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Pants Homology for Surfaces

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To my family

Preface

In this thesis we present two recent results about surfaces that play a central role in the proofs of both the Ehrenpreise Conjecture and the Surface Subgroup Theorem. Let S be a closed oriented surface and denote by $H_1(S)$ the first singular homology group with rational coefficients. Let γ be a closed curve immersed in S, such that $[\gamma]_{H_1(S)} = 0$. Thus there is a singular 2-chain which is bounded by γ. A priori there is no guarantee —and indeed will be false in general— that there exists an immersed surface T bounded by a representative of the homotopy class of γ, that is an orientation preserving immersion $f : T \longrightarrow S$ such that $\partial f := f|_{\partial T}$ is an homeomorphism between ∂T and a representative of the homotopy class of γ. As we said such f doesn't exists in general. However, if we restrict to the case of genus $g > 1$ and we relax the hypothesis on ∂f to be a covering map instead of an homeomorphism such f can be found (see Section 2.2 for counter examples without such hypothesis). In this case we say that γ *virtually bounds* T. More in general, given an integral formal sum $C = \sum_{i=1}^{k} n_i\gamma_i$ of closed curves of S we can ask, as well, if there is an orientation preserving immersion $f : T \longrightarrow S$ such that ∂f is a covering on an homotopy representative of the curves $(\gamma_1^{n_1}, \dots, \gamma_k^{n_k}) : \bigsqcup_{i=1}^{k} S^1 \longrightarrow S$. The answer is again positive under the same assumptions on the genus. This fact was first proven by Danny Calegari in the work [Cal09]. Moreover the work of Calegari implies the following fact. By *pair of pants* we mean a topological space homeomorphic to a sphere with three holes. Then we have that every 1-chain C, trivial in $H_1(S)$, bounds a 2-chain of immersed pair of pants. This can be interpreted by saying that

the *Pants Homology* is the same theory of the classical Homology.
The theory of Pants Homology has a geometric counterpart, called Good Pants Homology, where objects are required to satisfy some geometric bounds. If we start with an hyperbolic surface Σ, one can define the set $\Gamma_{\epsilon,R}$ of the closed geodesics of Σ with length in the interval $[2R-\epsilon, 2R+\epsilon]$, and the set $\Pi_{\epsilon,R}$ of the immersed pants with boundary components in $\Gamma_{\epsilon,R}$. Then one can ask if, given a chain $C = \sum r_i\gamma_i$ with $r_i \in \mathbb{Q}$ and $\gamma_i \in \Gamma_{\epsilon,R}$ such that $[C]_{H_1(\Sigma)} = 0$, there is a rational formal sum of pants $P = \sum s_i\Pi^i$, $s_i \in \mathbb{Q}$ and $\Pi^i \in \Pi_{\epsilon,R}$ such that $C = \partial P$.
This question rise up along the recent proof of the Ehrenpreise Conjecture, which states that for every $\epsilon > 0$ and for any two hyperbolic surfaces S and T, we can find two finite covers $\hat{S}$ and $\hat{T}$ of S and T respectively, such that $\hat{S}$ and $\hat{T}$ are $(1+\epsilon)$ quasi-isometrics. With $(1+\epsilon)$ quasi-isometric we mean that the two surfaces are homeomorphic and that the two metric induced by the hyperbolic structure differs of a multiplicative factors that goes to 1 when $\epsilon \to 0$. J. Kahn and V. Markovic prove such conjecture in their work [KM13] following the structure of their proof of the Surface Subgroup Theorem published in [KM12]. However the proof of the Ehrenpreise Conjecture is harder and the additional difficulties can be thought in terms of the Good Pants Homology, that is more precisely described as the homology theory of the curves in $\Gamma_{\epsilon,R}$ and the boundaries in $\partial\Pi_{\epsilon,R}$. One of the main question about this theory is if the Good Pants Homology is equivalent to classical Homology $H_1(S)$. The answer is again positive (here the fact that we have supposed the surfaces hyperbolic is sufficient to satisfy all the necessary conditions). The proof is not simple and relies also on deep results on Ergodic Theory and Dynamical Systems. In this thesis we focus on the geometric aspects of the proof. We will follow the original approach of [KM13]. We want to remark that both in the Calegri's Pants Homology and Kahn-Markovic's Good Pants Homology the harder part is to prove that, given $X, Y \in \pi_1(S)$ the class $[XY]$ in the homology theory in question is the same of the sum of classes $[X] + [Y]$. This fact is proven in the Pants Homology giving an

explicit construction of an immersed surface in an example over a once punctured torus, and the proof of the general situation is constructed from this example glueing suitable pair of pants to the surface constructed. The Good Pants Homology require a more deeper study of the curves in $\Gamma_{\epsilon,R}$, since in general an homotopy class $X \in \pi_1(S)$ has geodesic representative not in $\Gamma_{\epsilon,R}$. What we do is define a way to assign to each $X \in \pi_1(S)$ a closed geodesic $(X)_T \in \Gamma_{\epsilon,R}$[1] which is not homotopic to X, but it is homologically equivalent to $[X]_{H_1(S)}$. Finally, with some work, we will see that $(XY)_T = (X)_T + (Y)_T$, and then the final equivalence between the homologies. We remark that the definition of $(\cdot)_T$ is not trivial, and that it is sufficiently powerful to give the above equation for products.

In the first chapter we give some preliminaries about hyperbolic and conformal geometry without proofs. In Chapter 2 we give the complete proof of the Calegari's Immersion Theorem, and then that the Pants Homology is equivalent to the classical Homology. In Chapter 3 we prove that also the Good Pants Homology is equivalent to the standard homology. This is the main chapter of the thesis. Finally in Chapter 4 we present the framework in which is developed the proof of the Ehrenpreise Conjecture and of the Surface Subgroup Theorem. In particular we sketch a proof of them stressing when and how the Good Pants Homology is used.

[1]The definition of $(\cdot)_T$ and its basic properties is one of the main subjects of Section 3.2

Introduzione

In questa tesi presentiamo due recenti risultati riguardanti la teoria delle superfici che giocano un importante ruolo nella dimostrazione della Congettura di Ehrenpreise e del Teorema del Sottogruppo di Superficie. Sia S una superficie chiusa e orientata e denotiamo con $H_1(S)$ il primo gruppo di Omologia Singolare a coefficienti razionali. Sia γ una curva chiusa in S tale che $[\gamma]_{H_1(S)} = 0$. Quindi γ è bordo di una 2-catena. Non c'è garanzia a priori —e sarà falso in generale— che esista una superficie T immersa in S della quale γ è bordo, ovvero che esista una immersione $f : T \longrightarrow S$ che preserva l'orientazione e tale che $\partial f := f|_{\partial T}$ è un omeomorfismo tra ∂T ed un rappresentante della classe di omotopia di γ. Come abbiamo detto in generale tale f non esiste. Tuttavia, se ci restringiamo al caso di superfici di genere $g > 1$ e rilassiamo le condizioni su ∂f ad essere un rivestimento anzichè un omeomorfismo tale immersione esiste (si veda la Sezione 2.2 per dei controesempi in assenza di tali ipotesi). In tal caso diciamo che γ *borda virtualmente* T. Più in gnerale, data una somma formale intera $C = \sum_{i=1}^{k} n_i\gamma_i$ di curve chiuse di S, ci chiediamo se esiste una immersione $f : T \longrightarrow S$ che preserva l'orientazione e tale che ∂f è un rivestimento su un rappresentante omotopico delle curve $(\gamma_1^{n_1}, \dots, \gamma_k^{n_k}) : \bigsqcup_{i=1}^{k} S^1 \longrightarrow S$. Anche in questo caso sarà possibile trovare una tale f, sotto le stesse ipotesi sul genere. L'esistenza di queste immersioni è stata dimostrata per la prima volta da Danny Calegari nel lavoro [Cal09]. Il lavoro di Calegari implica anche la seguente cosa. Con il termine *paio di pantaloni* intendiamo uno spazio topologico omeomorfo ad una sfera con tre fori. Allora abbiamo che ogni 1-catena C, banale in $H_1(S)$,

è bordo di una 2-catena di paia di pantaloni. Questo può essere interpretato dicendo che l'*Omologia dei Panataloni* è la stessa cosa dell'Omologia classica. La teoria dell'Omologia dei Pantaloni ha un corrispettivo geometrico, chiamato Omologia dei Pantaloni Buoni, che richiede delle limitazioni quantitative su alcune proprietà geometriche dei suoi oggetti. Se prendiamo una superficie iperbolica Σ, possiamo definire l'insieme $\Gamma_{\epsilon,R}$ delle geodetiche chuse di Σ di lunghezza compresa nell'intervallo $[2R-\epsilon, 2R+\epsilon]$, e l'insieme $\Pi_{\epsilon,R}$ the pantaloni immersi in Σ con bordo in $\Gamma_{\epsilon,R}$. Quindi ci possiamo chiedere se, data una catena $C = \sum r_i\gamma_i$ con $r_i \in \mathbb{Q}$ e $\gamma_i \in \Gamma_{\epsilon,R}$ tale che $[C]_{H_1(\Sigma)} = 0$, esiste una somma razionale formale di pantaloni $P = \sum s_i\Pi^i$, con $s_i \in \mathbb{Q}$ e $\Pi^i \in \Pi_{\epsilon,R}$, tale che $C = \partial P$.
Questa domanda sorge durante la prova della Congettura di Ehrenpreise, la quale afferma che, per ogni $\epsilon > 0$ e per ogni S e T superfici iperboliche, possiamo rivestire S e T con due rivestimenti finiti, rispettivamente $\hat{S}$ e $\hat{T}$, in modo tale che $\hat{S}$ e $\hat{T}$ siano $(1+\epsilon)$ quasi-isometrici. Con $(1+\epsilon)$ quasi-isometrici intendiamo che le due superfici sono omeomorfe e che le rispettive metriche iperboliche differiscono per un fattore moltiplicativo che tende ad 1 quando $\epsilon \to 0$. J. Kahn e V. Markovic hanno dimostrato questa congettura nel loro lavoro [KM13], seguendo la struttura della (sempre loro) dimostrazione del Teorema del Sottogruppo di Superficie pubblicata in [KM12]. Tuttavia la dimostrazione della Congettura di Ehrenpreise è più difficile di quest'ultima e la difficoltà aggiuntiva va affrontata attraverso l'Omologia dei Pantaloni Buoni, più precisamente descritta come la teoria omologica delle curve in $\Gamma_{\epsilon,R}$ e dei bordi in $\partial\Pi_{\epsilon,R}$. Una delle principali domande che ci si pone è se l'Omologia dei Pantaloni Buoni è equivalente all'Omologia Singolare $H_1(S)$. Avremo una risposta affermativa anche in questo caso (e qui supporre la superficie iperbolica sarà una condizione sufficiente) ma dimostrarlo non è affatto semplice e si utilizzano profondi risultati di Teoria Ergodica e Sistemi Dinamici. Noi ci focalizzeremo sugli aspetti geometrici della dimostrazione. Seguiremo l'approccio originale del lavoro [KM13]. Va sottolineato che sia nell'Omologia dei Pantaloni di Calegari che nell'Omologia dei Pantaloni Buoni di Kahn e

Markovic il fatto principale da dimostrare è che, dati $X, Y \in \pi_1(S)$, la classe $[XY]$ nella teoria omologica in questione è la stessa cosa della somma delle classi $[X] + [Y]$. Questa cosa nell'Omologia dei Pantaloni si dimostra partendo da un esempio di costruzione esplicita di una superficie immersa in un toro forato, che poi verrà utilizzata in una superficie generiaca, incollandoci gli eventuali pantaloni necessari. L'Omologia dei Pantaloni Buoni richiede una comprensione più profonda delle curve in $\Gamma_{\epsilon,R}$, infatti, in generale, il rappresentante geodetico della classe di omotopia di $X \in \pi_1(S)$ non è una curva in $\Gamma_{\epsilon,R}$. Perciò troveremo un modo per assegnare una geodetica $(X)_T \in \Gamma_{\epsilon,R}$[2] ad ogni classe $X \in \pi_1(S)$ in maniera tale che $(X)_T$ non sarà necessariamente omotopica a X, ma sarà equivalene in omologia a $[X]_{H_1(S)}$. Così, con un pò di lavoro, potremo dimostrare che $(XY)_T = (X)_T + (Y)_T$ e, successivamente, che le due teorie omologiche sono equivalenti. Dare la definizione di $(\cdot)_T$ non sarà semplice (ne sarà ben posta la definizione in genarale), tuttavia la definizione sarà abbastanza potente da portare all'equazione per il prodotto di classi d'omotopia.
Nel primo capitolo descriveremo i concetti preliminari necessari ai capitoli successivi, omettendo le dimostrazioni, con pricipale riguardo alla geometria iperbolica e conforme. Nel Capitolo 2 daremo la prova completa del Teorema di Immersione di Calegari, quindi proveremo che l'Omologia dei Pantaloni è la stessa cosa dell'Omologia classica. Nel Capitolo 3 dimostreremo che anche l'Omologia dei Pantaloni Buoni è equivalente all'Omologia standard. Di fatto, questo è il capitolo principale della tesi. Infine nel Capitolo 4 presenteremo il contesto in cui si sviluppano la prova della Congettura di Ehrenpreise e del Teorema del Sottogruppo di Superficie. In particolare daremo un breve schema della dimostrazione sottolineando come e quando viene utilizzata l'Omologia dei Pantaloni Buoni.

[2]Dare la definizione di $(\cdot)_T$ e provare le sue prime proprietà è uno degli scopi principali della sezione 3.2

Contents

Chapter 1

Preliminaries

In this chapter we gives the definitions and the theorems that will be useful in the following chapters without proving anything. We start with concepts about Riemannian geometry. A good treatment of this subject can be found in [Pet06] and [dC92]. We omit the definitions and basic facts about standard algebraic topology. For these subjects we refer to the book [Hat02].

Definition 1.0.1 (Riemannian Metric)**.** A *Riemannian metric* over a smooth manifold M is a family of inner products

$$g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$$

for $p \in M$, such that, given two smooth vector fields X and Y, the map

$$p \longmapsto g_p(X(p), Y(p))$$

is C^∞. The couple (M, g) is called a *Riemannian manifold*.

When the metric is understood we can write only M for (M, g). We observe that an immersion of smooth manifolds $f : M \longrightarrow N$ can pull back an eventual metric g over N, defining g^M as $g^M_p(u, v) := g^N_{f(p)}(T_pf(u), T_pf(v))$.

Definition 1.0.2 (Length)**.** Given a curve $c : [a, b] \longrightarrow M$ in a Riemannian manifold we define the length of c as

$$l^b_a(c) = \int_a^b \sqrt{g(c'(t), c'(t))}dt.$$

Proposition 1.0.1. *Any connected Riemannian manifold M is a metric space with metric function given by*

$$d(p,q) = \inf l(c)$$

where the infimum vary over all the curve from p to q. The eventual (and not necessarily unique) γ which realize such distance is called geodesic.

1.1 Conformal Geometry

The theory of conformal geometry, in particular in dimension 2, is the right framework to study a lot of properties that touches different points of view, such as topology, hyperbolic geometry, Riemannian geometry, holomorphic functions and dynamical systems. Intuitively conformal maps preserves angles, and so conformal geometry is a geometry "up to scalar multiplication". A good and elementary introduction to conformal geometry can be found in [BP92] and [Rat06].

Definition 1.1.1 (Conformal structures)**.** Two Riemannian metrics g and h over a smooth manifold M are said *conformally equivalent* if exists a real smooth function $\lambda : M \longrightarrow \mathbb{R}$ such that $g = \lambda h$. The equivalence class of conformally equivalent Riemannian metrics of a manifold is a *conformal structure* for M.

Definition 1.1.2 (Conformal Map)**.** A diffeomorphism between two Riemannian manifolds is said a conformal map if pulls-back conformally equivalent metrics. We call $\mathrm{Conf}(M,N)$ the set of conformal maps from M to N.

A very important case for conformal geometry is the one given by the plane. In fact it happens that:

Proposition 1.1.1. *Given two oriented and connected Riemann surfaces X and Y (complex manifolds of dimension 1) the set $\mathrm{Conf}(X,Y)$ is the set of all holomorphisms and anti-holomorphisms between X and Y.*

So, in particular, holomorphic maps on the plane or on regions of the plane are conformal maps. The following result is fundamental for the theory of such maps

Theorem 1.1.2 (Riemann mapping theorem)**.** *Let* Ω *be a simply connected region of the plane different from the whole* $\mathbb{C}$*. Then there exists a biholomorphism (a bijective holomorphism)* $f : \Omega \longrightarrow D^2$ *where* $D^2 = \{z \in \mathbb{C} : |z| < 1\}$*.*

More generally, we have the celebrated Uniformization Theorem

Theorem 1.1.3 (Uniformization Theorem)**.** *Let* X *be a* 2*-dimensional simply connected manifold with a fixed conformal structure. Then* X *is conformally equivalent to one the following three Riemann surfaces:*
(1) *the sphere* $S^2 = \mathbb{C} \cup \{\infty\} = \hat{\mathbb{C}}$*,*
(2) *the plane* $\mathbb{C}$*,*
(3) *the unit disk* D^2*.*

These theorems are classically subjects of the theory of Riemann Surfaces. They are treat in every book about this subject, for example [For91]

Quasi-Conformal maps

We are going to introduce the concept of quasi-conformal map, that is a generalization of a conformal map: a quasi-conformal map is a conformal map which not only rescales lengths but also permits bounded distortions on angles. There are several equivalent definitions of quasi-conformal maps and such equivalences are not trivial at all. We give definitions and theorems that we need without proofs. Most of the basics facts and a classical and good introduction to this subject can be found in [Ahl06] and we refer to it for the proofs of most of the facts we introduce.

Definition 1.1.3. A rectangle R in the plane $\mathbb{C}$ is the bounded set $R(a, b) = \{z \in \mathbb{C} : \mathrm{Re}(z) \leq a, \mathrm{Im}(z) \leq b\}$. The ratio $a : b$ is called the *module* of R written $m(R)$.

The module of a rectangle is a conformal invariant. Let Ω be an open region of $\mathbb{C}$

Definition 1.1.4. A *quadrilateral* is a Jordan region (a region bounded by a Jordan curve) Q, with $\bar{Q} \subset \Omega$, together with a pair of disjoint closed arcs (called b-arcs) in ∂Q and a conformal map of $\bar{Q}$ to a rectangle $R(a, b)$ which sends the b-arcs to the veritcal sides of the rectangle.

This definition depends on the existence of the conformal maps, which is not trivial since depends on the Riemann Mapping Theorem. However this gives the definition of module of a quadrilateral $m(Q) = a : b$. Let Ω and Ω' be two regions of the complex plane and $f : \Omega \longrightarrow \Omega'$ an orientation preserving homeomorphism.

Definition 1.1.5 (Quasi-conformal map)**.** f is a K-Quasi-conformal (K -q.c. from now on) map if

$$\frac{1}{K} m(Q) \leq m(f(Q)) \leq K m(Q)$$

for every quadrilateral in Ω.

Some basic properties are:

Proposition 1.1.4. *If f is K-q.c. then:*
(i) f^{-1} is K-q.c.,
(ii) the composition with a H-q.c. map gives a KH-q.c. map,
(iii) if $K = 1$, f is conformal.

Q.c. maps of the unitary disk D^2 has some important properties.

Theorem 1.1.5 (Mori's Theorem)**.** *Every K-q.c. map of the open unitary disk onto itself extends to an homeomorphism on the boundary. Furthermore if we think to $S^1 = \mathbb{R} \cup \{\infty\}$ and we call the induced homeomorphism of the boundary as $h : S^1 \longrightarrow S^1$, we have that exist a constant $M(K) < \frac{1}{16} e^{\pi K}$ such that*

$$M^{-1} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq M$$

Such condition is called M-condition. It is also true the inverse:

Theorem 1.1.6 (M-Condition Sufficiency). *Every map h satisfying an M-condition can be extended to a K-q.c. map, with K depending only on M.*

Consider the distributional derivatives f_z and $f_{\bar{z}}$. We call *complex dilatation* the measurable function $\mu_f = \frac{f_{\bar{z}}}{f_z}$. If f is a homeomorphism we have that it is a K-q.c. map if satisfy the following two conditions:
(i)f has locally integrable distributional derivatives,
(ii)$|f_{\bar{z}}| \leq k|f_z|$ for some $k < 1$. The map f is K-q.c. with $K = \frac{1+k}{1-k}$

This leads to the problem of searching quasi-conformal solutions for the Beltrami equation

$$f_{\bar{z}} = \mu f_z \tag{1.1}$$

given the measurable complex-valued function μ with $||\mu||_\infty < 1$. The solution of this problem is called the *measurable Riemann mapping theorem.* Consider the quasi-conformal maps from the extended plane $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$ to itself normalized by fixing points 0, 1 and ∞. We have

Theorem 1.1.7 (Measurable Riemann Theorem). *There is a one to one correspondence between normalized quasi-conformal maps of $\hat{\mathbb{C}}$ and solutions of the Beltrami equation varying μ. Moreover the normalized solution f^μ depends holomorphically on μ*

We notice that we can give the definition of K-quasi-conformal homeomorphism between two Riemann Surfaces X and Y as follows: $f : X \longrightarrow Y$ is a K-q.c. homeomorphism if it is an homeomorphism and if the lift $\tilde{f} : \tilde{X} \longrightarrow \tilde{Y}$ between the universal covers is K-q.c.
We notice that the universal covers are one of the three individuated by the Uniformization Theorem.

1.2 Hyperbolic Geometry

We want to present the fundamental facts of hyperbolic geometry. As in all the preliminaries we don't prove anything. A complete introduction to the subject can be found in [Rat06] and in [BP92]

Definition 1.2.1 (Hyperbolic Space $\mathbb{H}^n$)**.** Consider $\mathbb{R}^{n+1}$ with coordinates $x_1, \ldots, x_{n+1}$. Then we call $\mathbb{H}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + \cdots + x_n^2 - x_{n+1}^2 = -1, x_{n+1} > 0\}$ with the Riemannian metric given by the restriction on it of the metric tensor

$$dx^2 = dx_1^2 + dx_2^2 + \ldots dx_n^2 - dx_{n+1}^2.$$

It can be proved that $\mathbb{H}^n$ is a simply connected n-Riemannian manifold. The geodesics here can be found by intersecting $\mathbb{H}^n$ with planes through the origin.

There are many others models for hyperbolic space, some of them really useful in some situations. We present them for future uses.

Example 1.2.1 (The Poincaré disk model)**.** Consider the unitary n disk D^n in the n-Euclidean space. If we consider the interior of D^n with the metric tensor given by

$$ds^2 = (\frac{1}{1-r^2})^2 dx^2$$

where r is the distance from the origin, and dx^2 is the Euclidean metric tensor, we obtain a simply connected surface with constant curvature -1 which is isometric to $\mathbb{H}^n$.
In this model the geodesics for the metric ds^2 are the euclidean spherical arcs orthogonal to ∂D^n, as well hyperbolic k-plane are euclidean k-spheres orthogonal to the boundary. We also consider the limit cases of circles and planes which are lines and planes through the origin of D^n. Finally in this model the boundary of the disk has an interpretation in terms of hyperbolic geometry: every geodesic in $\mathbb{H}^n$ can be uniquely identified by two points of $\mathbb{S}^{n-1}$ and every two such points uniquely determines a geodesic in $\mathbb{H}^n$. Such points are exactly the intersection points of the circle which represent the geodesic with ∂D^n. By this interpretation is well defined $\partial\mathbb{H}^n := \partial D^n$ In the particular case $n = 3$ we usually identify $\mathbb{S}^2 = \partial\mathbb{H}^3$ with $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$.

Example 1.2.2 (The upper half-space model)**.** Consider the upper half space $\mathbb{R}^n_+ = \{x_n > 0\}$ with the metric tensor $ds^2 = (\frac{1}{x_n})^2 dx^2$. It is a model for $\mathbb{H}^n$ (it is a simply connected space with constant curvature -1, isometric to $\mathbb{H}^n$) where geodesics are lines and circles orthogonal to the hyperplane $\{x_n = 0\}$. Note that the boundary $\partial\mathbb{H}^n$, as defined in the disk model, here can be thought as the $\{x_n = 0\} \cup \{\infty\}$ Finally, in the particular case of $\mathbb{H}^2$ we use the model given by $\mathbb{C}_{\mathrm{Im}(z)>0}$, whereas for $\mathbb{H}^3$ we usually identify the boundary $\{x_3 = 0\}$ with $\mathbb{C}$.

Remark 1.2.1. The last two model for $\mathbb{H}^n$ that we have discussed, with the conformal structure inherited by the Euclidean space, are conformally equivalent to $\mathbb{H}^n$. That is, we can compute the angles in these models to make computations for $\mathbb{H}^n$. Moreover we can pass from one to another if we speak about conformal invariants.

There are two important formulas useful to make geometric computation in $\mathbb{H}^2$. Let A,B,$C \subseteq \mathbb{H}^2$ be the three geodesic edges of a triangle in $\mathbb{H}^2$. Let α,β and γ the internal angles of the triangle, respectively, opposite to A,B and C. We have the following two formulas.

Hyperbolic sine formula

$$\frac{\sinh l(A)}{\sin\alpha} = \frac{\sinh l(B)}{\sin\beta} = \frac{\sinh l(C)}{\sin\gamma}. \tag{1.2}$$

Hyperbolic cosine formula

$$\cosh l(A) = \cosh l(B)\cosh l(C) - \sinh l(B)\sinh l(C)\cos\alpha. \tag{1.3}$$

for quadrilaterals we have the following Theorem

Theorem 1.2.1 (Saccheri's quadrilaterals)**.** *Let B,D_1,D_2, and S be four geodesic arcs in $\mathbb{H}^2$ which are edges of a quadrilateral. Suppose that D_1 and D_2 are opposite sides and that $l(D_1) = l(D_2) = d$. Moreover suppose that the two adjacent internal angles where B meets the D_i's are orthogonal. Then the other two angles are acute and equal, and we have*

$$\sinh(\frac{l(S)}{2}) = \sinh(d)\sinh(\frac{l(B)}{2}).$$

A quadrilateral as above is named a *Saccheri's quadrilateral.*

Definition 1.2.2 (Hyperbolic Manifold)**.** Let (M, h) a n-Riemannian manifold. We say that it is an hyperbolic manifold if there is a cover $\mathbb{H}^n \longrightarrow M$ which is a local isometry.

We remark that usually this is a theorem and not a definition for hyperbolic manifolds.

In the followings we want to describe the isometries of $\mathbb{H}^n$. Denote by $\text{Isom}(\mathbb{H}^n)$ the space of such maps.

Proposition 1.2.2 (Trace of Isometries)**.** *Every isometry of $\mathbb{H}^n$ extends to an homeomorphism of $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \partial\mathbb{H}^n$.*
Furthermore every isometry can be determined by his trace on $\partial\mathbb{H}^n$.

Such dependence is the basic fact for the understanding of $\text{Isom}(\mathbb{H}^n)$. In particular it can be proved that:

$$\text{Isom}(\mathbb{H}^n) = \text{Conf}(D^n)$$

Which, in the cases $n = 2$ and $n = 3$ leads to the following equalities:

$$(1)\ \text{Isom}(\mathbb{H}^2) = PSL(2, \mathbb{R}) = \frac{SL(2, \mathbb{R})}{\{\pm I\}},$$
$$(2)\ \text{Isom}(\mathbb{H}^3) = PSL(2, \mathbb{C}) = \frac{SL(2, \mathbb{C})}{\{\pm I\}}.$$

In particular isometries in the hyperbolic space are that maps which preserve cross-ratio.

Definition 1.2.3 (cross-ratio)**.** Given 4 points in u, v, p and q we define their cross-ratio as

$$[u, v, p, q] = \frac{(u - p)(v - q)}{(v - p)(u - q)}$$

We remark that this means that to an isometry of $\mathbb{H}^n$ is only required to preserve angles, or, equivalently, that a unit of measure is intrinsically definite in $\mathbb{H}^n$.

Recall that, by the Brower Fixed Point Theorem (see [Mil97]), every element of $\mathrm{Isom}(\mathbb{H}^n)$ has a fixed point in $\overline{\mathbb{H}^n}$. This implies a classification of the elements of $\mathbb{H}^n$

Theorem 1.2.3 (Isometry Classification)**.** *Every $\phi \in \mathrm{Isom}(\mathbb{H}^n)$ falls in one of the following mutually excluding possibilities:*
(elliptic) ϕ has one fixed point in $\mathbb{H}^n$,
(parabolic) ϕ has no fixed point in $\mathbb{H}^n$ and exactly one point in $\partial\mathbb{H}^n$,
(hyperbolic) ϕ has no fixed point in $\mathbb{H}^n$ and exactly two fixed points in $\partial\mathbb{H}^n$.

If $\phi \in \mathrm{Isom}(\mathbb{H}^n)$ is Hyperbolic, then the two fixed points at the boundary are the endpoints of a geodesic γ_ϕ that is an invariant subset for ϕ. We call γ_ϕ the *axis* of γ.

Definition 1.2.4. (i) A discrete subgroup of $PSL(2, \mathbb{C})$ is called *Kleinian* group,
(ii) a discrete subgroup of $PSL(2, \mathbb{R})$ is called *Fuchsian* group.

Proposition 1.2.4 (Limit Set)**.** *Let $x \in \mathbb{H}^3$ (resp. $\mathbb{H}^2$), let G be a Kleinian (resp. Fuchsian) group and let $G_x \subseteq \mathbb{H}^3$ (resp.$\mathbb{H}^2$) the orbit of x with respect to the action of G on $\mathbb{H}^3$ (resp.$\mathbb{H}^2$). Then the subset $L_G \subseteq \partial\mathbb{H}^3$ (resp.$\partial\mathbb{H}^2$) of the accumulation points for G_x does not depend on the choice of x.*

We notice that if $\mathbb{H}^n/G$ is a closed manifold ($n = 2$ or 3) then L_G has to be all $\partial\mathbb{H}^n$.

Definition 1.2.5. If G is a group acting on a locally compact space X, then the action is said *properly discontinuous* if, for every compact $K \subseteq X$, there are at most finitely many $g \in G$ such that $gK \cap K \neq \emptyset$.

A Kleinian group G is *non elementary* if L_G has more then 3 points. Denote with $H(L_G)$ the convex hull of L_G in $\overline{\mathbb{H}^n}$. We have the following:

Proposition 1.2.5. *Let $n = 2, 3$ Suppose that G in non elementary and acts freely in $\overline{\mathbb{H}^n} \setminus L_G$. Then*
(i) $L_G = \bigcap U_i$ where the intersection vary over all the open $U_i \subseteq \partial\mathbb{H}^n$ such

that $GU_i \subseteq U_i$.

(ii) For every $G' \trianglelefteq G$, $G' \neq \{id\}$, *we have that* $L_{G'} = L_G$.

(iii) $H(L_G)$ *is* G*-invariant.*

(iv) G *acts properly discontinuous on* $\overline{\mathbb{H}^n} \setminus L_G$ *and on* $\partial\mathbb{H}^n \setminus L_G$.

(v) The following spaces are hyperbolic manifolds:

$M_G = H(L_G)/G$,

$N_G = \mathbb{H}^n/G$,

$O_G = (\overline{\mathbb{H}^n} \setminus L_G)/G$.

Chapter 2

Pants Homology

2.1 Geometric Subgroups of Surface Groups

In this section we will prove the following fact: given an homotopy class of closed curves in a surface it is possible to find a finite cover of the surface where the class has an embedded representatives. This fact will be used later in section 2.2. Practically we prove a more general result where the embeddings are generalized to a group condition and actually all the results have a group theoretic point of view. We need some definitions to state the main results:

Definition 2.1.1 (Incompressible Surface)**.** Let S be a surface. A compact subsurface X of S is said *incompressible* if no component of the closure of $S \setminus X$ is a disk with boundary contained in ∂X

Remark 2.1.1. It follow from the definition of incompressibility and Seifert-van Kampen's theorem that $\pi_1(X) \longrightarrow \pi_1(S)$ is injective.

Definition 2.1.2 (Geometric Subgroup)**.** A subgroup G of a group F is said *geometric* if there exist a surface S with $\pi_1(S) = F$ and an incompressible subsurface $X \subseteq S$ with $\pi_1(X) = G$.

This generalizes embeddings sufficiently:

Theorem 2.1.1 (Geometric Surface Subgroups, [Sco78])**.** *Let S be a surface, let F be a finitely generated subgroup of $\pi_1(S)$ and let $g \in \pi_1(S) \setminus F$. Then there exists a finite cover S_1 of S such that $\pi_1(S_1)$ contains F but not g and F is geometric in S_1*

Now we can answer the original question.

Corollary 2.1.2. *Let $\gamma \in \pi_1(S)$. Then there exists a finite cover S_1 of S such that γ can be represented by an embedded closed curve on S_1.*

Proof. By Theorem 2.1.1 (using some element different from γ as g) the infinite cyclic subgroup generated by γ is geometric in a suitable finite cover S_1. Which means that we can immerse homeomorphically a cylinder C into S_1 such that the loop generating $\pi_1(C)$ maps to γ. That is γ is embedded in S_1. □

Is better for us to deeply explain the group theoretic point of view and its connections with the geometric one, so we need others definitions and some results:

Definition 2.1.3 (Residually Finite Group)**.** A group G is said to be *residually finite* (RF) if for any non trivial element $g \in G$ there is a subgroup G_1 of finite index in G which does not contain g.

Definition 2.1.4 (S-Residually Finite Group)**.** A group G with a subgroup S is said to be *S-residually finite* (S-RF) if for any non trivial element $g \in G \setminus S$ there is a subgroup G_1 of finite index in G which contains S but not g.

Definition 2.1.5 (Extended Residually Finite Group)**.** A group G is said to be *extended residually finite* (ERF) if G is S-RF for every subgroup S of G.

Definition 2.1.6 (Locally Extended Residually Finite Group)**.** A group G is said to be *locally extended residually finite* (LERF) if G is S-RF for every finitely generated subgroup S of G

Lemma 2.1.3 (Stability of RF-properties)**.** *If G is RF or ERF or LERF, then any subgroup of G has the same property and so does any group K which contains G as a subgroup of finite index.*

Proof. The first part about subgroups is obvious. Let K be as in the statement. If G is not normal in K we consider $G_0 = \bigcap_{k \in K} k^{-1}Gk$ instead of G (note that G_0 is normal, has the same property of G and has finite index in K). So we suppose G to be normal in K and we can take the finite quotient $F = K/G$ and the projection $p : K \longrightarrow F$.
Case 1*: G is RF:* take $k \in K$, then if $k \in G$ we found G_1 by definition of RF for G and G_1 has finite index also in K; if $k \notin G$ then G itself is the subgroup we are searching for.
Case 2*: G is ERF:* let $S \leq K$ and $k \in K \setminus S$. $S \cap G$ is a normal subgroup of S and $F_1 = S/(S \cap G)$ is a subgroup of F. We have that $K_1 = p^{-1}(F_1)$ contains S and has finite index in K (K_1 contains G), so if k is not in K_1 we have done. Suppose $k \in K_1$, then we can write $k = gs$ with $g \in G$ and $s \in S$ ($p(k) \in F_1 = \frac{S}{S \cap G}$). Since $k \notin S$, g can't be in $S \cap G$. So there exist a subgroup G_2 of G which has finite index in it, contains $S \cap G$ and $g \notin G_2$ (ERF property of G). Note that $G_3 = \bigcap_{s \in S} s^{-1}G_2 s$ is also a subgroup of G of finite index which contains $S \cap G$ but not g. Note also that the normalizer $N_K(G_3)$ contains S. Let K_3 be the subgroup of K_1 generated by S and G_3. Then G_3 is normal in K_3 and the quotient is F_1. Now K_3 contains G_3 which has finite index in G, so K_3 has finite index in K and obviously contains S but not g (which is not contained in G_3). So also k can't be in K_3 and we have done.
Case 3*: G is LERF:* the proof is the same of the case 2: we have only to note that once we have S finitely generated, $S \cap G$ has finite index in S and so is finitely generated too. □

We recall that a cover $p : (\tilde{X}, \tilde{*}) \longrightarrow (X, *)$ is said *regular* (or *normal*) if for some $\tilde{*}_0 \in p^{-1}(*)$ we have that $p_*(\pi(\tilde{X}, \tilde{*}_0))$ is a normal subgroup of $\pi_1(X.*)$. For a detailed treatment of regular covers and their properties see [Hat02] 70-73.

Lemma 2.1.4. *If X is a PL manifold (possibly with boundary) with a regular covering $\tilde{X}$ and covering group G we have that $T_C := \{g \in G : gC \cap C \neq \emptyset\}$ is finite for every compact subset C of $\tilde{X}$.*

Proof. Since X is a PL manifold of dimension n also $\tilde{X}$ is. Moreover X admits a triangulation which made it in a simplicial complex K. The simplicial structure pulls-back to a simplicial structure $\tilde{K}$ for $\tilde{X}$. Now the group action of G on $\tilde{X}$ can be viewed as an action on $\tilde{K}$ since every covering transformation can be approximated with the corresponding simplicial map, and such simplicial approximation is homotopic to the original map. Since every map of G is in a different homotopy class the two action of G are the same. Now take a compact C in $\tilde{X}$ for which T_C is infinite. The compactness of C tell us that there are only finite many n-simplexes of $\tilde{K}$ meeting C and we call their union Δ. From the infinity of T_C follow the infinity of T_Δ. Recall that the hypothesis of regularity for the covering means that the action of G has to be free, that is $gx \neq hx$ for every $x \in \tilde{X}$ and $g,h \in G$. But the number of vertices in Δ is finite, and simplicial maps send vertices in vertices. This means that only a finite number of $g \in G$ can send vertices of Δ inside Δ. So T_Δ is finite and also T_C is. □

Lemma 2.1.5. *Let X be a PL manifold with the same hypothesis of Lemma 2.1.4. Then the following conditions are equivalent:*
(i) G is RF,
(ii) if C is a compact in $\tilde{X}$ then G has a subgroup G_1 of finite index such that $gC\cap = \emptyset$ for every non trivial element g of G_1,
(iii) if C is a compact in $\tilde{X}$ then the projection map $\tilde{X} \longrightarrow X$ factors through a finite covering X_1 of X such that C projects homeomorphically into X_1.

Proof. (ii) $\Longleftrightarrow$ (iii) is obvious taking $X_1 = \tilde{X}/G_1$.
Suppose (i) true and let C be compact in $\tilde{X}$, then by 2.1.4 we have that $T_C = \{g \in G : gC \cap C \neq \emptyset\}$ is finite. So take $G_1 = \bigcap_{t \in T_C} G_t$ where G_t is a group from the RF property which does not contains t. G_1 satisfy (ii) by definition.

Now suppose that (ii) hold, let g be a non trivial element of G and x be a point of $\tilde{X}$; use (ii) with $C = x \cup gx$, then $g \notin G_1$ and we have proved that G is RF □

Lemma 2.1.6. *Let X be a PL manifold with regular covering $\tilde{X}$ and covering group G. Then G is LERF $\iff$ for any given f.g. subgroup S of G and a compact subset C of $\tilde{X}/S$ there is a finite covering X_1 of X such that the projection $\tilde{X}/S \longrightarrow X$ factors through X_1 and C projects homeomorphically into X_1.*

Proof. ($\Leftarrow$). Let S be a f.g. subgroup and $g \in G \setminus S$, $x \in \tilde{X}$ and C in $\tilde{X}/S$ be the projection of $x \cup gx$.
By hypothesis we get $X_1 = \tilde{X}/G_1$ a finite cover of X and clearly G_1 can't contains g since $x \cup gx$ projects homeomorphically into X_1. Hence G is LERF.
($\Rightarrow$). Let S and C be as in the hypothesis, $p : \tilde{X} \longrightarrow \tilde{X}/S$ the projection and $Y = p^{-1}(C)$. From the definition of covering we can easily find a compact $D \subseteq Y$ with $p(D) = C$. So $T = \{g \in G : gD \cap D \neq \emptyset\}$ is finite by lemma 2.1.4. By LERF of G we can find G_1 which has finite index, contains S and $G_1 \cap T = S \cap T$ (this last requirements is the same of "excluding a finite set of elements from G_1", that is we can ask a finite number of elements of T to not be in G_1 by definition of LERF). Then $X_1 = \tilde{X}/G_1$ works (note that D is untouched by the new elements we added to S, so its image C go homeomorphically into X_1). □

Now we restrict to the case of surfaces. Recall that every topological surface has a PL structure.

Lemma 2.1.7 (Incompressible Subsurface Lemma)**.** *Let S be a surface such that $\pi_1(S)$ is finitely generated and let C be a compact subset of S. Then there is a compact, connected, incompressible subsurface Y of S which contains C such that the natural map $\pi_1(Y) \longrightarrow \pi_1(S)$ is an isomorphism.*

Proof. Choose a basepoint $*$ for S and a finite set of generators for $\pi_1(S, *)$. For each such generator we consider the pointed map $\sigma_i : (S^1, 0) \longrightarrow (S, *)$

that maps S^1 onto the generator and 0 to $*$ and we take a regular neighbourhood N of the union

$$C \cup (\bigcup_i \sigma_i)$$

which is a subsurface of S. We may need to attach to N some 1-handles to make it connected, and some 2-disks to make it incompressible. So we obtain the final subsurface Y for which $\pi_1(Y) \longrightarrow \pi_1(S)$ is surjective by construction and injective because Y is incompressible. □

Lemma 2.1.8 ((LERF $\leftrightarrow$ Geometric) Lemma)**.** *Let S be a surface. Then $\pi_1(S)$ is LERF if and only if given a finitely generated subgroup F of $\pi_1(S)$ and $g \in (\pi_1(S) - F)$ there is a finite cover S_1 of S such that $\pi_1(S_1)$ contains F but not g and F is geometric in S_1*

Proof. The if part is trivial. Now suppose that $\pi_1(S)$ is LERF and we are given a finitely generated subgroup F of $\pi_1(S)$ and $g \in (\pi_1(S) - F)$. Let S_F denote the covering of S corresponding to F. Pick $x \in \tilde{S}$ and let C denote the projection of $x \cup gx$ into S_F and use the Incompressible Surface Lemma 2.1.7 to obtain an incompressible $S_F \supseteq Y \supseteq C$ with $\pi_1(Y) = \pi_1(S_F)$. Now Proposition 2.1.6 tell us that S has a finite covering S_1 through which factorize the projection $S_F \longrightarrow S$ and Y projects homeomorphically into S_1. This concludes the proof. □

So by this lemma we have that Theorem 2.1.1 is equivalent to prove that every surface group is LERF, that is what we will prove. We start discussing a simple example, which contains the idea of the proof for the general case. Geometrically is the torus and Klein bottle case.

Lemma 2.1.9. *The group $G = \mathbb{Z} \times \mathbb{Z}$ is RF.*

Proof. G acts on the plane $\mathbb{R}^2$ as the group of translations. A fundamental domain for this action is a square Q with side 1 and the action of G on the plain tessellates it with infinite copies of Q. By Lemma 2.1.5 we have to prove that given a compact $C \subseteq \mathbb{R}^2$ there is a finite index subgroup G_1 such that for every non trivial $g \in G_1$ we have $gC \cap C = \emptyset$. Every compact $C \subseteq \mathbb{R}^2$ is contained in a finite union of fundamental domains X. Eventually joining finite others fundamental domains we can suppose X is contained in a big square in the sense that exists a $k \in \mathbb{N}$ such that $X \subseteq \{(x,y) \in \mathbb{R}^2 : -k \leq x, y \leq k\}$. So we consider such big square instead of X and take $G_1 = (2k+1)\mathbb{Z} \times (2k+1)\mathbb{Z}$ as subgroup. Now G_1 has obviously finite index in G and send our big square in disjoint copy of it.

