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Analysis of the Kohn Laplacian on the Heisenberg Group and on Cauchy–Riemann Manifolds

Tesi di Laurea in Analisi Matematica

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Riflettiamo ora su cos'è la matematica. Di per sé è un sistema astratto, un'invenzione dello spirito umano, che come tale nella sua purezza non esiste. E' sempre realizzato approssimativamente, ma - come tale - è un sistema intellettuale, e una grande, geniale invenzione dello spirito umano. La cosa sorprendente è che questa invenzione della nostra mente umana è veramente la chiave per comprendere la natura, che la natura è realmente strutturata in modo matematico e che la nostra matematica, inventata dal nostro spirito, è realmente lo strumento per poter lavorare con la natura, per metterla al nostro servizio, per strumentalizzarla attraverso la tecnica.

Papa Benedetto XVI
(Colloquio con i giovani di Roma, 6 aprile 2006)

Let us now reflect on what mathematics is: in itself, it is an abstract system, an invention of the human spirit which as such in its purity does not exist. It is always approximated, but as such is an intellectual system, a great, ingenious invention of the human spirit. The surprising thing is that this invention of our human intellect is truly the key to understanding nature, that nature is truly structured in a mathematical way, and that our mathematics, invented by our human mind, is truly the instrument for working with nature, to put it at our service, to use it through technology.

Pope Benedict XVI
(Meeting with youth of Rome, 6 April 2006)

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Introduction

The purpose of this study is to analyse the regularity of a distinguished differential operator, the so-called Kohn Laplacian $\Box_b$, in two settings: on the Heisenberg group $\mathbb{H}_n$ and on manifolds for which $\mathbb{H}_n$ is a model, that is, the strongly pseudo-convex CR manifolds.

Our work begins with the presentation of $\mathbb{H}_n$ and of its properties. The Heisenberg group is defined as $\mathbb{C}^n \times \mathbb{R}$ with the product

$$(z,t) * (z',t') = (z + z', t + t' + 2\,\mathrm{Im}(z\overline{z}'))$$

and it can be seen in two different ways: as a Lie group and as the boundary of the Siegel Upper-Half Space. In this last definition, $\mathbb{H}_n$ is an embedded manifold and $\Box_b$ can be seen as the restriction to $\mathbb{H}_n$ of a differential operator on $\mathbb{C}^{n+1}$. On the other hand, looking at the Lie Group, we can view $\Box_b$ as a sum of squares of the vector fields $\mathrm{Re}\, Z_1, \mathrm{Im}\, Z_1, \ldots, \mathrm{Re}\, Z_n, \mathrm{Im}\, Z_n$, where $Z_1, \ldots, Z_n, T$ form a left-invariant (complex) vector fields basis for the complexified tangent bundle $\mathbb{C}T(\mathbb{H}_n)$.

On $\mathbb{H}_n$ there exists "the CR complex" $\overline{\partial}_b$, defined as

$$\overline{\partial}_b f = \sum_{|J|=q+1}{}' \left(\sum_{|I|=q,k=1,\ldots,n}{}' \epsilon^J_{kI} \overline{Z}_k f_I \right) d\overline{z}^J,$$

where f is a $(0,q)$-form $f = \sum'_{|I|=q} f_I d\overline{z}^I$ and $d\overline{z}^I = d\overline{z}_{i_1} \wedge, \ldots, \wedge d\overline{z}_{i_q}$. We can also define its adjoint $\overline{\partial}_b^*$ with respect to the $L^2(dV)$ inner product, where dV is the Lebesgue measure in $\mathbb{C}^n \times \mathbb{R}$. It can be written as

$$\overline{\partial}_b^* f = \sum_{|J|=q-1}{}' \left(- \sum_{|I|=q,k=1,\ldots,n}{}' \epsilon^I_{kJ} Z_k f_I \right) d\overline{z}^J.$$

Then we can form the operator $\Box_b = \overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b$. It turns out that, on $(0,q)$-forms, $\Box_b$ acts diagonally, that is,

$$\Box_b \left(\sum_{|I|=q}{}' f_I d\overline{z}^I \right) = \sum_{|I|=q}{}' \left(\mathcal{L}_{n-2q} f_I \right) d\overline{z}^I,$$

where $\mathcal{L}_\alpha$ is a scalar differentiable operator of order two:

$$\mathcal{L}_\alpha = -\frac{1}{2} \sum_{k=1,\dots,n} \left(\overline{Z}_k Z_k + Z_k \overline{Z}_k\right) + i\alpha T.$$

Thus, for studying $\Box_b$, it suffices to study $\mathcal{L}_\alpha$. For these operatos we study the local solvability, hypoellipticity and regularity in L^p, Hölder and Sobolev norms.

In order to obtain the main estimates for $\Box_b$ and its inverse, we study the fundamental solution of $\mathcal{L}_\alpha$ on $\mathbb{H}_n$. To investigate its hypoellipticity, again, we first study the solvability and hypoellipticity of $\mathcal{L}_\alpha$, that is true when α is "admissible", namely $\alpha \neq \pm(n+2k)$ $\forall k \in \mathbb{N}$. That enables us to find solvability and hypoellipticity for $\Box_b$. Finally, to be able to state L^p and Hölder estimates for $\mathcal{L}_\alpha$, we take some time to talk about homogeneous distributions on the Heisenberg group.

In the second part we start working mainly with a manifold M of real dimension $2n+1$. We say that M is a Cauchy Riemann manifold (of hypersurface type) if there exists a subbundle, that we denote to be $T^{1,0}(M)$, of the complex tangent bundle $\mathbb{C}T(M)$ such that the complex dimension of $T^{1,0}(M)$ is n, $T^{1,0}(M) \cap \overline{T^{1,0}(M)} = \{0\}$ and where the integrability condition is true. More, we say that a CR manifold M is strongly pseudo-convex if the Levi form defined on M is positive defined.

Since we will show that the Heisenberg group is a model for the strongly pseudo-convex CR manifolds, we study how to extend to them the conclusions we found for $\mathbb{H}_n$. In particular, we look for an osculating Heisenberg structure in a neighborhood of a point in M, and we want this structure to change smoothly from a point to another. In order to do so, we define Normal Coordinates and we study their properties. More, we also examinate different Normal Coordinates in the case of a real hypersurface with an induced CR structure.

In the final part, we define again the tangential CR complex, its adjoint and the $\Box_b$ operator on M. Then we start studying these new operators showing some subelliptic estimates; first for $\overline{\partial}_b$ and $\overline{\partial}_b^*$, then for the $\Box_b$. To find this conclusions, we don't assume M to be pseudo-complex anymore, but we ask less, that is, the $Z(q)$ and the $Y(q)$ conditions on the eigenvalues of the Levi form of the defining function of M. This also provides local regularity theorems for $\Box_b$ and show its hypoellipticity on M.

Chapter 1

Preliminaries

Goal. In this chapter we want to give some definitions and results as a first view at the Cauchy–Riemann world. We will show some versions of the Hartogs Extension Theorem and we will talk about the Levi pseudocovexity in the complex space. Then we will also provide fundamental defintions, such as the notion of the CR manifolds, the CR tangent complex $\overline{\partial}_b$ and of the Levi form for CR manifolds.

1.1 *CR*-operators and Hartogs Theorem

Definition 1.1.1.
Let M be an open set in $\mathbb{C}^n$. Let $p \in M$ and let $U \in \mathcal{U}_p$, i.e, U is a neighborhood of p.
We say that M is a C^k *real hypersurface* in $\mathbb{C}^n$, with $k \in \mathbb{N}$, if $\exists \rho \in C^k(U, \mathbb{R})$ (namely, a *defining function*) such that

$$M \cap U = \{z \in U \ / \rho(z) = 0\} \quad \text{and} \quad d\rho(z) \neq 0 \quad \text{on} \quad M \cap U.$$

Observation 1.1.2. M so defined divides U into $U_+ := \{z \in U \ / \rho(z) > 0\}$ and $U_- := \{z \in U \ / \rho(z) < 0\}$.

Definition 1.1.3.
With the same notations, we now call *Tangential Cauchy–Riemann operator* a (0,1)-vector field on $M, \overline{L}$, such that

$$\overline{L} = \sum_{j=1}^{n} a_j(z) \frac{\partial}{\partial \overline{z}_j} \quad \text{on} \quad M \cap U$$

where a_j's satisfy

$$\sum_{j=1}^{n} a_j(z)\frac{\partial\rho}{\partial\bar{z}_j}(z) = 0, \text{ that is, } \overline{L}\rho(z) = 0.$$

We also call $\overline{L}u = 0$ the *Tangential Cauchy–Riemann equation*.

Notation 1.1.4. Given U an open set, take $\mathcal{O}(U)$ as the set of the holomorphic function on U.

Observation 1.1.5. If $f \in C^1(\overline{U}_-) \cap \mathcal{O}(U_-)$ then, by continuity, $\overline{L}f = 0$ on $M \cap U$.
This tells us that the restiction of a holomorphic function f to a hypersurface will automatically satisfy $\overline{L}f = 0$.

Definition 1.1.6.
Let M be a C^1 hypersurface in $\mathbb{C}^n$, $n \geq 2$.
$f \in C^1(M, \mathbb{C})$ is called a *CR-function* if f satisfies the homogeneous tangential Cauchy–Riemann equation:

$$\sum_{j=1}^{n} a_j \frac{\partial f}{\partial\bar{z}_j}(z) = 0$$

$\forall a = (a_1, \ldots, a_n) \in \mathbb{C}$ with $\sum_{j=1}^{n} a_j \frac{\partial\rho}{\partial\bar{z}_j}(z) = 0$, $z \in M$, and where ρ is a C^1 defining function for M.

Observation 1.1.7. From observation 1.1.5, we can now say that the restiction of a holomorphic function f to a hypersurface is a CR-function.

Is the opposite true? Namely, Given any CR-function f on M, can one extend f holomorphically into one side of M? In general the answer is no.

Example 1.1.8. Let M be the hypersurface defined by $\{y_1 = 0\}$ in $\mathbb{C}^n$:

$$M := \{z \in \mathbb{C}^n \ / \ y_1 = 0\}.$$

Consider $f \in C^\infty(M, \mathbb{R})$, $f(x_1, z_2, \ldots, z_n) = f(x_1)$ in $U \in \mathcal{U}_0$. Suppose $f(x_1)$ is not real analytic at the origin and note that, by hypothesis, $\frac{\partial f}{\partial\bar{z}_j} = 0$ for $j = 2, \ldots, n$.
Then f is a CR manifold on M, still f can't be holomorphically extended to some neighborhood of the origin, or to just one side of the hypersurface M.

Now we give a result about the so-called *inhomogeneous Cauchy–Riemann equation* $\overline{\partial}u = f$ and then about the extension theorems. The basic $\overline{\partial}$ in $\mathbb{C}^n$ is defined in 1.5[1].

Theorem 1.1.9.
Let $f = \sum_{j=1}^{n} f_j d\overline{z}_j$ and $f_j \in C_0^k(\mathbb{C}^n)$ with $n \geq 2,\ j = 1, \dots, n$ and $k \geq 1$ such that the *compatibility conditions* are satisfied, namely

$$\frac{\partial f_j}{\partial \overline{z}_i} = \frac{\partial f_i}{\partial \overline{z}_j} \quad \forall i, j : 1 \leq i < j \leq n.$$

Then there exists a function $u \in C_0^k(\mathbb{C}^n)$ such that $\overline{\partial} u = f$.

Proof. Set

$$u(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f_1(\zeta, z_2, \dots, z_n)}{\zeta - z_1} d\zeta \wedge d\overline{\zeta} = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f_1(\zeta + z_1, z_2, \dots, z_n)}{\zeta} d\zeta \wedge d\overline{\zeta}.$$

Then, from differentation under the integral sign, $u \in C^k(\mathbb{C}^n)$. And since f is compactly supported, $u(z) = 0$ when $|z_2| + \cdots + |z_n|$ is sufficiently large. By the properties of the Cauchy Integral Formula (2.1.2[1]), we have

$$\frac{\partial u}{\partial \overline{z}_1}(z) = f_1(z).$$

Using again the Cauchy Integral Formula (2.1.1[1]), we have

$$f_j(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\frac{\partial f_j}{\partial \overline{\zeta}}(\zeta, z_2, \dots, z_n)}{\zeta - z_1} d\zeta \wedge d\overline{\zeta}$$

and, on the other hand,

$$\frac{\partial u}{\partial \overline{z}_j}(z) = \frac{\partial}{\partial \overline{z}_j} \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f_1(\zeta, z_2, \dots, z_n)}{\zeta - z_1} d\zeta \wedge d\overline{\zeta} = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\frac{\partial f_1}{\partial \overline{z}_j}(\zeta, z_2, \dots, z_n)}{\zeta - z_1} d\zeta \wedge d\overline{\zeta}.$$

Using the compatibilty condition for $j > 1$, $\frac{\partial f_1}{\partial \overline{z}_j} = \frac{\partial f_j}{\partial \overline{z}_1}$, we obtain

$$\frac{\partial u}{\partial \overline{z}_j}(z) = f_j(z) \quad \forall j = 1, \dots, n.$$

Hence $\partial u = f$.
In particular, u is holomorphic on an unbounded component of the complement of the support of f (by definition of holomorphy). Since we already know that $u \equiv 0$ when $|z_2| + \cdots + |z_n|$ is sufficiently large, then u must be zero on the unbounded component of $(\mathrm{supp} f)^c$ (Identity Theorem 2.1.10[1]). Thus u is compactly supported and that proves the theorem. □

Theorem 1.1.10 (Hartogs Extension Theorem)**.**
Let D be a bounded domain in $\mathbb{C}^n$, $n \geq 2$, and $K \subset\subset D$ such that $D \setminus K$ is connected.
Then $f \in \mathcal{O}(D \setminus K) \Longrightarrow f \in \mathcal{O}(D)$.

Notation 1.1.11. By $f \in C^k_{(p,q)}(D)$ we mean that f is a (p,q)-form in D with C^k coefficients.

Proof. Let $\chi \in C_0^\infty(D)$ be a cut-off function such that $\chi \equiv 1$ in some neighborhood of K. Then we have that $-f(\overline{\partial}\chi) \in C^\infty_{(0,1)}(\mathbb{C}^n)$. This function has compact support and satisfies the compatibility condition. Thus, by the previous theorem, $\exists u \in C_0^\infty(\mathbb{C}^n)$ such that $\overline{\partial}u = -f(\overline{\partial}\chi)$ and $u = 0$ in some open neighborhood of $\mathbb{C}^n \setminus D$.
We now define $F := (1-\chi)f - u$. We see that F is the extension of f and it's holomorphic:

$$\overline{\partial}F = \overline{\partial}f \cdot (1-\chi) - f\overline{\partial}\chi - \overline{\partial}u = \overline{\partial}f \cdot (1-\chi) = 0.$$

That proves the theorem. □

Another version of the same theorem is:

Theorem 1.1.12.
Let $f \in C(D)$, $D \in \mathbb{C}^n$, $n \geq 2$, D domain. Let S be a smooth real hypersurface in $\mathbb{C}^n$.
Then $f \in \mathcal{O}(D \setminus S) \Longrightarrow f \in \mathcal{O}(D)$. Namely, f can be extended holomorphically to D.

Proof. In order to prove this statement, it would be enough to show that f is holomorphic near each $p \in D \cap S$. We will do it assuming p to be the origin. In this case, we can write S as a graph, $S = \{z \in \mathbb{C}^n /\ y_1 = \phi(x_1, z')\}$ where $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $z_j = x_j + iy_j$, for $j = 1, \ldots, n$, and ϕ is a smooth function such that $\phi(0) = 0$ and $d\phi(0) = 0$.
Hence $\forall \beta > 0 \;\; \exists \delta_\beta > 0$ and exists a polydisc U_β in $\mathbb{C}^{n-1}$ centered at the origin such that $|\phi(x_1, z')| < \beta \quad \forall x \in \mathbb{R} : |x| < \delta_\beta$ and $\forall z' \in U_\beta$.
Let now $\beta_1 > 0$ sufficiently small and $\beta_2 > \beta_1$ sufficiently close to β_1 such that we can assume $\{|x_1| < \delta_{\beta_1},\ \beta_1 < y_1 < \beta_2\} \times U_{\beta_1} \equiv V_0 \times U_{\beta_1} \subseteq D \setminus S$ (x_1 and z' can approch zero while y_1 can't, so the set is not too close to S).
Thus, by hypothesis, $f \in \mathcal{O}(V_0 \times U_{\beta_1})$.
Next, if we fix $z' \in U_{\beta_1}$, $f(z_1, z')$ is continuous on $V = \{z_1 \in \mathbb{C}/\ |x_1| < \delta_{\beta_1},\ |y_1| < \beta_2\}$ and holomorphic on V except for the smooth curve $\{z \in \mathbb{C}/\ y_1 = \phi(x_1, z')\}$.
Then, by Morera's Theorem in $\mathbb{C}$, $f(z_1, z')$ is holomorphic on $V \subset \mathbb{C}$.

Now we choose a contour of integration Γ in U_{β_1}, $\Gamma = \Gamma_2 \times \cdots \times \Gamma_n$, where $\Gamma = \{z_j \in \mathbb{C}/\ |z_j| = r_j\}$ for $j = 2, \ldots, n$ and such that $\Gamma \subset U_{\beta_1}$.
Now we define $F : \mathbb{C}^n \to \mathbb{C}$:

$$F(z_1, z') = \frac{1}{(2\pi i)^{n-1}} \int_\Gamma \frac{f(z_1, \zeta')}{(\zeta_2 - z_2) \cdots (\zeta_n - z_n)} d\zeta_2 \cdots d\zeta_n.$$

If, for every $j = 2, \ldots, n$, we call $D_j = \{z_j \in \mathbb{C}/|z_j| < r_j\}$ and $U = D_2 \times \cdots \times D_n$, then F is holomorphic on $V \times U$. More, for $(z_1, z') \in V_0 \times U$, $F(z_1, z') = f(z_1, z')$ (Cauchy Integral Formula for Polydiscs, 2.1.7[1]).
Then, by the Identity Theorem (2.1.10[1]), f is holomorphic on $V \times U$ and that completes the proof. □

Without giving the proofs, for whose we refer to 3.2[1], here we state a generalized version of the Hartogs Theorem.

Lemma 1.1.13. Let M be a hypersurface and r its C^k defining function. Let f be a CR-function of class C^k on M.
Then f can be extended to a C^{k-1} function $\tilde{f}$ in some open neighborhood on M such that $\overline{\partial}\tilde{f} = 0$ on M.

Theorem 1.1.14.
Let D be a bounded domain in $\mathbb{C}^n$, $n \geq 2$, with connected C^1 boundary. Let f be a CR-function of class C^1 on ∂D.
Then $\forall \epsilon > 0$ small, f extends holomorphically to a function $F \in C^{1-\epsilon}(\overline{D}) \cap \mathcal{O}(D)$ such that $F_{|\partial D} = f$.

1.2 Levi form and Levi Pseudoconvexity in $\mathbb{C}^n$

We will now see how a local one-side extension is related to the Levi Form of the Domain. Then we will define the pseudoconvexity in $\mathbb{C}^n$ and we are now going to state some results. However, since we'll not use them in the future, we'll not provide complete proofs of them, but the proofs are avaiable in 3.3 and 3.4[1].

Definition 1.2.1.
Let D be a bounded domain $\subseteq \mathbb{C}^n$, $n \geq 2$, and r its C^2 defining function. Let $p \in \partial D$. The Hermitian form:

$$\mathcal{L}_p(r, t) := \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}(p) t_j \overline{t_k},$$

defined $\forall t = (t_1, \dots, t_n) \in \mathbb{C}^n$ with $\sum_{j=1}^n \frac{\partial r}{\partial z_j}(p) t_j = 0$, is called the *Levi form of the function* r *at the point* p.

Observation 1.2.2. The Levi form is independent of the defining function up to a positive factor.
Then the number of positive or negative eigenvalues of the Levi form is independet of the choice of the defining function.

Proof. Let ρ be another C^2 defining function for D. Then $\rho = hr$, with $h \in C^1(\partial D)$, $h > 0$.
Hence

$$\mathcal{L}_p(\rho, t) = \sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p) t_j \overline{t_k} =$$

$$\sum_{j,k=1}^n \frac{\partial r}{\partial z_j}(p) \frac{\partial h}{\partial \overline{z}_k}(p) t_j \overline{t_k} + \sum_{j,k=1}^n \frac{\partial h}{\partial z_j}(p) \frac{\partial r}{\partial \overline{z}_k}(p) t_j \overline{t_k} + h(p) \sum_{j,k=1}^n \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}(p) t_j \overline{t_k} =,$$

since the first and the second sum are exactly the same,

$$= h(p) \sum_{j,k=1}^n \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}(p) t_j \overline{t_k} = h(p) \mathcal{L}_p(r, t).$$

□

Notation 1.2.3. For $p \in \partial D$ we call

$$T_p^{1,0}(\partial D) := \left\{ t \in \mathbb{C}^n \ / \ \sum_{j=1}^n \frac{\partial r}{\partial z_j}(p) t_j = 0 \right\},$$

that is the space of $(1, 0)$ vector fields which are tangent to ∂D at p.

Theorem 1.2.4 (Local, one-side, Extension Theorem for CR-functions)**.** Let M be a hypersurface, $p \in M$, r be a C^2 defining function for M in $U \in \mathcal{U}_p$. Let $\mathcal{L}_p(r, t) < 0$ for some $t \in T_p^{1,0}(M)$.
Then $\exists\, U' \subset U$ such that $\forall f \in C^2(M \cap U')$, f CR-function, $\exists\, F \in C^0(\overline{U'}_+)$, where $\overline{U'}_+ = \{z \in U' \ / \ r(z) \geq 0\}$, such that

$$\begin{cases} F_{|_{M \cap U'}} = f \\ \overline{\partial} F = 0 \quad \text{on} \quad U'_+ = \{z \in U' \ / \ r(z) > 0\} \end{cases}$$

Proof. By assuming $p = 0$ and via three changes of variables near it, we can write the Taylor expansion of r at 0 as:

$$r(z_1, 0, \ldots, 0) = \frac{\partial^2 r}{\partial z_1 \partial \overline{z}_1}(0)|z_1|^2 + O(|z_1|^3).$$

That will help us define a suitable set U' (and then U'_- and U'_+) and, using lemma 1.1.13, find the right extention. □

Observation 1.2.5. If $\mathcal{L}_p(r,t)$ has eigenvalues of opposite signs, then f can be extended holomorfocally to both sides (say F_+ and F_- respectively) such that $F_{+|_M} = F_{-|_M} = f$. Hence, F_+ and F_- can be patched together to form a holomorphic function defined in some open neighborhood of p.

Definition 1.2.6.
Let D be a bounded domain in $\mathbb{C}^n, n \geq 2$, $r \in C^2$ his defining function.

- D is called *pseudoconvex*, or *Levi pseudoconvex, at* $p \in \partial D$ if:

$$\mathcal{L}_p(r,t) \geq 0 \ \forall t \in T^{1,0}(\partial D).$$

- D is *strictly* (or *strongly*) *pseudoconvex at* p if

$$\mathcal{L}_p(r,t) > 0 \quad \forall t \in T^{1,0}(\partial D),\ t \neq 0.$$

- D is a *(strictly) pseudoconvex domain* if D is (strictly) pseudoconvex $\forall p \in \partial D$.

Definition 1.2.7.
Let D be an open set in $\mathbb{C}^n, n \geq 2$. $\phi : D \to [-\infty, +\infty)$ is *pluriharmonic* if

1. ϕ is *upper semicontinuous*, i.e., $\limsup_{z \to z_0} \phi(z) \leq \phi(z_0)$ with $z, z_0 \in D$.
2. $\forall z \in D, \forall w \in \mathbb{C}^n$, whenever $\{z + \tau w /\ \tau \in \mathbb{C}\} \subset D$, $\phi(z + \tau w)$ is *subharmonic* in τ, i.e., ϕ is continuous and the integral mean inequality holds.

Theorem 1.2.8 (Characterization theorem)**.**
Let D be a bounded domain in $\mathbb{C}^n, n \geq 2$.
$\phi : D \to \mathbb{R},\ \phi \in C^2$. ϕ is (strictly) plurisubharmonic $\Leftrightarrow$

$$\sum_{j,k=1}^{n} \frac{\partial^2 \phi}{\partial z_j \partial \overline{z}_k}(z) t_j \overline{t_k} \geq 0\ (> 0) \qquad \forall t \in \mathbb{C}^n,\ \forall z \in D$$

Theorem 1.2.9.
Let D be a bounded strongly pseudoconvex domain in $\mathbb{C}^n, n \geq 2$ with $r \in C^k$ his defining function, $k \geq 2$.
Then there exists a C^k strictly plurisubharmonic defining function for D.

Corollary 1.2.10. Let D be a bounded pseudoconvex domain with C^2 boundary in $\mathbb{C}^n, n \geq 2$.
D is strongly pseudoconvex $\Leftrightarrow$ D is locally biholomorphically equivalent to a strictly convex domain near every boundary point.

1.3 *CR*-manifolds

In this paragraph we give the definition of CR manifold, that is one of the main objects that we'll study in this paper. Once we describe the space, we'll be able to work on it and we'll start doing so with the definition of tangential CR complex, in the next paragraph.

Definition 1.3.1.
Let M be a real smooth manifold of dimension $2n+1$, $n \geq 2$.
We denote its *tangent bundle* as

$$T(M) := \bigcup_{x \in M} T_x(M),$$

where $T_x(M)$ is a tangent space of $x \in M$.
Then we also denote its *complex tangent bundle* as

$$\mathbb{C}T(M) := T(M) \otimes_{\mathbb{R}} \mathbb{C}.$$

Observation 1.3.2. Since $dim_{\mathbb{R}} M = 2n+1$, then also $dim_{\mathbb{R}} T_p(M) = 2n+1$. Locally we choose a chart, so we can write a basis of $T_p(M)$ as

$$\left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n} \right\}.$$

More, we can write

$$T(M) = \{(p, X)/\ p \in M,\ \ X \in T_p(M)\}$$

and

$$\mathbb{C}T(M) = \{(p, X) \otimes a + (q, Y) \otimes ib/\ a, b \in \mathbb{R},\ (p, X), (q, Y) \in T(M)\},$$

whose real-dimension is $4n+2$ (complex-dimension is $2n+1$).

Definition 1.3.3. Let M be a real smooth manifold of dimension $2n+1$, $n \geq 1$, and let $T^{1,0}(M)$ be a subbundle of $\mathbb{C}T(M)$.
Let us take $U \subset M$, open. We say that

$$\Gamma(U, T^{1,0}(M))$$

is the space of all smooth sections of $T^{1,0}(M)$ over U.

Definition 1.3.4 (Cauchy–Riemann Manifold)**.**
Let M be a real smooth manifold of dimension $2n+1$, $n \geq 1$ and let $T^{1,0}(M)$ be a subbundle of $\mathbb{C}T(M)$.
$(M, T^{1,0}(M))$ is a *Cauchy–Riemann manifold* with the *CR-structure* $T^{1,0}(M)$ if:

1. $dim_{\mathbb{C}} T^{1,0}(M) = n$
2. $T^{1,0}(M) \cap T^{0,1}(M) = \{0\}$ where $T^{0,1}(M) = \overline{T^{1,0}(M)}$
3. the *integrability condition* stands

where the integrability condition says that

$$\forall X_1, X_2 \in \Gamma(U, T^{(1,0)}(M)) \Rightarrow [X_1, X_2] \in \Gamma(U, T^{1,0}(M)).$$

Observation 1.3.5. One can note that in the case $n = 1$, the third condition in void.

Example 1.3.6. The most natural CR-manifold are those defined by smooth hypersurface in $\mathbb{C}^n$.
Let $\rho : \mathbb{C}^n \to \mathbb{R}$ smooth, with $d\rho \neq 0$ on $M = \{z \in \mathbb{C}^n / \rho(z) = 0\}$. Then M is a smooth manifold with $dim_{\mathbb{R}} = 2n+1$.
Take $T^{1,0}(\mathbb{C}^n)$ as a subbundle of $\mathbb{C}T(\mathbb{C}^n)$; if we define $T^{1,0}(M) := T^{1,0}(\mathbb{C}^n) \cap \mathbb{C}T(M)$, subbundle of $\mathbb{C}T(M)$, then $(M, T^{1,0}(M))$ is a CR manifold with the CR structure induced from $\mathbb{C}^n$.

Definition 1.3.7.
Let $(M, T^{1,0}(M))$ and $(N, T^{1,0}(N))$ be two CR manifolds. Let $\varphi : M \to N$ be a smooth function and $\varphi_* : T^{1,0}(M) \to T^{1,0}(N)$ its pushforward operator.

- We say that φ is a *CR mapping* if $\varphi_* L$ is a smooth section of $T^{1,0}(N)$ $\forall L$ smooth section of $T^{1,0}(M)$.
- If φ^{-1} exists and is also a CR mapping, then we say that $(M, T^{1,0}(M))$ is *CR diffeomorphic to* $(N, T^{1,0}(N))$.

Lemma 1.3.8. Let $(M, T^{1,0}(M))$ be a CR manifold and N a manifold.
If $\varphi : M \to N$ is diffeomorphic, then φ induces a CR structure on N: $T^{1,0}(N) := \varphi_* T^{1,0}(M)$.
Therefore φ becomes a CR diffeomorphism.

Proof. The first and second point of the definition of CR manifold are easly satisfied. To check the compatibiliy condition is also easy. Indeed:
if $X_1, X_2 \in T^{1,0}(M)$, then $[X_1, X_2] \in T^{1,0}(M)$ and $\varphi_* X_1, \varphi_* X_2 \in T^{1,0}(N)$.
So $[\varphi_* X_1, \varphi_* X_2] = \varphi_*[X_1, X_2] \in T^{1,0}(N)$ by definition of φ_*. That proves the lemma. □

Definition 1.3.9.
A smooth function g defined on a CR manifold $(M, T^{1,0}(M))$ is called a *CR function* if $\overline{L}g = 0 \ \ \forall \overline{L}$ smooth sections in $T^{0,1}(M)$.

1.4 The tangential *CR* complex, $\overline{\partial}_b$

We now want to discuss the definition of the tangential CR complex $\overline{\partial}_b$.
There are two ways to start:

- if M is a hypersurface sitting in $\mathbb{C}^{n+1}$, $\overline{\partial}_b$ can be defined extrinsically via the ambient complex structure $\overline{\partial}$.
- if M is a CR manifold, we can define $\overline{\partial}_b$ intrinsically without referring to the ambient space.

1.4.1 Extrinsic approach

Notation 1.4.1. Let M be a smooth hypersurface on $\mathbb{C}^{n+1}$ and r a defining function for M.
We take U open neighborhood of M and $I^{p,q}$ ideal in $\Lambda^{p,q}(\mathbb{C}^{n+1})$, $0 \leq p, q \leq, n$, s.t $\forall z \in U$, the fiber $I^{p,q}_z$ is generated by r and $\overline{\partial}r$, i.e., $I^{p,q}_z = \{rH_1 + \overline{\partial}r \wedge H_2\}$, where H_1 is a smooth (p, q)-form and H_2 a smooth $(p, q-1)$-form.

Definition 1.4.2.
As we denote by $\Lambda^{p,q}(\mathbb{C}^{n+1})_{|_M}$ and $I^{p,q}{}_{|_M}$ the restrictions of $\Lambda^{p,q}(\mathbb{C}^{n+1})$ and $I^{p,q}$ respectively to M, we can define

$$\Lambda^{p,q}(M) := \text{ the orthogonal complement of } I^{p,q}{}_{|_M} \text{ in } \Lambda^{p,q}(\mathbb{C}^{n+1})_{|_M}$$

and

$$\mathcal{E}^{p,q}(M) := \text{ space of smooth sections of } \Lambda^{p,q}(M) \text{ over } M$$

i.e. $\mathcal{E}^{p,q}(M) = \Gamma(M, \Lambda^{p,q}(M))$.
Before heading on, we give another definition that we will use soon enough.
We define a map
$$\tau : \Lambda^{p,q}(\mathbb{C}^{n+1}) \to \Lambda^{p,q}(M)$$
by restricting a (p,q)-form ϕ in $\mathbb{C}^{n+1}$ to M, then projecting the restriction to $\Lambda^{p,q}(M)$.

Observation 1.4.3.

- Note that, with this definition $\Lambda^{p,q}(M)$ is not intrinsic on M.
- One can also note that $\mathcal{E}^{p,n}(M) = \{0\}$.

Definition 1.4.4.
Now we can define the *tangential Cauchy–Riemann operator* as
$$\overline{\partial}_b : \mathcal{E}^{p,q}(M) \to \mathcal{E}^{p,q+1}(M)$$
$$\phi \mapsto \tau\overline{\partial}\phi_1$$
where ϕ_1 is a (p,q)-form in $\mathbb{C}^{n+1}$ such that $\tau\phi_1 = \phi$.
So we can write
$$\overline{\partial}_b\phi = \overline{\partial}_b\tau\phi_1 := \tau\overline{\partial}\phi_1.$$

Observation 1.4.5. It's good to note that the definition is indipendent by the choice of ϕ_1.

Proof. Let ϕ_1, ϕ_2 be (p,q)-forms in $\mathbb{C}^{n+1}$ such that $\tau\phi_1 = \phi$ and $\tau\phi_2 = \phi$. Then $\phi_1 - \phi_2 = rg + \overline{\partial}r \wedge h$, for some (p,q)-form g and $(p,q-1)$-form h and $\overline{\partial}(\phi_1-\phi_2) = r\overline{\partial}g + \overline{\partial}r\wedge g - \overline{\partial}r\wedge\overline{\partial}h$. Hence, by the definition of τ, $\tau\overline{\partial}(\phi_1-\phi_2) = 0$ and this completes the proof. □

Observation 1.4.6. As a final observation we see that the followings hold

- $\overline{\partial}_b^2 = 0$
- $0 \to \mathcal{E}^{p,0}(M) \xrightarrow{\overline{\partial}_b} \mathcal{E}^{p,1}(M) \xrightarrow{\overline{\partial}_b} \dots \xrightarrow{\overline{\partial}_b} \mathcal{E}^{p,n-1}(M) \to 0$

1.4.2 Intrinsic approach

Notation 1.4.7. Let now $(M, T^{1,0}(M))$ be an orientable CR manifold with $dim_{\mathbb{R}} = 2n+1,\ n \geq 1$. Note that a real smooth manifold is said *orientable* if there exists a non-vanishing top degree form on it.
Here we assume M to be equipped with a Hermitian metric $<,>$ on $\mathbb{C}T(M)$ such that $T^{1,0}(M)$ orthogonal to $T^{0,1}(M)$.

Definition 1.4.8.
With these notations, we define:

$$\eta(M) := \text{ orthogonal complement of } T^{1,0}(M) \oplus T^{0,1}(M).$$

It comes immediately that $\eta(M)$ is a line bundle over M.
We also denote:

$$T^{1,0}(M)^* := \text{ dual bundle of } T^{1,0}(M)$$

and

$$T^{0,1}(M)^* := \text{ dual bundle of } T^{0,1}(M).$$

Observation 1.4.9. By definition, it means that forms in $T^{1,0}(M)^*$ annihilate vectors in $T^{0,1}(M) \oplus \eta(M)$.

Definition 1.4.10.
Now, taken $0 \le p, q \le n$, we define

$$\Lambda^{p,q}(M) := \Lambda^p(T^{1,0}(M)^*) \otimes \Lambda^q(T^{1,0}(M)^*).$$

$\Lambda^{p,q}(M)$ can be identified with a subbundle of $\Lambda^{p,q}\mathbb{C}(T(M))^*$.

Observation 1.4.11. According to this definition, note that $\Lambda^{p,q}(M)$ is intrinsic to M (different from the extrinsic approach).

Definition 1.4.12.
Exactly as in the extrinsic case, we also define

$$\mathcal{E}^{p,q}(M) := \text{ space of smooth sections of } \Lambda^{p,q}(M) \text{ over } M,$$

i.e. $\mathcal{E}^{p,q}(M) = \Gamma(M, \Lambda^{p,q}(M))$.

Definition 1.4.13.
Now we define the *tangential Cauchy–Riemann operator* $\overline{\partial}_b$,

$$\overline{\partial}_b : \mathcal{E}^{p,q}(M) \to \mathcal{E}^{p,q+1}(M)$$

as follows.

- If $\phi \in \mathcal{E}^{p,0}(M)$, then

  $$< \overline{\partial}_b \phi, (V_1 \wedge \cdots \wedge V_p) \otimes \overline{L} >= \overline{L} < \phi, V_1 \wedge \cdots \wedge V_p >$$

  $\forall\ V_1, \ldots, V_p \in T^{1,0}(M)$, $\overline{L} \in T^{0,1}(M)$.

- Then $\overline{\partial}_b$ is extended to $\mathcal{E}^{p,q}(M)$, $q > 0$, as a derivation. Namely, if $\phi \in \mathcal{E}^{p,q}(M)$, then

$$< \overline{\partial}_b \phi, (V_1 \wedge \cdots \wedge V_p) \otimes (\overline{L}_1 \wedge \cdots \wedge \overline{L}_{q+1}) >=$$

$$= \frac{1}{q+1} \left\{ \sum_{j=1}^{q+1} (-1)^{j+1} \overline{L}_j < \phi, (V_1 \wedge \cdots \wedge V_p) \otimes (\overline{L}_1 \wedge \cdots \wedge \widehat{\overline{L}_j} \wedge \cdots \wedge \overline{L}_{q+1}) > \right\} +$$

$$+ \frac{1}{q+1} \left\{ \sum_{i<j} (-1)^{i+j} < \phi, (V_1 \wedge \cdots \wedge V_p) \otimes ([\overline{L}_i, \overline{L}_j] \wedge \overline{L}_1 \wedge \cdots \wedge \widehat{\overline{L}_i} \cdots \wedge \widehat{\overline{L}_j} \wedge \cdots \wedge \overline{L}_{q+1}) > \right\}$$

Observation 1.4.14. If we define the projection

$$\pi_{p,q} : \Lambda^{p,q}\mathbb{C}(T(M))^* \hookrightarrow \Lambda^{p,q}(M),$$

then $\overline{\partial}_b = \pi_{p,q} \circ d$, where d is the exterior derivative of M.

