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MEAN CURVATURE FLOW IN SE(2) AND APPLICATIONS TO VISUAL PERCEPTION

Tesi di Laurea in Analisi Matematica Superiore

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À Marisa, Maria, Antonella,
Roberta, Martina, Marta, Beatrice et Giulia,
les femmes de ma vie.
C'est à vous que je dédie mon travail,
parce que c'est pour vous que tout ça a été réalisable.

Introduction

Our goal in this thesis is to provide a result of existence of the degenerate non-linear, non-divergence PDE which describes the mean curvature flow in the Lie group $SE(2) = \mathbb{R}^2 \times S^1$ equipped with a sub-Riemannian metric.
The research is motivated by problems of visual completion and models of the visual cortex. Indeed the first layer of the mammalian visual cortex has been modelled as the fiber bundle of $SE(2)$ by Petitot and Tondut in [21], Citti and Sarti in [4] [5]. The strongly anysotropic structure of the cortex is described through a subriemannian metric, which is totally degenerate at every point. In this setting perceptual phenomena such as the formation of subjective surfaces are described as sub-Riemannian mean curvature flows and minimal surfaces.
When we look to the image in Fig(1) we have the clear perception of a zebra below the grating. This means that our visual system is able to integrate the visual signals and completes the missing part of the image, partially occluded by other objects.
According to the models proposed in [4] [5] the reconstruction process can be modelled as a mean curvature motion which leads to the propagation of the visual signal and fills the gap in the image.

Figure 1: An example of visual completion

A sub-Riemannian metric in $SE(2)$ is induced by the choice of two vector fields X_1 and X_2 at every point, which, together with their commutator, span the tangent space at every point. A riemannian metric g is defined on the plane spanned by $\{X_1, X_2\}$, which is totally degenerate, since it is not defined outside this plane. In this space a notion of control distance has been defined. All the differential properties of the space have been defined in terms of the vector fields X_1 and X_2. For example a function is of class C^1 with respects to the metric g if its gradient $\nabla_g f = (X_1 f, X_2 f)$ is continuous. Analogously we can define second order operators as the sub-Riemannian Laplace operator. The notion of surface in this setting has been introduced around 2002 by Franchi, Serapioni and Serra Cassano as 0-level set of a C^1-function f in [11].

The notion of mean curvature is known only at points where the gradient ∇f does not vanish. Just like in the euclidean setting it is possible to define a surface flowing by curvature as a surface whose points (x, t) move along the normal direction to the surface (with respect to the sub-Riemannian metric g), with a speed which is proportional to the intrinsic mean curvature. This problem has been studied in the Euclidean setting in the celebrated Evans

and Spruck's paper [8], who made a formalization of the model implemented by Osher and Sethian. In the sub-Riemannian setting we quote the paper of Capogna and Citti [3], Dirr, Dragoni and M. von Renesse [6] and Ferrari, Liu and Manfredi [9] in which a probabilistic approach is used.
However the problem of existence of a solution of the mean curvature flow was still open in the $SE(2)$ space. Hence we focus on this fact and provide an existence result for viscosity solution of the mean curvature flow in this setting, using a Riemannian approximation of the sub-Riemannian problem.
The thesis is organized as follows:

- in chapter 1 we will introduce the main perceptual phenomena studied by Gestalt psychology and in particular the problem of completion. Then we describe the structure of the primary visual cortex (V1), which is the region of the brain responsible for these visual tasks. We will describe in particular the functional architecture of this layer of the cortex and the aspect of the problem, which allows us to model the cortex as a Lie group.

- In chapter 2 we introduce the notion of sub-Riemannian metric on a Lie group: the definition of a sub-Riemannian manifold, the horizontal tangent space, all the geometric properties and the distance of the space. In this way we will build a geometric environment which will models the functional architecture of V1. We conclude the chapter with a detailed description of cortical properties which allows us to describe the cortex as a sub-Riemannian manifold.

- In chapter 3 we will describe the differential geometry of a Riemannian surface. We introduce the notion of affine connection, curvature, and mean curvature. We also show that the analogous sub-Riemannian objects can be recovered as limit of the correspondent Riemannian ones.

We also provide an overview of the mean curvature motion with the level sets method and the associated PDE. In particular minimal surfaces arise as 0-level set of the viscosity solution u.

- Chapter 4 contains the main result of the thesis, and provide the proof of the existence of a mean curvature flow. Since the PDE describing the problem is non-linear and degenerate, we will look for viscosity solutions. We first introduce a Riemannian non-degenerate approximation of the solution for which the existence of a smooth solution $u^{\delta,\epsilon,\sigma}$ is known. However the solution depends on the approximating parameters ϵ, δ and σ. We have now to establish estimates uniform in these parameters. The non-commutativity of the vector fields X_i does not allow us to repeat the classical prove of Evans and Spruck. The main idea of the proof is to introduce a new family of vector fields $\{Y_i\}_{i=1,2,3}$ which commute with $\{X_i\}_{i=1,2,3}$ and to obtain a new equation for the gradient of the solution. The maximum principle leads to estimates for the gradient of $u^{\delta,\epsilon,\sigma}$, uniform in all parameters, and we obtain a Lipshitz continous viscosity solution when we pass to the limit.

- Finally in chapter 5 we will study some applications to visual perception: an algorithm of diffusion which will give us the possibility to build minimal surfaces and simulate the behavior of V1 for what concerns completion phenomena.

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Chapter 1

Perceptual completion phenomena and the visual cortex

Our aim in this chapter is to introduce the concepts studied in Gestalt psychology which result in the perceptual completion phenomena and to present the basic structures of the functional architecture of the primary visual cortex (V1). The basic idea is that neural interaction strongly depends on the organization and connectivity of neurons in the cortex. We will restrict our attention to the structures relevant to the model presented in the later chapters, in other words those involved in boundary coding: receptive fields and receptive profiles of simple cells in V1 are fundamental for this process. Then we will give a description of the main structures involved in the perceptual completion of the functional architecture of V1. Finally the connectivity pattern between simple cells will be considered. This will lay the foundation for correctly modelling the structures and connectivity from a mathematical point of view and will enable us to show that these form the basis for the perceptual completion of contours

1.1 Gestalt psychology and perceptual completion phenomena

Visual perception is not a simple acquisition of the real stimulus, but is the result of a series of complex processes which mediate between the physical stimuli and the phenomenological organization of such stimuli. According to Gaetano Kanizsa, one of the main exponents of the Gestalt psychology, *"Perception consists of an active construction by means of which sensory data are selected, analyzed and integrated with properties not directly noticeable but only hypothesized, deduced, or anticipated, according to available information and intellectual capacities."* The basic idea of the Gestalt theory is that there exist laws which allow figural emergence without any mediation by past experience. These characteristics are defined as laws that describe the influence of global context in the perception of local features. Elements tend to be perceptually grouped and made salient in case of *proximity, similarity, closure, good continuation and alignment*. More than one grouping law at a time can contribute to the perception of a complex object. For example phenomena in which there is a phenomenological presence of boundaries without a physical stimulus (such as in the famous Kanizsa-Triangle) describe the mechanisms of modal and amodal completion, which are examples of grouping according to *good continuation and alignment*. We will shortly consider in figure (1.1) a clear example of amodal and modal completion, which was studied in depth by Kanizsa.

A point underlined by these studies is that in both cases of completion the occluding and the occluded objects are perceived at the same time in the scene and therefore there are points in the input stimulus corresponding to more than one figure at the perceptual level. This suggests that the phenomenological space has a higher dimension than that of the physical space, as in this example of a two dimensional image.

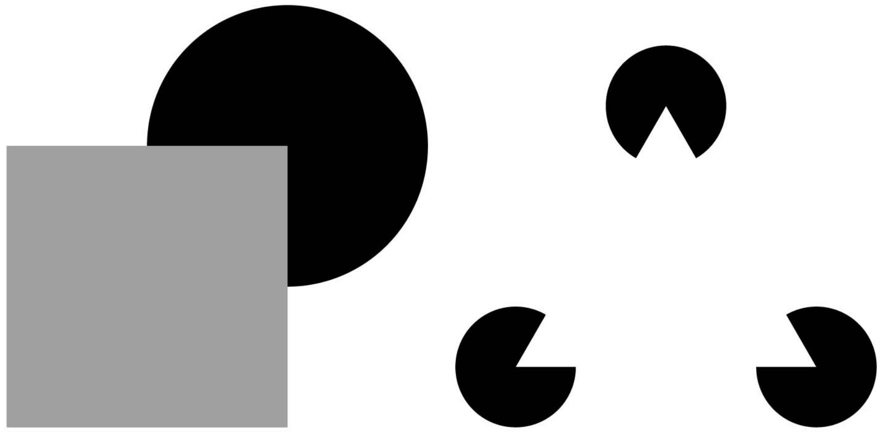


Figure 1.1: (Left) An example of *amodal completion.* The figure is perceived as a black circle occluded by a gray square. The circle is present in the visual field, but the completion is performed without an illusory contour. (Right) The Kanizsa triangle. A white triangle occluding three black disks is phenomenologically perceived. There is an apparent contour separating the triangle from the figure, indeed the interior looks whiter than the background. There is also a stratification of figures, the triangle emerges and seems to be above the disks. This type of phenomenon is classified by Kanizsa as *modal completion.*

1.2 The visual cortex

In order to describe from a mathematical point of view the previous phenomena in which we are interested, we first need to focus on the functional architecture of the primary visual cortex and in its basic structures. We will consider only the structures that are relevant to the model presented in the later chapters, those involved in boundary coding. The main idea behind this model is that neural computations strictly depend on the organization and connectivity of neurons in the cortex.

1.2.1 The cerebral cortex and the visual pathway

The cerebral cortex is the outermost layer of neural tissue in the two cerebral hemispheres. It plays a central role in sensory and cognitive processing since most of the neurons responsible for these processes are located here. It is commonly divided in three parts: sensory, motor, and association. We are interested in the first of these, which is the part of the cortex that receives sensory inputs. In particular the visual cortex is the area that serves the sense of vision and receives the optical information from the visual path (see figure (1.2)). Light enters the eyes and arrives to the retina, which is composed of

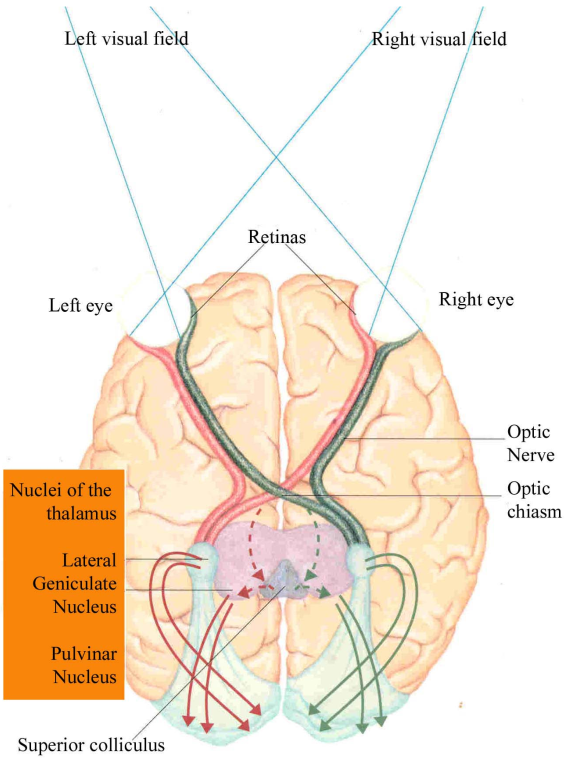


Figure 1.2: The visual path of the brain

ten thin layers of brain tissue where the neural processing of visual stimuli begins. These layers are formed mainly of photoreceptors, and the final layer consists of ganglion cells which have the role of sending the final output of the retina (in the form of action potentials) away from the eyes using their long axons. These axons form the optic nerve, which transmits the visual signals to the lateral geniculate nucleus (LGN) of the thalamus, a structure in the middle of the brain which connects the sensory organs to their main sensory processing cortical areas. From the LGN the signal goes to various destinations: the most important is the visual cortex, situated in the back of the head, where the larger part of the visual processing is performed. The primary visual cortex (V1) is the area to which most of the retinal output first arrives and is the most widely studied visual area.

1.2.2 V1: primary visual cortex

As the axons of the ganglion cells project a detailed spatial representation of the retina to the LGN, the LGN projects a similar representation to the primary visual cortex. More precisely each cell in V1 is characterized by its receptive field, the portion of the retinal plane which responds to visual stimulation: the action of light alters the firing of the neuron. Classically a receptive field is subdivided into *ON* and *OFF* areas. The area is considered *ON* if the cell spikes responding to a positive signal and *OFF* if it spikes responding to a negative signal. Hence it is possible to define the receptive profile of a neuron as a function $\psi(x, y)$ measuring the response of the cell, $\psi : \mathrm{D} \to \mathbb{R}$ where D is the receptive field and (x, y) are retinal coordinates. This function models the neural output of the cell in response to a punctual stimulus in the 2 dimensional point (x, y). The characterization given by Hubel and Wiesel in ([14]),([15]) classifies the cells in V1 according to their responses. Cells which have separate *ON/OFF* zones are called *simple cells*, all the others *complex cells*. Simple cells have directional receptive profiles (they respond to orientation) and they are sensitive to the boundaries of images. To understand the processing of the image operated by these cells, it is necessary to consider the functional structures of the primary visual cortex: the layered, the retinotopic and the hypercolumnar structure.

1.2.3 The functional architecture of V1

We refer to the functional architecture as the spatial organization and the connectivity between neurons inside a cortical area. In V1 we can identify three structures we mentioned before.

- *The layered structure* indicates that the cortex is formed of 6 horizontal layers.

- *The retinotopic structure* has a particular kind of topographic organization implying that there exists a topology preserving mapping between the retina and the cortex. For this reason cells in each structure can be seen as forming a map of the visual field: simple cell receptive fields form a mosaic that covers the retina. In other words: what is near in the visual field is near in the cortex. From the image processing point of view retinotopic mapping introduces a simple deformation of the stimulus image that we will not consider here.

- *The hypercolumnar structure* organizes the cortical cells in columns corresponding to parameters such as orientation, ocular dominance, color, etc. For the simple cells, sensitive to orientation, columnar structure means that to every retinal position is associated a set of cells (hypercolumn) sensitive to all the possible orientations.

At a certain scale and resolution, for each point of the retina (x, y) there exists a whole set of neurons in V1 which respond maximally to every possible local orientation θ. Since ideally the position on the retina takes values in the plane $\mathbb{R}^2$ and the orientation preference in the circle S^1, the visual cortex can be locally modelled as the product space $\mathbb{R}^2 \times S^1$. Each point (x, y, θ) of this 3D space, represents a column of cells in the cortex associated to a retinal position (x, y), all of which are tuned to the orientation given by the angle θ.
Fig(1.3) shows a schematic representation of the visual cortex. The hypercolumns are drawn vertically. The different colors represent different orientations. The coordinates (x, y, θ) of this 3D space isomorphic to $\mathbb{R}^2 \times S^1$ are

the parameters of the receptive fields (RPs): (x, y) is the retinotopic position and θ the angle of tuning.

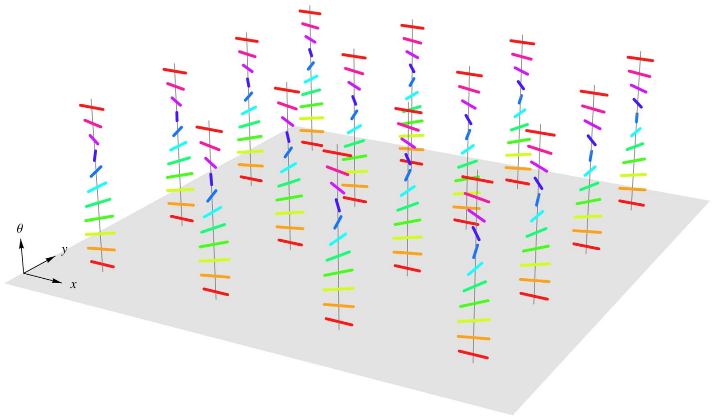


Figure 1.3: The visual cortex modelled as a set of hypercolumns. Over each retinotopic point (x, y) there is a set of cells coded for the set of orientations $\{\theta \in \mathbb{S}^1\}$ and generating the 3D space $\mathbb{R}^2 \times S^1$. Each bar represents a possible orientation.

The fundamental consideration here is that V1 is modelled as a 3D space of positions and orientations, while the cortex is infact a 2D layer. The structure of the cortex allows us to code 3D information in a 2D structure: this dimensional collapse has been illustrated visually by the pinwheel structure, a fascinating configuration observed by William Bosking et al. using optical imaging techniques in which the cells' orientation preference is color-coded and every hypercolumn is represented by a pinwheel (see [2]). Figure (1.4)

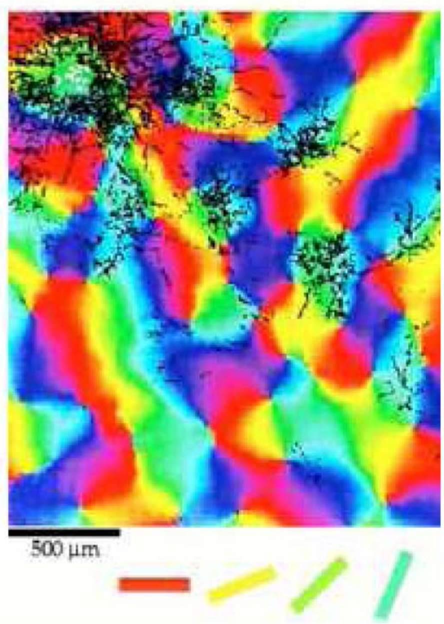


Figure 1.4: A marker is injected in the cortex, at a specific point, and it diffuses mainly in regions with the same orientation as the point of injection (marked with the same color in figure)

To conclude our study of the functional architecture of V1 we need to discuss the connectivity between neurons inside the structure we have seen. In the hypercolumnar structure we can identify two types of communication between neurons which play a central role in the model we want to present:

- *The intracortical circuitry* is able to select the hypercolumns orientation of maximum output in response to a visual stimulus and to suppress all the others. The mechanism able to produce this selection is called non-maximal suppression or orientation selection.

- *The horizontal or cortical connectivity* takes place in the connectivity structure, the part of the visual cortex which ensures connectivity between hypercolumns. The horizontal connections connect cells with the same orientation belonging to different hypercolumns.

