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PHYSICS

Renormalized Energy-Momentum Tensor of Quantized Scalar Perturbations for Compact Objects

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Abstract

In this thesis, we study the Hadamard renormalization of the energy-momentum tensor of quantized scalar perturbations on static spherically symmetric spacetimes. We reduce the problem to an auxiliary scalar field theory in the two-dimensional conformally flat time-radial sector and approximate the Regge-Wheeler-type effective potential as a finite potential well. The field is quantized in terms of normal modes characterized by definite parity, a discrete low-energy spectrum below the potential height, and a continuous high-energy spectrum above it. Owing to the step discontinuity of the potential, the renormalized expectation value of the energy-momentum tensor operator (REVENTO) is explicitly constructed piecewise in the interior and exterior regions following Hadamard's renormalization and by imposing matching conditions on the field and its first derivative at the well's surfaces. Inside the well, the REVENTO has the structure of a Zeldovich stiff fluid with homogeneous proper energy density receiving low- and high-energy contributions. Without the Hadamard parametrix, the low-energy contribution remains finite and regular, but the high-energy contribution becomes ultraviolet divergent. Outside, the low-energy contribution has a dust structure with proper energy density decaying exponentially away from the well. In addition, the exterior REVENTO contains a state-independent contribution with a vacuum-energy structure induced by the potential. It remains to be established whether the exterior high-energy contribution ensures the REVENTO's regularity across the well's surfaces without spurious divergences, and whether these effective perfect-fluid structures persist for more realistic effective-potential models.

Notations & Conventions

Unless explicitly stated otherwise, we adopt the following notations and conventions.

- The pair $(\mathcal{M}, g)$ shall be called a **spacetime** and denotes a real, oriented, connected, pseudo-Riemannian, metric manifold of $\dim \mathcal{M} = D = 1 + d$ spacetime dimensions, d spatial dimensions, and mostly-positive Lorentzian signature $\text{sgn } g = (-, +, \dots, +)$.
- The trio $(\mathcal{M}, g, T)$ denotes a solution of the Einstein field equations (EFE), $G = \kappa^2 T$, where $G = R - \frac{1}{2} Rg$ is the Einstein tensor field and T is the energy-momentum tensor field (EMT).
- Greek indices run from zero $\mu, \nu, \dots \in \{0, 1, \dots, d\}$ and Latin indices from one $i, j, \dots \in \{1, \dots, d\}$.
- The Einstein summation convention is assumed throughout, e.g., $A^\mu B_\mu = \sum_{\mu=0}^d A^\mu B_\mu$.
- All tensor fields of nonzero rank are expressed in bold regardless of their rank, e.g., $u, g, T, \dots$.
- Given a coordinate chart $x^\mu : \mathcal{M} \supseteq \mathcal{A} \mapsto \mathbb{R}$, the Lorentzian inner product, norm, and trace induced by the metric tensor field are respectively denoted by $v \cdot w := g(v, w) = g_{\mu\nu} v^\mu w^\nu = v^\mu w_\mu$, $v^2 := v \cdot v$, and $T := \text{tr}_g T = g_{\mu\nu} T^{\mu\nu} = T^\mu_\mu$. The determinant of the metric is denoted by $g := \det(g_{\mu\nu})$.
- We adopt Planck units $\{\ell_P, m_P, \dots\}$, taking the Planck length ℓ_P and Planck mass m_P as the fundamental units of length and mass, respectively. We also set the speed of light in vacuum dimensionless and equal to unity, $c = 1$. In these units, $[\text{time}] = [\text{length}] = \ell_P$ and $[\text{energy}] = [\text{mass}] = m_P$. Notably, we explicitly retain $\hbar = m_P \ell_P$ and $\kappa^2 \propto G_N = m_P^{-1} \ell_P^{D-3}$ dimensionful.
- Round and square brackets surrounding the indices of a tensor denote the symmetrized and anti-symmetrized part of a tensor, respectively. E.g., for a rank-2 tensor: $C_{(\mu\nu)} := \frac{1}{2!} (C_{\mu\nu} + C_{\nu\mu})$ and $C_{[\mu\nu]} := \frac{1}{2!} (C_{\mu\nu} - C_{\nu\mu})$.

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Chapter 1

Introduction

In this introductory chapter, we motivate and formulate the central problem of this thesis: the Hadamard renormalization of the expectation value of the energy-momentum tensor quantum operator for scalar quantum perturbations on static spherically symmetric spacetimes, idealizing the Regge-Wheeler effective potential as a finite potential well. The SSS spacetimes are of particular interest because they serve as base models for describing the spacetime sourced by compact objects, particularly when rotation and geometrical deformations can be neglected. Meanwhile, the quantization of scalar field perturbations under the framework of QFTCST can be conceived as a first step toward understanding the possible quantum nature of gravity by studying the regime where gravitational effects and the quantum effects of matter cannot both be neglected simultaneously.

1.1 Compact Objects

Among the fundamental interactions of Nature, gravity is distinguished by the fact that the dynamical field is the spacetime geometry itself. Our current best description of this interaction is Einstein's theory of gravity, whose total action is given by:

$$S[g, \Phi] = S_{\text{EH}}[g] + S_{\text{M}}[g, \Phi] = \int_{\Omega \subseteq \mathcal{M}} d^D x \sqrt{-g} \left(\frac{R}{2\kappa^2} + \mathcal{L}_{\text{M}} \right) \quad (1.1)$$

By the principles of general relativity, the Lagrangian density of the matter sources $\mathcal{L}_{\text{M}}$ must be *locally covariant*. That is, constructed from contractions of the matter tensor fields, the metric tensor field, and their respective covariant derivatives [19, 30, 23]. Matter sources gravity by means of its EMT.

Definition 1.1 EMT

Let the components of the EMT T of the matter action $S_{\text{M}}[g, \Phi]$ in eq. (1.1) be defined as:

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{M}}}{\delta g^{\mu\nu}} \quad (1.2)$$

In our units:

$$\begin{aligned} [S] &= m_{\text{P}} \ell_{\text{P}}, & [d^D x \sqrt{-g}] &= \ell_{\text{P}}^D, & [R] &= \ell_{\text{P}}^{-2} \\ [\kappa^2] &= m_{\text{P}}^{-1} \ell_{\text{P}}^{D-3}, & [\mathcal{L}] &= [\mathbf{T}] = m_{\text{P}} \ell_{\text{P}}^{-(D-1)} \end{aligned} \quad (1.3)$$

Imposing the action principle, extremization of eq. (1.1) with respect to (w.r.t.) the metric tensor field g yields the EFE:

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = \kappa^2 T_{\mu\nu} \quad (1.4)$$

If the matter Lagrangian in eq. (1.1) depends at most on the first covariant derivative of the matter tensor field, then extremization of the matter action w.r.t. Φ yields the **Euler-Lagrange** equations of motion (e.o.m.):

$$\frac{\partial \mathcal{L}_M}{\partial \Phi} - \nabla_\mu \left[\frac{\partial \mathcal{L}_M}{\partial (\nabla_\mu \Phi)} \right] = 0 \quad (1.5)$$

For $\Phi = \{\Phi^1, \dots, \Phi^N\}$ independent matter fields, we shall have N Euler-Lagrange e.o.m., which are coupled whenever the matter fields interact with each other. As we shall explore in detail in chapter 2, the linearity of eq. (1.5) is entirely dependent on the structure of the matter Lagrangian. In contrast, the EFE are inherently nonlinear. Solving eq. (1.4) without imposing isometries a priori is virtually impossible in any practical sense.

In this thesis, our interest lies in the static spherically symmetric class of spacetimes. Static meaning $\exists K \in \text{iso}(\mathcal{M}, g)$ that is timelike and hypersurface-orthogonal in an open subset $\mathcal{U} \subseteq \mathcal{M}$, and spherically symmetric meaning there exists a subalgebra $\mathfrak{so}(D-1) \subset \text{iso}(\mathcal{M}, g)$ of spacelike Killing vector fields (KVF) whose orbits are the $(D-2)$ -sphere $\mathcal{S}^{D-2}$. These include the well-known EFE solutions of Minkowski, de Sitter, anti de Sitter, Schwarzschild, Reissner-Nordström, and Bartnik-McKinnon¹.

Proposition 1.1 Static Spherically Symmetric Metric

The metric of a **static spherically symmetric (SSS) spacetime** in the adapted coordinate chart $x^\mu = (t, r, \theta^a)$ where $a \in \{1, \dots, D-2\}$ is given by:

$$ds^2 = -f(r) dt^2 + h(r) dr^2 + r^2 d\Omega_{D-2}^2 \quad (1.6)$$

Furthermore, let the **Regge-Wheeler tortoise coordinate** r_* be defined as:

$$r_* := \int dr \sqrt{\frac{h(r)}{f(r)}} \quad (1.7)$$

In **Regge-Wheeler coordinates** $X^\mu = (t, r_*, \theta^a)$, the metric of a SSS spacetime reads:

$$ds^2 = f(r_*) (-dt^2 + dr_*^2) + r^2(r_*) d\Omega_{D-2}^2 \quad (1.8)$$

Proposition 1.1 follows straightforwardly by adapting the time coordinate as the parameter along the static KVF $K_t = d/dt$, the angular coordinates to the induced metric of the unit $(D-2)$ -sphere, $d\Omega_{D-2}^2 = \Omega_{ab}(\theta^c) d\theta^a d\theta^b$, and exploiting the Lie derivative identity $\mathcal{L}_K g = 0$.

The Regge-Wheeler coordinates show that the spacetime contains the isometry subalgebra $\mathbb{R} \oplus \mathfrak{so}(d) \subseteq \text{iso}(\mathcal{M}, g)$ and that the time-radial sector is conformally flat. In this chart, the areal radius r is regarded as a scalar function obtained by inverting the relation $r_* = r_*(r)$ in eq. (1.7), which is only well defined outside the static Killing horizon. For SSS spacetimes, the static Killing horizon coincides with the static limit ergosurface and, if any, with the event horizon (under suitable regularity and analytical conditions).

¹The Bartnik-McKinnon metric is a particularly useful example for which $h(r) \neq f^{-1}(r)$ in general. This spacetime is a solution of the EFE sourced by an $\text{SU}(2)$ Yang-Mills field in $D = 4$ [3, 29].

SSS spacetimes are of particular interest because they are the base model for describing the spacetime sourced by **compact objects**—intending any astrophysical self-gravitating body from stars to black holes—concretely when their rotation and geometrical deformations can be neglected. As a first approximation, the compact object itself can be modeled as a **perfect fluid**. Namely, a fluid in local thermodynamic equilibrium that is free of anisotropic stresses and dissipative fluxes.

Definition 1.2 Perfect Fluid's EMT

If u, ρ, p denote the timelike normalized D -velocity, proper energy density, and isotropic proper pressure, respectively, let the components of a **perfect fluid's EMT** be defined as:

$$T_{\mu\nu} = pg_{\mu\nu} + (\rho + p) u_\mu u_\nu \quad (1.9)$$

In the rest frame of the perfect fluid, $u^{\mu'} = (1, 0, \dots, 0)$. By the equivalence principle, $g_{\mu'\nu'} = \eta_{\mu'\nu'}$ in a sufficiently small neighborhood. Therefore:

$$T_{\mu'\nu'} = \text{diag}(\rho, p, \dots, p) \quad (1.10)$$

If the metric is on-shell, then the EFE enforce the source's EMT to be locally conserved, $\text{div } T = 0$, and to inherit the spacetime isometries, $\mathcal{L}_K T = 0$ for all² $K \in \text{iso}(\mathcal{M}, g, T)$. When definition 1.2 sources a SSS spacetime, the fluid is said to be in *hydrostatic equilibrium*: the fluid elements move along the congruence of the static KVF, $u \parallel K_t$, and the fluid itself is spherically symmetric, $\rho(r), p(r)$. One of the simplest models for the $(\mathcal{M}, g, T)$ of a compact object is then as follows.

Proposition 1.2 SSS & Homogeneous Perfect Fluid Sourced Spacetime

Consider a perfect fluid with a homogeneous proper energy density $\rho_0 \in \mathbb{R}$ within a finite region of areal radius $r = R$ and vanishing proper pressure at the boundary:

$$\rho = \rho_0 \Theta(R - r) \quad (1.11a)$$

$$p(R) = 0 \quad (1.11b)$$

If this perfect fluid uniquely sources a static and spherically symmetric spacetime via the EFE, then the solution to proposition 1.1 is given by:

$$f(r) = \begin{cases} \frac{1}{4} \left[(D-1)\sqrt{1 - \varrho R^2} - (D-3)\sqrt{1 - \varrho r^2} \right]^2, & 0 \leq r \leq R \\ 1 - \frac{\varrho R^{D-1}}{r^{D-3}}, & r \geq R \end{cases} \quad (1.12a)$$

$$h(r) = \begin{cases} (1 - \varrho r^2)^{-1}, & 0 \leq r \leq R \\ \left(1 - \frac{\varrho R^{D-1}}{r^{D-3}}\right)^{-1}, & r \geq R \end{cases} \quad (1.12b)$$

And the pressure profile is given by:

²The converse statement need not be true: the source's EMT may (and often does) admit symmetries that are not isometries.

$$p(r) = \rho_0(D-3) \frac{\sqrt{1-\varrho r^2} - \sqrt{1-\varrho R^2}}{(D-1)\sqrt{1-\varrho R^2} - (D-3)\sqrt{1-\varrho r^2}} \quad (1.13)$$

Where^a:

$$\varrho = \frac{2\kappa^2}{(D-1)(D-2)}\rho_0 \quad (1.14)$$

^aThe pole at $D = 2$ shows the “degenerate” nature of Einstein’s gravity in two dimensions where $G = 0$ identically. Strictly, this result is valid for $D \geq 3$.

Proposition 1.2 follows from the tt - and rr -components of the EFE combined with $\text{div } T = 0$, imposing the pressure boundary condition and metric regularity at the center $h(r=0) = 1$, and matching the interior solution with the exterior vacuum solution. Cf. [9] for an explicit proof. In our units:

$$[u] = 1, \quad [\rho] = [p] = m_P \ell_P^{-(D-1)}, \quad [\varrho] = \ell_P^{-2} \quad (1.15)$$

The solution in the interior region $r \leq R$ is the D -dimensional **interior Schwarzschild metric**:

$$ds^2(r \leq R) = -\frac{1}{4} \left[(D-1)\sqrt{1-\varrho R^2} - (D-3)\sqrt{1-\varrho r^2} \right]^2 dt^2 + \frac{1}{1-\varrho r^2} dr^2 + r^2 d\Omega_{D-2}^2 \quad (1.16)$$

While the solution in the exterior region $r \geq R$ is the D -dimensional Schwarzschild metric, where the mass parameter has been specified in terms of the perfect fluid’s density:

$$ds^2(r \geq R) = - \left(1 - \varrho R^2 \frac{R^{D-3}}{r^{D-3}} \right) dt^2 + \left(1 - \varrho R^2 \frac{R^{D-3}}{r^{D-3}} \right)^{-1} dr^2 + r^2 d\Omega_{D-2}^2 \quad (1.17)$$

The exterior spacetime is asymptotically flat, and in both regions we recover Minkowski spacetime when no matter is present, $\varrho \propto \rho_0 \rightarrow 0$.

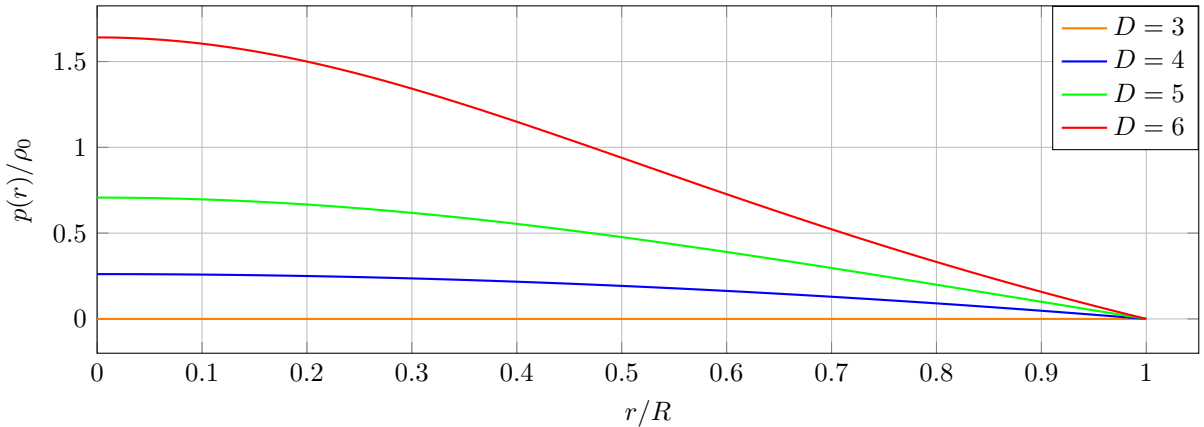


Figure 1.1: Normalized pressure profile $p(r)/\rho_0$ of proposition 1.2 for $D \in \{3, 4, 5, 6\}$ at fixed compactness $\mathcal{C} = 0.5$ in terms of the areal radius coordinate normalized by the fluid’s radius R . In $D = 3$, the fluid is everywhere pressureless.

As shown in fig. 1.1, the pressure profile of eq. (1.13) is a monotonically decreasing function that is minimal at the surface, $p(r = R) = 0$, and maximal at the center, $p(r = 0)$. If the pressure of proposition 1.2 is to be non-negative and finite, $0 \leq p(r) < \infty$, then the **compactness** $\mathcal{C}$ of the source must satisfy the **Buchdahl bound**:

$$\mathcal{C} := \varrho R^2 < \frac{4(D-2)}{(D-1)^2} =: \mathcal{C}_B \quad (1.18)$$

Figure 1.2 shows the divergence of the pressure at the **Buchdahl limit** $\mathcal{C} = \mathcal{C}_B$. Beyond the Buchdahl limit $\mathcal{C} > \mathcal{C}_B$, the central pressure becomes negative, and there is a pole where the pressure diverges before reaching the surface. I.e., the Buchdahl bound $\mathcal{C} < \mathcal{C}_B \leq 1$ is the condition on the free parameters ρ_0, R for proposition 1.2 to describe a regular configuration.

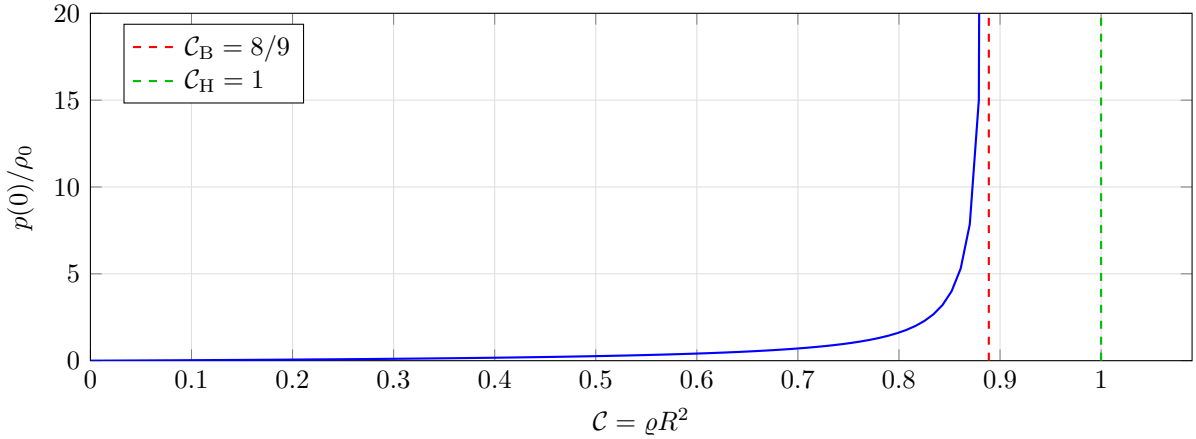


Figure 1.2: Normalized central pressure $p(0)/\rho_0$ of proposition 1.2 as a function of the compactness $\mathcal{C}$ in $D = 4$. The central pressure diverges as the compactness approaches the Buchdahl limit from the left $\mathcal{C} \rightarrow \mathcal{C}_B = 8/9$. The value $\mathcal{C}_H = 1$ corresponds to the location of the static Killing horizon coinciding with the fluid's surface.

1.2 Scalar Perturbations

Beyond more accurate modeling for the spacetime metric and matter EMT of compact objects, the next level of complexity is to consider a **perturbation** on the **background spacetime** $(\mathcal{M}, g, T)$. The perturbation has an associated energy content δT and thus modifies the spacetime geometry by $g \rightarrow g + \delta g$. The field theory of the perturbation is computed on the background spacetime, and then the perturbed spacetime is determined by solving the EFE $G[g + \delta g] = \kappa^2 (T + \delta T)$. The simplest perturbation is a free point particle, which has been studied extensively [24]. In this thesis, we will instead consider perturbations induced by a real scalar field on the background spacetime defined by proposition 1.1.

Definition 1.3 Scalar Field's Action

Let the action of a real scalar field $\Phi = \Phi(x)$ on $(\mathcal{M}, g)$ be defined as:

$$S[g, \Phi] = \int_{\Omega \subseteq \mathcal{M}} d^D x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi - V(\Phi, g, x) \right] \quad (1.19)$$

In definition 1.3 we have taken the canonical kinetic term, $\mathcal{L} \supset \mathcal{K}(\Phi) = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi = -\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi$. We allow for the potential energy density $\mathcal{L} \supset V(\Phi, g, x)$ to depend on spacetime points, on the metric tensor

field and its derivatives, and on the scalar field but not its derivatives. Furthermore, we restrict the potential to be real, and require that any dependence on the metric be locally covariant. The physical dimensions of the scalar field are:

$$[\Phi] = \sqrt{m_P \ell_P^{-(D-3)}} \quad (1.20)$$

Proposition 1.3 Scalar Field's E.O.M.

The variation of the action in definition 1.3 w.r.t. the scalar field Φ yields the e.o.m.,

$$\square \Phi - \frac{\partial V}{\partial \Phi} = 0 \quad (1.21)$$

where the d'Alembert operator $\square$ is given by:

$$\square := g^{\mu\nu} \nabla_\mu \nabla_\nu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu) \quad (1.22)$$

Proposition 1.3 is immediate from combining definition 1.3 and eq. (1.5). The explicit form of the d'Alembert operator follows by exploiting the compatibility of the metric and Jacobi's formula.

Proposition 1.4 Scalar Field's EMT

If the potential does not depend on the metric, $V = V(\Phi, x)$, then the variation of the action in definition 1.3 w.r.t. the metric tensor field g yields the EMT:

$$T_{\mu\nu} = \nabla_\mu \Phi \nabla_\nu \Phi - g_{\mu\nu} \left[\frac{1}{2} g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi + V(\Phi, x) \right] \quad (1.23)$$

Proof. Using the variation of the metric determinant, $\delta_g \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}$, the variation of the action w.r.t. g is:

$$\delta_g S = \int d^D x [\delta_g (\sqrt{-g}) \mathcal{L} + \sqrt{-g} (\delta_g \mathcal{L})] = \int d^D x \sqrt{-g} \left[-\frac{1}{2} g_{\mu\nu} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} \right] \delta g^{\mu\nu}$$

Definition 1.1 then yields the standard structure of a minimally coupled matter field in which there is no explicit dependence on curvature tensors beyond the metric:

$$T_{\mu\nu} = -2 \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} + g_{\mu\nu} \mathcal{L}$$

The desired result follows directly from $\mathcal{L} = -\frac{1}{2} g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi - V(\Phi, x)$ and $\partial \mathcal{L} / \partial g^{\mu\nu} = -\frac{1}{2} \nabla_\mu \Phi \nabla_\nu \Phi$. $\square$

Taking the Lie derivative of proposition 1.4 shows a general feature of the EMT: there is no guarantee that the symmetries of $T(\Phi)$ are also symmetries of the matter field Φ and vice versa. Cf. [27] for an interesting discussion of this phenomenon.

To compute the contribution to the EMT of the potential's metric dependence, the relation $V = V(g)$ must be specified. As an example, if $V(\Phi, g) = \frac{1}{2} \xi R[g] \Phi^2$, then one must add to proposition 1.4 the term:

$$\begin{aligned}
T_{\mu\nu}^\xi &= \xi (G_{\mu\nu} - \nabla_\mu \nabla_\nu + g_{\mu\nu} \square) \Phi^2 \\
&= 2\xi \left(\frac{1}{2} G_{\mu\nu} \Phi^2 - \nabla_\mu \Phi \nabla_\nu \Phi - \Phi \nabla_\mu \nabla_\nu \Phi + g_{\mu\nu} g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi + g_{\mu\nu} \Phi \square \Phi \right)
\end{aligned} \tag{1.24}$$

The perturbation problem is determined by solving proposition 1.3 for the chosen background, plugging the solution into proposition 1.4, and then solving (at least a linearization of) the EFE for the perturbed spacetime. A subtler situation is encountered for **quantum perturbations**, which is our case of interest.

1.3 Quantum Perturbations

A first step toward understanding the elusive quantum nature of gravity is to study the regime where gravitational effects and the quantum effects of matter cannot both be neglected simultaneously. Using the well-established formalisms of quantum field theory (QFT) and Einstein's theory of gravity, **quantum field theory in curved spacetimes (QFTCST)** provides a framework for describing such a regime. QFTCST is a background-field approach where we construct the QFT of a given classical field theory $S[\Phi]$ on a fixed and classical background spacetime $(\mathcal{M}, g, T_{\text{bg}})$. The field Φ is treated as a perturbation whose EMT $T(\Phi)$ does not contribute to the background's matter distribution T_{bg} and thus does not source the background EFE. If the perturbation of the metric and its quantization are considered, it is treated as an additional dynamical field on the background spacetime that does not determine its causal structure [7, 2].

The field quantization $\Phi \rightarrow \hat{\Phi}$ leads to the EMT quantization $T(\Phi) \rightarrow \hat{T}(\hat{\Phi})$. Since the tensor field $T(\Phi)$ is generally nonlinear in the classical field and its derivatives, the quantum operator $\hat{T}(\hat{\Phi})$ inherits this nonlinear dependence on the quantum field. However, a quantum field is an operator-valued distribution, and the product of distributions at the same spacetime point is generally ill-defined. To solve this issue, we adopt the **point-splitting** approach where products of tensor fields are turned into bitensor fields, $\Phi^2(x) \rightarrow \Phi(x) \Phi(x')$. The classical EMT is recovered by acting on the bitensors with a bitensorial differential operator and then taking the coincidence limit $x' \rightarrow x$. Upon quantization, the Feynman propagator provides the two-point function for constructing the quantized EMT.

Definition 1.4 Feynman Propagator

Let t, t' be the chosen time coordinates for the spacetime events $x, x' \in \mathcal{M}$. We define the **Feynman propagator** $G^{\text{F}}(x, x')$ of a bosonic quantum field $\hat{\Phi}(x)$ w.r.t. a given quantum state as:

$$G^{\text{F}}(x, x') := i \langle T \hat{\Phi}(x) \hat{\Phi}(x') \rangle = i \left[\Theta(t - t') \langle \hat{\Phi}(x) \hat{\Phi}(x') \rangle + \Theta(t' - t) \langle \hat{\Phi}(x') \hat{\Phi}(x) \rangle \right] \tag{1.25}$$

Here, T denotes the *time ordering* and Θ Heaviside's step function^a.

^aCf. definition A.4.

In definition 1.4 we have denoted the expectation value of a quantum operator $\hat{f}$ w.r.t. a quantum state $|\psi\rangle$ as $\langle \hat{f} \rangle = \langle \psi | \hat{f} | \psi \rangle$. Although our notation suppresses $|\psi\rangle$, the state-dependence of the Feynman propagator in the absence of a preferred vacuum state is crucial for the arguments that follow.

Definition 1.5 EVENTO

Let $\mathcal{T}_{\mu\nu}(x, x')$ be defined as the bitensorial differential operator constructed by the point splitting of the classical EMT. The expectation value of the EMT quantum operator $\hat{T}_{\mu\nu}(x)$ w.r.t. a given quantum state is defined as:

$$\langle \hat{T}_{\mu\nu}(x) \rangle := \lim_{x' \rightarrow x} \mathcal{T}_{\mu\nu}(x, x') [-iG^F(x, x')] \quad (1.26)$$

Definitions 1.4 and 1.5 have dimensions:

$$[G^F] = m_P \ell_P^{-(D-3)}, \quad [\langle \hat{\mathbf{T}} \rangle] = m_P \ell_P^{-(D-1)} \quad (1.27)$$

Alas, in general the Feynman propagator is singular in the coincidence limit, $G^F(x, x') \xrightarrow{x' \rightarrow x} \infty$, and definition 1.5 inherits this feature. To obtain a UV-finite expression, a suitable regularization and renormalization prescription $\langle \hat{\mathbf{T}}(\hat{\Phi}) \rangle \rightarrow \langle \hat{\mathbf{T}}(\hat{\Phi}) \rangle_{\text{ren}}$ is consequently required. We shall follow the **Hadamard renormalization** procedure, described in detail in section 3.3. The general idea is to construct the theory's regularized propagator by expanding the Feynman propagator near the coincidence limit, identifying the *Hadamard parametrix* which encodes the UV divergences, and subtracting this singular structure. Henceforth, we call the renormalized expectation value of the EMT quantum operator, $\langle \hat{\mathbf{T}}(\hat{\Phi}) \rangle_{\text{ren}}$, the **REVEMTO**.

Only after the theory's REVEMTO has been computed can we determine the **backreaction** of the quantum perturbation on the spacetime geometry, $g \rightarrow \tilde{g} = g + \delta g$. Indeed, the energy content of the quantum field $\hat{\Phi}$ is postulated to be small relative to that of the background fields, but it is certainly not zero. In QFTCST, the coupling between the metric and the quantum field is described by the **semiclassical EFE**:

$$G(\tilde{g}) = \kappa^2 T_{\text{bg}}(\tilde{g}) + \kappa^2 \langle \hat{\mathbf{T}}(\hat{\Phi}, \tilde{g}) \rangle_{\text{ren}} \quad (1.28)$$

The QFTCST postulates hold as long as the quantum contribution to the backreacted EMT remains sufficiently small such that the correction δg does not invalidate the classical background approximation itself. This requires that the system's characteristic length and energy scales lie sufficiently far away from the Planck scale, where the possible quantum nature of gravity is expected to emerge. Within the regime $L \gg \ell_P$ and $E \ll m_P$, QFTCST provides an effective description of quantum matter interacting with classical gravity, leaving the quantization of gravity beyond its scope.

1.4 Scalar Fields in Static Spherically Symmetric Spacetimes

The general objective of this thesis can now be stated concretely: we seek to construct the QFT of a real scalar field in a SSS spacetime and determine the REVEMTO.