□

A similar argument of the previous proof can be used to prove

Proposition 2.1.10. *$G = \mathbb{Z} \times \mathbb{Z}$ is LERF.*

Proof. We want apply Lemma 2.1.6. Let S a finitely generated subgroup of G. Recall that the only subgroups of $\mathbb{Z}$ are of the form $k\mathbb{Z}$ with $k = 0, 1, 2, \dots,$ so the only subgroups of G with infinite index are of the form $k\mathbb{Z} \times \{0\}$ or are the trivial subgroup. So the space $\mathbb{R}^2/S$ is a cylinder and has non compact fundamental domain that is a vertical or horizontal strip of the plain that we can suppose $F = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq k\}$. Let C be a compact subset of the cylinder $\mathbb{R}^2/S$. Since the map $p : \mathbb{R}^2 \longrightarrow \mathbb{R}^2/S$ is a covering we can find a compact $D \subseteq \mathbb{R}^2$ with $P(D) = C$ and $D \subseteq F$. Now since D is compact we can find an $h \in \mathbb{N}$ such that $D \subseteq \bar{F} = \{(x,y) \in F : 0 \leq y \leq h\}$. Now we have $G_1 := k\mathbb{Z} \times h\mathbb{Z} \geq k\mathbb{Z} \times \{0\} = S$ so the map

$$\mathbb{R}^2/S \longrightarrow \frac{\mathbb{R}^2}{\mathbb{Z} \times \mathbb{Z}}$$

factors through the space $\mathbb{R}^2/G_1$. Furthermore $D \subseteq \bar{F} \subseteq F$ so C maps homeomorphically in our new quotient space as required by Lemma 2.1.6.

□

Now we are going to study free groups or, equivalently, non-closed surfaces. This case is easier than the closed case and we prove it first to give a simpler version of the proof which motivates the final proof.

Non-closed Surfaces

Theorem 2.1.11 (Geometric non-closed surface subgroup theorem). *Let S be a non closed surface, let F be a finitely generated subgroup of $\pi_1(S)$ and let $g \in \pi_1(S) \setminus F$. Then there is a finite covering S_1 of S such that $\pi_1(S_1)$ contains F but not g and F is geometric in S_1. Further F is a free factor of $\pi_1(S_1)$.*

By Lemma 2.1.8 we have the equivalence of the previous theorem with the following:

Theorem 2.1.12 (LERF for non-closed surfaces). *Let G be a free group, let F be a finitely generated subgroup of G and let $g \in G \setminus F$. Then G has a subgroup G_1 of finite index which contains F but not g. Further F is a free factor of G_1.*

Proof of Theorem 2.1.11 First we note that we cn suppose the surface S to be compact. Indeed once we proved Theorem 2.1.12 for groups of compact surfaces we have that it holds for every finitely generated free group and, then, it holds for every free group (this implication is obvious). Then also Theorem 2.1.11 holds for every surface (compact or not). So let S be a compact surface with boundary. We can think to S as a 2-disk D^2 with identifications in pairs on arcs of the boundary. In particular D^2 can be made to be a fundamental region of the universal covering space of S. That is ∂D^2 divided in $4n$ arcs, where n is the rank of the free group $\pi_1(S)$, identified such that arcs lying in the interior of S are alternates with arcs lying in ∂S. The arcs of the first type need to be oriented so that the identifications will be orientation preserving, and we label these arcs with integers from $-n$ to n so that the arc i is identified with $-i$. Now every cover of S is a countable collection of copy of D^2 with all the labelled edges identified (coherently with the orientations) in pairs with the same label of opposite sign. The converse is also true: every such collection is a projection to S which is the standard projection restricted to every copy of D^2. Now let F be a finitely generated subgroup of $\pi_1(S)$ and $g \in \pi_1(S)) \setminus F$. Then there is a based covering

$p : S_F \longrightarrow S$ with $p_\#(\pi_1(S_F)) = F < \pi_1(S)$. Take a path l in S_F starting in the basepoint of S_F and such that $p(l)$ represent g. Note that since $g \notin F$, l must be not closed. By the incompressible surface lemma (2.1.7) we find a subsurface Y of S_F containing l (an l compact as described above can be always found easily since $p(l)$ is compact). Now let X be the union of the finite many copies of D in S_F which meets Y. We can restrict the covering $S_F \longrightarrow S$ to the projection $X \longrightarrow S$ which is not a covering since ∂X can have labelled edges not glued with anything. However we can glue them in allowable pairs since every copy of D has exactly one edge i and one $-i$ and they can only be glued in pairs. We obtain a finite covering S_1 containing Y thus $\pi_1(S_1)$ contains F and F is geometric in S_1. Obviously $g \notin \pi_1(S_1)$ since l is not closed. To see that F is a free factor of $\pi_1(S_1)$ we recall that Y is incompressible in S_F and $Y \subseteq X \subseteq S_F$ so we have $\pi_1(X) = \pi_1(Y) = \pi_1(S_F)$. We recall from homotopy theory that if we glue two edges of the boundary surface so that it remains non-closed, we are adding a free factor of rank one to its fundamental group (in general if we glue the vertices of a segment to a topological space we are adding a free generator to the π_1 of the space). So when we glue the edges of ∂X we add free generator to the fundamental group of Y

Closed surfaces

Consider the hyperbolic plane $\mathbb{H}^2$ and fix a point $*$ on it. Consider five geodesic rays γ_i, starting from $*$ with direction v_i, $i \in \frac{\mathbb{Z}}{5\mathbb{Z}}$. Suppose that the smaller unsigned angle between v_i and v_{i+1} is exactly $\frac{2\pi}{5}$ for every i. Then let p_i be the point of γ_i at distance r from $*$ for an arbitrary $r > 0$. Taking the geodesic arcs between p_i and p_{i+1} we have constructed a regular pentagon P_r in $\mathbb{H}^2$. By the Gauss-Bonnet theorem for polygons we have $\text{Area}(P_r) = 3\pi - 5\alpha$ where α is an internal angle of P_r. Since we can get pentagons small as we wish taking r small, we can find pentagons with internal angles near to the euclidean case, i.e. we can find all the five angles greater then $\frac{\pi}{2}$. However letting $r \to \infty$, we can go near as we wish to an ideal pentagon, that is a pentagon with every vertex in $\partial\mathbb{H}^2$, and then P_r have all angles near 0. So in the middle there is an r_0 for which we get a right angled pentagon $P_{r_0} = P$. Define Γ the subgroup of $\mathrm{Isom}(\mathbb{H}^2)$ generated by the five reflections in the sides of P (precisely in the geodesics individuated by the sides). Since every internal angle of P is $\pi/2$ the translates of P by the action of Γ tessellates $\mathbb{H}^2$. Furthermore P is a fundamental region for this action. Finally denote with L the set of all the geodesics obtained translating by translations of Γ from the sides of P.

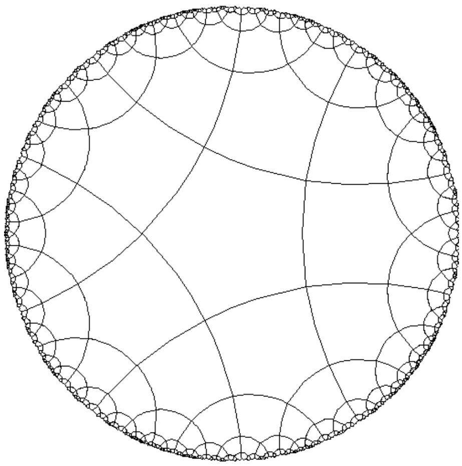

Figure 2.1: The set L.

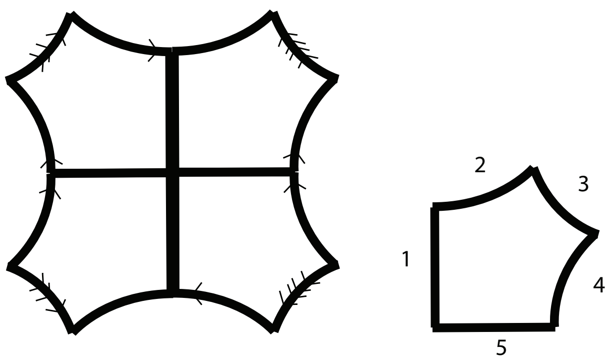

Figure 2.2: Right-angled pentagon on the left and a right angled octagon made of pentagons on the right.

Now we can glue four pentagons isomorphic to P to obtain a right angled octagon as in figure 2.2, that, with the indicated identifications, represent a closed surface F with $\chi(F) = -1$. Define the following subgroup G of Γ: called x_1, x_2, x_3, x_4 and x_5 the five reflections which generate Γ, consider the subgroup generated by the four elements $x_1x_2x_5$, x_1x_4, x_3x_5 and $x_1x_3x_1x_5$. This group G acts on the octagon in figure 2.2 identifying the four pairs of sides as shown in the figure so the octagon is a fundamental region for G. Furthermore the isometry $x_1x_2x_3$ is orientation reversing (since it is the composition of three reflection) whereas all the other generators are orientation preserving, that means that F is non orientable. By the classification theorem for topological surfaces we have determined uniquely the topology of F that is the connected sum of a torus with a real projective plane. Note that since we used four pentagons we have that the index of G in Γ is 4 (in particular is finite). Now recall that the fundamental group of F can be calculated from the Seifert - Van Kampen theorem adding a Moebius strip to a punctured torus, and obtaining

$$G = \pi_1(F) =< a, b, z | [a, b]z^{-2} > .$$

Lemma 2.1.13. *Every closed surface S with $\chi(S) < 0$ is a covering space for F.*

Proof. Every such surface is a connected sum of many tori T and projective planes U, with at least one torus. First we see the lemma for $S = T\#T$, the closed orientable surface of genus 2 with

$$\pi_1(S) =< a_1, b_1, a_2, b_2 | [a_1, b_1][a_2, b_2] > .$$

Consider the subgroup $< a, b, z^{-1}az, z^{-1}bz >$ of $\pi_1(F)$ and note that

$$[a, b][z^{-1}bz, z^{-1}az] = [a, b]z^{-2} = \mathrm{id},$$

so it is isomorphic to the fundamental group of S. So there exist a cover $\tilde{F}$ of F with $\pi_1(\tilde{F})$ mapped isomorphically to that subgroup. But the topology of a surface is determined by his fundamental group and $\tilde{F}$ must be homeomorphic to S. Now every orientable surface with $\chi < 0$ is a cover of $T\#T$ as well every non-orientable surface with $\chi < 0$ is a cover of F. □

We want prove that every surface group is LERF that, since Lemma 2.1.8, is equivalent to prove Theorem 2.1.1. Suppose that the group $G = \pi_1(F)$ is LERF, then by Lemma 2.1.13 and the topological classification of surfaces, we have done (remember the stability of the LERF property in Lemma 2.1.3).

Theorem 2.1.14. *The group $G = \pi_1(F)$ is LERF*

Proof. Let S be a finitely generated subgroup of G, C be a compact in $\mathbb{H}^2/S$ and $p : \mathbb{H}^2 \longrightarrow \mathbb{H}^2/S$ be the covering map. Let D be a compact of $\mathbb{H}^2$ such that $p(D) = C$. We are searching for a subgroup G_1 with $S \subseteq G_1 \subseteq G$ with finite index in G and such that for every $g \in G_1$ is verified that $g \in S$ every time that $gD \cap D \neq \emptyset$. By Lemma 2.1.6 will come that G is LERF since C will project homeomorphically into $\mathbb{H}^2/G_1$. From the Incompressible Surface Lemma 2.1.7, we can find an incompressible subsurface C_1 of $\mathbb{H}^2/S$ such that $C \subseteq C_1$ and $\pi_1(C_1) \longrightarrow \pi_1(\mathbb{H}^2/S)$ is the natural isomorphism induced by the inclusion. Let Y denote $p^{-1}(C_1)$ which is a connected surface in $\mathbb{H}^2$, since C_1 is incompressible. Recall the set L defined by translations of P. Consider a line $l \in L$ which does not meet Y. Then Y is completely contained in one of the two half-planes individuated by l. Consider the intersection of all such half-planes varying $l \in L$ and denote it with $\bar{Y}$. It is convex and union of pentagons by definition. Now note that Y is S-invariant by definition and so $\bar{Y}$ too since S is a subgroup of Γ and $\bar{Y}$ is the smallest set of pentagons containing Y. It follow that $p|_{\bar{Y}}$ is a cover onto its image $p(\bar{Y}) \subseteq \mathbb{H}^2/S$ which will be union of pentagons. Now consider the projection of the lines $p(L)$. By definitions of Y and $\bar{Y}$ if an $l \in L$ meets the interior of $\bar{Y}$, then $p(l)$ has to meet C_1. But C_1 is compact and so only finitely many such projections of lines meet it. So $p(\bar{Y})$ is compact. Now let X be a fundamental region in $\bar{Y}$ for the action of S on it. Define Γ_2 the group generated by the reflections in the sides of $\bar{Y}$. Since all the vertices of $\bar{Y}$ have internal angle equal to $\frac{\pi}{2}$, it must be a (not necessarily compact) fundamental region for the action of Γ_2 onto $\mathbb{H}^2$. Take $\Gamma_1 < \mathrm{Isom}(\mathbb{H}^2)$ the subgroup generated by Γ_2 and S. Since every $s \in S$ leaves $\bar{Y}$ invariant it conjugates elements of Γ_2 to other elements of it. So Γ_2 is normal in Γ_1 and the quotient is isomorphic to S which implies

$\mathbb{H}^2/\Gamma_1 = (\mathbb{H}^2/\Gamma_2)/S$. Since Γ_2 is generated by reflections on sides of $\bar{Y}$, it follow that X must be a compact fundamental region for the action of Γ_1 on $\mathbb{H}^2$. In particular Γ_1 has finite index in Γ. Let D be a compact in the interior of $\bar{Y}$ which verify $p(D) = C \subseteq C_1$ and let $g \in \Gamma_1$ such that gD meets D. Then $g(\text{int}(\bar{Y}))$ meets $\text{int}(\bar{Y})$, so g is not generated by reflections of Γ_2 and it must be in S. Then $G_1 = \Gamma_1 \cap G$ has all the required properties.

□

2.2 Immersed Surfaces

Let M and N be two oriented, not necessarily connected, manifolds (possibly with boundary) with $\dim N \leq \dim M$.

Definition 2.2.1 ((Positive) Immersion)**.** A continuous function $f : N \longrightarrow M$ is an immersion if it is a local homeomorphism on its image. It is a *positive* immersion if it is orientation preserving.

Note that the restriction of f to the interior of N is a local homeomorphism on M, as well the restriction to ∂N.

Let S, T compact connected surfaces, possibly with boundary.

Definition 2.2.2 (Bounded Immersion)**.** Let $\gamma : \bigsqcup_i S^1 \longrightarrow S$ be an immersion of an oriented 1-manifold in S. We say that γ *bounds* a positive immersion $f : T \longrightarrow S$ if there exists an orientation preserving homeomorphism $\partial f : \partial T \longrightarrow \bigsqcup_i S^1$ such that the following diagram commutes

$$\begin{array}{ccc} \partial T & \hookrightarrow & T \\ \downarrow{\scriptstyle \partial f} & & \downarrow{\scriptstyle f} \\ \bigsqcup_i S^1 & \xrightarrow{\gamma} & S \end{array}$$

We say that γ *rationally bounds* (or *virtually bound*) f if we allow the map ∂f to be an orientation preserving covering map of finite degree instead of an homeomorphism.

We can extend the definition of rationally bounded immersion to a rational sum of closed curves. Let $C = \sum_{i=0}^{n} r_i \gamma_i$, for some $r_i \in \mathbb{Q}$ and γ_i immersed 1-manifolds that we suppose connected for simplicity. Taking a common multiple we can suppose each r_i to be integer. Then say that C rationally bounds if $\gamma = (\gamma_1^{r_1}, \ldots, \gamma_n^{r_n}) : \bigsqcup_i S^1 \longrightarrow S$ rationally bounds as defined above.

Now we can state the main result of this section. It was originally proved in [Cal09]. When we write $H_1(S)$ we mean the first homology group of S with rational coefficients.

Theorem 2.2.1 (Calegari's Immersion Theorem). *Let S be a compact, connected orientable surface (possibly with boundary) with $\chi(S) < 0$. Let $C = \sum r_i g_i$ be a finite rational sum of homotopy classes of oriented closed curves g_i such that C is trivial in $H_1(S)$. Then there exists a rational number R_0 such that $\forall R > R_0$, $R \in \mathbb{Q}$ the 1-manifold identified by $R\partial S + \sum r_i g_i$ virtually bounds an immersed surface.*

Since every topological surface S as in the statement admits at least an hyperbolic structure, we can realize every immersion of 1-manifolds in the statement with geodesic representatives.

The following example remarks the relation between negative characteristic and Theorem 2.2.1 and stress the non triviality of the statement.

Example 2.2.1 (Necessity of the negative Euler Characteristic). Consider the torus S_1 without punctures, with $H_1(S_1)$ generated by a and b. It has Euler characteristic $\chi(S_1) = 0$. Then there is no positive immersed surface $f : T \longrightarrow S_1$ rationally bounded by three closed curves represented by $a + b - ab$.

Proof.

Fact 2.2.2. *f can be taken surjective.*

Proof. We will prove that given an f as in the hypothesis we can find another surjective immersion $\tilde{f}$ in the same homotopy class. First of all note that a connected boundary component δ of T is a circle so a neighbourhood of it in T (which is the union $\bigcup_{x\in\delta} N_x$ with N_x a neighbourhood of x in T) is homeomorphic to a cylinder C_δ. So a new map $\tilde{f}$ which is the same of f outside C_δ and is homotopic to f on C_δ is in the same homotopy class of f as a map of all T. In particular this means that if we have another representative of the homotopy class of $f(\delta)$ we can homotope f to have a new immersion bounded by this new representative. Take a rectangular fundamental domain with oriented pairs of opposite sides representing a and b. We can think to this rectangle as the compact region of $\mathbb{R}^2$ defined as $R = \{0 \leq x \leq A, 0 \leq y \leq B\}$ with $A, B \geq 0$, with orientation on the sides given by the increasing direction of the axis x and y. In this representation the diagonal of the rectangle oriented from (A, B) to the point $(0, 0)$ is in the homotopy class of $-ab$. So we can start with f bounded exactly by this curve plus the sides a and b. Since the immersion is positive we can suppose that the points of R satisfying $y \leq \frac{B}{A}x$ are in the image of f. Now consider the family of curves all homotopic to $-ab$ represented in R by the equations $y = \frac{B}{A}x + q$ varying $q \in [0, B]$. Note that for $q \neq 0$ we need also to consider the segment in R individuated by $y = \frac{B}{A}x - B + q$ in order to have a representative of $-ab$ inside R. So we can homotope f along this family of curves for the boundary $-ab$: this ensure at every stage of the homotopy that all the points satisfying $y \leq \frac{B}{A}x + q$ are in the image of the immersion. In particular at the end of the homotopy we have a srjective immersion □

Fact 2.2.3. *T must have at least 3 boundary components. In particular $\chi(T) < 0$*

Proof. Let γ_a ,γ_b and γ_{-ab} be the three immersions from S^1 to S_1 which are homotopic to a, b and $-ab$ respectively and bounds $f(T)$. Then by Definition 2.2.2 ∂f must be a cover over the disjoint union of the three copy of S^1 mapped as above. So there must be at least 3 boundary components on T. □

Now take a triangulation of S_1 such that a, b and $-ab$ are edges and the degree (or valence) of every vertex is 6. Such a triangulation exists, since

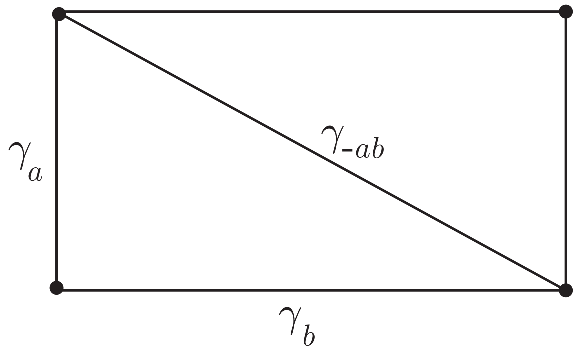

Figure 2.3: A Triangulation of the torus with one vertex of degree 6

one can be found cutting S_1 along γ_a and γ_b so that we obtain a rectangle, and then considering the two triangle individuated by the diagonal of the rectangle, that can be thought as γ_{-ab} since what we seen in 2.2.2 (see Figure 2.3).
Now f pulls-back the triangulation over T: for it, first we can pull-back every vertex to a disjoint union of points. Since f is a local homeomorphism we can pull-back every path in a union of disjoint paths. Finally we have the 1-skeleton so we can fill the triangles accordingly with f (note that we are using the surjectivity of f 2.2.2). So we have a triangulation of T with the property that in every vertex in the interior of T has valence 6 (since f is a local homeomorphism) whereas the vertices in the boundary of T have valence 4, in fact the boundary is a part of the triangulation and map to an edge of the triangulation over S_1 but the three edges intersecting the vertex and cutting themselves, so exactly two edges are in each side of the third.
Take two distinct but equal copies of T and glue them along the boundary components. We obtain a new surface $2T$ without boundary and with a new triangulation with every vertex of valence 6. If we call V the number of vertices in $2T$, E the number of edges, and D the number of triangles we have from the discussion above

$$6V = 2L \tag{2.1}$$

while the following equation is true for every triangulation

$$3D = 2L \tag{2.2}$$

Together (2.1) and (2.2) gives the Euler characteristic $\chi(2T) = 0$. But $2T$ is a closed surface and χ classify closed surfaces: $2T$ has to be a genus one surface whereas fact 2.2.7 say that we have glued two surface with at least three boundary components, and so the genus of $2T$ has genus at least 2. □

We want to stress te fact that the hypothesis of *virtually bounds* of Theorem 2.2.1 can't be relaxed.

Example 2.2.2 (Necessity of the virtually hypothesis)**.** Consider the oriented topological surface S with genus 2 and 1 boundary component. We fix the basis for the group $\pi_1(S,*) =< a_1, b_1, a_2, b_2 >$. Now consider the free homotopy class of $\gamma = [a_1, b_1]^2[a_2, b_2]$(in Figure 2.4 we can see an oriented representative of γ). Obviously it is trivial in homology so by our theorem it virtually bounds some subsurface. However none of its representative *bounds* any immersed subsurface.

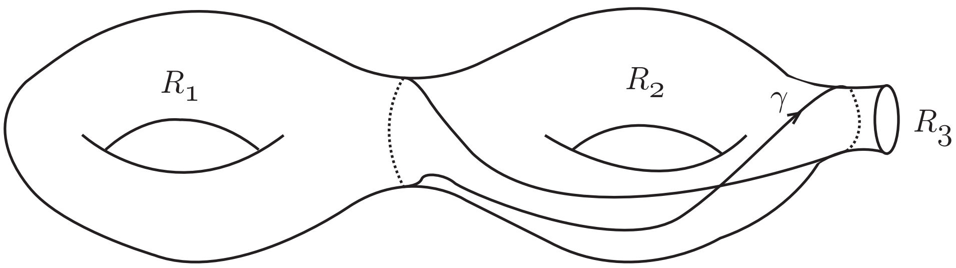

Figure 2.4: A surface S of genus 2 with 1 punctures. The curve γ divides the surface in three regions.

To prove it suppose $f : T \longrightarrow S$ a positive immersion bounded by a geodesic representative of γ for an hyperbolic metric g (with abuse of notation we call gamma both the representative and the homotopy class). We notice that γ have a self intersection point p. Then $S \setminus \gamma$ is made of three connected components R_1, R_2 and R_3 (we refer to Figure 2.4 for the notation), and $\gamma \setminus p$ is made of two arcs γ_1 (between R_1 and R_2) and γ_2 (between R_2 and R_3). We fix a triangulation $\Delta(S) = (V, E, F)$ of S with the condition that $p \in V$ and the two components of $\gamma \setminus p$ are two edges of E. In this way the triangulation $\Delta(S)$ restricts to a triangulation $\Delta(R_i)$ on every R_i's. We observe that ∂T has exactly 1 boundary component which wraps once around γ (because the immersion is bounded by γ), then each R_i is completely contained in $f(T)$, or meets $f(T)$ only in the boundary ($R_i \cap f(T) \subseteq \gamma$). Then T is made as union of many copies of some R_i's and we can pull-back the triangulation $\Delta(R_i)$ on T. In particular if we consider the orientation of γ given in Figure 2.4 we found that R_3 is not in the image of f, since T has only 1 boundary component. Now we want compute the Euler Characteristic of T from the pull-back triangulation. We proceed as follows: the preimage of γ is one to one on ∂T except in the point p which has two preimages $\tilde{p}_1$ and $\tilde{p}_2$. If we go along γ_1 starting in p and following the orientation of γ we have a copy of R_1 on the left. So we can pull-back a copy of the triangulation $\Delta(R_1)$ on T. Now we go along γ_2, where on the left we have R_2, then we can pull-back a copy of $\Delta(R_2)$ on T. So we are passed over all the boundary ∂T on T. However there may be other preimages of γ_1, since a priori we don't know what happen on T away from the boundary. For any vertex of $\Delta(R_2)$ we can find a concatenation of edges of $\Delta(R_2)$ that ends on a vertex in γ_1. Then every time we pulls-back γ_1, except the one that lift into ∂T, we have to pulls-back an entire copy of $\Delta(R_2)$. We know that the triangulation $\Delta(R_2)$ pulls-back one to one on T. Then we have exactly a copy of $f^{-1}(\gamma_1)$ not belonging to ∂T. In a similar way we see that any copy of $\Delta(R_1)$ that pulls-back on T is on the left of a pull-back of γ_1. Then we have that the triangulation on T is made of two copy of $\Delta(R_1)$ and one copy of $\Delta(R_2)$. Then we can compute $\chi(T)$ as the sum of the χ of the pull-back's, and this gives $\chi(T) = \chi(R_1) + \chi(R_1) + \chi(R_2) = -1 - 1 - 2 = -4$. However ∂T has only one connected component, then $\chi(T)$ has to be a number congruent to $1 \mod 2$. This proves that can not exists a positive immersion of a surface bounded by γ.

In order to prove the main result we need the following technical lemmas.

Lemma 2.2.4 (Common Extension Lemma)**.** *Consider a compact, oriented and connected surface T with genus $g \geq 1$ and $\chi(T) < 0$. Let $\delta \subseteq \partial T$ be a disjoint union of boundary component of T and $f : \delta \longrightarrow \gamma$ be an immersion in a 1-manifold γ, with positive degree n_i in each component of δ. Let N be a common multiple of all n_i. Then there exists a finite covering $\pi : \tilde{T} \longrightarrow T$ such that $\pi \circ f = \tilde{f} : \tilde{\delta} \longrightarrow \gamma$ has degree N in each component of $\tilde{\delta}$.*

Proof. There exists a double cover T' of T which doubles the number of components of ∂T. We write δ_i for the connected component of δ which maps with degree n_i on γ. δ_i has exactly two lifts $\epsilon_{i,1}$ and $\epsilon_{i,2}$ on $\partial T'$. We denote with δ' the preimage on T' of δ. We want to define a map

$$\Phi : \pi_1(T') \longrightarrow \frac{\mathbb{Z}}{N\mathbb{Z}}$$

such that $\Phi(\epsilon_{i,1}) = n_i$ and $\Phi(\epsilon_{i,2}) = -n_i$. Obviously we have $\Phi(\delta') = 0$ and so (eventually extending to 0 on other components) we can ask $\Phi(\partial T') = 0$. So we can extend Φ to the whole $\pi_1(T')$. Finally we found the finite cover $p : \tilde{T} \longrightarrow T'$ defined by

$$p_*(\pi_1(\tilde{T})) = \ker \Phi.$$

Note that $\Phi(\epsilon_{i,j}^{N/n_i}) = \pm \frac{N}{n_i} n_i = 0 \mod N$ that is $\frac{N}{n_i} < \epsilon_{i,j} > \subseteq \ker \Phi$. Then the $\epsilon_{i,j}$'s lift to boundary components $\tilde{\epsilon}$ which maps with degree N to γ. □

Remark 2.2.1. For each T with $\chi(T) < 0$, eventually passing to a cover, we can suppose genus$(T) > 0$. For it a 4-holed sphere is double covered by a 4-punctured torus (see figure in the left of 2.5), and we can do the same for n-punctured spheres, $n \geq 4$, which is covered by a $(2n-4)$-punctured torus. The 3-holed sphere is covered by the 4-holed sphere (see figure in the right of 2.5).

Lemma 2.2.5 (Immersion Sum Lemma)**.** *Let S be a compact, oriented and connected surface with $\chi(S) < 0$ Suppose that C_1 and C_2 are two chains*

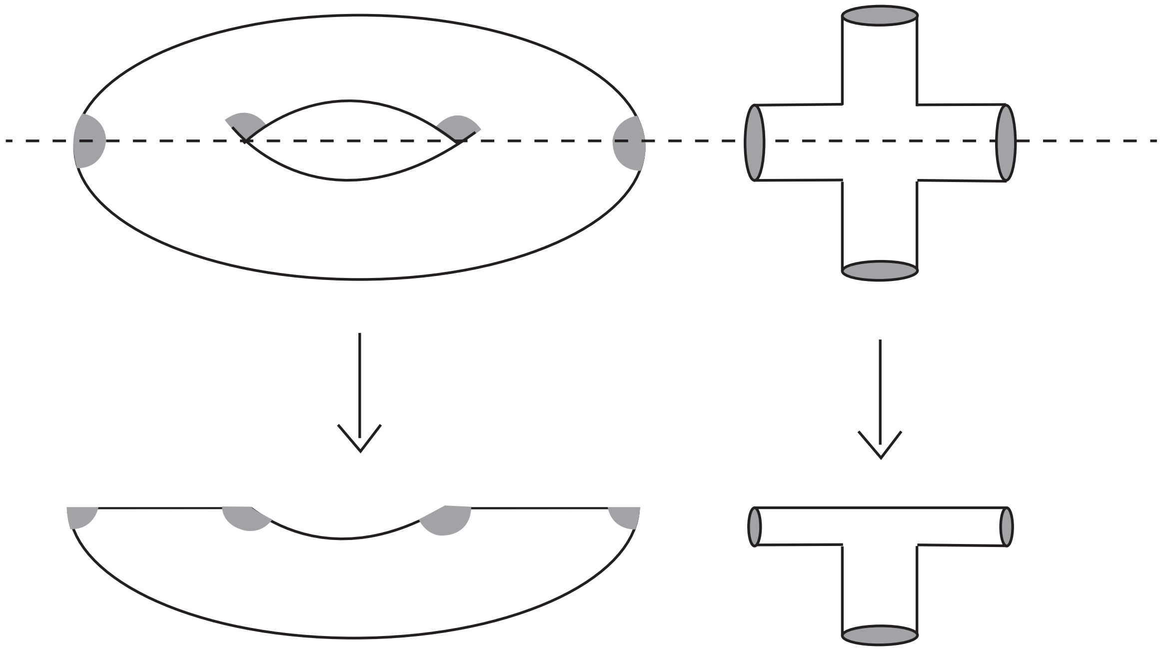

Figure 2.5: On the left there is a 4-holed torus covering a 4-holed sphere. On The right there is a 4-holed sphere covering a 3-holed sphere.

which virtually bounds two positive immersed surfaces T_1 *and* T_2 *satisfying* $\chi(T_1)$,$\chi(T_2) < 0$. *Then the chain* C_1+C_2 *virtually bounds a positive immersed surface in* S, *which has negative* χ.

Proof. x an hyperbolic structure over S. Write $C_1 = \sum r_i g_i$ and $C_2 = \sum s_j h_j$. If $h_j \neq g_i$, $\forall i, j$ we can take the disjoint union of the two surface and there is nothing to be proved. Actually we can also consider the sum surface if $g_i = h_j$ and r_i has the same sign of s_j. So let g be an homology class of the remaining case, with geodesic representative γ (such a representative is not unique but exists always for $\chi(S) < 0$). Let $\delta_1 \subseteq \partial T_1$ and $\delta_2 \subseteq \partial T_2$ be the connected components which maps on γ. Eventually applying lemma 2.2.4 to both T_1 and T_2 we can assume that exists $N \in \mathbb{N}$ such that every component of δ_i maps on γ with degree exactly N. So we can glue the two surfaces along these components (recall that we are in the case with degree of opposite signs) providing a new immersion bounded by $C_1 + C_2$ (Actually we have to do this for all g with opposite sign in the two sums). Note that since we are glueing or considering the disjoint union of the surfaces T_i or of finite covering of them, the Euler characteristic of all immersed surface we have constructed is negative. □

The punctured torus case

Now we study, as an example of Theorem 2.2.1, the case of the n-punctured torus $S_{1,n}$. We fix the notation for the generators $\{[a],[b]\}$ of $\frac{H_1(S_{1,n},\mathbb{Z})}{H_1(\partial S_{1,n},\mathbb{Z})}$. We start with the once punctured torus $S_{1,1}$. In the homotopy class of a and b we can find two generators for $\pi_1(S_{1,1},*)$. We notice that $[a,b]$ is, then, in the homotopy class of $\partial S_{1,1}$. See figure 2.6 for a picture of the once punctured torus we are considering.

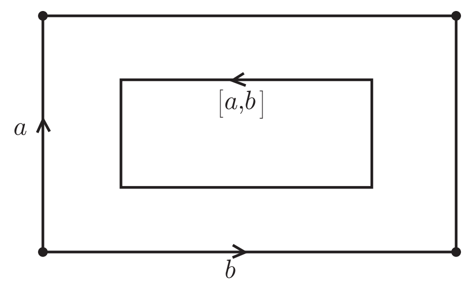

Figure 2.6: A non simply connected fundamental domain for the once punctured torus

Proposition 2.2.6. *The chain $C = a + b - ab + 2[a,b]$ bounds a positive immersed surface $f : T \longrightarrow S_{1,1}$.*

Proof. We will give an explicit construction of the surface. Fix an hyperbolic structure on $S_{1,1}$. Consider the 4-holed sphere T with $\pi_1(T)$ freely generated by x, y and z. We define the map $f_* : \pi_1(T) \longrightarrow \pi_1(S_{1,1})$ sending

$$x \mapsto bab^{-1},$$
$$y \mapsto a^{-1}ba,$$
$$z \mapsto b^{-1}a^{-1}.$$

We observe that $f_*(xyz) = [a, b]^2$, then every boundary components is mapped in one of the free homotopy classes that appear in the sum C. This assignment correspond to the following construction: consider the boundary of T subdivided in arcs as in figure 2.7, then labels every edge with a letter between a,b,a^{-1},b^{-1} as in the figure; then f send every boundary component of T in the geodesic representative of the free homotopy class determined by the letter in the labels along the component. We observe that the boundary components in the Figure 2.7 represent the four homotopy class in the chain C: the "external" component is mapped to $2[a, b]$, the three "internal", respectively, a, b and $-ab$. This identify an unique homotopy class for the map $f : T \longrightarrow S_{1,1}$. We need such class to be the class of an immersion, which is a local property. Having the homotopy class of f is the same thing to have an immersion for the spine Σ of T into $S_{1,1,}$. We can draw such spine as the graph on Figure 2.8. In such graph the numbered vertices are that points of the spine where meets path that are mapped in different homotopy class in $S_{1,1}$. By the natural immersion $\Sigma \hookrightarrow T$, we can consider four *crosses* of T, that are contractible neighbourhoods of the vertices inside T, not intersecting each other. They are numbered in Figure 2.7 with numbers inside circles. In order to see that we can find an immersion in the class of $f : T \longrightarrow S_{1,1}$ it is sufficient to see that f maps every cross in figure 2.7 homeomorphically in a cover of $S_{1,1}$. Figure 2.9 exhibits an explicit immersion in a cover of $S_{1,1}$, where all the crosses are immersed. This is sufficient to see that f is in the homotopy class of an immersion.

□

Remark 2.2.2. In this case we don't need to use a virtual bounding.

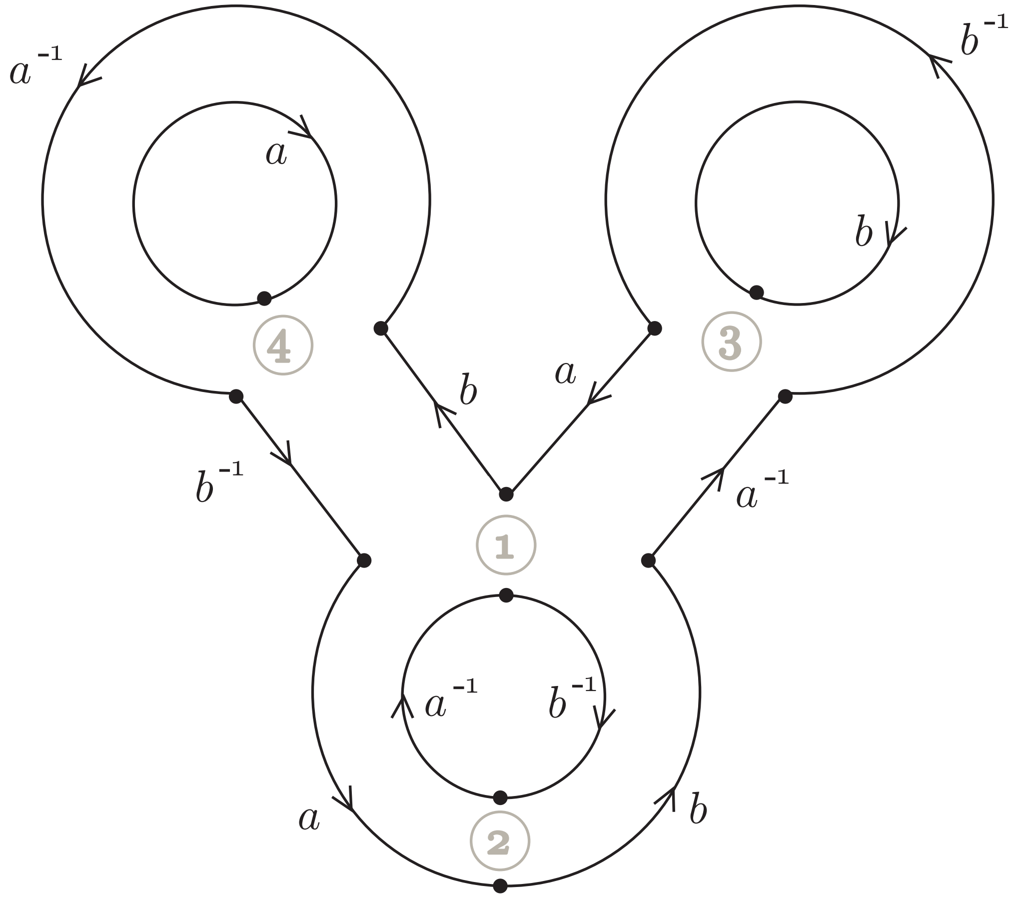

Figure 2.7: The surface T as a 4-holed sphere with the boundary subdivided into arcs with labels.

Remark 2.2.3. From the Immersion Sum Lemma 2.2.5 and the obvious fact that ∂S bounds S, it follows that $a + b - ab + n[a,b]$ rationally bounds immersed surface in $S_{1,1}$ for $n \geq 2$.

Fact 2.2.7. *Let γ be a closed curve in $S_{1,1}$ such that there exist p, $q \in \mathbb{Z}$ such that $[\gamma] = p[a] + q[b]$ in $H_1(S_{1,1}, \mathbb{Z})$. Let d_γ denote a positive sum of boundary components of $S_{1,1}$. Then there exists an $n \in \mathbb{Z}$ such that $\gamma - pa - qb + nd_\gamma$ virtually bounds a positive immersed surface in $S_{1,1}$.*

Proof. Represent the torus like a rectangle with two sides labelled a and other two labelled b. Since we have a punctured torus we need to remove

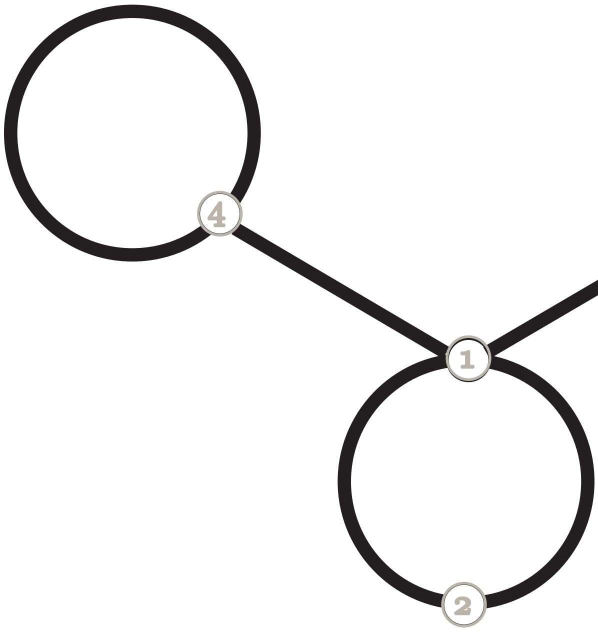

Figure 2.8: The spine Σ of the surface T realized as a graph.

an open disc inside the rectangle. Now consider the cover represented by de rectangle with sides pa and qb, with pq disks D_i removed (it is a torus with pq punctures). Put all these punctures close each other and consider a closed simple curve δ such that all the punctures are in the same side of δ. So we can orient δ and the D_i to have $\delta = \sum D_i$ in homology (in other words $\delta - \sum D_i$ bounds a positive immersed surface). Obviously the diagonal $\tilde{\gamma}$ of the pq-cover is a lift of γ and from 2.2.6 we have that $\tilde{\gamma} - pa - pq + 2\delta$ bounds an immersed surface in the pq-cover. Now using the Immersed Sum Lemma 2.2.5 (between this last surface found and the surface bounded by $\delta - \sum D_i$) we obtain an immersed surface in the pq-cover that can be projected to an immersed in $S_{1,1}$. □

Proposition 2.2.8. *Let $[\gamma] = \alpha[a] + \beta[b] + \partial_\gamma$ be the representative of the homology class of an embedded curve γ in $H_1(S_{1,n}, \mathbb{Z})$ where $n \geq 1$ and ∂_γ is represented by a sum of positive components of $\partial S_{1,n}$. Then there exist another positive sum of boundary components D_γ such that $\gamma - \alpha a - \beta b + D_\gamma$*

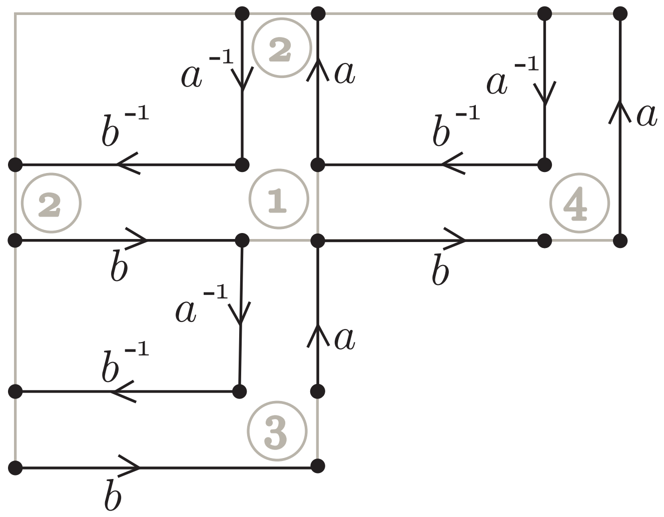

Figure 2.9: This is an immersed view of the surface in Figure 2.7 in a cover of the once punctured torus. For clarity we have omit the boundary components of the torus inside the fundamental domains. The circled numbers refer to the cross of the figure to help to recognize the structure of T (compare with Figure 2.7)

virtually bounds a positive immersed surface in $S_{1,n}$.

