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Horizon entropy from scratch

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Riassunto della tesi

Scopo di questo lavoro di tesi è lo studio di alcune proprietà delle teorie generali della gravità in relazione alla meccanica e la termodinamica dei buchi neri. In particolare, la trattazione che seguirà ha lo scopo di fornire un percorso autoconsistente che conduca alla nozione di entropia di un orizzonte descritta in termini della carica di Noether associata all'invarianza del funzionale d'azione, che descrive la teoria gravitazionale in considerazione, per trasformazioni di coordinate generali. Si presterà particolare attenzione ad alcune proprietà geometriche della Lagrangiana, proprietà che sono indipendenti dalla particolare forma della teoria che si sta prendendo in considerazione; trattasi cioè non di proprietà dinamiche, legate cioè alla forma delle equazioni del moto del campo gravitazionale, ma piuttosto caratteristiche proprie di qualunque varietà rappresentante uno spaziotempo curvo. Queste caratteristiche fanno sì che ogni teoria generale della gravità possieda alcune grandezze definite localmente sullo spaziotempo, in particolare una corrente di Noether e la carica ad essa associata. La forma esplicita della corrente e della carica dipende invece dalla Lagrangiana che si sceglie di adottare per descrivere il campo gravitazionale. Il lavoro di tesi sarà orientato prima a descrivere come questa corrente di Noether emerge in qualunque teoria della gravità invariante per trasformazioni generali e come essa viene esplicitata nel caso di Lagrangiane particolari, per poi identificare la carica ad essa associata come una grandezza connessa all'entropia di un orizzonte in qualunque teoria generale della gravità.

Lo schema della tesi è il seguente:

Capitolo 1: Viene ricavata l'identità di Bianchi generalizzata per teorie generali della gravità invarianti per diffeomorfismi. Viene sottolineato che l'identità di Bianchi è una relazione off-shell unicamente dovuta all'invarianza dell'azione sotto trasformazioni arbitrarie delle coordinate che esprime nient'altro che la covarianza generale della teoria, proprietà questa assolutamente indipendente dalla forma della Lagrangiana.

Capitolo 2: Viene ricavata l'espressione della corrente conservata associata all'invarianza dell'azione per diffeomorfismi generali. Si discute nel dettaglio la proprietà per cui questa corrente è conservata off-shell e come ciò viene interpretato alla luce del teorema di Noether.

Capitolo 3: Si dà l'espressione esplicita per la corrente ricavata nel capitolo precedente nel caso di Lagrangiane generali con dipendenza arbitraria dal tensore di Riemann ma non dalle sue derivate e per la Lagrangiana di Hilbert-Einstein. La carica associata viene calcolata esplicitamente per la Relatività Generale su un orizzonte a simmetria sferica di metrica assegnata.

Capitolo 4: Viene fornita un'interpretazione fisica alla carica calcolata nel capitolo 3, precisamente andando on-shell, ossia utilizzando le equazioni del moto per il campo gravitazionale. Si affronterà il caso di Lagrangiane generali e quindi in Relatività Generale verrà mostrato che la carica associata ad un orizzonte a simmetria sferica coincide con l'entropia di Bekenstein-Hawking.

Abstract

In this thesis, some features of general theories of gravity will be reviewed in relation to the mechanics and the thermodynamics of black holes. In particular, the entropy associated to the event horizon of a black hole can be described in terms of the conserved charge that comes from the invariance of the action functional describing the theory under general coordinate transformations. The attention will be focused especially on the general geometric properties of the Lagrangian, which are independent on the theory of gravity taken in account, i.e. they are not dynamical properties of the theory but rather intrinsic properties of the manifold representing curved spacetime. These properties make any general theory of gravity to possess quantities which are locally defined on the spacetime, in particular a Noether current and the corresponding charge. The explicit form of the current and the charge depend of the form of the Lagrangian chosen for describing the gravitational field. The thesis will be first describe how the current comes up in any diffeomorphism invariant theory of gravity, eventually its form will be given in the case of particular Lagrangians, and afterwards the charge will be identified as a quantity connected to the horizon entropy in any general theory of gravity. The scheme of the thesis is the following

Chapter 1: The generalized Bianchi identity will be derived for diffeomorphism invariant general theories of gravity. It will be pointed out that the Bianchi identity is an off-shell relation that comes from the variation of the action under arbitrary transformations of the coordinates. It expresses nothing but the general covariance of the theory, hence it is independent of the form of the Lagrangian.

Chapter 2: The conserved current associated to the diffeomorphism invariance of the theory will be studied. In particular, it will be stressed that such a current is off-shell conserved. We will face this feature in relation with the Noether's theorem.

Chapter 3: The particular form of the conserved current will be given here for general gravitational actions with arbitrary dependence on curvature tensor but not on its derivatives and Hilbert-Einstein action. The associated conserved charge will be computed in the case of General Relativity, on a spherically symmetric horizon of given background metric.

Chapter 4: A physical interpretation will be provided for the charge computed in chapter 3 going on-shell, i.e. using the equations of motion for the gravitational field. After a discussion involving general theories of gravity, it will be shown that the charge for a spherically symmetric horizon in General Relativity is the Bekenstein-Hawking entropy.

Introduction

Undoubtedly, one of the most remarkable developments in theoretical physics to have occurred during the past twenty five years was the discovery of a close relationship between certain laws of black hole physics and the ordinary laws of thermodynamics. The existence of this close relationship between these laws may provide us with a key to our understanding of the fundamental nature of black holes in a quantum theory of gravity, as well as to our understanding of some aspects of the nature of thermodynamics itself.

It was first pointed out by Bekenstein [2] that a close relationship might exist between certain laws satisfied by black holes in classical general relativity (GR) and the ordinary laws of thermodynamics. The area theorem of classical GR [6] states that the area, A , of a black hole can never decrease in any process

$$\Delta A \geq 0 \tag{1}$$

Bekenstein noted that this result is closely analogous to the statement of ordinary second law of thermodynamics: The total entropy, S , of a closed system never decreases in any process

$$\Delta S \geq 0 \tag{2}$$

Thus, Bekenstein proposed that the area of a black hole (times a constant of order unity in Planck units) should be interpreted as its physical entropy. Indeed if the black hole did not have an own entropy the second law of thermodynamic would easily violated. In fact it easy to think of a situation in which we take some matter with some entropy, and put it into the black hole. Since nothing can come out of the black hole, we conclude that the entropy of the universe has reduced, hence the change in entropy, $\delta S \leq 0$. Therefore the second law has been violated. The way to save this apparent violation of the second law is to associate some entropy with the black hole, $S_{\text{black hole}}$. This entropy, will then increase when some matter goes into the black hole. Then we may be able to show that the net change of entropy is not negative, i.e, $\delta S + \delta S_{\text{black hole}} \geq 0$. In GR, thanks to the work by Bardeen, Carter and Hawking [1] and the discovery by Hawking of the black body thermal emission of a black hole, it was proved that the entropy of a spherically symmetric black hole is

$$S_{BH} = \frac{A}{4G} \tag{3}$$

which is the Bekenstein-Hawking entropy. The original derivation of this formula for the black hole entropy in GR used many detailed properties of the Einstein field equations and, thus, appeared to be very special to GR. In this thesis we would like to answer the following question: *is it possible to introduce a notion of black hole entropy, or, more generally, an horizon entropy which is based on quantities locally defined on the horizon which are common to any theory of gravity?*. Actually, we know that in 1993 Robert M. Wald answered *yes* to this question introducing the notion of what is now known as Wald entropy [16] constructed a new derivation of the first law of black hole mechanics for any theory which is invariant under diffeomorphisms (i.e., coordinate transformations). In this construction, the black hole entropy is related to the Noether charge of diffeomorphisms under the Killing vector field which generates the horizon in the stationary black hole background. Further, the Wald entropy can always be expressed as a local geometric density integrated over a space-like cross-section of the horizon.

The aim of this thesis is to convince the reader, following a path different from the one constructed by Wald, that such a notion of horizon entropy expressed in terms of locally defined quantities over the horizon (that are independent of how the gravitational theory is built) really exists. We could say that we will try to give a formulation of the Wald entropy starting from the very beginning, as if we were not aware of Wald's results, i.e we will discuss everything in the thesis *from scratch*. In doing this, the starting point will be the project 8.1 in [11, p. 394].

The formula for the entropy as connected to the Noether charge which will be provided in the thesis will coincide with the Bekenstein-Hawking entropy for a spherically symmetric horizon when the action functional of the theory is the Hilbert-Einstein action in a $D = 4$ spacetime. The key point we would like to focus on is that we will recover a notion of entropy that is specific of a certain gravitational theory, i.e. GR, starting from quantities that can be defined also for theories described by different, and completely general, action functionals. Thus, one expects the entropy formula we will found in the thesis to represents the horizon entropy in any diffeomorphism invariant general theory of gravity.

Chapter 1

Bianchi identity in general theories of gravity

1.1 Derivation of the generalized Bianchi identity

In this section we will derive the analogous of the Bianchi identity in GR, but in the case of general class of diffeomorphism invariant theories of gravity. These theories, as well as GR, will be treated as classical field theories, i.e. the dynamical variables will be functions of spacetime, their dynamics being governed by a proper action functional, as we will soon see. In this thesis we will always refer to a spacetime that can be represented as a D -dimensional spacetime with $D \geq 4$. We will consider the case $D = 4$ only when we will deal with GR. In describing a general theory of gravity we will start from some basics principles that can be already found in GR, that is the simpler and the most elegant and tested, at least into our solar system, gravitational theory. In a general theory of gravity, the gravitational field will be characterized by the 10 components of the symmetric metric tensor $g_{ab}(x)$ defined via

$$ds^2 = g_{ab}(x)dx^a dx^b \tag{1.1}$$

where ds^2 is the spacetime interval that represents the distance between two infinitesimally separated events of spacetime. In general, even though (1.1) defines an intrinsic curved spacetime, it is always possible to find a locally inertial frame (or Lorentzian frame) in which the metric $g_{ab}(x)$ reduces to the point independent Minkowskian metric η_{ab} with Lorentzian signature $\text{diag}(-1, 1, 1, 1)$. This is nothing but the *equivalence principle*, that leads in a quite naturally way to a geometrical description of all gravitational effects. However, in an intrinsic curved spacetime the metric appearing in (1.1) cannot be reduced globally to a given background form and thus all the coordinate systems have to be treated equally, none of them has a privileged status in describing the physics of gravitational systems. Hence, the laws of physics must be the same in any arbitrary

frame of reference (i.e. coordinate systems). It follows that we need to formulate our theories in a manner such that the equations are covariant under *arbitrary* coordinate transformations [11]. Hereafter, we will always consider metric theories of gravity in the sense just explained. The general covariance principle strongly constrains the form of the action functional. We will consider an action functional of the form

$$S = \int d^D x \sqrt{-g} L_g(g^{ab}, R^a_{bcd}, \nabla_k R^a_{bcd}, \dots) \quad (1.2)$$

where \dots stands for the higher derivatives of the curvature tensor and there are no other dynamical fields apart from the metric. The form of the Lagrangian in (1.2) is the one we would expect to find in a geometrical description of a general theory of gravity. The Lagrangian is a scalar under general transformations $x'^a = x'^a(x^b)$, made of tensors, namely the metric, its first and, at least, second derivatives (no non trivial scalar can be made of the metric and its first derivative alone) enclosed into the curvature tensor, and this fact leads to law of physics written in a tensorial form and thus valid in any arbitrary frame of reference. Extremizing (1.2) respect to the metric g^{ab} leads to the gravitational field equations. One can show that it is possible to build theories of gravity which have equations of motion involving derivatives of second order in the dynamical variables even though higher derivatives of the curvature tensor appear into the Lagrangian [11], [13]. It can be shown that the variation of the action (1.2) under an arbitrary transformation of the dynamical variables can be always cast in the form

$$\delta S = \int d^D x \delta(\sqrt{-g} L_g) = \int d^D x \sqrt{-g} (E_{ab} \delta g^{ab} + \nabla_a \delta v^a) \quad (1.3)$$

where the term $\nabla_a \delta v^a$ leads to a surface term. We will prove this in the next section, first assuming, for the sake of simplicity, the Lagrangian depends on the metric and curvature tensor but not on its derivatives and we will eventually consider the general case in which also the derivatives of the curvature tensor enter the Lagrangian. The first term in (1.3) contains all the terms rising from the variation of the metric alone, instead the second term is built by terms rising from the variation of the derivatives of the metric, and thus E_{ab} and δv^a result unambiguously defined.

A very important action functional belonging to the wider class of actions represented by the general form (1.2) is the following

$$S_{HE} = \frac{1}{16\pi C} \int d^D x \sqrt{-g} R \quad (1.4)$$

known as the *Hilbert-Einstein action*, R being the Ricci scalar, where C is a general coupling constant that reduces to the Newton constant when $D = 4$, i.e. for GR. However, this action is defined, in general, on a D -dimensional spacetime. The multiplication constant has been chosen in such a way that setting $D = 4$ one recovers the familiar Hilbert-Einstein action of GR. When dealing with (1.4), (1.3) is written as

$$\delta S_{HE} = \frac{1}{16\pi C} \int d^D x \sqrt{-g} (G_{ab} \delta g^{ab} + \nabla_a \delta v^a) \quad (1.5)$$

where

$$G_{ab} = \frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g}R)}{\partial g^{ab}} \quad (1.6)$$

is the Einstein tensor. Thus, $E_{ab} = G_{ab}$ for the Hilbert-Einstein action. In general theories of gravity, extremizing (1.3) with respect to δg^{ab} lead to the equations of motion for the gravitational field. If one adds to the action describing the pure gravitational field an action describing the effects of matter, $S \rightarrow S_g + S_M$, the equations of motion would be $2E_{ab} = T_{ab}$, where T_{ab} is the energy-momentum tensor of matter. When the Hilbert-Einstein action is extremized in the presence of matter the equations of motion read $G_{ab} = 8\pi C T_{ab}$, that reduces to the Einstein equations $G_{ab} = 8\pi G T_{ab}$ when $D = 4$ (this fact, again, motivates the choice of the multiplication constant). In absence of matter, the equations of motion reduces to $E_{ab} = 0$ and this must valid for any diffeomorphism invariant theory of gravity, in particular we have $G_{ab} = 0$ for the Hilbert-Einstein theory. However, we are not interested in discussing the features of the equations of motion anymore here, but rather in the well known Bianchi identity

$$\nabla_a G^{ab} = 0 \quad (1.7)$$

Using (1.6) the Bianchi identity can be written as

$$\nabla_a \left(R^{ab} - \frac{1}{2} g^{ab} R \right) = 0 \quad (1.8)$$

Even though all the dynamics of the metric is governed by G_{ab} , the previous relation is an identity that holds independently of the equations of motion. Bianchi identity is rather a relation coming up from the algebraic properties of the curvature tensor. In fact, it can be shown that (1.8) is equivalent to $R^a{}_{b[cd;k]} = 0$, where $[abc]$ stands for the sum over the cyclic permutations of the indexes a, b and c . This last identity is easily proved in a local inertial frame and, after some manipulations, it can be cast in the form (1.8). It is important to stress that if the Bianchi identity did not hold, it could not be possible to express the 20 components of a general tensor with the same algebraic properties of $R^a{}_{bcd}$ as functions of the 10 components of a given metric. Thus, Bianchi identity is a necessary and sufficient condition for a general tensor with the same algebraic symmetries of the curvature tensor to be considered as a curvature tensor of some metric. This feature is purely geometric, due only to the fact that we are considering an intrinsic curved manifold with a metric, and it is independent from the choice of the coordinate system and from the form of the metric (the "way" the spacetime is curved), i.e. from the equations of motion. Hence the content of Bianchi identity it is not dynamical, but instead it is strictly connected to the very geometrical nature of spacetime. To be more specific, the Bianchi identity (1.7) emerges as a direct consequence of the general covariance of the theory, expressed by the invariance of the Lagrangian under general transformations.