Observation 1.4.15. Again we can say that the followings hold

- $\overline{\partial}_b^2 = 0$
- $0 \to \mathcal{E}^{p,0}(M) \xrightarrow{\overline{\partial}_b} \mathcal{E}^{p,1}(M) \xrightarrow{\overline{\partial}_b} \ldots \xrightarrow{\overline{\partial}_b} \mathcal{E}^{p,n-1}(M) \to 0$

Observation 1.4.16. Note that p plays no role in the definition of $\overline{\partial}_b$. So it suffices to consider the action of $\overline{\partial}_b$ on $(0,q)$-forms, $0 \leq q \leq n-1$.
As final observation, when $(M, T^{1,0}(M))$ is embedded as a smooth hypersurface in $\mathbb{C}^{n+1}$ with the CR structure $T^{1,0}(M)$ induced from the ambient space, then $\overline{\partial}_b$ can be defined both ways and the definition are isomorphic.

Observation 1.4.17. We can now consider the inhomogeneous $\overline{\partial}_b$ equation

$$\overline{\partial}_b u = f$$

where u is a $(0,q)$-form and f is a $(0,q+1)$-form. Since $\overline{\partial}_b^2 = 0$, it's necessary that $\overline{\partial}_b f = 0$, that is called compatibily condition.
This problem is not easy since we change the space from $\mathcal{E}^{0,q}(M)$ to $\mathcal{E}^{0,q+1}(M)$ and we have to satisfy the compatibility condition.
We'll give a solution of this problem in observation 3.3.5.

1.5 Levi Form and Levi Pseudoconvexity for CR manifolds

In this paragraph, finally, we will talk about pseudoconvexs CR manifolds. These are the manifold that can resemble the Heisenberg group, as we will see in the next chapter.

Notation 1.5.1. Let $(M, T^{1,0}(M))$ be an orientable CR manifold with $dim_{\mathbb{R}} M = 2n+1,\ n \geq 1$.
Let $L_1, \ldots, L_n$ be a local basis for smooth sections of $T^{1,0}(M)$ over $U \subset M$, U open. Then $\overline{L}_1, \ldots, \overline{L}_n$ is a local basis for $T^{0,1}(M)$ over U.
We now choose a local section T of $\mathbb{C}T(M)$ such that $L_1, \ldots, L_n, \overline{L}_1, \ldots, \overline{L}_n, T$ span $\mathbb{C}T(M)$ over U (we assume T is purely immaginary).
Then we can write $\mathbb{C}T(M) =< L_1, \ldots, L_n, \overline{L}_1, \ldots, \overline{L}_n, T >$.

Definition 1.5.2.
We say that the Hermitian matrix $(c_{ij})_{i,j=1,\ldots,n}$ defined by the condition

$$[L_i, \overline{L}_i] = c_{ij} T \text{ mod } T^{1,0}(M) \oplus T^{0,1}(M)$$

is called the *Levi form associated with the given CR structure.*

Observation 1.5.3. Chosen $p \in M$ and a system of local coordinates $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}$, note that $c_{ij}(p) = c_p(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j})$ where c_p is an Hermitian inner product in $T_p^{1,0}(M)$.

Observation 1.5.4. The number of not-zero eigenvalues and $|sign(c_{ij})_{i,j}|$ (absolute value of the signature of the matrix) are independent of the choice of $L_1, \ldots, L_n, \overline{L}_1, \ldots, \overline{L}_n$ and T.
Then, after eventually changing T to $-T$, it makes sense to consider whether $(c_{ij})_{i,j}$ is positive definite.

Definition 1.5.5.

- The CR structure is called *(strictly) pseudoconvex at $p \in M$* if the matrix $(c_{ij}(p))_{i,j}$ is positive (definite) semidefinite after an appropiate choice of T.
- If the CR structure is (strictly) pseudoconvex at every point of M, then M is called a *(strictly) pseudoconvex CR manifold.*
- If the Levi form vanishes completely on an open set $U \subset M$, i.e. $c_{ij} = 0$ on U for $1 \leq i, j \leq n$, then M is called *Levi flat.*

Theorem 1.5.6.
Let $D \subset \mathbb{C}^{n+1}$, $n \geq 1$, a bounded domain with C^∞ boundary.
Then

- D is (strictly) pseudoconvex $\Leftrightarrow$ $M := \partial D$ is a (strictly) pseudoconvex CR manifold.
- locally, a CR manifold in $\mathbb{C}^{n+1}$ is pseudoconvex $\Leftrightarrow$ it is the boundary of a smooth pseudoconvex domain from one side.

Proof. Let r be a defining funcion for D and $p \in \partial D$. We can assume $\frac{\partial r}{\partial z_n}(p) \neq 0$, hence we define

$$L_k := \frac{\partial r}{\partial z_n}\frac{\partial}{\partial z_k} - \frac{\partial r}{\partial z_k}\frac{\partial}{\partial z_n}, \quad \text{for } k = 1, \dots, n$$

Then $L_1, \dots, L_n$ is a local basis for the tangential $(1,0)$ vector fields near p on the boundary.
If now

$$L = \sum_{j=1}^{n+1} a_j \frac{\partial}{\partial z_j}$$

is another tangential $(1,0)$ vector fields near p, then

$$L(r) = \sum_{j=1}^{n+1} a_j \frac{\partial r}{\partial z_j} = 0.$$

We can rewrite it as

$$\sum_{j=1}^{n} a_j \frac{\partial r}{\partial z_j} = -a_{n+1}\frac{\partial r}{\partial z_{n+1}}. \qquad (*)$$

Moreover, we can easily see that

$$L = \left(\frac{\partial r}{\partial z_{n+1}}\right)\sum_{j=1}^{n} a_j L_j;$$

it is done writing down this expression and using the previous equality in this form .
Now we set $\eta = \partial r - \overline{\partial} r = \sum_{j=1}^{n}(\frac{\partial r}{\partial z_j} - \frac{\partial r}{\partial \overline{z}_j})$ and we compute:

$$\sum_{i,j=1}^{n} c_{ij} a_i \overline{a}_j =$$

using the definition of c_{ij} given in 1.5.2,

$$= \sum_{i,j=1}^{n} < \eta, [L_i, \overline{L}_j] > a_i \overline{a}_j =$$

$$= \sum_{i,j=1}^{n} (L_i < \eta, \overline{L}_j > - L_j < \eta, \overline{L}_i > -2 < d\eta, L_i \wedge \overline{L}_j >) a_i \overline{a}_j =$$

since the first two terms cancel

$$= \sum_{i,j=1}^{n} 4 < \partial\overline{\partial} r, L_i \wedge \overline{L}_j > a_i \overline{a}_j =$$

and by $(*)$

$$= 4 \left| \frac{\partial r}{\partial z_{n+1}} \right|^2 < \partial\overline{\partial} r, L \wedge \overline{L} >= 4 \left| \frac{\partial r}{\partial z_{n+1}} \right|^2 \sum_{i,j=1}^{n+1} \frac{\partial^2 r}{\partial z_i \partial \overline{z}_j} a_i \overline{a}_j .$$

Finally, looking at the two definitions, it's now clear that one implies the other. □

We complete now this section with the next Corollary.

Corollary 1.5.7. Any compact strongly pseudoconvex CR manifold $(M, T^{1,0}(M))$ is orientable.

Proof. Let $L_1, \dots, L_n, T$ (T chosen such that the Levi form is positive definite) defined as above and let the dual one forms be $\omega_1, \dots, \omega_n, \eta$. Then we consider the $2n+1$ form $\eta \wedge \omega_1 \wedge \dots \wedge \omega_n \wedge \overline{\omega}_1 \wedge \dots \wedge \overline{\omega}_n$.
Note that, if we change basis, the form will differ only by a positive function. Now a partion of unity argument will give the desired not-vanishing $2n+1$ form on M. That proves the lemma. □

Chapter 2

The Heisenberg Group $\mathbb{H}_n$

Goal. In this capther we are going to present the Heisenberg group, that is the main example of speudoconvex CR manifolds. The ideas, the methods and the results here used will be useful to understand the behaviour of the more general speudoconvex CR manifolds.

2.1 Definition and Lie structure of $\mathbb{H}_n$

2.1.1 Definition of $\mathbb{H}_n$

Definition 2.1.1.
We define the *Heisenberg Group*, $\mathbb{H}_n$, as

$$\mathbb{H}_n := (\mathbb{C}^n \times \mathbb{R}, *)$$

where $*$ is the following product:

$$(z, t) * (z', t') := (z + z', t + t' + 2\text{Im}(z\overline{z'}))$$

and where $z\overline{z} = \sum_{j=1}^{n} z_j \overline{z}_j$.
If we write $z = x + iy$ and $z' = x' + iy' \in \mathbb{C}^n$, it comes immediately that $z\overline{z} = (x+iy)(x'+iy') = xx'+yy'-i(xy'-x'y)$ and so $\text{Im}(z\overline{z'}) = -(xy'-x'y)$. Then we can rewrite the product as

$$(z, t) * (z', t') = (z + z', t + t' - 2(xy' - x'y))$$

Again, if we identify $\mathbb{C}^n \times \mathbb{R} = \mathbb{R}^{2n} \times \mathbb{R}$ and write the product just with real variables, we get

$$(x, y, t) * (x', y', t') = (x + x', y + y', t + t' - 2(xy' - x'y)).$$

Observation 2.1.2. We can easily see that

- $\mathbb{H}_n$ is a non-commutative group.
- The neutral element is $(0,0)$.
- The inverse of (z,t) is $(-z,-t)$.
- The center of the group, namely the elements that commute with all the elements of the group, is $\{(0,t) \in \mathbb{C}^n \times \mathbb{R}\}$.

Proof. With an easy computation, we'll just prove the third and fourth statements.
For the third one we have,

$$(z,t)(-z,-t) = (z - z, t - t + 2\mathrm{Im}(z(\overline{-z}))) = 0$$

because

$$z(\overline{-z}) = (x+iy)(\overline{-x-iy}) = (x+iy)(-x+iy) = -x^2-y^2+i(xy-xy) = -x^2-y^2.$$

On the other hand

$$(-z,-t)(z,t) = (-z+z, -t+t+2\mathrm{Im}(-z\overline{z})) = 0,$$

so $(-z,-t)$ is the inverse of (z,t).
For the last one, let (z_0,t_0) be in the center of the group. $\forall (z,t) \in \mathbb{H}_n$:

$$(z,t)(z_0,t_0) = (z_0,t_0)(z,t) \Leftrightarrow xy_0 - x_0y = x_0y - xy_0 \Leftrightarrow 2xy_0 = 2x_0y \Leftrightarrow$$

$$\Leftrightarrow xy_0 = x_0y \Leftrightarrow y_0 = x_0 = 0$$

since (z,t) is generic. Then the center of the group is exactly the set of elements of this type: $(0,t) \in \mathbb{C}^n \times \mathbb{R}$. And the proof is complete. □

Definition 2.1.3.
On $\mathbb{H}_n$ there exists two different groups of automorphisms.

- The first one is the group of the *anisotropic dilatations* δ_r, with $r \in \mathbb{R}^+$.

$$\delta_r : \mathbb{H}_n \to \mathbb{H}_n$$
$$(x,y,t) \mapsto (rx, ry, r^2t)$$

  These functions form a 1- parameter subgroup of the set of automorphisms $\mathrm{Aut}(\mathbb{H}_n)$. They will play a fundamental role in the study of CR analysis.

- The second one is the *sympletic group* $Sp(2n, \mathbb{R})$, the group of linear maps that preserve the sympletic form $\omega((x,y),(x',y')) := xy' - x'y$, i.e.,

$$\forall A \in Sp(2n,\mathbb{R}),\ \omega(A(x,y), A(x',y')) = \omega((x,y),(x',y')).$$

Observation 2.1.4. First we note that, if we write $(x,y) = z$ and $(x',y') = z'$, then $\omega((x,y),(x',y')) = xy' - x'y = \mathrm{Im}(z\overline{z}')$.
If now, with an abuse of notation, we set $A(x,y,t) := (A(x,y),t)$, then we have $A((x,y,t) * (x',y',t')) = A(x,y,t) \cdot A(x',y',t') \in \mathbb{C}^n \times \mathbb{R}$.

Proof.

$$A((x,y,t) * (x',y',t')) = A(x+x', y+y', t+t' - 2(xy' - x'y)) =$$

by definition

$$= (A(x+x', y+y'), t+t'-2(xy'-x'y)) = (A(x,y)+A(x',y'), t+t'-2(xy'-x'y)) =$$

by hypothesis

$$= (A(x,y) + A(x',y'), t+t' - 2\omega(A(x,y), A(x',y'))) =$$

calling $A(x,y) = z$ and $A(x',y') = z'$

$$= (z+z', t+t' - 2\omega(z,z')) = (z+z', t+t' + 2\mathrm{Im}(zz')) =$$

$$= (z,t) * (z',t') = (A(x,y),t) * (A(x',y'),t') = A(x,y,t) \cdot A(x',y',t')$$

□

2.1.2 Lie structure of $\mathbb{H}_n$

We will now show that the Heisenberg group is, in fact, a Lie group; this will lead to important conseguences in the study of its structure.

Recall 2.1.5. $G \equiv (G, *)$ is a *Lie Group* if

- G is a differentiable manifold,
- $g_1, g_2 \mapsto g_1 * g_2 = g_1 g_2$ is differentiable,
- $g_1 \mapsto g_1^{-1}$ is differentiable.

In this case, $\forall g, g' \in G$ the *left-invariant operator* τ_g is the operator such that

$$\begin{aligned} &\tau_g : C^\infty(G) \to \mathbb{C} \\ &\tau_g(f)(g') := f(g^{-1}g'). \end{aligned}$$

Then we also have the following property: $\tau_{g_2}\tau_{g_1} = \tau_{g_2g_1}$.

Proof.

$$\tau_{g_2}(\tau_{g_1}(f))(g) = \tau_{g_1}(f)(g_2^{-1}g) = f(g_1^{-1}g_2^{-1}g) = f((g_2g_1)^{-1}g) = \tau_{g_2g_1}(f)(g)$$

□

Definition 2.1.6.
It will be useful only later, but we can already give the definition of *right-invariant operator* τ^g, $g \in G$, in the same way:

$$\begin{aligned} &\tau^g : C^\infty(G) \to \mathbb{C} \\ &\tau^g(f)(g') := f(g'g^{-1}). \end{aligned}$$

And we also introduce the *reflection operator* J as

$$\begin{aligned} &J : C^\infty(G) \to \mathbb{C} \\ &J(f)(g) = f(g^{-1}). \end{aligned}$$

Definition 2.1.7.
We say that a vector field X on a Lie group G is *left-invariant* if X commutes with τ_g, i.e.,

$$\forall f \in C^\infty(G),\ \forall g, g' \in G, \text{ we have } X(\tau_g(f))(g') = \tau_g(X(f))(g')$$

Definition 2.1.8.
Now we are ready to define a basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, T\}$ for the tangent space $T(\mathbb{H}_n)$ of left invariant vector fields so that it is
$\{\partial_{x_1}, \ldots, \partial_{x_n}, \partial_{y_1}, \ldots, \partial_{y_n}, \partial_t\}$ at the origin.
We define it as:

$$\begin{cases} X_j &= \partial x_j + 2y_j \partial t \quad \text{for } \ j = 1, \ldots, n \\ Y_j &= \partial y_j - 2x_j \partial t \quad \text{for } \ j = 1, \ldots, n \\ T &= \partial t \end{cases}$$

Observation 2.1.9. This definition is made so that the following property is true $\forall (x,y,t) \in \mathbb{H}_n$:

$$X_j f(x,y,t) = \tau_{(x,y,t)^{-1}}(X_j f)(0,0,0) \quad \text{for} \quad j = 1,\dots,n$$
$$Y_j f(x,y,t) = \tau_{(x,y,t)^{-1}}(Y_j f)(0,0,0) \quad \text{for} \quad j = 1,\dots,n$$
$$T f(x,y,t) = \tau_{(x,y,t)^{-1}}(T f)(0,0,0)$$

Proof. $\forall j = 1,\dots,n$. Since X_j is left invariant:

$\tau_{(x,y,t)^{-1}}(X_j f)(0,0,0) = X_j(\tau_{(x,y,t)^{-1}} f)(0,0,0) = X_j(f((x,y,t)*(x',y',t')))_{|(z',t')=0} =$

Note that here $(x',y',t') = z'$ are temporary variables. By definition of X_j at the origin

$= \partial_{x'_j}(f((x,y,t)*(x',y',t')))_{|(z',t')=0} = \partial_{x'_j}(f(x+x',y+y',t+t'-2(xy'-yx')))_{|(z',t')=0} =$

$$\partial_{x_j}(f(\dots))_{|(z',t')=0} + 2y\partial_t(f(\dots))_{|(z',t')=0} = \partial_{x_j}(f(x,y,t)) + 2y\partial_t(f(x,y,t))$$

Repeating the same argument for Y_j and T completes the proof. □

Observation 2.1.10. The only non-trivial commutators of the vector fields X_j, Y_j and T are

$$[X_j, Y_j] = -4T \quad \text{for} \quad j = 1,\dots,n.$$

This immediately tells us that all the higher-order commutators are zero.

Proof. $\forall j = 1,\dots,n$

$$[X_j, Y_j] = [\partial_{x_j} + 2y_j\partial_t, \partial_{y_j} - 2x_j\partial_t] = [\partial_{x_j}, \partial_{y_j}] - 2[\partial_{x_j}, x_j\partial_t] + 2[y_j\partial_t, \partial_{y_j}] +$$
$$-4[y_j\partial_t, x_j\partial_t] = \partial_{x_j y_j} - \partial_{y_j x_j} - 2\partial_{x_j}(x_j\partial_t) + 2x_j\partial_{t x_j} + 2y_j\partial_{t y_j} - 2\partial_{y_j}(y_j\partial_t) +$$
$$-4y_j\partial_t(x_j\partial_t) + 4x_j\partial_t(y_j\partial_t) =$$

cancelling the two terms at the beginning and at the end of the line,

$$= -2\partial_t - 2x_j\partial_{x_j t} + 2x_j\partial_{t x_j} + 2y_j\partial_{t y_j} - 2\partial_t - 2y_j\partial_{y_j t} = -4\partial_t = -4T.$$

□

Proposition 2.1.11.
If we look at the complexified tangent bundle, $\mathbb{C}T(\mathbb{H}_n)$, it's easy to prove that we can obtain a basis of left-invariant vector fields:

$$\{Z_1,\dots,Z_n,\overline{Z}_1,\dots,\overline{Z}_n,T\}$$

where $Z_j := \frac{1}{2}(X_j - iY_j)$ and $\overline{Z}_j := \frac{1}{2}(X_j + iY_j)$.
If we want to compute them exactly, we find:

$$Z_j = \partial_{z_j} + i\overline{z}_j\partial_t \qquad \text{and} \qquad \overline{Z}_j = \partial_{\overline{z}_j} - iz_j\partial_t$$

where $z_j = x_j + iy_j \in \mathbb{C}$.

Observation 2.1.12. In this case the only non-trivial commutators of these vector fields are

$$[Z_j, \overline{Z}_j] = -2iT \quad \text{for} \quad j = 1, \ldots, n.$$

And the proof is the same of observation 2.1.10.

Proposition 2.1.13.
Let us name $\mathcal{L} := span\{Z_1, \ldots, Z_n\} =< \{Z_1, \ldots, Z_n\} >$. $\mathcal{L}$ is a subbundle of $\mathbb{C}T(\mathbb{H}_n)$.
Then $(\mathbb{H}_n, \mathcal{L})$ is a (strongly pseudoconvex) CR manifold of CR dimension n.

Proof. In order to prove that $(\mathbb{H}_n, \mathcal{L})$ is a CR manifold, we should satisfy the three conditions of definition 1.3.4.

1. $dim_{\mathbb{C}}\mathcal{L} = \frac{1}{2}(dim_{\mathbb{C}}\mathbb{H}_n + 1) = \frac{1}{2}(2n) = n$
2. $\mathcal{L} \cap \overline{\mathcal{L}} = \{0\}$ where $\overline{\mathcal{L}} =< \{\overline{Z}_1, \ldots, \overline{Z}_n\} >$
3. $[Z_i, Z_j] = 0 \ \forall i, j = 1, \ldots, n.$

Finally, to prove that $\mathbb{H}_n$ is strongly pseudoconvex, it will be enough to use theorem 1.5.6 and prove that the Siegel Upper-Half space, that we will define soon, is a strongly pseudoconvex domain. We will prove it in proposition 2.2.8. □

Definition 2.1.14.
Let us now endow the tangent bundle $T(\mathbb{H}_n)$ with a scalar multiple of the standard inner product in $\mathbb{R}^{2n+1}$:

$$< V, V' >:= c \sum_{j=1}^{2n+1} V_j V_j' \quad \text{for } j = 1, \ldots, n$$

with c chosen such that $\{Z_1, \ldots, Z_n, \overline{Z}_1, \ldots, \overline{Z}_n, T\}$ is an orthonormal basis in $\mathbb{C}T(\mathbb{H}_n)$.

Observation 2.1.15. With this product we can define a norm in the obvious sense and we have that

$$|Z_j| :=< Z_j, Z_j >= 1 \quad \text{and } |X_j| = |Y_j| = \sqrt{2}$$

for j=1,...,n.

Observation 2.1.16. The vector fields $\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$ (as well as $\{Z_1, \dots, Z_n, \overline{Z}_1, \dots, \overline{Z}_n\}$) are homogeneous of order 1 with respect to the dilatation $\delta_r,\ r \in \mathbb{R}^+$, i.e.,

$$V(f \circ \delta_r) = rV(f) \circ \delta_r$$

for V any of the vector fields above.
On the other hand, the vector field T is homogeneous of order 2:

$$T(f \circ \delta_r) = r^2 T(f) \circ \delta_r.$$

Definition 2.1.17.
We now define $T(\mathbb{H}_n)^*$ as the dual bundle of $T(\mathbb{H}_n)$, which inherits an inner product from the one in $T(\mathbb{H}_n)$.

Proposition 2.1.18.
For $T(\mathbb{H}_n)^*$ we need a dual basis of 1-forms $\{\omega_1, \dots, \omega_{2n+1}\}$, i.e., we ask

$$< \omega_j, V_k >= \delta_{jk}$$

for V_k an element of the basis and for $j, k = 1, \dots, n$.
Then it comes that the dual basis of $\{Z_1, \dots, Z_n, \overline{Z}_1, \dots, \overline{Z}_n, T\}$ is given by

$$\{dz_1, \dots, dz_n, d\overline{z}_1, \dots, d\overline{z}_n, \theta\}$$

where

$$\theta = dt + i\sum_{j=1}^{n}(z_j d\overline{z}_j - \overline{z}_j dz_j) = dt + 2\sum_{j=1}^{n}(x_j dy_j - y_j dx_j).$$

2.2 Siegel Upper-Half Space

Now that we know the Heisenberg group, we will see that it can be obtained as an embedded CR manifold in $\mathbb{C}^{n+1}$; more precisely, as the bound of a domain.

Definition 2.2.1.
Let $\mathcal{U}_{n+1}$ be the *Siegel upper-half space*:

$$\mathcal{U}_{n+1} := \left\{(z, w) \in \mathbb{C}^n \times \mathbb{C}/\ \mathrm{Im}w > |z|^2\right\}$$

If we define $\rho(z, w) = \mathrm{Im}w - |z|^2$ (that is called his *defining function*), then

$$\mathcal{U}_{n+1} = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}/\ \rho(z, w) > 0\}$$

Observation 2.2.2. We can note that $\mathcal{U}_{n+1}$ is biholomorphic to the unit ball in $\mathbb{C}^{n+1}$ via the Caley transform:

$$C : B_{n+1} \to \mathcal{U}_{n+1}$$
$$C(z,w) = \left(\frac{z}{1-w}, i\frac{1+w}{1-w} \right)$$

Notation 2.2.3. We can now consider the boundary of the Siegel upper-half space, $\partial\mathcal{U}_{n+1}$:

$$\partial\mathcal{U}_{n+1} = \left\{ (z,w) \in \mathbb{C}^n \times \mathbb{C} / \ \mathrm{Im} w = |z|^2 \right\} = \{(z,w) \in \mathbb{C}^n \times \mathbb{C} / \ \rho(z,w) = 0\}$$

Definition 2.2.4.
In order to study $\partial\mathcal{U}_{n+1}$, we want to parametrize it. We write its elements as $[z,t] \in \mathbb{C}^n \times \mathbb{R}$, $[z,t] = (z, t + i|z|^2)$.
For every point on the boundary, $(z', t' + i|z'|^2) \in \partial\mathcal{U}_{n+1}$, we can also define a function:

$$F_{(z',t'+i|z'|^2)} : \overline{\mathcal{U}}_{n+1} \to \overline{\mathcal{U}}_{n+1}$$
$$F_{(z',t'+i|z'|^2)}(z,w) := \left(z + z', w + t' + i|z'|^2 + 2iz \cdot \overline{z'} \right)$$

where $z \cdot \overline{z'} = \sum_{j=1}^{n} z_j \cdot \overline{z'}_j$.
The function $F_{(z',t'+i|z'|^2)}$ is a biholomorphic map.

Observation 2.2.5. We can parametrize the points in $\overline{\mathcal{U}}_{n+1}$ as:

$$(z, t + i|z|^2 + ih)$$

where $h \geq 0$.
Note that the following identity holds:

$$h = \rho(z, t + i|z|^2 + ih).$$

At this point we can say that $F_{(z',t'+i|z|^2)}(z,w)$ preserve ρ in the following way:

$$\rho\left(F_{(z',t'+i|z'|^2)}(z, t + i|z|^2 + ih) \right) = h = \rho(z, t + i|z|^2 + ih)$$

Proof. First we prove that $h = \rho(z, t + i|z|^2 + ih)$ is true. Indeed

$$\rho(z, t + i|z|^2 + ih) = \mathrm{Im}(t + i|z|^2 + ih) - |z|^2 = |z|^2 + h - |z|^2 = h.$$

For the second equality:

$$\rho\left(F_{(z',t'+i|z'|^2)}(z, t + i|z|^2 + ih) \right) = \rho(z + z', t + i|z|^2 + ih + t' + i|z'|^2 + 2iz\overline{z'}) =$$
$$= |z|^2 + h + |z'|^2 + 2\mathrm{Re}(z\overline{z'}) - |z + z'|^2 = h.$$

□

Definition 2.2.6.
Finally we can define a product on $\partial\mathcal{U}_{n+1}$ using the function $F_{(z',t'+i|z|^2)}$:

$$[z',t'] * [z,t] := F_{(z',t'+i|z'|^2)}(z,t+i|z|^2)$$

Proposition 2.2.7.
The previous product can be written as

$$[z',t'] * [z,t] = [z+z', t+t'+2\mathrm{Im}(z\overline{z'})]$$

That shows us that we re-obtained the Heisenberg group and it is $(\partial\mathcal{U}_{n+1}, *)$ Now that we know this fact, we leave the notation "$[z,t]$" and we will use only the notation "(z,t)" of definition 2.1.1.

Proof. First we recall

$$|z+z'|^2 = |z|^2 + |z'|^2 + 2\mathrm{Re}(z\overline{z'}) \qquad \text{and} \qquad z\overline{z'} = \mathrm{Re}(z\overline{z'}) + i\mathrm{Im}(z\overline{z'}).$$

Then

$$i|z|^2 + i|z'|^2 + 2iz\overline{z'} = i|z+z'|^2 - 2i\mathrm{Re}(z\overline{z'}) + 2iz\overline{z'} = i|z+z'|^2 - 2\mathrm{Im}(z\overline{z'}).$$

Now

$$[z',t'] * [z,t] = F_{(z',t'+i|z'|^2)}(z,t+i|z|^2) = (z+z', t+i|z|^2+t'+i|z'|^2+2iz\overline{z'}) =$$

$$= \left(z+z', t+t'+i|z+z'|^2 - 2\mathrm{Im}(z\overline{z'})\right) =$$

since $\mathrm{Im}\left(z\overline{z'}\right) = -\mathrm{Im}\left(\overline{z\overline{z'}}\right) = -\mathrm{Im}\left(z'\overline{z}\right)$

$$= [z+z', t+t'+2\mathrm{Im}(z'\overline{z})]$$

□

Now that we proved this, we have to cancel the debt of proposition 2.1.13, so:

Proposition 2.2.8.
We will now prove that $\mathcal{U}_{n+1}$ is a strictly pseudoconvex domain.
Then, from theorem 1.5.6, it comes that $\mathbb{H}_n = \partial\mathcal{U}_{n+1}$ is a strongly pseudo-convex CR manifold. So the debt is paid.

Proof. We know that the defining function of $\mathcal{U}_{n+1}$ is $\rho(z,w) = \text{Im}w - |z|^2$, with $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Just for the time of this proof, we'll change the notation and write

$$\rho(z) = \rho(\tilde{z}, z_{n+1}) = \text{Im}z_{n+1} - |\tilde{z}|^2,$$

with $z = (\tilde{z}, z_{n+1}) \in \mathbb{C}^n \times \mathbb{C}$.
Recalling definition 1.2.1, can calculate the double derivatives of ρ.
For $k = n+1$,

$$\frac{\partial \rho}{\partial \overline{z}_{n+1}} = \frac{\partial \text{Im}z_{n+1}}{\partial \overline{z}_{n+1}} - \frac{\partial |\tilde{z}|^2}{\partial \overline{z}_{n+1}} = \frac{\partial \text{Im}z_{n+1}}{\partial \overline{z}_{n+1}} = \frac{1}{2}\left(\frac{\partial y_{n+1}}{\partial x_{n+1}} + i\frac{\partial y_{n+1}}{\partial y_{n+1}}\right) = \frac{i}{2}$$

Then, $\forall j = 1, \ldots, n+1$, we get:

$$\frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_{n+1}} = 0$$

Now, for $k \neq n+1$,

$$\frac{\partial \rho}{\partial \overline{z}_k} = \frac{\partial \text{Im}z_{n+1}}{\partial \overline{z}_k} - \frac{\partial |\tilde{z}|^2}{\partial \overline{z}_k} = -\frac{\partial |\tilde{z}|^2}{\partial \overline{z}_k} = -\frac{\partial (|z_1|^2 + \ldots + |z_n|^2)}{\partial \overline{z}_k} =$$

$$= -\frac{\partial (z_1\overline{z_1} + \ldots + z_n\overline{z_n})}{\partial \overline{z}_k} = z_k$$

Then $\forall j = 1, \ldots, n+1;\ j \neq k$,

$$\frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} = 0$$

and

$$\frac{\partial^2 \rho}{\partial z_k \partial \overline{z}_k} = 1 > 0.$$

So the proof is complete. □

2.3 Integration on $\mathbb{H}_n$

Here we want to add information on $\mathbb{H}_n$ giving definitions about norm, metric, topology, measure, integrals. We'll use them starting chapter 3.

Definition 2.3.1.
We define a homogeneous norm on $\mathbb{H}_n$ by setting

$$|(z,t)| := \left((|z|^4 + t^2\right)^{\frac{1}{4}}$$

with $(z,t) \in \mathbb{C}^n \times \mathbb{R}$. We can write it as

$$|(x,y,t)| = \left((x^2+y^2)^2 + t^2\right)^{\frac{1}{4}}$$

with $(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$.

Observation 2.3.2. This norm satisfies the following properties:

1. $|(z,t)| \leq 0,\ |(z,t)| = 0 \Leftrightarrow (z,t) = 0$
2. $|(z,t) * (z',t')| \leq |(z,t)| + |(z',t')|$
3. $\delta_r\left(|(z,t)|\right) = r|(z,t)|$
   Namely, the norm is homogeneous of degree 1 w.r.t. δ_r.

where δ_r has been defined in 2.1.3.

Observation 2.3.3. During this study, while we almost always use the homogeneous norm, we shall occasionally use the euclidean norm

$$||u|| = ||(x,y,t)|| = (|x|^2 + |y|^2 + t^2)^{\frac{1}{2}}.$$

When we do, we also shall denote the vector addition by $+$.
We can write down the following obvious inequality:

$$||u|| \leq |u| \leq ||u||^{\frac{1}{2}} \quad \text{when } |u| \leq 1.$$

Observation 2.3.4. [Triangle inequality]
There exists a constant $c \geq 1$ such that, $\forall u, v \in \mathbb{H}_n$,

$$|u+v| \leq c(|u| + |v|)$$

Proof. By homogeneity, we may assume that $|u| + |v| = 1$. Then the set of pairs $(u,v) \in \mathbb{H}_n \times \mathbb{H}_n$ satisfying this equation is compact, so we can take c to be the larger of the maximums values of $|u+v|$ on this set. □

Observation 2.3.5. We also notice that the topology induced by the metric

$$d\left((z,t),(z',t')\right) := \left|(z,t) * (z',t')^{-1}\right| = |(x-x', y-y', t-t' + 2(xy' - x'y))|$$

is equivalent to the Euclidean topology on $\mathbb{R}^{2n} \times \mathbb{R}$.
$\mathbb{H}_n$ becomes, then, a locally compact topological group. As such, it has the *right-invariant* and the *left-invariant Haar measure.*

Recall 2.3.6. We call a measure μ the *right-invariant* or *left-invariant Haar measure*, on a locally compact Hausdorff topological group G, if the followings are satisfied:

- μ is left-invariant: $\mu(gE) = \mu(E)\ \forall E \subseteq G,\ g \in G,\ gE = \{ga/\ a \in E\}$
  or
  μ is right-invariant: $\mu(Eg) = \mu(E)\ \forall E \subseteq G,\ g \in G,\ Eg = \{ag/\ a \in E\}$
- $\mu(K) < \infty\ \forall K \subset\subset G$
- μ is outer regular: $\mu(E) = \inf\{\mu(U)/\ E \subseteq U \subseteq G, U \text{ open}\}\ \forall E \subseteq G$
- μ is inner regular: $\mu(E) = \sup\{\mu(U)/\ K \subseteq E \subseteq G, K \text{ compact}\}\ \forall E \subseteq G$

Proposition 2.3.7.
The ordinary Lebesgue measure on $\mathbb{R}^{2n+1}$ is invariant under both left and right translations on $\mathbb{H}_n$. In other words, the Lebesgue measure is both a left and right invariant Haar measure on $\mathbb{H}_n$.

Proof. For the right translation:

$$\int_{\mathbb{H}_n} f\left((x,y,t)(x',y',t')\right) dxdydt =$$

$$= \int_{\mathbb{H}_n} f\left(x+x', y+y', t+t' - 2(xy' - x'y)\right) dxdydt =$$

by the obvious change of variables, we have that the Jacobian is equal to

$$\begin{vmatrix} 1 & 0 & -2y' \\ 0 & 1 & 2x' \\ 0 & 0 & 1 \end{vmatrix} = 1,$$

then

$$= \int_{\mathbb{H}_n} f(x,y,t) dxdydt.$$

□

Observation 2.3.8. It's easy to see that, denoting

$$B(0,r) := \{(z,t) \in \mathbb{H}_n/\ |(z,t)| < r\}$$

the ball of radius $r > 0$, we have

$$|B(0,r)| = \int_{B(0,r)} dxdydt = r^{2n+2} \int_{B(0,1)} dxdydt = r^{2n+2} |B(0,1)|.$$

$2n+2$ is called the *homogeneous dimension of* $\mathbb{H}_n$.

Observation 2.3.9. Since $< dx_j, dx_j > = < dy_j, dy_j > = \frac{1}{2}$ for $j = 1, \ldots, n$, the *volume element* is

$$dV = \frac{1}{2^n} dxdydt = 2^{-n} dxdydt.$$

Definition 2.3.10.
Given $f, g \in L^1(\mathbb{H}_n)$, we also define the *convolution* $f * g$ as

$$f * g(x, y, t) := \int_{\mathbb{H}_n} f(x, y, t) g((x, y, t)^{-1}(x', y', t')) dx'dy'dt'$$

Observation 2.3.11. The following property is easy to check:

$$f * g(x, y, t) = \int_{\mathbb{H}_n} f(x, y, t) g((x, y, t)^{-1}(x', y', t')) dx'dy'dt' =$$

$$= \int_{\mathbb{H}_n} f((x, y, t)(x', y', t')^{-1}) g(x', y', t') dx'dy'dt' =$$

$$= \int_{\mathbb{H}_n} f(x - x', y - y', t - t' - 2(xy' - x'y)) g(x', y', t') dx'dy'dt'.$$

Observation 2.3.12. If we set $\check{g}(x, y, t) := g((x, y, t)^{-1})$, we can also prove that

$$\int_{\mathbb{H}_n} (f * g)(x, y, t) h(x, y, t) dxdydt = \int_{\mathbb{H}_n} f(x, y, t)(h * \check{g})(x, y, t) dxdydt$$

provided that both sides make sense.

2.4 CR operators on $\mathbb{H}_n$

We are now going to define the operators we will work with in the future: the tangent complex $\overline{\partial}_b$, its formal adjoint $\overline{\partial}_b^*$, the Kohn Laplacian $\square_{b,q}$ and the operator $\mathcal{L}_\alpha$.

2.4.1 The tangential *CR* complex $\overline{\partial}_b$ on $\mathbb{H}_n$ and its formal adjoint $\overline{\partial}_b^*$

Notation 2.4.1. We denote:

- $d\overline{z}^I := d\overline{z}_{i_1} \wedge \cdots \wedge d\overline{z}_{i_q}$ a $(0, q)$-form, where $I = (i_1, \ldots, i_q)$ is a strictly increasing multi-index.

- $\sum_I'$ the summation restricted to strictly increasing multi-indices.
- $C^\infty_{(0,q)}(\mathbb{H}_n)$, $L^2_{(0,q)}(\mathbb{H}_n), \dots$ the spaces of $(0,q)$-forms with coefficients smooth, L^2, etc.