Proof. The case $n = 1$ is proved in 2.2.7. So suppose $n > 1$ and fix an hyperbolic structure on $S_{1,n}$. Call S' the once punctured torus obtained by adding a disk D_i on every boundary components except one. Represent γ with the relative embedded geodesic in $S_{1,n}$, so it is not geodesic in S'. It is possible that $S' - \gamma \cup a \cup b$ has some bi-gons which contain one or more D_i. We remove such bi-gons substituting, in $S_{1,n}$, the geodesic γ with a new embedded curve γ' such that $\gamma - \gamma' + \delta_\gamma$ bounds a positive immersed surface in $S_{1,n}$ (here δ_γ is a positive sum of boundary components of $S_{1,n}$), and with γ and γ' homotopic in S'. This substitution is obtained with an iterative procedure that, at each step removes a component corresponding to a D_i from a bi-gon. This is realized considering the curve γ^i that is a geodesic homotopic to gamma in $S_{1,n} \cup D_i$, but is not homotopic to γ in $S_{1,n}$.
We notice that γ', a and b are geodesics in S' for an opportune hyperbolic

structure. Now apply Fact 2.2.7 to γ' in S' to have that $\gamma' - \alpha a - \beta b + d_{\gamma'}$ bounds, then use the Immersion Sum Lemma 2.2.5 and put together all the boundary components in D_γ.

□

Remark 2.2.4. The immersed surface of 2.2.6 has negative Euler characteristic. The holed disk which contains the boundary components of 2.2.7 ha also negative characteristic and all the others immersed surface we have constructed in this subsection by the Immersion Sum Lemma 2.2.5.

Proof of theorem 2.2.1 in the general case $S_{g,n}$

Now we start the proof of the main Theorem 2.2.1. Recall that S is an oriented, compact, connected hyperbolic surface of genus g and with n disks removed.

Since ∂S bounds S it is sufficient prove the theorem for a particular R rational. Multiplying all by a natural number we can assume the r_i all integer and by replacing g_i with g_i^{-1} to be all positive. Let γ_i be a geodesic representative for g_i. Then by Corollary 2.1.2 we can assume γ_i embedded if we work with one of them at time. First decompose S along a union of embedded closed geodesics $\delta = \{\delta_1, \dots, \delta_{k-1}\}$ such that every connected component S_k of such decomposition has genus 1. So, for every k fix a standard basis a_k , b_k for $\frac{H_1(S_k),\mathbb{Z}}{H_1(\partial S_k,\mathbb{Z})}$ Then a_1, b_1,. . . , a_g, b_g is a standard basis for $\frac{H_1(S,\mathbb{Z})}{H_1(\partial S,\mathbb{Z})}$ and we can write

$$[\gamma_i] = \sum_j \alpha_{i,j}[a_j] + \beta i, j[b_j] - D_i$$

where $D_i \in H_1(\partial S, \mathbb{Z})$. Note that we can always assume that D_i is a *positive* sum of boundary components because $[\partial S]_{H_1(S)} = 0$ so we can use this equation to adjust the signs. Now

$$0 = [C]_{H_1(S)} = \sum_i r_i[\gamma_i]_{H_1(S)} = \sum_i r_i(\sum_j(\alpha_{i,j}[a_j] + \beta_{i,j}[b_j]) - D_i) =$$

$$= \sum_j((\sum_i r_i\alpha_{i,j})[a_j] + (\sum_i r_i\beta_{i,j})[b_j]) - \sum_i r_iD_i$$

which means

$$\sum_i r_i D_i = \sum_j ((\sum_i r_i \alpha_{i,j})[a_j] + (\sum_i r_i \beta_{i,j})[b_j]) \tag{2.3}$$

so if we prove that

$$C_i := \gamma_i - \sum_j (\alpha_{i,j} a_j + \beta_{i,j} b_j) + \partial_i \tag{2.4}$$

virtually bounds a positive immersed surface in S (∂_i is an opportune positive sum of boundary components) we can conclude the proof as follow: by the Immersion Sum Lemma 2.2.5 $\sum r_i C_i$ bounds, but (using equations (2.3) and (2.4)) we have

$$\begin{aligned}\sum r_i C_i &= C + \sum_i r_i (D_i + \partial_i) \\ &= C + \sum_i \partial'_i.\end{aligned}$$

Now the chain in the last expression is homologically trivial (since it bounds an immersed surface) but also C is homologically trivial, so $[\sum_i \partial'_i]_{H_1(S)} = 0$ and we know that a positive sum of positive boundary components is trivial only if it is a multiple of ∂S. So we have that $C + R_0 \partial S$ virtually bounds a positive immersed surface for some positive R_0.

Then we are left to prove that C_i virtually bounds a positive immersed surface.

Let $\epsilon > 0$, we define *tubular neighbourhood* of a curve $\gamma \subseteq S$ an open subset $T_\gamma(\epsilon) = \bigcup_{x \in \gamma} U_x(\epsilon)$ where $U_x(\epsilon) = \{y \in S : d(x, y) < \epsilon\}$. Sometimes we suppress the ϵ if it is not important to keep track of its value.

Lemma 2.2.9. *Exists a chain $\gamma_i - \gamma''_i + \partial''_i$ virtually bounding a positive immersed surface and such that γ''_i is a positive sum of embedded closed geodesic such that $\gamma''_i \cap \delta = \emptyset$ while ∂''_i are sums of positive boundary components.*

Proof. We write γ instead of γ_i. If γ is a component of δ then there is nothing to be proved. There are two components of the subdivision in S_k

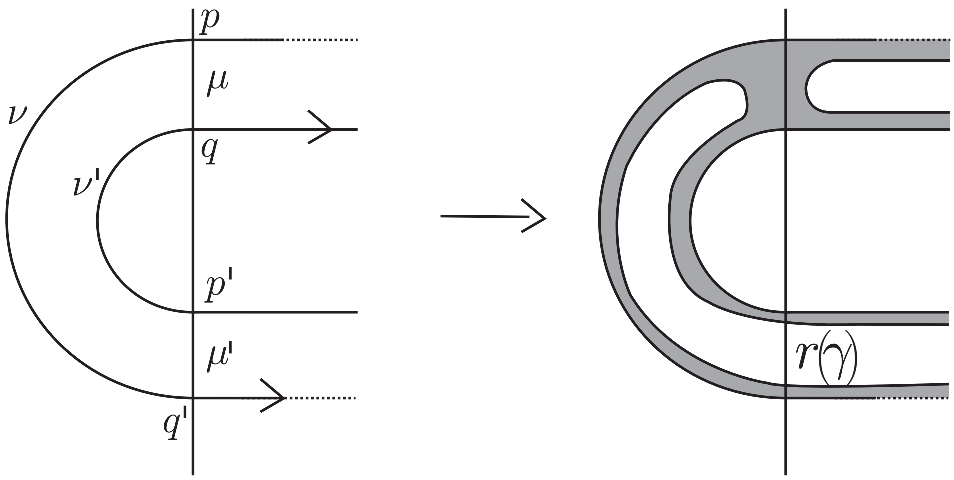

Figure 2.10: A general intersection between δ_j and γ

which have only one boundary component in common with δ. Le S_k one of them, and δ_j the boundary component in common with δ. Then $\gamma \cap S_k$ can be an embedded loop or a collection of geodesic arcs which are essentials (they don't bound a disk with an arc of δ). In the first case we have done. In the second case observe first that, since δ_j disconnect S, for every point where γ intersect δ oriented in the direction from outside S_k to inside there must be a point where γ intersect δ_j from inside to outside. Moreover the intersection points must be finite since δ_j and γ are both embedded in S. Consider two such intersection points p and q such that one of the two connected components of $\delta_j - \{p,q\}$, has no other intersection points with γ, and γ intersect δ_j in in opposite directions in p and q. Such two points exists for what we said above. Call μ the connected component of $\delta_j \setminus \{p,q\}$ without other intersections points. Let $T_\gamma(\epsilon)$ a tubular neighbourhood of γ. Then $T_\gamma \setminus \gamma$ has two connected components and exactly one of them intersect μ. we denote such component as T^*_γ. Let $T_\mu(\epsilon)$ be a tubular neighbourhood for μ. So we can consider the positive immersed surface $\Pi = T^*_\gamma \cup T_\mu$ taken with geodesic boundary, and we define the *resolution* of γ, $r(\gamma) = \partial\Pi \setminus \gamma$. So $\gamma - r(\gamma)$ bounds a positive immersed surface. Moreover $r(\gamma)$ doesn't intersect

δ_j at p or q so

$$\#(r(\gamma) \cap \delta_j) = \#(\gamma \cap \delta_j) - 2 < \#(\gamma \cap \delta_j).$$

This process can be visualised in Figures 2.11 and 2.10.

Iterating this process a finite number of time we obtain a collection of em-

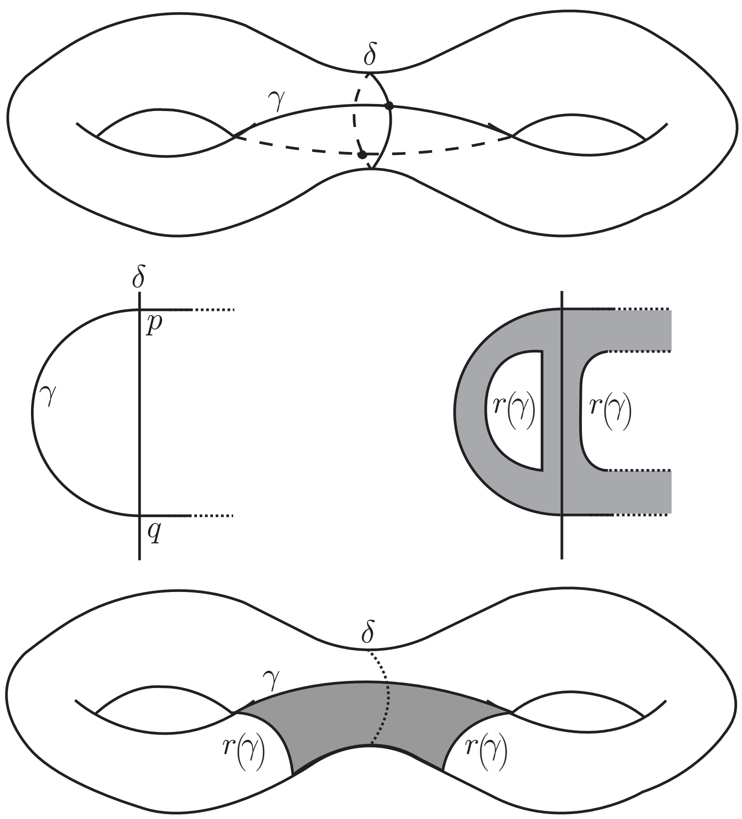

Figure 2.11: How to split a geodesics in two components.

bedded loops γ_i' such that $\gamma_i' \cap \partial S_k = \emptyset$ and $\gamma - \gamma_i' + \tilde{\partial}_i$ virtually bonds an immersed positive surface (here $\tilde{\partial}_i$ is an opportune positive sum of boundary components of S_k, that arise since some components of $r(\gamma)$ can be boundaries). Passing to the other components $S_{k'}$ of the decomposition we use the same process one component by one until we obtain γ_i'' as in the statement. Note that we implicitly use at every step the Immersion Sum Lemma. □

Now fix an arbitrary component S_k of the decomposition. Let ε be a closed loop of γ_i'' embedded in S_k. We can apply proposition 2.2.8 to all such $\varepsilon = \alpha_\varepsilon a_k + \beta_\varepsilon b_k$ and sum (using Lemma 2.2.5) together all the surfaces obtained in one S_k to obtain a surface bounded by $\gamma_{i,k}'' - \alpha_{i,k} a_k - \beta_{i,k} b_k + \partial_{i,k}$ where $\gamma_{i,k}''$ is the sum of components of γ_i'' embedded in S_k, whereas $\alpha_{i,k} = \sum_{\varepsilon \subseteq \gamma_{i,k}''} \alpha_\varepsilon$ and similar for the β's . The boundary components $\partial_{i,k}$ are made by positive sum of boundaries components of S_k, however adding to the immersed surface many copies of $S - S_k$ we can suppress all the boundaries which don't bounds S. Summing up all the resulting surfaces from each S_k we obtain the positive immersed surface bounded by the chain (2.4). The coefficients$\alpha_{i,j}$'s and $\beta_{i,j}$'s of (2.4) are the same we have found here summing all the $\alpha_{i,k}$'s and $\beta_{i,k}$ since in $H_1(S)/H_1(\partial S)$ we have $[\gamma_i] = [\gamma_i'']$.

The Pants Homology Theorem

Let S an oriented compact surface. With the term pair of pants we mean a topological sphere with three punctures. Let Γ be the set of all the free homotopy class of oriented closed curves of S. Then denote with $\mathbb{Q}\Gamma$ the $\mathbb{Q}$ vector space generated by the elements of γ with the identification $\gamma^{-1} = -\gamma$. For every positive immersions $f : P \longrightarrow S$ such that P is a 3-holed sphere, we define $\partial f : \partial P \longrightarrow S$ as the restriction of f to the boundary. Let Π be the set of all homotopy classes of such f with the additional hypothesis that $f_* : \pi_1(P) \longrightarrow \pi_1(S)$ is injective. Let $\mathbb{Q}\Pi$ denote the $\mathbb{Q}$-vector space generated by elements in Π. Finally we linearly extends $\partial : \mathbb{Q}\Pi \longrightarrow \mathbb{Q}\Gamma$.
We can now define the *Pants Homology* as the quotient space

$$H_\Pi(S) = \mathbb{Q}\Gamma/\partial(\mathbb{Q}\Pi).$$

We observe that:

Fact 2.2.10. *Every compact connected oriented surface T with $\chi(T) < 0$ admits a pants decomposition, that is we can cut T along a finite number of closed curves γ_i $i = 1,\ldots,m$ so that $S \setminus \bigcup_i \gamma_i$ is a disjoint union of pair of pants*

Proof. If T has genus 0 and $n \geq 3$ punctures, it is simple to found $\lceil \frac{n}{2} \rceil - 1$ closed curves which separates T in $\lceil \frac{n}{2} \rceil$ pants. If T has genus g then we only have to cut along g curves to reduce again to the punctured sphere case. $\square$

Then the Calegari's Theorem 2.2.1 now implies the following

Corollary 2.2.11 (Pants Homology Theorem)**.** *Let S be an oriented compact surface with $\chi(S) < 0$. Let $H_1(S)$ denotes the first standard homology group with rational coefficients. Then we have*

$$H_\Pi(S) = H_1(S).$$

Chapter 3

Good Pants Homology

3.1 Inefficiency Theory

We denote with $T^1\mathbb{H}^2$ the unit tangent bundle of $\mathbb{H}^2$ while, for a $p \in \mathbb{H}^2$, $T^1_p\mathbb{H}^2$ is the set of unitary tangent vectors of at p. For u, $v \in T^1_p\mathbb{H}^2$ we denote with $\Theta(u, v)$ the unoriented smaller angle between them. Smaller means that $\Theta(u, v) \in [0, \pi]$. Given a unit speed geodesic segment $\alpha : [a, b] \longrightarrow \mathbb{H}^2$ we write $i(\alpha) = \alpha'(a)$ and $t(\alpha) = \alpha'(b)$. From now on all parametrizations of geodesic arcs will be supposed to have unit speed if we don't say anything else. However sometimes we can repeat such assumption to avoid confusion. Let $\alpha_1, \dots \alpha_n$ be piecewise geodesic arcs such that the endpoint of α_i is the initial point of α_{i+1}. Denote with $\alpha_1\alpha_2 \dots \alpha_n$ the concatenation of the arcs while, if the endpoint of α_n is the initial point of α_1, $[\alpha_1 \dots \alpha_n]$ denote the corresponding closed curve. We write $l(\alpha)$ for the length of an arc α.

Definition 3.1.1 (Inefficiency of Arcs)**.** Let α be an arc on a surface. Let γ be a geodesic arc with the same endpoints of α and in the same homotopy class. We call *inefficiency* of α the real number $I(\alpha) = l(\alpha) - l(\gamma)$

Fact 3.1.1 (Monotonicity)**.** *Let α, β and γ be three piecewise geodesic arcs in $\mathbb{H}^2$, with endpoints such that $\alpha\beta\gamma$ is a well defined piecewise geodesic arc. Then $I(\alpha\beta\gamma) > I(\beta)$.*

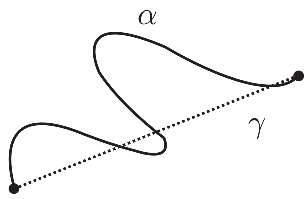

Figure 3.1: The inefficiency of α compared to γ.

Proof. Let η be the geodesic arc homotopic to $\alpha\beta\gamma$ relative to de endpoints, and let β' the geodesic arc with the endpoints of β. We have

$$\begin{aligned} I(\alpha\beta\gamma) =& l(\alpha\beta\gamma) - l(\eta) \\ & \geq l(\alpha\beta\gamma) - l(\alpha\beta'\gamma) \\ & = l(\beta) - l(\beta') \\ & = I(\beta). \end{aligned}$$

□

Consider an hyperbolic triangle with edges A, B and C, and denote with the same three letters the length of the edges. We fix the vertex where A and B intersect, and consider the triangle when A and B go to infinity. Let θ be the external angle between them.

By the hyperbolic cosine rule for triangles we have

$$\cosh C = \cosh A \cosh B + \cos\theta \sinh A \sinh B$$

we have

$$\frac{\cosh C}{e^{A+B}} = \frac{\cosh A}{e^A}\frac{\cosh B}{e^B} + \cos\theta \frac{\sinh A}{e^A}\frac{\sinh B}{e^B}.$$

Recalling that $\lim_{x\to\infty} \frac{\cosh x}{e^x} = \lim_{x\to\infty} \frac{\sinh x}{e^x} = \frac{1}{2}$ we see that

$$(e^{C-A-B})(\frac{\cosh C}{e^C}) = \frac{\cosh C}{e^{A+B}} \to \frac{1}{4}(\cos\theta + 1),$$

when A, $B \to \infty$. That means

$$e^{C-A-B} \to \frac{1}{2}(\cos\theta + 1) = \cos^2(\theta/2),$$

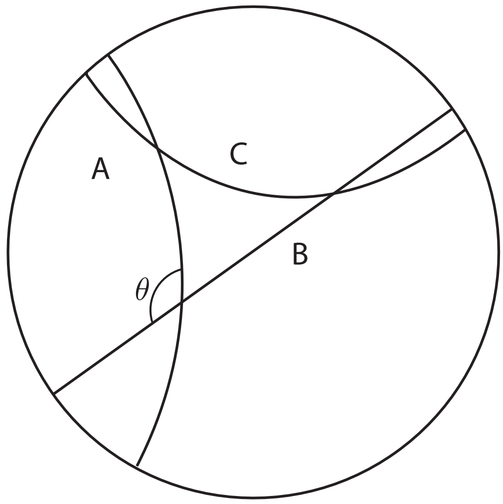

Figure 3.2: The ABC triangle in $\mathbb{H}^2$.

and finally we have

$$A + B - C \to 2\log(\sec\frac{\theta}{2}). \tag{3.1}$$

This computation permit to define an inefficiency for the angles: fix two geodesic rays a_∞ and b_∞ in $\mathbb{H}^2$ with a common initial point and different end point at infinity. Let θ be the exterior angle between them at their meeting point. Call a_r and b_s the geodesic subsegments of, respectively, a_∞ and b_∞ having lengths r and s. Then equation (3.1) gives us a way to define

$$I(\theta) = \lim_{r,s\to\infty} I(a_r^{-1}b_s) = 2\log\sec\frac{\theta}{2}. \tag{3.2}$$

Remark 3.1.1. In the following lemmas we often compare the inefficiency of a concatenation of two geodesic arcs with the inefficiency of the bending angle. We make now a general computation. With the notation above fix two real numbers $r_0 \le r$ and $s_0 \le s$. Then let η_0 be the geodesic arc between the final endpoints of a_{s_0} and b_{r_0} and let η be the geodesic arc between the final endpoints of a_s and b_r. Also let $a'_{s'}$ be the geodesic ray starting at the final endpoint of a_{s_0} and having length s'. We use the same notation $b'_{r'}$ for b. Then we have, by monotonicity $I(a_r^{-1}b_s) \ge I(a_{r_0}^{-1}b_{s_0})$ so, taking the limit,

Figure 3.3: The external angle θ.

$I(\theta) > I(a_{s_0}^{-1} b_{r_0})$.
We also have $I(a_s^{-1} b_s) = l(a_s) + l(b_r) - l(\eta)$, $I(a_{s_0}^{-1} b_{r_0}) = l(a_{s_0}) + l(b_{s_0}) - l(\eta_0)$ so

$$I(a_r^{-1} b_s) = I(a_{r_0}^{-1} b_{s_0}) + I(a_r'^{-1} \eta_0 b_s'),$$

and then

$$I(\theta) = I(a_{r_0}^{-1} b_{s_0}) + I(a_\infty'^{-1} \eta_0 b_\infty').$$

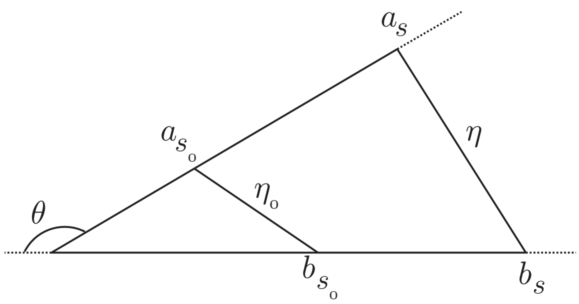

Figure 3.4: the construction of remark 3.1.1.

Now we prove some lemmas on inefficiency of arcs, which are the basic instruments we will use along all the chapter.

Lemma 3.1.2. *Let α be an arc on the hyperbolic surface S and γ be the appropriate geodesic arc homotopic to α relatively to the endpoints. Choose lifts of α and γ on $\mathbb{H}^2$ with the same endpoints and call them $\tilde{\alpha}$ and $\tilde{\gamma}$. Denote* *with* $\pi : \tilde{\alpha} \longrightarrow \tilde{\gamma}$ *the nearest point projection. Define the quantity*

$$E(\alpha) = \sup_{x \in \tilde{\alpha}} d(x, \pi(x)).$$

Then

$$E(\alpha) \leq \frac{I(\tilde{\alpha})}{2} + \log 2.$$

Proof. Consider the case of the piecewise geodesic arc α made of the concatenation of two geodesic arcs α^- and α^+ which meet in the point x_0.

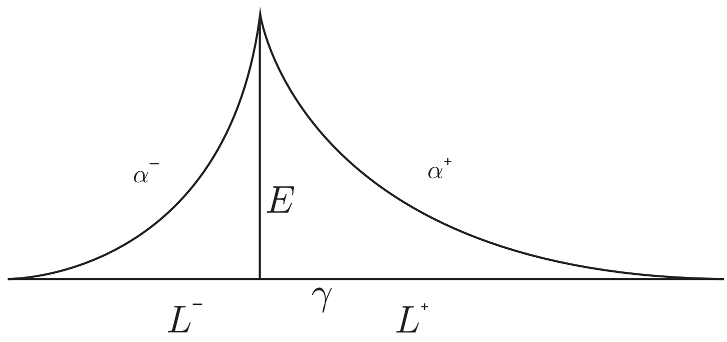

Figure 3.5: The case of a minimally inefficient α with a point at distance E.

Then, in the chosen lift, $\pi(\tilde{x_0})$ divide $\tilde{\gamma}$ in two subsegments of lengths L^- and L^+. Call E the number $d(\tilde{x_0}, \pi(\tilde{x_0}))$ (note that coincides with $E(\alpha)$). Then we have two right angled triangles sharing the edge of length E and we can use the inefficiency for angles and what seen in remark 3.1.1 to get

$$E + L^- - l(\alpha^-) < I(\frac{\pi}{2}),$$

and

$$E + L^+ - l(\alpha^+) < I(\frac{\pi}{2}).$$

Summing we have

$$E < \frac{I(\alpha)}{2} + I(\frac{\pi}{2}) = \frac{I(\alpha)}{2} + \log 2.$$

Finally note that every arc α having γ as corresponding geodesic arc and with $E(\alpha) = E$ as inefficiency greater or equal of the piecewise geodesic we considered. □

Lemma 3.1.3 (New Angle Lemma)**.** *In* $\mathbb{H}^2$ *let* α *be a piecewise geodesic arc and* β *be a geodesic arc such that their concatenation* $\alpha\beta$ *is well defined. Suppose* γ *is a geodesic arc with the same endpoints of* $\alpha\beta$ *and denote with* θ *the smaller angle between* γ *and* β.
Then for every δ, $\Delta > 0$ *there exists a constant* $L = L(\delta, \Delta) > 0$ *such that if* $l(\beta) > L$ *and* $I(\alpha\beta) < \Delta$ *then we have* $\theta \leq \delta$.

Proof. If $\theta = 0$ there is nothing to be proved.

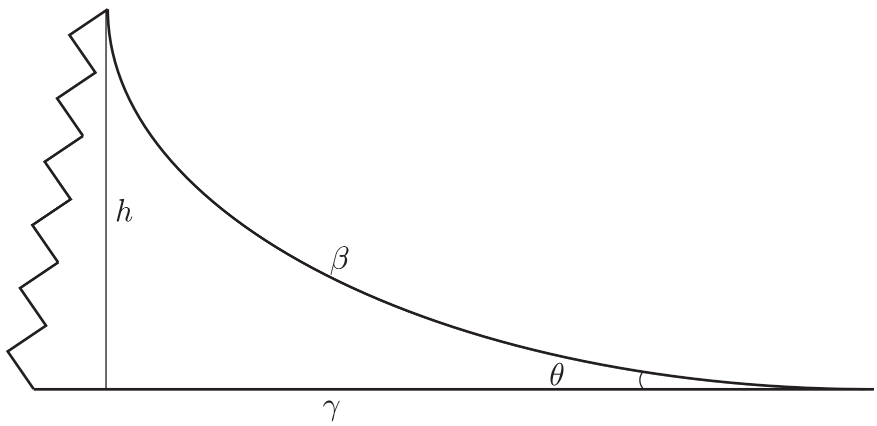

Figure 3.6: The New Angle Lemma.

Suppose $\theta \neq 0$ and let h be the distance from the non meeting endpoints of β and γ. By the hyperbolic sine rule fore right angled triangle we have

$$\frac{\sinh(h)}{\sin(\theta)} = \sinh(l(\beta)) > \sinh(L).$$

From the previous Lemma 3.1.2 we have

$$h < E(\alpha\beta) < I(\alpha\beta)/2 + \log 2 < \Delta/2 + \log 2.$$

So we conclude

$$\sin\theta < \frac{\sinh(h)}{\sinh(L)} < \frac{\sinh(\Delta/2 + \log 2)}{\sinh(L)},$$

that is $\theta \leq \delta = \arcsin(\frac{\sinh(\Delta/2+\log 2)}{\sinh(L)})$. □

Lemma 3.1.4. *Let $\alpha\beta\gamma$ be a concatenation of three geodesic arcs in $\mathbb{H}^2$, and let $\theta_{\alpha\beta}$ and $\theta_{\beta\gamma}$ be the two bending (so the external) angles between the arcs. Suppose both of them to be lesser then $\pi/2$. then we have*

$$I(\alpha\beta\gamma) \leq \log(\sec(\theta_{\alpha\beta})) + \log(\sec(\theta_{\beta\gamma})).$$

Proof. Let η be the appropriate geodesic segment for $\alpha\beta\gamma$. Let e_1 be the geodesic orthogonal to β passing through $\beta\cap\alpha$ and e_2 the geodesic orthogonal to β passing through $\beta\cap\gamma$. Let A_α be the geodesic arc orthogonal to e_1 at the point p and starting at the initial point of α, and let A_γ be the geodesic arc orthogonal to e_2 at the point q and starting at the initial point of γ. The geodesic arc β' from p to q connect e_1 with e_2, so $l(\beta') \geq l(\beta)$ since the last is the common orthogonal arc between e_1 and e_2. Follows that $l(\eta) \leq l(A_\alpha) + l(A_\gamma) + l(\beta)$.

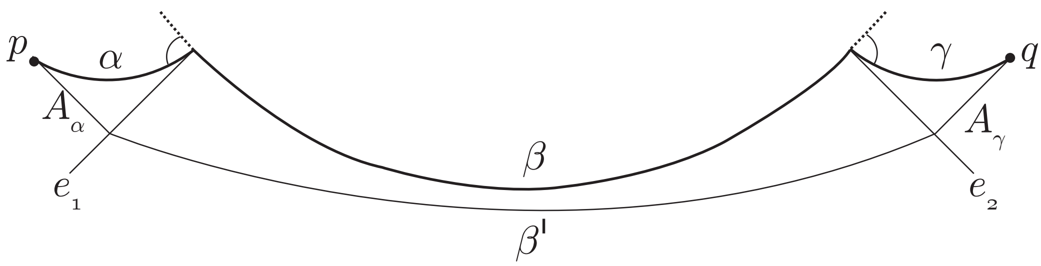

Figure 3.7: The construction of Lemma 3.1.4.

We notice that the internal angle between α and e_1 is exactly $\frac{\pi}{2} - \theta_{\alpha\beta}$. From the hyperbolic sine rule for right-angled triangles, and recalling that

$\sinh^{-1}(z) = \log(z + \sqrt{z^2+1})$ we have

$$\begin{aligned} l(\alpha) &= \sinh^{-1}(\sinh l(\alpha)) \\ &= \sinh^{-1}(\frac{\sinh l(A_\alpha)}{\cos\theta_{\alpha\beta}}) \\ &= \log(\frac{\sinh l(A_\alpha)}{\cos\theta_{\alpha\beta}} + \sqrt{(\frac{\sinh l(A_\alpha)}{\cos\theta_{\alpha\beta}})^2 + 1}) \\ &\leq l(A_\alpha) - \log\sec\theta_{\alpha\beta}. \end{aligned}$$

Same computations gives $l(\gamma) \leq l(A_\gamma) - \log\sec\theta_{\beta\gamma}$.
These three inequalities together gives

$$\begin{aligned} I(\alpha\beta\gamma) <& l(\alpha) + l(\beta) - l(A_\alpha) - l(A_\beta) \\ &< \log\sec\theta_{\beta\gamma} + \log\sec\theta\alpha\beta \end{aligned}$$

□

Lemma 3.1.5 (Long Segment Lemma For Angles)**.** *Let δ, $\Delta > 0$. There exists a constant $L = L(\delta, \Delta) > 0$ such that, if α and β are oriented geodesics in $\mathbb{H}^2$ with the terminal point of α equal to the initial point of β and such that $I(\alpha\beta) \leq \Delta$ and $l(\alpha)$, $l(\beta) > L$, then $I(\alpha\beta) < I(\Theta(t(\alpha), i(\beta))) < I(\alpha\beta) + \delta$.*

Proof. The first of the two inequalities of the statement follows immediately from the monotonicity of inefficiency and the definition of inefficiency for angles (see also remark 3.1.1). Let α'_∞ and β'_∞ be the two geodesic rays starting at the final point of α and β. We also denote $\alpha_\infty = \alpha \cup \alpha'_\infty$ and $\beta_\infty = \beta \cup \beta'_\infty$. Let η be the geodesic arc with the same endpoints of $\alpha\beta$ and η_1 the geodesic ray with the same endpoints of $\alpha_\infty\beta$. Finally let θ_0 the angle between α and η, θ_1 the angle between η and β and θ_2 the one between η and η_1.

By the New Angle Lemma the angles θ_0, θ_1 are smaller then δ provided L sufficiently large (and then $l(\alpha)$ and $l(\beta)$ large). Then, the triangle with edges η, η_2 and α'_∞ has an ideal vertex and two non-zero angles : θ_2 and

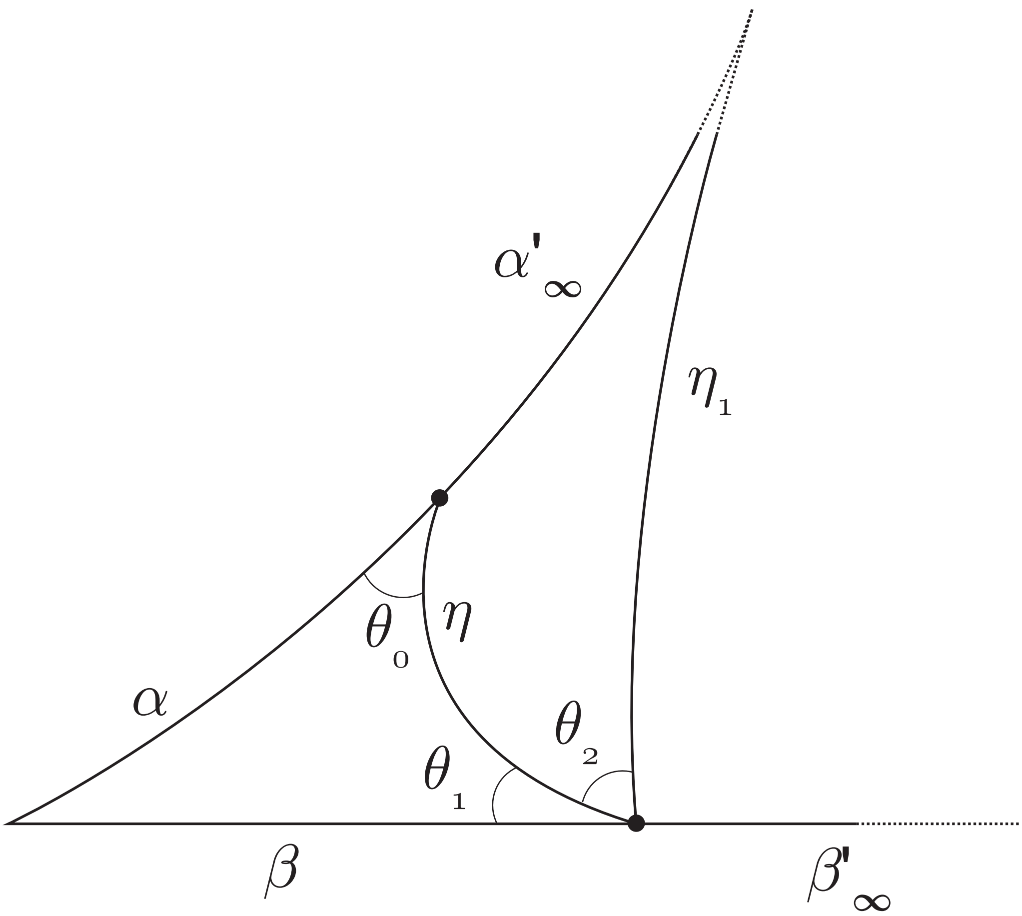

Figure 3.8: The Long Segment Lemma for Angles.

$\pi - \theta_0 > \pi - \delta$. Since their sum has to be lesser then π, θ_2 has to be strictly lesser then δ . So we have

$$\begin{aligned} I(\Theta(t(\alpha), i(\beta))) &= \lim_{r,s\to\infty} I(\alpha_r \beta_s) \\ &\leq \lim_{r,s\to\infty} (I(\alpha_r \beta)) + I(\eta_1^{-1} \beta'_\infty) \\ &\leq I(\alpha\beta) + I(\alpha'^{-1}_\infty \eta) + I(\theta_1 + \theta_2) \\ &\leq I(\alpha\beta) + I(\theta_0) + I(\theta_1 + \theta_2) \\ &< I(\alpha\beta) + 6 \log \sec(\frac{3}{2}\delta). \end{aligned}$$

and, since angle can be controlled by δ also the inefficiency can. □

Lemma 3.1.6 (Long Segment Lemma For Arcs)**.** *Let $\Delta > 0$. In $\mathbb{H}^2$ let α and γ be piecewise geodesic arcs and let β be a geodesic arc, such that the concatenation $\alpha\beta\gamma$ is well defined. Suppose that $I(\alpha\beta) + I(\beta\gamma) < \Delta$. Then, for any $\delta > 0$, there exist an $l = l(\delta) > 0$ such that if $l(\beta) > l$ then*

$$|I(\alpha\beta) + I(\beta\gamma) - I(\alpha\beta\gamma)| < \delta. \tag{3.3}$$

Proof. Replacing α and γ with the respectively geodesic arcs doesn't change $I(\alpha\beta)+I(\beta\gamma)-I(\alpha\beta\gamma)$, and decrease the value of $I(\alpha\beta)+I(\beta\gamma)$. So we can suppose both be geodesic arcs. Let p be the midpoint of β that divide it in the subsegments β^- and β^+ (so that $\alpha\beta\gamma = \alpha\beta^-\beta^+\gamma$). Call η the appropriate geodesic arc for $\alpha\beta\gamma$, $\hat{\alpha}$ the appropriate for $\alpha\beta^-$ and $\hat{\gamma}$ the appropriate for $\beta^+\gamma$. Also denote with θ^- the angle between β^- and $\hat{\alpha}$, and with θ^+ the angle between $\hat{\gamma}$ and β^+.

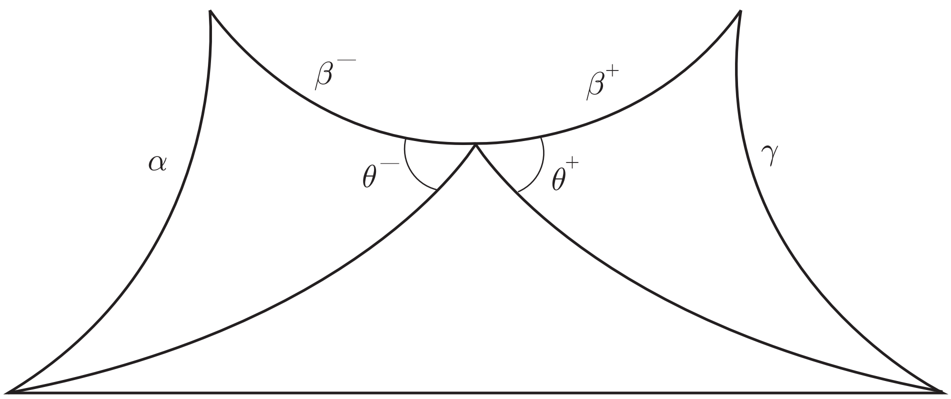

Figure 3.9: The Long Segment Lemma for Arcs.

We have

$$0 \leq I(\hat{\alpha}\hat{\gamma}) = I(\alpha\beta\gamma) - I(\alpha\beta^-) - I(\beta^+\gamma). \tag{3.4}$$

Since $I(\alpha\beta) + I(\beta\gamma)$ is bounded above we have, from the New Angle Lemma, that θ^- and θ^+ are smaller then a constant σ provided l sufficiently large. Then

$$I(\hat{\alpha}\hat{\gamma}) \leq I(\theta^- + \theta^+) < 2\log\sec(\sigma). \tag{3.5}$$

Now we note that $I(\alpha\beta) - I(\alpha\beta^-) = I(\hat{\alpha}\beta^+) \leq I(\theta^-)$ which gives

$$I(\alpha\beta) - I(\alpha\beta^-) \leq 2\log\sec(\sigma). \tag{3.6}$$

Similarly

$$I(\beta\gamma) - I(\beta^+\gamma) \leq 2\log\sec(\sigma). \tag{3.7}$$

Now equation (3.3) is exactly equation (3.4) with substitutions given by (3.5), (3.6) and (3.7) where $\delta = 6\log\sec(\sigma)$. Since σ can be taken as small as we want, provided l sufficiently large (see the statement of the New Angle Lemma), also δ can be as small as we want, provided same assumptions. □

Now we are going to consider the inefficiency theory for closed curves. So consider a concatenation of piecewise geodesic arcs $\alpha_1 \dots \alpha_n$ with the property that the terminal point of α_n is the initial point of α_1.
By $[\alpha_1 \dots \alpha_n]$ we denote the closed curve corresponding to the concatenation.

Definition 3.1.2 (Inefficiency For Closed Curves)**.** Let α be a closed curve on a closed hyperbolic surface S. Denote by γ the closed geodesic freely homotopic to α.
We define the inefficiency of α as

$$I(\alpha) = l(\alpha) - l(\gamma)$$

Lemma 3.1.7. *Let α be a closed curve in S and let γ be the appropriate homotopic geodesic. Choose lifts of α and γ on the universal cover $\mathbb{H}^2$ with the same endpoints at infinity, and call them $\tilde{\alpha}$ and $\tilde{\gamma}$. Denote with $\pi : \tilde{\alpha} \longrightarrow \tilde{\gamma}$ the nearest point projection. Define the quantity*

$$E(\alpha) = \sup_{x \in \tilde{\alpha}} d(x, \pi(x)).$$

Then, there exist a universal constant $L_0 > 0$ *such that, if* $l(\gamma) > L_0$*, we have*

$$E(\alpha) < \frac{I(\alpha)}{2} + 2.$$

Proof. Let $E > 0$. We prove the Lemma first for a particular α: fix a point $p \in S$ at distance E from γ and let α be the closed geodesic arc starting from p and freely homotopic to γ. In particular α is locally geodesic outside p.

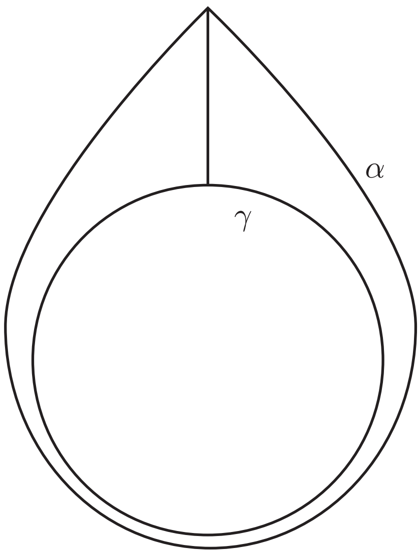

Figure 3.10: The minimally inefficient α with a point at distance E.

In the universal cover p has many different lifts: let $\tilde{p}_1$ one of them, and define $\tilde{p}_2 = g_\gamma \tilde{p}_1$ where g_γ is the isometry of $\mathbb{H}^2$ that corresponds to the action of the homotopy class of γ on it. In particular such two points are at distance E from $\tilde{\gamma}$ and the distance is realized by two geodesic arcs $\tilde{\eta}_1$ and $\tilde{\eta}_2$ from, respectively, $\tilde{p}_1$ and $\tilde{p}_2$ to $\tilde{\gamma}$. The $\tilde{\eta}_i$'s individuates a subsegment of $\tilde{\gamma}$ the that has length $l(\gamma)$ and that we will call $\tilde{\gamma}$ too from now since we won't use more the entire geodesic. The geodesic arc between $\tilde{p}_1$ and $\tilde{p}_2$ projects via π onto α and has length $l(\alpha)$.

