamma^i \nabla_i^{\rho,\omega}\psi \end{aligned} \tag{2.2.23}$$

Now, what remain to be proved is invariance of the equation induced by this operator, $\not{D}_{\rho,\omega}\psi = 0$ under the local action of the Lorentz group (a possible mass term would be trivially invariant). Following [Nak03] we prove invariance for the spin representation generated by the elements $\Sigma_{ij} := \frac{1}{4} i \left[\gamma_i, \gamma_j\right]$, introduced in Example 2.2.5. We split the operator action defined by (2.2.23) can be split in two parts, which can be analysed separately:

$$\underbrace{\gamma^i\langle \hat{V}_i, \mathrm{d}\psi\rangle}_{i.} + \underbrace{\gamma^i\langle \hat{V}_i, \omega^{kl}\rangle}_{ii.} \Sigma_{kl}\psi$$

Now, if we look at the transformation under local Lorentz action of the part (i.), we get

$$\gamma^i\langle \hat{V}_i, \mathrm{d}\psi\rangle \mapsto \gamma^j\Lambda^i_j\langle \hat{V}_i, \mathrm{d}(\rho^V(\Lambda)\psi)\rangle \tag{2.2.24}$$

$$= \gamma^j\Lambda^i_j\langle \hat{V}_i, \mathrm{d}(\rho^V(\Lambda))\psi\rangle + \gamma^j\Lambda^i_j\langle \hat{V}_i, \rho^V(\Lambda)\mathrm{d}\psi\rangle\rangle. \tag{2.2.25}$$

The second term of the latter, recalling that $\gamma^j \Lambda^i_j \rho^V(\Lambda) = \rho^V(\Lambda)\gamma^i$, can be calculated as

$$\rho(\Lambda)\gamma^i \langle \hat{V}_i, \mathrm{d}\psi\rangle$$

while we expect the first term to be deleted by part (ii.) of the splitting. Then we proceed expanding (ii.) on the coordinate basis and denoting $\widetilde{\omega} = \omega^{kl}\Sigma_{kl}$ in order to simplify the calculations. Hence, we have $\gamma^i\langle \hat{V}_i, \omega^{kl}\rangle\psi = \gamma^i e^r_i \widetilde{\omega}_r \psi$. Now, we claim that the coefficients $\widetilde{\omega}_r$ transform this way[3]

$$\widetilde{\omega}_r \mapsto \rho^V(\Lambda)\widetilde{\omega}_r \rho^V(\Lambda)^{-1} - \partial_r \rho^V(\Lambda)^{-1}$$

and we look at part (ii.),

$$\gamma^i e^r_i \widetilde{\omega}_r \psi \mapsto \gamma^i \Lambda^j_i e^r_j \rho^V(\Lambda)\widetilde{\omega}_r \rho^V(\Lambda)^{-1}\rho^V(\Lambda)\psi - \gamma^i e^r_i \partial_r \rho^V(\Lambda)\rho^V(\Lambda)^{-1}\rho^V(\Lambda)\psi \tag{2.2.26}$$
$$= \rho^V(\Lambda)\gamma^i e^r_i \widetilde{\omega}_r \psi - \gamma^i \langle \hat{V}_i \mathrm{d}\rho^V(\Lambda)\psi\rangle. \tag{2.2.27}$$

Noting the expected cancellation we can write the transformation of the whole equation of motion

$$\not{D}_{\rho,\omega}\psi = 0 \mapsto \rho^V(\Lambda)\not{D}_{\rho,\omega}\psi = 0. \tag{2.2.28}$$

Remark 2.2.6. We conclude noting that the operator just introduced is Green-hyperbolic. Indeed, it is easy to see that acting twice on a spinor field with the operator defined by (2.2.23), we a normally hyperbolic operator. Then, we can use Proposition 1.2.12.

2.3 The axiomatic framework for quantum field theories

In the previous sections we presented two key examples of quantum field theories, in order to underline singular aspects of the formulation of quantum field theories in curved backgrounds. In particular, we have shown that quantizing classical fields on a general curved Lorentzian manifold M means associating a $*$-algebra of quantum observables with the three property of isotony, causality and 'time slicing'. As an alert reader should have

[3]The proof of this formula can be done studying infinitesimally all the transformations involved. A detailed presentation of these calculations can be found in [Nak03, Ch. 7, at the end of Sec. 7.10]

noticed, all the results and constructions of the previous section are independent from the chosen manifold and operator. In fact, what turns out to be fundamental are global hyperbolotcity of the background and Green-hyperbolicity of the operator. A very general approach to quantum field theories in curved backgrounds is based on the generally covariant locality principle (GCLP) and its formulation as appeared for the first time in [BFV03]. This GCLP provides a scheme for the formulation of the QFTs emphasising the common features of the quantization methods on different general curved spacetimes, encoding in the mathematical formalism of category theory the covariance property that any theory has to satisfy to be physical. Indeed, the GCLP is formulated in terms of locally covariant quantum field theories (LCQFT): the central topic of this chapter.

2.3.1 The generally covariant locality principle

The GCLP is formulated using the language of category theory. We recommend the reader to look at the Appendix or at a thorough exposition on the subject like [ML78]. We are ready now to start with the definition of the two categories at the basis of the axiomatic formulation.

Definition 2.3.1. We call $\mathfrak{Ghs}$ the category such that:

- Objects are GHSTs of dimension d.
- Morphisms, $\mathsf{Mor}_{\mathfrak{Ghs}}(\mathbf{M}, \mathbf{N})$, are all the isometric embeddings $\chi : M \to N$ preserving orientation and time orientation, such that $\chi(M) \subset N$ is causally convex respect to the causal structure of $\mathbf{N}$.
- The composition law for morphisms is the usual composition of functions.

Remark 2.3.2. Now we proceed with the discussion of Remark 1.1.16 in order to give the preliminary background to check the category axioms for $\mathfrak{Ghs}$. Given two spacetimes $(M, g, \mathfrak{o}, \mathfrak{t})$ and $(N, h, \mathfrak{p}, \mathfrak{v})$, and an element $\chi \in \mathsf{Mor}_{\mathfrak{ghs}}(\mathbf{M}, \mathbf{N})$ we have that $\chi(M) \subset N$ is a causally compatible submanifold of N, hence we are induced to consider the restriction to the image of χ, $\overline{\chi} : M \to \chi(M)$, that is a diffeormophism and can be used to push forward the structure of $\mathbf{M}$. We can construct this way the spacetime $(\chi(M), \overline{\chi}_* g, \overline{\chi}_* \mathfrak{o}, \overline{\chi}_* \mathfrak{t})$ that is equivalent to $\left(\chi(M), h|_{\chi(M)}, \mathfrak{p}|_{\chi(M)}, \mathfrak{v}|_{\chi(M)}\right)$, and this classify $\overline{\chi}$ as a morphism from $\mathbf{M}$ to $\chi(\mathbf{M})$. Then, recalling that, given a manifold $\chi(M)$, the inclusion map $\iota_{\chi(M)} : \chi(M) \to N$ is naturally a morphism from $\left(\chi(M), h|_{\chi(M)}, \mathfrak{p}|_{\chi(M)}, \mathfrak{v}|_{\chi(M)}\right)$ to $(N, h, \mathfrak{p}, \mathfrak{v})$, hence we can conclude writing χ with the decomposition $\iota_{\chi(M)} \circ \overline{\chi}$

We check here the category axioms:

- For each object $(M, g, \mathfrak{o}, \mathfrak{t})$ the identity morphism is defined by the map $\mathrm{id}_M : M \to M$, $p \mapsto p$.
- The composition law for morphisms is associative because associative is the composition of functions.
- Given two morphisms $\chi_1 \in \mathsf{Mor}_{\mathfrak{ghs}}(\mathbf{M}, \mathbf{N})$ and $\chi_2 \in \mathsf{Mor}_{\mathfrak{ghs}}(\mathbf{N}, \mathbf{O})$, where $\mathbf{M}, \mathbf{N}, \mathbf{O}$, the map $\chi_1 \circ \chi_2$ is trivially an embedding preserving metric, orientation and time-orientation. What needs to be proven is that the submanifold $(\chi_1 \circ \chi_2)(M)$ is causally convex in $\mathbf{O}$. Given two points $p, q \in (\chi_1 \circ \chi_2)(M)$ and the curve $\gamma_{pq} : I \to \chi_1 \circ \chi_2)(M)$ from p to q. We take the decomposition of Remark 2.3.2 noticing that both $\overline{\chi_2}$ and $\overline{\chi_1}$ are diffeomorphism, respectively from M to $\chi_2(M)$ and from $\chi_2(M)$ to $\chi_1(\chi_2(M))$. Hence, we can define the curve $\widetilde{\gamma} = \overline{\chi_1}^{-1} \circ \gamma : I \to \chi_2(M)$ connecting $\overline{\chi_1}^{-1}(p)$ and $\overline{\chi_1}^{-1}(q)$, points in $\chi_2(M)$. The latter is included in $\chi_2(M)$ because $\chi_2(M)$ is causally convex, and so is $\overline{\chi_1} \circ \overline{\chi_1}^{-1} \circ \gamma = \gamma$

Now we define the second category needed to proceed with the axiomatic formulation

Definition 2.3.3. We define the category $\mathfrak{Alg}$ as follow:

- Objects are $*$-algebras.
- The set of morphism $\mathrm{Mor}_{\mathfrak{Alg}}(\mathcal{A}, \mathcal{B})$ are all the $*$-homomorphisms between $\mathcal{A}$ and $\mathcal{B}$.
- The composition law for morphisms is the usual composition of functions.

The check for this category to satisfy the category axioms is trivial. We have now all the element for the definition of the locally covariant quantum field theories

Definition 2.3.4. A *locally covariant quantum field theory* is a covariant functor from the category $\mathfrak{ghs}$ to the category $\mathfrak{Alg}$

Now the generally convariant locality principle (GCLP) can be formulated saying that on a globally hyperbolic spacetime, any quantum field theory must be formulated as a *locally covariant quantum field theory* (*LCQFT*). This statement is motivated by the fact that LCQFTs seem to be the perfect object to implement the covariance of general relativity in the physics of quantum field theories. Furthermore, the functorial nature of these quantum fields imposes the geometrical locality at the level of quantum observables, i.e. the elements of the $*$-algebra. In fact, this is exactly the implementation of the *property of isotony* that we noticed in the *algebras of local observables* for the classical scalar field: considering a LCQFT $\mathscr{A}:\mathfrak{ghs}\to\mathfrak{Alg}$, given a globally hyperbolic spacetime $\mathbf{M}=(M,g,\mathfrak{o},\mathfrak{t})$ and a causally convex connected open subset O of M, look at the inclusion $\iota_O:O\to M$ that is a morphism of the category $\mathfrak{ghs}$. Hence we consider the element of $\mathrm{Mor}_{\mathfrak{Alg}}(\mathscr{A}(\mathbf{O}),\mathscr{A}(\mathbf{M}))$, defined by $\mathscr{A}(\iota_O)$, we, then, recall that the spacetime $\mathbf{O}$ can be identified with the spacetime $(O,g|_O,\mathfrak{o}|_O,\mathfrak{t}|_O)$ and we know that the algebra $\mathscr{A}(\iota_O)((\mathscr{A}(\mathbf{O})))$ is a unital sub-$*$-algebra of $\mathscr{A}(\mathbf{M})$. Hence a LCQFT $\mathscr{A}$ associates a causally convex connected open subset of a globally hyperbolic spacetime to a unital $*$-algebra satisfying the inclusion $\mathscr{A}(\iota_O)((\mathscr{A}(\mathbf{O})))\subseteq\mathscr{A}(\mathbf{M})$.

Isotony and locality and their interpretation at the level of quantum fields are not the unique property we need to have an interesting theory. Indeed, two other property are often required to be satisfied by a locally covariant quantum field theory

i. A LCQFT $\mathscr{A}$ is called *causal* if for all triple of objects in $\mathfrak{ghs}$, $\mathbf{M}_1,\mathbf{M}_2,\mathbf{M}$, and all pair of morphisms $\chi_1\in\mathrm{Mor}_{\mathfrak{ghs}}(\mathbf{M}_1,\mathbf{M})$ and $\chi_2\in\mathrm{Mor}_{\mathfrak{ghs}}(\mathbf{M}_2,\mathbf{M})$ such that $\chi_1(M_1)$ and $\chi_2(M_2)$ are causally separated with respect to the causal structure of $\mathbf{M}$, then the following holds:
$$[\mathscr{A}(\psi_1)(\mathscr{A}(\mathbf{M}_1)),\mathscr{A}(\psi_2)(\mathscr{A}(\mathbf{M}_2))]=\{0\},$$

ii. $\mathscr{A}$ fulfils the *time slice axiom* if for all pairs of objects of $\mathfrak{ghs}$, $\mathbf{O},\mathbf{M}$, and each $\chi\in\mathrm{Mor}_{\mathfrak{ghs}}(\mathbf{O},\mathbf{M})$ such that the image of O through χ in M, $\chi(M)$, contains a smooth spacelike Cauchy surface Σ for $\mathbf{M}$, then
$$\mathscr{A}(\psi)(\mathscr{A}(\mathbf{M}))=\mathscr{A}(\mathbf{N}).$$

2.3.2 Construction of locally covariant QFTs

In this section we show how, on the light of the GCLP, a quantum field thoery can be constructed as a covariant functor. As we already underlined, in Section 2.1 we have given a first example of locally covariant quantum field theory. Indeed the algebra $\mathscr{A}^K(M)$ defined before turns out to be the image of a functor $\mathscr{A}^K:\mathfrak{ghs}\to\mathfrak{Alg}$. In order to achieve that construction we will pass through the category of symplectic vector spaces (yet to be defined here, but its meaning is clear from the context). Moreover we will show how a general approach to algebraic quantization is possible, and how the construction of quantum field theories can be performed according to the GCLP with a given set of initial data. In

[BFV03], the authors anticipated the possibility of giving slightly changed versions of the categories involved in the definition of locally covariant quantum field theories. Hence, following the work of [BG12], we introduce the category $\mathfrak{Ghs}$ and we will call LCQFT any functor $\mathscr{F}:\mathfrak{Ghs}\to\mathfrak{Alg}$.

Definition 2.3.5. The category $\mathfrak{Ghs}$ is formed taking

- as objects the triples consisting of a globally hyperbolic spacetime (M),a real vector bundle $\pi : E \to M$ endowed with a non-degenerate inner product $\langle\cdot\,,\cdot\rangle_E$ and a formally self-adjoint Green-hyperbolic operator $P:\Gamma(M,E)\to\Gamma(M,E)$.

- Morphisms are given by pairs $(\chi, F)\in \mathrm{Mor}_{\mathfrak{Ghs}}((M_1,E_1,P_1),(M_2,E_2,P_2))$, where $\chi: M_1\to M_2$ is an isometric embedding preserving orientation and time orientation, with $\chi(M_1)$ being a causally compatible open subset of M_2, and F is a fiberwise isometric vector bundle isomorphism over χ such that the following diagram commutes:

$$\begin{array}{ccc} \Gamma(M_2,E_2) & \xrightarrow{P_2} & \Gamma(M_2,E_2) \\ {\scriptstyle \mathrm{res}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}} \\ \Gamma(M_1,E_1) & \xrightarrow{P_1} & \Gamma(M_1,E_1) \end{array}$$

Diagram 2.3.1

where $\mathrm{res}(u) := F^{-1}\circ u\circ\chi$ is the formal definition of the restriction operator for every $u\in\Gamma(M_2,E_2)$, the definition of which can be understood more clearly looking at the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{F} & E_2 \\ {\scriptstyle \mathrm{res}(v)}\uparrow & & \uparrow{\scriptstyle v} \\ M_1 & \xrightarrow{\chi} & M_2 \end{array}$$

Diagram 2.3.2

Let us define also the operator ext acting on sections $\Gamma_c(M_1,E_1)$ this way $\mathrm{ext}(u) = F\circ u\circ f^{-1}$. It trivially holds the identity: $\mathrm{res}\circ\mathrm{ext}(u) = u, \forall u\in\Gamma_c(M_1,E_1)$ and $\mathrm{ext}\circ\mathrm{res}(v) = v, \forall v\in\Gamma_c(M_2,E_2)$. From this identity we deduce also $\mathrm{ext}\circ P_1 = P_2\circ\mathrm{ext}$. With this in mind, we present a Lemma to be used together with results of the Chapter 1: Theorem 1.2.15 and Propositions 1.2.16, 1.2.13[4]

[4]We present it here in the form used in [BG12], where a complete proof can be found

Lemma 2.3.6. *Let (χ, F) be a morphism between two objects (M_1, E_1, P_1) and (M_2, E_2, P_2) in the category $\mathfrak{Ghs}$ and let $(G_1)_\pm$ and $(G_2)_\pm$ be the respective Green's operators for P_1 and P_2. Then*

$$\mathrm{res} \circ (G_2)_\pm \circ \mathrm{ext} = (G_1)_\pm.$$

Definition 2.3.7. We call $\mathfrak{Symp}$ the category which objects are the pairs (V, τ), where V is a vector space and τ a symplectic form, i.e. a non degenerate, skew-symmetric bilinear map. Morphism of $\mathfrak{Symp}$ are linear maps preserving the symplectic structure. Given two pairs (V_1, τ_1), (V_2, τ_2), then $\alpha : V_1 \to V_2$ linear, such that $\alpha^*\tau_2 = \tau_1$ where the pullback is

$$\alpha^*\tau_2 : V_1 \times V_1 \to V_1$$
$$\alpha^*\tau_2 = \tau_2 \circ (\alpha \otimes \alpha)$$

Now all the preliminary results and definitions for the construction of bosonic LCQFT have been presented, we remind that the construction we did for K in Section 2.1 has been based on the same arguments, and we will recall them in the following, without repeating the proofs. We know that the exact sequence for the operator P and the associated causal propagator G^P allows to build up a symplectic space. We denote with

$$\mathcal{E}^P_{M,E} = \Gamma_c(M, E)/\mathrm{Im}\left(P|_{\Gamma_c(M,E)}\right) \tag{2.3.1}$$

the vector space associated to the triple $(\mathbf{M}, E, P)$ and with $[\varphi]$ the class of elements generated by $\varphi \in \Gamma_c(M, E)$. Hence, the symplectic form (2.1.18) can be rewritten in this context as

$$\tau^P_{M,E}([u], [v]) = \int_M \mathrm{dvol}\, \langle u, G^P v\rangle_E \tag{2.3.2}$$

Furthermore, we notice that the map ext, defined above, induces a linear map $\rho_{\chi,F}$: $\mathcal{E}^{P_1}_{M_1,E_1} \to \mathcal{E}^{P_2}_{M_2,E_2}$, $[\phi] \mapsto [\mathrm{ext}(\phi)]$. Hence, we are ready to prove the fundamental statement of this section on the functorial nature of the latter definitions.

Proposition 2.3.8. *Given the categories $\mathfrak{Ghs}$ and $\mathfrak{Symp}$, the association*

$$(\mathbf{M}, E, P) \mapsto \mathcal{E}^P_{M,E} \quad (\chi, F) \mapsto \rho_{\chi,F} \tag{2.3.3}$$

defines a functor $\mathcal{E} : \mathfrak{Ghs} \to \mathfrak{Symp}$.

Proof. The fact that $\mathcal{E}((\mathbf{M}, E, P)) = (\mathcal{E}^P_{M,E}, \tau^P_{M,E})$ is a symplectic vector space, i.e an object in $\mathfrak{Symp}$, has been analyzed in the previous chapter and nothing but the notation should be changed to present the general result. More interesting is the proof that, given two objects $(\mathbf{M}_1, E_1, P_1)$, $(\mathbf{M}_2, E_2, P_2)$ and the morphism (χ, F), the map

$$\mathcal{E}((\chi, F)) = \rho_{\chi,F} : \mathcal{E}^{P_1}_{M_1,E_1} \to \mathcal{E}^{P_2}_{M_2,E_2}$$

is a linear map preserving symplectic form in the sense of Definition 2.3.7. We first check the well-definiteness of the map $\rho_{\chi,F}$. Taking a class $[\varphi]$ and two elements in this class, φ and $\varphi + P_1 h$, we have $\rho_{\chi,F}([\varphi]) = [\text{ext}(\varphi)]$ and

$$\rho_{\chi,F}([\varphi]) = [\text{ext}(\varphi) + \text{ext}(P_1 h)] = [\text{ext}(\varphi) + P_2\text{ext}(h)] = [\text{ext}(\varphi)].$$

The proof that this map is a symplectomorphism is an application of Lemma 2.3.6. Indeed for $[u], [v] \in \mathcal{E}^{P_1}_{M_1,E_1}$, we get

$$\tau^{P_2}_{M_2,E_2}([\text{ext}(u)], [\text{ext}(v)]) = \int_{M_2} \text{dvol}_{M_2} \left\langle \text{ext}(u), G^{P_2} \circ \text{ext}(v) \right\rangle_{E_2} \tag{2.3.4}$$

but the formula for the change of variable for the integral of a section $h \in \Gamma(M_1, E_1)$, induced by the map $f : M_1 \to M_2$, says that

$$\int_{M_2} \text{dvol}_{M_2}\, h = \int_{M_1} \text{dvol}_{M_1}\, res(h).$$

Hence (2.3.4) becomes

$$\int_{M_1} \text{dvol}_{M_1} \left\langle u, \text{res} \circ G^{P_2} \circ \text{ext}(v) \right\rangle_{E_1} = \int_{M_1} \text{dvol}_{M_1} \left\langle u, G^{P_1} v \right\rangle_{E_1} = \tau^{P_1}_{M_1,E_1}([u], [v])$$

concluding the proof. □

In the previous section, we have found a technique for the construction of a $*$-algebra from a symplectic space. Here, in the form of a proposition we claim that this construction is functorial. Hence, we denote with $V_{\mathbb{C}}$ the complexification of the real symplectic vector space (V, τ), $\mathcal{T}^{\mathbb{C}}V = \bigoplus_{k\in\mathbb{N}_0} V_{\mathbb{C}}^{\otimes k}$ the algebra built up on the tensor product of V. As showed before, elements in this algebra can be interpreted as sequences of the form $\{v_n \in V_{\mathbb{C}}^{\otimes n}\}_{n\in\mathbb{N}_0}$ the operation of involution is given by involution for each element of the sequence, i.e. $\{v_n\}^* = \{(v_n)^*\}$, where the latter, for $v_n \in V_{\mathbb{C}}^{\otimes n}$, is defined as

$$(v_n)^* = (v_{i_1} \otimes \cdots \otimes v_{i_n})^* = (\overline{v_{i_n}} \otimes \cdots \otimes \overline{v_{i_1}})$$

where the bar indicates complex coniugation. In conclusion we define $\mathcal{J}_\tau^{Bos}$ as the ideal generated by the elements of the form $(u \otimes v - v \otimes u - i\hbar\tau(u,v))$, where the superscript *Bos* means bosonic. Hence, denoting with $[\{v_k\}]$ the equivalence classes in $\mathcal{T}^{\mathbb{C}}V/\mathcal{J}_\tau^{Bos}$. We are ready to prove the functoriality of what we have just presented.

Proposition 2.3.9. *Given an object in the category $\mathfrak{Symp}$ of symplectic vector spaces (V,τ), and an element $L \in \mathrm{Mor}_{Symp}((V_1,\tau_1),(V_2,\tau_2))$, then the association*

$$(V,\tau) \mapsto \mathscr{Q}^{Bos}((V,\tau)) := \mathcal{T}^{\mathbb{C}}V/\mathcal{J}_V^{Bos}, \tag{2.3.5}$$

together with the definition of the action on morphisms, for which given a symplectic map $L : (V_1,\tau_1) \to (V_2,\tau_2)$

$$L \mapsto \mathscr{Q}^{Bos}(L) : \mathscr{Q}^{Bos}((V_1,\tau_1)) \to \mathscr{Q}^{Bos}((V_2,\tau_2)) \tag{2.3.6}$$

$$[\{v_k\}] \mapsto [\{L(v_k)\}], \tag{2.3.7}$$

defines a functor $\mathscr{Q}^{Bos} : \mathfrak{Symp} \to \mathfrak{Alg}$.

