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DYNAMICAL CASIMIR EFFECT AND THE STRUCTURE OF VACUUM IN QUANTUM FIELD THEORY

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To Sara, my better half.

Sommario

Alcuni dei fenomeni più interessanti che derivano dagli sviluppi dalla fisica moderna sono sicuramente le fluttuazioni di vuoto. Queste si manifestano in diversi rami della fisica, quali Teoria dei Campi, Cosmologia, Fisica della Materia Condensata, Fisica Atomica e Molecolare, ed anche in Fisica Matematica.

Una delle più importanti tra queste fluttuazioni di vuoto, talvolta detta anche "energia di punto zero", nonché uno degli effetti quantistici più facili da rilevare, è il cosiddetto effetto Casimir.

Le finalità di questa tesi sono le seguenti:

- Proporre un semplice approccio ritardato per effetto Casimir dinamico, quindi una descrizione di questo effetto di vuoto, nel caso di pareti in movimento.

- Descrivere l'andamento della forza che agisce su una parete, dovuta alla autointerazione con il vuoto.

Abstract

Some of the most interesting phenomena that arise from the developments of the modern physics are surely vacuum fluctuations. They appear in different branches of physics, such as Quantum Field Theory, Cosmology, Condensed Matter Physics, Atomic and Molecular Physics, and also in Mathematical Physics.

One of the most important of these vacuum fluctuations, sometimes called "zero-point energy", as well as one of the easiest quantum effect to detect, is the so-called Casimir effect.

The purposes of this thesis are:

- To propose a simple retarded approach for dynamical Casimir effect, thus a description of this vacuum effect when we have moving boundaries.

- To describe the behaviour of the force acting on a boundary, due to its self-interaction with the vacuum.

Contents

Chapter 1

Introduction

We know that a large number of problems in physics lead back to an harmonic oscillator problem. The radiation field is the most famous example.
In fact, in the vacuum, in absence of charges, so with null current density $j = 0$ and obviously null charge density $\rho = 0$, Maxwell equations are written [1]

$$\nabla \cdot \mathbf{E} = 0 \,, \qquad \nabla \cdot \mathbf{B} = 0 \,,$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \,, \qquad \nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \,,$$

therefore the electric field $\mathbf{E}$, the magnetic field $\mathbf{B}$ and the potential vector $\mathbf{A}$ satisfy the characteristic wave equation

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = 0$$
$$\frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = 0$$
$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = 0 \,.$$

For completness, we know that in electromagnetism, and then in electrodynamics, we can always make a gauge transformation which does not change our equations and their solutions. We choose a framework where the density of current and the density of charge are null, so respectevely $j = 0$ and $\rho = 0$. In this case, it is convenient to make the so called *radiation gauge*, where $\mathbf{A}$, named vector potential in literature, is linked to the fields thanks to the equations

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \,,$$
$$\mathbf{B} = \nabla \times \mathbf{A} \,,$$

while the scalar potential A_0, sometimes called ϕ always vanishes. The constant $\mathrm{c} = 299\,792\,458\,ms^{-1}$ is the speed of light in the vacuum [2], which corresponds to the speed of propagation of electromagnetic waves.
We can also write these relations more briefly, using tensorial notation [3], once defined in the chosen gauge the differential form $A_i = g_{ij}A^j$, where $A^j = (0, \mathbf{A})$ and g_{ij} is the metric tensor. The electromagnetic tensor, which contains electric field and magnetic field, is

$$F_{ij} = \partial_i A_j - \partial_j A_i\,, \tag{1.1}$$

such that the following statements are valid

$$F^{0\alpha} = -\frac{E_\alpha}{\mathrm{c}}\,, \qquad F^{\alpha\beta} = -\epsilon^{\alpha\beta\gamma}B_\gamma\,, \qquad F^{ii} = 0$$

and also, from the last one, the antisymmetry relation $F^{ij} = -F^{ji}$ [4], as we can clearly observe from (1.1).
According to the notation used, latin indices have time-spatial nature, while greek indices are purely spatial indices, and 0 is undoubtedly purely time index. E_α and B_α stand for the α-component of, respectively, $\mathbf{E}$ and $\mathbf{B}$.

With this background, calling $F_{ij} = g_{ik}F^{kl}g_{lj}$, where g_{ij} is, again, the metric tensor, that in our case is simply the Minkowskian metric tensor,

$$g^{ij} = \eta_{ij} = \mathrm{diag}(1, -1, -1, -1)\,,$$

Maxwell equation, in absence of charges, appear

$$\partial_i F^{ij} = 0$$
$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0$$

where the first one includes Gauss law and Ampère law, and the second, named Bianchi identity, holds Gauss law for magnetism and Faraday-Neumann-Lenz equation. Wave equations become, shortly

$$\Box A^i = 0\,, \qquad \Box E^i = 0\,, \qquad \Box B^i = 0\,,$$

where the box operator is $\Box = \partial_k\partial^k = \dfrac{\partial^2}{\partial(x^0)^2} - \dfrac{\partial^2}{\partial(x^\alpha)^2}$, with $x_0 = \mathrm{c}t$.

Maxwell equations have wave solutions for the fields, with the (non)dispersive relation $\omega = \mathrm{c}|\mathbf{k}|$, where $\mathbf{k}$ is clearly the wave vector which identifies the direction of the wave propagation [5], but we are interested in another thing: defining the conjugate momentum $p(t) = \dot{q}(t)$, and $q(t)$ satisfies the characteristic equation of an harmonic oscillator

$$\ddot{q}(t) + \omega^2 q(t) = 0\,,$$

let's calculate the energy of electromagnetic field. We find, trivially, an expression strongly equivalent to the energy of an harmonic oscillator

$$H = \frac{1}{2}\left(p^2 + \omega^2 q^2\right) .$$

The next natural step is the quantization.
Like every elementary quantum mechanics textbook, we associate an hermitian quantum operator to the variable q and to its conjugate momentum p, introducing $\hat{q}$ and $\hat{p}$, which fulfill the canonic commutation relation

$$[\hat{q}, \hat{p}] = \mathrm{i}\hbar\hat{1} , \tag{1.2}$$

where we name the reduced Planck constant $\hbar = \dfrac{h}{2\pi} = 1.054\,571\,726(47) \cdot 10^{-34}\, Js$ [2]. In order to solve the harmonic oscillator problem, it is very helpful to introduce the ladder operators, the creation operator $\hat{a}^\dagger$ and the annihilation operator $\hat{a}$, with their algebra

$$\left[\hat{a}, \hat{a}^\dagger\right] = \hat{1} ,$$

and we can definitely write, after a few computation, the following expressions for the operators

$$\begin{cases} \hat{a} = \dfrac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + \mathrm{i}\hat{p}) \\ \hat{a}^\dagger = \dfrac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} - \mathrm{i}\hat{p}) \end{cases}$$

and, reversing the formulas above, operator associated to position and momentum in function of these ladder operators ensue

$$\begin{cases} \hat{q} = \sqrt{\dfrac{\hbar}{2\omega}}(\hat{a}^\dagger + \hat{a}) \\ \hat{p} = \sqrt{\dfrac{\hbar\omega}{2}}\mathrm{i}(\hat{a}^\dagger - \hat{a}) \end{cases}$$

so, the hamiltonian operator $\hat{H} = H(\hat{p}, \hat{q})$ will be

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) , \tag{1.3}$$

or, expliciting the number operator $\hat{N} = \hat{a}^\dagger\hat{a}$, it is written

$$\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right) . \tag{1.4}$$

From the application of the hamiltonian operator on Fock's states $|n\rangle$ follows the eigenvalue relation [7]

$$\hat{H}|n\rangle = E_n|n\rangle \tag{1.5}$$

with the energy eigenvalue

$$E_n = \hbar\omega\left(n_+\frac{1}{2}\right) ,$$

and where $|n\rangle$ are eigenstates of the number operator $\hat{N}$, defined as

$$|n\rangle = \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}|0\rangle , \tag{1.6}$$

where $|0\rangle$ is called vacuum state and it is defined by the condition $\hat{a}|0\rangle = 0$.
The integer number n stands for the number of quantons[1], so, in the electromagnetic case, we are talking about photons.

We are able to write the energy of a generical quantized system, summing over all different polarizations of the wave, labelled by the r-index, and over the wave vector $\mathbf{k}$ already appointed,

$$H(n_{\mathbf{k},r}) = \sum_{\mathbf{k},r}\hbar\omega_{\mathbf{k}}\left(n_{\mathbf{k},r} + \frac{1}{2}\right) .$$

The expression for quantum energies of electromagnetic radiation has a nonzero minimum value in absence of photons, when $n_{\mathbf{k},r} = 0$, the so-called zero-point energy. This particular behavior of the minimum arises because of fields quantum fluctuations of the vacuum state $|0\rangle$, so something new is occurring compared with classical electrodynamics.
In fact, there exist an infinite modes of radiation, each of them owns an energy $\frac{\hbar\omega}{2}$, so that the total vacuum energy diverges, unless we reach to exclude high-frequency modes [2] [8]. In other words, quantum field theory, in the vacuum, assigns half a quantum to each degree of the infinitely many degrees of freedom [23].

From a qualitative point of view, it is useful to think about a real harmonic oscillator. The relation (1.2) prevents the *simultaneous* vanishing of the kinetic

[1] According to Jean-Marc Lévy-Leblonde, theoretical physicist, who I heard in occasion of a conference in March 2014, in Bologna, quantum entities do not have to be considered neither as waves, nor particles. So, talking about wave-particle duality seems inaccurate, it is much better to introduce the term **quanton**, with the -on suffix like every element dealt with quantum mechanics [10].

[2] *Ultraviolet* divergence is the contribution of the high-frequency modes, opposite to *infrared* divergence, which features low frequences [9].

energy, proportional to the momentum squared p^2, and the potential energy, proportional to the position squared q^2 [7]. So, the lowest allowed energy is a compromise between these two energies, and consequentially the energy of the ground state can not be equal to zero.

Firstly, in order to bypass the problem of this nonzero value of vacuum energy, one could employ the follow reasoning: in physics, the significant things are differences of energy, and this latter can be redefined less than a constant value, so that we can remove zero-point energy, even if it diverges.

Obviously, it cannot be the correct solution to our problem, because, according to the General Theory of Relativity, not only a variation of energy assumes a key role, but also the total energy of the universe [11].

Many of infinities, in quantum field theory, are removed by means of renormalization procedures [8]. Moreover, zero-point energy and vacuum quantum fluctuations give rise to observable effects.

Some examples of the manifestation of the vacuum energy are surely the spontaneus emission of radiation from excited atoms, at first proposed by Einstein[3] for a correct energy balance of radiation [13], and the Casimir effect[4], briefly, an attractive force between two parallel conductor plates due to vacuum energy fluctuations in dependence from the distance between the planes, as shown in Figure 1.1. This latter phenomenon will be widely discuss in the hereinafter of this work in its different facets, starting from a stationary model and, subsequently, adventuring in a moving boundary one.

The Casimir effect is a noteworthy topic, because it is a multidisciplinary subject. It is reflected in many fields of physics, such as Quantum Field Theory, Condensed Matter Physics, Atomic and Molecular Physics, Gravitation and Cosmology, Mathematical Physics [23].

The purpose of this thesis is to study vacuum fluctuations due to the Casimir effect, with different boundary conditions, and then we investigate their applications to a moving wall.

In order to be more specific, in *Chapter* 2: *Simple models of stationary Casimir Effect* we present a brief review of the Casimir effect in the simplest case, thus with stationary boundary conditions, already known from literature.

[3] Albert Einstein (1879-1955), at the beginning of his career, investigated problems concerning statistical mechanics, with main applications in what, nowadays, we would call quantum domain, in particular his analysis of energy fluctuations in blackbody radiation led him to become the first to state, in 1909, long time before the discovery of quantum mechanics, that the theory of the future ought to be based on a dual description in terms of particles and waves. In 1916, he proposed a new law of Planck's blackbody radiation, and in the course of this last work, he observed a lack of Newtonian causality in the process called spontaneous emission [12].

[4] Hendrik Brugt Gerhard Casimir (1909-2000), Dutch physicist, was the first who predicted, in 1948, this purely quantum effect, an attraction between neutral, parallel conducting plates. In fact, there is no force acting between neutral plates in classical electrodynamics.

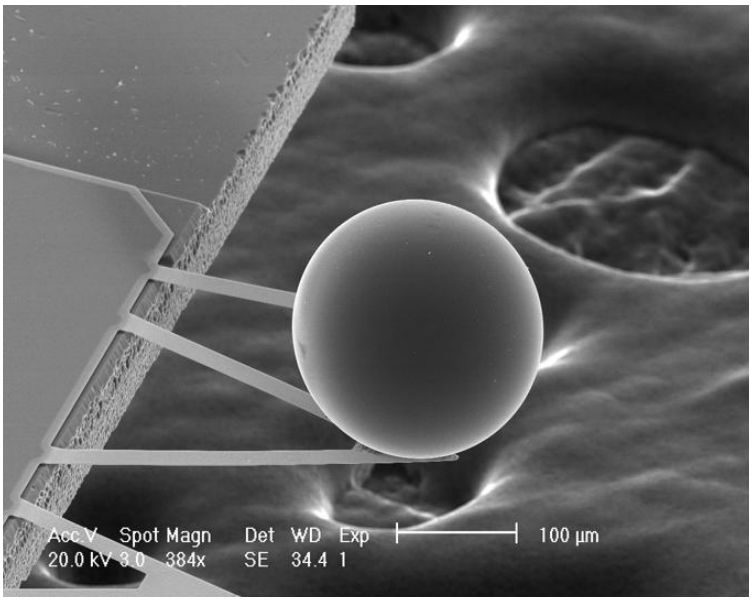

Figure 1.1: A microscopic sphere is brought to a conducting plane, which is not shown in the picture, up to a distance of about 100 nm, for measuring the Casimir effect [15].

In *Chapter* 3: *Moving Boundaries*, we present a description of the behaviour of these vacuum fluctuations, in (1+1)-dimensions, when one boundary is moving. We consider both motion of the wall with constant velocity and motion of the wall that oscillates around the initial position.

In *Chapter* 4: *A retarded Approach*, we describe the physics of *Chapter* 3 by means of a "naive" retarded model, highlighting the differences between the two approaches.

Finally, in *Chapter* 5: *A self-consistent law of motion*, we try to describe the motion of the moving wall attached to some kind of a spring.

Chapter 2

Simple models of stationary Casimir Effect

We are now going to introduce some models of the Casimir effect, in the case of steady boundaries, the so called stationary Casimir effect.

2.1 Quantized scalar field in a hole

We start with a real scalar field $\phi(t, x)$ defined on an interval $0 \leq x \leq a$ and obeying boundary conditions, introduced in [23]

$$\phi(t, 0) = \phi(t, a) = 0\,. \tag{2.1}$$

Figure 2.1: The topology of the hole.

The scalar field equation follows, as usual, the Klein-Gordon equation [9]

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi(t, x) - \frac{\partial^2}{\partial x^2}\phi(t, x) + \frac{m^2c^2}{\hbar^2}\phi(t, x) = 0\,, \tag{2.2}$$

which could be derived using Euler-Lagrange equation of motion

$$\partial_i \frac{\delta \mathcal{L}}{\delta \partial_i \phi} - \frac{\delta \mathcal{L}}{\delta \phi} = 0 \tag{2.3}$$

where the Lagrangian density $\mathcal{L}$ is

$$\mathcal{L} = \frac{1}{2}\partial_i\phi\,\partial^i\phi - \frac{1}{2}\frac{m^2c^2}{\hbar^2}\phi^2\,, \tag{2.4}$$

defined $i = (t, x)$ with t time-index and x space-index.

The indefinite scalar product, associated with the field equation, is

$$\langle f, g\rangle = \mathrm{i}\int_0^a \mathrm{d}x\,(f^*\partial_{x_0}\,g - \partial_{x_0}f^*\,g) = \mathrm{i}\int_0^a \mathrm{d}x f^*\overleftrightarrow{\partial_{x_0}}g\,, \tag{2.5}$$

where f and g are solutions of (2.2), and $x_0 = ct$.

It is not hard to check that positive-frequency and negative-frequency solutions are

$$\phi_n^{\pm}(t, x) = \left(\frac{\mathrm{c}}{a\omega_n}\right)^{\frac{1}{2}} \exp(\pm\mathrm{i}\omega_n t)\sin(k_n x)\,, \tag{2.6}$$

where is valid the dispersive relation

$$\omega_n = \sqrt{\frac{m^2\mathrm{c}^4}{\hbar^2} + \mathrm{c}^2k_n^2}\,, \tag{2.7}$$

and the wave vector is quantized according to the law

$$k_n = \frac{\pi n}{a}\,, \qquad n \in \mathbb{N}\,. \tag{2.8}$$

The (2.6) is true only if the (2.1) is. Otherwise, there are also terms containing $\cos(k_n x)$.

According to the scalar product (2.5), these solutions satisfy the orthonormalization relations

$$\langle\phi_m^{\pm}, \phi_n^{\pm}\rangle = \mp\delta_{m\,n}\,, \qquad \langle\phi_m^{\pm}, \phi_n^{\mp}\rangle = 0\,.$$

We begin by considering a free field in (1+1)-dimensional space-time.
The standard quantization of this field is performed by the expansion

$$\phi(t, x) = \sqrt{\hbar\mathrm{c}}\sum_n \left[\phi_n^-(t, x)a_n + \phi_n^+(t, x)a_n^\dagger\right]\,, \tag{2.9}$$

where the operators a_n, $a_n^\dagger$ are respectively the annihilation and creation operators, obeying the commutation relations

$$\left[a_m, a_n^\dagger\right] = \delta_{m\,n}\,, \qquad [a_m, a_n] = 0 = \left[a_m^\dagger, a_n^\dagger\right]\,. \tag{2.10}$$

As mentioned in Chapter 1, the vacuum state is defined, repetita iuvant, by the relation

$$a_n\,|0\rangle = 0 \qquad \forall n \in \mathbb{N}\,. \tag{2.11}$$

We are interesting in investigating the energy of this vacuum state $|0\rangle$.
The operator associated with the energy density is given by the 00-component of the energy-momentum tensor of the scalar field, defined, by means of the Lagrangian density,

$$T_{ij} = \frac{\delta\mathcal{L}}{\delta\partial^i\phi}\partial_j\phi - g_{ij}\mathcal{L}\,, \tag{2.12}$$

with the result

$$T_{00}(t,x) = \frac{1}{2}\left\{\frac{1}{\mathrm{c}^2}\left(\frac{\partial\phi(t,x)}{\partial t}\right)^2 + \left(\frac{\partial\phi(t,x)}{\partial x}\right)^2 + \frac{m^2\mathrm{c}^2}{\hbar^2}\phi^2\right\}\,, \tag{2.13}$$

for a massive scalar field.

The energy density results

$$\langle 0|T_{00}(x)|0\rangle = \frac{\hbar}{2a}\sum_{n=1}^{\infty}\omega_n - \frac{m^2\mathrm{c}^4}{2a\hbar}\sum_{n=1}^{\infty}\frac{\cos(2k_n x)}{\omega_n}\,, \tag{2.14}$$

and the total vacuum energy in the integration of this latter expression in the interval $[0,a]$, so that the second term does not contribute to the result because of periodicity of the trigonometric function

$$\mathcal{E}_0(a) = \int_0^a \mathrm{d}x\,\langle 0|T_{00}(x)|0\rangle = \frac{\hbar}{2}\sum_{n=1}^{\infty}\omega_n \tag{2.15}$$

Proof. First of all we write down the expression for the scalar field

$$\phi(t,x) = \sqrt{\hbar\mathrm{c}}\sum_n\left(\frac{\mathrm{c}}{a\omega_n}\right)^{\frac{1}{2}}\sin\left(k_n x\right)\left[\exp(-\mathrm{i}\omega_n t)a_n + \exp(\mathrm{i}\omega_n t)a_n^\dagger\right]\,. \tag{2.16}$$

The derivatives of (2.16) will be

$$\frac{\partial}{\partial t}\phi = \mathrm{i}\sqrt{\hbar\mathrm{c}}\sum_n\left(\frac{\mathrm{c}\,\omega_n}{a}\right)^{\frac{1}{2}}\sin(k_n x)\left[\exp(\mathrm{i}\omega_n t)a_n^\dagger - \exp(-\mathrm{i}\omega_n t)a_n\right]\,,$$

$$\frac{\partial}{\partial x}\phi = \sqrt{\hbar\mathrm{c}}\sum_n\left(\frac{\mathrm{c}}{a\omega_n}\right)^{\frac{1}{2}}\cos(k_n x)k_n\left[\exp(-\mathrm{i}\omega_n t)a_n + \exp(\mathrm{i}\omega_n t)a_n^\dagger\right]\,,$$

and their second power are

$$\left(\frac{\partial}{\partial t}\phi\right)^2 = \hbar \mathrm{c}\sum_n\sum_m\left(\frac{\mathrm{c}}{a}\right)(\omega_n\omega_m)^{\frac{1}{2}}\sin(k_n x)\sin(k_m x)(-1)\times$$
$$\times\Big[\exp(\mathrm{i}\omega_n t)\exp(\mathrm{i}\omega_m t)a_n^\dagger a_m^\dagger + \exp(-\mathrm{i}\omega_n t)\exp(-\mathrm{i}\omega_m t)a_n a_m +$$
$$-\exp(\mathrm{i}\omega_n t)\exp(-\mathrm{i}\omega_m t)a_n^\dagger a_m - \exp(-\mathrm{i}\omega_n t)\exp(\mathrm{i}\omega_m t)a_n a_m^\dagger\Big] =$$
$$= \hbar \mathrm{c}\sum_n\sum_m\left(\frac{\mathrm{c}}{a}\right)(\omega_n\omega_m)^{\frac{1}{2}}\sin(k_n x)\sin(k_m x)\times$$
$$\times\Big[-\exp(\mathrm{i}\omega_n t + \mathrm{i}\omega_m t)a_n^\dagger a_m^\dagger - \exp(-\mathrm{i}\omega_n t - \mathrm{i}\omega_m t)a_n a_m +$$
$$+\exp(\mathrm{i}\omega_n t - \mathrm{i}\omega_m t)a_n^\dagger a_m + \exp(-\mathrm{i}\omega_n t + \mathrm{i}\omega_m t)\left(\delta_{mn} + a_m^\dagger a_n\right)\Big],$$

$$\left(\frac{\partial}{\partial x}\phi\right)^2 = \hbar \mathrm{c}\sum_n\sum_m\left(\frac{\mathrm{c}}{a}\right)\frac{1}{(\omega_n\omega_m)^{\frac{1}{2}}}\cos(k_n x)\cos(k_m x)k_n k_m\times$$
$$\times\Big[\exp(\mathrm{i}\omega_n t)\exp(\mathrm{i}\omega_m t)a_n^\dagger a_m^\dagger + \exp(-\mathrm{i}\omega_n t)\exp(-\mathrm{i}\omega_m t)a_n a_m +$$
$$+\exp(\mathrm{i}\omega_n t)\exp(-\mathrm{i}\omega_m t)a_n^\dagger a_m + \exp(-\mathrm{i}\omega_n t)\exp(\mathrm{i}\omega_m t)a_n a_m^\dagger\Big] =$$
$$= \hbar \mathrm{c}\sum_n\sum_m\left(\frac{\mathrm{c}}{a}\right)\frac{1}{(\omega_n\omega_m)^{\frac{1}{2}}}\cos(k_n x)\cos(k_m x)k_n k_m\times$$
$$\times\Big[\exp(\mathrm{i}\omega_n t + \mathrm{i}\omega_m t)a_n^\dagger a_m^\dagger + \exp(-\mathrm{i}\omega_n t - \mathrm{i}\omega_m t)a_n a_m +$$
$$+\exp(\mathrm{i}\omega_n t - \mathrm{i}\omega_m t)a_n^\dagger a_m + \exp(-\mathrm{i}\omega_n t + \mathrm{i}\omega_m t)\left(\delta_{mn} + a_m^\dagger a_n\right)\Big],$$

while the field squared is

$$\phi^2 = \hbar \mathrm{c}\sum_n\sum_m\left(\frac{\mathrm{c}}{a}\right)\frac{1}{(\omega_n\omega_m)^{\frac{1}{2}}}\sin(k_n x)\sin(k_m x)\times$$
$$\times\Big[\exp(\mathrm{i}\omega_n t + \mathrm{i}\omega_m t)a_n^\dagger a_m^\dagger + \exp(-\mathrm{i}\omega_n t - \mathrm{i}\omega_m t)a_n a_m +$$
$$+\exp(\mathrm{i}\omega_n t - \mathrm{i}\omega_m t)a_n^\dagger a_m + \exp(-\mathrm{i}\omega_n t + \mathrm{i}\omega_m t)\left(\delta_{mn} + a_m^\dagger a_n\right)\Big].$$

The substitution of these relations in the (2.13) gives an operator, whose action on the vacuum state is

$$\langle 0|T_{00}|0\rangle = \frac{\hbar \mathrm{c}}{2}\sum_n\Big\{\frac{1}{\mathrm{c}^2}\frac{\mathrm{c}}{a}\omega_n\sin^2(k_n x) + \frac{\mathrm{c}}{a\omega_n}\cos^2(k_n x)k_n^2 +$$
$$+\frac{m^2\mathrm{c}^2}{\hbar^2}\left(\frac{\mathrm{c}}{a\omega_n}\right)\sin^2(k_n x)\Big\},$$

where we substituted in the second term the dispersive relation (2.7), written as $k_n^2 = \frac{\omega_n^2}{c^2} - \frac{m^2c^2}{\hbar^2}$, and, with some calculations, we prove the (2.14)

$$\begin{aligned}\langle 0|T_{00}|0\rangle = \frac{\hbar c}{2}\sum_n \Big\{ \frac{\omega_n}{ac}\sin^2(k_n x) + \frac{\omega_n}{ac}\cos^2(k_n x) + \\ -\frac{m^2c^3}{a\hbar^2\omega_n}\cos^2(k_n x) + \frac{m^2c^3}{a\hbar^2 a\omega_n}\sin^2(k_n x)\Big\} = \\ = \frac{\hbar c}{2}\sum_n \Big\{ \frac{\omega_n}{ac} - \frac{m^2c^3}{a\hbar^2\omega_n}\left[\cos^2(k_n x) - \sin^2(k_n x)\right] \Big\} = \\ = \frac{\hbar}{2a}\sum_{n=1}^{\infty}\omega_n - \frac{m^2c^4}{2a\hbar}\sum_n \frac{\cos(2k_n x)}{\omega_n} .\end{aligned}$$

□

The equation (2.15) for the vacuum state energy of a quantized field between boundaries is the key point for the formulation of the theory of the Casimir effect. Clearly, it is immediatly viewable that the quantity $\mathcal{E}_0(a)$ is infinite, but there are sundry ways to regularize it [24], as we will explain subsequently.