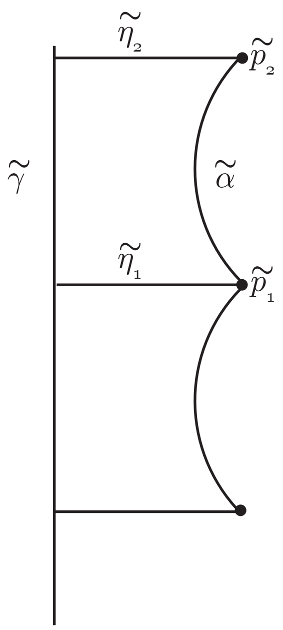

Figure 3.11: The situation in $\mathbb{H}^2$.

By the Long Segment Lemma For Arcs with $\delta = \frac{1}{7}$ and the monotonicity we have

$$I(\tilde{\eta}_1^{-1}\tilde{\gamma}\tilde{\eta}_2) < I(\tilde{\eta}_1^{-1}\tilde{\gamma}) + I(\tilde{\gamma}\tilde{\eta}_2) + \frac{1}{7} < 2I(\frac{\pi}{2}) + \frac{1}{7} < 2$$

where is needed $l(\tilde{\gamma}) > L_0$ for a universal constant L_0. So we have proved that

$$l(\gamma) + 2E - l(\alpha) = I(\tilde{\eta}_1^{-1}\tilde{\gamma}\tilde{\eta}_2) = 2.$$

That is $E = \frac{I(\alpha)}{2} + 1$. For any $\hat{\alpha}$ homotopic to the same γ and with $E(\hat{\alpha}) = E$ we observe that the inefficiency is higher then the inefficiency of α.

□

Lemma 3.1.8 (Long Segment Lemma For Closed Curves)**.** *Let $\Delta > 0$. Let $\alpha\beta$ be a concatenation of a piecewise geodesic arc α and a geodesic arc β such that the endpoint of β is the initial of α. Suppose $I(\alpha\beta) < \Delta$.*
Then for every $\delta > 0$ there exist an $l > 0$ such that, if $l(\beta) > l$, we have

$$|I([\alpha\beta]) - I(\beta\alpha\beta)| < \delta.$$

Proof. The proof is the same of Lemma 3.1.6. □

The following are the Sum of Inefficiency Lemmas. They are direct consequences of the Long Segment Lemma for Closed Curves.

Lemma 3.1.9. *Let δ, $\Delta > 0$ and $n \in \mathbb{N}$. There exists $L = L(\delta, \Delta, n) > 0$ such that the following holds. Let $\alpha_1, \dots, \alpha_{n+1} = \alpha_1, \beta_1, \dots, \beta_n$ be geodesic arcs on the surface S such that $\alpha_1\beta_1 \dots \alpha_n\beta_n$ is a piecewise geodesic arc on S. If $I(\alpha_i\beta_i\alpha_{i+1}) \leq \Delta$, and $l(\alpha_i) \geq L$ then*

$$|I([\alpha_1\beta_1 \dots \alpha_n\beta_n]) - \sum_{i=1}^{n} I(\alpha_i\beta_i\alpha_{i+1})| \leq \delta$$

More generally, but essentially the same is the following.

Lemma 3.1.10 (Sum of Inefficiencies Lemma)**.** *Let δ, $\Delta > 0$ and $n \in \mathbb{N}$. There exists $L = L(\delta, \Delta, n) > 0$ such that the following holds. Let $\alpha_1, \dots, \alpha_{n+1} = \alpha_1$ and $\beta_{1,1}, \dots, \beta_{1,j_1}, \dots, \beta_{n,1}, \dots, \beta_{n,j_n}$ be geodesic arcs on the surface S such that $\alpha_1\beta_{1,1} \dots \beta_{1,j_1} \dots \alpha_n\beta_{n,1} \dots \beta_{n,j_n}$ is a piecewise geodesic arc on S. If $I(\alpha_i\beta_{i,1} \dots \beta_{i,j_i}\alpha_{i+1}) \leq \Delta$, and $l(\alpha_i) \geq L$ then*

$$|I([\alpha_1\beta_{1,1} \dots \beta_{1,j_1} \dots \alpha_n\beta_{n,1} \dots \beta_{n,j_n}]) - \sum_{i=1}^{n} I(\alpha_i\beta_{i,1} \dots \beta_{i,j_i}\alpha_{i+1})| \leq \delta$$

Remark 3.1.2. If in Lemma 3.1.9 we take the β_i's trivial (simply a point) we can rewrite its conclusion as

$$|I([\alpha_1\alpha_2 \dots \alpha_n]) - \sum_{i=1}^{n} I(\alpha_i\alpha_{i+1})| \leq \delta.$$

If we apply to it the Long Segment Lemma for Angles 3.1.5, we obtain

$$|I([\alpha_1\alpha_2 \dots \alpha_n]) - \sum_{i=1}^{n} I(\theta_i)| \leq \delta,$$

where $\theta_i = \Theta(i(\alpha_{i+1}), t(\alpha_i))$.

3.2 Square Lemmas and Applications

From now on S is a fixed hyperbolic surface (however we can repeat such assumption in some important statement for completeness). We need to recall some theory on T^1S. For details of the constructions we mainly refer to [BM00]. The hyperbolic surface S is given by the group action of G on the oriented hyperbolic plane $\mathbb{H}^2$. We recall that the unitary tangent bundle $T^1\mathbb{H}^2$ can be identified with $PSL(2,\mathbb{R})$ and so $G\backslash PSL(2,\mathbb{R})$ can be identified with T^1S.

Remark 3.2.1. We remark that the identification depends on the choice of a point in the tangent unit bundle. In fact given $(p,u) \in T^1\mathbb{H}^2$ we uniquely determine te oriented geodesic line λ through p with direction u and end-points at infinity (λ^-,λ^+), so we can found a $g \in PSL(2,\mathbb{R})$ asking (in the upper half plane model) $g(i) = p$ (i is the imaginary unit), $g(0) = \lambda^-$ and $g(\infty) = \lambda^+$ (where the last two conditions are about the action of g in $\partial\mathbb{H}^2$). We observe, if an orientation of $\mathbb{H}^2$ is fixed and we are always in that case, that the condition $g(i) = p$ can be replaced by a unique condition on $\partial\mathbb{H}^2$: taking $v \in T^1_p\mathbb{H}^2$ as the unit vector obtained by a positive rotation of $\frac{\pi}{2}$ of the vector u, then we find a geodesic ray σ starting at p with direction v and ending at $\sigma^+ \in \partial\mathbb{H}^2$. Then the condition $g(i) = p$ is replaced with $g(-1) = \sigma^+$. Since three condition in $\partial\mathbb{H}^2$ uniquely determine $g \in PSL(2,\mathbb{R})$ we have finished. Obviously one have to carefully verify that this identification is diffeomorphic. Observe that we have chosen as basepoint in $T^1\mathbb{H}^2$ the couple $(i, \frac{\partial}{\partial y})$.

We also remark that this identification gives an action on the right of $PSL(2,\mathbb{R})$ on $T^1\mathbb{H}^2$ given by the change the basepoint: if $g \in PSL(2,\mathbb{R})$ comes, by the identification above, from the point (p,u) while h comes from (q,v) then $(p,u)\cdot h$ is that element of $PSL(2,\mathbb{R})$ corresponding to (p,u) by an identification that is analogue to the one above but with (q,v) as basepoint, instead of $(i, \frac{\partial}{\partial y})$. (In particular $(i, \frac{\partial}{\partial y})$ is identified with $\mathrm{id} \in PSL(2,\mathbb{R})$).
Finally there is also an action on the left of $PSL(2,\mathbb{R})$ on $T^1\mathbb{H}^2$: given $g \in$

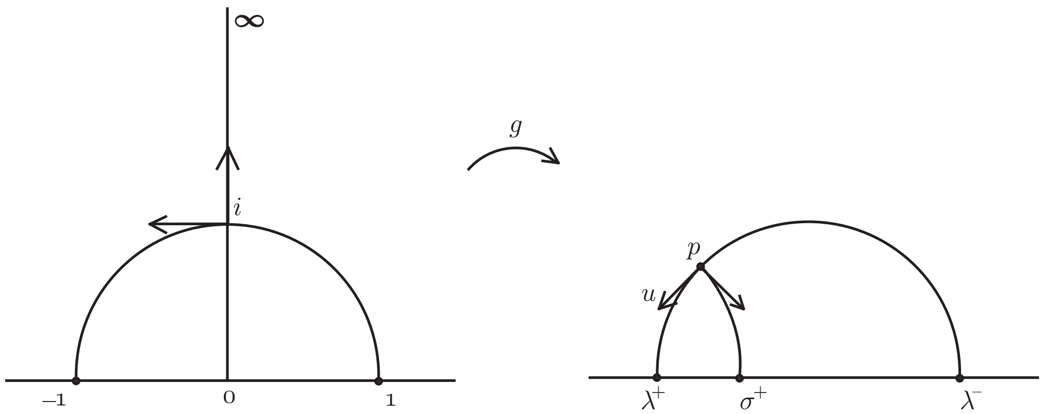

Figure 3.12: The isometry g acting on T^1S.

$PSL(2,\mathbb{R})$ defined by $g(z) = \frac{az+b}{cz+d}$, it act on $T^1\mathbb{H}^2$ by $g\cdot(p,v) = (g(p), \frac{v}{(cp+d)^2})$. Such actions are isometric, and are compatible with the action given by the identification above, that is the composition of isometries of $PSL(2,\mathbb{R})$.

We also need a measure on T^1S. The natural measure of $\mathbb{H}^2$ in the upper half plane model is given by $d\mu = \frac{dxdy}{y^2}$, where x and y are, respectively, the real and the imaginary part of a point $z \in \mathbb{H}^2$. One can verify that μ is invariant by the action of $PSL(2,\mathbb{R})$. If θ denote the standard Lebesgue measure over the unit circle S^1 we define the measure λ by $d\lambda(x,w) = d\mu(x)d\theta(w)$ in $T^1\mathbb{H}^2$. This is well defined, invariant for the action of $PSL(2,\mathbb{R})$ on $T^1\mathbb{H}^2$, and induces a measure (also denoted with λ) in T^1S. Recall that $T^1_p\mathbb{H}^2$ has a natural complex structure, once we have oriented $\mathbb{H}^2$, then given $u \in T^1_p$, we call $\sqrt{-1}u$ the vector rotated by $\frac{\pi}{2}$ in the positive direction. We recall that the geodesic flow is defined as a map $g_t : T^1S \longrightarrow T^1S$, for $t > 0$, $g_t(p,u) = (q,v)$, where q is the point along the geodesic starting at p with direction u, that is exactly at distance t along such geodesic (we mean that the length of the geodesic arc from p to q that we have defined has exactly length t). v is the parallel transportation of u along the same geodesic arc. It can be proved that g_t act as an isometry of S and preserve the measure λ in T^1S. We need the following result about the mixing property of the geodesic flow. It is the Exponentially Mixing Theorem for geodesic flow, and it is proved essentially in [Rat87] and [Moo87].

Theorem 3.2.1 (Exponentially Mixing for Geodesic Flow)**.** *Let S be a closed hyperbolic surface, T^1S be the unitary tangent bundle of S and F, $G \in C_0^\infty(T^1S)$. Let g_t be the geodesic flow in T^1S. Define*

$$\rho(t) = |\lambda(T^1S)\int_{T^1S} F(g_t(p,u))G(p,u)d\lambda(p,u) - \int_{T^1S} Fd\lambda \int_{T^1S} Gd\lambda|.$$

Then there exists $C > 0$, dependent only on $||F||_{C^1}$ and $||G||_{C^1}$, and a $q = q(S) > 0$ such that

$$|\rho(t)| \leq Ce^{-qt}.$$

Definition 3.2.1 (E-Nearly Homotopic)**.** Let $E \geq 0$. We say that two geodesic segments A and B in $\mathbb{H}^2$ are E-nearly homotopic if their endpoints are at distance at most E.

Two geodesic arcs in a hyperbolic surface S are said E-nearly homotopic if they have lifts on $\mathbb{H}^2$ which are E-nearly homotopic.

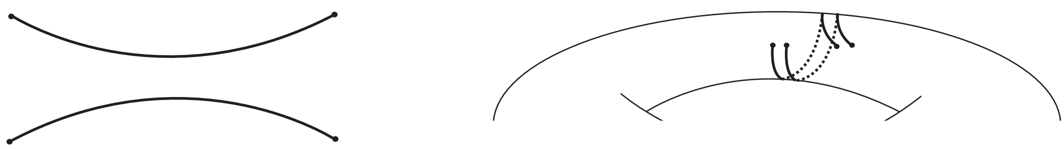

Figure 3.13: E-nearly homotopic geodesic arcs

Definition 3.2.2 (Connection)**.** Let $L > 0$. We say that a unit speed geodesic $\gamma : [0, l] \longrightarrow S$ is in the set $\text{Conn}_{\epsilon,L}((p,u),(q,v))$ if

(i)$\gamma(0) = p$ and $\gamma(l) = q$,

(ii)$|L - l| < \epsilon$,

(iii)$\Theta(\dot\gamma(0), u)$,$\Theta(\dot\gamma(l), v) < \epsilon$.

With $|\,\text{Conn}_{\epsilon,L}((p,u),(q,v))|$ we denote the cardinality of the set of connections. We notice that in hyperbolic geometry such set is finite.

Lemma 3.2.2 (Connection Lemma)**.** *Let* (p, u), $(q, v) \in T^1S$ *and let* $\epsilon > 0$. *There exists* $L_0 = L_0(S, \epsilon)$ *such that for any* $L > L_0$

$$|\operatorname{Conn}_{\epsilon,L}((p, u), (q, v))| \geq e^{L-L_0}.$$

Proof. We can find a neighbourhood U of the identity in $PSL(2, \mathbb{R})$ such that for $m \in U$ we have $\operatorname{dis}((q, v), (q, v) \cdot m) < \frac{\epsilon}{16}$. Let $f : U \longrightarrow [0, \infty)$ be a bump C^∞ function with compact support in U and $\int_U f = 1$. Let $N_U(q, v) = \{(q, v) \cdot m : m \in U\}$, define $f_{(q,v)}((q, v) \cdot m) = f(m)$ on $N_U(q, v)$ and extend to 0 outside it. Note that the C^k norm of $f_{(q,v)}$ doesn't depend on (q, v). Let $g_t : T^1S \longrightarrow T^1S$ be the geodesic flow. By Mixing for Geodesic Flow exists a constant $V = \lambda(T^1S) > 0$ such that

$$\int_{T^1S} f_{(q,v)}(g_t(x, w)) f_{(p,u)}(x, w) d(x, w) \to \frac{1}{V},$$

uniformly in (p, u) and (q, v) when $t \to \infty$. This means the there exists many $(x, w) \in N_U(p, u)$ with $g_t(x, w) \in N_U(q, v)$. The segment $g_{[0,t]}(x, w)$ is then ϵ-nearly homotopic to a unique geodesic α connecting p and q. Moreover they have the initial and terminal tangent vector ϵ-near, in fact α, in the universal cover of S corresponding to the tangent plane T_pS, is a line starting from p and passing through a lift of q. However the precise lift of q is determined by the homotopy class of α and so by $g_{[0,t]}(x, w)$. This means that also the initial and final vector of α and $g_{[0,t]}(x, w)$ are ϵ near. But $(x, w) \in N_U(p, u)$ and $g_t(x, w) \in N_U(q, v)$. So we have

$$\alpha \in \operatorname{Conn}_{\epsilon,t}((p, u), (q, v)).$$

Now consider the set $E_\alpha \subseteq N_U(p, u)$ of (x, w) such that $g_t(x, w) \in N_U(q, v)$ and $g_{[0,t]}(x, w)$ is ϵ-nearly homotopic to α. Take again the universal cover pointed at p of S and lift α. This naturally fix the lift of the point p, and reduce the lifts of q to those are along α. If $(x, w) \in E_\alpha$ then $d(x, p) < \epsilon$. Fix such an x, and take the couple (x, w) with $g_t(x, w) \in N_U(q, v)$. Then $g_t(x)$ has distance less then ϵ from q. Let $B_\epsilon(q)$ the disk centred in q of radius ϵ. The length of a circumference of radius t growth as e^t when $t \to \infty$. The set of geodesics starting from x is in bijective correspondence with the unit tangent vectors at x. The set of geodesics starting at x and ending in $B_\epsilon(q)$ with length t decrease with respect to t as e^{-t} in fact it is in correspondence with the intersection of the circumference of radius t with $B_\epsilon(q)$ which is a fraction depending only on ϵ of the total length of the circumference which goes as e^t. If we allow the length to be ϵ near to t we don't change type of dependence on t. We have proved that exists a constant $C = C(\epsilon, S) > 0$ such that $\lambda(E_\alpha) \leq Ce^{-t}$. So we have

$$\int_{E_\alpha} f_{(q,v)}(g_t(x,w))f_{(p,u)}(x,w)d(x,w) \leq (\sup_U f)^2 \int_{E_\alpha} d(x,w) \leq K(f)Ce^{-t},$$

and so

$$\begin{aligned}\int_{T^1S} f_{(q,v)}(g_t(x,w))f_{(p,u)}(x,w)d(x,w) &= \sum_{\alpha\in \mathrm{Conn}_{\epsilon,t}} \int_{E_\alpha} f_{(q,v)}(g_t(x,w))f_{(p,u)}(x,w)d(x,w)\\ &\leq |\,\mathrm{Conn}_{\epsilon,t}\,|K(f)C(\epsilon,S)e^{-t}.\end{aligned}$$

Since the first term converges for $t \to \infty$ we have the result for t sufficiently large depending on ϵ and S. □

Recall that for u, $u' \in T^1_p\mathbb{H}^2$ we denote by $\Theta(u,u')$ the smaller unoriented angle between them, while $d(p,q)$ is the hyperbolic distance between two points p, $q \in \mathbb{H}^2$ Let (p,u) and $(q,v) \in T^1\mathbb{H}^2$. By $u@q$ we mean the parallel transport of the vector u along the geodesic connecting p and q. We define

$$\mathrm{dis}((p,u),(q,v)) = \max(\Theta(u@q,v), d(p,q)).$$

We don't require dis to be a norm. For ϵ, $R > 0$ we denote with $\Gamma_{\epsilon,R}$ the set of the closed oriented geodesics in S whose half length is in the interval $[R-\epsilon, R+\epsilon]$. $\Pi_{\epsilon,R}$ denotes the set of oriented immersed pair of pants with all the three boundary components in $\Gamma_{\epsilon,R}$ (we call such components cuffs from now on). With $\mathbb{R}\Gamma_{\epsilon,R}$ (resp. $\mathbb{R}\Pi_{\epsilon,R}$) we mean the real vector space generated by $\Gamma_{\epsilon,R}$ (resp. $\Pi_{\epsilon,R}$) with the convention that $\gamma^{-1} = -\gamma$ for $\gamma \in \Gamma_{\epsilon,R}$ (resp. $\Pi^{-1} = -\Pi$ for $\pi \in \Pi_{\epsilon,R}$). For $W = \sum r_i\Pi_i$ an element of $\mathbb{R}\Pi_{\epsilon,R}$, we define

$\partial W = \sum r_i \partial \Pi_i$ as an element in $\mathbb{R}\Gamma_{\epsilon,R}$ (observe that ∂, as the restriction of the classical boundary operator, here is well defined). Observe that if $M \geq 1$ then $\Pi_{M\epsilon,R} \subseteq \Pi_{\epsilon,R}$

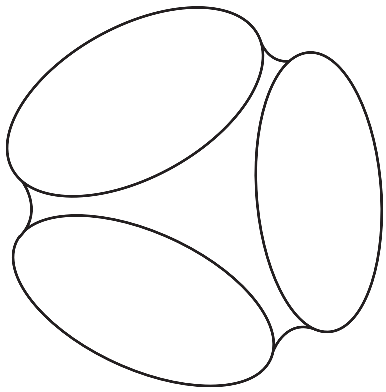

Figure 3.14: A typical pair of pant in $\Pi_{\epsilon,R}$ for R large enough.

Definition 3.2.3 (Good Pants Homology)**.** Let ϵ, $R > 0$. For every $M \geq 1$ we can define the real vector space

$$H^{M\epsilon,R}(S) = \frac{\mathbb{R}\Gamma_{\epsilon,R}}{\partial \mathbb{R}\Pi_{M\epsilon,R}}.$$

Such space is called $(M\epsilon, R)$- Pants Homology.

The pants $\Pi \in \Pi_{\epsilon,R}$ are called in literature *Good Pants*. Their theory became important for their application in [KM12] and [KM13]. In particular in the second is proved the following theorem that is the main result for this Chapter. Let $H_1(S) = H_1(S, \mathbb{Q})$ be the first singular homology group of the surface S.

Theorem 3.2.3 (Good Pants Homology Theorem)**.** *Let S be an hyperbolic closed surface of genus n. Let $\epsilon > 0$. There exists $R_0 = R_0(\epsilon, S)$ such that for every $R > R_0$ the following holds. There exists a set $H = \{h_1,, \ldots, h_{2n}\} \subseteq \mathbb{Q}\Gamma_{\epsilon,R}$ od linearly independent elements of $H_1(S, \mathbb{Q})$, such that for*

every γ
in$\Gamma_{\epsilon,R}$ *there are* $a_i \in \mathbb{Q}$ *so that*

$$\gamma = \sum_{i=1}^{2n} a_i h_i$$

in $H^{\epsilon,R}(S)$.

Shortly it means that we have the equality

$$H_1(S, \mathbb{Q}) = H^{\epsilon,R}(S).$$

The remainder of this chapter is devoted to the proof of this theorem.

Generating Essential Immersed Pants in a S

We will give an efficient way for generates pants in a surface, given a closed geodesic. First of all, we need to recall the Ping-Pong Lemma

Lemma 3.2.4 (the Ping-Pong Lemma)**.** *Let* X *be a space, and let* $g : X \longrightarrow X$ *and* $h : X \longrightarrow X$ *be two maps one-to-one and onto. If* A *and* B *are two non-empty subsets of* X*, such that* $A \nsubseteq B$ *and if*

$$g^n(A) \subseteq B \, \textit{for every } n \in \mathbb{Z} \setminus \{0\}, \tag{3.8}$$

$$h^m(B) \subseteq A \, \textit{for every } m \in \mathbb{Z} \setminus \{0\}, \tag{3.9}$$

then g *and* h *generate a free subgroup of rank* 2 *in the group of one-to-one onto maps* $X \longrightarrow X$.

Proof. Let $w = g^{n_1} h^{m_1} g^{n_2} h^{m_2} \dots g^{n_k}$ for n_i,$m_j \in \mathbb{Z} \setminus \{0\}$ be a word in the group generated by g and h. Then $w(A) \subset B$. So $w \neq \text{id}$. Every other word u of the group, is conjugated with a word of the type w by a power of g. But the only conjugates of the identity is the identity itself. So the group is free. □

Let Π be a topological pair of pants. The homotopy type is the same of the topological space T made of two points connected with three distinct edges. So we can immerse $T \longrightarrow \Pi$ with a π_1-isomorphism. In this way we have found in Π two points a and b and three paths connecting them, γ_i, $i = 0, 1, 2$ with $\gamma_i\gamma_{i+1}^{-1}$ in the free homotopy class of a boundary component of Π. We call this image of T inside Π the *spine* of Π. Let p and q be two points in S and let α_i, $i = 0, 1, 2$ be three distinct geodesic arcs connecting them and oriented from p to q. Let $i(\alpha_i) = \dot{\alpha}_i(p)$ and $t(\alpha_i) = \dot{\alpha}_i(q)$. Suppose that the triples of vectors $(i(\alpha_0), i(\alpha_1), i(\alpha_2))$ and $t(\alpha_0), t(\alpha_1), t(\alpha_2))$ have opposite cyclic order.

First we see that the three piecewise geodesic arcs $\alpha_i\alpha_{i+1}^{-1}$, $i \in \frac{\mathbb{Z}}{3\mathbb{Z}}$, are non

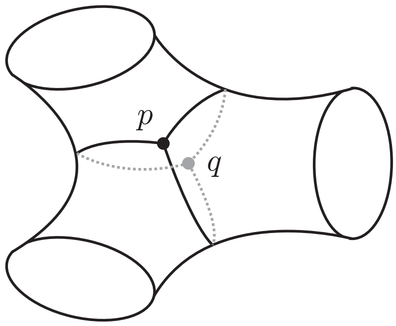

Figure 3.15: The immersed spine in a pair of pants.

trivial in homotopy since for every two points in an hyperbolic surface there is exactly one geodesic arc connecting them for every homotopy class of arcs. By construction is also satisfied the condition of a pants group that is:

$$(\alpha_0\alpha_1^{-1})(\alpha_1\alpha_2^{-1})(\alpha_2\alpha_0^{-1}) = 1.$$

Define $a = \alpha_0\alpha_1^{-1}$, and $b = \alpha_1\alpha_2^{-1}$. We want see that the subgroup of $\pi_1(S, p)$ generated by a and b is free. We consider the natural action of $\pi_1(S, p)$ on $\mathbb{H}^2$ as a discrete group of isometries. As isometries, a and b are loxodromics with axes the lift l_1 of $\alpha_0\alpha_1^{-1}$ and l_2 of $\alpha_1\alpha_2^{-1}$ respectively, and with translation length equal to the length of the curves they represents in S, say δ_1 for l_1 and δ_2 for l_2. We observe that, since S is closed and the l_i's cover two closed geodesics in two distinct free homotopy classes, l_1 and l_2 can't have a common endpoint in $\partial\mathbb{H}^2$ or there must be a cusp on S. We also note that since the subgroup generated by a and b is the same of the subgroup generated by a and ab^n, we can suppose δ_1 and δ_2 larger then an arbitrary constant $M > 0$. The action of a and b on $\mathbb{H}^2$ naturally induces an action of on $\partial\mathbb{H}^2 = S^1$ as the action of $PSL(2,\mathbb{R})$ on $\mathbb{RP}^1$. Call l_i^- and l_i^+ the repulsive and attracting endpoints of the lines l_i's, and $U_i^\pm$ their not intersecting neighbourhoods in S^1. For $k \in \mathbb{Z} \setminus \{0\}$ and M sufficiently large we have

$$a^k(U_1^-)^c \subseteq U_1^+,$$

$$b^k(U_2^-)^c \subseteq U_2^+.$$

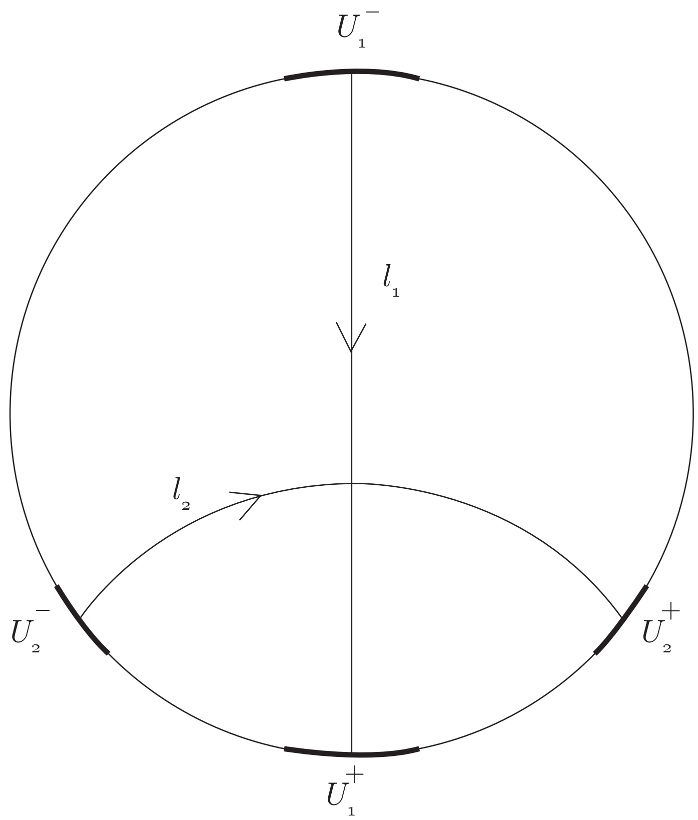

Figure 3.16: Apply the Ping-Pong Lemma to a pair of pants.

So, taking $A = U_2^- \cup U_2^+$, $B = U_1^- \cup U_1^+$, $g = a$ and $h = b$ we can apply the Ping Pong Lemma to conclude that the fundamental group of the topological space $\bigcup_{i=0}^{3} \alpha_i$ generates a subgroup in $\pi_1(S)$ isomorphic to $\pi_1(\Pi)$ (see figure 3.16). Now we can consider the immersion of the spine T of a pant in S sending each γ_i onto α_i. We seen that this will be π_1-injective. Then consider the fattening of T that is the union of three strip G_i each one containing γ_i. Topologically this is a pair of pants. Let $g_i : G_i \longrightarrow S$ be the map which send G_i in a tubular neighbourhood o α_i. The condition on the tangent vectors in p and q enables us to define the g_i's so that $g := g_i = g_j$ on $G_i \cap G_j$. So we have constructed an immersed pair of pants generated by α_i, $i = 0, 1, 2$.

Lemma 3.2.5 (Counting Pants). *Let* ϵ,$R_0 = R_0(\epsilon, S) > 0$ *and* $\gamma \in \Gamma_{\epsilon,R}$. *Let* $\Pi_{\epsilon,R}(\gamma)$ *denotes the subset of* $\Pi_{\epsilon,R}$ *of pants with* γ *as a cuff. Then there exist* $c_1(\epsilon, S)$, $c_2(\epsilon, S) > 0$ *such that for every* $R > R_0$ *we have*

$$c_1 Re^R \leq |\Pi_{\epsilon,R}(\gamma)| \leq C_2 Re^R.$$

Proof. Let F_γ be a set of $\lceil 2R \rceil$ evenly distributed points of γ. For $\Pi \in \Pi_{\epsilon,R}(\gamma)$ let α be the geodesic arc of minimal length in Π that is orthogonal to γ at its endpoints. Then consider the geodesic arc α' with endpoints in F_γ and $\frac{1}{2}$-nearly homotopic to α. Then $l(\alpha) \leq l(\alpha') + 1$. Note that α' is uniquely determined by α by construction. Moreover same α' means same Π since $\alpha' \cup \gamma$ is a spine for Π. If we fix two diametrically opposite points of F_γ, by a standard argument similar to the one given in Lemma 3.2.2, we found at most Ne^R such geodesic arcs α', with N depending only on S. The couple of points we have to consider are less or equal then $R + 1$. So we have proved that $|\Pi_{\epsilon,R}(\gamma)| \leq C_2 Re^R$.

For the other inequality start fixing two diametrically opposite points p, q on γ. Then Connection Lemma 3.2.2 say that we can find at least $N_1(\epsilon, R)e^R$ geodesics segments $\hat{\alpha} \in \mathrm{Conn}_{\frac{\epsilon}{100},R}((p, \sqrt{-1}\dot{\gamma}(p)), (q, \sqrt{-1}\dot{\gamma}(q)))$. For any $\hat{\alpha}$ we find an α orthogonal to γ and $\frac{\epsilon}{10}$-homotopic to $\hat{\alpha}$. Every such α determine a unique $\Pi \in \Pi_{\epsilon,R}(\gamma)$. Furthermore different α's determine different pants.

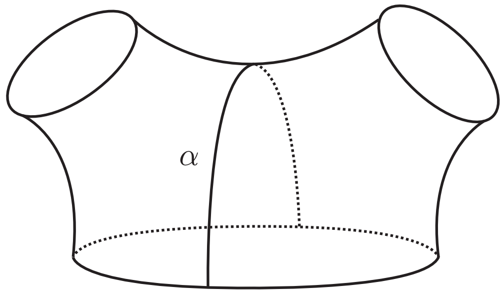

Figure 3.17: How to find pair of pants having γ as a cuff.

Two $\hat{\alpha}$ determining the same α have endpoints $\frac{\epsilon}{10}$ near. So we have:

$$\frac{2RN(S,\epsilon)}{10\epsilon}e^R \leq |\Pi_{\epsilon,R}(\gamma)|.$$

□

Corollary 3.2.6. *Let $M > 1$. Under the assumption of the previous Lemma let $X_\gamma(M)$ the set of pants in $\Pi_{\epsilon,R}$ with γ as a cuff and the other two cuffs in $\Gamma_{\frac{\epsilon}{M},R}$. Then $|X_{\gamma(M)}| \asymp Re^{-R}$.*

Proof. The upper bound follows directly from the previous Counting Pants Lemma. The other bound follows from the same proof given in the previous Lemma asking $\hat{\alpha}$ to have length within $\frac{\epsilon}{100M}$ to $2R - \frac{l(\gamma)}{2}$. Then the two cuffs generated in this way will have the desired property, while, from the Connection Lemma, the number of such $\hat{\alpha}$ goes asymptotically as $N(\epsilon, M, S)e^R$. □

Lemma 3.2.7 (Convergence Lemma)**.** *Let $E > 0$. Suppose $\hat{a}$ and $\hat{b}$ are oriented geodesics in $\mathbb{H}^2$ that have E-nearly homotopic geodesic subsegents A and B, unit speed parametrized by*

$$a : [-\frac{l(A)}{2}, \frac{l(A)}{2}] \longrightarrow \mathbb{H}^2,$$

$$b : [-\frac{l(B)}{2}, \frac{l(B)}{2}] \longrightarrow \mathbb{H}^2.$$

Suppose that $l(A) \leq l(B)$ *and set* $l = \frac{1}{2}l(A), l(B)$. *Then there exists* $0 \leq t_0 \leq E$, *such that for* $t \in [-l, l]$ *the following holds*

$$\mathrm{dis}((a(t), \dot{a}(t)), (b(t+t_0), \dot{b}(t+t_0))) \leq e^{|t|+E+2-l}.$$

Proof. We suppose that a and b parametrize the maximal subsegments A and B E-nearly homotopic, that is, we can't extends the parametrization along $\hat{a}$ and $\hat{b}$ without relaxing the E-nearly homotopic hypothesis.
Let O be the geodesic segment orthogonal to $\hat{a}$ and $\hat{b}$. Since $l(O) \leq E$, O has endpoints in A and B, say $a(\tau)$ and $b(\tau + t_0)$, for some τ,$t_0 \in [-l, l]$ not yet determined. Let s_t be the geodesic segment between $a(t)$ and $b(t+t_0)$. Then O, s_t, $a([\tau, t])$, $b([\tau+t_0, t+t_0])$ are the edges of a Saccheri quadrilateral. Then we have, for $t \geq 0$,

$$\sinh(\frac{l(s_t)}{2}) = \cosh(|t-\tau|)\sinh(\frac{l(O)}{2}),$$

which, for $t \to 0$, gives $\cosh(|t-\tau|) \to 1$, and, then, $\tau = 0$. Then we can write, for every $t \in [-l, l]$

$$\sinh(\frac{l(s_t)}{2}) = \cosh(|t|)\sinh(\frac{l(O)}{2}), \tag{3.10}$$

and, for $t = \pm l$,

$$\sinh(\frac{E}{2}) \geq \sinh(\frac{l(s_{\pm l})}{2}) = \cosh(l)\sinh(\frac{l(O)}{2}). \tag{3.11}$$

Putting together (3.10) and (3.11) we get

$$\begin{aligned} l(s_t) &\leq 2\sinh(\frac{l(s_t)}{2}) \\ &\leq 2\frac{\cosh|t|}{\cosh l}\sinh(\frac{E}{2}) \\ &\leq 2e^{|t|-l+E}(\frac{1+e^{-2|t|}}{1+e^{-2l}}) \\ &\leq 4e^{|t|-l+E} \\ &\leq e^{|t|-l+E+2}. \end{aligned}$$

This prove the inequality for the lengths. For the angles between the tangent vectors it sufficient to note that in a Saccheri's quadrilateral the non-right angles are acute and equal. Then the above inequality extends to the difference of the parallel transport of the tangent vectors.
The fact that $t_0 \leq E$ comes as follows,

$$t_0 \leq l(B)/2 - t \leq \frac{l(B) - l(A)}{2} \leq E.$$

□

Lemma 3.2.8. *Let $L > 0$. There exists a constant $\epsilon'(L)$ with the following properties. Suppose that $\alpha : [a_0, a_1] \longrightarrow \mathbb{H}^2$ and $\beta : [b_0, b_1] \longrightarrow \mathbb{H}^2$ are ϵ-nearly homotopic for some $0 < \epsilon < 1$. Suppose $a_1 - a_0 > L$. Then*

$$\text{dis}(\dot{\alpha}(a_i), \dot{\beta}(b_i)) < \epsilon(1 + \epsilon'(L))$$

and $\epsilon' \to 0$ when $L \to \infty$.

Proof. Let O be the common perpendicular between two geodesic arcs with endpoints O_A on α, and O_B on β. Let e_i, $i = 0, 1$, be the geodesic arc with endpoints $\alpha(a_i)$and $\beta(b_i)$. Consider the quadrilateral Q_i with O and e_i as two opposite edges and $\alpha_i = \alpha([O_A, a_i])$ and $\beta_i = \beta([O_B, b_i])$ (with the opportune orientation of the parametrizing segments) as the other two subsegments. By the statement $l(e_i) \leq \epsilon$. If we prove that when $L \to \infty$ we have $\frac{l(\alpha_i)}{l(\beta_i)} \to 1$ then the quadrilateral Q_i is near to a Saccheri's quadrilaterals for L sufficiently large. Then we will have that there exists an $\epsilon' = \epsilon'(L) > 0$ such that the internal angle γ_a^i between e_i and α_i and the internal angle γ_b^i between e_i and β_i satisfy

$$|\gamma_a^i - \gamma_b^i| < \epsilon\epsilon'.$$

The inequality of the statement for the unit tangent vectors then, follows from this (the inequality for the length is obvious). To prove that $\frac{l(\alpha_i)}{l(\beta_i)} \to 1$, first we note that when $L \to \infty$ also $l(\alpha_i) \to \infty$. Then we have from the triangle inequality

$$l(\alpha_i) \leq l(\beta_i) + \epsilon + l(O), \tag{3.12}$$

which imply that also $l(\beta_i) \to \infty$ when $L \to \infty$. Then, from equation (3.12), we can deduce

$$\frac{l(\alpha_i)}{l(\beta_i)} \leq \frac{\epsilon}{l(\beta_i)} + \frac{l(O)}{l(\beta_i)},$$

Then we have done since $l(O) \leq \epsilon \leq 1$ □

Now we are going to prove the main technical tool of this section (and also of this chapter), the Geometric Square Lemma. However it is better to prove before a simpler version in which we add an hypothesis (the number (5)).

In these following lemmas every parametrization of a closed geodesic γ is thought in the universal cover as a unit speed map $C : \mathbb{R} \longrightarrow S$. Such maps factorize through maps $\mathbb{R}/l(\gamma)\mathbb{Z} \longrightarrow S$ for which we use the same name C with an abuse of notation. So $C(x + kl(\gamma)) = C(x)$ for any $k \in \mathbb{Z}$. For $x \leq y$ real numbers, we write $C[x, y]$ for the image of the interval $[x, y] \subseteq \mathbb{R}$ in S mapped by C. Obviously we have $C[x, y] = C[x + kl(\gamma), y + kl(\gamma)]$. With abuse of notation, to simplify the notation, we often use only C to indicate both the parametrization and the geodesic γ.

Lemma 3.2.9 (Preliminary Geometric Square Lemma)**.** *There exists an $\hat{\epsilon} > 0$ such that the following holds. Let $\hat{\epsilon} > \epsilon > 0$, and let $E > 0$.There exist constants $K = K(\epsilon, E) > 0$ and $R_0(S, \epsilon, E) > 0$ with the following properties. Let $R > R_0$. Suppose that we are given four oriented closed geodesics $C_{ij} \in \Gamma_{\epsilon,R}$, i,j $= 0, 1$, and for each double index ij we are given 4 real numbers*

$$x^-_{ij} < x^+_{ij} < y^-_{ij} < y^+_{ij} < x^-_{ij} + l(C_{ij}).$$

Assume that

(1) $x^+_{ij} - x^-_{ij} > K$, $y^+_{ij} - y^-_{ij} > K$,

(2) *the segments $C_{ij}[x^-_{ij}, x^+_{ij}]$ and $C_{hk}[x^-_{hk}, x^+_{hk}]$ are E-nearly homotopic, and likewise the segments $C_{ij}[y^-_{ij}, y^+_{ij}]$ and $C_{hk}[y^-_{hk}, y^+_{hk}]$ are E-nearly homotopic, for any i, j, h, $k \in \{0, 1\}$,*

(3)*the segments $C_{0j}[x^-_{0j}, y^+_{0j}]$ and $C_{1j}[x^-_{1j}, y^+_{1j}]$ are E-nearly homotopic,*

(4) *the segments $C_{i0}[y^-_{i0}, x^+_{i0} + l(C_{i0})]$ and $C_{i1}[y^-_{i1}, x^+_{i1} + l(C_{i1})]$ are E-nearly*

homotopic,
(5) $y_{00}^{+} - x_{00}^{-} \geq R + K$, *and* $x_{00}^{+} + l(C_{00}) - y_{00}^{-} \geq R + K$.
Then we have

$$\sum_{i,j=0,1} (-1)^{i+j} C_{ij} = 0$$

in $H^{10\epsilon,R}(S)$.

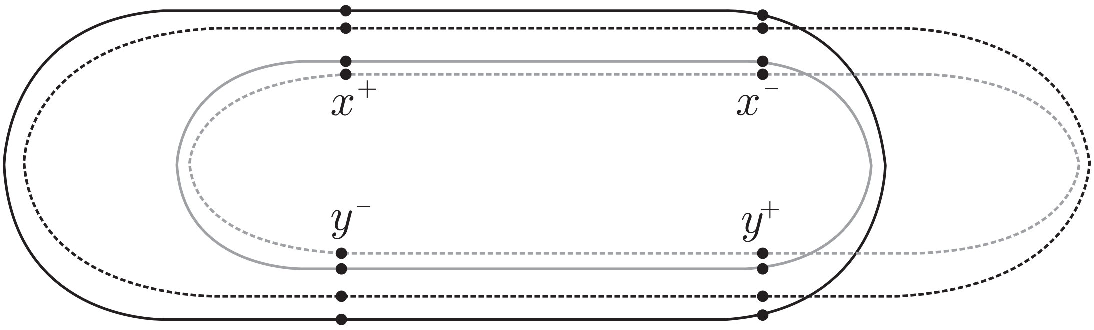

Figure 3.18: How we see the four curves C_{ij} thanks to assumptions (1)-(4).