Definition 2.4.2.
Now we remind that $\{Z_1, \dots Z_n, \overline{Z}_1, \dots \overline{Z}_n, T\}$ is an orthonormal basis of $\mathbb{C}T(\mathbb{H}_n)$ and we take a $(0,q)$-form $\phi = \sum'_{|I|=q} \phi_I d\overline{z}^I$, $\phi_I \in C^\infty(\mathbb{H}_n)$. We define the *tangential CR complex* $\overline{\partial}_b$ *on* $\mathbb{H}_n$ as:

$$\overline{\partial}_b \phi := \sum_{|I|=q}' \sum_{k=1}^{n} \overline{Z}_k(\phi_I) d\overline{z}_k \wedge d\overline{z}^I$$

Observation 2.4.3. We can rewrite it as

$$\overline{\partial}_b \phi = \sum_{|J|=q+1}' \left(\sum_{\substack{k=1,\dots,n \\ |I|=q}}' \epsilon^J_{kI} \overline{Z}_k(\phi_I) \right) d\overline{z}^J$$

where

$$\epsilon^J_{kI} = \begin{cases} 0, \text{ if } J \neq \{k\} \cup I \\ \text{parity of the permutation that rearranges } (k, i_1, \dots, i_q) \\ \qquad\qquad\qquad \text{in increasing order, if } J = \{k\} \cup I. \end{cases}$$

Definition 2.4.4.
Let now $dzdt$ denote the left-invariant Haar measure on $\mathbb{H}_n$ defined in recall 2.3.6. On the space $L^2_{(0,q)}(\mathbb{H}_n)$ we consider the inner product:

$$< \phi, \psi >_{L^2_{(0,q)}} = \int_{\mathbb{H}_n} (\phi(z,t), \psi(z,t)) dzdt, \quad \phi, \psi \in L^2_{(0,q)}(\mathbb{H}_n)$$

This is, in fact, the integral of the inner product

$$(\phi(z,t), \psi(z,t)) = \sum_{|I|=q} \left(\phi_I(z,t) \overline{\psi}_I(z,t) \right).$$

Definition 2.4.5.
Take a $(0,q)$-form $\phi = \sum'_{|I|=q} \phi_I d\overline{z}^I$, $\phi_I \in C^\infty(\mathbb{H}_n)$ and g a $(0,q-1)$-form $g = \sum'_{|J|=q-1} g_J d\overline{z}^J$ and $g_J \in C^\infty_0(\mathbb{H}_n)$.
The *formal adjoint* $\overline{\partial}_b^*$ of $\overline{\partial}_b$ is defined as the operator such that

$$< \overline{\partial}_b^* \phi, g > = < \phi, \overline{\partial}_b g > .$$

Proposition 2.4.6.
We can actually compute $\overline{\partial}_b^*$ and find that, for a $(0,q)$-form $\phi = \sum'_{|I|=q} \phi_I d\overline{z}^I$, $\phi_I \in C^\infty(\mathbb{H}_n)$, we get:

$$\overline{\partial}_b^*(\phi) = \overline{\partial}_b^* \left(\sum'_{|I|=q} \phi_I d\overline{z}^I \right) = \sum'_{|J|=q-1} \left(- \sum'_{\substack{k=1,\dots,n \\ |I|=q}} \epsilon^I_{kJ} Z_k(\phi_I) \right) d\overline{z}^J$$

Proof. With a $(0,q)$-form $\phi = \sum'_{|I|=q} \phi_I d\overline{z}^I$, $\phi_I \in C^\infty(\mathbb{H}_n)$ and g a $(0,q-1)$-form $g = \sum'_{|J'|=q-1} g_{J'} d\overline{z}^{J'}$ and $g_{J'} \in C_0^\infty(\mathbb{H}_n)$, we get:

$$< \overline{\partial}_b^* \phi, g >=< \phi, \overline{\partial}_b g >=< \sum'_{|I|=q} \phi_I d\overline{z}^I, \sum'_{|J|=q} \left(\sum'_{\substack{k=1,\dots,n \\ |J'|=q-1}} \epsilon^J_{kJ'} \overline{Z}_k(g_{J'}) \right) d\overline{z}^J >=$$

computing the inner product

$$= \sum'_{|I|=q} \left(\int_{\mathbb{H}_n} \phi_I \overline{\left(\sum'_{k=1,\dots,n \ |J'|=q-1} \epsilon^I_{kJ'} \overline{Z}_k(g_{J'}) \right)} dz dt \right) =$$

$$= \sum'_{|I|=q} \left(\sum'_{k=1,\dots,n \ |J'|=q-1} \epsilon^I_{kJ'} \int_{\mathbb{H}_n} \phi_I Z_k(\overline{g}_{J'}) dz dt \right) =$$

integrating by parts

$$= \sum'_{|I|=q} \left(\sum'_{k=1,\dots,n \ |J'|=q-1} \epsilon^I_{kJ'} \int_{\mathbb{H}_n} -(Z_k\phi_I) \overline{g}_{J'} dz dt \right) =$$

$$= \sum'_{|J'|=q-1} \left(\int_{\mathbb{H}_n} \left(- \sum'_{k=1,\dots,n |I|=q} \epsilon^I_{kJ'} Z_k \phi_I \right) \overline{g}_{J'} dz dt \right) =$$

$$=< \sum'_{|J|=q-1} \left(\sum'_{\substack{k=1,\dots,n \\ |I|=q}} \epsilon^I_{kJ} Z_k \phi_I \right) d\overline{z}^J, \sum'_{|J'|=q-1} g_{J'} d\overline{z}^{J'} >=$$

$$= < \sideset{}{'}\sum_{|J|=q-1} \left(\sideset{}{'}\sum_{\substack{k=1,\dots,n \\ |I|=q}} \epsilon^I_{kJ} Z_k \phi_I \right) d\overline{z}^J, g >$$

And the proof is complete.

□

2.4.2 The operator $\Box_{b,q}$ and the operators $\mathcal{L}_\alpha$

Definition 2.4.7.
We can finally define the equivalent of the laplacian for the Heisenberg group, the *Kohn Laplacian* for $(0,q)$-forms:

$$\Box_{b,q} := \overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b.$$

In order to study the Kohn Laplacian, it will be very useful to define immediately another operator:

$$\mathcal{L}_\alpha := -\frac{1}{2} \sum_{k=1}^{n} (\overline{Z}_k Z_k + Z_k \overline{Z}_k) + i\alpha T$$

for $\alpha \in \mathbb{C}$.

Proposition 2.4.8.
With respect to the fixed orthonormal basis $\{dz_1, \dots, dz_n, d\overline{z}_1, \dots, d\overline{z}_n, \theta\}$ on $\mathbb{C}T(\mathbb{H}_n)^*$ (see proposition 2.1.18) and given $\phi = \sum'_{|I|=q} \phi_I d\overline{z}^I$ a smooth $(0,q)$-form, the operator $\Box_{b,q}$ is given by

$$\Box_{b,q}(\phi) = \sideset{}{'}\sum_{|I|=q} \mathcal{L}_{n-2q} \phi_I d\overline{z}^I$$

Observation 2.4.9. $\Box_{b,q}$ is diagonal on the space of $(0,q)$-forms with respect to the basis $\{d\overline{z}^I\}$.

$$\Box_{b,q} \begin{pmatrix} f_{i_1} \\ \vdots \\ f_{i_n} \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{n-2q} f_{i_1} \\ \vdots \\ \mathcal{L}_{n-2q} f_{i_n} \end{pmatrix}$$

i.e.

$$\Box_{b,q} = \begin{pmatrix} \mathcal{L}_{n-2q} & & \\ & \ddots & \\ & & \mathcal{L}_{n-2q} \end{pmatrix} = \mathcal{L}_{n-2q} I$$

Proof. We now prove the proposition.
Take $f = \sum'_{|K|=q} f_K d\overline{z}^K$, $f_K \in C^\infty(\mathbb{H}_n)$ a $(0,q)$-form. From proposition 2.4.6 we get:

$$\overline{\partial}_b(\overline{\partial}_b^* f) = \overline{\partial}_b \left(\overline{\partial}_b^* \left(\sum_{|K|=q}' f_K d\overline{z}^K \right) \right) = \overline{\partial}_b \left(\sum_{|J|=q-1}' \left(- \sum_{k=1,\dots,n |K|=q}' \epsilon_{kJ}^K Z_k f_K \right) d\overline{z}^J \right) =$$

using observation 2.4.3 for a $(0,q-1)$-form (and with different names for the indices)

$$= \sum_{|L|=q}' \left(\sum_{\substack{l=1,\dots,n \\ |J|=q-1}}' \epsilon_{lJ}^L \overline{Z}_l \left(- \sum_{k=1,\dots,n |K|=q}' \epsilon_{kJ}^K Z_k f_K \right) \right) d\overline{z}^L =$$

$$= - \sum_{|L|=q}' \left(\sum_{\substack{k=1,\dots,n \ |K|=q \\ l=1,\dots,n \ |J|=q-1}}' \epsilon_{kJ}^K \epsilon_{lJ}^L \overline{Z}_l \left(Z_k f_K \right) \right) d\overline{z}^L$$

On the other hand, using 2.4.3 and then 2.4.6 again,

$$\overline{\partial}_b^*(\overline{\partial}_b f) = \overline{\partial}_b^* \left(\sum_{|H|=q+1}' \left(\sum_{j=1,\dots,n |K|=q}' \epsilon_{jK}^H \overline{Z}_j (f_K) \right) d\overline{z}^H \right) =$$

$$= - \sum_{|L|=q}' \left(\sum_{\substack{j=1,\dots,n \ |K|=q \\ i=1,\dots,n \ |H|=q+1}}' \epsilon_{jK}^H \epsilon_{iL}^H Z_i \left(\overline{Z}_j f_K \right) \right) d\overline{z}^L$$

Hence

$$\Box_{b,q} f = \overline{\partial}_b \overline{\partial}_b^* f + \overline{\partial}_b^* \overline{\partial}_b f =$$

$$= - \sum_{|L|=q}' \sum_{|K|=q}' \left(\sum_{l,\ k,\ |J|=q-1}' \epsilon_{kJ}^K \epsilon_{lJ}^L \overline{Z}_l Z_k + \sum_{i,\ j,\ |H|=q+1}' \epsilon_{jK}^H \epsilon_{iL}^H Z_i \overline{Z}_j \right) f_K d\overline{z}^L =$$

and we write it as

$$= - \sum_{|L|=q}' \sum_{|K|=q}' (-\Box_{LK}) f_K d\overline{z}^L = \sum_{|L|=q}' \sum_{|K|=q}' (\Box_{LK}) f_K d\overline{z}^L.$$

Then we need to evaluate

$$\Box_{LK} = -\left(\sum_{l,\ k,\ |J|=q-1}' \epsilon^K_{kJ}\epsilon^L_{lJ}\overline{Z}_l Z_k + \sum_{i,\ j,\ |H|=q+1}' \epsilon^H_{jK}\epsilon^H_{iL} Z_i \overline{Z}_j \right).$$

We defined the coefficient ϵ's in observation 2.4.3. We remind

$$\epsilon^K_{kJ}\epsilon^L_{lJ} \neq 0 \Leftrightarrow K = \{k\} \cup J \text{ and } L = \{l\} \cup J$$

and that $|J| = q-1$, $|K| = |L| = q$.
Moreover,

$$\epsilon^H_{jK}\epsilon^H_{iL} \neq 0 \Leftrightarrow H = \{j\} \cup K \text{ and } H = \{i\} \cup L$$

and $|H| = q+1$.

Step I
Let's consider for a second what happens when $\epsilon^K_{kJ}\epsilon^L_{lJ} \neq 0$ and $\epsilon^H_{jK}\epsilon^H_{iL} \neq 0$. In this case we can say that

$$K = L \Leftrightarrow k = l,\ i = j.$$

Indeed, if $K = L$, then $\epsilon^K_{kJ} \neq 0$ forces $K = \{k\} \cup J$ and $\epsilon^L_{lJ} \neq 0$ forces $L = \{k\} \cup J$. Thus $k = l$.
In the same way $\epsilon^H_{jK} \neq 0$ and $\epsilon^H_{iL} \neq 0$ force $H = \{j\} \cup K$ and $H = \{i\} \cup L$. Hence $i = j$.
The reverse arrow is absolutely trivial.

Step II
First we suppose $K = L$. By hypothesis we know that, when the coefficient ϵ's are not zero, than we know $k = l,\ i = j$.
We also observe that $\epsilon^K_{kJ}\epsilon^K_{kJ} = 1$ and $\epsilon^H_{jK}\epsilon^H_{jK} = 1$. So we can write:

$$\Box_{KK} = -\left(\sum_{k\in L}' \overline{Z}_k Z_k + \sum_{j\notin L}' Z_j \overline{Z}_j \right) =$$

using the same index k

$$= -\frac{1}{2}\sum_{k=1}^{n} \left(\overline{Z}_k Z_k + Z_k \overline{Z}_k\right) - \frac{1}{2}\left(\sum_{k\in L}' [\overline{Z}_k, Z_k] + \sum_{k\notin L}' [Z_k, \overline{Z}_k] \right) =$$

using $[Z_k, \overline{Z}_k] = -2iT$

$$= -\frac{1}{2}\sum_{k=1}^{n} \left(\overline{Z}_k Z_k + Z_k \overline{Z}_k\right) - \frac{1}{2}\left(2iqT - 2i(n-q)T\right) =$$

$$= -\frac{1}{2}\sum_{k=1}^{n}\left(\overline{Z}_k Z_k + Z_k\overline{Z}_k\right) + i(n-2q)T = \mathcal{L}_{n-2q}$$

This proves the statement for the terms along the diagonal.

Step III

Therfore we are left to prove that the remaining off-diagonals terms are all zero. In order to do so, we suppose $K \neq L$.

Again, when at least some coefficient ϵ's are not zero, we know something: $k \neq l$ and $j \neq i$ but still $\{k, j\} = \{l, i\}$. So $k = i$ and $l = j$. Also, it tells us that $|K \cap L| = q - 1$.

Indeed, if $|K \cap L| = q$, then $K = L$, that is impossible here. And $|K \cap L| < q - 1$ is also impossible because K and L have J in common and $|J| = q - 1$.

Notice that, given K and L, J, k and l are uniquely determined.

Hence

$$\square_{LK} = -\sum_{l,\ k,\ |J|=q-1}^{\prime} \epsilon^K_{kJ}\epsilon^L_{lJ}\overline{Z}_l Z_k - \sum_{i,\ j,\ |H|=q+1}^{\prime} \epsilon^H_{jK}\epsilon^H_{iL} Z_i\overline{Z}_j =$$

the summations disappear because all the indices are determined,

$$= -\epsilon^K_{kJ}\epsilon^L_{lJ}\overline{Z}_l Z_k - \epsilon^H_{jK}\epsilon^H_{iL} Z_i\overline{Z}_j = -\epsilon^K_{kJ}\epsilon^L_{lJ}\overline{Z}_l Z_k - \epsilon^H_{lK}\epsilon^H_{kL} Z_k\overline{Z}_l.$$

And, using again the definition in observation 2.4.3, we observe that

$$\epsilon^K_{kJ}\epsilon^L_{lJ} = -\epsilon^H_{lK}\epsilon^H_{kL}.$$

Finally we get that, since $k \neq l$

$$\square_{LK} = \pm[Z_k, \overline{Z}_l] = 0.$$

That completes the proof. $\square$

Chapter 3

Study on the $\mathcal{L}_\alpha$ and $\Box_b$ operators on $\mathbb{H}_n$

Goal. Now that we know the operators $\Box_b$ and $\mathcal{L}_\alpha$, we want informations about their behaviors. So we are going to study them and, via the fundamental solutions for $\mathcal{L}_0$ and $\mathcal{L}_\alpha$, show that they are hypoelliptic on $\mathbb{H}_n$.

3.1 Fundamental Solution for $\mathcal{L}_0$

Recall 3.1.1. Remind that $\mathcal{D}(\mathbb{H}_n)$ and $\mathcal{D}'(\mathbb{H}_n)$ are, respectively, the set of smooth function on $\mathbb{H}_n$ and the set of distributions on $\mathbb{H}_n$.
We say that $E \in \mathcal{D}'(\mathbb{H}_n)$ is a *fundamental solution for* an operator P if

$$PE = \delta$$

in the distribution sense. That is,

$$< PE, \phi >=< \delta, \phi >= \phi(0) \quad \forall \phi \in \mathcal{D}(\mathbb{H}_n)$$

where δ is the Dirac distribution.

Definition 3.1.2.
As an example, we'll start studying the operator

$$\mathcal{L}_0 = -\frac{1}{2}\sum_{k=1}^{n}(\overline{Z}_k Z_k + Z_k \overline{Z}_k).$$

In general, this second order term is called the *sub-Laplacian on the stratified Lie group* $\mathbb{H}_n$.

Recall 3.1.3. A Lie group $g = (G, *)$ is *stratified* if

- it is nilpotent (namely, it possesses a central series)
- it is simply connected (informally, without holes)
- g admits a vector space decomposition

$$g = V_1 \oplus \cdots \oplus V_m$$

such that $[V_1, V_j] = V_{j+1}$ for $1 \le j < m$ and $[V_1, V_m] = \{0\}$.

Observation 3.1.4. In our case, we observe that $\mathbb{H}_n$ is a step-two stratified nilpotent Lie group; namely the Lie algebra is stratified with $m = 2$ where

$$V_1 =< Z_1, \dots, Z_n, \overline{Z}_1, \dots, \overline{Z}_n > \quad \text{and} \quad V_2 =< T > .$$

Observation 3.1.5. By theorems 8.2.3 and 8.2.5[1] about hypoellipticity and estimates of sums of squares of vector fields, it follows immediately that $\mathcal{L}_0$ satisfies a subelliptic estimate of order $\frac{1}{2}$ and is hypoelliptic.

We now want to construct an explicit fundamental solution φ_0 for $\mathcal{L}_0$.

Observation 3.1.6. Recalling the definitions of nonisotropic dilatation δ_r in 2.1.3 and of norm $|(z,t)|$ in 2.3.1, we can say that $\mathcal{L}_0$ is homogeneous of degree 1 with respect to δ_r.
It is also reasonable to guess that a fundamental solution φ_0 for $\mathcal{L}_0$ should be given by some homogeneous function.

In fact,

Theorem 3.1.7.
Set

$$\varphi_0 := |(z,t)|^{-2n} = (|z|^4 + t^2)^{-\frac{n}{2}}$$

and let δ be the Dirac distribution. Then

$$\mathcal{L}_0 \varphi_0 = c_0 \delta$$

where

$$c_0 = n^2 \int_{\mathbb{H}_n} ((|z|^2 + 1)^2 + t^2)^{-\frac{n}{2}-1} dz dt.$$

Therefore $c_0^{-1}\varphi_0$ is a fundamental solution for $\mathcal{L}_0$.

Proof. For $\epsilon > 0$ we define

$$\varphi_{0,\epsilon}(z,t) := ((|z|^2+\epsilon^2)^2+t^2)^{-\frac{n}{2}}$$

A simple calculation shows that

$$\mathcal{L}_0\varphi_{0,\epsilon}(z,t) = n^2\epsilon^2((|z|^2+\epsilon^2)^2+t^2)^{-\frac{n}{2}-1} =$$

$$= n^2\epsilon^2\left(\left(\left|\frac{z}{\epsilon}\right|^2\epsilon^2+\epsilon^2\right)^2+t^2\right)^{-\frac{n}{2}-1} =$$

$$= n^2\epsilon^2\left(\left(\left|\frac{z}{\epsilon}\right|^2+1\right)^2\epsilon^4+\left(\frac{t}{\epsilon}\right)^2\epsilon^4\right)^{-\frac{n}{2}-1} =$$

$$= n^2\epsilon^2\left(\left(\left|\frac{z}{\epsilon}\right|^2+1\right)^2+\left(\frac{t}{\epsilon}\right)^2\right)^{-\frac{n}{2}-1}\epsilon^{-2n-4} =$$

$$= n^2\epsilon^{-2n-2}\left(\left(\left|\frac{z}{\epsilon}\right|^2+1\right)^2+\left(\frac{t}{\epsilon}\right)^2\right)^{-\frac{n}{2}-1} =$$

defining $\phi(z,t) := n^2\left((|z|^2+1)^2+t^2\right)^{-\frac{n}{2}-1}$

$$= \epsilon^{-2n-2}\phi\left(\frac{1}{\epsilon}(z,t)\right).$$

Then

$$\int_{\mathbb{H}_n}\mathcal{L}_0\varphi_{0,\epsilon}(z,t)dzdt = \int_{\mathbb{H}_n}\epsilon^{-2n-2}\phi\left(\frac{1}{\epsilon}(z,t)\right)dzdt =$$

with the change of variables $(z',t') = \frac{1}{\epsilon}(z,t)$, $dz'dt' = \epsilon^{-2n-2}dzdt$

$$= \int_{\mathbb{H}_n}\phi\left((z',t')\right)dz'dt' \equiv c_0.$$

Hence

$$\lim_{\epsilon\to 0}\mathcal{L}_0\varphi_{0,\epsilon} = c_0\delta \quad \text{in the distribution sense.}$$

On the other hand,

$$\lim_{\epsilon\to 0}\mathcal{L}_0\varphi_{0,\epsilon} = \mathcal{L}_0\varphi_0 \quad \text{in the distribution sense.}$$

Then

$$\mathcal{L}_0\varphi_0 = c_0\delta$$

and the theorem is proved. □

3.2 Fundamental Solution for $\mathcal{L}_\alpha$

We now proceed to search for an explicit fundamental solution in $\mathbb{H}_n$ for the operator

$$\mathcal{L}_\alpha = -\frac{1}{2}\sum_{k=1}^{n}(\overline{Z}_k Z_k + Z_k\overline{Z}_k) + i\alpha T, \quad \alpha \in \mathbb{C}.$$

Observation 3.2.1. Observe that $\mathcal{L}_\alpha$ has the same homogeneity properties as $\mathcal{L}_0$ with respect to δ_r on $\mathbb{H}_n$ and that $\mathcal{L}_\alpha$ is invariant under unitary trasformations in the z-variable (since the norms ask just for $|z|$).
Then we can expect that the fundamental solution will have other similar important properties.

From these observations, we intend to look for a fundamental solution $\varphi_\alpha(z,t)$ of the form

$$\varphi_\alpha(z,t) = |(z,t)|^{-2n} f(t|(z,t)|^{-2})$$

After a routine, but lenghty, calculation, in order for φ_α to be a solution of $\mathcal{L}_\alpha\varphi_\alpha = 0$ away from the pole, we see that f must satisfy the following ordinary second order differential equation:

$$(1-\omega^2)^{\frac{3}{2}} f''(\omega) - ((n+1)\omega(1-\omega^2)^{\frac{1}{2}} + i\alpha(1-\omega^2))f'(\omega) + in\alpha\omega f(\omega) = 0$$

where $\omega = t|(z,t)|^{-2}$.
By setting $\omega = cos\theta$, we get $f(\omega) = g(\theta)$, $0 \le \theta \le \pi$. Then the equation is reduced to

$$\left(sen\theta\frac{d}{d\theta} + ncos\theta\right)\left(\frac{d}{d\theta} + i\alpha\right) g(\theta) = 0$$

which has two linear indipendent solutions:

- $g_1(\theta) = e^{-i\alpha\theta}$
- $g_2(\theta) = e^{-i\alpha\theta}\int_0^\pi \frac{e^{i\alpha\theta}}{(sen\theta)^n} d\theta$

The only bounded solutions for $0 \le \theta \le \pi$ are $g(\theta) = ce^{-i\alpha\theta}$ with $c \in \mathbb{C}$. It follows that

$$f(\omega) = c\left(\omega - i\sqrt{1-\omega^2}\right)^\alpha = c\left(\frac{t - i|z|^2}{|(z,t)|^2}\right)^\alpha$$

If $c = i^\alpha$, then we get:

$$\varphi_\alpha(z,t) = |(z,t)|^{-2n} i^\alpha \left(\frac{t - i|z|^2}{|(z,t)|^2}\right)^\alpha = |(z,t)|^{-2n} i^\alpha \frac{(t-i|z|^2)^\alpha}{|(z,t)|^{2\alpha}} =$$

$$= |(z,t)|^{-2(n+\alpha)} \left(i(t - i|z|^2)\right)^\alpha = |(z,t)|^{-2(n+\alpha)} \left(|z|^2 + it\right)^\alpha =$$

since $|(z,t)|^2 = (|z|^4 + it^2)^{\frac{1}{2}} = ((|z|^2 + it)(|z|^2 - it))^{\frac{1}{2}}$,

$$= \left(|z|^2 + it\right)^\alpha \left((|z|^2 + it)(|z|^2 - it)\right)^{-(n+\alpha)\frac{1}{2}} =$$

$$= \left(|z|^2 + it\right)^\alpha (|z|^2 + it)^{-\frac{n+\alpha}{2}} (|z|^2 - it)^{-\frac{n+\alpha}{2}} = (|z|^2 + it)^{-\frac{n-\alpha}{2}} (|z|^2 - it)^{-\frac{n+\alpha}{2}}.$$

Definition 3.2.2.
Then, for $\alpha \in \mathbb{C}$, we define

$$\varphi_\alpha : \mathbb{H}_n \to \mathbb{C}$$
$$\varphi_\alpha(z,t) := (|z|^2 - it)^{-\frac{n+\alpha}{2}} (|z|^2 + it)^{-\frac{n-\alpha}{2}}$$

and, for $\epsilon > 0$,

$$\varphi_{\alpha,\epsilon} : \mathbb{H}_n \to \mathbb{C}$$
$$\varphi_{\alpha,\epsilon}(z,t) := (|z|^2 + \epsilon^2 - it)^{-\frac{n+\alpha}{2}} (|z|^2 + \epsilon^2 + it)^{-\frac{n-\alpha}{2}}$$

Observation 3.2.3. We observe that

- $\varphi_\alpha \in C^\infty(\mathbb{H}_n \setminus \{0\})$ and locally integrable in $\mathbb{H}_n$ (and hence it defines a distribution)
- $\varphi_{\alpha,\epsilon} \in C^\infty(\mathbb{H}_n)$

Proof. Here we prove that φ_α is locally integrable. First we observe that

$$|\varphi_\alpha(z,t)| = \left| (|z|^2 - it)^{-\frac{n}{2}} (|z|^2 + it)^{-\frac{n}{2}} (|z|^2 - it)^{-\frac{\alpha}{2}} \left(\frac{1}{|z|^2 + it}\right)^{-\frac{\alpha}{2}} \right| =$$

$$= \left| (|z|^4 + it^2)^{-\frac{n}{2}} \left(\frac{|z|^2 - it}{|z|^2 + it}\right)^{-\frac{\alpha}{2}} \right| =$$

since $\left|\frac{|z|^2 - it}{|z|^2 + it}\right| = 1$,

$$= \left|(|z|^4 + it^2)^{-\frac{n}{2}}\right| = \left||z|^4 + it^2\right|^{-\frac{n}{2}} = |(z,t)|^{-2n}.$$

Now, given $B(0,1)$ the unit ball with respect to the distance given by the nonisotropic norm on $\mathbb{H}_n$, we can integrate on $B(0,1)$, switch to polar coordinates and find out that the integrate is finite. That completes the proof. □

Observation 3.2.4. Moreover, using the Lebesgue's dominated convergence theorem, we see that $\varphi_{\alpha,\epsilon}$ converges in the distribution sense to φ_α. That is,

$$\int_{\mathbb{H}_n} \varphi_{\alpha,\epsilon}(z,t)\psi(z,t)dzdt \to \int_{\mathbb{H}_n} \varphi_\alpha(z,t)\psi(z,t)dzdt, \quad \forall\psi \in C_0^\infty(\mathbb{H}_n).$$

Definition 3.2.5. We recall the Euler Γ function as

$$\Gamma : \mathbb{C} \setminus \mathbb{N}_0^- \to \mathbb{R}$$
$$\Gamma(s) := \int_0^\infty e^{-t}t^{s-1}dt$$

where $\mathbb{N}_0^- = \{0, -1, -2, \dots\}$.
It's possible to show that this function satisfy the property $\Gamma(s) = s\Gamma(s-1)$.

Theorem 3.2.6.
We are now ready to state that, for $\alpha \in \mathbb{C}$,

$$\mathcal{L}_\alpha\varphi_\alpha = c_\alpha\delta_0$$

where

$$c_\alpha = \frac{2^{4-2n}\pi^n}{\Gamma(\frac{n+\alpha}{2})\Gamma(\frac{n-\alpha}{2})}$$

Then $\Phi_\alpha := c_\alpha^{-1}\varphi_\alpha$ is a fundamental solution for $\mathcal{L}_\alpha$

Remark 3.2.7. Notice that $c_\alpha \neq 0 \Leftrightarrow$ the denominator doesn't have a pole, i.e., $\alpha \neq \pm(n+2k), \quad k \in \mathbb{N} \cup \{0\}$.
Therefore we will call the numbers $\pm(n+2k)$ *non-admissible values.*

Proof of remark. Let $k \in \mathbb{N} \cup \{0\}$

$$\frac{n \pm \alpha}{2} \neq -k \;\Leftrightarrow\; n \pm \alpha \neq -2k \;\Leftrightarrow\; \pm\alpha \neq -n - 2k \;\Leftrightarrow$$
$$\Leftrightarrow\; \alpha \neq \pm(-n-2k) \;\Leftrightarrow\; \alpha \neq \pm(n+2k).$$

□

Proof of theorem. Set $\zeta_\epsilon(z,t) := |z|^2 + \epsilon^2 - it$ for $\epsilon > 0$. Then $\varphi_{\alpha,\epsilon}(z,t) = \zeta_\epsilon^{-\frac{n+\alpha}{2}}\overline{\zeta}_\epsilon^{-\frac{n-\alpha}{2}}$.
Recalling that $Z_j = \partial_{z_j} + i\overline{z}_j\partial_t$ and that $\overline{Z}_j = \partial_{\overline{z}_j} - iz_j\partial_t$, for $1 \leq j \leq n$, all the following properties are true:

- $Z_j\zeta_\epsilon^a = 2a\overline{z}_j\zeta_\epsilon^{a-1}$ and $\overline{Z}_j\overline{\zeta}_\epsilon^a = 2az_j\overline{\zeta}_\epsilon^{a-1}$

- $Z_j\overline{\zeta}_\epsilon^a = 0$ and $\overline{Z}_j\zeta_\epsilon^a = 0$
- $T\zeta_\epsilon^a = -ia\zeta_\epsilon^{a-1}$ and $T\overline{\zeta}_\epsilon^a = ia\overline{\zeta}_\epsilon^{a-1}$

We check the first property of every point:

$$Z_j\zeta_\epsilon^a = (\partial_{z_j} + i\overline{z}_j\partial_t)(z\overline{z} + \epsilon^2 - it)^a = a\zeta_\epsilon^{a-1}(\overline{z}_j - i^2\overline{z}_j) = a\zeta_\epsilon^{a-1}2\overline{z}_j$$

$$Z_j\overline{\zeta}_\epsilon^a = (\partial_{z_j} + i\overline{z}_j\partial_t)(z\overline{z} + \epsilon^2 + it)^a = a\zeta_\epsilon^{a-1}(\overline{z}_j + i^2\overline{z}_j) = 0$$

$$T\zeta_\epsilon^a = \partial_t(z\overline{z} + \epsilon^2 - it)^a = a\zeta_\epsilon^{a-1}\partial_t(z\overline{z} + \epsilon^2 - it) = -ia\zeta_\epsilon^{a-1}.$$

We also note that

$$\varphi_{\alpha,\epsilon}(z,t) = (|z|^2 + \epsilon^2 - it)^{-\frac{n+\alpha}{2}}(|z|^2 + \epsilon^2 + it)^{-\frac{n-\alpha}{2}} =$$

$$= \left(\left(\left|\frac{z}{\epsilon}\right|^2 + 1 - i\frac{t}{\epsilon^2}\right)\epsilon^2\right)^{-\frac{n+\alpha}{2}} \left(\left(\left|\frac{z}{\epsilon}\right|^2 + 1 + i\frac{t}{\epsilon^2}\right)\epsilon^2\right)^{-\frac{n-\alpha}{2}} =$$

$$= \left(\left|\frac{z}{\epsilon}\right|^2 + 1 - i\frac{t}{\epsilon^2}\right)^{-\frac{n+\alpha}{2}} \epsilon^{-n-\alpha} \left(\left|\frac{z}{\epsilon}\right|^2 + 1 + i\frac{t}{\epsilon^2}\right)^{-\frac{n-\alpha}{2}} \epsilon^{-n+\alpha} =$$

$$= \epsilon^{-2n}\left(\varphi_{\alpha,1}\circ\delta_{\frac{1}{\epsilon}}\right)(z,t)\,.$$

Since $\mathcal{L}_\alpha$ is an homogeneous operator of order -2 with respect to $|\cdot|$, we have that

$$\mathcal{L}_\alpha\varphi_{\alpha,\epsilon}(z,t) = \mathcal{L}_\alpha\left(\epsilon^{-2n}\left(\varphi_{\alpha,1}\circ\delta_{\frac{1}{\epsilon}}\right)\right)(z,t) = \epsilon^{-2n-2}\mathcal{L}_\alpha\left(\varphi_{\alpha,1}\circ\delta_{\frac{1}{\epsilon}}\right)(z,t)$$

Therefore, for $\psi \in C_0^\infty(\mathbb{H}_n)$,

$$< \mathcal{L}_\alpha\varphi_\alpha(z,t), \psi >= \lim_{\epsilon\to 0+} < \mathcal{L}_\alpha\varphi_{\alpha,\epsilon}(z,t), \psi >=$$

$$= \lim_{\epsilon\to 0+}\int_{\mathbb{H}_n} \mathcal{L}_\alpha\varphi_{\alpha,\epsilon}(z,t)\psi(z,t)dV =$$

$$= \lim_{\epsilon\to 0+}\epsilon^{-2n-2}\int_{\mathbb{H}_n} \mathcal{L}_\alpha\left(\varphi_{\alpha,1}\circ\delta_{\frac{1}{\epsilon}}\right)(z,t)\,\psi(z,t)dV =$$

with the change of variables $(\tilde{z},\tilde{t}) = \delta_{\frac{1}{\epsilon}}(z,t)$,

$$= \lim_{\epsilon\to 0+}\int_{\mathbb{H}_n} \mathcal{L}_\alpha\varphi_{\alpha,1}\left(\tilde{z},\tilde{t}\right)(\psi\circ\delta_\epsilon)(\tilde{z},\tilde{t})dV = \lim_{\epsilon\to 0+}\int_{\mathbb{H}_n} \mathcal{L}_\alpha\varphi_{\alpha,1}\left(z,t\right)(\psi\circ\delta_\epsilon)(z,t)dV =$$

since we have uniformly convergence on ϕ domain, we compute the limit as $\epsilon \to 0$ under the integral sign and we get

$$= \psi(0,0) \int_{\mathbb{H}_n} \mathcal{L}_\alpha \varphi_{\alpha,1}\,(z,t)\,dV = c_\alpha \psi(0,0),$$

if we call

$$c_\alpha := \int_{\mathbb{H}_n} \mathcal{L}_\alpha \varphi_{\alpha,1}\,(z,t)\,dV.$$

Now it only remains to compute c_α.
A straightforward calculation show that

$$\mathcal{L}_\alpha \varphi_{\alpha,1}(z,t) = (n^2-\alpha^2)(|z|^2+1-it)^{-\frac{n+\alpha+2}{2}}(|z|^2+1+it)^{-\frac{n-\alpha+2}{2}}.$$

Then

$$c_\alpha = \int_{\mathbb{H}_n} \mathcal{L}_\alpha \varphi_{\alpha,1}\,(z,t)\,dV = \frac{1}{2^n}\int_{\mathbb{H}_n} \mathcal{L}_\alpha \varphi_{\alpha,1}\,(z,t)\,dzdt =$$

$$= \frac{1}{2^n}\int_{\mathbb{H}_n} \left(n^2-\alpha^2\right)(|z|^2+1-it)^{-\frac{n+2+\alpha}{2}}(|z|^2+1+it)^{-\frac{n+2-\alpha}{2}}dxdydt =$$

by setting $a = \frac{n+2+\alpha}{2}$ and $b = \frac{n+2-\alpha}{2}$,

$$= \frac{1}{2^n}\int_{\mathbb{H}_n} \left(n^2-\alpha^2\right)(|z|^2+1-it)^{-a}(|z|^2+1+it)^{-b}dxdydt =$$

$$= \frac{n^2-\alpha^2}{2^n}\int_{\mathbb{C}^n}(|z|^2+1)^{-n-1}dxdy\int_{\mathbb{R}}(1-it)^{-a}(1+it)^{-b}dt.$$

Let's compute the two integral separately. The first integral is

$$\int_{\mathbb{C}^n}(|z|^2+1)^{-n-1}dxdy =$$

using polar coordinates and reminding $|S(0,1)| = \frac{2\pi^n}{\Gamma(n)}$,

$$= \frac{2\pi^n}{\Gamma(n)}\int_0^\infty \frac{r^{2n-1}}{(1+r^2)^{n+1}}dr =$$

$t = 1+r^2,\ \frac{dt}{2} = dr$

$$= \frac{\pi^n}{\Gamma(n)}\int_1^\infty t^{-n-1}(1-t)^{n-1}dt =$$

$s = t^{-1}$

$$= \frac{\pi^n}{\Gamma(n)} \int_0^1 (1-s)^{n-1} ds =$$

by the definition of $\Gamma(n)$

$$= \frac{\pi^n}{\Gamma(n+1)}.$$

To compute the second integral, we start with some assumptions. We assume that $-n \leq \alpha \leq n$. So we have that

$$n + \alpha \geq 0 \Rightarrow n + 2 + \alpha \geq 2 \Rightarrow a = \frac{n+2+\alpha}{2} \geq 1$$

and

$$n - \alpha \geq 0 \Rightarrow n + 2 - \alpha \geq 2 \Rightarrow b = \frac{n+2-\alpha}{2} \geq 1.$$

Given $s \in \mathbb{C}$, if $\text{Re}(s) > 0$ the following formula is true:

$$\int_0^\infty e^{-xs} x^{b-1} dx = \Gamma(b) s^{-b}.$$

Set $s = 1 + it$, then we define the function

$$\widehat{f}(t) := \Gamma(b)(1+it)^{-b} = \int_0^\infty e^{-x(1+it)} x^{b-1} dx = \int_0^\infty e^{-x} e^{-ixt} x^{b-1} dx.$$

This function is the Fourier transform of

$$f(x) = \begin{cases} e^{-x} x^{b-1} & , \text{ for } x > 0 \\ 0 & , \text{ for } x \leq 0. \end{cases}$$

Similarly we obtain that

$$\widehat{g}(t) = \Gamma(a)(1-it)^{-a} = \int_0^\infty e^{-x(1-it)} x^{a-1} dx = \int_0^\infty e^{-x} e^{ixt} x^{a-1} dx = \int_{-\infty}^0 e^{-|x|} e^{-ixt} |x|^{a-1} dx$$

is the Fourier transform of

$$g(x) = \begin{cases} 0 & , \text{ for } x \geq 0 \\ e^{-|x|} |x|^{a-1} & , \text{ for } x < 0. \end{cases}$$

Hence,

$$\Gamma(a)\Gamma(b) \int_{-\infty}^\infty (1+it)^{-b}(1-it)^{-a} dt = \int_{-\infty}^\infty \widehat{f}(t) \widehat{g}(t) dt =$$

by the Plancherel theorem (Parseval identity)

$$= 2\pi \int_{-\infty}^{\infty} f(x)g(-x)dx = 2\pi \int_{-\infty}^{\infty} e^{-2x}x^{a+b-2}dx =$$

changing $y = 2x$ and using $a + b - 2 = n$

$$= \frac{\pi\Gamma(n+1)}{2^n}.$$

This implies

$$\int_{-\infty}^{\infty} (1+it)^{-b}(1-it)^{-a}dt = \frac{2^{-n}\pi\Gamma(n+1)}{\Gamma(a)\Gamma(b)} \qquad (*)$$

for $-n \leq \alpha \leq n$.
In fact, the left-hand side of this equality defines an entire function of α from the following equality:

$$\int_{-\infty}^{\infty} (1+it)^{-b}(1-it)^{-a}dt = \int_{-\infty}^{\infty} (1+it)^{-\frac{n+2}{2}} e^{i\alpha \tan^{-1} t}dt.$$

Thus $(*)$ holds for all $\alpha \in \mathbb{C}$.
Hence,

$$c_\alpha = \frac{n^2-\alpha^2}{2^n} \int_{\mathbb{C}^n} (|z|^2+1)^{-n-1}dxdy \int_{\mathbb{R}} (1-it)^{-a}(1+it)^{-b}dt =$$

$$= \frac{n^2-\alpha^2}{2^n} \frac{\pi^n}{\Gamma(n+1)} \frac{2^{-n}\pi\Gamma(n+1)}{\Gamma(\frac{n+2+\alpha}{2})\Gamma(\frac{n+2-\alpha}{2})} = \frac{\pi^{n+1}2^{-2n}}{\Gamma(\frac{n+\alpha}{2})\Gamma(\frac{n-\alpha}{2})}.$$

That completes the proof of the theorem. □

3.3 Hypoellipticity of $\mathcal{L}_\alpha$ and $\Box_b$

Now, using theorem 3.2.6, we show that $\mathcal{L}_\alpha$ and $\Box_b$ are, in fact, hypoelliptic.