This can be better seen considering (1.7) as a collection of 4 constraints of the theory. Thus, it simply means that the equation of motions are not linearly independent and not all the 10 components of the metric g_{ab} are true dynamical variables. Since (1.9) provides four conditions (constraints) that can be used to fix 4 of the 10 components of the metric, there are only 6 components left whose time evolution can be obtained by solving the equations of motion. In general, if one wants to preserve the manifest general covariance of the theory, all the ten components of the metric have to be treated on the same footing, even if four of them evolve in time through arbitrary functions of time. The arbitrariness of these 4 components agrees perfectly with the general covariance that always gives the freedom to change the coordinates system, $x^a \rightarrow x'^a$, without changing the physics of the theory. Thus, we expect a Bianchi-like identity should hold in any diffeomorphism invariant theory. What happens is that the following identity holds

$$\nabla_a E^{ab} = 0 \quad (1.9)$$

for any E_{ab} given as in (1.3). To see it, we will proceed as follows: we will consider the variation of the Lagrangian density $\sqrt{-g}L_g$ under specific coordinate transformations of the form $x^a \rightarrow x'^a = x^a + \xi^a(x)$, where $\xi^a(x)$ are arbitrary, infinitesimal quantities.

Using the explicit expression (1.3) we have

$$\int d^D x \delta_\xi(\sqrt{-g}L_g) = \int d^D x \sqrt{-g}(E_{ab} \delta_\xi g^{ab} + \nabla_a(\delta_\xi v^a)) \quad (1.10)$$

In order to move further it is necessary to compute the local variation of the metric tensor under the general diffeomorphism $x^a \rightarrow x'^a = x^a + \xi^a(x)$, i.e. $\delta_\xi g^{ab} = g'^{ab}(x) - g^{ab}(x)$ that is the variation of the functional form of the metric tensor at a given location. Since we are considering the contravariant components of the metric tensor, we can start from its very definition, i.e. its transformation law under a change in the coordinates

$$g'^{ab}(x') = \frac{\partial x'^a}{\partial x^k} \frac{\partial x'^b}{\partial x^l} g^{kl}(x) \quad (1.11)$$

that for the coordinate transformation we are dealing with becomes

$$\begin{aligned} g'^{ab}(x') &= (\delta^a_k + \partial_k \xi^a)(\delta^b_l + \partial_l \xi^b) g^{kl}(x) \\ &\approx g^{ab}(x) + \partial^a \xi^b + \partial^b \xi^a \end{aligned} \quad (1.12)$$

where we have dropped the terms in ξ of higher order than the first. Now we have to compare $g'^{ab}(x')$ and $g^{ab}(x)$, that is the comparison has to be made at the same spacetime point. To do this we can expand the left member of the above equation in a Taylor series, stopping it at the linear terms in ξ . Then we get

$$\begin{aligned} g'^{ab}(x') &\approx g^{ab}(x) + \partial_k g^{ab}(x) \xi^k = g^{ab}(x) + \partial^a \xi^b + \partial^b \xi^a \Rightarrow \\ \delta_\xi g^{ab} &= g'^{ab}(x') - g^{ab}(x) = -\partial_k g^{ab}(x) \xi^k + \partial^a \xi^b + \partial^b \xi^a \end{aligned} \quad (1.13)$$

It is possible to give a more compact expression for the above local variation of the metric tensor recalling that the covariant derivative of the latter is identically zero, $\nabla_k g^{ab} = \partial_k g^{ab} + \Gamma^a_{kl} g^{bl} + \Gamma^b_{kl} g^{al} = 0$, from which we get $\partial_k g^{ab} = -\Gamma^a_{kl} g^{bl} - \Gamma^b_{kl} g^{al}$ and then

$$\begin{aligned} \delta_\xi g^{ab} &= \Gamma^a_{kl} g^{bl} \xi^k + \Gamma^b_{kl} g^{al} \xi^k + \partial^a \xi^b + \partial^b \xi^a \\ &= \partial^a \xi^b + \Gamma^{ab}_k \xi^k + \partial^b \xi^a + \Gamma^{ba}_k \xi^k \\ &= \nabla^a \xi^b + \nabla^b \xi^a = \mathcal{L}_\xi g^{ab} \end{aligned} \quad (1.14)$$

where $\mathcal{L}_\xi g^{ab}$ is the Lie derivative of the metric respect to ξ . Let us come back to the variation of the action. Putting into (1.10) the above expression for the local variation of the metric we get

$$\int d^D x \delta_\xi (\sqrt{-g} L_g) = \int d^D x \sqrt{-g} [E_{ab} (\nabla^a \xi^b + \nabla^b \xi^a) + \nabla_a (\delta_\xi v^a)] \quad (1.15)$$

From (1.3) we expect the tensor E_{ab} to be symmetric or, to better say, it would be useless considering its antisymmetric part since this would vanish in the contraction with δg^{ab} , that is obviously symmetric. As we will verify later, E_{ab} is built from the derivatives of the Lagrangian with respect the metric and the derivatives of the metric and hence it is straightforward symmetric. Using this fact we get

$$\begin{aligned} \int d^D x \delta_\xi (\sqrt{-g} L_g) &= \int d^D x \sqrt{-g} [2E_{ab} \nabla^a \xi^b + \nabla_a (\delta_\xi v^a)] \\ &= \int d^D x \sqrt{-g} [2E^{ab} \nabla_a \xi_b + \nabla_a (\delta_\xi v^a)] \\ &= \int d^D x \sqrt{-g} [2\nabla_a (E^{ab} \xi_b) - 2\nabla_a E^{ab} \xi_b + \nabla_a (\delta_\xi v^a)] \end{aligned} \quad (1.16)$$

where in the third step we have performed an integration by parts. To reach our goal, it is worth to express the local variation of the scalar density $\sqrt{-g} L_g$ in a more useful form. Keeping in mind that the Lagrangian is a general scalar, i.e $L'_g(x') = L_g(x)$ when $x^a \rightarrow x'^a = x^a + \xi^a(x)$, we get

$$\begin{aligned} L'_g(x^a + \xi^a) &= L'_g(x) + \xi^a \partial_a L_g(x) = L_g(x) \Rightarrow \\ L'_g(x) - L_g(x) &= -\xi^a \partial_a L_g(x) \Rightarrow \\ \delta_\xi L_g &= -\xi^a \nabla_a L_g = -\mathcal{L}_\xi L_g \end{aligned} \quad (1.17)$$

where in the first equation we have replaced $\partial_a L'_g(x)$ with $\partial_a L_g(x)$ since ξ^a are infinitesimal quantities (the terms in ξ^a of higher order than the first have been dropped) and in the last step the covariant derivative has taken the place of the ordinary one since L_g is

a scalar. Thus

$$\begin{aligned}\delta_\xi(\sqrt{-g}L_g) &= \delta_\xi(\sqrt{-g})L_g + \sqrt{-g}\delta_\xi L_g \\ &= -\frac{1}{2}\sqrt{-g}g_{ab}\delta_\xi g^{ab}L_g + \sqrt{-g}\delta_\xi L_g \\ &= \sqrt{-g}(-g_{ab}\nabla^a\xi^bL_g + \delta_\xi L_g)\end{aligned}\quad (1.18)$$

where in the last step the symmetry of the metric has been implemented. In the second equality, $\delta_\xi(\sqrt{-g})$ has been computed as follows. Since

$$\delta_\xi(\sqrt{-g}) = -\frac{1}{2\sqrt{-g}}\delta_\xi g \quad (1.19)$$

we need an expression for the variation of g . Writing

$$g = \exp [Tr \ln g^{ij}] \quad (1.20)$$

we have

$$\delta_\xi g = \frac{\partial g}{\partial g^{ab}}\delta_\xi g^{ab} = \exp [Tr \ln g^{ij}] Tr \left[\frac{\partial \ln g^{ij}}{\partial g^{ab}} \right] \delta_\xi g^{ab} = g (g^{ab})^{-1} = g g_{ab} \delta_\xi g^{ab} \quad (1.21)$$

and thus

$$\delta_\xi(\sqrt{-g}) = -\frac{1}{2\sqrt{-g}}g g_{ab}\delta_\xi g^{ab} = -\frac{1}{2}\sqrt{-g}g_{ab}\delta_\xi g^{ab} \quad (1.22)$$

Using (1.17) in the above expression leads to

$$\delta_\xi(\sqrt{-g}L_g) = \sqrt{-g}(-\nabla_a\xi^aL_g - \xi^a\nabla_aL_g) = -\sqrt{-g}\nabla_a(L_g\xi^a) \quad (1.23)$$

and (1.16) becomes

$$-\int d^D x \sqrt{-g} \nabla_a(L_g \xi^a) = \int d^D x \sqrt{-g} [2\nabla_a(E^{ab}\xi_b) - 2\nabla_a E^{ab}\xi_b + \nabla_a(\delta_\xi v^a)] \quad (1.24)$$

Rearranging this, we get

$$\int d^D x \sqrt{-g} 2\nabla_a E^{ab}\xi_b = \int d^D x \sqrt{-g} \nabla_a (2E^{ab}\xi_b + \delta_\xi v^a + L_g \xi^a) \quad (1.25)$$

from which we see that the right member is equal to a volume integral that can be transformed, through Gauss's theorem, into an integral over the boundary of the term in round bracket.

$$\int d^D x \sqrt{-g} 2\nabla_a E^{ab}\xi_b = \int d\sigma_a \sqrt{h} (2E^{ab}\xi_b + \delta_\xi v^a + L_g \xi^a) \quad (1.26)$$

where $d\sigma_a$ is a component of a 3-surface of intrinsic metric h . At this point, we can use the arbitrariness of ξ^a to take it in such a way that the variation of the metric together with its derivatives vanish on the boundary. In this way the right member of (1.26) is zero and since the volume of spacetime, over which the integral is performed, is completely arbitrary, the integrand in the left member of (1.26) must vanish and hence (1.9) immediately follows. Let us summarize the main results of this section:

- the Bianchi identity $\nabla_b E^{ab} = 0$ is a direct consequence of the general covariance of the theory.
- the Bianchi identity constraints a tensor with the same algebraic properties of the curvature tensor to be a curvature tensor derivable from a metric.
- the Bianchi identity is an off-shell relation, i.e. no equations of motion have been used in its derivation.

All these three facts are consequences of the geometrical nature of the theory, of seeing all gravitational effects emerging as geometrical properties of a manifold rather than dynamical properties of fields.

1.2 Variation of the action: a deeper insight

We will now focus on the variation of the action in order to give some more mathematical details behind (1.3). In doing this we will consider a Lagrangian that depends on the metric and the curvature tensor, but not on its derivatives

$$S = \int d^D x \sqrt{-g} L_g(g^{ab}, R^a{}_{bcd}) \quad (1.27)$$

In all the incoming calculations, and elsewhere in the thesis unless differently specified, we will always work in a coordinate basis $\{\mathbf{e}_a\}$ with basis vectors $\mathbf{e}_a = \partial_a$. Since spacetime is a differential manifold, the very definition of manifold always allows us to refer, without loss of generality, to a coordinate basis when we have to express the variation of the metric and thus the variation of the action. Now, let us write (1.27) in such a manner that the dependence upon the covariant metric and its first and second derivatives appears more clearly

$$S = \int d^D x \sqrt{-g} L_g(g_{ab}, \partial_c g_{ab}, \partial_c \partial_d g_{ab}) \quad (1.28)$$

With our choice to work with a coordinate basis, the connections and the curvature tensor read

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ak} (g_{kc,b} + g_{kb,c} - g_{bc,k}) \quad (1.29)$$

and

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{mc} \Gamma^m{}_{bd} - \Gamma^a{}_{md} \Gamma^m{}_{bc} \quad (1.30)$$

respectively, where we have used the notation $A_{,i} = \partial_i A$. The above expressions allow us to write the action (1.28) as follows

$$S = \int d^D x \sqrt{-g} L_g(g_{ab}, \Gamma^a{}_{bc}, \partial_d \Gamma^a{}_{bc}) \quad (1.31)$$

Equation (1.30) (with one index up) permits to read the variation of curvature tensor entirely in terms of variations of Γ^a_{bc} and $\Gamma^a_{bc,d}$ alone (only Gammas, no metric); then, if the δR^a_{bcd} we have to consider is with $g^{ab} = \text{const.}$ all what we have to do is to compute the variations of the Γ through (1.29), keeping $g^{ab} = \text{const.}$ in it, and use them in the variation of (1.30). This suggests to consider g^{ab} and R^a_{bcd} as independent variables, and to compute any variation of R^a_{bcd} as variation of Γ s. Thus (1.27) is formally equivalent to

$$S = \int d^D x \sqrt{-g} L_g(g^{ab}, \Gamma^a_{bc}, \partial_d \Gamma^a_{bc}) \quad (1.32)$$

Hence, we have (from now on, we will not specify the arguments of the Lagrangian anymore to ease the notation)

$$\begin{aligned} \delta S &= \int d^D x \sqrt{-g} \left[\frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g} L_g)}{\partial g^{ab}} \delta g^{ab} + P_a^{bcd} \delta R^a_{bcd} \right] \\ &= \int d^D x \sqrt{-g} \left[\left(\frac{\partial L_g}{\partial g^{ab}} - \frac{1}{2} g_{ab} L_g \right) \delta g^{ab} + P_a^{bcd} \delta R^a_{bcd} \right] \end{aligned} \quad (1.33)$$

where in the second equality (1.22) has been used. We have also introduced the tensor

$$P_a^{bcd} = \left(\frac{\partial L_g}{\partial R^a_{bcd}} \right)_{g^{ab}} \quad (1.34)$$

which has the same algebraic properties of R^a_{bcd}

$$P_a^{bcd} = -P_b^{acd} = -P_a^{bdc}, \quad P_a^{bcd} = P_c^{dab}, \quad P_a^{[bcd]} = 0 \quad (1.35)$$

If the term coming from the variation of the Lagrangian respect to the metric is straightforward to express into a simple form, that is not true for the term with the variation of the curvature tensor that needs some more manipulations. To begin with, it is convenient to work in a locally inertial frame in which $\Gamma^a_{bc} = 0$. Again there are no troubles in making this choice, since we are dealing with a Lagrangian that is a general scalar and thus any relation involving it will be valid in any system of reference. Thus we can write

$$P_a^{bcd} \delta R^a_{bcd} = P_a^{bcd} (\partial_c \delta \Gamma^a_{bd} - \partial_d \delta \Gamma^a_{bc}) \quad (1.36)$$

We will now show that, even though Γ^a_{bc} is not a tensor, $\delta \Gamma^a_{bc}$ is a tensor. Let us take two infinitesimally separated spacetime points P and P' and a vector A^a defined at P . After a parallel displacement between P and P' , the vector will be $A'^a = A^a + \Gamma^a_{bc} A^b dx^c$. Instead, using the connection $\Gamma'^a_{bc} = \Gamma^a_{bc} + \delta \Gamma^a_{bc}$ the vector will change as $A''^a = A^a + \Gamma^a_{bc} A^b dx^c + \delta \Gamma^a_{bc} A^b dx^c$. Thus $\delta \Gamma^a_{bc} A^b dx^c$ will be the difference between two vectors at the same point P' , that is a vector, and since A^a and dx^c are vectors too, $\delta \Gamma^a_{bc}$ must be

a tensor. Hence, we can substitute the partial derivative with the covariant one in the last equation and get

$$\begin{aligned} P_a^{bcd} \delta R_{bcd}^a &= P_a^{bcd} [\nabla_c \delta \Gamma_{bd}^a - \{c \leftrightarrow d\}] \\ &= 2P_a^{bcd} \nabla_c \delta \Gamma_{bd}^a \end{aligned} \quad (1.37)$$

In deriving this, we have used $P_a^{bcd} = -P_a^{bdc}$ and the fact that c and d are dummy indexes. The variation of the connection immediately follows from (1.29)

$$\delta \Gamma_{bd}^a = \frac{1}{2} g^{al} (\partial_b \delta g_{dl} + \partial_d \delta g_{bl} - \partial_l \delta g_{bd}) \quad (1.38)$$

where we use of $\delta g^{ab} = 0$, and we look at $\delta \partial_c g_{ab}$ as $\partial_c \delta g_{ab}$ so that, even if $\delta g^{ab} = 0$, we get $\delta g_{ab} \neq 0$ from the variations of the Γ 's, where these variations are taken with g^{ab} fixed. Since $\delta \Gamma_{bd}^a$ is a tensor, we can replace the ordinary derivative with the covariant one and get

$$\delta \Gamma_{bd}^a = \frac{1}{2} g^{al} (\nabla_b \delta g_{dl} + \nabla_d \delta g_{bl} - \nabla_l \delta g_{bd}) \quad (1.39)$$