Proof. The fact that $\mathscr{Q}^{Bos}(V,\tau)$ is a $*$-algebra has been deeply analyzed in the previous section, hence we just have to check that the action on symplectic maps gives well-defined $*$-morfisms. We first look at the involution operation. We recall that the application is build up on a real vector spaces and the action on the complexification of V is given by

$$\forall v \in V \lambda \in \mathbb{C}, L(v \otimes \lambda) := L(v) \otimes \lambda.$$

Hence, $L(\overline{v}) = \overline{L(v)}$ and we can deduce that $\mathscr{Q}^{Bos}(L)$ is at least an algebra homomorphism preserving the involution operation, indeed, for an element in $V_{\mathbb{C}}^{\otimes n}$ we get:

$$\mathscr{Q}^{Bos}(L)((v_1 \otimes \cdots \otimes v_n)^*) = L(\overline{v_n}) \otimes \cdots \otimes L(\overline{v_1}) \tag{2.3.8}$$

$$= \overline{L(v_n)} \otimes \cdots \otimes \overline{L(v_1)} \tag{2.3.9}$$

$$= (\mathscr{Q}(L)(v_1 \otimes \cdots \otimes v_n))^*. \tag{2.3.10}$$

In order to conclude the proof, the next step is to prove the preservation of the structure induced by the quotient. We should have that $\mathscr{Q}^{Bos}(L)(\mathcal{J}_V^{Bos}) = \mathcal{J}_{L(V)}^{Bos}$. The right hand side is generated by the elements of the form:

$$L(u) \otimes L(v) - L(v) \otimes L(u) - i\hbar\tau_2(L(u),L(v))) = L(u) \otimes L(v) - L(v) \otimes L(u) - i\hbar L_*\tau_2(u,v)$$

where in the last equality we used the definition of pull-back induced by L. Looking then at the left hand side, we get:

$$\mathscr{Q}(L)\,(u\otimes v - v\otimes u - i\hbar\tau_1(u,v)) = L(u)\otimes L(v) - L(v)\otimes L(u) - i\hbar\tau_1(u,v).$$

Hence we can conclude recalling that the map L, as morphism in the category $\mathfrak{Symp}$, preserves the symplectic form when acting through the pullback map, i.e. $L_*\tau_2(u,v) = \tau_1(u,v)$. The composition law trivially holds true. □

The latter proposition allows us to claim that the functor $\mathscr{A}^{Bos} = \mathscr{Q}^{Bos}\circ\mathcal{E}$ is a locally covariant quantum field theory and, moreover, it satisfies also the causality condition in the quantum formulation and the time slice axiom. We can formalise this fact with the next theorem, as presented in [BG12].

Theorem 2.3.10. *The functor $\mathscr{A}^{Bos} = \mathscr{Q}^{Bos}\circ\mathcal{E} : \mathfrak{Ghs}\to\mathfrak{Alg}$ is a bosonic locally covariant quantum field theory and the following axioms hold:*

(i) ***Causality*** *Let (M_j, E_j, P_j) be objects in $\mathfrak{Ghs}$, $j = 1,2,3$, and (χ_j, F_j) morphisms from (M_j, E_j, P_j) to (M_3, E_3, P_3), $j = 1,2$, such that $\chi_1(M_1)$ and $\chi_2(M_2)$ are causally disjoint regions in M_3. Then the subalgebras*

$$\mathscr{A}^{Bos}(\chi_1,F_1)(\mathscr{A}^{Bos}(M_1,E_1,P_1)),\ \mathscr{A}^{Bos}(\chi_2,F_2)(\mathscr{A}^{Bos}(M_2,E_2,P_2)) \subseteq \mathscr{A}^{Bos}(M_3,E_3,P_3),$$

commute.

(ii) ***Time slice axiom*** *Let (M_j, E_j, P_j) be objects in $\mathfrak{Ghs}$, $j = 1,2$, and (χ, F) a morphism from (M_1, E_1, P_1) to (M_2, E_2, P_2) such that there is a Cauchy hypersurface $\Sigma\subset M_1$ for which $f(\Sigma)$ is a Cauchy hypersurface of M_2. Then*

$$\mathscr{A}^{Bos}(\chi,F) : \mathscr{A}^{Bos}(M_1,E_1,P_1)\to\mathscr{A}^{Bos}(M_2,E_2,P_2)$$

is an isomorphism.

Now we can conclude this section, in which it has been shown how a bosonic LCQFT can be constructed in a very general context. Physically meaningful properties of the theory, such as isotony and causality are naturally encoded in this construction and time slice aciom is satisfied. Analogously, a recipe for the construction of the fermionic LCQFTs can be given with slight changes in the techniques used before. We don't want to present here an extended treatment of this topic, but we simply underline that, as expexted, the classical fermionic theory finds a natural framework in the category of Hilbert spaces, because the anti-commuting quantum relations can be intended as the transposition at the quantum level of an hermitian scalar product between fields. When dealing with scalar product, additional hypotesis are needed to guarantee the positive-definiteness. Indeed, working on fermionic field theories bilinear forms like (2.1.19) fail to be physically sensible for a general Green-hyperbolic operator P. The solution to this problem can be found restricting the construction of LCQFTs to those fermionic operators P, of first order, for which the pairing

$$(\psi_1, \psi_2) := \int_\Sigma \langle i\sigma_P(n^\flat)\psi_1|_\Sigma, \psi_2|_\Sigma\rangle_E \mathrm{dvol}_\Sigma$$

is a positive definite hermitian scalar product on $\mathrm{Sol}^P_{sc}(\mathbf{M}, E) = \{\psi \in \Gamma_{sc}(M, E)\, |P\psi = 0\}$. An extended treatment of fermionic field theories can be found in [BG12] and [San10].

Chapter 3

Supergeometry and LCQFT

In the previous chapters we have set the basis for understanding locally covariant quantum field theory, presenting the axiomatic formulation and constructing general models of bosonic field theories. Now, inspired by those results, we can proceed presenting the most recent developments concerning the generalization of the *general covariance scheme* to field theories defined on supermanifolds. For this scope, we start this third chapter with the presentation of the basic mathematical instruments for a minimal comprehension of the geometry of supermanifolds: a series of fundamental definition and proposition, from the concept of super vector spaces to more advanced notions such as that of Berezin integration and super differential forms. Afterwards in Section 3.2, in two different steps we show how super field theories can be defined rigorously. At first we select a class of suitable frameworks that are called *super-Cartan structures* (collected in the category $\mathfrak{ghsCart}$) to which it is possible to associate *canonically* an ordinary globally hyperbolic spacetime, then we proceed showing how a functor from (a full subcategory of) $\mathfrak{ghsCart}$ (generally denoted by $\mathfrak{sLoc}$) to the category of super $*$-algebras can be constructed once a *super-Green's hyperbolic operator* is given. Unfortunately, in this latter construction *supersymmetry transformations* fails to be coherently encoded and, hence, we sketch the idea of the solution adopted in [HHS16]. Finally, we recollect and apply all the results presenting explicit calculations for super field theory models on super-Cartan supermanifolds of dimension $2|2$, defining the category $2|2$-$\mathfrak{sLoc}$ and imposing restrictive conditions called *supergravity supertorsion constraints* (see [WZ77] and [How79]), giving a classification of *supersymmetry transformations* and showing explicitly their action on the superalgebra of super fields.

3.1 Basics in supergeometry

The exposition on this subject is entirely based on reference [CCF11] and many insights are taken from [HHS16]. We will denote with $\mathbb{K}$ any algebraic field used in the following, meaning either $\mathbb{R}$ or $\mathbb{C}$. The basic definition to start from is that of *super vector space*.

Definition 3.1.1. A super vector space is a vector space V obtained as the direct sum of two vector spaces V_0, V_1 where a $\mathbb{Z}_2$-parity is assigned to non-zero elements following the conventions:

$$|v| := \begin{cases} 0,\ v \in V_0 \\ 1,\ v \in V_1. \end{cases} \tag{3.1.1}$$

We define the superdimension of $V = V_0 \oplus V_1$ as the pair $(\dim(V_0), \dim(V_1))$ and we denote it with $\dim(V) = \dim(V_0)|\dim(V_1)$. Elements in V_0, resp. V_1, are called *even*, resp. *odd*.

Example 3.1.2. A first seminal example is that of $\mathbb{K}^{n|m} := \mathbb{K}^n \oplus \mathbb{K}^m$ where $n, m \in \mathbb{N}$ and where $(\mathbb{K}^n \oplus \mathbb{K}^m)_0 = \mathbb{K}^n$ and $(\mathbb{K}^n \oplus \mathbb{K}^m)_1 = \mathbb{K}^m$. Usually, the base algebraic field $\mathbb{K}$ is denoted by $\mathbb{K}^{1|0}$.

Once super vector spaces have been defined, one can look at the maps between such objects. among all the linear maps an interesting subset is composed by taking those preserving parity. Given two super vector spaces V, W, the parity preserving linear maps are all the linear maps $L : V \to W$ such that

$$L(V_i) \subseteq W_i \text{ for } i = 0, 1 \tag{3.1.2}$$

This condition is required to take care of the parity structure of a super vector space when it is mapped into another super vector space: for example, $\mathbb{R}^{n+m|0}$ and $\mathbb{R}^{n|m}$ cannot be isomorphic as super vector spaces if we take as morphisms only the maps preserving parity. After this consideration, we are ready for the definition of the category $\mathfrak{sVec}$

Definition 3.1.3. We define $\mathfrak{sVec}$ as the category whose objects are super vector spaces and, given two objects (V, W), $\mathrm{Mor}_{\mathfrak{sVec}}(V, W)$ is the set of all the linear maps between V and W preserving parity.

This definition is sensible and well posed, but one should note an issue due to the restriction of the space of linear maps to those preserving parity. Indeed, when one tries to define a dual of a super vector space, borrowing the definition from the category of ordinary vector spaces, the set

$$V^* := \{ \text{ all the morphism } L : V \to \mathbb{K}\}$$

turns out to have only an element (the zero-map) if V is a purely odd vector space (i.e. $V = \{0\} \oplus V_1$). This is because $\mathbb{K}$ being a purely even vector space, all non-zero maps from V to $\mathbb{K}$ reverse parity. We have seen this key example, but this issue shows up also when one tries to endow the set of morphism between two objects in the category $\mathfrak{sVec}$ with a proper structure of super vector space. Indeed, whereas the set $\mathrm{Mor}_{\mathfrak{Vec}}(A, B)$, where A, B are ordinary vector spaces, is a vector space, an analogous condition fails to be true when A, B are super vector spaces. A solution for this problem is to look at the *internal morphisms* of the category (see Definition A.8 and the relative introductory discussion in the Appendix). Hence, we have that $\underline{\mathrm{Mor}}(V, W)$ (where we omitted the subscript $\mathfrak{sVec}$ to simplify the notation) consists of all the linear maps between V and W, and is the direct sum of

$$\begin{aligned}
(\underline{\mathrm{Mor}}(V, W))_0 &= \{L : V \to W |\text{ L preserves parity }\}\\
(\underline{\mathrm{Mor}}(V, W))_1 &= \{L : V \to W |\text{ L reverses parity }\}
\end{aligned}$$

and the dual is taken to be $V^* := \underline{\mathrm{Mor}}(V, \mathbb{K})$. We can also define a tensor product between super vector spaces, built up on the same algebraic field $\mathbb{K}$, assigning parity this way:

$$\begin{aligned}
(V \otimes W)_0 &:= (V_0 \otimes W_0) \oplus (V_1 \otimes W_1)\\
(V \otimes W)_1 &:= (V_0 \otimes W_1) \oplus (V_1 \otimes W_0).
\end{aligned}$$

These latest facts can be simply summarised saying that $\mathfrak{sVec}$ is a tensor category with internal morphism. In this context, we introduce the commutativity map, i.e. an isomorphism between $V \otimes W$ and $W \otimes V$, taking into account the graded structure (needed for all the physical developments of the theory):

$$\begin{aligned}
c_{v,w} :& V \otimes W \to W \otimes V\\
& v \otimes w \mapsto (-1)^{|v||w|} w \otimes v
\end{aligned} \tag{3.1.3}$$

that makes sense also when w and v are elements of not definite parity. In that case the product $v \otimes w$ shall be decomposed as sum of parity-defined elements, then transformed and recomposed. As usual, even thought super, a vector space structure is not enough to get something very interesting. The next step towards physical application is the definition of superalgebras and the category in which they appear as objects $\mathfrak{sAlg}$.

Definition 3.1.4. A unital superalgebra is a super vector space endowed with two morphisms $\mu_A : A \otimes A \to A$ and $\eta_A : \mathbb{K} \to A$ such that the following diagrams commute

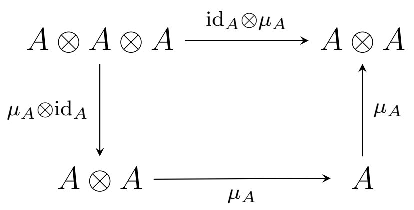


Diagram 3.1.1

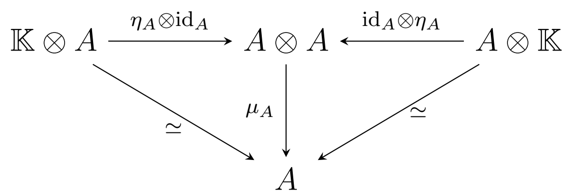


Diagram 3.1.2

Superalgebras are the objects of the category $\mathfrak{sAlg}$, in which morphisms are $\mathfrak{sVec}$-morphism preserving the products and unit.

Multiplication and unit are often denoted shortly with $\mu_A(a_1 \otimes a_2) = a_1 a_2$ and $\eta_A(1) = \mathbb{1}_A$. Also the category $\mathfrak{sAlg}$ can be seen as a tensor category if the tensor product is endowed with a suitable multiplication function:

$$\mu_{A\otimes B} := (\mu_A \otimes \mu_B) \circ (\mathrm{id}_A \otimes c_{B,A} \otimes \mathrm{id}_B)$$
$$\eta_{A\otimes B} := \eta_A \otimes \eta_B$$

that with the short notation introduced before becomes

$$\mu_{A\otimes B}(a_1 \otimes b_1, a_2 \otimes b_2) = (a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|a_1||a_2|} a_1 a_2 \otimes b_1 b_2$$
$$\mathbb{1}_{A\otimes B} = \mathbb{1}_A \otimes \mathbb{1}_B$$

Among all the superalgebras, we restrict our analysis to the subclass of the supercommutative superalgebras, i.e. those A for which the multiplication map μ_A satisfies $\mu_A \circ c_{A,A} = \mu_A$.

Example 3.1.5. A first fundamental example of supercommutative superalgebra is the free algebra generated over a field $\mathbb{K}$ by the even elements $\{t_i\}_{i=1,\dots,p}$ and the odd elements $\{\theta_j\}_{j=1,\dots,q}$, the latter called Grassmann variables, such that:

$$\theta_i \theta_j = -\theta_j \theta_i$$

for all i, j and hence $\theta_i^2 = 0 \forall i$. Formally, we can define the algebra $A = \mathbb{K}[t_1, \ldots, t_p] \otimes \Lambda(\theta_1, \ldots, \theta_q)$ where the second factor of the tensor product in the exterior algebra generated by $\theta_1, \ldots, \theta_q$. Elements of this algebra, using the multi-index notation, can be written as

$$f_0 + \sum_{k=1}^{q} \sum_{\{I_k\}} f_{I_k} \theta_{I_k}$$

where the second sum is performed over the set of multi-indexes $\{I_k\}$ composed by all the $k$thuple it is possible to build up with numbers from 1 to q, monotonically increasing, $f_0, f_{I_k} \in \mathbb{K}[t_1, \ldots, t_p]$ and the symbol θ_{I_k} denotes the product of Grassmann variables ruled by the multi-index I_k. Let's fix $p = q = 2$ to see how the notation and the product work. For $k = 1$ we have two multi-index (despite this time they are single indexes), $\{I_1\} = \{1, 2\}$, and for $k = 2$, $\{I_2\} = \{12\}$. Hence, a generic element takes the form:

$$f_0 + f_1\theta_1 + f_2\theta_2 + f_{12}\theta_1\theta_2$$

where $\theta_1\theta_2$ can be indifferently written as θ_{12}. We take now

$$\begin{aligned} a &= \alpha_0^1 t_1 + \alpha_0^2 t_2 + (\alpha_1^1 t_1 + \alpha_1^2 t_2)\theta_1 + (\alpha_2^1 t_1 + \alpha_2^2 t_2)\theta_2 + (\alpha_{12}^1 t_1 + \alpha_{12}^2 t_2)\theta_{12} \\ b &= \beta_0^1 t_1 + \beta_0^2 t_2 + (\beta_1^1 t_1 + \beta_1^2 t_2)\theta_1 + (\beta_2^1 t_1 + \beta_2^2 t_2)\theta_2 + (\beta_{12}^1 t_1 + \beta_{12}^2 t_2)\theta_{12} \end{aligned}$$

where $\theta_{12} = \theta_1\theta_2$. We can split each element of the superalgebra in even and odd part and then compute the product, for the odd part we have $a = (\alpha_1^1 t_1 + \alpha_1^2 t_2)\theta_1 + (\alpha_2^1 t_1 + \alpha_2^2 t_2)\theta_2$ and $b = (\beta_1^1 t_1 + \beta_1^2 t_2)\theta_1 + (\beta_2^1 t_1 + \beta_2^2 t_2)\theta_2$, together with the product:

$$a \cdot b = (\alpha_1^1 t_1 + \alpha_1^2 t_2)(\beta_2^1 t_1 + \beta_2^2 t_2)\theta_1\theta_2 + (\alpha_2^1 t_1 + \alpha_2^2 t_2)(\beta_1^1 t_1 + \beta_1^2 t_2)\theta_2\theta_1 = -b \cdot a$$

in accord with the parities $|a| = |b| = 1$ and $(-1)^{|a||b|} = -1$. Taking the product for a generic split element, one can easily see that we have just defined a supercommutative superalgebra

As in the ordinary quantum field theory, an operation of involution defined over the elements of the algebra gives the right basis for the construction of the quantum theory. Hence, we can endow a superalgebra with a *superinvolution* $*_A : A \to A$ satisfactory $*_A \circ \eta_A = \eta_A$ and $*_A \circ \mu_A = \mu_A \circ c_{A,A} \circ (*_A \otimes *_A)$, that may look quite cumbersome but its actually very intuitive when written as

$$(a_1 a_2)^* = (-1)^{|a_1||a_2|} {a_2}^* {a_1}^*$$

for two elements of definite parity a_1, a_2 and $(\mathbb{1}_A)^* = \mathbb{1}_A$. Extending trivially the superinvolution to the tensor product we can define the tensor category $*$-$\mathfrak{sAlg}$ in analogy with Definition 3.1.4. Another key example of superalgebra, following the example of ordinary geometry, is the tensor superalgebra built up on a super vector space. We define

$$\mathrm{T}(V) = \bigoplus_n V^{\otimes n} \quad \text{where } V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n\text{times}}$$

for which $(\mathrm{T}(V))_0 = \bigoplus_{n=\text{even}} V^{\otimes n}$ and $(\mathrm{T}(V))_1 = \bigoplus_{n=\text{odd}} V^{\otimes n}$, with the product defined as usual with the map $\phi_{r,s} : V^{\otimes r} \times V^{\otimes s} \to V^{\otimes r+s}$

$$\phi_{r,s}(v_{i_1} \otimes \dots v_{i_r}, w_{i_1} \otimes \dots w_{i_s}) = v_{i_1} \otimes \dots v_{i_r} \otimes w_{i_1} \otimes \dots w_{i_s}$$

We net define the concept of *super Lie-algebra*.

Definition 3.1.6. A super-Lie algebra is a super vector space $\mathfrak{v}$ together with a morphism $[\cdot,\cdot] : \mathfrak{v} \otimes \mathfrak{v} \to \mathfrak{v}$, called *super-Lie bracket*, satisfying:

i) $[X, Y] + (-1)^{|X||Y|} [Y, X]$;

ii) $[X, [Y, Z]] + (-1)^{|X||Y|+|X||Z|} [Y, [Z, X]] + (-1)^{|Z||Y|+|X||Z|} [Z, [X, Y]] = 0$

for $X, Y, Z \in \mathfrak{v}$. Equivalently the two conditions can be rewritten using the commutation morphism in the category $\mathfrak{sVec}$:

i) $[\cdot,\cdot](\mathrm{id}_v \otimes \mathrm{id}_v + c_{\mathfrak{v},\mathfrak{v}}) = 0$

ii) $[\cdot,\cdot](\mathrm{id}_v \otimes \mathrm{id}_v + c_{\mathfrak{v},\mathfrak{v}\otimes\mathfrak{v}} + c_{\mathfrak{v}\otimes\mathfrak{v},\mathfrak{v}}) = 0$

A morphism between super-Lie algebras is a linear map $L : \mathfrak{v} \to \mathfrak{v}'$ preserving the brackets, i.e. such that

$$[\cdot,\cdot]_{\mathfrak{v}'} \circ (L \otimes L) = L \circ [\cdot,\cdot]_{\mathfrak{v}} \tag{3.1.4}$$

These last constructions seems to be enough rich and complicate to build up a quantum theory of superfield in analogy with the ordinary case. But the goal of this section is not only to understand super vector spaces and superalgebras but to present the definition and then manage with supermanifolds. Before introducing them and the categorical structure with which their class can be endowed, we shall pass thorough *supermodules* defined on a superalgebra. This notion will be useful in defining some other supergeometric object.

Definition 3.1.7. Given a super algebra A, a left A-supermodule is a super vector space V together with a left A-action $l_V : A \otimes V \to V$ induced by A, such that the diagrams commute

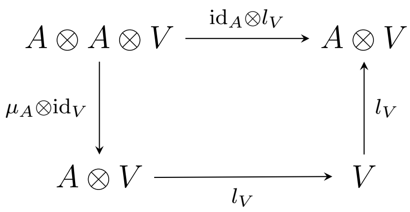


Diagram 3.1.3

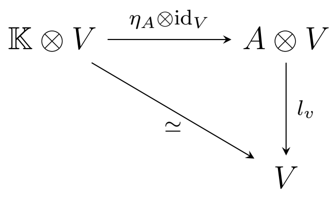


Diagram 3.1.4

Analogous definition can be provided, *mutatis mutandis*, for *right A-supermodules*, and *A-bisupermodules*, that are both right and left A-supermodules.

Let us first notice that if A is a supercommutative superalgebra then every left supermodules is also a right supermodules and that the abstract presentation of the Diagrams 3.1.3 and 3.1.4can be expanded writing the left action explicitly, for generic $a, b \in A$ and $x, y \in V$:

1. $a(x + y) = ax + ay$
2. $(a + b)x = ax + bx$
3. $(ab)x = a(bx)$
4. $\mathbb{1}_A x = x$

and this is useful to select the class of interesting morphism between supermodules, that are those $\phi : V \to W$ preserving parity and being linear with respect to the A-action, i.e. $\phi(ax) = a\phi(x), \forall a \in A$ and $\forall x \in V$, which formally shall be written as $\phi \circ l_V = l_W \circ (\mathrm{id}_A \otimes \phi)$, meaning that we ask the diagram to commute:

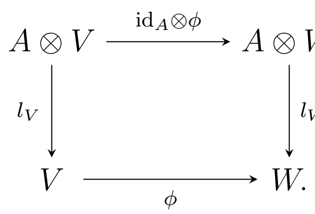


Diagram 3.1.5

Supermodules over the same supercommutative superalgebra A can be organized in a category, called $\mathfrak{sMod}$, endowed with a tensor product and internal morphism (cfr. [CCF11][pag.9]). At this point a legitimate question is whether is possible to recover the language of matrices and what meaning can be assessed to usual tools like determinant and trace. The first step in this direction is that of selecting a useful class of supemodules.