Proposition 1.5 Scalar Field in a SSS Spacetime

Consider the real scalar field action of definition 1.3 on a SSS spacetime where the metric is given in the form of proposition 1.1 and the potential is quadratic in the field, with $U(g, r)$ being a scalar function of the metric and the areal radius:

$$S[\Phi, g] = \int_{\Omega \subseteq \mathcal{M}} d^D x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} U(g, r) \Phi^2 \right] \quad (1.29)$$

The normal mode solutions to the e.o.m. in proposition 1.3 are given by:

$$\Phi_{\omega \ell \mathbf{m}}(t, r, \theta^a) = \frac{1}{r^{D/2-1}} Y_{\ell \mathbf{m}}(\theta^a) e^{-i\omega t} u_{\omega \ell}(r) \quad (1.30)$$

The angular component $Y_{\ell \mathbf{m}}$ denotes the hyperspherical harmonics, and the radial component $u_{\omega \ell}$ of the normal modes is given by the solutions of the following equation in the tortoise coordinate,

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V_\ell(r_*) \right] u_{\omega \ell}(r_*) = 0 \quad (1.31)$$

where the **effective potential** $V_\ell[r(r_*)]$ is given by:

$$V_\ell(r) = f(r) \left[U(g, r) + \frac{\ell(\ell + D - 3)}{r^2} \right] + \frac{D-2}{4} \frac{1}{r} \frac{d}{dr} \left[\frac{f(r)}{h(r)} \right] + \frac{(D-2)(D-4)}{4} \frac{1}{r^2} \frac{f(r)}{h(r)} \quad (1.32)$$

Proof. For the prescribed action, the potential is $V(\Phi, g, x) = \frac{1}{2} U(g, r) \Phi^2$. By proposition 1.3, the e.o.m. are $[\square - U(g, r)] \Phi = 0$. In the adapted spherical coordinates, the d'Alembertian of proposition 1.1's metric splits into three parts, one for each sector. The determinant factor is $\sqrt{-g} = r^{D-2} \sqrt{f(r)h(r)\Omega(\theta^a)}$, so the e.o.m. take the explicit form,

$$\left[-\frac{1}{f} \partial_t^2 + \frac{1}{r^{D-2} \sqrt{f} h} \partial_r \left(r^{D-2} \sqrt{\frac{f}{h}} \partial_r \right) + \frac{1}{r^2} \square_{S^{D-2}} - U(g, r) \right] \Phi(t, r, \theta^a) = 0 \quad (1.33)$$

where $\square_{S^{D-2}}$ denotes the d'Alembertian on the $(D-2)$ -sphere. Every scalar function constructed from contractions of the metric is invariant under isometries. In this chart, any scalar function of the areal radius is also invariant under the SSS isometries. Hence, $U(g, r)$ is invariant under the SSS isometries and, in this chart, it only depends on r . This implies eq. (1.33) is a linear and separable differential equation under the ansatz $\Phi(x) = A(r)T(t)\Theta(\theta^a)$. After some manipulations, we find:

$$-r^2 \left[-\frac{1}{Tf} \partial_t^2 T + \frac{1}{Ar^{D-2} \sqrt{f} h} \partial_r \left(r^{D-2} \sqrt{\frac{f}{h}} \partial_r A \right) - U(g, r) \right] = \frac{1}{\Theta} \square_{S^{D-2}} \Theta \equiv \lambda$$

The angular function must then be one of the hyperspherical harmonics, which are defined as the eigenfunctions of the d'Alembertian operator on the n -sphere:

$$\square_{S^{D-2}} \Theta = \lambda \Theta \implies \lambda = -\ell(\ell + D - 3) \text{ and } \Theta(\theta^a) = Y_{\ell \mathbf{m}}(\theta^a)$$

The time-radial part is separable as well:

$$f \left[\frac{1}{Ar^{D-2} \sqrt{f} h} \partial_r \left(r^{D-2} \sqrt{\frac{f}{h}} \partial_r A \right) - U(g, r) - \frac{\ell(\ell + D - 3)}{r^2} \right] = \frac{1}{T} \partial_t^2 T \equiv -\omega^2$$

With this choice of separation constant, the time function must solve the harmonic oscillator equation. A basis for the solutions is given by $\{e^{-i\omega t}, \text{c.c.}\}_{\omega>0}$, and thus:

$$\left(\frac{d^2}{dt^2} + \omega^2 \right) T = 0 \implies T(t) = e^{-i\omega t}$$

The remaining equation yields the radial part:

$$\sqrt{\frac{f}{h}} \frac{d}{dr} \left(r^{D-2} \sqrt{\frac{f}{h}} \frac{d}{dr} A \right) + \left(\omega^2 - f \left[U + \frac{\ell(\ell + D - 3)}{r^2} \right] \right) r^{D-2} A = 0$$

After the radial rescaling $A(r) = \frac{u(r)}{r^{D/2-1}}$ and switching to the tortoise coordinate such that $\frac{d}{dr_*} = \sqrt{\frac{f}{h}} \frac{d}{dr}$, a lengthy manipulation yields:

$$\frac{d^2 u}{dr_*^2} + \left[\omega^2 - f \left(U + \frac{\ell(\ell + D - 3)}{r^2} \right) - \frac{D-2}{4r} \frac{d}{dr} \left(\frac{f}{h} \right) - \frac{(D-2)(D-4)}{4r^2} \frac{f}{h} \right] u = 0$$

This is a one-dimensional time-independent Schrödinger-like equation $\left(-\frac{d^2}{dr_*^2} + V_\ell \right) u = \omega^2 u$ where the effective potential is given by eq. (1.32). $\square$

Given a SSS spacetime, the full solution of the scalar field's normal modes $\Phi_{\omega\ell m}$ is given by solving eq. (1.31) for the radial component $u_{\omega\ell}$ of the modes. The latter is a Schrödinger-like ODE in the tortoise coordinate, meaning that once $f(r)$ and $h(r)$ have been specified, we must invert the relation $r_* = r_*(r)$ in proposition 1.1 and express the effective potential in this variable, $V_\ell[r(r_*)]$. The structure of eq. (1.31) is known as the **Regge-Wheeler equation**, where analogous “master equations” arise for fields of different spin, including scalar ($s = 0$), electromagnetic ($s = 1$), and gravitational ($s = 2$) perturbations in SSS spacetimes [25, 8, 15, 5].

The dimensions of the potentials are:

$$[U(g, r)] = [V_\ell(r)] = \ell_P^{-2} \quad (1.34)$$

Equation (1.32) plays an analogous role to the effective potential experienced by a free-falling point particle. In particular, it comprises three contributions:

1. An **effective mass** term (as justified in the Klein-Gordon limit $g \rightarrow \eta$), which vanishes in the purely-kinetic theory:

$$V^{\text{mass}}(r) = f(r) U(g, r) \quad (1.35)$$

2. A **centrifugal barrier**, which vanishes for s -modes:

$$V^{\text{ang}}(r) = f(r) \frac{\ell(\ell + D - 3)}{r^2} \quad (1.36)$$

3. And a **background** term, which is independent of the parameters $\{U, \omega, \ell, m\}$ of the scalar field:

$$V^{\text{bg}}(r) = \frac{D-2}{4r} \left\{ \frac{d}{dr} \left[\frac{f(r)}{h(r)} \right] + \frac{(D-4)}{r} \frac{f(r)}{h(r)} \right\} \quad (1.37)$$

Provided the set $\{\Phi_{\omega\ell m}, \text{c.c.}\}$ is complete such that it spans the complexified space of solutions, and if the spectrum of ω is completely continuous, then:

$$\Phi(t, r, \theta^a) = \sum_{\ell, m} \int_0^\infty \frac{d\omega}{2\pi} [A_{\omega\ell m} \Phi_{\omega\ell m}(t, r, \theta^a) + A_{\omega\ell m}^* \Phi_{\omega\ell m}^*(t, r, \theta^a)] \quad (1.38)$$

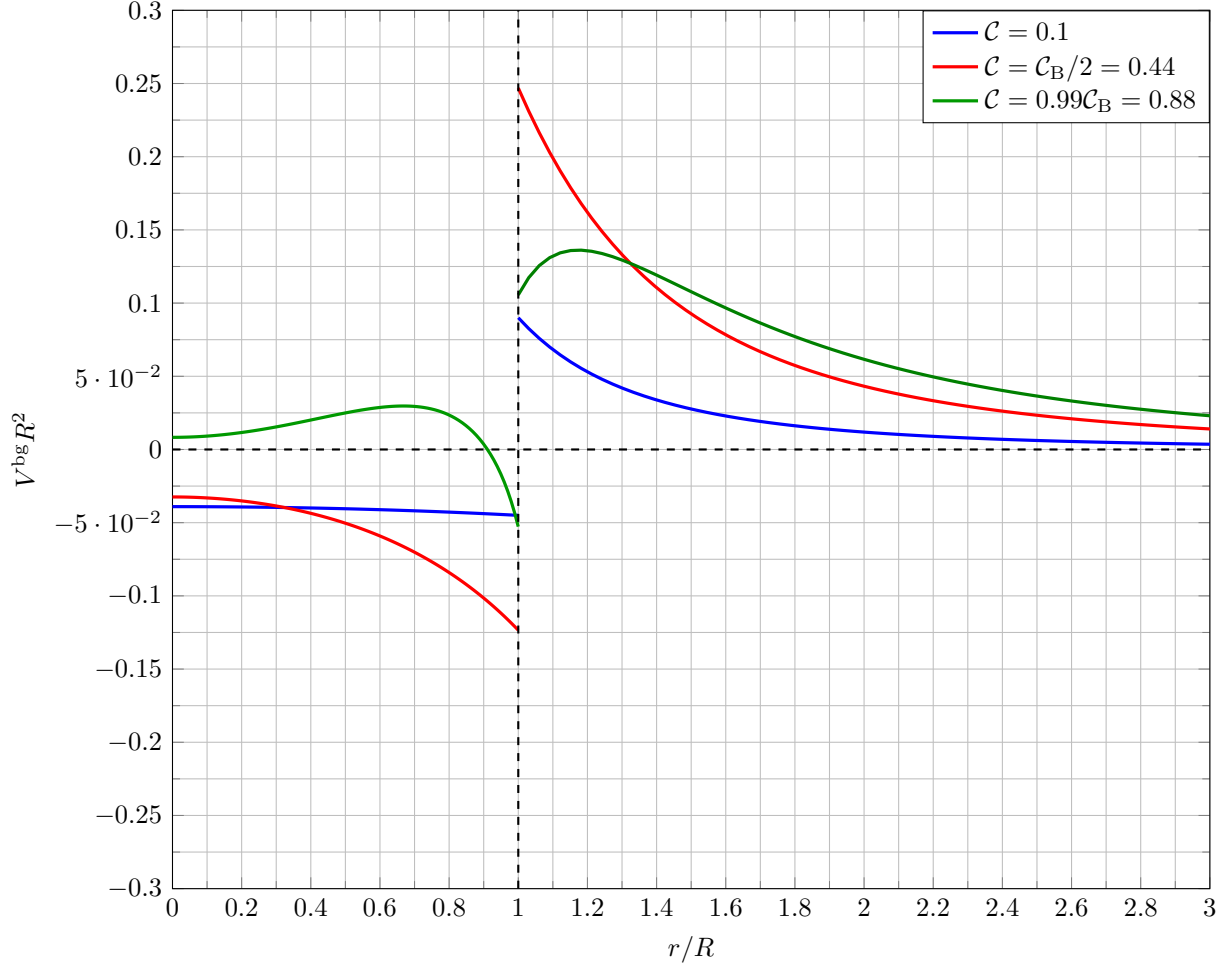


Figure 1.3: Normalized background effective potential $V^{\text{bg}} R^2$ for a scalar field propagating in the static spherically symmetric spacetime sourced by a homogeneous perfect fluid in $D = 4$. The three curves correspond to a low, intermediate, and high compactness relative to the Buchdahl limit $\mathcal{C}_B = 8/9$. The horizontal dashed line denotes the vanishing Minkowskian potential, while the vertical dashed line marks the fluid's surface at $r = R$.

Let the normal modes of proposition 1.5 be written as:

$$\Phi_{\omega \ell m}(t, r_*, \theta^a) = \frac{1}{r^{D/2-1}(r_*)} Y_{\ell m}(\theta^a) \psi_{\omega \ell}(t, r_*) \quad (1.39)$$

We will call $\psi_{\omega \ell}(t, r_*) := e^{-i\omega t} u_{\omega \ell}(r_*)$ the time-radial sector of the normal modes.

Definition 1.6 $(1+1)$ -dimensional Auxiliary Theory

For each ℓ , the time-radial sector of proposition 1.5 defines a scalar field theory for the **auxiliary field** Ψ_ℓ in the $(1+1)$ -dimensional Minkowski spacetime $(\mathcal{M}, \eta)$ and in $x^\mu = (t, r_*)$ coordinates:

$$S_\ell[\Psi_\ell, \eta] = \int dt dr_* \left[-\frac{1}{2} \eta^{\mu\nu} \partial_\mu \Psi_\ell \partial_\nu \Psi_\ell - \frac{1}{2} V_\ell(r_*) \Psi_\ell^2 \right] \quad (1.40)$$

The variation of definition 1.6 w.r.t. to the field Ψ_ℓ yields the e.o.m.:

$$[-\partial_t^2 + \partial_{r_*}^2 - V_\ell(r_*)] \Psi_\ell(t, r_*) = 0 \quad (1.41)$$

The normal modes of the auxiliary theory are precisely $\psi_{\omega\ell}(t, r_*)$. If the set $\{\psi_{\omega\ell}, \text{c.c.}\}$ is complete in the complexified auxiliary solution space, then:

$$\Psi_\ell(t, r_*) = \int_0^\infty \frac{d\omega}{2\pi} [B_{\omega\ell} \psi_{\omega\ell}(t, r_*) + B_{\omega\ell}^* \psi_{\omega\ell}^*(t, r_*)] \quad (1.42)$$

The dimensions of the auxiliary field are $[\Psi_\ell] = \sqrt{m_P \ell_P}$.

Proposition 1.6 SSS Propagator

If $G_\ell^{1+1}(t, r_*, t', r'_*)$ denotes the Feynman propagator of definition 1.6 w.r.t the vacuum state defined by the $\psi_{\omega\ell}(t, r_*)$ modes, then the Feynman propagator of eq. (1.38) w.r.t. to the vacuum state defined by the $\Phi_{\omega\ell\mathbf{m}}(t, r_*, \theta^a)$ modes is given by^a:

$$G^F(x, x') = \sum_{\ell=0}^\infty \frac{\sum_{\mathbf{m}} Y_{\ell\mathbf{m}}(\theta^a) Y_{\ell\mathbf{m}}^*(\theta^{a'})}{(rr')^{D/2-1}} G_\ell^{1+1}(t, r_*, t', r'_*) \quad (1.43)$$

^aAgain, assuming the sets of normal modes are complete in their respective spaces.

Proof. The quantization of a classical field theory on a curved spacetime will be described in detail in chapter 3, but the general procedure is to promote the field in eq. (1.38) to an operator(-valued distribution),

$$\hat{\Phi}(x) = \sum_{\ell, \mathbf{m}} \int_0^\infty \frac{d\omega}{2\pi} [\hat{A}_{\omega\ell\mathbf{m}} \Phi_{\omega\ell\mathbf{m}}(x) + \hat{A}_{\omega\ell\mathbf{m}}^\dagger \Phi_{\omega\ell\mathbf{m}}^*(x)]$$

and to impose the ladder algebra on the commutators of the expansion coefficients:

$$[\hat{A}_{\omega\ell\mathbf{m}}, \hat{A}_{\omega'\ell'\mathbf{m}'}^\dagger] = 2\pi\hbar \delta(\omega - \omega') \delta_{\ell\ell'} \delta_{\mathbf{m}\mathbf{m}'}$$

Taking the two-point function w.r.t. the quantum state defined by $\hat{A}_{\omega\ell\mathbf{m}} |0\rangle = 0$, the only non-vanishing term is the one proportional to $\langle \hat{A}_{\omega\ell\mathbf{m}} \hat{A}_{\omega'\ell'\mathbf{m}'}^\dagger \rangle = 2\pi\hbar \delta(\omega - \omega') \delta_{\ell\ell'} \delta_{\mathbf{m}\mathbf{m}'}$. Therefore:

$$\langle \Phi(x) \Phi(x') \rangle = \frac{\hbar}{(rr')^{D/2-1}} \sum_{\ell, \mathbf{m}} Y_{\ell\mathbf{m}}(\theta^a) Y_{\ell\mathbf{m}}^*(\theta^{a'}) \int_0^\infty \frac{d\omega}{2\pi} \psi_{\omega\ell}(t, r_*) \psi_{\omega\ell}^*(t', r'_*)$$

An analogous procedure yields:

$$\langle \Phi(x') \Phi(x) \rangle = \frac{\hbar}{(r'r)^{D/2-1}} \sum_{\ell, \mathbf{m}} Y_{\ell\mathbf{m}}(\theta^{a'}) Y_{\ell\mathbf{m}}^*(\theta^a) \int_0^\infty \frac{d\omega}{2\pi} \psi_{\omega\ell}(t', r'_*) \psi_{\omega\ell}^*(t, r_*)$$

By the addition theorem of the hyperspherical harmonics, the sum over $\mathbf{m}$ is a real and symmetric function of the angles $\sum_{\mathbf{m}} Y_{\ell\mathbf{m}}(\theta^a) Y_{\ell\mathbf{m}}^*(\theta^{a'}) = \sum_{\mathbf{m}} Y_{\ell\mathbf{m}}(\theta^{a'}) Y_{\ell\mathbf{m}}^*(\theta^a) \equiv \mathcal{Y}_\ell(\theta^a, \theta^{a'})$. Thus:

$$G_F(x, x') = \frac{-i\hbar}{(rr')^{D/2-1}} \sum_{\ell} \mathcal{Y}_{\ell}(\theta^a, \theta^{a'}) \int_0^{\infty} \frac{d\omega}{2\pi} [\Theta(t-t')\psi_{\omega\ell}(t, r_*)\psi_{\omega\ell}^*(t', r'_*) + \Theta(t'-t)\psi_{\omega\ell}^*(t, r_*)\psi_{\omega\ell}(t', r'_*)]$$

After imposing $[\hat{B}_{\omega\ell}, \hat{B}_{\omega'\ell}^{\dagger}] = 2\pi\hbar\delta(\omega - \omega')$, the integral over ω is the $(1+1)$ auxiliary propagator w.r.t. the vacuum state defined by $\hat{B}_{\omega\ell}|0\rangle_{1+1} = 0$. $\square$

Although proposition 1.6 shows that the D -dimensional propagator decomposes into a weighted sum of $(1+1)$ -dimensional propagators, obtaining an explicit expression for the D -dimensional REVMTO is by no means trivial. The weights $(rr')^{-D/2+1} \sum_m Y_{\ell m}(\theta^a) Y_{\ell m}^*(\theta^{a'})$ are UV finite in the coincidence limit for all ℓ . Hence, at least mode-wise, the singular structure is contained in the time-radial sector, and the regularization is required only for the $(1+1)$ -dimensional propagator $G_{\ell}^{1+1}(t, r_*) \rightarrow G_{\ell}^{1+1}(t, r_*)_{\text{reg}}$. The REVMTO of proposition 1.5 is then obtained from³:

$$\langle \hat{T}_{\mu\nu}(x) \rangle_{\text{ren}} = \lim_{x' \rightarrow x} \mathcal{T}_{\mu\nu}(x, x') \sum_{\ell=0}^{\infty} \frac{\sum_m Y_{\ell m}(\theta^a) Y_{\ell m}^*(\theta^{a'})}{(rr')^{D/2-1}} (-i) G_{\ell}^{1+1}(t, r_*, t', r'_*)_{\text{reg}} \quad (1.44)$$

The point-splitting operator acts on both the weights and the propagators. Thus, in general, the D -dimensional REVMTO *cannot* be expressed as a weighted sum of $(1+1)$ -dimensional REVMTOs. Moreover, the REVMTO of the auxiliary theory is computed based on the operator $\mathcal{T}_{\hat{\mu}\hat{\nu}}^{1+1}$ of the flat spacetime, *not* the D -dimensional $\mathcal{T}_{\mu\nu}^D$ of the SSS spacetime. Nevertheless, we can qualitatively argue that the fact we can define the auxiliary theory in the first place is due to the conformal flatness of SSS spacetimes in the time-radial sector, and we can naively expect $\langle \hat{T}_{\mu\nu}(t, r_*, \theta^a) \rangle_{\text{ren}}^D$ to be constructed from contributions involving $\langle \hat{T}_{\hat{\mu}\hat{\nu}}(t, r_*) \rangle_{\text{ren}}^{1+1}$. Yet another subtlety is the vacua on which these REVMTOs are built; the $\psi_{\omega\ell}$ normal modes are of course contained in $\Phi_{\omega\ell m}$, but in QFTCST the relationship between different vacua is not obvious, and the correspondence must be proven explicitly. Despite these difficulties, one thing is clear: the first step toward constructing the D -dimensional REVMTO is the $(1+1)$ -dimensional REVMTO.

1.5 Statement of the Problem

The overall message we have tried to convey in the previous section is the following: the classical and quantum theory of the scalar field Φ in a D -dimensional SSS reduces mode-wise to the classical and quantum theory of the “auxiliary” scalar field Ψ_{ℓ} in a $(1+1)$ -dimensional flat spacetime subject to the effective potential V_{ℓ} . The solution to the auxiliary problem therefore provides the natural first step toward constructing the full scalar quantum perturbation on the background SSS spacetime.

In going from definition 1.3 to proposition 1.5, we have quietly made a choice of the potential energy density for Φ . This choice will be justified in chapter 2, but it is already clear from eq. (1.33) that such a restriction is required to have a linear and separable e.o.m. in the first place. Even under this simplification, there is no known analytical solution of eq. (1.31) for a generic SSS metric, and consequently neither for eq. (1.41). Faced with this obstacle, we are therefore led to approximate V_{ℓ} . Figure 1.3 shows the background effective potential of proposition 1.2 in $D = 4$, corresponding to the SSS spacetime sourced by a homogeneous perfect fluid. Two features are immediately apparent:

1. The effective potential exhibits a finite step discontinuity at the surface.
2. The potential varies much less in the interior than in the exterior. Particularly for low compactness, the interior potential is approximately constant.

³Here we neglect the state-independent ambiguity term inherently associated with the Hadamard renormalization; see section 3.3 for details.

Motivated by these behaviors and with the ultimate objective of describing the effects of quantum perturbations in compact objects, we shall consider an idealized effective potential that is exactly constant inside a finite region and has a finite discontinuity at the boundary of said region. Concretely, we take V_ℓ as a **finite potential well**. This model captures the two qualitative features of the SSS effective potential most relevant to our purposes, while remaining sufficiently simple to allow for explicit analysis. Our final specific objective is to explicitly compute the REVEMTO of the corresponding $(1 + 1)$ -dimensional auxiliary QFT.

In chapters 2 and 3 we will describe the necessary techniques of classical and quantum field theories in curved spacetimes, respectively, for the development of the model. Experienced readers may proceed directly to chapter 4, where we will derive and present our results. This work is a direct continuation of the model initially developed by [26] in the context of studying the role of spacetime boundaries and dimensionality in Hadamard renormalization. In that work, they developed the *infinite potential well model* and laid the starting point for the finite potential well construction considered here, so we will also review those results in detail. Finally, we conclude with closing statements in chapter 5.

Chapter 2

Classical Field Theory in Curved Spacetimes

This chapter is devoted to the classical field theory of scalar fields in curved spacetimes, with particular emphasis on their solution spaces and their compatibility with spacetime isometries. When appropriate, we specialize the general formalism to the cases relevant for scalar fields on SSS spacetimes. We then show how, under suitable conditions on the scalar potential, the scalar field's EMT admits an effective perfect-fluid interpretation. The former provides the foundation for the canonical quantization developed in the following chapter, while the latter will be used to physically interpret our results.

2.1 Cosmic Fluids

Although highly ideal, perfect fluids are valuable for modeling not only astrophysical compact objects, but also the large-scale structure of matter in our Universe [14, 30].

Definition 2.1 Cosmic Fluid

Let $w \in \mathbb{R}$ be a real constant. We will call **cosmic fluid** to a perfect fluid (definition 1.2) with the equation of state (EoS):

$$w := \frac{p}{\rho} \quad (2.1)$$

If $0 \leq w \leq 1$, the cosmic fluid parameter can be physically interpreted as the squared ratio between the speed of sound in the fluid and the speed of light in vacuum, $c_s^2 := (\partial p / \partial \rho)_S = w$. If $w < 0$, its physical interpretation is best understood in terms of the energy conditions. Let us list several important cosmic fluids.

- **Vacuum energy** is defined by the cosmic fluid EoS:

$$w_\Lambda := -1 \quad (2.2)$$

This yields the EMT whose proper energy density is measured to be the same for all locally inertial observers:

$$T_{\mu\nu}^\Lambda = -\rho_\Lambda g_{\mu\nu} \quad (2.3)$$

When such an EMT uniquely sources the EFE, the solutions are given by the Einstein manifolds. The vacuum energy fluid models the “dark energy” which causes the accelerated expansion of the Universe in the Λ CDM cosmological model.

- **Dust** is defined by the EoS:

$$w_{\text{dust}} := 0 \quad (2.4)$$

This fluid models non-relativistic and pressure-less matter, $c_s^{\text{dust}} = 0$ and $p^{\text{dust}} = 0$, whose associated EMT is:

$$T_{\mu\nu}^{\text{dust}} = \rho_{\text{dust}} u_\mu u_\nu \quad (2.5)$$

In the locally inertial rest frame of dust, eq. (1.10) becomes:

$$T_{\mu'\nu'}^{\text{dust}} = \text{diag}(\rho_{\text{dust}}, 0, \dots, 0) \quad (2.6)$$

In the Λ CDM, this is the model for all large-scale massive (standard model and dark-) matter in the Universe.

- **Radiation** is defined $\forall D \neq 1$ by:

$$w_{\text{rad}} := \frac{1}{D-1} \quad (2.7)$$

This EoS is motivated by the fact that the EMT of a single particle satisfies $T_{\mu\nu} \propto p_\mu p_\nu$ and that of an ensemble $T_{\mu\nu} \propto \langle p_\mu p_\nu \rangle$. Hence, a fluid of massless particles always implies tracelessness, $T^{\text{rad}} = 0$. Such behavior is then inherited as the defining condition of the radiation cosmic fluid:

$$T_{\mu\nu}^{\text{rad}} = \frac{\rho_{\text{rad}}}{D-1} (g_{\mu\nu} + D u_\mu u_\nu) \quad (2.8)$$

In Λ CDM, this is the modeling of ultra-relativistic matter and massless matter (e.g., photons and neutrinos) in the Universe.

- A **stiff** fluid is defined by Zeldovich's EoS:

$$w_Z := 1 \quad (2.9)$$

The stiffness and compressibility of Zeldovich's fluid are the maximal values consistent with causality. Namely, acoustic perturbations propagate through the fluid at the speed of light, $c_s^Z = c$. The corresponding EMT reads:

$$T_{\mu\nu}^Z = \rho_Z (g_{\mu\nu} + 2u_\mu u_\nu) \quad (2.10)$$

In the locally inertial rest frame of a stiff fluid:

$$T_{\mu'\nu'}^Z = \rho_Z \delta_{\mu'\nu'} \quad (2.11)$$

In $D = 2$, the radiation EoS and EMT coincide with a Zeldovich stiff fluid. We recall that u describes the macroscopic fluid motion, not that of its microscopic constituents. Massless particles move along the spacetime's null geodesics and have no rest frame.

2.1.1 Energy Conditions

The energy conditions are motivated by contracting the EFE with a generic vector field Y ,

$$R_{\mu\nu}Y^\mu Y^\nu = \kappa^2 \left(T_{\mu\nu}Y^\mu Y^\nu - \frac{1}{D-2}T Y^2 \right) \quad (2.12)$$

where $D \geq 3$.

Definition 2.2 Energy Conditions

Let T be the EMT of the EFE solution $(\mathcal{M}, g, T)$. For all timelike U and lightlike k vector fields, we define the following.

- **Dominant energy condition (DEC):** For all future-directed U , the energy-momentum current $J^\mu := -T^{\mu\nu}U_\nu$ must be causal and future-directed^a:

$$g(J, J) \leq 0 \quad (2.13a)$$

$$J^0 \geq 0 \quad (2.13b)$$

- **Strong energy condition (SEC):**

$$T_{\mu\nu}U^\mu U^\nu - \frac{1}{D-2}TU^2 \geq 0 \quad (2.14)$$

- **Weak energy condition (WEC):**

$$T_{\mu\nu}U^\mu U^\nu \geq 0 \quad (2.15)$$

- **Null energy condition (NEC):**

$$T_{\mu\nu}k^\mu k^\nu \geq 0 \quad (2.16)$$

^aFuture-directed means lying in the future light-cone, which is an observer-independent notion. In some given frame, $J^0 > 0$ strictly, but we allow for $T = 0$ to satisfy all energy conditions trivially.

In general, $\text{DEC} \Rightarrow \text{WEC} \Rightarrow \text{NEC}$ and $\text{SEC} \Rightarrow \text{NEC}$. Since eq. (2.15) is the local energy density of T as measured by an observer whose tangent worldline is U , the WEC ensures a non-negative measurement. The DEC of eq. (2.13) generalizes this notion by enforcing that the measured energy-momentum flux is causal. The physical interpretation of eqs. (2.14) and (2.16) is given by the Raychaudhuri equation, where the SEC and the NEC ensure the focusing (either in the finite past or future) of timelike and null congruences, respectively.

Proposition 2.1 Perfect Fluid's Energy Conditions

For a perfect fluid (definition 1.2), the energy conditions of definition 2.2 become the following.

- The DEC holds if:

$$\begin{cases} \rho - |p| \geq 0 \\ \rho \geq 0 \end{cases} \quad (2.17)$$

- The SEC holds if:

$$\begin{cases} (D-3)\rho + (D-1)p \geq 0 \\ \rho + p \geq 0 \end{cases} \quad (2.18)$$

- The WEC holds if:

$$\begin{cases} \rho \geq 0 \\ \rho + p \geq 0 \end{cases} \quad (2.19)$$

- The NEC holds if:

$$\rho + p \geq 0 \quad (2.20)$$

Proof. The energy conditions are all scalar expressions. As such, they hold in any reference frame. Let us choose the locally inertial rest frame of the perfect fluid where eq. (1.10) holds, and drop the primed indices.

(NEC) Given $k^\mu = (k^0, k^i)$ and $k^2 = (k^0)^2 - \sum_{i=1}^d (k^i)^2 = 0$, then:

$$0 \leq T_{\mu\nu} k^\mu k^\nu = \rho (k^0)^2 + p \sum_{i=1}^d (k^i)^2 = (\rho + p) (k^0)^2 \implies \rho + p \geq 0, \forall k^0$$

(WEC) Given $U^\mu = (U^0, U^i)$ normalized to $U^2 = -(U^0)^2 + \sum_{i=1}^d (U^i)^2 = -1$, then:

$$0 \leq T_{\mu\nu} U^\mu U^\nu = \rho (U^0)^2 + p \sum_{i=1}^d (U^i)^2 = \rho + (\rho + p) \sum_{i=1}^d (U^i)^2 \implies \rho \geq 0 \text{ and } \rho + p \geq 0, \forall U^i$$

(SEC) From $g_{\mu\nu} g^{\mu\nu} = \delta^\mu_\mu = D$ and $u^2 = -1$, the trace of definition 1.2 is $T = -\rho + (D-1)p$. Thus:

$$\begin{aligned} 0 \leq T_{\mu\nu} U^\mu U^\nu - \frac{1}{D-2} T U^2 &= \rho + (\rho + p) \sum_{i=1}^d (U^i)^2 - \frac{1}{D-2} [-\rho + (D-1)p] (-1) \\ &= \frac{(D-3)\rho + (D-1)p}{D-2} + (\rho + p) \sum_{i=1}^d (U^i)^2 \end{aligned}$$

From which it follows that $(D-3)\rho + (D-1)p \geq 0$ and $\rho + p \geq 0, \forall U^i$.

(DEC) Finally, the current is given by $J_\mu = -T_{\mu\nu} U^\nu = -(\rho U^0, p U^i)$. In the locally inertial frame, U^μ is future-directed if $U^0 > 0$. Since we allow for the current to include the trivial case, then:

$$0 \leq J^0 = g^{0\mu} J_\mu = (-1)(-\rho U^0) = \rho U^0 \implies \rho \geq 0, \forall U^0$$

Furthermore:

$$0 \geq J^\mu J_\mu = -\rho^2 (U^0)^2 + p^2 \sum_i (U^i)^2 = -\rho^2 - (\rho^2 - p^2) \sum_i (U^i)^2 \implies \rho^2 \geq 0 \text{ and } \rho^2 - p^2 \geq 0, \forall U^i$$

If $p \geq 0$, then $\rho - |p| \geq 0 \implies \rho + p \geq 2p \geq 0$, and if $p < 0$ then $\rho + p \geq 0$. In both cases, the DEC implies the WEC and the NEC. $\square$

Proposition 2.1 shows that the energy conditions are natural properties satisfied by many matter systems that can be approximated as perfect fluids.

Proposition 2.2 Cosmic Fluid's Energy Conditions

For a cosmic fluid (definition 2.1), the energy conditions of definition 2.2 become the following^a.

1. If $\rho > 0$, then we have the following.

1.1. The DEC holds if:

$$|w| \leq 1 \quad (2.21)$$

1.2. The SEC holds if:

$$w \geq -\frac{D-3}{D-1} \quad (2.22)$$

1.3. Both the WEC and NEC hold if:

$$w \geq -1 \quad (2.23)$$

2. If $\rho < 0$, then we have the following.

2.1. Regardless of w , the DEC and WEC never hold.

2.2. The SEC and NEC hold if:

$$w \leq -1 \quad (2.24)$$

^aBy the cosmic EoS, $p \propto \rho = 0$ recovers the trivial case $T = 0$, which as we have stated, satisfies all ECs trivially.

Proof. We use proposition 2.1.

(DEC) One of the DEC requirements becomes $0 \leq \rho - |p| = \rho - |w\rho| = \rho(1 - |w|)$. If $\rho > 0$, then $|w| \leq 1$. If $\rho < 0$, the other condition of the DEC is automatically violated.

(SEC) The SEC requirements become $0 \leq (D-3)\rho + (D-1)p = \rho[D-3 + (D-1)w]$ and $0 \leq \rho + p = \rho(1+w)$. If $\rho > 0$, then $w \geq -\frac{D-3}{D-1} \geq -1$. If $\rho < 0$, then $w \leq -1 \leq -\frac{D-3}{D-1}$.