Evindently, physical results should not depend on the choice of the regularation procedure.

2.2 Regularizations in the vacuum

There are different possible ways to regularize the ultraviolet divergences in the vacuum. The most important of them are, without any doubt, *frequency cutoff regularization*, *zeta function regularization* and *point splitting regularization*, each of them will be dealt carefully.

2.2.1 Frequancy cutoff regularization

This could be considered the conceptually simplest scheme of regularization[5]. As was already said in the introduction, we want to remove the ultraviolet frequences. We put a damping function $\exp\{-\delta\omega_n\}$ in the sum

$$\mathcal{E}_0(a,\delta) = \frac{\hbar}{2}\sum_{n=1}^{\infty}\omega_n \exp\left(-\delta\omega_n\right) , \tag{2.17}$$

[5]The frequancy cutoff regularization was also used by Casimir in the original work [25].

that obviously comes to (2.15) when $\delta \to 0$.
Without losing generality, we consider the regularized vacuum energy of the interval for a massless field ($m = 0$), that results [24]

$$\mathcal{E}_0(a,\delta) = \frac{\hbar}{2}\sum_{n=1}^{\infty}\frac{c\pi n}{a}\exp\left(-\delta\frac{c\pi n}{a}\right) = \frac{\hbar c\pi}{8a}\sinh^{-2}\left(\frac{\delta c\pi}{2a}\right) . \tag{2.18}$$

Proof. We can prove the last passage of this latter expression using the relation, proved in A,

$$S_N = \sum_{n=1}^{N}\alpha n e^{-\alpha n} = \frac{\alpha e^{-\alpha N}\left[e^{\alpha(N+1)} + N - e^{\alpha}(N+1)\right]}{\left(e^{\alpha}-1\right)^2} , \tag{2.19}$$

and computing the limit for $N \to \infty$. In fact, the result is

$$\lim_{N\to+\infty}\frac{\alpha e^{-\alpha n}\left[e^{\alpha(N+1)} + N - e^{\alpha}(N+1)\right]}{\left(e^{\alpha}-1\right)^2} = \frac{\alpha e^{\alpha}}{\left(e^{\alpha}-1\right)^2} =$$
$$= \frac{\alpha}{\left(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}\right)^2} = \frac{\alpha}{4}\sinh^{-2}\left(\frac{\alpha}{2}\right) ,$$

that for $\alpha = \dfrac{\delta c\pi}{a}$, since we have $\sum_{n=1}^{\infty}\dfrac{\alpha}{\delta}\exp\left(-\alpha n\right)$, gives back the (2.18). □

In the limit of small δ we obtain

$$\mathcal{E}_0(a,\delta) = \frac{\hbar a}{2\pi c\delta^2} + \mathcal{E}(a) + \mathcal{O}(\delta^2) , \qquad \mathcal{E}(a) = -\frac{\hbar c\pi}{24a} , \tag{2.20}$$

so the vacuum energy is represented by a sum of singular terms, each of them with a finite contribution.

We now repeat the same calculation to the vacuum energy density in Minkoski vacuum $|0_M\rangle$

$$\langle 0_M|T_{00}|0_M\rangle = \frac{\hbar}{2\pi}\int_0^{+\infty} dk\,\omega , \tag{2.21}$$

and we evaluate the regularized vacuum energy for the interval $[0,a]$, that is

$$\mathcal{E}_{0M}(a) = \frac{\hbar a}{2\pi}\int_0^{+\infty} dk\,\omega . \tag{2.22}$$

At this point, we apply the same regularization as above, imposing an exponentially damping function $\exp\{-\delta\omega(k)\}$ under the integral, again considering the massless case, so that $\omega(k) = ck$

$$\mathcal{E}_{0M}(a,\delta) = \frac{\hbar c a}{2\pi}\int_0^{+\infty} dk\,k\exp\left(-\delta c k\right) = \frac{\hbar a}{2\pi c\delta^2} . \tag{2.23}$$

Proof. It is quickly to demonstrate the latter result, simply integrating by parts

$$\begin{aligned}\mathcal{E}_{0M}(a,\delta) =& \frac{\hbar c a}{2\pi}\int_0^{+\infty} \mathrm{d}k\, k e^{-\delta c k} = \left[-\frac{\hbar a}{2\pi\delta} k e^{-\delta c k}\right]_0^{+\infty} + \frac{\hbar a}{2\pi\delta}\int_0^{+\infty} \mathrm{d}k e^{-\delta c k} = \\ &= \left[-\frac{\hbar a}{2\pi c\delta^2} e^{-\delta c k}\right]_0^{+\infty} = \frac{\hbar a}{2\pi c\delta^2}\,.\end{aligned}$$

□

Consequently, the renormalized vacuum energy in the interval $[0, a]$, that we can call Casimir energy for the scalar field, is the difference between (2.18) and (2.23), in the limit $\delta \to 0$

$$\mathcal{E}_0^{ren}(a) = \lim_{\delta\to 0}\left[\mathcal{E}_0(a,\delta) - \mathcal{E}_{0M}(a,\delta)\right] = \lim_{\delta\to 0}\left[\mathcal{E}(a) + \mathcal{O}(\delta^2)\right] = -\frac{\hbar c\pi}{24a}\,. \tag{2.24}$$

So, in this simple case, the renormalization corresponds to removing a quantity equal to the vacuum energy of the unbounded space in the given interval. The renormalized energy $\mathcal{E}(a)$ monotonically decreases when boundary points approach each other and this points to the presence of an attractive force between the conducting planes, namely

$$F(a) = -\frac{\partial \mathcal{E}_0^{ren}(a)}{\partial a} = -\frac{\hbar c\pi}{24a^2}\,. \tag{2.25}$$

2.2.2 Zeta function regularization

ζ-function regularization method is the most elegant, and maybe even useful, in a large variety of cases, because of its pleasant mathematical properties. The basic idea is as simple as powerful, and it is based upon the analytical continuation of the Riemann ζ-function

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} \tag{2.26}$$

Firstly, we change the power of the frequency ω_n in the sum (2.15), and it lead to

$$\mathcal{E}_0(s) = \frac{\hbar}{2}\mu^{2s}\sum_{n=1}^{\infty}\omega_n^{1-2s}\,, \tag{2.27}$$

where μ is an arbitrary mass scale, introduced in order to keep the energy dimension of $\mathcal{E}_0$.
The physical expression is recovered on removing the regularization, thus evaluating the limit $\lim\limits_{s\longrightarrow 0}\mathcal{E}_0(s)$.

This latter series converges, in general, for $\mathrm{Re}(s) > \dfrac{D+1}{2}$, in a D-dimensional space. In our model, $D = 2$ because we are working with one time-dimension and one space-dimension.

So, we can evaluate the series $\sum_{n=1}^{\infty} \omega_n$, with $\omega_n = \dfrac{nc\pi}{a}$ in the cavity $[0, a]$. The energy density is

$$\langle T_{00}(x) \rangle = -\frac{\hbar c\pi}{24a^2} - \frac{m^2 c^4}{2a\hbar} \sum_{n=1}^{\infty} \frac{\cos(2k_n x)}{\omega_n}\,, \tag{2.28}$$

and the energy results

$$\mathcal{E}_0(a) = \frac{\hbar}{2} \sum_{n=1}^{\infty} \omega_n = \frac{\hbar}{2} \sum_{n=1}^{\infty} \omega_n = -\frac{\hbar c\pi}{24a}\,, \tag{2.29}$$

because of the relation $\sum_{n=1}^{\infty} n = -\dfrac{1}{12}$, proved in B.

This regularization procedure has an important property in common with the previos one: in both processes, the eigenvalue spectrum has to be determined explicitly.

2.2.3 Point splitting regularization

In this regularization procedure, we start representing the vacuum energy in terms of the Green's function, as follows

$$\mathcal{E}_0 = \mathrm{i} \int_V \mathrm{d}r \left. \frac{\partial^2 G(x, x')}{\partial x_0^2} \right|_{x'=x}\,, \tag{2.30}$$

and we introduce the regularization vector parameter ϵ, such that

$$\mathcal{E}_0(\epsilon) = \mathrm{i} \int_V \mathrm{d}r \left. \frac{\partial^2 G(x, x')}{\partial x_0^2} \right|_{x'=x+\epsilon}\,. \tag{2.31}$$

The only nonzero component of ϵ is the time one, so we keep $\epsilon = (\epsilon_0, \mathbf{0})$, with $\epsilon_0 \neq 0$. This technique was used in quantum field theory in operator product expansions and for quantum fields in curved backgrounds.

Anyway, point splitting regularization has been shown to be equivalent to the zeta function regularization, by Moretti in 1999 [18].

2.3 One dimensional space with a nontrivial topology

We continue considering a real scalar field on the interval $0 \leq x \leq a$, but we change the boundary conditions, imposing

$$\phi(t,0) = \phi(t,a)\,, \qquad \frac{\partial \phi}{\partial x}(t,0) = \frac{\partial \phi}{\partial x}(t,a)\,, \tag{2.32}$$

that is equivalent to identify the boundary points, $x = 0$ and $x = a$, as the same [23]. As a consequence, we want to get the scalar field on a flat manifold with topology of a circle S^1, as shown in Figure 2.2.

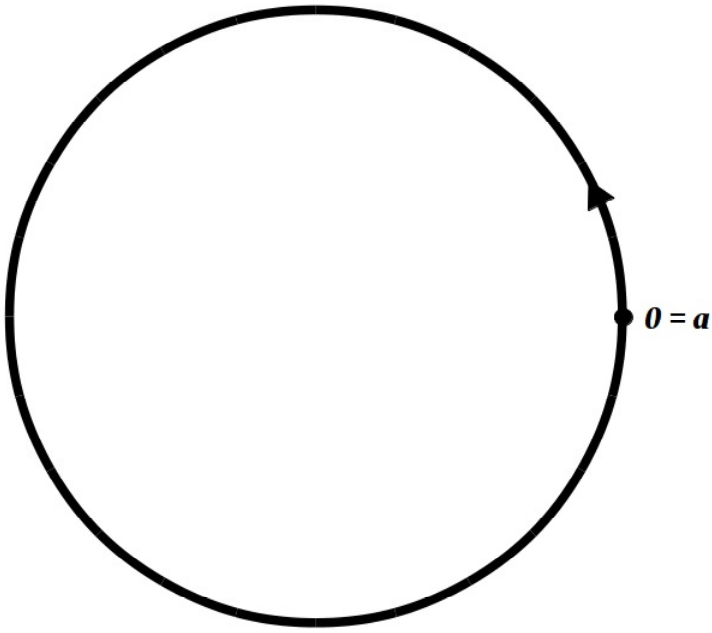

Figure 2.2: The topology of the manifold S^1.

Compared with (2.1), now solutions where $\phi \neq 0$ are possible at the point $x = 0$, likewise $x = a$. The orthonormal set of solutions of the field equation (2.2), with conditions (2.32), can be represented in the form

$$\phi_n^{(\pm)} = \left(\frac{\mathrm{c}}{2a\omega_n}\right)^{\frac{1}{2}} \exp\left[\pm \mathrm{i}(\omega_n t - k_n x)\right]\,, \tag{2.33}$$

with the following relations

$$\omega_n = \left(\frac{m^2\mathrm{c}^4}{\hbar^2} + \mathrm{c}^2 k_n^2\right)^{\frac{1}{2}}\,, \qquad k_n = \frac{2\pi n}{a}\,, \qquad n \in \mathbb{Z}\,.$$

Substituting these positive and negative frequency solutions in the field decomposition (2.9), and using its expression in order to calculate the density of energy, according to (2.13), we find the vacuum energy density of a scalar field defined on a one dimensional sphere

$$\langle 0|T_{00}(x)|0\rangle = \frac{\hbar}{2a}\sum_{n=-\infty}^{\infty}\omega_n\,. \tag{2.34}$$

The first comparison with (2.14) is the absence of an oscillating term, while the total vacuum energy turns out to be

$$\mathcal{E}_0(a,m) = \int_0^a \mathrm{d}x\,\langle 0|T_{00}(x)|0\rangle = \frac{\hbar}{2}\sum_{n=-\infty}^{\infty}\omega_n\,,$$

then,

$$\mathcal{E}_0(a,m) = \hbar\sum_{n=0}^{\infty}\omega_n - \frac{1}{2}mc^2\,. \tag{2.35}$$

The renormalization of this divergent quantity is performed by subtracting the contribution of the Minkowski space, as calculated previously in the accordance to (2.24), so it is, substituting (2.33), (2.35) and (2.22) into (2.24)

$$\begin{aligned}\mathcal{E}_0^{ren}(a,m) &= \hbar\left\{\sum_{n=0}^{\infty}\omega_n - \frac{a}{2\pi}\int_0^{\infty}\mathrm{d}k\omega(k)\right\} - \frac{mc^2}{2} = \\ &= \frac{2\hbar c\pi}{a}\left\{\sum_{n=0}^{\infty}\sqrt{\left(\frac{amc}{2\pi\hbar}\right)^2+n^2} - \int_0^{\infty}\mathrm{d}t\sqrt{\left(\frac{amc}{2\pi\hbar}\right)^2+t^2}\right\} - \frac{mc^2}{2}\,,\end{aligned} \tag{2.36}$$

with $t = \dfrac{ak}{2\pi}$.

Using the Abel-Plana formula [26]

$$\sum_{n=0}^{\infty}F(n) - \int_0^{\infty}\mathrm{d}tF(t) = \frac{1}{2}F(0) + \int_0^{\infty}\frac{\mathrm{d}t}{\mathrm{e}^{2\pi t}-1}\left[F(\mathrm{i}t) - F(-\mathrm{i}t)\right]\,, \tag{2.37}$$

we put

$$F(t) = \sqrt{\left(\frac{amc}{2\pi\hbar}\right)^2+t^2}$$

and since

$$F(\mathrm{i}t) - F(-\mathrm{i}t) = 2\mathrm{i}\sqrt{t^2-\left(\frac{amc}{2\pi\hbar}\right)^2}\quad\left(t\geq\frac{amc}{2\pi\hbar}\right)$$

for $F(z)$ analytic function in the right half-plane, we finally obtain

$$\mathcal{E}_0^{ren}(a,m) = -\frac{4\hbar c\pi}{a}\int_{\frac{amC}{2\pi\hbar}}^{\infty} \mathrm{d}t \frac{\sqrt{t^2 - \left(\frac{amC}{2\pi\hbar}\right)^2}}{\mathrm{e}^{2\pi t}-1} = -\frac{\hbar c}{\pi a}\int_{\mu}^{\infty} \mathrm{d}\xi \frac{\sqrt{\xi^2-\mu^2}}{\mathrm{e}^{\xi}-1}\,, \qquad (2.38)$$

where $\xi = 2\pi t$ and $\mu = \dfrac{mca}{\hbar}$.

In we consider the massless case, we have $\mu = 0$, and the result is

$$\mathcal{E}_0^{ren}(a,m) = -\frac{\hbar c}{\pi a}\int_0^{\infty} \mathrm{d}\xi \frac{\xi}{\mathrm{e}^{\xi}-1} = -\frac{\hbar c\pi}{6a}\,. \qquad (2.39)$$

Chapter 3

Moving boundaries

Taking in exam the wave equation

$$\frac{1}{\mathrm{c}^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = 0 \tag{3.1}$$

in a time-dependent domain $0 < x < L(t)$, where $L(t)$ is the right moving edge, constrained to the boundary conditions [21]

$$\phi(t,0) = \phi\left(t, L(t)\right) = 0\,. \tag{3.2}$$

A complete set of solutions to this problem can be written in the form [27]

$$\phi_n(t,x) = C_n\Big\{\exp\big[-\mathrm{i}\pi n R(\mathrm{c}t - x)\big] - \exp\big[-\mathrm{i}\pi n R(\mathrm{c}t + x)\big]\Big\}\,, \tag{3.3}$$

where the function $R(\xi)$ must satisfy the functional equation

$$R(\mathrm{c}t + L(t)) - R(\mathrm{c}t - L(t)) = 2\,. \tag{3.4}$$

The field ϕ is a linear combination of these solutions and their complex conjugate, namely

$$\phi(t,x) = \sqrt{\hbar \mathrm{c}}\sum_{n=1}^{\infty}\Big\{a_n\phi_n(t,x) + a_n^\dagger\phi_n^*(t,x)\Big\}\,, \tag{3.5}$$

where C_n is a normalization constant that will be evaluated in the next pages.

Proof. Let's prove the (3.1) calculating the derivatives of (3.3). If it is true for every n, then it is for their summation, too.

$$\begin{aligned}\frac{\partial\phi_n}{\partial t} &= C_n\Big\{\exp\big[-\mathrm{i}\pi nR(\mathrm{c}t-x)\big]\mathrm{c}\,(-\mathrm{i}\pi nR^{'}(\mathrm{c}t-x))+\\ &-\exp\big[-\mathrm{i}\pi nR(\mathrm{c}t+x)\big]\mathrm{c}\,(-\mathrm{i}\pi nR^{'}(\mathrm{c}t+x))\Big\}\,,\\ \frac{\partial^2\phi_n}{\partial t^2} &= C_n\Big\{\exp\big[-\mathrm{i}\pi nR(\mathrm{c}t-x)\big]\mathrm{c}^2\Big(-\pi^2n^2R^{'^2}(\mathrm{c}t-x)-\mathrm{i}\pi nR^{''}(\mathrm{c}t-x)\Big)+\\ &+\exp\big[-\mathrm{i}\pi nR(\mathrm{c}t+x)\big]\mathrm{c}^2\Big(\pi^2n^2R^{'^2}(\mathrm{c}t+x)+\mathrm{i}\pi nR^{''}(\mathrm{c}t+x)\Big)\Big\}\,.\end{aligned}$$

In the same way

$$\begin{aligned}\frac{\partial\phi_n}{\partial x} &= C_n\Big\{\exp\big[-\mathrm{i}\pi nR(\mathrm{c}t-x)\big]\mathrm{i}\pi nR^{'}(\mathrm{c}t-x)+\\ &+\exp\big[-\mathrm{i}\pi nR(\mathrm{c}t+x)\big]\mathrm{i}\pi nR^{'}(\mathrm{c}t+x)\Big\}\,,\\ \frac{\partial^2\phi_n}{\partial x^2} &= C_n\Big\{\exp\big[-\mathrm{i}\pi nR(\mathrm{c}t-x)\big]\Big(-\pi^2n^2R^{'^2}(\mathrm{c}t-x)-\mathrm{i}\pi nR^{''}(\mathrm{c}t-x)\Big)+\\ &+\exp\big[-\mathrm{i}\pi nR(\mathrm{c}t+x)\big]\Big(\pi^2n^2R^{'^2}(\mathrm{c}t+x)+\mathrm{i}\pi nR^{''}(\mathrm{c}t+x)\Big)\Big\}\,.\end{aligned}$$

It is possible to compute the same calculation for ϕ_n^*, and it will be very similar to ϕ's, in fact

$$\begin{aligned}\frac{\partial^2\phi_n^*}{\partial t^2} &= C_n^*\Big\{\exp\big[\mathrm{i}\pi nR(\mathrm{c}t-x)\big]\mathrm{c}^2\Big(-\pi^2n^2R^{'^2}(\mathrm{c}t-x)+\mathrm{i}\pi nR^{''}(\mathrm{c}t-x)\Big)+\\ &+\exp\big[\mathrm{i}\pi nR(\mathrm{c}t+x)\big]\mathrm{c}^2\Big(\pi^2n^2R^{'^2}(\mathrm{c}t+x)-\mathrm{i}\pi nR^{''}(\mathrm{c}t+x)\Big)\Big\}\,,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2\phi_n^*}{\partial x^2} &= C_n^*\Big\{\exp\big[\mathrm{i}\pi nR(\mathrm{c}t-x)\big]\Big(-\pi^2n^2R^{'^2}(\mathrm{c}t-x)+\mathrm{i}\pi nR^{''}(\mathrm{c}t-x)\Big)+\\ &+\exp\big[\mathrm{i}\pi nR(\mathrm{c}t+x)\big]\Big(\pi^2n^2R^{'^2}(\mathrm{c}t+x)-\mathrm{i}\pi nR^{''}(\mathrm{c}t+x)\Big)\Big\}\,.\end{aligned}$$

The first boundary condition is straightforward

$$\phi_n(t,0) = C_n\Big\{\exp\big[-\mathrm{i}\pi nR(\mathrm{c}t)\big]-\exp\big[-\mathrm{i}\pi nR(\mathrm{c}t)\big]\Big\} = 0\,,$$

while for demonstrate the second one we need the (3.4)

$$\phi_n(t,L(t)) = C_n\Big\{\exp\big[-\mathrm{i}\pi nR(\mathrm{c}t-L(t))\big]-\exp\big[-\mathrm{i}\pi nR(\mathrm{c}t+L(t))\big]\Big\} = 0\,.$$

□

Differentiating (3.4) with respect to the time variable t, we find the following expression for the velocity

$$\dot{L}(t) = \frac{R'[ct - L(t)] - R'[ct + L(t)]}{R'[ct - L(t)] + R'[ct + L(t)]}\mathrm{c}\,, \tag{3.6}$$

where the primes indicate the total derivative with respect to the argument. With these relation we can find the relation between the derivative of the function R, in particular

$$R'(\mathrm{c}t - L(t)) = R'(\mathrm{c}t + L(t))\frac{c + \dot{L}(t)}{c - \dot{L}(t)}\,, \tag{3.7}$$

or

$$R'(\mathrm{c}t + L(t)) = R'(\mathrm{c}t - L(t))\frac{c - \dot{L}(t)}{c + \dot{L}(t)}\,. \tag{3.8}$$

Now, we can find the value of the constant C_n

$$C_n = \frac{\mathrm{i}}{\sqrt{4\pi n}} = \frac{\mathrm{i}}{2\sqrt{\pi n}}\,. \tag{3.9}$$

according to [27], [28], too.