Example 3.1.8. Given a supercommutative superalgebra A, we define the tensor product $A^{p|q} = A \otimes \mathbb{K}^{p|q}$ where

$$(A^{p|q} = A \otimes \mathbb{K}^{p|q})_0 = A_0 \otimes \left(\mathbb{K}^{p|q}\right)_0 \oplus A_1 \otimes \left(\mathbb{K}^{p|q}\right)_1 \tag{3.1.5}$$

$$(A^{p|q} = A \otimes \mathbb{K}^{p|q})_1 = A_1 \otimes \left(\mathbb{K}^{p|q}\right)_0 \oplus A_0 \otimes \left(\mathbb{K}^{p|q}\right)_1 \tag{3.1.6}$$

In particular, what we are interested in is the class of all supermodules isomorphic to $A^{p|q}$, i.e. those super vector spaces M, with p elements in M_0, $\{\mathbf{e}_1, \dots, \mathbf{e}_p\}$, and q elements in M_1, $\{\epsilon_1, \dots, \epsilon_q\}$, such that

$$M_0 = \mathrm{span}_{A_0}\{\mathbf{e}_1, \dots, \mathbf{e}_p\} + \mathrm{span}_{A_1}\{\epsilon_1, \dots, \epsilon_q\} \tag{3.1.7}$$

$$M_1 = \mathrm{span}_{A_1}\{\mathbf{e}_1, \dots, \mathbf{e}_p\} + \mathrm{span}_{A_0}\{\epsilon_1, \dots, \epsilon_q\} \tag{3.1.8}$$

In the literature often refers to this costruction with the name of *free super module generated over* A by $\{\mathbf{e}_1, \dots, \mathbf{e}_p\}$ and $\{\epsilon_1, \dots, \epsilon_q\}$

Now we are ready to understand some interesting featuures of a matrix in a supergeometric context. Let's take a map L on the left free supermodule as in Example 3.1.8 built up on the same superalgebra A. Its action is fully determined by the action on the basis elements, i.e.

$$L(\mathbf{e}_h) = \sum_{i=1}^{p} t^i_h \mathbf{e}_i + \sum_{j=1}^{q} u^j_h \epsilon_j$$

$$L(\varepsilon_k) = \sum_{i=1}^{p} v^i_k \mathbf{e}_i + \sum_{j=1}^{q} w^j_k \epsilon_j$$

where additional attention shall be applied multiplying from the left (right) with coefficients defining L if the modules are intended as left (right) A-supermodules. The notation above can be summarised in the block matrix form

$$\begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} \tag{3.1.9}$$

where L_1 and L_4 are the even blocks, meaning that they preserve parity, built on the even elements $t^i_h, w^j_k \in A$, whereas L_2 and L_3 are the odd blocks containing the odd elements $u^j_h, v^j_k \in A$. Using the matrix representing L and another matrix representing on the same basis an application S, the action on a vector $x \in M$ can be written as the ordinary matrix product $L(x) := L \cdot x$ as well as the composition $S \cdot L$, taking care of the meaning of the multiplication when dealing with right rather than left A-supermodules.

Once the matrix notation has been recovered, the first natural attempt is to define two objects working with the same aim of the trace and the determinant. At this point, one should ask which properties of trace and determinant are interesting for the ordinary context and what of those properties is important to reproduce to develop the supergeometric theory. From the block form (3.1.9) for L one is tempted to define the supertrace as the sum of the ordinary trace of L_1, L_2, i.e. $\mathrm{tr}(L_1) + \mathrm{tr}(L_4)$, but this definitions fails to be independent from the choice of a basis for M. A good definition in order to preserve basis independence is

$$\mathrm{str}(L) := \mathrm{tr}(L_1) - \mathrm{tr}(L_4), \tag{3.1.10}$$

We don't give a proof here, since it is not needed for our purposes, but it can be found in more extended expositions such as [Del+99]. With the definition of supertrace in mind we can get some motivations for the definition of the concept of determinant updated to supermodules map. One of the most relevant properties of the determinant (even though it is not part of the axioms for its definition) is the so called *Binet rule*, i.e. $\det(AB) = \det(A)\det(B)$. Looking for something similar, one defines the *Berezinian* (named after Felix Berezin) by

$$\mathrm{Ber}(L) := \det(L_1 - L_2 L_4^{-1} L_3)\,(\det(L_4))^{-1} \tag{3.1.11}$$

where, for this definition we ask L_4 to be invertible. Actually, another formulation for $\mathrm{Ber}(L)$ can be provided, that is

$$\mathrm{Ber}(L) := \det(L_4 - L_3 L_1^{-1} L_2)\,(\det(L_1))^{-1} \tag{3.1.12}$$

that can be used requiring only the L_1 block to be invertible. As expected, when both L_1 and L_4 are invertible the two formulas coincide. Using one of these formulas the following rule holds true

$$\mathrm{Ber}(ST) = \mathrm{Ber}(S)\mathrm{Ber}(T).$$

We refer to [CCF11, pg.15] for the proof of the latter and for a deeper discussion of the properties of the Berezinian, as for example the well-know relation

$$\mathrm{Ber}(e^X) = e^{\mathrm{str}(X)}$$

of which we cannot give a detailed proof because the definition of exponential map in supergeometric context involve complicate instruments and techniques. Actually, the Berezinian can be seen as a group homomorphism from the group of A-automorphism of the supermodule of the matrices acting over M $(GL(M))$ and the invertible elements of A (denoted by $A_o^\times$).

3.1.1 Supermanifolds

The definition of these mathematical objects pass through concepts whose natural framework is that of the sheaf theory. In order to understand better the detail of some peculiar notions of supergeometry, we will try to give a brief exposition on ordinary geometry using a more abstract language. Indeed, understanding manifolds with the insight given by sheaf theory will make natural to proceed with the generalized definition of supermanifold. We start defining presheaves on a fixed topological space X. We take the category $\mathfrak{openTop}(X)$, where the objects are the opens sets $U \subseteq X$ and morphisms are the inclusions $i : U \to V$, for $U \subseteq V \subseteq X$. Then the notion of presheaf is easily given.

Definition 3.1.9. Considering a topological space X, the category $\mathfrak{openTop}(X)$ and a target category $\mathfrak{C}$ we define a presheaf as a functor $\mathscr{F} : \mathfrak{openTop}^{op}(X) \to \mathfrak{C}$.

In order to understand why such a definition turns out to be useful, we fix a topological space X and we focus our attention on all the open sets $U \subseteq X$. A presheaf assigns to each open set U an object $\mathscr{F}(U)$ in the category $\mathfrak{C}$. But as we know, a functor is more than an association of objects. Given, indeed, an open set V, such that $U \subseteq V$, we can take the inclusion $i_{U,V} : U \to V$, obtaining the morphism $\mathscr{F}(i_{U,V}) : \mathscr{F}(V) \to \mathscr{F}(U)$, usually denoted by $\mathrm{res}_{V,U}$, associating to an element $s \in \mathscr{F}(V)$ the element $\mathrm{res}_{V,U} := s|_U \in \mathscr{F}(U)$. Moreover, is important to underline that the composition rule for the functor entails

$$\mathscr{F}(i_{U,V}) \circ \mathscr{F}(i_{V,W}) = \mathscr{F}(i_{V,W} \circ i_{U,V}) = \mathscr{F}(i_{U,W}) \text{ or } \mathrm{res}_{W,U} = \mathrm{res}_{V,U} \circ \mathrm{res}_{W,V}$$

for $U \subseteq V \subseteq W$. A sheaf is a presheaf with two additional properties. Given an open set U and an open covering $\{U_i\}_{i\in I}$, then

i. given two elements $s, t \in \mathscr{F}(U)$ s.t. $s|_{U_i} = t|_{U_i}, \forall i \in I$, then $s = t$.

ii. Given a family $\{f_i\}_{i\in I}$, with $f_i \in \mathscr{F}(U_i), \forall i$, such that $f_i|_{U_i \cap U_j} = f_j|_{U_i\cap U_j}, \forall i, j \in I$, then exists unique f such that $f|_{U_i} = f_i, \forall i \in I$

As the notation (res) for the morphism in the target category suggests, sheaves have the principal aim of generalizing the concept of section over a manifold, and the morphism res take the role of the restrictions in the target category $\mathfrak{C}$. Hence, in order to make more concrete the definition above we show how it may look familiar. We take a differentiable manifold M, and identify the underlying topological space with X in Definition 3.1.9. Then, for every open set $U \subseteq M$, let $C^\infty(U)$ be the algebra of smooth functions. Given two open sets $U \subseteq V$, and $f \in C^\infty(V)$, then $\mathrm{res}_{V,U}(f) = f|_U$ is the usual restriction verifying all the requirements to be sheaf. We can give here a first example of map between sheaves and then provide a general definition. When dealing with two differentiable manifolds and a smooth map $\chi : M \to N$, the pullback induced by χ is a map $\chi^* : C^\infty_N(U) \to C^\infty_M(f^{-1}(U))$, assigned to each open set U, such that $\chi^*(f)(x) = f \circ \chi(x)$. This construction can be seen in the categorical formalism, noting that the association $U \mapsto C^\infty(U)$ is a functor from the category $\mathfrak{openTop}$ to the category $\mathfrak{alg}$, denoted by $C^\infty : \mathfrak{openTop}^{\mathrm{op}} \to \mathfrak{alg}$. We can get a bit more from this point of view, indeed we know that given two differentiable manifolds M, M' and an open embedding $\phi : M \to M'$, we can define a functor $\mathscr{F}_\phi : \mathfrak{openTop}(\phi(M)) \to \mathfrak{openTop}(M)$, such that $U' \mapsto \mathscr{F}_\phi(U') := \phi^{-1}(U')$, that transforms morphisms in $\mathfrak{openTop}(\phi(M))$, (a full subcategory of $\mathfrak{openTop}(M')$) as showed in the following diagram

$$\begin{array}{ccc} U' & \xrightarrow{\;i_{U',V'}\;} & V' \\ \phi^{-1}\big\downarrow & & \big\downarrow\phi^{-1} \\ \phi^{-1}(U') & \xrightarrow[i_{\phi^{-1}(U'),\phi^{-1}(V')}]{} & \phi^{-1}(V'). \end{array}$$

Diagram 3.1.6

for all $U', V' \subseteq M'$. Meaning that the nets of open sets can be transposed from a manifold M' to an embedded manifold M, seen as submanifold of M'. With this in mind, we recall that an embedding and its restriction to open sets naturally induces a map between the space of smooth functions associated to its domain and codomain: given $\phi : M \to M'$ we look at the pullback $\phi^* : C^\infty(M') \to C^\infty(M)$, $f \mapsto f \circ \phi$.

Applying this to the net of open sets of the categories $\mathfrak{openTop}(M')$ and $\mathfrak{openTop}(M)$, we derive, together with a rule to go from a net to the other, an association, given by ϕ^*, connecting the sheaves built up on these nets, respecting compatibility with morphisms (restrictions) in $\mathfrak{openTop}(M')$ and $\mathfrak{openTop}(M)$. We mean compatibility in the sense that the following diagram

$$\begin{array}{ccc} C^\infty(V') & \xrightarrow{\ \phi^*\ } & C^\infty(\phi^{-1}(V')) \\ {\scriptstyle \mathrm{res}_{V',U'}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{res}_{\phi^{-1}(V'),\phi^{-1}(U')}} \\ C^\infty(U') & \xrightarrow[\ \phi^*\ ]{} & C^\infty(\phi^{-1}(U')). \end{array}$$

Diagram 3.1.7

commutes for all U', V' and $i_{U',V'}$ in $\mathfrak{openTop}(M')$. This fact can be trivially proven, once $f \in C^\infty(V')$ is given and both paths of Diagram 3.1.7 are explicitly calculated. Denoting $\mathrm{res}_{V',U'}(f) = f|_{U'} \in C^\infty(U')$, holds:

$$\phi^* \circ \mathrm{res}_{V',U'}(f) = \phi^*(f|_{U'}) = f|_{U'} \circ \phi$$

that coincides with

$$\mathrm{res}_{\phi^{-1}(V'),\phi^{-1}(U')} \circ \phi^*(f) = f \circ \phi|_{\phi^{-1}(U')}$$

because the embedding ϕ is invertible in the image and then $\phi \circ \phi^{-1}(U') = U'$, then restricting $f \circ \phi$ to $\phi^{-1}(U')$ in M', is the same operation that restricting $f|_{U'}$ and composing it with ϕ.

Now, it has been clarified how an ordinary manifold can be seen as a sheaf of commuting algebras over a topological space. But, with a common scheme in the development of new mathematics and physics, commuting objects produce a too restrictive formalism and hence the next step towards the definition of supermanifolds is the trivial substitution of the sheaf of commuting algebras with a sheaf of supercommuting superalgebras. Starting from the easiest case - using as underlying topological space $\mathbb{R}^n$ - this new sheaf is defined as

$$U \mapsto C^\infty(U) \otimes \Lambda(\mathbb{R}^m). \tag{3.1.13}$$

Sometimes, $C^\infty(U) \otimes \Lambda(\mathbb{R}^m)$ is denoted by $C^\infty_{\mathbb{R}^n}\big|_U \otimes \Lambda(\mathbb{R}^m)$, meaning that the assignment

(3.1.13) can be obtained taking the restriction to U of the element of the global sections $C^\infty(\mathbb{R}^n) := C^\infty_{\mathbb{R}^n}$. This is the prototype of a supermanifold, and we have to give sense to the usual statement 'locally isomorphic to' in order to compare a generic association of a supercommuting sheaf to a topological space, with $C^\infty_{\mathbb{R}^n} \otimes \Lambda(\mathbb{R}^m)$. In order to formalise this idea, we give the auxiliary notion of *superspace* as presented in [HHS16, cf. pg. 8].

Definition 3.1.10. A superspace is a pair $M := (\widetilde{M}, \mathcal{O}_M)$, where $\widetilde{M}$ is a second countable and Hausdorff topological space and $\mathcal{O}_M$ is a sheaf of supercommutative superalgebras on $\widetilde{M}$. $\mathcal{O}_M$ is called the *structure sheaf* of the superspace M and the space of global sections of the structure sheaf $\mathcal{O}_M(\widetilde{M})$ is denoted by $\mathcal{O}(M)$.

A morphism between superspaces shall keep track of the additional structure introduced by the structure sheaf, recalling the construction summarized in the Diagram 3.1.7 for ordinary manifolds. The technical difference stands in the fact that, in more general situation, once a map between two topological space is given, a function between structure sheaves is not automatically defined. Hence, a morphism between superspaces is a datum of a pair of maps such that a compatibility condition with the restriction morphisms holds true.

Definition 3.1.11. A superspace morphism, between two superspaces M and N, is a pair $\chi := (\widetilde{\chi}, \chi^*)$, where $\widetilde{\chi} : \widetilde{M} \to \widetilde{N}$ and χ^* is a family of morphism $\{\chi_U\}_{U \subseteq M}$, $\chi^*_U : \mathcal{O}_{M'}(U) \to \mathcal{O}_M(\widetilde{\chi}^{-1}(U))$ such that the diagram

$$\begin{array}{ccc} \mathcal{O}_{M'}(V) & \xrightarrow{\chi^*_V} & \mathcal{O}_{M'}(\widetilde{\chi}^{-1}(V)) \\ {\scriptstyle \mathrm{res}_{V,U}} \downarrow & & \downarrow {\scriptstyle \mathrm{res}_{\widetilde{\chi}^{-1}(V), \widetilde{\chi}^{-1}(U)}} \\ \mathcal{O}_{M'}(U) & \xrightarrow[\chi^*_U]{} & \mathcal{O}_{M'}(\widetilde{\chi}^{-1}(U)). \end{array}$$

Diagram 3.1.8

commutes, for all $U \subseteq V \subseteq \widetilde{M}'$.

We can now define supermanifolds as superspaces modeled on the superspace $\mathbb{R}^{n|m} := \{\mathbb{R}^n, C^\infty_{\mathbb{R}^n} \otimes \Lambda(\mathbb{R}^m)\}$ defined above.

Definition 3.1.12. A supermanifold of dimension $n|m$ is a superspace $M = (\widetilde{M}, \mathcal{O}_M)$ such that for each $p \in \widetilde{M}$, $\exists V \subseteq \widetilde{M}$ neighbourhood of p and a homeomorphism of an open set $U \subseteq \mathbb{R}^n$ into V, $\varphi_U : U \to V \subseteq \mathbb{R}^n$ and an isomorphism of superalgebras

$\varphi^* : \mathcal{O}_M(V) \to C^\infty(U)\otimes\Lambda(\mathbb{R}^m)$. Morphism between supermanifolds are morphism between superspaces.

Looking at the definition of supermanifolds, the careful reader should notice that - because the structure sheaf defining a supermanifold consists of a tensor product one factor of which is the sheaf of smooth functions - once a supermanifold is given, there exists a naturally induced structure of ordinary manifold on the underlying topological space $\widetilde{M}$. And this is exactly the case. In the immediate following we are going to show how and in what sense we can relate an ordinary manifold to a given supermanifold. First we notice that looking at a section $f \in \mathcal{O}_M(V)$ we need a notion of value of the function at a pair $x \in V$. Can be proven (cf.[CCF11, Lemma 4.1.6]) that writing f as

$$f_0(x) + \sum_{k=1}^{m} \sum_{I_k} f_{I_k}(x)\theta_{I_k},$$

where $f_0(x), f_{I_k}(x)$ are elements in $C^\infty(V)$ (looking at Example 3.1.5 we simply replace the module $\mathbb{K}[t_1, \dots, t_n]$ with $C^\infty(V)$), $f_0(x)$ is the value of f at x, in the sense that the element $(f - f_0)(x) \in \mathcal{O}_M(V)$ is not invertible in any neighbourhood of x. Some statements in the immediate following are easier to be presented in terms of functors and categories, hence before proceeding let's quickly fix some facts concerning the categorical structure over the class of supermanifolds.

Definition 3.1.13. Let us denote by $\mathfrak{sMan}$ the category whose objects are supermanifolds and morphisms are superspace morphisms between supermanifolds. This category is a monoidal category with respect to a cartesian product defined taking as base topological space the cartesian product of the topological spaces, endowed with a sheaf assigned to rectangular open subsets $U \times V \subseteq \widetilde{M} \times \widetilde{M}'$, as the completed tensor product $\mathcal{O}_M\widehat{\otimes}\mathcal{O}_{M'}$. The unit object is the pair $\mathrm{pt} := (\{\cdot\}, \mathbb{R})$

An interesting results of characterization of the structure of a supermanifold says that a morphism between supermanifolds is already uniquely specified by the morphism between supercommutative superalgebras $\chi^* : \mathcal{O}_N(N) \to \mathcal{O}_M(M)$, defined as global sections of a structure sheaf.

Theorem 3.1.14. *The functor $\mathscr{F} : \mathfrak{sMan} \to \mathfrak{sAlg}$, assigning to each supermanifold M, the supercommutative superalgebra $\mathcal{O}(M)$ and to each morphism $(\widetilde{\phi}, \phi^*)$ the superalgebra map ϕ^*_M is a full and faithful functor.*

The proof of this statement can be found in the book [CCF11, Prop. 4.6.2]. The main point of the connections between the supermanifold structure and the ordinary properties we can associate to a supermanifold is maybe the possibility of associating uniquely and canonically an ordinary manifold to a supermanifold. As it's clear, this fact is interesting for all the applications to quantum field theories, because it allows to recover most of the formalism, ideas and results developed in the formulation of axiomatic QFT.

Dealing with (super)algebras a remarkable operation is that of taking quotients by ideals of the (super)algebra, bringing to the construction of new structures. Hence on a supermanifold we can define the relevant ideal sheaf

$$U \mapsto J_M(U) := \{ f \in \mathcal{O}_M(U) |\, f \text{ is nilpotent } \}.$$

Remark 3.1.15. We can easy characterize elements of this set as f such that $f_0(x) \equiv 0$. Indeed, we know that locally an element in $\mathcal{O}_M(U)$, can be represented as a $f(x) = f_0(x) + \sum_I f_I \theta^I$, hence if exists l such that $f^l = 0 \Rightarrow f_0^l(x) = 0 \Rightarrow f_0(x) = 0$; while the reverse is true because if every element of the sum comes "multiplied" by a Grassman coordinate (θ^j), then there exists certainly an integer number for which each θ^j comes twice in the product.

After this characterization of the elements in $J_M(U)$, one could notice that quotienting out the nilpotent elements give a direct relation with the set of ordinary smooth functions defined on the base manifold. Unfortunately, the quotient involving sheaves is not a painless operation: the natural assignment $U \mapsto \mathcal{O}_M(U)/J_M(U)$, in general, does not define a sheaf but a presheaf.

Definition 3.1.16. Let $M = (\widetilde{M}, \mathcal{O}_M)$ be a supermanifold, then we define the reduced manifold as the $n|0$-dimensional submanifold $(\widetilde{M}, \mathcal{O}_M/J_M)$, where the assignment $U \mapsto \mathcal{O}_M(U)/J_M(U) =: (\mathcal{O}_M/J_M)(U)$ defines the structure sheaf

Remark 3.1.17. $(\widetilde{M}, \mathcal{O}_M/J_M)$ is an object in the category of supermanifolds and exist a natural morphism from $(\mathrm{id}_{\widetilde{M}}, j^*) : (\widetilde{M}, \mathcal{O}_M) \to (\widetilde{M}, \mathcal{O}_M/J_M)$, where the j^*_U are the projection maps into $(\mathcal{O}_M/J_M)(U)$ associating $f \in \mathcal{O}(M)$ to the ordinary differentiable function on M, $(j^* f)(x) = f_0(x)$, (once f is expanded in local coordinates as above). We are not going to present here the proof of this claim, but we refer to [CCF11, ch. 3] for a detailed exposition. What is remarkable for the application to the following is that this functorial association exists and how it can be done. Formally, given any morphism

$\chi = (\widetilde{\chi}, \chi^*)$, then $\widetilde{\chi}$ is a morphism in $\mathfrak{Man}$. Moreover, the map j^* induces uniquely a map $j : \widetilde{M} \to M$ such that the following diagram commutes

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\ \widetilde{\chi}\ } & \widetilde{M'} \\ {\scriptstyle j_{\widetilde{M},M'}}\big\downarrow & & \big\downarrow{\scriptstyle j_{\widetilde{M},M'}} \\ M & \xrightarrow[\ \chi\ ]{} & M'. \end{array}$$

Diagram 3.1.9

Once the basic notions of supergeometry are fixed, different tools and structures defined over supermanifold should be studied. In the immediate following we will give meaning to notions such those of super vector fields, superderivations, superdifferential forms as the fundamental blocks for implementing differential equations on supermanifolds, and then to develop quantum field theories in the supergeometrical context. In ordinary geometry, studying analogous concepts the easiest scheme starts with the definition of tangent space at a point $x \in M$ (where M is a manifold) and continues defining vector fields as section of the tangent bundle, the latter obtained as union of the tangent spaces at each point, in order to proceed afterwards towards the definition of derivations an differential forms. Our approach will be different, following the presentation of [CCF11] and [HHS16]. At this stage one of the most remarkable difference between the ordinary and supergeometrical contexts is the following. When dealing with supermanifolds is not easy and natural to define certain objects passing through the concept of point and then to extend on open sets or to the whole manifold. This is usually resumed in the very popular claim that "on a supermanifold there are not enough points", meaning that the structure defined over the topological space is so rich that a pointwise description fails to be powerful. Hence, these new objects are assignment of sheaves respecting the structure of the supermanifold.

Let's proceed with the definition of *superderivation* and then *super vector field.*

Definition 3.1.18. A *super derivation* of a superalgebra A is an element in the space $\underline{\mathrm{Mor}}_{\mathbb{K}}(A, A)$, i.e. a $\mathbb{K}$-linear map to the algebra itself, satisfying a graded Leibniz rule

$$D(fg) = D(f)g + (-1)^{|D||f|} f D(g) \ f, g \in A.$$

Derivations form a left A-supermodule considering the relation $(fX)(g) := fX(g)$

Super vector fields shall be derivations acting on the structure sheaf, hence we give the following definition.