Chapter 2

Sub-Riemannian manifolds

In this chapter we will introduce the mathematical instruments that will allow us to model the cortical space introduced in the previous section. We are mainly interested in the structure of the cortex, which we know is responsible for the functionality of the cortex itself: the hypercolumnar structure of the primary visual cortex has been modelled as the principle fiber bundle of the Lie group SE(2) and its differential structure, crucial for explaining the orientation selection of V1, is sub-Riemannian. Instruments of Lie groups and differential geometry for the description of the visual cortex have been introduced by Hoffmann in [13], Zucker in [27], Petitot and Tondut in [21] and Duits and Franken in [7]. Before focusing on their models, we first need to review the definition and basic properties of differentiable manifold theory and Lie group theory, which are fundamental for explaining the simmetry and the organization of simple cells in the cortex, and the construction and the properties of a subriemannian manifold which explain the connectivity we have introduced.

2.1 Differentiable manifold theory

In order to introduce Lie groups and Subriemannian structure we need to first recall fundamental notions of differentiable manifold theory. All definitions and theorems can be found in [24].

2.1.1 Topological manifolds, charts and smooth manifolds

Definition 2.1. A topological space M is *locally Euclidean of dimension n* if every point $p \in M$ has a neighborhood U such that there is a homeomorphism ϕ from U onto an open subset $\mathbb{R}^n$. We call the pair $(U, \phi : U \to \mathbb{R}^n)$ a chart, where U is a coordinate *neighborhood* or a *coordinate map* or a *coordinate system* on U. We say that a chart (U, ϕ) is *centered* at $p \in U$ if $\phi(p) = 0$.

Definition 2.2. *A topological manifold* is a Hausdorff (T2), second countable, locally Euclidean space. It is said to be of *dimension n* if it is locally Euclidean of dimension n.

Suppose $(U, \phi : U \to \mathbb{R}^n)$ and $(V, \psi : V \to \mathbb{R}^n)$ are two charts of a topological manifold. Since $U \cap V$ is open in U and $\phi : U \to \mathbb{R}^n$ is a homeomorphism onto an open subset of $\mathbb{R}^n$, the image $\phi(U \cap V)$ will also be an open subset of $\mathbb{R}^n$. Similarly, $\psi(U \cap V)$ is an open subset of $\mathbb{R}^n$.

Definition 2.3. The two charts $(U, \phi : U \to \mathbb{R}^n)$ and $(V, \psi : V \to \mathbb{R}^n)$ of a topological manifold are C^∞-compatible if the two maps:

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V) \quad \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

are C^∞. These two maps are called the *transition functions* between the charts. If $U \cap V$ is empty, then the two charts are automatically C^∞-compatible. To simplify the notation, we sometimes write $U_{\alpha\beta}$ for $U_\alpha \cap V_\beta$.

Definition 2.4. A C^∞ *atlas* or simply an atlas on a locally Euclidean space M is a collection $\mathfrak{U} = \{(U_\alpha), \phi_\alpha\}$ of pairwise C^∞-compatible charts that cover M, i.e. $M = \bigcup_\alpha U_\alpha$.

An atlas $\mathfrak{U}$ on a locally Euclidean space is said to be *maximal* if it is not contained in a larger atlas; in other words, if $\mathfrak{M}$ is any other atlas containing $\mathfrak{U}$, then $\mathfrak{U} = \mathfrak{M}$.

Definition 2.5. A *smooth or C^∞ manifold* is a topological manifold M together with a maximal atlas. Tha maximal atlas is also called a *differentiable structure on M*. A manifold is said to have dimension n if all of its connected components have dimension n. A 1-dimensional manifold is also called a *curve*, a 2-dimensional manifold a *surface*, and a n-dimensional manifold is an n-manifold.

2.1.2 Tangent spaces, differential of a map, vector fields and integral curves

A basic principle in manifold theory is the linearization principle, according to which a manifold can be approximated near a point by its tangent space at that point. From mathematical literature we know that for any point p in an open set U in $\mathbb{R}^n$ there are two equivalent ways to define a tangent vector at p:

- as a column vector.[1]

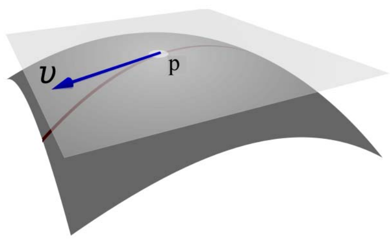


Figure 2.1: A tangent vector as an arrow

- as a point-derivation of C_p^∞, the algebra of germs[2] of C^∞ functions at p.

[1]Intuitively the tangent plane to a surface at p in $\mathbb{R}^n$ is the plane that just "touches" the surface at p. A vector at p is tangent to a surface if it lies in the tangent plane at p.

[2]We define a *germ* of a C^∞ function at p in $\mathbb{R}^n$ to be an equivalence class of smooth functions defined in a neighborhood at p in $\mathbb{R}^n$, the two functions being equivalent if they agree on some, possibly smaller, neighborhood of p. The set of germs of smooth real-valued

Both definitions generalize to a manifold. In the first approach, one defines a tangent vector at p in a manifold M by first choosing a chart (U, ϕ) at p and then denoting a tangent vector at p to be an arrow at $\phi(p)$ in $\phi(U)$. This approach, while more visual, is complicated to work with, since a different chart (V, ψ) at p would give rise to a different set of tangent vectors at p and one would have to decide how to identify the arrows at $\phi(p)$ in U with the arrows at $\psi(p)$ in $\psi(V)$. The cleanest and most intrinsic definition of a tangent vector at p in M is as a point-derivation, and this is the approach we adopt.

Definition 2.6. Generalizing a derivation at a point p in $\mathbb{R}^n$, we define a *derivation at a point* in a manifold M, or a point-derivation of C_p^∞ to be a linear map $D : C_p^\infty(M) \to \mathbb{R}$ such that

$$D(fg) = (Df)g(p) + f(p)Dg.^{3}$

functions at p in $\mathbb{R}^n$ is denoted by $C_p^\infty(\mathbb{R}^n)$, an unitary commutative ring. This concept generalizes to a manifold M using the local coordinates given by the atlas, for each point p in M

[3]The definition of tangent vector that we have seen in this chapter descends directly from the characterization of a tangent vector in $\mathbb{R}^n$. In calculus we visualize the tangent space $T_p(\mathbb{R}^n)$ at p in $\mathbb{R}^n$ as the vector space of all arrows emanating from p.

If f is C^∞ in a neighborhood of p in $\mathbb{R}^n$ and v is a tangent vector p, *the directional derivative* of f in the direction v at p is defined to be

$$D_v f = \lim_{t \to 0} \frac{f(c(t)) - f(p)}{t} = \left.\frac{d}{dt}\right|_{t=0} f(c(t))$$

where $c(t) = (p_1 + tv_1, \dots, p_n + tv_n)$ is the parametrization of the line through a point $p = (p_1, \dots, p_n)$ with direction $v = \langle v_1, \dots, v_n \rangle$ in $\mathbb{R}^n$. By the chain rule,

$$D_v f = \sum_{i=1}^{n} \frac{dc_i}{dt}(0) \frac{\partial f}{\partial x_i}(p) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p).$$

In the notation $D_v f$ it is to be understood that the partial derivatives are to be evaluated at p, since v is a vector at p. So $D_v f$ is a number, not a function. We write

$$D_v = \sum v_i \left.\frac{\partial}{\partial x_i}\right|_p$$

for the map that sends a function f to the number $D_v f$. To simplify the notation we often

Definition 2.7. A *tangent vector* at a point p in a manifold M is a derivation at p.

Definition 2.8. The tangent vectors at p form a vector space $T_p(M)$, called the tangent space of M at p. We also write T_pM.

Definition 2.9. A *vector field* on an open subset U of M is a function that assigns to each point p in U a tangent vector $X_p \in T_p(M)$. Since we can assign a basis $\{\partial/\partial x_i|_p\}$ to $T_p(M)$[4], where the elements of the basis are the n directional derivates which come from the local coordinates of U in $\mathbb{R}^n$, the vector X_p is a linear combination:

$$X_p = \sum a_i(p) \frac{\partial}{\partial x_i}\bigg|_p \quad p \in U, \quad a^i(p) \in \mathbb{R}$$

where a_i are smooth functions on U. The set of vector fields on a manifold M is denoted by $\mathfrak{X}(M)$.

omit the subscript p. The association $v \mapsto D_v$ of the directional derivative D_v to a tangent vector v offers a way to characterize tangent vectors as certain operators on functions.

For each tangent vector v at a point p in $\mathbb{R}^n$, the directional derivative at p gives a map of real vector spaces:

$$D_v : C_p^\infty(\mathbb{R}^n) \to \mathbb{R},$$

D_v is $\mathbb{R}$-linear and satisfies the Leibniz rule:

$$D_v(fg) = (D_v f)g(p) + f(p)D_v g$$

In general, any linear map $D : C_p^\infty(\mathbb{R}^n) \to \mathbb{R}$ satisfying the Leibniz rule is called a *derivation at p*. We can denote the set of all derivations at p by $\mathcal{D}_p(\mathbb{R}^n)$. This is a vector space, since the sum of the two derivations at p and a scalar multiple of a derivation at p are again derivations at p. Thus far we know that directional derivatives at p are all derivations at p so there is a map:

$$\phi : T_p(M) \to \mathcal{D}_p(M), v \mapsto D_v = \sum v_i \frac{\partial}{\partial x_i}\bigg|_p$$

This map is linear and is an isomorphism of vector spaces: it represents the reason why we can identify tangent vectors with derivations.

[4]This result is prooved in a theorem which states that $\{\frac{\partial}{\partial x_i}\big|_p\}_{i=1,\dots,n}$ form a basis for $T_p(M)$.

Observation 1. An equivalent definition is that a *vector field* X is a derivation on $C^\infty(M)$, i.e. $D : C^\infty(M) \to C^\infty(M)$ $\mathbb{R}$-linear which satisfies the Leibniz rule. This equivalence can be proved.

We will now define the concept of a smooth map between two manifolds in order to introduce the *differential* of a map:

Definition 2.10. Let N and M be manifolds of dimension n and m respectively. A map $F : N \to M$ is C^∞ at a point p in N if there are charts (V, ψ) about $F(p)$ in M and (U, ϕ) about p in N such that the composition $\psi \circ F \circ \phi^{-1}$, a map from the open subset $\phi(F^{-1}(V) \cap U)$ of $\mathbb{R}^n$ to $\mathbb{R}^m$, is C^∞ at $\phi(p)$. If F is C^∞ at every point of N, F is said to be smooth (C^∞).

Note that D denotes a derivation $\mathcal{D}$, the differential of a smooth map.

Definition 2.11. Let $F : N \to M$ be a C^∞ map between two manifolds. At each point $p \in N$, the map F induces a linear map of tangent spaces called its *differential at* p:

$$\mathcal{D} : T_p(N) \to T_{F(p)}M$$

If $X_p \in T_pN$,then $\mathcal{D}(X_p)$is the tangent vector in $T_{F(p)}M$ defined by:

$$(\mathcal{D}(X_p))f = X_p(f \circ F) \in \mathbb{R} \quad \text{for} \quad f \in C^\infty_{F(p)}(M).$$

Here f is a germ at $F(p)$, represented by a C^∞ function in a neighborhood of $F(p)$. Since the previous definition is independent of the representative of the germ, in practice we can be relaxed about the distinction between a germ and a representative function for the germ.

Observation 2. If $f : M \to \mathbb{R}$ is a C^∞-function, the differential of f is globally defined as:

$$df =: \mathfrak{X}(M) \to C^\infty(M)$$

such that for each vector field $X \in \mathfrak{X}(M)$:

$$df(X) := X(f)$$

It is clear that this definition descends directly from the general one.

Observation 3. If instead of N and M we consider a map F between $\mathbb{R}^n$ and $\mathbb{R}^m$ we discover with some computations that the matrix associated to the linear map: $\mathcal{D} : T_p(\mathbb{R}^n) \to T_p(\mathbb{R}^m)$ is precisely the Jacobian matrix of the derivative of F at p. Thus, the differential of a map between manifolds generalizes the derivative of a map between Euclidean spaces.

Definition 2.12. A *smooth curve* in a manifold M is by definition a smooth map $\gamma :]a, b[\to M$ from some open interval $]a, b[$ into M. Usually we assume $0 \in]a, b[$ and say that γ is a curve *starting at* p if $\gamma(0) = p$. The tangent vector(or velocity vector) $\gamma'(x)$ to the curve γ in $x \in (a, b)$ is defined to be:

$$\gamma'(x) = \mathcal{D}\left(\frac{d}{dt}\right) \in T_{\gamma(x)}M$$

Definition 2.13. We call γ *an integral curve* of the vector field X on M if $\gamma'(x) = X_{\gamma(x)}$, $\forall x \in (a, b)$, i.e. a smooth parametrized[5] curve γ whose tangent vector at any point coincides with the value of X at the same point. In local coordinates this means:

$$\begin{aligned} \gamma : (a, b) &\to \phi_u(U) \in \mathbb{R}^n \\ t &\mapsto (\gamma_1(t), \ldots, \gamma_n(t)) \end{aligned}$$

Observation 4. If we make some calculations we observe:

$$\left(D_\gamma \frac{\partial}{\partial t}\right)(f(x_1, \ldots, x_n)) = \frac{\partial}{\partial t} f(\gamma_1(t), \ldots, \gamma_n(t)) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \gamma_i'(t)$$

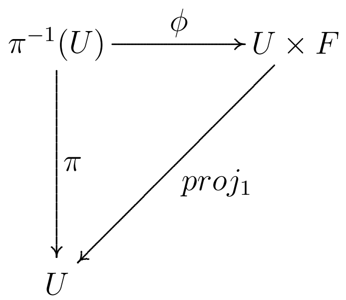
Hence $\gamma'(t) = \sum_{i=0}^n \gamma_i'(t) \frac{\partial}{\partial x_i}$. With respect to the basis $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$ we have $\gamma'(t) = (\gamma_1'(t), \ldots, \gamma_n'(t))$. Following the previous definition $\gamma'(x) = X_{\gamma(x)}$ this means $\sum \gamma_i'(t) \frac{\partial}{\partial x_i} = \sum a_i(\gamma_1(t), \ldots, \gamma_n(t)) \frac{\partial}{\partial x_i}$. Since $\{\frac{\partial}{\partial x_i}\}$ forms a basis, γ is an integral curve iff $\gamma_i'(t) = a_i(\gamma_1(t), \ldots, \gamma_n(t))$ for all i, i.e. $\gamma_1, \ldots, \gamma_n$ is the solution of the previous system of autonomous ODEs of the first order.

[5] A parametrization is the process of deciding and defining the parameters necessary for a complete or relevant specification (characterization) of a geometric object.

2.1.3 Fiber bundles and tangent spaces as vector bundles

The collection of tangent spaces in a manifold can be given the structure of a *vector bundle*; it is then called the *tangent bundle* of the manifold. Intuitively, a vector bundle over a manifold is a locally trivial family of vector spaces parametrized by points of the manifold. Vector fields may be viewed as sections of the tangent bundle over a manifold. *A fiber bundle* is intuitively a demonstration that a space *locally* looks like a certain product space, but *globally* may have a different topological structure.

Definition 2.14. A *fiber bundle* is a structure (E, B, π, F) where E, B and F are topological spaces and $\pi : E \to B$ is a continuous surjection satisfying the local triviality condition outlined below. The space B is called the *base space* of the bundle, E the *total space*, and F *the fiber*. The map π is called the *projection map*. We require that $\forall\ x \in E$ there is an open neighborhood $U \subset B$ of $\pi(x)$ (which will be called trivializing neighborhood) such that $\pi^{-1}(U)$ is homeomorphic to the product space $U \times F$, in such a way that π agrees with the projection onto the first factor. Thus the following diagram should commute:



where $proj_1 : U \times F \to U$ is the natural projection and $\phi : \pi^{-1}(U) \to U \times F$ is a homeomorphism. The set of all $\{(U_i, \phi_i)\}$ is called a *local trivialization* of the bundle. Thus for any p in B, the preimage $\pi^{-1}(\{p\})$ is homeomorphic to $F \times \{p\}$ (since $proj_1^{-1}(\{p\})$ clearly is p) and is called the *fiber over* p. Every fiber bundle $\pi : E \to B$ is an *open map*, since projections of products are open maps. Therefore B carries the *quotient topology* determined by the map π.

Example 2.1. The Moebius strip is the simplest example of a non-trivial bundle E. The base B is the circle S^1 and the fiber F is a line segment. Given $x \in B$, U is a small arc (neighborhood of x on the circle) and $\pi^{-1}(U)$ is homeomorphic to the square $U \times F$. Globally this is not true.

A special class of fiber bundles, called *vector bundles*, are those whose fibers are *vector spaces*.

Definition 2.15. Let M be a smooth manifold. Recall that at each point $p \in M$, the tangent space T_pM is the vector space of all point-derivations of $C_p^\infty(M)$, the algebra of germs of C^∞functions at p. The *tangent bundle* of M is the disjoint union of all tangent spaces of M:

$$TM = \bigsqcup_{p \in M} T_pM$$

In this definition the union is disjoint because for distinct points p and q in M, the tangent spaces T_pM and T_qM are already disjoint. TM has the structure of a differentiable manifold and the bundle structure is given by the natural map $\pi : TM \to M$ where $\pi^{-1}(p) \quad, \forall p \in M$, is the tangent space of the manifold M at the point p (or equivalently $\pi(v) = p$ if $v \in T_pM$), and this map does not depend on the choice of atlas or local coordinates for M. As a matter of notation, sometimes a tangent vector $v \in T_pM$ can be identified by the pair (p, v), to make explicit the point $p \in M$ at which v is a tangent vector.

Observation 5. We can observe that any fiber bundle is identified by the couple $(V, \pi : V \to M)$ where V is a differentiable manifold and π induces a diffeomorphism (and not only a homeomorphism) between $\pi^{-1}(U)$ and $U \times \mathbb{R}^n$, where $U \subset M$.

Another special class of fiber bundles, called *principal bundles*, are those bundles on whose fibers there is a free and transitive action[6] by a group G

[6] If G is a group and X is a set, then a *(right) group action* of G on X is a function

$$X \times G \to X$$
$$(x, g) \mapsto x \cdot xg$$

is given. The bundle is often specified along with the group by referring to it as a principal G-bundle. The group G is also the structure group of the bundle. As we will see we are interested in this definition because *principal fiber bundles* are used in our model to describe the visual cortex.

Definition 2.16. A *principle fiber G-bundle*, where G denotes any topological group[7], is a *fiber bundle* $\pi : P \to X$ together with a continuous right action $P \times G \to P$ such that G preserves the fibers of P and acts freely and transitively on them. This implies that the fiber of the bundle is homeomorphic to the group G itself. Usually one requires the base space X to be Hausdorff and possibly paracompact[8].