Proof. We want to calculate the value of the constant C_n imposing the normalization with the scalar product (2.5), as in the stationary model.
Firstly, we compute the time-derivative of the field

$$\begin{aligned}\frac{1}{\mathrm{c}}\frac{\partial \phi_n}{\partial t} =& \frac{1}{\mathrm{c}}C_n(-\mathrm{i}\pi n)\Big\{\exp\big[-\mathrm{i}\pi n R(\mathrm{c}t - x)\big]R'(\mathrm{c}t - x)\mathrm{c} + \\ &- \exp\big[-\mathrm{i}\pi n R(\mathrm{c}t + x)\big]R'(\mathrm{c}t + x)\mathrm{c}\Big\} = \\ =& (-\mathrm{i}\pi n)C_n\Big\{\exp\big[-\mathrm{i}\pi n R(\mathrm{c}t - x)\big]R'(\mathrm{c}t - x) + \\ &- \exp\big[-\mathrm{i}\pi n R(\mathrm{c}t + x)\big]R'(\mathrm{c}t + x)\Big\}\,,\end{aligned}$$

and, in a similar way, we compute the time-derivative of the complex conjugate of the field

$$\begin{aligned}\frac{1}{\mathrm{c}}\frac{\partial \phi_m^*}{\partial t} =& (\mathrm{i}\pi m)C_m^*\Big\{\exp\big[\mathrm{i}\pi m R(\mathrm{c}t - x)\big]R'(\mathrm{c}t - x) + \\ &- \exp\big[\mathrm{i}\pi m R(\mathrm{c}t + x)\big]R'(\mathrm{c}t + x)\Big\}\,.\end{aligned}$$

Now, we evaluate the expression of the scalar product, and we want it to be normalized according to

$$\langle \phi_m, \phi_n \rangle = \delta_{mn}\,.$$

$$
\begin{aligned}
\langle\phi_m,\phi_n\rangle =& \int_0^{L(t)} \mathrm{d}x \Bigg\{ \left[\phi_m^* \left(\frac{\mathrm{i}}{\mathrm{c}}\frac{\partial\phi_n}{\partial t}\right) - \left(\frac{\mathrm{i}}{\mathrm{c}}\frac{\partial\phi_m^*}{\partial t}\right)\phi_n\right] = \\
=&\pi C_m^* C_n \int_0^{L(t)} \mathrm{d}x\, n \left(\mathrm{e}^{\mathrm{i}\pi m R(\mathrm{c}t-x)} - \mathrm{e}^{\mathrm{i}\pi m R(\mathrm{c}t+x)}\right) \times \\
&\times \left(\mathrm{e}^{-\mathrm{i}\pi n R(\mathrm{c}t-x)} R'(\mathrm{c}t-x) - \mathrm{e}^{-\mathrm{i}\pi n R(\mathrm{c}t+x)} R'(\mathrm{c}t+x)\right) + \\
&+ m \left(\mathrm{e}^{\mathrm{i}\pi m R(\mathrm{c}t-x)} R'(\mathrm{c}t-x) - \mathrm{e}^{\mathrm{i}\pi m R(\mathrm{c}t+x)} R'(\mathrm{c}t+x)\right) \times \\
&\times \left(\mathrm{e}^{-\mathrm{i}\pi n R(\mathrm{c}t-x)} - \mathrm{e}^{-\mathrm{i}\pi n R(\mathrm{c}t+x)}\right) \Bigg\} = \\
=&\pi C_m^* C_n \int_0^{L(t)} \mathrm{d}x \Bigg\{ n\Big(\mathrm{e}^{\mathrm{i}\pi(m-n)R(\mathrm{c}t-x)} R'(\mathrm{c}t-x) + \\
&+ \mathrm{e}^{\mathrm{i}\pi(m-n)R(\mathrm{c}t+x)} R'(\mathrm{c}t+x) - \mathrm{e}^{\mathrm{i}\pi(mR(\mathrm{c}t+x)-nR(\mathrm{c}t-x))} R'(\mathrm{c}t-x) + \\
&- \mathrm{e}^{\mathrm{i}\pi(mR(\mathrm{c}t-x)-nR(\mathrm{c}t+x))} R'(\mathrm{c}t+x)\Big) + \\
&+ m\Big(\mathrm{e}^{\mathrm{i}\pi(m-n)R(\mathrm{c}t-x)} R'(\mathrm{c}t-x) + \\
&+ \mathrm{e}^{\mathrm{i}\pi(m-n)R(\mathrm{c}t+x)} R'(\mathrm{c}t+x) - \mathrm{e}^{\mathrm{i}\pi(mR(\mathrm{c}t-x)-nR(\mathrm{c}t+x))} R'(\mathrm{c}t-x) + \\
&- \mathrm{e}^{\mathrm{i}\pi(mR(\mathrm{c}t+x)-nR(\mathrm{c}t-x))} R'(\mathrm{c}t+x)\Big) \Bigg\} = \\
=&\pi C_m^* C_n \int_0^{L(t)} \mathrm{d}x \Bigg\{ (n+m) R'(\mathrm{c}t-x) \mathrm{e}^{\mathrm{i}\pi(m-n)R(\mathrm{c}t-x)} + \\
&+ (m+n) R'(\mathrm{c}t+x) \mathrm{e}^{\mathrm{i}\pi(m-n)R(\mathrm{c}t+x)} + \\
&- \left(nR'(\mathrm{c}t-x) - mR'(\mathrm{c}t+x)\right) \mathrm{e}^{\mathrm{i}\pi(mR(\mathrm{c}t+x)-nR(\mathrm{c}t-x))} + \\
&- \left(nR'(\mathrm{c}t+x) - mR'(\mathrm{c}t-x)\right) \mathrm{e}^{\mathrm{i}\pi(mR(\mathrm{c}t-x)-nR(\mathrm{c}t+x))} . \Bigg\}
\end{aligned}
$$

We can see that the first two terms, integrated on the space, give a delta-funcion $\delta(n-m)$, while the last two terms, integrated by parts, vanish.
Then, we obtain the following espression

$$|C_n|^2\, 4\pi n = 1\,,$$

that is satisfied by (3.9). □

Substituting in the previous expressions this value of the constant (3.9), we find that the scalar field (3.5) is written as

$$\phi = \sum_n \frac{\mathrm{i}}{2}\sqrt{\frac{\hbar \mathrm{c}}{\pi n}}\Big\{ \Big(\exp\big[-\mathrm{i}\pi n R(\mathrm{c}t-x)\big] - \exp\big[-\mathrm{i}\pi n R(\mathrm{c}t+x)\big]\Big)\, a_n + \\ - \Big(\exp\big[\mathrm{i}\pi n R(\mathrm{c}t-x)\big] - \exp\big[\mathrm{i}\pi n R(\mathrm{c}t+x)\big]\Big)\, a_n^\dagger \Big\} . \quad (3.10)$$

It is also possible to write the density of energy, or density of the Hamiltonian operator, that results

$$T_{00}(t,x) = \frac{1}{2}\left[\left(\frac{1}{\mathrm{c}}\frac{\partial \phi}{\partial t}\right)^2 + \left(\frac{\partial \phi}{\partial x}\right)^2\right] , \quad (3.11)$$

which results

$$T_{00}(t,x) = \hbar \mathrm{c} \sum_{n,m=1}^{\infty} \frac{\pi}{4}\sqrt{nm}\Big\{ a_n a_m \Big[R'^2(\mathrm{c}t-x)\exp\left[-\mathrm{i}\pi(n+m)R(\mathrm{c}t-x)\right] + \\ + R'^2(\mathrm{c}t+x)\exp\left[-\mathrm{i}\pi(n+m)R(\mathrm{c}t+x)\right]\Big] + \\ + a_n^\dagger a_m^\dagger \Big[R'^2(\mathrm{c}t-x)\exp\left[\mathrm{i}\pi(n+m)R(\mathrm{c}t-x)\right] + \\ + R'^2(\mathrm{c}t+x)\exp\left[\mathrm{i}\pi(n+m)R(\mathrm{c}t+x)\right]\Big] + \\ + a_n^\dagger a_m \Big[R'^2(\mathrm{c}t-x)\exp\left[-\mathrm{i}\pi(m-n)R(\mathrm{c}t-x)\right] + \\ + R'^2(\mathrm{c}t+x)\exp\left[-\mathrm{i}\pi(m-n)R(\mathrm{c}t+x)\right]\Big] + \\ + \left(\delta_{mn} + a_m^\dagger a_n\right)\Big[R'^2(\mathrm{c}t-x)\exp\left[-\mathrm{i}\pi(n-m)R(\mathrm{c}t-x)\right] + \\ + R'^2(\mathrm{c}t+x)\exp\left[-\mathrm{i}\pi(n-m)R(\mathrm{c}t+x)\right]\Big]\Big\} \quad (3.12)$$

Proof. We can prove the expression for the density of energy in two different ways. Starting from the Lagrangian density (2.4), by means of (2.12) we find, as shown previously, the expression (2.13), that turns into (3.11) when $m = 0$, so in case of massless field.
We can obtain the same result in a different way, through a Legendre transform, where, defined ϕ as canonical coordinate, $\dot{\phi} = \frac{1}{\mathrm{c}}\frac{\partial \phi}{\partial t}$, the conjugate momentum is

$$p = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{1}{\mathrm{c}}\frac{\partial \phi}{\partial t} = \dot{\phi} ,$$

so the Hamiltonian density is

$$\mathcal{H} = p\dot{\phi} - \mathcal{L} = \frac{1}{2}\left[\left(\frac{1}{\mathrm{c}}\frac{\partial\phi}{\partial t}\right)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2\right]. \tag{3.13}$$

Let's write the derivatives of the field (3.10)

$$\frac{1}{\mathrm{c}}\frac{\partial\phi}{\partial t} = \frac{\sqrt{\hbar\mathrm{c}\pi}}{2}\sum_n \sqrt{n}\Big\{\Big(R'(\mathrm{c}t-x)\mathrm{e}^{-\mathrm{i}\pi nR(\mathrm{c}t-x)} - R'(\mathrm{c}t+x)\mathrm{e}^{-\mathrm{i}\pi nR(\mathrm{c}t+x)}\Big)a_n + \\ + \Big(R'(\mathrm{c}t-x)\mathrm{e}^{\mathrm{i}\pi nR(\mathrm{c}t-x)} - R'(\mathrm{c}t+x)\mathrm{e}^{\mathrm{i}\pi nR(\mathrm{c}t+x)}\Big)a_n^\dagger\Big\},$$

$$\frac{\partial\phi}{\partial x} = -\frac{\sqrt{\hbar\mathrm{c}\pi}}{2}\sum_n \sqrt{n}\Big\{\Big(R'(\mathrm{c}t-x)\mathrm{e}^{-\mathrm{i}\pi nR(\mathrm{c}t-x)} - R'(\mathrm{c}t+x)\mathrm{e}^{-\mathrm{i}\pi nR(\mathrm{c}t+x)}\Big)a_n + \\ + \Big(R'(\mathrm{c}t-x)\mathrm{e}^{\mathrm{i}\pi nR(\mathrm{c}t-x)} - R'(\mathrm{c}t+x)\mathrm{e}^{\mathrm{i}\pi nR(\mathrm{c}t+x)}\Big)a_n^\dagger\Big\},$$

then

$$\left(\frac{1}{\mathrm{c}}\frac{\partial\phi}{\partial t}\right)^2 = \frac{\hbar\mathrm{c}\pi}{4}\sum_{n,m}\sqrt{nm}\Big[\Big(\mathrm{e}^{-\mathrm{i}\pi nR(\mathrm{c}t-x)}R'(\mathrm{c}t-x) - \mathrm{e}^{-\mathrm{i}\pi nR(\mathrm{c}t+x)}R'(\mathrm{c}t+x)\Big)a_n + \\ + \Big(\mathrm{e}^{\mathrm{i}\pi nR(\mathrm{c}t-x)}R'(\mathrm{c}t-x) - \mathrm{e}^{\mathrm{i}\pi nR(\mathrm{c}t+x)}R'(\mathrm{c}t+x)\Big)a_n^\dagger\Big]\Big[\big(\mathrm{e}^{-\mathrm{i}\pi mR(\mathrm{c}t-x)}R'(\mathrm{c}t-x) + \\ - \mathrm{e}^{-\mathrm{i}\pi mR(\mathrm{c}t+x)}R'(\mathrm{c}t+x)\big)a_m + \Big(\mathrm{e}^{\mathrm{i}\pi mR(\mathrm{c}t-x)}R'(\mathrm{c}t-x) - \mathrm{e}^{\mathrm{i}\pi mR(\mathrm{c}t+x)}R'(\mathrm{c}t+x)\Big)a_m^\dagger\Big] = \\ = \frac{\hbar\mathrm{c}\pi}{4}\sum_{n,m}\sqrt{nm}\Big\{a_n a_m\Big[R'^2(\mathrm{c}t-x)\mathrm{e}^{-\mathrm{i}\pi(n+m)R(\mathrm{c}t-x)} + R'^2(\mathrm{c}t+x)\mathrm{e}^{-\mathrm{i}\pi(n+m)R(\mathrm{c}t+x)} + \\ - R'(\mathrm{c}t-x)R'(\mathrm{c}t+x)\Big(\mathrm{e}^{-\mathrm{i}\pi[nR(\mathrm{c}t-x)+mR(\mathrm{c}t+x)]} + \mathrm{e}^{-\mathrm{i}\pi[nR(\mathrm{c}t+x)+mR(\mathrm{c}t-x)]}\Big)\Big] + \\ + a_n^\dagger a_m^\dagger\Big[R'^2(\mathrm{c}t-x)\mathrm{e}^{\mathrm{i}\pi(n+m)R(\mathrm{c}t-x)} + R'^2(\mathrm{c}t+x)\mathrm{e}^{\mathrm{i}\pi(n+m)R(\mathrm{c}t+x)} + \\ - R'(\mathrm{c}t-x)R'(\mathrm{c}t+x)\Big(\mathrm{e}^{\mathrm{i}\pi[nR(\mathrm{c}t-x)+mR(\mathrm{c}t+x)]} + \mathrm{e}^{\mathrm{i}\pi[nR(\mathrm{c}t+x)+mR(\mathrm{c}t-x)]}\Big)\Big] + \\ + a_n a_m^\dagger\Big[R'^2(\mathrm{c}t-x)\mathrm{e}^{-\mathrm{i}\pi(n-m)R(\mathrm{c}t-x)} + R'^2(\mathrm{c}t+x)\mathrm{e}^{-\mathrm{i}\pi(n-m)R(\mathrm{c}t+x)} + \\ - R'(\mathrm{c}t-x)R'(\mathrm{c}t+x)\Big(\mathrm{e}^{-\mathrm{i}\pi[nR(\mathrm{c}t-x)+mR(\mathrm{c}t+x)]} + \mathrm{e}^{-\mathrm{i}\pi[nR(\mathrm{c}t+x)+mR(\mathrm{c}t-x)]}\Big)\Big] + \\ + a_n^\dagger a_m\Big[R'^2(\mathrm{c}t-x)\mathrm{e}^{-\mathrm{i}\pi(m-n)R(\mathrm{c}t-x)} + R'^2(\mathrm{c}t+x)\mathrm{e}^{-\mathrm{i}\pi(m-n)R(\mathrm{c}t+x)} + \\ - R'(\mathrm{c}t-x)R'(\mathrm{c}t+x)\Big(\mathrm{e}^{-\mathrm{i}\pi[mR(\mathrm{c}t+x)-nR(\mathrm{c}t-x)]} + \mathrm{e}^{-\mathrm{i}\pi[mR(\mathrm{c}t-x)-nR(\mathrm{c}t+x)]}\Big)\Big]\Big\},$$

and

$$\left(\frac{\partial\phi}{\partial x}\right)^2 = \frac{\hbar c\pi}{4}\sum_{n,m}\sqrt{nm}\Big[\left(e^{-i\pi nR(ct-x)}R'(ct-x) - e^{-i\pi nR(ct+x)}R'(ct+x)\right)a_n +$$
$$+\left(e^{i\pi nR(ct-x)}R'(ct-x) - e^{i\pi nR(ct+x)}R'(ct+x)\right)a_n^\dagger\Big]\Big[\left(e^{-i\pi mR(ct-x)}R'(ct-x)+\right.$$
$$\left.- e^{-i\pi mR(ct+x)}R'(ct+x)\right)a_m + \left(e^{i\pi mR(ct-x)}R'(ct-x) - e^{i\pi mR(ct+x)}R'(ct+x)\right)a_m^\dagger\Big] =$$
$$= \frac{\hbar c\pi}{4}\sum_{n,m}\sqrt{nm}\Big\{a_n a_m\Big[R'^2(ct-x)e^{-i\pi(n+m)R(ct-x)} + R'^2(ct+x)e^{-i\pi(n+m)R(ct+x)}+$$
$$+ R'(ct-x)R'(ct+x)\left(e^{-i\pi[nR(ct-x)+mR(ct+x)]} + e^{-i\pi[nR(ct+x)+mR(ct-x)]}\right)\Big]+$$
$$+ a_n^\dagger a_m^\dagger\Big[R'^2(ct-x)e^{i\pi(n+m)R(ct-x)} + R'^2(ct+x)e^{i\pi(n+m)R(ct+x)}+$$
$$+ R'(ct-x)R'(ct+x)\left(e^{i\pi[nR(ct-x)+mR(ct+x)]} + e^{i\pi[nR(ct+x)+mR(ct-x)]}\right)\Big]+$$
$$+ a_n a_m^\dagger\Big[R'^2(ct-x)e^{-i\pi(n-m)R(ct-x)} + R'^2(ct+x)e^{-i\pi(n-m)R(ct+x)}+$$
$$+ R'(ct-x)R'(ct+x)\left(e^{-i\pi[nR(ct-x)+mR(ct+x)]} + e^{-i\pi[nR(ct+x)+mR(ct-x)]}\right)\Big]+$$
$$+ a_n^\dagger a_m\Big[R'^2(ct-x)e^{-i\pi(m-n)R(ct-x)} + R'^2(ct+x)e^{-i\pi(m-n)R(ct+x)}+$$
$$+ R'(ct-x)R'(ct+x)\left(e^{-i\pi[mR(ct+x)-nR(ct-x)]} + e^{-i\pi[mR(ct-x)-nR(ct+x)]}\right)\Big]\Big\}.$$

Adding these two terms as requested by (3.11) and using the algebra (2.10) so that

$$a_n a_m^\dagger = \delta_{mn} + a_m^\dagger a_n$$

we find the relation (3.12). □

Evaluating this latter expression of the density of energy in the vacuum, instead, there is only one nonvanishing term, because

$$\langle n|m\rangle = \delta_{nm}\,, \tag{3.14}$$

and it reads

$$T_{00}(t,x) = \sum_n \frac{\hbar c\pi n}{4}\Big\{R'^2(ct-x) + R'^2(ct+x)\Big\}\,, \tag{3.15}$$

or

$$T_{00}(t,x) = -\frac{\hbar c\pi}{48}\Big\{R'^2(ct-x) + R'^2(ct+x)\Big\}\,. \tag{3.16}$$

We can complete our collection writing the pressure, that corresponds to the two space-index of the energy momentum tensor T_{ij}, when they are equivalent. We can conclude that it is equal to the density of energy for a one dimensional model,

$$P = T_{\alpha\alpha} = T_{00} = -\frac{\hbar c\pi}{48}\left\{R'^2(ct-x) + R'^2(ct+x)\right\}, \tag{3.17}$$

in fact we may have expected that

$$T_{\alpha\alpha} = NT_{00}\,,$$

where N is the dimension of the purely spatial space.

Proof. Using the general equation (2.12), it is easy to see that

$$\begin{aligned} T_{\alpha\alpha} &= \frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial x} + \frac{1}{2c^2}\frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial t} - \frac{1}{2}\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial x} = \\ &= \frac{1}{2}\left[\left(\frac{1}{c}\frac{\partial\phi}{\partial t}\right)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2\right], \end{aligned}$$

that leads to (3.11). □

3.1 Uniform motion with a constant velocity

The simplest law is, obviously, the uniform motion

$$L(t) = L_0(1+\alpha t) \qquad \text{or} \qquad L(t) = L_0 + vt\,, \tag{3.18}$$

where $v = \alpha L_0$. The first exact solution was written by Havelock [16], and it was developed by Nicolai [17], and it has brought to a R-function depending on α according with the following formula [30]

$$R_\alpha(\xi) = \frac{2\ln\left|1+\dfrac{\alpha\xi}{c}\right|}{\ln\left|\dfrac{c+v}{c-v}\right|}\,. \tag{3.19}$$

Since in this paper we will use "physical units", we can define the ξ-parameter in two ways: or with space dimension $[\xi] = [L]$ as we did before, so writing $\xi = ct \pm x$, or with time dimension $[\xi] = [T]$, when we would express it as $\xi = t \pm \frac{x}{c}$ and the (3.19) ensues

$$R_\alpha(\xi) = \frac{2\ln|1+\alpha\xi|}{\ln\left|\dfrac{c+v}{c-v}\right|}\,. \tag{3.20}$$

The equation (3.19) can also be written, in literature, as follows

$$R_\alpha(\xi) = \frac{1}{\tanh^{-1}\left(\frac{v}{\mathrm{c}}\right)} \ln\left|1 + \frac{\alpha\xi}{\mathrm{c}}\right|, \tag{3.21}$$

since $\tanh^{-1}(z) = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$, but we prefer to use the previous notation.

Anyway, we can put the (3.19) into the (3.4) and verify its correctness.

Proof. Remembering that $v = \alpha L_0$ and that $\ln\left|\frac{\mathrm{c}+v}{c-v}\right| = \ln\left|\frac{1+v/\mathrm{c}}{1-v/\mathrm{c}}\right|$, the calculation is straightforward

$$R(\mathrm{c}t + L(t)) - R(\mathrm{c}t - L(t)) =$$

$$= \frac{2\ln\left|1 + \frac{\alpha}{\mathrm{c}}\left[\mathrm{c}t + L_0(1+\alpha t)\right]\right|}{\ln\left|\frac{\mathrm{c}+v}{c-v}\right|} - \frac{2\ln\left|1 + \frac{\alpha}{\mathrm{c}}\left[\mathrm{c}t - L_0(1+\alpha t)\right]\right|}{\ln\left|\frac{\mathrm{c}+v}{c-v}\right|} =$$

$$= \frac{2}{\ln\left|\frac{\mathrm{c}+v}{c-v}\right|}\ln\left|\frac{1+\alpha t + \frac{v}{\mathrm{c}}t + \frac{v\alpha}{\mathrm{c}}t}{1+\alpha t - \frac{v}{\mathrm{c}}t - \frac{v\alpha}{\mathrm{c}}t}\right| = \frac{2}{\ln\left|\frac{1+v/\mathrm{c}}{1-v/\mathrm{c}}\right|}\ln\left|\frac{(1+\alpha t)(1+v/\mathrm{c})}{(1+\alpha t)(1-v/\mathrm{c})}\right| = 2\,.$$

□

We can also check that $\dot{L}(t) = v$, by means of the general formula (3.6).

