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On the birational classification of the moduli space of curves

Tesi di Laurea in Geometria Algebrica

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Guidance by abstract ideas
is a dangerous business
when not controlled by
strong personal relations.

Paul Feyerabend

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Introduction

The classification of proper algebraic curves is a problem rooted in the works of Riemann in the nineteenth century, who was the first to understand that complex algebraic curves of a given topological genus g are not all equivalent to each other, but depend on a certain number of continuous parameters, which he himself called *moduli.* In particular, for curves of genus $g \geq 2$, he computed that the number of these parameters, namely the degrees of freedom required to uniquely determine the isomorphism class of such a curve, is exactly $3g - 3$.
From this intuition, algebraic geometers have tried for decades to formalise the idea of a *moduli space* in which every single point corresponds to a specific isomorphism class of a smooth curve of genus g. If we manage to endow the set of all curves with the structure of an algebraic variety, we can use the geometric tools to study the universe of curves as a whole. However, formalising this concept hides profound obstacles, particularly related to the existence of curves with a non-trivial automorphism group, which prevent the existence of a *fine moduli space* in the classical sense of scheme theory, i.e., a scheme encoding all the information on this classification problem.

The Deligne-Mumford compactification

Even assuming the construction of the space $\mathcal{M}_g$ parametrising smooth curves of genus g, a second fundamental problem quickly emerges: this space is not proper (the algebro-geometric equivalent to being compact and separated). Geometrically, this means that a family of smooth curves can degenerate, in the limit, to a singular curve. Since $\mathcal{M}_g$ parametrises only smooth curves, the limit point does not belong to the space. In order to study the global properties of $\mathcal{M}_g$, and in particular to apply the tools of birational geometry, we need to work with a proper space. However, including arbitrary singular curves would result in losing the separatedness property.
The first solution to this problem was introduced in 1969 by Deligne and Mumford in their famous article [DM69]. The key idea was to include only a very special class of degenerations: *stable curves.* A proper curve is stable if its only singularities are *nodes*, i.e., étale locally we see $xy = 0$, and if it has a finite automorphism

group. By adding these “limit” curves to the original space, we obtain the moduli space $\overline{\mathcal{M}}_g$. The strength of this construction lies in the Stable Reduction Theorem, which guarantees that every family of smooth curves admits a unique degeneration to a stable curve, making $\overline{\mathcal{M}}_g$ a proper space.

Enlarging the algebraic category

As previously mentioned, the existence of automorphisms makes it impossible to treat $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ as schemes. For this reason, the modern language used to approach the problem is that of *stacks*, introduced by Grothendieck in his seminal works, FGAs and SGAs. Stacks allow us to keep track of the automorphism information at every point, resolving the formal pathologies and enabling us to do geometry in a rigorous way.

To study the local properties of a stack coming from moduli problems, e.g., to study its smoothness and compute the dimension of its tangent space at a point, we rely on *Deformation Theory.* In our case, this machinery explains what happens when we infinitesimally “perturb” a stable curve. Thanks to this theory, we can confirm Riemann’s intuition and formally prove that the dimension of the space $\overline{\mathcal{M}}_g$ is exactly $3g-3$.

Algebraic spaces form the simplest class of stacks. They are a slight generalisation of schemes, being in fact schemes over a dense subset. When discussing stable curves in [DM69], Deligne and Mumford also introduced a suitable class of stacks, nowadays called *Deligne-Mumford stacks*, having in particular finite automorphism group at every point. It turns out that a DM stack has a very handy structure. More precisely, even if it is not a scheme, under suitable assumptions, the Keel-Mori Theorem implies the existence of the *coarse moduli space*, i.e., an algebraic space that in some sense best approximates this stack, thereby allowing us to deduce some properties of the stack by studying its coarse moduli space. Furthermore, the coarse moduli space of a smooth DM stack has finite quotient singularities, i.e., locally they are quotients of an affine space by a finite group.

Birational classification

Once the compactified stack $\overline{\mathcal{M}}_g$ has been constructed, one shows that its coarse moduli space $\overline{M}_g$ is even projective, implying that it is not only an algebraic space but a scheme. The ultimate goal of the thesis is to explore the global nature of the moduli space of stable curves $\overline{M}_g$ over $\mathbb{C}$ from the perspective of *birational geometry.* In algebraic geometry, two varieties are birationally equivalent if they contain isomorphic dense open subsets. It is a less rigid form of classification compared to strict isomorphism, but extremely powerful. The most important invariant used in birational geometry is the *Kodaira dimension.* Intuitively, the

Kodaira dimension measures how "curved" a variety is. A Kodaira dimension of $-\infty$ indicates spaces that resemble, in a sense, projective space, being then positively curved. Conversely, if the Kodaira dimension reaches its maximum possible value, i.e., the dimension of the variety itself, the variety is said to be *of general type*, which implies that the geometry of the space is extremely rigid and constrained, and in this case the curvature is negative.
Historically, Severi thought that $\mathcal{M}_g$ had always Kodaira dimension $-\infty$ for every g. It was a groundbreaking result by Harris, Mumford, and later Eisenbud to show that, actually, for a sufficiently large g, $\overline{\mathcal{M}}_g$ is a variety of general type [HM82; EH87]. This drastic change in behaviour was proved by studying the intersection ring of the moduli space above, finding in particular the relations between natural classes, such as the Hodge bundle and the boundary divisors, being the loci parametrising singular stable curves, and the canonical class.

Structure of the thesis

The work is structured to gradually provide the reader with all the technical and conceptual tools necessary to tackle the problem. More specifically, the thesis is structured as follows.

Firstly, Chapter 1 serves as a technical introduction, recalling the notion of a moduli problem and the fundamental concepts of birational geometry, with a particular focus on the Kodaira dimension.

Chapter 2 is dedicated to the objects we aim to parametrise: algebraic curves. Starting from the basic properties of smooth curves, we introduce nodal singularities and formally define the stability condition.

In Chapter 3, we discuss deformation theory, which is the essential tool for the local study of moduli spaces. In particular, we discuss infinitesimal deformations, and the construction of the miniversal family.

Chapter 4 contains all the necessary notions of the language of stack theory. Starting from the concept of a Grothendieck topology, we define stacks, and discuss the algebraicity condition, ultimately focusing on Deligne-Mumford stacks.

Finally, in Chapter 5, all elements above merge in the study of the moduli space of stable curves. We discuss its global properties such as properness and projectivity, and we compute its Picard group by describing the classes of the boundary divisors and the Hodge bundle. This ultimately leads to the technical core of this thesis, the computation of its Kodaira dimension carried out by Harris and Mumford in their article [HM82].

Chapter 1

Preliminaries

This thesis focuses on the birational study of the moduli space of stable curves of given genus. We begin by discussing what a moduli problem is and how we formalise it, and then we briefly talk about birational geometry.

1.1 Moduli problems

A moduli problem essentially is a classification problem, i.e., we want to classify the isomorphism classes of mathematical objects of a given type. The objects we want to study are smooth projective curves of genus g, and in order to do this we will generalise our study to stable curves, which we will introduce in the next chapter.

The aim of moduli theory is to find a suitable *moduli space*, namely an object in the category in which we are working whose points are in natural bijection with isomorphisms classes of objects we are considering. One of the main challenges is formalising precisely what this means. The usual approach is to define the notion of family of objects over an arbitrary scheme S and require that every family over S corresponds uniquely to a map from S to the moduli space.
We are focused on working with algebraic structures, hence we look for a moduli space with the structure of an algebraic variety, and ideally a projective one.

The functorial approach

The first attempt to formalise a moduli problem in the algebro-geometric context is to define a *moduli functor*, namely a functor

$$F\colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Sets}$$

sending a scheme S to the set of isomorphism classes of families of objects over S, and for a map $f\colon S \to T$, its image $F(f)\colon F(T) \to F(S)$ is often called the *pullback* map and simply denoted f^*.

This approach allows us to provide a more precise formulation of a moduli space M as a scheme *representing* the functor F, i.e., there is a natural isomorphism between F and $h_M := \mathrm{Mor}_{\mathrm{Sch}}(-, M)$. If this happens, we call M a *fine moduli space*. Note that if such a space exists, a family of objects $\mathcal{C}$ over a scheme S defines a set-theoretic map $S \to M$ such that $s \mapsto [\mathcal{C}_s]$, where $\mathcal{C}_s$ is the pullback of $\mathcal{C}$ under the inclusion $\{s\} \hookrightarrow S$. We desire that the map $S \to M$ is algebraic.

We will study objects by studying maps to it. This is essentially a corollary to the following simple but crucial result.

Lemma 1.1.1 (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a category and X be an object. For every functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$, the function*

$$\mathrm{Mor}(h_X, F) \to F(X) \quad \textit{defined by} \quad \alpha \mapsto \alpha_X(\mathrm{id}_X)$$

is bijective and functorial with respect to both X and F, where the left hand side denotes the set of natural transformations $h_X \to F$ and α_X denotes the induced function $h_X(X) = \mathrm{Mor}_{\mathcal{C}}(X, X) \to F(X)$.

Proof. Define the map $F(X) \to \mathrm{Mor}(h_X, F)$ given by

$$F(X) \ni \eta \mapsto \left\{ \begin{aligned} \theta_Y\colon h_X(Y) &\to F(Y) \\ f &\mapsto F(f)(\eta) \end{aligned} \right\}_{Y \in \mathrm{ob}\,\mathcal{C}} \in \mathrm{Mor}(h_X, F).$$

It is easy to check that this defines an inverse for the map in the statement, and functoriality is easily verified as well. □

Definition 1.1.2. We say that a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ is *representable* if there exists an object X in $\mathcal{C}$ and an isomorphism of functors $F \xrightarrow{\sim} h_X$. If this is the case, we say that F is *represented by X*.

Remark. If the functor F is represented by an object X, then by the Yoneda Lemma X is unique up to unique isomorphism.

Example. By [Har77, Thm II.7.1], we can see the projective space $\mathbb{P}^n_{\mathbb{Z}}$ as the scheme representing the functor $\mathrm{Sch}^{\mathrm{op}} \to \mathrm{Sets}$ defined by

$$S \mapsto \left\{ (L, s_0, \dots, s_n) \;\middle|\; \begin{array}{l} L \text{ line bundle on } S \text{ globally} \\ \text{generated by } s_0, \dots, s_n \in \Gamma(S, L) \end{array} \right\} \Big/ \sim,$$

where $(L, s_i) \sim (L', s_i')$ if there exists an isomorphism $\alpha\colon L \to L'$ such that $s_i = \alpha^* s_i'$ for all i.

Definition 1.1.3. If $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ is a functor represented by an object X in $\mathcal{C}$ via an isomorphism $\alpha\colon F \xrightarrow{\sim} h_X$, then the object $u \in F(X)$ corresponding under α to the identity $\mathrm{id}_X \in h_X(X)$ is called the *universal element* of F.

If u is the universal element of a functor F represented by X, and $a \in F(T)$ is an object over a scheme T corresponding to a map $f_a\colon T \to X$, then a is the pullback of u under f_a, namely $a = F(f_a)(u) = f_a^*(u)$.

Obstacles to representability

There may be different things interfering with the representability of a moduli functor, and consequently with the existence of a fine moduli space. Our main problem will be dealing with curves with non-trivial automorphism group, and we will see that in the next chapter.
The following result gives us a necessary condition for a moduli functor of schemes over $\mathbb{C}$ to be representable.

Proposition 1.1.4. *Let $F\colon\ \mathrm{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{Sets}$ be a moduli functor. Suppose that there is a family of objects $\mathcal{E}$ over a variety S such that*

(i) the fibres $\mathcal{E}_s$ are isomorphic for $s \in S(\mathbb{C})$, and

(ii) the family $\mathcal{E}$ is non-trivial, i.e., is not equal to the pullback of an object $E \in F(\mathbb{C})$ along the structure map $S \to \operatorname{Spec}\mathbb{C}$,

then F is not representable.

Proof. Suppose for a contradiction that F is represented by a $\mathbb{C}$-scheme X, hence the family $\mathcal{E}$ corresponds to a unique morphism $f\colon S \to X$.
The first condition implies that the fibre $E := \mathcal{E}_s$ is independent of the point $s \in S(\mathbb{C})$ and thus the compositions

$$\operatorname{Spec}\mathbb{C} = \operatorname{Spec}\kappa(s) \hookrightarrow S \xrightarrow{f} X$$

define a unique point $x \in X(\mathbb{C})$. We claim that the induced homomorphism

$$f_x^{\#}\colon \mathcal{O}_{X,x} \to (f_*\mathcal{O}_S)_x = \Gamma(S, \mathcal{O}_S)$$

factors through $\mathcal{O}_{X,x}/\mathfrak{m}_x \simeq \mathbb{C}$. Indeed, the image of a section in $\mathfrak{m}_x$ is a global section $t \in \Gamma(S, \mathcal{O}_S)$ such that $t_s \in \mathfrak{m}_s$ for every $s \in S(\mathbb{C})$. On an affine open $U \subseteq S$ where $U = \operatorname{Spec} A$, this means that

$$t|_U \in \bigcap_{s \in U} \mathfrak{m}_s \subseteq A,$$

and since S is a variety, we get

$$\bigcap_{s\in U} \mathfrak{m}_s = \mathrm{Jac}(A) = \mathrm{Nil}(A) = 0,$$

hence that $t|_U = 0$ for every affine open, implying that $t = 0$.
Therefore, f factors as $S \to \operatorname{Spec}\mathbb{C} \xrightarrow{x} X$, and this implies that the family $\mathcal{E}$ is the pullback under the constant map $S \to \operatorname{Spec}\mathbb{C} \xrightarrow{x} X$, namely it is a trivial family, which contradicts the second condition. □

1.2 Birational geometry

Rational maps are an important tool for the classification of varieties. The idea of birational geometry is to classify varieties with respect to an equivalence relation weaker than being isomorphic. Essentially, two varieties are birational if there is an isomorphism between two open Zariski subsets of theirs.

More precisely, recall that a *rational map* $f : X \dashrightarrow Y$ between two varieties is an equivalence class of pairs (U, f_U) where $U \subseteq X$ is a nonempty open and $f_U : U \to Y$ is a morphism. Two pairs (U, f_U) and (V, f_V) are equivalent if there exists a nonempty open $W \subseteq U \cap V$ such that $f_U|_W = f_V|_W$. The rational map f is said to be *dominant* if for some (or equivalently every) pair (U, f_U), the image of f_U is dense in Y.
A *birational map* is a rational map $f : X \dashrightarrow Y$ which admits an inverse, i.e., a rational map $g : Y \dashrightarrow X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$ as rational maps. In this case, we say that X and Y are *birational*.

In particular, when X is birational to $\mathbb{P}^n$ for some n, we say that X is *rational*.

Kodaira dimension

When studying the birational geometry of a projective variety, a useful tool is the Kodaira dimension. Here we briefly discuss the nature and properties of this important birational invariant.

In the following, X will denote unless otherwise stated an irreducible projective variety.
Let L be a line bundle on X. The *semigroup* of L is

$$\mathbf{N}(L) \coloneqq \{m \geq 0 \mid H^0(X, L^{\otimes m}) \neq 0\}.$$

Given $m \in \mathbf{N}(L)$, we have a rational mapping

$$\phi_m : X \dashrightarrow \mathbb{P}(L^{\otimes m})$$

associated to the complete linear series $|L^{\otimes m}|$; see [Laz04, §1.1.B]. We denote by Y_m the closure of $\phi_m(X)$.

Clearly, these definitions and further ones can be generalised for a Cartier divisor on X by considering $\mathcal{O}_X(D)$.

Definition 1.2.1. Assume X to be normal. The *Iitaka dimension* of a line bundle L on X is defined as

$$\kappa(L) \coloneqq \max_{m\in\mathbf{N}(L)}\{\dim Y_m\}$$

if $\mathbf{N}(L) \neq \{0\}$. Otherwise, we set $\kappa(L) \coloneqq -\infty$.
If X is not normal, we pass to its normalisation $\nu\colon X' \to X$ and define

$$\kappa(L) \coloneqq \kappa(\nu^* L).$$

As the following result states, Iitaka dimension can be also viewed as the order of growth of sections.

Proposition 1.2.2. *Let X be an irreducible normal projective variety, and let L be a line bundle on X. Then there are constants $a, A > 0$ such that*

$$am^\kappa \leq h^0(X, L^{\otimes m}) \leq Am^\kappa$$

where $\kappa = \kappa(L)$, for all sufficiently large $m \in \mathbf{N}(L)$.

Proof. See [Laz04, Cor. 2.1.37]. □

For a smooth projective variety X, we define its *Kodaira dimension* as the Iitaka dimension of its canonical bundle

$$\kappa(X) \coloneqq \kappa(\omega_X)$$

where $\omega_X = \det \Omega_X$. Recall that ω_X is also the dualizing sheaf [Har77, Cor. 7.12]. The Kodaira dimension of a singular projective variety X is defined to be the Kodaira dimension of any *smooth model*, i.e., a smooth projective variety X' birational to X.

Recall that, by Proposition 1.2.2, we can view the Kodaira dimension as the order of growth of $h^0(X, \omega_X)$.

Proposition 1.2.3. *The Kodaira dimension is a birational invariant.*

Proof. If X and Y are two birational smooth projective varieties, then a classical result states that $h^0(X, \omega_X) = h^0(Y, \omega_Y)$; see [Har77, Thm II.8.19]. The proposition then follows immediately. If X and Y are not smooth, then their smooth models are birational as well, hence we conclude the proof. □

Note that the result above also shows that the Kodaira dimension of singular projective variety is well-defined.

Note that by definition,

$$\kappa(L) \in \{-\infty, 0, \ldots, \dim X\}.$$

A line bundle L is said to be *big* if $\kappa(L) = \dim X$. A smooth projective variety X is called *of general type* if ω_X is big.

Example (Kodaira dimension of smooth curves). Let C be a smooth connected projective curve over a field k, i.e., a smooth connected projective variety of pure dimension 1. By Riemann-Roch and Serre Duality ([Har77, Thm IV.1.3 and Cor. III.7.12])

$$h^0(C, \omega_C^{\otimes m}) - h^0(C, (\omega_C^{\otimes m-1})^\vee) = m(2g-2) + 1 - g = (2m-1)(g-1)$$

where $g = h^0(C, \omega_C)$ is the arithmetic genus of C. We now distinguish three cases:

- if $g = 0$, then $\deg(\omega_C^{\otimes m}) < 0$ for all $m > 0$, and this implies $h^0(C, \omega_C^{\otimes m}) = 0$ (see [Har77, Lemma IV.1.2]);
- if $g = 1$, then $\omega_C = \mathcal{O}_C$, hence $h^0(C, \omega_C^{\otimes m}) = 1$ for all $m > 0$; and
- if $g \geq 2$, then $\deg((\omega_C^{\otimes m-1})^\vee) < 0$ for all $m > 0$, and together with the formula above, this implies $h^0(C, \omega_C^{\otimes m}) = (2m-1)(g-1)$.

This shows that the Kodaira dimension of a smooth connected projective curve C depends only on its genus g as follows.

g	$\kappa(C)$
0	$-\infty$
1	0
≥ 2	1

In particular, smooth curves of genus $g \geq 2$ are of general type.

Characterisation of bigness

Lemma 1.2.4. *Let X be an irreducible projective variety. A Cartier divisor D on X is big if and only if there is a constant $C > 0$ such that*

$$h^0(X, \mathcal{O}_X(mD)) \geq Cm^{\dim X}$$

for all sufficiently large $m \in \mathbf{N}(D)$.

Proof. See [Laz04, Lemma 2.2.3]. □

Proposition 1.2.5 (Kodaira's Lemma). *Let D be a big Cartier divisor and F an effective Cartier divisor on X. Then*

$$H^0(X, \mathcal{O}_X(mD - F)) \neq 0$$

for all sufficiently large $m \in \mathbf{N}(D)$. In particular, for those m the divisor $mD - F$ is effective.

Proof. Let $\dim X = n$, and consider the exact sequence

$$0 \to \mathcal{O}_X(mD - F) \to \mathcal{O}_X(mD) \to \mathcal{O}_F(mD) \to 0.$$

Since D is big, there is $C > 0$ such that $h^0(X, \mathcal{O}_X(mD)) \geq Cm^n$ for all sufficiently large $m \in \mathbf{N}(D)$. On the other hand, since F is a closed subscheme of codimension 1, the growth of $h^0(F, \mathcal{O}_F(mD))$ is at most like m^{n-1}. Hence, for large $m \in \mathbf{N}(D)$

$$h^0(X, \mathcal{O}_X(mD)) > h^0(F, \mathcal{O}_F(mD)),$$

and we conclude the proof by exploiting the sequence above. □

Kodaira's Lemma leads us to an important characterisation of big divisors.

Corollary 1.2.6 (Big = Ample + Effective). *Let D be a divisor an an irreducible projective variety X. Then the following are equivalent:*

(1) D is big.

(2) For any ample divisor H on X, there exist a positive integer $m > 0$ and an effective divisor E on X such that $mD \equiv_{\text{lin}} H + E$.

(3) There exist a positive integer $m > 0$, an ample divisor H, and an effective divisor E such that $mD \equiv_{\text{lin}} H + E$.

Proof. For simplicity, we simply denote linear equivalence as an equality. Assuming (1), since H is ample, there exists an $r \gg 0$ such that rH and $(r+1)H$ are effective. Apply Kodaira's Lemma (Proposition 1.2.5) with $F = (r+1)H$ to find a positive integer m and an effective divisor N such that

$$mD = (r+1)H + N = H + (rH + N).$$

Calling $E = rH + N$ proves (2). The implication (2) $\implies$ (3) is obvious. Finally, assuming (3), we get $mD - E$ ample. After possibly pass to a larger multiple of D, we can then assume that $mD = H' + E'$, where H' is very ample and E' is effective. By effectiveness and ampleness, we get $\kappa(D) \geq \kappa(H) = \dim X$, hence D is big. □

Chapter 2

Curves

In this chapter, we do a quick review of smooth curves, discuss the theory of nodal curves, and finally characterise the stability condition. Although smooth curves are the objects we aim to parametrise, studying this problem effectively requires us to consider stable curves. Their properties ensure that we work with a parameter space that possesses much better geometric characteristics.

Let k be a field. A *curve over* k is a pure one dimensional scheme C of finite type over k. Recall that if X is a proper scheme over k with $\dim(X) \leq 1$, then X is projective [SP, Tag 0A26]; in particular, proper curves over k are projective. For a projective curve C, define the *arithmetic genus* (or simply the *genus*) as

$$p_a(C) \coloneqq 1 - \chi(\mathcal{O}_C),$$

which is equal to $h^1(C, \mathcal{O}_C)$ if C is geometrically connected and reduced.
Recall that for any integral curve, the normalization process yields a smooth curve $\widetilde{C}$, which is birational to C; see [Liu02, Prop. 4.1.22]. For a reduced curve, we define its normalization to be the disjoint union of the normalizations of its irreducible components. We define the *geometric genus* of a reduced curve C as the arithmetic genus of its normalization $\widetilde{C}$.

Theorem 2.0.1 (Riemann-Roch)**.** *Let C be a connected, smooth, and projective curve of genus g over an algebraically closed field k. If L is a line bundle on C, then*

$$\chi(C, L) = \deg L + 1 - g.$$

Proof. See [Har77, Thm IV.1.3]. □

2.1 Smooth curves

If C is a smooth curve over k, then the sheaf of Kähler differentials $\Omega_{C/k}$ is a line bundle, and it plays a fundamental role in studying C.

Theorem 2.1.1 (Serre Duality)**.** *If C is a geometrically connected smooth projective curve over a field k, then Ω_C is the dualizing sheaf, i.e., there is a linear map* $\mathrm{tr}\colon H^1(C, \Omega_{C/k}) \to k$ *such that for every coherent sheaf F on C, we have a natural perfect pairing*

$$\mathrm{Hom}_{\mathcal{O}_C}(F, \Omega_{C/k}) \times H^1(C, F) \to H^1(C, \Omega_{C/k}) \xrightarrow{\mathrm{tr}} k.$$

In particular, if F is locally free, then $H^1(C, F)^\vee \simeq H^0(C, F^\vee \otimes \Omega_{C/k})$.

Proof. See [Har77, Cor. III.7.12] and [SP, Tag 0FVV]. □

We denote $\Omega_{C/k}$ simply by Ω_C. Actually, by the theorem above, on a smooth curve this line bundle is also the dualizing sheaf, hence it is often denoted by ω_C. However, in the next section we will see that for singular curves Ω_C is not the dualizing sheaf.

Remark. By using Serre Duality, the Riemann-Roch formula becomes

$$h^0(C, L) - h^0(C, L^\vee \otimes \Omega_C) = \deg L + 1 - g.$$

In particular, it follows that $\deg \Omega_C = 2g - 2$.

Positivity of line bundles

The following is a classical criterion to determine whether a line bundle is base point free, ample, or very ample.

Proposition 2.1.2. *Let C be a smooth, connected, projective curve over an algebraically closed field k, and let L be a line bundle on C. Then*

(1) if $\deg L < 0$*, then* $h^0(C, L) = 0$*;*

(2) if $\deg L > 0$*, then L is ample;*

(3) if $\deg L \geq 2g$*, then L is base point free;*

(4) if $\deg L > 2g$*, then L is very ample.*

Proof. See [Har77, Cor. IV.3.2]. □

Families

Definition 2.1.3. A *family of smooth curves (of genus g)* over a scheme S is a flat, proper, and finitely presented morphism $\mathcal{C} \to S$ of schemes such that for every point $s \in S$, the geometric fibre $\mathcal{C}_{\overline{s}} := \mathcal{C} \times_S \operatorname{Spec} \overline{\kappa(s)}$ is a connected smooth curve (of genus g) over $\overline{\kappa(s)}$.

Recall that requiring the geometric fibre over $s \in S$ to be connected is equivalent to requiring that the fibre $\mathcal{C}_s = \mathcal{C} \times_S \operatorname{Spec} \kappa(s)$ is geometrically connected over $\kappa(s)$.

Remark. Since every fibre is smooth, we get that $\mathcal{C} \to S$ is actually smooth. Moreover, if the base scheme S is connected and noetherian, we do not need to require that the genus is constant along fibres, since flatness and projectivity of curves already imply it; see [Har77, Cor. III.9.10].

As for any morphism, we can consider the relative sheaf of differentials $\Omega_{\mathcal{C}/S}$. In our setting, this is a line bundle on $\mathcal{C}$ such that for $s \in S$, its restriction to $\mathcal{C}_s$ is exactly $\Omega_{\mathcal{C}_s}$. We also have the following important properties.

Proposition 2.1.4. *Let $\pi\colon \mathcal{C} \to S$ be a family of smooth curves of genus $g \geq 2$.*

(1) $\pi_*\mathcal{O}_{\mathcal{C}} = \mathcal{O}_S$*;*

(2) The pushforward $\pi_*(\Omega_{\mathcal{C}/S}^{\otimes k})$ is a vector bundle of rank

$$r(k) := \begin{cases} g & \text{if} \quad k = 1 \\ (2k-1)(g-1) & \text{if} \quad k > 1 \end{cases}$$

whose construction commutes with base change, i.e., for a morphism of schemes $f\colon T \to S$, it holds $f^*\pi_*(\Omega_{\mathcal{C}/S}^{\otimes k}) \simeq (\pi_T)_*(\Omega_{\mathcal{C}_T/T}^{\otimes k})$ where $\pi_T\colon \mathcal{C}_T \to T$ is the base change morphism;

(3) $\mathrm{R}^1\,\pi_*\Omega_{\mathcal{C}/S}^{\otimes k}$ *is isomorphic to $\mathcal{O}_S$ if $k = 1$ and zero if $k > 1$; and*

(4) For $k \geq 3$, $\Omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample with respect to π. In particular, $\mathcal{C} \to S$ is projective.

Proof. See [Alp27, Prop. 6.1.16]. □

The sheaf $\Omega_{\mathcal{C}/S}$ is also a relative dualizing sheaf; see [Liu02, §6.4].

Automorphisms

If C is a smooth curve over a field k, and $p_1, \ldots, p_n \in C(k)$, we let $\mathrm{Aut}(C, p_i)$ denote the abstract group of automorphisms of $(C, p_1, \ldots, p_n)$ over k, namely automorphisms $\alpha \colon C \xrightarrow{\sim} C$ over k such that $\alpha(p_i) = p_i$ for all $i = 1, \ldots, n$.

Example. The projective line $C = \mathbb{P}^1_k$ is a smooth curve of genus 0 and we know that $\mathrm{Aut}(\mathbb{P}^1_k) \simeq \mathrm{PGL}(2, k)$. Moreover, it acts transitively on triples of points, hence if p_1 and p_2 are distinct points, up to automorphisms we may suppose that $p_1 = [0 : 1]$ and $p_2 = [1 : 0]$. One can easily see that $\mathrm{Aut}(\mathbb{P}^1, [0 : 1]) \simeq k \rtimes k^*$ and $\mathrm{Aut}(\mathbb{P}^1, [0 : 1], [1 : 0]) \simeq k^*$. Thus, if k is infinite, these groups are clearly infinite.

The following is a classical result of Hurwitz.

Proposition 2.1.5. *Let C be a smooth, connected, and projective curve of genus g over an algebraically closed field k, and let $p_1, \ldots, p_n \in C(k)$ be distinct points. The group $\mathrm{Aut}(C, p_i)$ is finite if $2g - 2 + n > 0$, i.e., either $g \geq 2$, or $g = 1$ and $n \geq 1$, or $g = 0$ and $n \geq 3$. Moreover, if $g \geq 2$ and $\mathrm{char}(k) = 0$, then $|\mathrm{Aut}(C)| \leq 84(g - 1)$.*

Proof. See [Mir95, Thm III.3.9]. □

Branched coverings

Let $f \colon C \to D$ be a finite morphism of smooth, connected, and projective curves over an algebraically closed field k. Assume that f is separable, i.e., the induced map $K(D) \to K(C)$ on function fields is a separable extension. For a point $p \in C(k)$ with image $q \in D(k)$, the *ramification index at p* is the integer e_p such that the induced map $f_p^{\#} \colon \mathcal{O}_{D,q} \to \mathcal{O}_{C,p}$ is $t \mapsto us^{e_p}$, where s and t are uniformizers and u is a unit. We say that f is

- *ramified at p* if $e_p > 1$;
- *tamely ramified at p* if $e_p > 1$ and either $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) \nmid e_p$;
- *unramified at p* if $e_p = 1$.

Note that if f is unramified at p, then the fibre over q at p is isomorphic to $\mathrm{Spec}\, \kappa(p)$, hence this definition agrees with the usual one of unramified ([Liu02, Def. 4.3.17]). Furthermore, since f is flat, unramifiedness at p is equivalent to étaleness at p.

In general, there is a short exact sequence

$$0 \to f^*\Omega_D \to \Omega_C \to \Omega_{C/D} \to 0. \tag{2.1}$$

Indeed, the right-exactness is always true. Moreover, since $f^*\Omega_D$ and Ω_C are line bundles, the left map is injective if and only if it is nonzero. As $K(D) \to K(C)$ is separable, we get $\Omega_{C/D} \otimes K(C) = \Omega_{K(D)/K(C)} = 0$, thus $f^*\Omega_D \to \Omega_C$ is nonzero at the generic point.
Using the same notation, if we look at the sequence above at the stalks over $p \in C(k)$, the differential dt maps to $d(us^{e_p}) = e_p us^{e_p-1}ds + s^{e_p}du$. If f is tamely ramified at p, then $(\Omega_{C/D})_p \simeq \mathcal{O}_{C,p}\langle ds\rangle/(s^{e_p-1}ds)$ and its length is exactly $e_p - 1$.

Definition 2.1.6. Let k be an algebraically closed field. A *branched covering* is a finite and separable morphism $f\colon C \to D$ of smooth, connected, and projective curves over k. Such a covering is said to be *simple* if there is at most one ramification point in every fibre and every ramification point $p \in C(k)$ is tamely ramified with ramification index $e_p = 2$.
For a branched covering $f\colon C \to D$ we define the *ramification divisor* to be

$$R = \sum_{p\in C(k)} \mathrm{length}(\Omega_{C/D})_p \cdot p.$$

Note that if f is at worst tamely ramified at every point, the formula become $R = \sum_{p\in C(k)}(e_p - 1) \cdot p$.

Theorem 2.1.7 (Riemann-Hurwitz)**.** *If $f\colon C \to D$ is a branched covering with ramification divisor R, then $\Omega_C \simeq f^*\Omega_D \otimes \mathcal{O}_C(R)$ and*

$$2g(C) - 2 = \deg(f)(2g(D) - 2) + \deg R.$$

Proof. The statement follows directly from the sequence (2.1) observing that $\Omega_{C/D}$ is exactly the coherent sheaf $\mathcal{O}_R$ supported on the ramification divisor R. In fact, tensoring the sequence (2.1) with $\Omega_C^\vee$ yields

$$0 \to f^*\Omega_D \otimes \Omega_C^\vee \to \mathcal{O}_C \to \mathcal{O}_R \to 0,$$

then $f^*\Omega_D \otimes \Omega_C^\vee$ is the ideal sheaf $\mathcal{O}_C(-R)$ of R. Finally, computing degrees concludes the proof. □

There is a notion of holomorphic (resp., topological) branched covering between connected and compact Riemann surfaces (resp., connected, Hausdorff, and compact topological spaces), but if the base space is $D = \mathbb{P}^1$ this is equivalent to consider algebraic branched coverings defined above; see [Alp27, Prop. 6.7.7].

Example. A smooth curve of genus g is called *hyperelliptic* if it is a simple branched double covering of $\mathbb{P}^1$, thus by Riemann-Hurwitz the number of ramification points will be

$$r = 2g + 2.$$

If $\operatorname{char} k \neq 2$, locally on a chart of $\mathbb{P}^1$, such a curve is the zero locus of

$$y^2 - f(x)$$

where f has degree $2g+2$, and the double covering map to $\mathbb{P}^1$ is defined locally as $(x, y) \mapsto x$. We then may consider the *hyperelliptic involution* sending $(x, y) \mapsto (x, -y)$, hence we can see that hyperelliptic curves always have non-trivial automorphism group.
Therefore, for any genera we can find curves with non-trivial automorphism group.

Moduli functor of smooth curves

As a first attempt to parametrise smooth projective curves, we define the *moduli functor of smooth curves of genus g*

$$\begin{aligned} F_{M_g} \colon \operatorname{Sch}/\mathbb{C} &\to \operatorname{Sets} \\ S &\mapsto \{\text{families of smooth curves } \mathcal{C} \to S \text{ of genus } g\}/\sim \end{aligned}$$

where two families $\mathcal{C} \to S$ and $\mathcal{C}' \to S$ are isomorphic if there is an isomorphism $\mathcal{C} \to \mathcal{C}'$ over S. Moreover, the pullback of a family $\mathcal{C} \to S$ via a morphism $T \to S$ is defined as the family $\mathcal{C} \times_S T \to T$.

However, Proposition 1.1.4 tells us that we have no hope for this functor to be representable by a scheme over $\mathbb{C}$. Indeed, let C be a curve with a non-trivial automorphism $\alpha \in \operatorname{Aut}(C)$, and let N be the nodal cubic curve $V(y^2z - x^3 - x^2z) \subseteq \mathbb{P}^2$. We can think of N as $\mathbb{P}^1$ with the points 0 and ∞ glued together, and we can then construct a non-trivial family $\mathcal{C} \to N$ by taking $C \times \mathbb{P}^1 \to \mathbb{P}^1$ and gluing the fibre $\pi^{-1}(0)$ with $\pi^{-1}(\infty)$ via α.

2.2 Nodal curves

If we want to parametrize smooth curves, in order to obtain a proper space, we also need to study singular curves. Later will be clear that it suffices to consider curves with the simplest kind of singularities, namely nodal curves. While we introduce the notion of nodal curve over an arbitrary field, when talking about stable curves we will always work over an algebraically closed field, unless otherwise specified.

Definition 2.2.1 (Nodal point)**.** Let C be a curve over a field k.

- We say that $p \in C(k)$ is a *split node* if there is an isomorphism

$$\widehat{\mathcal{O}}_{C,p} \simeq k[\![x, y]\!]/(xy).$$

- We say that a closed point $p \in C$ is a *node* if there exists a split node $\overline{p} \in C_{\overline{k}}$ over p.

We say that a curve C is *nodal* if every closed point is either smooth or nodal.

Example.
- Clearly, $C = \operatorname{Spec} k[x, y]/(xy)$ has a split node at 0.
- If $\operatorname{char} k \neq 2$, the nodal cubic $C = \operatorname{Spec} k[x, y]/(y^2 - x^2(x+1))$ has a split node at $p = 0$. Indeed, in $k[\![x, y]\!]$ we can write $y^2 - x^2(x+1) = (y - xt)(y + xt)$, where $t = \sqrt{1 + x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots \in k[\![x, y]\!]$ is an invertible element.
- The curve $C = \operatorname{Spec} \mathbb{R}[x, y]/(x^2 + y^2)$ has a node in 0, which is not split, as the quadratic form $x^2 + y^2$ does not split into linear factors.

Remark. Recall that if C is a curve over a field k, the singular locus $\operatorname{Sing}(C)$ is usually endowed with the schematic structure provided by the *first fitting ideal* of Ω_C; see [Alp27, §A.3.4].

Proposition 2.2.2 (Characterisation of Nodes)**.** *Let C be a curve over k, and let $p \in C$ be a closed point. The following are equivalent:*

(1) p is a node;

(2) C is a local complete intersection at p, and $\operatorname{Sing}(C)$ *is unramified over k at p;*

(3) $k \to \kappa(p)$ is separable, $\mathcal{O}_{C,p}$ is reduced, $\dim_{\kappa(p)} \mathfrak{m}/\mathfrak{m}^2 = 2$*, and there is a nondegenerate quadratic form $q \in$* $\operatorname{Sym}^2 \mathfrak{m}/\mathfrak{m}^2$ *mapping to 0 in* $\mathfrak{m}^2/\mathfrak{m}^3$*;*

(4) $k \to \kappa(p)$ is separable and $\widehat{\mathcal{O}}_{C,p} \simeq \kappa(p)[\![x, y]\!]/(q)$ where q is a nondegenerate quadratic form; and

(5) there exists a finite separable extension $k \to k'$ and a point $p' \in C_{k'}$ such that $\widehat{\mathcal{O}}_{C_{k'},p'} \simeq k'[\![x, y]\!]/(xy)$.

Proof. See [Alp27, Prop. 6.2.4]. □

Proposition 2.2.3 (Local Structure)**.** *Let C be a curve over a field k. If $p \in C$ is a node, then there exists a finite separable extension $k \to k'$ and étale neighborhoods*

$$\begin{tikzcd} & (U, u) \arrow[dl, "\acute{e}t"'] \arrow[dr, "\acute{e}t"] & \\ (C, p) & & (\operatorname{Spec} k'[x, y]/(xy), 0). \end{tikzcd}$$

Proof. Using the above characterisation of nodes, there is a finite separable extension $k \to k'$ such that $\widehat{\mathcal{O}}_{C,p} \otimes_k k' \simeq k'[\![x, y]\!]/(xy)$. Now the result follows from Artin Approximation; see [Alp27, Cor. B.5.21]. □

Example. If we consider the nodal cubic $C = \operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^2(x + 1))$, the operations we performed in the last example showed how to find such a common étale neighborhood. Essentially, we add a square root $t = \sqrt{x+1}$, then $y^2 - x^3 - x^2 = (y - xt)(y + xt)$. Therefore, we can take the neighborhood $U = \operatorname{Spec} \mathbb{C}[x, y, t]_t/(y^2 - x^2(x + 1), t^2 - x - 1) \to C$ given by $(x, y, t) \mapsto (x, y)$, and $U \to \operatorname{Spec} \mathbb{C}[x, y]/(xy)$ given by $(x, y, t) \mapsto (y - xt, y + xt)$.

Genus and dualizing sheaf

The genus of a nodal curve is strictly related to the genus of its normalization.