An equivalent definition of a principal G-bundle is as a G-bundle $\pi : P \to X$ with a fiber G where the structure group acts on the fiber by left multiplication[9]. Since right multiplication by G on the fiber commutes with

that satisfies the following two axioms:

- *Compatibility* $x \cdot (gh) = (x \cdot g) \cdot h, \quad \text{for all} \quad g, h \in G, \, x \in X$
- *Identity* $x \cdot e = x \quad \text{for all} \; x \in X$

An action is *free* if, given $g, h \in G$, the existence of an $x \in X$ with $x \cdot g = x \cdot h$ implies $g = h$. Equivalently: if g is a group element and there exists an $x \in X$ with $x \cdot g = x$ (that is, if g has at least one fixed point), then g is the identity. An action is *transitive* if X is non-empty and if for any $x, y, \in X$ there exists a g in G such that $x \cdot g = y$.

[7] A topological group is a group G together with a topology on G such that the group's binary operation and the group's inverse function are continuous functions with respect to the topology. It is a mathematical object with both an algebraic structure and a topological structure

[8] A paracompact space is a topological space in which every open cover has an open refinement that is *locally finite*

[9] As we have seen before and we will see also for Lie groups in the next section, a group G acts by *left* multiplication on an X set if there is a function

$$G \times X \to X$$
$$(g, x) \mapsto g \cdot xx$$

that satisfies the following two axioms:

- *Compatibility* $(gh) \cdot x = g \cdot (h \cdot x), \quad \text{for all} \quad g, h \in G, \, x \in X$

the action of the structure group, there exists an invariant notion of right multiplication by G on P. The fibers of π then become right G-torsors[10] for this action.

Definition 2.17. Let $\pi : E \to M$ be a vector bundle on M. We call a section of the vector bundle a map $\phi : M \to E$ such that $\pi \circ \phi = Id_M$

Definition 2.18. A vector field X on a manifold M is a function that assigns a tangent vector $X_p \in T_pM$ to each point $p \in M$. In terms of the tangent bundle, a vector field on M is simply a section of the tangent bundle $\pi : TM \to M$ and the vector field is smooth if it is smooth as a map from M to TM.

2.2 Lie groups and their properties

In this section we will provide some basic definitions of the Lie group theory. Definitions and theorems can be found in [25].

2.2.1 Definition

Definition 2.19. A *Lie Group* is a group which also carries the structure of a differentiable manifold in such a way that both the group operation

$$\cdot : G \times G \to G, \quad (g, h) \longmapsto g \cdot h \,\text{for}\, g, h \in G$$

and the inversion

$$i : G \to G\,, i(g) = g^{-1}, g \in G$$

are smooth maps.

Examples of Lie Groups are:

- The Euclidean space $\mathbb{R}^n$, with the usual sum as group law.

- *Identity* $x \cdot e = x$ for all $x \in X$

[10] Let G be a group. A G-torsor is a set on which G acts freely and transitively.

- The set of square matrices $n \times n$, with the determinant different from 0. In this set we consider the standard product of matrices, and the existence of an inverse is ensured by the condition on the determinant. Note that this group is not commutative.
- The circle $S^1 \subset \mathbb{C}$ of angles mod 2π, with the standard sum of angles.
- The group of rotations and translations on the plane $SE(2)$ which will be described in detail in the following pages.

2.2.2 Properties

Definition 2.20. For two vector fields (or two derivations) X and Y in $\mathfrak{X}(M)$, their *Lie bracket* (or *commutator*) is defined by their action on functions $f : M \to \mathbb{R}$:

$$[X, Y](f) = X(Y(f)) - Y(X(f))$$

Note that the Lie bracket is a measurement of the non-commutativity of the operators; it is defined as the difference of appying them in reverse order. In particular $[X, Y]$ is identically 0 if X and Y commute.

Definition 2.21. Let G be a Lie group. For any element $g \in G$, we define the *left-multiplication* (or *left-translation*) $L_g : G \to G$ by:

$$L_g(h) = g \cdot h \quad \text{for all} \quad g \in G$$

where $\cdot$ denotes the group operation in G.

Definition 2.22. A vector field X on G is called *left-invariant* if:

$$X(f \circ L_g) = (Xf) \circ L_g \quad \text{for all} \quad g \in G$$

Definition 2.23. The *Lie Algebra* of a Lie group G is the vector space of all left-invariant vector fields on G:

$$Lie(G) := \{X \in \mathfrak{X}(M) : X \text{ is left invariant, i.e. } X(f \circ L_g) = (Xf) \circ L_g\}$$

for all $g \in G$ and f smooth on M.

Observation 6. A theorem states that the Lie algebra associated to a Lie group encodes its differential structure, and it is identified as the tangent space at the identity of the group e, i.e.

$$Lie(G) \cong T_e G$$

2.3 Sub-Riemannian manifold

So far we have dealt with differentiable objects. Now we will introduce objects which depend on a *metric (or inner product)*, an instrument which allows to measure the length of any vector of the tangent space:

Definition 2.24. Let V be a vector space. An *inner product (or metric) on* V is a bilinear form, symmetric and positively defined, i.e.
$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ such that:

(i) $\langle u_1 + u_2, v \rangle = \langle u_1, v \rangle + \langle u_2, v \rangle \; \forall \, u_1, u_2, v \in V$;

(ii) $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle \; \forall \, u, v \in V, \forall \, \lambda \in \mathbb{R}$;

(iii) $\langle u, v \rangle = \langle v, u \rangle \; \forall \, u, v \in V$.

(iv) $\langle u, u \rangle \geq \; 0 \; \forall \, u \in V, con \; \langle u, u \rangle = 0 \Leftrightarrow u = 0$.

We will now establish a notation to introduce the concept of the sub-Riemannian metric, a tool which allows us to describe the connections between the hypercolumns in our model. Let us start from the definition of *distribution*, which is still an object which does not depend on the metric:

Definition 2.25. Let M be a $\mathcal{C}^\infty$ manifold of dimension m, and let $n \leqslant m$. Suppose that for each $x \in M$, we assign an n-dimensional subspace $\Delta_x \subset T_x(M)$ of the tangent space in such a way that for a neighborhood $N_x \subset M$ of x there exist n linearly independent smooth vector fields $X_1, \ldots, X_n$ such that for any point $y \in N_x$ we have $X_1(y), \ldots, X_n(y)$ span Δ_y. We let Δ refer to the collection of all the Δ_x for all $x \in M$ and we the call Δ a *distribution* of dimension n on M. The set of smooth vector fields $\{X_1, \ldots, X_n\}$ is called a *local basis* of Δ

Definition 2.26. A *sub-Riemannian manifold* is a smooth manifold M, a smooth constant rank distribution $HM \subset TM$ and a smooth inner product $\langle \cdot, \cdot \rangle$ on HM. The bundle HM is known as the horizontal bundle.

We remark here that we are not assuming any conditions about the horizontal bundle other than the constant rank.

Definition 2.27. A sub-Riemannian manifold with a *complement*, henceforth a *sRC manifold*, is a sub-Riemannian manifold together with a smooth bundle VM such that $HM \oplus VM = TM$. The bundle VM is known as the vertical bundle. The two sRC-manifolds M, N, are sRC-isometric if there exists a diffeomorphism $\pi : M \to N$ such that $\pi_* HM = HN$, $\pi_* VM = VN$ and $\langle \pi_* X, \pi_* Y \rangle_N = \langle X, Y \rangle_M$ for all horizontal vectors X, Y.

Observation 7. We can now recall the definition of a Riemannian manifold, which is a smooth n-dimensional manifold with a Riemannian metric g, where g is defined as a function which associates to each $p \in M$ an inner product g_p, defined on the tangent space T_pM, which smoothly depends on p (i.e. for each couple of vector fields X, Y, the map $p \to g_p(X_p, Y_p)$ is differentiable). The definition of a sub-Riemannian manifold is more general and a Riemannian manifold can be seen as a sub-Riemannian manifold in which the smooth rank distribution has the same dimension as the manifold, i.e. $HM = TM$ (this implies that the vertical bundle is null). Equivalently a sub-Riemannian manifold can be seen as a Riemannian manifold in which some generators of the tangent bundle collapse, i.e. a sub-Riemannian metric can be seen as the *limit* of a Riemannian metric.

Observation 8. If we consider a Riemannian manifold (M, g_p) and $f \in C^1(M)$ a function, for each $p \in M$ we define the *gradient of f in p* as the vector field ∇f satisfying:

$$d_p f(v) = g_p(\nabla f, v) \quad \forall\ v \in T_pM$$

The Riemannian gradient has the same useful properties as the gradient of the Euclidean calculus[11], such as it *vanishes* in the extremal point for f. We can also write the formula for the gradient in local coordinates:

$$\nabla f(x) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} g^{ij}(x) \frac{\partial f}{\partial x_i} \right) \frac{\partial}{\partial x_i} \tag{2.1}$$

where g^{ij} are the local expressions of the inverse of the matrix of the metric.

Definition 2.28. A sub-Riemannian manifold with a complement $(M, HM, VM, \langle \cdot, \cdot \rangle)$ is *r-graded* if there are r smooth constant rank bundles $V^{(j)}$, with $0 < j \leq r$, such that:

$$VM = V^{(1)} \oplus \ldots \oplus V^{(r)}$$

and we have:

$$HM \oplus V^{(j)} \oplus [HM, V^{(j)}] \subseteq HM \oplus V^{(j)} \oplus V^{(j+1)}$$

for each $0 \leq j \leq r$. Here we have adopted the convention that $V^{(0)} = HM$ and $V^{(k)} = 0$ for $k > r$.

Definition 2.29. The *grading* is *j-regular* if

$$HM \oplus V^{(j)} \oplus [HM, V^{(j)}] = HM \oplus V^{(j)} \oplus V^{(j+1)}$$

and *equiregular* if it is j-regular for all $0 \leq j \leq r$.

Let us now define a metric extension:

Definition 2.30. A metric extension for an *r-graded vertical complement* is a Riemannian metric g of $\langle \cdot, \cdot \rangle$ that makes the split

$$TM = HM \bigoplus_{1 \leq j \leq r} V^{(j)}$$

orthogonal.

[11]The Euclidean gradient is defined as the vector of the partial derivatives of a function f with respect to the set of coordinates.

For convenience of notation, we shall denote a section $V^{(k)}$ by $X^{(k)}$ and a set:

$$\hat{V}^{(j)} = \bigoplus_{k \neq j} V^{(k)}$$

Definition 2.31. From the previous observations we can define the horizontal gradient as $\nabla_0 = (X_1, \ldots, X_m)$ where $\{X_1, \ldots, X_m\}$ span the horizontal bundle. In the same way if a metric extension (which is a Riemannian metric) has been chosen we can denote the gradient as $\nabla = (X_1, \ldots, X_m, X_{m+1}, \ldots, X_n)$ where $\{X_{m+1}, \ldots, X_n\}$ span the vertical bundle.

Observation 9. If a metric extension has been chosen then $\hat{V}^{(j)} = (V^{(j)})^{\perp}$ is the orthogonal complement of $V^{(j)}$. For convenience, we shall often also extend the notation $\langle \cdot, \cdot \rangle$ to the whole tangent space using it interchangeably with g.

Observation 10. Every sRC-manifold that admits an r-grading also admits k-gradings for all $1 \leq k < r$ by setting:

$$\tilde{V}^{(j)} = V^{(j)} \quad 0 \leq j < k, \quad \tilde{V}^{(k)} = \bigoplus_{j \geq k} V^{(j)}$$

Definition 2.32. The unique 1-grading on each sRC-manifold, $V^{(1)} = VM$ is known as the *basic grading.*

Example 2.2. A Carnot group (of step r) is a Lie group, whose Lie algebra $\mathfrak{g}$ is stratified in the sense that:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_{r-1}$$

and

$$[\mathfrak{g}_0, \mathfrak{g}_j] = \mathfrak{g}_{j+1} \quad j = 1 \ldots r, \quad \mathfrak{g}_r = 0$$

together with a left-invariant metric $\langle \cdot, \cdot \rangle$ on HM. The vertical bundle VM consists of the left translates of $\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_{r-1}$.

2.4 V1 as the principle fiber bundle of SE(2) with a sub-Riemannian structure

If we recall the description of the functional architecture of the primary visual cortex that we saw in the previous chapter and at the beginning of this one, we underlined its symmetries and the organization of its cells. The *Rototranslation group* is the fundamental mathematical structure used in this thesis to model V1 and its physiological properties. In literature it is also known as the *2D Euclidean motion group SE(2)*. It is the 3D group of rigid motions in the plane or equivalently the group of elements invariant to rotations and translations. The aim of this section is to show that the visual cortex at a certain level is naturally modelled as the Rototranslation group, which is a Lie group whose tangent bundle naturally assumes the structure of the principal fiber bundle, with a sub-Riemannian metric.

2.4.1 The group law

In the previous chapter we saw that the visual cortex can be locally modelled as the product space $\mathbb{R}^2 \times S^1$, where $(x, y) \in \mathbb{R}^2$ represents the position on the retina and the orientation preference takes values in S^1. A way of visualizing this space is illustrated in Fig(2.2): the half-white/half-black circles represent the oriented receptive profiles of odd simple cells, where the angle of the axis is the angle of tuning. Every possible receptive profile is obtained from the origin by translating it through the vector (x_1, y_1) and rotating it over itself by an angle θ. We denote T_{x_1,y_1} as the translation of the vector (x_1, y_1) and R_θ a the rotation matrix of angle θ:

$$R_\theta = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$

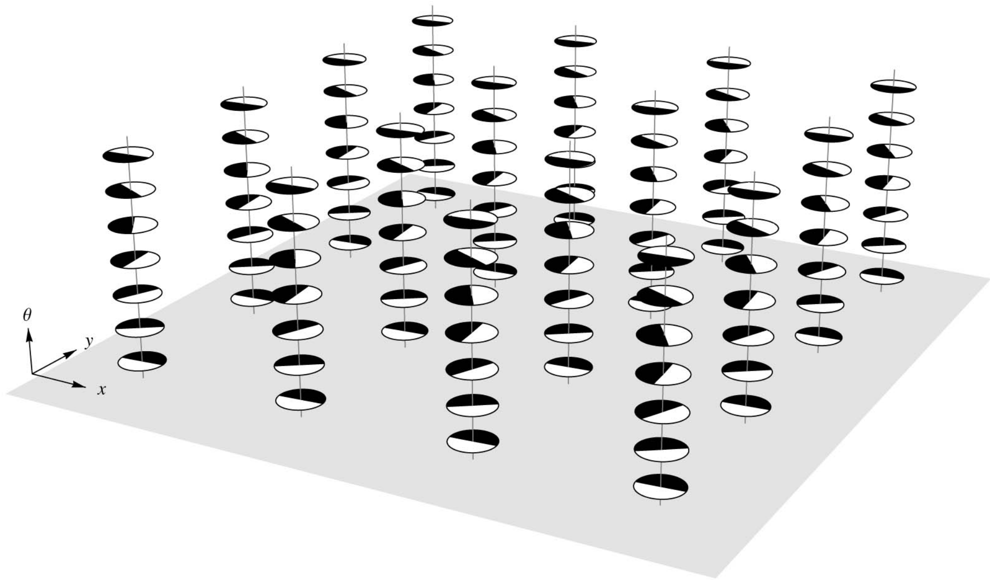


Figure 2.2: The visual cortex modelled as the group invariant under translations and rotations

A general element of the $SE(2)$ group is of the form $A_{x_1,y_1,\theta} = T_{x_1,y_1} \circ R_\theta$ and applied to a point (x, y) it yields:

$$A_{x_1,y_1,\theta_1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + R_{\theta_1} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

All the profiles can be interpreted as: $\phi(x_1, y_1, \theta_1) = \phi_0 \circ A_{x_1,y_1,\theta_1}$. The set of all parameters $\{g_i = (x_i, y_i, \theta_i)\}$ form a group with the operation induced by the composition $A_{x_1,y_1,\theta_1} \circ A_{x_2,y_2,\theta_2}$. This turns out to be:

$$g_1 \circ g_2 = (x_1, y_1, \theta_1) +_R (x_2, y_2, \theta_2) = \left(\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + R_{\theta_1} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right)^T, \theta_1 + \theta_2 \right)$$

Being induced by the composition law, one can easily check that $+_R$ verifies the group operation axioms, where the inverse of a point $g_1 = (x_1, y_1, \theta)$ is induced by the rototranslation

$$A^{-1}_{x_1,y_1,\theta} = R^{-1}_\theta \circ T^{-1}_{x_1,y_1}$$

and the identity element is given by the trivial point $e = (0, 0, 0)$. The group generated by the operation $+_R$ in the space $\mathbb{R}^2 \times S^1$ is called Rototranslation group or equivalently $SE(2)$. A structured space with the symmetries described above allows for the cortex to be invariant to rotations and translations in the representation of a retinal image; the signals will be identical no matter what their position or orientation in the phenomenological space.

2.4.2 The principle fiber bundle of SE(2)

What distinguishes a Lie group like $SE(2)$ from a topological group is the existence of a differential structure. The tangent space to $SE(2)$ has the structure of *a principal fiber bundle*: using the fundamental results of the Lie groups theory we can characterize the local structure of a Lie group by its associated Lie algebra, which a theorem we have seen states is identified with the tangent space calculated in the identity e of the group. For this reason in Citti and Sarti's model proposed in [4],[5] the visual cortex is also seen as the principle fiber bundle of SE(2), where the base space of the fibration is the retina and there is a map associating to each retinotopic position $(x, y) \in \mathbb{R}^2$ a fiber, which is a copy of the whole possible set of orientations (the hypercolumn). To be more specific: the base space is implemented in the retinal space and the total space in the cortical space.