Proof. We first calculate the derivative of (3.19)

$$R'_\alpha(\xi) = \frac{2\alpha}{\ln\left(\frac{\mathrm{c}+v}{\mathrm{c}-v}\right)\mathrm{c}} \cdot \frac{1}{1+\frac{\alpha}{\mathrm{c}}\xi}\,,$$

so

$$\dot{L}(t) = \frac{R'[ct - L(t)] - R'[ct + L(t)]}{R'[ct - L(t)] + R'[ct + L(t)]}\mathrm{c} =$$

$$= \frac{\frac{2\alpha}{\ln\left(\frac{\mathrm{c}+v}{\mathrm{c}-v}\right)\mathrm{c}}\left[\frac{1}{1+\frac{\alpha}{\mathrm{c}}(\mathrm{c}t - L(t))} - \frac{1}{1+\frac{\alpha}{\mathrm{c}}(\mathrm{c}t + L(t))}\right]}{\frac{2\alpha}{\ln\left(\frac{\mathrm{c}+v}{\mathrm{c}-v}\right)\mathrm{c}}\left[\frac{1}{1+\frac{\alpha}{\mathrm{c}}(\mathrm{c}t - L(t))} + \frac{1}{1+\frac{\alpha}{\mathrm{c}}(\mathrm{c}t + L(t))}\right]}\mathrm{c}\,,$$

simplifying the constants and calculating the common denominator, it is not hard to find

$$\dot{L}(t) = \mathrm{c}\frac{\frac{\alpha}{\mathrm{c}}(\mathrm{c}t + L_0 + vt - \mathrm{c}t + L_0 + vt)}{2 + \frac{\alpha}{\mathrm{c}}(\mathrm{c}t - L_0 - vt + \mathrm{c}t + L_0 + vt)} = \alpha\frac{(L_0 + vt)}{1+\alpha t} = v\,,$$

using once more the relation $v = \alpha L_0$. □

In this model, where we call $D = \ln\dfrac{\mathrm{c}+v}{c-v}$ for brevity, our scalar field reads

$$\phi = \sum_n \frac{\mathrm{i}}{2}\sqrt{\frac{\hbar \mathrm{c}}{\pi n}}\Bigg\{ \left(\exp\left[-\mathrm{i}\frac{2\pi n \ln\left|1+\alpha t-\frac{\alpha x}{\mathrm{c}}\right|}{D}\right] - \exp\left[-\mathrm{i}\frac{2\pi n \ln\left|1+\alpha t+\frac{\alpha x}{\mathrm{c}}\right|}{D}\right]\right) a_n + - \left(\exp\left[\mathrm{i}\frac{2\pi n \ln\left|1+\alpha t-\frac{\alpha x}{\mathrm{c}}\right|}{D}\right] - \exp\left[\mathrm{i}\frac{2\pi n \ln\left|1+\alpha t+\frac{\alpha x}{\mathrm{c}}\right|}{D}\right]\right) a_n^\dagger \Bigg\}, \tag{3.22}$$

while the density of energy (3.16) can be expressed in the following way,

$$T_{00}(t,x) = -\frac{\hbar\pi\alpha^2}{12D^2\mathrm{c}}\left\{\frac{1}{\left[1+\alpha t-\dfrac{\alpha x}{\mathrm{c}}\right]^2} + \frac{1}{\left[1+\alpha t+\dfrac{\alpha x}{\mathrm{c}}\right]^2}\right\}, \tag{3.23}$$

or, in an alternative form, writing $\beta = \dfrac{v}{\mathrm{c}}$ and substituting $D = \ln\dfrac{\mathrm{c}+v}{c-v}$ and $\alpha = \dfrac{v}{L_0}$

$$T_{00}(t,x) = -\frac{\hbar\mathrm{c}\pi}{12L_0^2}\frac{\beta^2}{\ln^2\left(\dfrac{1+\beta}{1-\beta}\right)}\left\{\frac{1}{\left[1+\dfrac{\beta}{L_0}(\mathrm{c}t-x)\right]^2} + \frac{1}{\left[1+\dfrac{\beta}{L_0}(\mathrm{c}t+x)\right]^2}\right\}. \tag{3.24}$$

We could also operate the limit for small velocity, recovering for the density of energy of the static case for $\dfrac{v}{\mathrm{c}} = \beta \ll 1$,

$$\lim_{\beta\longrightarrow 0} -\frac{\hbar\mathrm{c}\pi}{12L_0^2}\frac{\beta^2}{\ln^2\left(\dfrac{1+\beta}{1-\beta}\right)}\left\{\frac{1}{\left[1+\dfrac{\beta}{L_0}(\mathrm{c}t-x)\right]^2} + \frac{1}{\left[1+\dfrac{\beta}{L_0}(\mathrm{c}t+x)\right]^2}\right\} = -\frac{\hbar\mathrm{c}\pi}{24L_0^2}. \tag{3.25}$$

3.2 Oscillating boundary

Let us consider a massless real scalar field in a one dimensional vibrating cavity, with the left boundary fixed at $x = 0$, as in the previous case, while the right one performs an oscillatory motion around the equilibrium position $x = L_0$, described by the law

$$L(t) = L_0 \left[1 + \varepsilon \sin\left(\frac{2\pi}{L_0} ct\right)\right] , \tag{3.26}$$

and its behaviour is well shown in Figure 3.1

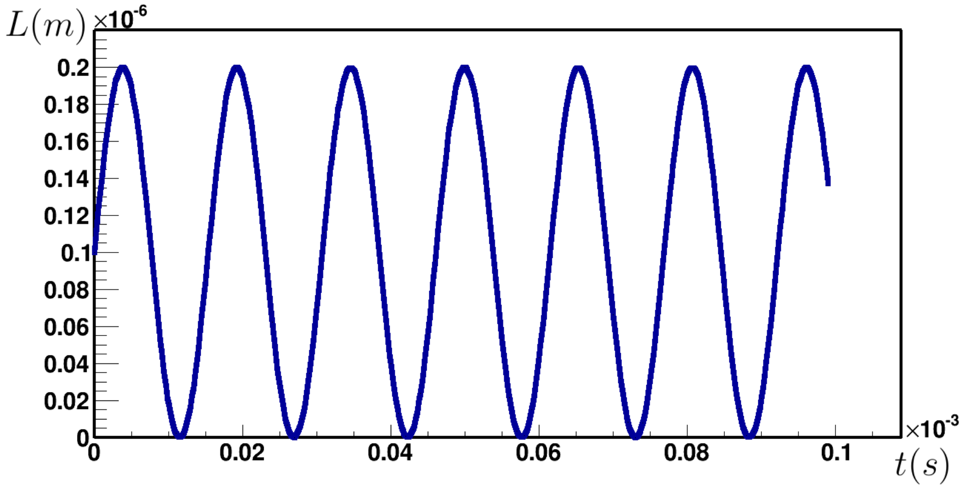

Figure 3.1: Position of the right boundary in function of time, following the oscillating law $L(t) = L_0 \left[1 + \varepsilon \sin\left(\frac{2\pi}{L_0} ct\right)\right]$.

Unfortunately, an exact solution can not be found, unlike in the previous case, so we have to deal with this problem in a different way.

The presence of a vibrating mirror can change the fluctuations of the field at a given spatial point time dependently.

This local fluctations are characterized by the energy density of the field, again,

$$\langle T_{00}(t,x)\rangle = \frac{1}{2}\left\{\frac{1}{c^2}\left\langle\left(\frac{\partial\phi(t,x)}{\partial t}\right)^2\right\rangle + \left\langle\left(\frac{\partial\phi(t,x)}{\partial x}\right)^2\right\rangle\right\} , \tag{3.27}$$

where the expectation value is obviously taken with respect to the vacuum state. We already know that $\langle T_{00}\rangle$ is a divergent quantity, but in this case we can regularize it by means of the "point-splitting" regularization procedure.

The "regular" part, physically meaningful and cut-off independent, is [28]

$$\langle T_{00}(t,x)\rangle_{reg} = -f(ct + x) - f(ct - x) , \tag{3.28}$$

and f is a particular function satisfying

$$\frac{24\pi f}{\hbar c} = \frac{R'''}{R'} - \frac{3}{2}\left(\frac{R''}{R'}\right)^2 + \frac{\pi^2}{2}\left(R'\right)^2 , \tag{3.29}$$

but only the last term is the Casimir energy contribution [22].

In order to get this result, we can treat the problem perturbatively, expanding $R(\mathrm{c}t) = R_0(\mathrm{c}t) + \varepsilon R_1(\mathrm{c}t)$ in term of the small (non dimensional) amptitude ε, getting

$$R_0(\mathrm{c}t + L_0) - R_0(\mathrm{c}t - L_0) = 2\,, \tag{3.30}$$

$$R_1(\mathrm{c}t + L_0) - R_1(\mathrm{c}t - L_0) = -L_0 \sin\left(q\frac{2\pi \mathrm{c}t}{L_0}\right)\left[R_0'(\mathrm{c}t + L_0) + R_0'(\mathrm{c}t - L_0)\right]. \tag{3.31}$$

The general solution reads [28]

$$R(\mathrm{c}t) = \frac{\mathrm{c}t}{L_0} + \varepsilon(-1)^{q+1}\left[\frac{\mathrm{c}t}{L_0}\sin\left(q\frac{2\pi \mathrm{c}t}{L_0}\right) - \frac{z}{L_0}\sin\left(q\frac{2\pi z}{L_0}\right)\right] \tag{3.32}$$

where q is an integer number that allows us to distinguish the "semi-resonant" case ($q = 1$), when no exponential amplification of the energy density is obtained, from the "resonant cases" ($q \geq 2$), when T_{00} exponentially increases [29]. Then, z is a position inside the hole $z \in [0, L_0]$, that we can write in terms of t as

$$z = \mathrm{c}t - pL_0 \tag{3.33}$$

with $p = \frac{1}{2}\mathrm{int}\left[\frac{\mathrm{c}t}{L_0}\right]$ or $p = \frac{1}{2}\mathrm{int}\left[\frac{\mathrm{c}t}{L_0} + 1\right]$, for $\frac{\mathrm{c}t}{L_0}$ even or odd, respectively.

This naive perturbative solution to the dynamical Casimir effect does not satisfies the correct boundary conditions, so it is neccesary to split the solution

$$R(\mathrm{c}t) = R_s(\mathrm{c}t) + R_{np}(\mathrm{c}t)\,. \tag{3.34}$$

We now write the explicit expression for $R_s(\mathrm{c}t)$ and $R_{np}(\mathrm{c}t)$, while a more technical discussion is postponed to the Appendix C

$$R_s(\mathrm{c}t) = \frac{\mathrm{c}t}{L_0} - \frac{2}{\pi q}\mathrm{Im}\left\{\ln\left[\frac{1 + \xi + (1 - \xi)\exp\left(\mathrm{i}q\frac{2\pi \mathrm{C}t}{L_0}\right)}{-(1 + \xi) + (1 - \xi)\exp\left(-\mathrm{i}q\frac{2\pi \mathrm{C}t}{L_0}\right)}\right]\right\}, \tag{3.35}$$

or, using the relation, $\tan^{-1}(x) = \frac{1}{2\mathrm{i}}\ln\left(\frac{x - \mathrm{i}}{x + \mathrm{i}}\right)$

$$R_s(\mathrm{c}t) = \frac{\mathrm{c}t}{L_0} - \frac{1}{\pi q}\tan^{-1}\left[\frac{\sin\left(q\frac{2\pi \mathrm{C}t}{L_0}\right)}{\frac{1+\xi}{1-\xi} + \cos\left(q\frac{2\pi \mathrm{C}t}{L_0}\right)}\right], \tag{3.36}$$

and

$$R_{np}(\mathrm{ct}) = (-1)^q \frac{\epsilon z}{L_0} \sin\left(q\frac{2\pi z}{L_0}\right) \left[\frac{2\xi}{1+\xi^2+(1-\xi^2)\cos\left(q\frac{2\pi z}{L_0}\right)}\right] , \tag{3.37}$$

where $0 \leq z \leq L_0$ has been defined previously, while

$$\xi = \exp\left[\frac{(-1)^{q+1} q 2\pi\varepsilon \mathrm{c}t}{L_0}\right] . \tag{3.38}$$

For the short time limit, so when $\mathrm{c}t \ll \dfrac{L_0}{\varepsilon}$, these functions result

$$R_s(\mathrm{c}t) \approx \frac{\mathrm{c}t}{L_0} - (-1)^q \frac{\varepsilon \mathrm{c}t}{L_0} \sin\left(q\frac{2\pi \mathrm{c}t}{L_0}\right) , \tag{3.39}$$

$$R_{np}(\mathrm{c}t) \approx (-1)^q \frac{\varepsilon z}{L_0} \sin\left(q\frac{2\pi z}{L_0}\right) , \tag{3.40}$$

which lead to the perturbative solution (3.32).

Proof. Let us start from R_s. First of all, we say that $\cos\left(q\dfrac{2\pi \mathrm{c}t}{L_0}\right) \approx 1$. Then, the argument of the arctangent becomes

$$\frac{\sin\left(q\frac{2\pi \mathrm{C}t}{L_0}\right)}{\frac{1+\xi}{1-\xi}+1} = \frac{\sin\left(q\frac{2\pi \mathrm{C}t}{L_0}\right)(1-\xi)}{1+\xi+1-\xi} ,$$

and now, expanding in series

$$\xi = \exp\left[\frac{(-1)^{q+1} q 2\pi\varepsilon \mathrm{c}t}{L_0}\right] \approx 1 + \frac{(-1)^{q+1} q 2\pi\varepsilon \mathrm{c}t}{L_0}$$

it is easy to obtain

$$\begin{aligned} R_s(\mathrm{c}t) &\approx \frac{\mathrm{c}t}{L_0} - \frac{1}{\pi q} \tan^{-1}\left[(-1)^q q\pi\varepsilon \frac{\mathrm{c}t}{L_0} \sin\left(q\frac{2\pi \mathrm{c}t}{L_0}\right)\right] \\ &\approx \frac{\mathrm{c}t}{L_0} - (-1)^q \frac{\varepsilon \mathrm{c}t}{L_0} \sin\left(q\frac{2\pi \mathrm{c}t}{L_0}\right) . \end{aligned}$$

In second place, if we consider the same approximation for $\cos\left(q\frac{2\pi \mathrm{C}t}{L_0}\right)$ and for ξ, the result of R_{np} is straightforward. □

For the long time limit $ct \gg \frac{L_0}{\varepsilon}$, R_{np} is negligible, but we prove this statement after having discussed the behaviour of R_s. The function R_s has a first term, linear in time, and a second one, that for late times oscillates.

Let us calculate the first derivative of this function, namely

$$R_s'(\mathrm{ct}) = \frac{2\xi}{L_0}\left[\frac{1}{1+\xi^2+(1-\xi^2)\cos\left(q\frac{2\pi \mathrm{Ct}}{L_0}\right)}\right], \tag{3.41}$$

excluding a negligible term.

Proof.

$$\begin{aligned}
R_s'(\mathrm{ct}) &= \frac{1}{L_0} - \frac{1}{q\pi}\left[1+\frac{\sin^2\left(q\frac{2\pi \mathrm{Ct}}{L_0}\right)}{\left[\frac{1+\xi}{1-\xi}+\cos\left(q\frac{2\pi \mathrm{Ct}}{L_0}\right)\right]^2}\right]^{-1}\frac{1}{\left[\frac{1+\xi}{1-\xi}+\cos\left(q\frac{2\pi \mathrm{Ct}}{L_0}\right)\right]^2}\times \\
&\times\left\{\frac{q2\pi}{L_0}\cos^2\left(q\frac{2\pi \mathrm{ct}}{L_0}\right)+\frac{q2\pi}{L_0}\frac{(1+\xi)}{(1-\xi)}\cos\left(q\frac{2\pi \mathrm{ct}}{L_0}\right)-\sin\left(q\frac{2\pi \mathrm{ct}}{L_0}\right)\times\right. \\
&\quad\left.\times\left[\frac{\xi'(1-\xi)+\xi'(1+\xi)}{(1-\xi)^2}-\frac{q2\pi}{L_0}\sin\left(q\frac{2\pi \mathrm{ct}}{L_0}\right)\right]\right\} = \\
&= \frac{1}{L_0}-\frac{1}{q\pi}\frac{1}{\left[\left(\frac{1+\xi}{1-\xi}\right)^2+2\frac{(1+\xi)}{(1-\xi)}\cos\left(q\frac{2\pi \mathrm{ct}}{L_0}\right)+1\right]}\times \\
&\times\left\{\frac{q2\pi \mathrm{ct}}{L_0}+\frac{q2\pi \mathrm{ct}}{L_0}\frac{(1+\xi)}{(1-\xi)}\cos\left(q\frac{2\pi \mathrm{ct}}{L_0}\right)-\frac{2\xi'}{(1-\xi)^2}\sin\left(q\frac{2\pi \mathrm{ct}}{L_0}\right)\right\} = \\
&= \frac{1}{L_0}-\frac{1}{q\pi}\frac{\frac{q2\pi}{L_0}\left[1+\xi^2-2\xi+(1-\xi^2)\cos\left(q\frac{2\pi \mathrm{ct}}{L_0}\right)\right]-2\xi'\sin\left(q\frac{2\pi \mathrm{ct}}{L_0}\right)}{2+2\xi^2+2(1-\xi^2)\cos\left(q\frac{2\pi \mathrm{Ct}}{L_0}\right)}.
\end{aligned}$$

We are ready to express ξ' as the derivative of (3.38) with respect to its argument $\mathrm{c}t$,

$$\xi' = \frac{(-1)^{q+1}q2\pi}{L_0}\varepsilon\exp\left[\frac{(-1)^{q+1}q2\pi\varepsilon \mathrm{c}t}{L_0}\right] = \frac{(-1)^{q+1}q2\pi}{L_0}\varepsilon\xi\,,$$

and substitute it in the demonstration, so that

$$R_s'(\mathrm{c}t) = \frac{1}{L_0} - \frac{1+\xi^2-2\xi+(1-\xi^2)\cos\left(q\frac{2\pi\mathrm{C}t}{L_0}\right)}{L_0\left[1+\xi^2+(1-\xi^2)\cos\left(q\frac{2\pi\mathrm{C}t}{L_0}\right)\right]} +$$

$$- \frac{4(-1)^{q+1}\varepsilon\xi\sin\left(q\frac{2\pi\mathrm{C}t}{L_0}\right)}{L_0\left[1+\xi^2+(1-\xi^2)\cos\left(q\frac{2\pi\mathrm{C}t}{L_0}\right)\right]} =$$

$$= \frac{1}{L_0}\left\{\frac{2\xi+4(-1)^q\varepsilon\xi\sin\left(q\frac{2\pi\mathrm{C}t}{L_0}\right)}{\left[1+\xi^2+(1-\xi^2)\cos\left(q\frac{2\pi\mathrm{C}t}{L_0}\right)\right]}\right\} =$$

$$= \frac{2\xi}{L_0\left[1+\xi^2+(1-\xi^2)\cos\left(q\frac{2\pi\mathrm{C}t}{L_0}\right)\right]}\left[1+2(-1)^q\varepsilon\sin\left(q\frac{2\pi\mathrm{c}t}{L_0}\right)\right].$$

In the final equation (3.41), we eliminate the last term. □

Since $\frac{\mathrm{d}\xi}{\mathrm{d}(\mathrm{c}t)} = \frac{(-1)^{q+1}q2\pi}{L_0}\varepsilon\xi$, we could differentiate the function R_s, with respect to its argument, considering ξ as a constant. So, the first derivative would result

$$R_s'(\mathrm{c}t) = \frac{2\xi}{L_0}\left[\frac{1}{1+\xi^2+(1-\xi^2)\cos\left(q\frac{2\pi\mathrm{C}t}{L_0}\right)}\right], \tag{3.42}$$

as in the general case.

Proof. Using again the relation (3.36), we have only to calculate the derivative

$$R_s'(\mathrm{c}t) = \frac{1}{L_0} - \frac{1}{q\pi}\left[1+\frac{\sin^2\left(q\frac{2\pi\mathrm{C}t}{L_0}\right)}{\left[\frac{1+\xi}{1-\xi}+\cos\left(q\frac{2\pi\mathrm{C}t}{L_0}\right)\right]^2}\right]^{-1} q\frac{2\pi}{L_0}\frac{1+\frac{1+\xi}{1-\xi}\cos\left(q\frac{2\pi\mathrm{C}t}{L_0}\right)}{\left[\frac{1+\xi}{1-\xi}+\cos\left(q\frac{2\pi\mathrm{C}t}{L_0}\right)\right]^2} =$$

$$= \frac{1}{L_0} - \frac{2}{L_0}\frac{1+\frac{1+\xi}{1-\xi}\cos\left(q\frac{2\pi\mathrm{C}t}{L_0}\right)}{\left(\frac{1+\xi}{1-\xi}\right)^2+2\left(\frac{1+\xi}{1-\xi}\right)\cos\left(q\frac{2\pi\mathrm{C}t}{L_0}\right)+1} =$$

$$= \frac{1}{L_0}\left[1-\frac{1-2\xi+\xi^2+(1-\xi^2)\cos\left(q\frac{2\pi\mathrm{C}t}{L_0}\right)}{1+\xi^2+(1-\xi^2)\cos\left(q\frac{2\pi\mathrm{C}t}{L_0}\right)}\right] =$$

$$= \frac{2\xi}{L_0}\left[\frac{1}{1+\xi^2+(1-\xi^2)\cos\left(q\frac{2\pi\mathrm{C}t}{L_0}\right)}\right].$$

□

Now we are ready to analyze the function R_{np}. Again, if we consider that ξ is a constant when we derive with respect to its argument ct, then we can express R_{np} in terms of the first derivative of R_s, as follows

$$R_{np} = (-1)^q \varepsilon z \sin\left(q\frac{2\pi \mathrm{c}t}{L_0}\right) R_s'(\mathrm{c}t)\,, \tag{3.43}$$

since trigonometric functions *sine* and *cosine*, computed in $q\dfrac{2\pi z}{L_0}$, are equivalent to these same functions evaluated in $q\dfrac{2\pi \mathrm{c}t}{L_0}$. In fact, reminding (3.33), this conclusion is immediate thanks to the periodicity of these trigonometric functions.
Consequently, R_{np} is a correction of ϵ-order to the second term of (3.36).

Anyway, we can write the density of energy by means of (3.16), substituting the latter equation for $R_s'(\mathrm{c}t)$ found in (3.41)

$$T_{00} = -\frac{\hbar \mathrm{c}\pi}{12L_0^2}\frac{1}{\left[\xi+\xi^{-1}-(\xi-\xi^{-1})\cos\left(q\frac{2\pi \mathrm{C}t}{L_0}\right)\right]^2}\,, \tag{3.44}$$

that substituting the definition (3.38) of ξ, become

$$T_{00} = -\frac{\hbar \mathrm{c}\pi}{24L_0^2}\left\{\frac{1}{\left[\cosh\left(\frac{(-1)^{q+1}q2\pi\varepsilon(\mathrm{C}t+x)}{L_0}\right)-\sinh\left(\frac{(-1)^{q+1}q2\pi\varepsilon(\mathrm{C}t+x)}{L_0}\right)\cos\left(q\frac{2\pi(\mathrm{C}t+x)}{L_0}\right)\right]^2}+\right.$$
$$\left.+\frac{1}{\left[\cosh\left(\frac{(-1)^{q+1}q2\pi\varepsilon(\mathrm{C}t-x)}{L_0}\right)-\sinh\left(\frac{(-1)^{q+1}q2\pi\varepsilon(\mathrm{C}t-x)}{L_0}\right)\cos\left(q\frac{2\pi(\mathrm{C}t-x)}{L_0}\right)\right]^2}\right\}. \tag{3.45}$$

3.2.1 Another particular law

However, it is possible to obtain other exact solutions for some resonant trajectories.