Proposition 2.2.4 (Genus Formula)**.** *Let C be a connected, nodal, and projective curve over an algebraically closed field k with δ nodes and ν irreducible components. Let $C = \bigcup_{i=1}^{\nu} C_i$ be the irreducible decomposition and let $\widetilde{C}_i$ be the normalization of C_i with genus $g_i = p_a(\widetilde{C}_i) = p_g(C_i)$. The genus g of C satisfies*

$$g = \sum_{i=1}^{\nu} g_i + \delta - \nu + 1. \tag{2.2}$$

Proof. Observe first that if C is a proper curve and $C = \bigcup_{i=1}^{\nu} C_i$ is its decomposition in *connected* components, then $\mathcal{O}_C = \bigoplus_i (j_i)_* \mathcal{O}_{C_i}$ where $j_i \colon C_i \to C$ is the inclusion, and thus we get

$$p_a(C) = 1 - \chi(\mathcal{O}_C) = 1 - \sum_i \chi(\mathcal{O}_{C_i}) = 1 - \nu + \sum_i p_a(C_i).$$

Using the notation in the statement, if we let $\widetilde{C}$ be the disjoint union of the $\widetilde{C}_i$, then the genus formula (2.2) becomes

$$p_a(C) = p_a(\widetilde{C}) + \delta.$$

To prove this, let $p_1, \dots, p_\delta \in C$ be the nodes of C. We claim that the normalization $\pi \colon \widetilde{C} \to C$ induces a short exact sequence

$$0 \to \mathcal{O}_C \to \pi_* \mathcal{O}_{\widetilde{C}} \to \bigoplus_{i=1}^{\delta} \kappa(p_i) \to 0.$$

Since $\kappa(p_i)$ is a skyscraper sheaf over the point p_i, it suffices to verify this sequence étale-locally around the nodes, and by the local structure of nodes (Theorem 2.2.3), we can assume that $C = \operatorname{Spec} k[x,y]/(xy)$. Thus $\widetilde{C} = \operatorname{Spec}(k[x] \times k[y])$ and the sequence above corresponds to

$$0 \to k[x,y]/(xy) \to k[x] \times k[y] \to k \to 0,$$

which is exact. This sequence induces a long exact sequence in cohomology

$$0 \to H^0(C, \mathcal{O}_C) \to H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}) \to \bigoplus_i \kappa(p_i) \to H^1(C, \mathcal{O}_C) \to H^1(\widetilde{C}, \mathcal{O}_{\widetilde{C}}) \to 0$$

where $H^j(C, \pi_*\mathcal{O}_{\widetilde{C}}) = H^j(\widetilde{C}, \mathcal{O}_{\widetilde{C}})$ because π is finite, hence affine ([SP, Tag 0BXR] and [Har77, Exercise III.8.2]). Finally, by taking the alternate sum of the dimensions, the statement follows. $\square$

Furthermore, since a nodal curve C over k is locally a complete intersection (l.c.i.), it admits a dualizing line bundle ω_C with a trace map $\operatorname{tr}_C \colon H^1(C, \omega_C) \to k$; see [ACG11, Lemma X.2.4] and [Har77, Thm III.7.11].
This sheaf will be very important in the study of stable curves. We provide an explicit description of it for a curve defined over an algebraically closed field k.
Let $\Sigma \coloneqq \operatorname{Sing}(C)$ be the singular locus and $U \coloneqq C \setminus \Sigma$. Let $\pi \colon \widetilde{C} \to C$ be the normalization of C, and let $\widetilde{\Sigma}$ and $\widetilde{U}$ be the preimages of Σ and U, fitting in the diagram

$$\begin{tikzcd}
\widetilde{U} \arrow[r, hook] \arrow[d, "\wr"] & \widetilde{C} \arrow[d, "\pi"] & \widetilde{\Sigma} \arrow[l, hook'] \arrow[d] \\
U \arrow[r, hook] & C & \Sigma; \arrow[l, hook']
\end{tikzcd}$$

in particular, note that $\widetilde{\Sigma}$ is a reduced effective Cartier divisor on $\widetilde{C}$. Let $\Sigma = \{z_1, \dots, z_n\}$ be an ordering of the nodes and denote $\pi^{-1}(z_i) = \{p_i, q_i\}$. The curve $\widetilde{C}$ is smooth, hence $\Omega_{\widetilde{C}}$ is the dualizing bundle. If we tensor with $\Omega_{\widetilde{C}}(\widetilde{\Sigma})$ the exact sequence

$$0 \to \mathcal{O}_{\widetilde{C}}(-\widetilde{\Sigma}) \to \mathcal{O}_{\widetilde{C}} \to \mathcal{O}_{\widetilde{\Sigma}} \to 0$$

we get

$$0 \to \Omega_{\widetilde{C}} \to \Omega_{\widetilde{C}}(\widetilde{\Sigma}) \to \mathcal{O}_{\widetilde{\Sigma}} \to 0.$$

We interpret sections of $\Omega_{\widetilde{C}}(\widetilde{\Sigma})$ as rational sections of $\Omega_{\widetilde{C}}$ with at worst simple poles along $\widetilde{\Sigma}$. Evaluating the sequence above on an open $\widetilde{V} \subseteq \widetilde{C}$ yields

$$0 \to \Gamma(\widetilde{V}, \Omega_{\widetilde{C}}) \to \Gamma(\widetilde{V}, \Omega_{\widetilde{C}}(\widetilde{\Sigma})) \xrightarrow{h} \bigoplus_{y \in \widetilde{V} \cap \widetilde{\Sigma}} \kappa(y),$$

where the last map h can be interpreted as taking a rational section of $\Omega_{\widetilde{C}}$ to the tuple whose coordinate at $y \in \widetilde{V} \cap \widetilde{\Sigma}$ is defined as the *residue* $\mathrm{res}_y(s)$ *of* s *at* y.
We define the subsheaf $\omega_C \subseteq \pi_*\Omega_{\widetilde{C}}(\widetilde{\Sigma})$, whose sections over an open $V \subseteq C$ are sections of $\Omega_{\widetilde{C}}(\widetilde{\Sigma})$ over $\pi^{-1}(V)$ such that for every node $z_i \in \Sigma \cap V$ with preimages $p_i, q_i \in \pi^{-1}(V)$, $\mathrm{res}_{p_i}(s) + \mathrm{res}_{q_i}(s) = 0$. In other words, the sheaf ω_C sits in the following exact sequences:

$$\begin{aligned} 0 \longrightarrow \omega_C \longrightarrow \pi_*\Omega_{\widetilde{C}}(\widetilde{\Sigma}) &\longrightarrow \bigoplus_{z_i \in \Sigma} \kappa(z_i) \longrightarrow 0 \\ s &\longmapsto (\mathrm{res}_{p_i}(s) + \mathrm{res}_{q_i}(s)) \end{aligned}$$

$$\begin{aligned} 0 \longrightarrow \pi_*\Omega_{\widetilde{C}} \longrightarrow \omega_C &\longrightarrow \bigoplus_{z_i \in \Sigma} \kappa(z_i) \longrightarrow 0 \\ s &\longmapsto (\mathrm{res}_{p_i}(s)). \end{aligned}$$

Proposition 2.2.5. *(Dualizing sheaf) Let C be a connected, nodal, and projective curve over an algebraically closed field k, and let $\pi\colon \widetilde{C} \to C$ be the normalization. Let ω_C be the dualizing sheaf defined above.*

(1) If $f\colon C' \to C$ is étale, then $f^*\omega_C \simeq \omega_{C'}$;

(2) ω_C is a line bundle;

(3) ω_C is a dualizing sheaf; and

(4) If $T \subseteq C$ is a subcurve, i.e., a closed subscheme which is itself a curve, we define its complement as $T^c \coloneqq \overline{C \setminus T}$, then

$$\omega_C|_T = \omega_T(T \cap T^c).$$

Proof. See [Ols16, Prop. 13.2.9]. □

Again, we obtain a duality pairing like in the smooth case, thus for any locally free sheaf F on C

$$H^i(C, F) \simeq H^{1-i}(C, F^\vee \otimes \omega_C).$$

Positivity of line bundles

Let C be a nodal, and projective curve, and let $\pi\colon \widetilde{C} \to C$ be its normalization. Let also $\widetilde{C}_i$ denote the normalization of each irreducible component C_i of C.
If L is a line bundle on C, the *multidegree* of L is the tuple $\mathbf{deg}(L) = (d_i)$, where

$d_i = \deg_{C_i}(L) \coloneqq \deg_{\widetilde{C}_i}(\pi^* L)$ is the degree of the pullback of L to $\widetilde{C}_i$; the *total degree* of L is $\deg(L) \coloneqq \sum_i d_i$. If $\mathbf{d} = (d_i)$ and $\mathbf{e} = (e_i)$ are multidegrees, we write $\mathbf{d} \geq \mathbf{e}$ to mean that $d_i \geq e_i$ for every i, and $\mathbf{d} > \mathbf{e}$ to mean that $\mathbf{d} \geq \mathbf{e}$ and $d_i > e_i$ for at least one i.
As in the case of smooth curves, the existence of a dualizing sheaf has useful consequences.

Proposition 2.2.6. *Let C be a connected projective nodal curve, and let L be a line bundle on C.*

(1) If $\mathbf{deg}(L) > \mathbf{deg}(\omega_C)$, then $H^1(C, L) = 0$;

(2) L is ample if and only if $\deg_{C_i}(L) > 0$ for every irreducible component C_i.

Proof. For (1), Serre duality says that $H^1(C, L)$ vanishes if and only if $H^0(C, L^\vee \otimes \omega_C)$ does. On the other hand, the degrees of $L^\vee \otimes \omega_C$ on the irreducible components of C are all nonpositive, and at least one of them is negative. Thus, any section of $L^\vee \otimes \omega_C$ vanishes identically on one component and is constant on the remaining ones. Since C is connected, it must vanish identically. For (2), observe that L is ample if and only if the line bundle $L|_{C_i}$ on C_i is ample for every irreducible component C_i. Indeed, for a finite surjective morphism $f \colon Y \to X$ of projective schemes, a line bundle M on X is ample if and only if $f^* M$ is a ample on Y [Laz04, Prop. 1.2.13 and Cor. 1.2.28]. If we let $\widetilde{C}$ be the disjoint union of the normalizations of the C_i's, then the natural map $\nu \colon \widetilde{C} \to C$ is surjective and finite, hence L is ample if and only if $\nu^* L$ is. Clearly, $\nu^* L$ is ample if and only if this holds on every connected component of $\widetilde{C}$. Since these are smooth curves, ampleness on them is equivalent to having positive degree. Therefore, the statement follows. □

Families

Similarly to the smooth case, we define a *family of nodal curves* over a scheme S as a flat, proper, and finitely presented morphism $\mathcal{C} \to S$ of schemes such that every geometric fibre $\mathcal{C}_{\overline{s}}$ is a connected nodal curve over $\overline{\kappa(s)}$.

Theorem 2.2.7. *(Local structure of families) Let $\pi \colon \mathcal{C} \to S$ be a family of nodal curves. Let $p \in \mathcal{C}$ be a node in the fibre $\mathcal{C}_s$ over a point $s \in S$. There is a commutative diagram*

$$\begin{tikzcd}
(\mathcal{C}, p) \arrow[d] & (\mathcal{C}', p') \arrow[l, "\text{ét}"'] \arrow[r, "\text{ét}"] \arrow[d] & (\operatorname{Spec} A[x,y]/(xy - f), (s', 0)) \arrow[dl] \\
(S, s) & (\operatorname{Spec} A, s'), \arrow[l, "\text{ét}"'] &
\end{tikzcd}$$

where $f \in A$ is a function vanishing at s'.

Proof. See [SP, Tag 0CBY]. □

2.3 Stable curves

In this section we introduce the notion of stability for a curve. Later in Chapter 5 we will see that stable curves allow us to work with a moduli space $\overline{M}_{g,n}$ having nice properties such as properness.

If n is a natural number, we define an *n-pointed nodal curve* to be the datum of a nodal curve C over an algebraically closed field k together with an ordered collection of distinct smooth points $p_1, \dots, p_n \in C(k)$, called *marked points*. More generally, if P is a finite set, a *P-pointed nodal curve* is the datum of a nodal curve C as above together with an injective map from P to the smooth locus of C. A point $q \in C$ is called *special* if it is either marked or a node. We often view the set of marked points $\{x_p\}_{p \in P}$ as a divisor $D = \sum_p x_p$ on C, and the line bundle $\omega_C(D)$ will be referred to as the *log-canonical sheaf* on (C, D).

Dual graph

To any projective P-pointed nodal curve we can associate a graph representing its structure, which is a useful tool to study these objects from a more combinatorial viewpoint.

Definition 2.3.1. A *dual graph* (or simply graph) Γ is the datum of:

- a finite nonempty set $V = V(\Gamma)$, whose elements are called the *vertices*, with an assignment of a nonnegative integer weight g_v to each $v \in V$;
- a finite set $L = L(\Gamma)$, whose elements are called the *half-edges*;
- an involution ι of L;
- a partition of L indexed by V, i.e., the assignment to each $v \in V$ of a subset L_v of L such that $L = \bigcup_{v \in V} L_v$ and $L_v \cap L_w = \emptyset$ if $v \neq w$.

A pair of distinct elements of L interchanged by ι is called an *edge* of the graph, while a fixed point of ι is called a *leg*. The set of edges of Γ is denoted $E = E(\Gamma)$. The *genus* of Γ is

$$g(\Gamma) := \sum_{v \in V} g_v + 1 - \chi(\Gamma),$$

where $\chi(\Gamma) = |V| - |E|$. A dual graph endowed with a one-to-one correspondence between a finite set P and the set of its legs is said to be *P-marked*.

The notion of isomorphism between dual graphs is the obvious one: a bijection between vertices and a bijection between half-edges respecting the structure defined above.

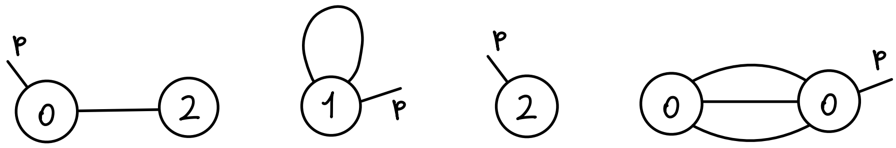


Figure 2.1: Examples of $\{p\}$-marked dual graphs of genus 2. We represent vertices as circles with the associated weight.

The *geometric realization* $|\Gamma|$ of a graph Γ is the one-dimensional CW-complex obtained by realizing each half-edge $\ell \in L_v$ as a segment $|\ell|$, having a point $|v|$ as one of its endpoints, and then identifying the free endpoint of $|\ell|$ with the free endpoint of $|\ell'|$ if and only if $\iota(\ell) = \ell'$, namely if ℓ and ℓ' form an edge. A graph is said to be *connected* if its realization is; moreover, note that $\chi(|\Gamma|) = \chi(\Gamma)$.

Let now $(C, \{x_p\}_{p\in P})$ be a P-pointed nodal curve. We can associate a dual graph $\mathrm{Graph}^S(C, \{x_p\}_{p\in P})$ to the datum of C together with a set of nodes $S \subseteq \Sigma$ of C. The vertices correspond to the connected components of the partial normalization C' of C at S, the weight of a vertex is the genus of the corresponding component of C', the half-edges issuing from a vertex correspond to the intersection of the corresponding component of C' with the preimage in C' of $\{x_p\}_{p\in P} \cup S$. Clearly, the edges are pairs of half-edges mapping to the same node in C, and the legs are half-edges coming from marked points. This graph is naturally P-marked in the obvious way. When $S = \Sigma$ is the set of all nodes, we simply write $\mathrm{Graph}(C, \{x_p\}_{p\in P})$.
Note that the genus of $\mathrm{Graph}^S(C, \{x_p\}_{p\in P})$ is equal to the genus of C.

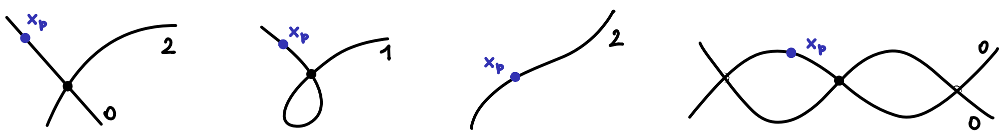


Figure 2.2: Examples of genus 2 $\{p\}$-pointed nodal curves. The numbers stands for the geometric genus of each irreducible component. Their dual graphs clearly are those in Figure 2.1.

Types of nodes

We can classify the nodes of a connected P-pointed nodal curve $(C, \{x_p\}_{p\in P})$ of genus g according to the combinatorics of its partial normalization at the node. Let z be a singular point of C, and let e be the corresponding edge of the dual graph $\Gamma = \mathrm{Graph}(C, \{x_p\}_{p\in P})$.
We say that z is a *nonseparating node* or that e is a *nondisconnecting edge* if the

partial normalization of C at z is connected, or equivalently if the graph obtained from Γ by removing e is connected. In this case, the partial normalization at z will be a connected nodal curve of genus $g-1$ with $|P|+2$ marked points, where the exceeding two points are the preimages of z.
The other possibility is that the normalization of C at q has two connected components C_1 and C_2 of genera a and b adding to g. In this case P is partitioned in two subsets A and B indexing, respectively, marked points on C_1 and marked points on C_2. We then say that z is a *separating node* (or that e is a *disconnecting edge*) *of type* $\mathcal{P}$, where $\mathcal{P} = \{(a, A), (b, B)\}$ is called a *bipartition* of (g, P). For simplicity, we usually say that z is a separating node of type (a, A), or equivalently of type $(b, B) = (g-a, P \setminus A)$.

For simplicity, often we will only talk about n-pointed nodal curves, but everything can be easily generalized to P-pointed ones.

Stability

If we hope to obtain a separated moduli space, we need to restrict to a subclass of nodal curves by imposing some conditions. Deligne and Mumford in [DM69] introduced the stability condition, which turned out to be a correct one to impose in order to obtain a suitable compactification of the moduli space of smooth curves.

Definition 2.3.2. An n-pointed connected, nodal, and projective curve (C, x_i) of genus g over an algebraically closed field k is *stable* (resp., *semistable*) if

(1) C is not of genus 1 without marked points, i.e., $(g, n) \neq (1, 0)$, and

(2) every smooth irreducible rational subcurve $\mathbb{P}^1 \subseteq C$ contains at least 3 (resp., 2) special points.

If only (1) holds, we say that (C, x_i) is *prestable*.

Remark. Since we are working over an algebraically closed field k, every smooth irreducible rational curve is isomorphic to $\mathbb{P}^1$ [SP, Tag 0C6U]. On the other hand, condition (1) makes sense also over a not algebraically closed field k.
Note also that there are no n-pointed stable curves of genus g if

$$(g, n) \in \{(0,0), (0,1), (0,2), (1,0)\},$$

or equivalently if $2g-2+n \leq 0$. Therefore, we often impose that $2g-2+n > 0$.

We may define a connected dual graph to be *stable* (resp., *semistable*) if for every vertex v,

$$2g_v - 2 + l_v > 0 \quad (\text{resp.,} \quad 2g_v - 2 + l_v \geq 0)$$

where g_v is the weight associated to v and $l_v = |L_v|$ is its valence, i.e., the number of half-legs issuing from v.

Proposition 2.3.3 (Characterisation of stability). *Let (C, x_i) be a projective n-pointed nodal curve, and let $D = \sum_{i=1}^n x_i$ be the divisor associate to the marked points. The following are equivalent:*

(1) (C, x_i) *is stable;*

(2) the log-canonical sheaf $\omega_C(D)$ *is ample;*

(3) $\mathrm{Aut}(C, x_i)$ *is finite; and*

(4) $\mathrm{Graph}(C, x_i)$ *is stable.*

Proof. Denote $\Gamma = \mathrm{Graph}(C, x_i)$ for simplicity. Observe that there is a natural group homomorphism from $\mathrm{Aut}(C, x_i)$ to $\mathrm{Aut}(\Gamma)$, whose kernel K is given by automorphisms of (C, x_i) that do not interchange irreducible components and fix nodes (apart from fixing marked points). Since $\mathrm{Aut}(\Gamma)$ is finite, the finiteness of $\mathrm{Aut}(C, x_i)$ is equivalent to that of K. On the other hand, K is the product of the groups K_v, as v varies in $V(\Gamma)$, and K_v is the automorphism group of the corresponding irreducible component (C_v, L_v), where L_v denotes the nodes and marked points in C_v corresponding to the set of half-edges issuing from v. Hence, the finiteness of K is equivalent to that of all K_v. Moreover, by the universal property of normalization, any automorphism of an irreducible nodal curve lifts to an automorphism of its normalization; more precisely, an automorphism of $(X, x_1, \ldots, x_n)$ that fixes the nodes $z_1, \ldots, z_k$ will lift to an automorphism of $(\widetilde{X}, \{\widetilde{x}_i\}_{i=1,\ldots,n} \cup \{\widetilde{p}_j, \widetilde{q}_j\}_{j=1,\ldots,k})$, where $\{\widetilde{p}_j, \widetilde{q}_j\}$ is the preimage of z_j. The last curve is called the *pointed normalization* and it is easy to show that properties (1)–(3) hold if and only if they hold the pointed normalization of every irreducible component of C. By definition, also (4) can be checked on vertices.
Thus it suffices to prove these equivalences for a smooth, irreducible n-pointed curve (C, x_i). The statement then follows using classic results for smooth curves and Proposition 2.1.5. □

Example. The curves depicted in Figure 2.2 are all stable except for the first one. As Proposition 2.3.3 states, we can also tell this by looking at their dual graphs in Figure 2.1.

We may also characterise stability through rational tails and bridges.

Definition 2.3.4. Let (C, p_i) be an n-pointed prestable curve over (possibly not algebraically closed) field k. We say that an irreducible smooth subcurve $E \subseteq C$ of genus 0 with nonempty complement $E^c = \overline{C \setminus E}$ is

- a *rational tail* if the scheme-theoretic intersection $E \cap E^c$ is a single reduced point x, $H^0(E, \mathcal{O}_E) \simeq \kappa(x)$, and E contains no marked points;

- a *rational bridge* if the scheme-theoretic intersection $E \cap E^c$ has degree 2 over $H^0(E, \mathcal{O}_E)$ and E contains no marked points, or if E contains a single marked point p_j and E is a rational tail of $(C, p_1, \dots, \hat{p_j}, \dots, p_n)$.

Note that if k is algebraically closed, such a subcurve E is isomorphic to $\mathbb{P}^1$; moreover, E is

- a rational tail if $E \cdot E^c = 1$ and E has no marked points;
- a rational bridge if either $E \cdot E^c = 2$ and E has no marked points or if $E \cdot E^c = 1$ and E has precisely one marked point.

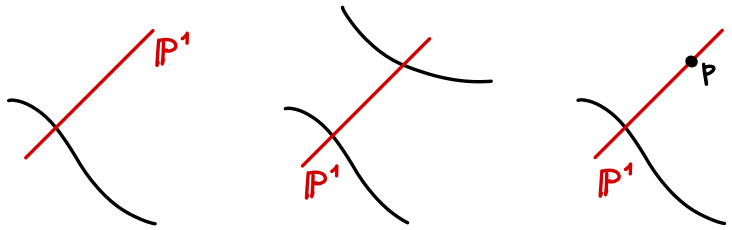


Figure 2.3: Rational tail (on the left) and rational bridges.

Over an arbitrary field k, the classification above becomes more subtle, but if K/k is an extension, then (C, p_i) has a rational tail (resp., bridge) if and only if $(C \times_k K, p_i \times_k K)$ has a rational tail (resp., bridge) [Alp27, Exc. 6.3.14].

It is clear from the definition that, over an algebraically closed field, stability (resp., semistability) is equivalent to not containing a rational tails or bridges (resp., rational tails). Therefore, the following statement follows directly from the observation above.

Proposition 2.3.5. *Let (C, p_i) be a prestable curve of genus g over an arbitrary field k. If K is n algebraically closed field containing k, then $(C \times_k K, p_i \times_k K)$ is stable (resp., semistable) if and only if (C, p_i) contains no rational tails or rational bridges (resp., rational tails).*

Families

Definition 2.3.6. A *family of n-pointed nodal curves* is the datum of a family $f\colon \mathcal{C} \to S$ of nodal curves, plus disjoint sections $\sigma_i\colon S \to \mathcal{C}$, $i = 1, \dots, n$, which do not meet the singular locus of any one of the fibres $\mathcal{C}_s$ of f. If $(f'\colon \mathcal{C}' \to S', \sigma_i')$ is another family of n-pointed nodal curves, a *morphism* from the first family to the second is a cartesian square

$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & \mathcal{C}' \arrow[d, "f'"] \\ S \arrow[r, "h"'] & S' \end{tikzcd}$$

compatible with sections, i.e., such that $H \circ \sigma_i = \sigma_i' \circ h$ for every $i = 1, \dots, n$.

Note that if $(\mathcal{C} \to S, \sigma_i)$ is a family of n-pointed nodal curves, then each fibre $(\mathcal{C}_s, \sigma_i(s))$ is naturally n-pointed.
A *family of stable* (resp., *semistable, prestable*) *n-pointed curves* is a family of P-pointed nodal curves as above, plus the requirement that each geometric fibre $(\mathcal{C}_{\overline{s}}, \{\sigma_p(s) \mid p \in P\})$ is a stable (resp., semistable, prestable) curve.
We will often use the symbol $\mathcal{O}(\sum \sigma_p)$ to mean the divisor $\mathcal{O}(\sum_p \sigma_p(S))$ in $\mathcal{C}$.

Remark. If $(\mathcal{C} \to S, \sigma_i)$ is a family of pointed prestable curves, then $\mathcal{C} \to S$ is a local complete intersection morphism [ACG11, Lemma X.2.4]. Thus there is a relative dualizing sheaf $\omega_{\mathcal{C}/S}$ compatible with base change, and in particular it restricts to the dualizing line bundle $\omega_{\mathcal{C}_s}$ on every fibre $\mathcal{C}_s$; see [Liu02, §6.4]. The image of each section σ_i is a divisor contained in the smooth locus, hence we can consider the line bundle $\omega_{\mathcal{C}/S}(\sum \sigma_i)$ on $\mathcal{C}$.

Proposition 2.3.7. *Let $(\mathcal{C} \to S, \sigma_i)$ be a family of n-pointed prestable curves of genus g over a scheme S. The following are equivalent:*

(1) $(\mathcal{C} \to S, \sigma_i)$ is a family of stable curves;

(2) $\omega_{\mathcal{C}/S}(\sum \sigma_i)$ is relatively ample over S.

In particular, a family of n-pointed stable curves is a projective morphism.

Proof. Both conditions can be checked on geometric fibres so the statement follows from Proposition 2.3.3. □

The following generalization of Proposition 2.1.4 is proven in the same way using the very ampleness of $\omega_C(\sum p_i)^{\otimes 3}$ for a stable n-pointed curve.

Proposition 2.3.8. *Let $(\mathcal{C} \xrightarrow{\pi} S, \sigma_i)$ be a family of n-pointed stable curves of genus g, and set $L \coloneqq \omega_{\mathcal{C}/S}(\sum \sigma_i)$. If $k \geq 3$, then $L^{\otimes k}$ is relatively very ample with respect to π and $\pi_*(L^{\otimes k})$ is a vector bundle of rank $(2k-1)(g-1) + kn$.*

Lemma 2.3.9 (Openness of stability)**.** *Let $(\mathcal{C} \xrightarrow{\pi} S, \sigma_i)$ be a family of n-pointed curves. The locus of points $s \in S$ such that $(\mathcal{C}_s, \sigma_i(s))$ is stable (resp., semistable, prestable, nodal) is open in S.*

Proof. First, we prove that the nodal locus is open. This follows from the local structure of nodes (Theorem 2.2.3), since the locus $\mathcal{C}^{\leq \mathrm{nod}} \subseteq \mathcal{C}$ of points which are smooth or nodal in their fibre is open, and thus the locus of points $s \in S$ such that $\mathcal{C}_s$ has worse than nodal singularities is identified with the closed subset $\pi(\mathcal{C} \setminus \mathcal{C}^{\leq \mathrm{nod}})$.
Since the locus in S where $\{\sigma_i(s)\}$ are distinct and smooth points in $\mathcal{C}_s$ is open, the condition that $\mathcal{C}_s$ is prestable is open. Moreover, assuming prestability, the locus of points $s \in S$ where $\mathcal{C}_s$ is semistable is identified with the open locus over which $\pi^*\pi_*\omega_{\mathcal{C}/S} \to \omega_{\mathcal{C}/S}$ is surjective; see [SP, Tag 0E6Y].
Finally, assuming again prestability, since ampleness is an open condition ([Laz04, Thm 1.2.17, Thm 1.7.8]), the locus of points s in S such that the line bundle $\omega_{\mathcal{C}/S}(\sum \sigma_i)|_{\mathcal{C}_s} \simeq \omega_{\mathcal{C}_s}(\sum \sigma_i(s))$ is ample is open, and this condition characterises stability by Proposition 2.3.3. $\square$

Stable model

Let g and n be such that stable n-pointed genus g curves exist, that is, such that $2g-2+n>0$. Then there is a canonical way of attaching a stable n-pointed genus g curve to a prestable one (C, x_i). Indeed, recall that irreducible components which prevent (C, x_i) from being stable are exactly rational tails and bridges. The connected components of their union are chains of smooth rational curves, and we refer to them as *exceptional chains*. By collapsing to a point each of these chains, we obtain an n pointed nodal curve (C', x_i') which is clearly stable, and it is called the *stable model* of (C, x_i).

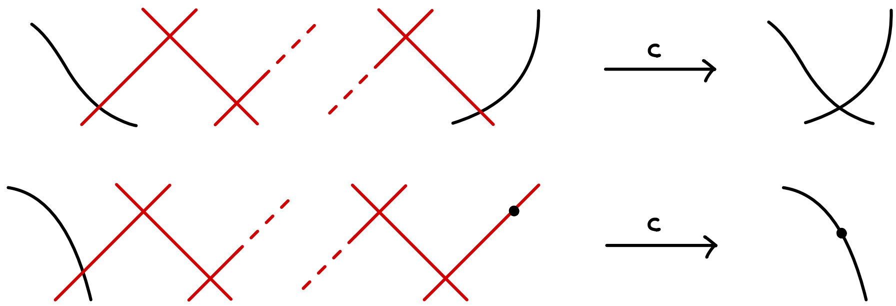


Figure 2.4: Contracting exceptional chains.

Proposition 2.3.10. *Let (C, p_i) be an n-pointed prestable curve of genus g over k such that $2g-2+n>0$. Then there is a canonical morphism*

$$c\colon C \to C'$$

contracting all exceptional chains to points, such that $(C', c(p_i))$ is an n-pointed stable curve of genus g, $c_*(\mathcal{O}_C) = \mathcal{O}_{C'}$ and $R^1c_*\mathcal{O}_C = 0$.

Proof. See [Alp27, Cor. 6.6.3]. □

The following is a generalization of Proposition 2.2.6.

Proposition 2.3.11. *Let (C, x_i) be an n-pointed semistable curve of genus g, and let V be the set of vertices of its graph. Suppose that $2g - 2 + n > 0$. For each $v \in V$, denote by C_v the corresponding component of C, by g_v the geometric genus of C_v, and by l_v the number of half-edges issuing from v. Let L be a line bundle on C. Then*

(1) $H^1(C, L) = 0$ if $\deg_{C_v} L \geq 2(2g_v - 2 + l_v)$ for all v;

(2) L is base point free if $\deg_{C_v} L \geq 2(2g_v - 2 + l_v)$ for all v;

(3) L is very ample if C is stable and $\deg_{C_v} L \geq 3(2g_v - 2 + l_v)$ for all v.

Proof. Denote $d_v = \deg_{C_v} L$. By assumptions, $d_v \geq 0$ for all v, and in fact $d_v > 0$ unless C_v is a smooth rational curve and $l_v = 2$, which can happen only if C is not stable. In this case, if $d_v = 0$, then L is trivial on C_v. Now, denoting by (C', x'_i) the curve obtained by contracting C_v to a point and by $\varphi\colon C \to C'$ the contraction map, $L = \varphi^* L'$ for some line bundle L' on C'. The curve (C', x'_i) and the line bundle L' still satisfy the assumptions of the proposition; moreover,

$$H^i(C, L) = H^i(C', L')$$

since by Proposition 2.3.10 the higher direct image $R^1\varphi_*\mathcal{O}_C$ is zero. Hence, iterating the contraction procedure, we are reduced to proving the statement under the assumption that $d_v > 0$ for all v. Item (1) then follows from Proposition 2.2.6, since $d_v > \deg_{C_v} \omega_C$ for every vertex v. Indeed, when $2g_v - 2 + l_v > 0$ we have $d_v \geq 2(2g_v - 2 + l_v) > \deg_{C_v} \omega_C$, while $d_v > 0 = \deg_{C_v} \omega_C$ when $2g_v - 2 + l_v = 0$.
For (2), we must show that $H^1(C, I_x \otimes L)$ vanishes for every $x \in C$, where I_x is the ideal sheaf of $\{x\}$; see [Har77, Prop. IV.3.1]. We distinguish two cases. Suppose that x is a smooth point of C, thus $I_x \otimes L$ is a line bundle. If C_v does not contain x, then $\deg_{C_v}(I_x \otimes L) = d_v > \deg_{C_v} \omega_C$. If $x \in C_v$, then

$$\deg_{C_v}(I_x \otimes L) = d_v - 1 \geq \deg_{C_v} \omega_C.$$

If equality holds, we are in one of the following cases. Either $2g_v - 2 + l_v = 0$, and hence $g_v = 0, l_v = 2$ or else $2g_v - 2 + l_v = 1$, and C_v does not contain marked points. The latter can happen only for $g_v = 0, l_v = 3$ and for $g_v = 1, l_v = 1$. In all cases, C_v cannot be the only component of C. Thus Proposition 2.2.6 applies to $I_x \otimes L$.
If instead x is a singular point of C. We let $\alpha\colon C' \to C$ be the partial normalization at x. If C_v is a component of C, we denote by C'_v the corresponding component

of C'. The sheaf $I_x \otimes L$ is of the form $\alpha_* L'$, where L' is a line bundle on C', hence $H^1(C, I_x \otimes L) = H^1(C', L')$. We also recall that $\alpha_* \omega_{C'} = I_x \otimes \omega_C$. Note also that

$$\deg_{C'_v} L' - \deg_{C_v} L = \deg_{C'_v} \omega_{C'} - \deg_{C_v} \omega_C, \tag{2.3}$$

thus

$$\begin{aligned} \deg_{C'_v} L' &\geq \deg_{C'_v} \omega_{C'} - \deg_{C_v} \omega_C + 2(2g_v - 2 + l_v) \\ &\geq \deg_{C'_v} \omega_{C'} + 2g_v - 2 + l_v \end{aligned}$$

which is strictly greater than $\deg_{C'_v} \omega_{C'}$ when $2g_v - 2 + l_v > 0$, while

$$\deg_{C'_v} L' \geq d_v - 1 > -1 = \deg_{C'_v} \omega_{C'}$$

when $2g_v - 2 + l_v = 0$; again Proposition 2.2.6 proves the vanishing of $H^1(C', L')$, and thus that of $H^1(C, I_x \otimes L)$.
For (3), we must show, for any choice of points $x, y \in C$, the vanishing of $H^1(C, I_x \otimes I_y \otimes L)$. We again distinguish cases. First, if both x and y are smooth points, then $I_x \otimes I_y \otimes L$ is a line bundle, and its degree on C_v is at least $3(2g_v - 2 + l_v) - 2 \geq 2g_v - 2 + l_v \geq \deg_{C_v} \omega_C$, since C is stable. Equality occurs only when both x and y belong to C_v, $2g_v - 2 + l_v = 1$, and C_v does not contain marked points. This can happen only for $g_v = 0, l_v = 3$ and for $g_v = 1, l_v = 1$. In both cases, C_v cannot be the only component of C and, if C_w is any other component, the degree of $I_x \otimes I_y \otimes L$ on it is strictly greater than $\deg_{C_w} \omega_C$, so that Proposition 2.2.6 applies.
In the remaining cases, at least one among x, y is singular. Let $\alpha \colon C' \to C$ be the partial normalization at those points of $\{x, y\}$ that are nodes. Using the same notation as above, we have $I_x \otimes I_y \otimes L = \alpha_* L'$, where L' is a line bundle on C', and hence $H^1(C, I_x \otimes I_y \otimes L) = H^1(C', L')$. If $x \neq y$ and they are both singular, one proceeds as in the second part of (2). We get

$$\deg_{C'_v} L' \geq \deg_{C'_v} \omega_{C'} + 2(2g_v - 2 + l_v) > \deg_{C'_v} \omega_{C'},$$

and we conclude again applying Proposition 2.2.6.
A similar argument covers the case where x is singular and y is not. The last case to be considered is the one where $x = y$ is singular. Instead of (2.3), we get

$$\deg_{C'_v} L' - \deg_{C_v} L = 2 \deg_{C'_v} \omega_{C'} - 2 \deg_{C_v} \omega_C,$$

which gives

$$\begin{aligned} \deg_{C'_v} L' &\geq 2 \deg_{C'_v} \omega_{C'} - 2 \deg_{C_v} \omega_C + 3(2g_v - 2 + l_v) \\ &\geq 2 \deg_{C'_v} \omega_{C'} + 2g_v - 2 + l_v. \end{aligned}$$

Moreover, if we get equalities, then $\deg_{C_v} \omega_C = 2g_v - 2 + l_v$, i.e., C_v does not contain marked points. Observe that $\deg_{C'_v} \omega_{C'} = \deg_{C_v} \omega_C - h_v$, where h_v is 2 if C_v is the only component containing x, 1 is x belongs to C_v and to another component, and 0 if $x \in C_v$; however, $h_v \leq l_v$. Hence, $\deg_{C'_v} \omega_{C'} + 2g_v - 2 + l_v$ is nonnegative, and vanishes if and only if $g_v = 0$, $l_v = 3$, and $h_v = 2$. Thus $\deg_{C'_v} L' \geq \deg_{C'_v} \omega_{C'}$, and if we have an equality, then C_v contains no marked points, $g_v = 0$, $l_v = 3$, and $h_v = 2$. If this happens, C' is connected and has components other C'_v, as $l_v - h_v = 1$ and C_v contains no marked points. Since the degree of L' on any component of C' different from C'_v is strictly greater than the one of $\omega_{C'}$, Proposition 2.2.6 shows that $H^1(C', L') = H^1(C, I_x \otimes I_x \otimes L) = 0$, and this concludes the proof. □

Corollary 2.3.12. *Let (C, x_i) be as in Proposition 2.3.11. Let $L = \omega_C(\sum x_i)$ be the log-canonical sheaf. Then*

$$H^0(C, L^2) \otimes H^0(C, L^k) \to H^0(C, L^{k+2})$$

is surjective for every $k \geq 4$.

Proof. The proof is a standard argument. Note that L^h satisfies the assumptions of parts (1) and (2) of Proposition 2.3.11 for $h \geq 2$, and those of part (3) for $h \geq 3$. In particular, we can choose sections s and t of L^2 which do not have common zeros. Let E be the divisor of s. By (1) of Proposition 2.3.11, from the short exact sequence

$$0 \to L^{i-2} \xrightarrow{\times s} L^i \to L^i|_E \to 0$$

for $i = k+2, k$ we deduce the exact sequence

$$H^0(C, L^k) \xrightarrow{\times s} H^0(C, L^{k+2}) \to H^0(E, L^{k+2}|_E) \to 0$$

and the surjectivity of $H^0(C, L^k) \to H^0(E, L^k|_E)$. Thus the composite map

$$H^0(C, L^k) \to H^0(E, L^k|_E) \xrightarrow{\times t} H^0(E, L^{k+2}|_E)$$

is surjective, since s and t have no common zeros. But then we see that

$$s \otimes H^0(C, L^k) + t \otimes H^0(C, L^k) = H^0(C, L^{k+2}).$$

□

The collapsing operation described above can actually be performed simultaneously for all fibres of any family of prestable curves.

Theorem 2.3.13. *Consider a family F of prestable, n-pointed, genus g curves*

$$f\colon \mathcal{C} \to S, \quad \sigma_i\colon S \to \mathcal{C},\ i = 1, \dots, n$$

such that $2g - 2 + n > 0$. Then there exists a family F' of stable n-pointed genus g curves

$$f'\colon \mathcal{C}' \to S, \quad \sigma_i'\colon S \to \mathcal{C},\ i = 1, \dots, n,$$

and a morphism $c\colon \mathcal{C} \to \mathcal{C}'$ over S such that:

(1) $\sigma_i' = c \circ \sigma_i$ for every $i = 1, \dots, n$;

(2) $\mathcal{O}_{\mathcal{C}'} = c_*\mathcal{O}_{\mathcal{C}}$ and $R^1 c_* \mathcal{O}_{\mathcal{C}} = 0$;

(3) the construction of c is compatible with base change; and

(4) for each closed point $s \in S$, the induced morphism $c_s\colon (\mathcal{C}_s, \sigma_i(s)) \to (\mathcal{C}_s', \sigma_i'(s))$ is the contraction of exceptional chains as in Proposition 2.3.10.

The pair (F', π) in unique up to a unique isomorphism. Moreover, if F is a semistable family, then $\omega_{\mathcal{C}/S}(\sum_i \sigma_i) = c^*\omega_{\mathcal{C}'/S}(\sum_i \sigma_i')$.

Proof. See [Alp27, Thm 6.6.6]. ☐

Using the proposition above, we have associated to the family F a new family

$$\mathrm{StMd}(F) \coloneqq F' \;=\; \begin{tikzcd} \mathcal{C}' \arrow[d, "f'"'] \\ S \arrow[u, bend right, "\sigma_i'"'] \end{tikzcd} \quad i = 1, \dots, n$$

plus the collapsing map $c_F : \mathcal{C} \to \mathcal{C}'$. The family $\mathrm{StMd}(F)$ is called the *stable model* of F.

Remark. From the functoriality of the log-canonical sheaf under morphisms of families of n-pointed nodal curves, we deduce that $\mathrm{StMd}(F)$ and c_F are functorial in F. More precisely, if we are given two families F_1 and F_2 of prestable P-pointed genus g curves and a morphism

$$\mathbf{H} \;=\; \begin{tikzcd} \mathcal{C}_1 \arrow[r, "H"] \arrow[d] & \mathcal{C}_2 \arrow[d] \\ S_1 \arrow[r, "h"'] & S_2 \end{tikzcd}$$

between them. Then there is a natural morphism from $\mathrm{StMd}(F_1)$ to $\mathrm{StMd}(F_2)$

$$\mathrm{StMd}(\mathbf{H}) \;=\; \begin{tikzcd} \mathcal{C}'_1 \arrow[r, "H'"] \arrow[d] & \mathcal{C}'_2 \arrow[d] \\ S_1 \arrow[r, "h'"'] & S_2 \end{tikzcd}$$

such that $H' \circ c_{F_1} = c_{F_2} \circ H$. Moreover, $\mathrm{StMd}(\mathbf{HK}) = \mathrm{StMd}(\mathbf{H})\,\mathrm{StMd}(\mathbf{K})$ whenever $\mathbf{HK}$ is defined.

Clutching

We introduce an important construction used later in Chapter 5 to study the structure of the moduli space of stable curves. First, we describe it for a single curve, and then explain how it can be generalised to families.

Let P be a finite set, and let Γ be a P-marked dual graph; for each $p \in P$, denote ℓ_p the corresponding leg in Γ. For each $v \in V = V(\Gamma)$, recall that L_v is the set of half-edges issuing from v. Suppose that we are given an L_v-pointed genus g_v nodal curve C_v for each vertex v. We may then construct a new P-pointed curve C by identifying two points of the disjoint union $\coprod_v C_v$ if and only if they are marked points labelled by the two halves of an edge of Γ, and by labelling with $p \in P$ the point originally labelled by ℓ_p.

Clearly, the curve $\coprod C_v$ is the partial normalisation of C at the nodes introduced by the identifications. Hence, the process we have just described is a sort of inverse to partial normalisation. Note also that if Γ is connected and all the C_v are connected, then C is connected as well. Conversely, the connectedness of C implies that of Γ. If not further specified, we will assume that Γ and all C_v are connected.

The genus of C is given by

$$g(C) = \sum_{v \in V} g_v + 1 - \chi(\Gamma) = \sum_{v \in V} g_v + h^1(\Gamma).$$

Since we pass from C to $\coprod C_v$ by a partial normalisation, C is stable if and only if all the C_v are, as observed in the proof of Proposition 2.3.3. Stability is then equivalent to

$$2g_v - 2 + l_v > 0,$$

where $l_v = |L_v|$.

The curve C we have constructed is said to be obtained from the C_v by *clutching along* Γ.