2.4.3 X_1,X_2,X_3, vector fields which generate SE(2) principal fiber bundle

If we consider a real stimulus, represented as an image $I : D \to \mathbb{R}$, we can assume that cells over each point $(x, y) \in D$ can code the *direction* of

the level lines[12] of I (which is so determined by its level lines), without a preferred direction. Let us consider as example a contour in a 2D-image.
A contour could be represented in the 2D plane as a regular curve (Figure 2.3)

$$\gamma_{2D}(t) = (x(t), y(t))$$

and almost everywhere we can assume that its tangent vector is non-vanishing, so that it can be identified by an orientation $\theta(t) : D \subset R \to S^1$, ($R$ is the retinical plane, D the image domain.) i.e. we are able to parametrize the curve by arc-length[13]:

$$(x'(t), y'(t)) = (\cos(\theta(t)), \sin(\theta(t))). \quad (2.2)$$

This means that the vector field

$$X_1(t) = \cos(\theta(t))\partial_x + \sin(\theta(t))\partial_y$$

is tangent to the level lines of I at the point $(x(t), y(t))$, and its normal direction is given by the gradient $\nabla I/|\nabla I| = (-\sin\theta, cos\theta)$.
The function θ takes values from the whole circle, in order to represent the polarity of the contours: two contours with the same orientation but with opposite contrasts are represented through opposite angles on the unit circle.
The action of the receptive profiles is to associate for every point $(x(t), y(t))$ the orientation $\theta(t)$ through the intracortical circuitry, which selects the hypercolumn's orientation of maximum output in response to the visual stimulus and supresses all the others, so that the variable associated to the hypercolumn will be an angle: the maximal response is our orientation $\theta(t)$.
In this way the two dimensional *retinical* curve γ_{2D} is *lifted* to a new curve

[12] Mathematically a level set is defined as a set of the form

$$\Gamma_c(\phi) = \{(x_1, \ldots, x_n)|\phi(x_1, \ldots, x_n) = c\}$$

i.e. a set where the function takes on a given constant value c

[13] A curve is parametrized by arc-length if its velocity vector, given a metric, is constantly equal to 1. It is always possible to give such a parametrization.

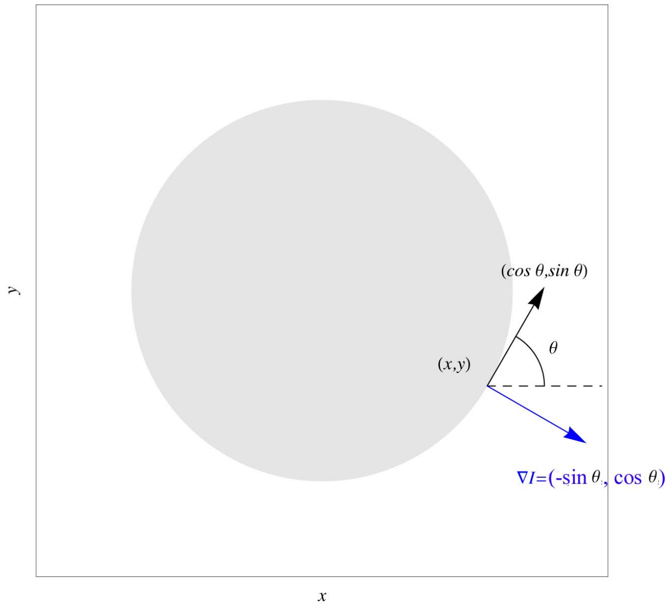


Figure 2.3: A contour in a 2D image can be modelled as a curve whose tangent is the vector $(\cos\theta, \sin\theta)$ and its normal direction is $\nabla I/|\nabla I| = (-\sin\theta, cos\theta)$ as indicated in the figure.

$\gamma(t)$ in the 3D *cortical* space:

$$(x(t), y(t)) \rightarrow (x(t), y(t), \theta(t)). \tag{2.3}$$

We call an *admissible curve* a curve in $\mathbb{R}^2 \times S^1$ if it is the lifting of a contour (identified by a planar curve). In Fig.(2.4) we can see an illustration of the lifting process. By the parametrization we have chosen before in (2.2) for the curve γ_{2D} (the blue curve in Fig.(2.4)) we can immediately express the value of θ:

$$\theta = -arctan(\frac{\dot{y}}{\dot{x}}).$$

The tangent vector to the lifted curve can be represented as a linear combination of the vectors $X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y$ given by the arc-parametrization and

$$X_2 = \partial_\theta$$

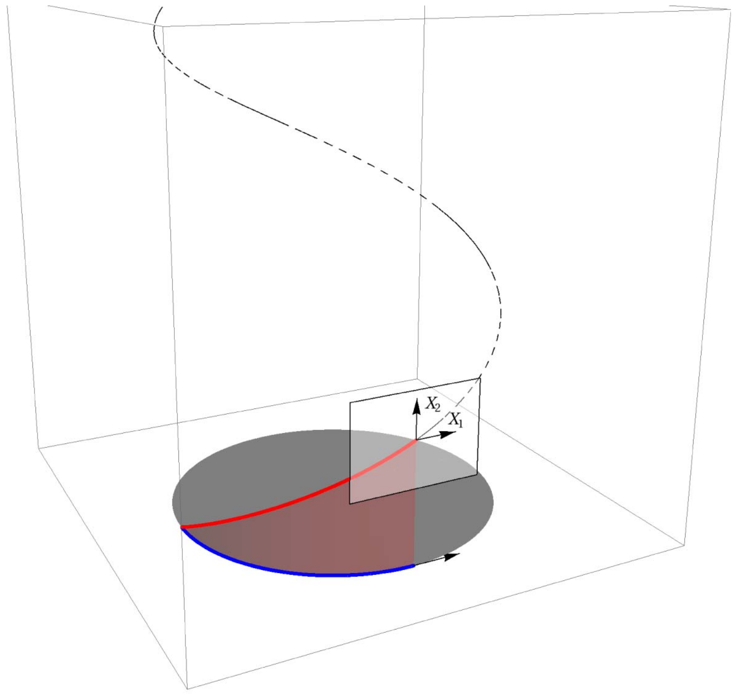


Figure 2.4: A contour represented by the curve $\gamma_{2D}(t)$ is lifted into the rototranslation group obtaining the red curve $\gamma(t)$. The tangent space of the rototranslation group is spanned by the vectors X_1 and X_2.

which descends from the orientation selectivity mechanism. For this reason the lifting γ (red curve in Fig(2.4)) of the curve γ_{2D} previously seen in (2.3) can be also expressed by (x, y, θ) where

$$\gamma' = (x', y', \theta') = (\cos(\theta), \sin(\theta), \theta') = X_1 + \theta' X_2$$

It immediately follows that $\gamma'(t)$ has a non-vanishing component in the direction X_1 and a second component θ' in the direction of X_2. In particular, admissible curves are integral curves of the two vector fields in a 3D (cortical) space, and cannot have components in the orthogonal direction given by the gradient $\nabla I/|\nabla I|$, which is $X_3 = -\sin(\theta)\partial_x + \cos(\theta)\partial_y$. From biological and neurophysiological evidence we have mathematically identified these three directions, developing a model which can be extended to all retinical images. *In fact, as we can define a retinal image through its level lines (this is due* *to all parameters, such orientation, color, etc, involved in the construction of a retinical image), the lifting of a surface can be performed repeating the previous analysis for each level line.*

2.4.4 Lie Algebra and Subriemannian Structure

We explicitly note that the vector fields X_1, X_2 and X_3 are *left invariant* with respect to the group law of rotations and translations, so that they are the generators of the associated *Lie algebra.* Moreover, the algebra is *stratified* in the sense we have seen in previous section in(2.2), i.e.

$$X_3 = [X_1, X_2] = -\sin(\theta)\partial_x + \cos(\theta)\partial_y.$$

In other words, we can say that the Hörmander condition is satisfied:

Definition 2.33. We say that the Hörmander condition is satisfied if X_1,X_2 and their commutators of any order span the Euclidean tangent space at every given point.

In fact in the present case X_1,X_2 and $X_3 = [X_1, X_2]$ are linearly independent and span the tangent space to $\mathbb{R}^2 \times S^1$ at each point.
In the standard Euclidean settings, the tangent bundle to $\mathbb{R}^2 \times S^1$ has three dimension at each point. Here the set of vectors

$$\{a_1X_1 + a_2X_2\}$$

defines a plane and every lifted curve is tangent to a vector of the plane, while there is not a natural curve with a non-vanishing component in the direction X_3, which should not be considered as a tangent direction. This means that only a two-dimensional subspace of the tangent space is selected as a model of the connectivity in V1. This is the reason why Citti and Sarti in [4], [5] proposed to endow the $\mathbb{R}^2 \times S^1$ with a sub-Riemannian structure, where X_1 and X_2 generate the horizontal bundle (plane) of the principal fiber bundle of $SE(2)$.
A metric, as we have seen, is simply the choice of the length of any vector

of the tangent space. Hence once we have defined our tangent space, we can immediately perform a choice of the metric. We will call the norm of the vector $\alpha_1 X_1 + \alpha_2 X_2$:

$$\|\alpha_1 X_1 + \alpha_2 X_2\|_g = \sqrt{\alpha_1^2 + \alpha_2^2}$$

The metric is clearly sub-Riemannian so, as we have seen, we can perform a Riemannian completion of the metric such as:

$$\|\alpha_1 X_1 + \alpha_2 X_2 + \varepsilon\alpha_3 X_3\|_g = \sqrt{\alpha_1^2 + \alpha_2^2 + \varepsilon^2\alpha_3^2}$$

and it is clear that we obtain the previous expression, as $\varepsilon \to 0$. We can give an expression for the inverse of the completed Riemannian metric which is useful for our calculations, since it does not blow up for $\varepsilon \to 0$

$$\left(g^{ij}\right) = \begin{pmatrix} \cos^2(\theta) & \cos(\theta)\sin(\theta) & 0 \\ \cos(\theta)\sin(\theta) & \sin^2(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This allows us to give an expression of the horizontal (and global) gradient in coordinates, as we have seen in (2.1), which are fundamental tools for modelling the completion phenomena as we will see in the next chapters.

Chapter 3

Mean curvature motion

In the previous chapter we built a geometric space inspired by the functional geometry of the primary visual cortex. In the sub-Riemannian space of the cortex the completion phenomena are accomplished in two main mechanisms, the first one extracting the existing information (real boundaries, image gradients and complex features) and the second one completing the missing information. The first process is carried out by simple cells in V1 and extracts information about the orientation, as we have seen before. The second mechanism propagated extracted information in an orientation specific modality by means of long-range horizontal connections that were defined in chapter 1. In this setting, the formation of subjective contours is explained as the meeting of two neural activation flows shooted by the boundary inducers and closing missing information between the existing boundaries. The specificity of this information propagation is described by the *association fields* (see Field, Heyes and Hess in ([10])) that indicate boundary collinear directions as privileged diffusion directions to the detriment of orthogonal ones. Furthermore they also describe the mechanism of local induction, which explains how in the cortical space the integrative process allows us to connect local tangent vectors to form integral curves. (see Figure 3.1). The experiments of Bosking ([2]) prove that the diffusion of a marker in the cortex are in perfect agreement with the association fields. This means, in particular,

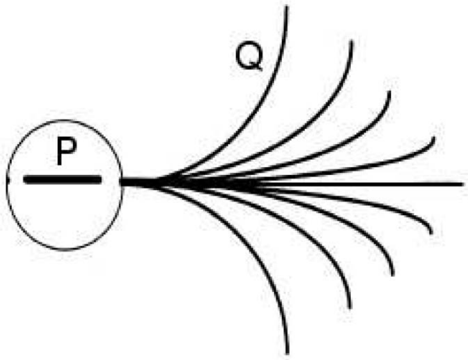


Figure 3.1: The stimulus in the central position P can be joined with other stimuli tangent to the lines in the figure, but cannot be joined with stimuli with different directions.

that the diffusion of the visual signal occurs along these curves. The PDE which describes the mean curvature flow shows a special diffusion along the integral curves of the vector fields X_1 and X_2, which are responsible for the phenomena of completion. [1] In particular, a minimal regular surface (with respect to the sub-Riemannian metric) is a subset which can be locally represented as the zero level set of a function which we will identify as the solution of the mean curvature flow.[2]

[1]The relation between diffusion and curvature equation goes back to paper of Bence,Merriman and Osher in [1] who describes the evolution of surface by mean curvature in terms of heat diffusion

[2]Let's give a more specific overview: a model for perceptual completion of *boundaries* has been provided by Citti and Sarti in [4] and [5]. They boundaries developped by the primary visual cortex are described as integral curves of the geometric structure we have built also in this work, in particular they are the geodesics of the sub-Riemannian manifold, i.e. they are minimum of the length functional defined throgh the sub-Riemannian metric. In this way they develop a model which is based on the functional architecture of the cortex, which is in accord with the results obtained by Mumford who instead modelled the countours of the completion phenomena as minimum of the elastica functional

$$\int_\gamma (1 + k^2) ds$$

3.1 Differential instruments for introducing the mean curvature of an hypersurface

3.1.1 Affine connection

In order to introduce the notion of curvature we need to define derivatives of an order higher than one on a manifold, which are expressed in the concept of *affine connection*. For making this tractation more understandable, we will focus on the definition of affine connection on a Riemannian manifold, which represents a particular case of the definition of a connection on a sub-Riemannian manifold. Details of this fact are discussed in [12], where is proved the existence and uniqueness of a connection $\nabla^{(r)}$ on a sub-Riemannian manifold with an r-graded complement, in which a metric extension (Riemannian) g is given. Furthermore $\nabla^{(r)}$ will coincide on the horizontal bundle(HM) with the Levi-Civita connection, the unique affine connection compatible with the metric on a Riemannian manifold. Hence the properties and the definition we will provide for the Levi-Civita connection hold on the horizontal bundle, and since we are interested in integral curves of vector fields in HM, showing results for the Levi-Civita connection on a Riemannian manifold will be sufficient.

where k is the derivative of θ, the angle of the orientation (the curvature). Citti and Sarti obtained their result modifying the elastica functional: in this way they reconduct the problem of founding the minimum of elastica to the research of the geodesic of the structure. It's easy to understand that the problem of founding a minimal surface is a generalization of the completion of boundaries, we just go higher with the dimension. The problem of building a minimal surface is reconducted to a diffusion problem (which is linked to the study of the evolution of the surface as we have said before), because the diffusion problem satisfies the condition to be foliated by geodesic, which means we want all the curves lying on the surface be geodesic, in order to respect the experimental evidence. This also means that the PDE of mean curvature flow we are going to introduce gives the solution to the research of minimal surfaces in two way: the first is solving the diffusion problem of dimension 2, the second is solving the linearized equation associated to the diffusion problem of dimension 2 in order to diffuse along the horizontal level lines lying on the surface.

Definition 3.1. An *affine connection* on an n-dimensional manifold M is a map $\nabla^M : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that for each $X, Y, Z \in \mathfrak{X}(M)$:

- ∇^M is $C^\infty(M)$ linear on the first argument, i.e. for all $f, g \in C^\infty(M)$:
$$\nabla^M_{fX+gY} Z = f\nabla^M_X Z + g\nabla^M_Y Z$$
- ∇^M is $\mathbb{R}$-linear on the second argument, i.e. for all $\alpha, \beta \in \mathbb{R}$:
$$\nabla^M_X(\alpha Y + \beta Z) = \alpha\nabla^M_X Y + \beta\nabla^M_X Z$$
- ∇^M satisfies Leibniz's rule on the second argument, i.e. for all $f \in C^\infty(M)$:
$$\nabla^M_X fY = df(X)Y + f\nabla^M_X Y$$

Moreover, a connection is said to be *torsion-free* if it is well-behaved with Lie parenthesis:
$$[X, Y] = \nabla^M_X Y - \nabla^M_Y X$$

Definition 3.2. A connection is said to be *compatible* with the Riemannian metric g (or equivalently $\langle \cdot, \cdot \rangle$) on M if it satisfies $X\langle Y, Z\rangle = \langle \nabla^M_X Y, Z\rangle + \langle Y, \nabla^M_X Z\rangle$ for all $X, Y, Z \in \mathfrak{X}(M)$.

An important result is that for a given a metric on a manifold, there is precisely one torsion-free connection which is compatible with the metric.

Theorem 3.1.1. *Given $(M, \langle \cdot, \cdot \rangle)$ a Riemannian manifold, there exists a unique torsion-free connection compatible with the Riemannian metric. This connection is called the Levi-Civita connection of M. Moreover, this connection satisfies for all $X, Y, Z, W \in \mathfrak{X}(M)$:*

$$2\langle \nabla^M_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle + \langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle$$

For the proof we refer, for example, to [20] [Chapter 2, Theorem 1.1]. This last theorem states that the metric of a manifold completely determines its connection.
Since it is useful to write the Levi-Civita connection in local coordinates, we introduce the Christoffel's Symbols, which allow us to do this.

Definition 3.3. Let p be a point in M with local coordinates $(x_1, \ldots, x_n)$in its neighborhood. For each $1 \leq i, j \leq n$ we define:

$$\nabla^M_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{n} \Gamma^k_{ij} \frac{\partial}{\partial x_k}$$

The functions Γ^k_{ij} are the Christoffel's symbols of the Levi-Civita connection ∇_M in the basis given by $(x_1, \ldots, x_n)$. Since the connection is torsion-free, these functions are symmetric in the lower indexes. Moreover, if g_{ij} is the expression of the metric in local coordinates, we have the following formula:

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{h=1}^{n} g^{kh} \left(\frac{\partial g_{hi}}{\partial x_j} + \frac{\partial g_{hj}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_h} \right)$$

From now on when we consider a Riemannian manifold with its connection, we will automatically imply that it is the Levi-Civita connection. The definition of this connection is what we will use to define calculus objects on Riemannian manifolds. We will now go through some definitions fundamental for introducing the concept of curvature.

Definition 3.4. Consider $X \in \mathfrak{X}(M)$. We define the divergence[3] of X as the trace of the linear function $Y \to \nabla^M_Y X$. The trace of a linear function on a vector space does not depend on the base with respect to which it is calculated. So we have:

$$div(X) = \sum_{i=1}^{n} \langle \nabla^M_{E_i} X, E_i \rangle$$

provided E_i is a local orthonormal frame for the tangent bundle of M.

Definition 3.5. Consider $f \in C^\infty(M)$. We define the laplacian[4] of f as the divergence of its gradient:

$$\Delta f = div(\nabla f)$$

[3]In the Euclidean space we define the divergence of a continously differentiable vector field $X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$ as the scalar-valued function $div(X) = \frac{\partial a_1}{\partial x_1} + \ldots + \frac{\partial a_n}{\partial x_n}$

[4]In the Euclidean space we define the laplacian of a twice differentiable real-valued function f as $\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$

One can easily check that the expression of the laplacian in local coordinates is

$$\Delta f = \frac{1}{\sqrt{det(g)}}\left(\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\sqrt{det(g)}\sum_{j=1}^{n} g^{ij}\frac{\partial f}{\partial x_j}\right)\right)$$

3.1.2 Curvature

Intuitively *curvature* is the amount by which a geometric object deviates from being *flat*.