If we consider, in particular, the following law of motion for the right moving boundary [36],

$$L(t) = L_0 + \frac{L_0}{2\pi}\left\{\sin^{-1}\left[\sin\theta\cos\left(\frac{2\pi \mathrm{c}t}{L_0}\right)\right]-\theta\right\}\,, \qquad t \geq 0\,, \tag{3.46}$$

where ϵ determines the amptitude of the oscillations, and it appear in θ, that is defined $\theta = \tan^{-1}\left(\dfrac{\epsilon\pi}{L_0}\right)$. This law leads to the exact expression for $R(t)$

$$R(2nL_0+\chi)=2n+\frac{1}{2}-\frac{1}{\pi}\tan^{-1}\left[\cot\left(\frac{\pi\chi}{L_0}\right)-\frac{2n\epsilon\pi}{L_0}\right], \tag{3.47}$$

where $n \geq 1$ is any positive integer and $\chi \in (-L_0, L_0]$ is a spatial variable. When $\frac{\epsilon}{L_0}$ is a small parameter,

$$L(t)=L_0-\epsilon\sin^2\left(\frac{\pi ct}{L_0}\right). \tag{3.48}$$

Proof. In the approximation $\frac{\epsilon}{L_0} \ll 1$ we have that

$$\sin\theta \approx \theta \approx \frac{\epsilon\pi}{L_0},$$
$$\sin^{-1}\left[\pi\frac{\epsilon}{L_0}\cos\left(\frac{2\pi ct}{L_0}\right)\right] \approx \pi\frac{\epsilon}{L_0}\cos\left(\frac{2\pi ct}{L_0}\right),$$

so, consequently,

$$\begin{aligned}L(t)&=L_0+\frac{\epsilon}{2}\cos\left(\frac{2\pi ct}{L_0}\right)-\frac{\epsilon}{2}=\\&=L_0-\epsilon\frac{1-\cos\left(\frac{2\pi ct}{L_0}\right)}{2}=L_0-\epsilon\sin^2\left(\frac{\pi ct}{L_0}\right),\end{aligned}$$

that prove the previous equation. □

This particular solution has the advantage that is valid at an arbitrary time, so we have not split the function $R(ct)$, as already done for purely harmonic oscillations.

Using the exact solution for $R(ct)$ (3.47), we find

$$f(2nL_0+\chi)=\hbar c\left\{\frac{\pi}{12L_0^2}-\frac{\pi}{16L_0^2D_n^2(\chi)}+\frac{\pi\epsilon}{6L_0^2}\delta(\chi-L_0)\right\}, \tag{3.49}$$

defining $D_n(\xi)$

$$D_n(\xi)=\alpha+\beta\sin\left(\frac{2\pi\chi}{L_0}\right)+\gamma\cos\left(\frac{2\pi\chi}{L_0}\right), \tag{3.50}$$

with $\alpha = 1 + 2\left(\frac{n\epsilon}{L_0}\right)^2$, $\beta = -2\frac{n\epsilon}{L_0}$, $\gamma = -2\left(\frac{n\epsilon}{L_0}\right)^2$.

The δ-function term, which appeared also for the law of motion (3.26) because of the discontinuity of $R'(ct)$, is due to the initial wall velocity, that accelerates rapidly at $t = 0$. This term is not relevant in the resonant evolution of the system, so we shall not discuss it further and we will not consider anymore the singularity in this section.

We are only interested in the Casimir term of the density of energy, so, according to the last of (3.29), we need the first derivative of R in order to calculate T_{00}, as expressed in (3.16). This way, the result is

$$T_{00} = -\frac{\hbar c\pi}{48L_0^2}\left\{\frac{1}{\left[1-\frac{2n\epsilon\pi}{L_0}\sin\left(2\pi\frac{ct+x}{L_0}\right)+\frac{4n^2\epsilon^2\pi^2}{L_0^2}\sin^2\left(\pi\frac{ct+x}{L_0}\right)\right]^2}+\right.$$
$$\left.+\frac{1}{\left[1-\frac{2n\epsilon\pi}{L_0}\sin\left(2\pi\frac{ct-x}{L_0}\right)+\frac{4n^2\epsilon^2\pi^2}{L_0^2}\sin^2\left(\pi\frac{ct-x}{L_0}\right)\right]^2}\right\}. \quad (3.51)$$

We can also decide to reject terms $\mathcal{O}\left(\left(\frac{\epsilon}{L_0}\right)^4\right)$, so that we can write

$$T_{00} = -\frac{\hbar c\pi}{48L_0^2}\left\{\frac{1}{\left[1-\frac{2n\epsilon\pi}{L_0}\sin\left(2\pi\frac{ct+x}{L_0}\right)\right]^2}+\frac{1}{\left[1-\frac{2n\epsilon\pi}{L_0}\sin\left(2\pi\frac{ct-x}{L_0}\right)\right]^2}\right\}, \quad (3.52)$$

or, with some more calculation,

$$T_{00} = -\frac{\hbar c\pi}{12L_0^2}\frac{1}{\left[1-\frac{2n\epsilon\pi}{L_0}\sin\left(2\pi\frac{ct+x}{L_0}\right)\right]^2\left[1-\frac{2n\epsilon\pi}{L_0}\sin\left(2\pi\frac{ct-x}{L_0}\right)\right]^2}\times$$
$$\times\left\{1-4n\pi\frac{\epsilon}{L_0}\sin\left(2\pi\frac{ct}{L_0}\right)\cos\left(2\pi\frac{x}{L_0}\right)+\right.$$
$$\left.+4\pi^2n^2\frac{\epsilon^2}{L_0^2}\left[\sin^2\left(2\pi\frac{ct}{L_0}\right)\cos^2\left(2\pi\frac{x}{L_0}\right)+\cos^2\left(2\pi\frac{ct}{L_0}\right)\sin^2\left(2\pi\frac{x}{L_0}\right)\right]\right\}. \quad (3.53)$$

Proof. Let us start chaging variable, with the imposition $2nL_0 - \chi = ct \pm x$. so we find

$$\begin{aligned} R(2nL_0+\chi) &= R(ct\pm x) = \\ &= 2n+\frac{1}{2}-\frac{1}{\pi}\tan^{-1}\left[\cot\left(\frac{\pi(ct\pm x)}{L_0}-2n\pi\right)-\frac{2n\epsilon\pi}{L_0}\right] = \\ &= 2n+\frac{1}{2}-\frac{1}{\pi}\tan^{-1}\left[\cot\left(\frac{\pi(ct\pm x)}{L_0}\right)-\frac{2n\epsilon\pi}{L_0}\right]. \end{aligned}$$

Now, we calculate the first derivative, with respect to the argument

$$
\begin{aligned}
R'(\mathrm{c}t \pm x) &= \frac{1}{L_0} \cdot \frac{1}{\sin^2\left(\pi \frac{\mathrm{c}t \pm x}{L_0}\right)} \cdot \frac{1}{1 + \left[\cot\left(\pi \frac{\mathrm{c}t \pm x}{L_0}\right) - \frac{2n\epsilon\pi}{L_0}\right]^2} = \\
&= \frac{1}{L_0} \cdot \frac{1}{\sin^2\left(\pi \frac{\mathrm{c}t \pm x}{L_0}\right) + \left[\cos\left(\pi \frac{\mathrm{c}t \pm x}{L_0}\right) - \frac{2n\epsilon\pi}{L_0} \sin\left(\pi \frac{\mathrm{c}t \pm x}{L_0}\right)\right]^2} = \\
&= \frac{1}{L_0} \cdot \frac{1}{1 - \frac{4n\epsilon\pi}{L_0} \sin\left(\pi \frac{\mathrm{c}t \pm x}{L_0}\right) \cos\left(\pi \frac{\mathrm{c}t \pm x}{L_0}\right) + \frac{4n^2\epsilon^2\pi^2}{L_0^2} \sin^2\left(\pi \frac{\mathrm{c}t \pm x}{L_0}\right)} = \\
&= \frac{1}{L_0} \cdot \frac{1}{1 - \frac{2n\epsilon\pi}{L_0} \sin\left(2\pi \frac{\mathrm{c}t \pm x}{L_0}\right) + \frac{4n^2\epsilon^2\pi^2}{L_0^2} \sin^2\left(\pi \frac{\mathrm{c}t \pm x}{L_0}\right)} .
\end{aligned}
$$

□

We can recover the static limit for $\frac{\epsilon}{L_0} \to 0$, and the result is $T_{00} = -\frac{\hbar \mathrm{c} \pi}{24 L_0^2}$, as we expect.

Chapter 4

A retarded approach

In a one-dimensional hole delimited by the interval $[0, L]$ we write a wave function vanishing at the boundary

$$\psi_n(t,x) = sin\left(\frac{\pi n x}{L}\right), \tag{4.1}$$

where the frequency ω_n must satisfy

$$\omega_n = \frac{\mathrm{c} n \pi}{L} \tag{4.2}$$

with $n \in \mathbb{N}$.
We put this expression for the frequency in the definition of energy

$$\mathcal{E}_n = \frac{\hbar\omega}{2} = \frac{\hbar \mathrm{c} n \pi}{2L}, \tag{4.3}$$

and we obtain the Casimir energy

$$\mathcal{E}_{Cas} = \frac{\hbar}{2}\sum_{n=1}^{\infty}\omega_n = \frac{\hbar \mathrm{c}\pi}{2}\sum_{n=1}^{\infty} n = -\frac{1}{24}\frac{\hbar \mathrm{c}\pi}{L}, \tag{4.4}$$

as already has been seen previously.
The differentiation of this latter expression for Casimir Energy brings to

$$\mathrm{d}\mathcal{E}_{Cas} = \frac{\hbar \mathrm{c}\pi}{24L^2}\mathrm{d}L, \tag{4.5}$$

that definitely gives the Casimir force $F = -\frac{\mathrm{d}\mathcal{E}_{Cas}}{\mathrm{d}L}$

$$F_{Cas} = -\frac{\hbar \mathrm{c}\pi}{24L^2}. \tag{4.6}$$

What if the right border is bonded to a spring?
If the border is moving, of course something is going to change.

It seems reasonable to think that the solution of the equation will be given by an effective distance $L_{eff}(t)$, and we make the following hypothesis

$$L_{eff}(t) = L\left(t - \frac{L(t)}{\mathrm{c}}\right). \tag{4.7}$$

In this model, we consider a photon travelling from the left boundary ($L = 0$), to the right one. It has to travel for

$$\mathrm{c}(t - t_0) = L(t) \quad \Rightarrow \quad t_0 = t - \frac{L(t)}{\mathrm{c}}.$$

Thus, in the expression of the force, we replaced $L(t)$ with $L(t_0)$, using the retarded distance (4.7).
We write the delayed Casimir force

$$F_{Cas} = -\frac{\hbar \mathrm{c} \pi}{24 L^2 \left(t - \frac{L(t)}{\mathrm{c}}\right)}. \tag{4.8}$$

4.1 The Casimir energy

Casimir energy can be found analitically, so that comes true the following non trivial implication

$$F_{Cas} = -\frac{\hbar \mathrm{c} \pi}{24 L^2 \left(t - \frac{L(t)}{\mathrm{c}}\right)} \qquad \Longrightarrow \qquad V_{Cas} = \frac{\hbar \mathrm{c} \pi}{24 L \left(t - \frac{L(t)}{\mathrm{c}}\right)} \tag{4.9}$$

Proof.

$$V_{Cas} = \int F_{Cas} \mathrm{d}L = -\frac{\hbar \mathrm{c}\pi}{24} \int \frac{\frac{\mathrm{d}L}{\mathrm{d}t}\mathrm{d}t}{L^2\left(t - \frac{L(t)}{\mathrm{c}}\right)} = -\frac{\hbar \mathrm{c}\pi}{24} \int L^{-2}\left(t - \frac{L(t)}{\mathrm{c}}\right) \dot{L}(t)\, \mathrm{d}t$$

We now operate a change of variable calling

$$\alpha(t) = t - \frac{L(t)}{\mathrm{c}}, \tag{4.10}$$

so that

$$\frac{\dot{L}(t)}{\mathrm{c}} = 1 - \dot{\alpha}(t) \qquad \text{and} \qquad \mathrm{d}t = \frac{\mathrm{d}\alpha}{1 - \frac{\dot{L}(t)}{\mathrm{c}}} = \frac{\mathrm{d}\alpha}{\dot{\alpha}} \, .$$

Let's continue

$$V_{Cas} = -\frac{\hbar \mathrm{c} \pi}{24} \int \frac{\mathrm{c}(1-\dot{\alpha})L^{-2}(\alpha)}{\dot{\alpha}} \mathrm{d}\alpha = -\frac{\hbar \mathrm{c}^2 \pi}{24} \left\{ \int \frac{1}{\dot{\alpha} L^2(\alpha)} \mathrm{d}\alpha - \int \frac{1}{L^2(\alpha)} \mathrm{d}\alpha \right\} .$$

Now it is easy to see that

$$\mathrm{d}\alpha = \mathrm{d}t - \frac{\mathrm{d}L}{\mathrm{c}} \quad \Rightarrow \quad \frac{1}{\mathrm{c}}\frac{\mathrm{d}L}{\mathrm{d}\alpha} = \frac{\mathrm{d}t}{\mathrm{d}\alpha} - 1 \quad \Rightarrow \quad \frac{\mathrm{d}t}{\mathrm{d}\alpha} = \frac{1}{\dot{\alpha}} = \frac{1}{\mathrm{c}}\frac{\mathrm{d}L}{\mathrm{d}\alpha} + 1 \, .$$

so, substituting $\frac{1}{\dot{\alpha}} = \frac{\mathrm{d}L}{\mathrm{d}\alpha} + 1$ in the first integral, we find

$$\begin{aligned} V_{Cas} &= \frac{\hbar \mathrm{c}^2 \pi}{24} \left\{ -\int \frac{\mathrm{d}L(\alpha)}{\mathrm{c}\, L^2(\alpha)} - \int \frac{1}{L^2(\alpha)} \mathrm{d}\alpha + \int \frac{1}{L^2(\alpha)} \mathrm{d}\alpha \right\} = \\ &= -\frac{\hbar \mathrm{c} \pi}{24} \int \frac{\mathrm{d}L(\alpha)}{L^2(\alpha)} = \frac{\hbar \mathrm{c} \pi}{24} \frac{1}{L(\alpha)} = \frac{\hbar \mathrm{c} \pi}{24} \frac{1}{L\left(t - \frac{L(t)}{\mathrm{c}}\right)} \, , \end{aligned}$$

that prove (4.9) □

We can also check that if we compute the spatial derivative of the potential, we come back to the expression of the force.

Proof.

$$\frac{\mathrm{d}V_{Cas}}{\mathrm{d}L} = \frac{\hbar \mathrm{c} \pi}{24} \left(-\frac{\mathrm{c}}{L^2\left(t - \frac{L(t)}{\mathrm{c}}\right)} \right) \left(-\frac{1}{\mathrm{c}} \right) \, ,$$

that turns right since $F_{Cas} = -\frac{\mathrm{d}V_{Cas}}{\mathrm{d}L}$. □

So, our hypothesis catches some features of the Casimir energy, for example, it is analytically well-defined). Anyway, we will conclude that this approach is not exactly correct.

Following this approach, we will analize the numerical results for the implication expressed in (4.9) in the next Chapter.

4.2 Right boundary with uniform velocity

We wrote the retarded lenght as $L\left(t-\dfrac{L(t)}{\mathrm{c}}\right)$. We now check that approach for the simplest law: while holding the left border, consider the right one moving with a constant velocity, so, as described by (3.18), the law of motion is

$$L(t) = L_0 + vt\,. \tag{4.11}$$

We can immediatly compute that the law of motion does not change when $|v| \ll \mathrm{c}$, so $\beta \ll 1$ i.e.

$$L\left(t-\frac{L(t)}{\mathrm{c}}\right) = (L_0 + vt)\left(1-\frac{v}{\mathrm{c}}\right)\,. \tag{4.12}$$

Proof. In a few steps, it is easy to verify

$$L\left(t-\frac{L(t)}{\mathrm{c}}\right) = L_0 + v\left(t-\frac{L_0+vt}{\mathrm{c}}\right) = L_0 + v\left(t\left(1-\frac{v}{\mathrm{c}}\right)-\frac{L_0}{\mathrm{c}}\right) =$$
$$= L_0\left(1-\frac{v}{\mathrm{c}}\right) + vt\left(1-\frac{v}{\mathrm{c}}\right) = (L_0+vt)\left(1-\frac{v}{\mathrm{c}}\right)\,,$$

as written above. □

We want to evaluate, from a point $x \in [0, L(t)]$, the detection of a signal bounced on $L(t)$, the right plate, at the time $t = t_1$, while when the right boundary is in $L(t)$ the signal is discovered in x, such that

$$\frac{L(t_1)-x}{\mathrm{c}} = t - t_1 \qquad \Longrightarrow \qquad L(t_1) = x + \mathrm{c}(t-t_1)\,, \tag{4.13}$$

as we can perceive from Figure 4.1.

Figure 4.1: A displayed description of the delay.

Now, since

$$L(t_1) = L_0 + vt_1\,, \tag{4.14}$$

it is trivial to find

$$t_1 = \frac{x + \mathrm{c}t - L_0}{\mathrm{c}+v}\,, \tag{4.15}$$

that, put again in (4.14), gives the result

$$L(t_1) = \frac{L_0 \mathrm{c}}{\mathrm{c}+v} + \frac{v\mathrm{c}t}{\mathrm{c}+v} + \frac{vx}{\mathrm{c}+v} \, . \tag{4.16}$$

We now use this latter expression for the distance in order to calculate the density of energy in the same way as the force, because this is a one-dimensional model. So using the identity

$$L(t_1) = L\left(t - \frac{L(t)}{\mathrm{c}}\right) \tag{4.17}$$

obtaining the following expression

$$T_{00} = -\frac{\hbar \mathrm{c}\pi}{24 L^2(t_1)} = -\frac{\hbar \mathrm{c}\pi}{24} \frac{(\mathrm{c}+v)^2}{(L_0 \mathrm{c} + v\mathrm{c}t + vx)^2} \, , \tag{4.18}$$

that we can write also in the following manner

$$T_{00} = -\frac{\hbar\pi}{24 \mathrm{c} L_0^2} \frac{(\mathrm{c}+v)^2}{\left(1 + \alpha t + \dfrac{\alpha x}{\mathrm{c}}\right)^2} \, , \tag{4.19}$$

or, expliciting $\beta = \frac{v}{\mathrm{c}}$,

$$T_{00} = -\frac{\hbar \mathrm{c}\pi}{24 L_0^2} \frac{(1+\beta)^2}{\left(1 + \dfrac{\beta}{L_0}(\mathrm{c}t + x)\right)^2} \, . \tag{4.20}$$

Now, if we operate the limit $\beta \to 0$, we recover the static result

$$\lim_{\beta \to 0} = -\frac{\hbar \mathrm{c}\pi}{24 L_0^2} \frac{(1+\beta)^2}{\left(1 + \dfrac{\beta}{L_0}(\mathrm{c}t + x)\right)^2} = -\frac{\hbar \mathrm{c}\pi}{24 L_0^2} \, . \tag{4.21}$$

We can note that the (4.20) is not exactly the same as (3.24).
First of all, in the previous case we have two terms, one depending on $\mathrm{c}t + x$ and one depending $\mathrm{c}t - x$, while here only the the previous dependence appeares.

Then, the constants are not the same. In fact, imposing our hypothesis of retarded approach, we are not able to recover $\dfrac{\beta^2}{\ln^2\left(\dfrac{1+\beta}{1-\beta}\right)}$ with such a naive approach.