(WEC) For the WEC, $0 \leq \rho(1+w)$ implies that if $\rho > 0$ then $w \geq -1$. If $\rho < 0$, then the other requirement is violated.

(NEC) For the NEC, $0 \leq \rho(1+w)$ implies that if $\rho > 0$ then $w \geq -1$, and if $\rho < 0$ then $w \leq -1$. $\square$

Figure 2.1 shows which energy conditions are satisfied by the aforementioned cosmic fluids. All energy conditions can be analytically continued to $D \rightarrow 2$, except the SEC, whose formulation requires further care because the

D -dimensional EFE becomes singular in this limit.

Cosmic Fluid	w	ρ	NEC	WEC	DEC	SEC
Vacuum energy	-1	> 0	$\checkmark^*$	$\checkmark^*$	$\checkmark^*$	$\times$
		< 0	$\checkmark^*$	$\times$	$\times$	$\checkmark^*$
Dust	0	> 0	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
		< 0	$\times$	$\times$	$\times$	$\times$
Radiation	$\frac{1}{D-1}$	> 0	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
		< 0	$\times$	$\times$	$\times$	$\times$
Stiff	$+1$	> 0	$\checkmark$	$\checkmark$	$\checkmark^*$	$\checkmark$
		< 0	$\times$	$\times$	$\times$	$\times$

Figure 2.1: Energy conditions for the cosmic fluids. The symbol $\checkmark$ indicates that the corresponding energy condition is satisfied, $\checkmark^*$ denotes saturation of at least one of its defining bounds, and $\times$ indicates violation. For $\rho < 0$, the DEC and WEC are always violated.

2.2 Scalar Field Theory

A classical field theory is said to be **free** or **non-interacting** if the theory's action is at most quadratic in the dynamical fields and their derivatives. For definition 1.3, this translates into the condition $\partial_\Phi^n V(\Phi, g, x) = 0$ for all $n \geq 3$. Expanding in powers of the field, the potential energy acquires the structure:

$$V(\Phi, g, x) = \Lambda(g, x) + J(g, x)\Phi + \frac{1}{2}U(g, x)\Phi^2 \quad (2.25)$$

We allow the scalar functions to depend (covariantly) on spacetime position, and on the metric and its covariant derivatives. Proposition 1.3 then becomes:

$$[\square - U(g, x)]\Phi = J(g, x) \quad (2.26)$$

This is the essential feature of a free theory: the e.o.m. are linear, and the set of solutions thus forms a vector space.

Suppose Φ is one of the matter sources of the EFE, $\mathcal{L}_\Phi \subset \mathcal{L}_M$. If $\Lambda(g, x) = \Lambda(x)$, then this term represents a shift in the cosmological constant of the vacuum energy cosmic fluid. If $\Lambda(g, x) \propto R(g)$, this represents a shift in the coupling constant κ^2 of Einstein's gravity. Finally, if $\Lambda(g, x)$ depends on curvature tensors of higher order than the Ricci scalar, we enter the domain of modified gravity beyond Einstein's theory. All of these cases are interesting in their own right, but the dynamics of Φ are independent of $\Lambda(g, x)$. Unless Φ sources the EFE, there is no coupling between Φ and $\Lambda(g, x)$, and the EMT contribution of the latter in proposition 1.4 can be treated independently. Since we are interested in perturbations that do not source a *fixed* background, we henceforth set $\Lambda(g, x) = 0$.

Equation (2.26) is linear, but not homogeneous. The term $J(g, x)$ acts as a source for the scalar field, and the general solution forms an affine space modeled on the vector space of solutions to the corresponding homogeneous equation. This additional complexity is not required for our purposes: we can restrict ourselves to backgrounds for which $\Phi_0 = 0$ is an equilibrium solution, which requires $\partial_\Phi V|_{\Phi_0=0} = J(g, x) = 0$, so Φ can be identified directly with the fluctuation about the fixed background.

2.2.1 Klein-Gordon Product

The previous discussion partially justifies our choice of the potential energy in proposition 1.5. We will then solely consider a free real scalar field theory which, up to boundary terms, the action of such a theory can be written as:

$$S[\Phi, g] = \int_{\Omega \subseteq \mathcal{M}} d^D x \sqrt{-g} \frac{1}{2} \Phi \hat{\mathcal{D}} \Phi \quad (2.27)$$

Where:

$$\hat{\mathcal{D}} = \square - U(g, x) \quad (2.28)$$

And the corresponding e.o.m. are:

$$\hat{\mathcal{D}} \Phi = 0 \quad (2.29)$$

The vector space of solutions is real and given by the kernel of the e.o.m. operator:

$$\mathcal{S}_{\mathbb{R}} := \ker \hat{\mathcal{D}} = \left\{ \Phi \in C^\infty(\mathcal{M} \mapsto \mathbb{R}) \mid \hat{\mathcal{D}} \Phi = 0 \right\} \quad (2.30)$$

For the canonical quantization of the theory, it is convenient to complexify the solution space:

$$\mathcal{S}_{\mathbb{C}} := \{ \Phi_1 + i\Phi_2 \mid \Phi_1, \Phi_2 \in \mathcal{S}_{\mathbb{R}} \} = \left\{ \Phi \in C^\infty(\mathcal{M} \mapsto \mathbb{C}) \mid \hat{\mathcal{D}} \Phi = 0 \right\} \quad (2.31)$$

It should be emphasized that we are complexifying the *solutions*, not the manifold nor the action—we still have $U^*(g, x) = U(g, x)$. This is analogous to the complex quantum fields considered in QFT in real Minkowski spacetime.

Definition 2.3 Klein-Gordon Product

- For $\Phi_1, \Phi_2 \in \mathcal{S}_{\mathbb{C}}$, let the **Noether current density** $\mathbf{J}(\Phi_1, \Phi_2^*)$ be defined as:

$$\mathbf{J}(\Phi_1, \Phi_2^*) = -i \Phi_1 \overleftrightarrow{\nabla} \Phi_2^* = -i (\Phi_1 \nabla \Phi_2^* - \Phi_2^* \nabla \Phi_1) \quad (2.32)$$

- For a hypersurface $\Sigma \subset \mathcal{M}$ with normal $\mathbf{n}$, induced metric determinant g_Σ , and integration measure $d\Sigma = d^{D-1}x \sqrt{|g_\Sigma|} \mathbf{n}$, let the complexified solution space $\mathcal{S}_{\mathbb{C}}$ be endowed with the map $\mathcal{S}_{\mathbb{C}} \times \mathcal{S}_{\mathbb{C}} \mapsto \mathbb{C}$ defined by the flux:

$$(\Phi_1, \Phi_2) = \int_{\Sigma} d\Sigma \cdot \mathbf{J}(\Phi_1, \Phi_2^*) \quad (2.33)$$

We will call eq. (2.33) the **Klein-Gordon (KG) product**. For this product to be physically meaningful, Σ must be a spacelike Cauchy surface with a future-directed unit normal. In other words, $(\mathcal{M}, g)$ must be globally hyperbolic, ensuring the existence of at least one Cauchy surface and its foliation thereof. The causal nature of the hypersurface may be relaxed, but global hyperbolicity is required for a well-posed global Cauchy problem. The bold notation in eq. (2.32) is symbolic for components since the covariant derivative is not a tensor, but it emphasizes that the KG product in eq. (2.33) is coordinate-independent.

There are two ways to justify definition 2.3: (1) By complexifying a bilinear current density $J(\Phi_1, \Phi_2)$, a real symplectic form $\Omega(\Phi_1, \Phi_2)$, and the real symplectic space $(S_{\mathbb{R}}, \Omega)$. (2) As the name suggests, eq. (2.32) is the conserved Noether current associated with the $U(1)$ internal symmetry of the complex scalar field theory $S[\Phi, \Phi^*, g]$. The complexification of the $S[\Phi, g]$ solutions inherits this conserved current.

Proposition 2.3 Klein-Gordon Product Properties

If $\Phi_j \in S_{\mathbb{C}}$ and $a_j \in \mathbb{C}$, then definition 2.3 enjoys the following properties.

- Conjugate symmetric, linear in the first argument, and antilinear in the second:

$$(\Phi_1, \Phi_2) = (\Phi_2, \Phi_1)^* \quad (2.34a)$$

$$(a_1 \Phi_1 + a_2 \Phi_2, \Phi_3) = a_1 (\Phi_1, \Phi_3) + a_2 (\Phi_2, \Phi_3) \quad (2.34b)$$

$$(\Phi_1, a_2 \Phi_2 + a_3 \Phi_3) = a_2^* (\Phi_1, \Phi_2) + a_3^* (\Phi_1, \Phi_3) \quad (2.34c)$$

- For complex-conjugated arguments:

$$(\Phi_1^*, \Phi_2^*) = -(\Phi_1, \Phi_2)^* \quad (2.35)$$

- If $(\mathcal{M}, g)$ is globally hyperbolic, then definition 2.3 is independent of the Cauchy surface up to a boundary term:

$$(\Phi_1, \Phi_2)_{\Sigma_1} = (\Phi_1, \Phi_2)_{\Sigma_2} \quad (2.36)$$

Proof. (a) Since the measure is real, conjugate symmetry is inherited from the conjugate symmetry of the current:

$$J(\Phi_1, \Phi_2^*) = -i \Phi_1 \overleftrightarrow{\nabla} \Phi_2^* = \left(-i \Phi_2 \overleftrightarrow{\nabla} \Phi_1^* \right)^* = [J(\Phi_2, \Phi_1^*)]^*$$

(b) Linearity in the first argument follows from the linearity of the covariant derivative:

$$J(a_1 \Phi_1 + a_2 \Phi_2, \Phi_3^*) = -i (a_1 \Phi_1 + a_2 \Phi_2) \overleftrightarrow{\nabla} \Phi_3^* = -a_1 i \Phi_1 \overleftrightarrow{\nabla} \Phi_3^* - a_2 i \Phi_2 \overleftrightarrow{\nabla} \Phi_3^* = a_1 J(\Phi_1, \Phi_3^*) + a_2 J(\Phi_2, \Phi_3^*)$$

(c) Antilinearity in the second argument follows from the linearity of ∇_μ and the field conjugation in the second argument of the current.

(d) For two complex-conjugated arguments:

$$J(\Phi_1^*, \Phi_2) = -i \Phi_1^* \overleftrightarrow{\nabla} \Phi_2 = - \left(-i \Phi_1 \overleftrightarrow{\nabla} \Phi_2^* \right)^* = -[J(\Phi_1, \Phi_2^*)]^*$$

Hence:

$$(\Phi_1^*, \Phi_2^*) = \int_{\Sigma} d\Sigma \cdot J(\Phi_1^*, \Phi_2) = - \int_{\Sigma} d\Sigma \cdot [J(\Phi_1, \Phi_2^*)]^* = -(\Phi_1, \Phi_2)^*$$

(e) The divergence of the Noether current is:

$$\nabla \cdot \mathbf{J}(\Phi_1, \Phi_2^*) = -i \nabla \cdot \left(\Phi_1 \overset{\leftrightarrow}{\nabla} \Phi_2^* \right) = -i \Phi_1 \overset{\leftrightarrow}{\mathcal{D}} \Phi_2^*$$

The proof then follows from integrating this divergence over a spacetime volume $\Omega \subseteq \mathcal{M}$ whose boundary is given by a timelike hypersurface and two distinct spatial Cauchy surfaces, one being in the future of the other, $\partial\Omega = \Pi \cup \Sigma_1 \cup \Sigma_2$. Imposing suitable boundary conditions on the scalar fields, the current can be made to vanish at the timelike boundary. If the fields are on-shell, then $\nabla \cdot \mathbf{J} = 0$ and the divergence theorem yields the equality of the KG products on Σ_1, Σ_2 . For an explicit proof, cf. [7]. $\square$

Proposition 2.3 identifies the KG product as a sesquilinear form, which for globally hyperbolic spacetimes is globally conserved. Although the norm induced by definition 2.3 is real, $(\Phi, \Phi) \in \mathbb{R}$, it is not positive-definite for all scalar fields. As an example, if $(\Phi, \Phi) > 0$, then $(\Phi^*, \Phi^*) = -(\Phi, \Phi) < 0$, which also shows that the real subspace $\mathcal{S}_{\mathbb{R}} \subset \mathcal{S}_{\mathbb{C}}$ has zero norm.

2.2.2 Normal Modes

A special class of complex solutions is that of **normal modes**, obtained by separating the e.o.m. and labeled by the separation constants and eigenvalues of operators that preserve the solution space. We shall say that a set of normal modes is **complete** if its linear span is $\mathcal{S}_{\mathbb{C}}$, so that every complex solution can be expanded in terms of these modes.

Definition 2.4 Positive- & Negative-Frequency

Let $(\mathcal{M}, g)$ be a stationary spacetime whose Killing vector field $\mathbf{K}$ is timelike inside a given region $\mathcal{U} \subseteq \mathcal{M}$. A normal mode $u_\omega \in \mathcal{S}_{\mathbb{C}}$ is said to have a **positive-frequency** w.r.t. $\mathbf{K}$ whenever

$$\mathcal{L}_{\mathbf{K}} u_\omega = -i\omega u_\omega, \quad \omega > 0 \quad (2.37)$$

and **negative-frequency** w.r.t. $\mathbf{K}$ whenever:

$$\mathcal{L}_{\mathbf{K}} u_\omega = +i\omega u_\omega, \quad \omega > 0 \quad (2.38)$$

Definition 2.4 is clearly inspired by the plane waves of Minkowski spacetime and asymptotically flat spacetimes. In $D = 2$ and adapted coordinates $K^\mu = \delta_t^\mu$, we see that:

$$\mathcal{L}_{\mathbf{K}} e^{-i\omega t + ikx} = K^\mu \nabla_\mu e^{-i\omega t + ikx} = \partial_t e^{-i\omega t + ikx} = -i\omega e^{-i\omega t + ikx} \quad (2.39)$$

Complex conjugation maps positive-frequency normal modes to negative-frequency ones:

$$\mathcal{L}_{\mathbf{K}} u_\omega^* = +i\omega u_\omega^* \quad (2.40)$$

Hence, if a set $\{u_\lambda\}$ spans the positive-frequency normal modes, then $\{u_\lambda^*\}$ spans the negative-frequency ones. *Provided* the set $\{u_\lambda, \text{c.c.}\}$ is complete, $\mathcal{S}_{\mathbb{C}} = \text{span}_{\mathbb{C}} \{u_\lambda\} \oplus \text{span}_{\mathbb{C}} \{u_\lambda^*\}$, every solution $\Phi \in \mathcal{S}_{\mathbb{C}}$ can be expanded as:

$$\Phi = \sum_{\lambda} (\alpha_\lambda u_\lambda + \beta_\lambda u_\lambda^*) \quad (2.41)$$

If the mode labels λ are continuous, the sums are replaced by appropriate integrals.

Definition 2.5 Orthonormal Modes

We will call a complete set of positive-frequency normal modes and their complex conjugates $\{u_\lambda, \text{c.c.}\}$ **orthonormal** if w.r.t. definition 2.3 the set satisfies:

$$(u_\lambda, u_{\lambda'}) = \delta_{\lambda\lambda'} \quad (2.42a)$$

$$(u_\lambda^*, u_{\lambda'}^*) = -\delta_{\lambda\lambda'} \quad (2.42b)$$

$$(u_\lambda, u_{\lambda'}^*) = 0 \quad (2.42c)$$

If the labels λ, λ' are continuous, the Dirac delta distribution replaces the Kronecker delta symbol.

If the set is orthonormal, the subspace spanned by the positive-frequency normal modes carries a positive-definite KG product. Orthonormality also allows us to determine the Fourier coefficients in the linear expansion.

Proposition 2.4 Fourier Coefficients

If $\{u_\lambda, \text{c.c.}\}$ is orthonormal, then the Fourier coefficients in the expansion of eq. (2.41) are given by:

$$\alpha_\lambda = (\Phi, u_\lambda) \quad (2.43a)$$

$$\beta_\lambda = -(\Phi, u_\lambda^*) \quad (2.43b)$$

Proof. By the linearity in the first argument, $(\Phi, u_\lambda) = (\sum_{\lambda'} (\alpha_{\lambda'} u_{\lambda'} + \beta_{\lambda'} u_{\lambda'}^*), u_\lambda) = \sum_{\lambda'} \alpha_{\lambda'} (u_{\lambda'}, u_\lambda) + \sum_{\lambda'} \beta_{\lambda'} (u_{\lambda'}^*, u_\lambda) = \sum_{\lambda'} \alpha_{\lambda'} \delta_{\lambda'\lambda} = \alpha_\lambda$. Analogously for the other coefficient. $\square$

For a real solution, $\Phi^* = \Phi \implies \beta_\lambda = \alpha_\lambda^*$, hence:

$$\Phi = \sum_{\lambda} [(\Phi, u_\lambda) u_\lambda - (\Phi, u_\lambda^*) u_\lambda^*] \quad (2.44)$$

Proposition 2.5 Lie Derivative Commutator

If $K \in \text{iso}(\mathcal{M}, g)$ exists and $[U(g, x), \mathcal{L}_K] = 0$, then the solution space $\mathcal{S}_C$ of eq. (2.27) is invariant under the isometry generated by the KVF^a.

^aThe commutators are understood to act on scalar functions.

Proof. (1) The first step of the proof is to show that the d'Alembertian commutes with the Lie derivative operator. Consider a generic diffeomorphism generator $V \in \text{diff}\mathcal{M}$. For a test scalar field f , we have $[\square, \nabla_\mu] f = R_{\mu\nu} \nabla^\nu f$, hence:

$$[\square, \mathcal{L}_V] f = [\nabla^\mu \nabla_\mu, V^\alpha \nabla_\alpha] f = [\square, V^\alpha] \nabla_\alpha f + V^\alpha [\square, \nabla_\alpha] f = [2(\nabla^\mu V^\alpha) \nabla_\mu + \square V^\alpha + V^\beta R_\beta^\alpha] \nabla_\alpha f$$

Now let $V = K$, where the Killing equation is $\nabla^{(\mu} K^{\nu)} = 0$. The first term vanishes by virtue of $\nabla^\mu K^\nu = \nabla^{[\mu} K^{\nu]}$ being antisymmetric and $\nabla_\mu \nabla_\alpha f$ being symmetric. Contracting the Killing equation with ∇_μ yields:

$$-\square K^\nu = \nabla_\mu \nabla^\nu K^\mu = [\nabla_\mu, \nabla^\nu] K^\mu = R^\nu_\beta K^\beta$$

The second term also vanishes and therefore:

$$[\square, \mathcal{L}_K] f = 0$$

(2) The e.o.m. operator is $\hat{\mathcal{D}} = \square - U(g, x)$, so if $[U, \mathcal{L}_K] = 0$, then $[\hat{\mathcal{D}}, \mathcal{L}_K] = 0$.

(3) If Φ is a solution, then $\hat{\mathcal{D}}(\mathcal{L}_K \Phi) = \mathcal{L}_K(\hat{\mathcal{D}}\Phi) = 0$ and $\mathcal{L}_K \Phi$ is also a solution. Ergo, the solution space is invariant under the isometry generated by the KVF, $\mathcal{L}_K|_{\mathcal{S}_\mathbb{C}} : \mathcal{S}_\mathbb{C} \mapsto \mathcal{S}_\mathbb{C}$. $\square$

Proposition 2.5 is the requirement to admit K -adapted normal modes in the first place, but it does not guarantee completeness.

Proposition 2.6 Skew-Adjointness of the Lie Derivative

Suppose $K \in \text{iso}(\mathcal{M}, g)$ exists and proposition 2.5 holds. If $K^{[\mu} J^{\nu]}$ vanishes on $\partial\Sigma$, then $\mathcal{L}_K$ is skew-adjoint w.r.t. the Klein-Gordon product:

$$(\mathcal{L}_K \Phi_1, \Phi_2) = -(\Phi_1, \mathcal{L}_K \Phi_2) \quad (2.45)$$

Proof. Using eq. (2.32) and $\mathcal{L}_K \Phi = K^\alpha \nabla_\alpha \Phi$, consider:

$$\begin{aligned} J^\mu (\mathcal{L}_K \Phi_1, \Phi_2^*) + J^\mu (\Phi_1, (\mathcal{L}_K \Phi_2)^*) &= -i(K^\alpha \nabla_\alpha \Phi_1)(\nabla^\mu \Phi_2^*) + i\Phi_2^* \nabla^\mu (K^\alpha \nabla_\alpha \Phi_1) \\ &\quad - i\Phi_1 \nabla^\mu (K^\alpha \nabla_\alpha \Phi_2^*) + i(K^\alpha \nabla_\alpha \Phi_2^*)(\nabla^\mu \Phi_1) \\ &= -iK^\alpha (\nabla_\alpha \Phi_1)(\nabla^\mu \Phi_2^*) - iK^\alpha \Phi_1 (\nabla_\alpha \nabla^\mu \Phi_2^*) \\ &\quad + iK^\alpha (\nabla_\alpha \Phi_2^*)(\nabla^\mu \Phi_1) + iK^\alpha \Phi_2^* (\nabla_\alpha \nabla^\mu \Phi_1) \\ &\quad - i(\nabla^\mu K^\alpha)(\Phi_1 \nabla_\alpha \Phi_2^*) + i(\nabla^\mu K^\alpha)(\Phi_2^* \nabla_\alpha \Phi_1) \\ &= K^\alpha \nabla_\alpha J^\mu (\Phi_1, \Phi_2^*) + (\nabla^\mu K^\alpha) J_\alpha (\Phi_1, \Phi_2^*) \end{aligned}$$

We can factorize the covariant derivative by exploiting the Killing equation $\nabla^{(\mu} K^{\alpha)} = 0 \Leftrightarrow \nabla_\alpha K^\alpha = 0$ and the conservation $\nabla_\alpha J^\alpha = 0$ on-shell:

$$\nabla_\alpha (K^\alpha J^\mu - K^\mu J^\alpha) = K^\alpha \nabla_\alpha J^\mu - (\nabla_\alpha K^\mu) J^\alpha = K^\alpha \nabla_\alpha J^\mu + (\nabla^\mu K^\alpha) J_\alpha$$

Therefore:

$$J^\mu (\mathcal{L}_K \Phi_1, \Phi_2^*) + J^\mu (\Phi_1, (\mathcal{L}_K \Phi_2)^*) = 2\nabla_\alpha (K^{[\alpha} J^{\mu]})$$

And then:

$$(\mathcal{L}_K \Phi_1, \Phi_2) + (\Phi_1, \mathcal{L}_K \Phi_2) = 2 \int_\Sigma d\Sigma_\mu \nabla_\alpha (K^{[\alpha} J^{\mu]})$$

The RHS will vanish if either $K^{[\alpha} J^{\mu]} = 0$ identically or $s_\alpha n_\mu K^{[\alpha} J^{\mu]}$ vanishes at $\partial\Sigma$, where $n \perp \Sigma$ and $s \perp \partial\Sigma$. $\square$

It follows from proposition 2.6 that the operator $\hat{K} := i\mathcal{L}_K$ is self-adjoint w.r.t. the KG product, provided the boundary conditions are met. If the spacetime and the e.o.m. admit them, then the standard quantum spectral theory holds in the Hilbert space spanned by the positive-frequency orthonormal modes. The potential in proposition 1.5 has been chosen to be invariant under the isometries of the SSS spacetime and to allow a definite-frequency normal-mode decomposition.

2.2.3 Parity

Continuous spacetime symmetries are generated by elements of the isometry algebra $\text{iso}(\mathcal{M}, g)$. If the spacetime symmetry is **discrete**, then it is disconnected from the identity of the isometry group $\text{Iso}(\mathcal{M}, g)$, and there is no KVF to describe the transformation. One of the discrete isometries of Minkowski spacetime is parity, which in Cartesian coordinates corresponds to the time-orientation preserving, spatial-orientation reversing, and involutive map $\mathcal{P} : (t, x^i) \mapsto (t, -x^i)$.

Definition 2.6 Parity

Let $(\mathcal{M}, g)$ be a stationary spacetime whose KVF K is timelike inside a given region $\mathcal{U} \subseteq \mathcal{M}$. We will define **parity** as the discrete isometry $\mathcal{P} \in \text{Iso}(\mathcal{M}, g)$ with the properties^a:

$$\mathcal{P} \circ \mathcal{P} = \mathbb{I} \quad (2.46a)$$

$$\mathcal{P}_* K = K \quad (2.46b)$$

^aAdmittedly, this definition is too broad since any inversion of a single spatial direction will count as parity and we need all of them to reverse. Moreover, one can also define parity in the n -sphere, and there is no notion of time in $\mathcal{S}^n$. Notwithstanding, any stricter definition of parity in $1 + d$ curved spacetimes will have to conform to these two criteria, which for our purposes are sufficient.

In operator terms, the condition $\mathcal{P}_* K = K \Leftrightarrow [\hat{\mathcal{P}}, \mathcal{L}_K] = 0$ ensures that parity preserves the stationary time orientation defined by the congruence of K and therefore excludes time-reversal.

The involution property $\mathcal{P}^2 = \mathbb{I} \Leftrightarrow \hat{\mathcal{P}}^{-1} = \hat{\mathcal{P}}$ implies that for parity eigenfunctions, $\hat{\mathcal{P}}\Phi_\pi = \pi\Phi_\pi$, the spectrum is:

$$\Phi_\pi = \hat{\mathcal{P}}^2\Phi_\pi = \pi^2\Phi_\pi \implies \pi = \{-1, +1\} \quad (2.47)$$

In general, parity acts on scalar fields $\Phi \in \mathcal{C}^\infty(\mathcal{M} \mapsto \mathbb{C})$ as:

$$(\hat{\mathcal{P}}\Phi)(x) = \Phi(\mathcal{P}^{-1}x) = \Phi(\mathcal{P}x) \quad (2.48)$$

Definition 2.7 Definite Parity

We shall say that a normal mode $u_\lambda \in \mathcal{S}_\mathbb{C}$ has a **definite parity** if u_λ is an eigenfunction of the parity operator $\hat{\mathcal{P}}$. That is:

$$\hat{\mathcal{P}}u_\lambda = \pi_\lambda u_\lambda \quad (2.49)$$

If $\pi_\lambda = +1$, then the normal mode is said to have an **even parity**, and if $\pi_\lambda = -1$ an **odd parity**.

Proposition 2.7 Parity Decomposition of the Solution Space

Let the **even-** and **odd-parity** sectors $\mathcal{S}_{\mathbb{C}}^{\pi} \subset \mathcal{S}_{\mathbb{C}}$ be defined as the definite-parity solutions:

$$\mathcal{S}_{\mathbb{C}}^{\pi} := \left\{ \Phi_{\pi} \in \mathcal{S}_{\mathbb{C}} \mid \hat{\mathcal{P}}\Phi_{\pi} = \pi\Phi_{\pi} \right\} \quad (2.50)$$

If $[\hat{\mathcal{P}}, \hat{\mathcal{D}}] = 0$, then:

$$\mathcal{S}_{\mathbb{C}} = \mathcal{S}_{\mathbb{C}}^{+} \oplus \mathcal{S}_{\mathbb{C}}^{-} \quad (2.51)$$

Proof. If $[\hat{\mathcal{P}}, \hat{\mathcal{D}}] = 0$ and $\Phi \in \mathcal{S}_{\mathbb{C}}$, then $\hat{\mathcal{D}}(\hat{\mathcal{P}}\Phi) = \hat{\mathcal{P}}(\hat{\mathcal{D}}\Phi) = 0$ and thus $\hat{\mathcal{P}}\Phi \in \mathcal{S}_{\mathbb{C}}$. This means that the parity operator leaves invariant the solution space:

$$[\hat{\mathcal{P}}, \hat{\mathcal{D}}] = 0 \implies \hat{\mathcal{P}} : \mathcal{S}_{\mathbb{C}} \mapsto \mathcal{S}_{\mathbb{C}}$$

Defining the projectors $\hat{\Pi}_{\pi} := \frac{1}{2}(\mathbb{I} + \pi\hat{\mathcal{P}})$, their algebra is then $\hat{\Pi}_{\pi}\hat{\Pi}_{\pi'} = \delta_{\pi\pi'}\hat{\Pi}_{\pi}$ and $\hat{\Pi}_{+} + \hat{\Pi}_{-} = \mathbb{I}$. Given $\pi^2 = 1$, we have:

$$\hat{\mathcal{P}}(\hat{\Pi}_{\pi}\Phi) = (\hat{\mathcal{P}}\hat{\Pi}_{\pi})\Phi = \pi(\hat{\Pi}_{\pi}\Phi) \implies \hat{\Pi}_{\pi}\Phi \in \mathcal{S}_{\mathbb{C}}^{\pi}$$

Finally:

$$\Phi = \mathbb{I}\Phi = (\hat{\Pi}_{+} + \hat{\Pi}_{-})\Phi = (\hat{\Pi}_{+}\Phi) + (\hat{\Pi}_{-}\Phi)$$

If $\Phi \in \mathcal{S}_{\mathbb{C}}^{+} \cap \mathcal{S}_{\mathbb{C}}^{-}$, then $\hat{\mathcal{P}}\Phi = \Phi = -\Phi \implies \Phi = 0$. Given $\mathcal{S}_{\mathbb{C}}^{+} \cap \mathcal{S}_{\mathbb{C}}^{-} = \{0\}$, the decomposition is a direct sum. $\square$

Proposition 2.7 shows that if parity is a symmetry of the spacetime and the e.o.m., then the solution space decomposes into even- and odd-parity sectors. If the KG product is invariant under parity, $(\hat{\mathcal{P}}\Phi_1, \hat{\mathcal{P}}\Phi_2) = (\Phi_1, \Phi_2)$, meaning adapting the hypersurfaces Σ to the congruence of $\mathbf{K}$, then the parity sectors are orthogonal:

$$(\Phi_{+}, \Phi_{-}) = (\hat{\mathcal{P}}\Phi_{+}, \hat{\mathcal{P}}\Phi_{-}) = (+\Phi_{+}, -\Phi_{-}) = -(\Phi_{+}, \Phi_{-}) \implies (\Phi_{+}, \Phi_{-}) = 0 \quad (2.52)$$

Moreover, since parity commutes with the stationary Lie derivative (by definition!) and the latter defines the definite-frequency normal modes, parity preserves the definite-frequency sectors. If the spacetime and the solution space admit both, normal modes may then be labeled as $\lambda = (\omega, \pi, \dots)$, where $\omega > 0$ and $\pi \in \{-1, +1\}$. In terms of eq. (2.28), the minimal conditions are $\mathcal{L}_{\mathbf{K}}U(g, x) = 0$ and $\hat{\mathcal{P}}U(g, x) = U(g, x)$.

2.2.4 Effective Perfect Fluid

Let us now show how the energy-momentum distribution of scalar fields mimics that of perfect fluids. We return to the generic potential of definition 1.3 and only impose independence from the metric.

Proposition 2.8 Scalar Field EMT's Perfect Fluid Structure

If $V = V(\Phi, x)$ only and $(\nabla\Phi)^2 < 0$, then proposition 1.4 has the same structure as definition 1.2 with D -velocity, proper density, and proper pressure given by:

$$u_\mu = \frac{\nabla_\mu \Phi}{\sqrt{-(\nabla\Phi)^2}} \quad (2.53a)$$

$$\rho = -\frac{1}{2}(\nabla\Phi)^2 + V(\Phi, x) \quad (2.53b)$$

$$p = -\frac{1}{2}(\nabla\Phi)^2 - V(\Phi, x) \quad (2.53c)$$

Proof. Consider the normalized “velocity” of the scalar field $u_\mu := (-\nabla\Phi \cdot \nabla\Phi)^{-1/2} \nabla_\mu \Phi$. The prefactor is only well defined for $\nabla\Phi \cdot \nabla\Phi < 0$ and ensures the timelike normalization $u^2 = -1$. In terms of the kinetic energy $\mathcal{K} = -\frac{1}{2}\nabla\Phi \cdot \nabla\Phi$, we can write $\nabla_\mu \Phi = \sqrt{2\mathcal{K}} u_\mu$. Substituting into the scalar field’s EMT, proposition 1.4:

$$T_{\mu\nu} = 2\mathcal{K}u_\mu u_\nu - g_{\mu\nu}(-\mathcal{K} + V) = ([\mathcal{K} + V] + [\mathcal{K} - V])u_\mu u_\nu + (\mathcal{K} - V)g_{\mu\nu}$$

This is the EMT of a perfect fluid with $\rho = \mathcal{K} + V$ and $p = \mathcal{K} - V$. □

Comparing with definition 1.3, we see that the proper pressure of a scalar field under the conditions of proposition 2.8 is given by the Lagrangian density, $p = \mathcal{L}$. This is consistent with a perfect fluid’s Lagrangian $\mathcal{L}_{\text{PF}} = p(n, s)$, where n is the particle number density and s the entropy density.