Let us see how this procedure generalizes to families. Let Γ be as above. Suppose that for each $v \in V$, we are given a family

$$F_v \quad = \quad \begin{tikzcd} \mathcal{C}_v \arrow[d, "f_v"'] \\ S \arrow[u, bend right, "\sigma_\ell"'] \end{tikzcd}, \qquad \ell \in L_v$$

of stable L_v-pointed genus g_v curves, all over the same base scheme S. Let $\mathcal{C}$ be the disjoint union of the $\mathcal{C}_v$, and $f\colon \mathcal{C} \to S$ the morphism whose restriction to $\mathcal{C}_v$ is f_v. This gives an L-pointed family of nodal curves over S, which we denote by F. If $m \in L$, taking residues[1] along σ_m yields a surjective homomorphism

$$\omega_f\left(\sum \sigma_\ell\right) \to \mathcal{O}_{\sigma_m(S)},$$

hence surjective homomorphisms

$$\omega_f^k\left(k\sum \sigma_\ell\right) \to \mathcal{O}_{\sigma_m(S)}$$

for all $k > 0$. These give homomorphisms

$$R_\ell^{(k)}\colon f_*\left(\omega_f^k\left(k\sum \sigma_\ell\right)\right) \to \mathcal{O}_S,$$

which are surjective for $k \geq 2$, by Corollary 2.3.12. We can then construct homomorphisms

$$R^{(k)}\colon f_*\left(\omega_f^k\left(k\sum \sigma_\ell\right)\right) \to \mathcal{O}_S^{\oplus E}$$

by defining its component indexed by the edge $\{\ell, \ell'\}$ to be $R_\ell^{(k)} + (-1)^{k-1}R_{\ell'}^{(k)}$. This homomorphism has constant rank, indeed the kernel of its fibre at $s \in S$ can be identified with $H^0\left(\mathcal{C}'_s, \omega_{\mathcal{C}_s}^k\left(k\sum_p \sigma_{\ell_p}(s)\right)\right)$, where $\mathcal{C}'_s$ is the curve obtained from $\mathcal{C}_s$ by clutching along Γ, and hence its dimension is independent of s. It follows that the kernel $\mathcal{S}_k$ of $R^{(k)}$ is locally free. It also follows from Corollary 2.3.12 that the graded $\mathcal{O}_S$-algebra $\bigoplus_{k\geq 0}\mathcal{S}_k$ is locally finitely generated. We set

$$\mathcal{C}' := \operatorname{Proj}\left(\bigoplus_{k\geq 0}\mathcal{S}_k\right).$$

As a consequence of Proposition 2.3.11, the fibre of $\mathcal{C}' \to S$ at s is precisely the curve obtained from $\mathcal{C}_s$ by clutching along Γ. For each $p \in P$, we define $\sigma'_p : S \to \mathcal{C}'$ to be the composition of σ_{ℓ_p} with the natural map $\mathcal{C} \to \mathcal{C}'$. The family

[1] Defined in 2.2 to describe the dualizing sheaf of a nodal curve.

$$F' \quad = \quad \begin{tikzcd} \mathcal{C}' \arrow[d, "f'"'] \\ S \arrow[u, bend right, "\sigma'_p"'] \end{tikzcd}, \qquad p \in P$$

is said to be obtained from $F = (F_v)_{v \in V}$ by clutching along Γ. As the stable model, the clutching is functorial.

Chapter 3

Deformation Theory

Deformation theory is essentially the study of the local geometry of a moduli space, and this will be later clear. We are interested in two classical deformation problems:

- *Deformations* of a scheme X_0 over k.
- *Embedded deformations* of a closed subscheme Z_0 in a fixed projective scheme X over k.

Here, we focus mainly on the first problem, since studying it in the context of curves provides local information on the moduli space $\overline{\mathcal{M}}_{g,n}$. In particular, after discussing infinitesimal deformations, we will briefly talk about general ones and the concept of a Kuranishi family. Finally, we discuss the second problem, which leads to the construction of the Hilbert scheme parametrising these type of deformations. Throughout this thesis, we will exploit the Hilbert scheme in several proofs.

3.1 Infinitesimal deformations

First-order deformations

For a field k, denote the *dual numbers* by $k[\varepsilon] := k[\varepsilon]/(\varepsilon^2)$.

Definition 3.1.1. Let X_0 be a scheme over a field k. A *first-order deformation of* X_0 is a scheme X flat over $k[\varepsilon]$ together with an isomorphism $\alpha\colon X_0 \to X \times_{k[\varepsilon]} k$, or in other words a cartesian diagram

$$\begin{tikzcd} X_0 \arrow[r, hook, dashed] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, dashed, "\text{flat}"] \\ \operatorname{Spec} k \arrow[r, hook] & \operatorname{Spec} k[\varepsilon]. \end{tikzcd}$$

A *morphism* of first-order deformations $(X, \alpha) \to (X', \alpha')$ is a morphism of schemes $f\colon X \to X'$ over $k[\varepsilon]$ such that $(f \times_{k[\varepsilon]} k) \circ \alpha = \alpha'$; in other words, considering X and X' in cartesian diagrams as above, we require the restriction of f to the central fibre X_0 to be the identity.

Note that there always exists a first-order deformation $X_0 \times_k k[\varepsilon]$ of X_0. We say that a first-order deformation X of X_0 is *trivial* if it is isomorphic (as first-order deformations) to $X_0 \times_k k[\varepsilon]$, and *locally trivial* if there exists a Zariski cover $X = \bigcup_i U_i$ such that U_i is a trivial first-order deformation of $U_i \times_{k[\varepsilon]} k \subseteq X_0$. If every first-order deformation of X_0 is trivial, we say that X_0 is *rigid*.
As a consequence of the following algebraic fact, every morphism of first-order deformations is actually an isomorphism.

Lemma 3.1.2. *Let A be a ring, $\mathfrak{n} \subseteq A$ be a nilpotent ideal, and $M \to N$ be a homomorphism of A-modules. Assume that N is flat over A. If the induced map $M/\mathfrak{n}M \to N/\mathfrak{n}N$ is an isomorphism, so is $M \to N$.*

Proof. If $C \coloneqq \operatorname{coker}(M \to N)$, then $C/\mathfrak{n}C = \operatorname{coker}(M/\mathfrak{n}M \to N/\mathfrak{n}N) = 0$. Since $\mathfrak{n}$ is nilpotent, $\mathfrak{n}^k = 0$ for some k, thus $C = \mathfrak{n}C = \cdots = \mathfrak{n}^k C = 0$. Similarly, if $K = \ker(M \to N)$, then $K/\mathfrak{n}K = \ker(M/\mathfrak{n}M \to N/\mathfrak{n}N) = 0$ thanks to the flatness hypothesis, hence this again implies that $K = 0$. □

Proposition 3.1.3. *Every first-order deformation of a smooth affine scheme X_0 over k is trivial.*

Proof. Let X be a first-order deformation of X_0. We thus have a commutative diagram

$$\begin{tikzcd} X_0 \arrow[r, "\mathrm{id}"] \arrow[d, hook] & X_0 \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} k \end{tikzcd},$$

and since $X_0 \to \operatorname{Spec} k$ is smooth, we can construct a lift $X \to X_0$. If we do the base change to $k[\varepsilon]$, then we obtain a morphism $X \to X_0 \times_k k[\varepsilon]$ which restricts to the identity on the central fibre X_0, and thus it is an isomorphism by Lemma 3.1.2. □

Remark. The hypotheses on X_0 cannot be removed. Indeed

- if $X_0 = \operatorname{Spec} k[x, y]/(xy)$, then $\operatorname{Spec} k[x, y, t\varepsilon]/(xy - \varepsilon) \to \operatorname{Spec} k[\varepsilon]$ such that is a non-trivial first-order deformation; and
- If $X_0 = V(y^2 z - x(x - z)(x - \lambda z)) \subseteq \mathbb{P}^2$ for $\lambda \neq 0, 1$, then the closed subscheme $V(y^2 z - x(x - z)(x - (\lambda + \varepsilon)z)) \subseteq \mathbb{P}^2_{k[\varepsilon]}$ defines a non-trivial first-order deformation.

For a locally trivial deformation X, there exists a Zariski open cover for which X locally trivializes. Therefore, to give a locally trivial deformation it suffices to specify how to glue trivial deformations of an affine open cover. This motivates us to understand automorphisms of the trivial deformation.

Lemma 3.1.4. *Let $X_0 = \operatorname{Spec} A$ be an affine scheme over k, and $X = \operatorname{Spec} A[\varepsilon]$ be the trivial first-order deformation. There are identifications*

$$\{\textit{automorphisms } X \to X \textit{ of first-order def.}\} \simeq \operatorname{Der}_k(A, A) \simeq \operatorname{Hom}_A(\Omega_{A/k}, A).$$

Proof. The second equivalence follows from the universal property of the module of differentials. Regarding the first one, such an automorphism of X corresponds to a $k[\varepsilon]$-algebra isomorphisms $\phi\colon A[\varepsilon] \to A[\varepsilon]$ that is the identity modulo ε. Therefore, ϕ is determined by the images $\phi(a) = a + d(a)\varepsilon$ of elements $a \in A$, where $d\colon A \to A$ is a k-linear map. Now the map ϕ is a ring homomorphism if and only if $\phi(ab) = \phi(a)\phi(b)$, namely

$$ab + d(ab)\varepsilon = (a + d(a)\varepsilon)(b + d(b)\varepsilon) = ab + (ad(b) + d(a)b)\varepsilon$$

for elements $a, b \in A$, which is the condition that $d\colon A \to A$ is a k-derivation. $\square$

If X_0 is a scheme over k, denote with $\operatorname{Def}(X_0)$ and $\operatorname{Def}^{\mathrm{lt}}(X_0)$ respectively the sets of isomorphism classes of first-order and locally trivial first-order deformations.

Proposition 3.1.5. *Let X_0 be a scheme of finite type over k with affine diagonal. There is a bijection*

$$\operatorname{Def}^{\mathrm{lt}}(X_0) \simeq H^1(X_0, T_{X_0})$$

under which the trivial deformation corresponds to $0 \in H^1(X_0, T_{X_0})$. If in addition X_0 is smooth over k, there is a bijection

$$\operatorname{Def}(X_0) \simeq H^1(X_0, T_{X_0}).$$

Proof. Let $X \to \operatorname{Spec} k[\varepsilon]$ be a locally trivial first-order deformation of X_0. Choose an affine cover $\{U_i\}$ of X such that U_i is a trivial deformation of $V_i := U_i \times_{k[\varepsilon]} k$, i.e., we have isomorphisms $\phi_i\colon V_i \times_k k[\varepsilon] \xrightarrow{\sim} U_i$. Denoting $V_{ij} = V_i \cap V_j$, we have automorphisms $\phi_j^{-1}|_{V_{ij}\times_k k[\varepsilon]} \circ \phi_i^{-1}|_{V_{ij}\times_k k[\varepsilon]}$ of the trivial deformation $V_{ij} \times_k k[\varepsilon]$, which corresponds by Lemma 3.1.4 to elements $\phi_{ij} \in T_{X_0}(V_{ij})$. Since $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ on $V_{ijk} = V_i \cap V_j \cap V_k$, we have that $\phi_{ij} + \phi_{jk} = \phi_{ik} \in T_{X_0}(V_{ijk})$. On the other hand, we can compute $H^1(X_0, T_{X_0})$ using the Čech complex

$$0 \to \bigoplus_i T_{X_0}(V_i) \xrightarrow{d_0} \bigoplus_{i,j} T_{X_0}(V_{ij}) \xrightarrow{d_1} \bigoplus_{i,j,k} T_{X_0}(V_{ijk})$$

$$(s_{ij}) \mapsto (s_{ij}|_{U_{ijk}} - s_{ik}|_{U_{ijk}} + s_{jk}|_{U_{ijk}})_{ijk}$$

and $\{\phi_{ij}\}$ is in the kernel of d_1, thus it defines am element of $H^1(X_0, T_{X_0})$. Conversely, given a representative $\{\psi_{ij}\} \in \ker d_1$ of an element of $H^1(X_0, T_{X_0})$, we can view each ψ_{ij} as an automorphism of the trivial deformation of V_{ij}, and we may glue together these trivial deformations via ψ_{ij} to construct a global deformation X of X_0. Finally, in the smooth case any first-order deformation is locally trivial by Proposition 3.1.3. $\square$

Example. Using the Euler sequence, one can easily show that $H^1(\mathbb{P}^n, T_{\mathbb{P}^n}) = 0$. By the proposition above, we get that $\mathbb{P}^n$ is rigid.

Higher-order deformations

If our scheme X_0 is defined over a ring A, and $A' \twoheadrightarrow A$ is a surjection of rings, we can define a deformation of X_0 over A' as for first-order deformations. However, it is not even clear in this situation if such a deformation exists, since in general we do not have a map from A to A'.

Definition 3.1.6. Let $A' \twoheadrightarrow A$ be a surjection of rings with nilpotent kernel and X be a flat scheme over A. An *(infinitesimal) deformation of* $X \to \operatorname{Spec} A$ *over* A' is a flat morphism $X' \to \operatorname{Spec} A'$ together with an isomorphism $\alpha \colon X \xrightarrow{\sim} X' \times_{A'} A$ over A, or in other words a cartesian diagram

$$\begin{tikzcd} X \arrow[r, hook, dashed] \arrow[d, "\text{flat}"] \arrow[dr, phantom, "\square"] & X' \arrow[d, dashed, "\text{flat}"] \\ \operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A'. \end{tikzcd}$$

A *morphism of deformations over* A' is a morphism over A' restricting to the identity on X.

Remark. Again by Lemma 3.1.2, every morphism of deformations is actually an isomorphism.

Proposition 3.1.7. *Let* X_0 *be a scheme of finite type over a field* k *such that* X_0 *is generically smooth and a local complete intersection. Let* $A' \twoheadrightarrow A$ *be a surjection of artinian local rings with residue field* k *such that* $\mathfrak{m}_{A'} J = 0$ *where* $J := \ker(A' \to A)$*. If* $X \to \operatorname{Spec} A$ *is a deformation of* X_0*, then:*

(1) The group of automorphisms of a deformation of $X \to \operatorname{Spec} A$ *over* A' *is bijective to* $\operatorname{Ext}^0_{\mathcal{O}_{X_0}}(\Omega_{X_0}, J)$*;*

(2) If there exists a deformation of $X \to \operatorname{Spec} A$ *over* A'*, then the set of isomorphisms classes of all such deformations is a torsor under* $\operatorname{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, J)$*; and*

(3) *There exists an element* $ob_X \in \mathrm{Ext}^2_{\mathcal{O}_{X_0}}(\Omega_{X_0}, J)$*, called the obstruction, with the property that there exists a deformation of* $X \to \operatorname{Spec} A$ *over* A' *if and only if* $ob_X = 0$*.*

In particular, if X_0 *is smooth, then the informations above are encoded in the cohomology groups* $H^i(X_0, T_{X_0}) \otimes_k J$ *for* $i = 0, 1, 2$*.*

Proof. See [Vis99, Thm. 4.4]. □

Formal deformations

Let X be a scheme over a field k. As for the infinitesimal case, a *deformation* of X over a scheme T of finite type over k is a cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\text{flat}"] \\ \operatorname{Spec} k \arrow[r] & T \end{tikzcd}$$

over k where the morphism $\mathcal{X} \to T$ is flat. Clearly, this means that if $\operatorname{Spec} k \to T$ represents a point $t_0 \in T(k)$, then the fibre of $\mathcal{X} \to T$ over t_0 is isomorphic to X. The notion of isomorphism of deformations is the obvious one as for the infinitesimal case.

Let Art_k be the category of artinian local k-algebras with residue field k. For a scheme X over k we define its *deformation functor* as

$$\begin{aligned} \mathrm{Def}_X \colon \mathrm{Art}_k &\to \mathrm{Sets} \\ A &\mapsto \{\text{isomorphism classes of deformations of } X \text{ over } \operatorname{Spec} A\}. \end{aligned}$$

Clearly, $\mathrm{Def}_X(k)$ is a singleton. Note that we may view Art_k as a subcategory of $\mathrm{Sch}^{\mathrm{op}}$ under the correspondence $A \mapsto \operatorname{Spec} A$, hence via Yoneda's Lemma an infinitesimal deformation of X over $A \in \mathrm{Art}_k$ corresponds to a morphism $\operatorname{Spec} A \to \mathrm{Def}_X$, where we are denoting by $\operatorname{Spec} A$ the functor $\mathrm{Mor}_{\mathrm{Sch}_k}(-, \operatorname{Spec} A)$ represented by it.

A natural question one may ask himself is whether for a scheme X there exists a "universal" deformation. More precisely, we search for a complete noetherian local k-algebra R with residue field k that *prorepresents* the deformation functor Def_X, i.e., there is an isomorphism $\mathrm{Def}_X \simeq h_R$ of functors $\mathrm{Art}_k \to \mathrm{Sets}$, where $h_R = \operatorname{Spf} R \coloneqq \mathrm{Hom}_{k-\mathrm{alg}}(R, -)$ is often called the *formal spectrum* of R.
Observe that we use the term prorepresentable since R is not an object of Art_k, hence it does not represent the deformation functor in the usual way.
Prorepresentability is a strong property, thus as one may expect, many functors satisfy only weaker properties.

Denote by Comp_k the category of complete noetherian local k-algebras with residue field k. Recall that the spectrum of such an algebra over $\mathbb{C}$ should be thought of as an algebraic version of a complex analytic germ.

Definition 3.1.8. Let X be a scheme over a field k. A *formal deformation* of X over $R \in \mathrm{Comp}_k$ is a commutative diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \mathcal{X}_1 \arrow[r, hook] \arrow[d] & \mathcal{X}_2 \arrow[r, hook] \arrow[d] & \dots \\ \operatorname{Spec} k \arrow[r, hook] & \operatorname{Spec} R/\mathfrak{m}^2 \arrow[r, hook] & \operatorname{Spec} R/\mathfrak{m}^3 \arrow[r, hook] & \dots \end{tikzcd} \tag{3.1}$$

where $\mathfrak{m}$ is the maximal ideal of R and every square is an infinitesimal deformation. We denote such a diagram by $\mathcal{X}_\bullet \to \operatorname{Spf} R$, where the target is called the formal spectrum of R.

As an infinitesimal deformation of X over $R/\mathfrak{m}^n$ can be viewed as a morphism $\operatorname{Spec} R/\mathfrak{m}^n \to \mathrm{Def}_X$, a formal deformation over R corresponds to a morphism $\mathcal{X}_\bullet \colon \operatorname{Spf} R \to \mathrm{Def}_X$. Recalling the Grothendieck's functorial characterisation of smoothness [SP, Tag 02H6], we may extend this definition to functors $\mathrm{Art}_k \to \mathrm{Sets}$. In particular, the morphism $\operatorname{Spf} R \to \mathrm{Def}_X$ induced by a formal deformation of X over R is *smooth* if for every surjection $A' \twoheadrightarrow A$ in Art_k and for every commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] \arrow[d] & \operatorname{Spf} R \arrow[d, "\mathcal{X}_\bullet"] \\ \operatorname{Spec} A' \arrow[r] \arrow[ur, dashed] & \mathrm{Def}_X \end{tikzcd}$$

of solid arrows, where the left vertical arrow is the closed embedding induced by the surjection $A' \twoheadrightarrow A$, there exists a morphism $\operatorname{Spec} A' \to \operatorname{Spf} R$ filling the diagram, often called a *lifting*.
Moreover, we define the *tangent space* to a functor $F\colon \mathrm{Art}_k \to \mathrm{Sets}$ as

$$TF := F(k[\varepsilon]).$$

Schlessinger in [Sch68] studied these functors and introduced four conditions regarding the properties of the natural map

$$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$$

as $A' \to A$ and $A'' \to A$ in Art_k vary; see [TV13] for a modern treatment. In particular, it is not difficult to see that a deformation functor Def_X of a k-scheme X satisfies the so called condition (H2), implying that its tangent space has a natural structure of k-vector space [Sch68, Rmk 2.13].

Definition 3.1.9. Let X be a scheme over a feld k and let $R \in \mathrm{Comp}_k$. A formal deformation $\mathcal{X}_\bullet \to \operatorname{Spf} R$ is said to be *versal* if the induced morphism $\operatorname{Spf} R \to \mathrm{Def}_X$ is smooth.

Note that if a versal deformation $\mathcal{X}_\bullet \to \operatorname{Spf} R$ of X/k exists, then it induces every infinitesimal deformation. Indeed, if $\mathcal{X}^A \in \mathrm{Def}_X(A)$ is an infinitesimal deformation of X over A, then the versality property applied to the surjection $A \to k$ implies that there exists a k-algebra homomorphism $\phi\colon R \to A$ such that $\mathcal{X}^A$ is isomorphic to the pullback of $\mathcal{X}_\bullet$ via ϕ.

Definition 3.1.10. A versal deformation $\mathcal{X}_\bullet \to \operatorname{Spf} R$ is called *miniversal* if

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim_k T\,\mathrm{Def}_X$$

where $\mathfrak{m}$ is the maximal ideal of R.

Remark. Note that also $T\operatorname{Spf} R$ has a structure of k-vector space as it is canonically isomorphic (as a k-algebra) to $\mathrm{Hom}_{k-\mathrm{alg}}(R, k[\varepsilon])$. Smoothness of a morphism $\operatorname{Spf} R \to \mathrm{Def}_X$ implies the surjectivity of the *Kodaira-Spencer* morphism, i.e., the induced map

$$T\operatorname{Spf} R \to T\,\mathrm{Def}_X$$

between tangent spaces, since any tangent vector $\operatorname{Spec} k[\varepsilon] \to \mathrm{Def}_X$ can be lifted to $\operatorname{Spec} k[\varepsilon] \to \operatorname{Spf} R$. It follows that

$$\dim_k T\,\mathrm{Def}_X \leq \dim_k T\operatorname{Spf} R = \dim_k \mathfrak{m}/\mathfrak{m}^2.$$

Therefore, miniversality means that the tangent spaces are isomorphic.

Theorem 3.1.11. *Let X be a scheme over a field k. If $T\,\mathrm{Def}_X$ is a finite-dimensional k-vector space, then a miniversal deformation of X exists and is unique up to isomorphism.*

Proof. See [TV13, Prop. 6.30 and 6.33]. □

The base $\operatorname{Spf} R$ of the miniversal deformation of X is sometimes called the *Kuranishi space* of X. However, this object can be arbitrarily bad [Vak06].

Effectiveness

Note that for an algebra $R \in \mathrm{Comp}_k$ it holds $R = \varprojlim R/\mathfrak{m}^n$. However, it is not clear whether a formal deformation $\mathcal{X}_\bullet \to \operatorname{Spf} R$ of a k-scheme X is induced by a deformation of X over R. In other words, are we able to complete the diagram (3.1) as below?

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d] & \mathcal{X}_1 \arrow[r, hook] \arrow[d] & \mathcal{X}_2 \arrow[r, hook] \arrow[d] & \dots \arrow[r, hook, dashed] & \mathcal{X} \arrow[d, dashed] \\
\operatorname{Spec} k \arrow[r, hook] & \operatorname{Spec} R/\mathfrak{m}^2 \arrow[r, hook] & \operatorname{Spec} R/\mathfrak{m}^3 \arrow[r, hook] & \dots \arrow[r, hook] & \operatorname{Spec} R
\end{tikzcd}$$

If this is the case, we say that the formal deformation $\mathcal{X}_\bullet \to \operatorname{Spf} R$ is *effective*. The Grothendieck's existence theorem [Alp27, Thm C.5.3] implies the following general result.

Proposition 3.1.12. *Let X be a scheme proper over k. Suppose that we are given a formal deformation $\mathcal{X}_\bullet \to \operatorname{Spf} R$ together with compatible line bundles L_i on $\mathcal{X}_i$ (where $\mathcal{X}_0 = X$) such that L_0 is ample. Then there exists a projective morphism $\mathcal{X} \to \operatorname{Spec} R$, together with a compatible ample line bundle L on $\mathcal{X}$, fitting in the diagram*

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d] & \mathcal{X}_1 \arrow[r, hook] \arrow[d] & \mathcal{X}_2 \arrow[r, hook] \arrow[d] & \dots \arrow[r, hook] & \mathcal{X} \arrow[d] \\
\operatorname{Spec} k \arrow[r, hook] & \operatorname{Spec} R/\mathfrak{m}^2 \arrow[r, hook] & \operatorname{Spec} R/\mathfrak{m}^3 \arrow[r, hook] & \dots \arrow[r, hook] & \operatorname{Spec} R.
\end{tikzcd}$$

Proof. See [Alp27, Cor. C.5.8]. □

In the situation above, we can take as Kuranishi space the scheme $\operatorname{Spec} R$, while in general this is only the formal scheme $\operatorname{Spf} R$.

3.2 About stable curves

In this section we apply the deformation theory developed so far to the case of stable curves. First, we look at the smooth case.

Corollary 3.2.1. *If C is a connected smooth projective curve of genus $g \geq 2$ over the field k, then*

$$\operatorname{Def}(C) \simeq H^1(C, T_C) \overset{SD}{\simeq} H^0(C, \Omega_C^{\otimes 2}),$$

which is a k-vector space of dimension $3g - 3$.

Proof. Isomorphisms follow from Proposition 3.1.5 and Serre Duality. Finally, the tangent sheaf is a line bundle of degree $\deg(T_C) = 2 - 2g < 0$, thus $h^0(C, T_C) = 0$ and by Riemann-Roch we compute $h^1(C, T_C) = -\deg(T_C) + g - 1 = 3g - 3$. □

Remark. For a curve C as above over an algebraically closed field k, we can compute the dimension of $\mathrm{Def}(C)$ also for genus $g = 0, 1$. For $g = 0$, we get $C \simeq \mathbb{P}^1$, then $T_C \simeq \mathcal{O}_{\mathbb{P}^1}(2)$ and consequently $h^1(T_C) = 0$. While for $g = 1$, we get $T_C \simeq \mathcal{O}_C$, hence $h^1(C, T_C) = 1$.

Since nodal curves are locally complete intersections [ACG11, Lemma X.2.4], by Proposition 3.1.7 their deformations are classified by $\mathrm{Ext}^i_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$. These results can be extended to the case of pointed nodal curves; see [Alp27, Prop. C.2.8].

Proposition 3.2.2. *Let $A' \twoheadrightarrow A$ be a surjection of artinian local rings with residue field k such that $J = \ker(A' \to A)$ satisfies $\mathfrak{m}_{A'} J = 0$. Let $(\mathcal{C} \to \operatorname{Spec} A, \sigma_i)$ be a family of n-pointed prestable curves over A, and let (C, p_i) be its central fibre, namely its base change to k. Then:*

(1) The group of automorphisms of a deformation of $(\mathcal{C} \to \operatorname{Spec} A, \sigma_i)$ over A' is $\mathrm{Ext}^0_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C \otimes_k J)$;

(2) The set of isomorphism classes of deformations of $(\mathcal{C} \to \operatorname{Spec} A, \sigma_i)$ over A' is a torsor under $\mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C \otimes_k J)$ and

$$\dim_k \mathrm{Ext}^1_{\mathcal{O}_C}\left(\Omega_C\left(\sum p_i\right), \mathcal{O}_C\right) - \dim_k \mathrm{Ext}^0_{\mathcal{O}_C}\left(\Omega_C\left(\sum p_i\right), \mathcal{O}_C\right) = 3g-3+n;$$

(3) There is no obstruction to deforming $(\mathcal{C} \to \operatorname{Spec} A, \sigma_i)$ over A'.

Moreover, if (C, p_i) is stable, then $\mathrm{Ext}^0_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C) = 0$, i.e., there are no non-trivial automorphisms of a deformation, thus

$$\dim_k \mathrm{Ext}^1_{\mathcal{O}_C}\left(\Omega_C\left(\sum p_i\right), \mathcal{O}_C\right) = 3g - 3 + n.$$

Proof. See [Alp27, Prop. 6.3.26]. $\square$

Miniversal family

By Theorem 3.1.11 we know that for a stable curve C over a field k the miniversal deformation exists. Moreover, if C is proper, Proposition 3.1.12 implies that such a deformation is effective, i.e., its Kuranishi space is a usual spectrum $\operatorname{Spec} R$, for some $R \in \mathrm{Comp}_k$.

Proposition 3.2.3. *Let X be a variety over a field k of characteristic zero such that either X is normal or X is reduced and lci. If $r := \dim_k T\,\mathrm{Def}_X < +\infty$, then the Kuranishi space of the miniversal deformation of X is $\operatorname{Spf} k[\![t_1, \ldots, t_r]\!]/I$ for some ideal I contained in $(t_1, \ldots, t_r)^2$.*

Proof. The Kuranishi space of X is $\operatorname{Spf} R$ for some $R \in \mathrm{Comp}_k$ and it follows $r = \operatorname{embdim} R$. By Cohen structure theorem [SP, Tag 032A] we get that R is a quotient of $k[\![t_1, \ldots, t_n]\!]$ for some n, and clearly $n \geq r$. But if $n > r$, since r is the embedding dimension of R, then there would be an isomorphism between R and a quotient of $k[\![t_1, \ldots, t_{n-1}]\!]$, hence we may assume that R is $k[\![t_1, \ldots, t_n]\!]/I$. Finally, if I were not contained in $(t_1, \ldots, t_r)^2$, we would have a contradiction as the embedding dimension of R is exactly r. □

Corollary 3.2.4. *Let C be an n-pointed stable curve of genus g over a field k. The miniversal deformation of C is given by a projective morphism*

$$\mathcal{C} \to \operatorname{Spec} k[\![t_1, \ldots, t_{3g-3+n}]\!].$$

Proof. First of all, C is projective, and given an ample line bundle on it, we may deform it in order to get the sequence in the assumptions of Proposition 3.1.12; see [Alp27, Rmk C.5.9]. Now since C is reduced and lci, and by Proposition 3.2.2

$$\dim_k T \operatorname{Def}_X = \dim_k \operatorname{Ext}^1_{\mathcal{O}_C}\left(\Omega_C\left(\sum p_i\right), \mathcal{O}_C\right) = 3g - 3 + n,$$

it follows from the proposition above that its Kuranishi space is

$$\operatorname{Spec} k[\![t_1, \ldots, t_{3g-3+n}]\!]/I$$

for some ideal I. Finally, by [TV13, Prop. 6.38] the minimal number of generators of I is at most the dimension of the obstruction space of C, but in Proposition 3.2.2 we have seen that there is no obstruction to deforming a nodal curve, hence $I = 0$. □

If $U = \operatorname{Spec} \mathbb{C}[\![t_1, \ldots, t_{3g-3+n}]\!]$ is the miniversal deformation of an n-pointed stable curve C, and we denote by u_0 the unique closed point of U, then the Kodaira-Spencer map provides an identification

$$T_{U,u_0} \xrightarrow{\sim} \operatorname{Ext}^1_{\mathcal{O}_C}\left(\Omega_C\left(\sum p_i\right), \mathcal{O}_C\right).$$

3.3 Embedded deformations

A classical problem in algebraic geometry is to parametrise closed subschemes of the projective space $\mathbb{P}^n_k$ over a field k with a fixed Hilbert polynomial $P \in \mathbb{Q}[z]$. This led to the definition of the Hilbert functor

$$\begin{aligned}\underline{\mathrm{Hilb}}^P(\mathbb{P}^n_S/S)\colon (\mathrm{Sch}/S)^{\mathrm{op}} &\to \mathrm{Sets} \\ (T \to S) &\mapsto \left\{\begin{matrix}\text{closed subschemes } Z \subseteq \mathbb{P}^n_T \text{ flat and finitely} \\ \text{presented over } T \text{ such that } Z_t \subseteq \mathbb{P}^n_{\kappa(t)} \text{ has} \\ \text{Hilbert polynomial } P \text{ for all } t \in T\end{matrix}\right\}\end{aligned}$$

It is a classical result that this functor is represented by a projective scheme over S, denoted by $\mathrm{Hilb}^P(\mathbb{P}^n_S/S)$; see [Alp27, §2] for a comprehensive discussion. Note that, as schemes, $\mathrm{Hilb}^P(\mathbb{P}^n_S/S) = \mathrm{Hilb}^P(\mathbb{P}^n_{\mathbb{Z}}/\mathbb{Z}) \times_{\mathbb{Z}} S$.

We may also generalise the functor above by considering closed subschemes of a fixed scheme X over a noetherian scheme S. However, if X is projective over S, then there is still a projective scheme $\mathrm{Hilb}^P(X/S)$ representing this functor.

The local study of the Hilbert scheme is provided by the theory of embedded deformations.

Definition 3.3.1. Let X be a scheme over a field k and $Z_0 \subseteq X$ be a closed subscheme. A *first-order embedded deformation of* $Z_0 \subseteq X$ is a closed subscheme $Z \subseteq X_\varepsilon := X \times_k k[\varepsilon]$ flat over $k[\varepsilon]$ such that $Z_0 = Z \times_{k[\varepsilon]} k$. In other words, a first-order embedded deformation is a filling of the diagram

$$\begin{tikzcd}
 & X \arrow[rr, hook] \arrow[dd] & & X_\varepsilon \arrow[dd] \\
Z_0 \arrow[ur, hook, "\mathrm{cl}"] \arrow[rr, hook, dashed] \arrow[dr] & & Z \arrow[ur, hook, dashed, "\mathrm{cl}"] \arrow[dr, dashed, "\mathrm{flat}"] & \\
 & \operatorname{Spec} k \arrow[rr, hook] & & \operatorname{Spec} k[\varepsilon]
\end{tikzcd}$$

with a scheme Z flat over $k[\varepsilon]$ and dotted arrows making the lower parallelogram cartesian.

Note that both Z_0 and $Z \times_{k[\varepsilon]} k$ are embedded in X, hence it makes sense to require them to be equal. Such an embedded deformation $Z \subseteq X_\varepsilon$ is said to be *trivial* if $Z = Z \times_k k[\varepsilon]$.

Remark. Rephrasing the definition above for a projective scheme X, the closed subscheme Z_0 is a k-point $[Z_0]$ of the Hilbert scheme $\mathrm{Hilb}^P(X/k)$, where P is the Hilbert polynomial of Z_0 with respect to a fixed embedding in a projective space. A first-order embedded deformation then corresponds to a morphism

$$\operatorname{Spec} k[\varepsilon] \to \mathit{Hilb}^P(X/k)$$

with image $[Z_0]$, namely a tangent vector in $T_{\mathit{Hilb}^P(X/k),[Z_0]}$.

Geometric intuition suggests us that a first order embedded deformation is given by a suitable choice of a "thickening direction" for each point of Z. The following classical result formalises this idea.

Proposition 3.3.2. *Let X be a scheme over a field k and $Z_0 \subseteq X$ be a closed subscheme. There is a bijection*

$$\{\textit{first order embedded deformations of } Z_0 \textit{ in } X\} \simeq H^0\left(Z_0, \mathcal{N}_{Z_0/X}\right)$$

where $\mathcal{N}_{Z_0/X}$ is the normal sheaf of Z_0 in X. Moreover, under this bijection, the trivial deformation corresponds to the zero section.

Proof. See [Alp27, Prop. C.1.3]. $\square$

As for deformations studied in the previous sections, often called *abstract deformations* to better distinguish them from embedded ones, we may consider high-order version of the latter.

Definition 3.3.3. Let $A' \twoheadrightarrow A$ be a surjection of rings with nilpotent kernel. Let X' be a scheme over A' and set $X \coloneqq X' \times_{A'} A$. If $Z \subseteq X$ be a closed subscheme flat over A, an *infinitesimal embedded deformation of $Z \subseteq X$ over A'* is a closed subscheme $Z' \subseteq X'$ flat over A' such that $Z' \times_{A'} A = Z$. In other words, such a deformation is a filling of the diagram

$$\begin{tikzcd}
 & X \arrow[rr, hook] \arrow[dd] & & X' \arrow[dd] \\
Z \arrow[ur, hook, "\mathrm{cl}"] \arrow[rr, hook, dashed] \arrow[dr, "\mathrm{flat}"'] & & Z' \arrow[ur, hook, dashed, "\mathrm{cl}"] \arrow[dr, dashed, "\mathrm{flat}"'] & \\
 & \operatorname{Spec} A \arrow[rr, hook] & & \operatorname{Spec} A'.
\end{tikzcd}$$

Similarly to abstract deformations, higher-order embedded ones are encoded in the Ext groups of a suitable sheaf.

Proposition 3.3.4. *Let X be a scheme over k with affine diagonal. Let $A' \twoheadrightarrow A$ be a surjection of artinian k-algebras with residue field k such that $\mathfrak{m}_{A'} J = 0$ where $J \coloneqq \ker(A' \to A)$. Let $Z \subseteq X_A = X \times_k \to A$ be a closed subscheme flat over A and let $Z_0 = Z \times_A k \subseteq X$ with ideal sheaf $I_{Z_0} \subseteq \mathcal{O}_X$. Then:*

(1) If there exists an embedded deformation $Z' \subseteq X_{A'}$ of Z over A', then the set of such deformations is a torsor under $\operatorname{Ext}^0_{\mathcal{O}_X}(I_{Z_0}, \mathcal{O}_{Z_0} \otimes_k J)$; and

(2) There exists an element $\operatorname{ob}_Z \in \operatorname{Ext}^1_{\mathcal{O}_X}(I_{Z_0}, \mathcal{O}_{Z_0} \otimes_k J)$, called the obstruction, with the property that there exists an embedded deformation of $Z \subseteq X$ over A' if and only if $\operatorname{ob}_Z = 0$.

Proof. See [Alp27, Prop. C.2.2]. $\square$

Observe that if $Z_0 \subseteq Z$ is a local complete intersection, i.e., the corresponding closed embedding is a lci morphism, then $I_{Z_0}/I_{Z_0}^2$ is locally free [Liu02, Cor. 6.3.8] and

$$\operatorname{Ext}^i_{\mathcal{O}_X}(I_{Z_0}, \mathcal{O}_{Z_0} \otimes_k J) = H^i(Z_0, \mathcal{N}_{Z_0/X}) \otimes_k J.$$

Chapter 4

Stack Theory

Grothendieck introduced stacks in [FGA] and [SGA1] as a way to package objects satisfying descent, e.g., quasi-coherent sheaves; see [Alp27, §3.1].

4.1 Grothendieck topologies

Firs of all, we need to extend the concept of a topology to categories.

Definition 4.1.1 (Sites)**.** A *Grothendieck topology* on a category $\mathcal{S}$ consists of the following data: for each object $X \in \mathcal{S}$, there is a set $\mathrm{Cov}(X)$ consisting of *coverings* of X, i.e., collections of morphisms $\{X_i \to X\}_{i \in I}$ in $\mathcal{S}$. We require that:

(i) if $X' \to X$ is an isomorphism, then $(X' \to X) \in \mathrm{Cov}(X)$;

(ii) if $\{X_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$ and $Y \to X$ is a morphism, then the fibred products $X_i \times_X Y$ exist in $\mathcal{S}$ for all $i \in I$, and $\{X_i \times_X Y \to Y\}_{i \in I} \in \mathrm{Cov}(Y)$; and

(iii) if $\{X_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$ and $\{X_{ij} \to X_i\}_{j \in J_i} \in \mathrm{Cov}(X_i)$ for each $i \in I$, then $\{X_{ij} \to X_i \to X\}_{i \in I, j \in J_i} \in \mathrm{Cov}(X)$.

A *site* is a category together with a Grothendieck topology.

Example. This definition in fact extends the notion of topological space. Indeed, if X is a topological space, we let $\mathrm{Op}(X)$ denote the category of open sets $U \subseteq X$, where there is a unique morphism $U \to V$ if and only if $U \subseteq V$. We say that an element of $\mathrm{Cov}(U)$ is a collection of open subsets $\{U_i\}_{i \in I}$ such that $U = \bigcup_{i \in I} U_i$. This defines a Grothendieck topology on $\mathrm{Op}(X)$. In particular, if X is a scheme, the Zariski topology on X defines a site X_{Zar}, called the *small Zariski site on* X.

Example. By replacing Zariski open immersions with étale morphisms, we obtain the *small étale site on* X. More precisely, it is the category $X_{\text{ét}}$ of étale morphisms $U \to X$, where a morphism $(U \to X) \to (V \to X)$ is simply an X-morphism $U \to V$ (which is necessarily étale). A covering of an object $(U \to X) \in X_{\text{ét}}$ is a colleciton of étale morphisms $\{U_i \to U\}$ such that $\coprod_i U_i \to U$ is surjective.

Example. The most important site in this thesis is the *big étale site* $\mathrm{Sch}_{\text{ét}}$, that is the category Sch of schemes where a covering of a scheme U is a collection of étale morphisms $\{U_i \to U\}$ in Sch such that $\coprod_i U_i \to U$ is surjective.
Replacing in the definition above étale morphisms with open immersions defines the *big Zariski site* $\mathrm{Sch}_{\mathrm{Zar}}$.

Example (fppf, fpqc). A morphism of schemes is *fppf* if it is surjective, flat, and locally of finite presentation. The *big fppf site* $\mathrm{Sch}_{\mathrm{fppf}}$ is the category Sch of schemes where a covering $\{U_i \to U\}$ is a collection of morphisms such that $\coprod_i U_i \to U$ is fppf.
A morphism of schemes is *fpqc* if it is surjective, flat, and every quasi-compact subset of the target is the image of a quasi-compact subset. The *big fpqc site* $\mathrm{Sch}_{\mathrm{fpqc}}$ is the category Sch of schemes where a covering $\{U_i \to U\}$ is a collection of morphisms such that $\coprod_i U_i \to U$ is fpqc.
Observe that

$$\text{étale} \implies \text{fppf} \implies \text{fpqc}.$$

Given a scheme S, by applying the constructions described above to the category Sch/S, we can define the relative versions $(\mathrm{Sch}/S)_{\mathrm{Zar}}$, $(\mathrm{Sch}/S)_{\text{ét}}$, $(\mathrm{Sch}/S)_{\mathrm{fppf}}$, and $(\mathrm{Sch}/S)_{\mathrm{fpqc}}$.

Sheaves

If X is a topological space, a presheaf of sets on X is simply a functor

$$F\colon \mathrm{Op}(X)^{\mathrm{op}} \to \mathrm{Sets}$$

and the sheaf axioms can be summarized in the condition that for each covering $U = \bigcup_i U_i$, the sequence

$$F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_{ij})$$

is an equalizer diagram, where $U_{ij} = U_i \cap U_j$, and the two maps are induced by restricting along the two inclusions.

These definitions can be generalized to sites.

Definition 4.1.2. A *presheaf* on a category $\mathcal{S}$ is a functor $\mathcal{S}^{\mathrm{op}} \to \mathrm{Sets}$.
A *sheaf* of a site $\mathcal{S}$ is a presheaf $F\colon \mathcal{S}^{\mathrm{op}} \to \mathrm{Sets}$ such that for every object X in $\mathcal{S}$ and covering $\{X_i \to X\}_i \in \mathrm{Cov}(X)$, the sequence

$$F(X) \to \prod_i F(X_i) \rightrightarrows \prod_{i,j} F(X_{ij})$$

is an equalizer diagram, where $X_{ij} = X_i \times_X X_j$, and the two maps are induced by the two maps $S_{ij} \to S_i$ and $S_{ij} \to S_j$.
A *morphism* of presheaves or sheaves is a natural transformation.

If F is a presheaf on $\mathcal{S}$ and $f : X \to Y$ is a morphism in $\mathcal{S}$, then the pullback $F(f)(a)$ of an element $a \in F(T)$ is often denoted by f^*a or $a|_S$.
We denote the categories of presheaves and sheaves respectively $\mathrm{Pre}(\mathcal{S})$ and $\mathrm{Sh}(\mathcal{S})$.

Example. If $\mathcal{S}$ is a category and X is an object of $\mathcal{S}$, then $h_X := \mathrm{Mor}_{\mathcal{S}}(-, X)$ defines a presheaf.