The vanishing of the covariant derivative of the metric does not imply any vanishing of $\nabla_c \delta g_{ab}$, as the quantities δg_{ab} are small arbitrary quantities, with arbitrary covariant derivatives. Hence, taking the covariant derivative of (1.39) leads to

$$\nabla_c \delta \Gamma_{bd}^a = \frac{1}{2} g^{al} \nabla_c [\nabla_b \delta g_{dl} + \nabla_d \delta g_{bl} - \nabla_l \delta g_{bd}] \quad (1.40)$$

Hence using (1.40), (1.37) becomes

$$2P_a^{bcd} \nabla_c \delta \Gamma_{bd}^a = P^{lbcd} \nabla_c [\nabla_b \delta g_{dl} + \nabla_d \delta g_{bl} - \nabla_l \delta g_{bd}] \quad (1.41)$$

$$\begin{aligned} &= P^{lbcd} \nabla_c [\nabla_b \delta g_{dl} - \nabla_l \delta g_{bd}] \\ &= 2P^{lbcd} \nabla_c \nabla_b \delta g_{dl} \end{aligned} \quad (1.42)$$

Again, we have used the antisymmetry of P^{lbcd} that makes the second term vanish in (1.41). The next step is to arrange (1.42) as follows

$$\begin{aligned} 2P^{lbcd} \nabla_c \nabla_b \delta g_{dl} &= \nabla_c (2P^{lbcd} \nabla_b \delta g_{dl}) - 2\nabla_c P^{lbcd} \nabla_b \delta g_{dl} \\ &= \nabla_c (2P^{lbcd} \nabla_b \delta g_{dl}) - 2\nabla_b (\nabla_c P^{lbcd} \delta g_{dl}) + 2\nabla_b \nabla_c P^{lbcd} \delta g_{dl} \\ &= \nabla_c (2P^{lbcd} \nabla_b \delta g_{dl} - 2\nabla_b P^{lbcd} \delta g_{dl}) + 2\nabla_b \nabla_c P^{lbcd} \delta g_{dl} \\ &= \nabla_c \delta v^c + 2\nabla_b \nabla_c P^{lbcd} \delta g_{dl} \end{aligned} \quad (1.43)$$

Now, we have to express this last equation in terms of δg^{dl} instead of δg_{dl} and this is not done by simply thinking of the former as the contravariant components of the latter, because a minus sign occurs. We can see this by noting that $\delta(\delta_l^a) = 0 = \delta(g^{ak} g_{kl}) =$

$\delta g^{ak} g_{kl} + g^{ak} \delta g_{kl} \Rightarrow g^{ak} \delta g_{kl} = -g_{kl} \delta g^{ak}$ and contracting this equation with g_{ad} we obtain $\delta g_{dl} = -g_{ad} g_{bl} \delta g^{ab}$. Collecting all these results together (1.33) becomes

$$\begin{aligned} \delta S &= \int d^D x \sqrt{-g} \left[\left(\frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g} L)}{\partial g^{ab}} - 2 \nabla^m \nabla^n P_{amnb} \right) \delta g^{ab} + \nabla_c \delta v^c \right] \\ &= \int d^D x \sqrt{-g} [E_{ab} \delta g^{ab} + \nabla_a \delta v^a] \end{aligned} \quad (1.44)$$

where

$$E_{ab} = \frac{\partial L_g}{\partial g^{ab}} - \frac{1}{2} g_{ab} L_g - 2 \nabla^m \nabla^n P_{amnb} \quad (1.45)$$

and

$$\delta v^a = (2P^{lbad} \nabla_b - 2\nabla_b P^{lbad}) \delta g_{dl} \quad (1.46)$$

As expected, E_{ab} is symmetric: the first two terms of (1.45) are symmetric because the metric is and the last one is symmetric too due to the algebraic properties of P_{amnb} . We will now focus our attention on the boundary term (1.46) in order to give a general expression for it that will be very useful later, entirely in terms of the variation of the metric and the variation of the connection. This can be done nothing that contracting $\delta \Gamma_{bd}^k$, as given by (1.39), with P^{lbad}

$$P^{lbad} \delta \Gamma_{bd}^k = \frac{1}{2} P^{lbad} [\nabla_b \delta g_{dl} + \nabla_d \delta g_{bl} - \nabla_l \delta g_{bd}] \quad (1.47)$$

Now, the above expression gets highly simplified thanks to the antisymmetry of P^{lbad} for the exchange $b \leftrightarrow l$. In fact the middle term vanishes since the δg_{bl} is symmetric and the third term is added to the first after using $P^{lbad} = -P^{blad}$ and renaming $b \leftrightarrow l$. Thus we are left with

$$P_l{}^{bad} \delta \Gamma_{bd}^l = P^{lbad} \nabla_b \delta g_{dl} \quad (1.48)$$

and (1.46) becomes

$$\delta v^a = 2P_l{}^{bad} \delta \Gamma_{bd}^l - 2\nabla_b P^{lbad} \delta g_{dl} \quad (1.49)$$

which can also be written as

$$\delta v^a = 2P_l{}^{bad} \delta \Gamma_{bd}^l + 2\nabla_b P_l{}^{ba}{}_{d} \delta g^{dl} \quad (1.50)$$

Equation (1.44), obtained for Lagrangians $L_g = L_g(g^{ab}, R^a{}_{bcd})$, corresponds to a variation of the action of the form (1.3). Now, we would like to show that the form (1.3) for the variation of the action functional is generic, that is it holds true for general theories of gravity, i.e. theories described by the general action functional (1.2). To do this, we rewrite (1.28) as

$$S = \int d^D x \sqrt{-g} L_g(g_{ab}, \partial_{k_1} g_{ab}, \partial_{k_1} \partial_{k_2} g_{ab}, \dots) \quad (1.51)$$

where ... means derivatives of the metric of order higher than the second. The variation (1.3) can be written as

$$\delta S = \int d^D x \sqrt{-g} (U^{ab} \delta g_{ab} + W^{k_1 ab} \delta \partial_{k_1} g_{ab} + W^{k_1 k_2 ab} \delta \partial_{k_1} \partial_{k_2} g_{ab} + \dots) \quad (1.52)$$

where $U^{ab} = -(\sqrt{-g})^{-1} \partial(\sqrt{-g} L_g) / \partial g_{ab}$ contains the terms coming from the variation of the metric, and $W^{k_1 ab} = \partial L_g / \partial(\partial_{k_1} g_{ab})$, $W^{k_1 k_2 ab} = \partial L_g / \partial(\partial_{k_1} \partial_{k_2} g_{ab})$, etc. etc, contain the terms coming from the variations of the derivatives of the metric, the derivatives of the derivatives of the metric, etc. etc.. Before moving further, we note that we have chosen to work with the variation of the metric written as δg_{ab} in (1.52) for our convenience, but when we will come back to δg^{ab} a minus sign will occur. Since we would like (1.52) to have *the same value* of (1.3), a minus sign has been introduced in the expression for U^{ab} . We shall now see that each term with the W s gives a contribution proportional to δg_{ab} , that will be summed up into U_{ab} , and a term that can be expressed as a divergence of a vector. For example, the first three terms are

$$A) \quad W^{k_1 ab} \partial_{k_1} \delta g_{ab} = -\partial_{k_1} W^{k_1 ab} \delta g_{ab} + \partial_{k_1} (W^{k_1 ab} \delta g_{ab})$$

$$\begin{aligned} B) \quad W^{k_1 k_2 ab} \partial_{k_1} \partial_{k_2} \delta g_{ab} &= \partial_{k_1} \partial_{k_2} W^{k_1 k_2 ab} \delta g_{ab} + \partial_{k_1} (W^{k_1 k_2 ab} \partial_{k_2} \delta g_{ab} - \partial_{k_2} W^{k_1 k_2 ab} \delta g_{ab}) \\ &= \partial_{k_1} \partial_{k_2} W^{k_1 k_2 ab} \delta g_{ab} + \partial_{k_1} (W^{k_1 k_2 ab} \partial_{k_2} \delta g_{ab} - \partial_{k_2} W^{k_1 k_2 ab} \delta g_{ab}) \end{aligned}$$

and

$$\begin{aligned} C) \quad W^{k_1 k_2 k_3 ab} \partial_{k_1} \partial_{k_2} \partial_{k_3} \delta g_{ab} &= -\partial_{k_1} \partial_{k_2} \partial_{k_3} W^{k_1 k_2 k_3 ab} \delta g_{ab} + \partial_{k_1} (W^{k_1 k_2 k_3 ab} \partial_{k_2} \partial_{k_3} \delta g_{ab} \\ &\quad - \partial_{k_2} W^{k_2 k_1 k_3 ab} \partial_{k_3} \delta g_{ab} + \partial_{k_2} \partial_{k_3} W^{k_2 k_1 k_3 ab} \delta g_{ab}) \\ &= -\partial_{k_1} \partial_{k_2} \partial_{k_3} W^{k_1 k_2 k_3 ab} \delta g_{ab} + \partial_{k_1} (W^{k_1 k_2 k_3 ab} \partial_{k_2} \partial_{k_3} \delta g_{ab} \\ &\quad - \partial_{k_2} W^{k_1 k_2 k_3 ab} \partial_{k_3} \delta g_{ab} + \partial_{k_2} \partial_{k_3} W^{k_1 k_2 k_3 ab} \delta g_{ab}) \end{aligned}$$

where we have used the fact that the W s are completely symmetric in $\{k_1, k_2, \dots\}$. Thus, by induction, the n th term is

$$\begin{aligned} W^{k_1 k_2 \dots k_n ab} \partial_{k_1} \partial_{k_2} \dots \partial_{k_n} \delta g_{ab} &= \partial_{k_1} \partial_{k_2} \dots \partial_{k_n} W^{k_1 k_2 \dots k_n ab} \delta g_{ab} \\ &\quad + \partial_{k_1} (W^{k_1 k_2 \dots k_n ab} \partial_{k_2} \dots \partial_{k_n} \delta g_{ab} - \partial_{k_2} W^{k_2 k_1 \dots k_n ab} \partial_{k_3} \dots \partial_{k_n} \delta g_{ab} \\ &\quad + \dots + \partial_{k_n} \dots \partial_{k_2} W^{k_n k_2 \dots k_1 ab} \delta g_{ab}) \end{aligned}$$

and we immediately read the term proportional to the variation of the metric

$$-(1)^n \partial_{k_1} W^{k_1 ab} + \partial_{k_1} \partial_{k_2} W^{k_1 k_2 ab} - (1)^{n-1} \partial_{k_1} \partial_{k_2} \partial_{k_3} W^{k_1 k_2 k_3 ab} + \dots = -\partial_{k_1} \frac{\delta L_g}{\delta \partial_{k_1} g_{ab}}$$

where we have used the definition of $W^{\dots ab}$ and the Euler derivative operator

$$\frac{\delta}{\delta\phi} = \frac{\partial}{\partial\phi} - \partial_a \frac{\partial}{\partial(\partial_a\phi)} + \partial_a \partial_b \frac{\partial}{\partial(\partial_a\partial_b\phi)} + \dots \quad (1.53)$$

The term inside the 4-divergence, instead, has the following structure

$$\delta v^{k_1} = F^{k_1 ab} \delta g_{ab} + F^{k_1 k_2 ab} \delta \partial_{k_2} g_{ab} + F^{k_1 k_2 k_3 ab} \delta \partial_{k_2} \partial_{k_3} g_{ab} + \dots \quad (1.54)$$

with

$$\begin{aligned} F^{k_1 ab} &= W^{k_1 ab} - \partial_{k_2} W^{k_2 k_1 ab} + \partial_{k_2} \partial_{k_3} W^{k_1 k_2 k_3 ab} + \dots = \frac{\delta L_g}{\delta(\partial_{k_1} g_{ab})} \\ F^{k_1 k_2 ab} &= W^{k_1 k_2 ab} - \partial_{k_3} W^{k_1 k_2 k_3 ab} + \partial_{k_3} \partial_{k_4} W^{k_1 k_2 k_3 k_4 ab} + \dots = \frac{\delta L_g}{\delta(\partial_{k_1} \partial_{k_2} g_{ab})} \\ F^{k_1 k_2 k_3 ab} &= W^{k_1 k_2 k_3 ab} - \partial_{k_4} W^{k_1 k_2 k_3 k_4 ab} + \partial_{k_4} \partial_{k_5} W^{k_1 k_2 k_3 k_4 k_5 ab} + \dots = \frac{\delta L_g}{\delta(\partial_{k_1} \partial_{k_2} \partial_{k_3} g_{ab})} \\ &\dots \\ F^{k_1 k_2 \dots k_n ab} &= \frac{\delta L_g}{\delta(\partial_{k_1} \partial_{k_2} \dots \partial_{k_n} g_{ab})} \end{aligned}$$

where we have used again the form of the W functions and the Euler derivative. Hence, the variation (1.52) turns out to be

$$\delta S = \int d^D x \sqrt{-g} (-E^{ab} \delta g_{ab} + \partial_{k_1} \delta v^{k_1}) \quad (1.55)$$

with

$$E^{ab} = -U^{ab} + \partial_c \frac{\delta L_g}{\delta \partial_c g_{ab}} \quad (1.56)$$

and the boundary term

$$\delta v^{k_1} = F^{k_1 ab} \delta g_{ab} + F_c^{k_1 ab} \delta \Gamma_{ab}^c + \dots + F_c^{k_1 k_2 \dots k_{n-1} ab} \delta \partial_{k_2} \dots \partial_{k_{n-1}} \Gamma_{ab}^c + \dots \quad (1.57)$$

In (1.55), what is in parenthesis must be a scalar since the left member is a variation a scalar which is a scalar. But we know that $E^{ab} \delta g_{ab}$ is a scalar and thus $\partial_{k_1} \delta v^{k_1}$ must be a scalar too. This fact allows us to replace the ordinary derivative with the covariant one and write

$$\delta S = \int d^D x \sqrt{-g} (-E^{ab} \delta g_{ab} + \nabla_{k_1} \delta v^{k_1}) \quad (1.58)$$

Since $\delta g_{ab} = -g_{ak'} g_{bl'} \delta g^{k'l'}$, we get

$$-E^{ab} \delta g_{ab} = \delta_{k'}^k \delta_{l'}^l E_{kl} \delta g^{k'l'} = E_{kl} \delta g^{kl} \quad (1.59)$$

recovering in this way exactly the variation (1.3).