Definition 3.1.19. Let's take an open set $U \subseteq \widetilde{M}$, a *super vector field* is an element in the super derivations of the algebra $\mathcal{O}_M(U)$. We denote the set of super vector fields by $\mathrm{Der}_M(U) \subseteq \underline{\mathrm{Mor}}(\mathcal{O}_M(U), \mathcal{O}_M(U))$ and we notice that it can be endowed with a super-Lie algebra structure taking the commutator as bracket

$$[D, D'] := D \circ D' - (-1)^{|D||D'|} D' \circ D$$

The assignment $U \mapsto \mathrm{Der}_M(U)$ defines a sheaf of super-Lie algebras (or left $\mathcal{O}_M$-supermodules) on $\widetilde{M}$, denoted by Der_M and called *superderivation sheaf.* As showed in [CCF11, Lemma 4.4.4], fixed an open set $U \subseteq M$, taking even local coordinates $\{x^i\}$, and odd local coordinates θ^i in a neighbourhood $U' \subseteq U$, a vector field on U can be represented in the form

$$V|_{U'} = \sum_{i=1}^{p} f_i(x^1, \ldots, x^p, \theta^1, \ldots, \theta^q) \frac{\partial}{\partial x^i} + \sum_{j=1}^{q} g_j(x^1, \ldots, x^p, \theta^1, \ldots, \theta^q) \frac{\partial}{\partial \theta^j}$$

The definition of vector field and derivation is naturally followed by that of differential operator. This definition can be provided recursively, starting from $\mathrm{Diff}_0(U) = \mathcal{O}_M(U)$, that acts via multiplication

$$\mathrm{Diff}_0(U) \ni f : \mathcal{O}_M(U) \to \mathcal{O}_M(U),\ g \mapsto f \cdot g$$

Hence, super differential operator of degree k are

$$\mathrm{Diff}_k(U) := \{D \in \underline{\mathrm{Mor}}(\mathcal{O}_M(U), \mathcal{O}_M(U)) | [D, f] \in \mathrm{Diff}_{k-1}(U),\ \forall f \in \mathcal{O}_M(U)\} \tag{3.1.14}$$

where $[D, f] = D \circ f - (-1)^{|D||f|} f \circ D$. Then we are ready to define $\mathrm{Diff}(U) = \bigcup_{k \geq 0} \mathrm{Diff}_k(U)$, that with a suitable restriction defines a superalgebra sheaf on $\widetilde{M}$ via the assignment $U \mapsto \mathrm{Diff}(U)$. We know from the previous chapter that issues connected to the support of the functions are of crucial importance when studying operators (differential or not) acting over the elements of the sheaf algebra. In supergeometry, failing the pointwise description, support of an element of $\mathcal{O}_M(U)$ can be defined "evaluating" the function in all the open sets. Given $f \in \mathcal{O}_M(V)$ for $V \subseteq \widetilde{M}$, we call $\mathrm{Null}_{f,V}$ the class of open sets for which $\mathrm{res_{V,U}}(f) = 0$ the support of f is the closed subset of $\widetilde{M}$

$$\mathrm{supp}(f) := V \setminus \bigcup \{U \in \mathrm{Null}_{f,V}\}$$

Let's notice that if $f \in C^\infty_M(U)$ then this definition perfectly matches the usual definition of support. Finally, we claim that the elements of the superalgebra sheaf don't change supports when acting on $\mathcal{O}_M(U)$, for $U \subseteq \widetilde{M}$.

We have set the basis for differential calculus on supermanifolds. The next step of this brief mathematical introduction consist of providing powerful and useful definition integration. First we define *super-differential forms*.

Definition 3.1.20. Given a supermanifold $M = (\widetilde{M}, \mathcal{O}_M)$, *the sheaf of super-differential forms* over $\widetilde{M}$, denoted by Ω'_M is the assignment

$$U \mapsto \Omega^1_M(U) := \underline{\mathrm{Mor}}_{\mathcal{O}_M(U)}(\mathrm{Der}_M(U), \mathcal{O}_M(U)).$$

Moreover, evaluation of differential forms can be used to define a pairing between super-differential forms and super-vector fields, where particular attention shall be paid in the sign convention. Hence one defines

$$\langle \cdot, \cdot \rangle := \mathrm{ev} \circ c_{\mathrm{Der}_M, \Omega^1_M} : \mathrm{Der}_M \otimes_{\mathcal{O}_M} \Omega^1_M \to \mathcal{O}_M. \tag{3.1.15}$$

Explicitly, fix $U \subseteq \widetilde{M}$, the pairing is the element $\langle X, \omega \rangle := (-1)^{|\omega||X|}\omega(X)$ in $\mathcal{O}_M(U)$. The *differential* of an element $f \in \mathcal{O}_M(U)$ is the morphism $f \mapsto df$, defined by the condition $\langle X, df \rangle := X(f)$.

Given an open set V, once a set of coordinates $\{x^1, \dots, x^p, \theta^1, \dots, \theta^q\} =: \{x^1, \dots, x^{p+q}\}$ has been fixed, an adapted basis for $\mathrm{Der}_M(V)$, $\{\partial_1, \dots, \partial_{p+q}\}$ defined by the relations $\partial_i x^j = \delta^j_i$, $(i, j = 1, \dots, p+q)$, and for $\Omega^1_M(V)$, $\{dx^1, \dots, dx^{p+q}\}$ is completely characterised by the duality relations $\langle \partial_i, dx^j \rangle = \delta^j_i$, $(i, j = 1, \dots, p+q)$. The exterior algebra of differential forms can be defined and it's denoted by $\Omega_M := \Lambda(\Omega^1_M)$.

Recalling the axiomatic definition of quantum field theories and the conditions required to set the categorical framework, we need an extension of the operation of "pulling back the geometrical structure" from a spacetime to another, that can be applied to a couple of superspaces in super-QFTs. In the development of the theory, we will focus on a subclass of supermanifolds whose structure are completely determined by the assignment of a super one form, we will perform pull-backs of these one-forms only for open embeddings of the respective reduced manifolds. Hence, given a morphism in $\mathfrak{sMan}$, $\chi : M \to M'$ such that $\widetilde{\chi}(\widetilde{M}) \subseteq \widetilde{M}'$ is open, we first define the *push forward* of a super vector field, $X : \mathcal{O}_M(U) \to \mathcal{O}_M(U)$, $U \subseteq \widetilde{M}$, in the natural way suggested by the diagram

$$\begin{array}{ccc} \mathcal{O}_{M'}(U) & \xrightarrow{\chi_{U*}(X)} & \mathcal{O}_{M'}(U) \\ \chi_U^* \big\downarrow & & \big\downarrow \chi_U^* \\ \mathcal{O}_M(U) & \xrightarrow[X]{} & \mathcal{O}_M(U) \end{array}$$

Diagram 3.1.10

leading to $\chi_{U*}(X) := {\chi_U^*}^{-1} \circ X \circ \chi_U^*$. Then, duality does the job: the pull-back of a super-one-forms ω is the pull-back of the evaluation of forwarded super vector fields $\chi_{U*}(X)$ on the form ω, i.e.

$$\langle X, \chi_U^*(\omega)\rangle := \chi_U^*(\langle \chi_{U*}(X), \omega\rangle) \tag{3.1.16}$$

From the definition we can deduce two relevant properties:

i. $\chi_U^*(f\omega) = \chi_U^*(f)\chi^*(\omega)$, for $f \in \mathcal{O}_{M'}(U), \omega \in \Omega^1_{M'}(U)$. Indeed, for every superderivation $X : \mathcal{O}_M \to \mathcal{O}_M$, we define the push-forward as in Diagram 3.1.10 and calculate starting from the definition

$$\begin{aligned} \langle X, \chi_U^*(f\omega)\rangle :=& \chi_U^*(\langle \chi_{U*}(X), f\omega\rangle) = \chi_U^* f\omega(\chi_{U*}(X)) \\ =& \chi_U^*(f)\chi_U^*(\omega(\chi_{U*}(X))) = \chi_U^*(f)\langle X, \chi_U^*(\omega)\rangle = \langle X, \chi_U^*(f)\chi_U^*(\omega)\rangle \end{aligned}$$

ii. $\chi_U^*(df) = d\chi_U^*(f), f \in \mathcal{O}_{M'}(U)$. Indeed

$$\begin{aligned} \langle X, \chi_U^*(df)\rangle :=& \chi_U^*(\langle \chi_{U*}(X), df\rangle) = \chi_U^*\left(\chi_{U*}(X)(f)\right) \\ =& \chi_U^* \circ {\chi_U^*}^{-1} \circ X \circ \chi_U^*(f) = X(\chi_U^* f) = \langle X, d\chi_U^*(f)\rangle \end{aligned}$$

Remark 3.1.21. At this stage, we shall study a notion of push-forward also for sections of the sheaves of sections over a supermanifold $\mathcal{O}_U(M)$. As already noticed, what we are tempted to call "pull-back" is a datum in the definition of morphism between supermanifold and encodes the transformation properties of the sheaf of sections from a supermanifold to the other. For compactly supported global sections the concept of push-forward can be successfully defined. We take two supermanifolds M, M' and a $\mathfrak{sMan}$ morphism $\chi : M \to M'$, with the additional requirement that when restricted to the image $\chi : M \to M'|_{\widetilde{M}}$ is an isomorphism. Hence, we have that the superalgebra morphism $\chi^*_{\widetilde{\chi}(\widetilde{M})}$ admits inverse, in particular when restricted compactly supported functions

$$\left(\chi^*_{\widetilde{\chi}(\widetilde{M})}\right)^{-1} : \mathcal{O}_{M,c}(\widetilde{M}) \to \mathcal{O}_{M',c}(\widetilde{\chi}(\widetilde{M}))$$

which can be composed with the *extension sheaf morphism*

$$\mathrm{ext}_{\widetilde{\chi}(\widetilde{M}),\widetilde{M}'} : \mathcal{O}_{M',c}(\widetilde{\chi}(\widetilde{M})) \to \mathcal{O}_{M',c}(\widetilde{M}')$$

to give the wanted push-forward

$$\chi_* := \mathrm{ext}_{\widetilde{\chi}(\widetilde{M}),\widetilde{M}'} \circ \left(\chi^*_{\widetilde{\chi}(\widetilde{M})}\right)^{-1} \tag{3.1.17}$$

With the intention to go straight towards the definition of integration, we first notice that in a supergeometrical context there is not any sensible meaning for the notion of top-degree (volume) forms and this is the main issue concerning integration on generic supermanifolds. The solution to this problem is very cumbersome and technical, but we try to sketch briefly the path, giving detailed references for the interested reader. We start doing a step back to free A-supermodules and we show a useful functorial construction. Once is taken a free left A-supermodule (V) and any adapted basis $\{e_1, \ldots, e_{p+q}\}$ we may assign a left A-supermodule ($\mathrm{Ber}(V)$) with the association

$$\{e_1, \ldots, e_{p+q}\} \mapsto [e_1, \ldots, e_{p+q}]$$

with the additional relation that for each automorphism L of V, relating two adpated basis,

$$[L(e_1), \ldots, L(e_{p+q})] = \mathrm{Ber}(L)[e_1, \ldots, e_{p+q}]. \tag{3.1.18}$$

Hence, declaring $[e_1, \ldots, e_{p+q}]$ even if q is even, or odd if q is odd, we have defined a free left A-supermodule of superdimension $1|0$ or $0|1$. As anticipated before, this assignment is functorial. For each $f : V \to V'$, we define $\mathrm{Ber}(f) : \mathrm{Ber}(V) \to \mathrm{Ber}(V')$ by setting

$$\mathrm{Ber}(f)([e_1, \ldots, e_{p+q}]) := [f(e_1), \ldots, f(e_{p+q})].$$

Then, due to the impossibility to define top forms on supermanifolds, integration is an operation involving different objects called *densities*.

Definition 3.1.22. Given U subsupermanifold of $\mathbb{R}^{p|q}$, we define densities as $\mathbb{R}$-linear,

$\mathbb{R}$-valued forms on compactly supported elements of $\mathcal{O}_M(U)$

$$\mu : \mathcal{O}_M(U) \to \mathbb{R} \qquad g = \sum_k g_{I_k}\theta^{I_k} \mapsto \int_U \mu(g) := \sum_k \int_{\widetilde{U}} \mu_{I_k}(x) g_{I_k}(x) \mathrm{dx}^1 \dots \mathrm{dx}^p \qquad (3.1.19)$$

for $\mu_{I_k}(x) \in C^\infty(U)$. The set of densities $\mathcal{D}(U)$ over U has a structure of left $\mathcal{O}_M(U)$-supermodule with the multiplication map

$$\begin{aligned} &\mathcal{O}_M(U) \times \mathcal{D} \to \mathcal{D} \\ &\qquad (u, \mu) \mapsto (u\mu) = \int (u\mu)(g) := \int \mu(u \cdot g). \end{aligned}$$

Remark 3.1.23. The dimension of the supermodule $\mathcal{D}$ as defined in the previous Definition turns out to be $1|0$ if the dimension (q) of the odd superspace is even and is $0|1$ if it is odd. We show this for a submanifold of $\mathbb{R}^{2|2}$: a remarkable case for the following and, however, a good example in order to trust the previous claim. The generalisation is trivial and has not to pass through all the amount of calculations we are going to present, but we believe that some explicit presentation could be useful to see how things concretely work. We take $U \subseteq \mathbb{R}^{2|2}$, with coordinates $(x^1, x^2, \theta^1, \theta^2)$ (let's fix $x = (x^1, x^2)$). Taking a generic density

$$\int \mu(g) := \int_{\widetilde{U}} (\mu_0(x)g_0(x) + \mu_1(x)g_1(x) + \mu_2(x)g_2(x) + \mu_{12}(x)g_{12}(x)$$

acting on $g = g_0(x) + g_1(x)\theta^1 + g_2(x)\theta^2 + g_{12}(x)\theta^1\theta^2$. We claim that each density of this kind is generated by the density

$$\int_U \overline{\mu}(g) = \int_U [\mathrm{dx}^1, \mathrm{dx}^2, \mathrm{d}\theta^1, \mathrm{d}\theta^2](g) := \int_{\widetilde{U}} g_{12}(x) \mathrm{dx}^1 \mathrm{dx}^2 \qquad (3.1.20)$$

where this notation is not *a priori* justified, but has been used as a first suggestion that the supermodule $\mathcal{D}$ is isomorphic to the $\mathcal{O}_M(U)$-supermodule, $\mathrm{Ber}(\Omega^1_M(U))$. Then, we have to find an element $u \in \mathcal{O}_M(U)$ such that

$$\int (u\overline{\mu})(g) = \int_{\widetilde{U}} (u\, g)_{12}(x) \mathrm{dx}^1 \mathrm{dx}^2.$$

is equal to the integral above. In this case the right choice turns out to be $u = \mu_{12}(x) + \mu_2(x)\theta^1 - \mu_1(x)\theta^2 + \mu_0(x)\theta^1\theta^2$.

Now referring to [Del+99, ch. 3.10] we notice that with minimal work can be proven that an isomorphism between open subsupermanifolds of $\mathbb{R}^{p|q}$, $\varphi : U^{p|q} \to V^{p|q}$ transforms densities over the domain in densities over the codomain, and this fact can be generalized with isomorphism from generic supermanifolds and open subsupermanifold of $\mathbb{R}^{p|q}$. Moreover, as anticipated before, given a supermanifold M an open $U \subseteq \widetilde{M}$ the *Berezinian* $\mathcal{O}_M(U)$-supermodule, built up on $\Omega^1_M(U)$ can be put in correspondence with $\mathcal{D}(U)$ (see [Del+99, Prop. 3.10.2, ch. 3.10, pg. 81]), leading to the definition of of *the local Berezin integral* over $M|_{\widetilde{U}}$

$$\int_{M|_U} [\mathrm{dx}^1, \ldots, \mathrm{dx}^p, \mathrm{d}\theta^1, \ldots, \mathrm{d}\theta^q] f \;:=\; \int_{\widetilde{U}} f_{I_q}\, \mathrm{dx}^1, \ldots, \mathrm{dx}^p \tag{3.1.21}$$

that can be globalized and gives a unique linear map

$$\int_M : \mathrm{Ber}(\Omega^1_M)_c(\widetilde{M}) \to \mathbb{R} \qquad v \mapsto \int_M v \tag{3.1.22}$$

We conclude showing how the change of variable formula works for an orientation preserving morphism between two supermanifolds, i.e. $\chi : M \to M'$ such that $\widetilde{\chi} : \widetilde{M} \to \widetilde{M}'$ preserves orientation. Taking two sets of local coordinates $(x'^1, \ldots, x'^{p+q})$ and $(x^1, \ldots, x^{p+q})$, fixed V' and $V = \widetilde{\chi}^{-1}(V)$, we denote the basis induced by the coordinates in $\Omega^1_{M'}$ and Ω^1_M, respectively by $\{\mathrm{d}x'^1, \ldots, \mathrm{d}x'^{n+m}\}$ and $\{\mathrm{d}x^1, \ldots, \mathrm{d}x^{n+m}\}$. When we pull-back the basis from $\Omega^1_{M'}$ to Ω^1_M, we define the basis $\left\{\mathrm{d}\chi^*_{V'}(x'^i)\right\}_{i=1}^{p+q}$. The automorphism mapping the two basis of $\Omega^1_M(V)$ one into each other, induced by χ, is denoted by $J(\chi)$ and acts on the basis elements in a matrix form

$$\mathrm{d}\chi^*_{V'}(x'^j) = \sum_j J(\chi)^i_j \mathrm{d}x^j$$

Then, we have the following isomorphism between Berezinian supermodules, $\chi^* : \mathrm{Ber}(\Omega^1_{M'}) \to \mathrm{Ber}(\Omega^1_M)$

$$\begin{aligned} \chi^*_{V'}\big([\mathrm{d}x'^1, \ldots, \mathrm{d}x'^{n+m}]\big) &= [\mathrm{d}\chi^*_{V'}(x'^1), \ldots, \mathrm{d}\chi^*_{V'}(x'^{n+m})] \\ &= \mathrm{Ber}(J(\chi))\,[\mathrm{d}x^1, \ldots, \mathrm{d}x^{n+m}]\,. \end{aligned} \tag{3.1.23}$$

The isomorphism is well-defined because the $\mathcal{O}_M(V')$-supermodule multiplication is mapped to the $\mathcal{O}_M(V)$-supermodule multiplication by the property i) of the pull-back map

$$\chi^*_U(v\, f) = \chi^*_U(v)\, \chi^*_U(f),$$

for all $v \in \mathrm{Ber}(\Omega^1_{M'})(U)$, $f \in \mathscr{O}_{M'}(U)$ and all open $U \subseteq \widetilde{M'}$. Hence, the equivalence

$$\int_M \chi^*(v) = \int_{M'} v \ , \tag{3.1.24}$$

holds for all $v \in \mathrm{Ber}(\Omega^1_{M'})_c(\widetilde{M'})$.

3.2 Super-QFTs

Following the approach to the exposition of ordinary locally covariant quantum field theories, all the mathematical notions and the technical tools for the development of super quantum field theories have been prepared in the previous section. Scope of this section is then to present the formal asset proposed in [HHS16] to give sense to a covariant locality principle for the treatment of super-QFTs. From the beginning, we declare our intention not to present all the results and the technical details of this construction, but to underline those aspects that can be used in analogy with ordinary QFTs and those which appear differently instead, with the only aim to depict the suitable framework for a brief presentation of 2|2-dimensional models.

3.2.1 Background geometry: super-Cartan supermanifolds

As first step of the exposition, we have to explain what is the best minimal datum for the development of field theories on supermanifolds. We already laid the foundations for the definition of the suitable structures in section 2.2 of the second chapter with the definition of Cartan structures (2.2.4). Construction of such objects was basically motivated by the need to make explicit the local action of Lorentz group and to use it to define a differential operator − and an equation of motion − for spinor fields. At this stage, we introduce the notion of *super-Cartan structure* on supermanifolds: the first thing to be fixed is what super-Lie algebra has to replace the Poincaré algebra which we used in the definition of vielbein and spin connection in Section 2.2: this super algebra is called the *super Poincaré* super-Lie algebra. We report here the definition of [HHS16], following the notation of the original version, fixing $W = \mathbb{R}^n$ and $g = \eta$, and hence $\mathrm{Spin}(g, W) = \mathrm{Spin}(1, n\text{-}1)$.

Definition 3.2.1. Given a fifthple $(\mathbb{R}^{1,n-1}, S, \rho, \rho^S, \Gamma)$, where S is a real vector space over which the spin group representation is defined by the action $\rho^S : \mathrm{Spin}(1, n\text{-}1) \times S \to S$, while $\rho : \mathrm{Spin}(1, n\text{-}1) \times \mathbb{R}^n \to \mathbb{R}^n$ is the action on $\mathbb{R}^n$, a super-Poincaré super-Lie algebra

$\mathfrak{sp}$ is the super vector space

$$\mathfrak{sp} := (\mathfrak{spin} \oplus \mathbb{R}^n) \oplus S \qquad \text{with} \qquad (\mathfrak{spin} \oplus \mathbb{R}^n) = (\mathfrak{sp})_0, S = (\mathfrak{sp})_1. \tag{3.2.1}$$

The super-Lie bracket is defined for $M_i \oplus v_i \oplus s_i \in \mathfrak{sp}$, for $i = 1, 2$ by

$$\begin{aligned}&[M_1 \oplus v_1 \oplus s_1, M_2 \oplus v_2 \oplus s_2] = \\ &[M_1, M_2]_{\mathfrak{spin}} \oplus (\rho_*(M_1 \otimes v_2 - M_2 \otimes v_1) - 2\Gamma(s_1, s_2)) \oplus \rho_*^S(M_1 \otimes s_1 - M_2 \otimes s_2)\end{aligned} \tag{3.2.2}$$

where we denoted by $\rho_* : \mathfrak{spin} \times \mathbb{R}^n \to \mathbb{R}^n$ and $\rho_*^S : \mathfrak{spin} \times S \to S$ the induced $\mathfrak{spin}$ action. The supertranslation super Lie algebra $\mathfrak{st}$ is given by the vector space

$$\mathfrak{st} := \mathbb{R}^n \oplus S \qquad \text{with} \qquad (\mathfrak{st})_0 = \mathbb{R}^n, (\mathfrak{st})_1 = S \tag{3.2.3}$$

Given a super-Poincaré super-Lie algebra and a translation algebra associated to the data $(\mathbb{R}^{1,n-1}, S, \rho, \rho^S, \Gamma)$, we take additionally a map $\varepsilon : S \otimes S \to \mathbb{R}$, being a metric with positive signature or a symplectic structure and a choice of orientation for $\mathbb{R}^n$ ($\mathfrak{o}$) and S ($\mathfrak{o}_S$). Moreover, we fix a positive cone of timelike vectors $C \subset \mathbb{R}^n$ and we choose Γ positive in the sense that $\Gamma(s, s)$ is in the closure of C for all $s \in S$ and is zero if and only if $s = 0$. We make explicit all data needed for the definition of *super-Cartan supermanifolds*

Definition 3.2.2. Given the set of data $(\mathbb{R}^{1,n-1}, S, \rho, \rho^S, \mathfrak{o}, \mathfrak{o}_S, \Gamma, C, \varepsilon)$ and supermanifold M of dimension $n|\dim(S)$, a *super-Cartan structure* over M is a pair (Ω, E), where Ω, the *super-spin connection*, is an even element of $\Omega^1(M, \mathfrak{spin})$ and E, the *supervielbein*, is an even element of $\Omega^1(M, \mathfrak{st})$, which is non-degenerate. The triple $\mathcal{M} = (M, \Omega, E)$ is a super-Cartan supermanifold.