Lemma 4.1.3. *Let F be a presheaf on* Sch. *The following are equivalent:*

(1) F is a sheaf on $\mathrm{Sch}_{\text{ét}}$ *(resp.,* $\mathrm{Sch}_{\mathrm{fppf}}$, $\mathrm{Sch}_{\mathrm{fpqc}}$*) ;*

(2) F sends coproducts to products and for every surjective étale (resp., fppf, faithfully flat) morphism $S' \to S$ of schemes, the sequence

$$F(S) \to F(S') \rightrightarrows F(S' \times_S S')$$

is an equalizer diagram;

(3) F is a sheaf on the big Zariski site $\mathrm{Sch}_{\mathrm{Zar}}$ *and for every surjective étale (resp., fppf, faithfully flat) morphism $S' \to S$ of affine schemes, the sequence $F(S) \to F(S') \rightrightarrows F(S' \times_S S')$ is an equalizer diagram.*

Proof. See [Alp27, Exc. 3.3.7]. □

Proposition 4.1.4. *If $(X \to S)$ is an object of* Sch/S*, then*

$$\mathrm{Mor}_S(-, X)\colon (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Sets}$$

is a sheaf on $(\mathrm{Sch}/S)_{\mathrm{fpqc}}$ *and therefore also a sheaf on* $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ *and* $(\mathrm{Sch}/S)_{\text{ét}}$.

Proof. As $\mathrm{Mor}_S(-, X)$ is a sheaf in the big Zariski topology, by Lemma 4.1.3 it suffices to show that if $T' \to T$ is a faithfully flat morphism of affine schemes over S, then the sequence

$$\mathrm{Mor}_S(T, X) \to \mathrm{Mor}_S(T', X) \rightrightarrows \mathrm{Mor}_S(T' \times_T T', X)$$

is an equalizer diagram, which is precisely Fpqc Descent for Morphisms; see [Alp27, Prop. 3.1.7]. □

Sheaves as defined above satisfy properties like in the classical case of (pre)sheaves defined on a topological space. First of all, we can glue them under the usual hypothesis [Alp27, Exc. 3.3.10]. There are also fibred products, and a sheafification functor [Alp27, Def. 3.3.12, Thm 3.3.15].

Finally, in Proposition 4.1.4 we have seen that a scheme is a sheaf on the site $\mathrm{Sch}_{\text{ét}}$. The following proposition gives us a criterion for a sheaf to be a scheme.

Proposition 4.1.5 (Descent Criterion for an Fppf Sheaf to be a Scheme)**.** *Let $\mathcal{P}$ be one of the following properties of morphisms of schemes: open immersion, closed immersion, affine, quasi-affine, or locally quasi-finite and separated. Let $X \to Y$ be a surjective smooth (resp., fppf) morphism of schemes. Let F be a sheaf on $(\mathrm{Sch}/Y)_{\text{ét}}$ (resp., $(\mathrm{Sch}/Y)_{\text{fppf}}$). Consider the fibred product*

$$\begin{tikzcd} F_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ F \arrow[r] & Y \end{tikzcd}$$

of sheaves. If F_X is a scheme and $F_X \to X$ has $\mathcal{P}$, then F is a scheme and $F \to Y$ has $\mathcal{P}$.

Proof. See [Alp27, Prop. 3.3.17]. □

4.2 Prestacks and stacks

When looking at a moduli problem, in order to keep track of automorphisms of objects, one is led to consider “functors”

$$F\colon \mathcal{S} \to \text{Groupoids},$$

where a *groupoid* is a category where all morphisms are invertible. This is how one can think about a prestack.

However, it is more convenient to define a prestack by packing all the groupoids $F(S)$ into one category $\mathcal{X}$ parametrizing them.

Let $\mathcal{S}$ be a category and $p\colon \mathcal{X} \to \mathcal{S}$ be a functor. A common and simple way to visualize this data is the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[d, "p"] & a \arrow[r, "\alpha"] \arrow[d, mapsto] & b \arrow[d, mapsto] \\ \mathcal{S} & S \arrow[r, "f"] & T \end{tikzcd}$$

where the lowercase letters a, b are objects of $\mathcal{X}$ and the uppercase letter S, T are objects of $\mathcal{S}$ such that $p(a) = S$, $p(b) = T$, and $p(\alpha) = f$. We say that a is over S and that α is over f.

Definition 4.2.1. A functor $p\colon \mathcal{X} \to \mathcal{S}$ is a *prestack over* $\mathcal{S}$ if

(i) (*pullbacks exist*) for every diagram

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[d, mapsto, dashed] & b \arrow[d, mapsto] \\ S \arrow[r] & T \end{tikzcd}$$

of solid arrows, there exists a morphism $a \to b$ over $S \to T$;

(ii) (*universal property*) for every diagram

$$\begin{tikzcd} c \arrow[drr, bend left] \arrow[dr, dashed] \arrow[dd, mapsto] & & \\ & a \arrow[r] \arrow[d, mapsto] & b \arrow[d, mapsto] \\ U \arrow[r] & S \arrow[r] & T \end{tikzcd}$$

of solid arrows, there exists a unique arrow $c \to a$ over $U \to S$ filling in the diagram.

Remark. Axiom (2) implies that the pullback in Axiom (1) is unique up to unique isomorphism. We often write f^*b or $b|_S$ to indicate a choice of a pullback.
A morphism $a \to b$ in $\mathcal{X}$ satisfying Axiom (2) is called *cartesian*.

Given a prestack $\mathcal{X}$ over $\mathcal{S}$, the *fibre category* $\mathcal{X}(S)$ over $S \in \mathcal{S}$ is the category of objects in $\mathcal{X}$ over S with morphisms over id_S. It is not difficult to show that the fibre category is a groupoid.

Example. If $F\colon \mathcal{S}^{\mathrm{op}} \to \mathrm{Sets}$ is a presheaf, we can associate to it a prestack $\mathcal{X}_F$ as the category of pairs (a, S), where $S \in \mathcal{S}$ and $a \in F(S)$. A morphism $(a, S) \to (b, T)$ is a map $f\colon S \to T$ such that $b = f^*a$, where f^* is shorthand for $F(f)$. The projection $\mathcal{X}_F \to \mathcal{S}$ is clearly defined by $(a, S) \mapsto S$. Observe that in this case the fibre category $\mathcal{X}_F(S)$ is equivalent to the set $F(S)$. We will often abuse notation by denoting $\mathcal{X}_F$ simply with F.

Example. If S is an object of a category $\mathcal{S}$, the restricted category $\mathcal{S}/S$ defines a prestack over $\mathcal{S}$, where the projection is given by $(T \to S) \mapsto T$. This is the same prestack obtained by applying the previous construction to the presheaf $\mathrm{Mor}_{\mathcal{S}}(-, S)$.
A scheme X then defines the prestack Sch/X. Again, we will often abuse notation by denoting this prestack simply X.

Definition 4.2.2 (Prestack of smooth curves)**.** We define the prestack $\mathcal{M}$ over Sch as the category of families of smooth curves $\mathcal{C} \to S$, i.e., smooth and proper morphisms such that every geometric fibre is a connected curve. A map

$$(\mathcal{C} \to S) \to (\mathcal{C}' \to S')$$

is the data of maps $\alpha\colon \mathcal{C} \to \mathcal{C}'$ and $f\colon S \to S'$ such that

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C}' \arrow[d] \\ S \arrow[r, "f"] & S \end{tikzcd}$$

is cartesian. The projection $\mathcal{M} \to \mathrm{Sch}$ is given by $(\mathcal{C} \to S) \mapsto S$. The prestack $\mathcal{M}_g$ is defined as the full subcategory of $\mathcal{M}$ consisting of families of smooth curves where every geometric fibre has genus g.

Morphisms of prestacks

Definition 4.2.3. Let $\mathcal{X}$ and $\mathcal{Y}$ be prestacks over $\mathcal{S}$.

(1) A *morphism of prestacks* is a functor $f\colon \mathcal{X} \to \mathcal{Y}$ such that the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr, "p_{\mathcal{X}}"'] & & \mathcal{Y} \arrow[dl, "p_{\mathcal{Y}}"] \\ & \mathcal{S} & \end{tikzcd}$$

strictly commutes, i.e., for every object a in $\mathcal{X}$, there is an equality

$$p_{\mathcal{X}}(a) = p_{\mathcal{Y}}(f(a))$$

of objects in $\mathcal{S}$, and the same for every morphism $a \to b$.

(2) If $f, g\colon \mathcal{X} \to \mathcal{Y}$ are morphisms of prestacks, a *2-isomorphism* $\alpha\colon f \to g$ is a natural transformation such that for every object a in $\mathcal{X}$, the morphism $\alpha_a\colon f(a) \to g(a)$ in $\mathcal{Y}$ is over the identity in $\mathcal{S}$. We describe this situation visually as

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{Y}. \end{tikzcd}$$

(3) We define the category $\mathrm{Mor}(\mathcal{X}, \mathcal{Y})$ whose objects are morphisms of prestacks and whose morphisms are 2-isomorphisms.

(4) A *2-commutative diagram* is a diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "f'"] \arrow[d, "g'"] & \mathcal{Y}' \arrow[d, "g"] \arrow[dl, phantom, "\swarrow\alpha"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

together with a 2-isomorphism $\alpha \colon g \circ f' \xrightarrow{\sim} f \circ g'$.

(5) A morphism $f \colon \mathcal{X} \to \mathcal{Y}$ of prestacks is a *monomorphism* (resp., *epimorphism*) if f is fully faithful (resp., essentially surjective), and f is an *isomorphism* if there exists a morphism $g \colon \mathcal{Y} \to \mathcal{X}$ of prestacks and 2-isomorphisms $g \circ f \xrightarrow{\sim} \mathrm{id}_{\mathcal{X}}$ and $f \circ g \xrightarrow{\sim} \mathrm{id}_{\mathcal{Y}}$.

Remark. If $f \colon \mathcal{X} \to \mathcal{Y}$ is a morphism of prestacks, one can easily show that:

- every 2-isomorphism is in fact an isomorphism of functors, hence $\mathrm{Mor}(\mathcal{X}, \mathcal{Y})$ is a groupoid;
- f is a monomorphism if and only if $f_S \colon \mathcal{X}(S) \to \mathcal{Y}(S)$ is fully faithful for every $S \in \mathcal{S}$; and
- f is an isomorphism if and only if f is fully faithful and essentially surjective.

A prestack $\mathcal{X}$ on $\mathcal{S}$ is *equivalent to a presheaf* if there exists a presheaf F on $\mathcal{S}$ and an isomorphism between $\mathcal{X}$ and $\mathcal{X}_F$.

The 2-Yoneda Lemma

The Yoneda Lemma [SP, Tag 001P] states that for a presheaf $F \colon \mathcal{S}^{\mathrm{op}} \to \mathrm{Sets}$ and an object $S \in \mathcal{S}$, there is a bijection $\mathrm{Mor}(h_S, F) \xrightarrow{\sim} F(S)$. In particular, there is a fully faithful embedding $\mathcal{S} \to \mathrm{Pre}(\mathcal{S})$ in the category of presheaves on $\mathcal{S}$, given by $S \mapsto h_S$.
Recall that an object S in $\mathcal{S}$ can be viewed as a prestack over $\mathcal{S}$, which we also denote by S.

Lemma 4.2.4 (2-Yoneda). *Let $\mathcal{X}$ be a prestack over $\mathcal{S}$ and $S \in \mathcal{S}$. The functor*

$$\operatorname{Mor}(S, \mathcal{X}) \to \mathcal{X}(S), \quad f \mapsto f_S(id_S)$$

is an equivalence of categories.

Proof. See [Alp27, Lemma 3.4.21]. □

We will often use the 2-Yoneda Lemma in passing between morphisms $S \to \mathcal{X}$ and objects of $\mathcal{X}(S)$. Furthermore, if $f, g \colon S \to \mathcal{X}$ are morphisms, corresponding to objects $a, b \in \mathcal{X}(S)$, then a 2-isomorphism $\alpha \colon f \xrightarrow{\sim} g$ corresponds to an isomorphism $a \xrightarrow{\sim} b$ in $\mathcal{X}(S)$.

Remark. If $F \colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Sets}$ is the functor $h_M = \operatorname{Mor}(-, M)$ representable by a scheme M, the classical Yoneda Lemma (Lemma 1.1.1) gives a bijection

$$\operatorname{Mor}(h_M, F) \simeq F(M)$$

and the object $u \in F(M)$ corresponding to the identity map is the universal family. The 2-Yoneda Lemma does not immediately apply to give a universal family as the category $\mathcal{X}(\mathcal{X})$ needs to be defined. Later, we will prove a Generalized 2-Yoneda Lemma and use it to define universal families.

Fibred products

Let $f \colon \mathcal{X} \to \mathcal{Z}$ and $g \colon \mathcal{Y}' \to \mathcal{Z}$ be morphisms of prestacks over a category $\mathcal{S}$. Define the prestack $\mathcal{X} \times_{\mathcal{X}} \mathcal{Y}$ over $\mathcal{S}$ as the category of triples (x, y, γ) where $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ are objects over the same object $p_{\mathcal{X}}(x) = p_{\mathcal{Y}}(y) \in \mathcal{S}$, and $\gamma \colon f(x) \xrightarrow{\sim} g(y)$ is an isomorphism in $\mathcal{Z}(S)$. A morphism $(x_1, y_1, \gamma_1) \to (x_2, y_2, \gamma_2)$ consists of a triple (h, λ, μ) where $h \colon p_{\mathcal{X}}(x_1) = p_{\mathcal{Y}}(y_1) \to p_{\mathcal{Y}}(y_2) = p_{\mathcal{X}}(x_2)$ is a morphism in $\mathcal{S}$, and $\lambda \colon x_1 \to x_2$ and $\mu \colon y_1 \to y_2$ are morphisms in $\mathcal{X}$ and $\mathcal{Y}$ over h such that

$$\begin{tikzcd} f(x_1) \arrow[r, "f(\lambda)"] \arrow[d, "\gamma_1"] & f(x_2) \arrow[d, "\gamma_2"] \\ g(y_1) \arrow[r, "g(\mu)"] & g(y_2) \end{tikzcd}$$

commutes. Let also $p_1 \colon \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{X}$ and $p_2 \colon \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{Y}$ denote the natural projections $(x, y, \gamma) = y$, and define the 2-isomorphism $\alpha \colon f \circ p_1 \xrightarrow{\sim} g \circ p_2$, which is defined on $(x, y, \gamma) \in \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ by setting $\alpha_{(x,y,\gamma)} := \gamma$. This clearly yields a 2-commutative diagram

$$\begin{tikzcd}
\mathcal{X}\times_{\mathcal{Z}}\mathcal{Y} \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\alpha\Nearrow"] & \mathcal{Y} \arrow[d, "g"] \\
\mathcal{X} \arrow[r, "f"] & \mathcal{Z}.
\end{tikzcd}$$

Theorem 4.2.5. *The prestack $\mathcal{X}\times_{\mathcal{Z}}\mathcal{Y}$ together with the projections p_1 and p_2 and the 2-isomorphism α as above satisfy the following universal property: for every 2-commutative diagram*

$$\begin{tikzcd}
 & & \mathcal{Y} \arrow[dr, "g"] & \\
\mathcal{T} \arrow[urr, bend left, "q_2"] \arrow[drr, bend right, "q_1"'] \arrow[r, phantom, "\tau\Uparrow"] & \mathcal{X}\times_{\mathcal{Z}}\mathcal{Y} \arrow[ur, "p_2"] \arrow[dr, "p_1"'] \arrow[rr, phantom, "\alpha\Uparrow"] & & \mathcal{Z} \\
 & & \mathcal{X} \arrow[ur, "f"'] &
\end{tikzcd}$$

with 2-isomorphisms $\tau\colon f\circ q_1 \xrightarrow{\sim} g\circ q_2$, there exists a morphism $h\colon \mathcal{T}\to\mathcal{X}\times_{\mathcal{Z}}\mathcal{Y}$ and 2-isomorphisms $\rho\colon q_1\to p_1\circ h$ and $\sigma\colon q_2\to p_2\circ h$ yielding a 2-commutative diagram

$$\begin{tikzcd}
 & & \mathcal{Y} \arrow[dr, "g"] & \\
\mathcal{T} \arrow[urr, bend left, "q_2"] \arrow[drr, bend right, "q_1"'] \arrow[r, "h"] \arrow[ur, phantom, "\sigma\Downarrow"] \arrow[dr, phantom, "\rho\Uparrow"] & \mathcal{X}\times_{\mathcal{Z}}\mathcal{Y} \arrow[ur, "p_2"] \arrow[dr, "p_1"'] \arrow[rr, phantom, "\alpha\Uparrow"] & & \mathcal{Z} \\
 & & \mathcal{X} \arrow[ur, "f"'] &
\end{tikzcd}$$

such that the diagram

$$\begin{tikzcd}
f\circ q_1 \arrow[r, "f(\rho)"] \arrow[d, "\tau"] & f\circ p_1\circ h \arrow[d, "\alpha\circ h"] \\
g\circ q_2 \arrow[r, "g(\sigma)"] & f\circ p_2\circ h
\end{tikzcd}$$

commutes. Moreover, the data (h,ρ,σ) is unique up to unique isomorphism.

Proof. We define h on objects by $\mathcal{T}\ni t\mapsto (q_1(t), q_2(t), \tau_t)$ and on morphisms as $(\phi\colon t\to t')\mapsto (p_{\mathcal{T}}(\phi), q_1(\phi), q_2(\phi))$. In this way, there are equalities of functors $q_1 = p_1\circ h$ and $q_2 = p_2\circ h$ so we can define σ and ρ as the identity natural transformations. We omit the remaining tedious details. $\square$

We say that a 2-commutative diagram

$$\begin{tikzcd}
\mathcal{X}' \arrow[r] \arrow[d] & \mathcal{Y}' \arrow[d] \\
\mathcal{X} \arrow[r] \arrow[ur, phantom, "\overset{\alpha}{\nearrow}"] & \mathcal{Y}
\end{tikzcd}$$

is *cartesian* if it satisfies the universal property of Theorem 4.2.5, and we often write such a diagram simply as

$$\begin{tikzcd}
\mathcal{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y}' \arrow[d] \\
\mathcal{X} \arrow[r] & \mathcal{Y}.
\end{tikzcd}$$

Remark. Let $\mathcal{X}$ be a prestack over a category $\mathcal{S}$, and let S, T be two objects in S which can view as prestacks over S. For all morphisms $a\colon S \to \mathcal{X}$ and $b\colon T \to \mathcal{X}$, there is a cartesian diagram

$$\begin{tikzcd}
S \times_{\mathcal{X}} T \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \times T \arrow[d, "a \times b"] \\
\mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X};
\end{tikzcd}$$

see [Alp27, Exc. 3.4.38].

The following is an important example of presheaf, and we will use it later to better describe the structure of a (pre)stack.

Example (Isomorphism presheaf). Let $\mathcal{X}$ be a prestack over $\mathcal{S}$, and fix $T \in \mathcal{S}$. Then it is not hard to show that:

(1) for objects a and b of $\mathcal{X}(T)$, the functor

$$\begin{aligned}
\underline{\mathrm{Isom}}_{\mathcal{X}(T)}(a, b)\colon \mathcal{S}/T &\to \mathrm{Sets} \\
(T' \xrightarrow{f} T) &\mapsto \mathrm{Mor}_{\mathcal{X}(T')}(f^*a, f^*b)
\end{aligned}$$

defines a presheaf on $\mathcal{S}/T$;

(2) there is a cartesian diagram

$$\begin{tikzcd}
\underline{\mathrm{Isom}}_{\mathcal{X}(T)}(a, b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \arrow[d, "{(a,b)}"] \\
\mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X};
\end{tikzcd}$$

(3) the presheaf $\underline{\mathrm{Aut}}_{\mathcal{X}(T)}(a) \coloneqq \underline{\mathrm{Isom}}_{\mathcal{X}(T)}(a, a)$ is actually a presheaf in groups; and

(4) the prestack $\mathcal{X}$ is equivalent to a presheaf if and only if the diagonal $\mathcal{X} \xrightarrow{\Delta} \mathcal{X} \times \mathcal{X}$ is fully faithful.

Stacks

In a few words, a stack over a site $\mathcal{S}$ is a prestack $\mathcal{X} \to \mathcal{S}$ where the objects and morphisms glue uniquely according to the Grothendieck topology.
Given a covering $\{S_i \to S\}$ in a site, we will denote by S_{ij} the product $S_i \times_S S_j$ and by S_{ijk} the product $S_i \times_S S_j \times_S S_k$.

Definition 4.2.6. A prestack $\mathcal{X}$ over a site $\mathcal{S}$ is a *stack* if the following conditions hold for all coverings $\{S_i \to S\}$ of any object $S \in \mathcal{S}$:

(i) (morphisms glue) For objects a and b over S and morphisms $\phi_i \colon a|_{S_i} \to b$ such that $\phi_i|_{S_{ij}} = \phi_j|_{S_{ij}}$ as displayed in the diagram

$$\begin{tikzcd} & a|_{S_i} \arrow[dr] \arrow[drr, bend left, "\phi_i"] & & & & & S_i \arrow[dr] & \\ a|_{S_{ij}} \arrow[ur] \arrow[dr] & & a \arrow[r, dashed] & b & \text{over} & S_{ij} \arrow[ur] \arrow[dr] & & S \\ & a|_{S_j} \arrow[ur] \arrow[urr, bend right, "\phi_j"'] & & & & & S_j \arrow[ur] & \end{tikzcd}$$

there is a unique morphism $\phi \colon a \to b$ over id_S such that $\phi|_{S_i} = \phi_i$.

(ii) (objects glue) For objects a_i over S_i and isomorphisms $\alpha_{ij} \colon a_i|_{S_{ij}} \to a_j|_{S_{ij}}$, as displayed in the diagram

$$\begin{tikzcd} & a_i \arrow[dr, dashed] & & & & S_i \arrow[dr] & \\ a_i|_{S_{ij}} \arrow[ur] \arrow[d, "\alpha_{ij}"] & & a & \text{over} & S_{ij} \arrow[ur] \arrow[dr] & & S \\ a_j|_{S_{ij}} \arrow[r] & a_j \arrow[ur, dashed] & & & & S_j \arrow[ur] & \end{tikzcd}$$

satisfying the cocycle condition $\alpha_{jk}|_{S_{ijk}} \circ \alpha_{ij}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$ on S_{ijk}, there exists an object a over S and isomorphisms $\phi_i \colon a|_{S_i} \to a_i$ over id_{S_i} such that $\phi_j|_{S_{ij}} = \alpha_{ij} \circ \phi_i|_{S_{ij}}$ on S_{ij}.

A *morphism* of stacks is a morphism of prestacks.

Remark. Axiom (i) is equivalent to requiring that for all objects a and b in $\mathcal{X}(S)$, the presheaf $\underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a,b)$ is a sheaf on $\mathcal{S}/S$.

Remark. Using definitions, one can check that the fibred product of two stacks over a site $\mathcal{S}$ is also a stack over $\mathcal{S}$. Moreover, given a prestack $\mathcal{X}$, there is a procedure yielding a stack $\mathcal{X}^{\mathrm{st}}$ called the *stackification* of $\mathcal{X}$, and having properties analogous to the sheafification; see [Alp27, Thm 3.5.18].

Example. Recall that a presheaf F on a site $\mathcal{S}$ defines a prestack $\mathcal{X}_F$ over $\mathcal{S}$. This prestack is a stack if and only if F is a sheaf. In particular, since a scheme X is a sheaf on $\mathrm{Sch}_{\text{ét}}$ by Proposition 4.1.4, the prestack Sch/X is a stack over $\mathrm{Sch}_{\text{ét}}$, often denoted simply as X.

Remark. One can generalize Lemma 4.1.3 from sheaves to stacks, showing that a prestack $\mathcal{X}$ over Sch is a stack over $\mathrm{Sch}_{\text{ét}}$ (resp., $\mathrm{Sch}_{\mathrm{fppf}}$, $\mathrm{Sch}_{\mathrm{fpqc}}$) if and only if $\mathcal{X}$ is a stack over $\mathrm{Sch}_{\mathrm{Zar}}$ and stack axioms hold for a surjective étale (resp., fppf, fpqc) morphism $\operatorname{Spec} A' \to \operatorname{Spec} A$ of affine schemes.

Recall the prestack $\mathcal{M}_g$ of smooth curves of genus g introduced in Definition 4.2.2.

Theorem 4.2.7 (Moduli stack of smooth curves)**.** *If $g \geq 2$, then $\mathcal{M}_g$ is a stack over* $\mathrm{Sch}_{\text{ét}}$.

Proof. It suffices to check that $\mathcal{M}_g$ is a stack over $\mathrm{Sch}_{\mathrm{fpqc}}$. Moreover, since smooth curves and their morphisms glue in the Zariski topology, it suffices by the remark above to verify stack axioms with respect to an fpqc map $S' \to S$.

Axiom (1): for families of smooth curves $\mathcal{C} \to S$ and $\mathcal{D} \to S$ of genus g, every commutative diagram

$$\begin{tikzcd}
\mathcal{C}_{S'\times_S S'} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C}_{S'} \arrow[r] \arrow[d] \arrow[rr, bend left, "f'"] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[r, dashed, "f"] \arrow[d] & \mathcal{D} \arrow[dl] \\
S'\times_S S' \arrow[r, shift left] \arrow[r, shift right] & S' \arrow[r] & S &
\end{tikzcd}$$

of solid arrows can uniquely filled in. The existence and uniqueness of f follow from Fpqc Descent for Morphisms [Alp27, Prop. 3.1.7]. Since the base change $\mathcal{C}_{S'} \to \mathcal{D}_{S'}$ is an isomorphism, then again by descent properties (see [Alp27, Prop. 3.1.26]) $f\colon \mathcal{C} \to \mathcal{D}$ is also an isomorphism.

Axiom (2): we must show that given a diagram

$$\begin{tikzcd}
p_1^*\mathcal{C}' \arrow[r, "\alpha"] \arrow[d] \arrow[rr, bend left] & p_2^*\mathcal{C}' \arrow[r] \arrow[dl] \arrow[dr, phantom, "\square"] & \mathcal{C}' \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d, dashed] \\
S' \times_S S' \arrow[rr, shift left, "p_1"] \arrow[rr, shift right, "p_2"'] & & S' \arrow[r] & S,
\end{tikzcd}$$

where $\mathcal{C}' \to S'$ is a family of smooth curves and $\alpha\colon p_1^*\mathcal{C}' \to p_2^*\mathcal{C}'$ is an isomorphism satisfying the cocycle condition $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$ on $S' \times_S S' \times_S S'$, there is a family of smooth curves $\mathcal{C} \to S$ and an isomorphism $\phi\colon \mathcal{C}|_{S'} \to \mathcal{C}'$ such that $p_1^*\phi = p_2^*\phi \circ \alpha$. By Proposition 2.1.4, $\Omega_{\mathcal{C}'/S'}^{\otimes 3}$ is relatively very ample on S and $E' \coloneqq \pi_*(\Omega_{\mathcal{C}'/S'}^{\otimes 3})$ is a vector bundle on S' of rank $5g-5$ whose construction commutes with base change. Thus $\Omega_{\mathcal{C}'/S'}^{\otimes 3}$ defines a closed immersion $\mathcal{C}' \hookrightarrow \mathbb{P}(E')$ over S'. The isomorphism α induces an isomorphism $\beta\colon p_1^*E' \to p_2^*E'$ satisfying the cocycle condition $p_{23}^*\beta \circ p_{12}^*\beta = p_{13}^*\beta$ on $S' \times_S S' \times_S S'$. Fpqc Descent for Quasi-coherent Sheaves (see [Alp27, Prop. 3.1.4]) yields a quasi-coherent sheaf E on S and isomorphisms $\psi\colon E' \to E|_{S'}$ such that $p_1^*\psi = p_2^*\psi \circ \beta$, and again by descent properties [Alp27, Prop. 3.1.18], it follows that E is a vector bundle as well.

$$\begin{tikzcd}
 & & \mathbb{P}(E') \arrow[r] \arrow[d, bend left] & \mathbb{P}(E) \arrow[d, bend left] \\
p_1^*\mathcal{C}' \arrow[r, "\alpha"] \arrow[d] \arrow[rr, bend left] & p_2^*\mathcal{C}' \arrow[r] \arrow[dl] & \mathcal{C}' \arrow[r, dashed] \arrow[d] \arrow[u, hook] & \mathcal{C} \arrow[d, dashed] \arrow[u, hook, dashed] \\
S' \times_S S' \arrow[rr, shift left, "p_1"] \arrow[rr, shift right, "p_2"'] & & S' \arrow[r] & S
\end{tikzcd}$$

Under the identifications of $\mathbb{P}(p_1^*E')$ and $\mathbb{P}(p_2^*E')$ with $\mathbb{P}(E) \times_S (S' \times_S S')$, the preimages $p_1^*\mathcal{C}'$ and $p_2^*\mathcal{C}'$ are equal. Applying Fpqc Descent for Closed Subschemes [Alp27, Prop. 3.1.11], we find a closed subscheme $\mathcal{C} \in \mathbb{P}(E)$ pulling back to $\mathcal{C}'$. Finally, smoothness and properness are fpqc local on the target (see [Alp27, Prop. 3.1.26]), and thus $\mathcal{C} \to S$ is smooth and proper, and every geometric fibre of $\mathcal{C} \to S$ is identified with a geometric fibre of $\mathcal{C}' \to S'$, thus it is a connected genus g curve. $\square$

Quotient stacks

If X is a scheme and G is a group acting on it, we can always consider the topological quotient X/G, but only in some particular cases we can provide a natural scheme structure for X/G. However, we can always define the quotient stack.

First, for a smooth affine group scheme[1] $G \to S$, a *principal G-bundle over an S-scheme T* is a morphism $P \to T$ of schemes with an action of G on P via $\sigma\colon G \times_S P \to P$ such that $P \to T$ is a G-invariant smooth morphism and

$$(\sigma, p_2)\colon G \times_S P \to P \times_T P, \quad (g, p) \mapsto (gp, p)$$

is an isomorphism; in other words, G acts freely and transitively on P with quotient T.

Definition 4.2.8. Let $G \to S$ be a smooth affine group scheme acting on a scheme U over S. The *quotient prestack* $[U/G]^{\mathrm{pre}}$ is the category over Sch/S consisting of pairs (T, u) where T is an S-scheme and $u \in U(T)$. A morphism $(T, u) \to (T', u')$ is the data of a map $f\colon T \to T'$ of S-schemes and an element $g \in G(T')$ such that $f^*u' = g \cdot u$.
The *quotient stack* $[U/G]$ is the category over Sch/S consisting of diagrams

$$\begin{tikzcd} P \arrow[r] \arrow[d] & U \\ T & \end{tikzcd}$$

where $P \to T$ is a principal G-bundle and $P \to U$ is a G-equivariant morphism of S-schemes. A morphism $(T \leftarrow P \to U) \to (T' \leftarrow P' \to U)$ consists of a morphism $T \to T'$ and a G-equivariant morphism $P \to P'$ of schemes such that the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[rr, bend left] \arrow[dr, phantom, "\square"] & P' \arrow[r] \arrow[d] & U \\ T \arrow[r] & T' & \end{tikzcd}$$

is commutative and the left square is cartesian.
The *classifying stack* BG is the quotient stack $[S/G]$.

The quotient stack $[U/G]$ is in fact a stack over the big étale site $\mathrm{Sch}_{\text{ét}}$, and it is the stackification of the quotient prestack $[U/G]^{\mathrm{pre}}$; see [Alp27, Prop. 3.5.13, Exc. 3.5.21].

Remark. Given an action of a group G on a set X, we define the *quotient groupoid* $[X/G]$ to be the category with objects all elements $x \in X$ and by declaring $\mathrm{Mor}(x, y) = \{g \in G \mid g \cdot x = y\}$.
Note that in the previous example the fibre category $[U/G]^{\mathrm{pre}}(T)$ is identified with the quotient groupoid $[U(T)/G(T)]$.

[1] See [SP, Tag 022R].

The following will be used in the next section to prove the algebraicity of the quotient stack.

Proposition 4.2.9. *Let $G \to S$ be a smooth affine group scheme acting on an S-scheme U via $\sigma \colon G \times_S U \to U$. Let $T \to [U/G]$ be a morphism corresponding via the 2-Yoneda Lemma to a principal G-bundle $P \to T$ and a G-equivariant map $f \colon P \to U$. There is a cartesian diagram*

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ T \arrow[r] & {[U/G]} \end{tikzcd} .$$

Moreover, there are cartesian diagrams

$$\begin{tikzcd} G \times_S U \arrow[r, "\sigma"] \arrow[d, "p_2"] \arrow[dr, phantom, "\square"] & U \arrow[d, "p"] \\ U \arrow[r, "p"] & {[U/G]} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \times_S U \arrow[r, "{(\sigma, p_2)}"] \arrow[d] \arrow[dr, phantom, "\square"] & U \times_S U \arrow[d, "p \times p"] \\ {[U/G]} \arrow[r, "\Delta"] & {[U/G] \times_S [U/G]} \end{tikzcd} .$$

Proof. See [Alp27, Exc. 3.4.37]. □

4.3 Algebraicity

Stacks Theory is very important to formalize in an abstract setting the concept of a moduli space. However, if we hope to do some geometry with these objects, we need to impose further conditions.
In this section we define three important classes of stacks over Sch, and we look at some of their properties. In order to do this, we first need to talk about the notion of representability of a morphism of prestacks.

A morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks over Sch is said to be *representable by schemes* if for every morphism $T \to \mathcal{Y}$ from a scheme, the fibred product $\mathcal{X} \times_{\mathcal{Y}} T$ is a scheme. Moreover, if $\mathcal{P}$ is a property of morphisms of schemes stable under base change, we say that a morphism $\mathcal{X} \to \mathcal{Y}$ representable by schemes *has* $\mathcal{P}$ if for every morphism $T \to \mathcal{Y}$ from a scheme, the morphism $\mathcal{X} \times_{\mathcal{Y}} T \to T$ of schemes has $\mathcal{P}$.

The following is the simplest generalisation of a scheme.

Definition 4.3.1. An *algebraic space* is a sheaf X on $\mathrm{Sch}_{\text{ét}}$ such that there exists a scheme U and a surjective étale morphism $U \to X$ representable by schemes.

Such a map is called an *étale presentation*. Every scheme is clearly an algebraic space. Using algebraic spaces, we can now further generalise the definitions above.

A morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks over Sch is said to be *representable* if for every morphism $T \to \mathcal{Y}$ from a scheme, the fibred product $\mathcal{X} \times_{\mathcal{Y}} T$ is an algebraic space. Moreover, if $\mathcal{P}$ is a property of morphisms of schemes stable under base change and étale local on the source (e.g. surjective, étale, or smooth), we say that a representable morphism $\mathcal{X} \to \mathcal{Y}$ *has* $\mathcal{P}$ if for every morphism $T \to \mathcal{Y}$ from a scheme and étale presentation $U \to \mathcal{X} \times_{\mathcal{Y}} T$, the composition $U \to \mathcal{X} \times_{\mathcal{Y}} T \to T$ has $\mathcal{P}$.

Definition 4.3.2. A *Deligne-Mumford stack* is a stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{ét}}$ such that there exists a scheme U and a surjective, étale, and representable morphism $U \to \mathcal{X}$.

Such a map is again called an étale presentation. Since a sheaf is a stack, every algebraic space is a Deligne-Mumford stack. Actually, it is also true that a Deligne-Mumford stack that is a sheaf is an algebraic space; see [Alp27, Thms 4.6.6 and 5.5.10]. We will often use DM as short for Deligne-Mumford.

Definition 4.3.3. An *algebraic stack* is a stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{ét}}$ such that there exists a scheme U and a surjective, smooth, and representable morphism $U \to \mathcal{X}$.

Such a map is called a *smooth presentation*. Clearly, every DM stack is an algebraic stack.

Remark. Fibred products exist for algebraic spaces, DM stacks, and algebraic stacks ([Alp27, Exc. 4.1.9]).

Definition 4.3.4. A substack $\mathcal{T} \subseteq \mathcal{X}$ of a stack over $\mathrm{Sch}_{\text{ét}}$ is called an *open* (resp., *closed*) *substack* if the inclusion morphism is representable by schemes and an open (resp., closed) immersion.

Algebraicity of $\mathcal{M}_g$

In order to show that the moduli stack of smooth curves $\mathcal{M}_g$ is algebraic, we first need to show that quotient stacks are algebraic.

Note that we have defined $[U/G]$ only for a scheme U, but the definition can be easily extended to algebraic spaces. An *action* of a smooth affine group scheme $G \to S$ on an algebraic space $U \to S$ is a morphism $\sigma\colon G \times_S U \to U$ satisfying the same axioms as in the case of schemes (see [Alp27, Def. B.1.10]), and we define the quotient stack $[U/G]$ as the stackification of the prestack $[U/G]^{\text{pre}}$, whose fibre category over an S-scheme T is the quotient groupoid $[U(T)/G(T)]$, as in Definition 4.2.8. In other words, objects of $[U/G]$ over an S-scheme T are principal G-bundles $P \to T$ together with a G-equivariant morphism $P \to U$. The argument to show that this a stack extends from the case of schemes.

Theorem 4.3.5 (Algebraicity of Quotient Stacks)**.** *If $G \to S$ is a smooth affine group scheme acting on an algebraic space $U \to S$, then the quotient stack $[U/G]$ is an algebraic stack over* $(\mathrm{Sch}/S)_{\text{ét}}$ *such that $U \to [U/G]$ is a principal G-bundle and in particular surjective, smooth, and affine.*

Proof. We use the natural projection $U \to [U/G]$ corresponding via the 2-Yoneda Lemma to the trivial principal G-bundle $p_2 \colon G \times U \to U$ and the G-equivariant map $\sigma \colon G \times U \to U$ given by the action. Let $T \to [U/G]$ be a morphism from an S-scheme corresponding to a principal G-bundle $P \to T$ and a G-equivariant map $P \to U$. By Proposition 4.2.9, there is a cartesian diagram

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ T \arrow[r] & {[U/G]} \end{tikzcd},$$

and since every base change is a principal G-bundle, so is $U \to [U/G]$. Moreover, the map $U \to [U/G]$ is a smooth presentation if U is a scheme. In general, if $U' \to U$ is an étale presentation with U' a scheme, the composition $U' \to U \to [U/G]$ provides a smooth presentation $\square$

Recall that every smooth, connected, projective curve C is tricanonically embedded in $\mathbb{P}^{5g-6}$ by the every ample line bundle $\Omega_C^{\otimes 3}$. This is basically the reason why the moduli stack $\mathcal{M}_g$ is algebraic. Indeed, we can consider the subscheme H' of the Hilbert scheme $\mathrm{Hilb}^P(\mathbb{P}^{5g-6}/\mathbb{Z})$ parametrising smooth families of tricanonically embedded curves, and this provides a smooth presentation.

Theorem 4.3.6 (Algebraicity of $\mathcal{M}_g$)**.** *If $g \geq 2$, then $\mathcal{M}_g \simeq [H'/\mathrm{PGL}_{5g-5}]$, where H' is a locally closed subscheme of a projective Hilbert scheme. In particular, the stack $\mathcal{M}_g$ over $\operatorname{Spec}\mathbb{Z}$ is algebraic.*

Proof. By Proposition 2.1.4, for a family of smooth curves $\pi \colon \mathcal{C} \to S$, $\Omega_{\mathcal{C}/S}^{\otimes 3}$ is relatively very ample on S and $\pi_*(\Omega_{\mathcal{C}/S}^{\otimes 3})$ is locally free of rank $5g-5$. Then $\Omega_{\mathcal{C}/S}^{\otimes 3}$ defines a closed immersion $\mathcal{C} \hookrightarrow \mathbb{P}(\pi_*(\Omega_{\mathcal{C}/S}^{\otimes 3}))$ over S. By Riemann-Roch (Theorem 2.0.1), the Hilbert polynomial of a fibre $\mathcal{C}_s \hookrightarrow \mathbb{P}_{\kappa(s)}^{5g-6}$ is

$$P(n) \coloneqq \chi(\mathcal{O}_{\mathcal{C}_s}(n)) = \deg(\Omega_{\mathcal{C}_s}^{\otimes 3n}) + 1 - g = (6n-1)(g-1),$$

and we define

$$H \coloneqq \mathrm{Hilb}^P(\mathbb{P}_{\mathbb{Z}}^{5g-6}/Z)$$

as the Hilbert scheme parametrising closed subschemes of $\mathbb{P}^{5g-6}$ with Hilbert polynomial P; see Section 3.3. Let $\mathcal{U} \hookrightarrow \mathbb{P}^{5g-6} \times H$ be the universal closed subscheme

and let $q\colon \mathcal{U} \to H$ be the projection.
See [Alp27, Thm 4.1.17] for the proof of the following statement. We can find a locally closed subscheme $H' \subseteq H$ such that the family $\mathcal{U}_{H'} \to H'$ satisfies

(a) for every $h \in H'$, $\mathcal{U}_h \to \operatorname{Spec} \kappa(h)$ is smooth and geometrically connected,

(b) the natural map $(p_2)_*\mathcal{O}_{\mathbb{P}^{5g-6}\times H'}(1) \to (p_2)_*\mathcal{O}_{\mathcal{U}_{H'}}(1)$ induced by the closed immersion $\mathcal{U}_{H'} \hookrightarrow \mathbb{P}^{5g-6} \times H'$ is an isomorphism, and

(c) the line bundles $\Omega^{\otimes 3}_{\mathcal{U}_{H'}/H'}$ and $\mathcal{O}_{\mathcal{U}_{H'}}(1)$ differ by a pullback of a line bundle from H'.

Moreover, if $T \to H$ is a morphism of schemes such that (a)–(c) hold for $\mathcal{U}_T \to T$, then $T \to H$ factors through H'; in particular, such an H' is unique.

Note that (a)–(c) imply that $\mathcal{U}_h \hookrightarrow \mathbb{P}^{5g-6}_{\kappa(h)}$ is embedded via $\Omega^{\otimes 3}_{\mathcal{U}_h/\kappa(h)}$ for all $h \in H$.
Now the group scheme $\underline{\mathrm{Aut}}(\mathbb{P}^{5g-6}_{\mathbb{Z}}) \simeq \mathrm{PGL}_{5g-5}$ over $\mathbb{Z}$ acts naturally on H via

$$g \cdot [\mathcal{D} \subseteq \mathbb{P}^{5g-6}_S] = [g(\mathcal{D}) \subseteq \mathbb{P}^{5g-6}_S],$$

where $g \in \mathrm{PGL}_{5g-5}$ and $[\mathcal{D} \subseteq \mathbb{P}^{5g-6}_S] \in H(S)$, and the closed subscheme H' is PGL_{5g-5}-invariant. Actually, by using properties (a)–(c), we can prove that $\mathcal{M}_g \simeq [H'/\mathrm{PGL}_{5g-5}]$; see [Alp27, Thm 4.1.17].
By applying Algebraicity of Quotient Stacks (Theorem 4.3.5), we conclude the proof. □

Universal families

For an algebraic stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{ét}}$, in order to define a universal family as in Definition 1.1.3, we would like to define $\mathcal{X}(\mathcal{X})$.