Definition 3.3.1.
Recalling definition 2.3.10 and observations 2.3.11 and 2.3.12 about the convolution, if $\alpha \neq \pm(n+2k)$ $\forall k \in \mathbb{N} \cup \{0\}$, we define

$$K_\alpha f := f * \Phi_\alpha \quad \forall f \in C_0^\infty(\mathbb{H}_n).$$

Observation 3.3.2. We can immediately say that $K_\alpha f \in C^\infty(\mathbb{H}_n)$.

Theorem 3.3.3.
If $f \in C_0^\infty(\mathbb{H}_n)$ and $\alpha \neq \pm(n+2k)\ \forall k \in \mathbb{N} \cup \{0\}$, then

$$\mathcal{L}_\alpha K_\alpha f = K_\alpha \mathcal{L}_\alpha f = f$$

Proof. Since $\mathcal{L}_\alpha$ is left-invariant (see definition 2.1.7), we have

$$\mathcal{L}_\alpha K_\alpha f = \mathcal{L}_\alpha (f * \Phi_\alpha) = f * \mathcal{L}_\alpha \Phi_\alpha = f * \delta = f.$$

On the other hand, we can take $g \in C_0^\infty(\mathbb{H}_n)$.
Note that $-\alpha \neq \pm(n+2k) \Leftrightarrow \alpha \neq \pm(n+2k)$. Then we just saw that $\mathcal{L}_{-\alpha} K_{-\alpha} f = f$ and we say

$$\int_{\mathbb{H}_n} g(u) f(u) dV(u) = \int_{\mathbb{H}_n} \left(\mathcal{L}_{-\alpha} K_{-\alpha} g\right)(u) f(u) dV(u) =$$

by integration by parts,

$$= \int_{\mathbb{H}_n} K_{-\alpha} g(u) \mathcal{L}_\alpha f(u) dV(u) = \int_{\mathbb{H}_n} \left(g * \Phi_{-\alpha}\right)(u) \mathcal{L}_\alpha f(u) dV(u) =$$

by observation 2.3.12

$$= \int_{\mathbb{H}_n} g(u) (\mathcal{L}_\alpha f * \check{\Phi}_{-\alpha})(u) dV(u) =$$

and since $\check{\Phi}_{-\alpha} \equiv \Phi_\alpha$ (that comes immediately by the definition of φ_α and c_α),

$$= \int_{\mathbb{H}_n} g(u) (\mathcal{L}_\alpha f * \Phi_\alpha)(u) dV(u) = \int_{\mathbb{H}_n} g(u) K_\alpha \mathcal{L}_\alpha f(u) dV(u).$$

Then $f = K_\alpha \mathcal{L}_\alpha f$ and the proof is complete. □

Theorem 3.3.4.

$$\mathcal{L}_\alpha \text{ is hypoelliptic } \Leftrightarrow \alpha \neq \pm(n+2k)\ \forall k \in \mathbb{N} \cup \{0\}$$

In particular,
$\Box_b$ is hypoelliptic on $\mathbb{H}_n$ for $(0,q)$-forms when $1 \leq q < n$.

Proof.
[$\Rightarrow$]
If $\alpha = \pm(n+2k)$, $k \in \mathbb{N} \cup \{0\}$, the function $\varphi_\alpha(z,t)$ defined in 3.2.2 has a pole and then is a nonsmooth solution of $\mathcal{L}_\alpha \varphi_\alpha = 0$. This proves the first arrow.
[$\Leftarrow$]

On the other hand, if $\alpha \neq \pm(n+2k)$, we take $f \in \mathcal{D}'(\mathbb{H}_n)$ such that $\mathcal{L}_\alpha f = g$ is smooth on some open set $U \in \mathbb{C}$. We want to prove that f is also smooth. Let $V \subset U$ be an open set relatively compact in U. Let also take a cut-off function $\zeta \in C_0^\infty(U)$, $\zeta = 1$ in some open neighborhood of $\overline{V}$.
Then ζg is smooth and, by the theorem 3.3.3, $\mathcal{L}_\alpha K_\alpha(\zeta g) = \zeta g$. Hence, to show that f is smooth on V, it's enough to show that $h := \zeta(f - K_\alpha(\zeta g))$ is smooth on V.
Since h is a distribution with compact support, we can say, using 3.3.3 again, that $\mathcal{L}_\alpha K_\alpha h = h$. Then, on V (where $\zeta \equiv 1$), we have:

$$\mathcal{L}_\alpha h = \mathcal{L}_\alpha f - \mathcal{L}_\alpha K_\alpha(\zeta g) = g - \zeta g = g - g = 0.$$

The fact that Φ_α is just singular at the origin garantees that $\mathcal{L}_\alpha K_\alpha h = \mathcal{L}_\alpha h * \Phi_\alpha$ is smooth on V. Then h is smooth on V and then f is so.
The hypoellipticity of $\Box_b$ on $(0,q)$-forms when $1 \leq q < n$ follows immediately from proposition 2.4.8:

$$\Box_b(\phi) = \sum_{|I|=q}{}' \mathcal{L}_{n-2q}\phi_I d\overline{z}^J.$$

This proves the theorem. □

Observation 3.3.5. We finish this chapter giving a solution for the problem stated in observation 1.4.17, namely finding a solution for the inhomogeneous $\overline{\partial}_b$ equation.
At the beginning the problem was stated in the form "Solve $\overline{\partial}_b u = f$ where u is a $(0,q)$-form and f is a $(0,q+1)$-form".
To find a suitable u is difficult because f and u don't belong to the same space and, more, we have to satisfy the compatibiliy condition $\overline{\partial}_b f = 0$.
Hence, it's easier to handle the operator $\Box_b$ that goes from $(0,q)$-forms to $(0,q)$-forms and doesn't have a compatibily condition. Since we already studied the $\Box_b$ operator, now we can state the following theorem and solve the problem.

Theorem 3.3.6.
Let $f \in C_{0(0,q)}(\mathbb{H}_n)$, $1 \leq q < n$.
If $\overline{\partial}_b f = 0$ in the distribution sense, then $u := \overline{\partial}_b^* K_{n-2q} f$ satisfies $\overline{\partial}_b u = f$ and $u \in C^{\frac{1}{2}}_{(0,q)}(\mathbb{H}_n, \text{loc})$.
Moreover, if $f \in C^k_{0(0,q)}(\mathbb{H}_n)$, $k \in \mathbb{N}$, then $u \in C^{k+\frac{1}{2}}_{(0,q)}(\mathbb{H}_n, \text{loc})$.

Chapter 4

The Generalized Heisenberg Group $\mathbb{H}_{n,k}$

Goal. In this chapter we define the *generalized Heisenberg group* $\mathbb{H}_{n,k}$, an extension of $\mathbb{H}_n$, and we provide a short presentation of the exetended results.

4.1 Extended definitions and results

Definition 4.1.1.
For $1 \leq k \leq n$, let

$\Omega_{n,k} := \left\{(z', z_{n+1}) \in \mathbb{C}^n \times \mathbb{C} / \ \mathrm{Im} z_{n+1} > |z_1|^2 + \ldots + |z_k|^2 - |z_{k+1}|^2 - \ldots - |z_n|^2\right\}$

The boundary of $\Omega_{n,k}$ is identified with the *generalized Heisenberg group* $\mathbb{H}_{n,k} := \mathbb{C}^n \times \mathbb{R}$ by

$$\pi : \partial\Omega_{n,k} \to \mathbb{H}_{n,k}$$

$$\left(z', t + i\left(\sum_{j=1}^{k} |z_j|^2 - \sum_{j=k+1}^{n} |z_j|^2\right)\right) \mapsto (z', t)$$

i.e., identifying $\partial\Omega_{n,k} \equiv \mathbb{H}_{n,k}$, we write

$$(z', t) \equiv \left(z', t + i\left(\sum_{j=1}^{k} |z_j|^2 - \sum_{j=k+1}^{n} |z_j|^2\right)\right).$$

Proposition 4.1.2.
Let $(z, t), (w, u) \in \mathbb{H}_{n,k}$.
The group structure of $\mathbb{H}_{n,k}$ is defined by

$$(z, w) * (w, u) = (z, w)(w, u) := \left(z + w, t + u + 2\mathrm{Im}\left(\sum_{j=1}^{k} z_j \overline{w}_j - \sum_{j=k+1}^{n} z_j \overline{w}_j\right)\right)$$

Observation 4.1.3. One can verify immediately that

$$\begin{aligned} Z_j &= \partial_{z_j} + i\overline{z_j}\partial_t, \quad \text{for } 1 \le j \le k \\ Z_j &= \partial_{z_j} - i\overline{z_j}\partial_t, \quad \text{for } k+1 \le j \le n \\ T &= \partial_t \end{aligned}$$

are left-invariant vector fields on $\mathbb{H}_{n,k}$ such that

$$[Z_j, \overline{Z}_j] = \begin{cases} -2iT, & \text{for } 1 \le j \le k \\ 2iT, & \text{for } k+1 \le j \le n \end{cases}$$

and that all the other commutators vanish.

Proposition 4.1.4.
From observation 4.1.3, it follows that the Z_j's define a non-degenerate CR structure on $\mathbb{H}_{n,k}$ such that the Levi form (defined in 1.5.2) has k positive eigenvalues and $n-k$ negative eigenvalues.

Definition 4.1.5.
Without loss of generality, we can assume $k \ge \frac{n}{2}$ and we shall call a CR structure as in 4.1.4 *k-strongly pseudoconvex.*

Observation 4.1.6. As we did for $\mathbb{H}_n$, we can fix a left-invariant metric on $\mathbb{H}_{n,k}$ which makes the basis $Z_1, \ldots, Z_n, \overline{Z}_1, \ldots, \overline{Z}_n, T$ orthonormal.
Its dual basis is given by

$$\{dz_1, \ldots, dz_n, d\overline{z}_1, \ldots, d\overline{z}_n, \tau\}$$

where $dz_j = x_j + iy_j$, $1 \le j \le n$ and

$$\tau = dt + 2\sum_{j=1}^{k}(x_j dy_j - y_j dx_j) - 2\sum_{j=k+1}^{n}(x_j dy_j - y_j dx_j).$$

More, as in observation 2.3.9, the volume element is

$$dV = \frac{1}{2^n} dx dy dt.$$

Definition 4.1.7.
Now we calculate $\Box_b$ on $\mathbb{H}_{n,k}$.
Let $K = \{1, \ldots, k\}$ and $K' = \{k+1, \ldots, n\}$. $\forall J$, such that $|J| = q$, we set

$$\alpha_J := |K \setminus J| + |K' \cap J| - |K \cap J| - |K' \setminus J|$$

Proposition 4.1.8.
Hence, if $f = \sum'_{|J|=q} f_J d\overline{z}^J \in C^\infty_{0(0,q)}(\mathbb{H}_{n,k})$ is a $(0,q)$-form with compact support on $\mathbb{H}_{n,k}$, we get:

$$\Box_b f = \left(\overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b * \overline{\partial}_b\right) \left(\sum_{|J|=q}' f_J d\overline{z}^J\right) = -\sum_{|J|=q}' \left(\left(\sum_{m \neq J} Z_m \overline{Z}_m \sum_{m \in J} \overline{Z}_m Z_m\right) f_J\right) d\overline{z}^J =$$

$$= \sum_{|J|=q}' \left(\left(-\frac{1}{2}\sum_{m=1}^{n}(Z_m \overline{Z}_m + \overline{Z}_m Z_m) + i\alpha_J T\right) f_J\right) d\overline{z}^J.$$

Observation 4.1.9. Note that $-n \leq \alpha_j \leq n$ and that

$$\alpha_J = n \Leftrightarrow J = K' \quad \text{and} \quad \alpha_J = -n \Leftrightarrow J = K.$$

Theorem 4.1.10.
$\Box_b$ is hypoelliptic for $(0,q)$-forms, $0 \leq q \leq n$, on $\mathbb{H}_{n,k}$ if $q \neq k$ and $q \neq n-k$. The conclusion of theorem 3.3.6 also holds for $\mathbb{H}_{n,k}$ when $q \neq k$ and $q \neq n-k$.

The proof of the first statement of the theorem follows from theorem 3.3.4 changing the coordinates z_j, $k+1 \leq j \leq n$, to $\overline{z}_j$. The proof of the second statement is the same as in theorem 3.3.6.

Chapter 5

Estimates for $\mathcal{L}_\alpha$ on $\mathbb{H}_n$

Goal. The goal of this chapter is to show some regularity theorems for $\mathcal{L}_\alpha$ on $\mathbb{H}_n$. In order to prove them, we need a lot of additional results. However, since some of them will rely on Calderón-Zygmund results and principal values theory, we will sometimes skip the details of the proofs. Every proof can be found in sections 8, 9 and 10[6].
In section 5.1 we'll just prepare our devices, then we'll arrive to state L^p and Hölder estimates for $\mathcal{L}_\alpha$.

5.1 Homogeneous and PV Distributions on $\mathbb{H}_n$

We already gave the definitions of left and right invariant operators and of the reflection operator in 2.1.6. Using them, we now define:

Definition 5.1.1.
Let $f \in C_0^\infty(\mathbb{H}_n)$ and $G \in \mathcal{D}'(\mathbb{H}_n)$, then we can define the functions

$$(G * f)(u) : \mathbb{H}_n \to \mathbb{C}$$
$$\left((G * f)(u)\right)(v) := (G(J\tau_u f))(v) = G(Jf(u^{-1}v)) = G(f(v^{-1}u))$$

and

$$(f * G)(u) : \mathbb{H}_n \to \mathbb{C}$$
$$((f * G)(u))(v) := (G(J\tau^u f))(v) = G(Jf(uv^{-1})) = G(f(vu^{-1}))$$

We can think at this " $*$ " as an extention of the convolution where, instead of two functions, we have a function and a distribution.

Observation 5.1.2. We can observe immediately that $(G*f)(u), (f*G)(u) \in C^\infty(\mathbb{H}_n)$.

Definition 5.1.3.
We already know $X_1, \dots, X_n, Y_1, \dots, Y_n, T$ is an orthonormal basis for left invariant vector fields and that, in a neighborhood of the origin, it is equal to $\partial_{x_1}, \dots, \partial_{x_n}, \partial_{y_1}, \dots, \partial_{y_n}, \partial_t$.
In the same way, we define a orthonormal basis for right invariant vector fields and we call it $R_1, \dots, R_{2n+1}$.
It will also be useful to denote $\{X_j, Y_j,\ 1 \leq j \leq n\}$ as $\{L_j,\ 1 \leq j \leq 2n\}$.

Observation 5.1.4. The operator $K_\alpha : C_0^\infty(\mathbb{H}_n) \to C^\infty(\mathbb{H}_n)$ (defined in 3.3.1) is continuous and can be extended as an operator $K_\alpha : \mathcal{E}'(\mathbb{H}_n) \to \mathcal{D}'(\mathbb{H}_n)$. We can do the same for $\mathcal{L}_\alpha$, L_j and R_j see (p.444 [6]).

Observation 5.1.5. So we have that X_j, Y_j commute with τ_u, $\forall j = 1, \dots, n$, and R_j commutes with τ^u, $\forall j = 1, \dots, 2n$.

Definition 5.1.6.
We also define the distribution $D_\theta \in \mathcal{E}'(\mathbb{H}_n)$, where θ is a general tangent vector at the origin, by

$$D_\theta(f) := - < df(0), \theta >= - < \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j}(0) + \frac{\partial f}{\partial y_j}(0) \right) + \frac{\partial f}{\partial t}(0), \theta > .$$

It is a diretional derivative calculated in zero.

Notation 5.1.7. In particular, for $1 \leq j \leq 2n$, we set

$$D_j := \begin{cases} D_{\partial x_j}, & \text{if } j = 1, \dots, n \\ D_{\partial y_j}, & \text{if } j = n+1, \dots, 2n \end{cases}$$

They are the diretional derivatives, calculated in zero, in the directions given by the coordinates.

Observation 5.1.8. If $f \in C_0^\infty(\mathbb{H}_n)$, we have the following properties:

- $R_j f = D_j * f$
- $L_j f = f * D_j$

Definition 5.1.9.
If $F \in \mathcal{D}'(\mathbb{H}_n)$ is a distribution, we also define

- $D_j * F := R_j F$

- $F * D_j := L_j F$

Since we want to talk about homogeneity, we define:

Definition 5.1.10.
If $f : \mathbb{H}_n \to \mathbb{R}$, we define

$$f_r(u) := f(ru) \;\; \forall u \in \mathbb{H}_n, \; \forall r > 0$$

and

$$f^r(u) := r^{-2n-2} f(r^{-1}u) \;\; \forall u \in \mathbb{H}_n, \; \forall r > 0.$$

Definition 5.1.11.
Given this notation, we obviously say that f is *homogeneous of degree* $\lambda \Leftrightarrow$

$$f_r(u) \equiv f(ru) = r^\lambda f(u) \;\; \forall u \in \mathbb{H}_n, \; \forall r > 0.$$

This notion extends naturally to distributions: $F \in \mathcal{D}'(\mathbb{H}_n)$ is *homogeneous of degree* $\lambda \Leftrightarrow$

$$F(g^r) = r^\lambda F(g) \;\; \forall g \in C_0^\infty(\mathbb{H}_n), \; \forall r > 0.$$

Example 5.1.12. δ is an homogeneous distribution of degree $-2n-2$.

Proof.

$$\delta(f^r) = \delta(r^{-2n-2} f(r^{-1}u)) = r^{-2n-2} f(0) = r^{-2n-2}\delta(f).$$

□

Proposition 5.1.13.
Let $F \in \mathcal{D}'(\mathbb{H}_n)$ be a homogeneous distribution of degree λ, then

$$L_j F, \; R_j F \;\; \text{are homogeneous of degree } \lambda - 1, \; \forall j : 1 \le j \le 2n.$$

Proof. The proposition is proved with a straightforward computation. □

Lemma 5.1.14. If f is a homogeneous function of degree λ, $\lambda \in \mathbb{R}$, $f \in C^1(\mathbb{H}_n \setminus \{0\})$, then there exists a constant $C > 0$ such that

$$|f(u) - f(v)| \le C|u-v||u|^{\lambda-1}, \quad \text{if } \; |u-v| \le \frac{1}{2}|u|,$$

and

$$|f(uw) - f(u)| \le C|w||u|^{\lambda-1}, \quad \text{if } \; |w| \le \frac{1}{2}|u|.$$

Proof. We may assume, by homogeneity, that $|u| = 1$. Then $|u - v| \leq \frac{1}{2}$ and so v is bounded away from zero. So, by the mean value theorem and by observation 2.3.3 respectively, we get

$$|f(u) - f(v)| \leq C||u - v|| \leq C|u - v|.$$

The same argument in the second case yields

$$|f(uw) - f(u)| \leq C||uw - u|| \leq$$

since $w \mapsto uw$ is smooth

$$\leq ||uw - u|| \leq C||w|| \leq C|w|.$$

□

We shall now be particularly concerned with functions and distributions which are homogeneous of degree $-2n-2$, since $2n+2$ is the homogeneous dimension of $\mathbb{H}_n$ (see observation 2.3.8). We start with the notion of "mean value".

Proposition 5.1.15.
Let $f : \mathbb{H}_n \to \mathbb{C}$ be an homogeneous function of degree $-2n-2$ and locally integrable away from the origin.
Then there exists a constant μ_f, that we name *mean value*, such that

$$\int_{\mathbb{H}_n} f(u) g(|u|) dV(u) = \mu_f \int_0^\infty g(r) r^{-1} dr$$

$\forall g : (0, +\infty) \to \mathbb{R}$ measurable so that either integral is defined.
Note that, strickly speaking, the mean value should be μ_f divided by $|B(0,1)|$.
This is a mean value kind of theorem.

Proof. We set

$$A_f(r) = \begin{cases} \int_{1\leq|u|\leq r} f(u) dV(u), & \text{for } r \geq 1 \\ -\int_{r\leq|u|\leq 1} f(u) dV(u), & \text{for } 0 < r < 1. \end{cases}$$

$A_f(r)$ is a continuous function on $(0, \infty)$ and it's possible to see that

$$A_f(rs) = A_f(r) + A_f(s).$$

Hence there exists a constant μ_f such that $A_f(r) = \mu_f \log r$. Without loss of generality, we suppose $r \geq 1$ and $1 \leq a \leq b \leq r$.

Taking $g = \chi(a,b)$, where $\chi(a,b)$ is the characteristic function of (a,b), we should prove that

$$\int_{a\le|u|\le b} f(u)dV(u) = \mu_f \int_a^b r^{-1}dr;$$

that is true because it is exactly $A_f(r) = \mu_f \log r$ restricted on (a,b). So the proposition is proved when g is the characteristic function of an interval.
Then the proof is completed by forming linear combinations and passing to limits to obtain general g's. □

Example 5.1.16. As a trivial example, we can denote $\mu_f = c_0$ and take $f(u) = |u|^{-2n-2}$ and $g(u) = r^\lambda \chi(a,b)$, $0 < a < b < \infty$.
Then we have

$$\int_{a\le|u|\le b} |u|^{\lambda-2n-2}dV(u) = \begin{cases} c_0\frac{1}{\lambda}(b^\lambda - a^\lambda), & \text{for } \lambda \neq 0 \\ c_0 \log\frac{b}{a}, & \text{for } \lambda = 0. \end{cases}$$

Definition 5.1.17.
If $f : \mathbb{H}_n \to \mathbb{C}$ is a homogeneous function of degree $-2n-2$, continuous away from the origin and its mean value μ_f is zero, then f defines the distribution:

$$(PVf) : C_0^\infty(\mathbb{H}_n) \to \mathbb{C}$$
$$(PVf)(g) := PV\int_{\mathbb{H}_n} f(u)g(u)dV(u) \equiv \lim_{\epsilon\to0+}\int_{|u|\ge\epsilon} f(u)g(u)dV(u)$$

and we will prove that the limit exists.

Proof. Since $\mu_f = 0$, we have

$$\int_{\mathbb{H}_n} f(u)h(|u|)dV(u) = 0$$

for all measurable h. We take $h(u) = \chi(\epsilon \le |u| \le 1)$ and we multiply everything for $g(0)$. Then we have

$$\int_{\epsilon\le|u|\le1} f(u)g(0)dV(u) = 0.$$

So

$$(PVf)(g) = \int_{|u|\ge1} f(u)g(u)dV(u) +$$

$$+\lim_{\epsilon\to0+}\left(\int_{\epsilon\le|u|\le1} f(u)g(u)dV(u) - \int_{\epsilon\le|u|\le1} f(u)g(0)dV(u)\right) =$$

$$= \int_{|u|\geq 1} f(u)g(u)dV(u) + \lim_{\epsilon\to 0+} \int_{\epsilon\leq|u|\leq 1} f(u)\left(g(u)-g(0)\right)dV(u) =$$

since $g(u)-g(0)=O(|u|)$, the integral is absolutely convergent

$$= \int_{|u|\geq 1} f(u)g(u)dV(u) + \int_{|u|\leq 1} f(u)\left(g(u)-g(0)\right)dV(u)$$

and we are done. □

Observation 5.1.18. From the definition it's evident that PVf is homogeneous of degree $-2n-2$.

Definition 5.1.19.
A distribution $F \in \mathcal{D}'(\mathbb{H}_n)$ is said to be *regular* if there exists $f \in C^\infty(\mathbb{H}_n \setminus \{0\})$ such that

$$F(g) = \int_{\mathbb{H}_n} f(u)g(u)dV(u) \quad \forall g \in C_0^\infty\left(\mathbb{H}_n \setminus \{0\}\right).$$

If that happens, we say that F *agrees* with f.

Proposition 5.1.20.
Let us take $F \in \mathcal{D}'(\mathbb{H}_n)$ a regular homogeneous distribution of degree λ which agrees with $f \in C^\infty(\mathbb{H}_n \setminus \{0\})$.
Then

1. f is homogeneous of degree λ
2. if $\lambda > -2n-2 \Rightarrow f \in \mathcal{D}'(\mathbb{H}_n \setminus \{0\})$ and $F = f$
3. if $\lambda = -2n-2 \Rightarrow \mu_f = 0$ and $F = (PVf) + c\delta$

Proof. 1.
The first point is obvious from the definition.
2.
For the second point, we observe that, since f is homogeneous of degree $\lambda > -2n-2$, then f is locally integrable and thus define a distribution (that we will call "f" as well). So $F-f$ is a distribution supported at 0, that is, a linear combination of δ and its derivatives.
But δ is homogeneous of degree $-2n-2$ and then we see that for any $g \in C_0^\infty(\mathbb{H}_n)$, with $(F-f)(g) \neq 0$, we have both

$$(F-f)(g^r) = O(r^{-2n-2}) \quad \text{as} \quad r \to \infty$$

and, by hypothesis,

$$(F - f)(g^r) = r^\lambda (F - f)(g),$$

which is a contradiction. Hence $F - f = 0$.
3.
If $\mu = 0$, then $F - PVf$ is homogeneous of degree $-2n-2$ ans supported at the origin, hence is a multiple of δ.
For the case $\mu \neq 0$, we just give the idea of the proof. If we set $\beta = \frac{\mu_f}{C_0}$, where C_0 is the mean value of the function $|\cdot|^{-2n-2}$, then it's possible to prove that both the distributions

$$F'(h) := F(h) - PV(f - \beta|\cdot|^{-2n-2})(h)$$

and

$$G(h) := \beta \int_{|u|\leq 1} (h(u) - h(0))|u|^{-2n-2} dV(u) + \beta \int_{|u|>1} h(u)|u|^{-2n-2} dV(u)$$

agree with $\beta|\cdot|^{-2n-2}$ away from the origin. Hence $F' = G + H$ where H is a linear combination of δ and its derivatives. From this we find a contradiction using the same homogeneity argument of point 2. □

The third point of this proposition allows us to give the following definition:

Definition 5.1.21. We call *PV distribution* a regular homogeneous distribution of degree $-2n-2$ and we include δ among the PV distributions.

PV distributions play the role of the classical singular integral kernels on $\mathbb{H}_n$ and here we have the analogue of the Calderón-Zygmund theorem:

Proposition 5.1.22.
If $F \in \mathcal{D}'(\mathbb{H}_n)$ is a PV distribution, then the mapping

$$\begin{aligned} C_0^\infty(\mathbb{H}_n) &\to C^\infty(\mathbb{H}_n) \\ g &\mapsto g * F \end{aligned}$$

extends to a bounded transformation on $L^p(\mathbb{H}_n)$, $1 < p < \infty$.

Proof. We refer to [8] for the case $p = 2$, and to [9] or [3] for the extension to the other values of p. In section 15 of [6] there is also the proof of a generalization of this theorem. □

A similar result is avaialable for kernels of higher homogeneity:

Proposition 5.1.23.
If $F \in \mathcal{D}'(\mathbb{H}_n)$ is a regular homogeneous distribution of degree λ, with $-2n-2 < \lambda < 0$.
Then the mapping

$$C_0^\infty(\mathbb{H}_n) \to C_0^\infty(\mathbb{H}_n)$$
$$g \mapsto g * F$$

extends to a bounded transformation from $L^p(\mathbb{H}_n)$ to $L^q(\mathbb{H}_n)$, $1 < p, q < \infty$, where

$$\frac{1}{q} = \frac{1}{p} - \frac{\lambda}{2n+2} - 1.$$

The same mapping also extends from $L^1(\mathbb{H}_n)$ to $L_{loc}^{-\frac{2n+2}{\lambda}-\epsilon}(\mathbb{H}_n)$, $\forall \epsilon > 0$.

Proof. This proposition is proved in the 15th section of [6]. □

Before proceeding, we need to make some remarks about the right-invariant version of the operator $\mathcal{L}_0$. We first note that $\mathcal{L}_0 = -\frac{1}{4}\sum_{j=1}^{n}(X_j^2 + Y_j^2) = -\frac{1}{4}\sum_{j=1}^{2n} L_j^2$, so the corresponding right-invariant operator is $\mathcal{R}_0 = -\frac{1}{4}\sum_{j=1}^{2n} R_j^2$

Lemma 5.1.24. We can recognize that

$$\mathcal{R}_0\Phi_0 = \delta$$

where Φ_0 is the fundamental solution of $\mathcal{L}_\alpha f = \delta$.

Proof. This comes from theorem 3.3.3. Indeed, if $\mathcal{D}_0 = -\sum_{j=1}^{2n} D_j * D_j$ is the distribution kernel of $\mathcal{L}_0$, then $\mathcal{L}_0\Phi_0 = \Phi_0 * \mathcal{D}_0$ and $\mathcal{R}_0\Phi_0 = \mathcal{D}_0 * \Phi_0$ and both of these expressions are equal to δ. □

More, since we can write $\mathcal{R}_0\Phi_0 = \mathcal{D}_0 * \Phi_0 = -\sum_{j=1}^{2n} D_j * D_j * \Phi_0$, we get:

$$-\sum_{j=1}^{2n} D_j * D_j * \Phi_0 = \delta$$

We now come to the deepest new result of this section, which provides us in effect with a set of "noncommutative Riesz transforms" with which to manipulate derivatives.

Theorem 5.1.25.
If $F \in \mathcal{D}'(\mathbb{H}_n)$ is a PV distribution, there exist regular homogeneous distributions $F_1, \ldots, F_{2n}$ of degree $-2n-1$ such that

$$F = \sum_{j=1}^{2n} D_j * F_j$$

Proof. Since this proof is quite long, we are going to skip some details. Using lemma 5.1.24 we may formally write

$$F = -\sum_{j=1}^{2n} D_j * D_j * \Phi_0 * F$$

and thus call $F_j := -D_j * \Phi_0 * F$. Therefore, to prove the theorem we have to show that $\Phi_0 * F$ can be defined as a regular homogeneous distribution of degree $-2n$ with the property that $\mathcal{R}_0(\Phi_0 * F) = F$. If we do that, we'll have that F_j is a regular homogeneous distribution of degree $-2n-1$ and then the thorem will be proved.
From now on, we'll divide the proof in steps.

Step a) If $F = \delta$, we have immediately that $\Phi_0 * F = \Phi_0 * \delta = \Phi_0$ is regular, homogeneous of degree $-2n$ and the property holds.

Step b.i) Thus, by the third point of proposition 5.1.20, we may assume that $F = PVf$ for some f with $\mu_f = 0$.
Given $v_0 \neq 0$, we choose $\epsilon > 0$ small such that $\{u \in \mathbb{H}_n : |u| < \epsilon \text{ and } |v_0 u^{-1}| < \epsilon\} = \varnothing$. Then, setting $w = v_0 u^{-1}$, we may define

$$\Phi_0 * F(v_0) := PV \int_{\mathbb{H}_n} \Phi_0(v_0 u^{-1}) f(u) dV(u) =$$

and using the same argument of 5.1.17, we get

$$= \int_{|u|<\epsilon} (\Phi_0(v_0 u^{-1}) - \Phi_0(v_0)) f(u) dV(u) +$$

$$+ \int_{|w|<\epsilon} \Phi_0(w) f(w^{-1} v_0) dV(u) + \int_{|u|\geq\epsilon, |w|\geq\epsilon} \Phi_0(v_0 u^{-1}) f(u) dV(u).$$

All these three integrals are absolutely convergent. The same formula, with v instead of v_0 but where still $w = v_0 u^{-1}$, can be used to define $\Phi_0 * F(v)$ for any v close to v_0 so that $vu^{-1} \neq 0$ when $|v| < \epsilon$ and $w^{-1}u \neq 0$ when $|w| < \epsilon$. Thus, differentiating under the integral sign, we can conclude that $\Phi_0 * F$ is C^∞ away from the origin and, with a straightforward calculation, that is homogeneous of degree $-2n$.

Step b.ii) Now we are left to prove that $\mathcal{R}_0(\Phi_0 * F) = F$ and this can be done with an approximation argument. Using the same ideas of section 3.2, we can define a function $\Phi_{0,\epsilon}$ such that

$$\mathcal{R}_{0,\epsilon}\Phi_{0,\epsilon} \to \delta \quad \text{as} \quad \epsilon \to 0.$$

We can also define a C^∞ function $\Phi_{0,\epsilon} * F(v)$ like the one above.
The main point is now to show that

$$\Phi_{0,\epsilon} * F \to \Phi_0 * F \quad \text{when} \quad \epsilon \to 0$$

as distributions; that would prove the theorem. By the Lebesgue dominated convergence theorem, it will suffice to show that $|\Phi_{0,\epsilon} * F(v)| \leq C|v|^{-2n}$. To prove that we would need a lot of calculation and the lemmas we previously stated. We skip this part. $\square$

5.2 L^p Estimates for $\mathcal{L}_\alpha$

We now introduce some Sobolev-type spaces which differ from the classical ones in that they take into account the characteristic splitting of directions on $\mathbb{H}_n$. Then we'll use them to provide our estimates.

Definition 5.2.1.
Let $\mathcal{A}$ be the algebra of all the left-invariant operators on $\mathbb{H}_n$.

Observation 5.2.2. Since $\{X_1, \dots, X_n, Y_1, \dots, Y_n\} = \{L_1, \dots, L_{2n}\}$ generate the Lie algebra of $\mathbb{H}_n$, together with the identity operator I, they generate $\mathcal{A}$ and induce a filtration $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots$ on $\mathcal{A}$.
Namely, for $k \in \mathbb{N} \cup \{0\}$, we define

$$\mathcal{B}_k := \{L_{a_1} L_{a_2} \cdots L_{a_i} \cdots L_{a_j} \ / \ 1 \leq a_i \leq 2n, \ i = 1, \dots, j, \ j \leq k\};$$

e.g. $\mathcal{B}_1 = \{L_1, L_2, \dots, L_{2n}\}$ and $\mathcal{B}_2$ is the set of all pairs of kind $L_i L_j$.
We also define

$$\mathcal{A}_k :=< \mathcal{B}_k \cup \{I\} >_{\mathbb{C}},$$

so $\mathcal{A}_k$ is the linear span of $\mathcal{B}_k \cup \{I\}$ over $\mathbb{C}$.

Definition 5.2.3.
For $1 \leq p \leq \infty$, we now define our Sobolev-type spaces:

$$S^{k,p}(\mathbb{H}_n) := \{f \in L^p(\mathbb{H}_n) \ s.t. \ Df \in L^p(\mathbb{H}_n) \ \forall D \in \mathcal{A}_k\}.$$

Observation 5.2.4. We can immediately state some properties about $S^{k,p}(\mathbb{H}_n)$.

- $S^{k,p}(\mathbb{H}_n)$ is a Banach space under the norm

$$||f||_{k,p} := ||f||_p + \sum_{D \in \mathcal{B}_k} ||Df||_p$$

- $C_0^\infty(\mathbb{H}_n) \subseteq S^{k,p}(\mathbb{H}_n)$ where $C_0^\infty(\mathbb{H}_n)$ is dense on $S^{k,p}(\mathbb{H}_n)$.
- Since $T = [L_{n+1}, L_1] \in \mathcal{A}_2$,

$$S^{k,p}(\mathbb{H}_n) \subseteq \{f \in L^p_{loc}(\mathbb{H}_n) \ / \ Df \in L^p_{loc}(\mathbb{H}_n) \ \forall D \in \mathcal{A}_j, 0 \leq j \leq k\}.$$

Notation 5.2.5. Let's suppose that the distributions we'll use in this chapter will always be regular. That is, we'll always have a function that agrees with the distribution.