4.3 Right oscillating boundary

We can solve the equation (4.13) for our retarded model for "slow time", thus when $\frac{\varepsilon\pi \mathrm{c}t}{L_0} \ll 1$ because the equation

$$x + \mathrm{c}t - \mathrm{c}t_1 = L_0 + L_0\varepsilon \sin\left(\frac{2\pi \mathrm{c}t_1}{L_0}\right) \tag{4.22}$$

in this approximation, turns out to be,

$$x + \mathrm{c}t - \mathrm{c}t_1 = L_0 + \varepsilon 2\pi \mathrm{c}t_1\,, \tag{4.23}$$

that in few steps carries the result

$$\mathrm{c}t_1 = \frac{x + \mathrm{c}t - L_0}{1 + 2\pi\varepsilon}\,. \tag{4.24}$$

As was already shown for the previous moving boundary, we substitute again this latter expression in the law of motion (3.26), obtaining

$$L(t_1) = L_0\left\{1 + \varepsilon \sin\left[\frac{2\pi}{1 + 2\pi\varepsilon}\left(\frac{x - L_0 + \mathrm{c}t}{L_0}\right)\right]\right\}\,. \tag{4.25}$$

Imposing again $L\left(t - \frac{L(t)}{\mathrm{c}}\right) = L(t_1)$, we find, using the (4.8), the expression for the density of energy

$$T_{00} = -\frac{\hbar \mathrm{c}\pi}{24L_0^2}\frac{1}{\left\{1 + \varepsilon \sin\left[\frac{2\pi}{1 + 2\pi\varepsilon}\left(\frac{x - L_0 + \mathrm{c}t}{L_0}\right)\right]\right\}^2}\,, \tag{4.26}$$

or, alternatively,

$$T_{00} = -\frac{\hbar \mathrm{c}\pi}{24L_0^2}\left\{1 + \varepsilon \sin\left(\frac{2\pi}{1 + 2\pi\varepsilon}\frac{x + \mathrm{c}t}{L_0}\right)\cos\left(\frac{2\pi}{1 + 2\pi\varepsilon}\right) + \right.$$
$$\left. - \varepsilon \cos\left(\frac{2\pi}{1 + 2\pi\varepsilon}\frac{x + \mathrm{c}t}{L_0}\right)\sin\left(\frac{2\pi}{1 + 2\pi\varepsilon}\right)\right\}^{-2}\,.$$

If we evaluate the limit for $\varepsilon \to 0$, we recover, from these equivalent latter expressions, the static density of energy

$$\lim_{\varepsilon\to 0} T_{00} = -\frac{\hbar \mathrm{c}\pi}{24L_0^2}\,. \tag{4.27}$$

4.3.1 Resonant law

We can also compute the same procedure with (3.48), imposing (4.13) even in this case, it is easy to find

$$x + \mathrm{c}t - \mathrm{c}t_1 = L_0 - \epsilon \sin^2\left(\frac{\pi \mathrm{c} t_1}{L_0}\right) , \tag{4.28}$$

that we approximate

$$x + \mathrm{c}t - \mathrm{c}t_1 = L_0 - \epsilon \frac{\pi^2 \mathrm{c}^2 t_1^2}{L_0^2} . \tag{4.29}$$

We want to solve this equation for t_1, that we write

$$\epsilon \frac{\pi^2}{L_0^2} \mathrm{c}^2 t_1^2 + \mathrm{c}t_1 + L_0 - x - \mathrm{c}t = 0 , \tag{4.30}$$

and it leads to the two different solutions

$$t_1^* = \frac{\mathrm{c}t + x - L_0}{\mathrm{c}} \qquad \text{and} \qquad t_1^{**} = \frac{L_0^2}{\pi^2 \mathrm{c}\epsilon} - (+\mathrm{c}t + x - L_0) = \frac{L_0^2}{\pi^2 \mathrm{c}\epsilon} - t_1^* . \tag{4.31}$$

Proof. The (4.30) is solvable by the simply evaluation of the Δ

$$\Delta = \mathrm{c}^2 \left[1 - 4\pi^2 \frac{\epsilon}{L_0} + 4\pi^2 \frac{\epsilon}{L_0^2}(\mathrm{c}t + x)\right] .$$

Now, if we consider

$$\begin{aligned}\sqrt{\Delta} &= \mathrm{c}\sqrt{1 - 4\pi^2 \frac{\epsilon}{L_0} + 4\pi^2 \frac{\epsilon}{L_0^2}(\mathrm{c}t + x)} \approx \mathrm{c}\left[1 + \frac{1}{2}\left(-4\pi^2 \frac{\epsilon}{L_0} + 4\pi^2 \frac{\epsilon}{L_0^2}(\mathrm{c}t + x)\right)\right] \\ &= \mathrm{c}\left[1 - 2\pi^2 \frac{\epsilon}{L_0} + 2\pi^2 \frac{\epsilon}{L_0^2}(\mathrm{c}t + x)\right] ,\end{aligned}$$

the solutions are trivial

$$t_1^* = \frac{-1 + 1 - 2\pi^2 \frac{\epsilon}{L_0} + 2\pi^2 \frac{\epsilon}{L_0^2}(\mathrm{c}t + x)}{2\pi^2 \mathrm{c} \frac{\epsilon}{L_0^2}} = \frac{\mathrm{c}t + x - L_0}{\mathrm{c}} , \tag{4.32}$$

$$t_1^{**} = \frac{-1 - 1 + 2\pi^2 \frac{\epsilon}{L_0} - 2\pi^2 \frac{\epsilon}{L_0^2}(\mathrm{c}t + x)}{2\pi^2 \mathrm{c} \frac{\epsilon}{L_0^2}} = \frac{L_0^2}{\pi^2 \mathrm{c}\epsilon} - \frac{(\mathrm{c}t + x - L_0)}{\mathrm{c}} . \tag{4.33}$$

□

Differently than the previous models, now we have two solutions. We proceed computing $L(t_1^*)$ and $L(t_1^{**})$, and superposing these two distances in the expression for the density of energy.
Firstly, we have

$$L(t_1^*) = L_0 - \epsilon \sin^2\left(\pi \frac{\text{ct} + x - L_0}{L_0}\right) = L_0 - \epsilon \sin^2\left(\pi \frac{\text{ct} + x}{L_0}\right) \tag{4.34}$$

and

$$L(t_1^{**}) = L_0 - \epsilon \sin^2\left(\frac{L_0}{\pi\epsilon} - \pi \frac{\text{ct} + x}{L_0} + \pi\right), \tag{4.35}$$

so we are now able to calculate the density of energy, that results

$$T_{00} = -\frac{\hbar c \pi}{48}\left\{\frac{1}{L^2(t_1^*)} + \frac{1}{L^2(t_1^{**})}\right\} =$$
$$= -\frac{\hbar c \pi}{48}\left\{\frac{1}{\left[L_0 - \epsilon \sin^2\left(\pi \frac{\text{ct}+x}{L_0}\right)\right]} + \frac{1}{L_0 - \epsilon \sin^2\left(\frac{L_0}{\pi\epsilon} - \pi\frac{\text{ct}+x}{L_0} + \pi\right)}\right\}. \tag{4.36}$$

Obviously, calculating the $\lim_{\epsilon \to 0} T_{00} = -\frac{\hbar c \pi}{24 L_0^2}$, as in the static case.

We could have considered the more general law of motion in case of resonating oscillating boundary. In fact, we can also solve (3.46) and we will find two solutions for this, too. In fact,

$$x + \text{ct} - \text{ct}_1 = L_0 + \frac{L_0}{2\pi}\sin^{-1}\left[\sin\theta \cos\left(\frac{2\pi \text{ct}_1}{L_0}\right) - \theta\right], \tag{4.37}$$

so, imposing the delay, we find, in a few steps, the relation

$$\cos\left(2\pi\frac{\text{ct}_1}{L_0}\right)\left[\sin\theta - \sin\left(2\pi\frac{ct+x}{L_0}\right)\right] + \sin\left(2\pi\frac{\text{ct}_1}{L_0}\right)\cos\left(2\pi\frac{ct+x}{L_0}\right) - \theta = 0. \tag{4.38}$$

To solve the (4.38) we have to operate the following change of variable

$$\gamma = \tan\left(\pi\frac{\text{ct}_1}{L_0}\right)$$

that allows the substitutions

$$\cos\left(2\pi\frac{\text{ct}_1}{L_0}\right) = \frac{1-\gamma^2}{1+\gamma^2}, \qquad \text{and} \qquad \sin\left(2\pi\frac{\text{ct}_1}{L_0}\right) = \frac{2\gamma}{1+\gamma^2}.$$

So, we are able to solve the polynomial equation in γ,

$$\left[\sin\left(2\pi\frac{ct+x}{L_0}\right) - sin\theta - \theta\right]\gamma^2 + 2\cos\left(2\pi\frac{ct+x}{L_0}\right)\gamma + \\ + \sin\theta - \sin\left(2\pi\frac{ct+x}{L_0}\right) - \theta = 0\,, \tag{4.39}$$

that has two solutions, slightly more complicated than the previous ones

$$\gamma_\pm = \frac{-\cos\left(2\pi\frac{ct+x}{L_0}\right) \pm \sqrt{1+\sin^2\theta - \theta^2 + 2\sin\theta\sin\left(2\pi\frac{ct}{L_0}\right)}}{\sin\left(2\pi\frac{ct+x}{L_0}\right) - \sin\theta - \theta}\,, \tag{4.40}$$

and the two solutions for the time t_1 are

$$t_1^\pm = \frac{L_0}{\pi c}\tan^{-1}(\gamma_\pm) = \\ = \frac{L_0}{\pi c}\tan^{-1}\left[\frac{-\cos\left(2\pi\frac{ct+x}{L_0}\right) \pm \sqrt{1+\sin^2\theta - \theta^2 + 2\sin\theta\sin\left(2\pi\frac{ct}{L_0}\right)}}{\sin\left(2\pi\frac{ct+x}{L_0}\right) - \sin\theta - \theta}\right]. \tag{4.41}$$

At this point, we are able to write the expressione for the retarded position $L(t_1^\pm)$

$$L(t_1^+) = L_0 + \frac{L_0}{2\pi}\sin^{-1}\Bigg\{\sin\theta\Bigg[2\sin^2\theta - 2 + 2\theta^2 + 2\theta\sin\theta - 2\theta\sin\left(2\pi\frac{ct+x}{L_0}\right) + \\ - 4\sin\theta\sin\left(2\pi\frac{ct+x}{L_0}\right) - \sin^2\theta\cos\left(2\pi\frac{ct+x}{L_0}\right) + \theta^2\cos\left(2\pi\frac{ct+x}{L_0}\right) + \\ - 2\sin\theta\sin\left(2\pi\frac{ct+x}{L_0}\right)\cos\left(2\pi\frac{ct+x}{L_0}\right)\Bigg]\times \\ \times\Bigg[2 + 2\sin^2\theta + 2\theta\sin\theta - 2\theta\sin\left(2\pi\frac{ct+x}{L_0}\right) + 2\cos\left(2\pi\frac{ct+x}{L_0}\right) + \\ + \sin^2\theta\cos\left(2\pi\frac{ct+x}{L_0}\right) - \theta^2\cos\left(2\pi\frac{ct+x}{L_0}\right) + \\ + 2\sin\theta\sin\left(2\pi\frac{ct+x}{L_0}\right)\cos\left(2\pi\frac{ct+x}{L_0}\right)\Bigg]^{-1} - \theta\Bigg\}\,, \tag{4.42}$$

and

$$L(t_1^-) = L_0 + \frac{L_0}{2\pi}\sin^{-1}\left\{\sin\theta\left[2\theta^2 - 2 + 2\theta\sin\theta + 2\sin^2\left(2\pi\frac{\mathrm{c}t+x}{L_0}\right) + \right.\right.$$
$$- 4\sin\theta\sin\left(2\pi\frac{\mathrm{c}t+x}{L_0}\right) - 2\theta\sin\left(2\pi\frac{\mathrm{c}t+x}{L_0}\right) + \cos\left(2\pi\frac{\mathrm{c}t+x}{L_0}\right) +$$
$$+ \sin^2\theta\cos\left(2\pi\frac{\mathrm{c}t+x}{L_0}\right) - \theta^2\cos\left(2\pi\frac{\mathrm{c}t+x}{L_0}\right) +$$
$$\left. + 2\sin\theta\sin\left(2\pi\frac{\mathrm{c}t+x}{L_0}\right)\cos\left(2\pi\frac{\mathrm{c}t+x}{L_0}\right)\right]\times$$
$$\times\left[2 + 2\sin^2\theta + 2\theta\sin\theta - 2\theta\sin\left(2\pi\frac{\mathrm{c}t+x}{L_0}\right) - \cos\left(2\pi\frac{\mathrm{c}t+x}{L_0}\right) + \right.$$
$$- \sin^2\theta\cos\left(2\pi\frac{\mathrm{c}t+x}{L_0}\right) + \theta^2\cos\left(2\pi\frac{\mathrm{c}t+x}{L_0}\right) +$$
$$\left.\left. - 2\sin\theta\sin\left(2\pi\frac{\mathrm{c}t+x}{L_0}\right)\cos\left(2\pi\frac{\mathrm{c}t+x}{L_0}\right)\right]^{-1} - \theta\right\}, \tag{4.43}$$

and we finally find

$$T_{00} = -\frac{\hbar \mathrm{c}\pi}{48}\left\{\frac{1}{L^2(t_1^+)} + \frac{1}{L^2(t_1^-)}\right\}. \tag{4.44}$$

Even in this case, as in every previous case, there is a limit that leads to the static solution, and it is $\lim_{\epsilon\to 0} T_{00} = -\frac{\hbar \mathrm{c}\pi}{24L_0^2}$.

4.4 Final remarks on the retarded approach

As we have already noted for the linear motion of the right boundary, even in the case of the oscillating boundary the results are not exactly the same.

In the simplest case, when right boundary moves with constant velocity, we have the same dependence from t and x, but the constants are not in accordance. In the case when the boundary oscillates, does not coincide neither the constants, nor the t and x dependence.

Thus, we make another observation about the first case. The second one is influenced by a large number of approximation, in both exact and delayed cases.

As was anticipated, following our approach, we have always only term depending on $\mathrm{c}t + x$, while in Chapter 3 we find terms with both, $\mathrm{c}t + x$ and $\mathrm{c}t - x$ dependence.

This lack could be due to the fact that we consider only the detection of photons coming from the right boundary, so all the regressive waves. We can easily expected that the extra term considers the photons coming from the left mirror, too, including so the progressive waves.

This kind of remark can improve our expression of T_{00}, adding the second term, but we are not definitively able to give the correct result, because this naive imposition of the delay, even if recover the static case in an appropriate limit, can not describe the physics inside a hole if one border is moving.

Chapter 5

A self-consistent law of motion

Finally, we write the law of motion of a relativistic harmonic oscillator, adding a term that contain the self-consistent Casimir force with the imposition of the delay, and the result is

$$\frac{M\ddot{L}(t)}{\sqrt{1-\frac{\dot{L}^2}{c^2}}} + M\omega_0^2\left(L(t)-L_0\right) = -\frac{\hbar c\pi}{24L^2\left(t-\frac{L(t)}{c}\right)} \quad , \tag{5.1}$$

where M is the mass and $\omega_0 = 2\pi\nu_0$, where ν_0 is the frequency, but we will also refer to ω_0 calling it frequency.

Formally, this equation is not so well-defined, it would be stricter multiplying the Casimir delayed term for Heaviside theta function, so the expression

$$\frac{M\ddot{L}(t)}{\sqrt{1-\frac{\dot{L}^2}{c^2}}} + M\omega_0^2\left(L(t)-L_0\right) = -\frac{\hbar c\pi}{24L^2\left(t-\frac{L(t)}{c}\right)}\theta\left(t-\frac{L(t)}{c}\right) \quad , \tag{5.2}$$

is more complete than (5.1) because if the signal is going to come without delay, the main equation is a relativistic harmonic oscillator, without any other term

$$\frac{M\ddot{L}(t)}{\sqrt{1-\frac{\dot{L}^2}{c^2}}} + M\omega_0^2\left(L(t)-L_0\right) = 0\,. \tag{5.3}$$

5.1 Solutions of the equation

For solving the equation (5.1) in numerical way it is worthwhile to consider the approximation

$$\frac{1}{\sqrt{1-\frac{\dot{L}^2}{c^2}}} \sim 1 + \frac{1}{2}\frac{\dot{L}^2}{c^2}\,, \tag{5.4}$$

that is a physical consequence of the fact that the speed of the boundary is much smaller compared with the speed the light $\frac{\dot{L}^2}{\mathrm{c}^2} \ll 1$, as said in [23]. So, we now have to solve

$$M\ddot{L}(t)\left(1+\frac{1}{2}\frac{\dot{L}^2}{\mathrm{c}^2}\right)+M\omega_0^2\left(L(t)-L_0\right)=-\frac{\hbar\mathrm{c}\pi}{24L^2\left(t-\frac{L(t)}{\mathrm{c}}\right)}\,. \tag{5.5}$$

In order to write a consistent algorithm, we have to approximate the derivatives of our function as follows [37]

$$\begin{aligned}\frac{L(i+1)-L(i-1)}{2\Delta t}&=\dot{L}(t_i)+\mathcal{O}(\Delta t^2)\,,\\ \frac{L(i+1)-2L(i)+L(i-1)}{\Delta t^2}&=\ddot{L}(t_i)+\mathcal{O}(\Delta t^2)\,,\end{aligned} \tag{5.6}$$

where i is the integer index for discretized time, and Δt is the minimum step of the time axis, so that $t_i = i\cdot\Delta t$. Adding two boundary conditions, it is straightforward to find the solution of a second order differential equation.

It is interesting to consider before the equation

$$M\ddot{L}(t)+M\omega_0^2\left(L(t)-L_0\right)=-\frac{\hbar\mathrm{c}\pi}{24L^2\left(t-\frac{L(t)}{\mathrm{c}}\right)}\,, \tag{5.7}$$

where the left hand side (LHS) finds out to be an harmonic oscillator.
In both cases, $L(i+1)$ depends by L in previous "slices" of time, i.e. $L(i)$, $L(i-1)$ and $L(j)$, with $j = i - \frac{L(i)}{c\Delta t}$ for the Casimir delayed term.
Harmonic oscillator is a simple case because, imposing (5.6), the (5.7) turns out to be an elementary equation immediatly solvable for $L(i+1)$. The solution of this equation is plotted in Figure 5.1.

Coming back to (5.5) and replacing continue-time with discrete-time whereby (5.6), we find a third grade equation for $L(i+1)$, which has only one acceptable solution that carry out a sensible correction to the equation (5.7). We can examinate different plots of this solution in dependence of the free parametres, such as the mass M and the self-consistent pulsation ω_0.
For example, if $M \sim 10^0\,kg$ [6], we obtain that the Casimir term is almost negligible and the right boundary follows a nearly harmonic motion, as shown in Figure 5.2.

[6] We put the value of M exactly equal to 1, but, in general, the important thing is the order of magnitude.

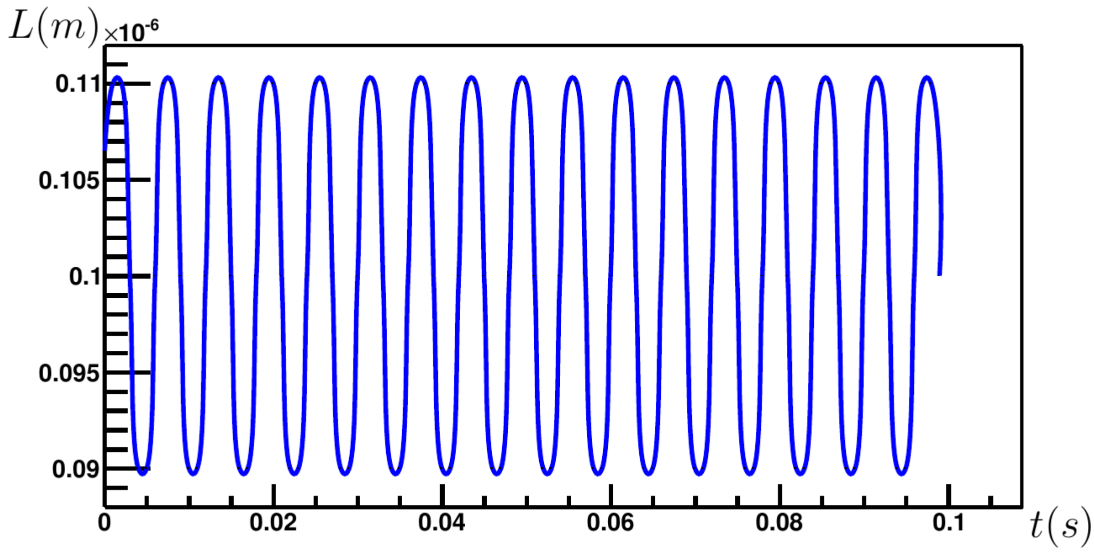

Figure 5.1: Position of the right boundary in function of time, without Casimir term.

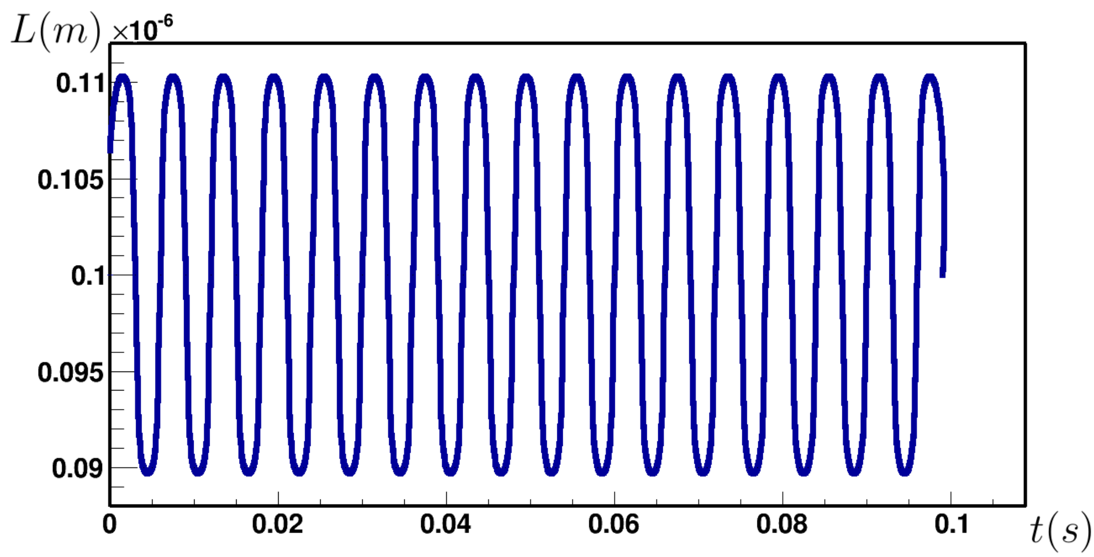

Figure 5.2: Position of the right boundary in function of time, $M = 1\,kg$ and $\omega_0 = 10^3\,Hz$.

A different result is found if the value of the mass is very small, for example we take $M = 10^{-9}\,kg$, we can see from picture Figure 5.3 that the amptitude of the position decreases as function of time, and, from the Figure 5.4 (that is equivalent to Figure 5.3, but for a longer time), we can easily guess that the steady position for a long time is the initial position, $L_0 = 100nm$ in this case.

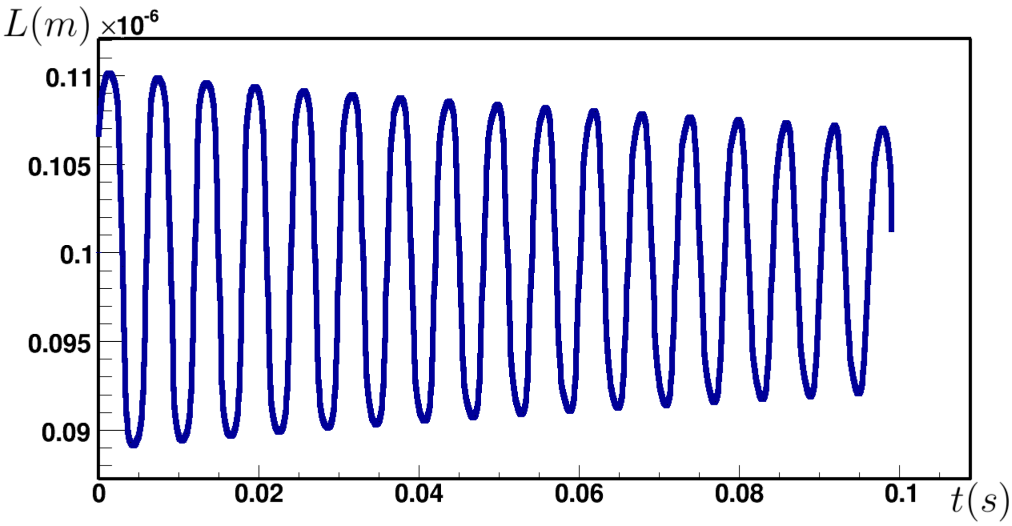

Figure 5.3: Position of the right boundary in function of time, $M = 10^{-9}\,kg$ and $\omega_0 = 10^3\,Hz$.

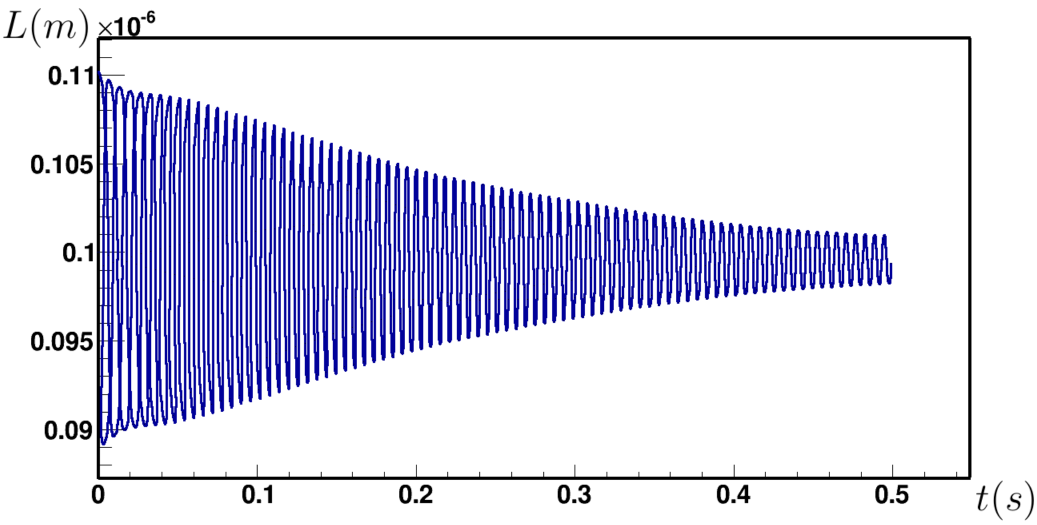

Figure 5.4: Position of the right boundary in function of time, $M = 10^{-9}\,kg$ and $\omega_0 = 10^3\,Hz$.