To end this section, we will see how the boundary term (1.57) changes when it is computed in the correspondence of an approximate Killing vector ξ^a . A Killing vector satisfies Killing's equation $\nabla_a \xi_b + \nabla_b \xi_a = 0$ from which it follows that $\nabla_a \nabla_b \xi_c = R^k{}_{abc} \xi_k$. The second equation follows from the very definition of the curvature tensor $[\nabla_a, \nabla_b] \xi_c = R^k{}_{cba} \xi_k$ in addition to its symmetry property $R^k{}_{[cba]} = R^k{}_{cba} + R^k{}_{acb} + R^k{}_{bac} = 0$. Combining these we eventually arrive to a relation involving the second derivatives of the Killing vector, namely

$$[\nabla_a, \nabla_b] \xi_c + [\nabla_b, \nabla_c] \xi_a + [\nabla_c, \nabla_a] \xi_b = 0 \quad (1.60)$$

which making use of the Killing's equation becomes

$$\begin{aligned} \nabla_a \nabla_b \xi_c + \nabla_b \nabla_c \xi_a - \nabla_c \nabla_b \xi_a &= 0 \Rightarrow \\ \nabla_a \nabla_b \xi_c &= -R^k{}_{acb} \xi_k = R^k{}_{abc} \xi_k \end{aligned} \quad (1.61)$$

We will now apply these properties of a Killing vector to find out how the boundary term transforms under infinitesimal coordinate transformation $x'^a = x^a + \xi^a(x)$, with $\xi^a(x)$ Killing vector. We get

$$\delta_\xi v^{k_1} = F^{k_1 ab} \delta_\xi g_{ab} + F_c{}^{k_1 ab} \delta_\xi \Gamma^c{}_{ab} + \dots + F_c{}^{k_1 k_2 \dots k_{n-1} ab} \partial_{k_2} \dots \partial_{k_{n-1}} \delta_\xi \Gamma^c{}_{ab} + \dots$$

It is immediate to write the variation of the first term since $\delta_\xi g_{ab} = \nabla_{(a} \xi_{b)} = 0$. The other terms involve the variation of a connection $\delta_\xi \Gamma$ which will be handled in the following way

$$\begin{aligned} \delta_\xi \Gamma^c{}_{ab} &= \frac{1}{2} g^{ck} (\nabla_a \delta_\xi g_{bk} + \nabla_b \delta_\xi g_{ak} - \nabla_k \delta_\xi g_{ab}) \\ &= -\frac{1}{2} g^{ck} (\nabla_a \nabla_b \xi_k + \nabla_a \nabla_k \xi_b + \nabla_b \nabla_a \xi_k + \nabla_b \nabla_k \xi_a \\ &\quad - \nabla_k \nabla_a \xi_b - \nabla_k \nabla_b \xi_a) \end{aligned} \quad (1.62)$$

where the fact that $\delta \Gamma$ is a tensor has been used to write the first row of the above expression. We can rearrange (1.62) into a more compact formula

$$\begin{aligned} \delta_\xi \Gamma^c{}_{ab} &= -\frac{1}{2} g^{ck} (R_{mbka} \xi^m + R_{makb} \xi^m + \nabla_{(a} \nabla_{b)} \xi_k) \\ &= \frac{1}{2} R^c{}_{(ab)k} \xi^k - \frac{1}{2} \nabla_{(a} \nabla_{b)} \xi^c \end{aligned} \quad (1.63)$$

where in the second step the symmetry properties of the curvature tensor has been implemented. Using (1.61), it is straightforward to see that $\delta_\xi \Gamma^c{}_{ab} = 0$. Thus, the boundary term, whatever the Lagrangian we are considering is, vanishes under a general diffeomorphism induced by a vector ξ^a that is a Killing vector at least inside the region of spacetime in which we are considering the variation of the metric and the variation of the connection and its derivatives, and thus the whole boundary term.

Hence, in this section,

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- we have found the explicit form of E_{ab} and δv^a for a Lagrangian depending upon the metric and the curvature tensor, but not on the derivatives of the latter. These expressions will be very useful in further calculations.
 - we have shown that, whatever the form of the Lagrangian is, the variation of an action functional describing a general theory of gravity is always casted in the form (1.3)
 - for general theories, the boundary term depends not only upon the variation of the metric and the variation of the connection, but also upon the derivatives of the latter. However, it vanishes when evaluating on a vector that is a Killing vector at least into a proper region of spacetime. This fact will be crucial in further discussions.

Chapter 2

A conserved current

2.1 Existence of a conserved current

We have seen that if one implies the variation of the action (1.2) to be the one derived in the previous chapter then the generalized Bianchi identity holds. The key point we have used in deriving it is the general covariance of the theory, i.e. that the Lagrangian is a scalar under general coordinate transformations. Another striking consequence of the general covariance and the Bianchi identity of the theory is the existence of a conserved current J^a , whose explicit form can be obtained equating the local variation of the Lagrangian density $\sqrt{-g}L_g$ under $x^a \rightarrow x^a + \xi^a(x)$ written in two different ways (eqs. (1.16) and (1.23)), namely

$$\delta_\xi(\sqrt{-g}L_g) = -\sqrt{-g}\nabla_a(L_g\xi^a) \quad (2.1)$$

and

$$\begin{aligned} \delta_\xi(\sqrt{-g}L_g) &= \sqrt{-g}[2E_{ab}\nabla_a\xi_b + \nabla_a(\delta_\xi v^a)] \\ &= \sqrt{-g}[\nabla_a(2E^{ab}\xi_b + \delta_\xi v^a) - 2\nabla_a E^{ab}\xi_b] \end{aligned} \quad (2.2)$$

that, making use of (1.9), leads to

$$\delta_\xi(\sqrt{-g}L_g) = \sqrt{-g}\nabla_a(2E^{ab}\xi_b + \delta_\xi v^a) \quad (2.3)$$

Equating (2.1) and (2.3) we get

$$\nabla_a(-L_g\xi^a) = \nabla_a(2E^{ab}\xi_b + \delta_\xi v^a) \quad (2.4)$$

Thus

$$\nabla_a(2E^{ab}\xi_b + L_g\xi^a + \delta_\xi v^a) = 0 \quad (2.5)$$

and the conserved current reads

$$J^a = \xi^a L_g + 2E^{ab}\xi_b + \delta_\xi v^a \quad (2.6)$$

At this point, it is very important to underline that the continuity equation

$$\nabla_a J^a = 0 \quad (2.7)$$

as well as the generalized Bianchi identity (1.9), is an off-shell relation, i.e. no equations of motion have been used to obtain that result. On one hand we have the current (2.6), crucially tied to the general Bianchi identity, which is conserved already off-shell; on the other hand, any symmetry in the action calls for a Noether current. For the Noether current, however, we have an on-shell conservation. It is, thus, somehow intriguing the off-shell conservation we find, and current (2.6) deserves some further investigation in order to clarify its relation with the conserved current as provided by Noether theorem.

2.2 Noether theorem

In this section we will use of the symmetries of the action (1.2), to construct the associated on-shell Noether currents, following the path provided by the usual proof of the Noether theorem itself. Our aim is to compare the conserved current extracted in this way with the current (2.6). Before facing this interesting fact we will give a general proof of the theorem. We recall briefly here how the proof of Noether theorem runs in the context relevant for us. We consider a field $\phi_A(x)$, which can be a thought of as scalar or can carry any collection of up and down, spacetime and/or internal space, indexes denoted collectively by A . Here x^a are, for the moment, Cartesian coordinates in a D -dimensional Minkowski spacetime. A formulation of the theorem is the following: *If an action describing a physical system is invariant under a continuous transformation of coordinates and fields, then always a locally conserved current exists, i.e. a combination of field functions and their derivatives exist which satisfies the continuity equation.* We consider then the action functional (hereafter we will omit the indexes A and B to ease the notation, but the field is always meant to be a general tensor)

$$S = \int d^D x L(\phi(x), \partial_a \phi(x)) \quad (2.8)$$

Extremizing this respect to $\phi(x)$ leads to the equations of motion. Now, let us introduce the following

- **total variation:** $\Delta\phi(x) = \phi'(x') - \phi(x)$
- **local variation:** $\delta\phi(x) = \phi'(x) - \phi(x)$

- **differential variation:** $d\phi(x) = \phi(x') - \phi(x)$

In a first order approximation we have

$$\phi'(x') = \phi'(x + \delta x) \approx \phi'(x) + \delta x^\mu \partial_\mu \phi(x)$$

where we have replaced $\delta x^\mu \partial_\mu \phi'(x)$ with $\delta x^\mu \partial_\mu \phi(x)$ since they differ in a second order infinitesimal. Thus we get

$$\Delta\phi(x) = \phi'_A(x') - \phi_A(x) \approx \delta\phi(x) + \delta x^\mu \partial_\mu \phi(x) = \delta\phi(x) + d\phi(x)$$

Now, we consider the following transformation

$$\begin{cases} x'^a = x^a + \delta x^a \\ \phi'(x') = \phi(x) + \Delta\phi(x) \end{cases} \quad (2.9)$$

and we suppose it to be a symmetry for the system we are dealing with. This is expressed by the condition

$$\Delta S = 0 \quad (2.10)$$

where

$$\Delta S = \int d^D x' L'(\phi'(x'), \partial_\mu \phi'(x')) - \int d^D x L(\phi(x), \partial_\mu \phi(x)) \quad (2.11)$$

In general, volume elements transform as

$$d^D x' = |J(x)| d^D x \quad (2.12)$$

where

$$J(x) = \det \left\| \frac{\partial x'^a}{\partial x^b} \right\| \quad (2.13)$$

is the Jacobian of the transformation. We have $x'^a = x^a + \delta x^a$ and thus, in the first order approximation, the Jacobian is

$$J(x) = \det(\delta_b^a + \partial_b \delta x^a) \quad (2.14)$$

To compute this determinant we will use the general formula

$$\det A = \exp(\text{Tr} \ln A) \quad (2.15)$$

In the adopted approximation $\ln(\delta_b^a + \partial_b \delta x^a) = \partial_b \delta x^a$ and thus $\text{Tr} \partial_b \delta x^a = \partial_a \delta x^a$. Hence

$$J(x) = \exp \partial_a \delta x^a = 1 + \partial_a \delta x^a \quad (2.16)$$

Thus $d^D x' = (1 + \partial_a \delta x^a) d^D x$ and the total variation of the action, at the first order, reads

$$\begin{aligned} \Delta S &= \int d^D x (1 + \partial_a \delta x^a) L'(\phi'(x'), \partial_a \phi'(x')) - \int d^D x L(\phi(x), \partial_a \phi(x)) \\ &\approx \int d^D x \Delta L(\phi(x), \partial_a \phi(x)) + \int d^D x \partial_a \delta x^a L(\phi(x), \partial_a \phi(x)) \end{aligned} \quad (2.17)$$

where we have written $\partial_a \delta x^a L'(\phi'(x'), \partial_a \phi'(x')) \approx \partial_a \delta x^a L(\phi(x), \partial_a \phi(x))$ since we are considering first order approximation (for the moment, it is not necessary to take the Lagrangian as a scalar under (2.9)). Implementing the total variation of the Lagrangian we get [14, pag.76]

$$\Delta S = \int d^D x \delta L + \int d^D x \delta x^a \partial_a L + \int d^D x \partial_a \delta x^a L = \int d^D x \delta L + \int d^D x \partial_a (\delta x^a L) \quad (2.18)$$

The first term in the right member is

$$\delta L = \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial \partial_a \phi} \delta \partial_a \phi = \left(\frac{\partial L}{\partial \phi} - \partial_a \frac{\partial L}{\partial \partial_a \phi} \right) \delta \phi + \partial_a \left(\frac{\partial L}{\partial \partial_a \phi} \delta \phi \right)$$

Thus, the total variation of the action becomes

$$\Delta S = \int d^D x \left[\left(\frac{\partial L}{\partial \phi} - \partial_a \frac{\partial L}{\partial \partial_a \phi} \right) \delta \phi + \partial_a \left(\frac{\partial L}{\partial \partial_a \phi} \delta \phi + L \delta x^a \right) \right] \quad (2.19)$$

Since, around any point in spacetime, the region of integration is arbitrary, the integrand must vanish for the symmetry condition to be fulfilled and hence

$$\left(\frac{\partial L}{\partial \phi} - \partial_a \frac{\partial L}{\partial \partial_a \phi} \right) \delta \phi + \partial_a \left(\frac{\partial L}{\partial \partial_a \phi} \delta \phi + L \delta x^a \right) = 0 \quad (2.20)$$

Implementing the equations of motion leads to

$$\partial_a \left(\frac{\partial L}{\partial \partial_a \phi} \delta \phi + L \delta x^a \right) = 0 \quad (2.21)$$

And the *on-shell* conserved current, i.e. the current to which the usual context of the derivation of Noether theorem brings, is

$$\hat{j}^a = \frac{\partial L}{\partial \partial_a \phi} \delta \phi + L \delta x^a \quad (2.22)$$

The key points of this derivation are the action functional written in the form (2.8), whose total variation under (2.9) must be zero, and the request to be on-shell, i.e. the current is made by combinations of fields that are solutions of the equations of motion. Now, we will try to generalize this derivation to the case of gravitational theories.

2.3 Noether theorem for gravitational actions

In this section we would like to going through the same path of the previous section but in the case of gravitational theories. We will take the field to be the metric tensor g_{ab} of curved spacetime and we will try to obtain an expression for the conserved current, following the Noether theorem. We already know that an action functional describing a general diffeomorphism invariant theory of gravity remains always unchanged under general transformations $x'^a \equiv x'^a(x^b)$. In fact taking the action

$$S = \int d^D x \sqrt{-g} L(g_{ab}, \partial_c g_{ab}) \quad (2.23)$$

where $L(g_{ab}, \partial_c g_{ab})$ is a general scalar depending upon the metric and its derivatives, it is straightforward to see that

$$\Delta S = \int d^D x' \sqrt{-g'} L'(g'_{ab}, (\partial_c g'_{ab})') - \int d^D x \sqrt{-g} L(g_{ab}, \partial_c g_{ab}) = 0 \quad (2.24)$$

since $d^D x' \sqrt{-g'} = d^D x \sqrt{-g}$ and $L'(g'_{ab}, (\partial_c g'_{ab})') = L(g_{ab}, \partial_c g_{ab})$ by construction. Hence, by Noether theorem, a conserved current is expected to exist. We will investigate this first for an unphysical system, which will only help us to establish a formal correspondence with the results of Noether theorem of the previous section, and then for a system described by the action functional (1.2).

2.3.1 Noether theorem for a gravitational toy model

Consider the following action functional

$$S = \int d^D x \sqrt{-g} L(g_{ab}, \partial_c g_{ab}) \quad (2.25)$$

Although such an action cannot describe a non trivial theory of gravity, it allows us to make some formal parallelisms with the case described by the action (2.8). As before, we have

$$\Delta S = \int d^D x \delta(\sqrt{-g}L) + \int d^D x \partial_a (\delta x^a \sqrt{-g}L) = 0 \quad (2.26)$$

with the only difference that the square root of the determinant of the metric appears in front of the Lagrangian. However, we eventually arrive at

$$\left(\frac{\partial(\sqrt{-g}L)}{\partial g_{kl}} - \partial_a \frac{\partial(\sqrt{-g}L)}{\partial \partial_a g_{kl}} \right) \delta g_{kl} + \partial_a \left(\frac{\partial(\sqrt{-g}L)}{\partial \partial_a g_{kl}} \delta g_{kl} + \sqrt{-g}L \delta x^a \right) = 0 \quad (2.27)$$

Going on-shell we get

$$\partial_a \left(\frac{\partial(\sqrt{-g}L)}{\partial \partial_a g_{kl}} \delta g_{kl} + \sqrt{-g}L \delta x^a \right) = 0 \quad (2.28)$$

These have the same structure of (2.20) and (2.21), the conserved current in this case being

$$\tilde{j}^a = \sqrt{-g} \left(\frac{\partial L}{\partial \partial_a g_{kl}} \delta g_{kl} + L \delta x^a \right) = \sqrt{-g} \hat{J}^a \quad (2.29)$$

The expression to the left in (2.28) is not generally covariant, and the quantities \tilde{J}^a are not a vector, but a vector density (the expression in round brackets in (2.29), that is \hat{J}^a , it is a vector). We need to move to express (2.28) in terms of a covariant divergence of a vector. This can be easily done by computing the covariant divergence of a vector A^a . Using the definition of covariant derivative we get

$$\nabla_a A^a = \partial_a A^a + \Gamma^a_{ab} A^b \quad (2.30)$$

but

$$\Gamma^a_{ab} = \frac{1}{2} g^{ak} (\partial_a g_{bk} + \partial_b g_{ak} - \partial_k g_{ab}) = \frac{1}{2} g^{ak} \partial_b g_{ak} = \frac{1}{\sqrt{-g}} \partial_b \sqrt{-g} \quad (2.31)$$

where we have used (1.22) for getting the last equality

$$\delta(\sqrt{-g}) = -\frac{1}{2} \sqrt{-g} g_{ab} \delta g^{ab} \Rightarrow \partial_k \sqrt{-g} \delta x^k = -\frac{1}{2} \sqrt{-g} g_{ab} \partial_k g^{ab} \delta x^k \quad (2.32)$$

and since $g_{ab} \partial_k g^{ab} = -g^{ab} \partial_k g_{ab}$ we get

$$\frac{1}{2} g^{ab} \partial_k g_{ab} = \frac{1}{\sqrt{-g}} \partial_k \sqrt{-g} \quad (2.33)$$

Hence

$$\nabla_a A^a = \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} A^a) \quad (2.34)$$

That is precisely what help us in writing the continuity equation in terms of a covariant divergence. In our case, we have $A^a = \hat{J}^a$ and thus we can write

$$\partial_a \tilde{J}^a = \partial_a (\sqrt{-g} \hat{J}^a) = \sqrt{-g} \nabla_a \hat{J}^a = 0 \quad (2.35)$$

meaning $\nabla_a \hat{J}^a = 0$ which is the generally covariant continuity equation. We see that \hat{J}^a , which coincides with (2.22), is the on-shell generally covariant conserved current emerging from the usual proof of Noether theorem for Lagrangian L in curved spacetime.