Definition 5.2.6.
If $U \subseteq \mathbb{H}_n$ is an open set, we define

$$S^{k,p}(U, loc) := \{F \in \mathcal{D}'(\mathbb{H}_n) \ s.t. \ \varphi F \in S^{k,p}(\mathbb{H}_n) \ \forall \varphi \in C_0^\infty(U)\}$$

where we say that a regular distribution F is L^p if the function that agrees with F is L^p.

Observation 5.2.7. It follows immediately that

$$S^{k,p}(\mathbb{H}_n) \subseteq S^{k,p}(\mathbb{H}_n, loc)$$

Moreover, if $U = \bigcup_{j=1}^\infty V_j$, where $V_j \subseteq \mathbb{H}_n$ open, then

$$S^{k,p}(U, loc) = \bigcap_{j=1}^{\infty} S^{k,p}(V_j, loc).$$

However, the virtue of the spaces $S^{k,p}(\mathbb{H}_n)$ becomes evident in the following proposition:

Proposition 5.2.8.
If $F \in \mathcal{D}'(\mathbb{H}_n)$ is a PV distribution, then the mapping

$$\begin{aligned} C_0^\infty(\mathbb{H}_n) &\to C^\infty(\mathbb{H}_n) \\ g &\mapsto g * F \end{aligned}$$

extends to a bounded operator on $S^{k,p}(\mathbb{H}_n)$, $1 < p < \infty$, $k \in \mathbb{N} \cup \{0\}$.

Proof. For each fixed p, we prove the proposition by induction on k. The case $k = 0$ is the proposition 5.1.22. Suppose now that the assertion is true for k. Let $g \in C_0^\infty$ and $h = g * F$. Then, by the definition of norm for $S^{k,p}(\mathbb{H}_n)$,

$$||h||_{k+1,p} \leq ||h||_{k,p} + \sum_{j=1}^{2n} ||L_j h||_{k,p} \leq$$

by the boundness in the inductive hypothesis $||h||_{k,p} \leq C||g||_{k,p}$, so

$$\leq C||g||_{k,p} + \sum_{j=1}^{2n} ||L_j h||_{k,p} \leq$$

and by the definition of norm again

$$\leq C||g||_{k+1,p} + \sum_{j=1}^{2n} ||L_j h||_{k,p}$$

By theorem 5.1.25, we may rewrite $F = \sum_{l=1}^{2n} D_l * F_l$, whence
$L_j h = h * D_j = g * F * D_j = \sum_{l=1}^{2n} g * (D_l * F_l) * D_j = \sum_{l=1}^{2n} (g * D_l) * (F_l * D_j) = = \sum_{l=1}^{2n} (L_l g) * (L_j F_l)$. By proposition 5.1.13, $L_j F_l$ is a PV distribution, so by inductive hypothesis,

$$||L_j h||_{k+1,p} \leq \sum_{l=1}^{2n} C_l ||L_l g||_{k,p} \leq C||g||_{k+1,p}.$$

The proof is complete. □

Finally, after these preliminares, we are ready to return to the operator $\mathcal{L}_\alpha$ and its fundamental solution Φ_α.

Proposition 5.2.9.
Suppose α is admissible, that is $\alpha \neq \pm(n+2k)$ with $k \in \mathbb{N} \cup \{0\}$, and let $f \in C_0^\infty(\mathbb{H}_n)$. Then

1. The mapping

$$\begin{aligned} C_0^\infty(\mathbb{H}_n) &\to C^\infty(\mathbb{H}_n) \\ f &\mapsto K_\alpha f = f * \Phi_\alpha \end{aligned}$$

   extends to a bounded mapping $L^p(\mathbb{H}_n) \to L^q(\mathbb{H}_n)$, where $1 < p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{n+1}$.
   The same mapping also extends to a bounded mapping
   $L^1(\mathbb{H}_n) \to L_{loc}^{\frac{n+1}{n-\epsilon}}(\mathbb{H}_n)\ \forall \epsilon > 0$.

2. The mappings

$$\begin{aligned} C_0^\infty(\mathbb{H}_n) &\to C^\infty(\mathbb{H}_n) \\ f &\mapsto L_j K_\alpha f = f * \Phi_\alpha * D_j, \quad 1 \leq j \leq 2n \end{aligned}$$

   extend to bounded mappings from $L^p(\mathbb{H}_n) \to L^r(\mathbb{H}_n)$, where $1 < p < r < \infty$ and $\frac{1}{r} = \frac{1}{p} - \frac{1}{2n+2}$.

3. The mappings

$$C_0^\infty(\mathbb{H}_n) \to C^\infty(\mathbb{H}_n)$$
$$f \mapsto L_i L_j K_\alpha f = f * \Phi_\alpha * D_j * D_i, \quad 1 \le i, j \le 2n$$

extend to bounded mappings on $S^{k,p}(\mathbb{H}_n)$ for $1 < p < \infty$ and $k \in \mathbb{N} \cup \{0\}$.

Proof. This proposition follows immediately from propositions 5.1.13, 5.1.22 and 5.1.23, since Φ_α is a regular homogeneous function of degree $-2n$. □

As a consequence, we obtain the following estimates for $\mathcal{L}_\alpha$:

Theorem 5.2.10.
If α is admissible, $1 < p < \infty$ and $k \in \mathbb{N} \cup \{0\}$, then

$$||f||_{k+2,p} \le c_{k,p} \left(||\mathcal{L}_\alpha f||_{k,p} + ||f||_p\right) \quad \forall f \in C_0^\infty(\mathbb{H}_n)$$

Proof. By theorem 3.3.3 we have that $f = K_\alpha \mathcal{L}_\alpha f$ and so, by point 3. in proposition 5.2.9 applied to $\mathcal{L}_\alpha f$,

$$||L_i L_j K_\alpha(\mathcal{L}_\alpha f)||_{k,p} \le c_{k,p} ||\mathcal{L}_\alpha f||_{k,p}$$

and so

$$||L_i L_j f||_{k,p} \le c_{k,p} ||\mathcal{L}_\alpha f||_{k,p}.$$

But we have that $||f||_{k+2,p} \le ||f||_p + \sum_{j=1}^{2n} ||L_j f||_p + \sum_{i,j=1}^{2n} ||L_i L_j f||_{k,p}$. So, to conclude, it suffices to show that

$$||L_j f||_p \le c_j (||L_j^2 f||_{k,p} + ||f||_p).$$

And this can be done by applying the Taylor's theorem to $f(u\gamma(t))$ where $\gamma(t)$ is a one-parameter subgroup generated by L_j and then using the translation invariance of $||\cdot||_p$ and Minkowski's inequality. □

Now we are going to state the main L^p regularity theorem, that will be complemented by theorem 5.3.10.

Theorem 5.2.11.
Let's take α admissible and $F, G \in \mathcal{D}'(\mathbb{H}_n)$ such that $\mathcal{L}_\alpha F = G$ on $U \subset \mathbb{H}_n$.

- If $G \in S^{k,p}(U, loc)$, then $F \in S^{k+2,p}(U, loc)$
- If $G \in L^p_{loc}(U)$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{n+1} > 0$, then

$$\begin{cases} F \in L^q(U, loc), & \text{if } p > 1 \\ F \in L^{q-\epsilon}(U, loc)\ \forall \epsilon > 0, & \text{if } p = 1. \end{cases}$$

Proof.
I)
Given any $V \subset\subset U$, choose $\varphi \in C_0^\infty(U)$ with $\varphi = 1$ on V and set $H := K_\alpha(\varphi G)$. Then if $G \in S^{k,p}(U, loc)$, we have by point 3. in proposition 5.2.9 that $L_i L_j H \in S^{k,p}(V)$, with $1 \le i, j \le 2n$.
We claim that also $L_j H$ and H are in $L^p(V)$, and hence $H \in S^{k+2,p}(V)$. Once it is shown, we have to note that $\mathcal{L}_\alpha(F - G) = G(1-\phi) = 0$ on V. So, since $\mathcal{L}_\alpha$ is hypoelliptic, $F - H \in C^\infty(V)$. Hence $F \in S^{k+2,p}(V)$ for all $V \subset\subset U$ and we are done.
II)
The second case is proved exactly in the same way, taking $G \in L^p_{loc}(U)$ and finding out that, with an assumpition on p, $H \in L^q(V)$ or $H \in L^{q-\epsilon}(V)$. After the same claim we arrive to say that $F \in L^q(V)$ or $F \in L^{q-\epsilon}(V)$.
To prove the claim, let

$$W := \{v^{-1}u : \ v \in \mathrm{supp}\varphi, u \in V\}.$$

W is bounded, therefore we may choose $\psi \in C_0^\infty(U)$, with $\psi = 1$ on W. Then, for $u \in V$,

$$H(u) = (\varphi G) * (\psi \Phi_\alpha)(u) \quad \text{and} \quad L_j H(u) = (\varphi G) * (\psi \Phi_\alpha * D_j)(u)$$

and so H and $L_j H$ are in $L^p(V)$, being convolutions of L^p and L^1 functions. That completes the claim and, hence, the proof. □

5.3 Hölder Estimates for $\mathcal{L}_\alpha$

Since here we want to talk about Hölder estimates, it is natural to introduce a family $\Gamma^\beta(\mathbb{H}_n)$ of Lipschitz and Hölder spaces on the Heisenberg Group which are defined in terms of the norm $|u| = |(z,t)| = (|z|^4 + t^2)^{\frac{1}{2}}$ instead of the Euclidean norm $||\cdot||$. We'll then show the estimates.

Definition 5.3.1.
We define the spaces:

- for $0 < \beta < 1$

$$\Gamma^\beta(\mathbb{H}_n) := \left\{ f \in L^\infty(\mathbb{H}_n) \cap C^0(\mathbb{H}_n) \ s.t. \ \sup_{u,v \in \mathbb{H}_n} \frac{|f(vu) - f(v)|}{|u|^\beta} < \infty \right\},$$

- for $\beta = 1$

$$\Gamma^1(\mathbb{H}_n) := \left\{ f \in L^\infty(\mathbb{H}_n) \cap C^0(\mathbb{H}_n) \ s.t. \ \sup_{u,v \in \mathbb{H}_n} \frac{|f(vu) + f(vu^{-1}) - 2f(v)|}{|u|} < \infty \right\},$$

- for $\beta = k + \beta'$ where $k \in \mathbb{N}$ and $0 < \beta' < 1$

$$\Gamma^\beta(\mathbb{H}_n) := \left\{ f \in L^\infty(\mathbb{H}_n) \cap C^0(\mathbb{H}_n) \ s.t. \ Df \in \Gamma^{\beta'}(\mathbb{H}_n) \ \forall D \in \mathcal{A}_k \right\}.$$

If $f \in \Gamma^\beta(\mathbb{H}_n)$, we'll call f a *Hölder function*.[1]

Observation 5.3.2. The spaces $\Gamma^\beta(\mathbb{H}_n)$ are Banach spaces, respectively, under the norms:

- for $0 < \beta < 1$

$$||f||_\beta := ||f||_\infty + \sup_{u,v \in \mathbb{H}_n} \frac{|f(vu) - f(v)|}{|u|^\beta},$$

- for $\beta = 1$

$$||f||_1 := ||f||_\infty + \sup_{u,v \in \mathbb{H}_n} \frac{|f(vu) + f(vu^{-1}) - 2f(v)|}{|u|},$$

- for $\beta = k + \beta'$ where $k \in \mathbb{N}$ and $0 < \beta' < 1$

$$||f||_\beta := ||f||_\infty + \sum_{D \in \mathcal{B}_k} ||Df||_{\beta'}.$$

The first theorem of this section says essentially that the convolution with a PV distribution is a bounded operator on $\Gamma^\beta(\mathbb{H}_n)$ $\forall \beta$, $0 < \beta < \infty$.

Theorem 5.3.3.
If $K_0 \in \mathcal{D}'(\mathbb{H}_n)$ is a PV distribution and $f \in \Gamma^\beta(\mathbb{H}_n)$, $0 < \beta < \infty$, has compact support, then

$$f * K_0 \in \Gamma^\beta(\mathbb{H}_n).$$

The proof will be accomplished by some results. For the purpose of the demostration, we call K the function which agrees with K_0 away from zero and we may assume $K_0 = PV(K)$.

Lemma 5.3.4. If $f \in \Gamma^\beta(\mathbb{H}_n)$ with compact support, $0 < \beta < 1$, then

$$f * K_0 \in \Gamma^\beta(\mathbb{H}_n)$$

[1] Futher notions about these spaces and their relationship with the standard Hölder spaces can be found in [6], chapter 20.

Proof of the Lemma. We call $g := f * K_0$. Then we have, with $u \in \mathbb{H}_n$

$$g(v) = f * K_0(v) = PV(K)\left(f\left(vu^{-1}\right)\right) = PV \int_{u \in \mathbb{H}_n} f(vu^{-1})K(u)dV(u) =$$

$$= \lim_{\epsilon \to 0} \int_{\epsilon \leq |u| \leq 1} f(vu^{-1})K(u)dV(u) + \int_{|u|>1} f(vu^{-1})K(u)dV(u) =$$

using exactly the same argument of 5.1.17,

$$= \lim_{\epsilon \to 0} \int_{\epsilon \leq |u| \leq 1} (f(vu^{-1}) - f(v))K(u)dV(u) + \int_{|u|>1} f(vu^{-1})K(u)dV(u) =$$

and then the limit disappear by definition of $\Gamma^\beta(\mathbb{H}_n)$,

$$= \int_{|u| \leq 1} (f(vu^{-1}) - f(v))K(u)dV(u) + \int_{|u|>1} f(vu^{-1})K(u)dV(u).$$

Hence, if we suppose that W_v is the support of the function $u \mapsto f(vu^{-1})$,

$$|g(v)| \leq C \int_{|u| \leq 1} |u|^\beta |u|^{-2n-2} dV(u) + C||f||_\infty \int_{|u|>1, u \in W_v} |u|^{-2n-2} dV(u).$$

Using example 5.1.16 and the fact that, for large v, W_v is contained in a set of the form $a_1|v| \leq |u| \leq a_2|v|$, with $a_1, a_2 > 0$, we see that the right-hand side is bounded uniformly in v, so that g is bounded.
Next we have, given $w \in \mathbb{H}_n$ and from the definition of PV,

$$g(v) = PV \int_{|u| \leq B|w|} f(vu^{-1})K(u)dV(u) + \int_{|u|>B|w|} f(vu^{-1})K(u)dV(u) \equiv$$

$$\equiv g_w(v) + g^w(v).$$

where $B \geq 2$. Now we can write

$$|g_w(v)| = \left| \int_{|u| \leq B|w|} \left(f(vu^{-1}) - f(v)\right) K(u)dV(u) \right| \leq$$

$$\leq C \int_{|u| \leq B|w|} |u|^{\beta-2n-2} dV(u) \leq C(B|w|)^\beta.$$

Thus, since v is arbitrary, we have

$$\frac{|g_w(vw) - g_w(v)|}{|w|^\beta} \leq CB^\beta.$$

On the other hand, we can set

$$K^\omega(u) := \begin{cases} K(u), & \text{if } |u| > B|\omega| \\ 0, & \text{if } |u| \le B|\omega| \end{cases}$$

and use the fact that $\mu_K = 0$ to have

$$g^w(vw) - g^w(v) = \int_{\mathbb{H}_n} f(vwu^{-1})K^w(u)dV(u) - \int_{\mathbb{H}_n} f(vu^{-1})K^w(u)dV(u) =$$

$$= \int_{\mathbb{H}_n} \big(f(vwu^{-1}) - f(vw)\big) K^w(u)dV(u) - \int_{\mathbb{H}_n} \big(f(vu^{-1}) - f(vw)\big) K^w(uw)dV(u) +$$

$$+ \int_{\mathbb{H}_n} \big(f(vu^{-1}) - f(v)\big) \left(K^w(uw) - K^w(u)\right) dV(u).$$

The first two integrals cancel each other by a change of variables and we divide the third one as

$$\int_{|u|\ge B|w|,|uw|\ge B|w|} + \int_{|u|\ge B|w|,|uw|\le B|w|} + \int_{|u|\le B|w|,|uw|\ge B|w|} \equiv I_1 + I_2 + I_3.$$

To estimate them we take B large enough so that the region of I_2 (respectively I_3) is contained in a set of the form $\{B|w| \le |u| \le B'|w|\}$ (respectively $\{b|w| \le |u| \le B|w|\}$) with $B', b > 0$. This is possible by lemma 2.3.4.
By example 5.1.16 and lemma 5.1.14, we have

$$|I_1| \le \int_{|u|\ge B|w|} \left|\big(f(vu^{-1}) - f(v)\big) \left(K(uw) - K(u)\right)\right| dV(u) \le$$

$$\le C \int_{|u|\ge B|w|} |u|^{\beta-2n-3}|w|dV(u) \le C|w| \left(B|w|\right)^{\beta-1} \le C|w|^\beta.$$

And

$$|I_2| \le \int_{B|w|\le|u|\le B'|w|} \left|\big(f(vu^{-1}) - f(v)\big) K(u)\right| dV(u) \le$$

$$\le C \int_{B|w|\le|u|\le B'|w|} |u|^{\beta-2n-2}dV(u) \le C|w|^\beta.$$

I_3 can be estimated likewise, since $|K(uw)| \le C|w|^{-2n-2}$ over his region of integration.
All together we get

$$\frac{|g^w(vw) - g^w(v)|}{|w|^\beta} \le C$$

and that completes the proof. □

Lemma 5.3.5. If $f \in \Gamma^\beta(\mathbb{H}_n)$, $1 < \beta < 2$, then

$$f * K_0 \in \Gamma^\beta(\mathbb{H}_n)$$

Proof of the Lemma. We call $g := f * K_0$ again and, thanks to lemma 5.3.4, we already know that $g \in \Gamma^{\beta-1}(\mathbb{H}_n)$. So we just have to prove that $L_j g \in \Gamma^{\beta-1}(\mathbb{H}_n)$, $1 \leq j \leq 2n$.
Using theorem 5.1.25, we can write $K_0 = \sum_{j=1}^{2n} D_j * K_j$, where K_j is a regular homogeneous distribution of degree $-2n-1$. Then

$$L_j g = f * K_0 * D_j = \sum_{l=1}^{2n} f * D_l * K_l * D_j = \sum_{l=1}^{2n} L_l f * L_j K_l.$$

Since $L_j K_l$ is a PV distribution and $L_l f \in \Gamma^{\beta-1}(\mathbb{H}_n)$, by lemma 5.3.4 we conclude that $L_j g \in \Gamma^{\beta-1}(\mathbb{H}_n)$. □

Proof of the Theorem. If we can show that $f \in \Gamma^1(\mathbb{H}_n)$ implies $f * K_0 \in \Gamma^1(\mathbb{H}_n)$, then the theorem will follow by induction on the greatest integer in β, by the argument in the proof of the last lemma.
But this boundness on $\Gamma^1(\mathbb{H}_n)$ will follow immediately from the last two lemmas together with the following proposition that provides a characterization of $\Gamma^1(\mathbb{H}_n)$. □

Proposition 5.3.6.
The following holds:

$$f \in \Gamma^1(\mathbb{H}_n) \iff$$
$$\forall \tau \geq 1\ \exists f_1 \in \Gamma^{\frac{1}{2}}(\mathbb{H}_n) \text{ and } \exists f_2 \in \Gamma^{\frac{3}{2}}(\mathbb{H}_n)$$
$$\text{with } ||f_1||_{\frac{1}{2}} \leq C\tau^{-1} \text{ and } ||f_2||_{\frac{3}{2}} \leq C\tau,\ C \text{ indipendent of } \tau,$$
$$\text{such that } f = f_1 + f_2.$$

For the proof of this proposition we need two other lemmas:

Lemma 5.3.7. If $f \in \Gamma^\beta(\mathbb{H}_n)$, $1 < \beta < 2$, then

$$\sup_{v,w \in \mathbb{H}_n} \frac{|f(vw) + f(vw^{-1}) - 2f(v)|}{|w|^\beta} < \infty.$$

Proof of the Lemma. Applying a uniformly smooth partition of unity, we may assume that f has compact support. We use theorem 5.1.25 to write

$$f = f * \delta = f * \left(\sum_{j=1}^{2n} D_j * H_j \right) = \sum_{j=1}^{2n} g_j * H_j$$

where $g_j = f * D_j = L_j f \in \Gamma^{\beta-1}(\mathbb{H}_n)$ and H_j is a regular homogeneous distribution of degree $-2n-1$.
If we set $\Phi_j^w(u) := H_j(uw) + H_j(uw^{-1}) - 2H_j(u)$, with $u, w \in \mathbb{H}_n$, we can say that

$$f(uw) + f(vw^{-1}) - 2f(v) = \sum_{j=1}^{2n} \int_{\mathbb{H}_n} g_j(vu^{-1})\Phi_j^w(u)dV(u).$$

By the same homogeneity argument as in the proof of observation 2.3.4, we have

$$|\Phi_j^w(u)| \le C|w|^2|u|^{-2n-3}, \text{ if } |w| \le \frac{1}{2}|u|.$$

In particular $\Phi_j^w \in L^1(\mathbb{H}_n)$.
It's also possible to prove that

$$\int_{\mathbb{H}_n} \Phi_j^w(u)dV(u) = 0.$$

In fact it would be obvious if $H_j \in L^1(\mathbb{H}_n)$. As it is, we consider $H_j\chi_r$, where χ_r is the characteristic function of $\{u \in \mathbb{H}_n \ / \ |u| < r\}$. Since $H_j\chi_r \in L^1(\mathbb{H}_n)$,

$$\int_{\mathbb{H}_n} \left[H_j\chi_r(uw) + H_j\chi_r(uw^{-1}) - 2H_j\chi_r(u)\right] dV(u) = 0$$

and

$$H_j\chi_r(uw) + H_j\chi_r(uw^{-1}) - 2H_j\chi_r(u) \to \Phi_j^w(u), \text{ as } r \to \infty.$$

By the Lebesgue dominated convergence theorem, we conclude that $\int_{\mathbb{H}_n} \Phi_j^w(u)dV(u) = 0$.
That enables us to write

$$f(uw) + f(vw^{-1}) - 2f(v) = \sum_{j=1}^{2n} \int_{\mathbb{H}_n} \left(g_j(vu^{-1}) - g_j(v)\right) \Phi_j^w(u)dV(u) =$$

$$= \int_{|u| \le 2|w|} + \int_{|u| > 2|w|} \equiv I_1 + I_2.$$

Applying our estimate about $|\Phi_j^w(u)|$, we get

$$|I_2| = \left|\int_{|u|>2|w|}\right| \le C \int_{|u|>2|w|} |u|^{\beta-1}|\omega|^2|u|^{-2n-3}dV(u) \le$$

$$\le C|w|^2(2|w|)^{\beta-1} \le C|w|^\beta.$$

On the other hand

$$|I_1| = \left| \int_{|u|\le 2|w|} \right| \le C(2|w|)^{\beta-1} \int_{|u|\le 2|w|} |\Phi_j^w(u)| dV(u) \le$$

and, since $|u| \le 2|w|$ implies that $|uw| \le B|w|$ and $|uw^{-1}| \le B|w|$ for some large B (see observation 2.3.4),

$$\le 4C|w|^{\beta-1} \int_{|u|\le B|w|} |H_j(u)| dV(u) \le C|w|^{\beta-1} \int_{|u|\le B|w|} |u|^{-2n-1} dV(u) \le$$

$$\le C|w|^{\beta-1}(B|w|) \le C|w|^\beta.$$

That proves the lemma. □

Lemma 5.3.8. If $g \in C^1(\mathbb{H}_n)$, then

$$\sup_{u,v\in\mathbb{H}_n} \frac{|g(vu) - g(v)|}{|u|} < C \sum_{j=1}^{2n} ||L_j g||_\infty.$$

Proof of the Lemma. Assume first $u = (z, 0)$, $z \in \mathbb{C}^n$. If L is the normalized generator of the one-parameter subgroup through u, then, by the mean value theorem, $|g(vu) - g(v)| \le C||u||\ ||Lg||_\infty$. Since $||u|| = |u|$ for $u = (z, 0)$, the lemma is proved in this case.
In general we have $u = (z, t)$. We choose $z_0 \in \mathbb{C}^n$ and write $u = u_0 u_1 u_2 u_1^{-1} u_2^{-1}$, where $u_0 = (z, 0)$, $u_1 = (\frac{1}{2} i z_0 \sqrt{t}, 0)$ and $u_2 = (\frac{1}{2} i z_0 \sqrt{t}, 0)$. Since $|u_0| = |z| \le |u|$ and $|u_1| = |u_2| = |t|^{1/2} \le |u|$, we can write $g(vu) - g(v)$ as a five-fold collapsing sum and paaly the the result we just established. □

Proof of the Proposition.
[⇐] Set

$$\Delta_w^2 f(v) := f(vw) + f(vw^{-1} - 2f(v)).$$

We suppose that, $\forall \tau \ge 1$,

$$f = f_1 + f_2, \text{ where } ||f_1||_{1/2} \le C\tau^{-1} \text{ and } ||f_2||_{3/2} \le C\tau.$$

Then we have $||\Delta_w^2 f_1||_\infty \le 2C|w|^{1/2}\tau^{-1}$ and, by lemma 5.3.7, $||\Delta_w^2 f_2||_\infty \le C|w|^{3/2}\tau$.
Hence

$$||\Delta_w^2 f_2||_\infty \le C(|w|^{1/2}\tau^{-1} + |w|^{3/2}\tau),$$

so we can take $\tau = |w|^{-1/2}$ and conclude that $f \in \Gamma^1(\mathbb{H}_n)$.

[$\Rightarrow$] Conversely, we suppose $f \in \Gamma^1(\mathbb{H}_n)$. We choose $\varphi \in C_0^\infty$ supported in $|v| \leq 1$ and satisfying $\varphi(v) = \varphi(v^{-1})$ and $\int_{\mathbb{H}_n} \varphi = 1$. Set $\varphi_k \in B(0, \frac{1}{2^k})$,

$$\varphi_k(v) := 2^{(2n+2)k}\varphi(2^k v).$$

So $\int_{\mathbb{H}_n} \varphi_k = 1\ \forall k$ and $\{\varphi_k\}$ is an approximation to the delta.
Set $f_k := f * \varphi_k$ and $g_k := f_k - f_{k-1}$. Since f is uniformly continuous, $f_k \rightrightarrows f$ uniformly, and we can write $f = f_0 + \sum_{k=1}^\infty g_k$.
It's long but possible to prove that $\exists C > 0$ indipendent of k such that

$$||g_k||_\infty \leq C2^{-k},\ ||L_j g_k||_\infty \leq C \ \text{ and } \ ||L_i L_j g_k||_\infty \leq C2^k$$

with $1 \leq i, j \leq 2n$.
From that and from lemma 5.3.8 we get that $||g_k||_{1/2} \leq C2^{-k/2}$ and $||g_k||_{3/2} \leq C2^{k/2}$.
Therefore we write $f = (f_0 + \sum_{k=1}^N g_k) + \sum_{k=N+1}^\infty g_k \equiv f_1 + f_2$ where

$$||f_1||_{3/2} = ||f_0 + \sum_{k=1}^N g_k||_{3/2} \leq C\sum_{k=1}^N 2^{k/2} \leq C2^{N/2}$$

and

$$||f_2||_{1/2} = ||\sum_{k=N+1}^\infty g_k||_{1/2} \leq C\sum_{k=N+1}^\infty 2^{-k/2} \leq C2^{-N/2}.$$

This provides a decompotition of f for $\tau = 2^{N/2}$, $N \in \mathbb{N}$ and the proposition 5.3.6 follows immediately. $\square$

We prove now a theorem on kernels with homogeneity higher then $-2n-2$. This result complements proposition 5.1.23.
For simplicity, we consider only integral degrees of homogeneity, which suffices for the applications.

Theorem 5.3.9.
Let K be a regular homogeneous distribution of degree $k - 2n - 2$, $k \in \mathbb{N}$, and let f be a function of compact support. Then

1. if $f \in \Gamma^\beta(\mathbb{H}_n)$, $0 < \beta < \infty$, then $f * K \in \Gamma_{loc}^{\beta+k}(\mathbb{H}_n)$
2. if $f \in L^p(\mathbb{H}_n)$, $\beta = k - \frac{2n+2}{p} > 0$, then $f * K \in \Gamma_{loc}^\beta(\mathbb{H}_n)$

where

$$\Gamma_{loc}^\beta(\mathbb{H}_n) := \left\{f \in C(\mathbb{H}_n)\ s.t.\ \varphi f \in \Gamma^\beta(\mathbb{H}_n)\ \forall \varphi \in C_0^\infty(\mathbb{H}_n)\right\}.$$

Proof. As usual we call $g = f * K$. If we take $D \in \mathcal{B}_k \setminus \mathcal{B}_{k-1}$, then

$$Dg = D(f * K) = f * DK$$

and DK is a PV distribution (because its degree is exactly $-2n-2$).
Hence, by theorem 5.3.3, $Dg \in \Gamma^\beta(\mathbb{H}_n)$ $\forall D \in \mathcal{B}_k \setminus \mathcal{B}_{k-1}$; so $g \in \Gamma^{\beta+k}_{loc}(\mathbb{H}_n)$ and the first point is proved.
To prove the second point we take

$$k_0 = \min\{k \in \mathbb{N} \text{ such that } k > (2n+2)/p\}$$

and

$$\beta_0 = k_0 - (2n+2)/p.$$

Note that $0 < \beta_0 \leq 1$.
As above, by considering derivatives on the kernel K, it suffices to prove the assertion for k_0 and β_0. Now we can divide the proof in two cases: $\beta_0 = 1$ and $0 < \beta_0 < 1$. Anyway, the second case is done as in the first case making a similar estimate.
So now we suppose $\beta_0 = 1$. Then $1 = k_0 - (2n+2)/p$. We have

$$|g(vw) - g(vw^{-1}) - 2g(v) - 2g(v)| \leq ||f||_p \left(\int_{\mathbb{H}_n} |K(uw) + K(uw^{-1}) - 2K(u)|^{p'} dV(u) \right)^{\frac{1}{p'}}$$

with $\frac{1}{p} + \frac{1}{p'} = 1$.
By the same homogeneity argument as in the proof of lemma 5.1.14, we get

$$|K(uw) + K(uw^{-1}) - 2K(u)| \leq C|w|^2|u|^{k-2n-4}.$$

Hence

$$\left(\int_{|u| \geq 2|w|} |K(uw) + K(uw^{-1}) - 2K(u)|^{p'} dV(u) \right)^{\frac{1}{p'}} \leq$$

$$\leq C|w|^2 \left(\int_{|u| \geq 2|w|} |u|^{(k-2n-4)p'} du \right)^{\frac{1}{p'}} = C|w|^2|w|^{k-2n-4+(2n+2)/p'}$$

whenever $k - 2n - 4 + (2n+2)/p' < 0$. However, since $1 = k_0 - (2n+2)/p$, it follows that $k - 2n - 4 + (2n+2)/p' = -1 < 0$.
Thus

$$\left(\int_{|u| \geq 2|w|} |K(uw) + K(uw^{-1}) - 2K(u)|^{p'} dV(u) \right)^{\frac{1}{p'}} \leq C|w|.$$

Since $|u| \leq 2|w|$ implies $|uw| \leq B|w|$, and $|uw^{-1}| \leq B|w|$ for some $B \geq 2$, by observation 2.3.4

$$\left(\int_{|u|\leq 2|w|} |K(uw) + K(uw^{-1}) - 2K(u)|^{p'} dV(u)\right)^{\frac{1}{p'}} \leq$$

$$\leq 4 \left(\int_{|u|\leq B|w|} |K(u)|^{p'} dV(u)\right)^{\frac{1}{p'}} \leq$$

$$\leq C \left(\int_{|u|\leq B|w|} |u|^{(k-2n-2)p'} dV(u)\right)^{\frac{1}{p'}} = C|w|^{k-2n-2+(2n+2)/p'}$$

if $k - 2n - 2 + (2n+2)/p' > 0$ (and it is actually 1). So, at the end,

$$\left(\int_{\mathbb{H}_n} |K(uw) + K(uw^{-1}) - 2K(u)|^{p'} dV(u)\right)^{\frac{1}{p'}} \leq C|w|.$$

The proof is complete. □

We can now state the Lipschitz regularity theorem for $\mathcal{L}_\alpha$:

Theorem 5.3.10.
α admissible and $F, G \in \mathcal{D}'(\mathbb{H}_n)$ satisfy $\mathcal{L}_\alpha F = G$ on $U \subset \mathbb{H}_n$.

- If $G \in \Gamma^{\beta}_{loc}(U)$, with $0 < \beta < \infty$, then $F \in \Gamma^{\beta+2}_{loc}(U)$.
- If $G \in L^{p}_{loc}(U)$, with $\beta = 2 - \frac{2n+2}{p} > 0$, then $F \in L^{\beta}_{loc}(U)$.

Proof. The proof proceeds just like the proof of theorem 5.2.11, using theorem 5.3.9. □

Chapter 6

CR-manifolds

Goal. In this chapter our goal is to define the coordinates that will allow us to see some CR manifolds as generalizations of the Heisenberg group. The purpose of doing so is to generalize to the CR manifolds the results we already found for the Heisenberg group about the Kohn Laplacian. We start defining our tools for the case of k-strongly pseudoconvex manifolds, then we will define the so-called Normal Coordinates for general pseudoconvex CR manifolds and in the specific case of an hypersurface. Finally, we will provide an example.

6.1 $\overline{\partial}_b$, $\overline{\partial}_b^*$ and $\Box_b$ on k-strongly speudoconvex *CR* Manifolds

To define $\Box_b$ on some CR manifolds, we must impose a Hermitian metric on M. We want to restrict ourselves to a class of metrics with respect to which the eigenvalues of the Levi form are ± 1.
In the first chapter we stated the general definition of Levi form. Here we see how to costruct a metric with this property on the eigenvalues. For that, we'll need two lemmas.

Definition 6.1.1.
Let us remind the definition of CR manifold at 1.3.4 and of Levi form at 1.5.2 and let M be a CR manifold of real dimension $2n+1$.
We can also say that the Levi form on $T^{1,0}(M)$, namely $<,>_L$, can be defined by:

$$< Z, W >_L := -i < d\tau, Z \wedge \overline{W} >$$

$\forall Z, W \in T^{1,0}(M)$ and where τ is a nonvanishing real one-form which annihilates $T^{1,0}(M) \oplus T^{0,1}(M)$.

Observation 6.1.2. The Levi form can be written as

$$< Z, W >_L = \frac{1}{2} i < \tau, [Z, \overline{W}] >$$

$\forall Z, W \in T^{1,0}(M)$.

Definition 6.1.3.
Let M be a CR manifold of real dimension $2n + 1$.
We say that M is *nondegenerate* if its Levi form, namely $<,>_L$, is nondegenerate for every point,
i.e.,

$$\nexists Z \in T^{1,0}(M) \ s.t. \ < Z, Z' >_L = 0 \ \forall Z' \in T^{1,0}(M).$$

Notation 6.1.4. We say that M is *strongly pseudoconvex* if the matrix generetad by the Levi form is positive definite.
We say that M is $k-$*strongly pseudoconvex* if the matrix generetad by the Levi form has k eigenvalues bigger than zero.

Lemma 6.1.5. Suppose M is a k-strongly pseudoconvex CR manifold, with $k < n$.
Then there exist smooth subbundles $E^+(M)$ and $E^-(M)$ of $T^{1,0}(M)$ such that

- $T^{1,0}(M) = E^+(M) \oplus E^-(M)$
- The Levi form is positive definite on $E^+(M)$ and negative definite on $E^-(M)$
- $E^+(M) \perp E^-(M)$ with respect to the Levi form, i.e.,

$$< e^+, e^- >_L = 0 \quad \forall e^+ \in E^+(M), \ \forall e^- \in E^-(M).$$

Proof. Choose an arbitrary Hermitian metric $<,>$ on $T^{1,0}(M)$. The Levi form determines, for each $\xi \in M$, a linear transformation $A_\xi : T^{1,0}(\xi) \to T^{1,0}(\xi)$ which is selfadjoint with respect to $<,>$ by the equation

$$< Z, A_\xi W > = < Z, W >_L, \ \forall Z, W \in T^{1,0}(\xi).$$

A_ξ is non singular, has k positive and $n - k$ negative eigenvalues for each ξ and varies smoothly with ξ.
Now we define the fiber of $E^+(M)$ at ξ, namely $E^+(\xi)$, to be the space spanned by the eigenvectors of A_ξ with positive eigenvalues and $E^-(\xi)$ to be the orthogonal complement of $E^+(\xi)$ with respect to the Levi form (not $<,>$). It is clear that $<,>_L$ is positive definite on $E^+(\xi)$ and negative definite

on $E^-(\xi)$.
Once we have them, we can build the fiber bundle $E^+(M)$ as $\bigcup_{\xi\in M} E^+(\xi)$ if we know that $E^+(\xi)$ varies smoothly with ξ; then we can do the same for $E^-(M)$.
To check the smoothness, we fix $\xi_0 \in M$ and we choose a neighborhood V of ξ_0 such that, $\forall \xi \in V$, the positive eigenvalues of A_ξ lie in some fixed interval (a, b), with $0 < a < b < \infty$. We may then define a projection $P_\xi : T^{1,0}_\xi(M) \to E^+(\xi)$ for $\xi \in V$ by

$$P_\xi = \frac{1}{2\pi i}\int_\gamma (z - A_\xi)^{-1} dz$$

where γ is a contour in the right half-plane enclosing (a, b). P_ξ varies smoothly with ξ, so $E^+(\xi)$ does ([10]). $\Box$

Lemma 6.1.6. Suppose M is a k-strongly pseudoconvex CR manifold, with $k \le n$.
Then there exists an Hermitian metric $<,>$ on $T^{1,0}(M)$ such that

$$\forall \xi \in M,\ \exists \text{ a basis } Z_1, \dots, Z_n \text{ for } T^{1,0}(M) \text{ near } \xi \text{ so that}$$

$$< Z_i, Z_j >= \delta_{ij} \quad \text{and} \quad < Z_i, Z_j >_L = \epsilon_i \delta_{ij}$$

where $\epsilon_i = \begin{cases} 1, & i \le k \\ -1, & i > k \end{cases}$.
From the proof it's clear that we can write the metric explicitely.