Based on proposition 2.8, the energy conditions (definition 2.2) translate into requirements on the choice of the potential based on the kinetic energy. As an example, the DEC holds iff $V(\Phi, x) \geq 0$. We can continue with the other energy conditions, but this is not particularly illuminating. Much more interesting for our purposes is the cosmic fluid structure $w(\Phi, x) = \frac{p(\Phi, x)}{\rho(\Phi, x)} = \frac{\mathcal{K}(\Phi) - V(\Phi, x)}{\mathcal{K}(\Phi) + V(\Phi, x)}$, which is in general a complicated function of the field and position. However, there are two notable asymptotics: the **kinetic-dominated regime** $|V(\Phi)| < \mathcal{K}(\Phi)$,

$$w(\Phi, x) = 1 + \mathcal{O}\left(\frac{V}{\mathcal{K}}\right) \quad (2.54)$$

and the **potential-dominated regime** $|V(\Phi)| > \mathcal{K}(\Phi)$:

$$w(\Phi, x) = -1 + \mathcal{O}\left(\frac{\mathcal{K}}{V}\right) \quad (2.55)$$

If $V(\Phi) = 0$ exactly, then definition 1.3 describes a scalar field with a purely-kinetic Lagrangian $\mathcal{L} = \mathcal{K}(\Phi)$ and e.o.m. $\square\Phi = 0$. The corresponding cosmic fluid parameter is $w(\Phi, x) = +1$, and thus the EMT of a purely-kinetic scalar field with timelike gradient is indistinguishable from that of a Zeldovich stiff fluid. By virtue of fig. 2.1, all energy conditions are satisfied if $\rho > 0$ and are violated if $\rho < 0$.

If $\mathcal{K}(\Phi) = 0$ exactly, then definition 1.3 takes the form $\mathcal{L} = -V(\Phi, x)$ and the e.o.m. yield $V = V(x)$. Given $w(\Phi, x) = -1$, the EMT of a solitary scalar potential is identical to that of a vacuum energy cosmic fluid. By fig. 2.1, $V(x) > 0$ saturates the DEC/WEC/NEC and violates the SEC for all $D > 3$. If $V(x) < 0$, the SEC/NEC hold while the DEC/WEC do not¹.

¹In cosmological inflation, the *inflaton field* is modeled by definition 1.3 on a flat FLRW spacetime background. The dynamics of the inflaton when the potential energy dominates the kinetic energy $V(\Phi) \gg \mathcal{K}(\Phi)$ such that $w \approx -1$ are called *slow-roll inflation* [4].

To reproduce the other cosmic fluids, we can enforce the on-shell condition of proposition 1.3, $\square\Phi = \partial_\Phi V(\Phi, x)$, such that the kinetic energy has the following spacetime average:

$$\begin{aligned}\bar{\mathcal{K}} &= -\frac{1}{2\mathcal{V}} \int_{\Omega} d^D x \sqrt{-g} \nabla_\mu \Phi \nabla^\mu \Phi = -\frac{1}{2\mathcal{V}} \int_{\Omega} d^D x \sqrt{-g} [\nabla_\mu (\Phi \nabla^\mu \Phi) - \Phi \square\Phi] \\ &= \frac{1}{2} \overline{\Phi \partial_\Phi V} - \frac{1}{4\mathcal{V}} \int_{\partial\Omega} d\Sigma \cdot \nabla (\Phi^2)\end{aligned}$$

The precise details on how this boundary term may vanish are discussed in [28, 20], but essentially they require coherent, bounded, rapid oscillations around a local minimum of the potential and the averaging region $\mathcal{V} = \int_{\Omega} d^D x \sqrt{-g} < \infty$ to contain many oscillations. Assuming we can discard the boundary term, and taking a monomial potential $V(\Phi, x) = V_n(x)\Phi^n$, we have the virial theorem:

$$\bar{\mathcal{K}} = \frac{1}{2} \overline{\Phi \partial_\Phi V} = \frac{n}{2} \bar{V}$$

The average over Ω of the proper energy density $\rho = F_+$ and the proper pressure $p = F_-$ are $\bar{F}_\pm = \left(\frac{n}{2} \pm 1\right) \bar{V}$, so the *effective* cosmic fluid parameter w_{eff} is given by²:

$$w_{\text{eff}} := \frac{\overline{p(\Phi, x)}}{\overline{\rho(\Phi, x)}} = \frac{n-2}{n+2} \quad (2.56)$$

- The limit $n \rightarrow 0$ corresponds to the potential-dominated regime where we recover the vacuum energy cosmic fluid $w_{\text{eff}} \rightarrow -1$.
- The limit $n \rightarrow \infty$ corresponds to the kinetic-dominated regime where we recover Zeldovich's stiff fluid $w_{\text{eff}} \rightarrow +1$.
- The dust cosmic fluid $w_{\text{eff}} = 0$ is reproduced for $n = 2$ and corresponds to averaging over the local minimum of a quadratic potential $V \propto \Phi^2$.
- Similarly, the scalar field's effective EoS averages to radiation $w_{\text{eff}} = (D-1)^{-1}$ for a conformal potential $V \propto \Phi^{\frac{2D}{D-2}}$ (we recall that if $D = 2$ then $w_{\text{rad}} = w_{\text{stiff}}$).

²Notice $w_{\text{eff}} \neq \bar{w}$.

Chapter 3

Quantum Field Theory in Curved Spacetimes

In this chapter, we describe the QFTCST framework in the canonical quantization scheme. We then introduce the aspects of point-splitting required for the Hadamard expansion of smooth bitensors near the coincidence limit. Finally, we present the Hadamard renormalization and an algorithmic procedure for constructing a UV-finite REVENTO for scalar fields, the principal observable of this thesis.

3.1 Scalar Field Quantization

Thus far we have described field theories in the Lagrangian formalism. In the Hamiltonian formalism, one singles out a particular coordinate as the time t (say, via the ADM decomposition) and defines the **canonical conjugate momentum density** field $\Pi(x)$ associated with the scalar field $\Phi(x)$ as:

$$\Pi(t, x^i) := \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial(\partial_t\Phi)} \quad (3.1)$$

Endowing (Φ, Π) with the Poisson brackets defines the phase space of the theory. A sensible Hamiltonian formulation of the theory requires $(\mathcal{M}, g)$ to be globally hyperbolic such that the time function t is globally defined and the foliation is made by spacelike Cauchy surfaces of $\Sigma : t = \text{cte}$.

Definition 3.1 Canonical Quantization

In the **canonical quantization** of the scalar field theory $S[\Phi, g]$ we promote the canonical fields to operator-valued distributions $\Phi \rightarrow \hat{\Phi}, \Pi \rightarrow \hat{\Pi}$ and impose the canonical commutators $[\cdot, \cdot]_\Sigma = i\hbar\{\cdot, \cdot\}_{\text{PB}}$ on the $\Sigma : t = \text{cte}$ hypersurfaces:

$$\left[\hat{\Phi}(t, x^j), \hat{\Pi}(t, y^j) \right]_\Sigma = i\hbar \delta^{(D-1)}(x^j - y^j) \quad (3.2a)$$

$$\left[\hat{\Phi}(t, x^j), \hat{\Phi}(t, y^j) \right]_\Sigma = 0 \quad (3.2b)$$

$$\left[\hat{\Pi}(t, x^j), \hat{\Pi}(t, y^j) \right]_\Sigma = 0 \quad (3.2c)$$

As operator-valued distributions, quantum fields are properly defined through their “smearings” with smooth test functions of compact support:

$$\hat{\Phi}[f] = \int_{\Omega} d^D x \sqrt{-g} \hat{\Phi}(x) f(x) \quad (3.3)$$

Suppose the solution space admits a complete set of positive-frequency orthonormal modes $\{u_{\lambda}, \text{c.c.}\}$. When the field is expanded as in eq. (2.41), the canonical quantization of definition 3.1 translates into promoting the Fourier coefficients to operators while leaving the modes as classical fields:

$$\hat{\Phi} = \sum_{\lambda} \left(\hat{a}_{\lambda} u_{\lambda} + \hat{a}_{\lambda}^{\dagger} u_{\lambda}^* \right) \quad (3.4)$$

The quantum versions are called **annihilation** $\hat{a}_{\lambda}$ and **creation** $\hat{a}_{\lambda}^{\dagger}$ operators.

Proposition 3.1 Ladder Algebra

The canonical quantization of definition 3.1 induces the **ladder algebra** on the commutators of the creation and annihilation operators:

$$\left[\hat{a}_{\lambda}, \hat{a}_{\lambda'}^{\dagger} \right]_{\Sigma} = \hbar \delta_{\lambda\lambda'} \quad (3.5a)$$

$$\left[\hat{a}_{\lambda}, \hat{a}_{\lambda'} \right]_{\Sigma} = 0 \quad (3.5b)$$

$$\left[\hat{a}_{\lambda}^{\dagger}, \hat{a}_{\lambda'}^{\dagger} \right]_{\Sigma} = 0 \quad (3.5c)$$

For continuous labels λ , the Kronecker deltas are replaced by the corresponding Dirac delta distributions.

We will prove proposition 3.1 explicitly for the infinite potential well.

The Fock space of the QFT is constructed from the vacuum state associated with the chosen normal mode decomposition.

Definition 3.2 Vacuum State

The **vacuum state** $|0\rangle$ associated with $\{u_{\lambda}, \text{c.c.}\}$ is defined as the normalized state annihilated by all annihilation operators:

$$\langle 0|0\rangle = 1 \quad (3.6a)$$

$$\hat{a}_{\lambda} |0\rangle = 0, \quad \forall \lambda \quad (3.6b)$$

Letting $\langle \hat{f} \rangle := \langle 0 | \hat{f} | 0 \rangle$, the **excited states** are constructed from the vacuum state by the successive applications of the creation operators:

$$|n_{\lambda}, \dots, n_{\lambda'}\rangle \propto \left(\hat{a}_{\lambda}^{\dagger} \right)^{n_{\lambda}} \cdots \left(\hat{a}_{\lambda'}^{\dagger} \right)^{n_{\lambda'}} |0\rangle \quad (3.7)$$

Definition 3.2 and proposition 3.1 identify the excited states as positive-norm states. As an example:

$$\langle 1_{\lambda} | 1'_{\lambda} \rangle = \langle \hat{a}_{\lambda} \hat{a}_{\lambda'}^{\dagger} \rangle = \hbar \delta_{\lambda\lambda'} \geq 0 \quad (3.8)$$

A crucial feature of QFTCST is that, in general, the excited states of a quantum field admit the physical interpretation of “*particles*” only in appropriate asymptotic regimes. Moreover, different choices of positive-frequency normal modes $\{u_\lambda, \text{c.c.}\}, \{\bar{u}_\lambda, \text{c.c.}\}$ with their respective vacua $|0\rangle, |\bar{0}\rangle$ are not guaranteed to define unitarily equivalent Fock representations. The observer-dependence and background-dependence of “*the*” vacuum and “*the*” particle number density give rise to the famous Hawking, Unruh, and Parker effects [21, 11, 31, 6, 22, 12].

3.2 Point-Splitting

Our local quantum observable of interest is $\langle \hat{T} \rangle$. As discussed in section 1.3, a proper definition of this quantity requires the point-splitting of fields into bitensors at the classical level.

A **bitensor** is a tensorial field $B(x, x') \in \mathcal{T}_x^{(p,q)} \mathcal{M} \otimes \mathcal{T}_{x'}^{(r,s)} \mathcal{M}$ that depends on two spacetime points $x, x' \in \mathcal{M}$. In a given coordinate chart, a bitensor may carry tensor indices associated with either spacetime point. Henceforth, let unprimed indices refer to the tangent space at x and primed indices to the one at x' . In general, the components of a rank- $(p + q + r + s)$ bitensor are:

$$B^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q}{}^{\alpha'_1 \dots \alpha'_r}_{\beta'_1 \dots \beta'_s}(x, x') \quad (3.9)$$

The limit $x' \rightarrow x$ is called the **coincidence limit** of the bitensor. We shall assume:

1. The point $x' \in \mathcal{N}(x)$ belongs to the *normal convex neighborhood* $\mathcal{N}$ of the point x . Namely, the set of points that are linked to x by a unique geodesic.
2. The coincidence limit is a unique tensor field at x and is independent of the direction in which the limit is taken.

Definition 3.3 Sygne World Function

Let $\beta : (\lambda_0, \lambda_1) \mapsto \mathcal{M}$ be the geodesic segment connecting x, x' and described by $z(\lambda)$, where λ is an arbitrary affine parameter such that $z(\lambda_0) := x$ and $z(\lambda_1) := x'$, whose tangent vector is $\dot{z} := dz/d\lambda$. We define the **Synge world function** $\sigma(x, x')$ as the biscalar:

$$\sigma(x, x') := \frac{1}{2}(\lambda_1 - \lambda_0) \int_{\lambda_0}^{\lambda_1} d\lambda \, g(\dot{z}, \dot{z}) \quad (3.10)$$

Definition 3.3 of dimensions $[\sigma] = \ell_P^2$ plays an important role in the Hadamard renormalization procedure.

Proposition 3.2 SWF Properties

General properties of the Synge world function (definition 3.3):

1. Argument symmetric, $\sigma(x, x') = \sigma(x', x)$.
2. Invariant under an affine reparameterization, $\lambda \rightarrow \bar{\lambda} = a\lambda + b$, where $a, b \in \mathbb{R}$.
3. If the geodesic is timelike, then $\lambda = \tau$ is the proper time and $\sigma = -\frac{1}{2}(\Delta\tau)^2 < 0$.
4. If the geodesic is spacelike, then $\lambda = l$ is the proper length and $\sigma = \frac{1}{2}(\Delta l)^2 > 0$.
5. If the geodesic is null, then $\sigma = 0$.
6. Letting $\sigma_\mu := \nabla_\mu \sigma$ and $\sigma_{\mu'} := \nabla_{\mu'} \sigma$, then:

$$\sigma = \frac{1}{2} g^{\mu\nu} \sigma_\mu \sigma_\nu = \frac{1}{2} g^{\mu'\nu'} \sigma_{\mu'} \sigma_{\nu'} \quad (3.11)$$

7. The coincidence limits are:

$$\lim_{x' \rightarrow x} \sigma = 0 \quad (3.12a)$$

$$\lim_{x' \rightarrow x} \sigma_\mu = \lim_{x' \rightarrow x} \sigma_{\mu'} = 0 \quad (3.12b)$$

$$\lim_{x' \rightarrow x} \sigma_{\mu\nu} = \lim_{x' \rightarrow x} \sigma_{\mu'\nu'} = g_{\mu\nu} \quad (3.12c)$$

$$\lim_{x' \rightarrow x} \sigma_{\mu\nu'} = \lim_{x' \rightarrow x} \sigma_{\mu'\nu} = -g_{\mu\nu} \quad (3.12d)$$

We refer to [24] for the proof, which is our main reference for this section. Proposition 3.2 shows that, in general, definition 3.3 is half the squared geodesic distance between the points x, x' .

Proposition 3.3 Flat SWF

In flat spacetime with Cartesian coordinates, definition 3.3 reads:

$$\sigma(x, x') = \frac{1}{2} \eta_{\mu\nu} (x - x')^\mu (x - x')^\nu \quad (3.13)$$

Therefore:

$$\sigma_\mu = \eta_{\mu\nu} (x - x')^\nu \quad (3.14a)$$

$$\sigma_{\mu'} = -\eta_{\mu\nu} (x - x')^\nu \quad (3.14b)$$

And:

$$\sigma_{\mu\nu} = \sigma_{\mu'\nu'} = \eta_{\mu\nu} \quad (3.15a)$$

$$\sigma_{\mu'\nu} = \sigma_{\mu\nu'} = -\eta_{\mu\nu} \quad (3.15b)$$

Proof. The tangent vector in definition 3.3 obeys the geodesic equation $\nabla_{\dot{z}} \dot{z}^\mu = \ddot{z}^\mu + \Gamma_{\alpha\beta}^\mu \dot{z}^\alpha \dot{z}^\beta = 0$. In Minkowski spacetime with Cartesian coordinates all Christoffel symbols vanish, and the solutions to the geodesic eq. are straight lines $\nabla_{\dot{z}} \dot{z} = \ddot{z} = 0 \implies z = a\lambda + b$, with a, b being constant vectors. The boundary conditions enforce:

$$x = z(\lambda_0) = a\lambda_0 + b$$

$$x' = z(\lambda_1) = a\lambda_1 + b$$

Subtracting the two equations, $a = \frac{x' - x}{\lambda_1 - \lambda_0}$, the norm of the tangent is $g(\dot{z}, \dot{z}) = a^2 = \frac{(x - x')^2}{(\lambda_1 - \lambda_0)^2} = \text{cte}$ and definition 3.3 becomes:

$$\sigma = \frac{1}{2} (\lambda_1 - \lambda_0) \frac{(x - x')^2}{(\lambda_1 - \lambda_0)^2} \int_{\lambda_0}^{\lambda_1} d\lambda = \frac{1}{2} (x - x')^2$$

Introducing coordinates, $\sigma = \frac{1}{2}\eta_{\alpha\beta}(x-x')^\alpha(x-x')^\beta$, the derivatives are straightforward:

$$\begin{aligned}\partial_\mu\sigma &= \frac{1}{2}\eta_{\alpha\beta}\delta_{\mu\alpha}(x-x')^\beta + \frac{1}{2}\eta_{\alpha\beta}(x-x')^\alpha\delta_{\mu\beta} = \eta_{\mu\gamma}(x-x')^\gamma \\ \partial_{\mu\nu}^2\sigma &= \eta_{\mu\gamma}\delta_{\nu\gamma} = \eta_{\mu\nu}\end{aligned}$$

An analogous procedure works for the primed indices. □

Near the coincidence limit, a generic bitensor can be expanded in powers of the Synge world function and its derivatives. As an example, for a rank-2 bitensor with indices in x , we have:

$$B_{\alpha\beta}(x, x') = A_{\alpha\beta} + A_{\alpha\beta\gamma}\sigma^\gamma + \frac{1}{2}A_{\alpha\beta\gamma\delta}\sigma^\gamma\sigma^\delta + \mathcal{O}(3) \quad (3.16)$$

The expansion coefficients A are rank- $(2+n)$ tensor fields at x , where n denotes the power of σ , and are determined by taking the coincidence limits of the bitensor's covariant derivatives. E.g., by virtue of proposition 3.2:

$$\lim_{x \rightarrow x'} B_{\alpha\beta}(x, x') = A_{\alpha\beta} \quad (3.17)$$

For bitensors with indices in x' or mixed indices, one usually employs the so-called **parallel propagator** $g^\mu{}_{\nu'}(x, x')$. As the name indicates, this rank-2 bitensor maps tensor fields at x' to x via parallel transport along the unique geodesic connecting the two points. The parallel propagator also enters the construction of the point-split EMT bitensorial differential operator in definition 1.5. As an example, for a purely kinetic scalar field theory (proposition 1.4 with $V = 0$), this operator reads:

$$\mathcal{T}_{\mu\nu}(x, x') = g_\nu{}^{\nu'}(x, x') \nabla_\mu \nabla_{\nu'} - \frac{1}{2}g_{\mu\nu}(x)g^{\alpha\beta'}(x, x') \nabla_\alpha \nabla_{\beta'} \quad (3.18)$$

3.3 Hadamard Renormalization

For a real scalar field, the Feynman propagator is identified as a biscalar distribution symmetric in its arguments. The singularity at the coincidence limit is treated in the Hadamard renormalization by expanding in powers of σ near the coincidence limit, isolating the state-independent structure from the state-dependent one, and subtracting the universal short-distance singularity from the point-split expectation value. In this section, we follow [10] closely and refer all proofs to this reference.

Proposition 3.4 Hadamard Expansions

With $\epsilon \rightarrow 0^+$, the **Hadamard expansions** of definition 1.4 are given by the following^a.

- For $D = 2$:

$$G^F(x, x') = \frac{i\alpha_2}{2} [V(x, x') \ln[\sigma(x, x') + i\epsilon] + W(x, x')] \quad (3.19)$$

- For even $D \neq 2$:

$$G^F(x, x') = \frac{i\alpha_D}{2} \left[\frac{U(x, x')}{[\sigma(x, x') + i\epsilon]^{D/2-1}} + V(x, x') \ln[\sigma(x, x') + i\epsilon] + W(x, x') \right] \quad (3.20)$$

- For odd D :

$$G^F(x, x') = \frac{i\alpha_D}{2} \left[\frac{U(x, x')}{[\sigma(x, x') + i\epsilon]^{D/2-1}} + W(x, x') \right] \quad (3.21)$$

^aThis expansion requires $\hat{\Phi}$ to be in a **Hadamard-type** normalized quantum state [16].

In proposition 3.4, the prefactor is given by:

$$\alpha_D := \begin{cases} \frac{1}{2\pi}, & D = 2 \\ \frac{\Gamma(D/2 - 1)}{(2\pi)^{D/2}}, & D \neq 2 \end{cases} \quad (3.22)$$

U, V, W are symmetric biscalars that are regular in the coincidence limit and which possess Taylor expansions in terms of definition 3.3. The relevant ones for us are:

$$W(x, x') = \sum_{n=0}^{+\infty} W_n(x, x') \sigma^n(x, x') = w(x) + \sum_{p=1}^{+\infty} \frac{(-1)^p}{p!} w_{(p)}(x, x') \quad (3.23a)$$

$$V(x, x') = \sum_{n=0}^{+\infty} V_n(x, x') \sigma^n(x, x') \quad (3.23b)$$

where $f_{(p)}(x, x') = f_{a_1 \dots a_p}(x) \nabla^{a_1} \sigma(x, x') \dots \nabla^{a_p} \sigma(x, x')$. The coefficients U_n, V_n, W_n are symmetric and regular biscalars as well, and can be Taylor expanded as:

$$V_n(x, x') = v_n(x) + \sum_{p=1}^{+\infty} \frac{(-1)^p}{p!} v_{n(p)}(x, x') \quad (3.24a)$$

By virtue of the coincidence limits of the Synge world function and its derivatives (proposition 3.2), the relevant expansion coefficients are given by the following.

Proposition 3.5 Coincidence Limits

For all dimensions:

$$w(x) = \lim_{x' \rightarrow x} W(x, x') \quad (3.25a)$$

$$w_{\mu\nu}(x) = \lim_{x' \rightarrow x} \nabla_\mu \nabla_\nu W(x, x') \quad (3.25b)$$

Generically, the coefficients in eq. (3.23a) are determined by substituting the Hadamard expansion into the Green function identity of the Feynman propagator and integrating the recursion relations along the unique geodesic connecting x, x' .

Definition 3.4 Regular & Singular Structures

For all dimensions, we identify the following two structures of the Feynman propagator's Hadamard expansion.

- The quantum state-dependent part that is regular in the coincidence limit is given by:

$$G_{\text{reg}}^{\text{F}}(x, x') := \frac{i\alpha_D}{2} W(x, x') \quad (3.26)$$

- The purely geometrical part that is singular in the coincidence limit is given by:

$$G_{\text{sing}}^{\text{F}}(x, x') := G^{\text{F}}(x, x') - G_{\text{reg}}^{\text{F}}(x, x') \quad (3.27)$$

From definition 3.4, we shall call the following rescaling the **regularized propagator**

$$W(x, x') = \frac{2}{i\alpha_D} G_{\text{reg}}^{\text{F}}(x, x') \quad (3.28)$$

and the following rescaling the **Hadamard parametrix**:

$$H(x, x') = \frac{2}{i\alpha_D} G_{\text{sing}}^{\text{F}}(x, x') \quad (3.29)$$

The Hadamard parametrix is the universal singular structure of the theory's Feynman propagator and physically encodes the local UV divergences.

In Hadamard's renormalization procedure, we replace $G^{\text{F}}(x, x') \rightarrow G_{\text{reg}}^{\text{F}}(x, x')$ and add a state-independent rank-2 tensor field $\tilde{\Theta}$ that only depends on the parameters of the theory and on the local geometry:

$$\left\langle \hat{T}_{\mu\nu}(x) \right\rangle_{\text{ren}} := \frac{\alpha_D}{2} \lim_{x' \rightarrow x} \mathcal{T}_{\mu\nu}(x, x') W(x, x') + \tilde{\Theta}_{\mu\nu} \quad (3.30)$$

Explicitly, we have the following.

Proposition 3.6 REVENTO

Consider a scalar field action (definition 1.3) with potential $V(\Phi, x) = m^2\Phi^2/2\hbar^2 + \xi R\Phi^2/2$ over all spacetime $\Omega = \mathcal{M}$ where $\partial\mathcal{M} = \emptyset$.

$$S[\Phi, g] = -\frac{1}{2} \int_{\mathcal{M}} d^D x \sqrt{-g} \left(g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi + \frac{m^2}{\hbar^2} \Phi^2 + \xi R \Phi^2 \right) \quad (3.31)$$

For an arbitrary rank-2 tensor field $\Theta(m, \xi, g)$ that is state-independent and locally conserved, $\nabla_\mu \Theta^{\mu\nu} = 0$, the REVENTO of eq. (3.31) in the Hadamard prescription is given by the following.

- For D even:

$$\begin{aligned} \left\langle \hat{T}_{\mu\nu}(x) \right\rangle_{\text{ren}} = \frac{\alpha_D}{2} \left[-w_{\mu\nu} + \frac{1}{2}(1 - 2\xi) \nabla_\mu \nabla_\nu w + \frac{1}{2} \left(2\xi - \frac{1}{2} \right) g_{\mu\nu} \square w + \right. \\ \left. + \xi R_{\mu\nu} w - g_{\mu\nu} v_1 \right] + \Theta_{\mu\nu} \end{aligned} \quad (3.32)$$

- For D odd:

$$\begin{aligned} \left\langle \hat{T}_{\mu\nu}(x) \right\rangle_{\text{ren}} = \frac{\alpha_D}{2} \left[-w_{\mu\nu} + \frac{1}{2}(1 - 2\xi) \nabla_\mu \nabla_\nu w + \frac{1}{2} \left(2\xi - \frac{1}{2} \right) g_{\mu\nu} \square w + \right. \\ \left. + \xi R_{\mu\nu} w \right] + \Theta_{\mu\nu} \end{aligned} \quad (3.33)$$

In proposition 3.6, the change from $\tilde{\Theta} \rightarrow \Theta$ was a dimension-dependent rescaling that ensured the conservation of the resulting expression, while still retaining the state-independent ambiguity term inherent to the Hadamard renormalization. The Θ -independent terms and the Θ term of the REVMTO are locally conserved independently of each other.

Proposition 3.7 REVMTO's Trace

If $\Theta = 0$ and $\xi_c(D) = \frac{1}{4} \frac{D-2}{D-1}$ denotes the conformal coupling constant, then the trace of the REVMTO in proposition 3.6 is given by the following.

- For D even:

$$\left\langle \hat{T} \right\rangle_{\text{ren}} = \frac{\alpha_D}{2} \left[-\frac{m^2}{\hbar^2} w + (D-1) (\xi - \xi_c) \square w + 2v_1 \right] \quad (3.34)$$

- For D odd:

$$\left\langle \hat{T} \right\rangle_{\text{ren}} = \frac{\alpha_D}{2} \left[-\frac{m^2}{\hbar^2} w + (D-1) (\xi - \xi_c) \square w \right] \quad (3.35)$$

Despite the power of the method and great generality of proposition 3.6, we still encounter difficulties:

- The REVMTO in proposition 3.6 is unique up to the addition of the tensor field Θ . This induces a serious issue for backreaction effects because the RHS of the semi-classical EFE becomes ambiguously defined. The structure of Θ can be restricted considerably if one assumes that this is the EMT generated by a general-relativistic Lagrangian $\mathcal{L}_\Theta$ of dimensions $[\mathcal{L}_\Theta] = [\Theta] = m_P \ell_P^{-(D-1)}$ and that depends solely on m, ξ, g .
- If the classical EMT is initially traceless, the REVMTO may acquire a **trace anomaly** after renormalization. For a conformally invariant scalar field theory, $m = 0$ and $\xi = \xi_c$, we see from proposition 3.7 that if D is odd, then there is no trace anomaly. However, if D is even, then the trace anomaly of proposition 3.6's REVMTO is given by:

$$\left\langle \hat{T} \right\rangle_{\text{ren}} = \alpha_D v_1 \quad (3.36)$$

- For an even D , in order to make the argument of the logarithm in proposition 3.4 dimensionless one must introduce a **renormalization length scale** l through $V(x, x') \ln \sigma(x, x') \rightarrow V(x, x') \ln [\sigma(x, x')/l^2]$. For an arbitrary reference scale l_0 , this implies that one must add to the REVMTO in eq. (3.32) the term:

$$\Theta_{\mu\nu}^l = -\frac{\alpha_D}{2} \left[-v_{0\mu\nu} + \frac{1}{2}(1-2\xi)\nabla_\mu\nabla_\nu v_0 + \frac{1}{2}\left(2\xi - \frac{1}{2}\right)g_{\mu\nu}\square v_0 + \xi R_{\mu\nu}v_0 - g_{\mu\nu}v_1 \right] \ln\left(\frac{l_0}{l}\right)^2 \quad (3.37)$$

The trace is¹:

$$g^{\mu\nu}\Theta_{\mu\nu}^l = -\frac{\alpha_D}{2} \left[-\frac{m^2}{\hbar^2}v_0 + (D-1)(\xi - \xi_c)\square v_0 \right] \ln\left(\frac{l_0}{l}\right)^2 \quad (3.38)$$

Example 3.1 Two-Dimensions

The Feynman propagator, regularized propagator, and the Hadamard parametrix share the same physical dimensions, which for $D = 2$ are $[G^F] = [W] = [H] = m_P \ell_P = \hbar$. The expansion of the singular part of the propagator in proposition 3.4 yields the Hadamard parametrix:

$$H(x, x') = V(x, x') \ln[\sigma(x, x') + i\epsilon] \quad (3.39)$$

The regularized propagator is then:

$$W(x, x') = \frac{4\pi}{i} G^F(x, x') - V(x, x') \ln[\sigma(x, x') + i\epsilon] \quad (3.40)$$

The required expansion order is:

$$V = V_0 + V_1\sigma + \mathcal{O}(\sigma^{3/2}) \quad (3.41)$$

The Hadamard coefficients are:

$$V_0 = v_0 - v_{0a}\nabla^a\sigma + \frac{1}{2!}v_{0ab}\nabla^a\sigma\nabla^b\sigma + \mathcal{O}(\sigma^{3/2}) \quad (3.42a)$$

$$V_1 = v_1 + \mathcal{O}(\sigma^{1/2}) \quad (3.42b)$$

While the Taylor coefficients are:

$$v_0 = -\hbar \quad (3.43a)$$

$$v_{0a} = 0 \quad (3.43b)$$

$$v_{0ab} = -\frac{\hbar}{12}Rg_{ab} \quad (3.43c)$$

$$v_1 = -\frac{m^2}{2\hbar} - \frac{\hbar}{2}\left(\xi - \frac{1}{6}\right)R \quad (3.43d)$$

Since D is even, the parametrix-dimensional term in eq. (3.37) reduces to:

¹Given that eq. (3.38) vanishes for $m = 0$ and $\xi = \xi_c$, then eq. (3.36) holds even after introducing l .

$$\Theta_{\mu\nu}^{l^2} = -\frac{m^2}{8\pi\hbar} \ln\left(\frac{l_0}{l}\right)^2 g_{\mu\nu} \quad (3.44)$$

If $m = 0$ and $\xi = \xi_c(2) = 0$, the trace anomaly of eq. (3.36) becomes:

$$\langle \hat{T} \rangle_{\text{ren}} = \frac{\hbar}{24\pi} R \quad (3.45)$$

Given $[m] = m_{\text{P}}$ and $[\xi] = [g] = 1$, the only possible non-trivial Lagrangian of dimensions $[\mathcal{L}_{\Theta}] = m_{\text{P}} \ell_{\text{P}}^{-1}$ is $\mathcal{L}_{\Theta} = m^2/\hbar$. Up to a dimensionless function $f(\xi)$ of ξ , then:

$$\Theta = f(\xi) \frac{m^2}{\hbar} g \quad (3.46)$$

Notice how eqs. (3.44) and (3.46) are locally conserved and vanish in the CFT limit $m \rightarrow 0$.

Chapter 4

Finite Potential Well

We now apply the formalisms developed in the preceding chapters to the finite-potential-well approximation of the Regge-Wheeler-type effective potential. We establish the classical and quantum $(1 + 1)$ -dimensional auxiliary theory and determine the corresponding REVMTO following Hadamard's renormalization.