2.3.2 Noether theorem for general theories of gravity

We will ask the following question: is it possible to use the Noether approach for a general action functional to reach a conserved current? A problem here is that the Lagrangian is not only a function of the field and its first derivatives, but necessarily also a function of the derivatives of the field of order higher than the first. Thus the

Noether ‘mechanism’ considered in previous sections cannot be straightforwardly applied. However, we already know how the local variation of the gravitational action looks like, for a generally covariant Lagrangian which depends upon derivatives of the metric of any order. Taking an action functional of the form (1.2) we have seen that its local variation is given by (1.55), with $E^{ab} = 0$ being the equations of motion (in vacuum). This structure is similar to the one of (2.19); we recognize in it the same two components: one, whose vanishing gives the equations of motion, and the other which contributes to give the on-shell Noether current considered in last sections. From (2.17), which holds true for derivatives of any order of the fields in L , we get

$$\begin{aligned}\Delta S &= \int d^D x \delta(\sqrt{-g}L_g) + \int d^D x \partial_a(\sqrt{-g}L_g \delta x^a) \\ &= \int d^D x \sqrt{-g} (E^{ab} \delta g_{ab} + \nabla_a \delta v^a) + \int d^D x \partial_a(\sqrt{-g}L_g \delta x^a)\end{aligned}\quad (2.36)$$

The second term in the last equality can be expressed in terms of the covariant divergence by mean of (2.34), and eventually we get

$$\Delta S = \int d^D x \sqrt{-g} [E^{ab} \delta g_{ab} + \nabla_a (\delta v^a + L \delta x^a)] \quad (2.37)$$

Since the total variation of the action is zero for general transformations we can write

$$E^{ab} \delta g_{ab} + \nabla_a (\delta v^a + L \delta x^a) = 0 \quad (2.38)$$

This expression is formally identical to the one we have encountered previously both in the toy model and in the canonical proof of the Noether theorem. If in this latter cases we have needed the equations of motion to arrive at the continuity equation for the current J^a , here something very special happens since we are able to express the term $E^{ab} \delta g_{ab}$, that would vanish on-shell, as a total derivative. In fact we know that the variation of the metric can be expressed in the form (1.14), being ξ^a the vector field which describes the infinitesimal coordinate transformations, and thus the above expression becomes

$$2E^{ab} \nabla_a \xi_b + \nabla_a (\delta_\xi v^a + L \xi^a) = -2\nabla_a E^{ab} \xi_b + \nabla_a (2E^{ab} \xi_b + \delta_\xi v^a + L \xi^a) = 0 \quad (2.39)$$

And thanks to the Bianchi identity

$$\nabla_a (2E^{ab} \xi_b + \delta_\xi v^a + L \xi^a) = 0 \quad (2.40)$$

from which we read the conserved current

$$J^a = 2E^{ab} \xi_b + \delta_\xi v^a + L \xi^a \quad (2.41)$$

Here we recognize the on-shell Noether expression for the conserved current, described in previous sections, in the 2nd and the 3rd term. The current which is conserved off-shell has however, we see, an additional off-shell component, given by the 1st term. Let us summarize the results we have obtained in this chapter:

- The off-shell conservation law (2.40) is a pure cinematic relation, i.e it is completely independent of the source that generates the gravitational fields. It is an intrinsic feature of all differential manifolds, independently of how they are curved by a gravitational source. We know that this same argument applies to the generalized Bianchi identity, and in fact, in general, equations like (2.40) are called *Noether identities* [5].
- The key factor that allows one to obtain the off-shell conserved current J^a is that the variation of the field, i.e. the metric, can be written in terms of a gradient of the variation of the coordinates.
- This special transformation of the metric under general diffeomorphisms has been already use to get the Bianchi identity.
- In general, everywhen a field transforms in this way due to a change in the coordinates, an off-shell conserved current appears, as well as relations analogous to the Bianchi identity, as one can verify by taking the Lagrangians considered in 2.2, 2.3.1 and 2.3.2.
- We expect that also for any non-gravitational Lagrangian, when the variations of the fields can be written as a gradient of the variation of the coordinates, one can obtain an off-shell conserved current. We will now show this in the case of the electromagnetic field.

2.3.3 Electromagnetic field and gauge symmetry

Let us take the 4-potential $A_a(x) = (-\phi, \mathbf{A})$ describing the electromagnetic field in terms of the scalar and vector potential on a 4-dimensional flat Minkowskian spacetime with constant metric η_{ab} , and the the following action functional

$$S = \int d^4 x \mathcal{L}_{em}(A_a, \partial_b A_a) \quad (2.42)$$

where the Lagrangian will be taken as a scalar under Lorentz transformations in order to have the laws of physics written in the same way in any reference system connected each others by a Lorentz transformation. Now, we will suppose that (2.42) is invariant under the local transformation

$$\begin{cases} x'^a = x^a \\ A'_a = A_a + \partial_a \theta(x) \end{cases} \quad (2.43)$$

where $\theta(x)$ is an arbitrary function of spacetime. If the above transformation is a symmetry of the action functional, then $\Delta S = 0$ when $\delta_\theta A_a = \partial_a \theta$ and $\delta x^a = 0$. The total

variation of (2.42), following (2.17) is

$$\begin{aligned}\Delta S &= \int d^4x (1 + \partial_a \delta x^a) L'(\phi'(x'), \partial_a \phi'(x')) - \int d^4x L(\phi(x), \partial_a \phi(x)) \\ &\approx \int d^4x \Delta L(\phi(x), \partial_a \phi(x)) + \int d^4x \partial_a \delta x^a L(\phi(x), \partial_a \phi(x)) \\ &= \int d^4x \delta L(\phi(x), \partial_a \phi(x)) = \delta S\end{aligned}\quad (2.44)$$

since we are considering transformations that leave the spacetime coordinates unchanged. Now, the local variation of the above action functional can be written in the form

$$\delta S = \int d^4x [E^a \delta A_a + \partial_a \delta v^a] \quad (2.45)$$

where

$$E^a = \frac{\partial \mathcal{L}_{em}}{\partial A_a} - \partial_k \frac{\partial \mathcal{L}_{em}}{\partial \partial_k A_a}, \quad \delta v^a = \frac{\partial \mathcal{L}_{em}}{\partial \partial_a A_k} \delta A_k \quad (2.46)$$

Thus, under the specific transformation (2.43), (2.45) will be

$$\delta_\theta S = 0 = \int d^4x [E^a \partial_a \theta + \partial_a \delta_\theta v^a] = \int d^4x [\partial_a (E^a \theta + \delta_\theta v^a) - \partial_a E^a \theta] \quad (2.47)$$

where in the second step we have performed an integration by parts. Hence

$$\int d^4x \partial_a E^a \theta = \int d^4x \partial_a (E^a \theta + \delta_\theta v^a) \quad (2.48)$$

and choosing a proper behaviour of $\theta(x)$ on the frontier in such a way that the boundary term vanishes in the second member of the above relation we are left with

$$\partial_a E^a = 0 \quad (2.49)$$

that is the analogous of the Bianchi identity in gravitational theories.

Now, implementing the "Noether's mechanism" introduced in the previous chapter we eventually get a relation identical to (2.20)

$$\left(\frac{\partial \mathcal{L}_{em}}{\partial A_a} - \partial_k \frac{\partial \mathcal{L}_{em}}{\partial \partial_k A_a} \right) \delta A_a + \partial_a \left(\frac{\partial \mathcal{L}_{em}}{\partial \partial_a A_k} \delta A_k \right) = 0 \quad (2.50)$$

where the first term on the left vanishes on-shell and the second represents the Noether's charge on-shell. However, using the variation of the 4-potential due to the specific transformation (2.43), we get

$$E^a \partial_a \theta + \partial_a \delta_\theta v^a = \partial_a (E^a \theta + \delta_\theta v^a) = 0 \quad (2.51)$$

where we have used the Bianchi identity (2.49). Hence, we immediately read the off-shell conserved current

$$J^a = E^a \theta + \delta_\theta v^a \quad (2.52)$$

satisfying

$$\partial_a J^a = 0 \quad (2.53)$$

identically. As in the case of general theories of gravity, the equations of motion play no role in finding this Noether identity. Note once again that the existence of this current is a direct consequence of the Bianchi identity.

All this arguments are straightforwardly applied in the case of the Maxwell Lagrangian

$$\mathcal{L}_{em} = -\frac{1}{4} F_{ab}(x) F^{ab}(x) \quad (2.54)$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$ is the Faraday's tensor. The action functional written for the Maxwell Lagrangian is symmetric under (2.43) since it is easily showed that the Maxwell Lagrangian is invariant under the same transformation. In fact one can immediately see that $\delta_\theta F_{ab} = 0$ when $\delta_\theta A_a = \partial_a \theta$ assuring $\delta_\theta \mathcal{L}_{em} = 0 \rightarrow \delta_\theta S \rightarrow \Delta S = 0$. Using the explicit form of the Maxwell Lagrangian we get

$$\begin{aligned} \frac{\partial \mathcal{L}_{em}}{\partial \partial_k A_a} &= -\frac{1}{2} F^{mn} \frac{\partial F_{mn}}{\partial \partial_k A_a} = -\frac{1}{2} F^{mn} (\delta^k_m \delta^a_n - \delta^k_n \delta^a_m) \\ &= -\frac{1}{2} F^{ka} + \frac{1}{2} F^{ak} = -F^{ka} \end{aligned} \quad (2.55)$$

and in the same way

$$\frac{\partial \mathcal{L}_{em}}{\partial \partial_a A_k} = F^{ka} \quad (2.56)$$

Hence

$$E^a = -\partial_k \frac{\partial \mathcal{L}_{em}}{\partial \partial_k A_a} = \partial_k F^{ka} \quad (2.57)$$

and

$$\delta v^a = \frac{\partial \mathcal{L}_{em}}{\partial \partial_a A_k} \delta A_k = F^{ka} \partial_k \theta \quad (2.58)$$

Thus the conserved current reads

$$J^a = \partial_k F^{ka} \theta + F^{ka} \partial_k \theta = \partial_k (F^{ka} \theta) \quad (2.59)$$

We will say more about the physical meaning of this current, as well as the current for the gravitational theories, in the next two chapters when we will face the problem of finding the charge associated to such currents.

Chapter 3

Expressing the current and the associated charge

In this chapter we want to give the general expression for the conserved charge Q associated to the off-shell conserved current J^a in general theories of gravity. Before treating this specific case, we will find out how to build a conserved charge associated to a Noether current J^a which satisfies $\partial_a J^a = 0$.

3.1 The charge

Consider in a D -dimensional spacetime a vector field $J^a(x)$. The Gauss's theorem states that the volume integral of the D -divergence of $J^a(x)$ all over the D -dimensional volume \mathcal{V} is equal to the $(D - 1)$ -surface integral of $J^a(x)$ all over the hypersurface $\partial\mathcal{V}$ that contains \mathcal{V}

$$\int_{\mathcal{V}} d^D x \partial_a J^a = \int_{\partial\mathcal{V}} d^{D-1} \sigma_a J^a \quad (3.1)$$

where $d^{D-1} \sigma_a$ is one of the components of the infinitesimal element of the considered $(D - 1)$ -hypersurface. In the view of considering scalars that remain constant in time, we can take the D -dimensional volume V to be whole enclosed into two $(D - 1)$ -dimensional spacelike hypersurfaces at fixed time, namely $\partial\mathcal{V}|_{t_1}$ and $\partial\mathcal{V}|_{t_2}$, and a $(D - 1)$ -dimensional timelike hypersurface $\partial\mathcal{V}|_r$. Thus the Gauss's theorem leads to

$$\int_{\mathcal{V}} d^D x \partial_a J^a = \int_{\partial\mathcal{V}|_{t_2}} d^{D-1} \sigma_a J^a - \int_{\partial\mathcal{V}|_{t_1}} d^{D-1} \sigma_a J^a + \int_{\partial\mathcal{V}|_r} d^{D-1} \sigma_a J^a \quad (3.2)$$

where the minus sign in front of the second term in the right member express that both the normal of the spacelike hypersurfaces point to the future.

We will now take the region \mathcal{V} in (3.1) to be a D -dimensional hypersphere of radius R . Hence $\partial\mathcal{V}$ is made of two spacelike hypersurfaces in the time interval $[t_1, t_2]$, ($r < R$) $_{|t=t_1}$

and $(r < R)|_{t=t_2}$, with $t_2 > t_1$, and a timelike hypersurface $\partial V|_{r=R}$. Thus (3.2) reads

$$\int_{\mathcal{V}} d^D x \partial_a J^a = \int_{(r < R)|_{t=t_2}} d^{D-1} \sigma_a J^a - \int_{(r < R)|_{t=t_1}} d^{D-1} \sigma_a J^a + \int_{\partial V_{r=R}} d^{D-1} \sigma_a J^a \quad (3.3)$$

If the vector field J^a satisfies $\partial_a J^a = 0$ all over the D -dimensional spacetime

$$\int_{(r < R)|_{t=t_2}} d^{D-1} \sigma_a J^a - \int_{(r < R)|_{t=t_1}} d^{D-1} \sigma_a J^a = \int_{\partial V_{r=R}} d^{D-1} \sigma_a J^a \quad (3.4)$$

from which we see that the variation of the quantity represented by the integral over the region inside the hypersphere is equal to the flux of J^a through the boundary of the hypersphere (as happens for the electric charge). This fact allows us to take the integral

$$Q(t) = \int_{(r < R)|_t} d^{D-1} \sigma_a J^a \quad (3.5)$$

to be the charge into the region $(r < R)$ at time t . Since $\partial_a J^a(x) = 0$ there must exist an antisymmetric second rank tensor J^{ab} such that $J^a = \partial_b J^{ab}$. Hence we get

$$Q(t) = \int_{(r < R)|_t} d^{D-1} \sigma_a \partial_b J^{ab} \quad (3.6)$$

and using the Stokes' theorem leads to

$$Q(t) = \int_{(r < R)|_t} d^{D-1} \sigma_a \partial_b J^{ab} = \frac{1}{2} \int_{\partial(r < R)|_t} d^{D-2} \sigma_{ab} J^{ab} \quad (3.7)$$

where $d^{D-2} \sigma_{ab}$ is the infinitesimal coordinate element of the $(D-2)$ -hypersurface $\partial(r < R)|_t$, i.e the hypersurface that "cuts" the region $(r < R)$ at time t .

Now, we have to see how the charge associated to a conserved current results defined in curved spacetime. In this case we have to integrate $\nabla_a J^a$ over the proper volume integral $d^D x \sqrt{-g}$ and thus (3.1) becomes

$$\int_{\mathcal{V}} d^D x \sqrt{-g} \nabla_a J^a = \int_{\mathcal{V}} d^D x \partial_a (\sqrt{-g} J^a) = \int_{\partial \mathcal{V}} d^{D-1} \sigma_a \sqrt{-g} J^a \quad (3.8)$$

where we have used (2.34) for getting the second equality. Instead, the last one comes from the application of Gauss's theorem. Hence, in the region $(r < R)$ at time t the charge turns out to be

$$Q(t) = \int_{(r < R)|_t} d^{D-1} \sigma_a \sqrt{-g} J^a = \int_{(r < R)|_t} d^{D-1} \sigma_a \sqrt{-g} \nabla_b J^{ab} \quad (3.9)$$

and using again (2.34), which is valid also for any antisymmetric second rank tensor, we get

$$Q(t) = \int_{(r < R)|_t} d^{D-1} \sigma_a \partial_b (\sqrt{-g} J^{ab}) = \frac{1}{2} \int_{\partial(r < R)|_t} d^{D-2} \sigma_{ab} \sqrt{-g} J^{ab} \quad (3.10)$$

Hence the $(D-2)$ -hypersurface integral all over the region $\partial(r < R)$ at time t of $\sqrt{-g} J^{ab}$ is conserved. For general theories of gravity, we will now show that if an antisymmetric tensor J^{ab} exists such that $J^a = \nabla_b J^{ab}$ (and we will always suppose that such a tensor exists) then $\nabla_a J^a = 0$. In fact

$$\nabla_a J^a = \nabla_a \nabla_b J^{ab} = \frac{1}{2} [\nabla_a, \nabla_b] J^{ab} \quad (3.11)$$

where we used the antisymmetry of J^{ab} in getting the last equality. By the definition of the curvature tensor we get

$$\frac{1}{2} [\nabla_a, \nabla_b] J^{ab} = \frac{1}{2} (R^a_{kab} J^{kb} + R^b_{kba} J^{ak}) = R_{ab} J^{ab} = 0 \quad (3.12)$$

due to the symmetry of R_{kb} . Thus $\nabla_a \nabla_b J^{ab} = 0$ and we have the continuity equation in terms of J^{ab} .