Proof. If $k = n$, $\epsilon_i \equiv 1$ and we simply take $<,>=<,>_L$.
If $k < n$, by lemma 6.1.5 we choose a plitting $T^{1,0}(M) = E^+(M) \oplus E^-(M)$ and define $<,>$ on $T^{1,0}_\xi(M)$ for each $\xi \in M$ by

$$< Z, W >=< Z^+, W^+ >_L - < Z^-, W^- >_L$$

where $Z, W \in T^{1,0}_\xi(M)$ and $Z = Z^+ + Z^-$ and $W = W^+ + W^-$ are the splittings fo Z and W.
Then we obtain a basis $Z_1, \dots, Z_n$ by choosing $Z_1, \dots, Z_k$ to be an orthonormal basis for $E^+(M)$ and $Z_{k+1}, \dots, Z_n$ to be an orthonormal basis for $E^-(M)$. $\Box$

Observation 6.1.7. From now on we will consider the class of Hermitian metrics $<,>$ on $\mathbb{C}T(M)$ such that

1. $<,>|_{T^{1,0}(M)}$ satisfies the conditions in lemma 6.1.6

2. $<,>_{|_{T^{1,0}(M)}}$ determines $<,>_{|_{T^{0,1}(M)}}$ by the equation $<\overline{Z},\overline{W}>=\overline{<Z,W>}$

3. $T^{1,0}(M)\perp T^{0,1}(M)$ with respect to $<,>$

4. $<\tau,\tau>=1$ in the induced metric on $\mathbb{C}T(M)^*$.

This metric can be found by fixing τ, choosing the splitting $T^{1,0}(M)=E^+\oplus E^-$ if $k<n$ and choosing an orthogonal complement for $T^{1,0}(M)\oplus T^{0,1}(M)$. We assume that M is equipped with a metric satisfying these propersties.

Notation 6.1.8. We denote by T the vector field dual to τ.
Then $Z_1,\dots,Z_n,\overline{Z}_1,\dots,\overline{Z}_n,T$ is an orthonormal basis for $\mathbb{C}T(M)$.
We also denote the dual basis on $\mathbb{C}T(M)^*$ by $\omega_1,\dots,\omega_n,\overline{\omega}_1,\dots,\overline{\omega}_n,\tau$.
That also means that every $(0,q)$-form ϕ can be written as

$$\phi=\sum_{|J|=q}\phi_J\overline{\omega}^J$$

where ϕ_J are complex valued functions on M.
Finally, we denote the volume element as $d(\cdot)$.

Definition 6.1.9.
Suppose M is a k-strongly pseudoconvex CR manifold, with $k\le n$. We now introduce the notation $\mathcal{E}$ for error terms. In the following propositions we will use it to regroup lower-order forms so to focus on the higher-order one. If $\phi=\sum_{|J|=q}\phi_J\overline{\omega}^J$ is a smooth $(0,q)$-form on M, then $\mathcal{E}$ will denote an expression of the form

$$\mathcal{E}(\phi)\cong\sum_{|J|=q,|K|=q}a_{JK}\phi_J\overline{\omega}^K$$

with $a_{JK}\in C^\infty(M)$.
More, if $Z\in T^{1,0}(M)$,

$$\mathcal{E}(Z\phi)\cong\sum_{\substack{|J|=q,|K|=q\\ l=1,\dots,n}}a_{JKl}(Z_l\phi_J)\overline{\omega}^K$$

with $a_{JKl}\in C^\infty(M)$.
Similar expressions can be written for $\mathcal{E}(\overline{Z}\phi)$ and $\mathcal{E}(T\phi)$.
Finally we denote

$$\mathcal{E}(A,B):=\mathcal{E}(A)+\mathcal{E}(B).$$

Example 6.1.10. For example, using observation 6.1.2, the equation $< Z_i, Z_j >_L = \epsilon_i\delta_{ij}$ at lemma 6.1.6 can be written as

$$[Z_i, \overline{Z}_j]f = -2i\epsilon_i\delta_{ij}Tf + \mathcal{E}(Zf, \overline{Z}f)$$

with $Z \in T^{1,0}(M)$ and $f \in C^\infty(M)$.

Notation 6.1.11. Let J, l be such that $|J| = q$ and $l = 1, \dots, n$.

$$\overline{\omega}_l \lrcorner \overline{\omega}^J = \begin{cases} 0, \text{ if } l \notin J \\ (-1)^{i-1}\overline{\omega}_{j_1} \wedge \cdots \wedge \overline{\omega}_{j_{i-1}} \wedge \overline{\omega}_{j_{i+1}} \wedge \cdots \wedge \overline{\omega}_{j_q}, \text{ if } l = j_i. \end{cases}$$

Observation 6.1.12. It's easy to see that we also have

$$\overline{\omega}_l \lrcorner (\overline{\omega}_l \wedge \overline{\omega}^J) = \begin{cases} \overline{\omega}^J, \text{ if } l \notin J \\ 0, \text{ if } l \in J \end{cases}$$

and

$$\overline{\omega}_l \wedge (\overline{\omega}_l \lrcorner \overline{\omega}^J) = \begin{cases} \overline{\omega}^J, \text{ if } l \in J \\ 0, \text{ if } l \notin J. \end{cases}$$

In the next three propositions we'll use the same arguments and ideas of paragraph 2.4.2.

Proposition 6.1.13.
Suppose M is a k-strongly pseudoconvex CR manifold, with $k \leq n$. Reminding the definition of CR complex $\overline{\partial}_b$ given in paragraph 1.4.2, we compute it. If $f \in C^\infty(M)$, we have

$$\overline{\partial}_b f = \sum_{j=1}^{n} (\overline{Z}_j f)\overline{\omega}_j$$

Hence, if $\phi = \sum_{|J|=q} \phi_J \overline{\omega}^J$ is a smooth $(0, q)$-form, then

$$\overline{\partial}_b \phi = \sum_{\substack{|J|=q \\ l=1,\dots,n}} (\overline{Z}_l \phi_J)\overline{\omega}_l \wedge \overline{\omega}^J + \sum_{|J|=q} \phi_J \overline{\partial}_b(\overline{\omega}^J) =$$

$$= \sum_{\substack{|J|=q \\ l=1,\dots,n}} (\overline{Z}_l \phi_J)\overline{\omega}_l \wedge \overline{\omega}^J + \mathcal{E}(\phi)$$

Definition 6.1.14.
Let us take a smooth $(0,q)$-form ϕ and a smooth $(0,q-1)$-form ψ. We now define the *formal adjoint* $\overline{\partial}_b^*$ of $\overline{\partial}_b$ as the operator such that

$$< \overline{\partial}_b^* \phi, \psi > = < \phi, \overline{\partial}_b \psi > .$$

Now we can even define the self-adjoint *Kohn Laplacian* $\Box_b$ on $(0,q)$-forms as

$$\Box_b := \overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b .$$

Proposition 6.1.15.
Suppose M is a k-strongly pseudoconvex CR manifold, with $k \leq n$. Now we can compute the complex adjoint and say that, $\forall \phi$ smooth $(0,q)$-form,

$$\overline{\partial}_b^* \phi = - \sum_{\substack{|J|=q \\ l=1,\dots,n}} (Z_l \phi_J) \overline{\omega}_l \lrcorner \overline{\omega}^J + \mathcal{E}(\phi).$$

Proposition 6.1.16.
Suppose M is a k-strongly pseudoconvex CR manifold, with $k \leq n$. We also have that, $\forall \phi \in C^\infty_{(0,q)}(M)$,

$$\overline{\partial}_b^* \overline{\partial}_b \phi = - \sum_{\substack{|J|=q,\ l=1,\dots,n \\ i=1,\dots,n}} (Z_l \overline{Z}_i \phi_J) \overline{\omega}_l \lrcorner (\overline{\omega}_i \wedge \overline{\omega}^J) + \mathcal{E}(Z\phi, \phi)$$

and

$$\overline{\partial}_b \overline{\partial}_b^* \phi = - \sum_{\substack{|J|=q,\ l=1,\dots,n \\ i=1,\dots,n}} (\overline{Z}_i Z_l \phi_J) \overline{\omega}_i \wedge (\overline{\omega}_l \lrcorner \overline{\omega}^J) + \mathcal{E}(\overline{Z}\phi, \phi).$$

Finally, we compute $\Box_b$ and we obtain

$$\Box_b \phi = \sum_{|J|=q} \left[\left(-\frac{1}{2} \sum_{l=1,\dots,n} (Z_l \overline{Z}_l + \overline{Z}_l Z_l) + i \alpha_{J,k} T \right) \phi_J \right] \overline{\omega}^J + \mathcal{E}(Z\phi, \overline{Z}\phi, \varphi).$$

where $\alpha_{J,k} = |\{1,\dots,k\} \setminus J| + |\{k+1,\dots,n\} \cap J| - |\{1,\dots,k\} \cap J| - |\{k+1,\dots,n\} \setminus J|$.

Observation 6.1.17. We note that $\Box_b$ is expressed by the same formula as on $\mathbb{H}_n$ modulo lower order error terms.

6.2 Normal Coordinates on Strongly Pseudoconvex *CR* Manifolds

Although it would be possible to speak about normal coordinates for k-strongly pseudoconvex (or nondegenerate) CR manifolds, from now on we will consider only the case $k = n$. That is, we'll talk about strongly pseudoconvex CR manifolds.
Our object here will be to find, for every point in M, an "osculating Heisenberg structure" to M at the point; i.e., $\forall \xi_0 \in M$, we want to find coordinates $z_1, \ldots, z_n, t$ for M near ξ_0 such that they vary smoothly with ξ_0 and

$$Z_j = \frac{\partial}{\partial z_j} + i\bar{z}_j \frac{\partial}{\partial t},\ j = 1, \ldots, n \ \text{ and } \ T = \frac{\partial}{\partial t}$$

modulo suitably small error terms near ξ_0.

6.2.1 Definition of Normal Coordinates

Notation 6.2.1. Reminding lemma 6.1.6, we write

$$X_j := Z_j + \overline{Z}_j \ \text{ and } \ Y_j := i(Z_j - \overline{Z}_j).$$

We also write

$$\Xi_j := X_j,\ \Xi_{j+n} := Y_j, \quad \text{for } j = 1, \ldots, n,$$

and

$$\Xi_0 := T.$$

In this way, when we consider the set $\{X_j, Y_j, T \ / \ j = 1, \ldots, n\}$, we will just write it as $\{\Xi_j \ / \ j = 0, \ldots, 2n\}$.
Accordingly, we denote its dual basis by $\{\sigma_j \ / \ j = 0, \ldots, 2n\}$. It follows that

$$\omega_j = \sigma_j + i\sigma_{j+n},$$

$j = 1, \ldots, n$, and

$$\tau = \sigma_0.$$

Further, we will write the standard coordinates on $\mathbb{R}^{2n+1}$ as $(x_1, \ldots, x_n, y_1, \ldots, y_n, t) = (u_1, \ldots, u_{2n}, u_0) = u$ and $z_j = x_j + iy_j = u_j + iu_{j+n}$.

Remark 6.2.2. We take a function $\gamma : [0,1] \to M$, $X \in T(M)$ and XI is the column matrix of the coefficients of X with respect to a basis of $T(M)$.
We call *integral curve* the solution γ of the following Cauchy Problem:

$$\begin{cases} \dot{\gamma}(s) = XI\,(\gamma(s)) \\ \gamma(0_{\mathbb{R}}) = \xi_M \end{cases}$$

Definition 6.2.3.
Suppose M is a strongly pseudoconvex CR manifold, fix $\xi \in M$ and take $u \in \tilde{U}_\xi \subset \mathbb{R}^{2n+1}$, where $\tilde{U}_\xi$ is a starshaped neighborhood of 0.
We define the *exponential map* E_ξ as

$$\begin{aligned} E_\xi : \tilde{U}_\xi &\to M \\ u &\mapsto E_\xi(u) \end{aligned}$$

so that $E_\xi(u)$ is the endpoint $\eta(1)$ of the integral curve $\eta(s)$, $0 \leq s \leq 1$, of the vector field $\sum_{j=0}^{2n} u_j \Xi_j$ with $\eta(0) = \xi$.

Observation 6.2.4. Wa can say that

- $E_\xi \in C^\infty(\tilde{U}_\xi)$
- $dE_\xi : T(\tilde{U}_\xi) \to T(M)$ and $(dE_\xi)\left(\frac{\partial}{\partial u_j}\right)_{|0} = \Xi_{j|\xi}$.

So E_ξ is a diffeomorphism

$$E_\xi : U_\xi \to V_\xi,$$

where $U_\xi \subset \tilde{U}_\xi$ is a neighborhood of 0 which can be assumed starshaped, while V_ξ is a neighborhood of ξ. It follows also that $E_\xi^{-1} : V_\xi \to U_\xi$ is a coordinate mapping on V_ξ.

Definition 6.2.5.
With the same notation of observation 6.2.4, we call *Normal Coordinates* the coordinates induced from $E_\xi^{-1} : V_\xi \to U_\xi$.

Definition 6.2.6.
Take f a function on V_ξ. For this function we define the *Heisenberg-type order* as follows.
We say that f is O^1, and we write $f = O^1$, $\Leftrightarrow$

$$f = O\left(\sum_{j=1}^{n}(|x_j(\eta)| + |y_j(\eta)|) + |t(\eta)|^{\frac{1}{2}}\right) \text{ as } \eta \to \xi,\ \eta \in M.$$

Inductively $f = O^k \Leftrightarrow f = O(O^1 \cdot O^{k-1})$.

Observation 6.2.7. If $f \in C^\infty(V_\xi)$, then

$$f = O^1 \;\Leftrightarrow\; f = O\left(\sum_{j=1}^{n}(|x_j(\eta)| + |y_j(\eta)|) + |t(\eta)|\right)$$

and

$$f = O^2 \;\Leftrightarrow\; f = O\left(\sum_{j=1}^{n}(|x_j(\eta)|^2 + |y_j(\eta)|^2) + |t(\eta)|\right).$$

Theorem 6.2.8.
Suppose M is a strongly speudoconvex CR manifold and the previous notations hold.
With respect to the coordinates $u = (x, y, t)$ on V_ξ defined by E_ξ^{-1}, we have:

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t} + \sum_{i=1}^{n} \left(O^1 \frac{\partial}{\partial x_i} + O^1 \frac{\partial}{\partial y_i} \right) + O^2 \frac{\partial}{\partial t},$$

$$Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t} + \sum_{i=1}^{n} \left(O^1 \frac{\partial}{\partial x_i} + O^1 \frac{\partial}{\partial y_i} \right) + O^2 \frac{\partial}{\partial t},$$

for $j = 1, \dots, n$. And

$$T = \frac{\partial}{\partial t} + \sum_{i=1}^{n} \left(O^1 \frac{\partial}{\partial x_i} + O^1 \frac{\partial}{\partial y_i} \right) + O^1 \frac{\partial}{\partial t},$$

The proof of this theorem is composed of three parts.

First step of the proof of the Theorem.
Let us write generally

$$\Xi_j = \sum_{k=0}^{2n} B_{jk} \frac{\partial}{\partial u_k}$$

with $j = 0, \dots, 2n$ and where B_{jk} are functions on M.
Since we already noted that $B_{jk}(0) = \delta_{jk}$ (see observation 6.2.4), we only need to verify that $B_{j0} = 2u_{j+n} + O^2$ and $B_{(j+n)0} = -2u_j + O^2$ for $1 \le j \le n$.
The idea we'll use is borrowed by chapter V of [2]. Let $(A_{jk})_{jk}$ be the inverse transpose of $(B_{jk})_{jk}$, that is,

$$(A_{jk})_{jk} = (B_{jk})_{jk}^{-H}.$$

Thus, reminding that E_ξ^* goes from $\mathcal{A}^1(U_\xi)$ to $\mathcal{A}^1(V_\xi)$ and that $\sigma_j \in \mathcal{A}^1(U_\xi)$ ($\{\sigma_j\}_j$ is the dual basis of $\{\Sigma_j\}_j$, see notation 6.2.1) and $du_k \in \mathcal{A}^1(V_\xi)$, $j = 1, \dots, 2n$, we have that $E_\xi^* \sigma_j = \sum_{k=1}^{2n} A_{jk} du_k$.

Lemma 6.2.9. If $u \in U_\xi$ and $s \le 1$, then

$$\sum_{k=1}^{n} A_{jk}(su) u_k = u_j.$$

Proof of the Lemma. For $u \in U_\xi$, consider the mapping $\mu_u : [-1,1] \to U_\xi$ defined by $\mu_u(s) = su$. Then, by definition of E_ξ, we have

$$d\left(E_\xi \circ \mu_u\right)\left(\frac{\partial}{\partial s}\right)_{|s} = \sum_{j=1}^{2n} u_j \Xi_{j|_{E_\xi(su)}}$$

and, dually,

$$\left(E_\xi \circ \mu_u\right)^* (\sigma_j)_{|_{E_\xi(su)}} = u_j ds_{|s}.$$

On the other hand, by definition of μ_u, $\mu_u^*(du_k) = u_k ds$ and, since $E_\xi^* \sigma_j = \sum_{k=1}^{2n} A_{jk} du_k$, the lemma follows immediately. □

Second step of the proof of the Theorem.
Let us now define the functions $c_{jkm} : U_\xi \to \mathbb{C}$ by

$$[\Xi_j, \Xi_k] = \sum_{m=0}^{2n} c_{jkm} \Xi_m,$$

with $j,k,m = 0,\ldots,2n$.
Since $2d\sigma_j(X \wedge Y) = X\sigma_j(Y) - Y\sigma_j(X) - \sigma_j([X,Y])$, we have the dual equation

$$d\sigma_m = -\frac{1}{2} \sum_{j,k=0}^{2n} c_{jkm} \sigma_j \wedge \sigma_k. \qquad (*)$$

For $u \in U_\xi$ and $-1 \leq s \leq 1$, we also define the matrices $\mathcal{A}(s,u)$ and $\Gamma(s,u)$ by

$$\mathcal{A}_{jk}(s,u) := sA_{jk}(su)$$

and

$$\Gamma_{km}(s,u) := \sum_{j=0}^{2n} c_{jmk}(su) u_j$$

Lemma 6.2.10. For $u \in U_\xi$ and $-1 \leq s \leq 1$,

$$\frac{\partial \mathcal{A}}{\partial s}(s,u) = I - \Gamma(s,u)\mathcal{A}(s,u)$$

Proof of the Lemma. Define $\mu : (-1,1) \times U_\xi \to U_\xi$ by $\mu(s,u) := \mu_u(s) = su$ and set $\sigma_j^{'} = (E_\xi \circ \mu)^* \sigma_j$. By lemma 6.2.9 we can say,

$$\sigma^{'}_{m|_{(s,u)}} = \sum_{l=0}^{2n} A_{ml}(s,u)(u_l ds + s du_l) = \sum_{l=0}^{2n} \mathcal{A}_{ml}(s,u) du_l + u_m ds. \qquad (**)$$

On the other hand, since exterior multipliction and differentiation are functional, $(*)$ implies that

$$d\sigma'_{m|_{(s,u)}} = -\frac{1}{2}\sum_{j,k=0}^{2n} c_{jkm}(su)(\sigma'_j \wedge \sigma'_k)_{|_{(s,u)}}. \qquad (***)$$

Substituting $(**)$ in $(***)$ and collecting the coefficients of $ds \wedge du_l$, we have

$$\frac{\partial \mathcal{A}_{ml}}{\partial s} - \delta_{ml} = -\frac{1}{2}\left(\sum_{j,k=0}^{2n} c_{jkm}\mathcal{A}_{kl}u_j - \sum_{j,k=0}^{2n} c_{jkm}\mathcal{A}_{jl}u_k\right)$$

and, since $c_{jkm} = -c_{kjm}$

$$\frac{\partial \mathcal{A}_{ml}}{\partial s} = \delta_{ml} - \sum_{k=0}^{2n} \Gamma_{mk}\mathcal{A}_{kl}.$$

This proves the lemma. □

Third and final step of the proof of the Theorem.
Now we have, by Taylor's theorem,

$$B(su) = I + sB^{(1)}(u) + s^2 B^{(2)}(u) + \ldots$$

since $B(0) = I$ and where $B^{(1)}, B^{(2)}, \ldots$ are certain matrices.
It is clear that $B(u) = I + B^{(1)}(u) + O^2$ as $u \to 0$, thus it will suffice to determine $B^{(1)}_{j0}$, $1 \leq j \leq 2n$.
But we may also write

$$A(su) = I + sA^{(1)}(u) + s^2 A^{(2)}(u) + \ldots$$

and the equation $BA^H = I$ implies that $B^{(1)} = -(A^{(1)})^H$. Moreover,

$$\mathcal{A}(su) = sI + s^2 A^{(1)}(u) + s^3 A^{(2)}(u) + \ldots$$

and, if we write $\Gamma(s,u) = \Gamma^{(0)}(u) + s\Gamma^{(1)}(u) + \ldots$, lemma 6.2.10 implies that $A^{(1)} = -\frac{1}{2}\Gamma^{(0)}$.
Hence $B^{(1)} = \frac{1}{2}(\Gamma^{(0)})^H$ or, since $\Gamma^{(0)}_{jk} = \sum_{m=0}^{2n} c_{mkj}(0)u_m$ and $u_0 = t = O^2$,

$$B^{(1)}_{(j0)} = \frac{1}{2}\sum_{m=0}^{2n} c_{mj0}(0)u_m = \frac{1}{2}\sum_{m=1}^{2n} c_{mj0}(0)u_m + O^2. \qquad (\star)$$

Therefore, it only remains to determinate the coefficients $c_{mj0}(0)$, with $1 \leq m, j \leq 2n$.
Now, by definition of c,

$$4[Z_j, Z_k] = [X_j, X_k] - [Y_j, Y_k] - i[X_j, Y_k] - i[Y_j, X_k] =$$

$$= \sum_{l=0}^{2n} (c_{jkl} - c_{(j+n)(k+n)l} - ic_{j(k+n)l} - ic_{(j+n)kl})\Xi_l$$

But $C^\infty(T^{1,0}(M))$ is closed under brackets, so the coefficient of $\Xi_0 = T$ must vanish. Hence $c_{jk0} = c_{(j+n)(k+n)0}$ and $c_{j(k+n)0} = -c_{(j+n)k0}$. Likewise,

$$4[Z_j, \overline{Z}_k] = \sum_{l=0}^{2n} (c_{jkl} + c_{(j+n)(k+n)l} + ic_{j(k+n)l} - ic_{(j+n)kl})\Xi_l$$

and, by the equation in example 6.1.10, the coefficient of $\Xi_0 = T$ in this expression is $-8i\delta_{jk}$. Hence here we have $c_{jk0} = -c_{(j+n)(k+n)0}$ and $c_{j(k+n)0} - c_{(j+n)k0} = -8\delta_{jk}$.
Solving these equations, we find that, for $1 \leq j, k \leq n$,

$$c_{jk0} = c_{(j+n)(k+n)0} = 0 \quad \text{and} \quad c_{(j+n)k0} = -c_{j(k+n)0} = 4\delta_{jk}.$$

Substituting in $(\star)$ for $1 \leq j \leq n$, we see that

$$B_{j0} = B^{(1)}_{j0} + O^2 = 2u_{j+n} + O^2 \quad \text{and} \quad B_{(j+n)0} = B^{(1)}_{(j+n)0} + O^2 = -2u_j + O^2,$$

and this completes the proof of the theorem. □

Corollary 6.2.11. With the same hypothesis of the theorem, we can easly say

$$Z_j = \frac{\partial}{\partial z_j} + i\overline{z}_j \frac{\partial}{\partial t} + \sum_{i=1}^{n} \left(O^1 \frac{\partial}{\partial z_i} + O^1 \frac{\partial}{\partial \overline{z}_i} \right) + O^2 \frac{\partial}{\partial t},$$

for $j = 1, \ldots, n$.

6.2.2 Smooth Behaviour of the Normal Coordinates

We now investigate what happens when the base poin ξ varies.

Definition 6.2.12.
We define the set

$$\Omega := \{(\xi, \eta) \in M \times V_\xi\}$$

and we remind that V_ξ is a neighborhood of ξ in M.

Observation 6.2.13. Ω is a neighborhood of the diagonal in $M \times M$.

Definition 6.2.14.
Further, reminding from observation 6.2.4 that $E_\xi^{-1} : V_\xi \subset M \to U_\xi \subset \mathbb{R}^{2n+1} = \mathbb{H}_n$, we denote by Θ_ξ the coordinate mapping E_ξ:

$$\begin{aligned} \Theta_\xi : V_\xi &\to U_\xi \subset \mathbb{R}^{2n+1} \equiv \mathbb{H}_n \\ \eta &\mapsto \Theta_\xi(\eta) := E_\xi^{-1}(\eta), \end{aligned}$$

Then we set

$$\begin{aligned} &\Theta : \Omega \to \mathbb{R}^{2n+1} = \mathbb{H}_n \\ &(\xi, \eta) \mapsto \Theta(\xi, \eta), \end{aligned}$$

where

$$\Theta(\xi, \eta) := \Theta_\xi(\eta) \in U_\xi \subset \mathbb{R}^{2n+1} = \mathbb{H}_n.$$

We call Θ our *Normal Coordinate Map.*

Observation 6.2.15. Note that, if $M = \mathbb{H}_n$, then we have $\Theta(\xi, \eta) = \xi^{-1}\eta$.

Notation 6.2.16. Accordingly with notation 6.2.1, we denote the coordinates $\Theta(\xi, \eta) \in \mathbb{H}_n$ as $u(\xi, \eta) = (x(\xi, \eta), y(\xi, \eta), t(\xi, \eta))$.
We also set $\rho(\xi, \eta) = |\Theta(\xi, \eta)|$, $|\cdot|$ being the Heisenberg norm.

We show the importance of this coordinates in the following theorem

Theorem 6.2.17.
Suppose M is a strongly pseudoconvex CR manifold. We have that all the following properties hold:

1. $\Theta(\xi, \eta) = -\Theta(\eta, \xi) = \Theta(\eta, \xi)^{-1} \in \mathbb{H}_n$.
2. $\Theta \in C^\infty(\Omega, \mathbb{H}_n)$.
3. $\Theta_\xi^* : \mathcal{A}^1(U_\xi) \to \mathcal{A}^1(V_\xi)$,
   $\Theta_\xi^*(dV)_{|\epsilon}$ is the volume element on M at ϵ.
4. Suppose ξ, η and ζ vary over a compact set in M so that $(\xi, \eta), (\xi, \zeta), (\eta, \zeta) \in \Omega$ and $\rho(\xi, \zeta) \le 1$ and $\rho(\xi, \eta) \le 1$.
   Then there exist $C_1, C_2 > 0$ constants such that
   $$|\Theta(\xi, \eta) - \Theta(\zeta, \eta)| \le C_1(\rho(\xi, \zeta) + \rho(\xi, \zeta)^{\frac{1}{2}}\rho(\xi, \eta)^{\frac{1}{2}})$$
   and
   $$\rho(\zeta, \eta) \le C_2(\rho(\xi, \zeta) + \rho(\xi, \eta)).$$

Proof. The first three points are easy to prove: 1. comes directly from the definition and 2. follows form theorems of ordinary differential equations on smooth dependance of solutions on parameters.
3. follows form the fact that Θ_ξ^* maps the orthonormal basis $\omega_1, \dots, \omega_n, \overline{\omega}_1, \dots, \overline{\omega}_n, \tau$ of $\mathbb{C}T_\xi(M)^*$ to the orthonormal basis $dz_1, \dots, dz_n, d\overline{z}_1, \dots, d\overline{z}_n, dt$ of $\mathbb{C}T_\xi(\mathbb{H}_n)^*$.
For 4., we first note that we can regard ξ as a function of $\zeta \in M$ and $u \in U_\xi \subset \mathbb{H}_n$ by the equation $\xi = E_\zeta(u)$. Hence we may write $\Theta(\xi, \eta) = f(\eta, \zeta, u)$, where $f(\eta, \zeta, 0) = \Theta(\zeta, \eta)$.
We now expand f in a Taylor series at 0: in coordinates we have

$$u_j(\xi, \eta) = u_j(\zeta, \eta) + \sum_{k=0}^{2n} a_{jk}(\eta, \zeta) u_k(\xi, \zeta) + O\left(\sum_{k=0}^{2n} |u_k(\xi, \zeta)|^2\right)$$

where the a_{jk} are smooth functions which vanish for $\eta = \zeta$.
Hence, by the mean value theorem and observation 2.3.3, we have that

$$|u_0(\xi, \eta) - u_0(\zeta, \eta)| \le C_\rho(\eta, \zeta)\rho(\xi, \zeta) \qquad (*)$$

and

$$|u_j(\xi, \eta) - u_j(\zeta, \eta)| \le C_\rho(\xi, \zeta), \quad \text{for } 1 \le j \le 2n. \qquad (*)$$

On the other hand, for any $u \in \mathbb{H}_n$, we have $|u_k| \le |u|$, for $1 \le k \le 2n$, and $|u_0| \le |u|^2$; conversely $|u| \le (2n+1)\max\{|u_0|^{\frac{1}{2}}, |u_1|, \dots, |u_{2n}|\}$.
In particular,

$$|\Theta(\xi, \eta) - \Theta(\zeta, \eta)| \le$$

$$\le (2n+1)\max\{|u_0(\xi, \eta) - u_0(\zeta, \eta)|^{\frac{1}{2}}, |u_1(\xi, \eta) - u_1(\zeta, \eta)|, \dots, |u_{2n}(\xi, \eta) - u_{2n}(\zeta, \eta)|\}.$$

Substituting $(*)$ here, we can say that

$$|\Theta(\xi, \eta) - \Theta(\zeta, \eta)| \le C_1(\rho(\xi, \zeta) + \rho(\xi, \zeta)^{\frac{1}{2}}\rho(\xi, \eta)^{\frac{1}{2}})$$

and the first inequality is proved.
To prove the second one it suffices to show that, if $\rho(\xi, \zeta) \le \epsilon$ and $\rho(\xi, \eta) \le \epsilon$, then $\rho(\zeta, \eta) \le C_2\epsilon$. To prove that, we write $\Theta(\zeta, \eta) = (\Theta(\zeta, \eta) - \Theta(\xi, \eta)) + \Theta(\xi, \eta)$ and we use the first inequality and observation 2.3.4 to get (c is a constant),

$$\rho(\zeta, \eta) \le c(|\Theta(\zeta, \eta) - \Theta(\xi, \eta)| + \rho(\xi, \eta)) \le$$

$$\le cC_1(\rho(\zeta, \xi) + \rho(\zeta, \xi)^{\frac{1}{2}}\rho(\xi, \eta)^{\frac{1}{2}}) + c\rho(\xi, \eta) \le C_2\epsilon.$$

That completes the proof. □

6.3 Normal Coordinates on a Hypersurface

Now we are going to talk about the special case in which M is an hypersurface embedded in a complex manifold. In this case we'll show a different construction for the normal coordinate map $\Theta(\xi, \eta)$ on M.

Definition 6.3.1.
Suppose V is a strongly pseudoconvex complex manifold with $dim_{\mathbb{C}} V = n+1$, and M is a real hypersurface with the induced CR structure.
We assume that there exist a real valued defining function r defined on a neighborhood of M so that $M = \{u \in V \ / \ r(u) = 0\}$ and $dr \neq 0$ on M.

Definition 6.3.2.
Following the path of section 6.1, we give here other definitions.

- Reminding the way we defined a general Levi form in definition 6.1.1, here we choose an explicit nonvanishing one-form

$$\tau := i(\overline{\partial} - \partial)r$$

  annihilating $T^{1,0}(M) \oplus T^{0,1}(M)$, in term of which our Levi form is defined (we can replace r by $-r$ if necessary to make the Levi form positive).

- Then, using lemma 6.1.6, we choose a metric $<,>$ on M so that the four conditions at observation 6.1.7 are satisfied.

- Finally we take T as the vector field dual to τ with respect to the metric.

Notation 6.3.3. Let V_0 be a coordinate chart on V with complex coordinates $\omega_0, \dots, \omega_n$ such that, on $M_0 := M \cap V_0$, there is an orthonormal basis $Z_1, \dots, Z_n$ for $T^{1,0}(M)$.

From now on we will construct normal coordinates for the region M_0 with respect to the basis $Z_1, \dots, Z_n, \overline{Z}_1, \dots, \overline{Z}_n, T$.

Definition 6.3.4.
Let

$$J : \mathbb{C}T(V) \to \mathbb{C}T(V)$$

be the almost-complex structure tensor on V (that is, a differentiable endomorphism such that $J^2 = -1$); and let

$$P : \mathbb{C}T(V) \to T^{1,0}(V)$$
$$P := \frac{1}{2}(I - iJ)$$

be the projection onto the holomorphic tangent bundle $T^{1,0}(V)$.
Now we define
$$Z_0 := P(T),$$
so $Z_0, \dots, Z_n$ form a basis for $T^{1,0}(V)$ along M_0.

Notation 6.3.5. For $0 \le j \le n$, we may write
$$Z_j = \sum_{k=0}^{n} c_{jk} \frac{\partial}{\partial \omega_k},$$
where $(c_{jk})_{j,k=0,\dots,n}$ is a smooth nonsingular matrix of functions on M_0.
Let $(d_{jk})_{jk} := (c_{jk})_{j,k}^{-H}$ be its inverse transpose, which is also smooth on M_0.

Definition 6.3.6.
We now fix $\xi \in M_0$ and, for $\eta \in V_0$, we define
$$\zeta_j := \sum_{k=0}^{n} d_{jk}(\xi)(\omega_k(\eta) - \omega_k(\xi)), \quad 0 \le j \le n.$$
Then
$$\{\zeta_j\}_{j=0,\dots,n} = \{\zeta_0, \dots, \zeta_n\}$$
is a holomorphic coordinate system for V_0 centered at ξ and, for $0 \le j \le n$,
$$\frac{\partial}{\partial \zeta_j}_{|\xi} = \sum_{k=0}^{n} c_{jk}(\xi) \frac{\partial}{\partial \omega_k}_{|\xi} = Z_{j|\xi}$$

We now need a couple of lemmas.

Lemma 6.3.7. With all the previous notations, we have that
$$dr_{|\xi} = -\mathrm{Im}\left(d\zeta_{0|\xi}\right)$$

Proof. First we note that
$$dr = \partial r + \overline{\partial} r = \partial r + \overline{\partial r} = 2\mathrm{Re}\partial r$$
since r is real.

So we have that
$$dr_{|\xi} = 2\mathrm{Re} \sum_{j=0}^{n} \frac{\partial r}{\partial \zeta_j}(\xi) d\zeta_j(\xi) =$$

using the last equality of definition 6.3.6,

$$= 2\mathrm{Re}\sum_{j=0}^{n}(Z_j r)(\xi)d\zeta_{j|_\xi} =$$

since Z_j is tangential to M, for $j = 1, \dots, n$,

$$= 2\mathrm{Re}\left((Z_0 r)(\xi)d\zeta_{0|_\xi}\right).$$

Since $Z_0 = \frac{1}{2}(T - iJT)$ and T is tangential, $\mathrm{Re}(Z_0 r) = 0$. Hence, it's clear that

$$2\mathrm{Re}\left((Z_0 r)d\zeta_0\right) = 2\mathrm{Re}(Z_0 r)\mathrm{Re}(d\zeta_0) - 2\mathrm{Im}(Z_0 r)\mathrm{Im}(d\zeta_0) = -2\mathrm{Im}(Z_0 r)\mathrm{Im}(d\zeta_0).$$

On the other hand, with a straighforward calculations,

$$2\mathrm{Im}(Z_0 r) = 2\mathrm{Im} < dr, Z_0 >= - < dr, JT >=$$

$$= - < J^* dr, T >=< i(\partial - \overline{\partial})r, T >=< \tau, T >= 1.$$

Thus

$$dr_{|_\xi} = -2\mathrm{Im}(Z_0 r_{0|_\xi})\mathrm{Im}(d\zeta_{0|_\xi}) = -\mathrm{Im}\left(d\zeta_{0|_\xi}\right)$$

and the proof is complete. □

In particular, $\zeta_1, \dots, \zeta_n, \mathrm{Re}\,(\zeta_0)$ form a coordinate system for a neighborhood of ξ in M_0, which is a first approximation to normal coordinates.

Lemma 6.3.8. Again, with the previous notations, we can say

$$\frac{\partial^2 r}{\partial\zeta_j\partial\overline{\zeta}_k}(\xi) = \delta_{jk}, \quad \text{for} \quad 1 \le j, k \le n.$$

Proof. Using again definition 6.3.6 and the fact that r is real, we can say

$$\frac{\partial^2 r}{\partial\zeta_j\partial\overline{\zeta}_k}(\xi) = 2 < \partial\overline{\partial}r, Z_j \wedge \overline{Z}_k > (\xi) =< d(\overline{\partial} - \partial)r, Z_j \wedge \overline{Z}_k > (\xi) =$$

$$= -i < d\tau, Z_j \wedge \overline{Z}_k > (\xi) = \delta_{jk}$$

by definition of Levi form, since the Z_k's are orthonormal. This prove the proposition. □

Proposition 6.3.9.
From lemmas 6.3.7 and 6.3.8, we see that the Taylor expansion of r at ξ in the coordinates $\{\zeta_j\}_{j=0,\dots,n}$ is

$$r = -\mathrm{Im}\zeta_0 + \sum_{j=1}^{n}|\zeta_j|^2 + \mathrm{Re}\sum_{j,k=0o1}^{n}\frac{\partial^2 r}{\partial\zeta_j\partial\zeta_k}(\xi)\zeta_j\zeta_k + O(|\zeta_0||\zeta| + |\zeta|^3)$$

Definition 6.3.10.
We now define new coordinates $z_0, \dots, z_n$ near ξ using the so called *Levi Procedure.* We set

$$z_j := \zeta_j, \quad \text{for} \quad 1 \leq j \leq n,$$

and

$$z_0 := \zeta_0 - i \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial \zeta_j \partial \zeta_k}(\xi)\zeta_j \zeta_k.$$

Observation 6.3.11. With the last definition, we can write that

$$\frac{\partial}{\partial z_j}_{|_\xi} = \frac{\partial}{\partial \zeta_j}_{|_\xi} = Z_{j|_\xi}, \quad \text{for all} \quad j = 0, \dots, n.$$

In particular we have that $\det \left(\frac{\partial z_j}{\partial \zeta_k}(\xi) \right)_{j,k=0,\dots,n} = 1$; so $\{z_j\}_{j=0,\dots,n} = \{z_0, \dots, z_n\}$ in fact form a coordinate system near ξ.