As we did in Definition 2.2.4 with formulas (2.2.18) and (2.2.19), we can assign to $\mathcal{M}$ a supertorsion $T_{\mathcal{M}} \in \Omega^2(M, \mathfrak{st})$ and supercurvature $R_{\mathcal{M}} \in \Omega^2(M, \mathfrak{spin})$ two forms. Once a bracket in the space of $\mathfrak{sp}$-valued forms is defined,

$$\begin{aligned}[\cdot, \cdot]_{\wedge, \mathfrak{sp}} :&\Omega^r(M, \mathfrak{sp}) \otimes \Omega^s(M, \mathfrak{sp}) \to \Omega^{r+s}(M, \mathfrak{sp}), \\ &(\omega_1 \otimes X_1) \otimes (\omega_2 \otimes X_2) \to (-1)^{|X_1||\omega_2|}\omega_1 \wedge \omega_2 \otimes [X_1, X_2]_{\mathfrak{sp}}\end{aligned}$$

the formulas for $T_{\mathcal{M}}$ and $R_{\mathcal{M}}$ are given by

$$T_{\mathcal{M}} := \mathrm{d}E + [\Omega, E]_{\wedge,\mathfrak{sp}} \tag{3.2.4}$$

$$R_{\mathcal{M}} := \mathrm{d}\Omega + [\Omega, \Omega]_{\wedge,\mathfrak{sp}} \,. \tag{3.2.5}$$

The first natural question is whether and in what sense the results of Chapter 1 can be reproduced when dealing with supermanifolds. The idea is to use the natural construction of the reduced manifold associated to a supermanifold, in order to take advantage of concept well defined for ordinary oriented and time-oriented manifold (such as the notion of causal structure) and hence select a subclass of all the super-Cartan supermanifold for which the definition of fields and their dynamics can be well posed. Then, we notice that the supertraslation algebra can be endowed with an adapted basis $\{p_0, \dots, p_{n\text{-}1}, q_1, \dots, q_{\dim(S)}\}$, where $p_\alpha \in \mathbb{R}^n$ ($\alpha = 0, \dots, n-1$) can be chosen to be orthonormal for $(\mathbb{R}, \eta)$, oriented with respect to $\mathfrak{o}$ and time-oriented in the sense that $p_0 \in C$, while $q_i \in S$ ($i = 1, \dots, \dim(S)$) can be chosen to be an orthonormal (or orthosymplectic) basis for (S, ε) and oriented with respect to $\mathfrak{o}_S$. Using this basis we can expand the supervielbein

$$E = \hat{e}^\alpha \otimes p_\alpha + \xi^a \otimes q_a \tag{3.2.6}$$

where the "coefficients" of the expansions, $\{\hat{e}^0, \dots, \hat{e}^{n-1}, \xi^1, \dots, \xi^{\dim(S)}\}$, are an adapted basis for $\Omega^1(M)$ due to the non-degeneracy assumption for E. Parity for the elements of this basis is assigned in order E to be even as by definition above. Since we learnt how to describe the geometry of an ordinary manifold from a set of one-forms, our aim is to extract a vielbein for the reduced manifold from the first part of the sum (3.2.6), then orientation on $\mathbb{R}^n$ and the positive cone C are used to induce orientation and time-orientation. Explicitly, we take the embedding $j_{\widetilde{M},M} : \widetilde{M} \to M$ defined in Remark 3.1.17 (where we make explicit domain and codomain) and using pull-back of one forms (cf. equation (3.1.16)), recalling that $\mathbb{R}^n$ with the trivial bracket is isomorphic to the standard translation algebra $\mathfrak{t}$, we define the $\mathfrak{t}$-valued one form

$$\widetilde{e} := j^*_{\widetilde{M},M}(E) \;\in\; \Omega^1(\widetilde{M}, \mathfrak{t}),$$

and can be expanded on the basis of $\mathfrak{t}$ $\{p_\alpha\}_{\alpha=1}^n$, giving $\widetilde{e} = \widetilde{e}^\alpha \otimes p_\alpha$.

Definition 3.2.3. Given a super-Cartan supermanifold $\mathcal{M} = (M, \Omega, E)$ we construct the following list of data:

Lorentzian metric: a metric is defined impicitly by

$$g_{\widetilde{M}} : \Gamma(T\widetilde{M}) \times \Gamma(T\widetilde{M}) \to C^\infty(M),$$
$$(X,Y) \mapsto g_{\widetilde{M},E}(X,Y) := \eta(\langle X, \widetilde{e}\rangle, \langle Y, \widetilde{e}\rangle).$$

Orientation: the volume form associated to the collection $\{\widetilde{e}^\alpha\}$ by $\mathrm{vol}(\widetilde{e}) = \widetilde{e}^0, \dots, \widetilde{e}^{n-1}$ defines an orientation $\mathfrak{o}_{\widetilde{M}}$ on the reduced manifold $\widetilde{M}$.

Time-orientation: recalling that $p_0 \in C$ by construction, we take the vector field $\widetilde{X}_0$, defined as a dual for $\widetilde{e}^0$ with respect to the relations $\langle \widetilde{X}_\alpha, \widetilde{e}^\beta\rangle$, represent a time orientation $\mathfrak{t}_{\widetilde{M}}$ induced by the cone $C \subset \mathbb{R}^n$.

The fourthple $(\widetilde{M}, g_{\widetilde{M}}, \mathfrak{o}_{\widetilde{M}}, \mathfrak{t}_{\widetilde{M}})$ hence is a spacetime according to Definition 1.1.7 and we call it *reduced spacetime* $\widetilde{\mathbf{M}}$ associated to $\mathcal{M}$.

This construction can be done for all super-Cartan supermanifolds, hence, at this stage, its useful to approach the subject with category theory and to establish in which context the association of a reduced spacetime is possible and turns out to be functorial. Before proceeding, we provide another source of motivation for the following definitions. We know that integration plays a relevant role in the construction of quantum field theory, in particular because a pairing between fields is needed for the implementation of super-commuting relation on the algebra of fields (in analogy with the ordinary construction of the algebra of fields).

Moreover, on a super-Cartan supermanifold a notion of integration is specified by the assigned supervielbein. Indeed, given $\mathcal{M} := (M, \Omega, E)$ we can associate a Berezinian to $E = \hat{e}^\alpha \otimes p_\alpha + \xi^i \otimes q_i$ (well defined since E is non-degenerate):

$$Ber(E) := [\hat{e}^0, \dots, \hat{e}^{n-1}, \xi^1, \dots, \xi^{\dim(S)}] \in Ber(\Omega^1(M))$$

Remark 3.2.4. The object $\mathrm{Ber}(E)$ does not depend on the choice of the adapted basis. Indeed, taking another adapted basis $\{\hat{e}'^0, \dots, \hat{e}'^{n-1}, \xi'^1, \dots, \xi'^{\dim(S)}\}$ (whose even part is orthonormal and whose odd part is orthonormal/orthosymplectic), we know that there exist a parity preserving linear map $T : \mathfrak{st} \to \mathfrak{st}$, in block form

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}, \tag{3.2.7}$$

where, in order to preserve orthonormality of the even part and orthonormality (orthosymplect condition) for the odd part, $T_1 \in \mathrm{SO}_0(n-1)$, $T_2 = T_3 = 0$, $T_4 \in \mathrm{SO}(\dim(S))$. Hence, the Berezinian as defined by (3.1.11)

$$\mathrm{Ber}(T) = \det(T_1 - T_2 T_4^{-1} T_3)\,(\det(T_4))^{-1} = \det(T_1) = 1$$

and looking at formula (3.1.18), we deduce that

$$\left[\hat{e}'^0, \ldots, \hat{e}'^{n-1}, \xi'^1, \ldots, \xi'^{\dim(S)}\right] = \left[\hat{e}^0, \ldots, \hat{e}^{n-1}, \xi^1, \ldots, \xi^{\dim(S)}\right]$$

Then, according to the discussion at the end of section 3.1, we can integrate elements of the sheaf of global compactly supported section $\mathcal{O}_c(M)$ over a super-Cartan supermanifold $\mathcal{M} = (M, \Omega, E)$, for $H \in \mathcal{O}_c(M)$ by

$$\int_M \mathrm{Ber}(E) H \tag{3.2.8}$$

and a pairing for global section with supports overlapping on a compact set ($H_1, H_2 \in \mathcal{O}(M)$) is easily defined by

$$\langle H_1, H_2 \rangle_{\mathcal{M}} := \int_M \mathrm{Ber}(E) H_1 H_2\ . \tag{3.2.9}$$

As usual, attention has to be paid in exchanging order of the integrands

$$\langle H_1, H_2 \rangle_{\mathcal{M}} = (-1)^{|H_1||H_2|} \langle H_2, H_1 \rangle_{\mathcal{M}}$$

Then, we proceed defining the category of super-Cartan supermanifolds ($\mathfrak{sCart}$) and prove that the assignment of the reduced spacetime is a functor to $\mathfrak{Loc}$.

Definition 3.2.5. We define the category of $\mathfrak{sCart}$, whose objects are super-Cartan supermanifolds $\mathcal{M} = (M, \Omega, E)$ and whose morphism $\chi : \mathcal{M} \to \mathcal{M}'$, are all the morphism in $\mathfrak{sMan}$ between the base supermanifolds $\chi : M \to M'$[1] such that:

i. $\widetilde{\chi}$ is an open embedding;

ii. $\chi : M \to M'|_{\widetilde{\chi}(\widetilde{M})}$ is an isomorphism;

[1]With a slight abuse of notation $\mathfrak{sMan}$ morphism and $\mathfrak{sCart}$ are denoted by the same symbol.

iii. the spin connection Ω and the supervielbein E are preserved by pull-back of one-forms, i.e $\chi^*(\Omega') = \Omega$ and $\chi^*(E') = E$

Proposition 3.2.6. *Given any object in* $\mathfrak{sCart}$ *the assignment of a reduced spacetime is functorial, i.e. we can define the functor* $\mathscr{R} : \mathfrak{sCart} \to \mathfrak{Loc}$ *by:*

- *for any super-Cartan supermanifold* $\mathcal{M} = (M, \Omega, E)$, $\mathscr{R}(\mathcal{M}) := (M, g_{\widetilde{M}}, \mathfrak{o}_{\widetilde{M}}, \mathfrak{t}_{\widetilde{M}})$;
- *for any pair of objects* $\mathcal{M}$, $\mathcal{M}'$, *taking a* $\chi \in \mathrm{Mor}(\mathcal{M}, \mathcal{M}')$ *we assign the morphism* $\mathscr{R}(\chi) = \widetilde{\chi} : \mathscr{R}(\mathcal{M}) \to \mathscr{R}(\mathcal{M}')$.

In order to simplify the notation we will denote $\mathscr{R}(\mathcal{M}) =: \widetilde{\mathcal{M}}$.

In the light of the last proposition, it is clear that we can define a causal structure for a super-Cartan supermanifold borrowing the causal structure of the reduced spacetime and then all the relevant properties due to causal structure can be used for characterisation of the objects in $\mathfrak{sCart}$. In particular we can define the category of globally hyperbolic super-Cartan supermanifolds, where the objects are super-Cartan supermanifold whose reduced spacetime is a GHST.

Definition 3.2.7. The category $\mathfrak{ghsCart}$ is the full subcategory of $\mathfrak{sCart}$ defined taking as objects all the objects whose reduced spacetime is globally hyperbolic.

3.2.2 Axioms and construction of super-QFTs: a quick review

In this subsection we show how the axioms for super-QFTs can be stated in order to include field theories on supermanifolds in the framework depicted by the *general covariance locality principle*. Unfortunately, the mathematical tools needed for a detailed and thorough exposition of the subject are too technical for the purpose of this thesis. Hence, the approach we have chosen is that of giving a schematic summary of what has been done in [HHS16], omitting most of the formal statements. In fact, on one side we believe that understanding how a scheme such that of locally covariant QFT can be reproduced successfully in other contexts only at a superficial level is already enough enlightening, on the other side our aim is not to spend great effort in theoretical study of heavy mathematical language but to give the recipe for a concrete definition of models of super-QFT.

From theoretic data to super field theories

As showed before, once a set of defining data $(\mathbb{R}^{1,n-1}, S, \varepsilon, \mathfrak{o}, \mathfrak{o}_S, \Gamma)$ is given in addition to a base supermanifold M, we can specify a super Poincaré super Lie-algebra $\mathfrak{sp}$, a super-translation algebra $\mathfrak{t}$ (cfr. eq Definition 3.2.1) and a reduced spacetime whose orientation and time orientation are induced by the super-Cartan structure based on this data set. This choice is enough to provide a sensible definition of super field theory, describing the geometry of the reduced spacetime and hence the scenario of the physical theory and fixing the amount of supersimmetries. Nevertheless, even though all these hypothesis − together with global hyperbolicity of the reduced spacetime − are enough for a theoretical treatment of the subject, it is a common practice to impose constraints on the objects defining the geometry, in order to get a sensible theory from a physical point of view and to reduce the great amount of degree of freedom that would require too effort in computations. This constraints are usually imposed on the quantity T_Ω (see eq. (3.2.4) above) and allow to select a full subcategory $\mathfrak{sLoc}$ of $\mathfrak{ghsCart}$. Two remarkable example are the supergravity supertorsion constraints proposed for the first time in [WZ77] and used in several following works (such as [How79]).

If we take an object $\mathcal{M} = (M, \Omega, E)$ of $\mathfrak{sLoc}$, inspired by motivations presented in Section 2.1 and by constructions performed in 2.3, we shall fix a *space of super field configuration*, that in our approach is taken to be the space of global section of the supermanifold M, denoted by $\mathcal{O}(\mathcal{M}) := \mathcal{O}(M)$. Then, with the aim to rule the dynamics of fields, we take a super differential operator (cf. eq. (3.1.14)), we denote it by $P_\mathcal{M}$ and we demand it to be super-self adjoint with respect to the pairing (3.2.9)

$$\langle F, P_\mathcal{M}(H)\rangle_\mathcal{M} = (-1)^{|F||H|}\langle P_\mathcal{M}(F), H\rangle.$$

As in ordinary field theory, in line with locally covariant approach, for super field theories we are not going to solve explicitly the equations of motion, but we prefer a quantization scheme which deals with Green's operator[2]. Then we define *retarded and advanced super Green's operator* (super-GO) and we build up the theory only for operators $P_\mathcal{M}$ admitting super-GO, called Green hyperbolic operator.

Definition 3.2.8. Given a super-Cartan supermanifold $\mathcal{M}$ and super-differential operator

[2] Existence of Green's operator can be proven composing Green's operator of well-know ordinary theory, after the super field has been suitable decomposed, using theorems and propositions on stability of Green's operator of section 1.2, in particular Proposition 1.2.11

$P_{\mathcal{M}} : \mathcal{O}(\mathcal{M}) \to \mathcal{O}(\mathcal{M})$, we define as *retarded/advanced super-Green's operator*, the linear maps $G^{\pm}_{\mathcal{M}} : \mathcal{O}_c(\mathcal{M}) \to \mathcal{O}_c(\mathcal{M})$:

1. $P_{\mathcal{M}} \circ G^{\pm}_{\mathcal{M}} = \mathrm{id}_{\mathcal{O}_c(\mathcal{M})}$;

2. $G^{\pm}_{\mathcal{M}} \circ P_{\mathcal{M}}|_{\mathcal{O}_c(\mathcal{M})} = \mathrm{id}_{\mathcal{O}_c(\mathcal{M})}$;

3. $\mathrm{supp}\left(G^{\pm}_{\mathcal{M}}(H)\right) \subseteq J^{\pm}_{\mathcal{M}}(\mathrm{supp}(H))$, for all $H \in \mathcal{O}_c(\mathcal{M})$.

An operator $P_{\mathcal{M}}$ is called *super-Green hyperbolic* if admits reatrded and advanced Green's operator. Moreover, we can define the *super causal propagator* associated to $P_{\mathcal{M}}$ by the formula

$$G_{\mathcal{M}} := G^{+}_{\mathcal{M}} - G^{-}_{\mathcal{M}}. \tag{3.2.10}$$

Super quantum field theories are formulated in term of a super-self adjoint super-Green's hyperbolic operator defined over a super-Cartan supermanifold: this in fact consist of all the initial data needed to perform the super quantum theory. Since the machinery for the construction of the superalgebra of quantum fields follows the path of what has been showed in section 2.3, we present here the main properties of super Green's operator (this statements strongly recall the analogous properties for ordinary Green hyperbolic operator) and then, in few steps, we give the definition of the functor defining a super quantum field theory satisfying the same proprieties listed in theorem 2.3.10.

Proposition 3.2.9. *Given any two objects $\mathcal{M}, \mathcal{M}'$ in $\mathfrak{ghsCart}$ and a super-self adjoint super Green's hyperbolic super differential operator $P_{\mathcal{M}}$ (defined for all objects of $\mathfrak{ghsCart}$), denoting by $G^{\pm}_{\mathcal{M}}(G^{\pm}_{\mathcal{M}'})$ two super Green's operators for $P_{\mathcal{M}}(P_{\mathcal{M}'})$, by $G_{\mathcal{M}}(G_{\mathcal{M}'})$ the induced super causal propagator and by $\langle\cdot,\cdot\rangle_{\mathcal{M}}$ the pairing defined in* (3.2.9)*, the following properties hold:*

i. $\langle H_1, G^{\pm}_{\mathcal{M}}(H_2)\rangle_{\mathcal{M}} = (-1)^{(|H_1|+|P_{\mathcal{M}}|)|P_{\mathcal{M}}|}\langle G^{\mp}_{\mathcal{M}}(H_1), H_2\rangle_{\mathcal{M}}$ *for all* $H_1, H_2 \in \mathcal{O}_c(\mathcal{M})$ *and the maps* $G^{\pm}_{\mathcal{M}}$ *are unique.*

ii. The sequence $O \to \mathcal{O}_c(\mathcal{M}) \xrightarrow{P_{\mathcal{M}}} \mathcal{O}_c(\mathcal{M}) \xrightarrow{G_{\mathcal{M}}} \mathcal{O}_{sc}(\mathcal{M}) \xrightarrow{P_{\mathcal{M}}} \mathcal{O}_{sc}(\mathcal{M})$ *is an exact complex everywhere.*

iii. Given a $\mathfrak{ghsCart}$ *morphism* $\chi : \mathcal{M} \to \mathcal{M}'$ *and the induced push-forward defined by* (3.1.17)*, then* $G^{\pm}_{\mathcal{M}} = \chi^* \circ G'^{\pm}_{\mathcal{M}} \circ \chi_*$.

These are the main results on super field theories, intended as an assignment of a super-self adjoint super Green's hyperbolic operator to each object of a category $\mathfrak{sLoc}$, and could be noticed that the sequence of the point (ii.) in particular can be used to define a map, directly on the quotient space $\mathscr{V}_{\mathcal{M}} := \mathcal{O}_c(\mathcal{M})/P_{\mathcal{M}}\left(\mathcal{O}_c(\mathcal{M})\right)$ (for any two representatives) [3]

$$\begin{aligned} \tau^{\mathcal{M}} &: \mathscr{V}_{\mathcal{M}} \otimes \mathscr{V}_{\mathcal{M}} \to \mathbb{R} \\ &[H_1] \otimes [H_2] \to \tau^{\mathcal{M}} \langle G_{\mathcal{M}}(H_1), H_2 \rangle_{\mathcal{M}}, \end{aligned} \tag{3.2.11}$$

whose symmetry property depends on the dimension of the space S (this is a remarkable difference with respect to the ordinary case, because, as anticipated, this means that the category we have to deal with has to be defined taking into account the datum S). We have:

$$\tau^{\mathcal{M}}([H_1],[H_2]) = (-1)^{|P_{\mathcal{M}}|+1}(-1)^{|H_1||H_2|}\tau_{\mathcal{M}}([H_2],[H_1])$$

and since $P_{\mathcal{M}}$ is even/odd if $\dim(S)$ is even/odd, the map is super-skew symmetric or super symmetric. This latter formula shows one of the differences with ordinary QFT, indeed the intermediate mathematical object between the background and the algebra of fields is neither a symplectic space nor an inner product space. Hence, before proceeding we shall define the right category to include vector spaces endowed with bilinear forms like 3.2.11.

Definition 3.2.10. We define $\mathfrak{V}^S$ as the category whose objects are pair (V, τ) consisting of a real super-vector space and a weakly non degenerate bilinear form satisfying

$$\tau(v_1, v_2) = (-1)^{\dim(S)+1}(-1)^{|v_1||v_2|}\tau(v_2, v_1),$$

and whose morphism are linear maps preserving the bilinear form.

In conclusion, we can assign to each $\mathcal{M}$ in $\mathfrak{sLoc}$ an object in $\mathfrak{V}^S$ given by the couple $(\mathscr{V}_{\mathcal{M}}, \tau_{\mathcal{M}})$ and to each morphism $\chi : \mathcal{M} \to \mathcal{M}'$ in $\mathfrak{sLoc}$ the morphism induced by the push-forward

$$[F] \mapsto [\chi_*(F)].$$

This assignment is a functor and the proof of this can be found in [HHS16, prop. 5.8, pg. 23]. In order to use uniform notation with the literature, we denote it by $\mathscr{L} : \mathfrak{sLoc} \to \mathfrak{V}^S$. Now, we can deal with super real vector spaces V and we can consider the tensor super

[3]Well-definiteness and non-degeneracy are due to the exactness of the sequence: see Theorem 1.2.15 for the analogous result on ordinary Green's hyperbolic operators and all the section 2.1 to understand motivations of what we are showing at this stage of the thesis.

algebra of the complexification, denoted by $\mathcal{T}_{\mathbb{C}}(V)$. This super algebra can be endowed with a super involution and the super commutation relation encoding the quantization of fields can be imposed using the bilinear form τ, quotienting the whole algebra by the ideal

$$\mathcal{I}^{\tau,S} = \{v_1 v_2 + (-1)^{\dim(S)+1}(-1)^{|v_1||v_2|} v_2 v_1 - \beta\tau(v_1, v_2)\} \tag{3.2.12}$$

with $\beta = i$ if $\dim(S)$ is evene and $\beta = 1$ if $\dim(S)$ is odd. Hence, the mathematical object

$$\mathscr{Q}_{V,\tau} = \mathcal{T}_{\mathbb{C}}(V)/\mathcal{I}^{\tau,S} \tag{3.2.13}$$

defines an object in $*\text{-}\mathfrak{sAlg}$, and the assignment of $\mathscr{Q}_{V,\tau}$ to any pair (V,τ) is functorial (cfr. [HHS16, prop. 5.10]). This functor is called *qunatization functor* and is usually denoted by $\mathscr{Q} : \mathfrak{V}^S \to *\text{-}\mathfrak{sAlg}$.

In conclusion, the composition of the functors just defined gives a functor

$$\mathscr{A} = \mathscr{Q} \circ \mathscr{L} : \mathfrak{sLoc} \to *\text{-}\mathfrak{sAlg} \tag{3.2.14}$$

fulfilling *locality, causality and the time-slice axiom* (cfr. [HHS16, prop. 5.11]).

We hence sketched how the wanted functor proceeding along the path developed by [BG12] for ordinary field theory and hence an interpretation of the super quantum field theory as a smearing field is possible: to each element $F \in \mathcal{O}_c(\mathcal{M})$ we can associate $\Phi_{\mathcal{M}}(F) \in \mathscr{A}(\mathcal{M})$ by $\Phi_{\mathcal{M}}(F) = [F]$. Fields are labelled by super-Cartan supermanifolds and satisfy covariance with respect to $\mathfrak{ghsCart}$-morphisms $\chi : \mathcal{M} \to \mathcal{M}'$ in the sense that the following diagram commutes (recalling that the morphism $\mathscr{A}(\chi)$ acts on elements $[F] \in \mathcal{O}_c(\mathcal{M})$ by $\mathscr{A}(\chi)(F) := [\chi_* F]$

$$\begin{array}{ccc} \mathcal{O}_c(\mathcal{M}) & \xrightarrow{\Phi_{\mathcal{M}}} & \mathscr{A}(\mathcal{M}) \\ \chi_* \downarrow & & \downarrow \mathscr{A}(\chi) \\ \mathcal{O}_c(\mathcal{M}') & \xrightarrow[\Phi_{\mathcal{M}'}]{} & \mathscr{A}(\mathcal{M}'). \end{array}$$

Diagram 3.2.1

Unfortunately, this result cannot be satisfactory if we want to implement supersimmetry transformation on the framework suggested by the generally covariant locality principle. The construction we just presented is made possible by suitable definition of all the categories involved, in particular the category $\mathfrak{sLoc}$ on which the functor $\mathscr{A}$ is defined. Then, looking at the morphism between objects in $\mathfrak{sLoc}$, we notice that all the maps $\chi : \mathcal{M} \to \mathcal{M}'$ are even, because even are the super algebra maps $\chi^* : \mathcal{O}(\mathcal{M}) \to \mathcal{O}(\mathcal{M})$, and the induced push forward $\chi_* : \mathcal{O}_c(\mathcal{M}) \to \mathcal{O}_c(\mathcal{M}')$. Hence, recalling that each element $F \in \mathcal{O}(\mathcal{M})$ can be splitted $F = F_0 \oplus F_1$ in an even and odd part, we can see that the splitting is transposed both at the level of the super algebra of fields, $\Phi_{\mathcal{M}}(F) = [F_0] \oplus [F_1]$, and when acting with a morphism on the background, $\Phi_{\mathcal{M}'}(\chi_* F) = [\chi_* F] = [(\chi_* F_0)] \oplus [\chi_* F_1]$. On the light of the last comments, this machinery for super-QFTs seems only a fancy construction to treat simultaneously bosonic and fermionic fields, without any chance to encode supersimmetry transformation of the fields. As announced before, the solution for the problems noticed can be found using advanced techniques in category theory (in particular the branch called enriched category theory) lying outside our background knowledge, hence we give only a sketch of the general idea followed in [HHS16]. A subclass of supermanifolds called *the class of superpoints* is introduced. The underlying topological space is taken to be a point and the sheaf is the assignment of the Grassmann algebra of dimension n, $\mathrm{pt}_n = (\{\bullet\}, \Lambda^n)$. Supermanifolds M and M' are extended to $\mathrm{pt}_n \times M$ and $\mathrm{pt}_n \times M'$ and morphisms $\chi : \mathrm{pt}_n \times M \to \mathrm{pt}_n \times M'$, at the level of sheaves $\chi^* : \Lambda^n \times \mathcal{O}(M) \to \Lambda^n \times \mathcal{O}(M')$, can be represented on elements of the form $\mathbb{1} \otimes F \in \Lambda^n \otimes \mathcal{O}(M')$ by (in multiindex notation):

$$\chi^*(\mathbb{1} \otimes F) = \sum_I \varepsilon^I \otimes \chi_I^*(F) \tag{3.2.15}$$

Transformation like this can reverse parity. Let us split $\mathbb{1} \otimes F$ in $\mathbb{1} \otimes F_0 \oplus F_1 = (\mathbb{1} \otimes F_0) \oplus (\mathbb{1} \otimes F_1)$ and take a transformation "parametrized" by the odd parameter ε, hence

$$\chi^*(\mathbb{1} \otimes F) = \mathbb{1} \otimes \chi_0^*(F) + \varepsilon \otimes \chi_\varepsilon^*(F) = \mathbb{1} \otimes \chi_0^*(F) + \varepsilon \otimes \chi_\varepsilon^*(F_0) \oplus \varepsilon \otimes \chi_\varepsilon^*(F_1) \tag{3.2.16}$$

and we can trivially notice parity of the term $F_{0/1}$ is reversed on the image of χ^*. Even though we skipped all the technical details of a thorough exposition, what is important to remark is that all this idea can be made rigorous introducing the concept of *enriched locally covariant quantum field theory* and to keep fixed the equation (3.2.15) when working on 2|2-dimensional models.