3.2 Current and charge for Lagrangians $L_g = L_g(g^{ab}, R^a_{bcd})$

In chapter 2, we have seen that the generalized Bianchi identity $\nabla_a E^{ab} = 0$ leads to an off-shell conservation law expressed by equation (2.40)

$$\nabla_a J^a = 0 \quad (3.13)$$

with

$$J^a = 2E^{ak} \xi_k + \delta_\xi v^a + L_g \xi^a \quad (3.14)$$

where E^{ak} and $\delta_\xi v^a$ are defined by (1.3) and ξ^a is the displacement vector defining the diffeomorphism $x'^a = x^a + \xi^a(x)$. The conservation of J^a implies that there exists an antisymmetric tensor J^{ab} such that $J^a = \nabla_b J^{ab}$. We are interested in finding an explicit formula for J^{ab} essentially because it will help us writing the conserved charge associated to the current J^a , as we have seen in 3.1. We note that the general expression (3.14) for the current is valid in any general theory of gravity; in the case of Lagrangians written in the form $L_g = L_g(g^{ab}, R^a_{bcd})$ we are able to give an explicit expression for J^a . In order to achieve this goal, we need to put the known formulas for E^{ak} and $\delta_\xi v^a$ into (3.14). We will begin computing J^{ab} for a theory of gravity described by a Lagrangian made of the metric g^{ab} and the curvature tensor R^a_{bcd} , and we will eventually give the expression for J^{ab} for the Hilbert-Einstein action.

3.2.1 The general case

We already know that for an action functional written in the form (1.27), an arbitrary variation of the dynamical variables leads to

$$E^{ak} = \frac{\partial L_g}{\partial g_{ak}} - \frac{1}{2} L_g g^{ak} - 2 \nabla_d \nabla_b P^{adbk} \quad (3.15)$$

and hence

$$2E^{ak}\xi_k = 2\frac{\partial L_g}{\partial g^{ak}}\xi_k - L_g\xi^a - 4\nabla_d\nabla_b P^{adbk}\xi_k \quad (3.16)$$

Noting that

$$\begin{aligned} \frac{\partial L_g}{\partial g^{ak}} &= \frac{\partial L_g}{\partial R_{np}^{lm}} \frac{\partial R_{np}^{lm}}{\partial g^{ak}} = P_{lm}^{np} \frac{\partial R_{np}^{lm}}{\partial g^{ak}} \\ &= P_{lm}^{np} R^l_{hnp} \frac{\partial g^{mh}}{\partial g^{ak}} = P_{lm}^{np} R^l_{hnp} \delta^{am} \delta^{kh} \\ &= P_a^{lnp} R_{klnp} \end{aligned} \quad (3.17)$$

(where we have use $R_{np}^{lm} = R^l_{hnp} g^{mh}$ and the fact that the derivation is made keeping the curvature tensor, which does not depend on the contravariant metric, fixed), we get

$$2E^{ak}\xi_k = 2P^{adbl} R^k_{dbl}\xi_k - L_g\xi^a - 4\nabla_d\nabla_b P^{adbk}\xi_k \quad (3.18)$$

Now, recalling (1.49), we get

$$\delta_\xi v^a = 2P_l^{bad} \delta_\xi \Gamma^l_{bd} - 2\nabla_b P^{lbad} \delta_\xi g_{dl} \quad (3.19)$$

where

$$\delta_\xi g_{dl} = -\nabla_{(d}\xi_{l)} \quad (3.20)$$

Thus the second term of the boundary term is

$$-2\nabla_b P^{lbad} \delta_\xi g_{dl} = 2\nabla_b P^{lbad} (\nabla_d \xi_l + \nabla_l \xi_d) \quad (3.21)$$

and after the exchange $l \leftrightarrow d$ in the second term of the last identity we get

$$-2\nabla_b P^{lbad} \delta_\xi g_{dl} = 2\nabla_b (P^{lbad} + P^{dbal}) \nabla_d \xi_l \quad (3.22)$$

The first term involves the variation of a connection we have already computed in 1.2, namely

$$\delta_\xi \Gamma^l_{bd} = \frac{1}{2} R^l_{(bd)k} \xi^k - \frac{1}{2} \nabla_{(b} \nabla_{d)} \xi^l \quad (3.23)$$

We can go further in manipualitng this and get

$$\begin{aligned} -\frac{1}{2} (\nabla_b \nabla_d \xi^l + \nabla_d \nabla_b \xi^l) &= -\frac{1}{2} (2\nabla_d \nabla_b \xi^l + [\nabla_b, \nabla_d] \xi^l) \\ &= -\nabla_d \nabla_b \xi^l + \frac{1}{2} R^l_{kdb} \xi^k \end{aligned} \quad (3.24)$$

and thus

$$\delta_\xi \Gamma^l_{bd} = -\nabla_d \nabla_b \xi^l + \frac{1}{2} (R^l_{kdb} + R^l_{bdk} + R^l_{dbk}) \xi^k \quad (3.25)$$

Adding and subtracting R^l_{bkd} in the term involving the sum of curvature tensors we get

$$\begin{aligned}\delta_\xi \Gamma^l_{bd} &= -\nabla_d \nabla_b \xi^l + \frac{1}{2}(R^l_{kdb} + R^l_{bdk} + R^l_{dbk} + R^l_{bkd} - R^l_{bkd})\xi^k \\ &= -\nabla_d \nabla_b \xi^l + \frac{1}{2}(R^l_{bdk} - R^l_{bkd} + \frac{1}{2}R^l_{[dbk]})\xi^k \\ &= -\nabla_d \nabla_b \xi^l + R^l_{bdk}\xi^k\end{aligned}\quad (3.26)$$

where $R^l_{[dbk]} = 2(R^l_{dbk} + R^l_{kdb} + R^l_{bkd}) = 0$ and the antisymmetry of the curvature tensor have been used to obtain the final result. Hence

$$\begin{aligned}2P_l^{bad}\delta_\xi \Gamma^l_{bd} &= 2P_l^{bad}(-\nabla_d \nabla_b \xi^l + R^l_{bdk}\xi^k) \\ &= 2P^{adbl}\nabla_d \nabla_b \xi_l - 2P^{adbl}R_{kdbl}\xi^k\end{aligned}\quad (3.27)$$

and the boundary term is

$$\begin{aligned}\delta_\xi v^a &= 2\nabla_b(P^{lbad} + P^{dbal})\nabla_d \xi_l + 2P^{adbl}\nabla_d \nabla_b \xi_l - 2P^{adbl}R_{kdbl}\xi^k \\ &= -2\nabla_b(P^{adbl} + P^{albd})\nabla_d \xi_l + 2P^{adbl}\nabla_d \nabla_b \xi_l - 2P^{adbl}R_{kdbl}\xi^k\end{aligned}\quad (3.28)$$

The conserved current now reads

$$\begin{aligned}J^a &= 2P^{adbl}R_{kdbl}\xi^k - L_g \xi^a - 4\nabla_d \nabla_b P^{adbk}\xi_k + L_g \xi^a \\ &\quad - 2\nabla_b(P^{adbl} + P^{albd})\nabla_d \xi_l + 2P^{adbl}\nabla_d \nabla_b \xi_l - 2P^{adbl}R_{kdbl}\xi^k \\ &= -2\nabla_b(P^{adbl} + P^{albd})\nabla_d \xi_l + 2P^{adbl}\nabla_d \nabla_b \xi_l - 4\nabla_d \nabla_b P^{adbl}\xi_l\end{aligned}\quad (3.29)$$

Now we can guess the form of J^{ab} simply observing that the above current depends upon the displacement ξ_l and its first and second derivatives. Thus we choose the following ansatz for J^{ab} (a similar calculation can be found in [4])

$$J^{ab} = A^{abdl}\nabla_d \xi_l + B^{abl}\xi_l\quad (3.30)$$

Differentiating it we get

$$\nabla_b J^{ab} = \nabla_b A^{abdl}\nabla_d \xi_l + A^{abdl}\nabla_b \nabla_d \xi_l + \nabla_b B^{abl}\xi_l + B^{abl}\nabla_b \xi_l\quad (3.31)$$

and comparing to (3.29) we can make the following identifications

$$A^{abdl} = 2P^{abdl}\quad (3.32)$$

and

$$\begin{aligned}\nabla_b B^{abl}\xi_l &= -4\nabla_d \nabla_b P^{adbl}\xi_l = -4\nabla_b \nabla_d P^{adbl}\xi_l \Rightarrow \\ B^{abl} &= -4\nabla_d P^{adbl} + V^{abl}\end{aligned}\quad (3.33)$$

with $\nabla_b V^{abl} = 0$. Moreover, the following identity must hold

$$\nabla_b A^{abdl} + B^{adl} = -2\nabla_b(P^{adbl} + P^{albd}) \quad (3.34)$$

and we will now verify it

$$\begin{aligned} 2\nabla_b P^{abdl} - 4\nabla_b P^{adbl} + V^{abl} &= 2\nabla_b(P^{abdl} - 2P^{adbl}) + V^{abl} \\ &= -2\nabla_b(P^{adbl} + P^{albd}) + V^{abl} \end{aligned} \quad (3.35)$$

where the symmetry property $P^{a[bdl]} = 0$ has been used to obtain the final equality. We see that, for inner consistency, it must be $V^{abl} = 0$. Thus J^{ab} reads

$$J^{ab} = 2P^{abdl}\nabla_d\xi_l - 4\nabla_d P^{abdl}\xi_l \quad (3.36)$$

J^{ab} is not unique, since any change $J^{ab} \rightarrow J^{ab} + V^{ab}$ with $\nabla_b V^{ab} = 0$ leads to the same conserved current J^a . We can now write down the charge in the region $r < R$ at time t for this general case, namely

$$Q(t) = \frac{1}{2} \int_{\partial(r < R)_t} d^2\sigma_{ab}\sqrt{-g} (2P^{abdl}\nabla_d\xi_l - 4\nabla_d P^{abdl}\xi_l) \quad (3.37)$$

In general

$$Q(t) = \frac{1}{2} \int_{\partial\Lambda_t} d^2\sigma_{ab}\sqrt{-g} (2P^{abdl}\nabla_d\xi_l - 4\nabla_d P^{abdl}\xi_l) \quad (3.38)$$

where Λ_t is any spacelike region at time t and $\partial\Lambda_t$ is its boundary. For further applications, it is worth to effort the case in which the displacement ξ^a is a Killing vector, at least into a region around a spacetime event, and see how the current J^a and the corresponding J^{ab} look like. If ξ^a is a Killing vector, it satisfies $\nabla_{(a}\xi_{b)} = 0$ and $\nabla_a\nabla_b\xi_c = R^k_{abc}\xi_k$. Thus, in (3.29) the first term vanishes by symmetry and we are left with

$$J^a_K = 2P^{abdl}\nabla_d\nabla_b\xi_l - 4\nabla_d\nabla_b P^{abdl}\xi_l \quad (3.39)$$

and using the second property of ξ^a recalled above

$$J^a_K = 2P^{abdl}R^k_{dbl}\xi_k - 4\nabla_d\nabla_b P^{abdl}\xi_k \quad (3.40)$$

and we see that when computed in the correspondence of a Killing vector, the current can be written in such a way it is just proportional to ξ . The above expression is also equal to

$$J^a_K = (2E^{ak} + L_g g^{ak})\xi_k \quad (3.41)$$

We could have reached immediately this expression for the current just remembering the form (3.14) and the fact that the boundary term vanishes in correspondence of a Killing vector, as we have seen in chapter 2. For an action functional

$$S = \frac{1}{16\pi C} \int d^D x \sqrt{-g} L_g(g^{ab}, R^a_{bcd}) \quad (3.42)$$

we get

$$Q(t) = \frac{1}{32\pi C} \int_{\partial(r < R)|_t} d^2 \sigma_{ab} \sqrt{-g} (2P^{abdl} \nabla_d \xi_l - 4\nabla_d P^{abdl} \xi_l) \quad (3.43)$$

and

$$Q(t) = \frac{1}{32\pi C} \int_{\partial\Lambda|_t} d^2 \sigma_{ab} \sqrt{-g} (2P^{abdl} \nabla_d \xi_l - 4\nabla_d P^{abdl} \xi_l) \quad (3.44)$$

where

$$P_a{}^{bcd} = \left(\frac{\partial L_g}{\partial R^a{}_{bcd}} \right)_{g^{ab}} \quad (3.45)$$

i.e. the P s remain defined via the Lagrangian L_g .

3.2.2 Hilbert-Einstein case

We will now specialize the previous discussion in the case of the Hilbert-Einstein action and we will see how J^{ab} looks like in this special case. We consider the Lagrangian

$$S = \int d^D x \sqrt{-g} R = \int d^D x \sqrt{-g} L_g \quad (3.46)$$

which can be at once written in terms of the curvature tensor. Since $R = g^{kl} R_{kl}$ we get

$$\begin{aligned} S &= \int d^D x \sqrt{-g} g^{kl} R_{kl} = \int d^D x \sqrt{-g} g^{kl} R^m{}_{kml} \\ &= \int d^D x \sqrt{-g} [g^{kl} \frac{1}{2} (R^m{}_{kml} - R^m{}_{klm})] \end{aligned} \quad (3.47)$$

From the action written this way, the tensor $P^a{}_{bcd}$, which is the key to write down the current J^a and the corresponding J^{ab} , is easily extracted. In fact we immediately get

$$\begin{aligned} P_a{}^{bcd} &= \frac{\partial L}{\partial R^a{}_{bcd}} = \frac{1}{2} \left[g^{ab} \frac{\partial R^m{}_{kml}}{\partial R^a{}_{bcd}} - \{l \leftrightarrow m\} \right] \\ &= \frac{1}{2} (g^{kl} \delta_a^m \delta_k^b \delta_m^c \delta_l^d - \{l \leftrightarrow m\}) = \frac{1}{2} (\delta_a^c g^{bd} - \delta_a^d g^{bc}) \end{aligned} \quad (3.48)$$

where in the second step the fact that g^{ab} is kept fixed while deriving the Lagrangian with respect the curvature tensor has been used. For our applications we need the full contravariant tensor P^{abdl} which is simply

$$P^{abdl} = g^{ak} P_k{}^{bdl} = \frac{1}{2} g^{ak} (\delta_k^d g^{bl} - \delta_k^l g^{bd}) = \frac{1}{2} (g^{ad} g^{bl} - g^{al} g^{bd}) \quad (3.49)$$

From this, we see that P^{abdl} is divergence-less in all its indexes. Hence, for the Hilbert-Einstein action, (3.36) reads

$$J^{ab} = 2P^{abdl}\nabla_d\xi_l = (g^{ad}g^{bl} - g^{al}g^{bd})\nabla_d\xi_l = (\nabla^a\xi^b - \nabla^b\xi^a) \quad (3.50)$$

(which is not unique for a given J^a) with the corresponding current

$$J^a = \nabla_b(\nabla^a\xi^b - \nabla^b\xi^a) \quad (3.51)$$

The charge in a spatial region Λ_t at time t , instead, is

$$Q(t) = \frac{1}{2} \int_{\partial\Lambda_t} d^{D-2}\sigma_{ab}\sqrt{-g}(\nabla^a\xi^b - \nabla^b\xi^a) \quad (3.52)$$

For

$$S = \frac{1}{16\pi C} \int d^Dx \sqrt{-g}R \quad (3.53)$$

we get

$$Q(t) = \frac{1}{32\pi C} \int_{\partial\Lambda_t} d^{D-2}\sigma_{ab}\sqrt{-g}(\nabla^a\xi^b - \nabla^b\xi^a) \quad (3.54)$$

3.3 Horizons in static spherically-symmetric metrics

For our purposes, we will consider horizons that come from a given background metric. In a general theory of gravity living in a D -dimensional spacetime, when one considers a spherically-symmetric mass distribution collapsed in such a way that it can be viewed as a pointlike source, the gravitational field outside the source in vacuum will be the one described by the following spacetime interval

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + dX_{\perp}^2 \quad (3.55)$$

where $f(r) = (1 - r_H/r)$ and dX_{\perp}^2 denotes the metric on the $t = \text{constant}$, $r = \text{constant}$ surface. The surface $H = r - r_H$ defines an horizon, i.e. a region of spacetime that behaves like a semipermeable membrane. No signal of any kind can cross the horizon from the inner region $r < r_H$ to reach the outside region $r > r_H$. On this surface, $f(r_H) = 0$ and the spacetime interval apparently diverges, but this behaviour it is not linked to the existence of a true singularity, i.e. a region of spacetime in which the curvature tensor becomes infinite, rather to a bad choice of our coordinates system. However, in physical terms this surface is an infinite redshift surface, i.e. a luminous signal moving from the inside of the horizon towards the outer space will take an infinite time to reach an observer placed outside the horizon, because the dilatation of time, as measured by the external observer, diverges when $r = r_H$. The metric defined by (3.55) describes a static gravitational field, as stated by the Birkhoff theorem.