Lemma 4.3.7 (Generalized 2-Yoneda Lemma)**.** *Let $\mathcal{X}$ be a stack over $\mathrm{Sch}_{\text{ét}}$. If $\mathcal{T}$ is an algebraic stack and $U \to \mathcal{T}$ is a smooth presentation, we denote by p_i the projections $U \times_{\mathcal{T}} U \to U$, and by p_{ij} the projections $U \times_{\mathcal{T}} U \times_{\mathcal{T}} U \to U \times_{\mathcal{T}} U$. We define the category $\mathcal{X}(\mathcal{T})$ whose objects are pairs (x, α) where $x \in \mathcal{X}(U)$ and $\alpha\colon p_1^*x \xrightarrow{\sim} p_2^*x$ is an isomorphism satisfying the cocycle condition $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$, while a morphism $(x, \alpha) \to (y, \beta)$ is the datum of a morphism $h\colon x \to y$ satisfying $p_2^*h \circ \alpha = \beta \circ p_1^*h$. In other words, we have an equalizer diagram*

$$\mathcal{X}(\mathcal{T}) \longrightarrow \mathcal{X}(U) \rightrightarrows \mathcal{X}(U \times_{\mathcal{T}} U) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \mathcal{X}(U \times_{\mathcal{T}} U \times_{\mathcal{T}} U).$$

There is a natural equivalence of categories

$$\mathrm{Mor}(\mathcal{T}, \mathcal{X}) \to \mathcal{X}(\mathcal{T});$$

in particular, $\mathcal{X}(\mathcal{T})$ is independent of the presentation.

Proof. Let $q\colon U \to \mathcal{T}$ denote the smooth presentation, then we have

$$\begin{tikzcd} U \times_{\mathcal{T}} U \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\swArrow\gamma"] & U \arrow[d, "q"] \\ U \arrow[r, "q"] & \mathcal{T}. \end{tikzcd}$$

Given a morphism $f\colon \mathcal{T} \to \mathcal{X}$, we let $x \in \mathcal{X}(U)$ be the object corresponding via the 2-Yoneda Lemma (Lemma 4.2.4) to the composition $f \circ q\colon U \to \mathcal{T} \to \mathcal{X}$. Composing $\gamma\colon q \circ p_2 \to q \circ p_1$ with f induces a 2-isomorphism $\alpha\colon p_1^* x \xrightarrow{\sim} p_2^* x$. We now define the map $\Phi\colon \operatorname{Mor}(\mathcal{T}, \mathcal{X}) \to \mathcal{X}(\mathcal{T})$ by sending f to $(x, \alpha) \in \mathcal{X}(\mathcal{T})$.
To show that Φ is an equivalence, we first assume that it holds for algebraic spaces; in particular, it holds for U, $U \times_{\mathcal{T}} U$, and $U \times_{\mathcal{T}} U_{\mathcal{T}} U$. Let $(x, \alpha) \in \mathcal{X}(\mathcal{T})$, thus $x \in \mathcal{X}(U)$ and by the 2-Yoneda Lemma there is a corresponding map $f_x\colon U \to \mathcal{X}$. We construct an inverse Ψ to Φ as follows. Let $y \in \mathcal{T}$ be an object over a scheme S corresponding to a map $f_y\colon S \to \mathcal{T}$, and consider the commutative diagram

$$\begin{tikzcd} S_2 \arrow[r, hook] \arrow[drr] & S_1 \arrow[r] & S_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[r, "f_x"] \arrow[d, "q"] & \mathcal{X} \\ & & S \arrow[r, "f_y"] & \mathcal{T} & \end{tikzcd}$$

where $S_0 = S \otimes_{\mathcal{T}} U$ is an algebraic space, $S_1 \to S_0$ is an étale presentation, and $S_2 \to S$ is an étale surjection of schemes factoring through S_1, which exists because smooth morphisms have sections étale locally (see [Alp27, Cor. A.3.5]). The composition in the first row defines an object z' of $\mathcal{X}(S_2)$. The isomorphism $\alpha\colon p_1^* x \xrightarrow{\sim} p_2^* x$ over $U \times_{\mathcal{T}} U$ induces an isomorphism $\beta\colon p_1^* z' \xrightarrow{\sim} p_2^* z'$ over $S_2 \times_S S_2$. Moreover, since α satisfies the cocycle condition and the equivalence in the statements holds for $U \times_{\mathcal{T}} U \times_{\mathcal{T}} U$, the map β also satisfies the cocycle condition, thus there is an object $z \in \mathcal{X}(S)$ pulling back to z'. We set $\Psi(x, \alpha) = z$ and it is a simple exercise to verify that this is in fact an inverse. Finally, when $\mathcal{T}$ is an algebraic space, we can apply the same argument as above to an étale presentation. $\square$

Definition 4.3.8. For an algebraic stack $\mathcal{X}$, we define the *universal family over* $\mathcal{X}$ to be the object $u \in \mathcal{X}(\mathcal{X})$ (unique up to unique isomorphism) corresponding to $\mathrm{id}_{\mathcal{X}} \in \operatorname{Mor}(\mathcal{X}, \mathcal{X})$ under the equivalence above.

We are now ready to describe the universal family of the moduli stack of smooth curves $\mathcal{M}_g$. Define a family of smooth curves $\mathcal{C} \to \mathcal{T}$ over an algebraic stack $\mathcal{T}$ as a morphism of algebraic stacks which is representable by schemes, proper, flat, and finitely presented such that for every geometric point $\operatorname{Spec} k \xrightarrow{s} \mathcal{T}$, the fibre $\mathcal{C}_s = \mathcal{C} \times_{\mathcal{T}} k$ is a smooth, connected, and projective curve of genus g. By Descent Theory (see [Alp27, §3.1]) the category of families of smooth curves over $\mathcal{T}$ is identified with $\mathcal{M}_g(\mathcal{T})$. We denote with $\mathcal{U}_g \to \mathcal{M}_g$ the family of curves corresponding to the universal family.

Representability and stabilisers

In Section 4.2, we have observed that a prestack $\mathcal{X}$ is equivalent to a presheaf if and only if the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is fully faithful, suggesting that the diagonal of a stack encodes some important properties.
In fact, the following theorem shows how algebraicity reflects on the diagonal.

Theorem 4.3.9 (Representability of the Diagonal)**.**

(1) The diagonal of an algebraic space is representable by schemes.

(2) The diagonal of an algebraic stack is representable.

Proof. See [Alp27, Thm 4.2.1]. $\square$

Corollary 4.3.10. *If the diagonal of a stack $\mathcal{X}$ is representable (resp., representable by a scheme), then every morphism $U \to \mathcal{X}$ from a scheme is representable (resp., representable by a scheme).*
In particular, every morphism from a scheme to an algebraic stack (resp., algebraic space) is representable (resp., representable by a scheme).

Proof. For the first part, recall that for any two morphisms $S_1 \to \mathcal{X}$ and $S_2 \to \mathcal{X}$ from schemes we have a cartesian diagram

$$\begin{tikzcd} S_1 \times_{\mathcal{X}} S_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S_1 \times S_2 \arrow[d] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X}. \end{tikzcd}$$

Given a morphism $V \to \mathcal{X}$ from a scheme, we want to show that $U \times_{\mathcal{X}} V$ is an algebraic space (resp., a scheme). The diagram above tells us that $U \times_{\mathcal{X}} V$ is isomorphic to $(U \times V) \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}$, which is an algebraic space (resp., a scheme) because the diagonal is representable (resp., representable by a scheme).
The second part follows directly fro the first one and Representability of the Diagonal (Theorem 4.3.9). $\square$

Recall that the first axiom of stack is equivalent to the isomorphism presheaves $\underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a, b)$ being all sheaves. Recall also that there is a cartesian diagram

$$\begin{tikzcd} \underline{\mathrm{Isom}}_{\mathcal{X}(T)}(a, b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \arrow[d, "{(a,b)}"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X}, \end{tikzcd}$$

thus for an algebraic stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{ét}}$ the isomorphism sheaf is actually an algebraic space by Theorem 4.3.9.

The diagonal indeed encodes the stabilisers group in the following sense.

Definition 4.3.11. Let $\mathcal{X}$ be an algebraic stack and $x\colon \operatorname{Spec} k \to \mathcal{X}$ be a field-valued point. We define the *stabiliser of* x as the group algebraic space[2]

$$G_x \coloneqq \underline{\mathrm{Aut}}_{\mathcal{X}(k)}(x).$$

The stabilisers naturally form a family as $x \in \mathcal{X}(k)$ varies.

Definition 4.3.12. The *inertia stack* of an algebraic stack $\mathcal{X}$ is the fibred product

$$\begin{tikzcd} I_{\mathcal{X}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\Delta"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X}. \end{tikzcd}$$

In the relative setting, for a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks, the *relative inertia stack* is $I_{\mathcal{X}/\mathcal{Y}} \coloneqq \mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X}$.

Clearly, the fibre of $I_{\mathcal{X}} \to \mathcal{X}$ over a field-valued point $x \in \mathcal{X}(k)$ is precisely G_x.

4.4 Properties, dimension and tangent space

In this section we briefly talk about properties of algebraic stacks and morphisms between them, then we extend to algebraic stacks the concepts of dimension and tangent space.

The notion of representability (by schemes) of a morphism is clearly stable under base change. Furthermore, representability (by schemes) is also stable under composition; see [Alp27, Lemma 4.3.1].

The following properties of morphisms of algebraic stacks are smooth local on the target: representable, isomorphism, open/closed immersion, affine, or quasi-affine. See [Alp27, Prop. 4.3.6].

Definition 4.4.1. Let $\mathcal{P}$ be a property of schemes étale (resp., smooth) local. We say that a DM stack $\mathcal{X}$ *has* $\mathcal{P}$ if for an étale (resp., smooth) presentation $U \to \mathcal{X}$ (equiv. for all presentations), the scheme U has $\mathcal{P}$.

As for schemes, separation properties for algebraic stacks are defined in terms of the diagonal.

Definition 4.4.2.

- A morphism of algebraic stacks $\mathcal{X} \to \mathcal{Y}$ has *affine diagonal* (resp., *quasi-affine diagonal*, *separated diagonal*) if the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is affine (resp., quasi-affine, separated).

[2] A group object in the category of algebraic spaces.

- A morphism of algebraic stacks $\mathcal{X} \to \mathcal{Y}$ is *quasi-separated* if the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ and second diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X}$ are quasi-compact.

- A morphism of algebraic stacks $\mathcal{X} \to \mathcal{Y}$ is *of finite presentation* if it is locally of finite presentation, quasi-compact, and quasi-separated.

- A representable morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *separated* if the diagonal, which is representable by schemes, is proper.

Conditions on the diagonal translates to conditions on the isomorphism sheaves. In particular, $\mathcal{X}$ has affine diagonal if and only if $\underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a, b)$ is affine for every scheme S and objects $a, b \in \mathcal{X}(S)$.

Lemma 4.4.3. *Let S be an affine scheme and $G \to S$ be a smooth affine group scheme acting on an algebraic space U over S. If U has (quasi-)affine diagonal, then so does $[U/G]$.*

Proof. In Theorem 4.3.5 we proved that $[U/G]$ is an algebraic stack, thus Representability of the Diagonal (Theorem 4.3.9) implies that $[U/G] \to [U/G] \times_S [U/G]$ is a representable morphism. Since being (quasi-)affine is smooth local on the target, thanks to the diagram

$$\begin{tikzcd} G \times_S U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \times_S U \arrow[d] \\ {[U/G]} \arrow[r] & {[U/G] \times_S [U/G]} \end{tikzcd}$$

of Proposition 4.2.9 it suffices to show that $G \times_S U \to U \times_S U$ is (quasi-)affine. Since G is affine, so is the composition $G \times_S U \to U \times_S U \xrightarrow{p_2} U$. On the other hand, since U has (quasi-)affine diagonal, so does p_2. This implies that $G \times_S U \to U \times_S U$ is (quasi-)affine. $\square$

For $g \geq 2$, from Algebraicity of $\mathcal{M}_g$ (Theorem 4.3.6) and the lemma above, it follows that $\mathcal{M}_g$ has affine diagonal.

Underlying topological space

We can associate a topological space $|\mathcal{X}|$ to every algebraic stack $\mathcal{X}$.

Definition 4.4.4. For an algebraic stack $\mathcal{X}$, we define the *topological space of* $\mathcal{X}$ as the set $|\mathcal{X}| := \{x \colon \operatorname{Spec} k \to \mathcal{X} \mid k \text{ field}\}/_\sim$ where two field-valued point $x \colon \operatorname{Spec} k \to \mathcal{X}$ and $x' \colon \operatorname{Spec} k' \to \mathcal{X}$ are identified if there exist field extensions $k \to K$ and $k' \to K$ such that $x|_{\operatorname{Spec} K} \simeq x'|_{\operatorname{Spec} K}$ in $\mathcal{X}(K)$. We define a subset $U \subseteq |\mathcal{X}|$ to be open if there exists an open substack $\mathcal{U} \subseteq \mathcal{X}$ such that $U = |\mathcal{U}|$.

Clearly, a morphism of algebraic stacks $\mathcal{X} \to \mathcal{Y}$ induces a continuous map $|\mathcal{X}| \to |\mathcal{Y}|$.

Given $x \in |\mathcal{X}|$, the stabiliser group depends on the choice of representative $x_1\colon \operatorname{Spec} k \to \mathcal{X}$ of x but its dimension is independent of this choice. Similarly, other properties (e.g. being smooth) are independent of this choice. See [Alp27, Exc. 4.3.20]. Therefore, it makes sense to define $\dim G_x$.
For a DM stack $\mathcal{X}$, we define the *geometric stabiliser of* x as the discrete group $G_{\overline{x}}(k)$ where $\overline{x}\colon \operatorname{Spec} k \to \mathcal{X}$ is a geometric point representing x, namely k is algebraically closed.

Definition 4.4.5. An algebraic stack $\mathcal{X}$ is said to be *quasi-compact*, *connected*, or *irreducible* if $|\mathcal{X}|$ is, and we say that $\mathcal{X}$ is *noetherian* if it is locally noetherian, quasi-separated, and quasi-compact.

Definition 4.4.6. A morphism of algebraic stacks $\mathcal{X} \to \mathcal{Y}$ is *quasi-compact* if for every morphism $\operatorname{Spec} R \to \mathcal{Y}$, the fibred product $\mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} R$ is quasi-compact. We say that $\mathcal{X} \to \mathcal{Y}$ is *of finite type* if it is locally of finite type and quasi-compact.

Example. The moduli stack $\mathcal{M}_g$ has affine diagonal and is thus quasi-separated [Alp27, Ex. 4.3.15]. Moreover, it is also noetherian and of finite type over $\mathbb{Z}$ [Alp27, Ex. 4.3.23].

Other properties

A morphism of schemes is *locally quasi-finite* if it is locally of finite type and every fibre is discrete. Since this property is étale local on source and target, we can extend it to morphisms of algebraic spaces.

Definition 4.4.7.

- A representable morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *locally quasi-finite* if for every morphism $T \to \mathcal{Y}$ from a scheme, the morphism of algebraic spaces $\mathcal{X} \times_{\mathcal{Y}} T \to T$ is locally quasi-finite.
- A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *locally quasi-finite* if it is locally of finite type, the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is locally quasi-finite, and for every morphism $\operatorname{Spec} k \to \mathcal{Y}$, the topological space $|\mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} k|$ is discrete.
- A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *quasi-finite* if it is locally quasi-finite and quasi-compact.

Finally, to define étale and unramified properties for a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks, we require that it is *relatively Deligne-Mumford*, i.e., for every morphism $T \to \mathcal{Y}$ from a scheme, the fibred product $\mathcal{X} \times_{\mathcal{Y}} T$ is a DM stack.

Definition 4.4.8. A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *étale* (resp., *unramified*) if it is relatively DM and for every smooth presentation $V \to \mathcal{Y}$ and étale presentation $U \to \mathcal{X} \times_{\mathcal{Y}} V$, the induced morphism $U \to V$ of schemes is étale (resp., unramified).

It follows from facts on schemes that a morphism is étale if and only if it is smooth and unramified, and it is unramified if and onlt if the diagonal is étale; see [Alp27, Thms A.3.2, A.3.3].

Dimension

Recall that for a scheme X, the dimension $\dim X$ is the dimension of its underlying topological space while the local dimension $\dim_x X$ at a point $x \in X$ is the minimum dimension of open subsets containing x, which is equal to $\dim \mathcal{O}_{X,x}$ when X is of finite type over a field and $x \in X$ is a closed point.

Definition 4.4.9.

- Let X be a noetherian algebraic space and $x \in |X|$. We define the *dimension of X at x* to be
$$\dim_x X = \dim_u U \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$$
where $U \to X$ is an étale presentation and $u \in |U|$ is a preimage of x.

- Let $\mathcal{X}$ be a noetherian algebraic stack and $x \in |\mathcal{X}|$. If $U \to \mathcal{X}$ is a smooth presentation, and $u \in |U|$ is a preimage of x, we define the *dimension of $\mathcal{X}$ at x* to be
$$\dim_x \mathcal{X} = \dim_u U - \dim_{\Delta(u)} R_u \in \mathbb{Z} \cup \{\infty\}$$
where $R = U \times_{\mathcal{X}} U$, the algebraic space R_u is the fibre of the projection $R \to U$ at u, and $\Delta \colon U \to R$ is the diagonal.

- If $\mathcal{X}$ is a noetherian algebraic stack, we define the *dimension of $\mathcal{X}$* to be
$$\dim \mathcal{X} = \sup_{x \in |\mathcal{X}|} \dim_x \mathcal{X} \in \mathbb{Z} \cup \{\infty\}.$$

The definition above is independent of the choices of a presentation and a preimage [Alp27, Prop. 4.5.2]. Note that the dimension of an algebraic stack can be negative.

Example. If G is a smooth affine algebraic group over a field k acting on a pure dimensional scheme U of finite type over k, then
$$\dim[U/G] = \dim U - \dim G.$$

Tangent space

Recall that for a scheme X, the tangent space at a point $x \in X$ is identified with the set of morphisms $\operatorname{Spec} \kappa(x)[\varepsilon] \to X$ having image x, which we view geometrically as tangent vectors.

Definition 4.4.10. Let $\mathcal{X}$ be an algebraic stack and $x \colon \operatorname{Spec} k \to \mathcal{X}$ be a point. We define the *tangent space of $\mathcal{X}$ at x* as

$$T_{\mathcal{X},x} := \left\{ \text{2-commutative diagrams} \quad \begin{tikzcd} \operatorname{Spec} k \arrow[d, hook] \arrow[dr, "x"] & \\ \operatorname{Spec} k[\varepsilon] \arrow[r, "\tau"'] \arrow[ur, phantom, "\Leftarrow\alpha"] & \mathcal{X} \end{tikzcd} \right\} \Big/ \sim ,$$

i.e., the set of pairs (τ, α) with $\tau \colon \operatorname{Spec} k[\varepsilon] \to \mathcal{X}$ and $\alpha \colon x \xrightarrow{\sim} \tau|_k$, where we identify two pairs $(\tau, \alpha) \sim (\tau', \alpha')$ if there is an isomorphism $\gamma \colon \tau \xrightarrow{\sim} \tau'$ in $\mathcal{X}(k[\varepsilon])$ such that $\alpha' = \beta|_k \circ \alpha$.

Example. If $[C] \in \mathcal{M}_g(k)$ is a smooth, connected, projective curve over k of genus $g \geq 2$, by Deformation Theory (Corollary 3.2.1) we get $T_{\mathcal{M}_g,[C]} \simeq H^1(C, T_C)$, which has dimension $3g - 3$.

Under suitable hypothesis, we can endow the tangent space with a structure of vector space.

Proposition 4.4.11. *If $\mathcal{X}$ is an algebraic space with affine diagonal and $x \in \mathcal{X}(k)$ where k is a field, then $T_{\mathcal{X},x}$ is naturally a k-vector space.*

Proof. Given $c \in k$ and $(\tau, \alpha) \in T_{\mathcal{X},x}$, we define the scalar multiplication $c \cdot (\tau, \alpha)$ as the composition $\operatorname{Spec} k[\varepsilon] \to \operatorname{Spec} k[\varepsilon] \xrightarrow{\tau} \mathcal{X}$, where the first map is induced by $\varepsilon \mapsto c\varepsilon$, with the same 2-isomorphism α. To define the sum, we will use the fact that there is an equivalence of categories

$$\mathcal{X}(k[\varepsilon_1] \times_k k[\varepsilon_2]) \to \mathcal{X}(k[\varepsilon_1]) \times_{\mathcal{X}(k)} \mathcal{X}(k[\varepsilon_2]),$$

which is proved in [Alp27, Prop. 4.5.10]. We define then the sum of (τ, α) and (τ', α') as the composition $\operatorname{Spec} k[\varepsilon] \to \operatorname{Spec}(k[\varepsilon_1] \times_k k[\varepsilon_2]) \to \mathcal{X}$, where the first map is induced by $(\varepsilon_1, 0), (0, \varepsilon_2) \mapsto \varepsilon$. It remains to verify that the operations defined above endow the tangent space with a structure of k-vector space. $\square$

Remark. For a scheme X, every point x has a corresponding residue field $\kappa(x)$ with a monomorphism $\operatorname{Spec} \kappa(x) \to X$ having image x. This is not true in general for an algebraic stack, but there is the notion of *residual gerbe* which generalises that of residue field; see [Alp27, §4.5.3].

4.5 Criteria for smoothness and properness

In this section we extend to algebraic stacks two important results of Scheme Theory known as the Infinitesimal Lifting Criterion and the Valuative Criterion.

Infinitesimal Lifting Criterion

For schemes we have functorial characterisations of smooth/étale/unramified morphisms provided by the Infinitesimal Lifting Criteria [SP, Tag 02H6, Tag 02HM, and Tag 02HE].

Theorem 4.5.1. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a locally of finite type morphism of locally noetherian algebraic stacks with quasi-compact and separated diagonals. Consider a 2-commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd} \tag{4.1}$$

of solid arrows where $A \twoheadrightarrow A_0$ is a surjection of artinian local rings with residue field k such that $\ker(A \to A_0) \simeq k$. *Then*

(1) f is smooth if and only if there exists a lifting of every diagram (4.1)*;*

(2) f is étale if and only if there exists a lifting, which is unique up to unique isomorphism, of every diagram (4.1)*;*

(3) f is unramified if and only if every two liftings of a diagram (4.1) *are uniquely isomorphic; and*

(4) f has unramified diagonal if and only if every automorphism of a lifting of a diagram (4.1) *is trivial.*

We can also interpret liftings of a diagram (4.1) using the map of groupoids

$$\Phi\colon \mathcal{X}(A) \to \mathcal{X}(A_0) \times_{\mathcal{Y}(A_0)} \mathcal{Y}(A);$$

more precisely, such a diagram defines an element (x, y, α) of $\mathcal{X}(A_0) \times_{\mathcal{Y}(A_0)} \mathcal{Y}(A)$ and a lifting of it corresponds to an element $z \in \mathcal{X}(A)$ such that $\Phi(z) = (x, y, \alpha)$. Therefore, the criterion in Theorem 4.5.1 states that f is smooth (resp., is étale, is unramified, has unramified diagonal) if and only if Φ is essentially surjective (resp., an equivalence, fully faithful, faithful).

Proof. See [Alp27, Thm 4.7.1]. □

Furthermore, under suitable assumptions, we have the following intuitive relation between dimensions.

Proposition 4.5.2. *Let $\mathcal{X}$ be a smooth noetherian algebraic stack over the field k, and $x \in |\mathcal{X}|$ be a finite type point, i.e., $x \in \mathcal{X}(L)$ for a finite field extension L/k. If x has smooth stabiliser G_x, then*

$$\dim_x \mathcal{X} = \dim T_{\mathcal{X},x} - \dim G_x.$$

Proof. See [Alp27, Prop. 4.7.6]. □

We are now able to prove the smoothness of $\mathcal{M}_g$ for $g \geq 2$.

Proposition 4.5.3. *For genus $g \geq 2$, the moduli stack $\mathcal{M}_g$ is smooth over $\mathbb{Z}$.*

Proof. Let $\operatorname{Spec} k \to \mathcal{M}_g$ the morphism corresponding to a smooth, connected, and projective curve C over k. Consider a diagram

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[rr, bend left, "{[C]}"] & \operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & \mathcal{M}_g \arrow[d] \\ & \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} \mathbb{Z} \end{tikzcd}$$

as in (4.1). The map $\operatorname{Spec} A_0 \to \mathcal{M}_g$ clearly corresponds to a family of curves $\mathcal{C}_0 \to \operatorname{Spec} A_0$, and a lifting in this diagram is the same as an extension of

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & \mathcal{C}_0 \arrow[r, hook, dashed] \arrow[d] & \mathcal{C} \arrow[d, dashed] \\ \operatorname{Spec} k \arrow[r, hook] & \operatorname{Spec} A_0 \arrow[r, hook] & \operatorname{Spec} A. \end{tikzcd}$$

By Proposition 3.1.7, there is a cohomology class $\mathrm{ob}_C \in H^2(C, T_C)$ such that the diagram above has an extension if and only if $\mathrm{ob}_C = 0$, but $H^2(C, T_C) = 0$ because C is a curve, hence such a lifting exists. □

Valuative Criterion

Recall that in Definition 4.4.2 we have defined separatedness for a representable morphism of algebraic stacks.

Definition 4.5.4. Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks.

(1) A morphism $\mathcal{X} \to \mathcal{Y}$ is *universally closed* if for every morphism $\mathcal{Y}' \to \mathcal{Y}$ of algebraic stacks, the morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{Y}'$ induces a closed map $|\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'| \to |\mathcal{Y}'|$.

(2) A representable morphism $\mathcal{X} \to \mathcal{Y}$ is *proper* if it is universally closed, separated, and of finite type.

(3) A morphism $\mathcal{X} \to \mathcal{Y}$ is *separated* if $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is proper.

(4) A morphism $\mathcal{X} \to \mathcal{Y}$ is *proper* if it is universally closed, separated and of finite type.

For schemes we have a functorial characterisation of properness provided by the Valuative Criterion [SP, Tag 0BX5].

Theorem 4.5.5. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of locally noetherian algebraic stacks. Consider a 2-commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow\alpha"] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] & \mathcal{Y} \end{tikzcd} \tag{4.2}$$

where R is a DVR with fraction field K. Then

(1) f is universally closed if and only if for every diagram (4.2)*, there exists an extension $R \to R'$ of DVRs with $K \to K' :=$* $\mathrm{Frac}(R)$ *of finite transcendence degree and a lifting*

$$\begin{tikzcd} \operatorname{Spec} K' \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[r] \arrow[d, dashed] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{Spec} R' \arrow[r] \arrow[urr, dashed] & \operatorname{Spec} R \arrow[r] & \mathcal{Y}; \end{tikzcd} \tag{4.3}$$

(2) f is proper of and only if is of finite type and for every diagram (4.2)*, there exists an extension $R \to R'$ of DVRs with the map $K \to K'$ between fraction fields having finite transcendence degree and a lifting as in* (4.3)*, which is unique up to unique isomorphism;*

(3) f is separated if and only if every two liftings of a diagram (4.2) *are uniquely isomorphic; and*

(4) f has separated diagonal if and only if every automorphism of a lifting of a diagram (4.2) *is trivial.*

Proof. See [Alp27, §4.8.3]. □

For moduli problems, this criterion has an important geometric meaning. We think of the DVR R as a curve and of its fraction field K as the punctured curve, hence we are asking ourselves whether an object over the latter extends uniquely to a family over the entire curve R.

Remark. While the Infinitesimal Criterion for stacks is essentially the same as that for schemes, in the Valuative Criterion for stacks we need to allow extensions of the DVR, which are not necessary for schemes. In fact, there are examples showing that for stacks this is necessary [Alp27, Ex. 4.8.17].

Remark. The separatedness of $\mathcal{M}_g$ is equivalent to showing that given two families of smooth curves over a DVR such that their pullbacks to the fraction field are isomorphic, then the whole families are isomorphic. This is true even when considering families of stable curves [Alp27, Prop. 6.5.20], hence the smooth case for $g \geq 2$ is an immediate corollary.
However, we can see that $\mathcal{M}_g$ is not proper over $\operatorname{Spec} \mathbb{Z}$. Indeed, there is plenty of examples of families of smooth curves that fail the Valuative Criterion for properness, e.g. if $R = \mathbb{C}[\![t]\!]$, then $K = \mathbb{C}((t))$ and we have the family of hyperelliptic curves over $\operatorname{Spec} K$ defined locally as

$$\begin{array}{c} V\left(y^2 - x(x-t)\prod_{i=1}^{2g}(x-\lambda_i)\right) \subseteq \mathbb{A}^2 \times \operatorname{Spec} K \\ \downarrow \\ \operatorname{Spec} K \end{array}$$

where the λ_i's are fixed distinct complex numbers vertical map is the projection. We claim that the family of smooth curves above cannot be extended to a family of smooth curves over $\operatorname{Spec} R$. In fact, such an extension must have necessarily an hyperelliptic central fibre [Løn76, Lemma 4.2]. Now the cross-ratio of the first four points is

$$r(t) = \frac{\lambda_1(\lambda_2 - t)}{(\lambda_1 - t)\lambda_2},$$

thus for $t = 0$ we get $r(0) = 1$, implying that two of those points in the central fibre collides, contradicting its smoothness.

4.6 Deligne-Mumford stacks

This section focuses on Deligne-Mumford stacks. We begin with a characterisation of this property which is useful to see that $\mathcal{M}_g$ is indeed DM. Then we

will explore the geometry of these spaces, defining classical notions such as quasi-coherent sheaves and cohomology. In particular, we will see that DM stacks are étale locally quotients of an affine scheme by the stabiliser group.
We will also introduce the notion of a coarse moduli space, and discuss its existence and properties.

Theorem 4.6.1 (Characterisation of Deligne-Mumford Stacks)**.** *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent:*

(1) the stack $\mathcal{X}$ is Deligne-Mumford;

(2) the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is unramified; and

(3) every point of $\mathcal{X}$ has a discrete and reduced stabiliser group.

In particular, a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is relatively Deligne-Mumford if and only if its diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is unramified.

Proof. See [Alp27, Thm 4.6.4]. □

Remark. Note that if $\mathcal{X}$ has quasi-compact diagonal, then item (3) is equivalent to requiring that every stabiliser is finite and reduced.

The following result directly follows from the characterisation above.

Corollary 4.6.2. *Let $G \to S$ be a smooth affine group scheme acting on an algebraic space U over S via $\sigma\colon G \times_S U \to U$. Then $[U/G]$ is Deligne-Mumford if and only if every point of U has a discrete and reduced stabiliser group, or equivalently if the map $(\sigma, p_2)\colon G \times U \to U \times U$ is unramified.*

Corollary 4.6.3. *For $g \geq 2$, the stack $\mathcal{M}_g$ is a Deligne-Mumford stack of finite type over* $\operatorname{Spec}\mathbb{Z}$ *with affine diagonal. Moreover, $\mathcal{M}_g \times_{\mathbb{Z}} k$ has pure dimension $3g-3$ for every field k.*

Proof. We have already seen that $\mathcal{M}_g$ has affine diagonal and is of finite type. By the Characterisation of DM Stacks (Theorem 4.6.1), it suffices to show that every smooth, connected, projective curve C over a field k has a discrete and reduced automorphism group scheme $\underline{\mathrm{Aut}}(C)$. We show that the dimension of $T_{\underline{\mathrm{Aut}}(C),\mathrm{id}_C}$ is zero. By looking at

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r, "\mathrm{id}_C"] \arrow[d, hook] & \underline{\mathrm{Aut}}(C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{M}_g \arrow[d] \\
\operatorname{Spec} k[\varepsilon] \arrow[r] \arrow[ur, "\tau"] & \operatorname{Spec} k \arrow[r, "{([C],[C])}"] & \mathcal{M}_g \times \mathcal{M}_g
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\operatorname{Spec} k \arrow[r, "{[C]}"] \arrow[d, hook] & \mathcal{M}_g \arrow[d] \\
\operatorname{Spec} k[\varepsilon] \arrow[r] \arrow[ur, bend left] \arrow[ur, bend right] \arrow[ur, phantom, "\Downarrow\tau"] & \mathcal{M}_g \times \mathcal{M}_g
\end{tikzcd}$$

we can see that a tangent vector $\tau \in T_{\underline{\mathrm{Aut}}(C),\mathrm{id}_C}$ of the left diagram translates on the right diagram to an automorphism τ of the trivial first-order deformation of C. By Proposition 3.1.7, the group of such automorphisms is identified with $H^0(C, T_C)$, and this vector space is zero since $\deg T_C = 2 - 2g < 0$ for $g \geq 2$ (see Proposition 2.1.2). Finally, we compute the dimension by applying Proposition 4.5.2, since $\dim \underline{\mathrm{Aut}}(C) = 0$. $\square$

Similarly, we have the following characterisation of algebraic spaces, which is the class of stacks that more resembles to schemes. In fact, one may show that quasi-separated algebraic spaces have a dense open subspace which is a scheme [Alp27, Thm 5.5.1].

Theorem 4.6.4. *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent:*

(1) the stack $\mathcal{X}$ is an algebraic space;

(2) the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a monomorphism; and

(3) every point of $\mathcal{X}$ has a trivial stabiliser.

In particular, if $G \to S$ is a smooth affine group scheme acting on an algebraic space U over S, then $[U/G]$ is an algebraic space if and only if every point of U has a trivial stabiliser group, or equivalently if the map $(\sigma, p_2)\colon G \times U \to U \times U$ is a monomorphism.

Proof. See [Alp27, Thm 5.5.10 and Cor. 4.6.8]. $\square$

Quasi-coherent sheaves

As for schemes, we define the *small étale site* of a DM stack $\mathcal{X}$ as the category $\mathcal{X}_{\text{ét}}$ of schemes étale over $\mathcal{X}$. A covering of an $\mathcal{X}$-scheme U is a collection of étale morphisms $\{U_i \to U\}$ over $\mathcal{X}$ such that $\coprod_i U_i \to U$ is surjective.

We denote by $\mathrm{Ab}(\mathcal{X}_{\text{ét}})$ the category of abelian sheaves on $\mathcal{X}_{\text{ét}}$. For an abelian sheaf $F \in \mathrm{Ab}(\mathcal{X}_{\text{ét}})$, the sections over an étale $\mathcal{X}$-scheme are denoted by $F(U \to \mathcal{X})$. We can extend this definition to a DM stack $\mathcal{U}$ étale over $\mathcal{X}$ as follows. Let $U \to \mathcal{U}$ and $R \to U \times_{\mathcal{U}} U$ be étale presentations and define $F(\mathcal{U} \to \mathcal{X})$ as the equaliser of

$$F(U \to \mathcal{X}) \rightrightarrows F(R \to \mathcal{X}).$$

It is not difficult to check that this definition is independent of the choice of presentations. In particular, we can discuss

$$\Gamma(\mathcal{X}, F) \coloneqq F(\mathcal{X} \xrightarrow{\mathrm{id}} \mathcal{X}).$$

We can generalise as well the classical constructions, e.g. *pushforward*, *inverse image*, and the *stalk*; see [Alp27, Excs. 5.1.6 and 5.1.10].

Example. We may generalise two classical sheaves.

- If $\mathcal{X}$ is a DM stack, its *structure sheaf* $\mathcal{O}_{\mathcal{X}}$ is defined on an étale $\mathcal{X}$-scheme U by $\mathcal{O}_{\mathcal{X}}(U \to \mathcal{X}) = \Gamma(U, \mathcal{O}_U)$.
- If $\mathcal{X}$ is a DM stack over a scheme S, the *relative sheaf of differentials* $\Omega_{\mathcal{X}/S}$ is defined by $\Omega_{\mathcal{X}/S}(U \to \mathcal{X}) = \Gamma(U, \Omega_{U/S})$.

Since the structure sheaf is a ring object in $\mathrm{Ab}(\mathcal{X}_{\text{ét}})$, we may define $\mathcal{O}_{\mathcal{X}}$*-modules* as module objects over $\mathcal{X}_{\text{ét}}$. We denote $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ the category formed by them. We extend in the obvious way the definitions of *tensor product* and *Hom sheaf.* Finally, there is a *pullback* functor, and it is the left adjoint to the pushforward one; see [Alp27, Excs. 5.1.14 and 5.1.15].

Given an $\mathcal{O}_{\mathcal{X}}$-module F on a DM stack $\mathcal{X}$, for an étale $\mathcal{X}$-scheme U we may consider the restriction $F|_{U_{\mathrm{Zar}}}$ to the small Zariski site of U, which is a usual $\mathcal{O}_U$-module.

Definition 4.6.5. Let $\mathcal{X}$ be a Deligne-Mumford stack. An $\mathcal{O}_{\mathcal{X}}$-module F is *quasi-coherent* if

(i) for every étale $\mathcal{X}$-scheme U, the restriction $F|_{U_{\mathrm{Zar}}}$ is a quasi-coherent $\mathcal{O}_U$-module, and

(ii) for every étale morphism $f\colon U \to V$ of étale $\mathcal{X}$-schemes, the natural morphism $f^*(F|_{V_{\mathrm{Zar}}}) \to F|_{U_{\mathrm{Zar}}}$ is an isomorphism.

If in addition $\mathcal{X}$ is locally noetherian, we say F is *coherent* (resp., a *vector bundle of rank* r, a *line bundle*) if $F|_{U_{\mathrm{Zar}}}$ is coherent (resp., a vector bundle of rank r, a line bundle) for every étale morphism $U \to \mathcal{X}$.

We denote by $\mathrm{QCoh}(\mathcal{X})$ and $\mathrm{Coh}(\mathcal{X})$ the categories of quasi-coherent and coherent sheaves.
As one might expect, for a scheme $\mathcal{X}$ this agrees with the usual definition [Alp27, Exc. 5.1.22].

We also define the *Picard group* $\mathrm{Pic}(\mathcal{X})$ as the set of isomorphism classes of line bundles. It is an abelian group under tensor product.

Remark. We can also see that a quasi-coherent sheaf on a DM stack $\mathcal{X}$ is equivalent to the data of a quasi-coherent sheaf F_U for every $\mathcal{X}$-scheme U and compatible isomorphisms $f^*F_V \to F_U$ for every map $f\colon U \to V$ over $\mathcal{X}$ [Alp27, Exc. 5.1.24]. This provides a useful functorial way to think about quasi-coherent sheaves on a moduli stack.

Clearly, the structure sheaf $\mathcal{O}_{\mathcal{X}}$ is a line bundle, and the relative sheaf of differentials $\Omega_{\mathcal{X}/S}$ is quasi-coherent; actually, $\Omega_{\mathcal{X}/S}$ is a vector bundle when $\mathcal{X} \to S$ is smooth.

Example. The *Hodge bundle* $\mathbb{E}$ on $\mathcal{M}_g$ for $g \geq 2$ is defined by

$$\mathbb{E}(U \to \mathcal{M}_g) = \Gamma(\mathcal{C}, \Omega_{\mathcal{C}/U})$$

for an étale morphism $U \to \mathcal{M}_g$ corresponding to a family $\mathcal{C} \to U$ of smooth curves. By Proposition 2.1.4, this is a vector bundle of rank g.
Following the remark above, the Hodge bundle can be viewed as the data of $\pi_*\Omega_{\mathcal{C}/S}$ for every family of smooth curves $\pi\colon \mathcal{C} \to S$.

We may also extend the cohomology theory on Deligne-Mumford stacks. As for schemes, one shows that the abelian categories $\mathrm{Ab}(\mathcal{X}_{\text{ét}})$ and $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ have enough injectives, and if $\mathcal{X}$ is quasi-separated, then it holds for $\mathrm{QCoh}(\mathcal{O}_{\mathcal{X}})$ as well [Alp27, Lemma 5.1.34].
For $F \in \mathrm{Ab}(\mathcal{X}_{\text{ét}})$ we can then define the *pth cohomology group* $H^p(\mathcal{X}_{\text{ét}}, F)$ as the pth derived functor of the global sections functor Γ.

An important tool to compute usual cohomology groups is Čech cohomology. This can be generalised in the DM context and allows us to extend several results already known for schemes; see [Alp27, §5.1.6].

Coarse moduli

Given an affine scheme $\operatorname{Spec} A$ and a finite abstract group G acting on it, we may define a quotient scheme $\operatorname{Spec} A^G$, where A^G is the invariant ring; see [Mum70, §II.7]. On the other hand, we have seen the construction of the quotient stack $[\operatorname{Spec} A/G]$, and one may ask himself how these two objects are related.

Definition 4.6.6. A morphism $\pi\colon \mathcal{X} \to X$ from an algebraic stack to an algebraic space is called a *coarse moduli space* if

(i) for every algebraically closed field k, the induced map $\mathcal{X}(k)/{\sim} \to X(k)$ is bijective, and

(ii) the morphism π is universal for maps to algebraic spaces, i.e., every map $\mathcal{X} \to Y$ to an algebraic space factors uniquely as

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"'] \arrow[dr] & \\ X \arrow[r, dashed] & Y. \end{tikzcd}$$

If G is an fppf group scheme[3] over a scheme S acting on an algebraic space U over S, a G-invariant morphism $U \to X$ is called a *geometric quotient* if the

[3]For instance, this happens if G is associated to a finite abstract group.

induced map $[U/G] \to X$ is a coarse moduli space. In this case, we write $X = U/G$ and we have the commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[dr, "\text{geometric quotient}"'] & {[U/G]} \arrow[d, "\text{coarse moduli}"] \\ & U/G. \end{tikzcd}$$

As these objects are defined properly, for a finite abstract group G acting on $\operatorname{Spec} A$ we get that $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is a geometric quotient [Alp27, Thm 5.2.11]. In particular, the scheme $\operatorname{Spec} A^G$ ia coarse moduli space for $[\operatorname{Spec} A/G]$.

What we have seen above is a first instance of how coarse moduli spaces are useful in the study of an algebraic stack. For many moduli problems we are unable to find a fine moduli space, i.e., the moduli stack $\mathcal{X}$ is not a scheme, but if we manage to find a suitable coarse moduli space X, then this will be the "best approximation" of $\mathcal{X}$ by algebraic spaces.
We will formalise these ideas above.

Lemma 4.6.7. *Let $\pi\colon \mathcal{X} \to X$ be a coarse moduli space such that for every étale morphism $U \to X$ from an affine scheme, the base change $\mathcal{X} \times_X U \to U$ is a coarse moduli space. Then the natural map $\mathcal{O}_X \to \pi_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism.*

Proof. Since π is universal for maps to algebraic spaces, the natural map

$$\operatorname{Mor}(X, \mathbb{A}^1) \to \operatorname{Mor}(\mathcal{X}, \mathbb{A}^1)$$

is bijective, and one can check that this means exactly that $\Gamma(X, \mathcal{O}_X) \simeq \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Since for every étale morphism $U \to X$ from an affine scheme, the base change $\mathcal{U} = \mathcal{X} \times_X U \to U$ is a coarse moduli space as well, the we can repeat what we have just done and get $\Gamma(U, \mathcal{O}_U) \simeq \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}})$. This concludes the proof. $\square$

Local structure and Keel-Mori

We will see that a Deligne-Mumford stack is étale locally a quotient of affine schemes by a finite group.