Proof. Write l_{ij} for $l(C_{ij})$. First of all we have to find an $x_{00} \in [x_{00}^{-} + \frac{K}{2}, x_{00}^{+} - \frac{K}{2} + 1]$ and an $y_{00} \in [y_{00}^{-} + \frac{K}{2}, y_{00}^{+} - \frac{K}{2}]$ such that $y_{00} - x_{00} = R$.
If $y_{00}^{-} \leq x_{00}^{-} + R$ we can fix $x_{00} = x_{00}^{-} + K/2$ and so $y_{00} := x_{00} + R \geq y_{00}^{-} + K/2$.
If $y_{00}^{-} \geq x_{00}^{-} + R$ we let $y_{00} := y_{00}^{-} + K/2$ and $x_{00} = y_{00} - R = y_{00}^{-} + K/2 - R \geq x_{00}^{-} + K/2$.
In both the case using (5) we can prove the upper bound for x_{00} and y_{00} (if (5) it is not true we can have $x_{00}^{-} + R \geq y_{00}^{+}$). Consider the restriction $C_{00} : [x_{00}^{-}, x_{00}^{+}] \longrightarrow S$ that can be reparametrized with unit speed from the interval $[-\frac{l}{2}, \frac{l}{2}]$ where is required $l \geq K$ by assumption (1). Call A this new parametrization. Then we have $C_{00}(x_{00}) = A(t)$ for a t such that $|t| \leq \frac{l-K+2}{2}$. Then, by the use of the Convergence Lemma 3.2.7 and the assumption (2), we can find $x_{ij}^{-} < x_{ij} < x_{ij}^{+}$ such that

$$\mathrm{dis}((C_{00}(x_{00}), \dot{C}_{00}(x_{00})), (C_{ij}(x_{ij}), \dot{C}_{ij}(x_{ij}))) \leq e^{|t|+E+2-\frac{l}{2}} \leq e^{E-\frac{K}{2}+3} \leq \epsilon, \tag{3.13}$$

where the last inequality holds if we choose a $K(\epsilon, E) > 2E + 6 - 2\log\epsilon$. By identical reasons we find $y_{ij}^- < y_{ij} < yij^+$ such that

$$\mathrm{dis}((C_{00}(y_{00}), \dot{C}_{00}(y_{00})), (C_{ij}(y_{ij}), \dot{C}_{ij}(y_{ij}))) \leq \epsilon. \tag{3.14}$$

From the assumptions (3) and (4), (3.13), (3.14), and the triangular inequality, we have that the geodesic segments $C_{0j}[x_{0j}, y_{0j}]$ and $C_{1j}[x_{1j}, y_{1j}]$ ($j = 0$,1) are 2ϵ-nearly homotopic, and likewise $C_{i0}[y_{i0}, x_{i0}+l_{i0}]$ and $C_{i1}[y_{i1}, x_{i1}+l_{i1}]$ ($i = 0$,1) are 2ϵ-nearly homotopic too. We will refer these properties as (3′) and (4′). Set $I_{ij} = y_{ij} - x_{ij}$ and $J_{ij} = x_{ij} + l_{ij} - y_{ij}$ so $I_{ij} + J_{ij} = l_{ij}$. Then $I_{00} = R$ and $|J_{00} - R| < 2\epsilon$, since $C_{ij} \in \Gamma_{\epsilon,R}$. By triangle inequality we have

$$|I_{10} - R| = |I_{10} - I_{00}| = |y_{10} - x_{10} - y_{00} + x_{00}| < 2\epsilon.$$

So also,

$$|J_{10} - R| \leq |I_{10} - R| + |l_{10} - 2R| < 4\epsilon.$$

Then (using(4′))

$$|J_{i1} - R| \leq |J_{i0} - R| + |J_{i1} - J_{i0}| < 8\epsilon$$

$$|I_{i1} - R| \leq |J_{i1} - R| + |l_{i1} - 2R| < 10\epsilon.$$

In general we have $|I_{ij} - R|$, $|J_{ij} - R| < 10\epsilon$ for i, $j \in \{0, 1\}$. By the Connection Lemma 3.2.2 we can find

$$\alpha_{00} \in \mathrm{Conn}_{\epsilon,R+\log 4}((C_{00}(x_{00}), \sqrt{-1}\dot{C}_{00}(x_{00})), (C_{00}(y_{00}), -\sqrt{-1}\dot{C}_{00}(y_{00}))).$$

Note that the use of the Connection Lemma implicitly defines the constant R_0 which is completely independent to the constant K. In fact, the existence of α_{00} require $R > L_0(\epsilon, S)$ for L_0 from the statement of the Connection Lemma, and so $R_0 > L_0$ became a defining inequality.
We can find geodesic arcs α_{ij} connecting x_{ij} with y_{ij} that are ϵ-nearly homotopic to α_{00}, in fact we can see the situation in the universal cover, where we choose some lifts of x_{ij}, x_{00}, y_{ij} and y_{00} such that equations (3.13) and (3.14) remains for the lifts; in such situation is simple to find a lift of α_{ij}

ϵ-nearly homotopic to a chosen lift of α_{00} and such that their projections are homotopic.

Then $\mathrm{dis}(i(\alpha_{ij}), i(\alpha_{00})) < 2\epsilon$ and $\mathrm{dis}(t(\alpha_{ij}), t(\alpha_{00})) < 2\epsilon$ by Lemma 3.2.8

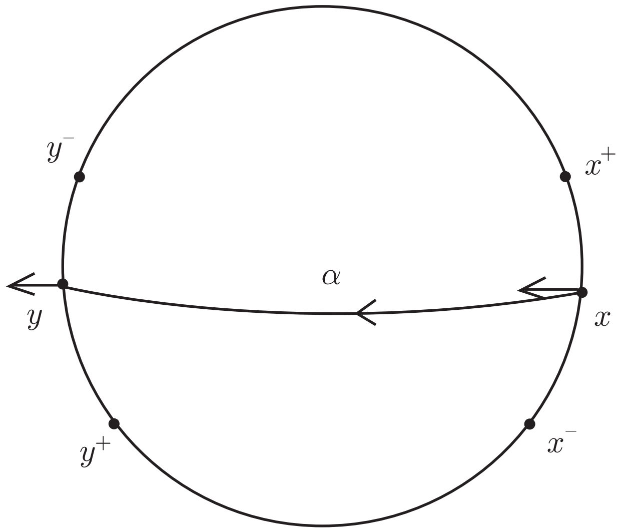

Figure 3.19: The resolution of the Preliminary Geometric Square Lemma

(to apply this Lemma we have to suppose that ϵ is bounded by a universal constant $\hat{\epsilon}$). Since

$$\mathrm{dis}((C_{00}(x_{00}), \dot{C}_{00}(x_{00}), (C_{ij}(x_{ij}), \dot{C}_{ij}(x_{ij})) \leq \epsilon$$

and

$$\mathrm{dis}((C_{00}(y_{00}), \dot{C}_{00}(y_{00}), (C_{ij}(y_{ij}), \dot{C}_{ij}(y_{ij})) \leq \epsilon,$$

we have (by triangular inequality)

$$\alpha_{ij} \in \mathrm{Conn}_{3\epsilon, R+\log 4}((C_{ij}(x_{ij}), \sqrt{-1}\dot{C}_{ij}(x_{i}j)), (C_{ij}(y_{ij}), -\sqrt{-1}\dot{C}_{ij}(y_{ij}))).$$

Consider $C_{ij} \cup \alpha_{ij}$. By construction the vectors of the three geodesic arcs, starting in x_{ij} and ending at y_{ij}, at these points have opposite cyclic order, so they generate an immersed pants Π_{ij} with geodetic cuffs A_{ij} freely homotopic to $C_{ij}[x_{ij}, y_{ij}]\alpha_{ij}^{-1}$, B_{ij} freely homotopic to $C_{ij}[y_{ij}, x_{ij} + l_{ij}]\alpha_{ij}$ and C_{ij}. Now, using the Sum of Inefficiency Lemma for Angles 3.1.2 , we have

$$|I([C_{ij}[x_{ij}, y_{ij}]\alpha_{ij}^{-1}]) - 2I(\pi/2)| < 2\epsilon,$$

which means (recall that $I(\pi/2)=\log 2$)

$$|l(A_{ij})-R-I_{ij}|<2\epsilon,$$

$$|l(A_{ij})-2R|<10\epsilon+2\epsilon<20\epsilon.$$

Similarly we found, $|l(B_{ij})-2R|<20\epsilon$. This isn't the best bound, however is enough for our purposes, in fact we have $\Pi_{ij}\in\Pi_{10\epsilon,R}$. Finally we see that (3′) imply $A_{0j}=A_{1j}$ while (4′) imply $B_{i0}=B_{i1}$. It follows that

$$0=\sum(-1)^{i+j}\partial\Pi_{ij}=\sum(-1)^{i+j}C_{ij}$$

in the $\Pi_{10\epsilon,R}$ homology. □

Lemma 3.2.10 (Geometric Square Lemma)**.** *There exists an $\hat{\epsilon}>0$ such that the following holds. Let $\hat{\epsilon}>\epsilon>0$, and let $E>0$.There exist constants $K_1=K_1(\epsilon,E,S)>0$ and $R_0(S,\epsilon,E)>0$ with the following properties.*
Let $R>R_0$. Suppose that we are given four oriented geodesics $C_{ij}\in\Gamma_{\epsilon,R}$, i,$j=0,1$, and for each double index ij we are given 4 real numbers

$$x_{ij}^-<x_{ij}^+<y_{ij}^-<y_{ij}^+<x_{ij}^-+l(C_{ij}).$$

Assume that
(1) $x_{ij}^+-x_{ij}^->K_1$, $y_{ij}^+-y_{ij}^->K_1$,
(2) the segments $C_{ij}[x_{ij}^-,x_{ij}^+]$ and $C_{hk}[x_{hk}^-,x_{hk}^+]$ are E-nearly homotopic, and likewise the segments $C_{ij}[y_{ij}^-,y_{ij}^+]$ and $C_{hk}[y_{hk}^-,y_{hk}^+]$ are E-nearly homotopic, for any i, j, h, $k\in\{0,1\}$,
(3)the segments $C_{0j}[x_{0j}^-,y_{0j}^+]$ and $C_{1j}[x_{1j}^-,y_{1j}^+]$ are E-nearly homotopic,
(4) the segments $C_{i0}[y_{i0}^-,x_{i0}^++l(C_{ij})]$ and $C_{i1}[y_{i1}^-,x_{i1}^++l(C_{ij})]$ are E-nearly homotopic.
Then we have

$$\sum_{i,j=0,1}(-1)^{i+j}C_{ij}=0$$

in $H^{100\epsilon,R}(S)$.

Proof. In the proof we will use some temporary constants. To avoid confusion in the dependence of such constants we give that now. $L_0 = L_0(S, \epsilon, E)$ and $K_0 = K_0(S, \epsilon, E)$ are constants whose values will be determined along the argument. $Q_0 = Q_0(L_0, K_0, \epsilon, E) > 0$ can depend on all that parameter, while the constants of the statements will be $K_1 = K_1(L_0, K_0, Q_0, \epsilon, S, E)$ and $R_0 = R_0(Q_0, L_0, K_0, \epsilon, S, E)$. It is important to not confuse K_0 and K_1 with $K = K(\epsilon, E)$, the constant coming from the PGSL 3.2.9.
If we can't apply directly the PGSL, then, eventually interchanging the roles of the x's and the y's, we have that

$$x_{ij}^+ \leq y_{ij}^- - l_{ij} + R + K < y_{ij}^- - R + K + 1, \tag{3.15}$$

for all i and j. We recall that $K > 2E + 6 - 2\log\epsilon$ from the proof of the PGSL, in particular we can assume we can assume $K > 10 + 2\log\frac{E}{\epsilon}$. Let $Q_0 > K + 11$ a constant, we set $y_{00} = y_{00}^- + Q_0$ and $w_{00} = y_{00} - R$. So by Equation (3.15)

$$w_{00} \geq y_{00}^- + Q_0 - R > x_{00}^+ + 10 \tag{3.16}$$

and (by assumption (1))

$$y_{00}^- + Q_0 \leq y_{00} \leq y_{00}^+ - K_1 + Q_0 \tag{3.17}$$

Since $Q_0 \geq \log\frac{E}{\epsilon} + 10$ and supposing $K_1 \geq Q_0 + \log\frac{E}{\epsilon}$ we have from (3.17)

$$y_{00}^- + \log\frac{E}{\epsilon} + 10 \leq y_{00} \leq y_{00}^+ - \log\frac{E}{\epsilon}. \tag{3.18}$$

Therefore we can use the Convergence Lemma, in the same way we used it in the proof of the PGSL, to find $y_{ij}^- \leq y_{ij} \leq y_{ij}^+$ such that

$$\mathrm{dis}((C_{00}(y_{00}), \dot{C}_{00}(y_{00}), (C_{ij}(y_{ij}), \dot{C}_{ij}(y_{ij})) \leq \epsilon.$$

Note that this imply also $|y_{ij} - (y_{ij}^- + Q_0)| < \epsilon + E$ by triangular inequality, (2) and definition of y_{00}.

We let $w_{ij} = y_{ij} - R \geq y_{ij}^- + Q_0 - \epsilon - E - R > y_{ij}^- + K + 11 - R \geq x_{ij}^+ + 10$ where the last inequality use (3.15). Then $w_{ij} > x_{ij}^- + K_1 + 10 \geq x_{ij}^- + \log\frac{E}{\epsilon} +$

10. On the other hang $w_{ij} \le y_{ij}^- - R + Q_0 + \epsilon + E \le y_{ij}^- - \log \frac{E}{\epsilon}$, provided $R_0(Q_0, \epsilon, E)$ sufficiently large. So we have

$$x_{ij}^- + \log \frac{E}{\epsilon} + 10 \le w_{ij} \le y_{ij}^- - \log \frac{E}{\epsilon}. \quad (3.19)$$

Furthermore (3.19) permit us to use the Convergence Lemma as well we used it before (the E-nearly homotopy is the one given by assumption (3)) to have

$$\text{dis}((C_{00}(w_{00}), \dot{C}_{00}(w_{00}), (C_{ij}(w_{ij}), \dot{C}_{ij}(w_{ij})) \le \epsilon,$$

and then, combining with the assumption (3), we obtain that $C_{0j}[w_{0j}, y_{0j}]$ and $C_{1j}[w_{1j}, y_{1j}]$ are ϵ-nearly homotopic, and also their first derivatives are ϵ near.
Fix a point $(q, v) \in T^1S$. The Connection Lemma 3.2.2 assure that there exists an $\tilde{L}(\epsilon, E)$ such that for every $L_0 > \tilde{L}(\epsilon, E)$ we can find

$$\beta_{0j} \in \text{Conn}_{\epsilon, L_0}(v, -\sqrt{-1}\dot{C}_{0j}(w_{0j})).$$

From now on we consider L_0 as a fixed constant.
Take $\alpha_{00} \in \text{Conn}_{\epsilon, R-L_0+\log 4}(\sqrt{-1}\dot{C}_{00}(y_{00}), v)$.
Note that taking α_{00} implicitly redefine the constant $R_0(\epsilon, E, L_0)$. Now, as we done for the α_{ij} of the proof of the PGSL, we found

$$\alpha_{ij} \in \text{Conn}_{3\epsilon, R-L_0+\log 4}(\sqrt{-1}\dot{C}_{ij}(y_{ij}), v),$$

and

$$\beta_{ij} \in \text{Conn}_{3\epsilon, L_0}(v, -\sqrt{-1}\dot{C}_{ij}(w_{ij})),$$

such that α_{00} and α_{ij} are ϵ-nearly homotopic as well β_{0j} and β_{ij}.

Consider the geodesic segments $C_{ij}[w_{ij}, y_{ij}]$ and $(C_{ij}[y_{ij}, w_{ij} + l_{ij}])^{-1}$, and the piecewise geodesic arc $\beta_{ij}^{-1}\alpha_{ij}^{-1}$. One can see that the three unit vectors of these three arcs at the points w_{ij} and y_{ij} have opposite cyclic order in the unit circle. So there exist a pair of pants generated by them named Π_{ij}. Let A_{ij} be the closed geodesic freely homotopic to $\alpha_{ij}\beta_{ij}C_{ij}[w_{ij}, y_{ij}]$ and

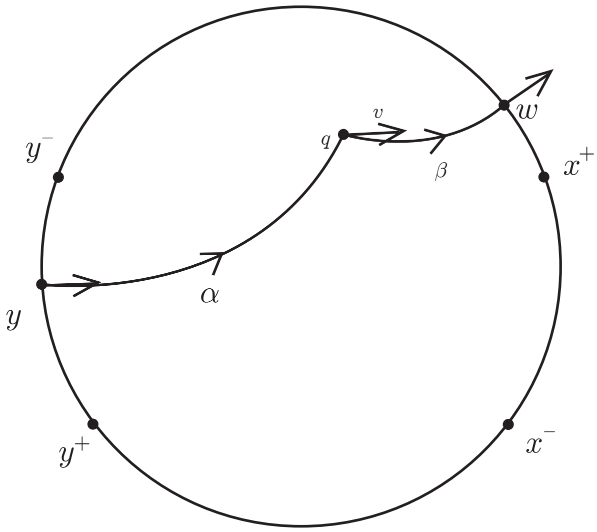

Figure 3.20: The resolution of the Geometric Square Lemma.

B_{ij} the closed geodesic freely homotopic to $C_{ij}[y_{ij}, w_{ij} + l_{ij}]\beta_{ij}^{-1}\alpha_{ij}^{-1}$. Then $\partial\Pi_{ij} = C_{ij} - A_{ij} - B_{ij}$. With computations similar to those made in the proof of PGSL for $l(A_{ij})$, using the Sum of Inefficiency Lemma for Angles 3.1.2 we have $|l(A_{ij}) - 2R| < 20\epsilon$ and $|l(B_{ij}) - 2R| < 20\epsilon$. So $\Pi_{ij} \in \Pi_{10\epsilon,R}$. We have, from the evolution of hypothesis (3) along the proof, that $A_{0j} = A_{1j}$. Then

$$\sum_{i,j=0,1}(-1)^{i+j}C_{ij} - \sum_{i,j=0,1}(-1)^{i+j}\partial\Pi_{ij} = \sum_{i,j=0,1}(-1)^{i+j}B_{ij}.$$

Now we want parametrize $B_{ij} : \mathbb{R} \longrightarrow B_{ij}$ in order to apply the PGSL to them. Working on lifts of B_{ij} and $C_{ij}[y_{ij}, w_{ij} + l_{ij}]\beta_{ij}^{-1}\alpha_{ij}^{-1}$ with same endpoints at infinity, we can consider the standard nearest point projection $\pi : B_{ij} \longrightarrow C_{ij}[y_{ij}, w_{ij} + 1]\beta_{ij}^{-1}\alpha_{ij}^{-1}$ and call $B_{ij}(a_{ij}^-) = \pi(q)$, $B_{ij}(a_{ij}^+) = \pi(C_{ij}(y_{ij}^+))$, $B_{ij}(b_{ij}^-) = \pi(C_{ij}(x_{ij}^-))$ and $B_{ij}(b_{ij}^+) = \pi(C_{ij}(x_{ij}^+))$ (see what we done in 3.1.7 for a more detailed construction, see also figure 3.21).

Observe that $I(B_{ij}) \leq 22\epsilon$ and, by Lemma 3.1.7, $d(x, \pi(x)) \leq 1 + 11\epsilon$.

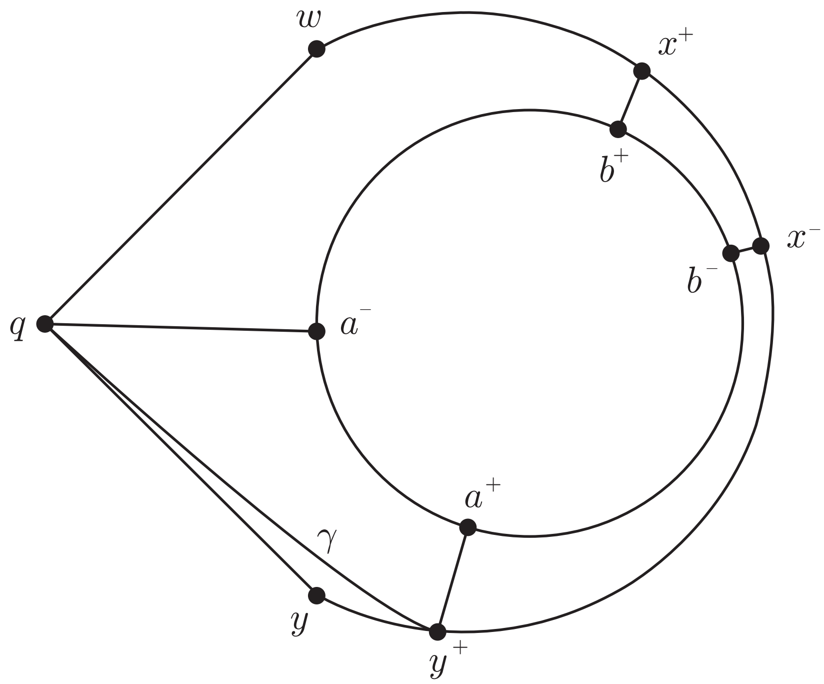

Figure 3.21: The parametrization of B_{ij}

Then, by positivity of inefficiency,

$$\begin{aligned} b^+_{ij} - b^-_{ij} &\geq x^+_{ij} - x^-_{ij} - 2 - 20\epsilon \\ &> K_1 - 2 - 20\epsilon \\ &> K_0, \end{aligned}$$

where we have supposed $K_1 > K_0 - 20\epsilon - 2 > 0$. For the other segment we recall that, from (3.17), assumption (2) and definition of y_{ij}, we have

$$y^+_{ij} - y_{ij} \geq y^+_{00} - y_{00} - E - \epsilon \geq K_1 - Q_0 - E - \epsilon.$$

We call simply $y = C_{ij}[y_{ij}, y^+_{ij}]$, γ the geodesic arc between the points q and $C_{ij}(y_{ij})$ and homotopic to $\alpha^{-1}_{ij} y$, $a = B_{ij}[a^-_{ij}, a^+_{ij}]$,and e_1 and e_2 the little geodesic arcs connecting the endpoints of a with the endpoints of γ. From the positivity of inefficiency we have

$$I(y\alpha_{ij}) = l(y\alpha_{ij}) - l(\gamma) \geq 0,$$

and

$$0 \leq I(e_1 a e_2) = l(e_1 a e_2) - l(\gamma) = l(e_1 a e_2) - l(y\alpha_{ij}) + I(y\alpha_{ij}),$$

which imply

$$l(e_1 a e_2) - l(y\alpha_{ij}) \geq 0.$$

Now we can see that this means

$$\begin{aligned} a_{ij}^+ - a_{ij}^- &\geq l(\gamma) + l(\alpha_{ij}) - l(e_1) - l(e_2) \\ &\geq (K_1 - Q_0 - E - \epsilon) + (R - L_0 + \log 4 - \epsilon) - 2 - 22\epsilon \\ &> K_0 + R, \end{aligned}$$

for a choice of $K_1 = K_1(K_0, Q_0, L_0, \epsilon, E) > 0$ sufficiently large. We want to apply PGSL to B_{ij} with the four points b_{ij} and a_{ij}. Suppose $K_0 > K(10\epsilon, E + 1 + 11\epsilon)$, where K is the constant given by the PGSL. First note that $a_{ij}^+ - a_{ij}^- \geq R + K_0$ is a sufficient condition for assumption (5) of PGSL, and that we have just proved assumption (1). Then the $(E+1+11\epsilon)$-nearly homotopies needed in PGSL, come from the E-nearly homotopies of GSL and the construction we made. Moreover $B_{ij} \in \Gamma_{10\epsilon,R}$, so

$$\sum (-1)^{i+j} \partial \Pi_{ij} = 0$$

in $H^{100\epsilon,R}(S)$. □

From now the surface over we work is supposed to be a pointed surface $(S, *)$. We are going to develop an algebraic language for the Good Pants Homology, so we need to fix some notation. If $A \in \pi_1(S, *)$, we denote with $[A]$ the closed geodesic which is freely homotopic to A. We let $\cdot A \cdot$ be the geodesic segment from $*$ to $*$ homotopic to $[A]$.

If $A_1, \ldots, A_n \in \pi_1(S, *)$ we let $\cdot A_1 \cdot A_2 \cdot \ldots \cdot A_n \cdot$ be the piecewise geodesic arc that is the concatenation of the $\cdot A_i \cdot$. We let $[\cdot A_1 \cdot A_2 \cdot \ldots \cdot A_n \cdot]$ be the closed piecewise geodesic made with the same piecewise geodesic of $\cdot A_1 \cdot A_2 \cdot \ldots \cdot A_n \cdot$ but with the difference that this last is considered as a closed curve.

Be warned to no make confusion with the notation $[A]$ where A is an element

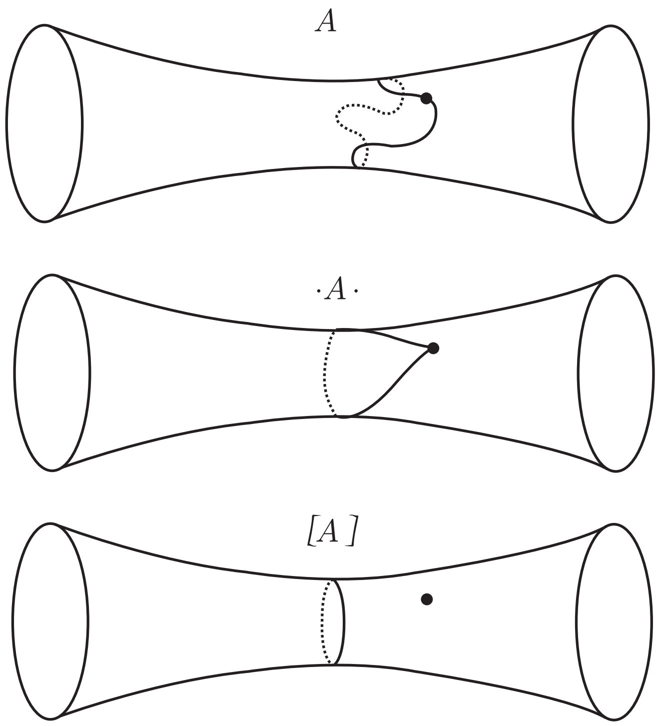

Figure 3.22: Three curves referring to A.

of the group $\pi_1(S,*)$. In the notation without dots we mean the geodesic representative *freely* homotopic, while in the dotted notation we have the endpoints of the geodesic arcs fixed (see the remark below). As before $l(\cdot)$ gives the length of the object in the argument.

Remark 3.2.2. Note that we have:

$$I(\cdot A_1\cdot\ldots\cdot A_n\cdot)=\sum l(\cdot A_i\cdot)-l(\cdot A_1A_2\ldots A_n\cdot),$$

and

$$I([\cdot A_1\cdot\ldots\cdot A_n\cdot])=\sum l(\cdot A_i\cdot)-l([A_1A_2\ldots A_n]).$$

Notice that we may have (and usually we have):

$$I([\cdot A_1\cdot\ldots\cdot A_n\cdot])>I(\cdot A_1\cdot\ldots\cdot A_n\cdot)$$

Now we translate the Geometric Square Lemma in this new algebraic language.

Lemma 3.2.11 (Algebraic Square Lemma)**.** *There exists an $\hat{\epsilon} > 0$ such that the following holds. Let $\hat{\epsilon} > \epsilon > 0$, and let $\Delta > 0$. There exist constants $K = K(S, \epsilon, \Delta) > 0$ and $R_0 = R_0(S, \epsilon, \Delta) > 0$ such that for $R > R_0$ the following holds. Let A_i, U, B_i, $V \in \pi_1(S, *)$, for $i = 0{,}1$. Assume that*
(1) $|l([A_i U B_i V]) - 2R| < 2\epsilon$, $i{,}j = 0,\ 1$,
(2) $I([\cdot A_i \cdot U \cdot B_j \cdot V \cdot]) < \Delta$,
(3) $l(\cdot U \cdot)$, $l(\cdot V \cdot) > K$.
Then

$$\sum_{ij} (-1)^{i+j} [A_i U B_i V] = 0$$

in $H^{100\epsilon, R}(S)$.

Proof. In the universal cover we use the nearest point projection π_{ij} of a lift of $[\cdot A_i \cdot U \cdot B_j \cdot V \cdot]$ onto a lift of $C_{ij} = [A_i U B_i V]$ (we suppose to have chosen two lifts with the same endpoints at infinity). By Lemma 3.1.7

$$d(\tilde{*}, \pi_{ij}(\tilde{*})) < \Delta/2 + 1 \tag{3.20}$$

for every $\tilde{*} \in \pi_{ij}^{-1}(*)$.
Let $C_{ij}(x_{ij}^{\pm})$ be such projections of $*$ before and after U and $C_{ij}(y_{ij}^{\pm})$ be the projections of $*$ before and after V. By assumption (1) $C_{ij} \in \Gamma_{\epsilon,R}$.
Let $K_1(S, \epsilon, \Delta{+}2)$ and $R_0(S, \epsilon, \Delta{+}2)$ be the constants determined by the GSL 3.2.10, define $K \geq K_1 + \Delta + 1$. We have $x_{ij}^- < x_{ij}^+ < y_{ij}^- < y_{ij}^+ < x_{ij}^- + l(C_{ij})$.
Assumptions (2) - (4) of the GSL follows from our definitions of C_{ij} and inequality (3.20). Assumption (1) of the GSL is exactly our assumption (3) and our definition of K.
Now the conclusion of the GSL also concludes our statement. □

The Sum of Inefficiency Lemmas 3.1.9 and 3.1.10 can be rewritten as follow

Lemma 3.2.12. *Let ϵ, $\Delta > 0$, and $n \in \mathbb{N}$. There exists $L = L(\epsilon, \Delta, n) > 0$ such that if $U_1, \ldots, U_{n+1} = U_1, X_1, \ldots, X_n \in \pi_1(S, *)$, and $I(\cdot U_i \cdot X_i \cdot U_{i+1} \cdot) \leq \Delta$, and $l(\cdot U_i \cdot) \geq L$, Then*

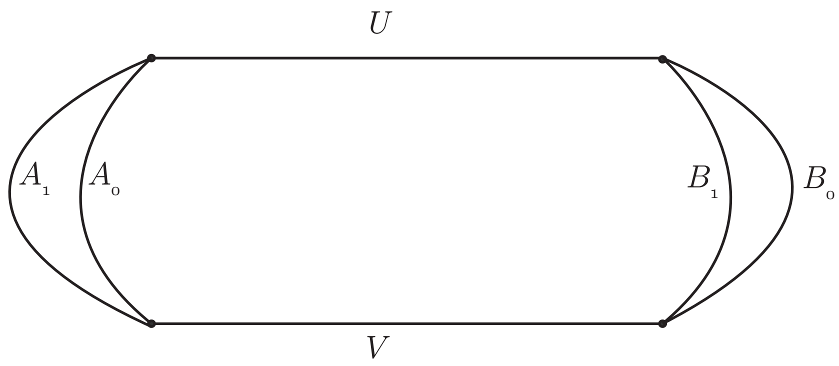

Figure 3.23: The Algebraic Square Lemma.

$$|I([\cdot U_1 \cdot X_1 \cdot U_2 \cdot X_2 \cdot \ldots \cdot U_n \cdot X_n \cdot]) - \sum_{i=1}^{n} I(\cdot U_i \cdot X_i \cdot U_{i+1} \cdot)| \leq \delta$$

Remark 3.2.3. In particular,under the same hypothesis but leaving out the X's, we have:

$$|I([\cdot U_1 \cdot U_2 \cdot \ldots \cdot U_n \cdot]) - \sum_{i=1}^{n} I(\cdot U_i \cdot U_{i+1} \cdot)| \leq \delta.$$

Moreover by the Long Segment Lemma for Angles we have

$$|I([\cdot U_1 \cdot U_2 \cdot \ldots \cdot U_n \cdot]) - \sum_{i} I(\theta_i)| \leq 2\delta,$$

where $\theta_i = \Theta(t(\cdot U_i \cdot), i(\cdot U_{i+1} \cdot))$.

Lemma 3.2.13. *Let δ, $\Delta > 0$, and $n \in \mathbb{N}$. There exists $L = L(\epsilon, \Delta, n) > 0$ such that if $U_1,\ldots,U_{n+1} = U_1 \in \pi_1(S,*)$, $X_{11},\ldots,X_{1j_1},\ldots,X_{n1},\ldots,X_{nj_n} \in \pi_1(S,*)$, and $I(\cdot U_i \cdot \cdot x_i \cdot \cdot U_{i+1} \cdot) \leq \Delta$, and $l(\cdot U_i \cdot) \geq L$, Then*

$$|I([\cdot U_1 \cdot X_{11} \cdot \cdots \cdot X_{1j_1} \cdot U_n \cdot X_{n1} \cdot \ldots \cdot X_{nj_n} \cdot]) - \sum_{i=1}^{n} I(\cdot U_i \cdot X_{i1} \cdot \cdots \cdot X_{ij_i} \cdot U_{i+1} \cdot)| \leq \delta.$$

For $X \in \pi_1(S,*)$. We write $\bar{X}$ for X^{-1}. What happen to the geodesic $[TA\bar{T}B]$ if we reverse A? We have the Flipping Lemma.

Lemma 3.2.14 (Flipping Lemma). *There exists an $\hat{\epsilon} > 0$ such that the following holds. Let $\hat{\epsilon} > \epsilon > 0$, and let $\Delta > 0$. There exists a constant $L = L(\epsilon, \Delta) > 0$ with the following properties. Suppose A, B, $T \in \pi_1(S)$ are such that*

$$I(\cdot T \cdot A \cdot \bar{T} \cdot), I(\cdot \bar{T} \cdot B \cdot T \cdot) \leq \Delta,$$

and $l(\cdot T \cdot) \geq L$. Then

$$|I([\cdot T \cdot A \cdot \bar{T} \cdot B \cdot]) - I([\cdot T \cdot \bar{A} \cdot \bar{T} \cdot B \cdot])| < \epsilon,$$

and therefore

$$|l([TA\bar{T}B]) - l([T\bar{A}\bar{T}B])| < \epsilon.$$

Proof. We can apply Lemma 3.2.12 to have

$$|I([\cdot T \cdot A \cdot \bar{T} \cdot B \cdot]) - I(\cdot T \cdot A \cdot \bar{T} \cdot) - I(\cdot \bar{T} \cdot B \cdot T \cdot)| \leq \frac{\epsilon}{2},$$

and,

$$|I([\cdot T \cdot \bar{A} \cdot \bar{T} \cdot B \cdot]) - I(\cdot T \cdot \bar{A} \cdot \bar{T} \cdot) - I(\cdot \bar{T} \cdot B \cdot T \cdot)| \leq \frac{\epsilon}{2}.$$

However $I(\cdot T \cdot A \cdot \bar{T} \cdot) = I(\cdot T \cdot \bar{A} \cdot \bar{T} \cdot)$ since are the same arc with opposite orientations. Then we have the statement. □

Remark 3.2.4. In the following lemmas many computations implicitly use the fact that $\log \sec(x) = O(x^2)$ for $x \to 0$. It simply follows from the Taylor expansion of the function near 0.

From now to the end of this section we will use $T \in \pi_1(S, *)$ and $\Delta > 0$ as parameters. However in every statement wi will redefine them to be more precise about their properties. We define the set $\mathcal{C}\,\mathrm{Conn}_{\epsilon,R}(A, T)$ as the set of all $B \in \pi_1(S, *)$ such that $[TA\bar{T}B]$,$[T\bar{A}\bar{T}B] \in \Gamma_{\epsilon,R}$ and $I(\cdot \bar{T} \cdot B \cdot T \cdot) < 1$.

Lemma 3.2.15 (Definition of A_T). *There exists an $\hat{\epsilon} > 0$ such that the following holds. Let $\hat{\epsilon} > \epsilon > 0$, and let $\Delta > 0$. There exist two constants $R_0 = R_0(\epsilon, \Delta, S)$ and $L = L(S, \epsilon, \Delta)$ such that if $R > R_0$, A, $T \in \pi_1(S, *)$, $I(\cdot \bar{T} \cdot A \cdot T \cdot) < \Delta$, $l(\cdot T \cdot) \geq L$, and $2R - l(\cdot A \cdot) - 2l(\cdot T \cdot) \geq L$, then*
(1) *$\mathcal{C}\,\mathrm{Conn}_{\epsilon,R}(A, T)$ is non empty and $|\mathcal{C}\,\mathrm{Conn}_{\epsilon,R}(A, T)| \geq e^{2R - l(\cdot A \cdot) - \Delta - 2l(\cdot T \cdot) - L}$,*

(2) $[TA\bar{T}B] - [T\bar{A}\bar{T}B] = [TA\bar{T}B'] - [T\bar{A}\bar{T}B']$ *in* $H^{100\epsilon,R}(S)$ *homology for any* $B,\ B' \in \mathcal{C}\,\mathrm{Conn}_{\epsilon,R}(A,T)$.
We then define

$$A_T = \frac{1}{2}([TA\bar{T}B] - [T\bar{A}\bar{T}B])$$

for an arbitrary $B \in \mathcal{C}\,\mathrm{Conn}_{\epsilon,R}(A,T)$.

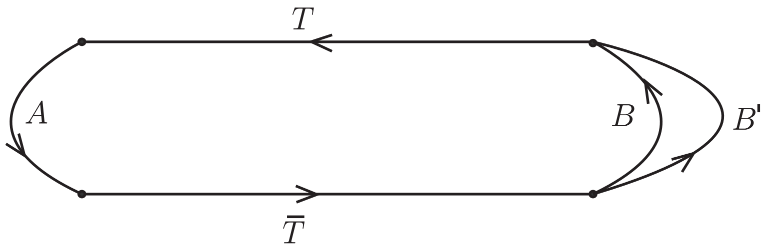

Figure 3.24: The Definition of A_T.

Proof. Let $\cdot B\cdot \in \mathrm{Conn}_{\epsilon,R'}((*, -i(\cdot T\cdot)), (*, i(\cdot T\cdot)))$, where

$$\begin{aligned} R' &= 2R - l(\cdot A\cdot) - 2l(\cdot T\cdot) - I(\cdot\bar{T}\cdot A\cdot T\cdot) \\ &\geq 2R - l(\cdot A\cdot) - 2l(\cdot T\cdot) + \Delta \\ &\geq L + \Delta. \end{aligned}$$

Such $\cdot B\cdot$ exists, if we choose a sufficiently large constant L, by the Connection Lemma 3.2.2. Notice that, by the Sum of Inefficiencies for Angles and the definition of connections, $I(\cdot\bar{T}\cdot B\cdot T\cdot) < \epsilon/2 + O(\epsilon^2)$ (see remark 3.2.4). We start estimates:

$$\begin{aligned} I([\cdot T\cdot A\cdot\bar{T}\cdot B\cdot]) &= l([TA\bar{T}B]) - l(A) - 2l(T) - l(B) \\ &= l([TA\bar{T}B]) - 2R + I(\cdot T\cdot A\cdot\bar{T}\cdot) \pm \epsilon. \end{aligned}$$

So, by the Sum of Inefficiency Lemma we have

$$|I([\cdot T\cdot A\cdot\bar{T}\cdot B\cdot]) - I(\cdot\bar{T}\cdot B\cdot T\cdot) - I(\cdot T\cdot A\cdot\bar{T}\cdot)| < \epsilon/2,$$

and then

$$|l([TA\bar{T}B]) - 2R| < \epsilon + O(\epsilon^2),$$

likewise

$$|l([T\bar{A}\bar{T}B]) - 2R| < \epsilon + O(\epsilon^2),$$

all provided $l(\cdot T \cdot)$ sufficiently large. Thus, we have proved that

$$B \in \mathcal{C}\,\mathrm{Conn}_{\epsilon,R}(A,T),$$

and, in particularly, (1).
(2) is exactly the application of the Algebraic Square Lemma. Obviously R_0 is equal to the homonymous constant from the ASL 3.2.11. □

Remark 3.2.5. In the standard homology $H_1(S)$, the curves $[A]$ and A_T are in the same class. The following Lemmas will stress more the fact that we are exploring a new point of view of the algebraic structure in H_1.

Lemma 3.2.16 (Simple Itemization Lemma)**.** *There exists an $\hat{\epsilon} > 0$ such that the following holds. Let $\hat{\epsilon} > \epsilon > 0$, and let $\Delta > 0$. There exist a constant $L = L(S, \epsilon, \Delta) > 0$ and a constant $R_0 = R_0(\epsilon, \Delta, S) > 0$ with the following properties. For any $R > R_0$ and A, B, $T \in \pi_1(S, *)$ such that $l(\cdot T \cdot)$, $l(\cdot A \cdot)$,$l(\cdot B \cdot) > L$, $[TA\bar{T}B] \in \Gamma_{\epsilon,R}$, and $I([\cdot T \cdot A \cdot \bar{T} \cdot B \cdot]) \leq \Delta$, we have $[TA\bar{T}B] = A_T + B_{\bar{T}}$ in $H^{100\epsilon,R}(S)$.*

Proof.

$$\begin{aligned}
[TA\bar{T}B] &= \frac{1}{2}([TA\bar{T}B] - [\bar{B}T\bar{A}\bar{T}]) \\
&= \frac{1}{2}([TA\bar{T}B] - [T\bar{A}\bar{T}\bar{B}]) \\
&= \frac{1}{2}([TA\bar{T}B] - [T\bar{A}\bar{T}B]) + \frac{1}{2}([\bar{T}BT\bar{A}] - [\bar{T}\bar{B}T\bar{A}]) \\
&= A_T + B_{\bar{T}}.
\end{aligned}$$

□

The following fact is a rewriting of the Sum f Inefficiency Lemma 3.2.12.

Fact 3.2.17. *Let ϵ, Δ. There exists $L = L(\epsilon, \Delta) > 0$ with the following properties. Let A_i, B_j, $T \in \pi_1(S,*)$, i,$j = 0$,1. If $l(\cdot T\cdot) > L$ and $I(\cdot\bar{T}\cdot A_i\cdot T\cdot)$,$I(\cdot T\cdot B_j\cdot\bar{T}\cdot) < \Delta$, then*

$$|l([A_0TB_0\bar{T}A_1TB_1\bar{T}]) - \sum_i l(\cdot\bar{T}A_iT\cdot) - \sum_j l(\cdot TB_j\bar{T}\cdot) + 4l(\cdot T\cdot)| < \epsilon.$$

Lemma 3.2.18 (The $ADCB$ Lemma)**.** *There exists an $\hat{\epsilon} > 0$ such that the following holds. Let $\hat{\epsilon} > \epsilon > 0$, and let Δ. There exist $L = L(S, \epsilon, \Delta) > 0$ and $R_0 = R_0(S, \epsilon, \Delta) > 0$ with the following properties. Let A,B,C,D,$T \in \pi_1(S,*)$ such that $l(\cdot B\cdot)$,$l(\cdot D\cdot)$, $l(\cdot T\cdot) > L$ and $[ATB\bar{T}CTD\bar{T}]$, $[ATD\bar{T}CTB\bar{T}] \in \Gamma_{\epsilon,R}$. If $R > R_0$ and*

$$I(\cdot T\cdot A\cdot\bar{T}\cdot),I(\cdot T\cdot B\cdot\bar{T}\cdot),I(\cdot T\cdot C\cdot\bar{T}\cdot),I(\cdot T\cdot D\cdot\bar{T}\cdot) < \Delta$$

then $[ATB\bar{T}CTD\bar{T}] = [ATD\bar{T}CTB\bar{T}]$ in $H^{200\epsilon,R}(S)$.

Proof. Let $\langle X, Y\rangle = [ATX\bar{T}CTY\bar{T}]$, for X,$Y \in \pi_1(S,*)$. We want to prove that

$$\langle X_0, Y_0\rangle - \langle X_0, Y_1\rangle = \langle Y_0, X_1\rangle - \langle Y_1, X_1\rangle \tag{3.21}$$

in $H^{100\epsilon,R}$ whenever all that curves are in $\Gamma_{\epsilon,R}$ and $I(\cdot TX_i\bar{T}\cdot)$,$I(\cdot TY_i\bar{T}\cdot) < \Delta$. Equation (3.21) is exactly the Algebraic Square Lemma with the choice (with reference to the notation of the statement 3.2.11) $A_i = Y_i$, $B_0 = ATX_0\bar{T}C$, $B_1 = CTX_1\bar{T}A$, $U = T$ and $V = \bar{T}$.