3.3 2|2 models

This section is devoted to the presentation of those super quantum field theories for which the background supermanifold is chosen to be of superdimension 2|2. Before proceeding towards a description of the theoretical representation data and towards a definition of a suitable full subcategory of $\mathfrak{ghsCart}$, we propose a quick overview of some remarkable features which characterise ordinary quantum field theories on two dimensional curved backgrounds.

Klein-Gordon and spinor field on two dimensional backgrounds

We first present a well-known fact concerning two dimensional Lorentzian manifolds. We recall that a Lorentzian manifold (M, g) is said to be *locally conformally flat* if at every point p is possible to find a chart (U, φ) $(p \in U)$, such that the metric coefficient in the coordinate basis can be written as $g_{ij}(p) = e^{2\sigma(p)}\eta_{ij}$. It can be proven that every two dimensional Lorentzian manifold is locally conformally flat (see for example [Nak03]). This property of two dimensional manifolds considerably simplify the description of the background geometry, also when it is given in terms of the data (M, g, ∇), defining a Lorentzian manifold which tangent bundle is endowed with the Levi-Civita connection. As explained in Section 2.2, the vielbein and the spin connection are defined as one-forms taking values in the translation algebra and in the Lorentz (or spin) algebra. Hence, before proceeding, we recall briefly the simple structure of those Lie-algebras in the following definition.

Definition 3.3.1. The two dimensional Lie algebra of translations is defined to be a vector space $\mathfrak{t}(2)$, on which is assigned the trivial Lie bracket. For an adapted basis $\{p_\alpha\}_{\alpha=0,1}$, we have $[p_\alpha, p_\beta] = 0$. The one dimensional Lie algebra $\mathfrak{so}(1, 1)$, (sometimes equivalently[4] denoted as $\mathfrak{spin}(1, 1)$), is trivially defined in term of one element K, which clearly commutes with itself. The Lie algebra consisting of the direct sum of them is called $2D$-Poincaré Lie algebra and is defined by

$$\mathfrak{poinc}(1, 1) = \mathfrak{spin}(1, 1) \oplus \mathfrak{t}(2) \tag{3.3.1}$$

$$[K, p_\alpha] = K_\alpha^\beta p_\beta \tag{3.3.2}$$

where $K_0^1 = K_1^0 = 1$ and $K_0^0 = K_1^1 = 0$. In the following we will sometime omit explicit references to the dimension and we will write $\mathfrak{t}$, $\mathfrak{spin}$, $\mathfrak{poinc}$

[4]The different notation for the same Lie algebra is due to the fact that the two dimensional proper orthocronus Lorentz group $SO_0(1, 1)$ and its double covering $Spin(1, 1)$ have the same associated Lie algebra (this a very well-known result in the theory of Lie groups and associated Lie algebra)

Remark 3.3.2. In order to enrich the framework depicted above, we can define two representations, both two dimensional, of $\mathfrak{so}(1,1)$. The first one is the standard representation on two dimensional vector spaces. Let's take a vector space V, with a basis $\{f_\alpha\}_{\alpha=0,1}$, recalling that a two dimensional vector space is always isomorphic (via an isomorphism preserving the Lie brackets) to $\mathfrak{t}(2)$

$$\begin{aligned} r_V : \mathfrak{so}(1,1) \times V &\to V \\ (\lambda K, v^\alpha f_\alpha) &\mapsto [\lambda K, v^\alpha f_\alpha] := \lambda v^\alpha \, K_\alpha^\beta f_\beta \end{aligned} \tag{3.3.3}$$

The second one is the spin representation and can be defined explicitly for $S = \mathbb{R}^2$ using the *Clifford algebra* of the gamma matrices on the complexification of S. They have to satisfy the relations

$$\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2\eta_{\alpha\beta} \mathrm{id}_{\mathbb{C}}. \tag{3.3.4}$$

Moreover, other elements of the algebra turn out to be useful, these are $\gamma_3 := \gamma_0\gamma_1$ and the antisymmetrized products

$$\sigma^{\alpha\beta} = \frac{1}{2}(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha) \tag{3.3.5}$$

In the following, we will take the representation defined by the matrices

$$\gamma_0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \gamma_1 = -\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{3.3.6}$$

Hence, the spin representation is given on a basis for S ($\{q_a\}_{a=1,2}$) by

$$\begin{aligned} \rho^S : \mathfrak{so}(1,1) \times S &\to S \\ (\lambda K, s^a q_a) &\mapsto \lambda s^a \frac{(\gamma_3)_a^b}{2} q_b \end{aligned} \tag{3.3.7}$$

Now, recalling the Definition 2.2.2 for the notion of vielbein, we can write

$$e = \hat{e}^\alpha \otimes p_\alpha = \mathsf{S}\delta_\mu^\alpha \mathrm{d}x^\mu \otimes p_\alpha \tag{3.3.8}$$

where S^2 is taken to be the always positive factor $e^{2\sigma}$ defining the equivalence above and S is called *conformal factor*. Then, solving the vanishing torsion constraint

$$\mathrm{d}\hat{e}^\alpha + K_\beta^\alpha \omega \wedge \hat{e}^\beta = 0$$

we can find an useful form for ω in term of S, expressed using the non-coordinate dual vector fields defined by $\langle \hat{V}_\alpha, \hat{e}^\beta \rangle = \delta_\alpha^\beta$

$$\omega = \hat{e}^\beta K_\beta^\alpha \hat{V}_\alpha(\mathsf{S}) \ . \tag{3.3.9}$$

Hence, using the representation given in Remark 3.3.2 we can write explicitly the field operators defined in Chapter 2 for the two dimensional case. We have that, for M two dimensional, the Klein-Gordon operator can be given using a sort of 'covariant differential' (in analogy with those for the spinor field) induced by the representation r_L and acting on object of the form $\hat{V}_\alpha(\phi)$ (with $\phi \in C^\infty(M)$) this way

$$\mathsf{d}_{r,\omega} \hat{V}_\alpha(\phi) := \mathrm{d}\hat{V}_\alpha(\phi) + \omega K_\alpha^\beta \hat{V}_\beta(\phi) \ . \tag{3.3.10}$$

Hence, we get

$$\begin{aligned} P_\square : C^\infty(M) &\to C^\infty(M) \\ \phi &\mapsto P_\square \phi := \eta^{\alpha\beta} \langle \hat{V}_\alpha, \mathsf{d}_{\mathsf{r},\omega} \hat{V}_\beta(\phi) \rangle, \end{aligned} \tag{3.3.11}$$

that after some manipulations becomes

$$P_\square \phi = \eta^{\alpha\beta} \hat{V}_\beta \left(\hat{V}_\alpha(\phi) \right) + \epsilon^{\alpha\gamma} \omega_\alpha \hat{V}_\gamma(\phi) \tag{3.3.12}$$

Analogously, we can give an explicit form for the operator (2.2.23) acting on $\psi \in \Gamma(M, M \times S)$:

$$(\not{D}_{\rho,\omega}\psi)_a = (\gamma^\alpha)_a^b \langle \hat{V}_\alpha, \mathrm{d}\psi_b \rangle + (\gamma^\alpha)_a^b \langle \hat{V}_\alpha, \rho^S(\omega)_b^c \psi_c \rangle \tag{3.3.13}$$

$$= (\gamma^\alpha)_a^b \hat{V}_\alpha(\psi_b) + (\gamma^\alpha)_a^b \omega_\alpha \frac{(\gamma_3)_a^c}{2} \psi_c \tag{3.3.14}$$

3.3.1 A $\mathfrak{sLoc}$ category for 2|2 dimensional spacetime

Here we present the application of the theoretical construction of Section 3.2 to 2|2-dimensional super-Cartan supermanifolds ($\mathcal{M}$). As established above, we first have to fix the set of theoretic data $(\mathbb{R}^{1,n-1}, S, \rho, \rho^S, \mathfrak{o}, \mathfrak{o}_S, \Gamma, C, \varepsilon)$. The first datum defines the spin and Lorentz group acting on fields and hence is fixed by the choice of the dimension for the supermanifold. We will work with $\mathbb{R}^{1,1}$ that is the vector space $\mathbb{R}^2$, endowed with the flat metric η. The canonical basis of $\mathbb{R}^2$ will be denoted by $\{p_\alpha\}_{\alpha=0,1}$, denoting the metric coefficients by $\eta_{\alpha,\beta} = \eta(p_\alpha, p_\beta)$. The orientation $\mathfrak{o}$ is fixed by the basis. Furthermore, we take as positive cone the set $C \subseteq \mathbb{R}^2$, defined by $C = \{v \in \mathbb{R}^2,\ \eta(v,v) > 0 \text{ and } v^0 > 0\}$ The spin representation is defined taking as vector space $S = \mathbb{R}^2$ and using gamma matrices defined in Remark 3.3.2. We list the other items below, point by point, checking the relevant properties.

1. The $\mathrm{SO}_0(1,1)$ group consist of all the linear maps Λ such that $\eta(w,w) = \eta(\Lambda w, \Lambda w)$. It is easy to see that $\mathrm{SO}_0(1,1)$ can be mapped via a Lie group isomorphism to $(\mathbb{R}, +)$. We can define hence a representation of its action on $\mathbb{R}^2$ fixing the canonical basis and representing the elements of the group by the matrices

$$\Lambda(\alpha) = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix}. \tag{3.3.15}$$

With this notation, for all $w \in \mathbb{R}^2$

$$\rho : \mathrm{SO}_0(1,1) \times \mathbb{R}^2 \to \mathbb{R}^2 \tag{3.3.16}$$

$$(\alpha, w) \mapsto \rho(\Lambda, w) = \Lambda(\alpha) w. \tag{3.3.17}$$

2. The spin representation ρ^S is defined taking $S = \mathbb{R}^2$, endowing it with the canonical basis $\{q_a\}$ (a=1,2) and describing the element of the algebra in matrix form

$$\Sigma(\alpha) = \begin{pmatrix} e^{-\frac{\alpha}{2}} & 0 \\ 0 & e^{\frac{\alpha}{2}} \end{pmatrix}. \tag{3.3.18}$$

The action is hence

$$\rho : \mathrm{Spin}_0(1,1) \times \mathbb{R}^2 \to \mathbb{R}^2$$

$$(\alpha, s) \mapsto \rho^S(\Sigma, w) = \Sigma(\alpha) s. \tag{3.3.19}$$

3. The pairing Γ is defined using the pairing of $\mathbb{R}^2$

$$\Gamma : S \otimes S \to \mathbb{R}^2 \tag{3.3.20}$$

$$(s_1, s_2) \mapsto \Gamma(s_1, s_2) = (i\gamma^\alpha)^{ab} {s_1}^a {s_2}^b p_\alpha. \tag{3.3.21}$$

Looking at this definition is easy to prove that the pairing is symmetric and positive with respect to the cone. Moreover, we can prove covariance with respect to the representations defined above.

Proposition 3.3.3. *The pairing* Γ *defined by equation* (3.3.20) *is covariant with respect to the* $\mathrm{SO}_0(1,1)$ *and* $\mathrm{Spin}(1,1)$ *representation (respectively defined in* (3.3.15) (3.3.18)*), i.e. the following formula holds true*

$$\Gamma\left(\Sigma_{(\alpha)}s, \Sigma_{(\alpha)}\tilde{s}\right) = \Lambda_{(\alpha)}\Gamma(s,\tilde{s}). \tag{3.3.22}$$

Proof. We explicit all the definition expanding vectors in column form, using the canonical basis of $\mathbb{R}^2$, hence $w = (w_1, w_2)^T$ and $s = (s_1, s_2)^T$. In this notation $\Gamma(s,\tilde{s}) = (s_1\tilde{s}_1 + s_2\tilde{s}_2, -s_1\tilde{s}_1 + s_2\tilde{s}_2)^T$ and we have

$$\Gamma(\Sigma_{(\alpha)}s, \Sigma_{(\alpha)}\tilde{s}) = \begin{pmatrix} e^{-\alpha}s_1\tilde{s}_1 + e^{\alpha}s_2\tilde{s}_2 \\ -e^{-\alpha}s_1\tilde{s}_1 + e^{\alpha}s_2\tilde{s}_2 \end{pmatrix}, \tag{3.3.23}$$

Calculating the right hand side of (3.3.22), we have that

$$\Lambda_\alpha\Gamma(s,\tilde{s}) = \begin{pmatrix} (\cosh(\alpha) - \sinh(\alpha))s_1\tilde{s}_1 + (\cosh(\alpha) + \sinh(\alpha))s_2\tilde{s}_2 \\ (-\cosh(\alpha) + \sinh(\alpha))s_1\tilde{s}_1 + (\cosh(\alpha) + \sinh(\alpha))s_2\tilde{s}_2) \end{pmatrix}.$$

and recalling the definition of $\cosh(\alpha) = (e^\alpha + e^{-\alpha})/2$ and $\sinh(\alpha) = (e^\alpha - e^{-\alpha})/2$, we can conclude the proof. □

4. The symplectic form on S is defined using the antisymmetric symbols $\varepsilon_{ab} = -\varepsilon_{ba}$, with $\varepsilon_{12} = 1$. We impose

$$\varepsilon : S \otimes S \to \mathbb{R} \tag{3.3.24}$$

$$(s,\tilde{s}) \mapsto \varepsilon(s,\tilde{s}) = \varepsilon_{ab}s^a\tilde{s}^b. \tag{3.3.25}$$

That is invariant under the action of the $\mathrm{Spin}(1,1)$ representation. Indeed, we have in column form

$$\varepsilon(\Sigma_{(\alpha)}s, \Sigma_{(\alpha)}\tilde{s}) = (e^{-\frac{\alpha}{2}}s_1, e^{\frac{\alpha}{2}}s_2)\begin{pmatrix} e^{\frac{\alpha}{2}}\tilde{s}_2 \\ -e^{-\frac{\alpha}{2}}\tilde{s}_1 \end{pmatrix} = s_1\tilde{s}_2 - \tilde{s}_1 s_2 = \varepsilon(s,\tilde{s})$$

Now we shall study the super Poincaré super-Lie algebra that can be constructed starting from this data following Definition 3.2.1. As first step, we calculate the action of

$\mathfrak{spin}(1,1)$ algebra induced by ρ and ρ^S. We have,

$$\begin{aligned}\rho_* :&\mathfrak{spin}(1,1)\times\mathbb{R}^2\to\mathbb{R}^2\\ &(\alpha,v)\mapsto L_{(\alpha)}v=\begin{pmatrix}0&\alpha\\ \alpha&0\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}\end{aligned}\tag{3.3.26}$$

and

$$\begin{aligned}\rho^S_* :&\ \mathfrak{spin}(1,1)\times\mathbb{R}^2\to\mathbb{R}^2\\ &(\alpha,s)\mapsto \sigma_{(\alpha)}s=\begin{pmatrix}-\frac{\alpha}{2}&0\\ 0&\frac{\alpha}{2}\end{pmatrix}\begin{pmatrix}s_1\\ s_2\end{pmatrix}\end{aligned}\tag{3.3.27}$$

Summing up, the algebra $\mathfrak{sp}=(\mathfrak{spin}(1,1)\oplus\mathbb{R}^2)\oplus S$, is composed by the set of generator $\{K,p_\mu,q_a\}$, the first one is the so called *boost generator*, while the p_μ label the *even translation generator* and the q_a label the *odd translation generator*. Using equation (3.2.2), after a straightforward but tedious calculation, the commutators defining the structure of the algebra can be calculated. We show here the list (let's notice that just in the following formulas, repetition of the index "a" is not intended as understood summation)

$$[K,K]=0\qquad [K,p_\mu]=k_\mu^\nu p_\nu\qquad [K,q_a]=\frac{(-1)^a}{2}\,q_a\tag{3.3.28}$$

$$[p_\mu,q_a]=0\qquad [q_a,q_b]=2\Gamma(q_a,q_b)=2\delta_{ab}(p_0+(-1)^a p_1)\tag{3.3.29}$$

Once the representation data have been fixed, we can use them to give a complete characterisation of the objects in the category of $\mathfrak{ghsCart}$ for 2|2-dimensional supermanifolds. In addition, we consider only super-Cartan supermanifolds $\mathcal{M}$ whose reduced manifold $\widetilde{M}$ is connected and whose structure sheaf $\mathcal{O}_M$ is globally isomorphic to $C^\infty_{\widetilde{M}}\otimes\Lambda(\mathbb{R}^2)$. Hence, we can find global odd coordinates and label them by θ^a, $a=1,2$. In order to simplify the notation we can set $\overline{\theta}:=\theta^1\theta^2$. As presented in Definition 3.2.2, the objects of this category are completely defined once a supervielbein $E\in\Omega^1(M,\mathfrak{st})$ is assigned together with a super spin connection $\Omega\in\Omega^1(M,\mathfrak{sp})$. At this stage we can give a general expression for both one-forms, expanding in the adapted basis of the super translation algebra

$\mathfrak{st}$, $\{p_\alpha, q_a\}$, hence $E = e^\alpha \otimes p_\alpha + \xi^a \otimes q_a$ where

$$e^\alpha = \widetilde{e}^\alpha + f^\alpha \overline{\theta} - \mathrm{d}\theta^b \theta^c h^\alpha_{bc} \tag{3.3.30}$$

$$\xi^a = g^a_b \theta^b + \mathrm{d}\theta^b (k^a_b + l^a_b \overline{\theta}) \tag{3.3.31}$$

with $f^\alpha, g^a_b \in \Omega^1(\widetilde{M})$ and $h^\alpha_{bc}, k^a_b, l^a_b \in C^\infty(\widetilde{M})$. Moreover, due to the condition of non-degeneracy for the supervielbein, functions k^a_b have to be invertible and then we can absorb them with a change of coordinates (this means substituting $k^a_b \leftrightarrow \delta^a_b$). The set of one-form $\widetilde{e}^\alpha$ can be used to compute the vielbein on the reduced manifold $\widetilde{e}^\alpha \otimes p_\alpha$. The inverse of the supervielbein can be calculated in terms of dual vector fields $\{\widetilde{X}_\alpha\}_{\alpha=0,1}$ and dual super derivation $\{\partial_b\}_{b=1,2}$ defined by the duality relations $\langle \widetilde{X}_\alpha, \hat{e}^\beta \rangle = \delta^\beta_\alpha$, $\langle \partial_b, \mathrm{d}\theta^a \rangle = \delta^a_b$, $\langle \partial_a, \hat{e}^\beta \rangle = 0$ and $\langle \widetilde{X}_\alpha, \mathrm{d}\theta^b \rangle = 0$. The result are four quantities $\{X_\alpha, D_a\}$ ($\alpha = 0, 1$, $a = 1, 2$), obtained as combination with coefficient in $C^\infty(\widetilde{M})$ of $\widetilde{X}_\alpha$ and ∂_a, fulfilling the relations

$$\langle X_\alpha, e^\alpha \rangle = \delta^\beta_\alpha \quad \langle X_\alpha, \xi^b \rangle = 0 \quad \langle D_a, \xi^b \rangle = \delta^b_a \quad \langle D_a, e^\beta \rangle = 0 \tag{3.3.32}$$

The super-spin connection, being the $\mathfrak{spin}$ algebra one dimensional, can be written in term of an even element $\omega \in \Omega^1(M)$. Hence,

$$\Omega = \omega \otimes K \tag{3.3.33}$$

In particular, the even one form ω can be reduced at the choice of three smooth functions, i.e.

$$\omega = \widetilde{e}^\alpha (\rho_\alpha + \sigma_\alpha \overline{\theta}) + \mathrm{d}\theta^a \theta^b \phi_{ab} \tag{3.3.34}$$

with in $\rho_\alpha, \sigma_\alpha, \phi_{ab} \in C^\infty(\widetilde{M})$. Before proceeding let's see the simplest example of $2|2$-dimensional super-Cartan supermanifold.

Example 3.3.4. We call *super Minkowski spacetime* $\mathbb{M}$ the super-Cartan supermanifold consisting of the following data: as underlying supermanifold we take $M = \mathbb{R}^{2|2}$, the super-spin connection Ω is set to be zero, and the supervielbein is given, using global coordinates of $\mathbb{R}^2$, by

$$E = \left(\mathrm{d}x^\alpha - \mathrm{d}\theta^b \gamma^\alpha_{ba} \theta^a\right) \otimes p_\alpha + \mathrm{d}\theta^a \otimes q_a \tag{3.3.35}$$

Dual odd and even superderivation take the form[5]:

$$\widetilde{X}_\alpha = \partial_\alpha \tag{3.3.36}$$

$$D_a^{\mathbb{M}} = \partial_a + \theta^b \gamma^\alpha_{ab} \partial_\alpha \tag{3.3.37}$$

Now, we already restricted the class of all the super-Cartan supermanifolds requiring additional hypothesis on the reduced manifold and the structure sheaf, in order to simplify expressions for the mathematical quantities involved. As we explained in the previous section, we are allowed to work with a full subcategory of $\mathfrak{ghsCart}$, hence this restrictions are perfectly compatible with the theoretical framework. In particular, we can add some constraints on the supertorsion $T_{\mathcal{M}}$ associated to each object $\mathcal{M} = (M, \Omega, E)$ (see eq. (3.2.4) after Definition 3.2.2). These constraints are the so called supergravity supertorsion constraints and are introduced with the aim to produce a reasonable theory with treatable calculations and physically sensible field operators. We will use, an adapted version of the supertorsion constraints proposed in [How79] that together with the parity requirements already imposed will provide an interesting class of supervielbein. Implementation of this constraints will be resumed in the following proposition.