While schemes are obtained by gluing affine schemes in the Zariski topology, the result above tells us that DM stacks are obtained by gluing quotient stacks in the étale topology.
This structure will then allow us to construct a coarse moduli space, provided by the Keel-Mori Theorem.

We define the *geometric stabiliser* of a point $x \in |\mathcal{X}|$ of a DM stack $\mathcal{X}$ as the abstract group given by the stabiliser of any geometric point with image x.
A point $x \in |\mathcal{X}|$ is *of finite type* if there exists a representative $\operatorname{Spec} k \to \mathcal{X}$ of x that is locally of finite type.

Theorem 4.6.8 (Local Structure of Deligne-Mumford Stacks)**.** *Let $\mathcal{X}$ be a quasi-separated Deligne-Mumford stack and $x \in |\mathcal{X}|$ be a finite type point with geometric stabiliser G_x. There exists an étale morphism*

$$f\colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$$

where w is a point of $[\operatorname{Spec} A/G_x]$ such that f induces an isomorphism of geometric stabiliser groups at w. Moreover, if $\mathcal{X}$ has separated (resp., affine) diagonal, then it can be arranged that f is representable and separated (resp., affine).

Proof. see [Alp27, Thm 5.3.1]. □

We now see how the result above implies the existence of a coarse moduli space.

Theorem 4.6.9 (Keel-Mori)**.** *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S. Then there exists a coarse moduli space $\pi\colon \mathcal{X} \to X$ with $\pi_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$ such that*

(1) the space X is separated and of finite type over S,

(2) the map π is a proper universal homeomorphism, and

(3) for every flat morphism $Y \to X$ of algebraic spaces, the base change

$$\mathcal{X} \times_X Y \to Y$$

is a coarse moduli space.

Moreover, X is proper over S if and only if $\mathcal{X}$ is.

Proof. We provide a partial proof, dealing first with the case when $S = \operatorname{Spec} R$ is affine. Note that if $\{\mathcal{X}_i\}$ is a Zariski open covering of $\mathcal{X}$ with coarse moduli spaces $\mathcal{X}_i \to X_i$, then since coarse moduli spaces are unique by definition, every intersection $\mathcal{X}_i \cap \mathcal{X}_j$ admits a coarse moduli space U_{ij} which is identified with an open subspace of both X_i and X_j, and gluing them yields an algebraic space X and a map $\mathcal{X} \to X$. Since being a coarse moduli space is an étale local property (this follows directly from definition), we constructed a coarse moduli space for $\mathcal{X}$. This shows that the problem is Zariski local on $\mathcal{X}$, hence it suffices to show that every closed point has an open neighbourhood that admits a coarse moduli space.

Let $x \in |\mathcal{X}|$ be a closed point. By the Local Structure of DM Stacks (Theorem 4.6.8), there exists an affine étale morphism

$$f\colon (\mathcal{W} = [\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$$

that induces an isomorphism of geometric stabilisers at w. We want to prove that the locus $\mathcal{U}$ of $z \in |\mathcal{W}|$ such that f induces such an isomorphism at z is open.

Since $\mathcal{X}$ is separated, the morphism $I_{\mathcal{X}} \to \mathcal{X}$ from the inertia stack is finite. The fibre over $z \in \mathcal{W}(k)$ of the natural morphism

$$\Psi\colon I_{\mathcal{W}} \to I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W}$$

of group schemes over $\mathcal{W}$ is exactly the morphism $G_z \to G_f(z)$ we want to study. We have a cartesian diagram

$$\begin{tikzcd}
I \times_{\mathcal{W}} \arrow[r, "\Psi"] \arrow[d] \arrow[dr, phantom, "\square"] & I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W} \arrow[d] \\
\mathcal{W} \arrow[r] & \mathcal{W} \times_{\mathcal{X}} \mathcal{W};
\end{tikzcd}$$

see [Alp27, Exc. 4.2.15]. Assumptions on f imply that its diagonal $\mathcal{W} \to \mathcal{W} \times_{\mathcal{X}} \mathcal{W}$ is both an open and closed immersion, hence so is Ψ. Finiteness of inertia $I_{\mathcal{X}} \to \mathcal{X}$ implies that $p_2\colon I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W} \to \mathcal{W}$ is finite as well. Therefore

$$p_2(|I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W}| \setminus |I_{\mathcal{W}}|) \subseteq |\mathcal{W}|$$

is closed, and its complement is $\mathcal{U}$, which is then open.

Let now $\pi_{\mathcal{W}}\colon \mathcal{W} \to W = \operatorname{Spec} A^{G_x}$ be the coarse moduli space. Choose an affine open subscheme $X_1 \subseteq \pi(\mathcal{U})$ containing $\pi_{\mathcal{W}}(w)$. Then $\mathcal{X}_1 = \pi_{\mathcal{W}}^{-1}(X_1)$ is isomorphic to $[\operatorname{Spec} A_1/G_x]$ for some ring A_1 such that $X_1 = \operatorname{Spec} A_1^{G_x}$. We then get an affine étale morphism

$$g\colon \mathcal{X}_1 \to \mathcal{X}$$

inducing a bijection on all geometric stabilisers.

It remains to prove that the open substack $\mathcal{X}_0 = \operatorname{im}(g)$ admits a coarse moduli space X_0 and it satisfies the three listed properties; see [Alp27, Thm 5.4.6].

The general case when S is a noetherian algebraic space reduces to the affine case essentially imitating the argument to étale locally construct the coarse moduli space. Finally, $\mathcal{X} \to X_0$ is proper by (2), hence the properness of $\mathcal{X}$ is equivalent to that of X. $\square$

Remark. Note that the theorem above not only states the existence of a coarse moduli space, but it also satisfies those three non-trivial properties. In general, it can be difficult to work with a coarse moduli space that only satisfies the defining properties, and it is not clear whether coarse moduli spaces satisfy $\pi_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$ or are stable under étale base change.

Corollary 4.6.10. *If G is an affine fppf group scheme over a noetherian scheme S acting properly on a separated algebraic space U, i.e., the map $G \times_S U \to U \times_S U$ is proper, such that all stabiliser groups are reduced, then there exists a geometric quotient $\pi\colon U \to X = U/G$ to an algebraic space X separated and of finite type over S such that $\mathcal{O}_X = (\pi_*\mathcal{O}_U)^G$.*

Proof. Since G is affine over S, the map $G \times_S U \to U \times_S U$ is affine, thus its properness is equivalent to the finiteness of U, and it implies that $[U/G]$ has finite diagonal. Also, the stabiliser groups are reduced, then $[U/G]$ is a separated DM stack, and we may apply Keel-Mori (Theorem 4.6.9). $\square$

Furthermore, coarse moduli spaces constructed from the Keel-Mori Theorem have an induced local structure coming from the DM stacks above them.

Corollary 4.6.11 (Local Structure of Coarse Moduli Spaces)**.** *Let $\mathcal{X}$ be a Deligne-Mumford stack of finite type and separated over a noetherian algebraic space S, and let $\pi\colon \mathcal{X} \to X$ be its coarse moduli space. For every closed point $x \in |\mathcal{X}|$ with geometric stabiliser group G_x, there exists a cartesian diagram*

$$\begin{tikzcd}
{[\operatorname{Spec} A/G_x]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\pi"] \\
\operatorname{Spec} A^{G_x} \arrow[r] & X
\end{tikzcd}$$

such that $\operatorname{Spec} A^{G_x} \to X$ *is an étale neighbourhood of* $\pi(x) \in |X|$.

Proof. The statement follows directly from Theorem 4.6.8 and the following fact. If $f\colon \mathcal{X} \to \mathcal{Y}$ is a morphism of DM stacks separated and of finite type over a noetherian algebraic space S fitting in a commutative diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "f"] \arrow[d, "\pi_{\mathcal{X}}"'] & \mathcal{Y} \arrow[d, "\pi_{\mathcal{Y}}"] \\
X \arrow[r] & Y
\end{tikzcd}$$

where $\pi_{\mathcal{X}}$ and $\pi_{\mathcal{Y}}$ are coarse moduli spaces and $x \in |\mathcal{X}|$ is a closed point such that f is étale at x and the induced map on geometric stabiliser groups is bijective, then there exists an open neighbourhood $U \subseteq X$ of $\pi_{\mathcal{X}}(x)$ such that $U \to X \to Y$ is étale and $\pi_{\mathcal{X}}(U) \simeq U \times_Y \mathcal{Y}$; see [Alp27, Exc. 5.4.14]. $\square$

Rephrasing the result above, we proved that the coarse moduli space of a smooth DM stack has *finite quotient singularities*, i.e., singularities that étale locally are described as quotients of smooth affine schemes by finite groups.

Tameness

When in presence of suitable conditions, the coarse moduli space is a useful tool for computing sheaf cohomology on a DM stack.

Definition 4.6.12. A Deligne-Mumford stack $\mathcal{X}$ is said to be *tame* if for every geometric point $x \in \mathcal{X}(k)$, the order of $\mathrm{Aut}_{\mathcal{X}(k)}(x)$ is invertible in $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. If in addition $\mathcal{X}$ admits a coarse moduli space, we say that it is a *tame coarse moduli space.*

Note that if $\mathcal{X}$ is defined over a field k, the hypothesis above means that the order of every geometric stabiliser group is prime to the characteristic of k. In particular, if $\mathrm{char}\, k = 0$, every DM stack is tame.

Having a tame coarse moduli space $\pi\colon \mathcal{X} \to X$ implies that the pushforward functor π_* is exact [Alp27, Lemma 5.4.22].

Corollary 4.6.13. *Let $\mathcal{X}$ be a Deligne-Mumford stack of finite type and separated over a noetherian algebraic space S. If $\pi\colon \mathcal{X} \to X$ is a tame coarse moduli space, then for every quasi-coherent sheaf F on $\mathcal{X}$,*

$$H^i(\mathcal{X}, F) = H^i(X, \pi_* F)$$

and is 0 *if* $i > \dim \mathcal{X}$.

Proof. The first statement follows from the fact that π_* is exact. The second one is also immediate by applying a generalised Grothendieck's Vanishing Theorem for algebraic spaces [Alp27, Exc. 5.1.48]. $\square$

Chapter 5

The Moduli Space of Stable Curves

In order to study further properties of the moduli stack $\mathcal{M}_g$ of smooth curves, Deligne and Mumford introduced in [DM69] a suitable compactification by considering stable curves. In this last chapter we discuss the properties of the moduli stack $\overline{\mathcal{M}}_{g,n}$ of stable n-pointed curves of genus g, particularly focusing in the last section on the foundational work of Harris and Mumford [HM82] regarding its birational geometry.

We will always assume $2g-2+n>0$ to avoid trivialities.
We define the prestack $\overline{\mathcal{M}}_{g,n}$ of stable n-pointed curves of genus g over Sch as the category of families of stable n-pointed curves of genus g and morphisms between them as in Definition 2.3.6.

As for $\mathcal{M}_g$, it is known that $\overline{\mathcal{M}}_{g,n}$ is an algebraic stack locally of finite type over $\operatorname{Spec}\mathbb{Z}$ [Alp27, Cor. 6.4.15]. One way to do this is to define the prestack of all curves $\mathcal{M}_g^{\text{all}}$ parametrising families of proper, one dimensional n-pointed schemes of finite type of genus g, and show that $\mathcal{M}_g^{\text{all}}$ is an algebraic stack locally of finite type over $\operatorname{Spec}\mathbb{Z}$; see [Alp27, §6.4]. Finally, Lemma 2.3.9 ensures us that $\overline{\mathcal{M}}_{g,n} \subseteq \mathcal{M}_g^{\text{all}}$ is an open substack, and we can then deduce the properties above.
Alternatively, one can adapt the proof of Theorem 4.3.6 to show directly the algebraicity of $\overline{\mathcal{M}}_{g,n}$ by explicitly presenting it as a quotient of a suitable subscheme of a Hilbert scheme.
Furthermore, recall that if (C,p_i) is an n-pointed stable curve over a field k, then $L \coloneqq (\omega_{C/k})^{\otimes 3}$ is very ample (Proposition 2.3.3), and the Hilbert polynomial P of the embedding $C \hookrightarrow \mathbb{P}^N$ via L is independent of $[(C,p_i)] \in \overline{\mathcal{M}}_{g,n}(k)$. We may then consider the closed subscheme

$$H \subseteq \operatorname{Hilb}^P(\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z}) \times_{\operatorname{Spec}\mathbb{Z}} (\mathbb{P}^N_{\mathbb{Z}})^n$$

parametrising pairs $(C \hookrightarrow \mathbb{P}^N, p_1, \ldots, p_n)$ such that $p_i \in C$. Since the Hilbert scheme is projective and in particular quasi-compact, so is H, and the image of the forgetful morphism $H \to \mathcal{M}_{g,n}^{\text{all}}$ contains $\overline{\mathcal{M}}_{g,n}$, hence we conclude that $\overline{\mathcal{M}}_{g,n}$ is quasi-compact.

We are now able to deduce further properties of $\overline{\mathcal{M}}_{g,n}$ thanks to the results discussed in the previous chapters.

Theorem 5.0.1. *For $2g-2+n>0$, the stack $\overline{\mathcal{M}}_{g,n}$ is a non-empty, quasi-compact, Deligne-Mumford stack smooth over* $\operatorname{Spec}\mathbb{Z}$*. Moreover, $\overline{\mathcal{M}}_{g,n} \times_{\mathbb{Z}} k$ has pure dimension $3g-3+n$ for every field k.*

Proof. We have seen that $\mathcal{M}_{g,n}$ is a quasi-compact algebraic stack, and clearly the hypotheses ensure that it is non-empty. By using Deformation Theory, we prove the remaining properties. More specifically, by Proposition 3.2.2 for a n-pointed stable curve (C, p_i) over k

$$\begin{aligned} \operatorname{Ext}^0_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C) &= 0, \\ \dim_k \operatorname{Ext}^1_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C) &= 3g-3+n, \text{ and} \\ \operatorname{Ext}^2_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C) &= 0. \end{aligned}$$

We already know by Proposition 2.3.3 that a stable curve has finitely many automorphism, and the first row implies that the group scheme $\underline{\operatorname{Aut}}(C, p_i)$ is finite, reduced, and étale over k, hence by Theorem 4.6.1 we conclude that $\overline{\mathcal{M}}_{g,n}$ is a DM stack.
The last row implies that there are no obstructions to deforming (C, p_i), therefore as in Proposition 4.5.3 the Infinitesimal Lifting Criterion (Theorem 4.5.1) proves that $\overline{\mathcal{M}}_{g,n}$ is smooth over $\operatorname{Spec}\mathbb{Z}$.
Moreover, since the set of isomorphism classes of first-order deformations of (C, p_i) is identified with the tangent space of $\overline{\mathcal{M}}_{g,n} \times_{\mathbb{Z}} k$ at the k-point corresponding to (C, p_i), the middle row implies that $\dim T_{\overline{\mathcal{M}}_{g,n},[(C,p_i)]} = 3g-3+n$. Finally, by Proposition 4.5.2 and finiteness of the stabiliser group we conclude that $\dim_{[(C,p_i)]} \overline{\mathcal{M}}_{g,n} = 3g-3+n$. □

5.1 Stable reduction

The moduli stack of stable curves was introduced to find a compactification of $\mathcal{M}_g$. We discuss here the Deligne and Mumford's fundamental result regarding the properness of $\overline{\mathcal{M}}_{g,n}$.

Theorem 5.1.1 (Stable Reduction)**.** *Let R be a DVR and K its fraction field. Set $\Delta = \operatorname{Spec} R$ and $\Delta^* = \operatorname{Spec} K$. If $(\mathcal{C}^* \to \Delta^*, \sigma_i^*)$ is a family of n-pointed stable curves, then there exists an extension of DVRs $R \to R'$ and an n-pointed family $(\mathcal{C} \to \Delta' = \operatorname{Spec} R', \sigma_i)$ of stable curves extending the base change of $(\mathcal{C}^* \to \Delta^*, \sigma_i^*)$ to $K' = \operatorname{Frac} R'$.*

Although this theorem holds for any DVR, we restrict our proof to the characteristic 0 case, following [Alp27, §6.5].
The general version was discussed first in [DM69], where the proof relies on the embedding of the generic fibre $\mathcal{C}^*$ into its Jacobian.
Later, in [AW71]employed a strategy similar to the one presented here, but with additional techniques required to handle positive characteristic.

In the following, denote by t the uniformiser of R, and by $0 = (t) \in \Delta$ the unique closed point. The strategy of the proof is summarised as follows.

(1) Reduce to the case where $\mathcal{C}^* \to \Delta^*$ is smooth.

(2) Find a flat extension $\mathcal{C} \to \Delta$.

(3) Replace $\mathcal{C}$ with a resolution of singularities such that the reduced central fibre $(\mathcal{C}_0)_{\mathrm{red}}$ is nodal.

(4) Take a ramified extension $\Delta' \to \Delta$ such that the central fibre of the normalisation of $\mathcal{C} \times_\Delta \Delta'$ is reduced and nodal.

(5) Arrange that the marked points of the central fibre are smooth and distinct.

(6) Contract rational tails and bridges in the central fibre.

Throughout the following proof, in the figures we carry out examples to clarify these steps.

Proof in characteristic 0.

Step 1: Reduce to the case where $\mathcal{C}^* \to \Delta^*$ is smooth. If $\mathcal{C}^*$ has δ nodes, then after replacing K and R with extensions, we may assume that the jth node is given by a K-point $n_j^* \in \mathcal{C}^*$ whose preimage under the normalisation $\widetilde{\mathcal{C}}^* \to \mathcal{C}^*$ consists of two K-points q_j^* and r_j^*. Recall that $(\widetilde{\mathcal{C}}^* \to \Delta^*, \sigma_i^*, q_j^*, r_j^*)$ is the so called pointed normalisation, and let $(\widetilde{\mathcal{C}}_k^* \to \Delta^*, q_{kl}^*)$ be its pointed connected components, where $\{q_{kl}^*\} = \{\sigma_i^*, q_j^*, r_j^*\} \cap \widetilde{\mathcal{C}}_k^*$. As already observed in Section 2.3, since $\mathcal{C}^*$ is stable, each $(\widetilde{\mathcal{C}}_k^* \to \Delta^*, q_{kl}^*)$ is stable as well.
Assuming that Stable Reduction holds when the generic fibre $\mathcal{C}^*$ is smooth, we can construct stable families $(\widetilde{\mathcal{C}}_k \to \Delta, q_{kl})$ extending $(\widetilde{\mathcal{C}}_k^* \to \Delta^*, q_{kl}^*)$ after replacing R

with an extension. For each $j = 1, \dots, \delta$, we now glue[1] the sections q_{il} and $q_{i'l'}$ corresponding to q_j^* and r_j^*. These sections are glued to a node, and this produces an n-pointed family $(\mathcal{C} \to \Delta, \sigma_i)$ of nodal curves with δ additional sections given by the nodes. The pointed normalisation of the central fibre $\mathcal{C}_0$ is the disjoint union of the central fibres $(\widetilde{\mathcal{C}}_k)_0$, which are stable, then it follows that $(\mathcal{C}_0, \sigma_i(0))$ is stable.

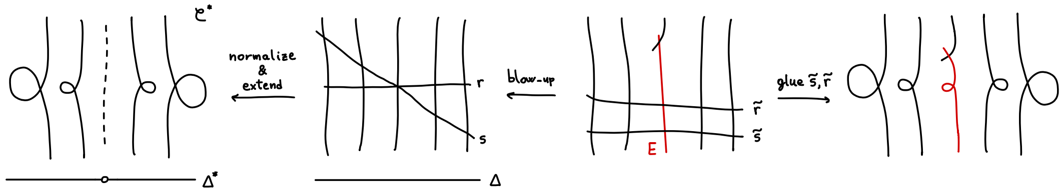


Figure 5.1: Here we perform Step 1 and stable reduction on the nodal family degenerating to a cuspidal curve, locally defined by $y^2 = x^3 + tx^2$. More precisely, we normalize along the nodes, extend the family so obtained, and blow-up where the two sections s and r intersect. Finally, we glue their proper transforms.

Step 2: Find a flat extension $\mathcal{C} \to \Delta$. By Step 1, we may assume that $\mathcal{C}^* \to \Delta^*$ is smooth. Using that $(\Omega_{\mathcal{C}^*/\Delta^*}(\sum_i \sigma_i^*))^{\otimes 3}$ is very ample (Proposition 2.3.8), we may embed $\mathcal{C}^*$ as a closed subscheme of $\mathbb{P}^N \times \Delta^*$. By the Flatness Criterion for Smooth Curves [Alp27, Prop. A.2.2], the closure $\mathcal{C}$ of $\mathcal{C}^* \subseteq \mathbb{P}^N \times \Delta$ is flat over Δ. Hence we get a family $\mathcal{C} \to \Delta$ extending $\mathcal{C}^* \to \Delta^*$. The sections σ_i^* extend to sections $\sigma_i \colon \Delta \to \mathcal{C}$ by the usual Valuative Criterion for Properness.
However, after this procedure, the central fibre $\mathcal{C}_0$ may have worse-than-nodal singularities. Even the marked points $\sigma_i(0)$ may be singular and non-distinct.

Step 3: Replace $\mathcal{C}$ with a resolution of singularities such that the reduced central fibre $(\mathcal{C}_0)_{\mathrm{red}}$ is nodal. By classical results of birational geometry of surfaces [Alp27, §B.2.1], there is a projective birational morphism $\widetilde{\mathcal{C}} \to \mathcal{C}$ from a regular scheme which is an isomorphism over Δ^* and such that the reduced central fibre $(\widetilde{\mathcal{C}}_0)_{\mathrm{red}}$ is nodal. Again by the Flatness Criterion for Smooth Curves, the morphism $\widetilde{\mathcal{C}} \to \Delta$ is flat. Replacing $\mathcal{C}$ with $\widetilde{\mathcal{C}}$ and the sections σ_i with their strict transform, we conclude this step.

Step 4: Take a ramified extension $\Delta' \to \Delta$ such that the central fibre of the normalisation of $\mathcal{C} \times_\Delta \Delta'$ is reduced and nodal. This is the step where the characteristic 0 assumption is used. Since $\mathcal{C}$ is regular (hence Cohen-Macauley) and $t \in R$ is not a zero divisor, the central fibre is Cohen-Macauley, and thus has no embedded points; see [SP, Tag 02JN and Tag 0BXG]. On the other hand, $(\mathcal{C}_0)_{\mathrm{red}}$ is an effective Cartier divisor and thus is locally cut out by a single equation.

[1] This is formally described with a Ferrand Pushout; see [Alp27, §B.4.1].

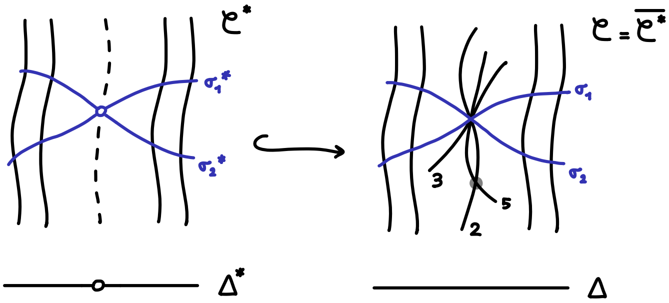


Figure 5.2: When proceeding with Step 2, the central fibre of the closure of the embedding of the family may be highly singular. Here the numbers represent the multiplicity of each component, and the thick dot stands for an embedded point. Moreover, in the central fibre the extended sections may overlap or meet the singular locus.

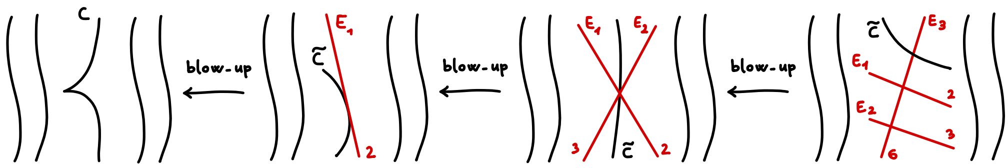


Figure 5.3: Here we perform as in Figure 5.1 the stable reduction on a family degenerating to the cuspidal cubic, only this time we consider the family over $\Delta = \operatorname{Spec} k[\![t]\!]$ locally given by $y^2 = x^3 + t$. In this case the generic fibre is smooth, thus applying Step 3 we may replace the central fibre with a curve whose reduced version is nodal.

For every $p \in \mathcal{C}_0$, there exists an étale neighbourhood and local coordinates such that the map $\mathcal{C} \to \Delta$ is given by:

- if $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is smooth, then $(x, y) \mapsto x^a$, i.e., the multiplicity of the irreducible component of $\mathcal{C}_0$ containing p is a, and

- if $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is a node, then $(x, y) \mapsto x^a y^b$, i.e., the two components containing p have multiplicities a and b. Moreover, if $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is a nonseparating node, then $a = b$.

Let N be the least common multiple of the multiplicities of the irreducible components of $\mathcal{C}_0$. After replacing R with an extension, we can assume that R contains a primitive Nth root of unity ζ. Let now $\Delta' = \operatorname{Spec} R' \to \operatorname{Spec} R = \Delta$ be a map induced by a totally ramified extension of DVRs of degree N, i.e., the uniformiser t is mapped to $t'^N \in R'$, where t' is the uniformiser of R'. Let $\mathcal{C}' := \mathcal{C} \times_\Delta \Delta'$ with $p' \in \mathcal{C}'$ being the unique preimage of p, and let $\widetilde{\mathcal{C}}'$ be the normalisation of $\mathcal{C}'$.

If $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is smooth, then $\mathcal{C}'$ is defined étale locally by $x^a = t^N$ near p'. Since we work in characteristic 0, we can factorise

$$x^a - t^N = \prod_{i=0}^{a-1} \left(x - (\zeta^i t)^{N/a}\right)$$

into distinct factors. In $\widetilde{\mathcal{C}}'$, the point p has a preimages, each étale locally given by $x = \zeta^i t^{N/a}$. Hence, each preimage is a smooth point in both the total family and the central fibre.
If instead $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is a node, then $\mathcal{C}'$ is defined étale locally at p' by $x^a y^b = t^N$. If $d = \gcd(a, b) > 1$, we can factorise

$$x^a y^b - t^N = \prod_{i=0}^{d-1} \left(x^{a/d} y^{b/d} - (\zeta^i t)^{N/d}\right),$$

thus étale locally at p, the normalisation factors as $\widetilde{\mathcal{C}}' \to \mathcal{C}'' \to \mathcal{C}'$ with p having d preimages in $\mathcal{C}''$, each described by $(x^{a/d} y^{b/d} - (\zeta^i t)^{N/d})$. Unless $d = a = b$, the central fibre of $\mathcal{C}'$ is still nonreduced, and its reduction is nodal at each preimage p but now the multiplicities of the two branches are relatively prime. We may then assume that $d = 1$. After possibly exchanging x and y, we may write $1 = \alpha a - \beta b$ for positive integers α and β. The normalisation of $R[x, y]/(x^a y^b - t^N)$ is thus given by

$$\begin{aligned} R[x, y]/(x^a y^b - t^N) &\hookrightarrow R[u, v]/(uv - t^{N/ab}) \\ x &\mapsto u^b \\ y &\mapsto v^a \end{aligned}$$

where $u = t^{\alpha N/b}/x^\beta y^\alpha$ and $u = x^\beta y^\alpha / t^{\beta N/a}$. The central fibre $\widetilde{\mathcal{C}}'_0$ is reduced with a node at the unique preimage of p. In $N = ab$, the total family $\widetilde{\mathcal{C}}'$ is smooth at this preimage; otherwise, it has an A_{n-1}-singularity where $n = N/ab$.
The central fibre is now reduced and nodal, but we have potentially introduced A_{n-1}-singularities into the total families. To fix this, we may repeatedly blow up each of these singularities, replacing them with a nodal union of $\lfloor n/2 \rfloor$ smooth rational curves [Alp27, Thm B.2.2]. We then get a new family $\mathcal{C}$ which is regular and whose central fibre is reduced and nodal.

Step 5: Arrange that the marked points of the central fibre are smooth and distinct. After again repeatedly blowing up closed points in the central fibre $\mathcal{C}_0$ where the marked points are singular or overlap, the strict transform of the sections become distinct and smooth points. This new family $(\mathcal{C} \to \Delta, \sigma_i)$ is prestable with regular total family.

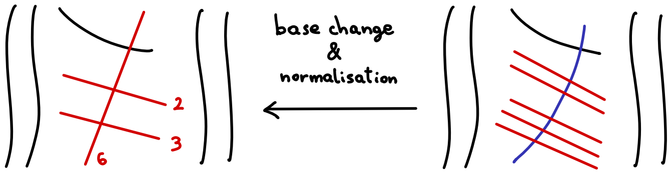


Figure 5.4: Resuming from Figure 5.3, we may apply Step 4. We thus perform the base change of the last family to $\Delta' = \operatorname{Spec} k[\![t]\!]$ induced by $t \mapsto t^6$. Normalising then yields a family with a nodal central fibre; see [Alp27, Ex. 6.5.13] for the explicit description of the computations carried out in this example, which result in some rational components (in red) and one of genus 1 (in blue).

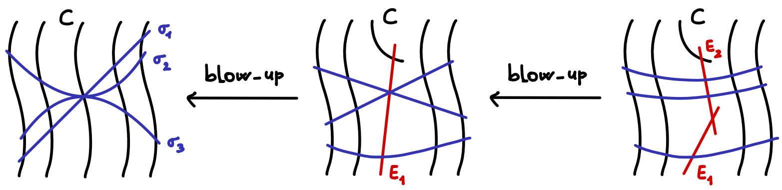


Figure 5.5: To better see what Step 5 is about, consider a constant family $C \times \Delta \to \Delta$ with three sections locally given by $\sigma_1 = -\sigma_2 = t^2$ and $\sigma_3 = t$. Blowing-up twice at their intersection, these sections become disjoint. However, since the exceptional component E_1 is rational with only two special points, the central fibre is not yet stable.

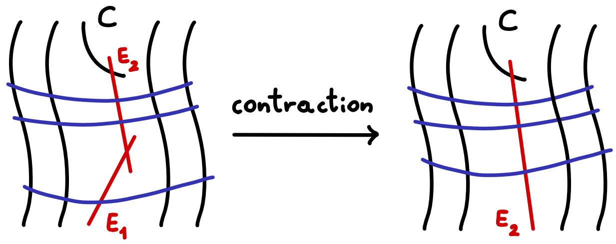


Figure 5.6: Resuming from Figure 5.5, if we contract the rational bridge E_1 yields a family of stable curves. Similarly, applying Step 6 to Figure 5.3 means to contract all the five rational tails, and then the central fibre becomes stable.

Step 6: Contract rational tails and bridges in the central fibre. After the previous steps, the central fibre of $(\mathcal{C} \to \Delta, \sigma_i)$ will be stable, unless there are rational tails and bridges. By Theorem 2.3.13, passing to the stable model yields a family $(\mathcal{C}', \sigma_i')$ of stable curves. $\square$

Observe that proceeding with the first five steps above and then contracting only rational tails yields a family of semistable curves. Furthermore, the total space will still be regular thanks to Castelnuovo's Contraction Theorem [Alp27, Thm B.2.6]. This proves the following variant.

Theorem 5.1.2 (Semistable reduction)**.** *Let R be a DVR with fraction field K, and set $\Delta = \operatorname{Spec} R$ and $\Delta^* = \operatorname{Spec} K$. If $(\mathcal{C}^* \to \Delta^*, \sigma_i^*)$ is a family of n-pointed smooth curves, then there exists an extension of DVRs $R \to R'$ and an n-pointed family $(\mathcal{C} \to \Delta, \sigma_i)$ of semistable curves with regular total family $\mathcal{C}$ extending the base change of $(\mathcal{C}^* \to \Delta^*, \sigma_i^*)$ to $K' = \operatorname{Frac} R'$.*

Remark. When pursuing explicitly the stable reduction, the main challenge is determining the normalisation $\widetilde{\mathcal{C}'}$ is Step 4. A simpler way to perform this computation can be found in [Alp27, §6.5.2].

Stable Reduction (Theorem 5.1.1) clearly implies the existence part of the Valuative Criterion for Properness (Theorem 4.5.5) for $\overline{\mathcal{M}}_{g,n}$, and uniqueness follows from [Alp27, Prop. 6.5.20] as for $\mathcal{M}_g$.

Theorem 5.1.3. *If $2g - 2 + n > 0$, the Deligne-Mumford stack $\overline{\mathcal{M}}_{g,n}$ is proper over* $\operatorname{Spec} \mathbb{Z}$*. Moreover, there is a coarse moduli space $\overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$, where $\overline{M}_{g,n}$ is a proper algebraic space over* $\operatorname{Spec} \mathbb{Z}$*.*

Proof. Since $\overline{\mathcal{M}}_{g,n}$ is quasi-separated [Alp27, Exc. 6.4.18], we may apply the Valuative Criterion for Properness. Now the Keel-Mori Theorem (Theorem 4.6.9) provides a coarse moduli space, which is proper as $\overline{\mathcal{M}}_{g,n}$ is. □

5.2 Universal family

In this section we see how the stable model procedure described in Section 2.3 induces a morphism $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ that is identified with the universal family.

Proposition 5.2.1 (Forgetful Morphism)**.** *There is a morphism of algebraic stacks*

$$\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n} \quad \textit{such that} \quad (C, p_1, \ldots, p_{n+1}) \mapsto (C', c(p_1), \ldots, c(p_n)),$$

where $c\colon C \to C'$ is the contraction morphism as in Proposition 2.3.10.

Proof. The desired map is constructed taking an $(n+1)$-pointed family of stable curves $(\mathcal{C} \to S, \sigma_1, \ldots, \sigma_{n+1})$ to the n-pointed family $(\mathcal{C} \to S, \sigma_1, \ldots, \sigma_n) = F$ of prestable curves, and then $\operatorname{StMd}(F) \in \overline{\mathcal{M}}_{g,n}$. □

By following the Generalised 2-Yoneda Lemma (Lemma 4.3.7), the identity morphism $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ corresponds to an n-pointed family

$$(\mathcal{U}_{g,n} \to \overline{\mathcal{M}}_{g,n}, \sigma_i^{\mathrm{un}})$$

of stable curves, called the universal family. More precisely, an object of $\mathcal{U}_{g,n}$ over a scheme S is an n-pointed family of stable curves $(\mathcal{C} \to S, \sigma_i)$ with an additional section $\tau\colon S \to \mathcal{C}$ that may land in the nodal locus or intersect with the σ_i. Moreover, an n-pointed family of stable curves $(\mathcal{C} \to S, \sigma_i)$ corresponds to a morphism $S \to \overline{\mathcal{M}}_{g,n}$ fitting in a cartesian diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{U}_{g,n} \arrow[d] \\ S \arrow[r] \arrow[u, bend right, "\sigma_i"'] & \overline{\mathcal{M}}_{g,n}. \arrow[u, bend right, "\sigma_i^{\mathrm{un}}"'] \end{tikzcd}$$

On the other hand, we may define a morphism $\overline{\mathcal{M}}_{g,n+1} \to \mathcal{U}_{g,n}$ taking an $(n+1)$-pointed stable family $(\mathcal{C} \to S, \sigma_i)$ to the n-pointed family $(\mathcal{C}' \to S, \sigma_1', \dots, \sigma_n')$ arising from the Forgetful Morphism (Proposition 5.2.1) plus the section $c \circ \sigma_{n+1}$, where $c\colon \mathcal{C} \to \mathcal{C}'$ is the contraction of the prestable family $(\mathcal{C} \to S, \sigma_1, \dots, \sigma_n)$. This, together with the forgetful morphism $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$, yields a commutative diagram

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n+1} \arrow[rr] \arrow[dr] & & \mathcal{U}_{g,n} \arrow[dl] \\ & \overline{\mathcal{M}}_{g,n}. & \end{tikzcd}$$

Proposition 5.2.2. *The morphism $\overline{\mathcal{M}}_{g,n+1} \to \mathcal{U}_{g,n}$ is an isomorphism over $\overline{\mathcal{M}}_{g,n}$, i.e., the forgetful morphism $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the universal family.*

Proof. See [Alp27, Prop. 6.6.12]. □

5.3 Boundary

The clutching operation defined in Section 2.3 induces the morphism of stacks described below.

Fix a stable n-marked, genus g dual graph Γ, and let g_v and L_v be respectively the genus associated to a vertex $v \in V = V(\Gamma)$ and the set of half-edges issuing from v. Set

$$\overline{\mathcal{M}}_\Gamma = \prod_{v \in V} \overline{\mathcal{M}}_{g_v, L_v},$$

and define the *clutching morphism*

$$\xi_\Gamma\colon \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}$$

as follows. An object F of $\overline{\mathcal{M}}_\Gamma$ over a scheme S is the datum of a family

$$F_v = (\pi_v \colon \mathcal{C}_v \to S)$$

of stable L_v-pointed genus g_v curves, and the clutching procedure yields a family

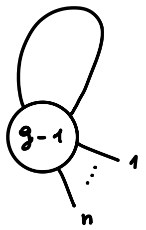
$$\xi_\Gamma(F) = f \colon \mathcal{C}' \to S$$

of stable n-pointed genus g curves. Since this construction is functorial, we obtain a morphism of DM stacks, which is also representable [ACG11, Prop. XII.10.25]. Moreover, it is not difficult to check that these morphisms are also finite.

We then define the closed substack $\mathcal{D}_\Gamma \subseteq \overline{\mathcal{M}}_{g,n}$ as the image of the clutching morphism ξ_Γ. Note that

$$\dim \mathcal{D}_\Gamma = \sum_{v \in V} (3g_v - 3 + l_v) = 3g - 3 + n - |E(\Gamma)|$$

where $l_v = |L_v|$ and $E(\Gamma)$ is the set of edges. In other words, its codimension is exactly $|E(\Gamma)|$.
We denote with ξ_0 the clutching morphism along the graph with one vertex and one edge as in Figure 5.7 on the left.
While for $1 \leq i \leq n$ and $I \subseteq \{1, \dots, n\} = [n]$, we denote with $\xi_{i,I}$ the clutching morphism along the graph with two vertices, one I-pointed of genus i and the other $([n] \setminus I)$-pointed of genus $g - i$, and one edge between them as in Figure 5.7 on the right. The geometric meaning behind this is to consider the loci of curves with a single node.



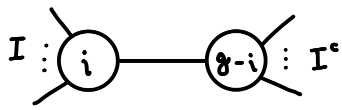


Figure 5.7: The two types of stable graphs with one edge.

Definition 5.3.1. Let $2g - 2 + n > 0$. The *boundary divisors* of $\overline{\mathcal{M}}_{g,n}$ are

$$\begin{aligned} \delta_0 &= \mathrm{im}(\xi_0), \\ \delta_{i,I} &= \mathrm{im}(\xi_{i,I}), \quad \text{and} \\ \delta &= \delta_0 \cup \bigcup_{i,I} \delta_{i,I}. \end{aligned}$$

with the convention that δ_0 is empty if $g = 0$.

Note that $\delta_{i,I} = \delta_{g-i,I^c}$. In particular, if $n = 0$, we simply denote $\delta_{i,\emptyset}$ by δ_i, and we get

$$\delta = \bigcup_{i=0}^{\lfloor g/2 \rfloor} \delta_i.$$

We will see that for any field k, the total boundary divisor δ is a normal crossings divisor over $\overline{\mathcal{M}}_{g,n} \otimes_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} k$. Recall that an effective Cartier divisor $Z \subseteq X$ of a regular scheme *has normal crossings* if for every $z \in Z$,

$$\widehat{\mathcal{O}}_{Z,z} \simeq \widehat{\mathcal{O}}_{X,z}/(f_1 \cdots f_k)$$

where $f_1, \ldots, f_k$ extend to a regular system $f_1, \ldots, f_n$.

We say that a closed substack $\mathcal{Z} \subseteq \mathcal{X}$ of a DM stack is an *effective Cartier divisor* (resp., a *normal crossings divisor*) if there is an étale presentation $U \to \mathcal{X}$ such that $\mathcal{Z} \times_{\mathcal{X}} U \subseteq U$ is an effective Cartier divisor (resp., a normal crossings divisor).

When it does not cause confusion, we will often denote $\overline{\mathcal{M}}_{g,n} \otimes_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} k$ simply by $\overline{\mathcal{M}}_{g,n}$.

Proposition 5.3.2. *Over a field k, the boundary $\delta \subseteq \overline{\mathcal{M}}_{g,n}$ is a normal crossings divisor.*

Proof. Clearly, the codimension of δ is 1. We omit the normal crossings part; see [Alp27, Prop. 6.6.19]. □

5.4 Irreducibility and projectivity

In this section, we introduce the Hurwitz scheme and use it to prove that $\overline{\mathcal{M}}_{g,n}$ is irreducible, then we briefly discuss the projectivity of the moduli space $\overline{M}_{g,n}$. Later in Section 5.6, we will see how Harris and Mumford in [HM82] exploited the Hurwitz scheme to compute the Kodaira dimension of $\overline{M}_g$.

Hurwitz scheme

Note that since $\overline{\mathcal{M}}_{g,n}$ is smooth over a field k, its irreducibility is equivalent to its connectedness. Moreover, recall that the universal family $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ has connected fibres, hence it suffices to prove the connectedness of $\overline{\mathcal{M}}_g$.

We wish to parametrise simply branched coverings $C \to \mathbb{P}^1$ of genus g and degree d defined in Section 2.1. Recall that by Riemann-Hurwitz (Theorem 2.1.7), if $C \to \mathbb{P}^1$ has b ramification points, then $b = 2g + 2d - 2$.

Definition 5.4.1. A *family* $(\mathcal{C} \to \mathcal{D} \to S, \sigma_i)$ *of coverings of* $\mathbb{P}^1$ *of genus* g *simply branched over* b *ordered points over a scheme* S is the datum of a family $\mathcal{C} \to S$ of smooth genus g curves and a b-pointed family $(\mathcal{D} \to S, \sigma_i)$ of smooth genus 0 curves together with a finite, flat morphism $\mathcal{C} \to \mathcal{D}$ over S of degree d such that for every geometric point s of S, $\mathcal{C}_s \to \mathcal{D}_s \simeq \mathbb{P}^1$ is simply branched over $\sigma_1(s), \ldots, \sigma_b(s)$.

We define the *Hurwitz functor* as $\mathrm{Hur}_{g,b}\colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Sets}$ sending

$$S \mapsto \left\{ \begin{matrix} \text{families } (\mathcal{C} \to \mathcal{D} \to S, \sigma_i) \text{ of coverings of } \mathbb{P}^1 \\ \text{of genus } g \text{ simply branched over } b \text{ points} \end{matrix} \right\} \Big/ \sim$$

where two families $(\mathcal{C} \to \mathcal{D} \to S, \sigma_i)$ and $(\mathcal{C}' \to \mathcal{D}' \to S, \sigma_i')$ over S are equivalent if there are isomorphisms $\mathcal{C} \xrightarrow{\sim} \mathcal{C}'$ and $\mathcal{D} \xrightarrow{\sim} \mathcal{D}'$ compatible with the sections and the structure morphisms to S.