Now, we would like to find the charge (3.10) associated to the horizon H . It is evident that one has a charge for each diffeomorphism $x'^a = x^a + \xi^a(x)$ i.e. a charge for each displacement vector $\xi^a(x)$. However, we will consider just one, particular vector ξ^a , namely the vector that generates an isometry of spacetime, i.e. a diffeomorphisms that leaves the metric unchanged, $\delta g_{ab} = 0$. Hence, in order to represent an isometry, the vector ξ^a must be a Killing vector, i.e. it must satisfy Killing's equation $\nabla_{(a}\xi_{b)} = 0$. Since the metric (3.55) is static, we will consider the isometry that represents the time-translation invariance of the metric in Schwarzschild geometry. Such an isometry is generated by the Killing vector $\xi = \partial/\partial t$. We will now show that this vector is normal to the horizon H . For doing this let us consider the gradient of H which is a covariant vector normal to H by construction whose components are

$$n_a = \partial_a H \quad (3.56)$$

The corresponding contravariant components are

$$n^a = g^{ab}n_b = g^{ab}\partial_b H \quad (3.57)$$

and since the only non vanishing component of n_a is n_r we get

$$n^r = g^{rr} = f(r) \quad (3.58)$$

and the contravariant normal vector is

$$\mathbf{n} = n^a \partial_a = f(r) \frac{\partial}{\partial r} \quad (3.59)$$

Now, as $r \rightarrow r_H$, $\xi \rightarrow \mathbf{n}$, as we will see. To show this, it is convenient to introduce the Kruskal-Szekeres coordinates

$$\begin{aligned} \kappa U &= -\exp(-\kappa u), & u &= t - r^* \\ \kappa V &= \exp(\kappa v), & v &= t + r^* \end{aligned} \quad (3.60)$$

where

$$r^* = \int \frac{dl}{f(l)} \quad (3.61)$$

is the so called tortoise-coordinate, and $\kappa = 1/2f'(r_H)$ is what we will later call the surface gravity. In terms of this new coordinates the Killing vector ξ becomes

$$\xi = \frac{\partial}{\partial t} = \frac{\partial U}{\partial t} \frac{\partial}{\partial U} + \frac{\partial V}{\partial t} \frac{\partial}{\partial V} = \kappa \left(-U \frac{\partial}{\partial U} + V \frac{\partial}{\partial V} \right)$$

The condition $U = 0$ ($t \rightarrow \infty$) defines the future horizon and the above vector on this horizon reads

$$\xi|_H = \kappa \left(V \frac{\partial}{\partial V} \right)_H \quad (3.62)$$

Now, if one writes the normal vector \mathbf{n} in the Kruskal-Szekeres coordinates finds that

$$\mathbf{n} = \kappa \left(U \frac{\partial}{\partial U} + V \frac{\partial}{\partial V} \right) \quad (3.63)$$

which on the future horizon becomes

$$\mathbf{n}|_H = f \left(\frac{\partial}{\partial r} \right)_H = \kappa \left(V \frac{\partial}{\partial V} \right)_H = \boldsymbol{\xi}|_H \quad (3.64)$$

Hence, on the horizon H the Killing vector $\boldsymbol{\xi} = \partial/\partial t$ is normal to H . Moreover, it is a null vector. In fact its norm is given by [11]

$$\xi^2 = C^2 \kappa UV, \quad C^2 = \frac{8r_H^2}{r} \quad (3.65)$$

and since the condition $UV = 0$ defines the horizon in the Kruskal-Szekeres coordinates, ξ^2 vanishes on the horizon. Hence the Killing vector $\boldsymbol{\xi} = \partial/\partial t$ is a vector normal to the horizon H whose norm vanishes on H . In order to apply the results of previous sections, we compute the charge associated to the horizon, Q_H , as the charge associated to any region ($r < R$) with $R > r_H$, and taking the limit $R \rightarrow r_H$.

3.4 The charge for the general case

Consider the action

$$S = \frac{1}{16\pi C} \int d^D x \sqrt{-g} L_g(g^{ab}, R^a_{bcd}, \nabla_k R^a_{bcd}, \dots) \quad (3.66)$$

that is the one given by (1.2) with a dimensional normalization factor, required to give the strength of the coupling between gravity and matter sectors in total Lagrangian. The charge associated to $\xi^a = (\partial/\partial t)^a$ is

$$Q = \frac{1}{32\pi C} \int_{\partial\Lambda} d^{D-2} \sigma_{ab} \sqrt{-g} J^{ab} \quad (3.67)$$

where J^{ab} is derived as in (3.36) from L_g . If the frontier $\partial\Lambda$ is taken to be a spherically symmetric hypersurface with metric given by (3.55) we get

$$Q_H = \lim_{R \rightarrow r_H} \left(\frac{1}{32\pi C} \int_{r=R} d^{D-2} \sigma_{ab} \sqrt{-g} J^{ab} \right) = \frac{1}{32\pi C} \int_{r=r_H} d^{D-2} \sigma_{ab} \sqrt{-g} J^{ab} \quad (3.68)$$

where now $d^{D-2} \sigma_{ab}$ is the infinitesimal coordinate area element of a $(D-2)$ -sphere of radius $R = r_H$. The charge associated to H does not depend upon time anymore since

the background metric is static. (3.68) is the more general expression we can provide for the charge associated to a horizon through $\xi^a = (\partial/\partial t)^a$. We can go a little more further considering Lagrangians of the form $L_g = L_g(g^{ab}, R^a{}_{bcd})$. In this case, the charge associated to the horizon can be expressed in terms of $P_a{}^{bcd} = (\partial L_g / \partial R^a{}_{bcd})$ and reads

$$Q_H = \frac{1}{32\pi C} \int_{r=r_H} d^{D-2} \sigma_{ab} \sqrt{-g} (2P^{abcd} \nabla_d \xi_l - 4\nabla_d P^{abcd} \xi_l) \quad (3.69)$$

3.5 The charge in General Relativity

We would like to apply the above strategy to compute the charge (3.68) associated to the conserved current J^a in the case of GR, i.e. for a theory of gravity described by the Hilbert-Einstein action (1.4), with $C = G$, in a $D = 4$ -dimensional spacetime. The spacial cross-section of the horizon is the 2-sphere at $r = r_H$ and we consider a slightly larger 2-sphere with $R = r_H + \epsilon$.

The charge associated to ($r < R$) is

$$Q = \frac{1}{32\pi G} \int_{r=R} d^2 \sigma_{ab} \sqrt{-g} J^{ab} = \frac{1}{32\pi G} \int_{r=R} d^2 S_{ab} J^{ab} \quad (3.70)$$

In the above expression we have introduced the proper infinitesimal area element

$$d^2 S_{ab} = \sqrt{-g} d^2 \sigma_{ab} = \frac{1}{2} \sqrt{-h} \sqrt{\gamma} [abcd] \left| \frac{\partial(x^c, x^d)}{\partial(\theta^1, \theta^2)} \right| \theta^1 d\theta^2 \quad (3.71)$$

where $[abcd]$ denotes the complete antisymmetric symbol, γ is the determinant of the intrinsic metric of the 2-sphere and h is the determinant of the metric of the $(t-r)$ plane orthogonal to the 2-sphere. This can be rewritten as

$$d^2 S_{ab} = \sqrt{-h} [ab] dS \quad (3.72)$$

where dS is the proper infinitesimal area element of the 2-sphere. The antisymmetric combination $[ab]$ can be written in terms of two covariant vectors noting that the 2-sphere has two normals belonging to the $(t-r)$ plane. We can implement this by considering the following covariant vectors

$$v_a = (-1, 0, 0, 0), \quad w_a = (0, 1, 0, 0) \quad (3.73)$$

and the combination

$$[ab] = -v_a w_b + v_b w_a \quad (3.74)$$

which is antisymmetric by construction. Now, the charge reads

$$Q = \frac{1}{32\pi G} \int_{r=R} dS (-v_a w_b + v_b w_a) J^{ab} \quad (3.75)$$

where we used the fact that $\sqrt{-h} = 1$ for the metric defined by (3.55). By virtue of the antisymmetry of J^{ab} we get

$$Q = -\frac{1}{16\pi G} \int_{r=R} dS v_a w_b J^{ab} \quad (3.76)$$

Inserting the form of J^{ab} given for the Hilbert-Einstein action by (3.50) into the above expression, leads to

$$Q = -\frac{1}{16\pi G} \int_{r=R} dS v_a w_b (\nabla^a \xi^b - \nabla^b \xi^a) = -\frac{1}{8\pi G} \int_{r=R} dS v_a w_b \nabla^a \xi^b \quad (3.77)$$

where in the last equality we have used the fact that ξ^a is a Killing vector. Now we see that the vectors v_a and w_a can be written in terms of ξ^a . In fact since $\xi = \partial/\partial t$ we have $\xi^a = (1, 0, 0, 0)$. Thus

$$v^a = g^{ab} v_b \Rightarrow v^a = \left(\frac{1}{f}, 0, 0, 0 \right) = \frac{1}{f} \xi^a \quad (3.78)$$

and

$$w^a = g^{ab} w_b \Rightarrow w^a = (0, f, 0, 0) = f \left(\frac{\partial}{\partial r} \right)^a \quad (3.79)$$

and the charge reads

$$Q = -\frac{1}{8\pi G} \int_{r=R} dS \frac{1}{f} \xi_a f \left(\frac{\partial}{\partial r} \right)_b \nabla^a \xi^b \quad (3.80)$$

Now, we have to compute the quantity $\xi_a \nabla^a \xi^b$. Consider a surface \mathbf{S} and vector l^a normal to \mathbf{S} . Since l^a is orthogonal to \mathbf{S} one can write $l_a = \mu(x) \partial_a \mathbf{S}$, where $\mu(x)$ is an arbitrary function of spacetime. Hence the product $l^a \nabla_a l^b$ is

$$\begin{aligned} l^a \nabla_a l^b &= l^a \nabla_a \mu(x) \partial^b \mathbf{S} + l^a \mu(x) \nabla_a \partial^b \mathbf{S} \\ &= l^a \mu^{-1}(x) \partial_a \mu(x) l^b + g^{bk} l^a \mu(x) \nabla_a \partial_k \mathbf{S} \\ &= \frac{d}{d\lambda} [\ln \mu(x)] l^b + g^{bk} l^a \mu(x) \nabla_k (\mu^{-1}(x) l_a) \\ &= \frac{d}{d\lambda} [\ln \mu(x)] l^b + \frac{1}{2} \partial^b l^2 - l^2 \partial^b [\ln \mu(x)] \end{aligned} \quad (3.81)$$

Now, we suppose that the vector l^a is null on \mathbf{S} , i.e. $l^2 = 0$ on \mathbf{S} . Hence, the last term in the above expression is trivially zero. In addition, since l^2 is constant on \mathbf{S} , $t_b \partial^b l^2 = 0$ for any vector t_b tangential to \mathbf{S} . Thus if one chose $t_b = l_b$ it follows that $\partial^b l^2 \propto l^b$ and $l^a \nabla_a l^b \propto l^b$. The function $\mu(x)$ can be chosen such that $l \cdot \nabla = 0$. Thus, let ξ^a a Killing vector normal to \mathbf{S} and l^a a vector normal to \mathbf{S} such that $l \cdot \nabla = 0$. Then, on \mathbf{S}

$$\xi^a = f l^a \quad (3.82)$$

for some function f , and thus it follows that

$$\xi^a \nabla_a \xi^b = \kappa \xi^b \quad (3.83)$$

where $\kappa = \xi \cdot \partial \ln |f|$ is called the *surface gravity*. We can provide a formula for κ in terms of quantity related to the metric of the horizon. Since ξ^a is normal to H , we can invoke the *Frobenius' theorem* which implies that

$$\xi_{[a} \nabla_b \xi_{c]} = 0 \quad (3.84)$$

where [...] indicates total antisymmetry in the indices a, b, c . For a Killing vector ξ^a , $\nabla_a \xi_b = \nabla_{[a} \xi_{b]}$ (symmetric part of $\nabla_a \xi_b$ vanishes). In this case (3.84) becomes

$$\xi_c \nabla_a \xi_b + (\xi_a \nabla_b \xi_c - \xi_b \nabla_a \xi_c) = 0 \quad (3.85)$$

Multiplying by $\nabla^a \xi^b$ we get

$$\xi_c (\nabla^a \xi^b) (\nabla_a \xi_b) = -2 (\nabla^a \xi^b) \xi_a (\nabla_b \xi_c) \quad (3.86)$$

and using (3.83)

$$\begin{aligned} \xi_c (\nabla^a \xi^b) (\nabla_a \xi_b) &= -2 \kappa \xi^b (\nabla_b \xi_c) \\ &= -2 \kappa^2 \xi_c \end{aligned} \quad (3.87)$$

Hence

$$\kappa^2 = \frac{1}{2} (\nabla^a \xi^b) (\nabla_a \xi_b) \quad (3.88)$$

It is important to note that κ is defined only by the Killing's field which is given by the metric, independently of how this metric has been generated. Let us come back to the charge. The relation (3.83) holds on the horizon. However, if we compute it explicitly we get

$$\xi^a \nabla_a \xi^b = \xi^a \xi^k \Gamma^b_{ak} = \Gamma^b_{00} = \frac{1}{2} g^{br} \frac{\partial g_{00}}{\partial r} \quad (3.89)$$

and since $g_{00} = -f(r)$

$$\xi^a \nabla_a \xi^b = -\frac{1}{2} f \partial_r f \delta^b_r \quad (3.90)$$

showing that $\xi^a \nabla_a \xi^b$ is in the direction of r . On the horizon, we know that it must have the direction of t because of (3.83). This is assured since $f(\partial/\partial r) \rightarrow (\partial/\partial t)$ when $R \rightarrow r_H$. On a larger horizon instead we can write

$$\xi^a \nabla_a \xi^b = \kappa f \left(\frac{\partial}{\partial r} \right)^b \quad (3.91)$$

Using this, (3.80) can be written as

$$Q = -\frac{\kappa}{8\pi G} \int_{r=R} dS f \left\| \left(\frac{\partial}{\partial \mathbf{r}} \right) \right\| \quad (3.92)$$

Now in the limit $R \rightarrow r_H$ we get

$$Q_H = \lim_{R \rightarrow r_H} \left(-\frac{\kappa}{8\pi G} \int_{r=R} dS f \left\| \left(\frac{\partial}{\partial \mathbf{r}} \right) \right\| \right) = -\frac{\kappa}{8\pi G} \int_{r=r_H} dS = -\frac{\kappa}{2\pi G} \left(\frac{A_H}{4} \right) \quad (3.93)$$

where in the second equality we have used the fact that the norm of $(\partial/\partial r)^a$ is f^{-1} . In the last equality, A_H represents the area of the whole horizon. The most important result which has to be stressed here is that in deriving never the horizon charge for the more general action (1.27) nor for the horizon charge in GR, we have invoked the equations of motion, i.e. both of these are off-shell conserved charges. This fact has the immediate consequence that the surface gravity κ contains no dynamical information, but on the contrary it is a pure cinematic quantity that comes up as a result of the choice of the background metric.