Proposition 3.3.5. *Let's consider an object in $\mathcal{M} = (M, \Omega, E)$ in $\mathfrak{ghsCart}$, with connected reduced manifold $\widetilde{M}$ and with the structure sheaf of M ($\mathcal{O}_M$) globally isomorphic to $C^\infty(\widetilde{M}) \otimes \Lambda(\mathbb{R}^2)$. Expanding the supervielbein in the form $E = e^\alpha \otimes p_\alpha + \xi^a \otimes q_a$ and considering the supertorsion defined by eq.*(3.2.4) *expanded as*

$$T_{\mathcal{M}} = \left(\mathrm{d}\xi^a \wedge \mathrm{d}\xi^b T^\alpha_{ab} + \hat{e}^\beta \wedge \mathrm{d}\xi^c T^\alpha_{\beta c} + \hat{e}^\beta \wedge \hat{e}^\gamma T^\alpha_{\beta\gamma}\right) \otimes p_\alpha \tag{3.3.38}$$

$$+ \left(\mathrm{d}\xi^a \wedge \mathrm{d}\xi^b T^a_{ab} + \hat{e}^\beta \wedge \mathrm{d}\xi^c T^a_{\beta c} + \hat{e}^\beta \wedge \hat{e}^\gamma T^a_{\beta\gamma}\right) \otimes q_a, \tag{3.3.39}$$

the supertorsion constraints

$$T^\alpha_{ba} = (\gamma^\alpha)_{ba}, \qquad T^\alpha_{\beta a} = 0, \qquad T^\alpha_{\beta\gamma} = 0\ , \tag{3.3.40}$$

[5] We notice that we use subscript greek letters in order to distinguish coordinate vector fields of $\{\partial_\alpha\}$ from dual vector fields ($\{\partial_a\}$) of $\{\mathrm{d}\theta^a\}$ (a=1,2).

are solved by E and $\Omega = \omega \otimes K$ defined by:

$$e^\alpha = \hat{e}^\alpha - \mathrm{d}\theta^a \theta^b (\gamma^\alpha)_{ba} \tag{3.3.41}$$
$$\xi^a = \mathrm{d}\theta^a - \frac{\omega}{2}\theta^b (\gamma_3)_b^a, \tag{3.3.42}$$

where ω and $\hat{e}^\alpha$ fulfil the ordinary torsion constraint $\mathrm{d}\hat{e}^\alpha + \omega \wedge K^\alpha_\beta \hat{e}^\beta = 0$.

Proof. We will not present the full amount of calculations needed to solve the constraint equations, but we will only show how the calculation can be done solving the constraints for T^α. From (3.2.4) we have:

$$\begin{aligned} T^\alpha &= \mathrm{d}e^\alpha + \omega \wedge K^\alpha_\beta e^\beta \\ &= \underbrace{(\mathrm{d}\hat{e}^\alpha + \omega \wedge K^\alpha_\beta \hat{e}^\beta)}_{=0} - \mathrm{d}(\mathrm{d}\theta^a \theta^b \gamma^\alpha_{ba}) - \omega \wedge K^\alpha_\beta \mathrm{d}\theta^a \theta^b \gamma^\beta_{ba} \end{aligned}$$

Hence, we write

$$T^\alpha = \mathrm{d}\theta^a \wedge \mathrm{d}\theta^b \gamma^\alpha_{ba} - \omega \wedge K^\alpha_\beta \mathrm{d}\theta^a \theta^b \gamma^\beta_{ba} \tag{3.3.43}$$

Now, we can calculate the supertorsion even part resulting from constraints. We have:

$$T^\alpha = \xi^a \wedge \xi^b \gamma^\alpha_{ba}$$

and substituting the ξ^a from (3.3.42) we get

$$T^\alpha = \mathrm{d}\theta^a \wedge \mathrm{d}\theta^b \gamma^\alpha_{ba} + \omega \wedge \mathrm{d}\theta^a \theta^b (\gamma_3 \gamma^\alpha)_{ba}$$

and, noting that $-K^\alpha_\beta (\gamma^\beta)_{ba} = (\gamma_3 \gamma^\alpha)_{ba}$, the latter matches exactly equation (3.3.43). □

Hence, thanks to the last proposition we can select an interesting class of super-Cartan super manifolds and give the definition of a suitable category.

Definition 3.3.6. We define the category $2|2$-$\mathfrak{sLoc}$ as the full subcategory of $\mathfrak{ghsCart}$ which objects are those super-Cartan supermanifold (M, Ω, E) satisfying the conditions:

i. The reduced manifold $\widetilde{M}$ associated to M is connected.

ii. The structure sheaf $\mathcal{O}_M$ of M is globally isomorphic to $C^\infty_{\widetilde{M}} \otimes \Lambda(\mathbb{R}^2)$.

iii. The super-spin connection and the supervielbein, expanded in the adapted basis of the algebra $\mathfrak{sp}$, take the form

$$\Omega = \omega \otimes K \tag{3.3.44}$$

$$E = \left(\hat{e}^\alpha - \mathrm{d}\theta^a\theta^b(\gamma^\alpha)_{ba}\right) \otimes p_\alpha + \left(\mathrm{d}\theta^a - \frac{\omega}{2}\theta^b(\gamma_3)^a_b\right) \otimes q_a, \tag{3.3.45}$$

whit the additional condition $\mathrm{d}\hat{e}^\alpha + \omega \wedge K^\alpha_\beta \hat{e}^\beta = 0$.

Remark 3.3.7. With those definition for the super-spin connection (3.3.44) and the supervielbein (3.3.45) the supercurvature two form $R_{\mathcal{M}} \in \Omega^1(M, \mathfrak{spin}(1,1))$ can be easily calculated. From (3.2.5)

$$R_{\mathcal{M}} := \mathrm{d}\Omega + [\Omega, \Omega]_{\wedge, \mathfrak{sp}} = \mathrm{d}\omega \otimes K + \omega \wedge \omega \otimes [K, K] = \mathrm{d}\omega \otimes K. \tag{3.3.46}$$

We claimed that the restriction of the admissible background geometries allows us to produce sensible super field theory. In order to convince the reader of this fact, we show now how a super Green's hyperbolic operator can be defined using the supervielbein (3.3.45) and the super-spin connection (3.3.44) and how it's possible to define a pairing for elements in $\mathcal{O}_c(\mathcal{M})$, using (3.2.9)

Inspired by the definition of the field operator for spinor fields in Section 2.2, we first give the inverse of the supervielbein coefficients

$$X_\alpha = \hat{e}_\alpha + \frac{\omega_\alpha}{2}\theta^b(\gamma_3)^a_b\partial_a \tag{3.3.47}$$

$$D_a = \partial_a - + \frac{\omega_\alpha}{2}(\gamma^\alpha\gamma_3)^n_a\overline{\theta}\partial_n + (\gamma^\alpha)_{as}\theta^s\hat{e}_\alpha \tag{3.3.48}$$

satisfying the relations (3.3.32). Then, for any object in 2|2-$\mathfrak{sLoc}$ we define the operator $P_{\mathcal{M}}$ acting on elements $\Phi \in \mathcal{O}(\mathcal{M})$ as

$$P_{\mathcal{M}} = \frac{1}{2}\left(\epsilon^{ab} D_{\Omega b} \circ D_a\right)(\Phi) \tag{3.3.49}$$

where the D_a acts as an usual derivation on Φ, while the action of $D_{\Omega b}$ is ruled by the super-spin connection, via the representation

$$\rho(\omega)^b_a := \frac{\omega}{2}(\gamma_3)^b_a\,, \tag{3.3.50}$$

becoming in conclusion

$$(D_{\Omega b} \circ D_a)(\Phi) = D_{\Omega b}(D_a(\Phi)) := D_b(D_a(\Phi)) - \langle D_b, \rho(\omega)^c_a \rangle D_c(\Phi). \tag{3.3.51}$$

Using the latter defining formula we can expand the operator $P_{\mathcal{M}}$ making explicit the role of the spin-connection

$$P_{\mathcal{M}}(\Phi) = \frac{1}{2}\epsilon^{ab} D_b(D_a(\Phi)) + \frac{\omega_\alpha}{4}\theta^s(\gamma^\alpha\gamma_3)^c_s D_c(\Phi) \tag{3.3.52}$$

In order to better understand the form of the super-differential operator and to obtain a comparison with well-known ordinary field theories (Klein-Gordon field and Dirac field), we check the action of $P_{\mathcal{M}}$ on elements of $\mathcal{O}(\mathcal{M})$ using the expansion

$$\Phi = \varphi + \psi_a\theta^a + \eta\overline{\theta}. \tag{3.3.53}$$

After some calculations we find:

$$P_{\mathcal{M}}(\varphi) = \frac{\overline{\theta}}{2}\left(\eta^{\alpha\beta}\widetilde{X}_\alpha\left(\widetilde{X}_\beta(\varphi)\right) + \omega_\alpha\varepsilon^{\beta\alpha}\widetilde{X}_\beta(\varphi)\right) \tag{3.3.54}$$

$$\begin{aligned} P_{\mathcal{M}}(\psi_a\theta^a) &= \frac{\theta^s}{2}\left((\gamma^\alpha)\widetilde{X}_\alpha(\psi_a) + \underbrace{K^\alpha_\beta(\gamma^\beta)^c_s}_{=(\gamma^\alpha\gamma_3)^c_s}\frac{\omega_\alpha}{2}\psi_c\right) \\ &= \frac{\theta^s}{2}\left(\gamma^\alpha\langle\widetilde{X}_\alpha, \mathrm{d}\psi\rangle + (\gamma^\alpha)^r_s\langle\widetilde{X}_\alpha, \rho(\omega)^c_r\rangle\psi_c\right) \end{aligned} \tag{3.3.55}$$

$$P_{\mathcal{M}}(\eta\,\overline{\theta}) = -\eta \tag{3.3.56}$$

Now, we can keep track of the "parity exchanges" induced by our super-differential operator and we are allowed to write $P_{\mathcal{M}}$ in the matrix form acting on vectors whose entries are the element of the decomposition (3.3.53), ordered by increasing degree, whereas the elements of the matrix are field operators from ordinary QFT in curved backgrounds:

$$P_{\mathcal{M}}(\Phi) = \begin{pmatrix} 0 & 0 & -\mathrm{id}_{C^\infty(\widetilde{M})} \\ 0 & \nabla\!\!\!\!/ & 0 \\ \Box & 0 & 0 \end{pmatrix}\begin{pmatrix} \varphi \\ \psi_a \\ \eta \end{pmatrix} \tag{3.3.57}$$

Then, we can look at self-adjointness of this operator with respect to the pairing induce by the supervielbein. The latter, on two expanded element of $\mathcal{O}_c(\mathcal{M})$

$$\Phi_1 = \varphi_1 + \psi_{1a}\theta^a + \eta_1\overline{\theta} \qquad \Phi_2 = \varphi_2 + \psi_{a2}\theta^a + \eta_2\overline{\theta},$$

takes the form

$$\langle \Phi_1, \Phi_2 \rangle_{\mathcal{M}} := \int_M \mathrm{Ber}(E)\Phi_1\Phi_2 = \int_{\widetilde{M}} \mathrm{dvol}_{\widetilde{M}} \left(\varphi_1\eta_2 + \epsilon^{ba}\psi_{1a}\psi_{2b} + \varphi_2\eta_1\right) \tag{3.3.58}$$

On the light of the last considerations we can prove the following proposition.

Proposition 3.3.8. *The operator $P_M : \mathcal{O}_c(\mathcal{M}) \to \mathcal{O}_c(\mathcal{M})$ is super-self adjoint with respect to the pairing defined by the formula* (3.3.58)*, i.e. for elements of definite parity the following equality holds true*

$$\langle \Phi_1, P_{\mathcal{M}}\Phi_2 \rangle_{\mathcal{M}} = (-1)^{|P_{\mathcal{M}}||\Phi_1|}\langle P_{\mathcal{M}}\Phi_1, \Phi_2 \rangle_{\mathcal{M}} \tag{3.3.59}$$

Proof. We give the proof only for the case of two even elements, showing a scheme that can be applied for all the other situations. Hence, we take

$$\Phi_1 = \varphi_1 + \overline{\theta}\eta_1 \qquad \Phi_2 = \varphi_2 + \overline{\theta}\eta_2 \tag{3.3.60}$$

$$P_{\mathcal{M}}\Phi_1 = -\eta_1 + \overline{\theta}\Box\varphi_1 \qquad P_{\mathcal{M}}\Phi_2 = -\eta_2 + \overline{\theta}\Box\varphi_2 \tag{3.3.61}$$

and then the two sides of (3.3.59) becomes

$$\langle \Phi_1, P_{\mathcal{M}}\Phi_2 \rangle_{\mathcal{M}} = -\int_{\widetilde{M}} \eta_2\eta_1 + \int_{\widetilde{M}} \varphi_1(\Box\varphi_2) \tag{3.3.62}$$

$$\langle P_{\mathcal{M}}\Phi_1, \Phi_2 \rangle_{\mathcal{M}} = -\int_{\widetilde{M}} \eta_1\eta_2 + \int_{\widetilde{M}} (\Box\varphi_1)\varphi_2. \tag{3.3.63}$$

And we can conclude recalling the self-adjointness of the d'Alembertian operator with respect to the standard pairing between smooth compactly supported functions. □

Furthermore, using the formalism involving matrix and vectors to represent the action of the super field operator we can built up super-Green's operators for $P_{\mathcal{M}}$, we have indeed that

$$G^{\pm}_{\mathcal{M}}(\Phi) = \begin{pmatrix} 0 & 0 & G^{\pm}_{\Box} \\ 0 & G^{\pm}_{\not\nabla} & 0 \\ -\mathrm{id}_{C^\infty(\widetilde{M})} & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi_a \\ \eta \end{pmatrix}, \tag{3.3.64}$$

where $G^{\pm}_{\Box}$ and $G^{\pm}_{\nabla\!\!\!/}$ are Green's operator for ordinary fields operator $\Box$ and $\nabla\!\!\!/$.

Remark 3.3.9. We showed how one of the simpler super field operator can be constructed and in principle one can use this prototype operator as a building block of many others. For example, a field operator of remarkable interest is defined as

$$P^m_{\mathcal{M}} := P_{\mathcal{M}} + m\,\mathrm{id}_{\mathcal{O}(\mathcal{M})} : \mathcal{O}(\mathcal{M}) \to \mathcal{O}(\mathcal{M}). \tag{3.3.65}$$

Self-adjointness of $P^m_{\mathcal{M}}$ is an easy consequence of the self-adjointness of $P_{\mathcal{M}}$ and super Green's operator can be calculated once the matrix formalism is recovered. We get the form for the super field operator

$$P^m_{\mathcal{M}} = \begin{pmatrix} m & 0 & -1 \\ 0 & \nabla\!\!\!/ + m & 0 \\ \Box + m^2 & 0 & m \end{pmatrix} \tag{3.3.66}$$

and, hence, the Green's operator can be easily proven to be

$${}^{m}G^{\pm}_{\mathcal{M}} = \begin{pmatrix} m\, G^{\pm}_{\Box+m^2} & 0 & G^{\pm}_{\Box+m^2} \\ 0 & G^{\pm}_{\nabla\!\!\!/+m} & 0 \\ -\Box \circ G^{\pm}_{\Box+m^2} & 0 & m\, G^{\pm}_{\Box+m^2} \end{pmatrix} \tag{3.3.67}$$

where $G^{\pm}_{\Box+m^2}$, $G^{\pm}_{\nabla\!\!\!/+m}$ are Green's operator for differential operator written as subscripts.

3.3.2 Something more than morphism for 2|2-$\mathfrak{sLoc}$

Now that all the feature of the super field theory defined by the operator $P_{\mathcal{M}}$ have been established, one can also prove that all the hypothesis for the definition of an *enriched field theory* in the sense proposed by [HHS16, pg. 34, Def. 6.15]. Hence, we can proceed with the study of supersymmetry transformations. Actually, what we show now is not a full characterization of the so called *superset of the enriched morphism* of two objects in 2|2-$\mathfrak{sLoc}$, but we look only at those transformation coming from the first order truncation of the expansion (3.2.15). Recalling that formula here

$$\chi^*(\mathbb{1} \otimes F) = \sum_I \varepsilon^I \otimes \chi^*_I(F)\ ,$$

we have, at the first order in the odd "parameter" ε,

$$\chi^*(\mathbb{1}\otimes F) = \mathbb{1}\otimes\chi_0^*(F) + \varepsilon\otimes\chi_1^*(F)\ . \tag{3.3.68}$$

Now, in order to give an useful condition for the practical research of supersimmetry transformations we force χ^* to be a superalgebra morphism, i.e. we want it to preserve the supercommutative products (in the following denoted by the same symbol "$\cdot$" for simplicity). Then for $F, G \in \mathcal{O}(\mathcal{M}')$

$$\begin{aligned}\chi^*(\mathbb{1}\otimes F\cdot G) =& \mathbb{1}\otimes\chi_0^*(F\cdot G) + \varepsilon\otimes\chi_1^*(F\cdot G) \qquad (3.3.69)\\ \chi^*(\mathbb{1}\otimes F)\cdot\chi^*(\mathbb{1}\otimes G) =& (\mathbb{1}\otimes\chi_0^*(F) + \varepsilon\otimes\chi_1^*(F))\cdot(\mathbb{1}\otimes\chi_0^*(G) + \varepsilon\otimes\chi_1^*(G))\\ =& \mathbb{1}\otimes\chi_0^*(F)\chi_0^*(G) + \varepsilon\otimes\left(\chi_1^*(F)\chi_0^*(G) + (-)^{|F|}\chi_0^*(F)\chi_1^*(G)\right)\ ,\end{aligned}$$

and the second expression perfectly fits the first if χ_0^* is induced by a $\mathfrak{sCart}$-morphism preserving the reduced vielbein and χ_1^* is an odd superderivation. From now on, we denote $\chi_1 := Q$ and we say that the morphsim χ is a *supersimmetry transformation generated by* Q, in order to recover the standard notation and language of super-QFT. Then, recalling that morphism and enriched morphism in the category 2|2-$\mathfrak{sLoc}$ or in its enriched version shall preserve the geometry of the super-Cartan supermanifolds involved, we find a set of necessary condition defining Q: the Lie-derivative of the geometrical quantities built up from the supervielbein, calculated along the odd superderivation Q should be vanishing, for example

$$\mathcal{L}_Q(e^\alpha) = 0\ , \qquad \mathcal{L}_Q(\xi^a) = 0\ , \tag{3.3.70}$$

$$\mathcal{L}_Q(R_\mathcal{M}) = 0\ , \qquad \mathcal{L}_Q(\Omega) = 0\ . \tag{3.3.71}$$

Now, if we try to implement the previous conditions we discover that an object $\mathcal{M}$ in the category 2|2-$\mathfrak{sLoc}$ has non trivial supersimmetry transformations generated by Q if and only if its supercurvature is vanishing. The latter condition is fulfilled only by the so called *super Minkowski spacetime*, for which we can understand symmetries in term of the odd superderivations.

Proposition 3.3.10. *Given an objects $\mathcal{M}, \mathcal{M}'$ in 2|2-$\mathfrak{sLoc}$, the class of supersymmetry transformation which can be written as a pair (χ_0, Q), where χ_0 is an $\mathfrak{sCart}$-morphism and Q is a superderivation acting as in equation* (3.3.69)*, is empty if $R_\mathcal{M}$ (or equivalently $R_{\mathcal{M}'}$) is a non zero element of $\Omega^1(M, \mathfrak{spin}(1,1))$. Whereas, if $\mathcal{M}, \mathcal{M}'$ are objects satisfying the* *additional constraint $R_{\mathcal{M}} = R_{\mathcal{M}'} = 0$, we find that the supersymmetry transformations are generated by the odd superderivations*

$$Q_a^{\mathsf{s}} = \mathsf{S}^{\frac{1}{2}}\left(\partial_a - \theta^b \gamma^\alpha_{ba}\partial_\alpha\right) \qquad \textit{for} \quad a = 1, 2\ . \tag{3.3.72}$$

Proof. We prove this statement using the necessary conditions listed in (3.3.70) and (3.3.71), where all the geometric quantities are those defining objects in 2|2-$\mathfrak{sLoc}$ (see Definition 3.3.6). Without loss of generality, we can make the ansatz for the superderivation[6]:

$$Q = \theta^a j_a^\alpha \widetilde{X}_\alpha + (h^b + k^c\,\overline{\theta})\partial_c. \tag{3.3.73}$$

where j_a^α, h^b, k^c are functions in $C^\infty(\widetilde{M})$. We first impose the condition $\mathcal{L}_Q(e^\alpha) = 0$, using the *Cartan formula* (see [Mor01, thm. 2.11, pg.74]) which holds true for a general super differential form

$$\mathcal{L}_Q(\rho) = \iota_Q(\mathrm{d}\rho) + \mathrm{d}(\iota_Q(\rho))\ .$$

Hence, we get that the following equations which have to be fulfilled simultaneously by j_a^α, h^b, k^c,

$$\begin{cases} \mathrm{d}\theta^r k^c \gamma^\alpha_{sa}\overline{\theta} = 0 \\ \theta^s\left(K^\alpha_\beta j^\beta_s \omega - j^\gamma_s \omega_\gamma K^\alpha_\beta \widetilde{X}^\beta + \mathrm{d}h^a(\gamma^\alpha)_{as}\right) = 0 \\ \mathrm{d}\theta^s\left(j^\alpha_s + h^a(\gamma^\alpha)_{as}\right) = 0 \end{cases} \tag{3.3.74}$$

The first of these equations indicates that $k^c = 0$ (for all $c = 1, 2$) and then we can look at the condition $\mathcal{L}_Q(R_{\mathcal{M}}) = 0$ with $Q = \theta^a j_a^\alpha \widetilde{X}_\alpha + h^b\partial_c$. Before, proceeding we notice that the second and the third equations link the two functions j_s^β and h^a, reducing the problem to the research of $j_s^\beta \in C^\infty(\widetilde{M})$ such that

$$\mathrm{d}j_a^\alpha + \frac{1}{2}\left(K^\alpha_\beta j^\beta_s \omega - j^\gamma_s \omega_\gamma k^\alpha_\delta \hat{e}^\delta\right) = 0 \tag{3.3.75}$$

Then, from (3.3.46) in Remark 3.3.7, we recall that $R_{\mathcal{M}} = \mathrm{d}\omega \otimes K$ and hence

$$\mathcal{L}_Q(R_{\mathcal{M}}) = \mathrm{d}(\iota_Q \mathrm{d}\omega) = 0$$

[6] The one presented is indeed the most general form for an odd superderivation expanded in the local basis $\{\widetilde{X}_\alpha, \partial_a\}$

which gives

$$\begin{cases} \mathrm{d}\theta^a \wedge \hat{e}^\sigma J^\alpha_a \left((\mathrm{d}\omega)_{\sigma\alpha} - (\mathrm{d}\omega)_{\alpha\sigma} \right) \\ \mathrm{d} \left(J^\alpha_a \left((\mathrm{d}\omega)_{\alpha\sigma} - (\mathrm{d}\omega)_{\sigma\alpha} \right) \right) \wedge \hat{e}^\sigma \, \theta^a = 0 \end{cases} \tag{3.3.76}$$

Still looking at the first equation we find that either $J^\alpha_a = 0$ or $(\mathrm{d}\omega)_{\alpha\sigma} = (\mathrm{d}\omega)_{\sigma\alpha}$. In the first case, looking at the third equation of the system (3.3.74) we deduce that also $h^c = 0$ for $c = 1, 2$ and hence $Q = 0$; whereas, if $(\mathrm{d}\omega)_{\alpha\sigma} = (\mathrm{d}\omega)_{\sigma\alpha}$ holds, since the coefficients of the ordinary two form should be antysymmetric, $\mathrm{d}\omega = 0$. Assuming the second, we proceed with $\mathcal{L}_Q(\omega) = \iota_Q \mathrm{d}\omega + \mathrm{d}(\iota_Q \omega)$. Last comments considerably simplify the calculations, which becomes:

$$\mathcal{L}_Q(\omega) = \mathrm{d}(\iota_Q(\omega)) = \mathrm{d} \left(\theta^a j^\alpha_a \omega_\alpha \right) = \mathrm{d}\theta^a j^\alpha_a \omega_\alpha + \theta^a \mathrm{d}(j^\alpha_a \omega_\alpha) = 0 \; . \tag{3.3.77}$$

Both terms of the sum shall be zero, but looking at the first term one gets $j^\alpha_a \omega_\alpha = 0$ meaning that

$$j^0_a \omega_0 + j^1_a \omega_1 = 0 \; .$$

Moreover, using $j^\alpha_a \omega_\alpha = 0$ in equation (3.3.75), one finds

$$\mathrm{d}j^\alpha_a + \frac{1}{2} K^\alpha_\beta j^\beta_s \omega \tag{3.3.78}$$

that brings with some manipulations to the system of equations for j^α_a

$$\begin{cases} \widetilde{X}_1(j^0_a) = \frac{1}{2} j^0_a \omega_1 \\ \widetilde{X}_1(j^1_a) = \frac{1}{2} j^1_a \omega_0 \end{cases} \tag{3.3.79}$$