Moreover, keeping the case of unitary mass, if we change a few the inital condition, we obtain a solution with the same behaviour of (3.26), pictured in Figure 3.1. We can see this similarity in Figure 5.5.

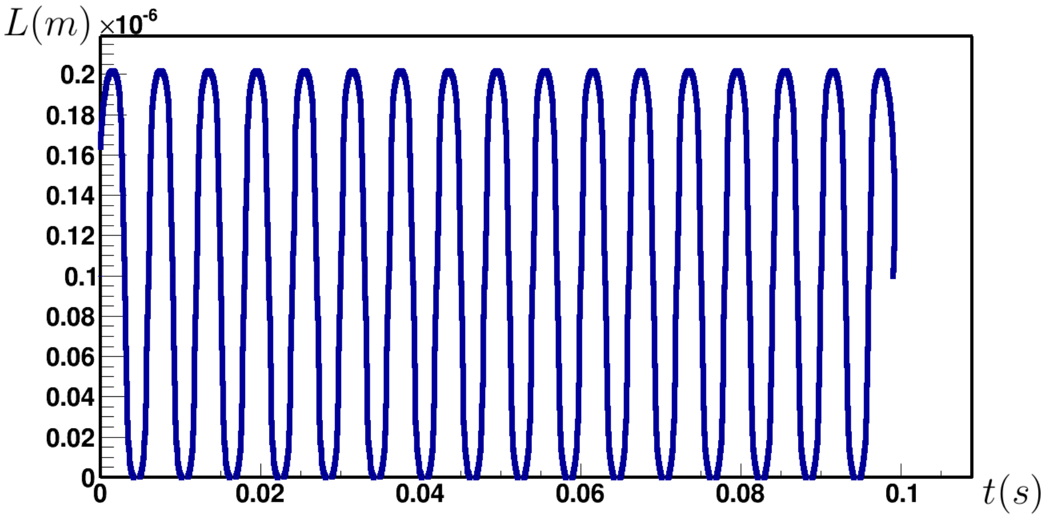

Figure 5.5: Position of the right boundary in function of time, with suitable boundary conditions, lead back to a known issue.

5.2 Casimir energy calculated numerically

Now, we obviously want to calculate the energy.
Proceeding via numerical way, so integrating stp-by-step through the formula

$$E(i) = \hbar c\pi \frac{L(i+1) - L(i)}{24L^2(j)}, \tag{5.8}$$

where $j = i - \dfrac{L(i)}{c\Delta t}$.
In this case, again, we show two different, but this time, being M-independent, the interesting variable is the frequency ω_0. In fact for $\omega_0 = 10^3\,Hz$ we have the trend illustrated in Figure 5.6, rapidly vanishing to zero, while for different incresing values, such as $\omega_0 = 10^6\,Hz$, a resonance effect come in and the energy behaviour has several changes, dispayed in Figure 5.7.

Then, we can evaluate the energy with the equation (4.9)

$$V_{Cas} = \frac{\hbar c\pi}{24L\left(t - \dfrac{L(t)}{c}\right)} \tag{5.9}$$

and using the position found in numerical way in the previous section.

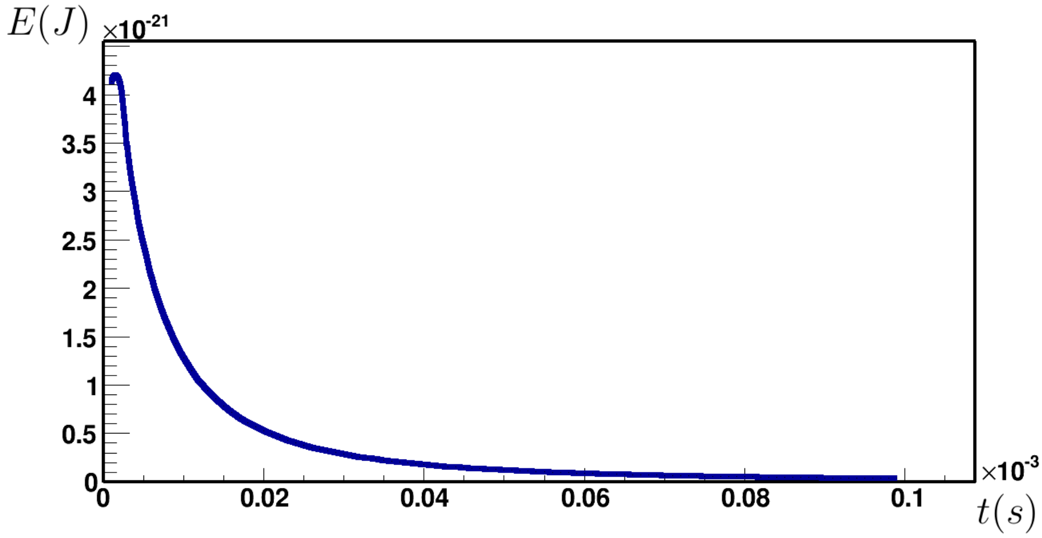

Figure 5.6: Numerical Casimir energy when $\omega_0 = 10^3\,Hz$.

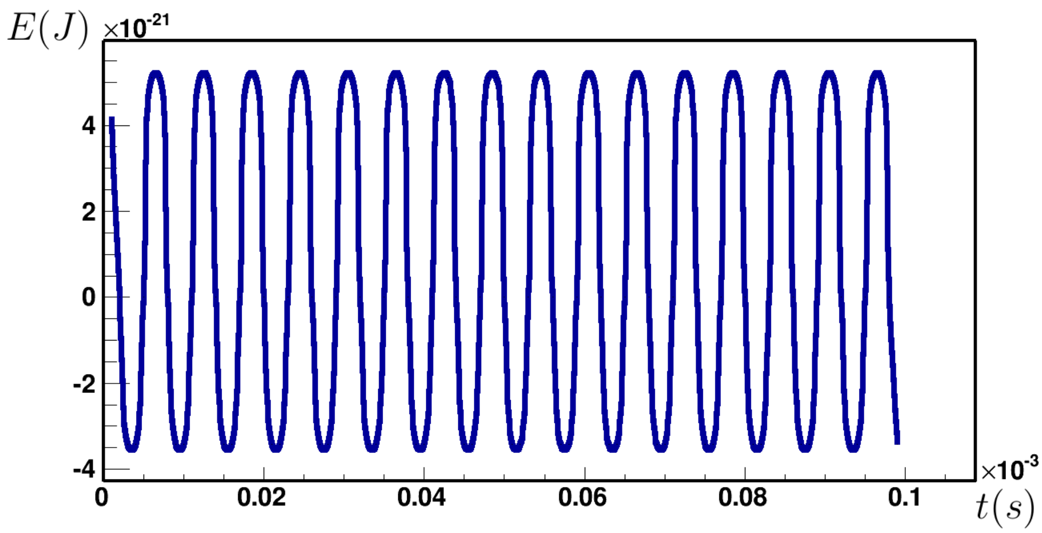

Figure 5.7: Numerical Casimir energy when $\omega_0 = 10^6\,Hz$.

Even in this case, we show two plots for different values of $\omega_0 = 10^3 Hz$, in Figure 5.8, and $\omega_0 = 10^6 Hz$, in Figure 5.9, in order to compare numerical solutions and analytical solutions.

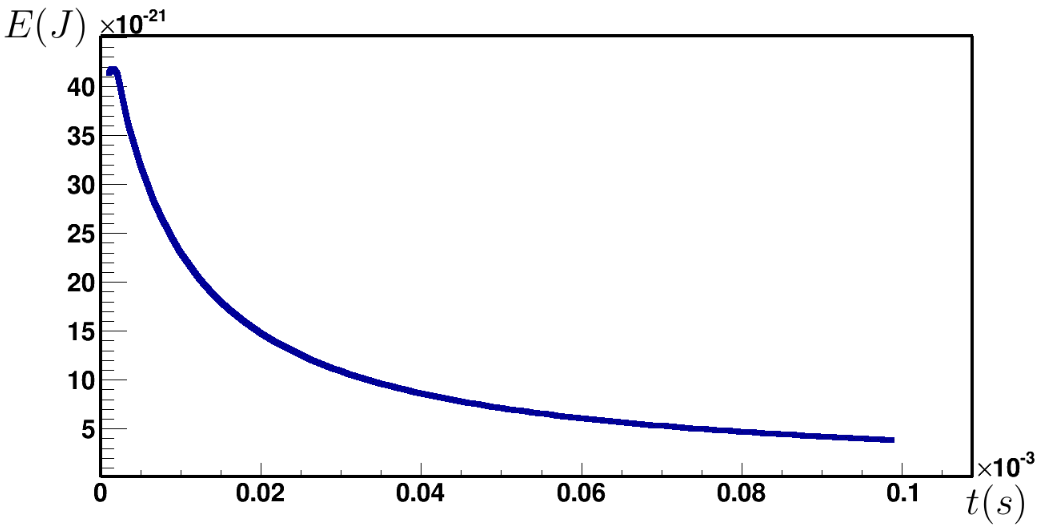

Figure 5.8: Analytical Casimir retarded energy for $\omega_0 = 10^3\,Hz$.

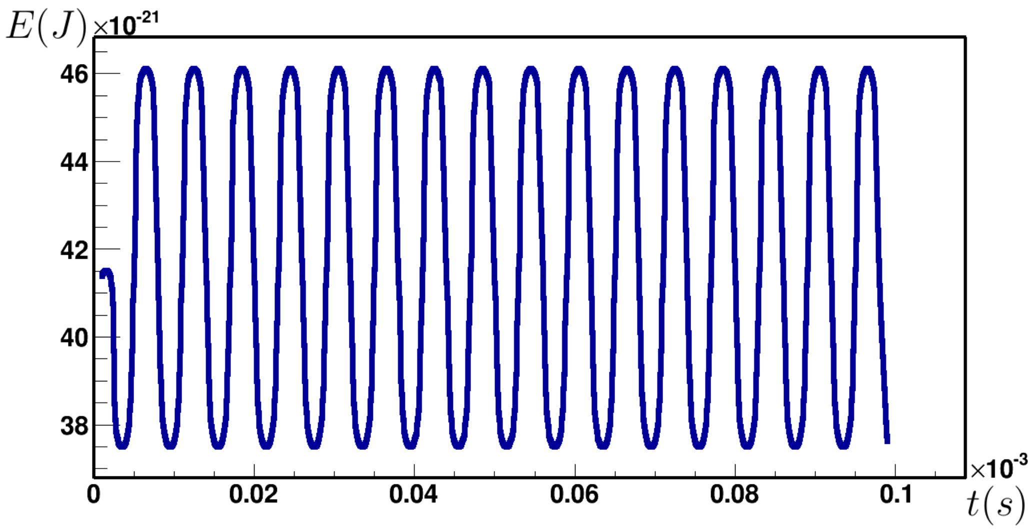

Figure 5.9: Analytical Casimir retarded energy for $\omega_0 = 10^6\,Hz$.

It is remarkable the similarity between the results. In fact, setting $\omega_0 = 10^3\,Hz$, we see graphically the similar behaviour between Figure 5.8 and the previous Figure 5.6.
But, if we set $\omega_0 = 10^6 Hz$, we get a resonance effect, as shown in Figure 5.9, and also amenable to Figure 5.7.

Despite of a similar trend, there is an important difference. We can note that the analytical results present one more order of magnitude than the numerical.

Another interesting thing is that, if we substitute the relation (3.26) as the position in the equation for V_{Cas} in (4.9), reported also in (5.9), we find a similar behaviour, as reported in Figure 5.10, where the energy oscillates, but with a different frequency.

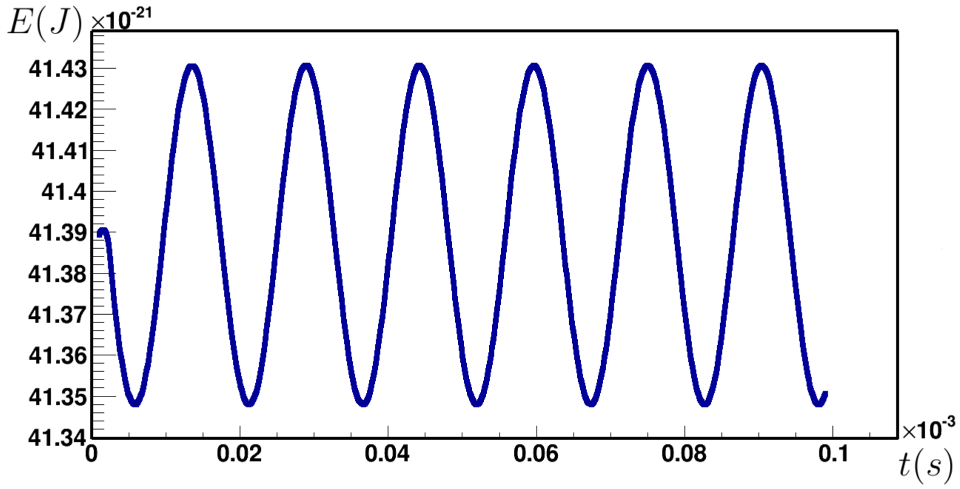

Figure 5.10: Analytical Casimir energy with position $L(t) = L_0 \left[1 + \varepsilon \sin\left(\frac{2\pi}{L_0} ct\right)\right]$.

5.3 About the self-consistent solution

In this subsection, we would deduce an expression for the self-consistent frequncy. In fact, in (5.1) we suppose the term linked to frequency to be linear with respect to $L(t)$. Anyway it could not be true.

We try to gain the behaviour of this term, useful for us, as the difference between the kinetic term $\frac{\ddot{L}(t)}{\sqrt{1-\frac{\dot{L}^2}{c^2}}}$ and the Casimir term, expressed as the density of energy $T_{00}(t,x)$, evaluated in $x = L(t)$.

This is not interesting for the linear constant velocity motion, because the kinetic term vanishes. In fact $\dot{L} = v$ does not contain time dependence and consequently $\ddot{L} = 0$.

For the oscillating law of motion, it is convenient to consider the exact resonating law (3.46).

$$\frac{\ddot{L}(t)}{\sqrt{1-\frac{\dot{L}^2}{c^2}}} = -\frac{2\pi c^2 \sin\theta}{L_0\sqrt{1-\theta^2-\sin^2\theta+2\theta\sin\theta\cos\left(\frac{2\pi c t}{L_0}\right)}}\times$$
$$\times\left\{\cos\left(\frac{2\pi ct}{L_0}\right)+\sin^2\left(\frac{2\pi ct}{L_0}\right)\left[\frac{\sin\theta\cos\left(\frac{2\pi ct}{L_0}\right)-\theta}{1-\left(\sin\theta\cos\left(\frac{2\pi ct}{L_0}\right)-\theta\right)^2}\right]\right\}. \tag{5.10}$$

The 00-component of the energy-momentum tensor is the expression (3.51). As was already mentioned before, we consider the energy of the oscillating mirror, so we consider T_{00} at the instant t in the position $L(t)$.
Then, declaring the mass of the moving boundary M, we are able to write the difference

$$M\frac{\ddot{L}(t)}{\sqrt{1-\frac{\dot{L}^2}{c^2}}} - T_{00}(t, L(t)) =$$
$$= -\frac{2M\pi c^2 \sin\theta}{L_0\sqrt{1-\theta^2-\sin^2\theta+2\theta\sin\theta\cos\left(\frac{2\pi ct}{L_0}\right)}}\times$$
$$\times\left\{\cos\left(\frac{2\pi ct}{L_0}\right)+\sin^2\left(\frac{2\pi ct}{L_0}\right)\left[\frac{\sin\theta\cos\left(\frac{2\pi ct}{L_0}\right)-\theta}{1-\left(\sin\theta\cos\left(\frac{2\pi ct}{L_0}\right)-\theta\right)^2}\right]\right\}+$$
$$+\frac{\hbar c\pi}{48L_0^2}\left\{\frac{1}{\left[1-\frac{2n\epsilon\pi}{L_0}\sin\left(2\pi\frac{ct+L(t)}{L_0}\right)+\frac{4n^2\epsilon^2\pi^2}{L_0^2}\sin^2\left(\pi\frac{ct+L(t)}{L_0}\right)\right]^2}+\right.$$
$$\left.+\frac{1}{\left[1-\frac{2n\epsilon\pi}{L_0}\sin\left(2\pi\frac{ct-L(t)}{L_0}\right)+\frac{4n^2\epsilon^2\pi^2}{L_0^2}\sin^2\left(\pi\frac{ct-L(t)}{L_0}\right)\right]^2}\right\}. \tag{5.11}$$

We want now to highlight the dependence from the position $L(t)$, so we have to invert the equation (3.46) in order to write ct in function of $L(t)$. The result is

$$ct = \frac{L_0}{2\pi}\cos^{-1}\left[\frac{\sin\left(2\pi\frac{L(t)-L_0}{L_0}\right)+\theta}{\sin\theta}\right] = \frac{L_0}{2\pi}\cos^{-1}\left[\frac{\sin\left(2\pi\frac{L(t)}{L_0}\right)+\theta}{\sin\theta}\right], \tag{5.12}$$

that put in the previous equation gives the following kinetic term

$$\frac{M\ddot{L}(t)}{\sqrt{1-\frac{\dot{L}^2}{c^2}}} = -\frac{2M\pi c^2}{L_0\sin\theta}\frac{1}{\cos^2\left(\frac{2\pi L(t)}{L_0}\right)\sqrt{1+\theta^2-\sin^2\theta+2\theta\sin\left(\frac{2\pi L(t)}{L_0}\right)}}\times$$
$$\times\left\{1-\theta^2+\theta\sin\theta+\sin\left(\frac{2\pi L(t)}{L_0}\right)(\sin\theta-\theta)+\right.$$
$$\left.-\theta\sin\theta\sin^2\left(\frac{2\pi L(t)}{L_0}\right)-\sin\theta\sin^3\left(\frac{2\pi L(t)}{L_0}\right)\right\}, \tag{5.13}$$

and the Casimir term

$$T_{00} = -\frac{\hbar c\pi}{48L_0^2}\left\{\left[1-\frac{2n\epsilon\pi}{L_0}\sin\left(\cos^{-1}\left[\frac{\sin\left(2\pi\frac{L(t)}{L_0}\right)+\theta}{\sin\theta}\right]+2\pi\frac{L(t)}{L_0}\right)+\right.\right.$$
$$\left.+\frac{4n^2\epsilon^2\pi^2}{L_0^2}\sin^2\left(\frac{1}{2}\cos^{-1}\left[\frac{\sin\left(2\pi\frac{L(t)}{L_0}\right)+\theta}{\sin\theta}\right]+\pi\frac{L(t)}{L_0}\right)\right]^{-2}+$$
$$\left[1-\frac{2n\epsilon\pi}{L_0}\sin\left(\cos^{-1}\left[\frac{\sin\left(2\pi\frac{L(t)}{L_0}\right)+\theta}{\sin\theta}\right]-2\pi\frac{L(t)}{L_0}\right)+\right.$$
$$\left.\left.+\frac{4n^2\epsilon^2\pi^2}{L_0^2}\sin^2\left(\frac{1}{2}\cos^{-1}\left[\frac{\sin\left(2\pi\frac{L(t)}{L_0}\right)+\theta}{\sin\theta}\right]-\pi\frac{L(t)}{L_0}\right)\right]^{-2}\right\}. \tag{5.14}$$

Proof. The inversion of (3.46) is straightforward, in fact the following statements are equivalent

$$L(t) = L_0+\frac{L_0}{2\pi}\left\{\sin^{-1}\left[\sin\theta\cos\left(\frac{2\pi ct}{L_0}\right)\right]-\theta\right\},$$
$$\sin\left(2\pi\frac{L(t)-L_0}{L_0}\right) = \sin\theta\cos\left(2\pi\frac{ct}{L_0}\right)-\theta\,,$$
$$\cos\left(2\pi\frac{ct}{L_0}\right) = \frac{\sin\left(2\pi\frac{L(t)-L_0}{L_0}\right)+\theta}{\sin\theta}, \tag{5.15}$$
$$ct = \frac{L_0}{2\pi}\cos^{-1}\left[\frac{\sin\left(2\pi\frac{L(t)-L_0}{L_0}\right)+\theta}{\sin\theta}\right], \tag{5.16}$$

where we can also write

$$\sin\left(2\pi\frac{L(t)-L_0}{L_0}\right) = \sin\left(2\pi\frac{L(t)}{L_0}\right) . \tag{5.17}$$

In the kinetic term, the easiest substitution in directly the (5.15), and its exact consequence

$$\sin^2\left(2\pi\frac{ct}{L_0}\right) = 1 - \frac{\left[\sin\left(2\pi\frac{L(t)-L_0}{L_0}\right) - \theta\right]^2}{\sin^2\theta} , \tag{5.18}$$

while in the Casimir term we substitute the (5.16). □

We have now a force F, expressed in function of a distance L. We can plot the behaviour of this force, given by the difference of (5.13) and (5.14), at the changing of the free parametres.

We can not consider all values of L, because the the law of motion used (3.46) and its inversion (5.16) have inverse trigonometric functions, which have a limited domain $[-1, 1]$. The conclusion is that we obtain acceptable values only if the position is near L_0.

We fixed $L_0 = 10^{-7}\,m$, $\epsilon = 10^{-10}m$, $n = 2$, and we have estimated the force, in function of the position, for different values of the mass M.
In particular, from Figure 5.11, we can see what happened when the kinetic term is much greater than T_{00}. The force is almost linear, getting close to L_0.

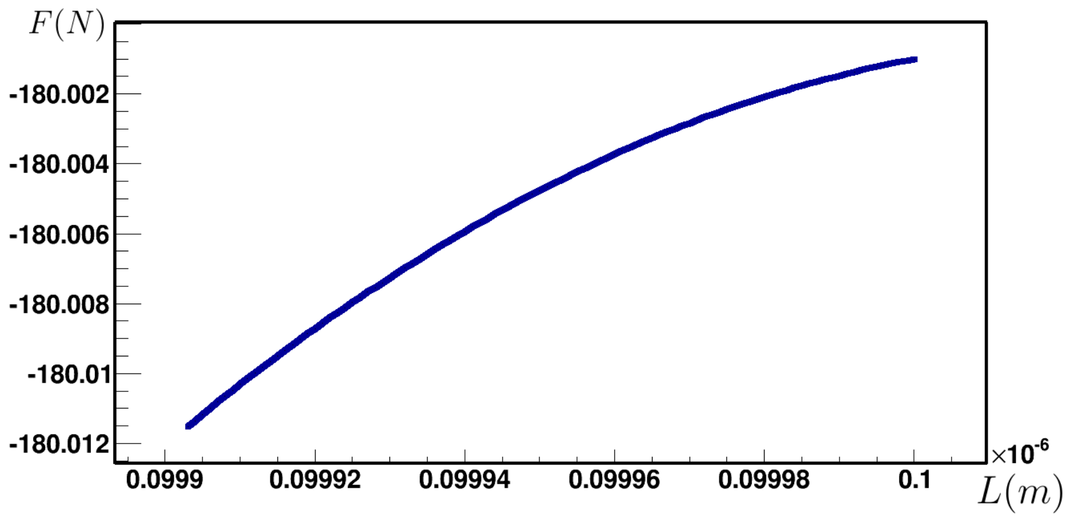

Figure 5.11: The force with negligible Casimir term.

Then, there is a damping that changes the previous regime. When the order of magnitude of $T_{00}\left(t, L(t)\right)$ becomes comparable with $\frac{M\ddot{L}(t)}{\sqrt{1-\frac{\dot{L}^2}{c^2}}}$, we obtain the result shown in Figure 5.12.