4.1 Infinite Potential Well

The first approach one can take to solve the e.o.m. of the auxiliary theory (definition 1.6) is to simply take the effective potential to be vanishing for all angular momenta:

$$V_\ell(r) = 0 \quad (4.1)$$

For an arbitrary D , this does not necessarily imply the SSS spacetime is Minkowskian. Indeed, even for a purely kinetic s -wave, $U(g, r) = 0$ and $\ell = 0$, the Minkowskian effective potential becomes $V^{\text{bg}}(r) = (D - 2)(D - 4)/(4r^2)$. For simplicity and contrast, consider the same setup. From the background effective potential of eq. (1.37), we can see that eq. (4.1) corresponds to a SSS spacetime for which,

$$\frac{f(r)}{h(r)} = C_1 r^{4-D} \quad (4.2a)$$

$$r_* = \frac{1/\sqrt{C_1}}{D/2 - 1} r^{D/2-1} + C_2 \quad (4.2b)$$

where $C_j \in \mathbb{R}$, and notice that in the $D \rightarrow 2$ limit, $r_* \propto \ln r$. Equation (4.1) also implies that the auxiliary field is purely kinetic and independent of the angular momenta, $\Psi_\ell = \Psi$. To reproduce the step-discontinuity behavior of fig. 1.3, we can consider the scalar field Φ to be confined within an **infinite potential well** of finite tortoise length $r_* = L$, which in areal-radius coordinates corresponds to the fluid's spherical surface. This requirement is implemented by imposing the Dirichlet boundary conditions:

$$\Psi(t, r_* = 0) = \Psi(t, r_* = L) = 0 \quad (4.3)$$

The REVMTO for this auxiliary theory has been computed in [18], and specifically in [26] following Hadamard's renormalization. Below we quote the results in our notation.

Proposition 4.1 Infinite Potential Well

Consider the auxiliary theory subject to the Dirichlet boundary conditions at $r_* = 0$ and $r_* = L$.

$$S[\Psi, \eta] = \int_{-\infty}^{+\infty} \int_0^L dt dr_* \left(-\frac{1}{2} \eta^{\mu\nu} \partial_\mu \Psi \partial_\nu \Psi \right) \quad (4.4)$$

For $n \in \mathbb{N} = \{1, 2, 3, \dots\}$, a complete basis $\{\psi_n, \psi_n^*\}$ of the e.o.m. solution space is given by the positive-frequency orthonormal modes:

$$\psi_n(t, r_*) = \frac{1}{\sqrt{\omega_n L}} e^{-i\omega_n t} \sin(\omega_n r_*), \quad \omega_n = \frac{\pi}{L} n \quad (4.5)$$

The corresponding quantum field reads:

$$\hat{\Psi}(t, r_*) = \sum_{n=1}^{+\infty} \frac{1}{\sqrt{\omega_n L}} \sin(\omega_n r_*) \left(\hat{a}_n e^{-i\omega_n t} + \hat{a}_n^\dagger e^{+i\omega_n t} \right) \quad (4.6)$$

As expected, the non-renormalized EVENTO w.r.t. the vacuum state defined by eq. (4.5) is divergent and non-symmetric:

$$\langle \hat{T}_{00} \rangle = \langle \hat{T}_{11} \rangle = \frac{\pi \hbar}{2L^2} \sum_{n=1}^{+\infty} n \quad (4.7a)$$

$$\langle \hat{T}_{01} \rangle = -\langle \hat{T}_{10} \rangle = -i \frac{\pi \hbar}{2L^2} \sum_{n=1}^{+\infty} n \sin(2\omega_n r_*) \quad (4.7b)$$

Following Hadamard's renormalization, the regularized propagator is:

$$W(t, r_*, t', r'_*) = 4\hbar \sum_{n=1}^{+\infty} \frac{1}{n} \sin(\omega_n r_*) \sin(\omega_n r'_*) e^{-i\omega_n |t-t'|} + \hbar \ln \left[\frac{-(t-t')^2 + (r_* - r'_*)^2}{2} + i\epsilon \right] \quad (4.8)$$

And the REVENTO of the theory reads:

$$\langle \hat{T}_{\mu\nu} \rangle_{\text{ren}} = -\frac{\pi \hbar}{24L^2} \delta_{\mu\nu} \quad (4.9)$$

The standard interpretation of proposition 4.1 is in terms of the **Casimir energy**, obtained by integrating the local energy density $\langle \hat{\rho} \rangle_Z := \langle \hat{T}_{00} \rangle_{\text{ren}}$ along $0 \leq r_* \leq L$. Taking the negative spatial gradient of the Casimir energy w.r.t. the well's width L shows that two parallel lines at the boundaries attract each other [17].

Comparing with the theory presented in section 2.1, we also see that the infinite potential well (1 + 1) REVENTO has the same structure as that of a Zeldovich stiff fluid in its locally inertial rest frame (LIRF) and with a homogeneous proper energy density:

$$\langle \hat{\rho} \rangle_Z = \langle \hat{p} \rangle_Z = -\frac{\pi \hbar}{24L^2} \quad (4.10)$$

Since $D = 2$, this is also the EMT structure of a radiation cosmic fluid. By virtue of fig. 2.1 and given that the

proper energy density is negative, the REVMTO in eq. (4.9) violates all energy conditions. As described in section 2.2.4, the EoS $\omega_Z = 1$ is to be expected at the classical level because the auxiliary scalar field is purely kinetic, $\mathcal{L}(\Psi) = \mathcal{K}(\Psi)$. Notably, we find that the QFT inherits this behavior but with an important distinction: recalling proposition 2.8, the effective perfect-fluid interpretation suffers from being physically meaningful only when the gradient is timelike $(\partial_\mu \Psi)^2 < 0$. As shown in fig. 4.1, this means that for the modes in eq. (4.5) and the chart $x^\mu = (t, r_*)$, the effective fluid 2-velocity of the classical EMT is not well-defined throughout the well. However, the components of the REVMTO identify the reference frame of coordinates $x^\mu = (t, r_*)$ itself as the LIRF of the effective perfect fluid, and the effective fluid 2-velocity as $u^\mu = (1, 0)$. This is everywhere timelike and requires no reference to a scalar-field gradient at all. The REVMTO's perfect-fluid interpretation does not inherit the timelike-gradient restriction of the classical EMT.

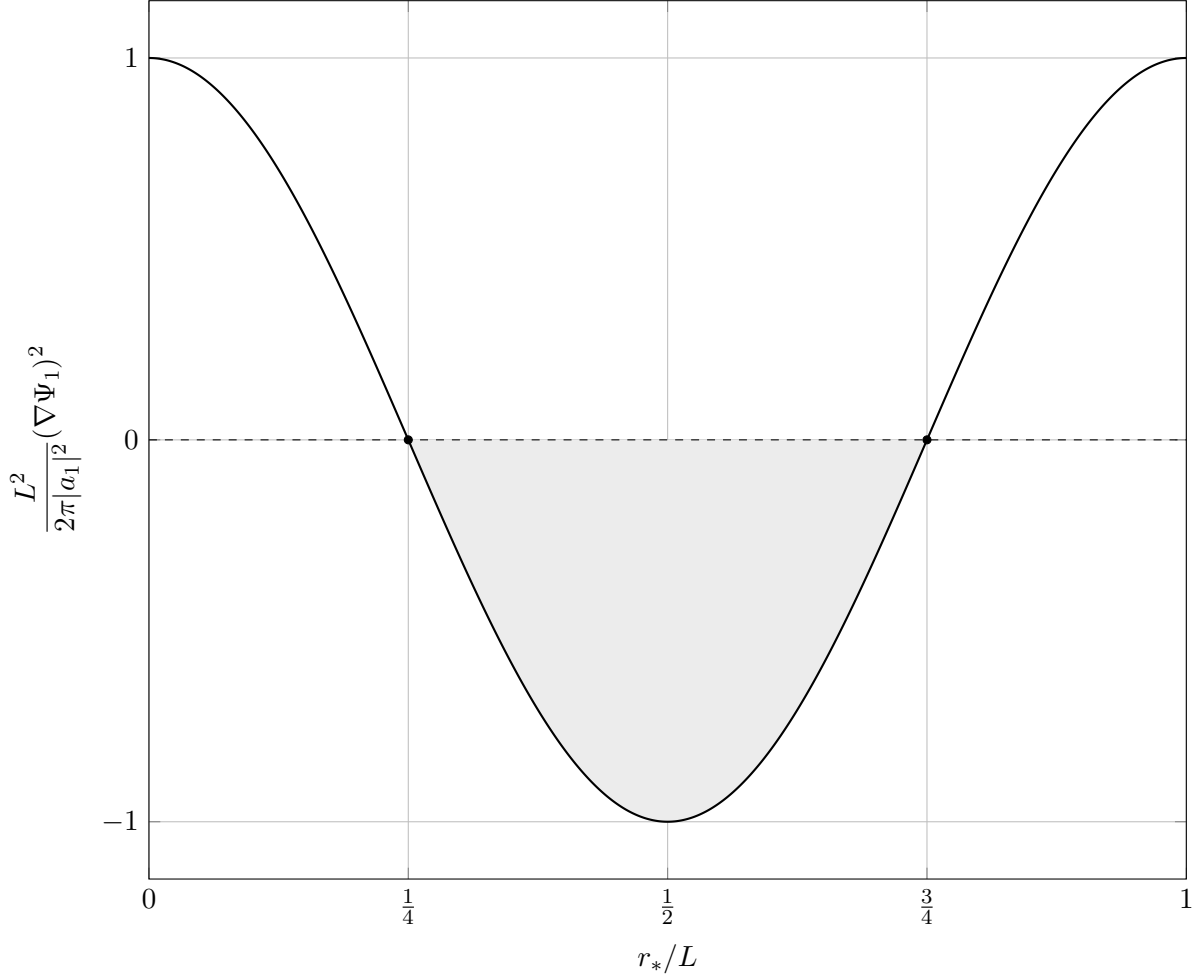


Figure 4.1: Causal nature of the normalized gradient of the classical real field $\Psi_1 = a_1 \psi_1 + a_1^* \psi_1^*$ constructed from the fundamental normal mode of proposition 4.1, evaluated on the time slice $t = \pi/(4\omega_1) = L/4$ for which $(\nabla \Psi_1)^2 = 2\pi L^{-2} |a_1|^2 \cos(2\pi r_*/L)$. The gradient is timelike in the shaded region, spacelike in the unshaded region, and null at $r_* = L/4$ and $r_* = 3L/4$.

4.2 Model Setting

To solve the auxiliary theory (definition 1.6) the next simplest case we can consider is an effective potential that is constant for all angular momenta:

$$V_\ell(r) = \vartheta^2 = \text{cte} \quad (4.11)$$

From the background effective potential of eq. (1.37), it follows that eq. (4.11) corresponds to a SSS spacetime for which:

$$\frac{f(r)}{h(r)} = \left(\frac{2\vartheta}{D-2} \right)^2 r^2 + C_1 r^{4-D} \quad (4.12a)$$

$$r_* = \frac{1}{\vartheta} \sinh^{-1} \left(\frac{1}{\sqrt{C_1}} \frac{2\vartheta}{D-2} r^{D/2-1} \right) + C_2 \quad (4.12b)$$

where $C_j \in \mathbb{R}$. The limit $D \rightarrow 2$ forces $\vartheta \rightarrow 0$ and thus the previous case. More generally, we see that fixing the effective potential determines one of the metric components.

To reproduce the step-discontinuity behavior of fig. 1.3, we shall consider that the auxiliary field is inside a well of finite depth and finite length. This amounts to taking $V_\ell(r)$ proportional to the Heaviside step function and imposing the smooth matching of the solutions inside and outside the well. The constant of proportionality is taken to be non-negative since in the Klein-Gordon limit the effective potential reduces to the squared mass of the scalar field, $V_\ell(r) = m^2 \geq 0$. The auxiliary theory is a scalar field theory in its own right, and thus all the theory developed in chapters 2 and 3 holds. Henceforth, for simplicity, we change our notation to:

$$\Psi_\ell(t, r_*) \rightarrow \Phi(t, x), \quad V_\ell(r_*) \rightarrow U(x) \quad (4.13)$$

The action of the model is then as follows.

Definition 4.1 Finite Potential Well Action

Consider the following action for a real scalar field $\Phi(x)$ in the $D = 2$ Minkowski spacetime $(\mathcal{M}, \eta)$ with coordinates $x^\mu = (t, x)$.

$$S[\Phi, \eta] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dt dx \left[-\frac{1}{2} \eta^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} U(x) \Phi^2 \right] \quad (4.14)$$

For $U_0 > 0$ and $L > 0$, let the potential $U(x)$ be given by:

$$U(x) := (U_0)^2 \Theta(|x| - L) = \begin{cases} 0, & |x| < L \\ (U_0)^2, & |x| > L \end{cases} \quad (4.15)$$

As shown in fig. 4.2, $U(x)$ describes a **finite potential well** of depth U_0^2 and half-width L . From a field theory perspective, definition 4.1 describes the dynamics of a real scalar field with a position-dependent mass that is finite in the **exterior region** $|x| > L$ and zero in the **interior region** $|x| < L$. In $D = 2$ and with our units, $[\mathcal{L}] = [\mathbf{T}] = m_P \ell_P^{-1}$, $[\Phi] = \sqrt{m_P \ell_P}$, and the two free parameters have $[L] = \ell_P$ and $[U_0] = \ell_P^{-1}$.

Proposition 4.2 Finite Potential Well E.O.M.

Varying the action of definition 4.1 w.r.t. the scalar field Φ and imposing the action principle yields the e.o.m.

$$[\square - U(x)] \Phi = 0 \quad (4.16)$$

which splits into:

$$\begin{cases} \square \Phi = 0, & |x| < L \\ (\square - U_0^2) \Phi = 0, & |x| > L \end{cases} \quad (4.17)$$

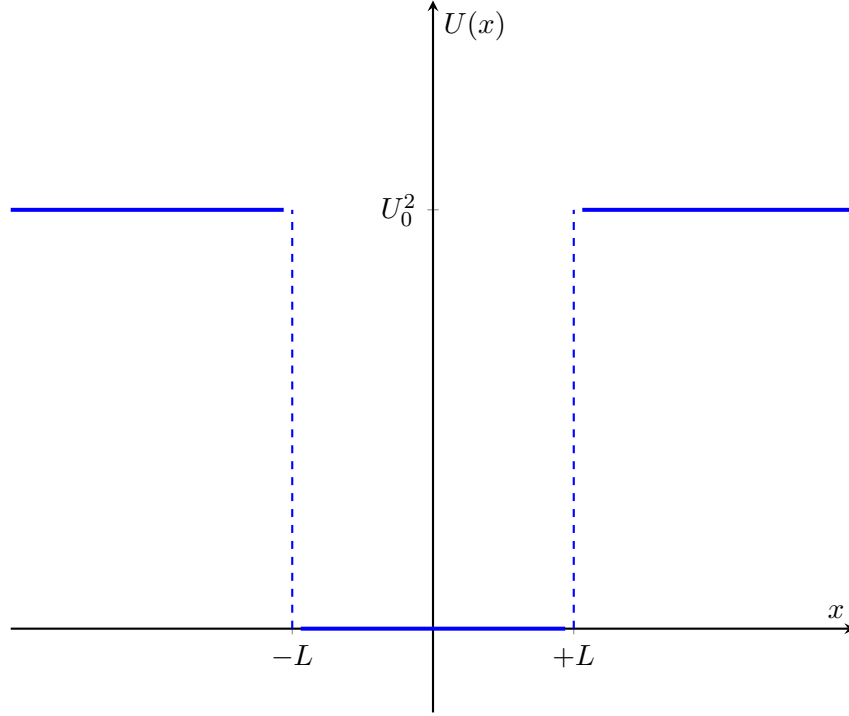


Figure 4.2: Finite potential well of definition 4.1.

Proposition 4.2 follows by direct substitution into proposition 1.3 or straightforward variation of the action. Any given solution to the e.o.m. will be defined for different spatial domains. Hence, we impose that the solution and its first derivative be continuous all over spacetime. Since the potential is discontinuous only at the surfaces $x = \pm L$, it suffices to enforce the following smoothness boundary conditions.

Definition 4.2 Matching Conditions

For any solution $\Phi(t, x)$ of proposition 4.2, we impose the **matching conditions**:

$$\Phi(t, x)|_{x=\pm L^-} = \Phi(t, x)|_{x=\pm L^+} \quad (4.18a)$$

$$\partial_x \Phi(t, x)|_{x=\pm L^-} = \partial_x \Phi(t, x)|_{x=\pm L^+} \quad (4.18b)$$

The $(1 + 1)$ -dimensional Minkowski spacetime is maximally symmetric with $2(2 + 1)/2 = 3$ KVFs corresponding to time translations, space translations, and a boost along the spatial direction. The discrete symmetries are time reversal and parity. Since the potential is invariant under time translations and parity, definite-frequency and definite-parity normal modes are allowed. A set of normal modes with these characteristics was explicitly constructed in [26].

Proposition 4.3 Low-Energy Normal Modes

Let the **low-energy normal modes** $\phi_n^\pm(t, x)$ be defined as:

$$\phi_n^+(t, x) := \begin{cases} B_n^+ e^{-i\omega_n^+ t} \cos(\omega_n^+ x), & |x| \leq L \\ B_n^+ e^{-i\omega_n^+ t} \cos(\omega_n^+ L) e^{-k_n^+ (|x|-L)}, & |x| \geq L \end{cases} \quad (4.19a)$$

$$\phi_n^-(t, x) := \begin{cases} B_n^- e^{-i\omega_n^- t} \sin(\omega_n^- x), & |x| \leq L \\ B_n^- e^{-i\omega_n^- t} \sin(\omega_n^- L) e^{-k_n^- (|x|-L)} \operatorname{sgn}(x), & |x| \geq L \end{cases} \quad (4.19b)$$

For $s \in \{+, -\}$, these modes satisfy the following properties.

- Discretized bounded frequencies $0 < \omega_n^s < U_0$ with:

$$n \in \left\{ 1, \dots, N^\pm = \left\lceil \frac{U_0 L}{\pi} + \frac{-1 \pm 1}{4} \right\rceil \right\} \quad (4.20)$$

- Spatial decay parameter:

$$k_n^s(\omega_n^s) = \sqrt{(U_0)^2 - (\omega_n^s)^2} \quad (4.21)$$

- Normalization constants:

$$B_n^s = \left[2\omega_n^s \left(L + \frac{1}{\sqrt{(U_0)^2 - (\omega_n^s)^2}} \right) \right]^{-1/2} \quad (4.22)$$

- Transcendental quantization conditions:

$$\tan(\omega_n^+ L) = \frac{k_n^+}{\omega_n^+} = \sqrt{\left(\frac{U_0}{\omega_n^+}\right)^2 - 1} \quad (4.23a)$$

$$\cot(\omega_n^- L) = -\frac{k_n^-}{\omega_n^-} = -\sqrt{\left(\frac{U_0}{\omega_n^-}\right)^2 - 1} \quad (4.23b)$$

- The set $\{\phi_n^s, \text{c.c.}\}$ is a basis for the vector subspace of bound low-energy solutions of proposition 4.2.
- The basis $\{\phi_n^s, \text{c.c.}\}$ is orthonormal under definition 2.3:

$$(\phi_n^s, \phi_{n'}^{s'}) = \delta^{ss'} \delta_{nn'} \quad (4.24a)$$

$$((\phi_n^s)^*, (\phi_{n'}^{s'})^*) = -\delta^{ss'} \delta_{nn'} \quad (4.24b)$$

$$(\phi_n^s, (\phi_{n'}^{s'})^*) = 0 \quad (4.24c)$$

- The $\phi_n^s(t, x)$ and their c.c. have a definite parity.
- The $\phi_n^s(t, x)$ have positive frequencies, while the c.c. have negative frequencies.

Proposition 4.4 High-Energy Normal Modes

Let the **high-energy normal modes** $\varphi_\omega^s(t, x)$ be defined as:

$$\varphi_\omega^+(t, x) := \begin{cases} B_\omega^+ e^{-i\omega t} \cos(\omega x), & |x| \leq L \\ B_\omega^+ e^{-i\omega t} (\cos(\omega L) \cos[k(|x| - L)] - \frac{\omega}{k} \sin(\omega L) \sin[k(|x| - L)]), & |x| \geq L \end{cases} \quad (4.25a)$$

$$\varphi_\omega^-(t, x) := \begin{cases} B_\omega^- e^{-i\omega t} \sin(\omega x), & |x| \leq L \\ B_\omega^- e^{-i\omega t} (\sin(\omega L) \cos[k(|x| - L)] + \frac{\omega}{k} \cos(\omega L) \sin[k(|x| - L)]) \operatorname{sgn}(x), & |x| \geq L \end{cases} \quad (4.25b)$$

Properties:

- Continuous frequencies $0 < U_0 < \omega \in \mathbb{R}$.
- Exterior wavenumber:

$$k(\omega) = \sqrt{\omega^2 - U_0^2} \quad (4.26)$$

- Normalization constants:

$$B_\omega^+ = \left[\frac{\sqrt{\omega^2 - U_0^2}}{\omega^2 - U_0^2 \cos^2(\omega L)} \right]^{1/2} \quad (4.27a)$$

$$B_\omega^- = \left[\frac{\sqrt{\omega^2 - U_0^2}}{\omega^2 - U_0^2 \sin^2(\omega L)} \right]^{1/2} \quad (4.27b)$$

- The set $\{\varphi_\omega^s, \text{c.c.}\}$ is a basis for the vector subspace of unbound high-energy solutions of proposition 4.2.
- The basis $\{\varphi_\omega^s, \text{c.c.}\}$ is orthonormal under definition 2.3:

$$(\varphi_\omega^s, \varphi_{\omega'}^{s'}) = \delta^{ss'} 2\pi \delta(\omega - \omega') \quad (4.28a)$$

$$((\varphi_\omega^s)^*, (\varphi_{\omega'}^{s'})^*) = -\delta^{ss'} 2\pi \delta(\omega - \omega') \quad (4.28b)$$

$$(\varphi_\omega^s, (\varphi_{\omega'}^{s'})^*) = 0 \quad (4.28c)$$

- The $\varphi_\omega^s(t, x)$ and their c.c. have a definite parity.
- The $\varphi_\omega^s(t, x)$ have positive frequencies, while their c.c. have negative frequencies.

Proof Sketch of Propositions 4.3 and 4.4.

(1) In the region $|x| < L$, the e.o.m. to solve is $\square \Phi = (-\partial_t^2 + \partial_x^2) \Phi(t, x) = 0$. Discarding the exponentially growing and decaying solutions in time, imposing the reality condition $\Phi^* = \Phi$, and imposing the definite-parity

condition $\hat{\mathcal{P}}\Phi = s\Phi$, yields the mode basis $\{u_\omega^s, (u_\omega^s)^*\}$, where the positive-frequency modes are:

$$u_\omega^s(t, |x| < L) = A_\omega^s e^{-i\omega t} (e^{+i\omega x} + s e^{-i\omega x}), \quad \omega > 0, A_\omega^s \in \mathbb{C}$$

(2) In the region $|x| > L$, the e.o.m. to solve is $(\square - U_0^2)\Phi = (-\partial_t^2 + \partial_x^2 - U_0^2)\Phi(t, x) = 0$. Setting the separation constant to $-\omega^2$ with $\omega > 0$ induces the parameter $k^2(\omega) = |U_0^2 - \omega^2|$.

For $0 < \omega < U_0$, discarding the exponential time solutions and exponentially growing spatial solutions, and imposing the reality and definite-parity conditions yields the basis $\{v_\omega^s, (v_\omega^s)^*\}$, where the positive-frequency modes are:

$$v_\omega^s(t, |x| > L) = \begin{cases} D_\omega^s e^{-i\omega t} e^{-k(|x|-L)}, & x > L \\ s D_\omega^s e^{-i\omega t} e^{-k(|x|-L)}, & x < -L \end{cases}$$

While for $\omega > U_0 > 0$, following the same procedure yields:

$$v_\omega^s(t, |x| > L) = \begin{cases} e^{-i\omega t} [B_\omega^s e^{-ik(|x|-L)} + C_\omega^s e^{+ik(|x|-L)}], & x > L \\ s e^{-i\omega t} [B_\omega^s e^{-ik(|x|-L)} + C_\omega^s e^{+ik(|x|-L)}], & x < -L \end{cases}$$

(3) For $\omega^s < U_0$, the matching of the modes at $x = L$ constrains the parameter $\omega^s \rightarrow \omega_n^s$ to the transcendental quantization condition, and allows us to write the positive-frequency modes as $\phi_n^s(t, x) = e^{-i\omega_n^s t} f_n^s(x)$, where:

$$\begin{aligned} f_n^+(x) &:= \begin{cases} 2A_n^+ \cos(\omega_n^+ x), & |x| \leq L \\ 2A_n^+ \cos(\omega_n^+ L) e^{-k_n^+ (|x|-L)}, & |x| > L \end{cases} \\ f_n^-(x) &:= \begin{cases} 2iA_n^- \sin(\omega_n^- x), & |x| \leq L \\ 2iA_n^- \sin(\omega_n^- L) e^{-k_n^- (|x|-L)} \operatorname{sgn}(x), & |x| > L \end{cases} \end{aligned}$$

(4) For $\omega > U_0$, the matching of the modes at $x = L$ yields the positive-frequency modes $\varphi_\omega^s(t, x) = e^{-i\omega t} f_\omega^s(x)$, where:

$$\begin{aligned} f_\omega^+(x) &:= \begin{cases} 2A_\omega^+ \cos(\omega x), & |x| \leq L \\ 2A_\omega^+ (\cos(\omega L) \cos[k(|x|-L)] - \frac{\omega}{k} \sin(\omega L) \sin[k(|x|-L)]), & |x| > L \end{cases} \\ f_\omega^-(x) &:= \begin{cases} 2iA_\omega^- \sin(\omega x), & |x| \leq L \\ 2iA_\omega^- (\sin(\omega L) \cos[k(|x|-L)] + \frac{\omega}{k} \cos(\omega L) \sin[k(|x|-L)]) \operatorname{sgn}(x), & |x| > L \end{cases} \end{aligned}$$

(5) The normalization factors A_ω^s, A_n^s are determined by imposing orthonormality under the KG product. The parity $s \neq s'$ and number $n \neq n'$ orthogonalities are verified automatically, so the coefficients are determined via the norm. For the low-energy modes:

$$(\phi_n^s, \phi_n^s) = |A_n^s|^2 8\omega_n^s \left(L + \frac{1}{k_n^s} \right) \stackrel{!}{=} 1 \quad \implies \quad |A_n^s|^2 = \frac{1}{8\omega_n^s \left(L + \frac{1}{k_n^s} \right)}$$

While for the high-energy modes, letting $\mathcal{O}$ denote terms regular at $\omega' \rightarrow \omega$, we find:

$$(\varphi_\omega^+, \varphi_\omega^+) = |A_\omega^+|^2 4k \left[\cos^2(\omega L) + \frac{\omega^2}{k^2} \sin^2(\omega L) \right] 2\pi\delta(\omega - \omega) + \mathcal{O} \stackrel{!}{=} 2\pi\delta(\omega - \omega)$$

$$(\varphi_{\omega}^-, \varphi_{\omega}^-) = |A_{\omega}^-|^2 4k \left[\sin^2(\omega L) + \frac{\omega^2}{k^2} \cos^2(\omega L) \right] 2\pi\delta(\omega - \omega) + \mathcal{O} \stackrel{!}{=} 2\pi\delta(\omega - \omega)$$

And thus:

$$|A_{\omega}^+|^2 = \left(4k \left[\cos^2(\omega L) + \frac{\omega^2}{k^2} \sin^2(\omega L) \right] \right)^{-1} = \frac{1}{4} \frac{\sqrt{\omega^2 - U_0^2}}{\omega^2 - U_0^2 \cos^2(\omega L)}$$

$$|A_{\omega}^-|^2 = \left(4k \left[\sin^2(\omega L) + \frac{\omega^2}{k^2} \cos^2(\omega L) \right] \right)^{-1} = \frac{1}{4} \frac{\sqrt{\omega^2 - U_0^2}}{\omega^2 - U_0^2 \sin^2(\omega L)}$$

The normalization coefficients are defined up to an arbitrary phase factor. Thus, we are free to perform the rescalings $A_n^+ \rightarrow B_n^+ = 2A_n^+$ and $A_n^- \rightarrow B_n^- = 2iA_n^-$, and similarly for A_{ω}^s .

(6) The maximum number N^s of low-energy solutions with parity s can be proven as follows. In the upper limits $\omega_n^s \rightarrow U_0$, the transcendental quantization conditions yield:

$$\tan(U_0 L) = 0 \implies U_0 L = \pi(n_{\max}^+ - 1) \implies n_{\max}^+ = \frac{U_0 L}{\pi} + 1$$

$$\cot(U_0 L) = 0 \implies U_0 L = \pi\left(n_{\max}^- - \frac{1}{2}\right) \implies n_{\max}^- = \frac{U_0 L}{\pi} + \frac{1}{2}$$

Since $\omega^s(n_{\max}^s) = U_0$ and the allowed values of n are strictly $\omega^s(n) < U_0$, then $n < n_{\max}^s$ and the maximum allowed values are:

$$N^s = \max \{n \in \mathbb{Z}^+ : n < n_{\max}^s\} = \lceil n_{\max}^s \rceil - 1 = \lceil n_{\max}^s - 1 \rceil < \infty$$

Here, $\lceil x \rceil = \min \{m \in \mathbb{Z}^+ : m \geq x\}$ denotes the ceiling function for $x \in \mathbb{R}$ (the floor function would yield $N^s = n_{\max}^s$ if the latter is an integer and this is not allowed). $\square$

In contrast to the **high-energy sector** $\omega > U_0 > 0$, in the **low-energy sector** $0 < \omega_n^s < U_0$ the frequency is quantized. In the **interior region** $|x| < L$, the dispersion relation is $-\omega^2 + k^2 = 0$, and the normal modes are massless. In the **exterior region** $|x| > L$ and high-energy sector $\omega > U_0$, the dispersion relation is $-\omega^2 + k^2 = -U_0^2$, and the normal modes have an effective mass $m = U_0$. In the exterior region and low-energy sector $\omega_n^s < U_0$, the normal modes are damped by the decay parameters k_n^s .

The total number of low-energy solutions is $N = N^+ + N^- = \lceil \frac{U_0 L}{\pi} \rceil + \lceil \frac{U_0 L}{\pi} - \frac{1}{2} \rceil = \lceil \frac{2LU_0}{\pi} \rceil$. Thus, if $U_0 L > 0$, then there always exists at least one solution $N \geq 1$.

Proposition 4.5 Low-Energy & High-Energy Bases Orthogonality

The low-energy basis $\{\phi_n^s, \text{c.c.}\}$ and high-energy basis $\{\varphi_{\omega}^s, \text{c.c.}\}$ of propositions 4.3 and 4.4 are orthogonal to each other under definition 2.3:

$$(\phi_n^s, \varphi_{\omega}^{s'}) = 0 \tag{4.29a}$$

$$(\phi_n^s, (\varphi_{\omega}^{s'})^*) = 0 \tag{4.29b}$$

$$\left((\phi_n^s)^*, (\varphi_\omega^s)^* \right) = 0 \quad (4.29c)$$

Proof. Showing the orthogonality in the frequencies is enough, so we ignore the parity index. Since the modes $\phi_n(t, x) = e^{-i\omega_n t} f_n(x)$ and $\varphi_\omega(t, x) = e^{-i\omega t} f_\omega(x)$ are solutions of the KG equation, they are eigenfunctions of the “Hamiltonian” operator $\hat{H} := -\partial_x^2 + U(x)$. Indeed:

$$\hat{H}\varphi_\omega(t, x) = [-\partial_x^2 + U(x)] \varphi_\omega(t, x) = -\partial_t^2 \varphi_\omega(t, x) = \omega^2 \varphi_\omega(t, x)$$

Along $\Sigma : t = \text{cst}$, the KG product reads:

$$\left(\phi_n, \hat{H}\varphi_\omega \right) = -i \int dx \phi_n \overset{\leftrightarrow}{\partial}_t \left(\hat{H}\varphi_\omega \right)^* = \left(\hat{H}\phi_n, \varphi_\omega \right) + i \left[\phi_n \overset{\leftrightarrow}{\partial}_x \varphi_\omega^* \right]_{-\infty}^{+\infty}$$

The exponential decay of the low-energy modes ensures the boundary terms vanish at both ends, $\phi_n \sim e^{-k_n|x|} \xrightarrow{x \rightarrow \pm\infty} 0$ and $\partial_x \phi_n \sim e^{-k_n|x|} \xrightarrow{x \rightarrow \pm\infty} 0$. Hence:

$$0 = \left(\hat{H}\phi_n, \varphi_\omega \right) - \left(\phi_n, \hat{H}\varphi_\omega \right) = (\omega_n^2 - \omega^2) (\phi_n, \varphi_\omega)$$

Since the frequency domain is non-intersecting, $0 < \omega_n < U_0 < \omega$, it follows that $\omega_n^2 \neq \omega^2$ and thus $(\phi_n, \varphi_\omega) = 0$. The same procedure for the inner product permutation with the complex conjugates holds, and this completes the proof. $\square$

Proposition 4.5 implies that although the full basis for the space $\mathcal{S}$ of KG solutions is $\{\phi_n^s, \varphi_\omega^s, \text{c.c.}\}$, the two energy sectors do not communicate with each other:

$$\mathcal{S} = \mathcal{S}_{\text{discrete}} \oplus \mathcal{S}_{\text{continuous}} \quad (4.30)$$

The general solution $\Phi(t, x)$ of proposition 4.2 is then given by the following linear combination of the normal modes in propositions 4.3 and 4.4:

$$\Phi(t, x) = \sum_{s=+,-} \sum_{n=1}^{N^s} [a_{s,n} \phi_n^s(t, x) + \text{c.c.}] + \sum_{s=+,-} \int_{U_0}^{\infty} \frac{d\omega}{2\pi} [b_{s,\omega} \varphi_\omega^s(t, x) + \text{c.c.}] \quad (4.31)$$

The Fourier coefficients $a_{s,n}, b_{s,\omega}, \text{c.c.}$, are determined as follows.