If one takes the explicit form for $\omega = \hat{e}^\alpha K^\beta_\alpha \widetilde{X}_\beta(\log \mathsf{S})$, given by the ordinary vanishing torsion constraints, two simply equations remain in order to characterise the odd superderviation Q:

$$\begin{cases} \theta^s K^\alpha_\beta \widetilde{X}_\alpha(j^\beta_s) = \frac{1}{2} K^\alpha_\beta j^\beta_s \widetilde{X}_\alpha(\log \mathsf{S}) \\ \mathrm{d}\theta^s \left(j^\alpha_s + h^a (\gamma^\alpha)_{as} \right) = 0 \end{cases} \tag{3.3.80}$$

which admits the solutions given by $\theta^s j^\alpha_s = \mathsf{S}^{\frac{1}{2}} \theta^s \gamma^\alpha_{as}$ (for $a = 1, 2$) and $H^a = \mathsf{S}^{\frac{1}{2}}$ (for $a = 1, 2$). Before claiming that a final form for Q has been found we have to check the last condition $\mathcal{L}_Q(\xi^a) = 0$. We rewrite the ansatz as $Q = h^c(-\theta^s(\gamma^\alpha)_{cs}\widetilde{X}_\alpha + \partial_c)$

$$\xi^a = \mathrm{d}\theta^a - \theta^s \frac{\omega}{2}(\gamma_3)^a_s \quad \text{and} \quad \mathrm{d}\xi^a = -\mathrm{d}\theta^s \wedge \omega \frac{(\gamma_3)^a_s}{2} - \mathrm{d}\omega\theta^s \frac{(\gamma_3)^a_s}{2} \ .$$

Hence,

$$\iota_Q(\mathrm{d}\xi^a) = \underbrace{\mathrm{d}\theta^r\theta^s \frac{(\gamma_3)^a_r}{2}\omega_\alpha h^c(\gamma^\alpha)_{cs}}_{1.} - \underbrace{\omega\frac{(\gamma_3)^a_c}{2}h^c}_{2.} + \underbrace{\hat{e}^\beta\theta^s\theta^l \frac{(\gamma_3)^a_l}{2}(\gamma^\alpha)_{cs}R_{\alpha\beta}}_{3.} \ .$$

Now, we can notice that term (1.) vanishes because $\omega_\alpha h^c(\gamma^\alpha)_{cs}) = -j^\alpha_s\omega_\alpha = 0$, while the term (3.) vanishes because $R_{\alpha\beta} = 0$. Proceeding, we get

$$\mathrm{d}\iota_Q(\xi^a) = \mathrm{d}(h^a + \theta^s\theta^l \frac{(\gamma_3)^a_l}{2}\omega_\alpha h^c(\gamma^\alpha_{cs}))$$

that becomes simply $\mathrm{d}\iota_Q(\xi^a) = \mathrm{d}h^a$ becaus, as before $\omega_\alpha h^c(\gamma^\alpha)_{cs} = -j^\alpha_s\omega_\alpha = 0$. Concluding this part we can write the equation for h^a

$$\mathrm{d}h^a - h^b\frac{\omega}{2}(\gamma_3)^a_b$$

that can be multiplied on the left by γ^α_{as} and, recalling the relation $(\gamma^\alpha\gamma_3)^a_b = -K^\alpha_\beta(\gamma^\beta)^a_b$, coincides exactly with equation (3.3.78). And we conclude writing in Q in the most general form

$$Q^{\mathsf{S}}_a = \mathsf{S}^{\frac{1}{2}}\left(\partial_a - \theta^b\gamma^\alpha_{ba}\partial_\alpha\right) \qquad \text{for} \quad a = 1,2 \tag{3.3.81}$$

That, for the super Minkowski spacetime ($\mathsf{S} = 1$) becomes

$$Q^{\mathbb{M}}_a = \partial_a - \theta^b\gamma^\alpha_{ba}\partial_\alpha \qquad \text{for} \quad a = 1,2 \ . \tag{3.3.82}$$

□

Now, recovering the interpretation of the super field as a functor from the enriched category[7] $2|2$-$\mathfrak{esLoc}$ to the category of $*$-$\mathfrak{esAlg}$, we can fix an object in $\mathcal{M}$ in $2|2$-$\mathfrak{esLoc}$,

[7] As we discussed in Section 3.2, for a sensible implementation of supersymmetries in locally covariant quantum field theory, we should make use of enriched categories rather than ordinary ones. We didn't give a definition, but we repeat here that the enriched category we have to deal with can be constructed preserving, in a precise mathematical sense, the same class of objects but enriching the sets of morphisms between two objects. Hence, referring to the theoretical framework of [HHS16, Sec.7], the transformations of Proposition 3.3.10 are exactly those enriched morphisms between objects in an enriched category.

such that $R_{\mathcal{M}} = 0$, and look at the enriched automorphisms in $\mathfrak{esLoc}$ (i.e. supersimmetry transformations) that can be written in term of the odd superderivations (3.3.81) and a constant parameter ε in the spinor vector space S. The super quantum field theory defined by $P_{\mathcal{M}}$ gives then a rule to associate an enriched automorphism of the object $\mathscr{A}(\mathcal{M})$ (where $\mathscr{A}$ on the objects acts exactly as the functor defined by (3.2.14) and, for this reason, we use the same notation) to each enriched morphism of $\mathcal{M}$. Practically, since we have defined $\Phi_{\mathcal{M}}(F) = [F] \in \mathcal{A}(\mathcal{M})$, we can regard an action on the fields algebra as an action of the elements generating the class $[F]$. Hence, we take the contraction $Q = \varepsilon^a Q_a$

$$\delta_{Q_\epsilon}\left(\Phi_{\mathcal{M}}(F)\right) := \Phi_{\mathcal{M}}\left(Q_\varepsilon(F)\right) = [Q_\varepsilon(F)] \tag{3.3.83}$$

where the action of Q on the element of $\mathcal{O}_c(\mathcal{M})$ is well defined and can be explicitly computed. Indeed, taking the expansion $F = f + s_a\theta^a + k\overline{\theta}$, we get

$$Q_\varepsilon(F) = \mathsf{S}^{\frac{1}{2}}\varepsilon^a s_a + \mathsf{S}^{\frac{1}{2}}\varepsilon^a\left(k\epsilon_{ac} - \gamma^\alpha_{ac}\widetilde{X}_\alpha(f)\right)\theta^c + \mathsf{S}^{\frac{1}{2}}\varepsilon^a(\gamma^\alpha)^c_a\widetilde{X}_\alpha(s_c)\overline{\theta} \tag{3.3.84}$$

Now, if we define the splitting on the super $*$-algebra elements $\Phi_{\mathcal{M}}(F) := \phi_{\mathcal{M}}(f) + \psi^a_{\mathcal{M}}(s_a) + \eta_{\mathcal{M}}(k)$, we can conclude writing

$$\delta_{Q_\epsilon}\left(\Phi_{\mathcal{M}}(F)\right) = \phi_{\mathcal{M}}(\mathsf{S}^{\frac{1}{2}}\varepsilon^a s_a) + \psi^c_{\mathcal{M}}\left(\mathsf{S}^{\frac{1}{2}}\varepsilon^a\left(k\epsilon_{ac} - \gamma^\alpha_{ac}\widetilde{X}_\alpha(f)\right)\right) + \eta_{\mathcal{M}}\left(\mathsf{S}^{\frac{1}{2}}\varepsilon^a(\gamma^\alpha)^c_a\widetilde{X}_\alpha(s_c)\right).$$

Now, analyzing this action for homogeneous elements of the algebra, as has been done in [HHS16, Sec. 8] for both concrete models presented, we can recover the form for supersymmetry transformations as proposed in standard literature, we get

$$\begin{aligned}
\delta_{Q_\epsilon}\left(\phi_{\mathcal{M}}(f)\right) &= -\psi^a_{\mathcal{M}}\left(\mathsf{S}^{\frac{1}{2}}\varepsilon^c\gamma^\alpha_{ca}\widetilde{X}_\alpha(f)\right)\\
\delta_{Q_\epsilon}\left(\psi^a_{\mathcal{M}}(s_a)\right) &= \phi_{\mathcal{M}}\left(\mathsf{S}^{\frac{1}{2}}\varepsilon^c s_c\right) + \eta_{\mathcal{M}}\left(\mathsf{S}^{\frac{1}{2}}\varepsilon^c(\gamma^\alpha)^c_a\widetilde{X}_\alpha(s_c)\right)\\
\delta_{Q_\epsilon}\left(\eta_{\mathcal{M}}(k)\right) &= \psi^c_{\mathcal{M}}\left(\mathsf{S}^{\frac{1}{2}}\varepsilon^c\epsilon_{ac}\, k\right).
\end{aligned}$$

Concluding the characterization of the category (enriched) 2|2-$\mathfrak{sLoc}$ and the associated super algebra of fields, induced by the super Green-hyperbolic super self-adjoint operator $P_{\mathcal{M}}$.

Conclusions and Outlook

In three chapters, as anticipated in the introduction, we tried to give a self-contained exposition on locally covariant quantum field theory as thought in the ordinary literature and of its extension to supergeometric backgrounds, with a particular care on the proofs of some useful properties for field theories on $2|2$-dimensional geometries. We devote now few lines to depict a brief outline of the state of the research and a possible outlook for the presented subject.

With the last developments, supergeometry was combined with LCQFT: a general construction for non-interacting super-QFT has been achieved and the set of axioms seems to be successful in including basic models, such as the superparticle, and interesting models, such as the Wess-Zumino supergravity model (see [HHS16, Sec. 8]). Yet, many steps towards a full understanding of super-QFT in this framework should be done. Especially, a satisfying description of interacting field theories, well understood in ordinary LCQFT ([BDF+09],[DHP09]), is still missing. A rigorous study of the regularization and renormalization properties of super-QFT could indeed lead to a rigorous proof of non-renormalization theorems. The starting point of such research is certainly the construction of super-Wick products, at least for the super scalar field. Of particular interest should be the restriction imposed by the *enriched morphisms* on the degree of freedom in the definition of ordinary Wick products. Indeed, it has been proven in [HW01, Th. 5.1] that *local Wick products* can be defined up to finite polynomials in the fields with coefficients depending on the geometric constant associated to the manifold. Moreover, already exist some examples of QFTs for which an exceptional amount of symmetries can be used to characterize those polynomials (see [Pin09]).

The analysis of the $2|2$ dimensional super-Cartan structures, proposed at the end of the thesis, goes exactly in this direction. Indeed, if those backgrounds reveal to be enough complex to provide interesting examples of non-renormalizable theories, the low (ordinary) dimension simplifies considerably the calculations of explicit Green's operator and, in the end, of *quantum states* which can be defined over the algebra of fields.

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Category theory

In the second half of the last century category theory started to be deeply used in many development of mathematical physics. In this thesis we used some key notions which could be new for a reader with background on different branch of physics, hence we believe that a quick presentation on this topic will be useful to give notions being fundamental to understand the mathematical framework in which all the locally covariant quantum field theories are naturally developed. We start fixing what is meant for a category.

Definition A.1. A *category* $\mathfrak{Cat}$ is a collection of *objects* $\mathsf{Obj}_{\mathfrak{Cat}}$ and for each pair of objects (A, B) a set of *morphisms* $\mathsf{Mor}_{\mathfrak{Cat}}(A, B)$, together with an application between different sets of morphisms: the *composition law*,

$$\begin{aligned} \circ : \mathsf{Mor}_{\mathfrak{Cat}}(B, C) \times \mathsf{Mor}_{\mathfrak{Cat}}(A, B) &\to \mathsf{Mor}_{\mathfrak{Cat}}(A, C) \\ (g, f) &\mapsto g \circ f \end{aligned}$$

for three given objects (A, B, C). This set of data shall respect the following axioms.

Identity law : for all objects A, the set $\mathsf{Mor}_{\mathfrak{Cat}}(A, A)$ contains an element id_A (there could me more than one, in general) such that, for any other object B, each $f \in \mathsf{Mor}_{\mathfrak{Cat}}(A, B)$ and each $h \in \mathsf{Mor}_{\mathfrak{Cat}}(B, A)$, the following equivalences hold:

$$f \circ \mathrm{id}_A = f,$$
$$\mathrm{id}_A \circ h = hg \ ;$$

Associativity : given four objects A, B, C, D, for all $f \in \mathsf{Mor}_{\mathfrak{Cat}}(A, B)$, for all $h \in \mathsf{Mor}_{\mathfrak{Cat}}(B, C)$ and $g \in \mathsf{Mor}_{\mathfrak{Cat}}(C, D)$ it holds that

$$g \circ (h \circ f) = (g \circ h) \circ f.$$

Once a category is given, one can build up other categories extracting a part of the collection of the objects, selecting a subset of each set of morphism or doing both the operations.

Definition A.2. Let $\mathfrak{Cat}$ be a category. Then a *subcategory* $\mathfrak{Sub}$ *of* $\mathfrak{Cat}$ is a category whose objects $\mathsf{Obj}_{\mathfrak{Sub}}$ are all objects of the category $\mathfrak{Cat}$ and for each couple of objects (A, B), the inclusion $\mathsf{Mor}_{\mathfrak{Sub}}(A, B) \subseteq \mathsf{Mor}_{\mathfrak{Cat}}(A, B)$ holds. Moreover, on the same line of the axioms above the following:

- the identity is preserved, i.e. for each object A of the category $\mathfrak{Sub}$ the identity morphism in the set $\mathsf{Mor}_{\mathfrak{Sub}}(A, A)$ must coincide with the identity morphism of $\mathsf{Mor}_{\mathfrak{Cat}}(A, A)$;
- the composition law is preserved: i.e. for each A, B, $C \in \mathsf{Obj}_{\mathfrak{Sub}}$, each $f \in \mathsf{Mor}_{\mathfrak{Sub}}(A, B)$ and each $g \in \mathsf{Mor}_{\mathfrak{Sub}}(B, C)$ the composition $h \circ g$ in $\mathfrak{Sub}$ coincides with the composition $h \circ g$ in $\mathfrak{Cat}$.

We say that a subcategory $\mathfrak{Sub}$ of the category $\mathfrak{Cat}$ is a *full subcategory of* $\mathfrak{Cat}$ if $\mathsf{Mor}_{\mathfrak{Sub}}(A, B) = \mathsf{Mor}_{\mathfrak{Cat}}(A, B)$ for each A, $B \in \mathsf{Obj}_{\mathfrak{Sub}}$.

Example A.3. Examples of categories are:

- the class of all sets form a category ($\mathfrak{Set}$), whose morphisms are functions $f : A \to B$ between two sets A, B, composed following the ordinary composition of functions;
- the category whose objects are vector spaces ($\mathfrak{Vec}$), whose morphisms are linear functions $L : V \to W$ between vector spaces, composed following the ordinary composition of functions.

We leave to the reader the easy check of the axioms for the this three axioms

A natural question is whether its possible to define something having the role of application connecting two different categories; this objects exist and are called functors. Some authors distinguish between two different kind of functor, *covariant and contravariant functors*, depending on how the composition law is transformed by its action. Unfortunately, some other authors prefer to avoid this distinction using a little notational expedient. We used both strategies indifferently throughout this work, hence after the classical definition we precise the notation in a remark.

Definition A.4. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two categories. A *covariant functor* $\mathscr{F}$ *from* $\mathfrak{A}$ *to* $\mathfrak{B}$ is an association of an object $\mathscr{F}(A)$ for the category $\mathfrak{B}$, once an object A for the category $\mathfrak{A}$ is given, and the class of maps

$$\{\mathscr{F} : \mathsf{Mor}_{\mathfrak{A}}(A, B) \to \mathsf{Mor}_{\mathfrak{B}}(\mathscr{F}(A), \mathscr{F}(B)) \text{ for } A, B \text{ objects of the category } \mathfrak{A}\}$$

such that the following axioms hold

- *preserved composition law*: for all A, B, C objects of $\mathfrak{A}$, each $f \in \mathsf{Mor}_{\mathfrak{A}}(A,B)$ and each $g \in \mathsf{Mor}_{\mathfrak{A}}(B,C)$ we have

$$\mathscr{F}(g \circ_{\mathfrak{A}} f) = \mathscr{F}(g) \circ_{\mathfrak{B}} \mathscr{F}(f),$$

  where we made explicit the difference between the composition laws on the two sides of the equation;

- *identity map preservation*: for all objects A of $\mathfrak{A}$, the identity map id_A is mapped to the identity map of the corresponding object $\mathscr{F}(A)$ of $\mathfrak{B}$, i.e.

$$\mathscr{F}(\mathrm{id}_A) = \mathrm{id}_{\mathscr{F}(A)}.$$

A *contravariant functor* $\mathscr{E}$ from $\mathfrak{A}$ to $\mathfrak{B}$ from $\mathfrak{A}$ to $\mathfrak{B}$ is an association of an object $\mathscr{E}(A)$ for the category $\mathfrak{B}$, once an object A for the category $\mathfrak{A}$ is given, and a collection of applications

$$\{\mathscr{E} : \mathsf{Mor}_{\mathfrak{A}}(A,B) \to \mathsf{Mor}_{\mathfrak{B}}(\mathscr{E}(B), \mathscr{E}(A)) \text{ for } A, B \in \mathsf{Obj}_{\mathfrak{A}}\}$$

such that the *preserved composition law axiom* is switched to the requirement

- *reversed composition law*: for each A, B, C objects of the category $\mathfrak{A}$, each $f \in \mathsf{Mor}_{\mathfrak{B}}(A,B)$ and each $h \in \mathsf{Mor}_{\mathfrak{A}}(B,C)$ we have

$$\mathscr{E}(h \circ_{\mathfrak{A}} f) = \mathscr{E}(f) \circ_{\mathfrak{B}} \mathscr{E}(h),$$

  where, as before, different composition laws have been expressed.

Remark A.5. As anticipated, sometimes the distinction between functors preserving or reversing the composition law is explicated differently. First, for all categories $\mathfrak{A}$ we can define the category $\mathfrak{A}^{\mathrm{op}}$, as a category with the same objects of the category and with reversed morphisms: i.e. given two objects A, B of $\mathfrak{A}$, and so of $\mathfrak{A}^{\mathrm{op}}$, the set of morphism $\mathsf{Mor}_{\mathfrak{A}^{\mathrm{op}}}(A,B) = \mathsf{Mor}_{\mathfrak{A}}(B,A)$. In this context, we define a functor in the same way we defined covariant functors above and, if we have contravariant functor $\mathscr{F} : \mathfrak{A} \to \mathfrak{B}$, we can take into account the reversion of the composition law considering $\mathscr{F}$ simply as a functor from the category $\mathfrak{A}$ to the category $\mathfrak{B}^{\mathrm{op}}$.

One can also study properties of functors. Taking two covariant functors $\mathscr{F} : \mathfrak{A} \to \mathfrak{B}$ and $\mathscr{E} : \mathfrak{B} \to \mathfrak{C}$ a new functor can be constructed defining a composition between functors: $\mathscr{E} \circ \mathscr{F}$ is the covariant functor from the category $\mathfrak{A}$ and the category $\mathfrak{B}$ such that

- each object A in the category $\mathfrak{A}$ are associated to $(\mathscr{E} \circ \mathscr{F})(A)) := \mathscr{E}(\mathscr{F}(A))$ in the objects of the category C;
- each morphism $\chi \in \mathrm{Mor}_{\mathfrak{A}}(A, B)$ is mapped to

$$(\mathscr{E} \circ \mathscr{F})(\chi)) := \mathscr{E}(\mathscr{F}(\chi)) \in \mathrm{Mor}_{\mathfrak{C}}(\mathscr{E}(\mathscr{F}(A)), \mathscr{E}(\mathscr{F}(A)))$$

This definition can be adapted respectively for composition of contravariant functors (the result is a covariant functor) and for composition of a covariant functor with a contravariant one (the result is a contravariant functor). Now, we proceed with the definition of *natural transformation.*

Definition A.6. Given two functors $\mathscr{F}, \mathscr{G} : \mathfrak{A} \to \mathfrak{B}$, a natural transformation $\eta : \mathfrak{A} \to \mathfrak{B}$ is an assignment of a class of maps $\{\eta_A : \mathscr{F}(A) \to \mathscr{E}(A)\}$ labeled by the objects in the category $\mathfrak{A}$, such that given two objects B, C of $\mathfrak{A}$, for all $\chi \in \mathrm{Mor}_{\mathfrak{A}}(B, C)$, the following diagram commutes

$$\begin{array}{ccc} \mathscr{F}(B) & \xrightarrow{\eta_B} & \mathscr{E}(B) \\ {\scriptstyle \mathscr{F}(\chi)}\big\downarrow & & \big\downarrow{\scriptstyle \mathscr{E}(\chi)} \\ \mathscr{F}(C) & \xrightarrow{\eta_C} & \mathscr{E}(C) \end{array}$$

Diagram A.1

Concluding this appendix, we present the construction of the object of *internal morphism* for a *monoidal category.* This part as not to be intended as self-consistent, because it is strongly interconnected with the issues raised in Section 3.1 on the definition of a super vector space structure on the set of linear maps between objects in $\mathfrak{sVec}$.

Definition A.7. Given a category $\mathfrak{C}$, we say that $\mathfrak{C}$ is a *monoidal category* if there exists a functor

$$\otimes : \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$$

together with

i. an isomorphism $(X \otimes Y) \otimes Z \xrightarrow{\simeq} X \otimes (Y \otimes Z)$, for any triple of objects X, Y, Z;

ii. an object I, called *identity object* for the pair $(\mathfrak{C}, \otimes)$, such that for any object X, two isomorphisms $I \otimes X \xrightarrow{\simeq} X$ and $X \otimes I \xrightarrow{\simeq} X$ exist.

Let's take now the category $\mathfrak{Set}$, then considering the cartesian product $\times$, $(\mathfrak{Set}, \times)$ is a monoidal category. Fixing any pair of objects X, Y and looking at the morphism $\mathrm{Mor}(X, Y) := \{\text{functions from} X \text{to} Y\}$, it is a trivial fact that $\mathrm{Mor}(X, Y)$ is a set and, hence, an object in $\mathfrak{Set}$. In particular, this means that fixed two objects X, Y, for any object S in the category $\mathfrak{Set}$ we can find an isomorphism

$$\mathrm{Mor}(S \times X, Y) \xrightarrow{\simeq} \mathrm{Mor}(S, \mathrm{Mor}(X, Y)) \tag{A.1}$$

indeed taking

$$f : S \times X \to Y \qquad (s, x) \mapsto f(s, x)$$

we define the map $f_s := f(s, \cdot)$ which can be used to associate bijectively

$$f_\cdot : S \to \mathrm{Mor}(X, Y) \qquad s \mapsto f_s : X \to Y$$

In general, given a monoidal category $\mathfrak{C}$ it is not possible to construct this association. The first issue is that the set of morphism $\mathrm{Mor}(\cdot, \cdot)$, in general, is not an object in the same category. A possible solution is to look for an object in $\mathfrak{C}$ that fulfils a relation analogous to (A.1). This is exactly what we need and hence we can present the definition of *internal morphism.*

Definition A.8. Given a monoidal category $(\mathfrak{C}, \otimes)$, for any triple of objects X, Y we define, if it exists, *the object of internal morphism* (or simply *the internal morphism*), $\underline{\mathrm{Mor}}(X, Y)$, that object such that exist an isomorphism

$$\mathrm{Mor}(Z \otimes X, Y) \xrightarrow{\simeq} \mathrm{Mor}(Z, \underline{\mathrm{Mor}}(X, Y)) \qquad \text{for all objects } Z \text{ in } \mathfrak{C}$$

Example A.9. Given two object in the category of sets, the internal morphism object coincide exactly with the morphism. The collection of all the morphism between two objects is in fact always a set, hence an object in $\mathfrak{Set}$. Considering the category of vector spaces ($\mathfrak{Vec}$) endowed with the standard tensor product $\otimes$, given V, W vector spaces, it can be proven that $\mathrm{Mor}(Z \otimes V, W) \simeq \mathrm{Mor}(Z, \mathrm{Mor}(V, W))$ and hence the usual morphism are also internal morphism. This is not the case when we deal with super vector spaces $\mathfrak{sVec}$, and it is indeed the reason why we need to introduce internal morphism: the object satisfying the wanted relation is not the set of morphism between two super vector spaces as defined in Definition 3.1.3.

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