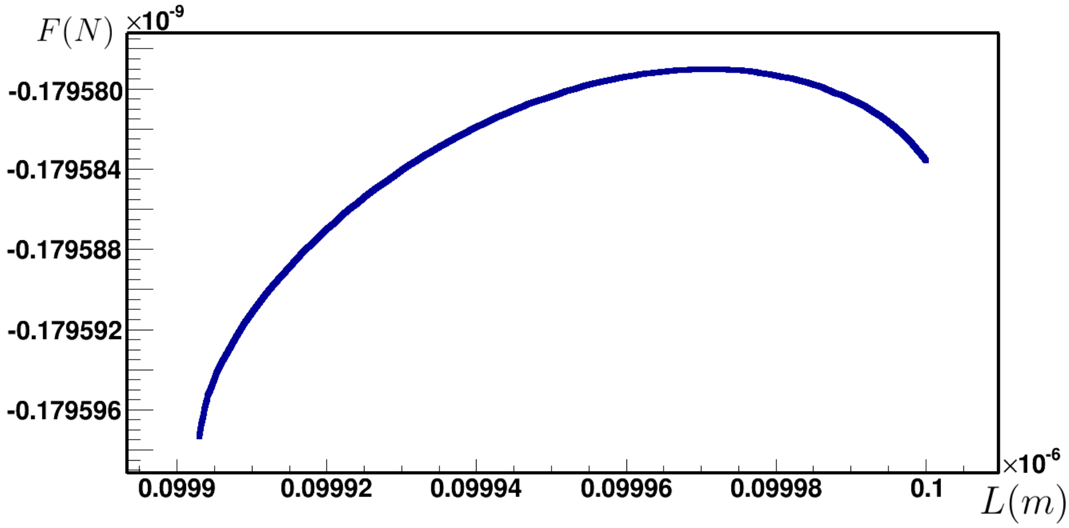

Figure 5.12: The force when order of magnitude of kinetic and Casimir contributions are comparable.

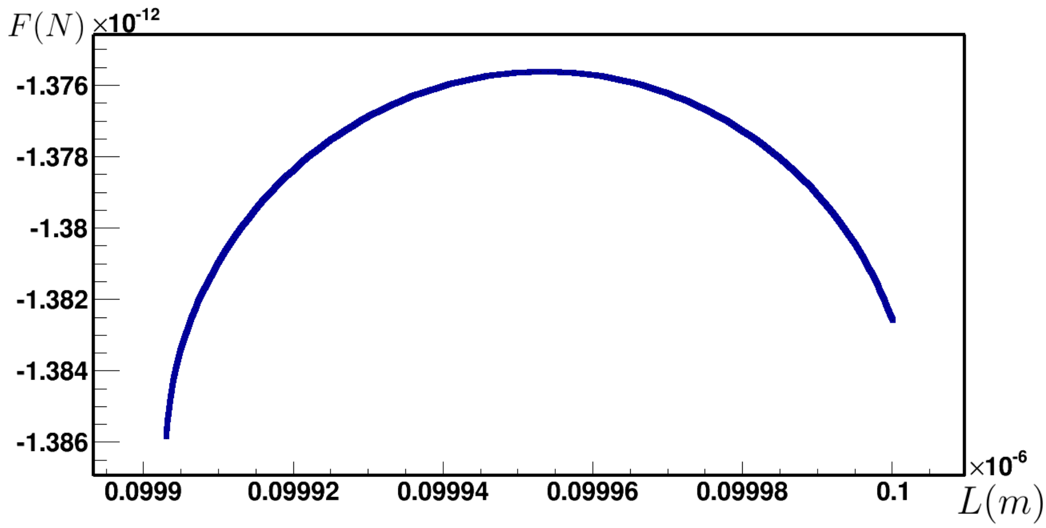

Figure 5.13: The force with negligible kinetic term.

This force, gradually stabilize,as in Figure 5.13, at the decreasing of the mass, when the leading term is the Casimir one. In this case, we can note that the behaviour is no more linear, but parabolic.

Finally, we can conclude that the force, due to self-interaction of the right boundary, is not linear when the Casimir term takes over, according to this model.

Chapter 6

Conclusions

The Casimir effect represents one of the most interesting manifestations of the vacuum.

The stationary case has already been widely studied, and we noted, in Chapter 2, that it obviously depends on the boundary conditions. Static models are also the simplest ones, because we are able to evaluate the energy density by means of the regularization procedures.

Then, in Chapter 3, we considered the same vacuum effect, with nonstationary boundary conditions. In particular, in our one dimensional model, we took the left boundary constrained in the origin, while the right one was moving according to different laws of motion.
When the wall was in a uniform motion we could calculate the energy density, that is the variable we are interested in. When the wall oscillated, we were not able to calculate it exactly, but we did some considerations which led us to an approximated expression for the energy density.

Thus, in Chapter 4, we tried to impose the delay condition, intuitively caused by the motion of the boundary. In the case of a uniform motion with constant velocity, the qualitative behaviour of the time and spatial variables t and x was the same as in the formal calculation, with the appropriate considerations. Unfortunately, we were not able to recover exactly the same constant, so this approach is not as rigorous as the first one.
This retarded approach was even less precise in the case with the oscillating boundary, because we had to use an approximation. The resulting expression for the energy density is very difficult to compare with the expression found previously with the scalar field approach.
Therefore, we have found that our simple retarded approach could give a qualitative behaviour of the solution in the simplest case, but we did not manage to recover the exact solution.

In Chapter 5, we tried to describe the motion of the moving boundary due

to its self-interaction with the vacuum. We first wrote the law of a harmonic oscillator, adding a term of the Casimir force. We had several free parametres, but we considered, in particular, how the position of the moving wall and the Casimir energy changed, for different values of the mass of the boundary. Numerical results, for both position and energy, were in accordance with the analytical procedure.
Then, we studied the force of this boundary, linked to its self-frequency, calculating the difference between the kinetic term and the Casimir term. The results were interesting. In fact, when the order of magnitude of the mass was so big to make the Casimir term negligible, then the behaviour near the boundary was almost linear, as we expected. Taking smaller values of the mass, until the kinetic term and the Casimir term become comparable, we found that the force was no more linear in dependence of the position, but it had a parabolic behaviour in our configuration.

In conclusion, vacuum fluctuations can be qualitatively described with the imposition of a delay, but it hardly can give exact results.
We can also conclude that the force, due to self-interaction of a moving boundary, changes its regime in presence of these vacuum fluctuations, and it has no more linear dependence on the position.

Appendix A

Series of frequency cut-off regularization

Demonstration of the value of convergence

We can prove the (2.19), namely

$$\sum_{n=1}^{N} \alpha n \mathrm{e}^{-\alpha n} = \frac{\alpha \mathrm{e}^{-\alpha N} \left[\mathrm{e}^{\alpha(N+1)} + N - \mathrm{e}^{\alpha}(N+1)\right]}{\left(\mathrm{e}^{\alpha} - 1\right)^2}, \tag{A.1}$$

proceeding by induction.

First of all we check the result in the case $N = 1$, so on the the left-hand side $S_1 = \alpha \mathrm{e}^{-\alpha}$, while on the right-hand side we have

$$S_1 = \frac{\alpha \mathrm{e}^{-\alpha} \left[\mathrm{e}^{2\alpha} + 1 - 2\mathrm{e}^{\alpha}\right]}{\left(\mathrm{e}^{\alpha} - 1\right)^2} = \frac{\alpha \mathrm{e}^{-\alpha} \left(\mathrm{e}^{\alpha} - 1\right)^2}{\left(\mathrm{e}^{\alpha} - 1\right)^2},$$

and it is true.

Check also for $N = 2$. Taking into account the left-hand side of the main equation we have

$$S_2 = \alpha \mathrm{e}^{-\alpha} + 2\alpha \mathrm{e}^{-2\alpha} = \alpha \mathrm{e}^{-\alpha} \left(1 + 2\mathrm{e}^{-\alpha}\right),$$

so we control that the same expression appears in the right-hand side

$$\begin{aligned} S_2 &= \frac{\alpha \mathrm{e}^{-2\alpha} \left[\mathrm{e}^{3\alpha} + 2 - 3\mathrm{e}^{\alpha}\right]}{\left(\mathrm{e}^{\alpha} - 1\right)^2} = \frac{\alpha \mathrm{e}^{-2\alpha}}{\left(\mathrm{e}^{\alpha} - 1\right)^2} \left[\mathrm{e}^{\alpha} \left(\mathrm{e}^{2\alpha} - 1\right) - 2\left(\mathrm{e}^{\alpha} - 1\right)\right] = \\ &= \frac{\alpha \mathrm{e}^{-\alpha}}{\left(\mathrm{e}^{\alpha} - 1\right)} \left[\mathrm{e}^{\alpha} - 1 + 2\mathrm{e}^{-\alpha}\left(\mathrm{e}^{\alpha} - 1\right)\right] = \alpha \mathrm{e}^{-\alpha}\left(1 + 2\mathrm{e}^{-\alpha}\right), \end{aligned}$$

and it is verified, too.

Supposing now to be true $k = N - 1$, then we check the correctness for $k = N$. Therefore we want to verify that

$$S_N = S_{N-1} + \alpha N \mathrm{e}^{-\alpha N} = \frac{\alpha \mathrm{e}^{-\alpha N}\left[\mathrm{e}^{\alpha(N+1)} + N - \mathrm{e}^{\alpha}(N+1)\right]}{\left(\mathrm{e}^{\alpha} - 1\right)^2}\,, \tag{A.2}$$

so let's compute

$$\begin{aligned}
S_{N-1} + \alpha N \mathrm{e}^{-\alpha N} &= \frac{\alpha \mathrm{e}^{-\alpha(N-1)}\left[\mathrm{e}^{\alpha N} + N - 1 - N\mathrm{e}^{\alpha}\right]}{\left(\mathrm{e}^{\alpha} - 1\right)^2} + \alpha N \mathrm{e}^{-\alpha N} = \\
=&\frac{\alpha \mathrm{e}^{-\alpha N}}{\left(\mathrm{e}^{\alpha} - 1\right)^2}\left[\mathrm{e}^{\alpha}\left(\mathrm{e}^{\alpha N} + N - 1 - \mathrm{e}^{\alpha N}\right) + N\mathrm{e}^{2\alpha} + N - 2N\mathrm{e}^{\alpha}\right] = \\
=&\frac{\alpha \mathrm{e}^{-\alpha N}}{\left(\mathrm{e}^{\alpha} - 1\right)^2}\left[\mathrm{e}^{\alpha(N+1)} - N\mathrm{e}^{\alpha} - \mathrm{e}^{\alpha} + N\right] = \\
=&\frac{\alpha \mathrm{e}^{-\alpha N}}{\left(\mathrm{e}^{\alpha} - 1\right)^2}\left[\mathrm{e}^{\alpha(N+1)} + N - \mathrm{e}^{\alpha}(N+1)\right]\,.
\end{aligned}$$

In conclusion, we find that the (A.2) is true, and, consequentially, the (2.19) is finally proved.

Appendix B

Analytical continuation of the Riemann Zeta function

The Riemann ζ-function is defined

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}\,. \tag{B.1}$$

We can write the following equalities

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{\Gamma(s)} \int_0^{\infty} \mathrm{d}t\, \mathrm{e}^{-nt} t^{s-1} = \frac{1}{\Gamma(s)} \int_0^{\infty} \mathrm{d}t\, t^{s-1} \sum_{n=1}^{\infty} \mathrm{e}^{-nt} = \\ &= \frac{1}{\Gamma(s)} \int_0^{\infty} \mathrm{d}t\, t^{s-1} \frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}}\,.\end{aligned}$$

Now, expanding the exponential, we find

$$\begin{aligned}\zeta(s) &= \frac{1}{\Gamma(s)} \int_0^{\infty} \mathrm{d}t\, t^{s-1} \frac{\mathrm{e}^{-t}}{1-\left(1-t+\frac{t^2}{2}-\frac{t^3}{6}+\dots\right)} = \\ &= \frac{1}{\Gamma(s)} \int_0^{\infty} \mathrm{d}t\, t^{s-1} \frac{\mathrm{e}^{-t}}{t-\frac{t^2}{2}+\frac{t^3}{6}} = \frac{1}{\Gamma(s)} \int_0^{\infty} \mathrm{d}t\, \mathrm{e}^{-t} \frac{t^{s-2}}{1-\left(\frac{t}{2}-\frac{t^2}{6}\right)} = \\ &= \frac{1}{\Gamma(s)} \int_0^{\infty} \mathrm{d}t\, \mathrm{e}^{-t} t^{s-2} \left(1+\frac{t}{2}-\frac{t^2}{6}+\frac{t^2}{4}+\dots\right) = \\ &= \frac{1}{\Gamma(s)} \int_0^{\infty} \mathrm{d}t\, \mathrm{e}^{-t} t^{s-2} \left(1+\frac{t}{2}-\frac{t^2}{12}\right) = \\ &= \frac{1}{\Gamma(s)} \left[\Gamma(s-1)+\frac{1}{2}\Gamma(s)+\frac{1}{12}\Gamma(s+1)\right]\,,\end{aligned}$$

that turns out to be, using the relation $\Gamma(s+1) = s\Gamma(s)$,

$$\zeta(s) = \frac{1}{s+1} + \frac{1}{2} + \frac{s}{12}\,.$$

We are interested in the case $s = -1$, so that we have

$$\zeta(-1) = \sum_{n=1}^{\infty} n = -\frac{1}{12}\,.$$

We can also note, in this case, that the terms of the expansion that we rejected contains a multiplication for the factor $(s+1)$, that vanish for $s = -1$, so this result is exact.

Appendix C

Perturbative solution of DCE

As introducted in Section 3.2, we proceed with a perturbative expansion of $R(ct)$, namely

$$R(\mathrm{ct}) = R_0(\mathrm{ct}) + \varepsilon R_1(\mathrm{ct})\,, \tag{C.1}$$

where ε is the small amptitude. Terms of the same order follow the relations

$$R_0(\mathrm{ct} + L_0) - R_0(\mathrm{ct} - L_0) = 2\,, \tag{C.2}$$

$$R_1(\mathrm{ct} + L_0) - R_1(\mathrm{ct} - L_0) = -L_0 \sin\left(q\frac{2\pi\mathrm{ct}}{L_0}\right)\left[R_0'(\mathrm{ct} + L_0) + R_0'(\mathrm{ct} - L_0)\right]\,, \tag{C.3}$$

as already written.

The general solution of (C.2) is

$$R_0(\mathrm{ct}) = r + \frac{\mathrm{ct}}{L_0} + \sum_{n=1}^{\infty}\left[X_n \cos\left(n\frac{2\pi\mathrm{ct}}{L_0}\right) + Y_n \sin\left(n\frac{2\pi\mathrm{ct}}{L_0}\right)\right]\,, \tag{C.4}$$

where r, X_n and Y_n are constants that we determine thanks to the boundary condition.

If we put this latter solution of $R_0(\mathrm{ct})$ into the equation for the first order perturbation (C.3), we get

$$-\frac{1}{2}\left[R_1(\mathrm{ct} + L_0) - R_1(\mathrm{ct} - L_0)\right] = \sin\left(q\frac{2\pi\mathrm{ct}}{L_0}\right) + \frac{\pi}{2}\sum_{n=1}^{\infty} n(-1)^n \times$$

$$\times \left\{X_n\left[\cos\left((q+n)\frac{2\pi\mathrm{ct}}{L_0}\right) - \cos\left((q-n)\frac{2\pi\mathrm{ct}}{L_0}\right)\right] + \right.$$

$$\left. + Y_n\left[\sin\left((q+n)\frac{2\pi\mathrm{ct}}{L_0}\right) + \sin\left((q-n)\frac{2\pi\mathrm{ct}}{L_0}\right)\right]\right\},$$

whose solution reads

$$R_1(\mathrm{c}t) = (-1)^{q+1}\frac{\mathrm{c}t}{L_0}\Bigg\{\sin\left(q\frac{2\pi \mathrm{c}t}{L_0}\right)+\frac{\pi}{2}\sum_{n=1}^{\infty}n\times$$
$$\times\left(X_n\left[\cos\left((q+n)\frac{2\pi \mathrm{c}t}{L_0}\right)-\cos\left((q-n)\frac{2\pi \mathrm{c}t}{L_0}\right)\right]+\right.$$
$$\left.+Y_n\left[\sin\left((q+n)\frac{2\pi \mathrm{c}t}{L_0}\right)+\sin\left((q-n)\frac{2\pi \mathrm{c}t}{L_0}\right)\right]\right)\Bigg\}+g(\mathrm{c}t)\,, \tag{C.5}$$

with an arbitrary periodic function $g(\mathrm{c}t)$. If we consider only the short time limit, $\varepsilon\frac{\mathrm{c}t}{L_0}<1$, the the boundary condition is satysfied by $R_0(\mathrm{c}t)$, so we need $g(\mathrm{c}t)$ in sucj the way that $R_1(\mathrm{c}t)=0$.
Therefore, our general solution takes place for $r = X_n = Y_n = 0$, and it is the result in (3.32), namely

$$R(\mathrm{c}t) = \frac{\mathrm{c}t}{L_0}+\varepsilon(-1)^{q+1}\left[\frac{\mathrm{c}t}{L_0}\sin\left(q\frac{2\pi \mathrm{c}t}{L_0}\right)-\frac{z}{L_0}\sin\left(q\frac{2\pi z}{L_0}\right)\right]. \tag{C.6}$$

We want to go beyond the short time limit.
We introduce an arbitrary time τ and split the time in $t = t+\tau-\tau$. The perturbative solution is can be written in the same form as already seen, where new parameters τ-depedent $r(\mathrm{c}\tau)$, $X_n(\mathrm{c}\tau)$ and $Y_n(\mathrm{c}\tau)$ replace the previous constants r, X_n and Y_n, respectively.

$$R(\mathrm{c}t) = r(\mathrm{c}\tau)+\frac{\mathrm{c}(t-\tau)}{L_0}+\sum_{n=1}^{\infty}\left[X_n(\mathrm{c}\tau)\cos\left(n\frac{2\pi \mathrm{c}t}{L_0}\right)+Y_n(\mathrm{c}\tau)\sin\left(n\frac{2\pi \mathrm{c}t}{L_0}\right)\right]+$$
$$+\varepsilon(-1)^{q+1}\frac{\mathrm{c}(t-\tau)}{L_0}\Bigg\{\sin\left(q\frac{2\pi \mathrm{c}t}{L_0}\right)+\frac{\pi}{2}\sum_{n=1}^{\infty}n\Bigg(X_n(\mathrm{c}\tau)\Big[\cos\left((q+n)\frac{2\pi \mathrm{c}t}{L_0}\right)+$$
$$-\cos\left((q-n)\frac{2\pi \mathrm{c}t}{L_0}\right)\Big]+Y_n(\mathrm{c}\tau)\left[\sin\left((q+n)\frac{2\pi \mathrm{c}t}{L_0}\right)+\sin\left((q-n)\frac{2\pi \mathrm{c}t}{L_0}\right)\right]\Bigg)\Bigg\}+$$
$$+g(\mathrm{c}t,\mathrm{c}\tau)+\mathcal{O}\left(\varepsilon^2\right)\,,$$

where it is interesting to note that $g(\mathrm{c}t,\mathrm{c}\tau)$ is no more a periodic function.

Imposing the RG equation

$$\left(\frac{\partial R}{\partial(\mathrm{c}\tau)}\right)_{\mathrm{c}t}=0 \tag{C.7}$$

leads, in our case, to three independent equations

$$\frac{\partial r(\mathrm{c}\tau)}{\partial(\mathrm{c}\tau)} = \frac{1}{L_0} + \mathcal{O}\left(\varepsilon^2\right) , \tag{C.8}$$

$$\frac{\partial X_n(\mathrm{c}\tau)}{\partial(\mathrm{c}\tau)} = \varepsilon\frac{\pi(-1)^{q+1}}{2L_0}\left[|n-q|X_{|n-q|} + (n+q)X_{n+q}\right] + \mathcal{O}\left(\varepsilon^2\right) , \tag{C.9}$$

$$\frac{\partial Y_n(\mathrm{c}\tau)}{\partial(\mathrm{c}\tau)} = \varepsilon\frac{(-1)^{q+1}}{L_0}\left[\delta_{nq} + \frac{\pi}{2}\left(|n-q|Y_{|n-q|} - (n+q)Y_{n+q}\right)\right] + \mathcal{O}\left(\varepsilon^2\right) . \tag{C.10}$$

Calling $\tilde{\tau} = \tau\frac{\varepsilon\pi(-1)^{q+1}}{2L_0}$, the equations (C.9) and (C.10) can be written in the form

$$\frac{\partial\tilde{X}_n}{\partial(\mathrm{c}\tilde{\tau})} = (n-q)X_{n-q} + (n+q)\tilde{X}_{n+q} + \mathcal{O}\left(\varepsilon^2\right) , \tag{C.11}$$

$$\frac{\partial Y_n(\mathrm{c}\tau)}{\partial(\mathrm{c}\tau)} = \frac{2}{\pi}\delta_{nq} + (n-q)\tilde{Y}_{n-q} - (n+q)\tilde{Y}_{n+q} + \mathcal{O}\left(\varepsilon^2\right) . \tag{C.12}$$

The solution of (C.8) is not complicated, $r(\mathrm{c}\tau) = \frac{\mathrm{c}\tau}{L_0} + \kappa$, where κ is a constant, that vanishes $\forall t$ imposing our boundary conditions. Thanks to inital conditions $r(0) = \tilde{X}_n(0) = \tilde{Y}_n(0) = 0$, too. In particular, even $\tilde{X}_n(\mathrm{c}t) = 0$ for all t.
The non-vanishing $\tilde{Y}_n$ terms are recovered for $n = qm$, with $m \in \mathbb{N}$, such that $\tilde{Y}_{n<0} = 0$ (which means that the coefficients $\tilde{Y}_n$'s are equal to the orginal Y_n's) and $\tilde{Y}_{qm} = \frac{1}{\pi qm}\tanh^m(q\tilde{\tau})$.

Now, setting $t = \tau$, the RG-improved solution is found. It results

$$R(\mathrm{c}t) = \frac{\mathrm{c}t}{L_0} + \sum_{j=1}^{\infty} Y_{qj}(\mathrm{c}t)\sin\left(qj\frac{2\pi\mathrm{c}t}{L_0}\right) + \varepsilon g(\mathrm{c}t, \mathrm{c}t) . \tag{C.13}$$

The non-periodic function $g(\mathrm{c}t, \mathrm{c}t)$ can be easily evaluated, once defined $\xi = \exp\left[\frac{(-1)^{q+1}q2\pi\varepsilon\mathrm{c}t}{L_0}\right]$,

$$g(\mathrm{c}t, \mathrm{c}t) = (-1)^q\frac{\epsilon z}{L_0}\sin\left(q\frac{2\pi z}{L_0}\right)\left[\frac{2\xi}{1+\xi^2+(1-\xi^2)\cos\left(q\frac{2\pi z}{L_0}\right)}\right] , \tag{C.14}$$

that obviously corresponds to (3.37).

Finally, the solution reads

$$R(\mathrm{c}t) = \frac{\mathrm{c}t}{L_0} - \frac{2}{\pi q}\mathrm{Im}\left\{\ln\left[\frac{1+\xi+(1-\xi)\exp\left(\mathrm{i}q\frac{2\pi \mathrm{C}t}{L_0}\right)}{-(1+\xi)+(1-\xi)\exp\left(-\mathrm{i}q\frac{2\pi \mathrm{C}t}{L_0}\right)}\right]\right\}+$$
$$+(-1)^q\frac{\epsilon z}{L_0}\sin\left(q\frac{2\pi z}{L_0}\right)\left[\frac{2\xi}{1+\xi^2+(1-\xi^2)\cos\left(q\frac{2\pi z}{L_0}\right)}\right], \tag{C.15}$$

that is simply $R(\mathrm{c}t) = R_s(\mathrm{c}t) + R_{np}(\mathrm{c}t)$.

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