Proposition 4.6 Fourier Coefficients

The Fourier coefficients in eq. (4.31) are given by:

$$a_{s,n} = (\Phi, \phi_n^s) \quad (4.32a)$$

$$a_{s,n}^* = -(\Phi, (\phi_n^s)^*) \quad (4.32b)$$

$$b_{s,\omega} = (\Phi, \varphi_\omega^s) \quad (4.32c)$$

$$b_{s,\omega}^* = -(\Phi, (\varphi_\omega^s)^*) \quad (4.32d)$$

Proof. By proposition 2.3, exploiting the linearity in the first argument of the KG product, consider:

$$(\Phi, f) = \sum_{s', n'} a_{s', n'} (\phi_{n'}^{s'}, f) + \sum_{s', n'} a_{s', n'}^* ((\phi_{n'}^{s'})^*, f) + \sum_{s'} \int \frac{d\omega'}{2\pi} b_{s', \omega'} (\varphi_{\omega'}^{s'}, f) + \sum_{s'} \int \frac{d\omega'}{2\pi} b_{s', \omega'}^* ((\varphi_{\omega'}^{s'})^*, f)$$

For $f = \phi_n^s$, by the basis orthogonality in eq. (4.24) and proposition 4.5, the only non-vanishing term is the first one:

$$(\Phi, \phi_n^s) = \sum_{s', n'} a_{s', n'} (\phi_{n'}^{s'}, \phi_n^s) = \sum_{s', n'} a_{s', n'} \delta_{s' s} \delta_{n' n} = a_{s, n}$$

Analogously:

$$\begin{aligned} (\Phi, (\phi_n^s)^*) &= \sum_{s', n'} a_{s', n'}^* ((\phi_{n'}^{s'})^*, (\phi_n^s)^*) = - \sum_{s', n'} a_{s', n'}^* \delta_{s' s} \delta_{n' n} = -a_{s, n}^* \\ (\Phi, \varphi_\omega^s) &= \sum_{s'} \int \frac{d\omega'}{2\pi} b_{s', \omega'} (\varphi_{\omega'}^{s'}, \varphi_\omega^s) = \sum_{s'} \int \frac{d\omega'}{2\pi} b_{s', \omega'} \delta_{s' s} 2\pi \delta(\omega' - \omega) = b_{s, \omega} \\ (\Phi, (\varphi_\omega^s)^*) &= \sum_{s'} \int \frac{d\omega'}{2\pi} b_{s', \omega'}^* ((\varphi_{\omega'}^{s'})^*, (\varphi_\omega^s)^*) = - \sum_{s'} \int \frac{d\omega'}{2\pi} b_{s', \omega'}^* \delta_{s' s} 2\pi \delta(\omega' - \omega) = -b_{s, \omega}^* \end{aligned}$$

□

We proceed with the canonical quantization of the classical scalar field theory by promoting the Fourier coefficients $a_{s, n}$, $b_{s, \omega}$, c.c. and the canonical fields $\Phi(t, x)$, $\Pi(t, x) = \partial_t \Phi(t, x)$ to operator-valued distributions and imposing the canonical commutators of definition 3.1 on $\Sigma : t = \text{cst}$. The non-vanishing one is:

$$[\hat{\Phi}(t, x), \hat{\Pi}(t, x')] = i\hbar \delta(x - x') \quad (4.33)$$

Explicitly, the quantum KG field reads:

$$\hat{\Phi}(t, x) = \sum_{s=\pm} \sum_{n=1}^{N^s} \hat{a}_{s, n} \phi_n^s(t, x) + \sum_{s=\pm} \int_{U_0}^{\infty} \frac{d\omega}{2\pi} \hat{b}_{s, \omega} \varphi_\omega^s(t, x) + \text{h.c.} \quad (4.34)$$

Proposition 4.7 Ladder Algebra

Equation (4.33) induces the following canonical commutators on the annihilation and creation operators of eq. (4.34):

$$[\hat{a}_{s, n}, \hat{a}_{s', n'}^\dagger] = \hbar \delta_{nn'} \delta_{ss'} \quad (4.35a)$$

$$[\hat{b}_{s, \omega}, \hat{b}_{s', \omega'}^\dagger] = 2\pi \hbar \delta(\omega - \omega') \delta_{ss'} \quad (4.35b)$$

All other commutators vanish.

Proof. Let $\hat{f}_j := (\hat{\Phi}, f_j)$. Using definition 2.3 on $\Sigma : t = \text{cte}$ and definition 3.1 yields:

$$\begin{aligned}
 [\hat{f}_1, \hat{f}_2] &= [(\hat{\Phi}, f_1), (\hat{\Phi}, f_2)] \\
 &= \left[-i \int dx \hat{\Phi}(x) \overleftrightarrow{\partial}_t f_1^*(x), -i \int dy \hat{\Phi}(y) \overleftrightarrow{\partial}_t f_2^*(y) \right] \\
 &= - \int dx dy \left(\partial_t f_1^*(x) \partial_t f_2^*(y) [\hat{\Phi}(x), \hat{\Phi}(y)] - \partial_t f_1^*(x) f_2^*(y) [\hat{\Phi}(x), \hat{\Pi}(y)] + \right. \\
 &\quad \left. - f_1^*(x) \partial_t f_2^*(y) [\hat{\Pi}(x), \hat{\Phi}(y)] + f_1^*(x) f_2^*(y) [\hat{\Pi}(x), \hat{\Pi}(y)] \right) \\
 &= i\hbar \int dx dy [\partial_t f_1^*(x) f_2^*(y) - f_1^*(x) \partial_t f_2^*(y)] \delta(x - y) \\
 &= -i\hbar \int dx f_1^*(x) \overleftrightarrow{\partial}_t f_2^*(x) \\
 &= \hbar(f_1^*, f_2)
 \end{aligned}$$

For the quantized versions of proposition 4.6, using proposition 4.3 we have:

$$\begin{aligned}
 [\hat{a}_{s,n}, \hat{a}_{s',n'}] &= \hbar \left((\phi_n^s)^*, \phi_{n'}^{s'} \right) = 0 \\
 [\hat{a}_{s,n}^\dagger, \hat{a}_{s',n'}^\dagger] &= \hbar \left(-\phi_n^s, -(\phi_{n'}^{s'})^* \right) = 0 \\
 [\hat{a}_{s,n}, \hat{a}_{s',n'}^\dagger] &= \hbar \left((\phi_n^s)^*, -(\phi_{n'}^{s'})^* \right) = \hbar \delta_{ss'} \delta_{nn'}
 \end{aligned}$$

Analogously, from proposition 4.4:

$$\begin{aligned}
 [\hat{b}_{s,\omega}, \hat{b}_{s',\omega'}] &= \hbar \left((\varphi_\omega^s)^*, \varphi_{\omega'}^{s'} \right) = 0 \\
 [\hat{b}_{s,\omega}^\dagger, \hat{b}_{s',\omega'}^\dagger] &= \hbar \left(-\varphi_\omega^s, -(\varphi_{\omega'}^{s'})^* \right) = 0 \\
 [\hat{b}_{s,\omega}, \hat{b}_{s',\omega'}^\dagger] &= \hbar \left((\varphi_\omega^s)^*, -(\varphi_{\omega'}^{s'})^* \right) = 2\pi\hbar \delta_{ss'} \delta(\omega - \omega')
 \end{aligned}$$

Lastly, the mixed commutators all vanish by virtue of proposition 4.5. □

The construction of the Fock space follows immediately. The vacuum state $|0\rangle$ associated with the normal modes of propositions 4.3 and 4.4 is given by the normalized state $\langle 0|0\rangle = 1$ annihilated by all the annihilation operators $\hat{a}_{s,n}, \hat{b}_{s,\omega}$. That is, $\forall s, \omega, n$, then:

$$\hat{a}_{s,n} |0\rangle = 0 \quad (4.36a)$$

$$\hat{b}_{s,\omega} |0\rangle = 0 \quad (4.36b)$$

Henceforth, all expectation values are taken w.r.t. this vacuum:

$$\langle \hat{f} \rangle := \langle 0 | \hat{f} | 0 \rangle$$

We conclude this section by noting that the Lie derivative w.r.t. the time-translation KVF $\mathbf{K} = d/dt$ can be rescaled to define the “Schrödinger operator”:

$$\hat{E} := i\hbar \mathcal{L}_{\mathbf{K}} = i\hbar \partial_t \quad (4.37)$$

The positive-frequency low- and high-energy normal modes of propositions 4.3 and 4.4 are eigenfunctions of this operator with eigenvalues:

$$\hat{E}\phi_{s,n} = \hbar\omega_{s,n}\phi_{s,n} \quad (4.38a)$$

$$\hat{E}\varphi_{s,\omega} = \hbar\omega\varphi_{s,\omega} \quad (4.38b)$$

The corresponding negative-frequency normal modes have the negative eigenvalues $-\hbar\omega_{s,n}$ and $-\hbar\omega$, respectively.

4.3 Interior Region

In this section, we construct the REVMTO by focusing on the interior region $|x| < L$.

4.3.1 Propagator

The construction of the REVMTO starts with the propagator.

Proposition 4.8 Interior Feynman Propagator

In the interior region $|x| < L$, the Feynman propagator of eq. (4.34) reads:

$$\begin{aligned} G^F(t, x, t', x') = & i\hbar \sum_{n=1}^{N^+} (B_n^+)^2 \cos(\omega_n^+ x) \cos(\omega_n^+ x') e^{-i\omega_n^+ |t-t'|} \\ & + i\hbar \sum_{n=1}^{N^-} (B_n^-)^2 \sin(\omega_n^- x) \sin(\omega_n^- x') e^{-i\omega_n^- |t-t'|} \\ & + i\hbar \int_{U_0}^{\infty} \frac{d\omega}{2\pi} (B_\omega^+)^2 \cos(\omega x) \cos(\omega x') e^{-i\omega |t-t'|} \\ & + i\hbar \int_{U_0}^{\infty} \frac{d\omega}{2\pi} (B_\omega^-)^2 \sin(\omega x) \sin(\omega x') e^{-i\omega |t-t'|} \end{aligned} \quad (4.39)$$

Proof. The expansion of the quantum field in terms of the normal modes in propositions 4.3 and 4.4 for $|x| < L$ reads:

$$\begin{aligned} \hat{\Phi}(t, x) = & \sum_{n=1}^{N^+} B_n^+ \cos(\omega_n^+ x) \left(\hat{a}_{n,+} e^{-i\omega_n^+ t} + \hat{a}_{n,+}^\dagger e^{+i\omega_n^+ t} \right) + \sum_{n=1}^{N^-} B_n^- \sin(\omega_n^- x) \left(\hat{a}_{n,-} e^{-i\omega_n^- t} + \hat{a}_{n,-}^\dagger e^{+i\omega_n^- t} \right) \\ & + \int_{U_0}^{\infty} \frac{d\omega}{2\pi} B_\omega^+ \cos(\omega x) \left(\hat{b}_{\omega,+} e^{-i\omega t} + \hat{b}_{\omega,+}^\dagger e^{+i\omega t} \right) + \int_{U_0}^{\infty} \frac{d\omega}{2\pi} B_\omega^- \sin(\omega x) \left(\hat{b}_{\omega,-} e^{-i\omega t} + \hat{b}_{\omega,-}^\dagger e^{+i\omega t} \right) \end{aligned}$$

To compute the two-point function $\langle \hat{\Phi}(t, x) \hat{\Phi}(t', x') \rangle$, we use the identities:

$$\langle \hat{a}_{n,s} \hat{a}_{n',s'}^\dagger \rangle = \hbar \delta_{ss'} \delta_{nn'} \quad (4.40a)$$

$$\langle \hat{b}_{\omega,s} \hat{b}_{\omega',s'}^\dagger \rangle = 2\pi \hbar \delta_{ss'} \delta(\omega - \omega') \quad (4.40b)$$

The only non-vanishing terms are proportional to these VEVs:

$$\begin{aligned} \langle \hat{\Phi}(t, x) \hat{\Phi}(t', x') \rangle &= \sum_{n,m=1}^{N^+} B_n^+ B_m^+ \cos(\omega_n^+ x) \cos(\omega_m^+ x') e^{-i(\omega_n^+ t - \omega_m^+ t')} \langle \hat{a}_{n,+} \hat{a}_{m,+}^\dagger \rangle + \\ &+ \sum_{n,m=1}^{N^-} B_n^- B_m^- \sin(\omega_n^- x) \sin(\omega_m^- x') e^{-i(\omega_n^- t - \omega_m^- t')} \langle \hat{a}_{n,-} \hat{a}_{m,-}^\dagger \rangle + \\ &+ \int_{U_0}^{\infty} \frac{d\omega d\omega'}{(2\pi)^2} B_\omega^+ B_{\omega'}^+ \cos(\omega x) \cos(\omega' x') e^{-i(\omega t - \omega' t')} \langle \hat{b}_{\omega,+} \hat{b}_{\omega',+}^\dagger \rangle + \\ &+ \int_{U_0}^{\infty} \frac{d\omega d\omega'}{(2\pi)^2} B_\omega^- B_{\omega'}^- \sin(\omega x) \sin(\omega' x') e^{-i(\omega t - \omega' t')} \langle \hat{b}_{\omega,-} \hat{b}_{\omega',-}^\dagger \rangle \end{aligned}$$

The only non-symmetric terms under the exchange $\langle \hat{\Phi}(t', x') \hat{\Phi}(t, x) \rangle$ are the exponentials, so the time-ordering T effectively induces an absolute value in their arguments. The Feynman propagator then follows immediately after enforcing the deltas in $G^F(t, x, t', x') = i \langle T \hat{\Phi}(t, x) \hat{\Phi}(t', x') \rangle$. $\square$

As detailed in section 3.3, the propagator decomposes as $G^F = G_{\text{reg}}^F + G_{\text{sing}}^F$. The former is the regular structure at the coincidence limit $(t', x') \rightarrow (t, x)$ and the latter is the divergent structure. Since $N^s < \infty$, the contribution from the discrete sum coming from the low-energy normal modes is finite and thus belongs to G_{reg}^F . We then expect G_{sing}^F to be contained solely in the contribution of the high-energy normal modes. This is verified in the following proposition.

Proposition 4.9 Interior Feynman Propagator Limits

Proposition 4.8 has the following temporal and spatial limits, where $\mathcal{O}$ denotes all the regular terms at the coincidence limit:

$$\lim_{t \rightarrow t'} G^F(t, x, t', x') = -\frac{i\hbar}{2\pi} \ln |U_0(x - x')| + \mathcal{O} \quad (4.41a)$$

$$\lim_{x \rightarrow x'} G^F(t, x, t', x') = -\frac{i\hbar}{2\pi} \ln |U_0(t - t')| + \mathcal{O} \quad (4.41b)$$

Proof. Consider the expansion in powers of ω of the high-energy normalization coefficients B_ω^s of proposition 4.4:

$$(B_\omega^s)^2 = \omega^{-1} + \mathcal{O}(\omega^{-3})$$

The integral $\int_{U_0}^{\infty} d\omega \omega^{-\alpha}$ diverges for $\alpha \leq 1$ and converges for $\alpha > 1$. Therefore, the UV divergence of proposition 4.8 only comes from the leading terms $(B_\omega^s)^2 \sim \omega^{-1}$, which yield logarithmic divergences. Letting the discrete sums and the $\mathcal{O}(\omega^{-3})$ contributions be all collected as $\mathcal{O}$, we then have the following.

$$\begin{aligned}
 \lim_{t' \rightarrow t} (i\hbar)^{-1} G^F - \mathcal{O} &= \int_{U_0}^{\infty} \frac{d\omega}{2\pi} \frac{1}{\omega} \cos(\omega x) \cos(\omega x') + \int_{U_0}^{\infty} \frac{d\omega}{2\pi} \frac{1}{\omega} \sin(\omega x) \sin(\omega x') = \frac{1}{2\pi} \int_{U_0}^{\infty} d\omega \frac{\cos[\omega(x-x')]}{\omega} \\
 &= -\frac{1}{2\pi} \text{Ci}[U_0(x-x')] = -\frac{1}{2\pi} \left(\gamma + \ln|U_0(x-x')| + \mathcal{O}[U_0^2(x-x')^2] \right) \\
 &= -\frac{1}{2\pi} \ln|U_0(x-x')| + \mathcal{O}(1)
 \end{aligned}$$

We have made use of the definition and expansion of the cosine integral (for details, see [1], chapter 13) and absorbed all regular terms under $x' \rightarrow x$ into $\mathcal{O}(1)$.

$$\begin{aligned}
 \lim_{x' \rightarrow x} (i\hbar)^{-1} G^F - \mathcal{O} &= \int_{U_0}^{\infty} \frac{d\omega}{2\pi} \frac{1}{\omega} \cos^2(\omega x) e^{-i\omega|t-t'|} + \int_{U_0}^{\infty} \frac{d\omega}{2\pi} \frac{1}{\omega} \sin^2(\omega x) e^{-i\omega|t-t'|} = \frac{1}{2\pi} \int_{U_0}^{\infty} d\omega \frac{e^{-i\omega|t-t'|}}{\omega} \\
 &= \frac{1}{2\pi} [-\gamma - \ln(iU_0|t-t'|) + \mathcal{O}[U_0(t-t')]] = -\frac{1}{2\pi} \ln|U_0(t-t')| + \mathcal{O}(1)
 \end{aligned}$$

We have made use of the definition and expansion of the exponential integral and absorbed all regular terms under $t' \rightarrow t$ into $\mathcal{O}(1)$. Combining all regular terms into a single $\mathcal{O}$ yields the desired result. $\square$

As expected, the Feynman propagator of the model diverges in the coincidence limit given by the combined limits $t' \rightarrow t$ and $x' \rightarrow x$. This implies that the EVEMTO of definition 1.5 is also divergent and must therefore be regularized and renormalized. We shall follow the Hadamard renormalization procedure of section 3.3.

Proposition 4.10 Interior Hadamard Parametrix

In the interior region $|x| < L$, the Hadamard parametrix for definition 4.1 is given by:

$$H(t, x, t', x') = -\hbar \ln \left[\frac{-(t-t')^2 + (x-x')^2}{2} + i\epsilon \right] \quad (4.42)$$

Proof. For $D = 2$, the general structure of the Hadamard parametrix is given in example 3.1, particularly eq. (3.39). Comparing the action of definition 4.1 with eq. (3.31), we see that $R_{\mu\nu} = 0$ and $R = 0$. In the interior region $U(|x| < L) = 0$ and thus $m = 0$. The final result follows $V(t, x, t', x') = -\hbar + \mathcal{O}(\sigma^{3/2})$ and $\sigma(t, x, t', x') = \frac{1}{2}[-(t-t')^2 + (x-x')^2]$. $\square$

The parametrix encodes the complete singular structure of the propagator by matching proposition 4.9. The dimensional inconsistency in the $\ln$ argument of proposition 4.10 will be treated by introducing the renormalization length scale l . At the level of the REVMTO, this amounts to adding the term in eq. (3.44).

Proposition 4.11 Interior Regularized Propagator

In the interior region $|x| < L$, the regularized propagator for definition 4.1 is given by:

$$\begin{aligned}
 W(t, x, t', x') &= 4\pi\hbar \sum_{n=1}^{N^+} (B_n^+)^2 \cos(\omega_n^+ x) \cos(\omega_n^+ x') e^{-i\omega_n^+ |t-t'|} + \\
 &\quad + 4\pi\hbar \sum_{n=1}^{N^-} (B_n^-)^2 \sin(\omega_n^- x) \sin(\omega_n^- x') e^{-i\omega_n^- |t-t'|} +
 \end{aligned}$$

$$\begin{aligned}
 & + \hbar \int_{U_0}^{+\infty} d\omega \left[(B_\omega^+)^2 - (B_\omega^-)^2 \right] \cos \omega(x+x') e^{-i\omega|t-t'|} + \\
 & + \hbar \int_{U_0}^{+\infty} d\omega \left[(B_\omega^+)^2 + (B_\omega^-)^2 \right] \cos \omega(x-x') e^{-i\omega|t-t'|} + \\
 & + \hbar \ln \left[\frac{-(t-t')^2 + (x-x')^2}{2} + i\epsilon \right]
 \end{aligned}$$

Proof. For $D = 2$, the regularized propagator is given by eq. (3.40) of example 3.1. The desired result then follows directly from propositions 4.8 and 4.10 and exploiting the double-angle trigonometric identities $2 \cos \alpha \cos \beta = \cos(\alpha - \beta) + \cos(\alpha + \beta)$ and $2 \sin \alpha \sin \beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$. $\square$

4.3.2 Low-Energy Contributions

Let us focus on the contribution solely coming from the interior low-energy modes (proposition 4.3). Since this is regular, we don't need the Hadamard parametrix, and we write:

$$\begin{aligned}
 W \supset W^{\text{low}}(t, x, t', x') &:= 4\pi\hbar \sum_{n=1}^{N^+} (B_n^+)^2 \cos(\omega_n^+ x) \cos(\omega_n^+ x') e^{-i\omega_n^+ |t-t'|} + \\
 &+ 4\pi\hbar \sum_{n=1}^{N^-} (B_n^-)^2 \sin(\omega_n^- x) \sin(\omega_n^- x') e^{-i\omega_n^- |t-t'|}
 \end{aligned} \tag{4.43}$$

We first determine the coincidence limits (proposition 3.5).

Proposition 4.12 Coincidence Limits: Interior & Low-Energy

The coincidence limits of the low-energy normal modes' contributions in proposition 4.11 in the interior region $|x| < L$ are given by:

$$w^{\text{low}}(x) = 4\pi\hbar \sum_{n=1}^{N^+} (B_n^+)^2 \cos^2(\omega_n^+ x) + 4\pi\hbar \sum_{n=1}^{N^-} (B_n^-)^2 \sin^2(\omega_n^- x) \tag{4.44a}$$

$$w_{tt}^{\text{low}}(x) = w_{xx}^{\text{low}}(x) = -4\pi\hbar \sum_{n=1}^{N^+} (B_n^+)^2 (\omega_n^+)^2 \cos^2(\omega_n^+ x) - 4\pi\hbar \sum_{n=1}^{N^-} (B_n^-)^2 (\omega_n^-)^2 \sin^2(\omega_n^- x) \tag{4.44b}$$

$$w_{xt}^{\text{low}}(x) = w_{tx}^{\text{low}}(x) = 0 \tag{4.44c}$$

Proof. Since the discrete sums are finite, we can commute the limits and the sums. The coincidence limit $w^{\text{low}}(t, x) = \lim_{(t, x') \rightarrow (t, x)} W^{\text{low}}(t, x, t', x')$ is immediate. Since the spacetime is flat, the other coincidence limits are $w_{\mu\nu}^{\text{low}}(t, x) = \lim_{(t, x') \rightarrow (t, x)} \partial_{\mu\nu}^2 W^{\text{low}}(t, x, t', x')$. Splitting the propagator into definite-parity sectors, $W^{\text{low}} = W^{\text{low},+} + W^{\text{low},-}$, the second derivatives of the low-energy regularized propagator for the even modes read:

$$\partial_t^2 W^{\text{low},+} = -4\pi\hbar \sum_{n=1}^{N^+} (B_n^+)^2 \omega_n^+ \cos(\omega_n^+ x) \cos(\omega_n^+ x') e^{-i\omega_n^+ |t-t'|} [\omega_n^+ \text{sgn}^2(t-t') + 2i\delta(t-t')]$$

$$\begin{aligned}\partial_{tx}^2 W^{\text{low},+} &= \partial_{xt}^2 W^{\text{low},+} = 4\pi\hbar i \sum_{n=1}^{N^+} (B_n^+)^2 (\omega_n^+)^2 \sin(\omega_n^+ x) \cos(\omega_n^+ x') \operatorname{sgn}(t-t') e^{-i\omega_n^+ |t-t'|} \\ \partial_x^2 W^{\text{low},+} &= -4\pi\hbar \sum_{n=1}^{N^+} (B_n^+)^2 (\omega_n^+)^2 \cos(\omega_n^+ x) \cos(\omega_n^+ x') e^{-i\omega_n^+ |t-t'|}\end{aligned}$$

Where we have used the distributional identities of proposition A.2 for the derivatives of the absolute value. We first perform the spatial limit $x' \rightarrow x$ and then take the temporal limit $t' \rightarrow t$, which when computed as an evaluation immediately yields the xx and tx derivatives using $\operatorname{sgn}(0) = 0$. For the tt derivative, we use the distributional identity $\operatorname{sgn}^2(z) = 1$. The contact term $\delta(t-t')$ is only defined by smearing against test functions and does not admit a pointwise evaluation at coincidence, whereas $w_{\mu\nu}$ is defined as a smooth coincidence limit. Thus, we henceforth retain only the non-contact contributions and obtain:

$$\begin{aligned}w_{tt}^{\text{low},+}(x) &= w_{xx}^{\text{low},+}(x) = -4\pi\hbar \sum_n (B_n^+)^2 (\omega_n^+)^2 \cos^2(\omega_n^+ x) \\ w_{tx}^{\text{low},+}(x) &= w_{xt}^{\text{low},+}(x) = 0\end{aligned}$$

Following the same procedure, we find the derivatives of $W^{\text{low},-}$, and the coincidence limits $w_{\mu\nu}^{\text{low},-}$. The desired result is then given by the additions $w_{\mu\nu}^{\text{low}} = w_{\mu\nu}^{\text{low},+} + w_{\mu\nu}^{\text{low},-}$. $\square$

Henceforth, all results for the REVMTO will be given without the state-independent ambiguity term Θ , inherent to the Hadamard renormalization, but with the inclusion of the state-independent renormalization length term Θ^l .

Proposition 4.13 REVMTO: Interior & Low-Energy

The components of the REVMTO in the interior region $|x| < L$ and from the low-energy normal modes' contributions in proposition 4.11 are given by:

$$\left\langle \hat{T}_{tt} \right\rangle_{\text{ren}}^{\text{low}} = \left\langle \hat{T}_{xx} \right\rangle_{\text{ren}}^{\text{low}} = \frac{\hbar}{2} \sum_{n=1}^{N^+} (B_n^+)^2 (\omega_n^+)^2 + \frac{\hbar}{2} \sum_{n=1}^{N^-} (B_n^-)^2 (\omega_n^-)^2 \quad (4.45a)$$

$$\left\langle \hat{T}_{tx} \right\rangle_{\text{ren}}^{\text{low}} = \left\langle \hat{T}_{xt} \right\rangle_{\text{ren}}^{\text{low}} = 0 \quad (4.45b)$$

Proof. The structure of the REVMTO is given in proposition 3.6, concretely eq. (3.32). Comparing with the action of definition 4.1 we have $D = 2$, $\alpha_D = 1/2\pi$, $R_{\mu\nu} = 0$, $R = 0$, $\xi = 0$, $m = 0$. From example 3.1 this implies that $v_1 = 0$ and $\Theta_{\mu\nu}^l = 0$. The interior REVMTO structure is therefore:

$$\left\langle \hat{T}_{\mu\nu}(t, x) \right\rangle_{\text{ren}}^{\text{int}} = \frac{1}{4\pi} \left(-w_{\mu\nu} + \frac{1}{2} \partial_{\mu\nu}^2 w - \frac{1}{4} \eta_{\mu\nu} \square w \right)^{\text{int}} \quad (4.46)$$

All terms admit the energy-sector split $w = w^{\text{low}} + w^{\text{high}}$. Here we concentrate on the low-energy one. From proposition 4.12 we see that $w^{\text{low}} = w^{\text{low}}(x)$ only. Thus, for the 00-component:

$$\begin{aligned}\left\langle \hat{T}_{tt}(t, x) \right\rangle_{\text{ren}}^{\text{low}} &= \frac{1}{4\pi} \left(-w_{tt} + \frac{1}{4} \partial_x^2 w \right)^{\text{low}} \\ &= \hbar \sum_n (B_n^+)^2 (\omega_n^+)^2 \left[\cos^2(\omega_n^+ x) - \frac{1}{2} \cos(2\omega_n^+ x) \right] + \hbar \sum_n (B_n^-)^2 (\omega_n^-)^2 \left[\sin^2(\omega_n^- x) + \frac{1}{2} \cos(2\omega_n^- x) \right]\end{aligned}$$

$$= \frac{\hbar}{2} \sum_n (B_n^+)^2 (\omega_n^+)^2 + \frac{\hbar}{2} \sum_n (B_n^-)^2 (\omega_n^-)^2$$

Where we have used the trigonometric identities $\cos^2(\theta) = \frac{1}{2} + \frac{1}{2} \cos(2\theta)$ and $\sin^2(\theta) = \frac{1}{2} - \frac{1}{2} \cos(2\theta)$. Furthermore, for the 11-component:

$$\left\langle \hat{T}_{xx}(t, x) \right\rangle_{\text{ren}}^{\text{low}} = \frac{1}{4\pi} \left(-w_{xx} + \frac{1}{4} \partial_x^2 w \right)^{\text{low}} = \frac{1}{4\pi} \left(-w_{tt} + \frac{1}{4} \partial_x^2 w \right)^{\text{low}} = \left\langle \hat{T}_{tt}(t, x) \right\rangle_{\text{ren}}^{\text{low}}$$

The mixed components vanish identically, and this completes the proof. $\square$

4.3.3 High-Energy Contributions

We now turn to proposition 4.11's contribution in the interior region arising from the high-energy modes:

$$\begin{aligned} W \supset W^{\text{high}}(t, x, t', x') &:= \hbar \int_{U_0}^{+\infty} d\omega \mathcal{N}_\omega^- \cos \omega(x + x') e^{-i\omega|t-t'|} + \hbar \int_{U_0}^{+\infty} d\omega \mathcal{N}_\omega^+ \cos \omega(x - x') e^{-i\omega|t-t'|} + \\ &+ \hbar \ln \left[\frac{-(t-t')^2 + (x-x')^2}{2} + i\epsilon \right] \end{aligned} \quad (4.47)$$

Where we have set:

$$\mathcal{N}_\omega^\pm := (B_\omega^+)^2 \pm (B_\omega^-)^2 \quad (4.48)$$

Proposition 4.14 Coincidence Limits: Interior & High-Energy

The coincidence limits of the contribution coming from the high-energy normal modes to the regularized propagator (proposition 4.11) in the interior region $|x| < L$ are given by^a:

$$w^{\text{high}}(x) = \hbar \int_{U_0}^{\infty} d\omega \left[(B_\omega^+)^2 - (B_\omega^-)^2 \right] \cos(2\omega x) + \hbar \int_{U_0}^{\infty} d\omega \left[(B_\omega^+)^2 + (B_\omega^-)^2 - \frac{2\omega}{\omega^2 - U_0^2} \right] + \mathcal{O}(1) \quad (4.49a)$$

$$w_{\mu\nu}^{\text{high}}(x) = -\hbar \left(\int_{U_0}^{\infty} d\omega \left[(B_\omega^+)^2 - (B_\omega^-)^2 \right] \omega^2 \cos(2\omega x) + \int_{U_0}^{\infty} d\omega \left[(B_\omega^+)^2 + (B_\omega^-)^2 - \frac{2}{\omega} \right] \omega^2 \right) \delta_{\mu\nu} \quad (4.49b)$$

^aHere, $\mathcal{O}(1)$ denotes constant terms that originate from the infrared divergence of the natural log integral representation as $\omega \rightarrow U_0$. However, the high-energy spectrum is strictly $\omega > U_0$, and the REVEMTO, which is of second order, is independent of this undetermined constant.

Proof. Recalling $(B_\omega^s)^2 = \omega^{-1} + \mathcal{O}(\omega^{-3})$, we see that the UV divergence comes only from the $\cos \omega(x - x')$ term since $\mathcal{N}_\omega^- = \mathcal{O}(\omega^{-3})$ while $\mathcal{N}_\omega^+ = 2/\omega + \mathcal{O}(\omega^{-3})$. Given this, we shall first take the temporal limits.