Proposition 6.3.12.
Moreover, using proposition 6.3.9 and the new coordinates, r now has the Taylor expansion

$$r = -\text{Im} z_0 + \sum_{j=1}^{n} |z_j|^2 + O(|z_0||z| + |z|^3) = -\text{Im}\zeta_0 + \sum_{j=1}^{n} |z_j|^2 + O^3$$

while O^3 has the same meaning as in definition 6.2.6, with z_0 repleacing t.

Observation 6.3.13. The significance of proposition 6.3.12 is that M is highly tangent at ξ to the hypersurface $\{z \in \mathbb{C}^{n+1} \ / \ \text{Im} z_0 = \sum_{j=1}^{n} |z_j|^2\}$, which is the geometric model for the Heisenberg group (see chapter 2.2).

Definition 6.3.14.
If we define x_j and y_j, $j = 1, \dots, n$, by

$$z_j = x_j + i y_j, \quad j = 1, \dots, n,$$

and

$$t := \text{Re} z_0,$$

then

$$\{x_1, \dots, x_n, y_1, \dots, y_n, t\}$$

form a coordinate system for a neighborhood of ξ on M. They will be our *Normal Coordinates.*

Now we proceed to show that they have the desired properties.

Notation 6.3.15. As in proposition 6.3.12, we use the notation O^k for functions on either M or V, with the understanding that z_0 replaces t in the latter case. We note that with this convention the restriction of an O^k function on V to M is again O^k.

Theorem 6.3.16.
We are finally ready to write

$$Z_j = \frac{\partial}{\partial z_j} + i\overline{z}_j \frac{\partial}{\partial t} + \sum_{k=1}^{n} O^1 \frac{\partial}{\partial z_k} + O^2 \frac{\partial}{\partial t}$$

and

$$T = \frac{\partial}{\partial t} + \sum_{k=1}^{n} O^1 \frac{\partial}{\partial z_k} + \sum_{k=1}^{n} O^1 \frac{\partial}{\partial \overline{z}_k} + O^1 \frac{\partial}{\partial t}$$

Proof. The assertion about T is simply that

$$T_{|\xi} = \frac{\partial}{\partial t}_{|\xi},$$

which is true since

$$\frac{\partial}{\partial t}_{|\xi} = 2\mathrm{Re}\left(\frac{\partial}{\partial z_0}_{|\xi}\right) = 2\mathrm{Re}\left(Z_{0|\xi}\right) = T_{|\xi}.$$

To prove the assertion about Z_j, we wish to construct a local basis $\{\tilde{Z}_1, \ldots, \tilde{Z}_n\}$ for $T^{1,0}(M)$ near ξ such that

$$\tilde{Z}_j = \frac{\partial}{\partial z_j} + i\overline{z}_j \frac{\partial}{\partial t} + O^2 \frac{\partial}{\partial t}.$$

If we can do it, we'll have

$$\tilde{Z}_{j|\xi} = \frac{\partial}{\partial z_j}_{|\xi} = Z_{j|\xi}$$

and hence

$$Z_j = \tilde{Z}_j + \sum_{k=1}^{n} \tilde{Z}_k$$

and the theorem follows.
What is left now is to actually construct such a basis. In order to find vector

fields which are both holomorphic and tangent to M, we introduuce the (non-holomorphic) coordinate system $(x_1, \ldots, x_n, y_1, \ldots, y_n, t, r)$ for V near ξ. In these coordinates, M is the hiperplane $\{r = 0\}$, so the vector fields $\frac{\partial}{\partial x_j}$, $\frac{\partial}{\partial y_j}$ and $\frac{\partial}{\partial t}$ restricted to M are identical with the vector fields $\frac{\partial}{\partial x_j}$, $\frac{\partial}{\partial y_j}$ and $\frac{\partial}{\partial t}$ on M in the normal coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n, t)$ on M. We can also say that a vector field on V is tangent to $M \Leftrightarrow$ the coefficients of $\frac{\partial}{\partial r}$ in it is zero. The almost-complex structure J is given in terms of the old coordinates of definition 6.3.10 $(z_0, \ldots, z_n) = (x_0, \ldots, x_n, y_0, \ldots, y_n)$ by the matrix

$$J_0 = \begin{pmatrix} 0 & -1 & 0 & \dots & \dots & 0 \\ 1 & 0 & -1 & \ddots & & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 & 0 \\ \vdots & & \ddots & 1 & 0 & -1 \\ 0 & \dots & \dots & 0 & 1 & 0 \end{pmatrix}.$$

Therefore, in the new (non-holomorphic) coordinate, it's given by AJ_0A^{-1}, where A is the Jacobian matrix of the transformation $(x_0, \ldots, x_n, y_0, \ldots, y_n) \to (t, r, x_1, \ldots, x_n, y_1, \ldots, y_n)$.
Since $t = x_0$ and $r = -y_0 + \sum_{j=1}^n (x_j^2 + y_j^2) + O^3$, a straightforward calcutalion shows that the projection $P = \frac{1}{2}(I - iJ)$ onto the holomorphic vectors is given by

$$\begin{aligned} P\left(\frac{\partial}{\partial t}\right) &= \frac{1}{2}\left(\frac{\partial}{\partial t} + i\frac{\partial}{\partial r}\right) + O^1\frac{\partial}{\partial t} + O^1\frac{\partial}{\partial r}, \\ P\left(\frac{\partial}{\partial r}\right) &= \frac{1}{2}\left(\frac{\partial}{\partial r} - i\frac{\partial}{\partial t}\right) + O^1\frac{\partial}{\partial t} + O^1\frac{\partial}{\partial r}, \\ P\left(\frac{\partial}{\partial x_j}\right) &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) + i\left(x_j\frac{\partial}{\partial t} - y_j\frac{\partial}{\partial r}\right) + O^2\frac{\partial}{\partial t} + O^2\frac{\partial}{\partial r}, \\ P\left(\frac{\partial}{\partial y_j}\right) &= \frac{1}{2}\left(\frac{\partial}{\partial y_j} + i\frac{\partial}{\partial x_j}\right) + i\left(y_j\frac{\partial}{\partial t} - x_j\frac{\partial}{\partial r}\right) + O^2\frac{\partial}{\partial t} + O^2\frac{\partial}{\partial r}, \end{aligned}$$

for $j = 1, \ldots, n$. We denote the coefficients of $\frac{\partial}{\partial r}$ in $P\left(\frac{\partial}{\partial r}\right)$ by a. Then, near ξ,

$$a^{-1} = 2 + O^1 \neq 0$$

and

$$a^{-1}P\left(\frac{\partial}{\partial r}\right) = \frac{\partial}{\partial r} - a^{-1}\left(\frac{1}{2}i + O^1\right)\frac{\partial}{\partial t}.$$

Also we can compute

$$\frac{1}{2}P\left(\frac{\partial}{\partial x_j}-i\frac{\partial}{\partial y_j}\right)=\frac{1}{2}\left(\frac{\partial}{\partial x_j}-i\frac{\partial}{\partial y_j}\right)+\frac{1}{2}\left((ix_j+y_j)\frac{\partial}{\partial t}+(-iy_j+x_j)\frac{\partial}{\partial r}\right)+$$
$$+O^2\frac{\partial}{\partial t}+O^2\frac{\partial}{\partial r}=\frac{\partial}{\partial z_j}+\frac{1}{2}i(\overline{z}_j+O^2)\frac{\partial}{\partial t}+\frac{1}{2}i(\overline{z}_j+O^2)\frac{\partial}{\partial r}$$

for all $j=1,\dots,n$. We call b the coefficients of $\frac{\partial}{\partial r}$ in this expression: $b=\frac{1}{2}i(\overline{z}_j+O^2)$.
Now we set

$$\tilde{Z}_j:=\frac{1}{2}P\left(\frac{\partial}{\partial x_j}-i\frac{\partial}{\partial y_j}\right)-ba^{-1}P\left(\frac{\partial}{\partial r}\right).$$

Then we can compute

$$\begin{aligned}\tilde{Z}_j&=\frac{\partial}{\partial z_j}+\frac{1}{2}i(\overline{z}_j+O^2)\frac{\partial}{\partial t}+ba^{-1}(\frac{1}{2}i+O^1)\frac{\partial}{\partial t}=\\&=\frac{\partial}{\partial z_j}+\frac{1}{2}i(\overline{z}_j+O^2)\frac{\partial}{\partial t}+\frac{1}{2}i(\overline{z}_j+O^2)(2+O^1)(\frac{1}{2}i+O^1)\frac{\partial}{\partial t}=\\&=\frac{\partial}{\partial z_j}+i\overline{z}_j\frac{\partial}{\partial t}+O^2\frac{\partial}{\partial t}.\end{aligned}$$

Thus $\tilde{Z}_j$ is tangent to M, has the required form and lies in $T^{1,0}(M)$ by construction. Then the claim, and so the theorem, is proved. □

As in paragraph 6.2.2, we wish now to let the base point ξ vary to study the regularity of these coordinates.

Notation 6.3.17. Exactly as in paragraph 6.2.2, taken $(\xi,\eta)\in M\times M$, we denote the coordinates $(z_1,\dots,z_n,t)$ of η with respect to ξ by

$$\Theta(\xi,\eta)=(z_1(\xi,\eta),\dots,z_n(\xi,\eta),t(\xi,\eta)).$$

Again, we call Θ our *Normal Coordinate Map.*

Observation 6.3.18. Looking back at the four properties of theorem 6.2.17, it's clear from the construction that Θ depends smoothly on $(\xi,\eta)\in M\times M$ (property 2.). Moreover, properties 3. and 4. of Θ in 14.10 are still valid, with the same proof. Property 1. is not quite true, but we have the following subsitute result.

Theorem 6.3.19.

$$z_j(\xi,\eta)=-z_j(\eta,\xi)+O^2$$

and

$$t(\xi,\eta) = -t(\eta,\xi) + O^3$$

where O^k means "O^k of η with respect to ξ" or viceversa; the same proof works both ways.

Proof. We refer to the definitions of z_j and t in terms of the original coordinates $\{\omega_j\}_{j=0,\dots,n}$ on V_0 (see notation 6.3.3). First, by the definition of ζ_j at 6.3.6, we have

$$z_j(\xi,\eta) + z_j(\eta,\xi) = \sum_{k=0}^{n} (d_{jk}(\xi) - d_{jk}(\eta))(\omega_k(\eta) - \omega_k(\xi)), \quad 0 \le j \le n.$$

which is O^2 in the sense that it vanishes to second order in all directions as η approches ξ. That proves the first equality.
Next, by definition 6.3.10, we write

$$z_0(\xi,\eta) = \zeta_0(\xi,\eta) - i\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial\zeta_j\partial\zeta_k}(\xi)\zeta_j(\xi,\eta)\zeta_k(\xi,\eta) =$$

adding the case $j,k=0$,

$$= \zeta_0(\xi,\eta) - i\sum_{j,k=0}^{n} \frac{\partial^2 r}{\partial\zeta_j\partial\zeta_k}(\xi)\zeta_j(\xi,\eta)\zeta_k(\xi,\eta) + O^3(\xi,\eta) =$$

using the fact that the quadratic form given by second derivatives is invariant under linear changes of coordinates,

$$= \sum_{k=0}^{n} d_{0k}(\xi)(\omega_k(\eta)-\omega_k(\xi)) - i\sum_{j,k=0}^{n} \frac{\partial^2 r}{\partial\omega_j\partial\omega_k}(\xi)(\omega_j(\eta)-\omega_j(\xi))(\omega_k(\eta)-\omega_k(\xi)) + O^3(\xi,\eta).$$

Therefore we get

$$z_0(\xi,\eta) + z_0(\eta,\xi) = \sum_{k=0}^{n} (d_{0k}(\xi) - d_{0k}(\eta))(\omega_k(\eta) - \omega_k(\xi)) +$$

$$-i\sum_{j,k=0}^{n} \left(\frac{\partial^2 r}{\partial\omega_j\partial\omega_k}(\xi) + \frac{\partial^2 r}{\partial\omega_j\partial\omega_k}(\eta) \right) (\omega_j(\eta) - \omega_j(\xi))(\omega_k(\eta) - \omega_k(\xi)) + O^3(\xi,\eta)$$

and we call this equality $(*)$. Next, by lemma 6.3.7,

$$dr_{|_\xi} = -\mathrm{Im} d\zeta_{0|_\xi} = -\mathrm{Im} \sum_{k=0}^{n} d_{0k}(\xi) d\omega_{k|_\xi} =$$

$$= \frac{1}{2i} \sum_{k=0}^{n} \left(\overline{d}_{0k}(\xi) d\overline{\omega}_k - d_{0k}(\xi) d\omega_k \right)_{k|_\xi},$$

which implies that

$$d_{0k}(\xi) = -2i \frac{\partial r}{\partial \omega_k}(\xi), \quad k = 0, \ldots, n.$$

But this is valid for any ξ, so

$$d_{0k} = -2i \frac{\partial r}{\partial \omega_k}, \quad k = 0, \ldots, n.$$

Expanding d_{0k} in a Taylor series at ξ, we have then

$$d_{0k}(\eta) - d_{0k}(\xi) = -2i \sum_{j=0}^{n} \frac{\partial^2 r}{\partial \omega_k \partial \omega_j}(\xi)(\omega_j(\eta) - \omega_j(\xi)) +$$

$$-2i \sum_{j=0}^{n} \frac{\partial^2 r}{\partial \omega_k \partial \overline{\omega}_j}(\xi)(\overline{\omega}_j(\eta) - \overline{\omega}_j(\xi)) + O^2(\xi, \eta).$$

On the other hand, expanding d_{0k} at η, we get

$$d_{0k}(\xi) - d_{0k}(\eta) = -2i \sum_{j=0}^{n} \frac{\partial^2 r}{\partial \omega_k \partial \omega_j}(\xi)(\omega_j(\xi) - \omega_j(\eta)) +$$

$$-2i \sum_{j=0}^{n} \frac{\partial^2 r}{\partial \omega_k \partial \overline{\omega}_j}(\xi)(\overline{\omega}_j(\eta) - \overline{\omega}_j(\xi)) + O^2(\eta, \xi).$$

But here the error $O^2(\eta, \xi)$ is second order in all directions,so it's also $O^2(\xi, \eta)$. Therefore, subtracting the first from the second and divinding by 2, we obtain

$$d_{0k}(\eta) - d_{0k}(\xi) = i \sum_{j=0}^{n} \left(\frac{\partial^2 r}{\partial \omega_k \partial \omega_j}(\xi) + \frac{\partial^2 r}{\partial \omega_k \partial \omega_j}(\eta) \right) (\omega_j(\eta) - \omega_j(\xi)) +$$

$$+ i \sum_{j=0}^{n} \left(\frac{\partial^2 r}{\partial \omega_k \partial \overline{\omega}_j}(\xi) + \frac{\partial^2 r}{\partial \omega_k \partial \overline{\omega}_j}(\eta) \right) (\overline{\omega}_j(\eta) - \overline{\omega}_j(\xi)) + O^2(\xi, \eta).$$

Substituting the last result in $(*)$, we have immediately

$$z_0(\xi, \eta) + z_0(\eta, \xi) =$$

$$i \sum_{j,k=0}^{n} \left(\frac{\partial^2 r}{\partial \omega_k \partial \overline{\omega}_j}(\xi) + \frac{\partial^2 r}{\partial \omega_k \partial \overline{\omega}_j}(\eta) \right) (\omega_k(\eta) - \omega_k(\xi))(\overline{\omega}_j(\eta) - \overline{\omega}_j(\xi)) + O^3(\xi, \eta)$$

But this sum is real, so

$$t(\xi, \eta) + t(\eta, \xi) = \mathrm{Re}\,(z_0(\xi, \eta) + z_0(\eta, \xi)) = O^3(\xi, \eta).$$

This completes the proof. □

6.4 An Example of Normal Coordinates

In this paragraph we want to show an explicit simple example about how we find our Normal Coordinates and how we use them to rewrite a basis of $T(M)$ in the form that resemble the Heisenberg group (see theorem 6.2.8).

Here we take $n = 1$ and $M = \mathbb{R}^3$. So we have that

$$(x, y, \theta) \in M$$

is a coordinate system and

$$\partial_x, \partial_y, \partial_\theta \in T(M)$$

is a basis of $T(M)$.
We fix a point $\xi_0 = (x_0, y_0, \theta_0) \in M$. Locally, close to ξ_0, we take another basis for $T(M)$:

$$\begin{cases} X_1 = (-\cos\theta_0 + (\theta - \theta_0)\sin\theta_0)\partial_x - (\sin\theta_0 + (\theta - \theta_0)\cos\theta_0)\partial_y \\ X_2 = \partial_\theta \\ X_3 = \frac{1}{4}\sin\theta_0\partial_x - \frac{1}{4}\cos\theta_0\partial_y \end{cases}$$

X_1, X_2 are obviously linearly indipendent and $X_3 = -\frac{1}{4}[X_1, X_2]$.
In fact,

$$[X_1, X_2] = X_1X_2 - X_2X_1 = (-\cos\theta_0 + (\theta - \theta_0)\sin\theta_0)\,\partial^2_{x\theta} - (\sin\theta_0 + (\theta - \theta_0)\cos\theta_0)\partial^2_{y\theta} +$$

$$-\partial_\theta(-\cos\theta_0\partial_x + (\theta - \theta_0)\sin\theta_0\partial_x - \sin\theta_0\partial_y - (\theta - \theta_0)\cos\theta_0\partial_y) =$$

$$= -\cos\theta_0\partial^2_{x\theta} + (\theta - \theta_0)\sin\theta_0\partial^2_{x\theta} - \sin\theta_0\partial^2_{y\theta} - (\theta - \theta_0)\cos\theta_0\partial^2_{y\theta} +$$

$$+\cos\theta_0\partial^2_{\theta x} - \sin\theta_0\partial_x - (\theta - \theta_0)\sin\theta_0\partial^2_{\theta x} + \sin\theta_0\partial^2_{\theta y} + \cos\theta_0\partial_y + (\theta - \theta_0)\cos\theta_0\partial^2_{\theta y} =$$

$$= -\sin\theta_0\partial_x + \cos\theta_0\partial_y = -4X_3.$$

Thus $\{X_1, X_2, X_3\}$ is a basis for $T(M)$.

We want an explicit calculation of the exponential map

$$\begin{aligned} E_{\xi_0} &: \tilde{U}_{\xi_0} \subset \mathbb{R}^3 \to M \\ u &= (a, b, c) \mapsto E_{\xi_0}(u) = E_{\xi_0}(a, b, c) \end{aligned}$$

where $E_{\xi_0}(a, b, c) = \gamma(1)$ and $\gamma(s)$, $0 \le s \le 1$, is the integral curve of the vector field $X = aX_1 + bX_2 + cX_3$ with $\gamma(0) = \xi_0$.

In order to compute it, first we have to solve the Cauchy Problem.
So now we look for the integral curve $\gamma : [0,1] \to M$, that is the solution of:

$$\begin{cases} \dot{\gamma}(s) = XI\,(\gamma(s)) \\ \gamma(0_{\mathbb{R}}) = \xi_0 \end{cases}$$

where

$$XI = \begin{pmatrix} a(-\cos\theta_0 + (\theta - \theta_0)\sin\theta_0) + \frac{c}{4}\sin\theta_0 \\ -a(\sin\theta_0 + (\theta - \theta_0)\cos\theta_0) - \frac{c}{4}\cos\theta_0 \\ b \end{pmatrix}.$$

We denote $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$. So

$$\dot{\gamma}(s) = XI\,(\gamma(s)) \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \dot{\gamma}_1(s) = -a\cos\theta_0 + a(\gamma_3(s) - \theta_0)\sin\theta_0 + \frac{c}{4}\sin\theta_0 \\ \dot{\gamma}_2(s) = -a\sin\theta_0 - a(\gamma_3(s) - \theta_0)\cos\theta_0 - \frac{c}{4}\cos\theta_0 \\ \dot{\gamma}_3(s) = b \end{cases}$$

using the boundary conditions,

$$\Rightarrow \begin{cases} \gamma_3(s) = bs + \theta_0 \\ \gamma_1(s) = -as\cos\theta_0 + a\frac{bs^2}{2}\sin\theta_0 + \frac{cs}{4}\sin\theta_0 + x_0 \\ \gamma_2(s) = -as\sin\theta_0 - a\frac{bs^2}{2}\cos\theta_0 - \frac{cs}{4}\cos\theta_0 + y_0. \end{cases}$$

Then, for $s = 1$ we have

$$(*) \begin{cases} x = \gamma_1(1) = x_0 - a\cos\theta_0 + \frac{1}{2}ab\sin\theta_0 + \frac{c}{4}\sin\theta_0 \\ y = \gamma_2(1) = y_0 - a\sin\theta_0 - \frac{1}{2}ab\cos\theta_0 - \frac{c}{4}\cos\theta_0 \\ \theta = \gamma_3(1) = b + \theta_0 \end{cases}.$$

At the end we really found

$$E_{\xi_0}(a,b,c) = \gamma(1) = (x, y, \theta) \in M.$$

We know that there exist $U_{\xi_0} \subset \mathbb{R}^3$ and $V_{\xi_0} \subset M$ such that

$$E_{\xi_0} : U_{\xi_0} \to V_{\xi_0}$$

is a diffeomorphism.
It follows also that

$$\begin{aligned} E_{\xi_0}^{-1} &: V_{\xi_0} \to U_{\xi_0} \\ E_{\xi_0}(a,b,c) &\mapsto (a,b,c) \end{aligned}$$

is a coordinate mapping on V_{ξ_0} and (a, b, c) are our Normal Coordinates. So we invert the system to see (a, b, c) explicitly.
From the system $(*)$, note that $b = \theta - \theta_0$, we get

$$\begin{cases} x - x_0 = -a\cos\theta_0 + \frac{1}{2}a(\theta - \theta_0)\sin\theta_0 + \frac{c}{4}\sin\theta_0 \\ y - y_0 = -a\sin\theta_0 - \frac{1}{2}a(\theta - \theta_0)\cos\theta_0 - \frac{c}{4}\cos\theta_0 \end{cases}$$

$$\Rightarrow \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \begin{pmatrix} -\cos\theta_0 + \frac{1}{2}(\theta - \theta_0)\sin\theta_0 & \frac{1}{4}\sin\theta_0 \\ -\sin\theta_0 - \frac{1}{2}(\theta - \theta_0)\cos\theta_0 & -\frac{1}{4}\cos\theta_0 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix}$$

and we name

$$A = \begin{pmatrix} -\cos\theta_0 + \frac{1}{2}(\theta - \theta_0)\sin\theta_0 & \frac{1}{4}\sin\theta_0 \\ -\sin\theta_0 - \frac{1}{2}(\theta - \theta_0)\cos\theta_0 & -\frac{1}{4}\cos\theta_0 \end{pmatrix}.$$

Then

$$\det A = \frac{1}{4}\cos^2\theta_0 - \frac{1}{8}(\theta - \theta_0)\sin\theta_0\cos\theta_0 + \frac{1}{4}\sin^2\theta_0 + \frac{1}{8}(\theta - \theta_0)\cos\theta_0\sin\theta_0 = \frac{1}{4}$$

and so

$$A^{-1} = \begin{pmatrix} -\cos\theta_0 & -\sin\theta_0 \\ 4\sin\theta_0 + 2(\theta - \theta_0)\cos\theta_0 & -4\cos\theta_0 + 2(\theta - \theta_0)\sin\theta_0 \end{pmatrix}.$$

Thus we have that

$$\begin{pmatrix} a \\ c \end{pmatrix} = A^{-1}\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} -\cos\theta_0 & -\sin\theta_0 \\ 4\sin\theta_0 + 2(\theta - \theta_0)\cos\theta_0 & -4\cos\theta_0 + 2(\theta - \theta_0)\sin\theta_0 \end{pmatrix}\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} a = -\cos\theta_0(x - x_0) - \sin\theta_0(y - y_0) \\ b = \theta - \theta_0 \\ c = (4\sin\theta_0 + 2(\theta - \theta_0)\cos\theta_0)(x - x_0) + (-4\cos\theta_0 + 2(\theta - \theta_0)\sin\theta_0)(y - y_0) \end{cases}.$$

Now that we found the normal coordinates, we want to show how X_1, X_2, X_3 react when we write them using the new coordinates.
So we take $f : \mathbb{R}^3 \to \mathbb{C}$, $f(a, b, c) = f(a(x, y, \theta), b(x, y, \theta), c(x, y, \theta))$ and close to ξ_0 we can write

$$(\star) \begin{cases} \partial_x f(a, b, c) = -\cos\theta_0 \partial_a f(a, b, c) + (4\sin\theta_0 + 2(\theta - \theta_0)\cos\theta_0)\,\partial_c f(a, b, c) \\ \partial_y f(a, b, c) = -\sin\theta_0 \partial_a f(a, b, c) + (-4\cos\theta_0 + 2(\theta - \theta_0)\sin\theta_0)\,\partial_c f(a, b, c) \\ \partial_\theta f(a, b, c) = \partial_b f(a, b, c) + (2\cos\theta_0(x - x_0) + 2\sin\theta_0(y - y_0))\,\partial_c f(a, b, c) \end{cases}.$$

Then we find

$$\begin{cases} X_1 f(a,b,c) = (\partial_a + 2b\partial_c)\, f(a,b,c) \\ X_2 f(a,b,c) = (\partial_b - 2a\partial_c)\, f(a,b,c) \\ X_3 f(a,b,c) = \partial_c f(a,b,c) \end{cases}$$

that actually is the basis of $T(\mathbb{H}_n)$ (see definition 2.1.8 and theorem 6.2.8).

In fact,

$$X_1 f(a,b,c) = (-\cos\theta_0 + (\theta-\theta_0)\sin\theta_0)\partial_x f(a,b,c) - (\sin\theta_0 + (\theta-\theta_0)\cos\theta_0)\partial_y f(a,b,c).$$

Substituting from $(\star)$, this expression has now terms in ∂_a and in ∂_c. We isolate them and deal with them separately.
∂_a)

$$(-\cos\theta_0 + (\theta-\theta_0)\sin\theta_0)(-\cos\theta_0)\partial_a - (\sin\theta_0 + (\theta-\theta_0)\cos\theta_0)(-\sin\theta_0)\partial_a =$$

$$= (\cos^2\theta_0 - (\theta-\theta_0)\sin\theta_0\cos\theta_0 + \sin^2\theta_0 + (\theta-\theta_0)\cos\theta_0\sin\theta_0)\partial_a = \partial_a$$

∂_c)

$$(-\cos\theta_0 + (\theta-\theta_0)\sin\theta_0)\,(4\sin\theta_0 + 2(\theta-\theta_0)\cos\theta_0)\,\partial_c +$$

$$-(\sin\theta_0 + (\theta-\theta_0)\cos\theta_0)\,(-4\cos\theta_0 + 2(\theta-\theta_0)\sin\theta_0)\,\partial_c =$$

$$= \big(-4\cos\theta_0\sin\theta_0 - 2(\theta-\theta_0)\cos^2\theta_0 + 4(\theta-\theta_0)\sin^2\theta_0 + 2(\theta-\theta_0)^2\sin\theta_0\cos\theta_0 +$$

$$+4\sin\theta_0\cos\theta_0 - 2(\theta-\theta_0)\sin^2\theta_0 + 4(\theta-\theta_0)\cos^2\theta_0 - 2(\theta-\theta_0)^2\sin\theta_0\cos\theta_0\big)\partial_c = +2b\partial_c.$$

Then

$$X_1 f(a,b,c) = (\partial_a + 2b\partial_c)\, f(a,b,c).$$

The second one is

$$X_2 f(a,b,c) = \partial_\theta f(a,b,c) = \partial_b f(a,b,c) + (2\cos\theta_0(x-x_0) + 2\sin\theta_0(y-y_0))\,\partial_c f(a,b,c).$$

Firse we compute

$$2\cos\theta_0(x-x_0) = 2\cos\theta_0\left(-a\cos\theta_0 + \frac{1}{2}ab\sin\theta_0 + \frac{c}{4}\sin\theta_0\right) =$$

$$= -2a\cos^2\theta_0 + ab\cos\theta_0\sin\theta_0 + \frac{1}{2}c\cos\theta_0\sin\theta_0,$$

and

$$2\sin\theta_0(y-y_0) = 2\sin\theta_0\left(-a\sin\theta_0 - \frac{1}{2}ab\cos\theta_0 - \frac{c}{4}\cos\theta_0\right) =$$

$$= -2a\sin^2\theta_0 - ab\cos\theta_0\sin\theta_0 - \frac{1}{2}c\sin\theta_0\cos\theta_0.$$

Then we get

$$X_2 f(a,b,c) = (\partial_b - 2a\partial_c)\, f(a,b,c).$$

Finally we look at X_3.

$$X_3 f(a,b,c) = \frac{1}{4}\sin\theta_0\partial_x f(a,b,c) - \frac{1}{4}\cos\theta_0\partial_y f(a,b,c).$$

Substituting again from $(\star)$, we have terms in ∂_a and in ∂_c. As for X_1, we isolate them and deal with them separately.

∂_a)

$$\frac{1}{4}\sin\theta_0(-\cos\theta_0)\partial_a - \frac{1}{4}\cos\theta_0(-\sin\theta_0)\partial_a = 0$$

∂_c)

$$\frac{1}{4}\sin\theta_0\,(4\sin\theta_0 + 2(\theta-\theta_0)\cos\theta_0)\,\partial_c - \frac{1}{4}\cos\theta_0\,(-4\cos\theta_0 + 2(\theta-\theta_0)\sin\theta_0)\,\partial_c =$$

$$= \left(\sin^2\theta_0 + \frac{1}{2}(\theta-\theta_0)\sin\theta_0\cos\theta_0 + \cos^2\theta_0 - \frac{1}{2}(\theta-\theta_0)\cos\theta_0\sin\theta_0\right)\partial_c = \partial_c.$$

Then

$$X_3 f(a,b,c) = \partial_c f(a,b,c)$$

and that completes the exericise.

Chapter 7

Subelliptic Estimates for $\Box_b$ on M

Goal. In this final chapter we want to study the $\overline{\partial}_b$-Laplacian $\Box_b$ on a CR manifold M. The $\overline{\partial}_b$-Laplacian is not elliptic, since it has a one-dimensional characteristic set. However, under certain conditions, it is possible to estabish a $\frac{1}{2}$-estimate for the $\Box_b$ operator and also to prove its hypoellipticity. At last, we will also show the existence and regularity theorems of the $\overline{\partial}_b$ equation.

7.1 Subelliptic Estimates for Q_b

Recall 7.1.1. Let $(M, T^{1,0}(M))$ be a compact orientable CR manifold of real dimension $2n+1$ with $n \geq 1$. We already described its structure and defined the tangent $\overline{\partial}_b$ operator in sections 1.3, 1.4 and 1.5. Then, in paragraph 6.1, we studied $\overline{\partial}_b$ and we defined it's adjoint $\overline{\partial}_b^*$ and the $\Box_b$ operator in the case of k-strongly pseudoconvex CR manifolds.

Observation 7.1.2. Although here we are not asking for the k-strongly pseudoconvexity, we can think about our operators as in definition 6.1.14 and propositions 6.1.15 and 6.1.16. In short we'll introduce a condition even weaker than the k-strongly pseudoconvexity.

Definition 7.1.3.
In order to give subelliptic estimates, we restrict $\overline{\partial}_b$ to the Hilbert spaces of $L^2_{0,q}(M)$ of $(0,q)$-forms with L^2 coefficients.
Then we can write the domain of $\overline{\partial}_b$ as

$$\mathrm{Dom}(\overline{\partial}_b) := \left\{ \phi \in L^2_{0,q}(M) \ / \ \overline{\partial}_b \phi \in L^2_{0,q+1}(M) \right\}.$$

We can now define $\overline{\partial}_b^*$ in the standard way (as in definition 6.1.14) and we say that

$$\mathrm{Dom}(\overline{\partial}_b^*) := \big\{\phi \in L^2_{0,q}(M) \ / \ \exists g \in L^2_{0,q-1}(M) \text{ s.t. } < g, \psi > = < \phi, \overline{\partial}_b \psi > \\ \forall \psi \in \mathrm{Dom}(\overline{\partial}_b), \ \psi \ (p, q-1) - \text{form}\big\}.$$

Finally the classical definition of $\Box_b := \overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b$ yields that the Laplacian is defined on

$$\mathrm{Dom}(\Box_b) := \Big\{\phi \in L^2_{0,q}(M) \ / \ \phi \in \mathrm{Dom}(\overline{\partial}_b) \cap \mathrm{Dom}(\overline{\partial}_b^*), \\ \overline{\partial}_b \phi \in \mathrm{Dom}(\overline{\partial}_b^*) \text{ and } \overline{\partial}_b^* \phi \in \mathrm{Dom}(\overline{\partial}_b)\Big\}.$$

Notation 7.1.4. We denote $W^s(U)$, $s > 0$, the Sobolev space defined by

$$W^s(U) = \{u \in L^2(U) \ / \ |\hat{u}(\xi)|(1 + |\xi|^2)^{\frac{s}{2}} \in L^2(U)\}$$

that is an Hilbert space with the norm

$$||u||_s = ||\hat{u}(\xi)(1 + |\xi|^2)^{\frac{s}{2}}||_{L^2}.$$

Accordingly, we call $W^s_{0,q}(U)$ the space of $(0, q)$-forms with coefficients in $W^s(U)$.

Definition 7.1.5.
We define an Hermitian form Q_b on smooth $(0, q)$-forms by

$$Q_b(\phi, \psi) := (\overline{\partial}_b \phi, \overline{\partial}_b \psi) + (\overline{\partial}_b^* \phi, \overline{\partial}_b^* \psi) + (\phi, \psi) = ((\Box_b + I)\phi, \psi)$$

for $\phi, \psi \in \mathcal{E}^{0,q}(M)$ (we recall this notation from chapter 1.4).

Recall 7.1.6. Locally we can assume we have an orthonormal basis $\{L_1, \ldots, L_n, \overline{L}_1, \ldots, \overline{L}_n, T\}$ for $\mathbb{C}T(M)$, and its dual basis on $\mathbb{C}T(M)^*$ $\{\omega_1, \ldots, \omega_n, \overline{\omega}_1, \ldots, \overline{\omega}_n, \tau\}$.
Then we can express a smooth $(0, q)$-form ϕ as

$$\phi = \sum_{|J|=q}{}' \phi_J \overline{\omega}^J$$

where ϕ_J's are smooth functions.
We remind, from proposition 6.1.13, that a direct computation yields

$$\overline{\partial}_b \phi = \sum_{|J|=q}{}' \sum_{j=1,\ldots,n} (\overline{L}_j \phi_J) \overline{\omega}_j \wedge \overline{\omega}^J + \mathcal{E}(\phi)$$

and, from proposition 6.1.15,

$$\overline{\partial}_b^* \phi = - \sum_{|K|=q-1}' \sum_{j=1,\dots,n} (L_j \phi_{jK}) \overline{\omega}^K + \mathcal{E}(\phi).$$

Notation 7.1.7. We also abbreviate $||\phi||_{L^2}$ with $||\phi||$ and write

$$||\phi||_L^2 := \sum_{k,J} ||L_k \phi_J||^2 + ||\phi||^2$$

and

$$||\phi||_{\overline{L}}^2 := \sum_{k,J} ||\overline{L}_k \phi_J||^2 + ||\phi||^2.$$

Recalling theorem 1.5.6, we first state a general result.

Definition 7.1.8.
Let D be a relatively compact subset with C^∞ boundary in a complex Hermitian manifold of complex dimension $n+1$, with $n \geq 1$.
D is said to satisfy the *condition* $Z(q)$, $1 \leq q \leq n$, if the Levi form associated with D has

at least $n+1-q$ positive eigenvalues

or

at least $q+1$ negative eigenvalues

at every boundary point.

Observation 7.1.9. Obviously condition $Z(q)$ is satisfied for all q with $1 \leq q \leq n$ on any strongly pseudoconvex domain.

Theorem 7.1.10.
Let D be a relatively compact subset with C^∞ boundary in a complex Hermitian manifold of complex dimension $n+1$, with $n \geq 1$. Suppose that condition $Z(q)$ holds for some q, $1 \leq q \leq n$.
Then we have

$$\int_{\partial D} |f|^2 d\sigma \leq C \left(||\overline{\partial} f||^2 + ||\overline{\partial}^* f||^2 + ||f||^2 \right)$$

for $f \in \mathcal{E}^{0,q}(\overline{D}) \cap \mathrm{Dom}(\overline{\partial}^*)$.
Furtheremore we have that

$$||f||_{\frac{1}{2}} \leq C \left(||\overline{\partial} f|| + ||\overline{\partial}^* f|| + ||f|| \right)$$

for $f \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*)$, where $C > 0$ is independent of f.