Definition 3.6. If we consider a Riemannian manifold M and $X, Y, Z \in \mathfrak{X}(M)$, one defines the *Riemannian curvature function* associated to M by setting:

$$R(X,Y)Z = \nabla^M_Y \nabla^M_X Z - \nabla^M_X \nabla^M_Y Z + \nabla^M_{[X,Y]} Z$$

The Riemannian curvature function is actually a tensor since it can be proved to be linear in each of its arguments and it is the only object related to the curvature which does not depend on the metric. If we consider one more field W we define the following notation involving the choice of a metric:

$$(X, Y, Z, W) = \langle R(X,Y)Z, W\rangle$$

The Riemannian curvature tensor satisfies some important properties such as the Bianchi Identity, and details of these and the following facts can be found in [26] and [23]. The important idea beyond the previous definition is that the curvature tensor can be understood by means of a simpler curvature, the *sectional curvature*, which intuitively measures the curvature of a Riemannian manifold along planes of the tangent space of M in p. The *sectional curvature* is defined as follows:

Definition 3.7. If $X, Y \in \mathfrak{X}(M)$ are two linearly independent vector fields, for each $p \in M$ we can denote as $\pi_p(X,Y) \subset T_pM$ the plane spanned by $X(p)$ and $Y(p)$. Let p be in M and X, Y vector fields non-zero in p. Then we define:

$$K_{sect}(X,Y)(p) = \frac{(X,Y,X,Y)}{|X|^2|Y|^2 - \langle X,Y\rangle}(p)$$

as the *sectional curvature*, which only depends on the plane spanned by X, Y and not on the specific choice of X and Y.

This last statement is proved in a theorem, the same as the proposition which proves that sectional curvature completely determines the Riemannian curvature tensor, see [23] [Proposition 11, Theorem 4].

3.1.3 Submanifolds and Mean Curvature

In this subsection we will consider submanifolds with the Riemannian structure induced from an ambient Riemannian manifold[5]. Let M be an n-dimensional Riemannian manifold and S a m-dimensional differentiable manifold, with $m \leq n$.

Definition 3.8. A smooth map $\Phi : S \to M$ is an *immersion of* S if its differential is non-singular at each point of S. If, in addition, Φ is injective and is an homeorphism onto its image, Φ is called an *embedding* of S. An embedding naturally gives to S the structure of an *embedded submanifold*[6] of M, which is a submanifold for which the inclusion map is a topological embedding. This means that the topology on S is the same as the subspace topology.

Definition 3.9. Suppose $\Phi : S \to M$ is an immersion. Then if the Riemannian structure of M is given by the metric g, Φ induces a Riemannian structure on S defined as follows: considering $p \in S$, for each $v, w \in T_pS$ we

[5] Note that hypersurfaces, curves and level sets of a function ϕ, these last ones defined as a set of the form

$$\Gamma_c(\phi) = \{(x_1, \dots, x_n) | \phi(x_1, \dots, x_n) = c\}$$

i.e. a set where the function takes on a given constant value c, can be considered as submanifolds. We will use level sets also in the next section, which is why we recall them and introduce the concept of submanifold.

[6] In general a submanifold S of M is a subset S of M which itself has the structure of a manifold, and for which the *inclusion map* $i : S \to M$ induces a topology and a differentiable structure on S

define

$$h_p(v, w) = g_{\Phi(p)}(d_p\Phi(v, d_p\Phi(w)))$$

It turns out that the Riemannian connection ∇^S is the projection of ∇^M on the tangent bundle of S. From now on we shall assume that all the hypersurfaces[7] that we take into account are orientable[8].

Definition 3.10. Let M be a Riemannian manifold and S an hypersurface with unit normal field η. For each $X, Y \in \mathfrak{X}(S)$ and $p \in S$ we define:

$$\nabla^S_X Y(p) = \left(\nabla^M_{\tilde{X}}\tilde{Y}\right)^{tangent}(p) = \left(\nabla^M_{\tilde{X}}\tilde{Y}\right)(p) - \langle\nabla^M_{\tilde{X}}\tilde{Y}, \eta\rangle\eta(p)$$

where $\tilde{X}$ and $\tilde{Y}$ are extensions of X and Y to a neighborhood of p in M.

Proposition 3.1.2. *Let us make some considerations about ∇^S:*

- *∇^S is well-defined, i.e. it does not depend on the extension of X and Y*
- *∇^S is the unique connection compatible with the metric induced on S by M*

This proof can be found in [20].

Definition 3.11. Let $S \subset M$ be an orientable hypersurface with the unit normal field η. Then for each $p \in S$ one defines the *shape operator*:

$$\begin{aligned} A : T_pS &\to T_pS \\ v &\mapsto \big(-\nabla^M_v \eta\big) \end{aligned}$$

[7]Suppose a manifold M has n dimensions; then any submanifold of M of n-1 dimensions is a hypersurface.

[8]Orientability is a property of surfaces in Euclidean space measuring whether it is possible to make a consistent choice of surface normal vectors at every point. This choice allows us to use the right-hand rule to define a "clockwise" direction of loops in the surface. This concept can be generalized to a manifold.

The operator we have just defined is linear. Since a proposition which can be proved states it is self-adjoint, we can say that this operator is the Riemannian generalization of the Euclidean differential of the Gauss map[9]. Intuitively the shape operator gives information about *the shape* of the hypersurface, and the theory behind it states that the shape evolves according to the curvature, see [23] [Chapter 1.5, Theorem 5]. We can finally introduce the fundamental object which describes the evolution of hypersurfaces:

Definition 3.12. Consider $S \subset M$, where M in an n-dimensional manifold, and S is an orientable hypersurface with a unit normal field η. For each $p \in S$ we define the mean curvature of S in p with respect to η as:

$$H(p) = -\frac{1}{n-1} tr(A)(p) = -\frac{1}{n-1} \sum_{i=1}^{n-1} \langle \nabla^M_{E_i} \eta, E_i \rangle \tag{3.1}$$

$$= -\frac{1}{n-1} div(\eta)(p) \tag{3.2}$$

where $\{E_1, \ldots, E_{n-1}\}$ is an orthonormal frame of $T_x S$ for x near p.

[9]In differential geometry the *Gauss map* maps a surface in Euclidean space $\mathbb{R}^3$ to the unit sphere S^2, i.e. given a surface X lying in $\mathbb{R}^3$, the Gauss map is a continuous map $N : X \to S^2$ such that $N(p)$ is a unit vector orthogonal to X at p, namely the normal vector to X at p. The Gauss map can always be defined locally, and its Jacobian determinant is equal to the Gaussian curvature, which is the product of the principal curvatures of a point on a surface. The *Gaussian curvature* is an intrinsic measure of curvature, since its value depends only on how distances are measured on the surface and not on the way it is isometrically embedded in space.
Let us give an idea recalling the example of a surface X in $\mathbb{R}^3$: at any point on a surface we can find a normal vector which is at right angles to the surface. The intersection of a plane containing the normal with the surface will form a curve called a *normal section* and the curvature of this curve is the *normal curvature*(the equivalent generalization is the sectional curvature we have defined before). For most points on most surfaces, different sections will have different curvatures; the maximum and minimum values of these are called *principal curvatures*, indicated by κ_1 and κ_2. The sign of the Gaussian curvature $K = \kappa_1 \cdot \kappa_2$ characterises the surfaces.

3.2 Mean curvature motion of hypersurfaces

The problem of investigating the evolution of a hypersurface moving according to its mean curvature has long been studied in the Euclidean setting using parametric methods of differential geometry. In this classical approach, we give at time 0 a smooth hypersurface Γ_0 which is the connected boundary of a bounded open subset of $\mathbb{R}^n$: as time progresses we allow the surface to evolve, by moving each point at a velocity $\vec{v}$ equal to $(n-1)$ times the mean curvature vector at that point:

$$\frac{\partial \vec{p}}{\partial t} = \vec{v} = (n-1)H(p) = div(\eta)$$

where η is the normal vector to the hypersurface and the second equality is obtained applying the definition of mean curvature we have seen before. Assuming the evolution is smooth, we define thereby for each $t > 0$ a new hypersurface Γ_t. For $n = 2$ the analysis has been successfully carried out in detail, but when $n \geq 3$ even if the initial surface Γ_0 is smooth, the smooth evolution cannot exist beyond some initial time interval. For these reasons it is necessary to study a different approach, examined by Osher and Sethian in ([19]), which consists in considering the initial hypersurface Γ_0 (as above) as 0-level set of some continuous function $g : \mathbb{R}^n \to \mathbb{R}$ so that

$$\Gamma_0 = \{x \in \mathbb{R}^n | g(x) = 0\}.$$

If we consider the previous expression for the mean curvature flow, the parabolic PDE[10] is obtained deriving it as is shown in ([19]):

$$u_t = (\delta_{ij} - u_{x_i}u_{x_j}/|\nabla u|^2)u_{x_i x_j} \quad \text{in} \quad \mathbb{R}^n \times [0, \infty) \tag{3.3}$$

[10] A *parabolic partial differential equation* is a type of second-order partial differential equation (PDE) of the form:

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = 0$$

which satisfies the condition

$$B^2 - AC = 0$$

i.e. all the eigenvalues of the operator are positive or negative, except for one which is equal to zero

$$u = g \quad \text{on} \quad \mathbb{R}^n \times \{t = 0\}. \tag{3.4}$$

For the unknown $u = u(x,t)$, $(x \in \mathbb{R}^n, t \geq 0)$, the PDE says that *each level set of u evolves according to its mean curvature, at least in regions where u is smooth and its spacial gradient ∇u does not vanish.* Consequently, focusing our attention on the set $\{u = 0\}$, it seems reasonable in view of the previous equations to define:

$$\Gamma_t = \{x \in \mathbb{R}^n | u(x,t) = 0\} \tag{3.5}$$

for each time $t > 0$. To resume: Osher and Sethian in ([19]) reconduct the study of the mean curvature motion of hypersurface to the motion of level sets by mean curvature. Evans and Spruck in [8] resolve the problem in this Euclidean case, providing a theoretical justification for this approach (existence and uniqueness of the weak solution to (3.3)(3.4)) and additionally checking that $\{\Gamma_t\}_{t\geq 0}$ so defined agrees with the classical notion of motion via mean curvature[Section 6 of [8]], over any time interval for which the latter exists. They also employ the PDE (3.3) to deduce geometric properties of $\{\Gamma_t\}_{t\geq 0}$. This *level set approach* has been extended by Ilmanen in ([16]) to include the study of the generalized flow of subsets in Riemannian manifolds, with the Riemannian notions we have provided in this chapter.

We are interested in studying the sub-Riemannian analogue of the mean curvature motion of level sets in SE(2), the *horizontal mean curvature flow*, a particular case of the result treated in [3]. Before going through this analysis we will briefly discuss the evolution of curves in $\mathbb{R}^2$ focusing on what consists the level sets method of Osher and Sethian, through a general analysis for a sub-Riemannian manifold, which explains the introduction of the PDE object of our studies, where we consider hypersurfaces and not only curves.

3.2.1 The evolution of curves in $\mathbb{R}^2$ and the evolution of implicit curves

Consider a family of curves $\Gamma(s,t) = (x(s,t), y(s,t)) : [0, L(t)] \times [0, T[\to \mathbb{R}^2$. We indicate with s the parametrization of the curve Γ at time t. Notice that the *domain* of the curve is characterized by the variables (s,t) and the codomain by the variables (x,y). This family of curves can be interpreted as the *evolution of the curve* $\Gamma(s) = (x(s), y(s))$ in time and can be described by the differential equation:

$$\Gamma_t = \vec{v}$$

where $v(s,t) : [0,L] \times [0,T[\rightarrow \mathbb{R}^2$ is the velocity vector. To this equation we add the initial conditions:

$$\Gamma(s,0) = \Gamma_0(s)$$

where $\Gamma_0(s)$ is the initial curve given. A generic point on the curve has

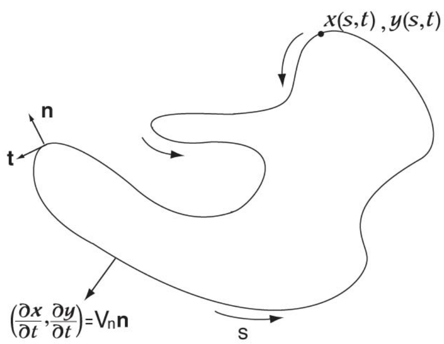


Figure 3.2: Parametrization of the family of curves $\Gamma(s,t)$

coordinates $(x(s,t), y(s,t))$ and it moves with velocity $\vec{v} = v_n\vec{n} + v_t\vec{t}$ where v_n and v_t are the normal(versor $\vec{n}$) and tangent(versor $\vec{t}$) components of the velocity vector. Hence it is possible to describe the evolution equation with these versors:

$$\Gamma_t = (v \cdot n)\vec{n} + (v \cdot t)\vec{t}, \quad \Gamma(s,0) = \Gamma_0(s).$$

Epstein-Gage Lemma proves that the tangent component of the velocity vector has effect only on the parametrization and not on the geometrical structure of the curve (i.e. the shape), so since in the evolution equation we are interested only in the shape evolution of the curve, we will consider only the normal component. For this reason we can reduce the evolution equation to:

$$\Gamma_t = v_n \vec{n}.$$

Consider a bi-dimensional space divided by the curve Γ in two sub-domains, and call Ω the region *in* the curve. Γ can be defined as a 0-level set of an implicit function $\phi(x, y)$:

$$\Gamma = \phi^{-1}(0).$$

If we observe the sign of ϕ calculated in a generic point (x_0, y_0) we can determine its position with respect to the region delimitated by Γ: if $\phi(x_0, y_0) < 0$, (x_0, y_0) is interior to Ω, if $\phi(x_0, y_0) > 0$ it is outside, and if $\phi(x_0, y_0) = 0$ this point is on Γ. The gradient of an implicit function is defined as:

$$\nabla\phi = (\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y})$$

is othogonal to the level sets of ϕ and it has direction along which ϕ grows. Hence, if (x_0, y_0) is a point belonging to the 0-level set of ϕ, $\nabla\phi$ evaluated in

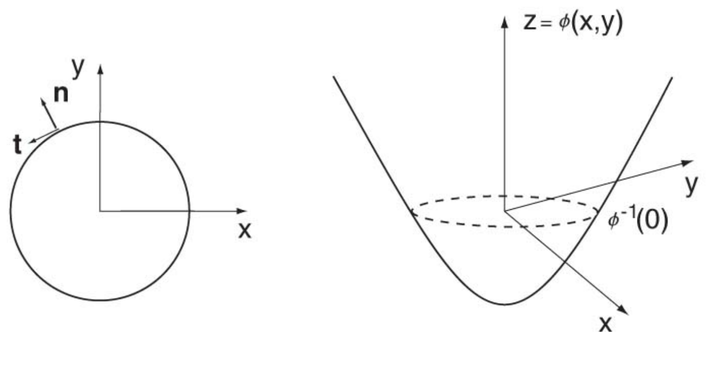


Figure 3.3: Representation of the curve Γ and of the function ϕ

(x_0, y_0) is a vector which points in the same direction as the normal versor $\vec{n}$ in the same point. The exterior normal can be expressed for each point in Γ

as:

$$\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}.$$

This equation can be used to define a *normal function* n in all our domain. The mean curvature of Γ is defined as the divergence of the normal:

$$K = div(\frac{\nabla\phi}{|\nabla\phi|}).$$

For a curve the mean curvature is geometrical equivalent to the inverse of the radius of curvature.

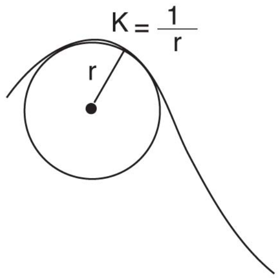


Figure 3.4: Geometric interpretation of the curvature of a curve

The motion of level sets, as we have said before, has been formulated by Osher and Sethian in ([19]), in order to overcome the difficulties we have expressed at the beginning of this section. The basic idea is to consider the curve $\Gamma(s,t)$ as implicitly represented by the *level set function* $\phi(x,y,t) : \mathbb{R}^2 \times [0,T) \to \mathbb{R}$. In this way, the 0-level set of the level set function $\phi(x,y,t) = 0$ is the set of points which form the curve $\Gamma(s,t)$. In other words, the evolution of the curve Γ at time t is given by the 0-level set of the function ϕ at time t:

$$\Gamma(t) = \phi(t)^{-1}(0) \tag{3.6}$$

The principal problem when we study an evolution equation is the way in which we make the function ϕ evolving in time, such that its 0-level set follows the movement of the curve $\Gamma(t)$. The evolution of the implicit curve

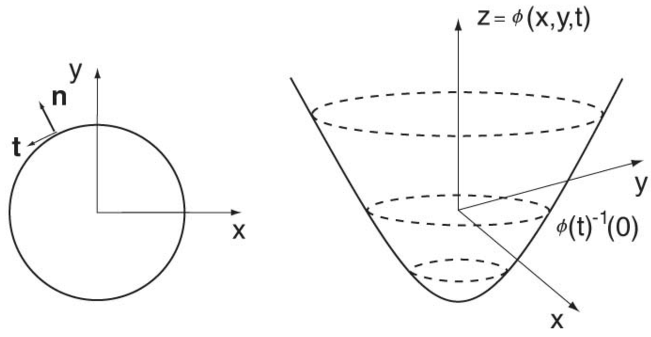


Figure 3.5: Representation of the evolution of the curve Γ through the function ϕ