One may check that $\mathrm{Hur}_{g,b}$ is a sheaf over $\mathrm{Sch}_{\text{ét}}$, hence a stack. Clearly, there are morphisms

$$\begin{tikzcd} & \mathrm{Hur}_{g,b} \arrow[dl] \arrow[dr] & \\ \mathcal{M}_g & & \mathcal{M}_{0,b} \end{tikzcd}$$

defined by sending a family $(\mathcal{C} \to \mathcal{D} \to S, \sigma_i) \in \mathrm{Hur}_{g,b}(S)$ to $(\mathcal{C} \to S) \in \mathcal{M}_g(S)$ and $(\mathcal{D} \to S, \sigma_i) \in \mathcal{M}_{0,b}(S)$.

Abusing of notation, when we are working over a field k, we simply write $\mathrm{Hur}_{g,b}$ meaning its base change to k. Also, we will interchange $\mathrm{Hur}_{g,b}$ and $\mathrm{Hur}_{g,d}$, keeping in mind the relation $b = 2g + 2d - 2$.

Remark. If $b \geq 3$, then $\mathcal{M}_{0,b} = M_{0,b}$ over a field k is actually a scheme. In fact, over an algebraically closed field k, a smooth genus 0 curve is isomorphic to $\mathbb{P}^1$, thus a point of $X = \mathcal{M}_{0,b} \times_k \overline{k}$ is given by a configuration of b distinct points on $\mathbb{P}^1$, up to automorphisms. By acting with the automorphism group PGL_2, we can fix the first three points to be 0, 1, and ∞, then we get

$$X \simeq (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{b-3} \setminus \Delta,$$

where Δ is the union of the pairwise diagonals. Therefore, X is even quasi-projective, and this implies that $\mathcal{M}_{0,b}$ is a scheme [Alp27, Prop. 5.5.27].

For $d \geq 2$, $\mathrm{Hur}_{0,b}$ is an algebraic space, and the map $\mathrm{Hur}_{g,b} \to M_{0,b}$ is an *algebraic covering space*, i.e., it is a finite, étale, and surjective morphism [Alp27, Exc. 6.7.13].

Over an algebraically closed field of characteristic 0, the other map $\mathrm{Hur}_{g,d} \to \mathcal{M}_g$ is surjective for sufficiently large d. This is an immediate corollary of the following.

Proposition 5.4.2. *Let C be a smooth, connected, and projective curve of genus g over an algebraically closed field k of characteristic 0, and let L be a line bundle on C of degree $d \geq 2g+3$. Then a general linear system $V \subseteq H^0(C, L)$ of dimension 2 induces a simply branched covering $C \to \mathbb{P}^1$.*

Proof. Since $\deg(\omega_C \otimes L^\vee) < 0$, we get $h^1(C, L) = h^0(C, \omega_C \otimes L^\vee) = 0$, thus by Riemann-Roch $h^0(C, L) = d + 1 - g$. By characteristic reasons, every finite morphism $C \to \mathbb{P}^1$ is automatically separable. If V does not induce a simply branched covering $C \to \mathbb{P}^1$, then it must happen one of the following:

(a) V has a base point,

(b) there is a ramification point with index greater than 2, or

(c) there exists two ramification points in the same fibre.

The Grassmannian $\mathrm{Gr}(2, H^0(C, L))$ of 2-dimensional subspaces of $H^0(C, L)$ has dimension $2(d-1-g)$. Let us prove that each of the conditions above is closed of codimension at least one.
For condition (a), if $p \in C(k)$ is a base point, then $V \subseteq H^0(C, L(-p))$, but $d \geq 2g$, thus $\deg(\omega_C \otimes (L(-p))^\vee) < 0$, implying as above that $h^0(C, L(-p)) = d - g$. We then get

$$\dim \mathrm{Gr}(2, H^0(C, L(-p))) = 2(d-g-2),$$

and varying p in C, we see that the locus defining (a) has dimension $2(d-g-2)+1$, which is exactly $\dim \mathrm{Gr}(2, H^0(C, L)) - 1$.
For condition (b), if p is a point with ramification index greater than 2, then $V \cap H^0(C, L(-3p)) \neq \emptyset$. Since $\deg(\omega_C \otimes (L(-3p))^\vee) < 0$, we get

$$\dim H^0(C, L(-3p)) = d - g - 2,$$

and the locus of 2-dimensional subspaces $V \subseteq H^0(C, L)$ which have nontrivial intersection with $H^0(C, L(-3p))$ has dimension

$$(\dim H^0(C, L(-3p)) - 1) + (\dim H^0(C, L) - 2) = 2(d-g-1) - 2.$$

Again varying p, we see that the locus of linear systems V having a point with ramification greater than two has codimension 1.
For condition (c), if p and q in C are ramification points in the same fibre, then $V \cap H^0(-2p-2q) \neq 0$. Arguing as above, we find that the dimension of the locus of 2-dimensional subspaces V intersecting properly $H^0(-2p-2q)$ is $2(d-g-1)-3$. Varying the two points p and q, we obtain again a codimension 1 locus. □

We may now deduce the irreducibility of $\mathcal{M}_g$ from the following classical result.

Theorem 5.4.3 (Clebsch-Hurwitz-Lüroth). *For $g \geq 2$ and $d \geq 2$, the Hurwitz moduli space $\mathrm{Hur}_{g,d}$ over $\mathbb{C}$ is connected.*

Proof. See [Alp27, Thm 6.7.19]. □

Corollary 5.4.4. *For any field of characteristic 0, $\mathcal{M}_g$ is irreducible.*

Proof. Since $\mathrm{Hur}_{g,d} \to \mathcal{M}_g$ is surjective for $d \geq 2g+3$, it follows immediately from the theorem above that over $\mathbb{C}$ the moduli stack $\mathcal{M}_g$ is connected. This implies that it is geometrically connected over $\mathbb{Q}$, hence over every characteristic 0 field k. Since $\mathcal{M}_g$ is smooth by Proposition 4.5.3, this implies its irreducibility. □

To prove that $\overline{\mathcal{M}}_{g,n}$ is irreducible, it suffices now to show that $\mathcal{M}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}$ is dense. This is true since every stable curve C over an algebraically closed field deforms to a smooth curve [Alp27, Exc. 6.7.25].
Actually, the converse is also true.

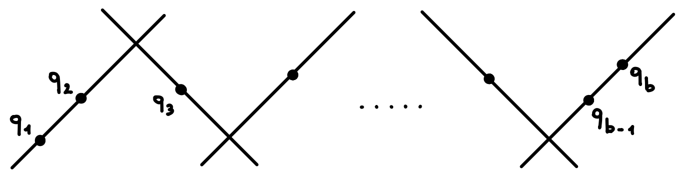
Proposition 5.4.5. *Let C be a smooth, connected, and projective curve of genus $g \geq 2$ over an algebraically closed field k of characteristic 0. There exists a family $\mathcal{C} \to S$ of stable curves over a smooth connected curve S over k with points $s, t \in S(k)$ such that $\mathcal{C}_s \simeq C$ and $\mathcal{C}_t$ is nodal.*

Sketch of proof. The proof relies on the following strategy.

(1) For $d \gg 0$, choose a simply branched covering $C \to \mathbb{P}^1$ of degree d branched over distinct points $q_1, \ldots, q_b \in \mathbb{P}^1$ where $b = 2g + 2d - 2$.

(2) Deform $C \to \mathbb{P}^1$ to $C' \to \mathbb{P}^1$ branched over general points q_i'.

(3) Degenerate $(\mathbb{P}^1, q_i')$ to the stable curve (D, q_i) as below.



(4) Degenerate $C' \to \mathbb{P}^1$ to $C'' \to D$ branched over q_i such that C'' is singular and stable.

See [Alp27, Prop. 6.7.26] for the complete proof. □

Note that the result above shows that every smooth curve has a nodal "limit", confirming that it suffices to include only nodal singularities.

In the last section, we will formalise the concept of degenerating a branched covering with the notion of an admissible covering.

Projectivity

Here we briefly discuss the approach introduced by Kollár in [Kol90]. Recall the universal family $\pi\colon \mathcal{U}_g \to \overline{\mathcal{M}}_g$ over $\operatorname{Spec}\mathbb{Z}$ and the Hodge line bundle $\lambda = \pi_*\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}$. Generalising this, define the *kth pluricanonical bundle* on $\overline{\mathcal{M}}_g$ as

$$\Lambda_k \coloneqq \pi_*\left(\omega^{\otimes k}_{\mathcal{U}_g/\overline{\mathcal{M}}_g}\right)$$

which by Proposition 2.3.8 is a vector bundle of rank g if $k = 1$, of rank $(2k-1)(g-1)$ if $k > 1$. Taking determinants, we obtain line bundles

$$\lambda_k \coloneqq \det \Lambda_k$$

on $\overline{\mathcal{M}}_g$.

The strategy is to show that for $k \gg 0$, a positive power of λ_k descends to an ample line bundle on $\overline{M}_g$, implying its projectivity. We omit further details.

Theorem 5.4.6. *For $g \geq 2$ and N sufficiently divisible, the line bundle $\lambda^{\otimes N}_{18g-18}$ on $\overline{\mathcal{M}}_g$ descends to a line bundle on $\overline{M}_g$ relatively ample over* $\operatorname{Spec}\mathbb{Z}$.

Proof. See [Alp27, Thm 6.9.3]. □

By using the fact that $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the universal family, one can also deduce the following general result.

Corollary 5.4.7. *If $2g - 2 + n > 0$, the moduli space $\overline{M}_{g,n}$ is projective over* $\operatorname{Spec}\mathbb{Z}$.

Proof. Clearly, $\overline{M}_{0,3} = \operatorname{Spec}\mathbb{Z}$, and it is not difficult to see that $\overline{M}_{1,1} \simeq \mathbb{P}^1$ since $M_{1,1} \simeq \mathbb{A}^1$ and the boundary of $\overline{\mathcal{M}}_{1,1}$ is a single point [Alp27, Ex. 5.4.15]. By the theorem above, $\overline{M}_g$ is also projective for $g \geq 2$. Now the universal family is a family of stable curves, hence $L = \omega_{\overline{\mathcal{M}}_{g,n+1}/\overline{\mathcal{M}}_{g,n}}$ is relatively ample over $\overline{\mathcal{M}}_{g,n}$. Moreover, some power of L descends to $\overline{M}_{g,n+1}$ and it is relatively ample on $\overline{M}_{g,n}$; see [Alp27, Exc. 5.4.29]. The morphism $\overline{M}_{g,n+1}/\overline{M}_{g,n}$ is then projective, and we conclude the proof by induction. □

5.5 Picard group

Here we discuss the structure of the Picard group of $\overline{\mathcal{M}}_{g,n}$ and its coarse moduli space $\overline{M}_{g,n}$.

Recall from Section 4.6 that a quasi-coherent sheaf on a Deligne-Mumford stack $\mathcal{X}$ is equivalent to the data of quasi-coherent sheaves over every $\mathcal{X}$-scheme with compatible isomorphisms between them. In particular, this means that to give a quasi-coherent sheaf F on $\overline{\mathcal{M}}_{g,n}$ is the same as giving, for every family of stable n-pointed curves $\xi = (\pi\colon \mathcal{C} \to S)$, compatible quasi-coherent sheaves F_ξ on S.

Recall also the example of the Hodge bundle $\mathbb{E}$ defined on $\mathcal{M}_g$ for $g \geq 2$ as

$$\mathbb{E}_{(\pi\colon \mathcal{C}\to S)} = \pi_*(\omega_\pi).$$

Generalising this, we define in the same way the *Hodge bundle* $\mathbb{E}$ on $\overline{\mathcal{M}}_{g,n}$, and the *Hodge line bundle*

$$\lambda := \det \mathbb{E} = \Lambda^g \mathbb{E}.$$

We abuse notation denoting with the same symbol the line bundle and its class.

When $n > 0$, other classical examples of line bundles over $\overline{\mathcal{M}}_{g,n}$ are provided by the *cotangent line classes.* Fixing $i \in \{1, \dots, n\}$, we define the *ith cotangent line class* $\psi_i \in \operatorname{Pic}(\overline{\mathcal{M}}_{g,n})$ as

$$(\psi_i)_\xi := \sigma_i^* \Omega_{\mathcal{C}/S}$$

over a family $\xi = (\mathcal{C} \to S, \sigma_i)$. We also set $\psi = \sum_i \psi_i$.

As for schemes, to a Cartier divisor $\mathcal{D}$ on a DM stack $\mathcal{X}$ one can attach a line bundle $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ by performing the usual construction on every $\mathcal{X}$-scheme S.
In particular, we can associate to the boundary divisors δ_i the corresponding line bundles, which we also denote δ_i.

If we consider the coarse moduli space $\pi\colon \overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$, we naturally get a pullback functor $\pi^*\colon \operatorname{QCoh}(\overline{M}_{g,n}) \to \operatorname{QCoh}(\overline{\mathcal{M}}_{g,n})$. We now see that in many cases this functor is not an equivalence, by showing that the Hodge bundle is not the pullback of a quasi-coherent sheaf on $\overline{M}_{g,n}$.
Clearly, it suffices to show this for the Hodge line bundle λ. Over an n-pointed stable curve $\xi = (C, p_i)$ of genus g, the line bundle λ_ξ is simply $\det H^0(C, \omega_C)$. If λ did come from $M_{g,n}$, any automorphism of C would act trivially on $\det H^0(C, \omega_C)$. But we can always find a curve where this is not satisfied:

- for odd g, an hyperelliptic curve C has its involution acting as -1 on $H^0(C, \omega_C)$, and fixing $2g + 2$ points;

- for even g, a ramified double covering C of an elliptic curve has the corresponding involution acting on $H^0(C, \omega_C)$ with eigenvalues 1 with multiplicity 1 and -1 with multiplicity $g - 1$, and fixing $2g - 2$ points.

Note that these examples work as soon as n is not greater than the number of points fixed by the involution. To generalise them, we need only a slight change: attach on such a fixed point a $\mathbb{P}^1$ and consider the automorphism as the involution on a component and the identity on the $\mathbb{P}^1$. Basically, we place the extra fixed points on another component, where we act trivially.

The discussion above implies that the pullback homomorphism

$$\mathrm{Pic}(\overline{M}_{g,n}) \to \mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$$

is not surjective. However, this map is injective with torsion cokernel [ACG11, Lemma 6.6].
Furthermore, the following sharper result holds.

Proposition 5.5.1. *The group* $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$ *is free abelian of finite rank, and* $\mathrm{Pic}(\overline{M}_{g,n})$ *is a subgroup of finite index.*

Proof. See [ACG11, Prop. XIII.6.7]. □

Mumford's formula

In order to compute the Kodaira dimension of $\overline{M}_g$ over $\mathbb{C}$, we will relate its canonical bundle with that of $\overline{\mathcal{M}}_g$, defined in the following. From now on, we simply denote by $\overline{\mathcal{M}}_{g,n}$ the moduli stack $\overline{\mathcal{M}}_{g,n} \times \mathbb{C}$ of stable curves over $\mathbb{C}$.
Recall that the sheaf of differentials $\Omega_{\overline{\mathcal{M}}_{g,n}} = \Omega_{\overline{\mathcal{M}}_{g,n}/\mathbb{C}}$ of $\overline{\mathcal{M}}_{g,n}$ is defined as

$$(\Omega_{\overline{\mathcal{M}}_{g,n}})_U = \Omega_{U/\mathbb{C}}$$

for any scheme U étale over $\overline{\mathcal{M}}_{g,n}$. Miniversal families (see Corollary 3.2.4) allow us to provide a more explicit description.

Given a stable n-pointed curve (C, p_i) of genus g, let $\mathcal{C} \to (U, u_0)$ be its miniversal family. We know that the Kodaira-Spencer map provides an identification

$$T_{U,u_0} \simeq \mathrm{Ext}^1_{\mathcal{O}_C}\left(\Omega^1_C, \mathcal{O}_C\left(-\sum p_i\right)\right),$$

thus since U is smooth, by Serre duality we get

$$(\Omega_{U,u_0}) \otimes_{\mathcal{O}_{U,u_0}} \kappa(u_0) = T^\vee_{U,u_0} \simeq H^0\left(C, \Omega_C \otimes \omega_C\left(\sum p_i\right)\right).$$

Therefore, for any family $\xi = (\pi\colon \mathcal{C} \to S, \sigma_i)$ of stable n-pointed genus g curves

$$(\Omega_{\overline{\mathcal{M}}_{g,n}})_\xi = \pi_*(\Omega_\pi \otimes \omega_\pi(D)),$$

where $D = \sum_i \sigma_i(S)$ is the divisor of sections. We recall that this is a vector bundle of rank g.
The *canonical bundle* is $K_{\overline{\mathcal{M}}_{g,n}} := \det \Omega_{\overline{\mathcal{M}}_{g,n}}$ as usual.

Other line bundles on $\overline{\mathcal{M}}_{g,n}$ can be defined using the machinery of the *Deligne pairing*. However, we are not focusing on these details, and we refer to [ACG11, §XIII.5] for a comprehensive treatment.

Soon we will see that the class of $K_{\overline{\mathcal{M}}_{g,n}}$ can be written as an integral combination of the classes of λ, δ and ψ. This is due to Mumford, who produced various similar relations exploiting the Grothendieck's Riemann-Roch (GRR) formula. In order to avoid a lengthy exposition, we briefly explain the standard machinery of Intersection Theory necessary to state the GRR formula, and we refer to [Ful84] for a comprehensive treatment.

Chow ring

In the following, schemes will always be separated and of finite type over a field k. A variety is then an integral scheme, and it is complete if it is proper over k.

Recall that if X is a variety, and if $Z \subseteq X$ is a subvariety, i.e., a subscheme that is a variety itself, of codimension 1, then there is a well-defined homomorphism

$$\operatorname{ord}_Z(-)\colon k(X)^* \to \mathbb{Z}$$

where $k(X)$ is the fraction field of X. Writing $f \in k(X)$ as $f = s/t$ with $s, t \in \mathcal{O}_{Z,Y}$, this map is defined by

$$\operatorname{ord}_Z(f) = \operatorname{length}_{\mathcal{O}_{Z,Y}}(\mathcal{O}_{Z,Y}/(s)) - \operatorname{length}_{\mathcal{O}_{Z,Y}}(\mathcal{O}_{Z,Y}/(t)),$$

and it should be thought of as the order of vanishing of the function f along the subvariety Z.

If X is a scheme, we define the group $Z_r(X)$ of *r-cycles* as the free abelian group generated by isomorphism classes of r-dimensional subvarieties of X. Moreover, for any $(r+1)$-dimensional subvariety $W \subseteq X$, and for any rational function $f \in k(W)^*$, let

$$[\operatorname{div}(f)] \coloneqq \sum_{V \subseteq W} \operatorname{ord}_V(f)[V]$$

where the sum is taken over all the codimension 1 subvarieties V of W. This is a finite sum, hence an element of $Z_r(X)$, and we denote by $\operatorname{Rat}_r(X)$ the subgroup of $Z_r(X)$ generated by those elements as $W \subseteq X$ and $f \in k(X)^*$ vary. Finally, we define

$$A_r(X) \coloneqq Z_r(X)/\operatorname{Rat}_r(X)$$

and the *Chow group* $A_\bullet(X) = \oplus_{r=0}^{\dim X} A_r(X)$. Note that if X is a variety, then $A_{n-1}(X)$ is the usual Weil divisor class group.

This construction satisfies the following functorial properties; see [Ful84, Thms 1.4 and 1.7][SP, Tag 02S0].

- If $f\colon X \to Y$ is a proper morphism, we define the *proper pushforward* $f_*\colon Z_r(X) \to Z_r(Y)$ by $f_*([V]) = \deg(V/f(V))[f(V)]$ for $V \subseteq X$, where

$\deg(V/f(V)) = [k(V) : k(f(V))]$ is the degree of the dominant morphism $V \to f(V)$. This induces an homomorphism

$$f_*\colon A_r(X) \to A_r(Y).$$

- Similarly, if $f\colon X \to Y$ is a flat morphism of relative dimension n, then defining the *flat pullback* $f^*\colon Z_r(Y) \to Z_{r+n}(X)$ by $f^*([V]) = [f^{-1}(V)]$ for $V \subseteq Y$, this map induces an homomorphism

$$f^*\colon A_r(Y) \to A_{r+n}(X).$$

Remark. If X is a complete variety over a field k, then the graded group $A_\bullet(\operatorname{Spec} k)$ is $\mathbb{Z}$ generated by the class of the unique point. The structure map f then induces a proper pushforward

$$f_*\colon A_\bullet(X) \to \mathbb{Z}.$$

In particular, for $\alpha \in A_\bullet(X)$, we denote the integer $f_*(\alpha)$ by $\int_X \alpha$ or $\deg(\alpha)$; note that $\int_X \alpha = 0$ if α is positive-dimensional.
Clearly, if X is a proper curve, and D a Weil divisor on X, then $\deg([D])$ is the usual degree.

The Chow group $A_\bullet(X)$ plays the same role as the homology groups in algebraic topology. Similarly, there is a Chow ring $A^\bullet(X)$ playing the role of the cohomology ring. However, if X is a smooth variety, there is a Poincaré duality theorem [Ful84, Cor. 17.4] providing a canonical isomorphism $A_p(X) = A^{n-p}(X)$ for all integers p, where $n = \dim X$. Since this is the only case we are interested in, we avoid technicalities defining $A^p(X) := A^{n-p}(X)$. Soon we will define a product on $A^\bullet(X) := \oplus_{p=0}^n A^p(X)$, endowing it with a structure of a graded associative commutative ring. The idea is to define the product of two "transverse" subvarieties as their intersection, considered as a subvariety.

Definition 5.5.2. Let X be a scheme, and let $V, W \subseteq X$ be two subvarieties of codimensions r an s respectively. By the Hauptidealsatz, the intersection $V \cap W$ has codimension at most $r+s$. We say that V and W are *dimentionally transverse* if every irreducible component of $V \cap W$ has maximal codimension $r+s$. We say that V and W are *generically transverse* if they satisfy the stronger condition that

$$\operatorname{codim}(T_\eta V \cap T_\eta W) = r + s$$

for every generic point η of each irreducible component of $V \cap W$. Ultimately, we say that two cycles $\sum_i V_i$ and $\sum_j W_j$ in $A_\bullet(X)$ are dimensionally (resp., generically) transverse if the property holds pairwise for the V_i and W_j.

Example. In $\mathbb{A}^2$, $\{y = 1\}$ and $\{y = x^2\}$ are generically transverse, while $\{y = 0\}$ and $\{y = x^2\}$ are only dimensionally transverse.

Let X be a smooth variety. There is a unique product structure on $A^\bullet(X)$ such that

$$[V] \cdot [W] = [V \cap W]$$

whenever V and W are generically transverse; see [Ful84, §8]. Moreover, this product is associative, commutative, and $A^\bullet(X)$ is graded by codimension with unit $[X]$, called the *fundamental class* of X.
Although Fulton constructed directly the product, if in addition X is quasi-projective, this result is historically a consequence of the moving lemma [Ful84, §11.4], stating that given any two cycles $\alpha, \beta \in A^\bullet(X)$, we can find rationally equivalent cycles α', β' which intersects generically transversally.

Remark. The intersection product makes the flat pullback a ring homomorphism between Chow rings, and if $f \colon X \to Y$ is flat and proper between smooth varieties, then we have the *push-pull formula*

$$f_*(f^*\beta \cdot \alpha) = \beta \cdot f_*\alpha \tag{5.1}$$

for all classes $\alpha \in A^\bullet(X)$ and $\beta \in A^\bullet(Y)$; see [Ful84, Prop. 8.3].

Chern classes

If X is a scheme, and E is a vector bundle on X, *Chern classes* of E are a collection of classes $c_i(E) \in A^i(X)$ reflecting the geometry of E.

Theorem 5.5.3. *Let X be a scheme. For any vector bundle E on X, there exist Chern classes $c_i(E) \in A^\bullet(X)$ which satisfy the following properties.*

- *(Vanishing) If $i > \operatorname{rk} E$, then $c_i(E) = 0$.*
- *(Commutativity) If F is a vector bundle on X and $\alpha \in A^\bullet(X)$, then for all i, j,*
$$c_i(E) \cdot (c_j(F) \cdot \alpha) = c_j(F) \cdot (c_i(E) \cdot \alpha).$$
- *(Projection formula) Let $f \colon Y \to X$ be a proper morphism. For all i and $\alpha \in A^\bullet(Y)$,*
$$f_*(c_i(f^*E) \cdot \alpha) = c_i(E) \cdot f_*(\alpha).$$
- *(Pull-back) Let $f \colon Y \to X$ be a flat morphism. For all i and $\alpha \in A^\bullet(X)$,*
$$c_i(f^*E) \cdot f^*\alpha = f^*(c_i(E) \cdot \alpha).$$

- *(Whitney sum) For any short exact sequence*
$$0 \to E \to F \to G \to 0,$$
of vector bundles on X, we have
$$c_k(F) = \sum_{i+j=k} c_i(E) \cdot c_j(G).$$

- *(Normalization) If D is a Weil divisor on X, then*
$$c_1(\mathcal{O}_X(D)) = [D].$$

Moreover, Chern classes are uniquely determined by the properties above.

Proof. See [Ful84, Thm 3.2]. □

We store all this information in the *Chern polynomial* $c(E) = 1 + c_1(E) + \dots$, then the Whitney sum formula becomes $c(F) = c(E) \cdot c(G)$.

Observe that if a vector bundle E of rank r admits a complete filtration
$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_r = E,$$
then setting $L_i = E_i/E_{i-1}$ for $1 \le i \le r$, Whitney sum implies that
$$c(E) = \prod_{i=1}^{r} c(L_i) = \prod_{i=1}^{r} (1 + \alpha_i),$$
where $\alpha_i = c_1(L_i)$ are called the *Chern roots* of E. The following result allows us to work with Chern roots even when a vector bundle does not admit such a filtration.

Proposition 5.5.4 (Splitting principle)**.** *Given a finite collection $\mathscr{E}$ of vector bundles on a scheme X, there is a flat morphism $f \colon X' \to X$ such that*

- $(f')^* \colon A^\bullet(X) \to A^\bullet(X')$ *is injective, and*
- *for any $E \in \mathscr{E}$, f^*E admits a filtration*
$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_r = f^*E.$$

Proof. See "*Splitting construction*" in [Ful84, Thm 3.2]. □

Let X be a scheme E be a vector bundle of rank r with Chern roots $\alpha_1, \ldots, \alpha_r$. Notice that ince $c(E) = \prod_{i=1}^r (1 + \alpha_i)$, Chern classes are exactly elementary symmetric polynomials in the Chern roots.
We now introduce two polynomials in the Chern classes which are crucial for the statement of the GRR formula. The *Chern character* is

$$\mathrm{ch}(E) \coloneqq \sum_{i=1}^r \exp(\alpha_i).$$

Since this expression is symmetric in the α_i, for what observed above, this defines a polynomial in the Chern classes.
The *Todd class* is

$$\mathrm{td}(E) \coloneqq \prod_{i=1}^r T(\alpha_i)$$

where

$$T(x) = \frac{x}{1 - e^{-x}}.$$

For the same reason, this is also a polynomial in the Chern classes. Moreover, observe that if we let

$$\mathrm{td}^\vee(E) \coloneqq \prod_{i=1}^r T(-\alpha_i),$$

it follows that $\mathrm{td}^\vee(E) = \mathrm{td}(E^\vee)$ since $c_i(E^\vee) = (-1)^i c_i(E)$.
Finally, if X is a smooth variety, we simply denote $\mathrm{td}(T_X)$ by $\mathrm{td}(X)$.

The Chern classes have also the following geometric interpretation.

Proposition 5.5.5. *Let $\tau_0, \ldots, \tau_{r-i}$ be global sections of E, where $r = \mathrm{rk}(E)$. If the vanishing set D of the global section $\tau_0 \wedge \cdots \wedge \tau_{r-i}$ of $\Lambda^{r-i+1}E$ has codimension i, then $c_i(E) = [D] \in A^i(X)$.*

Proof. See [EH16, Thm 5.3]. □

K-Theory

Let X be a scheme. Define the *Grothendieck group of vector bundles* $K^\bullet(X)$ over X as the free group on the isomorphism classes of vector bundles over X, modulo the relation $[F] = [E] + [G]$ for every short exact sequence

$$0 \to E \to F \to G \to 0$$

of vector bundles. Define a product on $K^\bullet(X)$ by $[E] \cdot [F] \coloneqq [E \otimes F]$. Note that this is well-defined since tensoring by a vector bundle preserves exactness, and this product endows $K^\bullet(X)$ with the structure of an associative, commutative

ring with unit $\mathcal{O}_X$.
If X is connected, the rank map induces a ring homomorphism $K^\bullet(X) \to \mathbb{Z}$. Moreover, any morphism of schemes $f\colon X \to Y$ induces a ring homomorphism

$$f^*\colon K^\bullet(Y) \to K^\bullet(X).$$

Similarly, we define the *Grothendieck group of coherent sheaves* $K_\bullet(X)$ over X as the free group on the isomorphism classes of coherent sheaves over X, modulo the relation $[\mathcal{F}] = [\mathcal{E}] + [\mathcal{G}]$ for every short exact sequence

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

of coherent sheaves. Since tensoring with a coherent sheaf is not exact in general, $K_\bullet(X)$ has only a natural structure of $K^\bullet(X)$-module given by $[E]\cdot[\mathcal{F}] \coloneqq [E\otimes\mathcal{F}]$ for a vector bundle E and a coherent sheaf $\mathcal{F}$ on X.
Finally, adopting original Grothendieck's notation, for any proper morphism of schemes $f\colon X \to Y$ we define a homomorphism $f_!\colon K_\bullet(X) \to K_\bullet(Y)$ by

$$f_!([\mathcal{F}]) \coloneqq \sum_{i\geq 0}(-1)^i[R^i f_*\mathcal{F}].$$

Here the properness assumption is necessary to ensure that $R^i f_*\mathcal{F}$ is coherent [SP, Tag 02O5]. By the long exact sequence in cohomology [Har77, Thm III.1.1A], the pushforward $f_!$ is well-defined.
For any scheme, there is a natural forgetful homomorphism $K^\bullet(X) \to K_\bullet(X)$ which takes a vector bundle to itself, and if X is smooth this is an isomorphism; see [Ful84, §15.1].

Therefore, if X is a smooth scheme, we identify $K^\bullet(X)$ and $K_\bullet(X)$, simply denoting it by $K(X)$.
Note that, setting $A(X)_\mathbb{Q} = A^\bullet(X)\otimes\mathbb{Q}$, the Chern character defines a map

$$\begin{aligned}\mathrm{ch}\colon K(X) &\to A(X)_\mathbb{Q}.\\ [E] &\mapsto [\mathrm{ch}(E)]\end{aligned}$$

This is well-defined since Chern character is additive on short exact sequences, and it is a ring homomorphism since $\mathrm{ch}(E\otimes F) = \mathrm{ch}(E)\cdot\mathrm{ch}(F)$. Both statements are easily proven looking at the relations between Chern roots in a short exact sequence and in a tensor product.
Moreover, by Theorem 5.5.3 Chern classes commutes with flat pullback, so does ch. However, it does not commute with proper pushforward.

Theorem 5.5.6 (Grothendieck's Riemann-Roch). *Let $f\colon X \to Y$ be a proper morphism of smooth schemes over a field k. Then, for all $\alpha \in K(X)$,*

$$\mathrm{ch}(f_!\alpha)\cdot\mathrm{td}(Y) = f_*(\mathrm{ch}(\alpha)\cdot\mathrm{td}(X)) \tag{5.2}$$

in $A(Y)_\mathbb{Q}$.

Proof. See [Ful84, Thm 15.2]. □

The content of theorem above is the relation between the homomorphism ch and the proper pushforward $f_!$.

Remark (HRR). If X is a smooth complete variety over a field k, then letting $f\colon X \to Y = \operatorname{Spec} k$ be the structure morphism, for any vector bundle E on X,

$$\mathrm{ch}(f_! E) = \mathrm{ch}\left(\sum_i (-1)^i H^i(X, E)\right) = \sum_i (-1)^i h^i(X, E) = \chi(X, E).$$

On the other hand,

$$f_*(\mathrm{ch}(E) \cdot \mathrm{td}(X)) = \int_X \mathrm{ch}(E) \cdot \mathrm{td}(X),$$

hence by the GRR formula we obtain the Hirzebruch-Riemann-Roch theorem

$$\chi(X, E) = \int_X \mathrm{ch}(E) \cdot \mathrm{td}(X).$$

Furthermore, restricting to proper curves, one may retrieve the usual Riemann-Roch theorem [Ful84, Ex. 15.2.1].

The following consequence of HRR will be useful in the computations at the end of the next section.

Corollary 5.5.7 (Noether's formula)**.** *Let X be a complete smooth surface, i.e., a variety of dimension 2. Then we get*

$$12\chi(X, \mathcal{O}_X) = \deg(K_X^2) + e(X)$$

where $K_X = -c_1(T_X)$ is the canonical divisor and $e(X)$ is the topological Euler characteristic in the analytic topology[2]*.*

Proof. By Hirzebruch-Riemann-Roch, we get

$$\chi(X, \mathcal{O}_X) = \int_X \mathrm{td}_2(X) = \frac{1}{12}\int_X (c_1(T_X)^2 + c_2(T_X)).$$

Recall now the Poincaré-Hopf theorem stating that the sum of the indices of any vector field with isolated zeroes on a compact manifold M is equal to the topological Euler characteristic of M. Therefore, by this and *Proposition* 5.5.5, we get that

$$e(X) = \int_X c_2(T_X).$$

This concludes the proof. □

[2]Since every complete smooth surface is projective (see [Har77, Rmk II.4.10.2]), the GAGA theorems of Serre allow us to pass from the algebraic category to the analytic one.

We may apply the GRR formula to a family of stable curves $(f\colon \mathcal{C} \to S, \tau_i)$, where $\mathcal{C}$ and S are smooth schemes over a field k. Recall that there is a canonical exact sequence

$$f^*(\Omega_S) \to \Omega_{\mathcal{C}} \to \Omega_f \to 0,$$

and Ω_S and $\Omega_{\mathcal{C}}$ are vector bundles, hence left map is injective if and only if it is nonzero. Moreover, the locus where f is smooth is open, thus the left map above is nonzero at the generic point, implying that we have a short exact sequence

$$0 \to f^*(\Omega_S) \to \Omega_{\mathcal{C}} \to \Omega_f \to 0.$$

Since the Todd class is multiplicative, we get

$$\mathrm{td}(\mathcal{C}) = \mathrm{td}^\vee(\Omega_{\mathcal{C}}) = \mathrm{td}^\vee(\Omega_f) \cdot f^*(\mathrm{td}^\vee(\Omega_S)).$$

The push-pull formula (5.1) then allows us to rewrite the GRR formula (5.2) as

$$\mathrm{ch}(f_!\mathcal{F}) = f_*(\mathrm{ch}(\mathcal{F}) \cdot \mathrm{td}^\vee(\Omega_f)) \tag{5.3}$$

for any coherent sheaf $\mathcal{F}$ over $\mathcal{C}$. Furthermore, by definition

$$\begin{aligned} \mathrm{ch}(\mathcal{F}) &= \mathrm{rk}(\mathcal{F}) + c_1(\mathcal{F}) + \frac{c_1(\mathcal{F})^2 - 2c_2(\mathcal{F})}{2} + \dots, \\ \mathrm{td}^\vee(\mathcal{F}) &= 1 + \frac{c_1(\mathcal{F})}{2} + \frac{c_1(\mathcal{F})^2 + c_2(\mathcal{F})}{12} + \dots, \end{aligned}$$

thus comparing the degree 1 terms in (5.2) yields

$$c_1(f_!\mathcal{F}) = f_*\left(\frac{c_1(\mathcal{F})^2}{2} - c_2(\mathcal{F}) - \frac{c_1(\mathcal{F})\cdot c_1(\Omega_f)}{2} + \frac{c_1(\Omega_f)^2 + c_2(\Omega_f)}{12}\right). \tag{5.4}$$

To further refine the equation above, we need to compute $c_1(\Omega_f)$ and $c_2(\Omega_f)$. Since $\mathcal{C}$ is smooth, we have a short exact sequence

$$0 \to \Omega_f \to \omega_f \to \omega_f \otimes \mathcal{O}_\Sigma \to 0 \tag{5.5}$$

where Σ is the nodal locus; see [ACG11, §X.2]. Soon we will compute the Chern classes of $\omega_f \otimes \mathcal{O}_\Sigma$ by applying the following particular case of GRR.

Corollary 5.5.8. *Let $j\colon Z \to Y$ be a codimension r closed immersion of smooth schemes, and let $\mathcal{G}$ be a coherent sheaf on Z. Then*

$$\begin{aligned} c_i(j_*\mathcal{G}) &= 0 \quad \textit{for} \quad i < r, \\ c_r(j_*\mathcal{G}) &= (-1)^{r-1}(r-1)!\,\mathrm{rk}(\mathcal{G})[Z], \end{aligned}$$

where $[Z] = j_*(\mathbf{1}_Z)$ denotes the fundamental class of Z in $A^\bullet(Y)$.

Proof. Since a closed immersion is proper, we may apply GRR (Theorem 5.5.6) to $j\colon Z \to Y$. Recall also that a closed immersion j is affine, hence $R^i j_* = 0$ for $i > 0$ [SP, Tag 01XC]. This implies that

$$j_! \mathcal{G} = \sum_{i \geq 0} (-1)^i R^i j_* \mathcal{G} = j_* \mathcal{G},$$

thus the GRR formula becomes

$$\mathrm{ch}(j_* \mathcal{G}) = j_*(\mathrm{ch}(\mathcal{G}) \cdot \mathrm{td}(Z)) \cdot \mathrm{td}(Y)^{-1}.$$

Now for codimension reasons, the right-hand side does not have terms in degree lower than r, and the degree r term is $\mathrm{rk}(\mathcal{G})[Z]$. On the other hand, for any coherent sheaf $\mathcal{H}$, the degree i component of the Chern character is

$$\mathrm{ch}_i(\mathcal{H}) = (-1)^{i-1} \frac{c_i(\mathcal{H})}{(i-1)!} + (\text{homog. poly. in } c_1, \ldots, c_{i-1}),$$

hence we conclude the proof. □

Applying the result above to $j\colon \Sigma \to \mathcal{C}$ and $\mathcal{G} = (\omega_f \otimes \mathcal{O}_\Sigma)|_\Sigma$, we get

$$\begin{aligned} &c_1(\omega_f \otimes \mathcal{O}_\Sigma) = 0, \quad \text{and} \\ &c_2(\omega_f \otimes \mathcal{O}_\Sigma) = -[\Sigma]. \end{aligned}$$

Therefore, from the sequence (5.5) we compute

$$\begin{aligned} &c_1(\Omega_f) = c_1(\omega_f), \quad \text{and} \\ &c_2(\Omega_f) = c_2(\omega_f) + [\Sigma] = [\Sigma] \end{aligned}$$

as ω_f is a line bundle. Using the same procedure, one can compute the first two Chern classes of the sheaf $\Omega_f \otimes \omega_f(D)$, where $D = \sum D_i$ is the divisor of sections τ_i, finding

$$\begin{aligned} &c_1(\Omega_f \otimes \omega_f(D)) = c_1(\omega_f^2(D)), \quad \text{and} \\ &c_2(\Omega_f \otimes \omega_f(D)) = [\Sigma]. \end{aligned}$$

Applying (5.4) to $\mathcal{F} = \omega_f$ then yields

$$c_1(f_* \omega_f) = f_* \left(\frac{c_1(\omega_f)^2}{2} - [\Sigma] - \frac{c_1(\omega_f) \cdot c_1(\omega_f)}{2} + \frac{c_1(\omega_f)^2 + [\Sigma]}{12} \right).$$

Denote by ξ the family $f\colon \mathcal{C} \to S$. We will make use of the following fact, whose proof is omitted; see [ACG11, §XIII.5]. Defining the two classes in $A(S)_\mathbb{Q}$

$$\begin{aligned} &(\kappa_1)_\xi := f_*(c_1(\omega_f(D))^2), \quad \text{and} \\ &(\widetilde{\kappa}_1)_\xi := f_*(c_1(\omega_f)^2), \end{aligned}$$

it follows that

$$(\widetilde{\kappa}_1)_\xi = (\kappa_1)_\xi - (\psi)_\xi,$$

where ψ is the line bundle on $\overline{\mathcal{M}}_{g,n}$ given by the sum of the cotangent line classes. Recalling that $(\lambda)_\xi = f_*\omega_f$ and $(\delta)_\xi = f_*[\Sigma]$, the formula above (omitting subscripts) thus become

$$\lambda = \frac{\kappa_1 - \psi + \delta}{12},$$

or equivalently

$$\kappa_1 = 12\lambda + \psi - \delta.$$

Actually, since these relations hold for any family over a scheme U étale over $\overline{\mathcal{M}}_{g,n}$, they hold $\operatorname{Pic}\overline{\mathcal{M}}_{g,n}$ as well, so writing them omitting subscripts is not an actual abuse of notation.

Theorem 5.5.9 (Mumford's formula)**.** *In* $\operatorname{Pic}(\overline{\mathcal{M}}_{g,n})$ *we have*

$$K_{\overline{\mathcal{M}}_{g,n}} = 13\lambda + \psi - 2\delta.$$

Proof. Let $f \colon \mathcal{C} \to S$ be a family of stable n-pointed genus g curves over a smooth scheme S. Applying (5.4) to $\mathcal{F} = \Omega_f \otimes \omega_f(D)$ and using relations found until now yields

$$c_1(f_*\mathcal{F}) = f_*\left(\frac{c_1(\omega_f^2(D))^2}{2} - [\Sigma] - \frac{c_1(\omega_f^2(D)) \cdot c_1(\omega_f)}{2} + \frac{c_1(\omega_f)^2 + [\Sigma]}{12}\right).$$

Note that

$$\begin{aligned} c_1(\omega_f^2(D))^2 &= c_1(\omega_f)^2 + 2c_1(\omega_f)c_1(\omega_f(D)) + c_1(\omega_f(D))^2 = \\ &= 3c_1(\omega_f)^2 + 2c_1(\omega_f)c_1(\mathcal{O}(D)) + c_1(\omega_f(D))^2, \end{aligned}$$

thus

$$f_*(c_1(\omega_f^2(D))^2) = 3\widetilde{\kappa}_1 + 2f_*(c_1(\omega_f)c_1(\mathcal{O}(D))) + \kappa_1.$$

Recall the ith cotangent line class $\psi_i = \tau_i^*(c_1(\Omega_f))$, and note also that

$$c_1(\mathcal{O}(D)) = \sum_i (\tau_i)_*(\mathbf{1}_S).$$

By applying the projection formula, we then get

$$\begin{aligned} c_1(\omega_f)c_1(\mathcal{O}(D)) &= \sum_i c_1(\omega_f)(\tau_i)_*(\mathbf{1}_S) = \sum_i (\tau_i)_*(\tau_i^*(c_1(\omega_f))\mathbf{1}_S) = \\ &= \sum_i (\tau_i)_*\psi_i, \end{aligned}$$

hence we conclude that

$$f_*(c_1(\omega_f)c_1(\mathcal{O}(D))) = \sum_i \psi_i = \psi$$

as $f \circ \tau_i = \mathrm{id}_S$. We finally get

$$\begin{aligned} c_1(f_*\mathcal{F}) &= f_*\left(\frac{c_1(\omega_f)^2 + c_1(\omega_f)c_1(\mathcal{O}(D))) + c_1(\omega_f(D))^2}{2} - [\Sigma] + \frac{c_1(\omega_f)^2 + [\Sigma]}{12}\right) = \\ &= \frac{\widetilde{\kappa}_1 + \psi + \kappa_1}{2} - \delta + \frac{\widetilde{\kappa}_1 + \delta}{12} = \kappa_1 - \delta + \lambda = \\ &= 13\lambda + \psi - 2\delta. \end{aligned}$$

□

5.6 Kodaira dimension

In this last section we will discuss the birational geometry of the moduli space $\overline{M}_g \times_{\mathbb{Z}} \operatorname{Spec}\mathbb{C}$ of stable curves of genus g over $\mathbb{C}$, simply denoted $\overline{M}_g$.
We will focus especially on the proof of the foundational result of Harris and Mumford [HM82] in which they showed that the moduli space $\overline{M}_g$ is of general type if g is odd and $g \geq 25$. Actually, a few years later Eisenbud and Harris proved that this holds for all $g \geq 24$ [EH87].