Chapter 4

Horizon entropy

A remarkable connection between thermodynamics and gravity arises in black hole physics, namely, black holes carry an intrinsic entropy. This result relies on the fundamental property that a black hole is a region of spacetime which is inaccessible to observations, and an essential role is played by the event horizon, the boundary between the regions observable and unobservable from infinity. Consider a box carrying some thermal systems, one may expect that its internal state will be taken out of equilibrium. According to the Second Law of thermodynamics, the subsequent evolution would then be characterized by a continued increase of the entropy, as the system returns to equilibrium. If the box were to fall into a black hole, it would move out of the region of spacetime in which measurements can be observed from infinity, and there would no longer be any evidence of the entropy carried by the box. The entropy in the observable spacetime would thus appear to have decreased, yielding an apparent violation of the Second Law. To restore the validity of the Second Law, one can assign an extra entropy to the black hole or to the horizon.

Similar reasoning led Bekenstein to make the bold conjecture, within GR, black holes carry an intrinsic entropy given by the surface area of the horizon measured in Planck units multiplied by a dimensionless number of order one [2]. This conjecture was also suggested by Christodoulou's works about the mechanical transformations involving black holes area and the subsequent Hawking's area theorem, which had shown that, like entropy, the horizon area can never decrease in classical GR (see [3] and [6]).

The next crucial insight came from Hawking while investigating quantum fields in a black hole spacetime. He found that external observers detect the emission of thermal radiation from a black hole with a temperature proportional to its surface gravity κ

$$T = \frac{\kappa}{2\pi} \tag{4.1}$$

What Hawking found is that if one faces the problem of describing a collapsing system in GR from the point of view of QFT, for example in terms of a scalar field, it happens

that when matter collapses to form a black hole, observers at large distances will receive thermal radiation of particles from the black hole at late times, with a thermal spectrum at the temperature (4.1), (see [8] and [7]). Due to the emission of particle a balancing relation of the kind $dM = dE$ should hold and since we are speaking of a thermal radiation one should have $dM = TdS_{BH}$, in accordance to the energy conservation principle. It seems natural to assume that the source of this energy radiated to infinity is the mass of the collapsing structure. In GR, given the temperature of a spherically symmetric black hole $T(E) = (8\pi GM)^{-1}$ as a function of the energy $E = M$, we can integrate the expression $dS_{BH} = dE/T(E)$ to define an entropy $S(E)_{BH}$ for the black hole

$$S_{BH} = \int_0^M dE(8\pi GE) = 4\pi GM^2 = \frac{1}{4} \left(\frac{A_H}{G} \right) \quad (4.2)$$

where A_H is the area of the $r = r_H = 2GM$, $t = \text{const}$ surface. This is the Bekenstein-Hawking entropy as appears in [2]. Hence, for getting the notion of horizon entropy in the Bekenstein-Hawking approach, it is crucial to have a black body radiation flux from the black hole. Trying to generalize this same approach to general theories of gravity, means to be able to manage QFT in D -dimensional, with $D > 4$, curved spacetimes of general background metric. This can result in a difficult or, depending on the theory under consideration, in a hopeless task. We should look to another way of introducing the notion of entropy of horizons, potentially free of the difficulty above. The stress on this point is precisely the main motivation of the thesis.

First of all, we notice that the association of a temperature to a horizon is conceptually distinct from the calculation of any radiation flux from it. We know that in Rindler spacetime, a temperature is assigned to the horizon (the accelerating observer feels herself immersed in a thermal bath) but no flux of radiation from the horizon is present. Any horizon which is locally approximated by a Rindler spacetime is naturally endowed with a notion of (Rindler horizon) temperature of the form (4.1). For any assigned metric with a horizon, the association (4.1) intended in this way is well-defined and is unrelated to the gravitational theory. In fact the surface gravity depends only upon quantities defined via the metric as appears clearly looking at (3.83) or (3.88). The association (4.1) results well defined even in flat spacetime in Rindler coordinates, i.e. even in conditions with no curvature at all. More details and examples about this interesting issue can be found in [15].

Thus, given a spacetime with an horizon of a given background metric, what we immediately have is a temperature associated to the horizon, whatever the gravitational theory is, i.e. whatever the theory which has the considered background metric as a solution of the equations of motion is.

We are now interested in joining this notion of temperature to the notion of Noether charge of the horizon as introduced in the previous chapter for general theories, to try to construct a sensible notion of entropy for the horizon. If we manage to do that, what we obtain is an expression S_{Noether} for the horizon entropy that is valid for any general

diffeomorphisms invariant theory of gravity, tackling this way the difficulty mentioned above with S_{BH} .

4.1 Horizon entropy

Let us summarize what we have obtained so far:

- For any general theory of gravity described by the action (1.2), for which the Lagrangian is a scalar under general diffeomorphisms, it is possible to extract a current (3.14), $J^a = 2E^{ak}\xi_k + L_g\xi^a + \delta_\xi v^a$ which satisfies $\nabla_a J^a = 0$ off-shell.
- Associated to J^a there is a charge Q whose expression is given by (3.5). This charge, as well as J^a , is off-shell conserved, in the sense explained in chapter 3.
- In GR, the charge associated to the Killing vector ξ^a representing the time translation invariance of the metric generated by a collapsed spherically symmetric mass distribution, i.e. the Schwarzschild's metric, when evaluated on the event horizon represented by a spacial 2-sphere of radius $R = r_s = 2M$, which ξ^a is normal to, is given by (3.93)

$$Q_H = -\frac{\kappa}{2\pi G} \left(\frac{A_H}{4} \right) \quad (4.3)$$

Hence, looking at (4.2), it seems natural to write

$$S_{\text{Noether}} \equiv -\left(\frac{2\pi}{\kappa} \right) Q_H \quad (4.4)$$

as it gives $S_{\text{Noether}} = S_{BH}$. Rewriting equation (4.2) as

$$S_{BH} = (8\pi GM) \frac{M}{2} = \left(\frac{2\pi}{\kappa} \right) \frac{M}{2} \quad (4.5)$$

we see that Q_H in (4.3) plays the role of $-(M/2)$ in GR. Indeed, we recognize in the formula (3.70) for the charge, with J^{ab} given by (3.50), minus half of the mass in the Komar expression for the latter [18, p. 289], [9]. This relation connecting the charge and the mass contained inside the horizon is crucial to recover the first law of thermodynamics written in terms of (4.4). In fact, in GR (4.5) holds and differentiating it we get

$$TdS_{BH} = dM \quad (4.6)$$

Thus, the relation $Q_H = -M/2$ allows to write

$$TdS_{\text{Noether}} = dM \quad (4.7)$$

which reproduces (4.6) in GR. We have now to understand better the general role of Q_H . Is the relation (4.4) an artifact of GR as a particular case? Or can be considered instead a general result? The crucial point is to verify if we can write in all generality a relation $-2dQ_H = dM$ across any (hyper)surface. This would imply $TdS_{\text{Noether}} = dM$ for any horizon, that is the first law of thermodynamics [1] as applied to horizons, indicating that the position (4.4) is a sensible definition of horizon entropy for general theories. In fact, from first law we know that $dS_{\text{Noether}}/dM = 1/T$, and thus it is independent of the theory taken in account. This implies that, even if both S_{Noether} and M do depend on the theory under consideration, their functional relationship must be independent of it, and will be identical to that found in GR. Hence, in any theory of gravity, we should have $TS_{\text{Noether}} = M/2$ and thus $Q_H = -M/2$, from which $-2dQ_H = dM$. We will now prove this relation considering general theories of gravity.

4.2 The charge and the matter flux

We will now see that (4.4) is really the entropy associated to horizon in a general diffeomorphism invariant gravitational theory. In order to do that, we will consider any $(D-1)$ -dimensional spacial hypersurface whose normal n^a is orthogonal to the the Killing vector field $\xi^a = (\partial/\partial t)^a$. We want now to evaluate the infinitesimal amount of matter dM that crosses the horizon and compare it to dQ_H . The former is

$$dM = -\sqrt{-g} T^{ab} \xi_b n_a dA dr \quad (4.8)$$

where the minus sign means that the matter flux occurs in the opposite verse respect to the normal to the hypersurface $n^a = (0, 1, 0, 0, \dots)$ which lies into the r -plane. In (4.8) ξ^a is the Killing vector $(\partial/\partial t)^a$. The normal is taken in such a way that $n_a \xi^a = 0$. Now, if we use the equations of motion $E^{ab} = 8\pi C T^{ab}$ we get

$$dM = -\frac{1}{16\pi C} \sqrt{-g} 2E^{ab} \xi_b n_a dA dr \quad (4.9)$$

From the expression for the current (2.6) we know that $2E^{ab} \xi_b = J^a - 2L_g \xi^a - \delta_\xi v^a$, with the boundary term $\delta_\xi v^a$ given by (1.57). In chapter 2 we have showed that the boundary term vanishes everywhen it is computed in correspondence of a Killing vector and this is true in any general theory of gravity. Thus $2E^{ab} \xi_b = J^a - 2L_g \xi^a$ and since $n_a \xi^a = 0$, (4.9) becomes

$$dM = -\frac{1}{16\pi C} \sqrt{-g} J^a n_a dA dr = -\frac{1}{16\pi C} \sqrt{-g} (\nabla_b J^{ab}) n_a dA dr \quad (4.10)$$

where in the second equality we have used $J^a = \nabla_b J^{ab}$. The above expression can be rewritten as

$$dM = -\frac{1}{16\pi C} \sqrt{-g} [\nabla_b (J^{ab} n_a) - J^{ab} \nabla_b n_a] dA dr \quad (4.11)$$

Now, $\nabla_b n_a = \partial_b n_a - \Gamma^k_{ba} n_k$ and hence $\nabla_{[b} n_{a]} = \partial_{[b} n_{a]}$, since the terms involving the connections cancels out by symmetry. But $n_a \equiv \partial_a$ since we are working in a coordinate basis. Thus, $\nabla_{[b} n_{a]} = 0 \Rightarrow \nabla_b n_a = \nabla_a n_b$ and the second term in the square bracket of (4.11) vanishes because of the antisymmetry of J^{ab} . Hence we are left with

$$dM = -\frac{1}{16\pi C} \sqrt{-g} \nabla_b (J^{ab} n_a) dA dr \quad (4.12)$$

which, using (2.34) applied to the vector $J^{ab} n_a$, becomes

$$dM = -\frac{1}{16\pi C} \partial_b (\sqrt{-g} J^{ab} n_a) dA dr \quad (4.13)$$

By the means of Gauss's theorem in the radial direction, we can write

$$\begin{aligned} dM &= -(16\pi C)^{-1} \sqrt{-g} (J^{ab}|_{r_f} - J^{ab}|_{r_i}) n_b n_a dA = -(16\pi C)^{-1} \sqrt{-g} \Delta J^{ab} n_a n_b dA \\ &= -(16\pi C)^{-1} \sqrt{-g} \Delta J^{ab} d^{D-2} \sigma_{ab} = -2[(32\pi C)^{-1} \sqrt{-g} \Delta J^{ab} d^{D-2} \sigma_{ab}] \end{aligned} \quad (4.14)$$

and recalling the charge written in terms of J^{ab}

$$Q_H = (32\pi C)^{-1} \int d^{D-2} \sigma_{ab} \sqrt{-g} J^{ab} \quad (4.15)$$

we are lead to

$$dM = -2dQ_H \quad (4.16)$$

Indeed, equation (4.16), in combination with (4.4), is equivalent to $TdS_{\text{Noether}} = dM$ locally [12] and (4.4) really can be taken as the notion of horizon entropy in any general diffeomorphism invariant theory of gravity.

4.3 Remarks

We shall summarize the main results of this chapter:

- The locally off-shell conserved current J^a leads to a charge, which is proportional to a quarter of the area of the horizon in GR. This charge is off-shell conserved.
- The charge is showed to be $-M/2$ and the quantity $S_{\text{Noether}} = Q_H/T$ is the physical horizon entropy *when the equations of motion are implemented*.
- The local equation of state $TdS_{\text{Noether}} = dM$ is obtained thanks to the equations of motion.

These observations have again an instructive parallelism with the electromagnetism. In section 2.3.3 we have seen that the locally off-shell conserved current corresponding to the invariance of the Maxwell Lagrangian under the gauge transformation $\delta_\theta A_a = \partial_a \theta$ is

$$J^a = \partial_k (F^{ka} \theta) \quad (4.17)$$

The associated conserved charge is

$$Q = \int_{\mathcal{V}} d^3x J^0 = \int_{\mathcal{V}} d^3x \partial_k (F^{k0} \theta) = - \int_{\mathcal{V}} d^3x \nabla \cdot (\mathbf{E} \theta) \quad (4.18)$$

If we impose $\partial_k F^{ak} = J^a$, that is if we impose equations of motion in which J^a , as given by (4.17), be the source, we must have

$$J^0 = \partial_k F^{k0} \theta + F^{k0} \partial_k \theta = \partial_k F^{0k} \quad (4.19)$$

This implies $\partial_k \theta = 0$ and $\theta = -1$. Thus the charge reads

$$Q = \int_{\mathcal{V}} d^3x \nabla \cdot \mathbf{E} = \int_{\mathcal{V}} d^3x \rho(x) = q \quad (4.20)$$

that is nothing but the electric charge contained inside the region of space with volume \mathcal{V} . Thus, when one implements the equations of motion (on-shell conserved current) the charge acquires a precise physical meaning. This is exactly what happens also in the case of gravitational theories. Finally, we stress that

- The horizon temperature, $T = (\kappa/2\pi)$, does not depend on the gravitational theory, instead, in general, the horizon entropy does.
- The horizon entropy depends on the curvature of the horizon, that in general changes point by point over the surfaces. This does not happen in GR, where the horizon entropy is always $A/4$, independently of the choice of the horizon patch.

Conclusions

In this thesis we have discussed some important features that we summarize here. In any diffeomorphism invariant theory of gravity there exists certain relations involving the dynamical variables (i.e. quantities describing the gravitational field) that are not linked to the particular form of the Lagrangian and consequently the equations of motion, but they are rather identities coming from peculiar geometrical aspects of the Lagrangian. Namely, we have discussed the generalized Bianchi identity (1.9) and the off-shell conservation of the current (2.6). Given an action functional written as (1.2) and its variation, which, as we have proved in the thesis, can be always cast in the form given by (1.3) for any general theory of gravity, Bianchi identity is derived using the fact that we are considering gravitational theories for which $L'(x') = L(x)$ under general infinitesimal diffeomorphisms. Strictly connected to the Bianchi identity is the existence of the current (2.6) that is defined locally on the spacetime and that is off-shell conserved thanks to the form that takes the local transformation of the metric under general infinitesimal diffeomorphisms. Again, the way the metric transforms locally is independent of the gravitational theory. Thus, for any diffeomorphism invariant general theory of gravity it is possible to define such a current. Its explicit form, instead, is determined by the form of the Lagrangian, and hence by the gravitational theory. This same argument applies to the associated charge. Any diffeomorphism invariant general theory of gravity has a conserved charge whose form can be explicitly established once the Lagrangian has been specified. In this thesis we have computed the charge associated to a spherically symmetric horizon in classical general relativity. What emerges is that this charge is proportional to a quarter of the area of the horizon. Using the notion of horizon temperature, which is well founded once the metric is defined, one can establish a connection between the charge of the horizon and the horizon entropy. We have shown that this connection can be generalized to any diffeomorphism invariant theory of gravity. In fact, using the equations of motion, one can see that $-2dQ_H = dM$ and consequently $TdS_{\text{Noether}} = dM$. Thus, the key result of the thesis is that the entropy given by (4.4) is really the horizon entropy in any diffeomorphism invariant theory of gravity.

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