Now we prove the $ADCB$ Lemma, that is $\langle B, D\rangle = \langle D, B\rangle$.
First we suppose $|l(\cdot TB\bar{T}\cdot) - l(\cdot TD\bar{T}\cdot)| < \frac{\epsilon}{4}$.
We remark that from $I(\cdot T\cdot B\cdot\bar{T}\cdot) < \Delta$ we have

$$\begin{aligned} l(\cdot TB\bar{T}\cdot) - 2l(\cdot T\cdot) &\geq l(\cdot B\cdot) - \Delta \\ &\geq L - \Delta, \end{aligned} \tag{3.22}$$

So for L large enough we can found, by the Connection Lemma 3.2.2,

$$\cdot E\cdot \in \mathrm{Conn}_{\frac{\epsilon}{8}, l(\cdot TB\bar{T}\cdot)-2l(\cdot T\cdot)}(-i(\cdot T\cdot), i(\cdot T\cdot)).$$

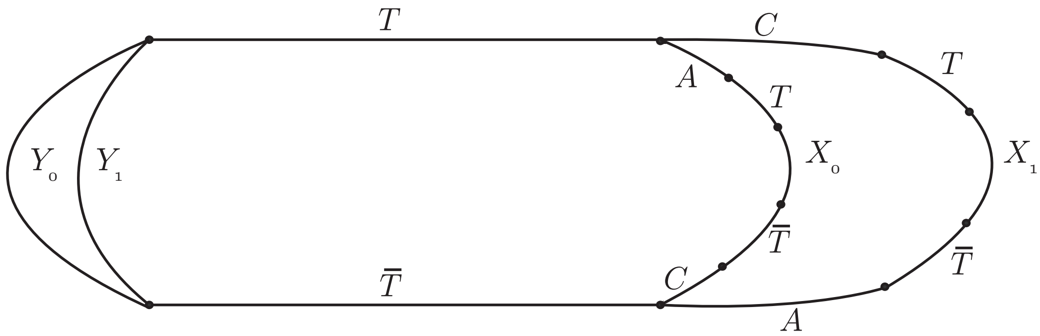

Figure 3.25: The equation (3.21) as an application of the Algebraic Square Lemma.

For any such E, by the Sum of Inefficiency for Angles 3.2.3 and remark 3.2.4, we have $I(\cdot T \cdot E \cdot \bar{T}\cdot) < \frac{\epsilon}{8} + O(\epsilon^2)$ and so $|l(\cdot TB\bar{T}\cdot) - l(\cdot TE\bar{T}\cdot)| < \frac{\epsilon}{4} + O(\epsilon^2)$. Since the fact that $[ATB\bar{T}CTD\bar{T}] \in \Gamma_{\epsilon,R}$ is taken as hypothesis, our estimates on E and the fact 3.2.17 say to us that $\langle B, E\rangle$,$\langle D, E\rangle$,$\langle E, B\rangle$ and $\langle E, D\rangle$ are in $\Gamma_{2\epsilon,R}$. We can use (3.21) to have

$$\langle B, D\rangle - \langle B, E\rangle - \langle D, B\rangle + \langle E, B\rangle = 0$$
$$\langle B, D\rangle - \langle E, D\rangle - \langle D, B\rangle + \langle D, E\rangle = 0$$
$$\langle D, E\rangle - \langle B, E\rangle - \langle E, D\rangle + \langle E, B\rangle = 0$$

in $H^{200\epsilon,R}(S)$. Hence we get

$$2\langle B, D\rangle - 2\langle D, B\rangle = 0$$

in $H^{200\epsilon,R}(S)$.

If we remove the assumption $|l(\cdot TB\bar{T}\cdot) - l(\cdot TD\bar{T}\cdot)| < \frac{\epsilon}{4}$, then let $k \in \mathbb{N}$ be the smallest natural such that

$$k > 4\frac{|l(\cdot TB\bar{T}\cdot) - l(\cdot TD\bar{T}\cdot)|}{\epsilon}.$$

We set

$$r_i = \frac{i}{2k}l(\cdot TD\bar{T}\cdot) + \frac{2k-i}{2k}l(\cdot TB\bar{T}\cdot) - 2l(\cdot T\cdot),$$

for $i = 0, 1, \ldots, 2k$. Since $r_i \geq L - \Delta$ (see equation 3.22) we can find

$$\cdot E_i \cdot \in \mathrm{Conn}_{\epsilon, r_i}(t(\cdot T \cdot), i(\cdot \bar{T} \cdot)).$$

In order to see that $\langle E_i, E_j \rangle$ is in $\Gamma_{2\epsilon, R}$ we need to verify that

$$|l(\cdot TB\bar{T}\cdot) + l(\cdot TD\bar{T}\cdot) - l(\cdot TE_i\bar{T}\cdot) - l(\cdot TE_j\bar{T}\cdot)| < \epsilon, \tag{3.23}$$

then fact 3.2.17 conclude the argument. From $I(\cdot T \cdot E_i \cdot \bar{T}\cdot) < \frac{\epsilon}{8} + O(\epsilon^2)$ and definitions of r_i we can prove inequality (3.23) for the cases $\langle E_i, E_{2k-i} \rangle$ and $\langle E_{i+1}, E_i \rangle$. This is sufficient to us for apply the Algebraic Square Lemma in the form (3.21), to obtain, in $H^{200\epsilon, R}$, the equations

$$\langle E_i, E_{2k-i} \rangle - \langle E_{i+1}, E_{2k-i} \rangle - \langle E_{2k-i}, E_i \rangle + \langle E_{2k-i}, E_{i+1} \rangle = 0,$$

for $0 \leq i < k$, and

$$\langle E_i, E_{2k-i+1} \rangle - \langle E_i, E_{2k-i} \rangle - \langle E_{2k-i+1}, E_i \rangle + \langle E_{2k-1}, E_i \rangle = 0,$$

for $0 < i \leq k$.
Adding these we get

$$\langle E_0, E_{2k} \rangle - \langle E_k, E_{k+1} \rangle - \langle E_{2k}, E_0 \rangle + \langle E_{k+1}, E_k \rangle = 0.$$

Now E_k and E_{k+1} can be used in the first part of the proof instead of B and D, since they satisfy the additional hypothesis:
$|l(\cdot TE_k\bar{T}\cdot) - l(\cdot TE_{k+1}\bar{T}\cdot)| < \epsilon/4$.
Then follows $\langle E_{k+1}, E_k \rangle - \langle E_k, E_{k+1} \rangle = 0$ in $H^{200\epsilon, R}$. So we also have $\langle E_0, E_{2k} \rangle = \langle E_{2k}, E_0 \rangle$. A last application of the ASL (3.21) with B, D, E_0 and E_{2k} gives the result.

□

Lemma 3.2.19 (Itemisation Lemma)**.** *There exists an $\hat{\epsilon} > 0$ such that the following holds. Let $\hat{\epsilon} > \epsilon > 0$, and let $\Delta > 0$. There exist $L = L(\epsilon, \Delta, S) > 0$ and $R_0 = R_0(\epsilon, \Delta, S)$ with the following properties. For every $R > R_0$ and for* *any* $A,B,C,D \in \pi_1(S,*)$ *such that* $l(\cdot T\cdot) > L$ *and* $I(\cdot A\cdot T\cdot B\cdot \bar{T}\cdot C\cdot T\cdot D\cdot \bar{T}\cdot) < \Delta$*, and that the curve* $[ATB\bar{T}CTD\bar{T}]$ *is in* $\Gamma_{\epsilon,R}$ *we have*

$$[ATB\bar{T}CTD\bar{T}] - [T\bar{D}\bar{T}\bar{C}T\bar{B}\bar{T}\bar{A}] = 2(A_{\bar{T}} + B_T + C_{\bar{T}} + D_T)$$

in $H^{200\epsilon,R}$*.*

Proof. It follow from the definition of A_T that

$$\begin{aligned}
[ATB\bar{T}CTD\bar{T}] - [\bar{A}TB\bar{T}CTD\bar{T}] &= 2A_{\bar{T}}\\
[\bar{A}TB\bar{T}CTD\bar{T}] - [\bar{A}T\bar{B}\bar{T}CTD\bar{T}] &= 2B_T\\
[\bar{A}T\bar{B}\bar{T}CTD\bar{T}] - [\bar{A}T\bar{B}\bar{T}\bar{C}TD\bar{T}] &= 2C_T\\
[\bar{A}T\bar{B}\bar{T}\bar{C}TD\bar{T}] - [\bar{A}T\bar{B}\bar{T}\bar{C}T\bar{D}\bar{T}] &= 2D_{\bar{T}}
\end{aligned}$$

in $H^{100\epsilon,R}(S)$ homology. So this plus the $ADCB$-Lemma 3.2.18 gives the result in $H^{200\epsilon,R}(S)$.

□

Remark 3.2.6. We notice that follows from the Itemization Lemma that

$$[TA\bar{T}BTC\bar{T}D] = A_T + B_{\bar{T}} + C_T + D_{\bar{T}}$$

in the $H^{200\epsilon,R}$. It is sufficient to see that

$$[TA\bar{T}BTC\bar{T}D] = \frac{1}{2}([TA\bar{T}BTC\bar{T}D] - [\bar{D}T\bar{C}\bar{T}\bar{B}T\bar{A}\bar{T}]).$$

3.3 The XY-Theorem and a Proof for the Good Pants Homology Theorem

In the following we call θ-graph an immersed spine of a pants with the two vertex coinciding. In particular we consider θ-graphs whose vertex is $*$ and the three arcs will be $\cdot X\cdot$, $\cdot Y\cdot$ and $\cdot Z\cdot$ for X,Y,$Z \in \pi_1(S,*)$. Then these θ-graph will generates an immersed pair of pants with cuffs $[XY]$,$[YZ]$ and $[ZX]$ if and only if the triples of unit vectors $i(\cdot X\cdot)$, $i(\cdot Y\cdot)$, $i(\cdot Z\cdot)$ and $t(\cdot X\cdot)$,$t(\cdot Y\cdot)$,$t(\cdot Z\cdot)$ have opposite cyclic orderings. We are going to prove the XY-theorem as a corollary of te two Rotation Lemmas.

Lemma 3.3.1 (First Rotation Lemma)**.** *There exists an* $\hat{\epsilon} > 0$ *such that the following holds. Let* $\hat{\epsilon} > \epsilon > 0$*, and let* $\Delta > 0$*. There exist constants* $K = K(\epsilon, \Delta) > 0$ *and* $R_0 = R_0(\epsilon, \Delta, S) > 0$ *with the following properties. For every* $R > R_0$ *let* W_i,S_i,$T \in \pi_1(S,*)$*,* $i = 0$,1,2*, and such that*
(1) $I(\cdot T\cdot W_i \cdot \bar{W}_{i+1} \cdot \bar{T}\cdot)$,$I(\cdot T\cdot S_i \cdot \bar{S}_{i+1} \cdot \bar{T}\cdot) < \Delta$*,*
(2) $l(\cdot T\cdot) \geq K$*,*
(3) $l(\cdot W_i\cdot) + l(\cdot S_i\cdot) + 2l(\cdot T\cdot) < R - K$*,*
(4) *The two triples of vectors* $(t(\cdot TW_i\cdot))$ *and* $(t(\cdot TS_i\cdot))$*, for* $i = 0$,1,2*, have opposite cyclic ordering in* $T^1_* S$*.*
Then
$$\sum_{i=0}^{2}(W_{i+1}\bar{W}_i)_T + \sum_{i=0}^{2}(S_i\bar{S}_{i+1})_T = 0$$
in $H^{300\epsilon,R}(S)$*.*

Proof. Hypothesis (1) and (3) let us ask that the system of equations

$$r_i + r_{i+1} = 2R - l(\cdot TW_{i+1}\bar{W}_i\bar{T}\cdot) - l(\cdot TS_i\bar{S}_{i+1}\bar{T}\cdot) \qquad (3.24)$$

for $i = 0$,1,2, has non-negative solutions r_i's, since the right hand is bigger then $2K - 2\Delta$. Take an arbitrary $A_i \in \mathrm{Conn}_{\epsilon,r_i}(-i(\cdot T\cdot), i(\cdot T\cdot))$ (such A_i exists provided K sufficiently large). We will show that the θ-graph with arcs $\cdot\bar{W}_i\bar{T}A_iTS_i\cdot$ generates an immersed pair of pants Π_A. The three cuffs will be the closed curves $[\bar{W}_{i+1}\bar{T}A_{i+1}TS_{i+1}\bar{S}_i\bar{T}\bar{A}_iTW_i]$ and we will also show that $\Pi_A \in \Pi_{3\epsilon,R}$. The statement will come from the equation $\partial\Pi_A = 0$ applying the Itemization Lemma 3.2.6 and noticing that $(A_i)_{\bar{T}} = -(\bar{A}_i)_{\bar{T}}$:

$$
\begin{aligned}
0 = \partial\Pi_A &= \sum_{i=0}^{2}[W_{i+1}\bar{W}_i\bar{T}A_iTS_i\bar{S}_{i+1}\bar{T}\bar{A}_{i+1}T] \\
&= \sum_{i=0}^{2}((W_{i+1}\bar{W}_i)_T + (A_i)_{\bar{T}} + (S_i\bar{S}_{i+1})_T + (\bar{A}_{i+1})_{\bar{T}}) \\
&= \sum_{i=0}^{2}(W_{i+1}\bar{W}_i)_T + \sum_{i=0}^{2}(S_i\bar{S}_{i+1})_T.
\end{aligned}
$$

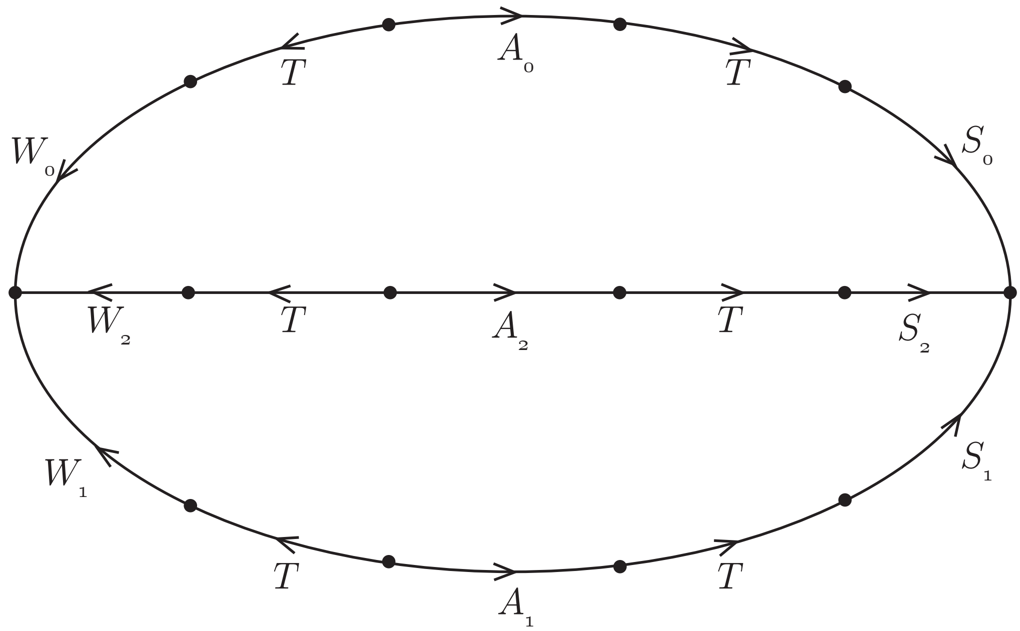

Figure 3.26: The θ-graph for the First Rotation Lemma

We now verify that $[\bar{W}_{i+1}\bar{T}A_{i+1}TS_{i+1}\bar{S}_i\bar{T}\bar{A}_iTW_i] \in \Gamma_{3\epsilon,R}$. By the New Angle Lemma 3.1.3, provided K sufficiently large (and then $l(\cdot T\cdot)$ sufficiently large by (2)) we have

$$\Theta(i(\cdot T \cdot W_{i+1} \cdot \bar{W}_i \cdot \bar{T}\cdot), i(\cdot TW_{i+1}\bar{W}_i\bar{T}\cdot)) < \frac{\epsilon}{2},$$

and
$$\Theta(t(\cdot T\cdot W_{i+1}\cdot \bar{W}_i\cdot \bar{T}\cdot), t(\cdot TW_{i+1}\bar{W}_i\bar{T}\cdot)) < \frac{\epsilon}{2}.$$
Same inequalities are satisfied replacing the W_1's with the S_i's. We also note that $i(\cdot T\cdot W_{i+1}\cdot \bar{W}_i\cdot \bar{T}\cdot) = i(\cdot T\cdot)$ as well $t(\cdot T\cdot W_{i+1}\cdot \bar{W}_i\cdot \bar{T}\cdot) = t(\bar{T})$. Again we have the same for S_i.
It follows that $\Theta(t(\cdot \bar{A}_{i+1}\cdot), i(\cdot TW_{i+1}\bar{W}_i\bar{T}\cdot)) < 2\epsilon$ since $t(\cdot \bar{A}_{i+1}\cdot) = -i(\cdot A\cdot_{i+1})$ and $\Theta(-i(\cdot A\cdot_{i+1}), i(\cdot T\cdot)) < \epsilon$ by definition of A_{i+1}. By the same reasons we have that all the internal angles of the closed piecewise geodesic arc $[\cdot TW_{i+1}\bar{W}_i\bar{T}\cdot A_i\cdot TS_{i+1}\bar{S}_i\bar{T}\cdot \bar{A}_{i+1}\cdot]$ are smaller then 2ϵ. Since equation (3.24) assure that r_i can be take large enough taking K enough large, we can apply the Sum of Inefficiency Lemma for Angles (see Remark 3.2.3) to have (see also remark 3.2.4)

$$|I([\cdot TW_{i+1}\bar{W}_i\bar{T}\cdot A_i\cdot TS_{i+1}\bar{S}_i\bar{T}\cdot \bar{A}_{i+1}\cdot]) - 4\log\sec(2\epsilon)| < \frac{\epsilon}{2},$$
that is
$$I([\cdot TW_{i+1}\bar{W}_i\bar{T}\cdot A_i\cdot TS_{i+1}\bar{S}_i\bar{T}\cdot \bar{A}_{i+1}\cdot]) < \frac{\epsilon}{2} + O(\epsilon^2).$$
On the other hand equation (3.24) gives

$$|2R - l(\cdot TW_{i+1}\bar{W}_i\bar{T}\cdot) - l(\cdot TS_i\bar{S}_{i+1}\bar{T}\cdot) - l(\cdot A_i\cdot) - l(\cdot A_{i+1}\cdot)| < 2\epsilon$$

Last two estimates together provide that the curves in question are in $\Gamma_{3\epsilon,R}$. Now we verify that the three geodesic arcs we have considered generates an immersed pair of pants Π_A. Recall that $I(\theta) = 2\log\sec\frac{\theta}{2}$ for every angle $\theta\in[0,\pi]$, by definition. So we can fix the unique θ_0 such that $I(\theta_0) = \Delta + 1$. By monotonicity and (1) we have that $I(\cdot\bar{W}_i\cdot\bar{T}\cdot) < I(\cdot T\cdot W_{i+1}\cdot \bar{W}_i\cdot \bar{T}\cdot) < \Delta$, from which follows $l(\cdot\bar{W}_i\bar{T}\cdot) > l(\cdot T\cdot) - \Delta > K - \Delta$. Follows that $l(\cdot\bar{W}_i\bar{T}\cdot)$ grows with K. So we can apply again the Sum of Inefficiency Lemma for Angles:

$$|I(\cdot TW_{i+1}\cdot \bar{W}_i\bar{T}\cdot) - I(\Theta(i(\cdot\bar{W}_i\bar{T}\cdot), i(\cdot\bar{W}_{i+1}\bar{T}\cdot)))| < 1,$$

which, together with

$$I(\cdot TW_{i+1}\cdot \bar{W}_i\bar{T}\cdot) \le I(\cdot T\cdot W_{i+1}\cdot \bar{W}_i\cdot \bar{T}\cdot) < \Delta,$$

gives

$$I(\Theta(i(\cdot\bar{W}_i\bar{T}\cdot), i(\cdot\bar{W}_{i+1}\bar{T}\cdot))) < \Delta + 1,$$

that is $\Theta(i(\cdot\bar{W}_i\bar{T}\cdot), i(\cdot\bar{W}_{i+1}\bar{T}\cdot)) > \theta_0$.

On the other hand from the New Angle Lemma (recall that $l(\cdot\bar{W}_i\bar{T}\cdot)$ is large provided K large) we have

$$\Theta(i(\cdot\bar{W}_i\bar{T}\cdot), i(\cdot\bar{W}_i\bar{T}A_iTS_i\cdot) < \frac{\theta_0}{2}.$$

So the three vectors $i(\cdot\bar{W}_i\bar{T}A_iTS_i\cdot)$ are alternatively bounded by the three vectors $i(\cdot\bar{W}_{i+1}\bar{T}\cdot)$ and then obligated in the same cyclic order. The same argument shows the same property for the terminal vectors $t((\cdot\bar{W}_i\bar{T}A_iTS_i\cdot))$ and $t(\cdot TS_i\cdot)$. This, together assumption (4), complete the proof since now Π_A is effectively a pair of pants. □

Remark 3.3.1. If the two triples of vectors $(t(\cdot TW_i\cdot))$ and $(t(\cdot TS_i\cdot))$, for i = 0,1,2, have the same cyclic order in T^1_*S, we can permute the W_i's in another cyclic order, then apply the lemma gives the equation

$$\sum_{i=0}^{2}(W_i\bar{W}_{i+1})_T + \sum_{i=0}^{2}(S_i\bar{S}_{i+1})_T = 0.$$

In particular the lemma can be applied to the case $W_i = S_i$.

Lemma 3.3.2 (Second Rotation Lemma)**.** *There exists a universal constant $\hat{\epsilon} > 0$ such that the following holds. Let $\hat{\epsilon} > \epsilon > 0$, and let $\Delta > 0$. There exist $K = K(\epsilon, \Delta) > 0$ and $R_0 = R_0(\epsilon, \Delta, S) > 0$ with the following properties. Let $R > R_0$ and let W_i,$T \in \pi_1(S, *)$, i = 0,1,2 such that*

(1) $I(\cdot T \cdot W_i \cdot \bar{W}_{i+1} \cdot \bar{T}\cdot) < \Delta$,

(2) $l(\cdot T\cdot) > K$.

Then

$$\sum_{i=0}^{2}(W_i\bar{W}_{i+1})_T = 0,$$

in $H^{300\epsilon,R}(S)$.

Proof. Let $v \in T^1_* S$ and let $\rho = e^{\frac{2\pi i}{3}}$. By the Connection Lemma for L sufficiently large $\mathrm{Conn}_{\epsilon,L}(t(\cdot T\cdot),(\rho^i v))$ is non empty for every $i = 0{,}1{,}2$. Choose $\cdot S_i \cdot \in \mathrm{Conn}_{\epsilon,L}(t(\cdot T\cdot),(\rho^i v))$. By the Sum of Inefficiency Lemma for Angles (see 3.2.3)

$$|I(\cdot T \cdot S_i \cdot \bar{S}_{i+1} \cdot \bar{T}\cdot) - 2\epsilon - I(\frac{\pi}{3}) - \epsilon| < 2\epsilon,$$

that is $I(\cdot T \cdot S_i \cdot \bar{S}_{i+1} \cdot \bar{T}\cdot) \leq O(\epsilon) + \log(\frac{4}{3}) \leq 1$, if ϵ is sufficiently small (this implicitly defines $\hat{\epsilon}$). So we are in condition to apply the First Rotation Lemma 3.3.1 with our S_i's as both the W_i's and the S_i's of that statement (see also remark 3.3.1). Then we have

$$2\sum_{i=0}^{2}(S_i\bar{S}_{i+1})_T = 0$$

in the $H^{300\epsilon,R}(S)$.

Now we reuse the First Rotation Lemma with both W_i's and S_i's from this lemma to have

$$\sum_{i=0}^{2}(W_i\bar{W}_{i+1})_T + \sum_{i=0}^{2}(S_i\bar{S}_{i+1})_T = 0,$$

that imply

$$2\sum_{i=0}^{2}(W_i\bar{W}_{i+1})_T = 0.$$

Note that we can be both in the situation of the First Rotation Lemma or in the situation of Remark 3.3.1 for the order of tangent vectors to the W_i's, however we can cancel the S_i's in both of them and then reorder the W_i's as in the statement. □

Theorem 3.3.3 (The XY Theorem)**.** *There exists a universal constant $\hat{\epsilon} > 0$ such that the following holds. Let $\hat{\epsilon} > \epsilon > 0$, and let $\Delta > 0$. There exist $K = K(\epsilon,\Delta) > 0$ and $R_0 = R_0(\epsilon,\Delta,S) > 0$ with the following properties. Let $R > R_0$, and let X,Y,$T \in \pi_1(S,*)$, such that*
(1) $I(\cdot T \cdot X \cdot Y \cdot \bar{T}\cdot)$,$I(\cdot T \cdot X \cdot \bar{T}\cdot)$,$I(\cdot T \cdot Y \cdot \bar{T}\cdot) < \Delta$,
(2) $l(\cdot T\cdot) \geq K$.
Then

$$(XY)_T = X_T + Y_T$$

in $H^{300\epsilon,R}$.

Proof. It is simply the Second Rotation Lemma with $W_0 = *$, $W_1 = X$ and $W_2 = \bar{Y}$. □

Lemma 3.3.4 (Good Direction Lemma)**.** *Let* $W \subseteq \pi_1(S,*)$ *be a finite subset. Then there exist* $\Delta = \Delta(S,W) > 0$*,* $L_0 = L_0(W,S) > 0$ *with the following properties. For every* $L > L_0$ *there exists* $T \in \pi_1(S,*)$ *with* $l(\cdot T \cdot) > L$ *such that* $I(\cdot T \cdot X \cdot \bar{T} \cdot) < \Delta$ *for every* $X \in W$*.*

Proof. Given $v \in T^1_* S$ and $t > 0$, we write $\alpha(tv)$ for the geodesic segment starting at $*$ in the direction v and of length t. Then we write $\alpha^{-1}(tv)$ for the same geodesic segment with the opposite orientation.
Let $X \in \pi_1(S,*)$ $X \neq *$ (the identity of the group). We consider an arc $\tilde{X}$ within two lifts of $*$ on lift of X in $\mathbb{H}^2$. Then we also have two different lifts of v, called $\tilde{v}_1$ and $\tilde{v}_2$ in the two endpoints of $\tilde{X}$. Then the geodesic arcs $\alpha(t\tilde{v}_i)$, $i = 1{,}2$, are isometric lifts of $\alpha(tv)$. We claim that the equation

$$\lim_{t\to\infty} I(\alpha^{-1}(tv) \cdot X \cdot \alpha(tv)) = \infty \tag{3.25}$$

is satisfied if and only if the endpoint $\alpha_\infty(\tilde{v}_1)$ in $\partial\mathbb{H}^2$ is the same of $\alpha_\infty(\tilde{v}_2)$.
Let γ be the geodesic arcs on $\mathbb{H}^2$ with endpoints $\alpha_\infty(\tilde{v}_1)$ and $\alpha_\infty(\tilde{v}_2)$. If such two points are the same then the inefficiency written above is infinite. Suppose such points different. Let $t > 0$, and consider, on $\mathbb{H}^2$, the piecewise geodesic arc $\alpha^{-1}(t\tilde{v}_1) \cdot X \cdot \alpha(t\tilde{v}_2)$. Then consider the geodesic arc O starting at the midpoint of $\tilde{X}$ and ending orthogonal to γ at the pint p. Let λ be the half length of $\tilde{X}$, let $S(t)$ be the geodesic segment between p and the endpoints of $\alpha(t\tilde{v}_2)$ which is not a lift of $*$. To prove the claim we are left to prove that he quantity $t + \lambda + l(O) - l(S(t))$ doesn't go to infinity when $t \to \infty$. To see that let $t_0 > 0$ such that the geodesic arc $S(t_0)$ is orthogonal to $\alpha(t\tilde{v}_2)$, and denote with θ the angle between $S(t_0)$ and $S(t)$. Then the hyperbolic sine rule gives

$$\sinh(l(S(t))) \geq \frac{\sinh(t - t_0)}{\sin\theta} \geq \sinh(t - t_0).$$

Then, using the monotonicity of sinh, we have that $S(t) \geq t - t_0$ and then $t + \lambda + l(O) - l(S(t))$ has to be bounded, since it can't be negative. This ends the proof of the previous claim.
The map from $T^1_* S$ to $\partial\mathbb{H}^2$ that send v to the endpoint at infinity $\alpha_\infty(\tilde{v})$, for a fixed lift $\tilde{v}$, is a Möbius transformation M, since that map is simply the identity if we lift $*$ to the centre of the unit disk model, and compose with a change of coordinates for $T^1\mathbb{H}^2$ in other cases.
Let M_i, $i = 1{,}2$ the Möbius transformation $T^1_* S \ni v \mapsto \alpha_\infty(\tilde{v}_i) \in \partial\mathbb{H}^2$. Define $M = M_2 M_1^{-1}$. Then (3.25) holds if and only if $\alpha_\infty(\tilde{v}_1)$ is a fixed point of M. Since M has at most two fixed points, or it is the identity, in order to prove that (3.25) is satisfied at most by two different v it is sufficient to prove that M is not the identity. By contradiction, suppose M the identity.
Since $\cdot X \cdot$ is a closed geodesic, we have $i(\cdot X \cdot) = t(\cdot X \cdot)$. Then we can consider $p \in \partial\mathbb{H}^2$ such that $M_1(\sqrt{-1} i(\cdot X \cdot)) = p$. Then from the equation $Mp = p$ we deduce that $M_2(\sqrt{-1} t(\cdot X \cdot)) = M_2(\sqrt{-1} i(\cdot X \cdot)) = p$. In the universal cover this last equation means that we have constructed a triangle (with vertices p and the two lifts of $*$) with sum of internal angles equal to π and two angle exactly equal to $\Pi/2$. But it is impossible. So $M \neq \mathrm{id}$ and (3.25) has only two solutions in v. We call such two directions the bad directions of X.
Then there are at most $2|W|$ bad directions for elements of W. Now the map

$$v \mapsto \lim_{t \to \infty} I(\alpha^{-1}(tv) \cdot X \cdot \alpha(tv))$$

is continuous as a map from $T^1_* S$ to $[0, \infty]$. Then we have that for any closed connected subset J of $T^1_* S$ disjoint from the set of the bad directions of W, there exists $\Delta > 0$ such that $I(\alpha^{-1}(tv) \cdot X \cdot \alpha(tv)) < \Delta$ for any $X \in W$, $t > 0$ and $v \in J$. We have proved that such a J exists.
Then let v_0 be the midpoint of J, and δ the half length of J. Let $\cdot T \cdot \in \mathrm{Conn}_{\delta, L+1}(v_0, v_0)$ (this determines L_0), then $L < l(\cdot T \cdot) < L + 2$ and $I(\cdot T \cdot X \cdot \bar{T} \cdot) < \Delta$ for all $X \in W$ □

Let n be the genus of S. Let $g_1, \ldots, g_{2n} \in \pi_1(S, *)$ be generators of the fundamental group. Then $[g_i]$ denote the closed curves corresponding to g_i.

We denote with $H_1(S)$ the first homology group of S with rational coefficients. For any oriented closed curve $\gamma \subset S$ exist unique $a_1,\dots,a_{2n} \in \mathbb{Q}$ such that $[\gamma] = \sum a_i[g_i]$ in H_1. So, defined Γ as the set of all oriented closed curves of S, we can define $q : \Gamma \longrightarrow \mathbb{R}\{g_1, \dots g_{2n}\}$ as $q(\gamma) = \sum a_i g_i$. Finally we can extend the definition to $q : \pi_1(S, *) \longrightarrow \mathbb{R}\{g_1, \dots g_{2n}\}$ by $q(X) = q([X])$.
Given $X \in \pi_1(S, *)$ the word length of X is the minimal number of generator g_i (counted with the multiplicity) needed to write X. For $h \in \mathbb{N}$, W_h is the subset of $\pi_1(S, *)$ of element with word length less or equal then h.

Theorem 3.3.5 (Good Pants Homology for Short Words)**.** *There exists a universal constant $\hat{\epsilon} > 0$ such that the following holds. Let $\hat{\epsilon} > \epsilon > 0$. For any $h \in \mathbb{N}$ there exists an $L_0(\epsilon, S) > 0$ such that for any $L > L_0$, exist $T \in \pi_1(S, *)$ and $R_0 > 0$ such that $l(\cdot T \cdot) > L$ and for every $R > R_0$ and every $X \in W_h$ we have*

$$X_T = (q(X))_T$$

in $H^{300\epsilon,R}(S)$.

Proof. Let $L_0(W_h, S)$, $\Delta = \Delta(W_h)$ and $T = T(W_h, L)$ given by the Good Direction Lemma 3.3.4. Then $l(\cdot T \cdot) > L$ and $I(\cdot T \cdot X \cdot \bar{T} \cdot) < \Delta$ for all $X \in W_h$.
If $X \in W_1$ then $q(X) = X$ or $q(X) = -\bar{X}$, therefore $(q(X))_T = (X)_T$. Suppose $X \in W_{k+1}$ and the theorem verified in W_i for $i \leq k$. Then $X = g_i^\sigma Y$ for some $i = 1,\dots,2n$, $Y \in W_k$ and $\sigma = \pm 1$. Then the Good Direction Lemma assure that we are satisfying the assumption of the XY-Theorem 3.3.3, so we can write $X_T = (g_i^\sigma)_T + Y_T$. Now we have the theorem verified for both the terms on the right. By induction we conclude the proof. □

Lemma 3.3.6 (First Cut Lemma)**.** *There exists a universal constant $\hat{\epsilon} > 0$ such that for every $\hat{\epsilon} > \epsilon > 0$ there exist an $L = L(\epsilon, S) > 0$, and an $R_0 = R_0(\epsilon, S) > 0$ with the following properties.*
For any $\gamma \in \Gamma_{\epsilon,R}$ and $T \in \pi_1(S, *)$ with $l(\cdot T \cdot) > L$ we can find X_0, $X_1 \in \pi_1(S, *)$ such that for any $R > R_0$
(1) $|l(\cdot X_i \cdot) - (R + 2L - \log 4)| < \frac{1}{2}$,

(2) $\Theta(t(\cdot T\cdot), i(\cdot X_i\cdot)), \Theta(t(\cdot X_i\cdot), i(\cdot\bar{T}\cdot)) < \frac{\pi}{6}$,
(3) $\gamma = (X_0)_T + (X_1)_T$ *in* $H^{300\epsilon,R}(S)$ *homology.*

Proof. Let $l = l(\gamma)/2$ the half length of γ. We think to γ as an isometric map $\gamma : S^1_\gamma \longrightarrow S$ where S^1_γ is $\frac{[-l,l]}{\{-l,l\}}$ (simply a circumference of length $2l$). Take two points x_0 and x_1 in S^1_γ at distance l and let $w_i \in T^1_{\gamma(x_i)}S$ be $-\sqrt{-1}\gamma'(x_i)$. We let γ_i be the subsegment of γ from x_i to x_{i+1}.
Let $\alpha_i \in \mathrm{Conn}_{\frac{\epsilon}{10},L}(t(\cdot T\cdot), w_i)$ where $L = L(\epsilon, S) > 0$ is determined in the Connection Lemma by the fact that the set is non empty.

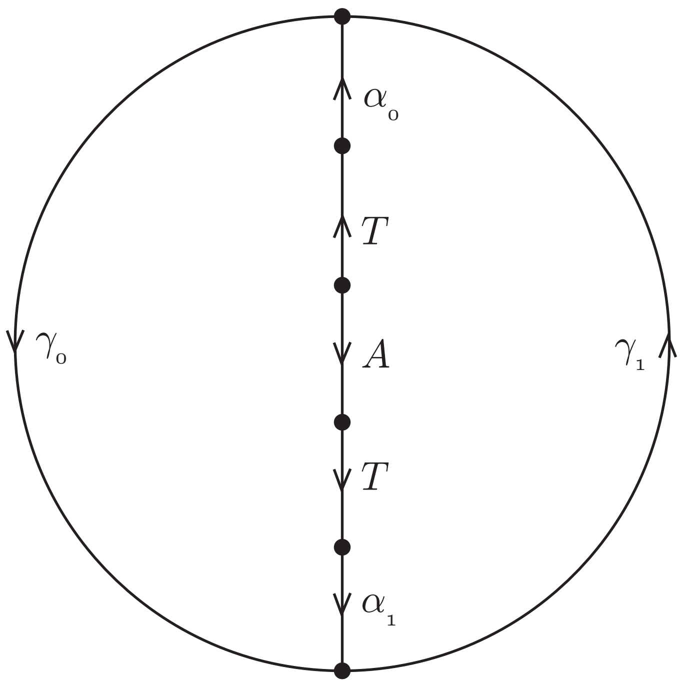

Figure 3.27: The First Cut Lemma

Then let $X_0 \in \pi_1(S, *)$ be the element corresponding to $\alpha_0\gamma_0\alpha_1^{-1}$ as well $X_1 \in \pi_1(S, *)$ corresponds to $\alpha_1\gamma_1\alpha_0^{-1}$. The property (1) follows the Sum of Inefficiency Lemma for Angles with $\delta = 1/4 - \frac{\epsilon}{2}$ (we suppose $\hat{\epsilon} < 1/2$) for the inefficiency of $I(\alpha_0\gamma_0\alpha_1^{-1})$ and $I(\alpha_1\gamma_1\alpha_0^{-1})$. In fact we have, for L and R

sufficiently large (in particular this define the R_0 of the statement)

$$|l(\cdot X_i \cdot) - 2L + \frac{\epsilon}{5} - R + \frac{\epsilon}{2} + 4\log\sec(\frac{\pi}{4} + \frac{\epsilon}{2})| < \frac{1}{2} - \epsilon,$$

that for ϵ sufficiently small gives (1). We notice that ϵ sufficiently small means that there exists an $\hat{\epsilon}$ bounding ϵ above.
It also follows from (1) that $I(\alpha_0\gamma_0\alpha_1^{-1})$ is bounded above. Since $l(\alpha_i) > L$, by the New Angle Lemma we have $\Theta(i(\cdot X_0 \cdot), i(\alpha_0))$ small as L growth. Combined with the inequality $\Theta(i(\alpha_0), t(\cdot T \cdot)) < \frac{\epsilon}{10}$ we can conclude (2) provided ϵ sufficiently small and L sufficiently large (the other case are similarly).
Let $\cdot A \cdot \in \mathrm{Conn}_{\frac{\epsilon}{10}, R+\log 4-2L-2l(\cdot T \cdot)}(-i(\cdot T \cdot), i(\cdot T \cdot))$. Again we use the Sum of Inefficiency Lemma for Angles for $I([\cdot X_0 \cdot \bar{T} \cdot \bar{A} \cdot T \cdot])$ and $I([\cdot X_1 \cdot \bar{T} \cdot A \cdot T \cdot])$ to have

$$|l([X_0\bar{T}\bar{A}T]) - 2R| < 2\epsilon,$$

$$|l([X_1\bar{T}AT]) - 2R| < 2\epsilon,$$

where we stress that the left side only contains the inefficiency of the piecewise geodesic, while the right side absorb also the inefficiency of the angles: in fact these depends only on the choice of ϵ, while the constant in the estimate of the Sum of Inefficiency Lemma can be reduced taking larger R and L.
Then $\gamma = [X_0\bar{T}\bar{A}T] + [X_1\bar{T}AT]$ in $H^{\epsilon,R}(S)$, since by construction they are the three cuffs of a pair pants. Now the Simple Itemization Lemma 3.2.16 gives $[X_0\bar{T}\bar{A}T] = (X_0)_T + (\bar{A})_{\bar{T}}$ and $[X_1\bar{T}AT] = (X_1)_T + A_{\bar{T}}$ in the $H^{100\epsilon,R}$.
The fact that $(\bar{A})_{\bar{T}} = -A_{\bar{T}}$ let us conclude

$$\gamma = (X_0)_T + (X_1)_T$$

in the $H^{300\epsilon,R}$ homology.

□

Definition 3.3.1. For T,$X \in \pi_1(S,*)$ with $X \neq *$ we define

$$\theta_X^T = \max\{\Theta(t(\cdot T \cdot), i(\cdot X \cdot)), \Theta(t(\cdot X \cdot), i(\cdot \bar{T} \cdot))\}$$

Lemma 3.3.7 (Second Cut Lemma)**.** *There exists a constant $L_0 = L_0(S) > 0$ such that for any $L > L_0$ and X,$T \in \pi_1(S,*)$ with $X \neq *$, we can write $X = X_0X_1$, for X_0,$X_1 \in \pi_1(S,*)$ such that*
(1) $|l(\cdot X_i \cdot) - (l(\cdot X \cdot)/2 + L - \log 2)| < \frac{1}{2}$,
(2) $I(\cdot X_0 \cdot X_1 \cdot) \leq 2L + 3$,
(3) $\theta^T_{X_i} \leq \max\{\theta^T_X + e^{L+4-l(\cdot X_i \cdot)}, \frac{\pi}{6}\}$

Proof. Let $\alpha : [0, l(\cdot X \cdot)] \longrightarrow S$ be a unit speed parametrization with $\alpha(0) = \alpha(l(\cdot X \cdot)) = *$. Fix $y = \frac{l(\cdot X \cdot)}{2}$. Then, by the Connection Lemma, there exists $L_0 > 0$ such that for every $L > L_0$ we can find

$$\beta \in \mathrm{Conn}_{\frac{1}{20},L}(t(\cdot T \cdot), \sqrt{-1}\alpha'(y)).$$

We observe that $\alpha[0, y]\beta^{-1}$ represent an $X_0 \in \pi_1(S,*)$ as well $\beta\alpha[0, l(\cdot X \cdot)]$ represent an $X_1 \in \pi_1(S,*)$. Obviously $X = X_0X_1$.

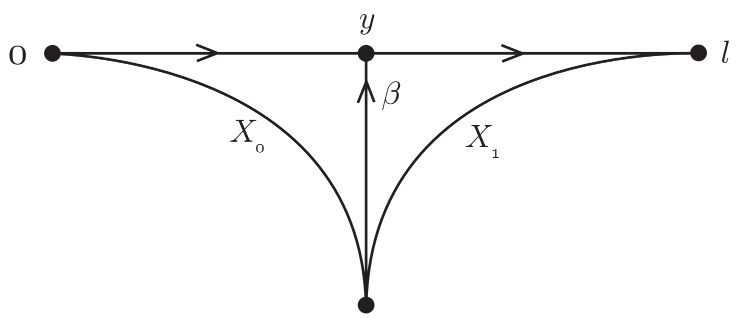

Figure 3.28: The Second Cut Lemma.