Proof. Let ρ be a defining function for D, let $x_0 \in D$ be a boundary point and let U be an open neighborhood of x_0. For any $f \in \mathcal{E}^{0,q}(\overline{D}) \cap \mathrm{Dom}(\overline{\partial}^*)$, with support in U, the proof of proposition 5.3.3[1] (with $\phi \equiv 0$) shows that

$$Q_b(f,f) = \sum_{|J|=q}{}' \sum_{k=1,\dots,n} ||\overline{L}_k f_J||^2 + \sum_{|J|=q}{}' \sum_{j,k=1,\dots,n} \int_{\partial D \cap U} \rho_{jk} f_{jJ} \overline{f}_{kJ} d\sigma +$$

$$+ O\left(\left(||\overline{\partial} f|| + ||\overline{\partial}^* f||\right) ||f|| + ||f||_{\overline{L}} ||f||\right).$$

We may assume that the Levi form is diagonal at x_0, namely $\rho_{jk}(x) = \lambda_j \delta_{jk} + b_{jk}(x)$ for $1 \le j,k \le n$, where λ_j's are the eigenvalues of the Levi form at x_0, δ_{jk} is the Kronecker delta and $b_{jk}(x_0) = 0$.
If follows that

$$\sum_{|J|=q}{}' \sum_{j,k=1,\dots,n} \int_{\partial D \cap U} \rho_{jk} f_{jJ} \overline{f}_{kJ} d\sigma = \sum_{|J|=q}{}' \left(\sum_{k\in J} \lambda_j\right) \int_{\partial D} |f_J|^2 d\sigma + cO\left(\sum_{|J|=q}{}' \int_{\partial D} |f_J|^2 d\sigma\right)$$

where $c > 0$ can be made arbitrary small if U is chosen sufficiently small. Integration by parts also shows

$$||\overline{L}_k f_J||^2 = -\left([L_k, \overline{L}_k] f_J, f_J\right) + ||L_k f_J||^2 + O\left(||f||_{\overline{L}} ||f||\right) \ge$$

$$\ge -\lambda_k \int_{\partial D} |f_J|^2 d\sigma - c\int_{\partial D} |f_J|^2 d\sigma + O\left(||f||_{\overline{L}} ||f|| + ||f||^2\right).$$

Hence, if condition $Z(q)$ holds on ∂D, then for each fixed J either there is $k_1 \in J$ such that $\lambda_{k_1} > 0$ or there is $k_2 \notin J$ such that $\lambda_{k_2} < 0$. Then, for $\epsilon > 0$ we have two cases:

$$Q_b(f,f) \ge \epsilon \sum_{|J|=q}{}' \sum_{k=1,\dots,n} ||\overline{L}_k f_J||^2 + \epsilon \sum_{|J|=q}{}' \left(\sum_{k\in J, \lambda_k<0} \lambda_k\right) \int_{\partial D} |f_J|^2 d\sigma +$$

$$+ \sum_{|J|=q}{}' \left((\lambda_{k_1} - c) \int_{\partial D} |f_J|^2 d\sigma\right)$$

$$+ O\left(\left(||\overline{\partial} f|| + ||\overline{\partial}^* f||\right) ||f|| + ||f||_{\overline{L}} ||f|| + ||f||^2\right)$$

or

$$Q_b(f,f) \ge \epsilon \sum_{|J|=q}{}' \sum_{k=1,\dots,n} ||\overline{L}_k f_J||^2 + \epsilon \sum_{|J|=q}{}' \left(\sum_{k\in J, \lambda_k<0} \lambda_k\right) \int_{\partial D} |f_J|^2 d\sigma +$$

$$+(1-\epsilon)\sum_{|J|=q}^{\prime}\left((-\lambda_{k_2}-c)\int_{\partial D}|f_J|^2 d\sigma\right)$$

$$+O\left(\left(||\overline{\partial}f||+||\overline{\partial}^* f||\right)||f||+||f||_{\overline{L}}||f||+||f||^2\right)$$

Then, choosing ϵ and c small enough and using small and large constants, we obtain the first statement. The second one follows by a partition of unity argument. □

Now we return to the subelliptic estimate for $\Box_b$ on $(0,q)$-forms on M.

Observation 7.1.11. If the CR manifold M is embedded as the boundary of a complex manifold D, topologically one can't distinguish whether M is the boundary of D or M is the boundary of the complement of D.
Thus, in order to obtain a subelliptic estimate for $(0,q)$-forms on M similar to theorem 7.1.10, we shall assume that condition $Z(q)$ holds on both D and its complement D^c. That's equivalent to say that conditions $Z(q)$ and $Z(n-q)$ hold on D.

Now we write this condition formally in terms of eigenvalues and we call it $Y(q)$.

Definition 7.1.12.
Let M be an oriented CR manifold of real dimension $2n+1$, with $n \geq 1$.
M is said to satisfy the *condition* $Y(q)$, $1 \leq q \leq n$, if the Levi form associated with M has at least either

$$\max\{n+1-q, q+1\} \text{ eigenvalues of the same sign}$$

or

$$\min\{n+1-q, q+1\} \text{ pairs of eigenvalues of opposite sign}$$

at every point of M.

Observation 7.1.13. It follows that condition $Y(q)$ holds on any strongly pseudoconvex CR manifold M when $1 \leq q \leq n-1$ (they satisfy the first condition). On the other hand $Y(n)$ is violated on any CR manifold.

Observation 7.1.14. From here to the end of this chapter we'll suppose almost everywhere that condition $Y(q)$, for some q with $1 \leq q \leq n$, holds on a compact, oriented, CR manifold $(M, T^{1,0}(M))$ of real dimension $2n+1$, $n \geq 1$.

Theorem 7.1.15.
Under the hypothesis of observation 7.1.14, we have

$$||\phi||^2_{\frac{1}{2}} \leq CQ_b(\phi,\phi)$$

uniformly $\forall \phi \in \mathcal{E}^{0,q}(M)$, $C > 0$.

Proof. Since condition $Y(q)$ implies that the vector fields $L_1, \dots, L_n, \overline{L}_1, \dots, \overline{L}_n$ and their Lie brackets span the whole complex tangent space, using a partition of unity, the proof is a consequence of 8.2.5[1] in case $m = 2$, and of the following theorem. $\Box$

Theorem 7.1.16.
Under the hypothesis of observation 7.1.14, for any $x_0 \in M$ there is a neighborhood V_{x_0} of x_0 s.t.

$$||\phi||^2_L + ||\phi||^2_{\overline{L}} + \sum_{|J|=q} |\operatorname{Re}(T\phi_J, \phi_J)| \leq CQ_b(\phi,\phi)$$

uniformly $\forall \phi \in \mathcal{E}^{0,q}(M)$ with support contained in V_{x_0}, $C > 0$.

Proof. From recall 7.1.6 we take $\phi = \sum'_{|J|=q} \phi_J \overline{\omega}^J$ with $\phi_J \in C^\infty(V_{x_0})$. That yields

$$\overline{\partial}_b \phi = \sum_{|J|=q}{}' \sum_{j=1,\dots,n} (\overline{L}_j \phi_J) \overline{\omega}_j \wedge \overline{\omega}^J + \mathcal{E}(\phi).$$

Then

$$||\overline{\partial}_b\phi||^2 = \sum_{|J|=q}{}' \sum_{j \notin J} ||\overline{L}_j \phi_J||^2 + \sum_{|J|,|L|=q}{}' \sum_{j,l=1,\dots,n} \epsilon^{jJ}_{lL} < \overline{L}_j\phi_J, \overline{L}_l \phi_L > +$$

$$+O(||\phi||_{\overline{L}}||\phi||).$$

where

$$\epsilon^{jJ}_{lL} := \begin{cases} 0, \text{ if } l \in L \text{ or } j \in J \text{ or } \{j\} \cup J \neq \{l\} \cup L \\ \text{sign of the permutation } \begin{pmatrix} jJ \\ lL \end{pmatrix}, \text{ otherwise} \end{cases}$$

(see also definition 2.4.3).
Using this fact, we rearrange the estimate:

$$||\overline{\partial}_b\phi||^2 = \sum_{|J|=q}{}' \sum_{j=1,\dots,n} ||\overline{L}_j \phi_J||^2 - \sum_{|K|=q-1}{}' \sum_{j,k=1,\dots,n} < \overline{L}_j \phi_{kK}, \overline{L}_k \phi_{jK} > +$$

$$+O(||\phi||_{\overline{L}}||\phi||).$$

Using integration by parts, we have

$$<\overline{L}_j\phi_{kK},\overline{L}_k\phi_{jK}>=<-L_k\overline{L}_j\phi_{kK},\phi_{jK}>+O(||\phi||_{\overline{L}}||\phi||)=$$

$$=<L_k\phi_{kK},L_j\phi_{jK}>+<[\overline{L}_j,L_k]\phi_{kK},\phi_{jK}>+O(||\phi||_L||\phi||+||\phi||_{\overline{L}}||\phi||).$$

Hence, using recall 7.1.6 again, we obtain

$$||\overline{\partial}_b\phi||^2=\sum_{|J|=q}{}' \sum_{j=1,\dots,n}||\overline{L}_j\phi_J||^2-||\overline{\partial}_b^*\phi||+\sum_{|K|=q-1}{}' \sum_{j,k=1,\dots,n}<[L_j,\overline{L}_k]\phi_{jK},\phi_{kK}>+$$

$$+O(||\phi||_L||\phi||+||\phi||_{\overline{L}}||\phi||).$$

To handle the commutator term, we assume the the Levi form is diagonal at x_0 and that $c_{11}(x_0)\neq 0$ (see definition 1.5.2), thanks to condition $Y(q)$. It follows that $c_{11}(x_0)=\frac{1}{C}>0$ for $x_0\in V_{x_0}$, if V_{x_0} is chosen to be small enough. Now, if f is a smooth function with $f(x_0)=0$ on M, we have

$$|\operatorname{Re}<T\phi_J,f\phi_L>|\leq\left|\operatorname{Re}<\frac{1}{c_{11}}[L_1,\overline{L}_1]\phi_J,f\phi_L>\right|+O(||\phi||_{\overline{L}}||\phi||)$$

$$\leq C\sup_{V_{x_0}}|f|\left(||\phi||_L^2+||\phi||_{\overline{L}}^2\right)+O(||\phi||_{\overline{L}}||\phi||).$$

Thus, if we denote the eigenvalues of the Levi form at x_0 by $\lambda_1,\dots,\lambda_n$, we can write

$$Q_b(\phi,\phi)=\sum_{|J|=q}{}' \sum_{j=1,\dots,n}||\overline{L}_j\phi_J||^2+\sum_{|J|=q}{}' \sum_{j\in J}\lambda_j\operatorname{Re}<T\phi_J,\phi_J>+$$

$$+cO(||\phi||_L^2+||\phi||_{\overline{L}}^2)+O(||\phi||_{\overline{L}}||\phi||)\qquad(*)$$

where $c=\sup_{V_{x_0}}|f|>0$ can be made arbitrary small, if necessary, by shrinking V_{x_0}.

Now we integrate by parts to get

$$||\overline{L}_j\phi_J||^2=||L_j\phi_J||^2-\lambda_j\operatorname{Re}<T\phi_J,\phi_J>+$$

$$+cO(||\phi||_L^2+||\phi||_{\overline{L}}^2)+O(||\phi||_L||\phi||+||\phi||_{\overline{L}}||\phi||).$$

Next, we set

$$\sigma(J):=\{j\ /\ \lambda_j>0\text{ if }\operatorname{Re}(<T\phi_J,\phi_J>)>0\quad\vee\quad\lambda_j<0\text{ if }\operatorname{Re}(<T\phi_J,\phi_J>)<0\}.$$

It follows that, for any small $\epsilon > 0$, we have

$$||\phi||_L^2 \geq \epsilon||\phi||_L^2 + (1-\epsilon)\sum_{|J|=q}' \sum_{j\in\sigma(J)} ||\overline{L}_j\phi_J||^2 \geq$$

$$\geq \epsilon||\phi||_L^2 - (1-\epsilon)\sum_{|J|=q}' \sum_{j\in\sigma(J)} \lambda_j \operatorname{Re} < T\phi_J, \phi_J > -c(||\phi||_L^2 + ||\phi||_{\overline{L}}^2) - C||\phi||^2.$$

Substituting what we just found in $(*)$, we obtain

$$Q_b(\phi,\phi) \geq \epsilon||\phi||_L^2 - (1-\epsilon)\sum_{|J|=q}' \sum_{j\in\sigma(J)} \lambda_j \operatorname{Re} < T\phi_J, \phi_J > +$$

$$+ \sum_{|J|=q}' \sum_{j\in J} \lambda_j \operatorname{Re} < T\phi_J, \phi_J > -c(||\phi||_L^2 + ||\phi||_{\overline{L}}^2) - O(||\phi||_{\overline{L}}||\phi||) =$$

$$= \epsilon||\phi||_L^2 + \sum_{|J|=q}' a_J \operatorname{Re} < T\phi_J, \phi_J > -c(||\phi||_L^2 + ||\phi||_{\overline{L}}^2) - O(||\phi||_{\overline{L}}||\phi||),$$

where

$$a_J = \sum_{j\in J\setminus\sigma(J)} \lambda_j - (1-\epsilon)\sum_{j\in\sigma(J)\setminus J} \lambda_j + \epsilon \sum_{j\in J\cap\sigma(J)} \lambda_j.$$

Note that, since $Y(q)$ holds at x_0, one of the following three cases must hold:

1. If the Levi form has $\max(n+1-q, q+1)$ eigenvalues of the same sign, then there exists a $j \in J$ and $k \notin J$ so that λ_j and λ_k are of the same sign which may be assumed to be positive, if necessary, by replacing T by $-T$.

2. If the Levi form has $\min(n+1-q, q+1)$ pairs of eigenvalues of opposite signs, then there are $j, k \notin J$ so that $\lambda_j > 0$ and $\lambda_k < 0$.

3. If the Levi form has $\min(n+1-q, q+1)$ pairs of eigenvalues of opposite signs, then there are $j, k \in J$ so that $\lambda_j > 0$ and $\lambda_k < 0$.

Then it's not too hard to verify that, by choosing $\epsilon > 0$ to be small enough, a_j can have the same sign of $\operatorname{Re} < T\phi_J, \phi_J >$ (when $\operatorname{Re} < T\phi_J, \phi_J > \neq 0$). Then we get,

$$Q_b(\phi,\phi) \geq C\left(||\phi||_L^2 + \sum_{|J|=q}' |\operatorname{Re} < T\phi_J, \phi_J > | - (sc)||\phi||_L^2 - (lc)||\phi||^2\right)$$

where (sc) is a small consant and (lc) is a large one.
Since

$$||L_j\phi_J||^2 \le C\left(||\overline{L}_j\phi_J||^2 + |\operatorname{Re} < T\phi_J, \phi_J > | + cO(||\phi||^2_L + ||\phi||^2_{\overline{L}}) + O(||\phi||_{\overline{L}}||\phi||)\right),$$

we can choose c and (sc) sufficiently small to obtain

$$||\phi||^2_L + ||\phi||^2_{\overline{L}} + \sum_{|J|=q} |\operatorname{Re}(T\phi_J, \phi_J)| \le CQ_b(\phi, \phi)$$

The proof is complete. □

Corollary 7.1.17 (Corollary of Theorem 7.1.15)**.**
Under the hypotesis of observation 7.1.14, Q_b is compact with respect to $L^2_{0,q}(M)$.

Proof. Using Friedrichs' lemma (see D.1[1]) and theorem 7.1.15, we obtain

$$Q_b(\phi, \phi) \ge C||\phi||^2_{\frac{1}{2}}$$

for $\phi \in \operatorname{Dom}(\overline{\partial}_b) \cap \operatorname{Dom}(\overline{\partial}^*_b)$. In particular, Q_b is compact with respect to $L^2_{0,q}(M)$. □

7.2 Subelliptic Estimates for $\Box_b + I$ and $\Box_b$

We now focus on the operator $\Box_b + I$. It's easy to see that it's injective on $L^2_{0,q}(M)$. We give here a Lemma and then an important theorem.

Lemma 7.2.1. Under the hypothesis of observation 7.1.14, let U be a local coordinate neighborhood and let $\{\zeta_k\}_{k=1,\dots,\infty}$ be a sequence of real smooth functions supported in U such that $\zeta = 1$ on the support of ζ_{k+1} for all k. Then, if $k = 1$, we have

$$||\zeta_1\phi||^2_{\frac{1}{2}} \le C||(\Box_b + I)\phi||^2$$

and, $\forall k > 1$,

$$||\zeta_k\phi||^2_{\frac{k}{2}} \le C||\zeta_1(\Box_b + I)\phi||^2_{\frac{k-2}{2}} + ||(\Box_b + I)\phi||^2$$

uniformly $\forall\phi \in \mathcal{E}^{0,q}(M)$ supported in U, with $C > 0$.

Proof. The lemma will be proved by induction. Here we identify the function ζ_1 with the operator of the product with ζ_1. For $k = 1$, by theorem 7.1.15, we have the inequality

$$||\zeta_1\phi||^2_{\frac{1}{2}} \leq CQ_b(\zeta_1\phi, \zeta_1\phi) = ||\overline{\partial}_b\zeta_1\phi||^2 + ||\overline{\partial}_b^*\zeta_1\phi||^2 + ||\zeta_1\phi||^2.$$

Reminding definition 2.4.4, we estimate the first piece on the right-hand side as follows:

$$||\overline{\partial}_b\zeta_1\phi||^2 =< \overline{\partial}_b\zeta_1\phi, \overline{\partial}_b\zeta_1\phi >=$$
$$=< \zeta_1\overline{\partial}_b\phi, \overline{\partial}_b\zeta_1\phi > + < [\overline{\partial}_b, \zeta_1]\phi, \overline{\partial}_b\zeta_1\phi >=$$
$$=< \overline{\partial}_b\phi, \overline{\partial}_b\zeta_1^2\phi > + < \overline{\partial}_b\phi, [\zeta_1, \overline{\partial}_b]\zeta_1\phi > + < [\overline{\partial}_b, \zeta_1]\phi, \overline{\partial}_b\zeta_1\phi >=$$
$$=< \overline{\partial}_b^*\overline{\partial}_b\phi, \zeta_1^2\phi > + < \overline{\partial}_b\zeta_1\phi, [\zeta_1, \overline{\partial}_b]\phi > + < [\zeta_1, \overline{\partial}_b]\phi, [\zeta_1, \overline{\partial}_b]\phi > + < [\overline{\partial}_b, \zeta_1]\phi, \overline{\partial}_b\zeta_1\phi > .$$

We can also calculate and note that

$$\operatorname{Re}\left(< \overline{\partial}_b\zeta_1\phi, [\zeta_1, \overline{\partial}_b]\phi > + < [\overline{\partial}_b, \zeta_1]\phi, \overline{\partial}_b\zeta_1\phi >\right) = 0.$$

A similar argument holds for $||\overline{\partial}_b^*\zeta_1\phi||^2$.
Thus we have

$$||\zeta_1\phi||^2_{\frac{1}{2}} \leq CQ_b(\zeta_1\phi, \zeta_1\phi) \leq C \operatorname{Re} < (\Box_b + I)\phi, \zeta_1^2\phi > +O(||\phi||^2) \leq$$

$$\leq C||(\Box_b + I)\phi|| \cdot ||\phi|| + O(||\phi||^2) \leq$$

since $||\phi||^2 \leq ||(\Box_b + I)\phi||$,

$$\leq C||(\Box_b + I)\phi||^2.$$

This establishes the initial step.
Let us assume that the assertion is true for all integers up to $k - 1$, then we prove it for k. With an pseudodifferential operator argument, we can find

$$||\zeta_k\phi||^2_{\frac{k}{2}} \leq C\left(||\zeta_1(\Box_b + I)\phi||^2_{\frac{k-2}{2}} + ||\zeta_{k-1}\phi||^2_{\frac{k-1}{2}}\right) \leq$$

then, by induction hypothesis,

$$\leq C\left(||\zeta_1(\Box_b + I)\phi||^2_{\frac{k-2}{2}} + ||\zeta_1(\Box_b + I)\phi||^2_{\frac{k-3}{2}} + ||(\Box_b + I)\phi||^2\right) \leq$$

$$\leq C\left(||\zeta_1(\Box_b + I)\phi||^2_{\frac{k-2}{2}} + ||(\Box_b + I)\phi||^2\right).$$

This completes the proof. □

Theorem 7.2.2.
Under the hypothesis of observation 7.1.14, given $\alpha \in L^2_{0,q}(M)$, let $\phi \in \mathrm{Dom}(\Box_b)$ be the unique solution of $(\Box_b + I)\phi = \alpha$.
If $U \subset M$ and $\alpha_{|_U} \in \mathcal{E}^{0,q}(U)$, then $\phi_{|_U} \in \mathcal{E}^{0,q}(U)$.
Moreover, if ζ and ζ_1 are two cut-off functions supported in U such that $\zeta_1 = 1$ on the support of ζ, then $\forall s > 0$ there is a constant C_s such that

$$||\zeta\phi||^2_{s+1} \leq C_s \left(||\zeta_1\alpha||^2_s + ||\alpha||^2\right).$$

Proof. If $\alpha_{|_U}$ is smooth then the estimate follows from lemma 7.2.1. Therefore it only remains to show that $\alpha_{|_U} \in \mathcal{E}^{0,q}(U)$.
Since Q_b is not elliptic, we shall here apply the technique of elliptic regularization to Q_b. The sketch of this proof can be found at 8.4.2[1], while the details about the elliptic regularization are at 5.2.1-5.2.5[1] (pages from 93 to 103). In order to prove the theorem, we will also need A.7 and A.8[1]. □

A few consequences follow immediately from this theorem.

Theorem 7.2.3.
Suppose the hypothesis of observation 7.1.14, given $\alpha \in L^2_{0,q}(M)$, let $\phi \in \mathrm{Dom}(\Box_b)$ be the unique solution of $(\Box_b + I)\phi = \alpha$. Let $U \subset M$ and let ζ and ζ_1 be two cut-off functions supported in U such that $\zeta_1 = 1$ on the support of ζ.
If $\alpha_{|_U} \in W^s_{0,q}(U)$ for some $s > 0$, then $\zeta\phi \in W^{s+1}_{0,q}(U)$ and

$$||\zeta\phi||^2_{s+1} \leq C \left(||\zeta_1\alpha||^2_s + ||\alpha||^2\right),$$

with $C > 0$.

Proof. Let ζ_0 be a cut-off function supported in U such that $\zeta_0 = 1$ on the support of ζ_1. Choose sequences of smooth $(0,q)$-forms $\{\beta_n\}_n$ and $\{\gamma_n\}_n$ with

$$\mathrm{supp}\beta_n \subset \mathrm{supp}\zeta_0 \quad \text{and} \quad \mathrm{supp}\gamma_n \subset \mathrm{supp}(1-\zeta_0)$$

such that

$$\beta_n \to \zeta_0\alpha \text{ in } W^s_{0,q}(M) \quad \text{and} \quad \gamma_n \to (1-\zeta_0)\alpha \text{ in } L^2_{0,q}(M).$$

Hence $\alpha_n := \beta_n + \gamma_n \to \alpha$ in $L^2_{0,q}(M)$ and $\zeta_1\alpha_n \to \zeta_1\alpha$ in $W^s_{0,q}(M)$.
Let $\phi_n \in \mathrm{Dom}(\Box_b)$ be the solution of $(\Box_b + I)\phi_n = \alpha_n$, so $\phi_n \to \phi$ in $L^2_{0,q}(M)$.
Then, theorem 7.2.2 shows

$$||\zeta(\phi_n - \phi_m)||_{s+1} \leq C||\zeta_1(\alpha_n - \alpha_m)||_s + ||\alpha_n - \alpha_m||_s.$$

It follows that $\zeta\phi_n$ is Cauchy in $W^{s+1}_{0,q}(M)$ and $\lim_{n\to\infty}\zeta\phi_n = \zeta\phi$ in $W^{s+1}_{0,q}(M)$.
Hence we have

$$||\zeta\phi||_{s+1} \leq C\left(||\zeta_1\alpha||_s + ||\alpha||\right).$$

This proves the theorem. □

Theorem 7.2.4.
Suppose the hypothesis of observation 7.1.14, given $\alpha \in L^2_{0,q}(M)$, let $\phi \in \text{Dom}(\Box_b)$ be the unique solution of $(\Box_b + I)\phi = \alpha$. Let $U \subset M$ and let ζ and ζ_1 be two cut-off functions supported in U such that $\zeta_1 = 1$ on the support of ζ.
If $\zeta_1\alpha \in W^s_{0,q}(M)$ for some $s > 0$, and if ϕ satisfies $(\Box_b + \lambda I) = \alpha$ for some constant λ, then $\zeta\phi \in W^{s+1}_{0,q}(M)$.
In other words, $\Box_b + \lambda I$ is hypoelliptic for every λ. Moreover, all the eigenforms of $\Box_b$ are smooth.

Proof. Let $\alpha' := \alpha + (1-\lambda)\phi$, then $(\Box_b + I) = \alpha'$. The assertion follows from theorem 7.2.3 and an induction argument. □

Theorem 7.2.5.
Suppose the hypothesis of observation 7.1.14 and $\phi \in \text{Dom}(\Box_b)$.
If $(\Box_b + I)\phi = \alpha$, with $\alpha \in W^s_{0,q}(M)$, $s \geq 0$, then $\phi \in W^{s+1}_{0,q}(M)$ and

$$||\phi||_{s+1} \leq C||\alpha||_s$$

where the constant C is independent of α.

Here there are some important conseguences:

Corollary 7.2.6. Suppose the hypothesis of observation 7.1.14.
The operator $(\Box_b + I)^{-1}$ is compact.

Proof. Since $(\Box_b + I)^{-1}$ is a bounded operator from $L^2_{0,q}(M)$ into $W^2_{0,q}(M)$, $s \geq 0$, the assertion follows from Rellich's lemma (see A.8[1]). □

Corollary 7.2.7. Suppose the hypothesis of observation 7.1.14.
The operator $\Box_b + I$ has a discrete spectrum with no finite limit point, and each eigenvalues occurs with finite moltiplicity. All eigenvalues are smooth. In particular, $\text{Ker}(\Box_b)$ is of finite dimension and consists of smooth forms.

Proof. By corollary 7.2.6, the spectrum of $(\Box_b + I)^{-1}$ is compact and countable with zero as its only possible limit point. Since $(\Box_b + I)^{-1}$ is injective, zero is not an eigenvalue of $(\Box_b + I)^{-1}$ and each eigenvalue has finite moltiplicity. Also λ is an eigenvalue of $\Box_b + I$ if and only if λ^{-1} is an eigenvalue of $(\Box_b + I)^{-1}$. This proves the corollary. □

Proposition 7.2.8.
Suppose the hypothesis of observation 7.1.14 and let $\phi \in \text{Dom}(\Box_b)$.
$\Box_b$ is hypoelliptic. Moreover, if $\Box_b\phi = \alpha$ with $\alpha \in W^s_{0,q}(M)$, $s \geq 0$, we have

$$||\phi||^2_{s+1} \leq C\left(||\alpha||^2_s + ||\phi||^2\right)$$

where the constant $C > 0$ is independent of α.

Proof. We show the estimate by an induction on s. If $s = 0$, theorem 7.2.5 implies

$$||\phi||_1^2 \leq C||(\Box_b + I)\phi||^2 \leq C\left(||\alpha||^2 + ||\phi||^2\right).$$

In general, if we assume the assertion holds up to step $s-1$, we have $\phi \in W^s_{0,q}(M)$.
We apply theorem 7.2.5 again and, using the induction hypothesis, get

$$||\phi||_{s+1}^2 \leq C||(\Box_b + I)\phi||_s^2 \leq C\left(||\Box_b\phi||_s^2 + ||\phi||_s^2\right) \leq C\left(||\alpha||_s^2 + ||\phi||_s^2\right).$$

□

7.3 Solvability and Regularity for the $\overline{\partial}_b$-Equation

Now that we found the hypoellipticity of $\Box_b$, we will use it to study the $\overline{\partial}_b$-equation on M and finish this chapter as we did in observation 3.3.5.

Definition 7.3.1.
Let $\mathcal{H}^b_{0,q}(M)$ denote the space of harmonic forms on M, i.e.,

$$\mathcal{H}^b_{0,q}(M) = \text{Ker}(\Box_b).$$

Thus $\mathcal{H}^b_{0,q}(M)$ consists of smooth harmonic (p,q)-forms and is of finite dimension.

Observation 7.3.2. Using corollary 7.2.7, $\Box_b$ is bounded away from zero on the orthogonal complement $\left(\mathcal{H}^b_{0,q}(M)\right)^\perp$, namely,

$$||\Box_b\phi|| \geq \lambda_1||\phi||$$

for all $\phi \in \text{Dom}(\Box_b) \cap \left(\mathcal{H}^b_{0,q}(M)\right)^\perp$, where λ_1 is the smallest positive eigenvalue of $\Box_b$.

It follows from theorem 7.2.3 and lemma 4.1.1[1] that the range of $\Box_b$, $\mathcal{R}(\Box_b)$, is closed. Also the following strong Hodge type decomposition holds on $L^2_{0,q}(M)$:

Proposition 7.3.3.
Suppose the hypothesis of observation 7.1.14.
$L^2_{0,q}(M)$ admits the strong orthogonal decomposition

$$L^2_{0,q}(M) = \mathcal{R}(\Box_b) \oplus \mathcal{H}^b_{0,q}(M) =$$
$$= \overline{\partial}_b\overline{\partial}_b^*\left(\text{Dom}(\Box_b)\right) \oplus \overline{\partial}_b^*\overline{\partial}_b\left(\text{Dom}(\Box_b)\right) \oplus \mathcal{H}^b_{0,q}(M).$$

where $\mathcal{R}(\Box_b)$ denotes the range of $\Box_b$.

Proof. Since $\mathcal{R}(\Box_b) = \left(\mathcal{H}^b_{0,q}(M)\right)^{\perp}$ and $\mathcal{R}(\overline{\partial}_b\overline{\partial}_b^*)\perp\mathcal{R}(\overline{\partial}_b^*\overline{\partial}_b)$, the decomposition follows. □

Definition 7.3.4.
We can thus define the boundary operator as follows

$$N_b : L^2_{0,q}(M) \to \mathrm{Dom}(\Box_b)$$

If $\alpha \in \mathcal{H}^b_{0,q}(M)$, we set

$$N_b\alpha := 0$$

If $\alpha \in \mathcal{R}(\Box_b)$,

$$N_b\alpha := \phi$$

where ϕ is the unique solution of $\Box_b\phi = \alpha$ with $\phi\perp\mathcal{H}^b_{0,q}(M)$.
Then we extend N_b by linearity.

Observation 7.3.5. It's easily seen that N_b is a bounded operator.

Notation 7.3.6. Let H^b denote the orthogonal projection from $L^2_{0,q}(M)$ into $\mathcal{H}^b_{0,q}(M)$.

Theorem 7.3.7.
Suppose the hypothesis of observation 7.1.14.
Then there exists an operator

$$N_b : L^2_{0,q}(M) \to L^2_{0,q}(M)$$

such that

1. N_b is a compact operator.
2. for any $\alpha \in L^2_{0,q}(M)$, $\alpha = \overline{\partial}_b\overline{\partial}_b^* N_b\alpha + \overline{\partial}_b^*\overline{\partial}_b N_b\alpha + H^b\alpha$.
3. $N_bH^b = H^bN_b = 0$.
   $N_b\Box_b = \Box_bN_b = I - H^b$ on $\mathrm{Dom}(\Box_b)$.
4. If N_b is also defined on $L^2_{p,q+1}(M)$, then $N_b\overline{\partial}_b = \overline{\partial}_bN_b$ on $\mathrm{Dom}(\overline{\partial}_b)$.
   If N_b is also defined on $L^2_{p,q-1}(M)$, then $N_b\overline{\partial}_b^* = \overline{\partial}_b^*N_b$ on $\mathrm{Dom}(\overline{\partial}_b^*)$.
5. $N_b(\mathcal{E}^{0,q}(M)) \subset \mathcal{E}^{0,q}(M)$ and, for each positive integer s, the estimate

   $$||N_b\alpha||_{s+1} \le C||\alpha||_s$$

   holds uniformly for all $\alpha \in W^s_{0,q}(M)$.

Proof. 1. follows from proposition 7.2.8 and Rellich lemma (see A.8[1]).
2. is just a restatement of proposition 7.3.3.
3. follows immediately from the definition of N_b.
For 4., if $\alpha \in \mathrm{Dom}(\overline{\partial}_b)$, we use both properties 2. and 3. to get

$$N_b\overline{\partial}_b\alpha = N_b\overline{\partial}_b(\overline{\partial}_b\overline{\partial}_b^*N_b + \overline{\partial}_b^*\overline{\partial}_bN_b)\alpha = N_b\overline{\partial}_b(\overline{\partial}_b^*\overline{\partial}_bN_b)\alpha =$$

$$= N_b(\overline{\partial}_b\overline{\partial}_b^*)\overline{\partial}_bN_b\alpha = N_b(\overline{\partial}_b\overline{\partial}_b^* + \overline{\partial}_b^*\overline{\partial}_b)\overline{\partial}_bN_b\alpha = N_b\Box_b\overline{\partial}_bN_b\alpha =$$

Using point 3.,

$$= \overline{\partial}_bN_b\alpha.$$

A similar equation holds for $\overline{\partial}_b^*$.
For 5., if $\alpha \in \mathcal{E}^{0,q}(M)$, then $\alpha - H^b\alpha \in \mathcal{E}^{0,q}(M)$ and we have

$$\Box_bN_b\alpha = \alpha - H^b\alpha.$$

Using proposition 7.2.8, $\Box_b$ is hypoelliptic and then $N_b\alpha \in \mathcal{E}^{0,q}(M)$. More, we have

$$\begin{aligned}||N_b\alpha||_{s+1} &\leq C\left(||\Box N_b\alpha||_s + ||N_b\alpha||\right) \leq \\ &\leq C\left(||\alpha||_s + ||H^b\alpha||_s + ||\alpha||\right) \leq \\ &\leq C||\alpha||\end{aligned}$$

where in the last step we use that $\mathcal{H}^b_{0,q}(M)$ is of finite dimension to conclude $||H^b\alpha||_s \leq C_s||H^b\alpha|| \leq C_s||\alpha||$. □

Corollary 7.3.8. Suppose the hypothesis of observation 7.1.14.
$\mathrm{Ran}(\overline{\partial}_b)$ is closed on $\mathrm{Dom}(\overline{\partial}_b) \cap L^2_{p,q-1}(M)$.

Proof. Since $\mathcal{R}(\overline{\partial}_b) \perp \mathrm{Ker}(\overline{\partial}_b^*)$, we have $\mathcal{R}(\overline{\partial}_b) = \overline{\partial}_b\overline{\partial}_b^*\left(\mathrm{Dom}(\Box_\mathrm{b})\right)$. □

Definition 7.3.9.
Let M be a compact orientable CR manifold. The *Szegö projection* S on M is defined to be the orthogonal projection $S = H^b$ from $L^2(M)$ into $\mathcal{H}^b(M) = \mathcal{H}^b_{0,0}(M)$.

Theorem 7.3.10.
Let M be a compact orientable CR manifold that satisfy condition $Y(1)$.
Then the *Szegö projection* S on M is given by

$$S = I - \overline{\partial}_b^*N_b\overline{\partial}_b.$$

Proof. According to theorem 7.3.7, there exists an operator N_b on $L^2_{0,1}(M)$.
The conclusion comes easily. □

Theorem 7.3.7 gives the following solvability and regularity theorem for $\overline{\partial}_b$.

Theorem 7.3.11.
Let M be a compact orientable CR manifold that satisfy condition $Y(1)$.
For any $\alpha \in L^2_{0,q}(M)$ with $\overline{\partial}_b\alpha = 0$ and $H^b\alpha = 0$, there is a unique solution ϕ of $\overline{\partial}_b\phi = \alpha$ with $\phi \perp \operatorname{Ker}(\overline{\partial}_b)$.
If $\alpha \in \mathcal{E}^{0,q}(M)$, then $\phi \in \mathcal{E}^{p,q-1}(M)$.
Furthermore, $\forall s \geq 0$, if $\alpha \in W^s_{0,q}(M)$, then $\phi \in W^{s+\frac{1}{2}}_{0,q}(M)$ and

$$||\phi||_{s+\frac{1}{2}} \leq C||\alpha||_s.$$

Proof. By point 2. of theorem 7.3.7 we have here $\alpha = \overline{\partial}_b\overline{\partial}_b^*N_b\alpha$. Then we simply take $\phi := \overline{\partial}_b^*N_b\alpha$ and ϕ is unique by the condition $\phi \perp \operatorname{Ker}(\overline{\partial}_b)$. The smoothness of ϕ follows from point 5. of theorem 7.3.7.
The estimate can be proved with a partial of unity and pseudodifferential operator argument. □

Bibliography

[1] Chen S.-C. and Shaw M.-C., *Partial Differential Equations in Several Complex Variables*, American Math. Society / International Press Vol. 19, Providence, RI, 2001

[2] Chevalley C., *Theory of Lie Groups*, Princeton Univ. Press, Princeton, N.J., 1946

[3] Coifman R. and Weiss G., *Analyse Harmonique Non-Commutative sur Certains Espaces Homogenes*, Lecture Notes in Math., 242, Springer-Verlag, Berlin-New York, 1971

[4] Cygan J., *Subadditivity of Homogeneous Norms on Certain Nilpotent Lie Groups*, American Math. Society 83 (1981), 69–70

[5] Evans L. C., *Partial Differential Equations*, American Math. Society, Providence, RI, 1998

[6] Folland G. B. and Stein E. M., *Estimates for the $\overline{\partial}_b$ Complex and Analysis on the Heisenberg Group*, Comm. Pure and Applied Math., 27 (1974), 429-522

[7] Garrett P., *Hartogs' Theorem: separate analyticity implies joint*, http://www.math.umn.edu/∼ garrett/m/complex/hartogs.pdf , 2005

[8] Knapp A. W. and Stein E. M., *Intertwining Operators for Semisimple Groups*, Ann. of Math., 93 (1971), 489-578

[9] Koranyi A. and Vagi S., *Singular Integrals on Homogeneous Spaces and Some Problems of Classical Analysis*, Ann. Scuola Norm. Sup. Pisa, 25 (1971), 575-648

[10] Nirenberg L., *On a question of Hans Lewy*, Russian Math. Surveys, 29 (1974), 251-262

[11] Peloso M. M., *An Introduction to Fourier Analysis of Linear Partial Differential Operators*, Course Notes, preprints

[12] Peloso M. M., *Analysis and Differential Operators on CR Manifolds*, Course Notes, preprints

[13] Peloso M. M., *Analysis on the Heisenberg Group*, Course Notes, preprints

[14] Phong D. H. and Stein E. M., *Hilbert integrals, Singular integrals, and Radon transforms II*, Invent. Math, 86 (1986), 75-113

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