ϕ is described by the following equation, also known as *the motion's level set equation*:

$$\phi_t + \vec{v} \cdot \nabla\phi = 0.$$

If we express the velocity vector through its components, we obtain:

$$\phi_t + (v_n\vec{n} + v_t\vec{t}) \cdot \nabla\phi = 0$$

Since the normal versor and the gradient point in the same direction, due to our previous considerations $\vec{t} \cdot \nabla\phi = 0$ for each tangent vector to the curve. Hence the level set equation becomes:

$$\phi_t + v_n\vec{n} \cdot \nabla\phi = 0.$$

If we solve the inner product on $\mathbb{R}^2$

$$\vec{n} \cdot \nabla\phi = \frac{\nabla\phi}{|\nabla\phi|} \cdot \nabla\phi = \frac{|\nabla\phi|^2}{\nabla\phi} = |\nabla\phi|$$

and we assign a regular function $\phi_0(x, y)$ such as $\phi_0^{-1}(0) = \Gamma_0$, we can rewrite the level set equation as follows:

$$\phi_t + v_n|\nabla\phi| = 0, \ \phi(x, y, 0) = \phi_0(x, y).$$

It is possible to present different kinds of evolution for the curve: the difference is represented by the expression of the normal component of the velocity vector.
If the normal component of the velocity vector v_n is equal to the mean curvature K (with sign changed) we obtain an expression for the evolution of a curve by curvature. The equation of motion is:

$$\phi_t = K|\nabla_\phi|$$

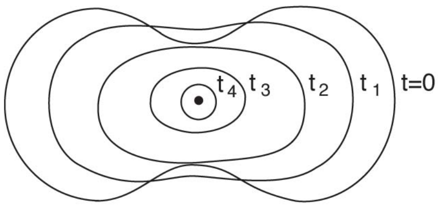


Figure 3.6: Representation of the evolution of the curve by mean curvature

3.2.2 Motion of level sets by horizontal mean curvature

Let M be a sub-Riemannian manifold and $g_\varepsilon(\langle\cdot,\cdot\rangle_\varepsilon)$ the Riemannian completion of the metric $g_0(\langle\cdot,\cdot\rangle_0)$ as we have seen in Chapter 2. If we consider a smooth hypersurface $S \subset M$, we will denote now with $\vec{n}^\varepsilon$ the unit normal in the metric g_ε and with $\vec{n}^0 = \sum_{i=1}^k (n^0)_i X_i$ its projection in the g_ε norm onto the horizontal plane, where k is the dimension of the horizontal bundle and n the dimension of the tangent bundle completed. Note that this definition does not depend on ε. The vector n^0 is called the *horizontal normal* and its *horizontal divergence* is:

$$K_0 = \sum_{i=1}^k X_i n_i^0,$$

also known from the previous considerations as *horizontal mean curvature*[11] of S, which will be affected only by the horizontal bundle. We study the flow $t \to \Gamma_t$ where a point $x \in \Gamma_t$ evolves with velocity $\partial_t x = -K_0 n^0$. The level set approach consists in studying a PDE, describing the evolution of a function $u(x,t)$ such that[12] $\Gamma_t = \{x \in M | u(x,t) = 0\}$. In this setting one has $n^\varepsilon = \nabla_\varepsilon u / |\nabla_\varepsilon u|$ and $n^0 = \nabla_0 u / |\nabla_0 u|$. Consequently, on a formal level, one has:

$$\partial_t u(x(t), t) = \langle \nabla_0 u(x(t)), \partial_t x(t) \rangle_0 + \partial_t u(x,t) \quad (3.7)$$

$$= -K_0 \langle \nabla_0 u, n^0 \rangle_0 + \partial_t u = -K_0 |\nabla_0 u| + \partial_t u = 0 \quad (3.8)$$

This problem is well approximated by the Riemannian mean curvature flows $\partial_t x = -K_\varepsilon n^\varepsilon$, where $K_\varepsilon = \sum_{i=1}^n X_i^\varepsilon n_i^\varepsilon$ is the g_ε mean curvature of M. The corresponding evolution PDE for the level sets is $\partial_t u^\varepsilon = K_\varepsilon |\nabla_\varepsilon u|$. We observe that for a given hypersurface, $n^\varepsilon \to n^0$ and $K_\varepsilon \to K_0$ as $\varepsilon \to 0$ (see footnote for explanations). The simple computation provided in the respective cases by Osher and Sethian in ([19]) that we extend also to our case shows that the mean curvature K_ε of the manifold $\{u(x) = 0\}$, entirely represented by the level sets of u as we have seen in the 2-dimensional case before, is given by the identity:

$$K_\varepsilon |\nabla_\varepsilon u| = \sum_{i,j=1}^n \left(\delta_{i,j} - \frac{X_i^\varepsilon u X_j^\varepsilon u}{(|\nabla_\varepsilon u|)^2} \right) X_i^\varepsilon X_j^\varepsilon u$$

The horizontal mean curvature K_0 is expressed as:

$$K_0 |\nabla_0 u| = \sum_{i,j=1}^k \left(\delta_{i,j} - \frac{X_i u X_j u}{(|\nabla_0 u|)^2} \right) X_i X_j u$$

[11]The curvature has been defined also through the divergence with respect to the horizontal vector fields. The observations made at the beginning of this chapter about the affine connection on a sub-Riemannian manifold, which extends to the affine connection on a Riemannian manifold, make it possible to restrict the definition seen for the divergence to the horizontal bundle of a sub-Riemannian manifold. The passage to the limit we are going to see is possible in virtue of this consideration, when we choose a Riemmanian extention of the metric depending on a parameter ε.

[12]When a manifold is defined as a level set, we assume that the gradient of the defining function does not vanish in a neighborhood of the manifold.

Consequently (3.8) can be rewritten more explicity as:

$$u_t = \sum_{i,j=1}^{k} \left(\delta_{i,j} - \frac{X_i u X_j u}{(|\nabla_0 u|)^2} \right) X_i X_j u \quad \text{for} \quad x \in M, t > 0$$

Note that with $\{X_i^\varepsilon\}_{i=1,2,3}$ we denote vector fields which are made orthonormal by the Riemannian metric extension.

3.2.3 The motion of level sets in $\mathbb{R}^2 \times S^1$

The equations in the previous subsection can be presented also in our Lie group $SE(2) = \mathbb{R}^2 \times S^1$, since it is equipped with a sub-Riemannian structure. In the previous chapter we saw that $X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y$, $X_2 = \partial_\theta$ and $X_3 = -\sin(\theta)\partial_x + \cos(\theta)\partial_y$ generate the principal fiber bundle of the rototraslation group *SE(2)*, where X_1 and X_2 belong to the horizontal bundle HM and εX_3 is the completion of the tangent bundle; X_3 is generated through the Lie bracket. For this reason we can describe the evolution of an hypersurface in $\mathbb{R}^2 \times S^1$ by the PDE:

$$u_t = \sum_{i,j=1}^{3} \left(\delta_{i,j} - \frac{X_i^\varepsilon u X_j^\varepsilon u}{(|\nabla^\varepsilon u|)^2} \right) X_i^\varepsilon X_j^\varepsilon u$$

where the ε identifies the metric. In the same way, the *horizontal mean curvature* is expressed as:

$$u_t = \sum_{i,j=1}^{2} \left(\delta_{i,j} - \frac{X_i u X_j u}{(|\nabla_0 u|)^2} \right) X_i X_j u.$$

This last equation is the one we are interested in, since the evolving surface can be represented as 0-level set of the *viscosity solution* $u(x,t)$, obtained as the limit of the viscosity solution $u^\varepsilon(x,t)$ of the previous one (the PDE which involves the Riemannian completion), when ε goes to 0. This evolved surface we obtain is the minimal surface involved in completion phenomena. For this reason, in the next chapters we will provide the definition of a viscosity solution and its existence for the PDE which describes the horizontal mean curvature flow in $SE(2)$.

Chapter 4

Existence of viscosity solutions

4.1 Viscosity solutions

Our goal is to provide a generalized solution of the degenerate nonlinear, non-divergence PDE:

$$u_t = \sum_{i,j=1}^{3} \left(\delta_{i,j} - \frac{X_i^\varepsilon u X_j^\varepsilon u}{|\nabla_\varepsilon u|^2} \right) X_i^\varepsilon X_j^\varepsilon u \tag{4.1}$$

$$u = g \quad \text{on} \quad \mathbb{R}^2 \times S^1 \times \{t = 0\} \tag{4.2}$$

Note that with $\{X_i^\varepsilon\}_{i=1,2,3}$ we denote vector fields which are made orthonormal by the Riemannian metric extension. The function $g : \mathbb{R}^2 \times S^1 \longmapsto \mathbb{R}$ given. To achieve our purpose as we have explained in the previous chapters we need to pass through the solution of:

$$u_t = \sum_{i,j=1}^{2} \left(\delta_{i,j} - \frac{X_i u X_j u}{|\nabla_0 u|^2} \right) X_i X_j u \tag{4.3}$$

As we can immediately observe the PDE becomes degenerate in the singularities of the horizontal gradient of the solution $u(.\,, t)$. Furthermore, just as in the Euclidean space, we cannot expect the smoothness of the solution to be preserved for all times. For this reason we need to introduce the analogous of a weak solution, called a viscosity solution, but since the right-hand side of the PDE cannot be put into divergence form, we are not able to define it in the classic sense by means of formal integration by parts of derivatives onto a smooth test function. Hence we want to define a notion of viscosity solutions to (4.3) in terms of pointwise behavior with respect to a smooth test function, in order to cover the possibility that $\nabla_0 u$ may vanish.

Definition 4.1. A function $u \in C(\mathbb{R}^2 \times S^1 \times [0,\infty))$ is a *viscosity subsolution* of (4.3) in $\mathbb{R}^2 \times S^1 \times [0,\infty)$ if for any (x,t) in $\mathbb{R}^2 \times S^1 \times [0,\infty)$ and any function $\phi \in C(\mathbb{R}^2 \times S^1 \times [0,\infty))$ such that $u - \phi$ has a local maximum at (x,t) then

$$\partial_t \phi \leq \begin{cases} \sum_{i,j=1}^{2} (\delta_{ij} - \frac{X_i\phi X_j\phi}{|\nabla_0\phi|^2}) X_i X_j \phi, & \text{if } |\nabla_0 \phi| \neq 0 \\ \sum_{i,j=1}^{2} (\delta_{ij} - p_i p_j) X_i X_j \phi, & \text{for some}\, p \in \mathbb{R}^2, |p| \neq 1, \text{ if}\, |\nabla_0\phi| = 0 \end{cases} \tag{4.4}$$

A function $u \in C(\mathbb{R}^2 \times S^1 \times [0,\infty))$ is a viscosity supersolution of (4.3) if:

$$\partial_t \phi \geq \begin{cases} \sum_{i,j=1}^{2} (\delta_{ij} - \frac{X_i\phi X_j\phi}{|\nabla_0\phi|^2}) X_i X_j \phi & \text{if } |\nabla_0 \phi| \neq 0 \\ \sum_{i,j=1}^{2} (\delta_{ij} - p_i p_j) X_i X_j \phi & \text{for some}\, p \in \mathbb{R}^2, |p| \neq 1, \text{ if}\, |\nabla_0\phi| = 0 \end{cases} \tag{4.5}$$

Definition 4.2. A *viscosity solution* of (4.3) is a function u which is both a viscosity subsolution and a viscosity supersolution.

Definition 4.3. A function $u \in C(\mathbb{R}^2 \times S^1 \times [0,\infty)) \cap \mathcal{L}^\infty(\mathbb{R}^2 \times S^1 \times [0,\infty))$ is a viscosity subsolution of equation (4.3) if whenever $(x,t) \in \mathbb{R}^2 \times S^1 \times [0,\infty)$ for every $yX \in \text{Lie}(\mathbb{R}^2 \times S^1)$ and $s \in \mathbb{R}$

$$u(exp(yX)(x), t+s) \leq u(x,t) + \sum_{i=1}^{2} p_i y_i + \frac{1}{2}\sum_{i,j=1}^{3} r_{ij} y_i y_j + qs + o((|y|)^2 + s^2) \tag{4.6}$$

for some $p \in HM \oplus VM$, $q \in \mathbb{R}$ and $R = (r_{ij}) \in \mathbb{R}^{3,3}$ then:

$$q \leq \begin{cases} \sum_{i,j=1}^{3} \left(\delta_{ij} - \frac{p_i p_j}{|p_H|^2}\right) r_{ij} & \text{if}\, |p_H| \neq 0 \\ \sum_{i,j=1}^{3} (\delta_{ij} - \eta_i \eta_j)\, r_{ij} & \text{for some}\, |\eta| \leq 1, \text{ if}\, |p_H| = 0 \end{cases} \tag{4.7}$$

Definition 4.4. An analogous definition is provided for a viscosity supersolution.

Theorem 4.1.1. *The two definitions are equivalent.*

4.2 Existence of viscosity solutions

In this section we will prove the existence of viscosity solutions for the initial value problem of (4.3), (4.2):

$$u_t = \sum_{i,j=1}^{2} \left(\delta_{i,j} - \frac{X_i u X_j u}{|\nabla_0 u|^2} \right) X_i X_j u$$
$$u = g \quad \text{on} \quad \mathbb{R}^2 \times S^1 \times \{t = 0\}$$

We assume that:

$$g \quad \text{is constant on} \quad \{\mathbb{R}^2 \times S^1\} \cap \{|x| \geq S\} \tag{4.8}$$

for some constant $S > 0$ and additionally, for the moment at least, g is smooth. Our intention is to approximate (4.1),(4.2) by the PDE:

$$\frac{\partial}{\partial t} u^{\epsilon,\delta} = \sum_{i,j=1}^{3} A_{ij}^{\epsilon,\delta}(\nabla_\epsilon u^{\epsilon,\delta}) X_i^\epsilon X_j^\epsilon u^{\epsilon,\delta} \quad \text{in}\, x \in \mathbb{R}^2 \times S^1, \quad t > 0 \tag{4.9}$$

$$u^{\epsilon,\delta} = g \quad \text{on} \quad \mathbb{R}^2 \times S^1 \times \{t = 0\} \tag{4.10}$$
$$\text{where} \quad A_{ij}^{\epsilon,\delta}(\xi) = \left(\delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2 + \delta} \right)$$

for $0 < \delta < 1$, $\epsilon, \sigma > 0$ for all $\xi \in \mathbb{R}^2 \times S^1$ and $1 \leq i, j \leq 3$. Note that $A_{ij}^{\epsilon,\delta}(\cdot)$ are the coefficients of the approximating equations, and

$$A_{ij}^{\epsilon,\delta,\sigma}(\xi) = A_{ij}^{\epsilon,\delta}(\xi) + \sigma \delta_{ij}$$

Our solution will result as the limit of solutions of this regularized parabolic equation. We will now specify the meaning of the variables we will use:

- ϵ referes to the metric; since we are working in a sub-Riemannian manifold passing through this limit means we restrict ourselves to the horizontal bundle, which is the result we are interested in as we have pointed out in the previous chapters.

- δ is the parameter which regularises the PDE and is linked to the geometric interpretation of the differential problem: as has been pointed out in the Euclidean case in ([8]) by Evans and Spruck the solution of the regularised equation evolves according to its mean curvature and depends on a factor δ, which influences the evolution of the level sets.

$$\Gamma_t^\delta = \{y = (x, x_{n+1}) \in \mathbb{R}^{n+1} | x_{n+1} = \delta^{-1} u^\delta(x,t)\}$$

is a graph and if Γ_0 is the boundary of a smooth, bounded, simply connected open set we select a smooth function g with $g = 0$ on Γ_0. So Γ_0^δ is the graph $\{x_{n+1} = \delta^{-1} g(x)\}$ as drawn in the next figure.

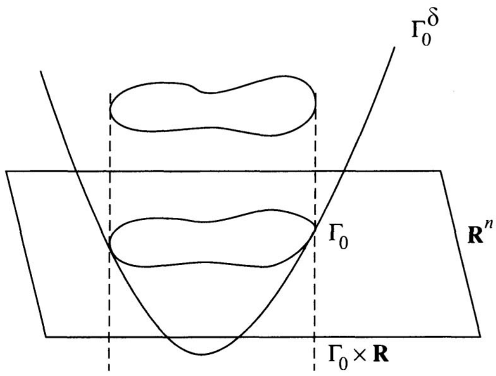


FIGURE 3

Figure 4.1: Geometrical interpretation of the regularisation through δ

For small δ, Γ_0^δ roughly approximates the cylinder $\Gamma_0 \times \mathbb{R}$, and the hope is that for moderate $t > 0$ and small $\delta > 0$ the smooth graph Γ_t^δ will be close to the cylinder $\Gamma_t \times \mathbb{R}$, where Γ_t denotes the evolution of Γ_0 via its mean curvature. The idea underlined in the Euclidean case by

Evans and Spruck and that we will generalize here is that the possibly singular behavior of $\{\Gamma_t\}_{t\geq 0}$ in $\mathbb{R}^n$will be approximated by the smooth evolution $\{\Gamma_t^\delta\}_{t\geq 0}$ in $\mathbb{R}^{n+1}$.

- σ represents a further regularisation which makes the differential operator parabolic. This condition makes the coefficients satisfy the coercivity condition, i.e. they are smooth and parabolic (we can estimate them from the low with the smaller eigenvalue, which is positive).

4.2.1 Analytical solution of the approximate equations

Before investigating the approximations (4.9),(4.10) analytically, we will state and prove a result that we will need for the proof of the existence:

Lemma 4.2.1. *Let X_1, X_2, X_3 be three vector fields which generate the Lie Algebra of $SE(2) = \mathbb{R}^2 \times S^1$:*

$$\begin{aligned} X_1 &= \cos\theta\partial_x + \sin\theta\partial_y \\ X_2 &= \partial_\theta \\ X_3 &= \sin\theta\partial_x - \cos\theta\partial_y \end{aligned}$$

Let Y_1, Y_2 and Y_3 be three other vector fields defined as follow:

$$\begin{aligned} Y_1 &= \partial_x \\ Y_2 = Y_1 + (x\cos\theta + y\sin\theta)X_3 &- (-\sin\theta x + y\cos\theta)X_1 \\ Y_3 &= \partial_y \end{aligned}$$

which also generate the Lie Algebra of $SE(2) = \mathbb{R}^2 \times S^1$. So X_i commutes with Y_i for $i = 1, 2, 3$.