(I) The second-order derivatives of eq. (4.47) without the parametrix read:

$$\partial_t^2 W^{\text{high}, \sigma}(t, x, t', x') = -i\hbar \int_{U_0}^{+\infty} d\omega \mathcal{N}_\omega^- \omega \cos \omega(x + x') [2\delta(t - t') - i\omega \operatorname{sgn}^2(t - t')] e^{-i\omega|t-t'|} +$$

$$\begin{aligned}
 & -i\hbar \int_{U_0}^{+\infty} d\omega \mathcal{N}_\omega^+ \omega \cos \omega(x-x') [2\delta(t-t') - i\omega \operatorname{sgn}^2(t-t')] e^{-i\omega|t-t'|}, \\
 \partial_x^2 W^{\text{high},\sigma}(t, x, t', x') &= -\hbar \int_{U_0}^{+\infty} d\omega \mathcal{N}_\omega^- \omega^2 \cos \omega(x+x') e^{-i\omega|t-t'|} + \\
 & -\hbar \int_{U_0}^{+\infty} d\omega \mathcal{N}_\omega^+ \omega^2 \cos \omega(x-x') e^{-i\omega|t-t'|}, \\
 \partial_{tx}^2 W^{\text{high},\sigma}(t, x, t', x') &= i\hbar \int_{U_0}^{+\infty} d\omega \mathcal{N}_\omega^- \omega^2 \sin \omega(x+x') \operatorname{sgn}(t-t') e^{-i\omega|t-t'|} + \\
 & + i\hbar \int_{U_0}^{+\infty} d\omega \mathcal{N}_\omega^+ \omega^2 \sin \omega(x-x') \operatorname{sgn}(t-t') e^{-i\omega|t-t'|}
 \end{aligned}$$

Following the same arguments as in the proof of proposition 4.12, the temporal $\lim_{t' \rightarrow t} W^{\text{high},\sigma}(t, x, t', x') = W^{\text{high},\sigma}(t, x, x')$, and similarly for the derivatives, read:

$$W^{\text{high},\sigma}(x, x') = \hbar \int_{U_0}^{\infty} d\omega \mathcal{N}_\omega^- \cos \omega(x+x') + \hbar \int_{U_0}^{\infty} d\omega \mathcal{N}_\omega^+ \cos \omega(x-x') \quad (4.50a)$$

$$W_{\mu\nu}^{\text{high},\sigma}(x, x') = -\delta_{\mu\nu} \hbar \int_{U_0}^{\infty} d\omega \mathcal{N}_\omega^- \omega^2 \cos \omega(x+x') - \delta_{\mu\nu} \hbar \int_{U_0}^{\infty} d\omega \mathcal{N}_\omega^+ \omega^2 \cos \omega(x-x') \quad (4.50b)$$

(II) To achieve a “spectral” integral representation of the distribution $H(\sigma) = \ln(\sigma + i\epsilon)$, we shall employ proposition A.6. However, the appropriate integration variable with a consistent interval is $0 < k = \sqrt{\omega^2 - U_0^2} < \infty$, not $U_0 < \omega < \infty$. Hence, with $dk = \frac{\omega}{\sqrt{\omega^2 - U_0^2}} d\omega$, the limit $t' \rightarrow t$ reads (as with the usual notation, the ϵ limit is left implicit):

$$\begin{aligned}
 \hbar^{-1} H(x, x') &= \ln \left[\frac{(x-x')^2}{2} + i\epsilon \right] \\
 &= -2 \int_0^{\infty} dk \frac{e^{-\epsilon k} \cos[(x-x')k]}{k} + \mathcal{O}(1) \\
 &= -2 \int_{U_0}^{\infty} d\omega \frac{\omega}{\omega^2 - U_0^2} e^{-\epsilon \sqrt{\omega^2 - U_0^2}} \cos \left[(x-x') \sqrt{\omega^2 - U_0^2} \right] + \mathcal{O}(1)
 \end{aligned} \quad (4.51)$$

We recall that $\mathcal{O}(1)$ denotes constant terms that regulate the infrared divergence of the natural log integral representation as $\omega \rightarrow U_0$. Combining eqs. (4.50) and (4.51), the $t' \rightarrow t$ limit of the high-energy regularized propagator reads:

$$\begin{aligned}
 W^{\text{high}}(x, x') &= \hbar \int_{U_0}^{\infty} d\omega \mathcal{N}_{\omega}^{-} \cos \omega(x + x') + \hbar \int_{U_0}^{\infty} d\omega \mathcal{N}_{\omega}^{+} \cos \omega(x - x') + \\
 &\quad - \hbar \int_{U_0}^{\infty} d\omega \frac{2\omega}{\omega^2 - U_0^2} \cos \left[(x - x') \sqrt{\omega^2 - U_0^2} \right] e^{-\epsilon \sqrt{\omega^2 - U_0^2}} + \mathcal{O}(1)
 \end{aligned}$$

After grouping all integrals, the coincidence limits $\epsilon \rightarrow 0$ and $x' \rightarrow x$ are immediate:

$$w^{\text{high}}(x) = \hbar \int_{U_0}^{\infty} d\omega \mathcal{N}_{\omega}^{-} \cos(2\omega x) + \hbar \int_{U_0}^{\infty} d\omega \left(\mathcal{N}_{\omega}^{+} - \frac{2\omega}{\omega^2 - U_0^2} \right) + \mathcal{O}(1)$$

(III) Let $X = (t, x)$ and $\Delta X = X - X'$. Using proposition 3.3, the derivatives of the parametrix are:

$$\hbar^{-1} \partial_{\mu\nu}^2 H(\sigma) = \partial_{\mu\nu}^2 \ln(\sigma + i\epsilon) = \frac{\partial_{\mu\nu}^2 \sigma}{\sigma + i\epsilon} - \frac{\partial_{\mu} \sigma \partial_{\nu} \sigma}{(\sigma + i\epsilon)^2} = \frac{\eta_{\mu\nu}}{\sigma + i\epsilon} - \frac{\eta_{\mu\alpha} \eta_{\nu\beta} (\Delta X)^{\alpha} (\Delta X)^{\beta}}{(\sigma + i\epsilon)^2}$$

From theorem A.1, after setting $\delta'(\sigma) = d\delta/d\sigma$ we have:

$$\hbar^{-1} \partial_{\mu\nu}^2 H(\sigma) = \eta_{\mu\nu} [\mathcal{P}(1/\sigma) - i\pi\delta(\sigma)] - \eta_{\mu\alpha} \eta_{\nu\beta} (\Delta X)^{\alpha} (\Delta X)^{\beta} [\mathcal{P}(1/\sigma^2) + i\pi\delta'(\sigma)]$$

Taking the first limit $\Delta t \rightarrow 0$ such that $\sigma = \Delta x^2/2$ and using the distributional identities of eq. (A.16) and proposition A.3, then:

$$\lim_{\Delta t \rightarrow 0} \hbar^{-1} \partial_{\mu\nu}^2 H(\sigma) = -2\delta_{\mu\nu} [\mathcal{P}(1/\Delta x^2) - i\pi\delta(\Delta x^2)]$$

Once again, $\delta(\Delta x^2)$ is a purely contact term that we discard. Employing proposition A.5 in the appropriate integration variable:

$$\hbar^{-1} \partial_{\mu\nu}^2 H(\Delta x^2) = 2\delta_{\mu\nu} \int_0^{\infty} dk k \cos(\Delta x k) e^{-\epsilon k} = \delta_{\mu\nu} \int_{U_0}^{\infty} d\omega 2\omega \cos \left(\Delta x \sqrt{\omega^2 - U_0^2} \right) e^{-\epsilon \sqrt{\omega^2 - U_0^2}} \quad (4.52)$$

Combining eqs. (4.50) and (4.52), the limit $\Delta t \rightarrow 0$ of the complete high-energy regularized propagator reads:

$$\begin{aligned}
 \partial_{\mu\nu}^2 W^{\text{high}}(x, x') &= -\delta_{\mu\nu} \hbar \int_{U_0}^{\infty} d\omega \mathcal{N}_{\omega}^{-} \omega^2 \cos[\omega(x + x')] + \\
 &\quad - \delta_{\mu\nu} \hbar \int_{U_0}^{\infty} d\omega \left(\mathcal{N}_{\omega}^{+} \omega^2 \cos[\omega(x - x')] - 2\omega \cos \left[(x - x') \sqrt{\omega^2 - U_0^2} \right] e^{-\epsilon \sqrt{\omega^2 - U_0^2}} \right)
 \end{aligned}$$

Taking the coincidence limit $\Delta x \rightarrow 0$ and $\epsilon \rightarrow 0$ completes the proof:

$$w_{\mu\nu}^{\text{high}}(x) = -\hbar \left[\int_{U_0}^{\infty} d\omega \mathcal{N}_{\omega}^{-} \omega^2 \cos(2\omega x) + \int_{U_0}^{\infty} d\omega (\mathcal{N}_{\omega}^{+} \omega^2 - 2\omega) \right] \delta_{\mu\nu}$$

□

Proposition 4.15 REVMTO: Interior & High-Energy

The components of the REVMTO (proposition 3.6) from the high-energy normal modes' contribution in the regularized propagator (proposition 4.11) in the interior region $|x| < L$ are given by:

$$\left\langle \hat{T}_{tt} \right\rangle_{\text{ren}}^{\text{high}} = \left\langle \hat{T}_{xx} \right\rangle_{\text{ren}}^{\text{high}} = \hbar \int_{U_0}^{\infty} \frac{d\omega}{2\pi} \left[\frac{(B_{\omega}^+)^2 + (B_{\omega}^-)^2}{2} - \frac{1}{\omega} \right] \omega^2 \quad (4.53a)$$

$$\left\langle \hat{T}_{tx} \right\rangle_{\text{ren}}^{\text{high}} = \left\langle \hat{T}_{xt} \right\rangle_{\text{ren}}^{\text{high}} = 0 \quad (4.53b)$$

Proof. The proof is completely analogous to that of proposition 4.13. Given eq. (4.46), the split $w = w^{\text{low}} + w^{\text{high}}$, and that $\partial_t w^{\text{high}}(x) = 0$, then:

$$\begin{aligned} \left\langle \hat{T}_{\mu\nu}(t, x) \right\rangle_{\text{ren}}^{\text{high}} &= \frac{1}{4\pi} \left(-w_{\mu\nu} + \delta_{\mu\nu} \frac{1}{4} \partial_x^2 w \right) \\ &= \delta_{\mu\nu} \frac{\hbar}{4\pi} \left[\int_{U_0}^{\infty} d\omega \mathcal{N}_{\omega}^- \omega^2 \cos(2\omega x) + \int_{U_0}^{\infty} d\omega (\mathcal{N}_{\omega}^+ \omega^2 - 2\omega) + \frac{1}{4} \int_{U_0}^{\infty} d\omega \mathcal{N}_{\omega}^- (-2\omega)(2\omega) \cos(2\omega x) \right] \\ &= \delta_{\mu\nu} \hbar \int_{U_0}^{\infty} \frac{d\omega}{2\pi} \left(\frac{\mathcal{N}_{\omega}^+}{2} - \frac{1}{\omega} \right) \omega^2 \end{aligned}$$

□

4.3.4 Interior REVMTO

Combining propositions 4.13 and 4.15, the REVMTO for definition 4.1 w.r.t. the normal modes of propositions 4.3 and 4.4 is given in the interior region $|x| < L$ by¹:

$$\begin{aligned} \left\langle \hat{T}_{\mu\nu}(|x| < L) \right\rangle_{\text{ren}} &= \delta_{\mu\nu} \frac{\hbar}{4} \sum_{n=1}^{N^+} \frac{\omega_n^+}{L + \frac{1}{\sqrt{(U_0)^2 - (\omega_n^+)^2}}} + \delta_{\mu\nu} \frac{\hbar}{4} \sum_{n=1}^{N^-} \frac{\omega_n^-}{L + \frac{1}{\sqrt{(U_0)^2 - (\omega_n^-)^2}}} + \\ &\quad + \delta_{\mu\nu} \hbar \int_{U_0}^{\infty} \frac{d\omega}{2\pi} \left[\frac{1}{2} \frac{\sqrt{\omega^2 - U_0^2}}{\omega^2 - U_0^2 \cos^2(\omega L)} + \frac{1}{2} \frac{\sqrt{\omega^2 - U_0^2}}{\omega^2 - U_0^2 \sin^2(\omega L)} - \frac{1}{\omega} \right] \omega^2 \end{aligned} \quad (4.54)$$

Remarks:

- As a sanity check, $[\omega] = [\omega_n^s] = [U_0] = [L]^{-1} = \ell_P^{-1}$ implies $[B_n^s] = 1$ and $[B_{\omega}^s] = \sqrt{\ell_P}$, so we obtain the right dimensions $[\langle \hat{T}_{\text{ren}} \rangle] = m_P \ell_P^{-1}$.
- The REVMTO is constant in the interior region of the well in fig. 4.2. Thus, eq. (4.54) is trivially locally conserved:

$$\nabla^{\mu} \left\langle \hat{T}_{\mu\nu}(|x| < L) \right\rangle_{\text{ren}} = 0 \quad (4.55)$$

¹We recall that the bound states ω_n^s are given by the solutions of the transcendental equation in eq. (4.23). Therefore, the REVMTO in the interior region is solely and completely determined by the two free parameters of the theory U_0 and L .

- Consider the power expansion of the high-energy normalization constants in proposition 4.4:

$$(B_\omega^\pm)^2 = \frac{1}{\omega} \left[1 \pm \frac{1}{2} \cos(2\omega L) \left(\frac{U_0}{\omega} \right)^2 + \frac{1}{4} \left(\pm \cos(2\omega L) + \frac{1}{2} \cos(4\omega L) \right) \left(\frac{U_0}{\omega} \right)^4 + \mathcal{O} \left(\frac{U_0^6}{\omega^6} \right) \right] \quad (4.56a)$$

$$\mathcal{N}_\omega^+ = \frac{2}{\omega} \left[1 + \frac{1}{8} \cos(4\omega L) \left(\frac{U_0}{\omega} \right)^4 + \mathcal{O} \left(\frac{U_0^6}{\omega^6} \right) \right] \quad (4.56b)$$

$$\mathcal{N}_\omega^- = \frac{\cos(2\omega L)}{\omega} \left[\left(\frac{U_0}{\omega} \right)^2 + \frac{1}{2} \left(\frac{U_0}{\omega} \right)^4 + \mathcal{O} \left(\frac{U_0^6}{\omega^6} \right) \right] \quad (4.56c)$$

The linear combination $\mathcal{N}_\omega^- \sim \omega^{-3} + \mathcal{O}(\omega^{-5})$ is convergent under integration, while $(B_\omega^s)^2 \sim \omega^{-1} + \mathcal{O}(\omega^{-3})$ and $\mathcal{N}_\omega^+ \sim 2\omega^{-1} + \mathcal{O}(\omega^{-5})$ are divergent. Without the Hadamard parametrix, the coincidence limit of the propagator and the REVMTO are logarithmically and quadratically divergent, respectively. The inclusion of the Hadamard parametrix into the renormalization procedure successfully removes all these pathologies: eq. (4.54) is UV-finite for both the discrete $0 < \omega_n^s < U_0$ and the continuous $\omega > U_0 > 0$ spectrum.

- The structure of the REVMTO is that of a Zeldovich stiff fluid in its locally inertial rest frame whose proper energy density and proper pressure are:

$$\langle \hat{\rho} \rangle_Z = \langle \hat{p} \rangle_Z = \hbar \sum_s \sum_{n=1}^{N^s} \frac{(B_n^s)^2}{2} (\omega_n^s)^2 + \hbar \int_{U_0}^{\infty} \frac{d\omega}{2\pi} \left[\frac{(B_\omega^+)^2 + (B_\omega^-)^2}{2} - \frac{1}{\omega} \right] \omega^2 \quad (4.57)$$

As in the infinite potential well model, the effective stiff-fluid structure is inherited from the classical theory, since the scalar field's Lagrangian is purely kinetic in the interior region. By the mode construction, $\omega_n^s > 0$, and thus the low-energy contribution is manifestly positive, $\langle \hat{\rho} \rangle_Z^{\text{low}}(|x| < L) > 0$. It follows that the energy conditions will be satisfied provided the high-energy contribution satisfies $\langle \hat{\rho} \rangle_Z^{\text{high}}(|x| < L) \geq -\langle \hat{\rho} \rangle_Z^{\text{low}}(|x| < L)$, and all violated otherwise.

4.4 Exterior Region

Now we focus on the exterior region $|x| > L$.

Proposition 4.16 Hadamard Parametrix: Exterior

In the exterior region $|x| > L$, the Hadamard parametrix of definition 4.1 is given by:

$$H(\sigma) = -\hbar \left(1 + \frac{1}{2} U_0^2 \sigma \right) \ln(\sigma + i\epsilon) \quad (4.58)$$

Proof. The proof is completely analogous to proposition 4.10 and follows from example 3.1. The main difference is that comparing eq. (3.31) and definition 4.1, in the outside region $U(|x| > L) = U_0^2$, so we identify $m/\hbar = U_0$. The expansion $V = V_0 + V_1\sigma + \mathcal{O}(\sigma^{3/2})$ has now the coefficients $V_0 = -\hbar + \mathcal{O}(\sigma^{3/2})$ and $V_1 = -\hbar U_0^2/2 + \mathcal{O}(\sigma^{1/2})$. $\square$

Proposition 4.17 Regularized Propagator: Exterior

In the exterior region $|x| > L$, the regularized propagator w.r.t. the normal modes of propositions 4.3

and 4.4 reads:

$$\begin{aligned}
 W(t, x, t', x') = & 4\pi\hbar \sum_{n=1}^{N^+} (B_n^+)^2 \cos^2(\omega_n^+ L) e^{-k_n^+ (|x|+|x'|-2L)} e^{-i\omega_n^+ |t-t'|} \\
 & + 4\pi\hbar \sum_{n=1}^{N^-} (B_n^-)^2 \sin^2(\omega_n^- L) e^{-k_n^- (|x|+|x'|-2L)} \text{sgn}(x) \text{sgn}(x') e^{-i\omega_n^- |t-t'|} \\
 & + 2\hbar \int_{U_0}^{\infty} d\omega (B_\omega^+)^2 \left(\cos(\omega L) \cos[k(|x| - L)] - \frac{\omega}{k} \sin(\omega L) \sin[k(|x| - L)] \right) \cdot \\
 & \cdot \left(\cos(\omega L) \cos[k(|x'| - L)] - \frac{\omega}{k} \sin(\omega L) \sin[k(|x'| - L)] \right) e^{-i\omega |t-t'|} \\
 & + 2\hbar \int_{U_0}^{\infty} d\omega (B_\omega^-)^2 \left(\sin(\omega L) \cos[k(|x| - L)] + \frac{\omega}{k} \cos(\omega L) \sin[k(|x| - L)] \right) \cdot \\
 & \cdot \left(\sin(\omega L) \cos[k(|x'| - L)] + \frac{\omega}{k} \cos(\omega L) \sin[k(|x'| - L)] \right) \text{sgn}(x) \text{sgn}(x') e^{-i\omega |t-t'|} \\
 & + \hbar \left[1 + \frac{1}{2} U_0^2 \sigma(t, x, t', x') \right] \ln[\sigma(t, x, t', x') + i\epsilon]
 \end{aligned} \tag{4.59}$$

Proof. The quantum field expansion in eq. (4.34) in terms of the exterior region normal modes of propositions 4.3 and 4.4 reads:

$$\begin{aligned}
 \hat{\Phi}(t, x) = & \sum_{n=1}^{N^+} \hat{a}_{n,+} B_n^+ e^{-i\omega_n^+ t} \cos(\omega_n^+ L) e^{-k_n^+ (|x|-L)} \\
 & + \sum_{n=1}^{N^-} \hat{a}_{n,-} B_n^- e^{-i\omega_n^- t} \sin(\omega_n^- L) e^{-k_n^- (|x|-L)} \text{sgn}(x) \\
 & + \int_{U_0}^{\infty} \frac{d\omega}{2\pi} \hat{b}_{\omega,+} B_\omega^+ e^{-i\omega t} \left(\cos(\omega L) \cos[k(|x| - L)] - \frac{\omega}{k} \sin(\omega L) \sin[k(|x| - L)] \right) \\
 & + \int_{U_0}^{\infty} \frac{d\omega}{2\pi} \hat{b}_{\omega,-} B_\omega^- e^{-i\omega t} \left(\sin(\omega L) \cos[k(|x| - L)] + \frac{\omega}{k} \cos(\omega L) \sin[k(|x| - L)] \right) \text{sgn}(x) \\
 & + \text{h.c.}
 \end{aligned}$$

The only non-vanishing terms in the two-point function are proportional to eq. (4.40) and thus:

$$\begin{aligned}
 \langle \hat{\Phi}(t, x) \hat{\Phi}(t', x') \rangle = & \hbar \sum_n^{N^+} (B_n^+)^2 \cos^2(\omega_n^+ L) e^{-k_n^+ (|x|+|x'|-2L)} e^{-i\omega_n^+ (t-t')} + \\
 & + \hbar \sum_n^{N^-} (B_n^-)^2 \sin^2(\omega_n^- L) e^{-k_n^- (|x|+|x'|-2L)} \text{sgn}(x) \text{sgn}(x') e^{-i\omega_n^- (t-t')} + \\
 & + \hbar \int_{U_0}^{\infty} \frac{d\omega}{2\pi} (B_\omega^+)^2 \left(\cos(\omega L) \cos[k(|x| - L)] - \frac{\omega}{k} \sin(\omega L) \sin[k(|x| - L)] \right) \cdot \\
 & \cdot \left(\cos(\omega L) \cos[k(|x'| - L)] - \frac{\omega}{k} \sin(\omega L) \sin[k(|x'| - L)] \right) e^{-i\omega (t-t')} + \\
 & + \hbar \int_{U_0}^{\infty} \frac{d\omega}{2\pi} (B_\omega^-)^2 \left(\sin(\omega L) \cos[k(|x| - L)] + \frac{\omega}{k} \cos(\omega L) \sin[k(|x| - L)] \right) \cdot \\
 & \cdot \left(\sin(\omega L) \cos[k(|x'| - L)] + \frac{\omega}{k} \cos(\omega L) \sin[k(|x'| - L)] \right) \text{sgn}(x) \text{sgn}(x') e^{-i\omega (t-t')}
 \end{aligned}$$

The rest of the proof proceeds as with propositions 4.8 and 4.11 by using $W(t, x, t', x') = 4\pi \langle T \hat{\Phi}(t, x) \hat{\Phi}(t', x') \rangle - H(t, x, t', x')$ with proposition 4.16. $\square$

Using proposition 3.6 and example 3.1, the structure of the REVMTO in the exterior region is:

$$\langle \hat{T}_{\mu\nu}(t, x) \rangle_{\text{ren}}^{\text{ext}} = \frac{1}{4\pi} \left(-w_{\mu\nu} + \frac{1}{2} \partial_{\mu\nu}^2 w - \frac{1}{4} \eta_{\mu\nu} \square w + \frac{\hbar U_0^2}{2} \eta_{\mu\nu} \right)^{\text{ext}} - \frac{\hbar U_0^2}{8\pi} \ln \left(\frac{l_0}{l} \right)^2 \eta_{\mu\nu} \quad (4.60)$$

In contrast to the interior region, we observe two contributions. A state-dependent part:

$$\langle \hat{T}_{\mu\nu}(t, x) \rangle_{\text{ren}}^{\text{ext}} \supset \frac{1}{4\pi} \left(-w_{\mu\nu} + \frac{1}{2} \partial_{\mu\nu}^2 w - \frac{1}{4} \eta_{\mu\nu} \square w \right)^{\text{ext}} \quad (4.61)$$

And a state-independent part:

$$\langle \hat{T}_{\mu\nu} \rangle_{\text{ren}}^{\text{ext}} \supset \frac{\hbar U_0^2}{8\pi} \left[1 - \ln \left(\frac{l_0}{l} \right)^2 \right] \eta_{\mu\nu} \quad (4.62)$$

We recall that l and l_0 are the renormalization length scale and reference, respectively, needed to make the $\ln$ argument in the parametrix dimensionless. Equation (4.62) can be interpreted as a quantum vacuum fluctuation induced by the non-vanishing external potential. The structure is that of a *vacuum energy cosmic fluid* with proper energy density:

$$\langle \hat{\rho} \rangle_{\Lambda}(|x| > L) = -\langle \hat{p} \rangle_{\Lambda}(|x| > L) = -\frac{\hbar U_0^2}{8\pi} \left[1 - \ln \left(\frac{l_0}{l} \right)^2 \right] \quad (4.63)$$

The choice $l_0^2/l^2 = e$ effectively eliminates this term from the exterior REVMTO.

4.4.1 Low-Energy REVMTO

We isolate the state-independent part of the REVMTO and focus on the state-dependent low-energy contribution. Since the sum is finite and regular, we dispense with the parametrix. The corresponding regularized propagator in the outside region reads:

$$\begin{aligned} W \supset W^{\text{low}}(t, x, t', x') &:= 4\pi\hbar \sum_n^{N^+} (B_n^+)^2 \cos^2(\omega_n^+ L) e^{-k_n^+ (|x|+|x'|-2L)} e^{-i\omega_n^+ |t-t'|} + \\ &+ 4\pi\hbar \sum_n^{N^-} (B_n^-)^2 \sin^2(\omega_n^- L) e^{-k_n^- (|x|+|x'|-2L)} \text{sgn}(x) \text{sgn}(x') e^{-i\omega_n^- |t-t'|} \end{aligned} \quad (4.64)$$

Comparing eqs. (4.43) and (4.64), we notice that the low-energy regularized propagators in the inside and outside regions coincide at $|x| = |x'| = L$. As expected from the matching condition of the modes!

Proposition 4.18 REVMTO: Exterior & Low-Energy

The components of the REVMTO from the low-energy modes' contributions in the regularized propagator of proposition 4.17 in the exterior region $|x| > L$ are given by:

$$\left\langle \hat{T}_{00}(|x| > L) \right\rangle_{\text{ren}}^{\text{low}} = \frac{\hbar}{2} \sum_s \sum_n^{N^s} \frac{\omega_n^s}{L + \frac{1}{\sqrt{(U_0)^2 - (\omega_n^s)^2}}} e^{-2\sqrt{(U_0)^2 - (\omega_n^s)^2}(|x|-L)} \quad (4.65a)$$

$$\left\langle \hat{T}_{xx}(|x| > L) \right\rangle_{\text{ren}}^{\text{low}} = 0 \quad (4.65b)$$

$$\left\langle \hat{T}_{tx}(|x| > L) \right\rangle_{\text{ren}}^{\text{low}} = 0 \quad (4.65c)$$

$$\left\langle \hat{T}_{xt}(|x| > L) \right\rangle_{\text{ren}}^{\text{low}} = 0 \quad (4.65d)$$

Proof. Taking the second derivatives of eq. (4.64) and discarding the temporal contact terms, the coincidence limits of proposition 4.17 and its second derivatives are:

$$\begin{aligned} w^{\text{low}}(x) &= 4\pi\hbar \sum_n^{N^+} (B_n^+)^2 \cos^2(\omega_n^+ L) e^{-2k_n^+ (|x|-L)} + 4\pi\hbar \sum_n^{N^-} (B_n^-)^2 \sin^2(\omega_n^- L) e^{-2k_n^- (|x|-L)} \\ w_{00}^{\text{low}}(x) &= -4\pi\hbar \sum_n^{N^+} (B_n^+)^2 (\omega_n^+)^2 \cos^2(\omega_n^+ L) e^{-2k_n^+ (|x|-L)} - 4\pi\hbar \sum_n^{N^-} (B_n^-)^2 (\omega_n^-)^2 \sin^2(\omega_n^- L) e^{-2k_n^- (|x|-L)} \\ w_{11}^{\text{low}}(x) &= 4\pi\hbar \sum_n^{N^+} (B_n^+)^2 [(k_n^+)^2 - 2k_n^+ \delta(x)] \cos^2(\omega_n^+ L) e^{-2k_n^+ (|x|-L)} \\ &\quad + 4\pi\hbar \sum_n^{N^-} (B_n^-)^2 [(k_n^-)^2 - 2k_n^- \delta(x) + 2\delta'(x) \text{sgn}(x)] \sin^2(\omega_n^- L) e^{-2k_n^- (|x|-L)} \\ w_{01}^{\text{low}}(x) &= 0 \\ w_{10}^{\text{low}}(x) &= 0 \end{aligned}$$

Given that $|x| > L > 0$, the spatial Dirac delta distribution and its derivative, which are only supported at $x = 0$, do not contribute when smeared against test functions whose support is contained entirely in the outside region (as consistent with the piecewise nature of the orthonormal mode basis), so henceforth we set all spatial contact terms to zero in the distributional sense. Therefore, given $\partial_t w(x) = 0$, the tt -component of the low-energy sector of eq. (4.61) is:

$$\begin{aligned} \hbar^{-1} \left\langle \hat{T}_{00}(t, x) \right\rangle_{\text{ren}}^{\text{low}} &\supset \frac{1}{4\pi} \left(-w_{00} + \frac{1}{4} \partial_x^2 w \right)^{\text{low}} \\ &= \sum_n^{N^+} (B_n^+)^2 (\omega_n^+)^2 \cos^2(\omega_n^+ L) e^{-2k_n^+ (|x|-L)} + \sum_n^{N^-} (B_n^-)^2 (\omega_n^-)^2 \sin^2(\omega_n^- L) e^{-2k_n^- (|x|-L)} \\ &\quad + \sum_n^{N^+} (B_n^+)^2 [U_0^2 - (\omega_n^+)^2] \cos^2(\omega_n^+ L) e^{-2k_n^+ (|x|-L)} + \sum_n^{N^-} (B_n^-)^2 [U_0^2 - (\omega_n^-)^2] \sin^2(\omega_n^- L) e^{-2k_n^- (|x|-L)} \\ &= \sum_n^{N^+} (B_n^+)^2 (\omega_n^+)^2 e^{-2k_n^+ (|x|-L)} + \sum_n^{N^-} (B_n^-)^2 (\omega_n^-)^2 e^{-2k_n^- (|x|-L)} \end{aligned}$$

Where we have used $U_0^2 \cos^2(\omega_n^+ L) = (\omega_n^+)^2$ and $U_0^2 \sin^2(\omega_n^- L) = (\omega_n^-)^2$, which follow from eq. (4.23). The mixed terms vanish identically, and the xx component vanishes by virtue of $\partial_x^2 w^{\text{low}} = 4w_{11}^{\text{low}}$. $\square$

Remarks:

- In contrast to the interior region, the low-energy REVENTO in the exterior region depends on the position relative to the center of the finite well. It is a locally conserved quantity nonetheless:

$$\nabla^\mu \left\langle \hat{T}_{\mu\nu}(|x| > L) \right\rangle_{\text{ren}}^{\text{low}} = 0 \quad (4.66)$$

The asymptotic limits are:

$$\lim_{x \rightarrow \pm\infty} \left\langle \hat{T}_{\mu\nu}(|x| > L) \right\rangle_{\text{ren}}^{\text{low}} = 0 \quad (4.67)$$

- Unlike the fields, the normal modes, and the propagator (and its coincidence limit), we see that the low-energy REVENTO is not continuous across the surfaces $|x| = L$. However, it is a finite discontinuity, which for the local energy density is given by:

$$\left\langle \hat{T}_{00}(|x| > L) \right\rangle_{\text{ren}}^{\text{low}} \Big|_{x=L} = 2 \left\langle \hat{T}_{00}(|x| < L) \right\rangle_{\text{ren}}^{\text{low}} \Big|_{x=L} \quad (4.68)$$

The matching conditions imposed on the field and the normal modes are of zeroth- and first-order. The EMT is of second-order, and thus sensitive to the step discontinuity of the potential via the e.o.m.

- The structure of proposition 4.18 is that of a *dust cosmic fluid* in its LIRF whose proper pressure is $\langle \hat{\rho} \rangle_d(|x| > L) = 0$ and whose proper energy density is:

$$\langle \hat{\rho} \rangle_d(|x| > L) = \frac{\hbar}{2} \sum_s \sum_n \frac{\omega_n^s}{L + \frac{1}{\sqrt{(U_0)^2 - (\omega_n^s)^2}}} e^{-2\sqrt{(U_0)^2 - (\omega_n^s)^2}(|x| - L)} \quad (4.69)$$

Once again, this effective perfect-fluid structure is consistent with the classical field theory of a real scalar field subject to a quadratic potential. Since $\langle \hat{\rho} \rangle_d(|x| > L) \geq 0$, this contribution satisfies all energy conditions (cf. fig. 2.1).