Now we use the Long Segment Lemma 3.1.5 for Angles to $I(\alpha[0, y]\beta^{-1})$ and $I(\beta\alpha[0, l(\cdot X \cdot)])$, which prove (1): note that the δ of the lemma absorb the errors of the lengths and the errors of the right angles, after that we choose the appropriate δ to have $\frac{1}{2}$ in the right side of the inequality.
The property (2) follows immediately from the definition of $I(\cdot X_0 \cdot X_1 \cdot)$ and the property (1).

Let $\theta = \Theta(i(\cdot X\cdot), i(\cdot X_0\cdot))$. By the hyperbolic law of sine $\sin\theta = \frac{\sinh(l(\beta))}{\sinh(l(\cdot X_0\cdot))}$ that gives, provided $l(\cdot X_0\cdot) > L+1$,

$$\sin\theta \leq \frac{\sinh(L+1)}{\sinh(l(\cdot X_0\cdot))} \leq e^{L+2-l(\cdot X_0\cdot)}$$

since $\frac{\sinh x}{\sinh y} \leq e^{x-y}$ if $x \leq y$. Therefore (using the fact that for θ small $\sin\theta \geq \frac{\theta}{e^2}$),

$$\Theta(t(\cdot T\cdot), i(\cdot X_0\cdot) \leq \Theta(t(\cdot T\cdot), i(\cdot X\cdot)) + e^{L+4-l(\cdot X_0\cdot)}.$$

Let $\varphi = \Theta(t(\cdot X_0\cdot), -i(\beta))$ then

$$\sin\varphi \leq \frac{\sinh(l(\cdot X\cdot)/2)}{\sinh(l(\cdot X_0\cdot))} \leq e^{2-L} < \frac{\pi}{12},$$

for L large enough. Then, since $t(\cdot\bar{T}\cdot) = -i(\cdot T\cdot)$ and $\Theta(-i(\beta), t(\cdot T\cdot)) < \frac{1}{20} < \frac{\pi}{12}$ we can conclude $\Theta(t(\cdot X_0\cdot), i(\cdot\bar{T}\cdot) \leq \frac{\pi}{6}$. The same argument apply for X_1. So also (3) is proved □

The following is essentially the Good Pants Homology theorem. We recall that given $g \in \pi_1(S,*)$, then $(g)_T \in \Gamma_{\epsilon,R}$

Theorem 3.3.8. *Let n be the genus of S and $g_1,\dots,g_{2n}$ be generators for $\pi_1(S,*)$.*
There exists an $\hat{\epsilon} > 0$ such that the following holds. Let $\hat{\epsilon} > \epsilon > 0$. There exist $R_0 = R_0(S,\epsilon) > 0$ and $T \in \pi_1(S,*)$, where T depends only on ϵ and S, with the following properties. For every $R > R_0$ and $\gamma \in \Gamma_{\epsilon,R}$ there exist $a_i \in \mathbb{Q}$, $i = 1,\dots,2n$, such that we have

$$\gamma = \sum_{i=1}^{2n} a_i (g_i)_T$$

in $H^{300\epsilon,R}(S)$.

Proof. The constant $\hat{\epsilon}$ is determined by the XY-Theorem and the First Cut Lemma. Let $\hat{\epsilon} > \epsilon > 0$. Let $L = L(\epsilon, S) > 0$ the maximum of the two homonymous constants from the two Cut Lemmas 3.3.6 and 3.3.7. Consider the $X \in \pi_1(S,*)$ such that $l(\cdot X\cdot) < 2L+5$, then there must exists

an $h \in \mathbb{N}$ such that $X \in W_h$ where W_h is the set of word long a most h, and so is finite for every h. By theorem 3.3.5 we find a $T \in \pi_1(S,*)$ with $l(\cdot T \cdot) > \max\{L, K(\epsilon, 2L+3)\}$, where the K is the homonymous constant determined by Theorem 3.3.3, and such that $X_T = (q(X))_T$ for $X \in W_h$. We take $R > R_0(S,\epsilon)$ with R_0 the constant determined by the First Cut Lemma 3.3.6, and we also ask $R > R_0(L,T,\epsilon)$ for R_0 determined by Theorem 3.3.5. Let $\gamma \in \Gamma_{\epsilon,R}$.
By the First Cut Lemma 3.3.6 we can find X_0,$X_1 \in \pi_1(S,*)$ satisfying assumptions $(1)-(2)$ of the Lemma, and $\gamma = (X_0)_T + (X_1)_T$ in $H^{300\epsilon,R}(S)$. Since the co-domain of q is abelian and recalling the definition of X_T (see 3.2.15), we have $q(\gamma) = q((X_0)_T + (X_1)_T) = q(X_0) + q(X_1)$. Then we use the Second Cut Lemma 3.3.7 to have $X_0 = X_{00}X_{01}$, where $l(\cdot X_{0i} \cdot) \in [\frac{R}{2} + 2L, \frac{R}{2} + 2L + 1]$ by Hypothesis (1) of both Lemmas 3.3.6 and 3.3.7. Same Decomposition can be done for X_1. Let $\chi_0 = \{X_0, X_1\}$, $\chi_1 = \{X_{00}, X_{01}, X_{10}, X_{11}\}$ and then iterate the definition applying the Second Cut Lemma 3.3.7 at every step, so that $\chi_k = \{X_{i_0,\dots,i_k} : i_j = 0,1\}$. We are interested in χ_k with $k \leq \lfloor \log_2 R \rfloor - 1$. Observe that $|\chi_k| = 2^{k+1}$.
Let $X \in \chi_k$. We have the recursive formula for the length

$$|l(\cdot X_{i_0,\dots,i_k} \cdot) - (\frac{l(\cdot X_{i_0,\dots,i_{k-1}} \cdot)}{2} + L - \log 2)| < \frac{1}{2}$$

which gives

$$l(\cdot X \cdot) \in [R2^{-k} + 2L, R2^{-k} + 2L + 1]. \tag{3.26}$$

Now we fix a sequence $\{Y_i\}_{i \leq k}$ with $Y_0 = X_0$ or X_1 and such that $Y_k = X$ and every $Y_{i+1} \in \chi_{i+1}$ is obtained from Y_i by the Second Cut Lemma. Equation (3.26) gives $l(\cdot Y_{i+1} \cdot) \leq l(\cdot Y_i \cdot) - \frac{R}{2^{i+1}} \leq l(\cdot Y_i \cdot) - \frac{R}{2^{\lfloor \log_2 R \rfloor - 1}} \leq l(\cdot Y_i \cdot) - 1$, and also $l(\cdot Y_i \cdot) \geq 2L$. These last inequalities together gives

$$l(\cdot Y_i \cdot) \geq 2L + (k - i).$$

By a recursive use of property (3) in 3.3.7, and noting that the sum of two number is greater then their maximum, we have

$$\begin{aligned}\theta^T_{Y_k} \leq& \frac{\pi}{6} + \sum_{i=0}^{k} e^{L+4-l(\cdot Y_i \cdot)} \\ \leq& \frac{\pi}{6} + \frac{e}{e-1} e^{4-L} \\ <& \frac{\pi}{3},\end{aligned}$$

where the last inequality can be satisfied by an L sufficiently large. Then $\theta^T_X < \frac{\pi}{3}$. We are near to apply the XY-Theorem: by Lemma 3.1.4 and the fact we have just proved we have $I(\cdot T \cdot X \cdot \bar{T} \cdot) \leq \log 4$ for any $X \in \chi_k$. Then theorem 3.3.3 provides

$$Y_T = (Y_0 Y_1)_T = (Y_0)_T + (Y_1)_T \tag{3.27}$$

for every $Y \in \chi_k$, where Y_i are the two element of χ_{k+1} generated by Y. A iterative use of (3.27) gives

$$\gamma = \sum_{X \in \chi_k} X_T,$$

in the $H^{300\epsilon,R}$. If k is maximal (that is $k = \lfloor \log_2 R \rfloor - 1$) we have that $l(\cdot X \cdot) \leq 2L + 5$ by (3.26), and so $X \in W_h$ by definition, and $X_T = (q(X))_T$. Then, in $H^{300\epsilon,R}(S)$:

$$\gamma = \sum_{X \in \chi_k} (q(X))_T = (q(\gamma))_T.$$

By the definition of q (see before the Good Pants Homology for Short Words Theorem 3.3.5) we obtain the result. □

This is essentially The Good Pants Homology Theorem but we have small ϵ and an amplification coefficient for ϵ in the Good Pants. We adjust these things with the following Lemma.

Lemma 3.3.9. *Let $\epsilon > 0$, and $M > 1$. There exists an $R_0 = R_0(\epsilon, M, S) > 0$ such that for every $R > R_0$ we can find a map $q_M : \Gamma_{\epsilon,R} \longrightarrow \mathbb{Q}^+\Pi_{\epsilon,R}$ and a constant $K = K(\epsilon, M, S) > 0$ such that for every $\gamma \in \Gamma_{\epsilon,R}$, $q_M(\gamma)$ is a positive sum of pants all of which have γ as one boundary cuff (with the appropriate orientation), the two other cuffs in $\Gamma_{\frac{\epsilon}{M},R}$, and $\gamma - \partial q_M(\gamma) \in \mathbb{Q}^+\Gamma_{\frac{\epsilon}{M},R}$.*

Proof. Let $\gamma \in \Gamma_{\epsilon,R}$. From the corollary after the Counting Pants Lemma 3.2.5 we have that the set $X_\gamma(M)$ of pants with γ as a cuff and the other two cuffs in $\Gamma_{\frac{\epsilon}{M},R}$ is finite with cardinality of the order Re^R. Define

$$q_M(\gamma) = \frac{1}{|X_{\gamma(M)}|} \sum_{X_\gamma(M)} \Pi.$$

Then the Lemma follows immediately. □

Theorem 3.2.3 then follows from Theorem 3.3.8 and Lemma 3.3.9. We use the notation form Theorem 3.3.8. Define $h_i = (g_i)_T$. Since

$$[h_i]_{H_1(S)} = [(g_i)_T]_{H_1(S)} = [g_i]_{H_1(S)},$$

and the g_i's are generators for $\pi_1(S,*)$, then the h_i's will be generators for $H_1(S,\mathbb{Q})$.

Then Theorem 3.3.8 assure that

$$\gamma = \sum a_i (g_i)_T = \sum a_i h_i.$$

Now we want to see that the equivalence $H^{300\epsilon,R} = H_1(S)$ implies the equivalence $H^{\epsilon,R} = H_1(S)$.

Let $q_{300} : \Gamma_{300\epsilon,R} \longrightarrow \mathbb{Q}^+\Pi_{300\epsilon,R}$ be the map in Lemma 3.3.9. For each h_i as above we can consider $h'_i = h_i - \partial q_{300}(h_i) \in \mathbb{Q}^+\Gamma_{\epsilon,R}$. Lemma 3.3.9 assure that

$$\gamma = \sum a_i h'_i,$$

in $H^{\epsilon,R}$.

Finally we want to suppress the hypothesis that ϵ has to be lesser then a universal $\hat{\epsilon}$.

Suppose $\gamma \in \Gamma_{E,R}$ with $E > \hat{\epsilon}$, then there exists $M > \hat{\epsilon}/E$ and such that $\gamma' = \gamma - \partial q_M(\gamma) \in \Gamma_{\epsilon,R}$ for an $\epsilon < \hat{\epsilon}$. Again by Lemma 3.3.9 we can work with γ' instead of γ.

This ends the proof of The Good Pants Homology Theorem 3.2.3.

Chapter 4

The Ehrenpreise Conjecture

4.1 Coordinates and Representations for Fuchsian and Quasi-Fuchsian Structures

Teichmuller Space

The Uniformization Theorem asserts that the only simply connected Riemann Surfaces up to biholomorphisms are the plane $\mathbb{C}$, the sphere $\hat{\mathbb{C}}$ and the unit disk D^2. Moreover we know that holomorphic structures, hyperbolic structures and conformal structures can be viewed as the same thing in dimension 2. We recall that the conformal structure on $\mathbb{H}^2$ is the same of D^2 as a Riemann Surface. Then, given a closed surface S of genus $g \geq 2$, we can ask how many hyperbolic structures such surface admits, that is in how many inequivalent way we can cover S with $\mathbb{H}^2$. Let $\mathcal{H}$ denotes the space of the Riemannian metrics h on S such that (S, h) is an hyperbolic surface. Let $\mathrm{Diff}^+(S)$ be the group of the orientation preserving diffeomorphisms of S, with the natural topology induced by S. Let $\mathrm{Diff}^+_0(S)$ the connected component of $\mathrm{Diff}^+(S)$ containing the identity map. The group $\mathrm{Diff}^+(S)$ acts on $\mathcal{H}$ by the push-forward of the metrics:

$$f_*(h)_p(u, v)) = h_{f^{-1}(p)}(d_p(f^{-1})u, d_p(f^{-1})(v)),$$

where $f \in \mathrm{Diff}^+(S)$,$p \in S$,$u, v \in T_pS$, $h \in \mathcal{H}$. The most natural space of hyperbolic structures one can consider is the so called Moduli Space:

$$\mathcal{M}(g) = \mathcal{H}/\,\mathrm{Diff}^+(S).$$

However it comes very hard to be studied.
A simpler space is the so called Teichmuller Space:

$$\mathcal{T}(g) = \mathcal{H}/\,\mathrm{Diff}_0^+(S).$$

These spaces are related by the so called Mapping Class Group

$$\mathrm{MCG}(g) = \mathrm{Diff}^+(S)/\,\mathrm{Diff}_0^+(S),$$

in the following way

$$\mathcal{M}(g) = \mathcal{T}(g)/\mathrm{MCG}(g).$$

In this section we present a way to parametrize the space $\mathcal{T}(g)$ with the Fenchel-Nielsen coordinates. These topics are well explained in details in [BP92].
Because of the importance of the Teichmuller space we explain how can be interpreted the quotient by Diff_0^+. Let $\varphi, \phi : S \longrightarrow S$ be two diffeomorphisms. We say that φ and ϕ are isotopic, if there is a smooth homotopy between them $F : S \times [0, 1] \longrightarrow S \times [0, 1]$ such that $F(x, t)$ is a diffeomorphism for every $t \in [0, 1]$. Then it can be proved that an orientation preserving homeomorphism is in the component Diff_0^+ if and only if it is isotopic to the identity.
$\mathcal{T}(g)$ can be thought as the space of the hyperbolic structures over S which not differs by a diffeomorphism isotopic to the identity. Likewise $\mathrm{MCG}(g)$ is the group of the isotopy class of diffeomorphisms of S.

Complex Distances

The following notation for the distances is fixed until the end of the chapter. For every subsets X and Y of the hyperbolic space $\mathbb{H}^n$ we denote with

$d(X,Y)$ the hyperbolic distance between the two sets. For p, $q \in \mathbb{H}^n$ and γ an oriented geodesic in $\mathbb{H}^n$ such that p, $q \in \gamma$, then $d_\gamma(p,q)$ denotes the *signed real distance* between the two point, that is the hyperbolic distance with a sign dependent on the orientation of γ, taken positive if γ goes from p to q. In particular $d_\gamma(p,q) = -d_\gamma(q,p)$.
As in the previous chapters, given two vectors v and u in $T^1_p\mathbb{H}^3$, we denote with $\Theta(u,v)$ the unoriented angles between them with values in $[0,\pi)$. Moreover, given $\boldsymbol{n} \in T^1_p\mathbb{H}^3$ orthogonal to the plane spanned by u and v, we define $\Theta_{\boldsymbol{n}}(u,v)$ as the angle between u and v, measured anticlockwise in the plane spanned by u and v and oriented by $\boldsymbol{n}$. We suppose $\Theta_{\boldsymbol{n}}$ takes values in $(-\pi,\pi]$.
If not specified, every parametrization of a geodesic in this chapter is considered with unit speed.
Now we introduce a complex valued function that,intuitively, can be thought as a distance between geodesics in $\mathbb{H}^3$. We work in $\mathbb{H}^3$ in the upper half space model with the identification $\partial\mathbb{H}^3 = \hat{\mathbb{C}} \cup \{\infty\}$.
Let α and β be two oriented geodesics in $\mathbb{H}^3$, and define γ as their common orthogonal oriented from α to β. We can find an orientation preserving isometry $B \in PSL(2,\mathbb{C})$ such that $B\gamma$ has endpoints $(0,\infty)$. Then the endpoints of $B\alpha$ and $B\beta$ will be, respectively, $(-q,q)$ and $(-p,p)$, for $q,p \in \mathbb{C}$.
We define the *signed complex distance* between α and β as $d^*_\gamma(\alpha,\beta) \in \mathbb{C}/2\pi i\mathbb{Z}$ by the formula

$$e^{d^*_\gamma(\alpha,\beta)}q = p.$$

Remark 4.1.1. Let α, β and γ be as above and suppose $x = \alpha\cap\gamma$, $y = \beta\cap\gamma$, $u = \dot{\alpha}(x)$, $v = \dot{\beta}(y)$, and $\boldsymbol{n} = \dot{\gamma}(y)$. Then

$$d^*_\gamma(\alpha,\beta) = d_\gamma(x,y) + i\Theta_{\boldsymbol{n}}(u@y, v).$$

In particular we notice that the signed complex distance d^*_γ coincide with the signed real distance d_γ if α and β are coplanar.

Obviously we have that $d^*_\gamma(\alpha,\beta) = -d^*_\gamma(\beta,\gamma) = -d^*_{(\gamma^{-1})}(\alpha,\beta)$ and that $d^*_\gamma(\alpha^{-1},\beta) = d^*_\gamma(\alpha,\beta^{-1}) = d^*_\gamma(\alpha,\beta) + i\pi$.

Finally we define the *unsigned complex distance* (we use the notation defined above)

$$\delta(\alpha,\beta)=\begin{cases} d^*_\gamma(\alpha,\beta) & \text{if}\,\mathrm{Re}(d^*_\gamma(\alpha,\beta))>0,\\ i|d^*_\gamma(\alpha,\beta)| & \text{if}\,\mathrm{Re}(d^*_\gamma(\alpha,\beta))=0,\\ -d^*_\gamma(\alpha,\beta) & \text{if }\ \mathrm{Re}(d^*_\gamma(\alpha,\beta))<0.\end{cases}$$

We observe that such definition doesn't depend on γ.
The unsigned complex distance is useful since it can be expressed in terms of the endpoints of α and β. Let (a_1,a_2) and $(b_1,b_2)\in\hat{\mathbb{C}}$ the endpoints of , respectively, α and β. Define $\chi=\frac{(a_1-a_2)(b_1-b_2)}{(a_1-b_2)(b_1-a_2)}$, then we have

$$\cosh\delta(\alpha,\beta)=\frac{1+\chi}{1-\chi}. \tag{4.1}$$

The definition of complex distance gives a definition of the *complex translation length* for an hyperbolic element $A\in PSL(2,\mathbb{C})$. Let γ be the axis of A oriented from the repelling to the attractive point. Let β an oriented geodesic of $\mathbb{H}^3$ orthogonal to γ. We define the *complex translation length* of A as $l(A)=d^*_\gamma(\beta,A(\beta))$. It can be verified that $l(A)$ does not depend on the choice of β. We notice that $\mathrm{Re}(l(A))>0$ by definition, so here isn't useful to make difference between the signed and unsigned case. We also observe that in $PSL(2,\mathbb{C})$, $\mathrm{Tr}(A)$ is defined up to multiplication of ±1 whereas $l(A)/2$ is defined up to addition of $i\pi$, then it is well defined the following formula for the complex length of an hyperbolic element

$$\mathrm{Tr}(A)=-2\cosh(l(A)/2), \tag{4.2}$$

For a complete introduction to complex distance and related topics, see [Ser01], [Tan94] and [Kou94].

Right-Angled Hexagons

A *skew hyperbolic right-Angled hexagon H* is a cyclically ordered set of oriented geodesic arcs L_i, $i\in\frac{\mathbb{Z}}{6\mathbb{Z}}$, such that L_i meets orthogonally L_{i+1}. We write $H = (L_1, L_2, L_3, L_4, L_5, L_6)$. Let $\lambda_n = \delta(L_{n-1}, L_{n+1})$. We have the following two formulas for a skew right-angled hexagon. They are consequence of (4.1), however it is possible to find a proof in [Kou94] and in [Fen89].

Proposition 4.1.1 (Cosh Rule for Right Angled Hexagon)**.**

$$\cosh(\lambda_n) = \frac{\cosh(\lambda_{n+3}) - \cosh(\lambda_{n+1})\cosh(\lambda_{n-1})}{\sinh(\lambda_{n+1})\sinh(\lambda_{n-1})}. \tag{4.3}$$

Proposition 4.1.2 (Sinh Rule for Right Angled Hexagon)**.**

$$\frac{\sinh\lambda_1}{\sinh\lambda_4} = \frac{\sinh\lambda_3}{\sinh\lambda_6} = \frac{\sinh\lambda_5}{\sinh\lambda_2}. \tag{4.4}$$

We notice that such formulas apply also in the degenerate case in which a line L_i has shrunk to a point at infinity (equivalently when L_{i-1} and L_{i+1} meets at infinity). However, from now on we don't consider degenerate cases. If the λ_i are all real, we observe that the hexagon is planar. These formulas are the main tools for the proof of the proposition below (see [Kou94] 1.6). We say that an hexagon H with edges coherently oriented is positively oriented. This condition is equivalent to ask $\delta(L_{n-1}, L_{n+1}) = d^*_\gamma(L_{n-1}, L_{n+1})$ for every $i \in \frac{\mathbb{Z}}{6\mathbb{Z}}$.
The following proposition says that, essentially, a positively oriented skew right angled hexagon is determined by three complex numbers (or by three of its edges).

Proposition 4.1.3. *Let* $\Sigma = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0,\ -\pi < \mathrm{Im}(z) \leq \pi\}$. *For any* $\lambda_1,\ \lambda_3,\ \lambda_5 \in \Sigma$ *there exist three oriented geodesics* L_2,L_4 *and* L_6, *such that every triple of oriented geodesics* (X_2, X_4, X_6) *satisfying*

$$\lambda_i = \delta(X_{i-1}, X_{i+1}),\ \textit{for}\ i = 1, 3, 5 \mod 6$$

is congruent by an orientation preserving isometry to one of (L_2, L_4, L_6) *or* $(-L_2, -L_4, -L_6)$.
All the four triples of complex numbers $(\lambda_1, \lambda_3, \lambda_5)$, $(\lambda_1, \lambda_3 + \pi i, \lambda_5 + \pi i)$, $(\lambda_1 + \pi i, \lambda_3, \lambda_5 + \pi i)$ *and* $(\lambda_1 + \pi i, \lambda_3 + \pi i, \lambda_5)$ *give the same three geodesics* L_2,L_4 *and* L_6 *but with different orientations.*

Then we can generate 8 *skew right angled hexagons individuated connecting the* L_i*'s by geodesic arcs of the required complex lengths, however exactly one of them is positively oriented.*

Remark 4.1.2. If we restrict our attention to the planar case the above Proposition say that given three positive real numbers λ_1, λ_2 and λ_3 we can find exactly one planar right angled hexagon, up to orientation preserving isometries of $\mathbb{H}^2$ with three alternates edges of length λ_i.

Pants and Fenchel-Nielsen Coordinates

Let $H = (C_1, a_2, C_3, a_1, C_2, a_3)$ be a positively oriented planar right-angled hexagon. In the planar case H is the boundary of a region of the plane. We denote such region with the boundary again with H. We can take two copy of the hexagon H with opposite orientations and then glue them along the arcs a_i's. What we get is a pair of pants (with geodetic boundary).
Now we start from a pair of pants Π with geodetic boundary. Let C_i, $i \in \frac{\mathbb{Z}}{3\mathbb{Z}}$, denotes the three boundary components. Let a_i be the common perpendicular geodesic segment between C_{i-1} and C_{i+1}. Cutting along the a_i's disconnect Π into two right angled hexagons, which are congruent by 4.1.3 since they share three alternating edges. So the C_j's are cut by the a_i's in two arcs of the same length. In particular we have that the three *half lengths* $\mathrm{hl}(C_i)$ of the C_i's determines a unique hyperbolic structure over a pair of pants.
Now we want to glue such pants along the boundaries to construct closed hyperbolic surfaces. Let S be a fixed closed oriented topological surface with genus $g > 1$. Every such surface admits a pant decomposition: we can find a finite set $\mathcal{C}$ of simple closed curves not homotopically equivalent, such that $S - \bigcup_{C \in \mathcal{C}} C$ is a disjoint union of some topological pair of pants. We recall that such decomposition can be done with exactly $3(g-1)$ curves. We denote with $\mathcal{P}$ the finite set of such pair of pants. Moreover we fix a diffeomorphism $\psi : S \longrightarrow \Sigma$ where $\Sigma = (S, h)$ is an hyperbolic surface.
Fix a $C \in \mathcal{C}$, then we have one of the following two mutually exclusive situ-

ations:
(i) There are exactly two pair of pants $\Pi'(C)$, $\Pi''(C) \in \mathcal{P}$ with C as boundary component. Notice that the orientation of C as boundary of Π' is the opposite of the orientation as boundary of Π''.
(ii)There is only one pair of pants $\Pi'(C)$ such that there are two components of $\partial\Pi'$ equal to C.

In the case (i) fix K_C as the free homotopy class of a closed simple curve in $\Pi' \cup \Pi'' \cup C$ which meets C in exactly two points, and that disconnect $\Pi' \cup \Pi'' \cup C$ into two topological pair of pants. When we consider $\psi(\Pi' \cup \Pi'' \cup C)$ we found a unique real number $\mathrm{hl}(C)$ determined by the length of the geodesic representative of C and a unique oriented geodesic representative for the class K_C, which meets C in the two points p_C and q_C. We can fix a lift $\tilde{K}_C$ of the geodesic representative of K_C in $\mathbb{H}^2$. In such lift we can fix three points $\tilde{p}_C^1$,$\tilde{q}_C^2$ and $\tilde{p}_C^3$ that are two consecutive lifts of p_C with a lift of q_C between them. $\tilde{K}_C$ meets a lift $\tilde{C}_i$ of C in $\tilde{p}_C^i$ and a lift $\tilde{C}_2$ on $\tilde{q}_C^2$. We observe that the orientation of $\tilde{C}_2$ is the opposite of $\tilde{C}_1$ and $\tilde{C}_3$. Let $\tilde{D}_i$ be the common perpendicular between $\tilde{C}_i$ and $\tilde{C}_{i+1}$, for $i = 1{,}2$. The orientations of $\tilde{D}_i$'s are from $\tilde{C}_i$ to $\tilde{C}_{i+1}$. We define the *twist parameter* for Σ at C as the real signed distance $t(C) = d_{\tilde{C}_2}(\tilde{D}_1, \tilde{D}_2)$.

In the case (ii) we fix K_C as the homotopy class of the closed simple curve of $\Pi' \cup C$ which meets C in only one point and doesn't disconnect Π'. The definition of $\mathrm{hl}(C)$ is equal to the case (i), for $t(C)$ we proceed as follows. We found the geodesic representative for K_C in $\psi(\Pi' \cup C)$ that intersect the geodesic representative γ of C at the point p_C. Then we can choose two lifts $\tilde{p}_C^1$ and $\tilde{p}_C^2$ in $\mathbb{H}^2$ of p_C such that they are consecutive along a lift $\tilde{K}_C$ of the geodesic representative for K_C. This construction define two lifts $\tilde{C}_1$ and $\tilde{C}_2$ of C at the points $\tilde{p}_C^1$ and $\tilde{p}_C^2$. Then we can consider $\tilde{D}$ as the common perpendicular between them, which meets $\tilde{C}_1$ and $\tilde{C}_2$ at the points $\tilde{d}_1$ and $\tilde{d}_2$ respectively. Such points projects to two (not necessarily distinct) points d_1 and d_2 on γ. We define $t(C)$ as the real signed distance $d_\gamma(d_1, d_2)$. Then we can assign to any hyperbolic surface $\Sigma = (S, h)$ the real parameters $(\mathrm{hl}(C), t(C))_{C \in (\mathcal{C})} \in \mathbb{R}_+^{3(g-1)} \times \mathbb{R}^{3(g-1)}$. These are coordinates for $\mathcal{T}(g)$.

Theorem 4.1.4 (Fenchel-Nielsen Coordinates)**.** *Let $g > 1$ and fix a closed topological surface S of genus g. Fix a pant decomposition $\mathcal{C}$ made by $3(g-1)$ curves. Let $\hat{\Psi} : \mathcal{H} \longrightarrow \mathbb{R}_+^{3(g-1)} \times \mathbb{R}^{3(g-1)}$ be the map that assign to every hyperbolic metric $h \in \mathcal{H}$ its Fenchel-Nielsen coordinates $(\mathrm{hl}(C), t(C))_{C \in \mathcal{C}}$. Then $\hat{\Psi}$ induces a map*

$$\Psi : \mathcal{T}(g) \longrightarrow \mathbb{R}_+^{3(g-1)} \times \mathbb{R}^{3(g-1)},$$

which is an homeomorphism.

Quasi-Fuchsian Groups

We recall that a Kleinian (resp. Fuchsian) group is a discrete subgroup of $PSL(2, \mathbb{C})$ (resp. $PSL(2, \mathbb{R})$). Let S be a closed topological surface an let G' be a Kleinian group such that $M = \mathbb{H}^3/G'$ is an hyperbolic 3-manifold homeomorphic to $S \times \mathbb{R}$. Recall that the action of Kleinian groups on $\mathbb{H}^3$ is determined by (and determines) an action $\zeta : G' \longrightarrow \mathrm{Aut}(\hat{\mathbb{C}})$, which maps each $g' \in G'$ to $\zeta(g) \in \mathrm{Aut}(\hat{\mathbb{C}})$.

Definition 4.1.1. In the situation above G' is said a *quasi-Fuchsian* group if the limit set $L_{G'} \subseteq \hat{\mathbb{C}}$ is a Jordan curve.

In fact $\hat{\mathbb{C}} \setminus L_{G'}$ will be two simply connected components.
If G is a Fuchsian group then it acts on $\partial\mathbb{H}^2$ and such action can be naturally extended to $\partial\mathbb{H}^3$, as an action $\eta : G \longrightarrow \mathrm{Aut}(\hat{\mathbb{C}})$, which maps each $g \in G$ in $\eta(g) \in \mathrm{Aut}(\hat{\mathbb{C}})$.

Proposition 4.1.5. *A Kleinian group G' is quasi-Fuchsian if and only if there is an isomorphism of groups $\sigma : G \longrightarrow G'$ with a Fuchsian G, and a quasi conformal map $f : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ such that, defined the actions of these groups on $\hat{\mathbb{C}}$ as the maps ζ, and η of G' and G,respectively, on* $\mathrm{Aut}(\hat{\mathbb{C}})$, *we* *have that*

$$f \circ (\zeta(\sigma(g))) = \eta(g) \circ f \text{ for every } g \in G.$$

In this case we say that G and G' are quasi-conformally conjugated by f.

In particular every Fuchsian group is quasi-Fuchsian.
If we call Ω_1 and Ω_2 the two components of $\partial\mathbb{H}^3 \setminus L_{G'}$ one can see that $\frac{\Omega_1}{G'}$, $\frac{\Omega_2}{G'}$ and S are all three homeomorphic. Moreover the homeomorphism between the $\frac{\Omega_i}{G'}$'s is given by the quotient of the map $f \circ c \circ f^{-1}$, where $c(z) = \bar{z}$.
Let $H(G')$ be the convex hull of $L_{G'}$, and let $H_\epsilon(G')$ be an ϵ neighbourhood of $H(G')$. Then $M_{G'} = \frac{H_\epsilon(G')}{G'}$ is homeomorphic to M. We notice that we need to use H_ϵ instead of H since if G' is itself Fuchsian then $H(G')$ is $\overline{\mathbb{H}^2} \subseteq \overline{\mathbb{H}^3}$ and then $\frac{H(G')}{G'}$ is a surface.
Let S be a closed surface, as before, with genus g and with fundamental group G. We want to describe the properties of the set of quasi-Fuchsian groups, that is the set of group G' founded as above. Let $\tilde{Q} \subset \mathrm{Hom}(G, PSL(2,\mathbb{C}))$ be the set of the injective homomorphisms with quasi-Fuchsian image. There is a natural action by conjugation of $PSL(2,\mathbb{C})$ on $\tilde{Q}$. We need to consider the quotient of $\tilde{Q}$ by such action since also Fuchsian groups inside $PSL(2,\mathbb{C})$ are defined modulo conjugation. We define such quotient of $\tilde{Q}$ as $Q(S)$, the set of all quasi-Fuchsian groups quasi-conformally congruent to G. In fact $Q(S)$ is a topological ball of dimension $12g - 12$ (see, for example, [Ber70]).
We are interested to give coordinates for $Q(S)$ in analogy to the Fenchel-Nielsen coordinates given for $\mathcal{T}(g)$. These will be the complex Fenchel-Nielsen coordinates, which parametrize $Q(S)$ with holomorphic coordinates (with respect to the natural complex structure on $Q(S)$ inherited from $PSL(2,\mathbb{C})$). The complex Fenchel-Nielsen coordinates were introduced in [Kou94] and [Tan94].

Skew Pants and Complex Fenchel-Nielsen Coordinates

We want to extend the construction given for pair of pants starting from planar right angled hexagons, to skew right-angled hexagons to have skew pair of pants. It is better to use a more algebraic argument. Let Π be a topological pair of pants as before. There exists a presentation of $\pi_1(\Pi)$ with three generators c_1, c_2 and c_3 such that $c_1c_2c_3 = \mathrm{id}$. The following is proved in [Kou94].

Proposition 4.1.6. *Let λ_1, λ_2, $\lambda_3 \in \mathbb{C}/2\pi i\mathbb{Z}$ such that* $\mathrm{Re}(\lambda_i) > 0$*. Then there exists a representation $\hat{\rho} : \pi_1(\Pi) \longrightarrow SL(2,\mathbb{C})$ such that* $\mathrm{Tr}(\hat{\rho}(c_i)) = -2\cosh\lambda_i$ *for $i = 1,2,3$. This homomorphism is unique up to conjugation in $SL(2,\mathbb{C})$, provided that the axis of the $\hat{\rho}(c_i)$'s are pairwise distinct.*

Then given the three half lengths $\mathrm{hl}(C_i) = \lambda_i$ we can determine a representation in $SL(2,\mathbb{C})$, unique up to conjugation.

Definition 4.1.2 (Skew pair of pants)**.** Let Π be a topological pair of pants. Let $\rho : \pi_1(\Pi) \longrightarrow PSL(2,\mathbb{C})$ be a discrete and faithful representation. We say that the conjugacy class $[\rho]$ is a *skew pair of pants.*

We also have a geometric point of view. A skew pair of pants ρ determine, up to homotopy, a map $f_\rho : \Pi \longrightarrow \mathbb{H}^3/\rho(\pi_1(\Pi)) = M_\rho$. We orient every boundary component $C_i \subset \partial\Pi$ in such a way that Π is on the left of C_i. For each $i \in \frac{\mathbb{Z}}{3\mathbb{Z}}$ there is exactly one closed oriented geodesic $\gamma_i \subset M_\rho$ freely homotopic to $f(C_i)$. Define $\eta_i \subseteq M_\rho$ as follows: let $\tilde{\gamma}_{i-1}$ and $\tilde{\gamma}_{i+1}$ be two lifts in $\mathbb{H}^3$ of γ_{i-1} and γ_{i+1}, respectively, then consider their common orthogonal $\tilde{\eta}_i$ and define η_i as its projection on M_ρ; η_i is a geodetic arc and doesn't depend on the chosen lifts of the γ_j's. In this way we have found a unique 1-complex determined by ρ and this leads to a description of a skew pair of pants in terms of skew right angled hexagons similar to the description made in the planar case. The hexagon in this case will be $H_\rho = (\gamma_1/2, \eta_3, \gamma_2/2, \eta_1, \gamma_3/2, \eta_2)$ where the $\gamma_i/2$ is the half of the arc γ_i, individuated by the endpoints of the η_j's on it.
Conversely three half lengths λ_i's determine a unique positive skew hexagon by Proposition 4.1.3. Then we can proceed with a similar argument to that we used for the planar case, to see how geometrically we can glue two oppositely oriented copies of skew hexagons in alternate edges, to construct the

1-complex of a skew pair of pants.

Let S be a fixed closed oriented topological surface with genus $g > 1$. As we done in the classical case we fix a pant decomposition $\mathcal{C}$ made of $3(g-1)$ curves, and we denote with $\mathcal{P}$ the set of pair of pants.
Let $\bar{\rho} : \pi_1(S) \longrightarrow PSL(2,\mathbb{C})$ be a quasi-Fuchsian representation for $\pi_1(S)$ (up to conjugacy). Then $\bar{\rho}$ is faithful and discrete. We want to give coordinates for $\bar{\rho}(\pi_1(S))$ inside $Q(S)$.
We recall the notation used in the description of the Fenchel-Nielsen coordinates. Fix a $C \in \mathcal{C}$, then there can be two situation.
(i) There are exactly two pair of pants $\Pi'(C)$, $\Pi''(C) \in \mathcal{P}$ with C as boundary component.
(ii) There is only one pair of pants $\Pi'(C)$ such that two components of $\partial\Pi'$ are C.

Suppose we are in (i).
Let K_C be the free homotopy class of a closed simple curve in $\Pi' \cup \Pi'' \cup C$ which meets C in exactly two points, and that disconnect $\Pi' \cup \Pi'' \cup C$ into two topological pair of pants. With a little abuse of notation, we use for every element of $\pi_1(S)$ the same name of a freely homotopic representative, for example we denote with C both a closed curves in S and the element of $\pi_1(S)$ freely homotopic to C (since $\bar{\rho}$ is determined up to conjugation, this abuse doesn't make confusion in the representation). We consider the pant $\Pi'(C)$ with $\partial\Pi' = C \cup C^2 \cup C^3$. Let $\lambda_1 = l(\bar{\rho}(C))/2$, $\lambda_2 = l(\bar{\rho}(C^2))/2$ and $\lambda_3 = l(\bar{\rho}(C^3))/2$ be the three complex number corresponding to half of the complex translation lengths. Such three complex numbers determines a representation $\rho' : \pi_1(\Pi'(C)) \longrightarrow SL(2,\mathbb{C})$ up to conjugation, by proposition 4.1.6.
We observe that is well defined the three manifold $M_{\rho'} = \mathbb{H}^3/\rho'(\pi_1(\Pi'))$, and an homotopy class of maps $f_{\rho'} : \Pi' \longrightarrow M_{\rho'}$. We can consider the intersection κ of a representative of K_C with $\Pi'(C)$. Then $f_{\rho'}(\kappa)$ is defined as geodesic

arc, if we take a geodesic representative for $f_{\rho'}(\kappa \cup C)$. We call p_C and q_C the two points of intersections in $M_{\rho'}$ of the geodesic representative of κ with the geodesic representative of C.
We can choose an axis $\tilde{C}_1$ of $\rho'(C)$ in $\mathbb{H}^3$ (it is determined up to orientation preserving isometries of $\mathbb{H}^3$). Chosen $\tilde{C}_1$ we determine $\tilde{C}_2$ as follows: let $\tilde{K}_C$ be the oriented geodesic arc in $\mathbb{H}^3$ meeting in a point $\tilde{p}_C^1$ with $\tilde{C}_1$ such that, when projected on $M_{\rho'}$, maps over $f_{\rho'}(\kappa)$ and $\tilde{p}_C^1$ is projected on p_C ; then there is a lift $\tilde{q}_C^2$ of the point q_C to the other endpoints of $\tilde{K}_C$, and with him a new lift $\tilde{C}_2$ of C passing through $\tilde{q}_C^2$. Finally this determines the oriented geodesic line $\tilde{D}_1$ in $\mathbb{H}^3$ as the common perpendicular from $\tilde{C}_1$ to $\tilde{C}_2$.
Now we proceed in a similar way to choose a representation $\rho'' : \Pi''(C) \longrightarrow SL(2, \mathbb{C})$ by proposition 4.1.6, with the restriction that we have to choose as axis of $\rho''(C)$ the geodesic $\tilde{C}_2$ (again, we take the half lengths values from the quasi-Fuchsian representation $\bar{\rho}$ in order to apply the Proposition 4.1.6). Then this determines, as a translated by $\tilde{K}_C$ in anology of what we done for Π', the geodesic $\tilde{C}_3$ up to orientation preserving isometries of $\mathbb{H}^3$. We can define $\tilde{D}_2$ as the common perpendicular from $\tilde{C}_2$ to $\tilde{C}_3$.
Since in both the representation ρ' and ρ'' we asked the half lengths of the images of C to be determined as $\bar{\rho}$ do, then we have that $\mathrm{Tr}(\rho'(C)) = -2\cosh(\lambda_1) = \mathrm{Tr}(\rho''(C))$. Then we can determine a representation of $\pi_1(\Pi' \cup \Pi'')$ in $SL(2, \mathbb{C})$ specifying the parameter $t(C) = d^*_{\tilde{C}_2}(\tilde{D}_1, \tilde{D}_2)$. We remark that this definition of $t(C)$ is now dependent only on $\bar{\rho}$ since the remaining geometry of the two pants is completely determined.

Suppose that we are in (ii).
Let K_C be the homotopy class of the closed simple curve of $\Pi' \cup C$ which meets C in only one point and doesn't disconnect Π'. Let $\partial\Pi'(C) = C \cup C^1 \cup C$. Thank to Proposition 4.1.6 we can find a representation $\rho' : \pi_1(\Pi') \longrightarrow SL(2, \mathbb{C})$ specifying $\lambda_3 = \lambda_1 = l(\bar{\rho}(C))/2$ ad $\lambda_2 = l(\bar{\rho}(C^2))/2$. Then we consider the three manifold $M_{\rho'} = \mathbb{H}^3/\rho'(\pi_1(\Pi'))$ and the homotopy class of maps $f_{\rho'} : \Pi' \longrightarrow M_{\rho'}$. We can find the geodesic representatives γ and κ

in $M_{\rho'}$ for $f_{\rho'}(C)$ and $f_{\rho'}(K_C)$ that intersect in the point p_C. Then we can proceed as we done for the (real) Fenchel-Nielsen coordinates, briefly: take an arc $\tilde{k}$ in $\mathbb{H}^3$ between two preimages of p_C and that projects onto κ, then we can find two covers $\tilde{C}_1$ and $\tilde{C}_2$ of C and their common orthogonal arc $\tilde{D}$ meeting in $\tilde{d}_1$ and $\tilde{d}_2$; projected these two points over d_1 and $d_2 \in C$ we can define $t(C) = d^*_C(d_1, d_2)$.
Thus we have determined a representation of $\pi_1(\Pi' \cup C)$ on $SL(2,\mathbb{C})$

Repeating the procedures for all $C \in \mathcal{C}$ we find a representation $\hat{\rho} : \pi_1(S) \longrightarrow SL(2,\mathbb{C})$ where for every C the equation $\mathrm{Tr}(C) = -\cosh(\mathrm{hl}(C))$ is satisfied. From this equations one can verify that the nearest Fuchsian group to $\hat{\rho}(\pi_1(S))$ in $PSL(2,\mathbb{C})$ satisfy $\mathrm{Tr}(C) < 0$.
One can ask if the family of complex numbers $(\mathrm{hl}(C), t(C))_{C\in\mathcal{C}}$ describe completely the structure of $Q(S)$. In fact they are coordinates for $Q(S)$, named *Complex Fenchel-Nielsen Coordinates* as the following Theorem states.