Proof. We will calculate their Lie bracket, then we will restrict ourselves to the identity $(x, y, \theta) = (0, 0, 0)$ of the group $SE(2)$:

$$\begin{aligned} \big([X_1, Y_1]\big)\Big|_0 &= (\cos\theta\partial_x + \sin\theta\partial_y)(\partial_x) - (\partial_x)(\cos\theta\partial_x + \sin\theta\partial_y) \\ &= \cos\theta\partial_{xx} + \sin\theta\partial_{yx} - \cos\theta\partial_{xx} - \sin\theta\partial_{xy} = 0 \end{aligned}$$

since when we give local Euclidean coordinates, second order derivatives commute for Cauchy-Schwarz.

$$\begin{aligned}
\big([X_2,Y_2]\big)\Big|_0 &= X_2(Y_2)-Y_2(X_2)\\
&= (\partial_\theta)(\partial_x + x\cos\theta\sin\theta\partial_x + y\sin^2\theta\partial_x - x\cos^2\theta\partial_y +\\
&- y\sin\theta\cos\theta\partial_y + x\sin\theta\cos\theta\partial_x - y\cos^2\theta\partial_x + x\sin^2\theta\partial_y +\\
&- y\cos\theta\sin\theta\partial_y) - (\partial_x + x\cos\theta\sin\theta\partial_x + y\sin^2\theta\partial_x +\\
&- x\cos^2\theta\partial_y - y\sin\theta\cos\theta\partial_y + x\sin\theta\cos\theta\partial_x +\\
&- y\cos^2\theta\partial_x + x\sin^2\theta\partial_y - y\cos\theta\sin\theta\partial_y)(\partial_\theta)\\
&= 2x\cos\theta\sin\theta\partial_y - y\cos^2\theta\partial_y + y\sin^2\theta\partial_y + x\cos^2\theta\partial_x +\\
&- x\sin^2\theta\partial_x + 2y\cos\theta\sin\theta\partial_x + 2x\cos\theta\sin\theta\partial_y +\\
&+ y\sin^2\theta\partial_y - y\cos^2\theta\partial_y\\
&= (-2y\cos(2\theta)\partial_y + x\cos(2\theta)\partial_x + 4x\cos\theta\sin\theta\partial_y +\\
&+ 2y\cos\theta\sin\theta\partial_x)\Big|_{(x,y,\theta)=(0,0,0)} = 0
\end{aligned}$$

$$\begin{aligned}
[X_3,Y_3] &= (\sin\theta\partial_x - \cos\theta\partial_y)(\partial_y) - (\partial_y)(\sin\theta\partial_x - \cos\theta\partial_y)\\
&= \sin\theta\partial_{xy} - \cos\theta\partial_{yy} - \sin\theta\partial_{xy} + \cos\theta\partial_{yy} = 0
\end{aligned}$$

□

Theorem 4.2.2. *For any $g \in C^\infty(\mathbb{R}^2\times S^1)$ there exists a unique solution $u^{\epsilon,\delta}\in C^{2,\alpha}(\mathbb{R}^2\times S^1\times[0,\infty))$ of the initial value problem (4.9),*

$$u^{\epsilon,\delta}(x,0) = g(x) \quad \textit{for all} \quad x\in\mathbb{R}^2\times S^1 \tag{4.11}$$

Moreover, for all $t>0$ one has:

$$\|u^{\epsilon,\delta}(\cdot,t)\|_{\mathcal{L}^\infty(\mathbb{R}^2\times S^1)} \le \|g\|_{\mathcal{L}^\infty(\mathbb{R}^2\times S^1)} \tag{4.12}$$

$$\|\tilde\nabla_\epsilon u^{\epsilon,\delta}(\cdot,t)\|_{\mathcal{L}^\infty(\mathbb{R}^2\times S^1)} \le \|\tilde\nabla_\epsilon g\|_{\mathcal{L}^\infty(\mathbb{R}^2\times S^1)} \tag{4.13}$$

where for $\tilde\nabla_\epsilon$ we denote $\tilde\nabla_\epsilon = (Y_1,Y_2,Y_3)$.

Before we start proving the theorem we will give some preliminary results for the cylinders which justify the use of the "parabolic maximum principle" in the proof. The idea is that for each closed ball $B(0,r)$ we can consider a parabolic cylinder $B(0,r) \times [0,T]$. For each cylinder we can assign an initial data on the lateral cover which does not depend on time: it is our initial data g for $t = 0$, which we have defined at the beginning. For this reason we obtain a parabolic cylinder in which the lateral cover has a data that we can estimate with the norm of g. We now have a set where the maximum principle is applicable[1], and since this gives a estimate which does not depend on the cylinder (and it does not depend on whether the operator is degenerate or not), it remains true also if we send the radium to ∞.

Definition 4.5. The Hölder space $C^{2,\alpha}, \alpha \in \{0,1\}$ is the set of functions having continous derivatives up to order 2 and such that the 2th partial derivatives are Hölder continous with exponent α, i.e. $\|f(x) - f(y)\|_a \leq C\|x-y\|_a^\alpha$. We pose $a = k + \alpha$, k a non-negative integer, $\alpha \in (0,1]$. Then:

$$\|f\|_a = \sum_{|\beta+2j \leq k|} sup\left|D_x^\beta D_t^j f\right| + [f]_a + \langle f\rangle_a$$

where

$$\langle f\rangle_a = \sum_{|\beta+2j=k-1|} \langle D_x^\beta D_t^j f\rangle_{\alpha+1}$$

$$[f]_a = \sum_{|\beta+2j=k|} \left[D_x^\beta D_t^j f\right]_\alpha$$

Note that the derivatives with respect to t weigh differently from the derivatives calculate with respect to x. $\|\cdot\|_a$ defines a norm, for further references see Lieberman [18] (pages 46-47). The spatial norm which referes to the variable x is the Riemannian we have defined in chapter 2 as

$$\|\alpha_1 X_1 + \alpha_2 X_2 + \varepsilon\alpha_3 X_3\|_g = \sqrt{\alpha_1^2 + \alpha_2^2 + \varepsilon^2\alpha_3^2}$$

[1]The maximum principle is a property of solutions to partial differential equations of the parabolic type. The result states that the maximum of a function in a domain is to be found on the boundary of that domain. In our case we have a bounded cylinder on which the operator is parabolic, so we can apply the maximum principle.

Theorem 4.2.3. *Consider a smoothed cylinder with base $B(0,r)$ and height $T>0$; we assign a $C^{2,\alpha}$, $(\alpha \in (0,1))$ initial data g on the lateral cover and on the base (in our setting we can assume g constant on the lateral cover), and M is a positive constant. Let us consider the homogeneous problem*

$$\sum_{i,j=1}^{3} A_{ij}^{\epsilon,\delta}(\nabla_\epsilon u^{\epsilon,\delta}) X_i^\epsilon X_j^\epsilon u^{\epsilon,\delta} - \frac{\partial}{\partial t} u^{\epsilon,\delta} = 0$$

$$\textit{where} \quad A_{ij}^{\epsilon,\delta}(\xi) = \left(\delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2 + \delta}\right)$$

$$\textit{and} \quad A_{ij}^{\epsilon,\delta,\sigma}(\xi) = A_{ij}^{\epsilon,\delta}(\xi) + \sigma \delta_{ij}$$

with the assigned initial data. Assume that

$$\|u^{\epsilon,\delta,\sigma}\|_\infty \leq M$$

Then the problem has a $C^{2,\alpha}$ solution $u^{\epsilon,\delta,\sigma}$.

Observation 11. Let us explicitly note that norms of $u^{\epsilon,\delta,\sigma}$ are not uniform in the parameters.

Corollary 4.2.4. *Note that in the previous hypothesis for a parabolic operator on a bounded smoothed cylinder is possible to apply the maximum principle, then we obtain that:*

$$M \leq \|g\|_\infty$$

Theorem 4.2.5. *Let $\xi \mapsto A_{ij}^{\epsilon,\delta,\sigma}(\xi)$ be the coefficients of our parabolic operator, defined in (4.9). If they are C^∞ in the variable ξ and g is defined on $B(0,r)$ and bounded by a positive constant M, then there exist a solution $u^{\epsilon,\delta,\sigma}$ which is smooth on the interior of the domain.*

Theorem 4.2.6 (Passage to the limit for $r \to \infty$ of the cylinders)**.** *Let us now consider a sequence of solutions $(u_r^{\epsilon,\delta,\sigma})_{r>0}$, each one defined on the cylinder $B(0,r) \times [0,T]$ such that*

$$\|u_r^{\epsilon,\delta,\sigma}\|_\infty \leq C$$

for each r. Then we can pass to the limit for $r \to \infty$, i.e. there exist a solution $u^{\epsilon,\delta,\sigma}$ defined on $\mathbb{R}^n \times [0,T]$ such that

$$\|u^{\epsilon,\delta,\sigma}\|_\infty \le C$$

Observation 12. We observe that C does not depend on ϵ, δ, σ. On the contrary all the other estimates depend on the parabolic coefficients and are not uniform in ϵ, δ, σ.

Let us now prove the first part of the existence theorem 4.2.2, which investigates the approximations (4.9),(4.10) analytically. The previous statements about the cylindric sets allow to generalize the estimates, based on our initial data, for the entire space $\mathbb{R}^2 \times S^1$.

Proof. We follow the analogue Euclidean result proved in [8] [theorem 4.1].

1. For $\sigma > 0$, consider the PDE:

$$\frac{\partial}{\partial t}u^{\epsilon,\delta,\sigma} = \sum_{i,j=1}^{3} A_{ij}^{\epsilon,\delta,\sigma}(\nabla_\epsilon u^{\epsilon,\delta,\sigma})X_i^\epsilon X_j^\epsilon u^{\epsilon,\delta,\sigma} \quad \text{in } x \in \mathbb{R}^2 \times S^1 \times [0,\infty) \tag{4.14}$$

   with initial data:

$$u^{\epsilon,\delta,\sigma}(x,0) = g(x), \quad \text{for all} \quad x \in \mathbb{R}^2 \times S^1 \tag{4.15}$$

   The smooth bounded coefficients $A_{ij}^{\epsilon,\delta,\sigma}$ also satisfy the uniform parabolic condition:

$$\sigma|\xi|^2 \le A_{ij}^{\epsilon,\delta,\sigma}(p)\xi_i\xi_j \qquad (\xi \in \mathbb{R}^3) \tag{4.16}$$

   for each $p \in \mathbb{R}^3$. As we have seen observation (4.2.5) shows the existence of a smooth bounded solutions $u^{\epsilon,\delta,\sigma}$ on varying σ (which are also unique). (For more references see Ladyzenskaja, Solonnikov, Ural'tseva [17]).

   From what we have said before based on the previous results we obtain[2]

$$\|u^{\epsilon,\delta,\sigma}(\cdot,t)\|_{\mathcal{L}^\infty(\mathbb{R}^2\times S^1)} \le \|g\|_{\mathcal{L}^\infty(\mathbb{R}^2\times S^1)} \tag{4.17}$$

[2]The L-infinite norm of a function is defined as:

$$\|g\|_\infty = inf\{C \ge 0 : |g(x)| \le 0 a.e.\}$$

2. As we have seen in Lemma (4.2.1), Y_1, Y_2, Y_3 commute with the left-invariant vector fields X_1, X_2, X_3 (which generate the Lie algebra of $\mathbb{R}^2 \times S^1$), so we can differentiate (4.14) along the directions $\{Y_i\}_{i=1,2,3}$ and obtain the new equation:

$$\frac{\partial}{\partial t} w = \sum_{i,j=1}^{3} \left[A_{i,j}^{\epsilon,\delta,\sigma} (\nabla_\epsilon u^{\epsilon,\delta,\sigma}) X_i^\epsilon X_j^\epsilon w + (\delta_{\xi_k} A_{i,j}^{\epsilon,\delta,\sigma})(\nabla_\epsilon u^{\epsilon,\delta,\sigma}) X_i^\epsilon X_j^\epsilon u^{\epsilon,\delta,\sigma} X_k w \right] \tag{4.18}$$

where $w = Y_i u^{\epsilon,\delta,\sigma}$, for all $i = 1, 2, 3$. The parabolic maximum principle applied to the previous equation yields:

$$\|\tilde{\nabla}_\epsilon u^{\epsilon,\delta,\sigma}(\cdot, t)\|_{\mathcal{L}^\infty(\mathbb{R}^2 \times S^1)} \leq \|\tilde{\nabla}_\epsilon g\|_{\mathcal{L}^\infty(\mathbb{R}^2 \times S^1)} \tag{4.19}$$

and since the $\{Y_i\}_{i=1,2,3}$ form the basis of the tangent bundle of $\mathbb{R}^2 \times S^1$, the previous estimate leads to:

$$\|\nabla_\epsilon u^{\epsilon,\delta,\sigma}(\cdot, t)\|_{\mathcal{L}^\infty(\mathbb{R}^2 \times S^1)} \leq C \|\tilde{\nabla}_\epsilon g\|_{\mathcal{L}^\infty(\mathbb{R}^2 \times S^1)} \tag{4.20}$$

where C is a positive constant depending only on $\mathbb{R}^2 \times S^1$.

3. The smooth bounded coefficients $\{A_{i,j}^{\epsilon,\delta,\sigma}\}$ satisfy the coercivity condition (4.16) , so that

$$\left(1 - \frac{M^2}{M^2 + \delta}\right)|\xi|^2 \leq \sum_{i,j=1}^{3} A_{i,j}^{\epsilon,\delta,\sigma}(\xi)\xi_i \xi_j \leq 3|\xi|^2$$

for $\xi \leq M$ uniformly in σ. Estimates (4.17),(4.19),(4.20) are extended by the theory of parabolic cylinders for all derivatives of $u^{\epsilon,\delta,\sigma}$ which are uniform in $0 < \sigma < 1$. We have used the parameter σ in order to regularise the equation, making the coefficients satisfy the coercivity condition which leads to estimates on the derivative which we use to conclude the proof.

If we consider the L-infinite norm on $\mathbb{R}^2 \times S^1$ we are considering it with respect to the norm defined on our space. In virtue of the previous consideration about cylinders, we can extend the estimate to the L-infinite norm on $\mathbb{R}^2 \times S^1$, because we are able to find such a constant.

4. We use (4.17) and (4.20) and the Ascoli-Arzelà theorem[3] to show that

$$u^{\epsilon,\delta,\sigma} \to u^{\epsilon,\delta}$$

uniformly in the $C^{1,0}$ (Lipschitz) norm when $\sigma \to 0$ for a smooth function $u^{\epsilon,\delta}$ solving (4.9),(4.10).

□

4.2.2 Passage to the limit

In order to extend to our setting Evans and Spruck's argument in the proof of [8][Theorem 4.1], after proving the existence of approximate solutions passing to the limit for $\sigma \to 0$, we need to pass to the limit for $\delta \to 0$ and $\epsilon \to 0$. The first limit guarantees the passage from approximate solutions to (4.1) and the second limit allows the passage to the horizontal bundle, (4.3). The advantage is that the estimate (4.20) is stable with respect to both $\delta \to 0$ and $\epsilon \to 0$.

Theorem 4.2.7. *Assume that $g \in C(\mathbb{R}^2 \times S^1)$ is continuous and satisfies (4.8). Then there exists a viscosity solution $u \in C^{1,0}$ of (4.3),(4.2) such that:*

$$u \textit{ is constant on } \quad \mathbb{R}^2 \times S^1 \times [0,\infty) \cap \{|x| + t \geq R\} \tag{4.21}$$

for $R > 0$, depending only on the constant S from (4.8).

Note that $A^\epsilon = (a^\epsilon_{ij})$ is the matrix of coefficients of $X^\epsilon_1, X^\epsilon_2, X^\epsilon_3$ in exponential coordinates, i.e. $X^\epsilon_i = \sum_{k=1}^3 a^\epsilon_{ik}\delta_{x_k}$.

[3]The Arzelà-Ascoli theorem is a fundamental result of mathematical analysis giving the necessary and sufficient conditions for deciding whether every sequence of a given family of real-valued continuous functions defined on a closed and bounded interval has a uniformly convergent subsequence (note that the group about which we are investigating the existence of a solution is compact)

Proof. 1. Since g is constant, from the previous consideration we can assume that ∇g is bounded, where ∇g denotes the Euclidean gradient of g. Employing estimates (4.12),(4.13) we can extract two sequences $\{\epsilon_k\}, \{\delta_k\} \to 0$ of positive numbers such that $\frac{\epsilon_k}{\delta_k} \to 0$ and for which we have a corresponding sequence of smooth solutions to (4.9): $\{u^k = u^{\epsilon_k,\delta_k}\}_{k\in\mathbb{N}} \subset \{u^{\epsilon,\delta}\}$. These solutions with initial data g are such that when $\epsilon_k, \delta_k \to 0$ we have $u^k \to u$, locally uniformly in δ,ϵ on $\mathbb{R}^2 \times S^1 \times [0,\infty)$, where u is a bounded, Lipschitz function (i.e. $\alpha = 1$, with respect to the distance we have defined).

2. The first argument we need to proove is that u is a viscosity solution of (4.3),(4.2). For this, let $\phi \in C^\infty(\mathbb{R}^2 \times S^1 \times [0,\infty))$ and suppose $u - \phi$ has a strict local maximum at a point $(x_0, t_0) \in \mathbb{R}^2 \times S^1 \times [0,\infty)$. As $u^k \to u$ uniformly near (x_0, t_0), $u^k - \phi$ has a local maximum at a point (x_k, t_k), with

$$(x_k, t_k) \to (x_0, t_0) \quad \text{as} \quad k \to \infty \tag{4.22}$$

Since u^k and ϕ are smooth, we have[4]:

$$\nabla_E u^k = \nabla_E \phi \ , \ \partial_t u^k = \delta_t \phi \ \text{ and } \ D^2_E(u^k - \phi) \leq 0 \ \text{ at } \ (x_k, t_k)$$

Thus (4.9) implies:

$$\partial_t \phi - \big(\delta_{ij} - \frac{X_i^{\epsilon_k}\phi X_j^{\epsilon_k}\phi}{|\nabla_{\epsilon_k}\phi|^2 + \delta_k^2}\big) X_i^{\epsilon_k} X_j^{\epsilon_k}\phi \leq 0 \ \text{ at } \ (x_k, t_k) \tag{4.23}$$

We substitute this expression with the one that involves the coefficients $A^{\epsilon,\delta}_{i,j}$ so that at (x_k, t_k)

$$\begin{aligned} \partial_t \phi \quad &- \quad A^{\epsilon_k,\delta_k}_{i,j}(\nabla_{\epsilon_k}\phi) X_i^{\epsilon_k} X_j^{\epsilon_k}\phi & (4.24)\\ &\leq \quad \partial_t u^k - A^{\epsilon_k,\delta_k}_{i,j}(\nabla_{\epsilon_k} u^k) X_i^{\epsilon_k} X_j^{\epsilon_k}(u^k + \phi - u^k) \leq 0 & (4.25) \end{aligned}$$

Suppose first $\nabla_0\phi(x_0, t_0) \neq 0$. Then $\nabla_0\phi(x_k, t_k) \neq 0$ for large k. We consequently may pass to the limits for $k \to \infty$ in (4.25), recalling

[4]Note that ∇_E and D^2_E are the Euclidean gradient and the Euclidean differential.