Proposition 4.19 Casimir Energy

Consider the contribution of the low-energy modes to the REVENTO in both the interior and exterior regions, $\left\langle \hat{T}_{\mu\nu}(x) \right\rangle^{\text{low}} := \left\langle \hat{T}_{\mu\nu}(|x| < L) \right\rangle^{\text{low}} + \left\langle \hat{T}_{\mu\nu}(|x| > L) \right\rangle^{\text{low}}$. The Casimir energy is:

$$E_C := \int_{-\infty}^{+\infty} dx \left\langle \hat{T}_{00}(x) \right\rangle^{\text{low}} = \frac{1}{2} \sum_{s=\pm} \sum_n^{N^s} \hbar \omega_n^s \quad (4.70)$$

Proof. Straightforwardly, from propositions 4.13 and 4.18 using the identities in proposition 4.3:

$$\begin{aligned} E_C &= \int_{-\infty}^{+\infty} dx \left\langle \hat{T}_{00}(x) \right\rangle^{\text{low}} = \left(\int_{-\infty}^{-L} + \int_{-L}^L + \int_L^{\infty} \right) dx \left\langle \hat{T}_{00}(x) \right\rangle^{\text{low}} \\ &= \int_{-L}^L dx \left\langle \hat{T}_{00}(|x| < L) \right\rangle^{\text{low}} + 2 \int_L^{\infty} dx \left\langle \hat{T}_{00}(|x| > L) \right\rangle^{\text{low}} \end{aligned}$$

$$\begin{aligned}
 &= \int_{-L}^L dx \frac{\hbar}{2} \sum_{s,n} (B_n^s)^2 (\omega_n^s)^2 + 2 \int_L^\infty dx \hbar \sum_{s,n} (B_n^s)^2 (\omega_n^s)^2 e^{-2k_n^s(|x|-L)} \\
 &= \hbar \sum_{s,n} (B_n^s)^2 (\omega_n^s)^2 L + \hbar \sum_{s,n} (B_n^s)^2 (\omega_n^s)^2 \frac{1}{k_n^s} \\
 &= \hbar \sum_{s,n} \frac{1}{2} \frac{\omega_n^s}{L + \frac{1}{\sqrt{(U_0)^2 - (\omega_n^s)^2}}} \left(L + \frac{1}{\sqrt{(U_0)^2 - (\omega_n^s)^2}} \right) = \sum_{s,n} \frac{\hbar \omega_n^s}{2}
 \end{aligned}$$

□

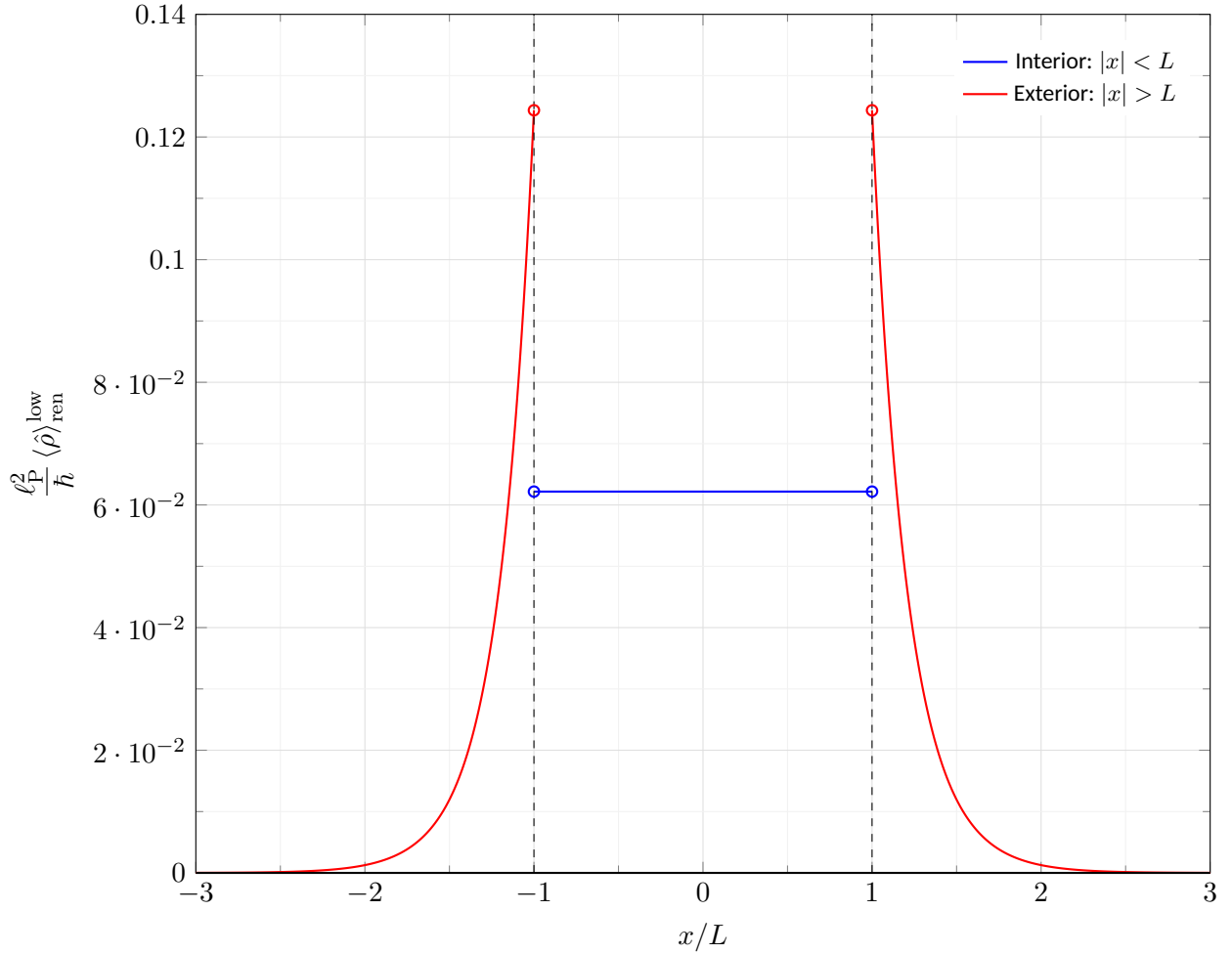


Figure 4.3: Low-energy contribution to the dimensionless proper energy density of the REVMTO for $L = \pi$ and $U_0 = 1$. The well admits two bound states with frequencies $\omega_1^+ = 0.377$ and $\omega_1^- = 0.736$. The vertical dashed lines mark the well's boundaries, while the open circles indicate the corresponding one-sided limits. The energy density is constant inside the well, $|x| < L$, and decays exponentially to zero outside, $|x| > L$.

4.4.2 Limits

Proposition 4.20 REVMTO Limits

The interior REVMTO of eq. (4.54) has the following limits:

$$\lim_{U_0 \rightarrow 0} \left\langle \hat{T}_{\mu\nu} (|x| < L) \right\rangle_{\text{ren}} = 0 \quad (4.71a)$$

$$\lim_{L \rightarrow 0} \left\langle \hat{T}_{\mu\nu} (|x| < L) \right\rangle_{\text{ren}} = \frac{\hbar U_0^2}{8\pi} \delta_{\mu\nu} \quad (4.71b)$$

While the exterior low-energy REVENTO of proposition 4.18 has the following limits:

$$\lim_{U_0 \rightarrow 0} \left\langle \hat{T}_{\mu\nu} (|x| > L) \right\rangle_{\text{ren}}^{\text{low}} = 0 \quad (4.72a)$$

$$\lim_{L \rightarrow 0} \left\langle \hat{T}_{\mu\nu} (|x| > L) \right\rangle_{\text{ren}}^{\text{low}} = 0 \quad (4.72b)$$

Proof. Since the REVENTO is completely regular, we can commute the sums and the integral with all limits. (i) In the limits $U_0 = 0$ and $L = 0$ there are no solutions for the discrete normal modes: $N(U_0 L = 0) = 0$ and $\omega_n^s = 0$. Hence, the corresponding limits of $\left\langle \hat{T} \right\rangle_{\text{ren}}^{\text{low}}$ vanish identically for both the interior and exterior region. (ii) The limit $U_0 \rightarrow 0$ of $\left\langle \hat{T} \right\rangle_{\text{ren}}^{\text{high}}$ in the interior is immediate. For $L \rightarrow 0$, we have:

$$\begin{aligned} \lim_{L \rightarrow 0} \hbar^{-1} \left\langle \hat{T}_{\mu\nu} \right\rangle_{\text{ren}}^{\text{high}} &= \delta_{\mu\nu} \lim_{a \rightarrow \infty} \int_{U_0}^a \frac{d\omega}{2\pi} \left[\frac{\omega^2}{2\sqrt{\omega^2 - U_0^2}} + \frac{\sqrt{\omega^2 - U_0^2}}{2} - \omega \right] \\ &= \frac{\delta_{\mu\nu}}{4\pi} \lim_{a \rightarrow \infty} \left(U_0^2 + a^2 \left[\sqrt{1 - (U_0/a)^2} - 1 \right] \right) \\ &= \delta_{\mu\nu} \frac{U_0^2}{8\pi} \end{aligned}$$

□

Not much can be said about these limits without an explicit computation of the exterior high-energy sector contribution. Notwithstanding, these partial limits are consistent with $U_0 \rightarrow 0$ corresponding to the *massless free theory limit*, and $L \rightarrow 0$ to both the *massive free theory limit* (exterior) and the *ultra-compact limit* (interior). For the latter limiting theory, we see that the interior REVENTO is not vanishing but still dust.

The limit $U_0 \rightarrow \infty$ with $L = \text{cst}$ corresponds to the **infinite potential well limit**. For the low-energy sector in both the interior and exterior, $N^s(U_0 \rightarrow \infty) \rightarrow \infty$ yields the total spectrum $\omega_n = \frac{\pi}{2L}n$, where $n \in \mathbb{Z}^+$. The exponential factor makes the exterior contribution vanish so that the total low-energy REVENTO becomes:

$$\lim_{U_0 \rightarrow \infty} \left\langle \hat{T}_{\mu\nu} \right\rangle_{\text{ren}}^{\text{low}} = \delta_{\mu\nu} \frac{\hbar\pi}{2(2L)^2} \sum_{n=1}^{\infty} n \quad (4.73)$$

This is the *unrenormalized* EVEMTO for an infinite potential well of length $2L$.

Chapter 5

Conclusions

We conclude with a brief overview of the principal ideas discussed thus far, a summary of our main results, and closing statements regarding the open questions and future directions stemming from this thesis.

5.1 Overview

Having introduced the general framework of scalar field theories on curved spacetimes at the classical and quantum levels, in this thesis we considered the free action describing the dynamics of a real scalar field $\Phi(t, r, \theta^a)$, and imposed that the quadratic potential shares the symmetries of the SSS spacetime in order to allow a complete normal mode decomposition. Under these hypotheses and in Regge-Wheeler coordinates r_* , the e.o.m. for the field reduce to a Regge-Wheeler type equation for the radial normal modes subject to an effective potential $V_\ell(r_*)$. We have shown that the time-radial sector of the D -dimensional theory defines a family of theories for an “auxiliary” scalar field $\Psi_\ell(t, r_*)$ in a $(1 + 1)$ -dimensional Minkowski spacetime and whose e.o.m. are precisely those of the positive-frequency and radial normal modes. Moreover, we showed that both the Feynman propagator and the REVENTO of the latter are essential intermediate ingredients of those of the former. Hence, to solve the D -dimensional classical and quantum theory of $\Phi(t, r, \theta^a)$, we must first solve the theories of $\Psi_\ell(t, r_*)$ in a $(1 + 1)$ -dimensional flat spacetime subject to $V_\ell(r_*)$. The effective potential of a SSS spacetime sourced by a homogeneous perfect fluid motivates the model studied in this thesis: the “background” contribution exhibits a finite step discontinuity at the surface of the compact object, and the interior profile is nearly constant.

Therefore, we have considered the auxiliary theory $\Psi_\ell(t, r_*) \rightarrow \Phi(t, x)$ with an idealized effective potential $V_\ell(r_*) \rightarrow U(x) = U_0^2 \Theta(|x| - L)$ corresponding to a finite well of depth $U_0^2 > 0$ and width $2L > 0$. Given the discontinuity of the potential at the surfaces $|x| = L$, we have imposed smoothness boundary conditions for the field, which are translated into matching conditions for the normal modes in the *interior* $|x| < L$ and *exterior* $|x| > L$ regions. The potential is static and parity-symmetric, so we have quantized $\Phi(t, x)$ in terms of a complete set of orthonormal modes $\{\phi_{sn}(t, x), \varphi_{sw}(t, x), \text{c.c.}\}$ with these properties. Said normal modes are characterized by a *low-energy sector* of discrete, parity-dependent, bound-state frequencies $0 < \omega_n^s < U_0$, and a *high-energy sector* of parity-independent continuous frequencies $\omega > U_0 > 0$.

This *finite-well model* illustrates the necessity of renormalization in a particularly transparent setting: the Feynman propagator of the theory is shown to be divergent at the coincidence limit, and the expectation value of the energy-momentum tensor quantum operator inevitably inherits this feature. Following *Hadamard’s renormalization procedure*, the theory’s *regularized propagator* has been constructed by computing and subtracting the *Hadamard parametrix*, and from its coincidence limits the REVENTO has been explicitly constructed.

5.2 Main Results

The main results of this thesis are eqs. (4.54) and (4.62) and proposition 4.18. Together, they give the REVMTO in the interior region of the finite potential well, and the low-energy sector and state-independent contributions to the REVMTO in the exterior region:

$$\langle \hat{T}_{\mu\nu}(x) \rangle_{\text{ren}} = \begin{cases} \begin{pmatrix} \langle \hat{\rho} \rangle_Z^{\text{low}} + \langle \hat{\rho} \rangle_Z^{\text{high}} & 0 \\ 0 & \langle \hat{\rho} \rangle_Z^{\text{low}} + \langle \hat{\rho} \rangle_Z^{\text{high}} \end{pmatrix}, & |x| < L \\ \begin{pmatrix} \langle \hat{\rho} \rangle_d^{\text{low}} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \langle \hat{\rho} \rangle_\Lambda & 0 \\ 0 & -\langle \hat{\rho} \rangle_\Lambda \end{pmatrix}, & |x| > L \end{cases} \quad (5.1)$$

In the interior region, the quantum perturbation's REVMTO is identical to that of a *Zeldovich stiff fluid* in its locally inertial rest frame with a homogeneous proper energy density given by the low- and high-energy contributions:

$$\langle \hat{\rho} \rangle_Z^{\text{low}} = \frac{\hbar}{4} \sum_{s=\pm} \sum_{n=1}^{N^s} \frac{\omega_n^s}{L + \frac{1}{\sqrt{(U_0)^2 - (\omega_n^s)^2}}} \quad (5.2a)$$

$$\langle \hat{\rho} \rangle_Z^{\text{high}} = \hbar \int_{U_0}^{\infty} \frac{d\omega}{2\pi} \left[\frac{1}{2} \frac{\sqrt{\omega^2 - U_0^2}}{\omega^2 - U_0^2 \cos^2(\omega L)} + \frac{1}{2} \frac{\sqrt{\omega^2 - U_0^2}}{\omega^2 - U_0^2 \sin^2(\omega L)} - \frac{1}{\omega} \right] \omega^2 \quad (5.2b)$$

In the exterior region and low-energy sector, the quantum perturbation's REVMTO is indistinguishable from that of a *dust cosmic fluid* in its locally inertial rest frame with an exponentially decaying proper energy density:

$$\langle \hat{\rho} \rangle_d^{\text{low}}(|x| > L) = \frac{\hbar}{2} \sum_{s=\pm} \sum_{n=1}^{N^s} \frac{\omega_n^s}{L + \frac{1}{\sqrt{(U_0)^2 - (\omega_n^s)^2}}} e^{-2\sqrt{(U_0)^2 - (\omega_n^s)^2}(|x| - L)} \quad (5.3)$$

The exterior REVMTO also contains a state-independent term that currently dominates the asymptotic region and is induced by the non-zero external potential and the renormalization length scale. Its structure is that of a *vacuum energy cosmic fluid* whose proper energy density is given by:

$$\langle \hat{\rho} \rangle_\Lambda = -\frac{\hbar U_0^2}{8\pi} \left[1 - \ln \left(\frac{l_0^2}{l^2} \right) \right] \quad (5.4)$$

Furthermore, the REVMTO is not continuous across the surfaces $|x| = L$. The discontinuity in the low-energy sector is, however, finite and quantified by:

$$\Delta \langle \hat{\rho} \rangle^{\text{low}} \Big|_{|x|=L} = \langle \hat{\rho} \rangle_Z^{\text{low}} \quad (5.5)$$

The most notable feature of eq. (5.1) is the emergence of effective perfect-fluid structures from the scalar quantum perturbation, where the pressures follow directly from the corresponding cosmic fluid equations of state. Furthermore, the low-energy contributions in both the interior and exterior are UV finite, whereas the high-energy interior contribution diverges if the Hadamard parametrix is not subtracted accordingly.

5.3 Future Directions

We have determined the interior REVMTO for both low- and high-energy sectors, but only the low-energy and state-independent contributions in the exterior region. The immediate extension of this work is then to compute the high-energy normal modes' contribution to the exterior REVMTO. This will complete the finite-well model across all energy sectors and regions, and will allow us to answer several open questions:

1. Does the exterior high-energy contribution ensure the REVMTO's regularity across the finite potential well's surfaces without introducing spurious divergences due to the potential's discontinuity?
2. Does the vacuum fluctuation contribution of eq. (5.4) survive in the complete exterior REVMTO, or is it canceled by a complementary term emerging from the high-energy normal modes? This is particularly relevant because this contribution currently dominates the asymptotic region and determines the sign of the energy density.
3. All energy conditions are satisfied in the interior region provided $\langle \hat{\rho} \rangle_Z^{\text{high}} \geq -\langle \hat{\rho} \rangle_Z^{\text{low}}$. Whether this feature holds in the exterior region can only be properly answered once the high-energy contribution is known.
4. The exterior high-energy contribution is also necessary to determine the limiting behavior of the complete model, such as the expected free theory limit $U_0 \rightarrow 0$, the ultra-compact limit $L \rightarrow 0$, and the infinite-well limit $U_0 \rightarrow \infty$. The simultaneous limits $L \rightarrow 0, U_0 \rightarrow \infty$ are of particular interest as toy models for the interior of highly compact objects such as black holes.

Once the $(1+1)$ -dimensional auxiliary theory is complete, the natural continuation is to compute the corresponding D -dimensional propagator, REVMTO, and ultimately solve the semiclassical EFE to determine the backreacted spacetime. Of course, this is easier said than done. A more approachable continuation is to solve the auxiliary theory using increasingly more accurate approximations of the SSS effective potential, in particular by restoring the radial and angular-momentum dependence. This would provide a crucial insight into whether the effective perfect fluid structures identified in our results are genuine features of quantized scalar perturbations in SSS spacetimes, or instead artifacts of the idealized infinite- and finite-well models.

Appendix A

Distributions

We briefly introduce the theory of distributions and the identities required for some proofs in chapter 4.

Definition A.1 Distribution

Let $\mathcal{D}(\mathbb{R}^n)$ denote the space of smooth functions with compact support^a on $\mathbb{R}^n$. A **distribution** T on $\mathbb{R}^n$ is a continuous linear functional on $\mathcal{D}(\mathbb{R}^n)$, namely:

$$T : \mathcal{D}(\mathbb{R}^n) \mapsto \mathbb{R} \quad (\text{A.1})$$

Let the space of distributions be denoted by $\mathcal{D}'(\mathbb{R}^n)$. The action of a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ on a test function $f \in \mathcal{D}(\mathbb{R}^n)$ is denoted by $T[f] \in \mathbb{R}$.

^aThe support of a function $f : \mathcal{X} \mapsto \mathbb{R}$ is the smallest closed subset $\mathcal{A} \subseteq \mathcal{X}$ containing all points not mapped to zero.

Outside of their domain, the functions $f \in \mathcal{D}(\mathbb{R}^n)$ are vanishing, and integrals inside the domain are finite.

Definition A.2 Distributional Derivative

The **distributional derivative** of $T \in \mathcal{D}'(\mathbb{R}^n)$ in the direction x_j is the distribution $\partial_j T \in \mathcal{D}'(\mathbb{R}^n)$ defined $\forall f \in \mathcal{D}(\mathbb{R}^n)$ as:

$$(\partial_j T)[f] = -T[\partial_j f] \quad (\text{A.2})$$

This definition can be justified by analogy with integration by parts, keeping in mind that test functions vanish at the boundary.

Definition A.3 Dirac Delta Distribution

The **Dirac delta** δ is the distribution defined $\forall f \in \mathcal{D}(\mathbb{R}^n)$ as:

$$\delta[f] = f(0) \quad (\text{A.3})$$

Every locally integrable function $g \in L(\mathbb{R}^n)$ defines a “tempered” distribution through:

$$T_g[f] = \int_{\mathbb{R}^n} dx g(x) f(x), \quad \forall f \in \mathcal{D}(\mathbb{R}^n) \quad (\text{A.4})$$

Definition A.3 is one of the distributions we shall encounter the most during the Hadamard renormalization procedure. In one dimension:

$$\delta[f] = \int_{-\infty}^{+\infty} dx \delta(x) f(x) = f(0)$$

The notation $\delta(x)$ should be understood distributionally rather than as an ordinary function.

Definition A.4 Heaviside Step Function

The **Heaviside step function** $\Theta(x)$ is the piecewise function defined on $\mathbb{R}$ as:

$$\Theta(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & x = 0 \\ 1, & x > 0 \end{cases} \quad (\text{A.5})$$

One can define the associated **Heaviside distribution** Θ as the one whose integration kernel corresponds to the Heaviside step function:

$$\Theta[f] = \int_{-\infty}^{+\infty} dx \Theta(x) f(x) = \int_0^{\infty} dx f(x) \quad (\text{A.6})$$

The kernel $\Theta(x)$ is discontinuous at $x = 0$ and therefore not differentiable in the ordinary sense. Having introduced the Heaviside distribution $\Theta[f]$, differentiation is now well-defined, but only in the distributional sense.

Proposition A.1 Heaviside Derivative

If $f \in \mathcal{D}(\mathbb{R})$, then:

$$\frac{d}{dx} \Theta[f] = \delta[f] \quad (\text{A.7})$$

Proof. Using definition A.2,

$$\frac{d}{dx} \Theta[f] = -\Theta \left[\frac{d}{dx} f \right] = - \int_0^{\infty} dx \frac{df}{dx} = f(0) - f(\infty) = f(0) = \delta[f]$$

where we have exploited the fact that test functions vanish at infinity (the boundary), $f(\infty) = 0$. □

Definition A.5 Absolute Value & Sign Functions

The **absolute value** $|x|$ of $x \in \mathbb{R}$ is the piecewise function defined as:

$$|x| := \begin{cases} +x, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -x, & \text{if } x < 0 \end{cases} \quad (\text{A.8})$$

The **sign function** $\text{sgn}(x)$ is the piecewise function of $x \in \mathbb{R}$ defined as:

$$\text{sgn}(x) := \begin{cases} +1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases} \quad (\text{A.9})$$

In 1D, the associated distributions are:

$$\text{AV}[f] = \int_{-\infty}^{+\infty} dx |x| f(x) = - \int_{-\infty}^0 dx x f(x) + \int_0^{+\infty} dx x f(x) \quad (\text{A.10a})$$

$$\text{SGN}[f] = \int_{-\infty}^{+\infty} dx \text{sgn}(x) f(x) = - \int_{-\infty}^0 dx f(x) + \int_0^{+\infty} dx f(x) \quad (\text{A.10b})$$

Having specified the value $\Theta(x=0) = 1/2$, then we can write:

$$\text{sgn}(x) = 2\Theta(x) - 1 \quad (\text{A.11})$$

Furthermore:

$$|x| = x \text{sgn}(x) = 2x\Theta(x) - x \quad (\text{A.12})$$

The differentiation of definition A.5 at $x = 0$ is well-defined only for the associated distributions and via these identities.

Proposition A.2 Absolute Value Derivatives

As distributions:

$$\frac{d}{dx} |x| = \text{sgn}(x) \quad (\text{A.13a})$$

$$\frac{d^2}{dx^2} |x| = \frac{d}{dx} \text{sgn}(x) = 2\delta(x) \quad (\text{A.13b})$$

Proof. The first result follows from the distributional differentiation of eq. (A.12),

$$\frac{d}{dx} |x| = \text{sgn}(x) + x \frac{d}{dx} \text{sgn}(x) = \text{sgn}(x) + x \frac{d}{dx} [2\Theta(x) - 1] = \text{sgn}(x) + 2x\delta(x) = \text{sgn}(x)$$

where we have used eq. (A.11) and proposition A.1, and also:

$$(x\delta)[f] = \delta[xf] = xf(x)|_{x=0} = 0$$

The second follows from differentiating once more and exploiting the same identities. $\square$

Proposition A.3 Delta Derivatives

As a distribution and for $a > 0$, we have:

$$x \frac{d}{dx} \delta(x) = -\delta(x) \quad (\text{A.14a})$$

$$\frac{d^n}{dx^n} \delta(ax) = a^{-(n+1)} \frac{d^n}{dx^n} \delta(x) \quad (\text{A.14b})$$

Proof. For the first one:

$$x \frac{d\delta}{dx}[f] = \frac{d\delta}{dx}[xf] = -\delta \left[\frac{d}{dx}(xf) \right] = -\delta[f] - \delta \left[x \frac{df}{dx} \right] = -\delta[f]$$

The second follows recursively. $\square$

Definition A.6 Cauchy Principal Value

The **Cauchy principal value** of $1/x$ is defined as the distribution $\mathcal{P} \left(\frac{1}{x} \right) \in \mathcal{D}'(\mathbb{R})$ for which:

$$\mathcal{P} \left(\frac{1}{x} \right) [f] := \lim_{\epsilon \rightarrow 0^+} \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) dx \frac{1}{x} f(x) \quad (\text{A.15})$$

For higher powers, we use the same notation $\mathcal{P} (1/x^n)$ for the corresponding distributions which recover definition A.6 for $n = 1$ and are given recursively by the properties:

$$\frac{d}{dx} \mathcal{P} \left(\frac{1}{x^n} \right) = -n \mathcal{P} \left(\frac{1}{x^{n+1}} \right) \quad n \geq 1 \quad (\text{A.16a})$$

$$x^m \mathcal{P} \left(\frac{1}{x^n} \right) = \mathcal{P} \left(\frac{1}{x^{n-m}} \right), \quad 0 \leq m < n \quad (\text{A.16b})$$

For a concrete definition of these distributions and the proof of the following theorem, cf. [13].

Theorem A.1 Sokhotski-Plemelj Theorem

We have the following distributional identities:

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{x \pm i\epsilon} = \mathcal{P} \left(\frac{1}{x} \right) \mp i\pi \delta(x) \quad (\text{A.17a})$$

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{(x \pm i\epsilon)^2} = \mathcal{P} \left(\frac{1}{x^2} \right) \pm i\pi \frac{d}{dx} \delta(x) \quad (\text{A.17b})$$

We will be mostly interested in theorem A.1 in order to find the integral representation of definitions A.3 and A.6.

Proposition A.4 Distributional Integral Representations I

In the limit $\epsilon \rightarrow 0^+$, we have the following distributional identities:

$$\lim_{\epsilon \rightarrow 0^+} \int_0^\infty dq e^{iq(x+i\epsilon)} = \pi\delta(x) + i\mathcal{P} \left(\frac{1}{x} \right) \quad (\text{A.18a})$$

$$\lim_{\epsilon \rightarrow 0^+} \int_0^\infty dq \cos(xq) e^{-\epsilon q} = \pi\delta(x) \quad (\text{A.18b})$$

$$\lim_{\epsilon \rightarrow 0^+} \int_0^\infty dq \sin(xq) e^{-\epsilon q} = \mathcal{P} \left(\frac{1}{x} \right) \quad (\text{A.18c})$$

Proof. The damping term $e^{-\epsilon q}$ is the regulator that guarantees the convergence for $\epsilon > 0$ of the integrals. Performing the integration and then using theorem A.1:

$$\lim_{\epsilon \rightarrow 0^+} \int_0^\infty dq e^{iq(x+i\epsilon)} = \lim_{\epsilon \rightarrow 0^+} \left. \frac{e^{q(ix-\epsilon)}}{ix - \epsilon} \right|_{q \rightarrow 0}^{q \rightarrow \infty} = \lim_{\epsilon \rightarrow 0^+} \frac{i}{x + i\epsilon} = i[\mathcal{P}(1/x) - i\pi\delta(x)]$$

Decomposing $e^{iq(x+i\epsilon)} = e^{-\epsilon q} (\cos xq + i \sin xq)$, the proof is completed by taking the real and imaginary parts. $\square$

Proposition A.5 Distributional Integral Representations II

In the limit $\epsilon \rightarrow 0^+$, we have the following distributional identities:

$$\lim_{\epsilon \rightarrow 0^+} \int_0^\infty dq q e^{iq(ix-\epsilon)} = -\mathcal{P} \left(\frac{1}{x^2} \right) - i\pi \frac{d}{dx} \delta(x) \quad (\text{A.19a})$$

$$\lim_{\epsilon \rightarrow 0^+} \int_0^\infty dq q \cos(xq) e^{-\epsilon q} = -\mathcal{P} \left(\frac{1}{x^2} \right) \quad (\text{A.19b})$$

$$\lim_{\epsilon \rightarrow 0^+} \int_0^\infty dq q \sin(xq) e^{-\epsilon q} = -\pi \frac{d}{dx} \delta(x) \quad (\text{A.19c})$$

Proof. Since the regulator $e^{-\epsilon q}$ guarantees convergence for $\epsilon > 0$, we may commute derivatives with the integral. Differentiating proposition A.4, suppressing the $\epsilon \rightarrow 0^+$ limit for simplicity, and using eq. (A.16), we then have:

$$\partial_x \int_0^\infty dq e^{q(ix-\epsilon)} = \int_0^\infty dq i q e^{q(ix-\epsilon)} = \pi \partial_x \delta(x) - i \mathcal{P} \left(\frac{1}{x^2} \right) = i \left[-\mathcal{P} \left(\frac{1}{x^2} \right) - i \pi \partial_x \delta(x) \right]$$

Decomposing into $i q e^{q(ix-\epsilon)} = q e^{-\epsilon q} (-\sin qx + i \cos qx)$ and taking the real and imaginary parts completes the proof. $\square$

Proposition A.6 Distributional Integral Representations III

In the limit $\epsilon \rightarrow 0^+$, we have the following distributional identities:

$$\lim_{\epsilon \rightarrow 0^+} \ln(x \pm i\epsilon) = - \lim_{\epsilon \rightarrow 0^+} \int_0^\infty dq \frac{e^{\pm i q (x \pm i\epsilon)}}{q} + \mathcal{O}(1) \quad (\text{A.20a})$$

$$\lim_{\epsilon \rightarrow 0^+} \ln(x^2 + i\epsilon) = -2 \lim_{\epsilon \rightarrow 0^+} \int_0^\infty dq \frac{e^{-\epsilon q} \cos(xq)}{q} + \mathcal{O}(1) \quad (\text{A.20b})$$

Here, $\mathcal{O}(1)$ is an x -independent constant required to regulate the infrared divergence of the integral representation.

Proof. (a) First, we notice that:

$$\lim_{\epsilon \rightarrow 0} \partial_x \ln(x \pm i\epsilon) = \lim_{\epsilon \rightarrow 0} \frac{1}{x \pm i\epsilon} = \mp i \lim_{\epsilon \rightarrow 0} \int_0^\infty dq e^{\pm i q (x \pm i\epsilon)}$$

Integrating over x , we have:

$$\lim_{\epsilon \rightarrow 0} \ln(x \pm i\epsilon) = \mp i \lim_{\epsilon \rightarrow 0} \int_0^\infty dq \int dx e^{\pm i q (x \pm i\epsilon)} = - \lim_{\epsilon \rightarrow 0} \int_0^\infty dq \frac{e^{\pm i q (x \pm i\epsilon)}}{q} + \mathcal{O}(1)$$

The convergence of the integrand guarantees that the integrals can be swapped. (b) For the second one, using proposition A.4:

$$\lim_{\epsilon \rightarrow 0} \partial_x \ln(x^2 + i\epsilon) = \lim_{\epsilon \rightarrow 0} \frac{2x}{x^2 + i\epsilon} = 2 \mathcal{P} \left(\frac{1}{x} \right) = 2 \lim_{\epsilon \rightarrow 0} \int_0^\infty dq e^{-\epsilon q} \sin(xq)$$

Integrating over x we get the desired result. $\square$

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