Theorem 4.1.7 (Complex Length Coordinates for quasi-Fuchsian Groups [Kou94],[Tan94])**.** *Given a pant decomposition $\mathcal{C}$ of a surface S, there are complex length functions*

$$(\mathrm{hl}(C), t(C))_{C\in\mathcal{C}} : Q(S) \longrightarrow \mathbb{C}^{3(g-1)} \times \mathbb{C}^{3(g-1)},$$

which form a global set of holomorphic coordinates for $Q(S)$ (with respect to the complex structure inherited from $PSL(2,\mathbb{C})$).

Remark 4.1.3. The complex half lengths $\mathrm{hl}(C)$ in this statement can have $\mathrm{Re}(l_C) < 0$. This is required in order to have holomorphic coordinates.

4.2 The Kahn-Markovic Theorem

The works [KM12] and [KM13] together proves a very strong result. Let $S(R)$ denote the hyperbolic surface of genus 2 with Fenchel-Nielsen coordinates $(\mathrm{hl}(C), t(C)) = (R, 1)$. With $O(R)$ we denote the quotient orbifold of $S(R)$ by the group $\mathrm{Isom}(S(R))$.

Definition 4.2.1 (Quasi-isometry)**.** Let (X, d), (Y, δ) be two metric spaces. We say that they are *quasi-isometric* if there is a continuous map $f : X \longrightarrow Y$ and two constant C,$A > 0$, such that the following two conditions are satisfied.
(1) $\frac{d(a,b)-A}{C} \leq \delta(f(a), f(b)) \leq C(d(a, b) + A)$,
for every a,$b \in X$.
(2) for every $y \in Y$ there exists an $x \in X$ such that $d(f(x), y) \leq A$.
Such an f is called *quasi-isometry.*

If f is a quasi-isometry with $A = 0$ we speak about C quasi-isometry. One can verify that a C quasi-isometry of $\mathbb{H}^2$ is a quasi-conformal homeomorphism too. We recall from the preliminaries that a quasi-conformal map of the open unit disk extends to an homeomorphism of the closed unit disk. Moreover there is a condition on the boundary, named M-condition. We have the converse: an homeomorphism of the boundary which satisfy an M-condition extends to an homeomorphism of the closed unit disk which is $K(M)$ quasi-conformal in the open disk. In [Ahl06] it is proved that such extension is a $C(K)$ quasi-isometry with respect to the hyperbolic metric for the unit disk.

The following is the Unified Kahn-Markovic Theorem.

Theorem 4.2.1 (J. Kahn, V. Markovic, Unified Theorem)**.** *Let M be an hyperbolic manifold of dimension n equal to 2 or 3. For any $\epsilon > 0$ there exists an $R_0 = R_0(\epsilon, M) > 0$ with the following properties. For every $R > R_0$ there exists an hyperbolic surface S which is a finite cover of $O(R)$ and an immersion $f : S \longrightarrow M$ such that the corresponding map in the universal cover $\tilde{f} : \tilde{S} \longrightarrow \tilde{M}$ is a $(1 + \epsilon)$ quasi-isometry.*

It has two famous corollaries:

Corollary 4.2.2 (Surface Subgroup Theorem)**.** *For any closed hyperbolic 3-manifold M^3 there exist a closed hyperbolic surface S and an immersion $f : S \longrightarrow M^3$ such that $f_* : \pi_1(S) \longrightarrow \pi_1(M^3)$ is an injective morphism.*

Proof. It follows from the Kahn-Markovic Theorem for $n = 3$. We don't give the details of the proof, however it is based on the fact that the geodesics on S are mapped by quasi isometries at bounded distance from geodesics on M. $\square$

Corollary 4.2.3 (Ehrenpreise Conjecture)**.** *Let S, T be two closed hyperbolic surfaces, and let $\epsilon > 0$. Then there exist two finite covers $\hat{S} \longrightarrow S$ and $\hat{T} \longrightarrow T$ and an $(1+\epsilon)$ quasi-conformal map $f : \hat{S} \longrightarrow \hat{T}$.*

Proof. Given an $\epsilon > 0$ say that $\psi : X \longrightarrow Y$ is an ϵ-good cover if X and Y are hyperbolic surfaces, ψ is a covering, and the lift $\tilde{\psi}$ of ψ on the universal covers is an $(1+\epsilon)$ quasi-isometry. From the Unified Kahn-Markovic Theorem we can find two ϵ-good covers $S_1 \longrightarrow S$ and $T_1 \longrightarrow T$ which are both finite locally isometric coverings of $O(R)$, for a real $R > 0$ sufficiently large. Then there exists a surface S_2 which is a finite cover both for S_1 and T_1, is a finite locally isometric cover for $O(R)$ and is an ϵ-good cover of S and T. Let $\hat{S}$ be the locally isometric cover of S which is the same topological cover of S_2. Then we can find an homeomorphism $\hat{S} \longrightarrow S_2$ which lift in the universal cover to a $(1+\epsilon)$ quasi-isometry of $\mathbb{H}^2$. In particular we find an homeomorphism $f_S : \hat{S} \longrightarrow S_2$ which is $K_S(\epsilon)$ quasi-conformal, with $K(\epsilon) \longrightarrow 1$ when $\epsilon \to 0$. In the same way we find a finite locally isometric cover $\hat{T} \longrightarrow T$ with a $K_T(\epsilon)$ quasi-conformal homeomorphism $f_T : \hat{T} \longrightarrow S_2$. Finally $f = f_T^{-1} f_S$ is the quasi-conformal homeomorphism desired (one have to adjust the constant, however $K_S(\epsilon)K_T(\epsilon) \to 1$ for $\epsilon \to 0$, so the constants are coherent with the statement). $\square$

Reduced Complex Fenchel-Nielsen coordinates

In their work [KM12] Kahn and Markovic use a slightly different description of the Fenchel-Nielsen coordinates. Let Π_0 be a topological pair of pants, and $\rho : \pi_1(\Pi_0) \longrightarrow PSL(2,\mathbb{C})$ a representative for a skew pair of pants, with the corresponding homotopy class of maps $f_\rho : \Pi \longrightarrow M_\rho = \mathbb{H}^3/\rho(\pi_1(\Pi_0))$. For such skew pair of pants and its 1-complex inside M_ρ we use the same notation introduced after the definition of skew pair of pants 4.1.2.
Fix an $i \in \{0, 1, 2\}$. Orient η_{i-1} and η_{i+1} in the direction pointing away from

γ_i. We conjugate ρ so that there is a lift $\tilde{\gamma}_i$ of γ_i on $\mathbb{H}^3$ that has endpoints at infinity 0 and ∞ (we use the half space model). Let $A_{\gamma_i} \in PSL(2,\mathbb{C})$ be the hyperbolic isometry with axes $\tilde{\gamma}_i$ that corresponds to the element of $\pi_1(\Pi_0)$ representing γ_i. Then A_{γ_i} extends to $\partial\mathbb{H}^3 = \hat{\mathbb{C}}$ as the map $z \mapsto e^{l(A_{\gamma_i})}z$ (We recall that $l(A_{\gamma_i})$ is the complex translation length as introduced after the complex distances). We observe that the lifts $\tilde{\eta}_{i-1}$ and $\tilde{\eta}_{i+1}$ of, respectively, η_{i-1} and η_{i+1}, that intersects $\tilde{\gamma}_i$ will alternate along $\tilde{\gamma}_i$. In particular we can define the half length of γ_i as

$$\mathrm{hl}(\gamma_i) = d_{\tilde{\gamma}_i^*}(\tilde{\eta}_{i-1}, \tilde{\eta}_{i+1}),$$

where the lifts $\tilde{\eta}_{i-1}$ and $\tilde{\eta}_{i+1}$ are chosen so that they intersect $\tilde{\gamma}_i$ consecutively. One can verify that this definition coincides with the previous. Moreover we can consider the hyperbolic element $\sqrt{A_{\gamma_i}} \in PSL(2,\mathbb{C})$ defined by its action on $\partial\mathbb{H}^3$ as $z \mapsto e^{\mathrm{hl}(\gamma_i)}z$. We have that $\sqrt{A_{\gamma_i}}$ maps $\tilde{\eta}_{i-1}$ to $\tilde{\eta}_{i+1}$.
We observe that the unit normal bundle $N^1(\tilde{\gamma}_i)$ is a cylinder on which acts the group $\mathbb{C}/2\pi i\mathbb{Z}$ by translations: every $v \in N^1(\tilde{\gamma}_i)$ is determined by a point $p \in \tilde{\gamma}_i$ and an angle $\theta \in (-\pi, \pi]$, then $z = x + iy \in \mathbb{C}/2\pi i\mathbb{Z}$ translates p by x and rotate θ by y. Observe that the action is free and transitive. This action pass to the quotients. On the unit normal bundle $N^1(\gamma_i)$ acts freely and transitively the group

$$\mathbb{C}/2\pi i\mathbb{Z}/\langle A_{\gamma_i}\rangle = \mathbb{C}/(2i\pi\mathbb{Z} + l(\gamma_i)\mathbb{Z}).$$

As well, if we define

$$N^1(\sqrt{\gamma_i}) = N^1(\tilde{\gamma}_i)/\langle\sqrt{A_{\gamma_i}}\rangle,$$

we have that

$$\mathbb{C}/2\pi i\mathbb{Z}/\langle\sqrt{A_{\gamma_i}}\rangle = \mathbb{C}/(2i\pi\mathbb{Z} + \mathrm{hl}(\gamma_i)\mathbb{Z})$$

acts on it freely and transitively. For $i, j = 0, 1, 2$ and $i \neq j$ let $n(i,j) \in N^1(\gamma_i)$ be the unit vector at $\gamma_i \cap \eta_j$ pointing along η_j. If we consider $N^1(\sqrt{\gamma_i})$ instead of $N^1(\gamma_i)$ we see that $n(i, i-1)$ and $n(i, i+1)$ are the same point on it, then we define such point as $\mathrm{foot}_{\gamma_i}(\rho)$ (sometimes we can write $\mathrm{foot}_{\gamma_i}(f)$

where $f = f_\rho$, recall that ρ where fixed as a skew pair of pants).
Consider a pant decomposition $\mathcal{C} = \{C_i\}$ for a topological closed oriented surface S with genus $g > 1$. We consider the curves C_i as non oriented. Let $\mathcal{C}^*$ be the set of oriented closed curves of $\mathcal{C}$, where every $C \in \mathcal{C}$ is taken with both the orientations in $\mathcal{C}^*$. A marked pair of pants for the pant decomposition $(S, \mathcal{C})$ is a couple (Π, C) where $C \in \mathcal{C}^*$, Π is a pant of the pant decomposition given by $\mathcal{C}$ with $C \subset \partial\Pi$, and C lies on the left of Π. For every marked pair of pants (Π, C) there is a unique (Π', C') such that $C = -C'$ (where the notation $-C'$ means C' with the opposite orientation). Let $\rho : \pi(S) \longrightarrow PSL(2, \mathbb{C})$ be a representation that is discrete and faithful when restricted to $\pi_1(\Pi)$, for each pair of pants Π from the pant decomposition $\mathcal{C}$. This means that each restriction $\rho|_{\pi_1(\Pi)}$ is a representative for a skew pair of pants.

Definition 4.2.2 (Viable Representation)**.** Let $(S, \mathcal{C})$ be a panted surface (a surface with a pant decomposition $\mathcal{C}$) such that the genus g of S is strictly grater then 1. We say that the representation

$$\rho : \pi_1(S) \longrightarrow PSL(2, \mathbb{C})$$

is viable if
(i) ρ is discrete and faithful when restricted to $\pi_1(\Pi)$ for each pair of pants $\Pi \in S \setminus \bigcup_{C \in \mathcal{C}} C$,
(ii) if we denote with $\mathrm{hl}_\Pi(C)$ the half length of C as defined by the restriction $\rho|_{\pi_1(\Pi)}$, we have that $\mathrm{hl}_\Pi(C) = \mathrm{hl}_{\Pi'}(C)$ for every marked pair of pants (Π, C) and $(\Pi', -C)$.

Remark 4.2.1. We observe that quasi-Fuchsian representations are in particular viable.

Given a viable representation $\rho : \pi_1(S) \longrightarrow PSL(2, \mathbb{C})$ it is well defined the three dimensional manifold $M(\rho) = \mathbb{H}^3/\rho(\pi_1(S))$. For every marked pair of pants (Π, C) we can consider the geodesic γ freely homotopic to a representative of $\rho(C)$ in $M(\rho)$. From the definition of viable representation follows that is well defined $\mathrm{hl}(C)$ as the half length of γ in $M(\rho)$, independently on the marked pair of pants.

For $\gamma' = \gamma^{-1}$ and $(\Pi', -C)$ the marked pair of pants having $-C$ has cuff, we define

$$s(C) = \mathrm{foot}_\gamma(\rho|_\Pi) - \mathrm{foot}_{\gamma'}(\rho|_{\Pi'}) - i\pi.$$

It follows that $s(C) \in \mathbb{C}/(2i\pi\mathbb{Z} + \mathrm{hl}(C)\mathbb{Z})$ and it is independent to the roles of (Π, C) and $(\Pi', -C)$. The couples of complex numbers $(\mathrm{hl}(C), s(C))_{C\in\mathcal{C}}$ are called *reduced complex Fenchel-Nielsen coordinates.*

Now fix a pair of pants Π of the pant decomposition individuated by $\mathcal{C}$, and denote with C_0, C_1 and C_2 the three cuffs of Π. Let c_0,$c_1 \in \pi_1(\Pi)$ be the elements representing the homotopy classes of C_0 and C_1. We say that a representative for the conjugacy class of $\rho : \pi_1(S) \longrightarrow PSL(2, \mathbb{C})$ is *normalized* if $\rho(c_0)$ has oriented axis $(0, \infty)$ in $\mathbb{H}^3$ and the repelling endpoint of $\rho(c_1)$ is 1.

For any $R > 0$ we define the set $\Omega = \Omega(R)$ of the coordinates $(z, w) = \{(z_C, w_C)\}_{C\in\mathcal{C}}$ satisfying:

(i) $z_C \in \mathbb{C}/2i\pi\mathbb{Z}$ with $|z_C - R| < 1$,

(ii) $w_C \in \mathbb{C}/(2i\pi\mathbb{Z} + z_C\mathbb{Z})$ with $|w_C - 1| < \frac{1}{2R}$.

If R is sufficiently large, it follows from Theorem 4.1.7 that for every $(z, w) \in \Omega$ there exists a viable normalised representation with reduced Fenchel-Nielsen coordinates (z, w).

Remark 4.2.2. We notice that, since the coordinates are reduced, the real part of $s(C)$ takes value in $\mathbb{R}/\mathrm{hl}(C)\mathbb{Z}$, then there are infinitely many (non-reduced) complex Fenchel-Nielsen coordinates which reduce to the same (z, w). This means that the normalized viable representation with reduced coordinates (z, w) is not unique in general.

The following theorem is a restatement of 4.1.7 in terms of viable representation and reduced complex Fenchel-Nielsen coordinates.

Theorem 4.2.4. *There exists an $R_0 = R_0(S) > 0$ such that the following holds for every $R > R_0$. Let $\rho' : \pi_1(S) \longrightarrow PSL(2, \mathbb{C})$ be a viable representation such that $|\mathrm{hl}(C) - R| < 1$ and $|s(C) - 1| < \frac{1}{2R}$ where $(\mathrm{hl}(C), s(C))_{C\in\mathcal{C}}$ are the reduced complex Fenchel-Nielsen coordinates for ρ'. Then, defined*

$z'_C = \mathrm{hl}(C)$ and $w'_C = s(C)$, we have that $(z', w') \in \Omega(R)$. Moreover we have that for each $(z, w) \in \Omega$ there exist a unique normalized viable representation $\rho_{z,w} : \pi_1(S) \longrightarrow PSL(2, \mathbb{C})$ such that:
(1) $z_C = \mathrm{hl}(C)$ and $w_C = s(C)$ where $(\mathrm{hl}(C), s(C))_{C \in \mathcal{C}}$ are the reduced complex Fenchel-Nielsen Coordinates for $\rho_{z,w}$.
(2) The family of representation $\rho_{z,w}$ varies holomorphically in (z, w).
(3) $\rho' = \rho_{z',w'}$.

Remark 4.2.3. We have the uniqueness of the viable representation since the normalization fix the representative for the conjugacy class of $\rho_{z,w}$ and, for R large enough, Ω is sufficiently small so that the $s(C)$ can be related to a unique (non-reduced) complex parameter, once we have fixed (z', w').

Definition 4.2.3. Fix $R > 0$. For every $C \in \mathcal{C}$ let $\zeta_C, \eta_C \in D^2 = \{z \in \mathbb{C} : |z| \leq 1\}$. For any $\tau \in D^2$ we define the normalized viable representation $\rho_\tau : \pi_1(S) \longrightarrow PSL(2, \mathbb{C})$ by its reduced complex Fenchel-Nielsen coordinates
$\mathrm{hl}(C)(\tau) = R + \tau \frac{\zeta_C}{2}$,
$s(C)(\tau) = 1 + \frac{\tau \eta_C}{2R}$.

Then ρ_τ is an holomorphic family of representation with respect to τ, by Theorem 4.2.4. The following Theorem is proved in [KM12]. It is used to control the constant K of the quasi-conformal map which conjugates a quasi-Fuchsian group to a Fuchsian, not so distant in terms of reduced complex Fenchel-Nielsen coordinates.

Theorem 4.2.5. *There exists $\hat{\epsilon} > 0$ such that the following holds. Let S be a closed topological surface with genus $g > 1$ and let $\mathcal{C}$ be a fixed pants decomposition. For any $\hat{\epsilon} > \epsilon > 0$ there exists $R_0(\epsilon, S) > 0$ such that the following holds. Let $\rho : \pi_1(S) \longrightarrow PSL(2, \mathbb{C})$ be a viable representation with reduced complex Fenchel-Nielsen coordinates satisfying*

$$|\,\mathrm{hl}(C) - R| < \epsilon, \text{ and } |s(C) - 1| < \frac{\epsilon}{R}.$$

Then there exists a viable representation $\rho_0 : \pi_1(S) \longrightarrow PSL(2,\mathbb{C})$ *with reduced coordinates* $\mathrm{hl}(C) = R$ *and* $s(C) = 1$*, and a* $K(\epsilon)$ *quasi-conformal map* $f : \partial\mathbb{H}^3 \longrightarrow \partial\mathbb{H}^3$*, such that* $K(\epsilon) \to 1$ *when* $\epsilon \to 0$*, and*

$$f^{-1}\rho(\pi_1(S))f = \rho_0(\pi_1(S)).$$

In particular the group $\rho(\pi_1(S))$ *is quasi-Fuchsian.*

Remark 4.2.4. The proof is based on the Wolpert-Kerckhoff-Series Formula which is the main tool to control geodesic lengths when we deform a quasi-Fuchsian group holomorphically. Such formula was proved by Caroline Series in [Ser01] and it is a generalizations of the previous works of Wolpert [Wol81], Kerckhoff [Ker83], and Kourouniotis [Kou92]. In particular the Theorem above follows from the following statement about the holomorphic family ρ_τ as defined in 4.2.3.

Theorem 4.2.6. *Let* S *be a closed topological surface with genus* $g > 1$ *and let* $\mathcal{C}$ *be a fixed pants decomposition. There exist* $\hat{\epsilon} > 0$ *and* $R_0(\hat{\epsilon}, S) > 0$ *such that the following holds. For each* $C \in \mathcal{C}$ *fix* ζ_C*,*$\eta_C \in D$*. For any* $R > R_0$*, and* $|\tau| < \hat{\epsilon}$ *the group* $\rho_\tau(\pi_1(S))$ *is quasi-Fuchsian, conjugated by the* K *quasi-conformal map* $f_\tau : \partial\mathbb{H}^3 \longrightarrow \partial\mathbb{H}^3$ *to* $\rho_0(\pi_1(S))$*. Moreover*

$$K = \frac{\hat{\epsilon} + |\tau|}{\hat{\epsilon} - |\tau|}.$$

We observe that the statement holds also if we are in a planar situation: we can estimate the quasi conformal constant between two Fuchsian structure in terms of reduced Fenchel Nielsen coordinates.
Now we see how this is used in order to prove the Theorem 4.2.1. It can be proved that the K quasi-conformal maps of $\partial\mathbb{H}^3$ correspond to $C(K)$ quasi-isometries of $\mathbb{H}^3$ (this is not a simple fact to be proved, see for example [Tuk94]). Then $C(K)$ will goes to 1 when $K \to 1$.
Finally we observe that the surface that comes up from Theorem 4.2.5 with exact reduced Fenchel-Nielsen coordinates is a finite cover of $O(R)$. In order to prove the Unified Kahn-Markovic Theorem one has left to find an

immersed surface in M (recall that it can be either an hyperbolic surface or 3 manifold) with a viable representation and reduced complex Fenchel-Nielsen coordinates as in the hypothesis of Theorem 4.2.5. In the following subsection we sketch how this can be done finding a big collection of skew pair of pants with the desired property inside M and then combining them accordingly to the requirements.

Measures on Skew Pants and the Equidistribution Theorems

From now on $M^3 = \mathbb{H}^3/\mathcal{G}$ is a fixed closed hyperbolic three manifold ($\mathcal{G}$ is a suitable Kleinian group). Let Γ (respectively Γ^*) be the set of unoriented (resp. oriented) closed geodesics in M^3. With $-\gamma^*$ we denote the curve $\gamma^* \in \Gamma^*$ with the opposite orientation. Let Π^0 be a topological pair of pants and an homotopy class of map $f : \Pi^0 \longrightarrow M^3$ induced by an injective homomorphism $\rho : \pi_1(\Pi^0) \longrightarrow \pi_1(M^3)$. This determines, with a little abuse of notation, a skew pants, that is a $\rho : \pi_1(\Pi^0) \longrightarrow PSL(2, \mathbb{C})$ up to conjugacy. Fix an orientation and a base point on Π^0. Let $\omega : \Pi^0 \longrightarrow \Pi^0$ be an orientation preserving homeomorphism of order 3 that permutes the three boundary components and fix the base point of Π^0. Then we can denote them with $\omega^i(C)$, $i = 0,1,2$. With ω we also denote the induced isomorphism of $\pi_1(\Pi^0)$. Then we can choose a $c \in \pi_1(\Pi^0)$ such that c correspond to a boundary component $C \subset \partial\Pi^0$ and $\omega^{-1}(c)c\omega(c) = 1$.

Definition 4.2.4 (Admissible Pants Representation)**.** let $\rho : \pi_1(\Pi^0) \longrightarrow PSL(2, \mathbb{C})$ be a faithful representation. We say that ρ is admissible if $\rho(\omega^i(c))$ is an hyperbolic Mobius transformation, and

$$\mathrm{hl}(\omega^i(C)) = \frac{l(\omega^i(C))}{2},$$

where we choose $l(\omega^i(C))$ so that $-\pi < \mathrm{Im}(l(\omega^i(C))) \leq \pi$.

Definition 4.2.5 (Admissible Skew Pants)**.** Let $\rho : \pi_1(\Pi^0) \longrightarrow PSL(2, \mathbb{C})$ be an admissible representation. We define the admissible skew pants Π^1 to be the conjugacy class $\Pi^1 = [\rho]$. The set of all admissible skew pants is denoted by Π. The set of all oriented admissible skew pants is denoted by Π^*.

Definition 4.2.6 (Good Skew Pants)**.** Let $\Pi^0 \in \Pi$, let ϵ,$R > 0$. We say that Π^0 is a Good Pair of Pants if

$$|\,\mathrm{hl}(\omega^i(C)) - R| < \epsilon.$$

We denote with $\Pi_{\epsilon,R}$ the set of the Good Pair of Pants in M^3. $\Pi^*_{\epsilon,R}$ is the set of the oriented Good Pants.

Definition 4.2.7. Let $\Pi^1 \in \Pi$, and let $\rho : \pi_1(\Pi) \longrightarrow \mathcal{G}$ be a representation such that $\Pi^1 = [\rho]$ and $\rho(\omega^i(c)) = A_i$. Define $\rho_1 : \pi_1(\Pi^1) \longrightarrow \mathcal{G}$ by $\rho_1(\omega^{-i}(c)) = A_i^{-1}$. Define $\mathcal{R}(\Pi^1) \in \Pi$ as $\mathcal{R}(\Pi^1) = [\rho_1]$.

One can verify that $\mathcal{R}$ is well defined and it is a fixed point free involution of Π. We observe that ,if $\gamma^*(\Pi^0, \omega^i(c))$ is the oriented geodesic representative of the homotopy class $\omega^i(c)$ with $c \in \rho_{\Pi^0}(\pi_1(\Pi^0))$ where $[\rho_{\Pi^0}] = \Pi^0$, we have that $\gamma^*(\Pi^0, \omega^i(c)) = -\gamma^*(\mathcal{R}(\Pi^0), \omega^{-i}(c))$.
We notice that all these definitions are still true if we replace M^3 with $M^2 = \mathbb{H}^2/\mathcal{F}$, a closed hyperbolic surface (here $\mathcal{F}$ is a suitable Fuchsian group). In particular admissible skew pants are immersed pants in the planar case (see their construction in section 3.2). Now we make some observations about $N^1(\sqrt{\gamma})$ for $\gamma \in \Gamma$.
First suppose that we are in the case of M^3. Then there is a natural identification of $N^1(\sqrt{\gamma})$ with $\mathbb{C}/(2i\pi\mathbb{Z} + \mathrm{hl}(\gamma)\mathbb{Z})$ (after we have fixed a point) which provides an euclidean structure to $N^1(\sqrt{\gamma})$.
If we are in the case of M^2 we have that $N^1(\gamma)$ has two connected components both isomorphic to γ. One can verify that, then also $N^1(\sqrt{\gamma})$ has two connected components (that we denote $N^1_-(\sqrt{\gamma})$ and $N^1_+(\sqrt{\gamma})$) both isomorphic to a circle of length $\mathrm{hl}(\gamma)$.
This is the main difference in between the cases M^2 and M^3. In both the cases $N^1(\sqrt{\Gamma})$ is the disjoint union of all the $N^1(\sqrt{\gamma})$ for $\gamma \in \Gamma$.

Given a space X, we denote with $\mathcal{M}_0(X)$ the space of the Borel measures with compact support. Moreover $\mathcal{M}_0^+(X) \subset \mathcal{M}_0(X)$ denotes the subspace

of positive measures. By $\mathcal{M}_0^{\mathcal{R}}(\Pi^*)$ we define the space of the positive Borel measures with finite support on the set of oriented skew pants Π invariant by the action of $\mathcal{R}$. Let $\lambda(\gamma)$ denote the Lebesgue on $N^1(\sqrt{\gamma})$.
For $A \subset X$ and $\delta > 0$ we define the δ-neighbourhood of A as $\mathcal{N}_\delta(A) = \{x \in X : \exists a \in A \,\text{such that}\, d(x,a) \le \delta\}$.

Definition 4.2.8. Let μ,$\nu \in \mathcal{M}_0^+(X)$ be two measures such that $\mu(X) = \nu(X)$, and let $\delta > 0$. Suppose that for every Borel set $A \subset X$ we have

$$\mu(A) \le \nu(\mathcal{N}_\delta(A)).$$

Then we say that μ and ν are δ-equivalent measures.

We need the following simple result about measures.

Theorem 4.2.7 (Hall's Marriage Theorem)**.** *Let A, B two finite sets with the same cardinality, and let (X,d) be a metric space. Let $\delta > 0$. Let Λ_A and Λ_B the counting measures on A and B respectively. Let $f : A \longrightarrow X$, $g : B \longrightarrow X$ be two maps. Suppose that $f_*\Lambda_A$ and $g_*\Lambda_B$ are δ-equivalent. Then one can find a bijection $h : A \longrightarrow B$ such that $d(f(a), g(h(a))) \le \delta$, for every $a \in A$.*

We define the operator

$$\hat{\partial} : \mathcal{M}_0(\Pi^*) \longrightarrow \mathcal{M}_0(N^1(\sqrt{\Gamma}))$$

as follows. The set Π is a countable set, so every $\mu \in \mathcal{M}_0(\Pi^*)$ is determined by its value on every $\Pi^1 \in \Pi^*$. Let $\gamma_i \in \Gamma$ for $i = 0$,1,2 denote the corresponding oriented geodesic so that (Π^1, γ_i) are marked pair of pants. If $\Pi^1 = [\rho]$ we denote with $\alpha_i^{\Pi^1} \in \mathcal{M}_0(N^1(\sqrt{\Gamma}))$ the atomic measure supported at the point $\text{foot}_{\gamma_i}(\rho) \in N^1(\sqrt{\gamma_i})$. Let

$$\alpha^{\Pi^1} = \sum_{i=0}^{2} \alpha_i^{\Pi^1},$$

then define

$$\hat{\partial}\mu = \sum_{\Pi^0 \in \Pi^*} \mu(\Pi^0)\alpha^{\Pi^0}.$$

When we write $\hat{\partial}\mu(\gamma)$ we mean the restriction of the measure $\hat{\partial}\mu$ on γ.

Remark 4.2.5. We notice that a measure $\mu \in \mathcal{M}_0(\Pi^*)$ can be viewed as a linear combination of $\mathbb{R}\Pi^*$, with the convention that a change of orientation change the sign of the pants.

The following is the main result about measures in skew pants.

Theorem 4.2.8 (Corrected Equidistribution Theorem). *Let M a closed 2 or 3 dimensional hyperbolic manifold. Let $\epsilon > 0$. There exist $q = q(M,\epsilon) > 0$, $R_0 = R_0(M,\epsilon) > 0$ and a polynomial $P(R)$ with coefficients dependent only on ϵ and M, such that for every $R > R_0$ there exists a measure $\mathcal{M}_0^{\mathcal{R}}(\Pi^*_{\epsilon,R})$ with the following properties. Let $\gamma \in \Gamma$ and let $\hat{\partial}\mu(\gamma)$ be the restriction of $\hat{\partial}\mu$ to $N^1(\sqrt{\gamma})$. If $\hat{\partial}\mu(\gamma)$ is not the zero measure then there exists a constant $K_\gamma > 0$ such that the measures $\hat{\partial}\mu(\gamma)$ and $K_\gamma\lambda(\gamma)$ are $P(R)e^{-qR}$-equivalent.*

The Corrected Equidistribution Theorem permits to conclude the proof of the Theorem 4.2.1. We briefly sketch how to conclude.

Sketch of the proof of the Unified Kahn-Markovic Theorem. We may assume that the measure has integer coefficients, and we may think to μ as a formal sum in $\mathbb{Z}\Pi^*_{\epsilon,R}$.

We want to use the Hall's Marriage Theorem to determine couples of skew pants and glue them along a common cuff.

$\mathcal{R}$ defines a (not uniquely determined) partition $\Pi_{\epsilon,R} = \Pi^+_{\epsilon,R} \cup \Pi^-_{\epsilon,R}$. Then, for Π^0 a good skew pants, $\text{foot}_{\gamma^*}(\Pi^0) = \text{foot}_{-\gamma^*}(\mathcal{R}(\Pi^0))$. Moreover the measures $\mu^+ \in \mathcal{M}_0^+(\Pi^+_{\epsilon,R})$ and $\mu^- \in \mathcal{M}_0^+(\Pi^-_{\epsilon,R})$ that are restriction of μ, satisfy $\hat{\partial}\mu = \hat{\partial}\mu^+ + \hat{\partial}\mu^- = 2\hat{\partial}\mu^+ = 2\hat{\partial}\mu^-$.

Fix a boundary $\gamma \in \Gamma$ such that $\hat{\partial}\mu(\gamma)$ is not the zero measure. Then the measure μ (which we think as a formal sum), once restricted to $\Pi_{\epsilon,R}(\gamma)$ (which means that we restrict the measure to the set of good pants with γ as a cuff) can be identified with multiset $X(\gamma)$ of the good pants in the formal sum, taken many times as their coefficients. Then the partition $\Pi_{\epsilon,R} = \Pi^+_{\epsilon,R} \cup \Pi^-_{\epsilon,R}$ gives a partition $X(\gamma) = X^+(\gamma) + X^-(\gamma)$.

One can easily found a coherent definition of $\hat{\partial} : X(\gamma) \longrightarrow N^1(\sqrt{\gamma})$. It can

be proved that the measures $\hat{\partial}\mu^{\pm}(\gamma)$ are exactly the push-forward measures by $\hat{\partial}$ of the counting measures on $X^{\pm}(\gamma)$ (denote such counting measures as $\sigma^{\pm}(\gamma)$). One can define a $g : X^{-}(\gamma) \longrightarrow N^1(\sqrt{\gamma})$ which is the $\hat{\partial}$ composed with an automorphism of $N^1(\sqrt{\gamma})$. We can choose such automorphism, such as send points within distance $1 + \frac{\epsilon}{R}$.
Then the push forwards $\hat{\partial}_*\sigma^{+}(\gamma)$ and $g_*\sigma^{-}(\gamma)$ can be proved to be $P(R)e^{-qR}$ equivalent, using the equality $2\hat{\partial}\mu^{+} = 2\hat{\partial}\mu^{-}$ and the fact that are $P(R)e^{-qR}$ equivalent to the a multiple of the Lebesgue measure. Now we can use the Hall's Marriage Theorem to found an $h : X^{+}(\gamma) \longrightarrow X^{-}(\gamma)$ such that the euclidean distance between $g(h(x))$ and $\hat{\partial}x$ is less then $P(R)e^{-qR}$.
Essentially we gave a pairing between the multisets $X^{+}(\gamma)$ and $X^{-}(\gamma)$. Now we can glue such pairs along γ. We notice that a good choice of the map g gives parameters $|s(C) - 1| \leq \frac{\epsilon}{R}$.

The Corrected Equidistribution Theorem has not the same proof in the two cases $M = M^2$ and $M = M^3$. In fact the proof in M^2 is harder, and the additional difficult will be The Good Pant Homology.
In the case M^3 Theorem 4.2.8 is proved using equidistributional and exponentially mixing arguments. It is the harder part of the work [KM12]. Now we state the strongest equidistributional result for M^2, that can be proved using similar type argument. We need some notation.
Recall that in M^2, the normal unit bundle $N^1(\gamma)$ as two connected components $N^1_{-}(\gamma)$ and $N^1_{+}(\gamma)$. A measure α on $\mathcal{M}^{+}_0 N^1(\sqrt{\gamma})$ decompose into two measures α_{+} and α_{-}, respectively on $N^1_{+}(\sqrt{\gamma})$ and $N^1_{-}(\sqrt{\gamma})$. In particular the Lebesgue measure $\lambda(\gamma)$ on $N^1(\sqrt{\gamma})$ can be thought as the sum $\lambda_{+}(\gamma) + \lambda_{-}(\gamma)$, as well every measure of the type $\hat{\partial}\mu$ decompose to $\hat{\partial}_{+}\mu$ and $\hat{\partial}_{-}\mu$. We denote with $\Gamma_{\epsilon,R} \subseteq \Gamma$ the set of the closed geodesics γ of M^2 such that $|\operatorname{hl}(\gamma) - R| < \epsilon$.

Theorem 4.2.9 (Equidistribution Theorem for Surfaces)**.** *Let $\epsilon > 0$. There exist constants $q = q(\epsilon, M^2) > 0$, $C = C(\epsilon, M^2) > 0$ and $R_0 = R_0(\epsilon, M^2)$ such that for every $R > R_0$ the following is true. Let μ be the measure on $\Pi_{\epsilon,R}$ that assigns to each pants in $\Pi_{\epsilon,R}$ the value* 1*. Then we have*

(1) $\mu(\Pi_{\epsilon,R}) \asymp e^{3R}$.
(2) *For every* $\gamma \in \Gamma_{\epsilon,R}$ *there exist two constants* $K_\gamma^{\pm}\lambda_{\pm}(\gamma)$, *satisfying*

$$|\frac{K_\gamma^+}{K_\gamma^-} - 1| < Ce^{-qR},$$

such the measure $\hat{\partial}_{\pm}\mu(\gamma)$ *is* Ce^{-qR}*-equivalent to* $K_\gamma^{\pm}\lambda_{\pm}(\gamma)$.
(3) *the constants satisfy* $K_\gamma^{\pm} \asymp Re^R$ *for every* $\gamma \in \Gamma_{\epsilon,R}$.

To prove the Correct Equidistribution Theorem for Surfaces we have to produce a measure on $\mathcal{M}_0^{\mathcal{R}}(\Pi_{\epsilon,R})$ such that for every $\gamma \in \Gamma_{\epsilon,R}$, $\hat{\partial}\mu(\gamma)$ is $P(R)e^{-qR}$-equivalent to $K_\gamma\lambda(\gamma)$ for some constant K_γ. The first remark is that we can work with $\mathcal{M}_0^{+}(\Pi_{\epsilon,R})$ instead of $\mathcal{M}_0^{\mathcal{R}}(\Pi_{\epsilon,R})$ since the decomposition $\Pi_{\epsilon,R} = \Pi_{\epsilon,R}^+ \cup \Pi_{\epsilon,R}^-$ is intrinsically given, around any $\gamma \in \Gamma_{\epsilon,R}$, by the two components of the normal unit bundle. Since λ is balanced with respect to the two components of $N^1(\sqrt{\gamma})$ we need that $\hat{\partial}\mu(\gamma)$ has the same total measure on both the components. If we define $\partial\mu = |\hat{\partial}_+\mu| - |\hat{\partial}_-\mu|$, the previous condition became $\partial\mu(\gamma) = 0$.
The Equidistribution Theorem 4.2.9 gives a measure μ_0 on $\Pi_{\epsilon,R}$, such that $\hat{\partial}\mu_0(\gamma)$ is Ce^{-qR}-equivalent to the measures $K_\gamma^{\pm}\lambda_{\pm}(\gamma)$, where the constants satisfy

$$|\frac{K_\gamma^+}{K_\gamma^-} - 1| < Ce^{-qR}.$$

If we think to μ_0 as a multi-set of pants, then we have almost the same number of pants on the two sides of γ. We want to replace μ_0 with $\mu_0 + X$ where $\partial X = -\partial\mu_0$.
Think to the general problem

$$\partial X = \alpha, \tag{4.5}$$

and ask:
(1) for which α there exists a solution X,
(2) If there is bound for X with respect to a bound for α.
If we think to α as a formal sum of curves of $\Gamma_{\epsilon,R}$, then the equation (4.5) imply that $[\alpha]_{H^1(M^2)} = 0$, since, as a set of curves, it is a boundary of a set of

pants. However Theorem 3.2.3 assure that also the converse is true if X is a measure on $\Pi_{\epsilon,R}$; that is, given $[\alpha]_{H^1(M^2)} = 0$, there exists an X which solves $\partial X = \alpha$. If $\alpha = \gamma \in \Gamma_{\epsilon,R}$ (which means that the measure is concentrate in only one geodesic), and $\partial X = \gamma$ then, by the Counting Pants Lemma 3.2.5, the average of the weights assigned by X to every pants in $\Pi_{\epsilon,R}$ adjacent to γ is at least $\frac{1}{Re^R}$. In general it is possible to solve (4.5) asking

$$||X||_\infty \leq P(R)e^{-R}||\alpha||_\infty,$$

where $P(R)$ is a polynomial in R with coefficients depending only on ϵ and M^2 (see the Good Correction Theorem 4.2.10). In all what follows we write $P(R)$ for a suitable polynomial on R.

Theorem 4.2.10 (Good Correction Theorem)**.** *Let $\epsilon > 0$. There exists $R_0 = R_0(\epsilon, M^2) > 0$,and a polynomial $P(R)$ with coefficients depending only on ϵ and M^2 such that for every $R > R_0$ there exists a set $H = \{h_1, \ldots, h_{2g}\} \subset \mathbb{Q}\Gamma_{\epsilon,R}$, and a map $\phi : \Gamma_{\epsilon,R} \longrightarrow \mathbb{Q}\Pi_{\epsilon,R}$ such that*
(1) $h_1, \ldots, h_{2g}$ is a basis for $H_1(S)$,
(2) $\partial(\phi(\gamma)) - \gamma \in \mathbb{Z}H$,
(3)

$$\sum_{\gamma\in\Gamma_{\epsilon,R}} |\phi(\gamma)(\Pi^1)| < P(R)e^{-R}.$$

Remark 4.2.6. The points (1) and (2) are equivalent to the Good Pants Homology Theorem 3.2.3. To prove the point (3) one need to use the Randomization theory developed in [KM13].

Remark 4.2.7. If we have $[\gamma]_{H_1(M^2)} = 0$ then γ is a boundary, then $\partial\phi(\gamma) = \gamma$ because of (2) and the fact that they differ only by boundaries. In particular we have that for any $\mu \in \mathbb{Q}\Pi_{\epsilon,R}$, we have

$$\partial\phi\partial\mu = \partial\mu.$$

We omit the proof of the following technical lemma.

Lemma 4.2.11. *Let λ denote the standard Lebesgue measure on $\mathbb{R}/2R\mathbb{Z}$. If there are δ, $K > 0$ such that a measure α is δ-equivalent to $K\lambda$ then $\alpha + \beta$ is $(\frac{|\beta|}{2K} + \delta)$.equivalent to $(K + \frac{|\beta|}{2R})\lambda$ on $\mathbb{R}/2R\mathbb{Z}$ for every measure β.*

Sketch of the proof of The Correct Equidistribution Theorem for Surfaces To any $\alpha(\gamma) \in \mathcal{M}(N^1(\sqrt{\gamma}))$ we associate the number $|\alpha|(\gamma) = |\alpha_+(\gamma)| + |\alpha_-(\gamma)|$. Let μ be the uncorrected measure from the Equidistribution Theorem 4.2.9. Define $\mu_1 \in \mathcal{M}(\Pi_{\epsilon,R})$. as $\mu_1 = \mu - \phi(\partial\mu)$ where ϕ is the function determined by the Good Correction Theorem 4.2.10. By construction and $(1)-(2)$ of 4.2.10 we have $\partial\mu_1 = 0$ (see the second remark after the statement). From 4.2.9 follows that the measures $\hat{\partial}_\pm\mu(\gamma)$ are Ce^{-qR} equivalents to $K^\pm_\gamma\lambda_\pm(\gamma)$, for some constants K^+_γ and K^-_γ that satisfy the inequality

$$|\frac{K^+_\gamma}{K^-_\gamma} - 1| < Ce^{-qR},$$

and such that $K^\pm_\gamma \asymp e^R$. From (3) of 4.2.10 we have, for any subset $I \subset N^1(\sqrt{\gamma})$

$$\begin{aligned}
|\hat{\partial}\phi(\partial\mu)(\gamma)(I)| &\le |P(R)e^{-R} \sum_{\Pi^1\in\Pi_{\epsilon,R}} \sum_{i=o}^{2} \mathrm{foot}_{\gamma_i}(\Pi^1)(I)| \\
&\le |P(R)e^{-R} \sum_{\Pi^1\in\Pi_{\epsilon,R}} \hat{\partial}_\pm\mu(\gamma)(I)| \\
&\le P(R)e^{-R}(Re^R)K^\pm_\gamma\lambda(\gamma)(\mathcal{N}_{Ce^{-qR}}(I)) \\
&\le P(R)e^R(Ce^{-qR} + \lambda(\gamma)(I)),
\end{aligned}$$

which in particular gives

$$|\hat{\partial}\phi(\partial\mu)|(\gamma) \le P(R)e^{(1-q)R}.$$

Now we can conclude the proof of 4.2.8 using the above Lemma to see that $\hat{\partial}\mu_1(\gamma)$ is $P(R)e^{-qR}$ equivalent to the Lebesgue measure λ on $N^1(\sqrt{\gamma})$.

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