(4.22) to deduce:

$$\partial_t \phi \leq \sum_{i,j=1}^{2} \big(\delta_{ij} - \frac{X_i\phi X_j\phi}{|\nabla_0\phi|^2}\big) X_i X_j \phi \text{ at } (x_0, t_0) \tag{4.26}$$

which means that u satisfies the definition of viscosity subsolution.
If $\nabla_0\phi(x_0, t_0) = 0$ then we set

$$\eta^k = \frac{\nabla_{\epsilon_k}\phi(x_k, t_k)}{\sqrt{|\nabla_{\epsilon_k}\phi(x_k, t_k)|^2 + \delta_k^2}}$$

There exists $\eta \in \mathbb{R}^n$ such that $\eta^k \to \eta$. Notice that for $j = m+1, \ldots, n$ one has:

$$|(\eta^k)_j| = \frac{\epsilon_k |X_j\phi(x_k, t_k)|}{\sqrt{|\nabla_{\epsilon_k}\phi(x_k, t_k)|^2 + \delta_k^2}} \leq \frac{(\epsilon_k/\delta_k)|X_j\phi(x_k, t_k)|}{\sqrt{(\epsilon_k/\delta_k)^2 \sum_{i=1}^2 (X_i\phi(x_k, t_k))^2 + 1}}$$

Since the expression vanishes as $k \to \infty$ we have $\eta_j = 0$ for $j = m+1, \ldots, n$. ($j = 3$ in our case, since m=2 and n=3 are the dimension of the horizontal and whole tangent bundle). The PDE (4.25) now reads as:

$$\partial_t\phi(x_k, t_k) - \sum_{i,j=1}^{3} (\delta_{ij} - \eta_i^k \eta_j^k) X_i^{\epsilon_k} X_j^{\epsilon_k} \phi(x_k, t_k) \leq 0$$

so as $k \to \infty$ we obtain

$$\partial_t\phi(x_0, t_0) \leq \sum_{i,j=1}^{2} (\delta_{ij} - \eta_i\eta_j) X_i X_j \phi(x_0, t_0) \tag{4.27}$$

concluding the proof for the case in which $u - \phi$ has a local strict maximum at point (x_0, t_0). If $u - \phi$ has a local maximum, but not necessarily a strict local maximum at (x_0, t_0), we can repeat the argument above replacing $\phi(x, t)$ with

$$\tilde{\phi}(x, t) = \phi(x, t) + |x - x_0|^4 + (t - t_0)^4$$

again to obtain (4.26),(4.27). Consequently u is a weak subsolution. That u is a weak supersolution follows analogously.

□

Chapter 5

Applications to visual perception

In this chapter we present an implementation of the perceptual completion model proposed by Citti and Sarti in [4][5] we have analyzed in the previous chapters. In Citti and Sarti model an image is lifted onto a surface in the $SE(2)$ space. The completion was achieved in related work as [22] by means of a propagation process modelled as a two step algorithm inspired by neural architectures. As we have noted in Chap. 3, the algorithm converges to a diffusion driven mean curvature flow in the sub-Riemannian settings and this is the reason why the mean curvature flow was proposed in order to provide completion. The result of the curvature flow we visualize through the level set method is a minimal surface in the sub-Riemannian metric.

5.1 Citti and Sarti cortical model

Let us start by recalling Citti and Sarti model. An image I can be represented as a bounded function defined on a domain $M \subset \mathbb{R}^2$, $I : M \to \mathbb{R}^+$. Points of M have coordinates (x, y). As we have previously seen, the output of the simple cells in response to a visual stimulus I is a function u defined on the 3D cortical space. This function can be interpreted as the cortical activity. The maximal selection mechanism then detects, at every point (x, y) pertaining to a level line of I, the orientation $\theta(x, y)$ of that level line. At every point of the image we detected the tangent direction to the level lines $(I_y, -I_x)$ where I_x and I_y are the components of the image gradient. If θ is the angle between the tangent and the x-axis the tangent can be rewritten as $\cos(\theta), \sin(\theta))$. Then

$$\theta(x, y) = -\arctan \frac{I_x}{I_y}, \quad \theta \in S^1$$

This surface Σ is the lifting of every level line in the image. This point of view allows us to understand a remarkable property of Σ, which is that since two level lines of an image never cross, neither do its lifted level lines.

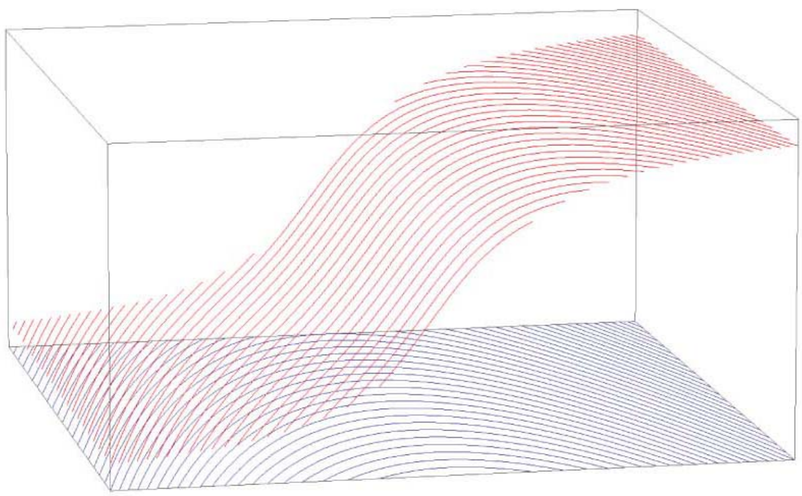


Figure 5.1: An image is lifted into the space of positions and orientations $\mathbb{R}^2 \times S^1$. The resulting surface is foliated by the lifting of the image level lines.

5.2 Level set method for mean curvature flow

We have already discuss the level set method in Chapter 3 for mean curvature flow. The level set approach consists in studying a PDE describing the evolution of a function $u(x, t)$ such that $\Gamma_t = \{x \in M | u(x, t) = 0\}$. The study of mean curvature motion of an hypersurface is reconducted to the motion of its level sets by mean curvature: this approach recostruct the evolution of an hypersurface u by mean curvature flow through the analysis of its level sets. When we want to complete an image, we first lift its level sets, as we can see in Fig 5.1, i.e. we lift the gradient orientation. In this process we lost information related to the color. For this reason we need to codify also the tone of gray: this means we introduce a supplementary surface defined on our lifted surface which contains the missing informations.

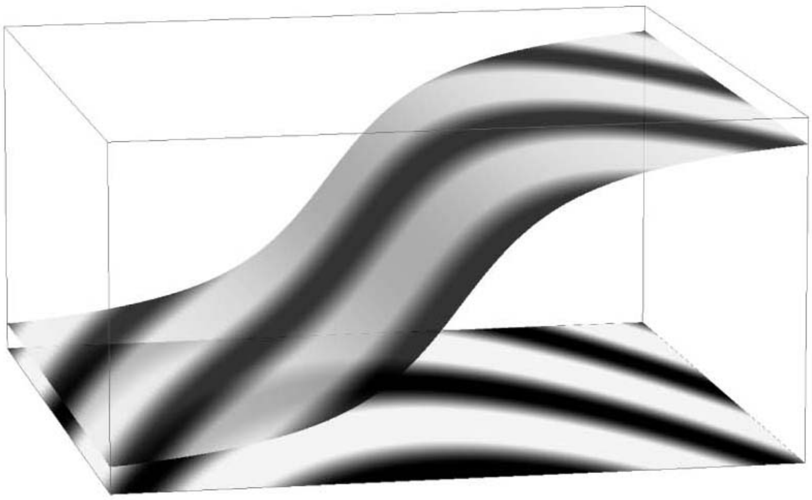


Figure 5.2: A supplementary surface which contains information about the color is defined under the lifted surface to complete missing data.

The algorithm is divided in two parts: we first lift the surface through the gradient orientation, simulating the cortical mechanisms of non-maxima suppression and visual signal propagation. This propagation can be modelled by a mean curvature flow equation for the surface. Then we need to complete the information we miss such as the color: for this reason we complete it applying the Laplace-beltrami operator on the lifted image.

5.3 Experiments and results

We apply this algorithm to a modified surface which needs to be completed. As we can see a hole is present at the center of the image. The first part of the algorithm we have described before completes the central part with a mean curvature flow.

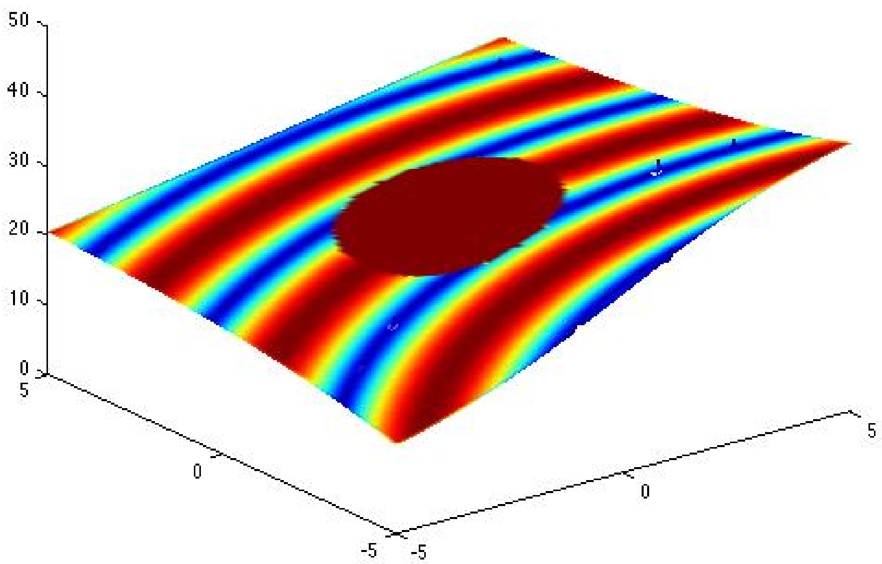


Figure 5.3: Surface with a missing hole which needs a completion process

We test the Riemannian approximation of the equation and also the sub-Riemannian expression. As we can see from the image the Riemannian approximation is indeed more stable, the other presents a noise which would propagate when we increase the number of iterations.

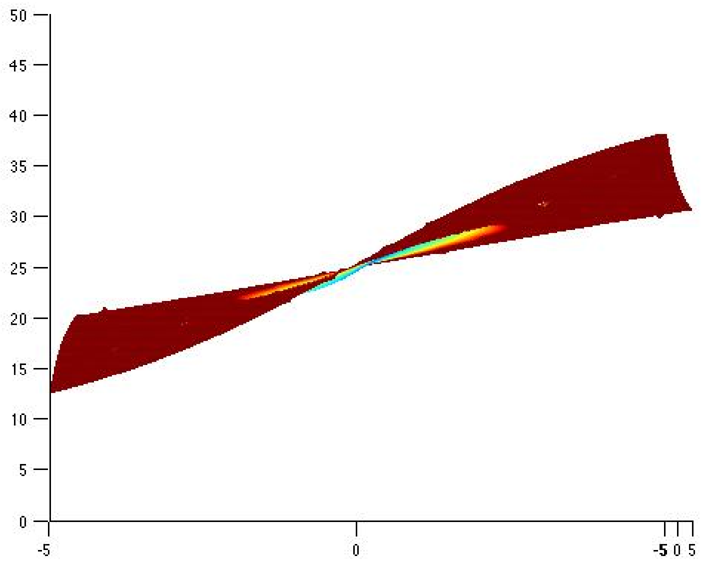


Figure 5.4: Mean curvature flow performed with a Riemannian approximation

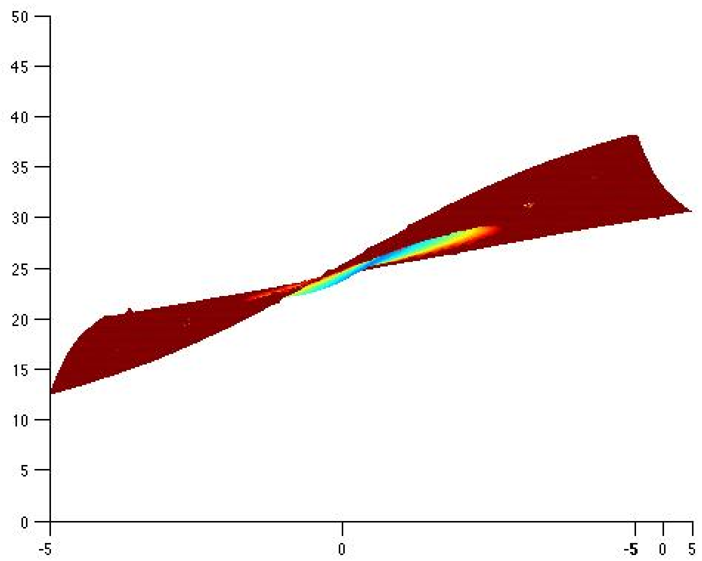


Figure 5.5: Mean curvature flow performed without a Riemannian approximation

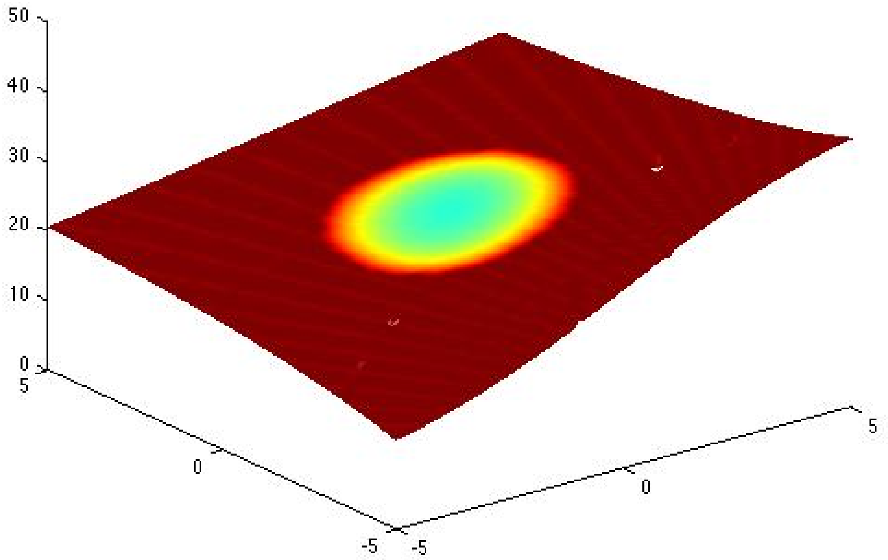


Figure 5.6: Completed surface

Last step will be the completion of the missed information for what concerns the color of this last figure, which will be performed through an implementation of the Laplace-Beltrami operator.

Bibliography

[1] J. Bence, B.Merriman and S.Osher, *Diffusion generated motion by mean curvature*, Computational Crystal growers Workshop,J. Taylor Sel. Taylor Ed, 1992.

[2] W. Bosking,Y. Zhang, B. Schofield, D. Fitzpatrick, *Orientation selectivity and the arrangement of horizontal connections in tree shrew striate cortex*, J. Neurosi, pag 2112-2117, 1997.

[3] L. Capogna and G. Citti,*Generalized mean curvature flow in Carnot Groups*, ArXiv.org, 26 Aug 2008.

[4] G. Citti and A. Sarti, *A cortical based model of perceptual completion in the roto-translation space*, in Proceeding of the Workshop on Second Orden Subelliptic Equations and Applications, 2003.

[5] G. Citti and A. Sarti,*A cortical based model of perceptual completion in the roto-translation space*, Journal of Mathematical Imaging and Vision, 24(3):307-326, may 2006.

[6] N. Dirr, F. Dragoni and M. von Renesse, *Evolution by mean curvature flow in sub-Riemannian geometries: a stochastic approach*, Commun. Pure Appl. Anal., 9, 307?326, 2010.

[7] R. Duits, E. Franken, *Left-invariant stochastic evolution equations on SE(2) and its applications to contour enhancement and contour completion via invertible orientation scores*, Arxiv e-prints, November 2007

[8] L.C. Evans and J. Spruck, *Motion of level sets by mean curvature*, in I.J, Diff. Geom. 33, 3, 635-681, 1991.

[9] F.Ferrari, Q. Liu and J. J. Manfredi, *On the horizontal mean curvature flow for axisymmetric surfaces in the Heisenberg group*, in Discrete and Continuous Dynamical Systems 34:7, 2779-2793, 1st December 2013.

[10] D. Field, A Heyes and R.F. Hess,*Contour integration by the human visual system: evidence for a local association field*, Vision Research, 33, pag 173-193, 1993.

[11] B. Franchi, R. Serapioni and F. Serra Cassano, *Regular hypersurfaces, Intrinsic perimeter and implicit function theorem in Carnot groups*, in Comm. Analysis and Geometry, 2002.

[12] R.K. Hladky,*Connections and curvature in a sub-Riemannian geometry*, Houston J. Math, 38, no. 4, 1107-1134, 2012.

[13] W.C. Hoffman, *The visual cortex is a contact bundle*, Applied Mathematics and Computation, vol 32, (89), 137-167, 1989.

[14] D. Hubel and T. Wiesel, *Receptive fiedls, binocular interaction and functional architecture in the cat's visual cortex*, The journal of physiology, 160: 106-154, January 1962.

[15] D. Hubel and T. Wiesel, *Ferrier lecture: Functional architecture of macaque monkey visual cortex*,in Royal Society of London Proceedings Series B, 198:1-59, May 1977.

[16] T. Ilmanen, *Generalized flow of sets by mean curvature on a manifold*, Indiana Univ. ath. J. 41,3, 147-171, 1992.

[17] O. A. Ladyženskaja, SOLONNIKOW,V.A. AND URAL'CEVA,N.N., *Linear and quasilinear equations of parabolic type*, Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1967

[18] G.M. Lieberman, *Second order Parabolic differential equations*, World scientific, 1996

[19] S. Oscher and J.A. Sethian,*Fronts propagating with curvature dependent speed: algorithms based on Hamilton-Jacobi formulations*, J.Computational Phys. 79, 12-49, 1988.

[20] P. Petersen,*Riemannian geometry*, Springer, 1998.

[21] J. Petitot, Y. Tondut, *Vers une Neuro-geometrie, Fibrations corticales, structures de contact et contours subjectifs modaux*, Mathematiques, Informatique et Sciences Humaines, EHESS, Paris, 145, 5-101, 1998.

[22] G. Sanguinetti, *Invariant models of vision bewteen phenomenology, image statistics and neurosciences*, Universidad de la Republica, Montevideo, 2011. Cortona, June, 15-21 2003.

[23] C. Senni Guidotti Magnani,*Prescribed mean curvature graphs on exterior domains of the hyperbolic plane*, Università di Bologna, Bologna, 2010.

[24] L. W. Tu,*An introduction to Manifolds*, Second Edition, Springer, 2011

[25] V.S. Varadarajan,*Lie groups, Lie Algebras, and Their Representations*, Springer Verlag, 1984

[26] D.R. Wilkins,*A course in Riemannian geometry*, David R. Wilkins, 2005

[27] S.W. Zucker, *The curve indicator random field: curve organization via edge correlation*, in Perceptual Organization for Artificial Vision Systems, Kluwer Academic, 2000, 265-288.

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