Recall that an irreducible algebraic variety X is said to be *unirational* if there exists a dominant rational map

$$f\colon \mathbb{P}^N \dashrightarrow X$$

for some N. Being unirational implies having negative Kodaira dimension, indeed without loss of generality we may assume X to be smooth, hence such an f induces an injective pullback map

$$f^*\colon H^0(X, \omega_X^{\otimes m}) \to H^0(\mathbb{P}^N, \mathcal{O}(-N-1)^{\otimes m}) = 0,$$

implying $h^0(X, \omega_X^{\otimes m}) = 0$ for every positive integer k.

In 1915, Severi proved that $\overline{M}_g$ is unirational for all $g \leq 10$ and conjectured that it could be true for all genera; see [ACG11, Thm 13.1]. In 1982, Harris and Mumford proved that this conjecture was false, completely changing the perspective on the birational geometry of this moduli space.
At the time of writing this thesis, $\overline{M}_g$ is known to be unirational for $g \leq 15$ and of general type for $g \geq 22$. In particular, the fact that $\overline{M}_{22}$ and $\overline{M}_{23}$ are of general type was enstablished in 2020 by Farkas-Jensen-Payne [FJP25], and for genera

between 16 and 21 the Kodaira dimension is still unknown.
The general situation is summarised `https://mgnbar.info`, where the present state of the research on the Kodaira dimension of $\overline{M}_{g,n}$ is displayed all at once in a dynamic table.

Low genus

First, we briefly discuss the unirationality results for low genera.

Genus 0. Recall that for $n < 3$, $\mathcal{M}_{0,n}$ is an algebraic stack not of Deligne-Mumford. On the other hand, as already observed in Section 5.4, for $n \geq 3$ the moduli space $M_{0,n}$ of smooth n-pointed genus 0 curves is even an open subscheme of a projective space, in fact

$$M_{0,n} \simeq (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \Delta$$

where Δ is the union of all pairwise diagonals; thus, the moduli space $M_{0,n}$ is rational.

Genus 1. In this case, the minimum number of marked points needed in order to define a moduli space is $n = 1$. The DM stack $\mathcal{M}_{1,1}$ of elliptic curves has a coarse moduli space $M_{1,1}$. Recall that every elliptic curve (E, p) can be described as a plane cubic

$$E_\lambda = V(y^2z - x(x - z)(x - \lambda z)) \subseteq \mathbb{P}^2$$

for some $\lambda \neq 0, 1$, where $p = [0 : 1 : 0] \in E_\lambda$. Moreover, this λ is not unique and determines a surjective map

$$\begin{aligned} \mathbb{A}^1 \setminus \{0, 1\} &\xrightarrow{6:1} M_{1,1}, \\ \lambda &\mapsto [E_\lambda] \end{aligned}$$

implying the unirationality of $M_{1,1}$. Actually, the j-invariant

$$j = 2^8 \frac{(\lambda^2 - \lambda - 1)^3}{\lambda^2(\lambda - 1)^2}$$

uniquely determines the elliptic curve, yielding a bijection $M_{1,1} \simeq \mathbb{A}^1$, hence implying the rationality of $M_{1,1}$. For a more detailed discussion on elliptic curves, see [Har77, §IV.4].

Genus 2. Smooth genus 2 curves are all hyperelliptic [Har77, Prop. 7.4.9], i.e., we can write them as a branched double covering of $\mathbb{P}^1$. By Riemann-Hurwitz (Theorem 2.1.7), these coverings are ramified over 6 points, hence we can write a smooth genus 2 curve as

$$y^2 = (x - a_1) \cdots (x - a_6)$$

for some distinct $a_1, \dots, a_6 \in \mathbb{P}^1$. After a change of coordinates on $\mathbb{P}^1$, we may assume that $(a_1, a_2, a_3) = (0, 1, \infty)$, thus we get a birational surjective map

$$(\mathbb{P}^1 \setminus \{0, 1, \infty\})^3 \setminus \Delta \to M_2$$

where Δ is again the union of pairwise diagonals. Hence M_2 is rational as well.

Genus 3. Given a non-hyperelliptic smooth curve C of genus 3, its canonical bundle provides an embedding in $\mathbb{P}^2$ as a quartic. Abusing notation, we then get a rational map

$$\Gamma(\mathbb{P}^2, \mathcal{O}(4)) \dashrightarrow M_3$$

having as image the non-hyperelliptic locus. Since this is a dense in M_3, we conclude that it is unirational.

Genus 4. Similarly, a non-hyperelliptic curve of genus 4 can be embedded via the canonical bundle into $\mathbb{P}^3$, only this time it can be realized as the intersection $C = Q \cap S$ of a quadric surface Q and a cubic surface S; see [Har77, Ex. IV.5.2.2]. We then get a rational map

$$\Gamma(\mathbb{P}^3, \mathcal{O}(2)) \times \Gamma(\mathbb{P}^3, \mathcal{O}(3)) \dashrightarrow M_4$$

whose image is the locus of non-hyperelliptic curves. As above, we conclude that M_4 is unirational.

A more general classical argument was used by Severi to prove the unirationality of the moduli space M_g for genera up to 10.

On the Kodaira dimension of the moduli space of curves

In the article [HM82] entitled as above, Harris and Mumford showed the following result.

Theorem 5.6.1. *The moduli space $\overline{M}_g$ of stable curves of genus g over $\mathbb{C}$ is of general type if g is odd and $g \geq 25$.*

The strategy of the proof relies on the geometric idea to consider the divisor given by the closure $\overline{D}_k \subseteq \overline{M}_g$ of the locus D_k of smooth curves which are k-fold covers of $\mathbb{P}^1$, where $k = (g+1)/2$, or equivalently the locus of smooth curves admitting a g^1_k, i.e., a complete linear system of degree k and projective dimension 1. In fact, the more general proof in Eisenbud-Harris [EH87] exploits the same idea but considering the so called Brill-Noether divisors, namely the locus of curves admitting a g^r_d for some suitable d, r.

The proof is broken down in six sections:

(1) Show that it is sufficient to study the canonical class of $\overline{M}_g$ in order to show that it is of general type, even if it is a singular variety.

(2) Compute the canonical class $K_{\overline{M}_g}$ as a sum of the Hodge class λ and the boundary divisors Δ_i.

(3) Compute the class of D_k.

(4) Describe $\overline{D}_k$ from a modular viewpoint.

(5) Determine the intersection of $\overline{D}_k$ with the boundary divisors.

(6) Compute the class of $\overline{D}_k$.

First, let us show how the results of these steps in fact prove that $\overline{M}_g$ is of general type for g odd and $g \geq 25$. Note that in order to define D_k, g must necessarily be odd. Computations of the classes above shows that

$$\begin{aligned} K_{\overline{M}_g} &= 13\lambda - 2\delta - \delta_1 \quad \text{and} \\ [\overline{D}_k] &= \frac{(2k-4)!}{k!(k-2)!}\left(6(k+1)\lambda - k\delta_0 - \sum_{i=1}^{k-1} 3i(2k-1-i)\delta_i\right). \end{aligned}$$

Let us quickly state a criterion to establish the ampleness of certain divisors on $\overline{M}_g$. We will use this result in the next proposition.
See [Alp27, §6.10] for a more detailed discussion about the ample cone of $\overline{M}_g$.

Theorem 5.6.2 (Cornalba–Harris)**.** *For any positive integers a and b, the divisor class $a\lambda - b\delta$ is ample on $\overline{M}_g$ if and only if $a > 11b$.*

Proof. See [CH88, Thm 1.3]. □

Proposition 5.6.3. *Let $D \subseteq \overline{M}_g$ be an effective divisor expressed in terms of the standard classes as*

$$[D] = a\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} b_i\delta_i,$$

for some $a \geq 0$ and $b_i > 0$ for all i. Suppose that for all i

$$\frac{a}{b_i} < \frac{13}{2} \quad \textit{and} \quad \frac{a}{b_1} < \frac{13}{3},$$

then $\overline{M}_g$ is of general type.

Proof. We want to prove that $K := K_{\overline{M}_g}$ is big, and this is equivalent by Corollary 1.2.6 to showing that $mK = H + E$ for some positive integer m, an ample divisor H, and an effective one E. Given a divisor $D = a\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} b_i\delta_i$ satisfying the assumptions above, we hope to find a positive $t \in \mathbb{Q}$ such that $K - tD = H' + E'$

where $H' = r\lambda - s\delta$ is ample and a multiple of $E' = \sum_i c_i\delta_i$ is effective. Taking a sufficiently large multiple of this identity will then imply that $mK = H + E$, where $H = mH'$ is ample and $E = mE' + mtD$ is effective. Writing

$$K = tD + H' + E'$$

and looking at the coefficients, we get

$$\begin{aligned} 13 &= ta + r, \\ 2 &= tb_i + s - c_i \quad \text{for} \quad i \neq 1, \quad \text{and} \\ 3 &= tb_1 + s - c_1. \end{aligned}$$

In order to get E' effective we impose that all c_i must be nonnegative. Moreover, by Theorem 5.6.2 to get H' ample we must impose $r > 11s$. Then we find

$$\begin{cases} 13 - ta > 11r \\ tb_i + r \geq 2 \quad (i \neq 1) \\ tb_1 + r \geq 3 \end{cases}, \quad \text{thus} \quad \begin{cases} r < \frac{13-ta}{11} \\ r \geq 2 - tb_i \quad (i \neq 1) \\ r \geq 3 - tb_1 \end{cases}.$$

We search for a $t \geq 0$ such that the system above has a solution $r > 0$. If such a t exists, it must satisfy

$$\begin{cases} \frac{13-ta}{11} > 0 \\ 2 - tb_i < \frac{13-ta}{11} \quad (i \neq 1) \\ 3 - tb_1 < \frac{13-ta}{11} \end{cases}.$$

If $a = 0$, we have nothing to check because such a t always exists. If instead $a > 0$, then the system of inequalities above is equivalent to

$$\begin{cases} t < \frac{13}{a} \\ t > \frac{9}{11b_i - a} \quad (i \neq 1) \\ t > \frac{20}{11b_1 - a} \end{cases}.$$

Note $11b_i - a$ is positive by assumptions on D. This last system has a solution if and only if

$$\begin{cases} \frac{9}{11b_i - a} < \frac{13}{a} \quad (i \neq 1) \\ \frac{20}{11b_1 - a} < \frac{13}{a} \end{cases} \iff \begin{cases} 2a < 13b_i \quad (i \neq 1) \\ 3a < 13b_1 \end{cases}.$$

Therefore, by using again assumptions on D, we conclude the proof. $\square$

We may then prove the main result of the article.

Proof of Theorem 5.6.1. We now want to apply Proposition 5.6.3 to $\overline{D}_k$. In order to do this, we must check whether

$$\begin{aligned} \frac{6(k+1)}{k} &< \frac{13}{2}, \\ \frac{6(k+1)}{3i(2k-1-i)} &< \frac{13}{2} \quad \text{for} \quad i \neq 0,1, \quad \text{and} \\ \frac{6(k+1)}{3(2k-2)} &< \frac{13}{3}. \end{aligned}$$

Since for all i

$$\frac{6(k+1)}{3i(2k-1-i)} \leq \frac{6(k+1)}{3(2k-1-(k+1))} = \frac{2(k+1)}{k},$$

it suffices to check

$$\frac{6(k+1)}{k} < \frac{13}{2},$$

which is equivalent to $k > 12$. We then conclude that for $g = 2k-1 \geq 23$ the effective divisor $\overline{D}_k$ satisfies the assumptions of Proposition 5.6.3, and consequently $\overline{M}_g$ is of general type. $\square$

In this last part of the thesis, we go through the six sections of the original paper, which are necessary to compute the classes used in the proof above, particularly $K_{\overline{M}_g}$ and $[\overline{D}_k]$.

Singularities

First of all, there is an important technical question that needs to be addressed. When dealing with a singular projective variety X, it may happen that the dualizing sheaf ω_X exists as a line bundle, but $\kappa(\omega_X) \neq \kappa(X)$. More generally, for a normal projective variety X, we may define the canonical sheaf ω_X as the unique one extending the canonical sheaf of the regular locus; this is well-defined by [Har94, Thm 1.12].

In order to study the Kodaira dimension, we wish for criteria to establish whether the singularities allow us to work directly with this canonical sheaf, i.e., whether $\kappa(\omega_X) = \kappa(X)$. Since this thesis does not focuses on these details, here we only give a brief idea.

A *resolution* of a projective variety X is a proper birational morphism $f\colon Y \to X$ such that Y is smooth. If we call E the set of points $y \in Y$ where f is not biregular, i.e., f^{-1} is not a morphism at $f(y)$, endowing E with the induced reduced structure yields the *exceptional divisor* of f.

We say that a normal projective variety X has *canonical singularities* if:

(i) the canonical divisor K_X is $\mathbb{Q}$-Cartier, i.e., for some integer $r \geq 1$ the Weil divisor rK_X is Cartier, and

(ii) if $f\colon X \to Y$ is a resolution of X and $\{E_i\}$ are the exceptional prime divisors of f, then
$$rK_Y = f^*(rK_X) + \sum_i a_i E_i \quad \text{with} \quad a_i \geq 0.$$

Remark. Recall that by definition X is of general type if a resolution Y is. If X has canonical singularities and $\kappa(rK_X) = \dim X$, then rK_Y is big and this proves that X is of general type. In other words, if we show that $\overline{M}_g$ has canonical singularities, then showing that $K_{\overline{M}_g}$ is big implies that $\overline{M}_g$ is of general type.

Since $\overline{\mathcal{M}}_g$ is a smooth Deligne-Mumford stack, then its coarse moduli space $\overline{M}_g$ has finite quotient singularities (see Corollary 4.6.11). The first section of [HM82] is devoted to showing that these singularities are canonical. The main tool used in the proof is the Reid-Tai Criterion, which establish conditions on the action of the finite cyclic group μ_r in order to get canonical quotient singularities $\mathbb{A}^n/\mu_r$; see [Rei87, §4.11].

The canonical class

The Mumford's formula (Theorem 5.5.9) provides the relation
$$K_{\overline{\mathcal{M}}_{g,n}} = 13\lambda + \psi - 2\delta.$$

Now we want to compute the canonical class of the coarse moduli space. Recall that by Proposition 5.5.1, $\mathrm{Pic}(\overline{M}_{g,n})$ is a finite index subgroup of $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$.

Corollary 5.6.4. *For $g \geq 1$ and $g + n \geq 4$, in $\mathrm{Pic}(\overline{M}_{g,n})$ we have*
$$K_{\overline{M}_{g,n}} = 13\lambda + \psi - 2\delta - \delta_1.$$

In particular, for $g \geq 4$ we get $K_{\overline{M}_g} = 13\lambda - 2\delta - \delta_1$.

Proof. The discrepancy between the two formulas comes from the coarse moduli map $\pi\colon \overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$. This morphism is ramified precisely at those points corresponding to stable curves with nontrivial automorphism group.
Letting Σ denote the locus parametrising these curves, then the assumptions on g and n ensure it is a proper closed subscheme and its only divisor component is the boundary divisor $\Delta_1 = \pi_*\delta_1$ [ACG11, Prop. XII.2.5]. The morphism π is then simply ramified along this divisor, implying that
$$\pi^*(\Delta_1) = 2\delta_1.$$

Recall that a morphism $f\colon X \to Y$ of schemes over k induces an exact sequence

$$f^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_f \to 0,$$

and this can be generalised to morphisms of DM stacks over k. In particular, we get

$$0 \to \pi^*\Omega_{\overline{M}_{g,n}} \to \Omega_{\overline{\mathcal{M}}_{g,n}} \to \Omega_\pi \to 0,$$

where the left exactness comes from the fact that π is generically étale (it is an isomorphism on the locus of curves with trivial automorphism group). Moreover, Ω_π is a sheaf supported in the ramification locus $R = \delta_1$. Actually, taking the determinant of this sequence yields a generalisation of the Riemann-Hurwitz formula; see [GH78]. At the level of divisors, we get

$$K_{\overline{\mathcal{M}}_{g,n}} = \pi^*K_{\overline{M}_{g,n}} + R,$$

hence

$$\pi^*K_{\overline{M}_{g,n}} = 13\lambda + \psi - 2\delta - \delta_1,$$

concluding the proof. □

The Hurwitz divisor

From now on, assume that the genus $g \geq 2$ is odd, and let $g = 2k - 1$. Recall the Hurwitz moduli space $\mathrm{H}_{g,k} = \mathrm{Hur}_{g,k}$ introduced in Definition 5.4.1, and the maps

$$\begin{tikzcd} & \mathrm{H}_{g,k} \arrow[dl] \arrow[dr] & \\ \mathcal{M}_g & & M_{0,b} \end{tikzcd} \tag{5.6}$$

where $b = 2g + 2k - 2 = 3g - 1$ is the number of ramification points. Moreover, we have seen that the map on the right is an algebraic covering space, hence

$$\dim \mathrm{H}_{g,k} = \dim M_{0,b} = b - 3 = 3g - 4.$$

Recall that for $g \geq 2$ the algebraic stack $\mathcal{M}_g$ is Deligne-Mumford, hence it admits a coarse moduli space M_g. We then have an induced map

$$\mathrm{H}_{g,k} \to M_g$$

and we define the *Hurwitz divisor* D_k as the image of this map. This is actually a divisor since Brill-Noether theory implies that D_k is a closed subscheme of dimension at least $3g - 4$ [ACG11, Thm XXI.3.21], but for a general curve C, the fibre of the morphism above is empty [Arb+85, Thm V.1.5], hence its dimension is exactly $3g-4$; we refer to [ACG11, §XXI] for the Brill-Noether theory on families of curves.

Moreover, the diagonal of $\mathcal{M}_g$ is finite, implying that the map from its inertia stack is finite as well. Removing now the component of the inertia stack corresponding to the identity, we get a closed substack mapping in $\mathcal{M}_g$ to the locus of curves with nontrivial automorphisms. The locus $\mathcal{M}_g^0$ of curves with trivial automorphism group is then an open substack, and it is nonempty as showed in [Bai62].
The purpose of this section of the article is to show that the Hurwitz divisor D_k over $\mathcal{M}_g^0$ has class

$$D_k \equiv_{\text{lin}} a\lambda$$

for some $a \in \mathbb{Q}$. We omit this computation by appealing to a later stronger result due to Harer stating that $\operatorname{Pic}(\mathcal{M}_g^0)$ is infinite cyclic [Har88].

Throughout, we will denote linear equivalence simply with an equality. Clearly, this is an actual equality only in the Chow group.

Corollary 5.6.5. *Denoting with $\overline{D}_k$ the closure of $D_k \subseteq \overline{M}_g$, there are integers $b_0, \ldots, b_{\lfloor g/2 \rfloor}$ such that on $\overline{M}_g$*

$$\overline{D}_k = a\lambda + \sum_{i=0}^{\lfloor g/2 \rfloor} b_i \delta_i.$$

Proof. For what we have just seen, $\overline{D}_k - a\lambda$ is an integral divisor on $\overline{M}_g$, trivial on the open set M_g, hence is a integral combination of boundary divisors. □

Admissible covers

In order to compute the coefficients in Corollary 5.6.5, we need a clearer description of the closure of D_k. To find which stable curves lie in this closure, Harris and Mumford introduced a moduli description for this divisor.

We want to compactify the Hurwitz space in a suitable way to extend the diagram (5.6) to

$$\begin{tikzcd} & \overline{\mathcal{H}}_{g,k} \arrow[dl] \arrow[dr] & \\ \overline{\mathcal{M}}_g & & \overline{M}_{0,b}. \end{tikzcd}$$

The idea is to generalise the concept of simply branched coverings where both source and target may be nodal curves. From now on, let $b = 2g + 2d - 2$.

An *admissible covering of genus g and degree d points* is a surjective finite morphism $f\colon C \to P$ from a prestable genus g curve to a stable b-pointed genus 0 curve (P, p_i) such that

(i) the preimage $f^{-1}(P^{\mathrm{sm}})$ of the smooth locus is the smooth locus C^{sm}, and the induced morphism between these loci is a simply branched covering of degree d over the points p_i; and

(ii) for every node $z \in P$ and every node $q \in C$ over z, the map f étale locally at q is of the form

$$\begin{gathered} k[u,v]/(uv) \to k[x,y]/(xy) \\ u \mapsto x^m \\ v \mapsto y^m \end{gathered}$$

for some integer $m \geq 1$, namely the two branches have equal ramification indices at the node.

This definition extends to families as follows.

Definition 5.6.6 (Admissible covering)**.** A family of admissible coverings of genus g and degree d

$$(\mathcal{C} \xrightarrow{\pi} \mathcal{D} \to S, \sigma_i)$$

over S is the datum of a family of prestable curves $\mathcal{C} \to S$ and a family of b-pointed stable curves $(\mathcal{D} \to S, \sigma_i)$ of genus 0, together with a morphism $\pi\colon \mathcal{C} \to \mathcal{D}$ which is étale except

- at smooth points over the sections σ_i, where it is a simply branched covering of degree d, and
- over the nodal locus of $\mathcal{D}$, where for each $s \in S$ and each point $x \in \mathcal{C}_s$ over a node $y \in \mathcal{D}_s$, x is a node and étale locally $\mathcal{C}$, $\mathcal{D}$, and π are described by

$$\begin{array}{llll} \mathcal{C}: & xy = a, & a \in \widehat{\mathcal{O}}_{S,s}, & x, y \text{ generate } \widehat{\mathfrak{m}}_x \subseteq \widehat{\mathcal{O}}_{\mathcal{C},x}, \\ \mathcal{D}: & uv = a^p, & & u, v \text{ generate } \widehat{\mathfrak{m}}_y \subseteq \widehat{\mathcal{O}}_{\mathcal{D},y}, \text{ and} \\ \pi: & u = x^p, \quad v = y^p & & \text{for some positive integer } p. \end{array}$$

We define the functor $\overline{\mathcal{H}}_{g,d}$ which associates to a scheme S the set of isomorphism classes of families of admissible coverings of genus g over b points $(\mathcal{C} \to \mathcal{D} \to S, \sigma_i)$ over S.

Theorem 5.6.7. *The functor $\overline{\mathcal{H}}_{g,d}$ is a proper Deligne-Mumford stack. Moreover, its coarse moduli space $\overline{\mathrm{H}}_{g,d}$ is a scheme finite over $\overline{M}_{0,b}$.*

Proof. We omit technicalities of the first part, only giving the main idea of the original proof; see [ACG11, §XVI.5-6] for a modern treatment.

In the original construction of this coarse moduli space [HM82, Thm 4], the authors exploited the rigidification technique described below. Firstly, observe that the morphism associated to the coarse moduli space $\overline{\mathcal{H}}_{g,d} \to \overline{\mathrm{H}}_{g,d}$ is universal for maps to algebraic spaces Definition 4.6.6, hence $\overline{\mathrm{H}}_{g,d}$ also lies over $\overline{M}_{0,b}$. The problem is then local on $\overline{M}_{0,b}$, so we may cover it by suitable open sets and construct the coarse moduli separately over each of them.
Taking $[D_0] \in \overline{M}_{0,b}(\mathbb{C})$, it may happen that every component of D_0 has at least one of the b marked points on it. If not, choose further smooth points $p_{b+1}, \ldots, p_c$ in D_0 to get one p_i in each component. Furthermore, in some neighbourhood $U \subseteq \overline{M}_{0,b}$ of $[D_0]$, we can choose smooth disjoint sections σ_i of the universal curve $\mathcal{U} \to \overline{M}_{0,b}$ through these points. This is possible because $\mathcal{U} = \overline{M}_{0,b+1}$ (Proposition 5.2.2) and there are birational morphisms $\mathcal{U} \to \mathbb{P}^1 \times \overline{M}_{0,b}$ which over $[D_0]$ take any one of the components of $\mathcal{U}_{[D_0]}$ isomorphically to $\mathbb{P}^1$, collapsing the remaining components to points; now $\mathbb{P}^1 \times \overline{M}_{0,b}$ has a section through any point of $\mathbb{P}^1 \times [D_0]$.
Choose then U small enough so that for all $[D] \in U$, the sections $\sigma_1, \ldots, \sigma_c$ meet every component of D. Moreover, we may arrange U's in such a way to have a local coordinate t_i on the fibres of $\mathcal{U}$ over U with $t_i = 0$ on the section $\sigma_i \colon U \to \mathcal{U}$ to first order.

In this setting, for all $[D] \in U$, if $\pi \colon C \to D$ is an admissible cover, $\mathcal{O}_C(\sum_{i=1}^{c} \pi^{-1} p_i)$ is ample on C; in particular, for some n, $\mathcal{O}_C(n \sum_{i=1}^{c} \pi^{-1} p_i)$ is very ample for all $\pi \colon C \to D$, since n only depends on the genus g of C. Define now a "rigidified" version of $\overline{\mathcal{H}}_{g,d}$ over U. Let $\overline{\mathcal{H}}^U_{g,d}$ be the functor parametrising admissible coverings $(C \xrightarrow{\pi} D, p_i)$, where $[D] \in U$, *plus* orderings of points in $\pi^{-1}(p_i)$:

$$
\begin{array}{rll}
p_{1,1}, \ldots, p_{1,k-1} \in C, & \text{over } p_1 \in D, & p_{1,1} \text{ ramified} \\
\cdots & \cdots & \cdots \\
p_{b,1}, \ldots, p_{b,k-1} \in C, & \text{over } p_b \in D, & p_{b,1} \text{ ramified} \\
p_{b+1,1}, \ldots, p_{b+1,k} \in C, & \text{over } p_{b+1} \in D, & \\
\cdots & \cdots & \\
p_{c,1}, \ldots, p_{c,k} \in C, & \text{over } p_c \in D, &
\end{array}
$$

where $k = (2 - 2g + b)/2$ is the degree of the covering, *plus* a choice of a square root

$$\sqrt{t_i} \in \widehat{\mathcal{O}}_{C,p_{i,1}} \quad \text{for} \quad 1 \leq i \leq b.$$

Note that, changing the choices above, the finite group

$$G := (\mu_2 \times \mathfrak{S}_{k-2})^b \times \mathfrak{S}_k^{c-b}$$

acts on $\overline{\mathcal{H}}^U_{g,d}$, and via this action

$$\overline{\mathcal{H}}^U_{g,d}(\mathbb{C})/G \subseteq \overline{\mathcal{H}}_{g,d}(\mathbb{C}).$$

It remains to prove that the functor $\overline{\mathcal{H}}^U_{g,d}$ is representable by a scheme; as usual, in [HM82, Thm 4] this is done finding a suitable Hilbert scheme.
Observe that if a scheme H represents $\overline{\mathcal{H}}^U_{g,d}$, then H/G is a coarse moduli space for the open substack of $\overline{\mathcal{H}}_{g,d}$ parametrising admissible covers $C \to D$ with $[D] \in U$, and moreover H itself represents the locus where the action of G is free. Precisely, a fixed point of $g \in G$ is a covering $\pi\colon C \to D$ such that C has an automorphism α commuting with π, i.e.

$$\begin{tikzcd} C \arrow[rr, "\alpha"] \arrow[dr, "\pi"'] & & C \arrow[dl, "\pi"] \\ & D & \end{tikzcd}$$

commutes, which permutes the finite sets $\pi^{-1}(p_i)$ and/or acts by $\sqrt{t_i} \mapsto -\sqrt{t_i}$ at the ramified points p_{i1}. Glueing then together the schemes H/G over the U's yields our coarse moduli space $\overline{\mathrm{H}}_{g,d}$.

Finally, the finiteness of $\overline{\mathrm{H}}_{g,d} \to \overline{M}_{0,b}$ is equivalent to being proper and with finite fibres [SP, Tag 02OG]. This map has clearly finite fibres, hence by Keel-Mori (Theorem 4.6.9) it suffices to check that $\overline{\mathcal{H}}_{g,d}$ satisfies the Valuative Criterion for properness. In other words, given an admissible covering

$$C \xrightarrow{\widetilde{\pi}} D \to \Delta^*$$

over the germ of the punctured disc $\Delta^* = \operatorname{Spec} \mathbb{C}((t))$, we want to prove that, possibly taking a suitable root $t^{1/n}$ of t, π extends to a family of admissible coverings

$$\mathcal{C} \xrightarrow{\pi} \mathcal{D} \to \Delta'$$

over the entire germ of the disc $\Delta' = \operatorname{Spec} \mathbb{C}[\![t^{1/n}]\!]$.
Since $\overline{M}_{0,b}$ is a proper scheme over $\mathbb{C}$, D extends to a family $\mathcal{D}_1$ of b-pointed stable curves of genus 0 over $\Delta = \operatorname{Spec} \mathbb{C}[\![t]\!]$. Let $\mathcal{C}_1$ be the partial normalisation of $\mathcal{D}_1$ in the fraction field of C given by

$$\mathcal{O}_{\mathcal{C}_1} = (\text{functions integral over } \mathcal{O}_{\mathcal{D}_1}, \text{ generically in } \mathcal{O}_C)\,.$$

The map $\mathcal{C}_1 \to \mathcal{D}_1$ may be ramified over one of the components of the central fibre $(\mathcal{D}_1)_0$. However, by Abhyankar's Lemma [SP, Tag 0BRM], replacing t by $t^{1/n}$ removes this issue. We get then a covering $\mathcal{C} \to \mathcal{D}$ ramified only over the sections $\sigma_i\colon \operatorname{Spec} \mathbb{C}[\![t^{1/n}]\!] \to \mathcal{D}$.

It remains to check that $\mathcal{C} \to \mathcal{D}$ is a family of admissible coverings by looking at what happens over the nodal locus of $\mathcal{D}$. If a node of the central fibre $\mathcal{D}_0$ lifts to the generic fibre $\mathcal{D}_\eta$, then étale locally $\mathcal{D}$ is given by

$$uv = 0$$

and $\mathcal{C}$ is a covering of the smooth branch $u = 0$ plus one of the smooth branch $v = 0$. Since the generic fibre $\mathcal{D}_\eta$ has branching order r over the curve $u = v = 0$ both on the surface $u = 0$ and on the surface $v = 0$, $\mathcal{C}$ must do the same. Otherwise, if a node of $\mathcal{D}_0$ does not lift to the generic fibre, then $\mathcal{D}$ has local equation

$$uv = s^m$$

where $s = t^{1/n}$, namely it has an A_{m-1}-singularity. Then the universal cover of $\mathcal{D}$, ramified only at the origin, is given by

$$x'y' = s, \qquad x' = u^{1/m}, \qquad y' = v^{1/m},$$

and thus $\mathcal{C}$ is necessarily given by

$$xy = x^h, \qquad x^l = u, \qquad y^l = v,$$

for some $hl = m$. This concludes the proof. $\square$

Remark (Automorphisms). The proof shows that the stabiliser group of a point $[\pi\colon C \to D] \in \overline{\mathcal{H}}_{g,d}$ is the subgroup

$$\{\alpha \in \mathrm{Aut}(C) \mid \pi \circ \alpha = \pi\} \leq \mathrm{Aut}(C).$$

As expected, we have morphisms

$$\begin{tikzcd} & \overline{\mathcal{H}}_{g,d} \arrow[dl] \arrow[dr] & \\ \overline{\mathcal{M}}_g & & \overline{M}_{0,b} \end{tikzcd} \quad \text{given by} \quad \begin{tikzcd} & (\mathcal{C} \to \mathcal{D} \to S, \sigma_i) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ \mathrm{StMd}(\mathcal{C} \to S) & & (\mathcal{D} \to S, \sigma_i), \end{tikzcd}$$

where $\mathrm{StMd}(\mathcal{C} \to S)$ is the stable model, as in Theorem 2.3.13. Note that the diagram above induces

$$\begin{tikzcd} & \overline{\mathrm{H}}_{g,d} \arrow[dl] \arrow[dr] & \\ \overline{M}_g & & \overline{M}_{0,b} \end{tikzcd}$$

between coarse moduli spaces. It is then clear that for $g = 2k - 1$ and $d = k$ the divisor $\overline{D}_k$ is the image of the morphism on the left.

Remark (Local description). Let $\pi\colon C \to D$ be a $\mathbb{C}$-point of $\overline{\mathrm{H}}_{g,d}$, and let $\{q_i\}_{i=1,\dots,m}$ be the nodes of D with $\pi^{-1}(q_i) = \{q_{i,1}, \dots, q_{i,r(i)}\}$. Say that the map π has ramification order $p_{i,j}$ at $q_{i,j}$. Let also

$$\begin{tikzcd} \mathcal{D} \arrow[r] & \operatorname{Spec} \mathbb{C}[\![t_1, \dots, t_{b-3}]\!] \arrow[l, bend left, "\sigma_i"] \end{tikzcd}$$

be the universal family of D, as in Corollary 3.2.4. If D at q_i is given by $u_i v_i = 0$, then for suitable choices of t_i (note that stability and genus 0 impose $m \le b-3$), we may assume that $\mathcal{D}$ locally at q_i is given by

$$u_i v_i = t_i.$$

Grothendieck's theory on étale coverings [SP, Tag 0BQN] implies that in order to lift the covering $\pi\colon C \to D$ to a covering $\widetilde{\pi}\colon \mathcal{C} \to \mathcal{D}$, it suffices to define $\mathcal{C}$ only near non-étale points of π. These points exactly corresponds to the sections σ_i, and letting s_i be a function on $\mathcal{D}$ vanishing to first order on the image of σ_i, we get that near this image the covering $\widetilde{\pi}$ is given by choosing a root $\sqrt{s_i}$. On the other hand, near each q_{ij} over a node, $\mathcal{C}$ must be given by

$$x_{ij} y_{ij} = t_{ij}$$

and $\widetilde{\pi}$ by

$$u_i = x_{ij}^{p_{ij}}, \quad v_i = y_{ij}^{p_{ij}},$$

and this implies $t_i = t_{ij}^{p_{ij}}$. In other words, we are defining a covering $\mathcal{C} \to \mathcal{D}$ over a base $\operatorname{Spec} \mathcal{O}_{[C\to D]}$, where

$$\mathcal{O}_{[C\to D]} = \mathbb{C}[\![t_1, \dots, t_{b-3}, t_{1,1}, \dots, t_{i,j}, \dots, t_{d,r(d)}]\!]/(t_{ij}^{p_{ij}} - t_i \mid i, j).$$

In particular, the completion of the local ring of the scheme $\overline{\mathrm{H}}_{g,d}$ at $[C \to D]$ is the ring of invariants of $\mathcal{O}_{[C\to D]}$ under the action of the stabiliser group. However, the rings $\mathcal{O}_{[C\to D]}$ might be messy, e.g. even not integrally closed.
In fact, this first approach to admissible coverings was later modernised, as in [ACV03], where a more general viewpoint leads to the description of the normalisation of $\overline{\mathcal{H}}_{g,d}$.

We now want to provide a description of what stable curves lie in $\overline{D}_k$.

Theorem 5.6.8. *Let C be a stable curve of genus $g = 2k-1$. Then*

(a) if C is irreducible, $[C] \in \overline{D}_k$ if and only if there exists a torsion-free rank 1 sheaf $\mathcal{F}$ on C such that

$$h^0(\mathcal{F}) \ge 2 \quad \textit{and}$$
$$\chi(\mathcal{F}) = 2 - k.$$

(b) if $C = C_1 \cup C_2$ is the irreducible decomposition, with $C_1 \cap C_2 = \{p\}$, then $[C] \in \overline{D}_k$ if and only if there exist torsion-free rank 1 sheaves $\mathcal{F}_1$ on C_1, $\mathcal{F}_2$ on C_2, and an integer ℓ such that for $i = 1, 2$

$$\begin{aligned} h^0(\mathcal{F}_i) &\geq 2, \\ h^0(\mathcal{F}_i(-\ell p)) &\geq 1, \quad \textit{and} \\ \chi(\mathcal{F}_1) + \chi(\mathcal{F}_2) &= 3 - k + \ell. \end{aligned}$$

Proof. For (a), we first observe that the theory of the compactification of $\mathrm{Pic}(C)$ [AK80] implies that there exists a space parametrising pairs $(C, \mathcal{F})$, where C is an irreducible genus g stable curve and $\mathcal{F}$ is a torsion-free rank 1 sheaf over C. Moreover, it also implies that this space is proper over the moduli space $\overline{M}_g$, and the conditions in (a) define a closed subset of this space.
If now C is smooth and belongs to D_k, let $\pi\colon C \to \mathbb{P}^1$ be the corresponding covering of degree k, and set $\mathcal{F} = \pi^*\mathcal{O}(1)$. We then get

$$\deg(\mathcal{F}) = \deg \pi + 1 - g = k + 1 - (2k - 1) = 2 - k,$$

implying that C satisfies the conditions in (a). Therefore, the locus of curves having such a bundle as in (a) contains the set of irreducible curves in $\overline{D}_k$.
Conversely, let C satisfy the conditions in (a). We claim that C lifts to a point of $\overline{\mathrm{H}}_{g,k}$, hence we get $[C] \in \overline{D}_k$. To prove the claim, letting C' be the normalisation of C, we will embed C' in a nodal curve C'' so that collapsing rational components of C'' gives back C, and we will construct at the same time an admissible covering $\pi\colon C'' \to P$ where P is a stable b-pointed curve of genus 0.

Choose global sections $s_1, s_2 \in \Gamma(C, \mathcal{F})$, and let $\mathcal{F}' \hookrightarrow \mathcal{F}$ be the subsheaf generated by them. Let Z be the set of nodes of C. At each $z \in Z$, the stalk $\mathcal{F}'_z$ is isomorphic either to $\mathcal{O}_{z,C}$ or to $\mathfrak{m}_z \subseteq \mathcal{O}_{z,C}$; call these subsets $Z_1, Z_2 \subseteq Z$ respectively. Denoting by $\widetilde{C}$ the partial normalisation of C at nodes in Z_2, there is an invertible sheaf $\widetilde{\mathcal{F}}$ on $\widetilde{C}$ such that if

$$C' \xrightarrow{\nu_2} \widetilde{C} \xrightarrow{\nu_1} C$$

are the normalisation maps, we get $\mathcal{F}' = (\nu_1)_*(\widetilde{\mathcal{F}})$. The function s_1/s_2 defines a morphism

$$\widetilde{\pi}\colon \widetilde{C} \to \mathbb{P}^1$$

thus

$$\pi' = \widetilde{\pi} \circ \nu_2\colon C' \to \mathbb{P}^1,$$

and

$$\begin{aligned}\#Z_2 + \deg(\pi') &= \#Z_2 + \deg(\widetilde{\pi})\\ &= \#Z_2 + \deg(\widetilde{\mathcal{F}})\\ &= \#Z_2 + \chi(\widetilde{\mathcal{F}}) + p_a(\widetilde{C}) - 1\\ &= \chi(\mathcal{F}') + \#Z_2 + p_a(\widetilde{C}) - 1\\ &= \chi(\mathcal{F}') + p_a(C) - 1\\ &\leq \chi(\mathcal{F}) + 2k - 2\\ &= k.\end{aligned}$$

To construct the covering $\pi\colon C'' \to P$, we start from π'. Let $S \subseteq \mathbb{P}^1$ the set of

- branch points of π,
- images of the points $z_{i,1}, z_{i,2} \in C'$ over the nodes $z_i \in Z$, and
- images of $k - \#Z_2 - \deg(\pi')$ further general points $w_i \in C'$.

We define P as the original $\mathbb{P}^1$, now called $(\mathbb{P}^1)_0$, with a *tail* $\mathbb{P}^1$ attached to each point of S. We are now ready to construct C''.

Step 1. If $x \in (\mathbb{P}^1)_0$ is a branching point, scheme-theoretically we write

$$(\pi')^{-1}(x) = \sum n_i y_i.$$

For each i, we glue to C' at y_i a copy of $\mathbb{P}^1$, mapped to the tail at x by a covering $\mathbb{P}^1 \to \mathbb{P}^1$ of degree n_i with ramification index n_i at y_i; see Figure 5.8 below.

Step 2. If $x \in (\mathbb{P}^1)_0$ is the image of $z_{i,1}$ and $z_{i,2}$, then necessarily $z_i \in Z_1$, and we denote by ℓ_1, ℓ_2 the ramification indexes of π' at $z_{i,1}$ and $z_{i,2}$ respectively. At other points in $(\pi')^{-1}(x)$, add further tails as in Step 1; see Figure 5.9.

Step 3. If $x_1, x_2 \in (\mathbb{P}^1)_0$ are the distinct images of $z_{i,1}$ and $z_{i,2}$, then this time $z_i \in Z_2$, and we denote by ℓ_1, ℓ_2 the ramification indexes of π' again. In this case, we add 3 copies of $\mathbb{P}^1$ to C': copy A over the tail at x_1, copy B over the tail at x_2, and copy C over $(\mathbb{P}^1)_0$. More precisely, copy C is mapped isomorphically to $(\mathbb{P}^1)_0$, copy B is mapped with degree $\ell_1 + 1$ and a ramification index of ℓ_1 at $z_{i,1}$, and copy C is mapped with degree $\ell_2 + 1$ and a ramification index of ℓ_2 at $z_{i,2}$. Copy A (resp., B) meets copy C at its remaining point over x_1 (resp., x_2). Finally, to cover all the other tails of $(\mathbb{P}^1)_0$, we attach to copy C more $\mathbb{P}^1$'s; see Figure 5.10.

Step 4. To increase the degree of this new map π up to k, we exploit the last $k - \#Z_2 - \deg(\pi')$ points added to S. More precisely, for such a point x we add $\deg(\pi')$ copies A_i of $\mathbb{P}^1$ over the tail at x, all but one mapping isomorphically to it, and one of degree 2 over it. This last copy introduces a new point over x, to which

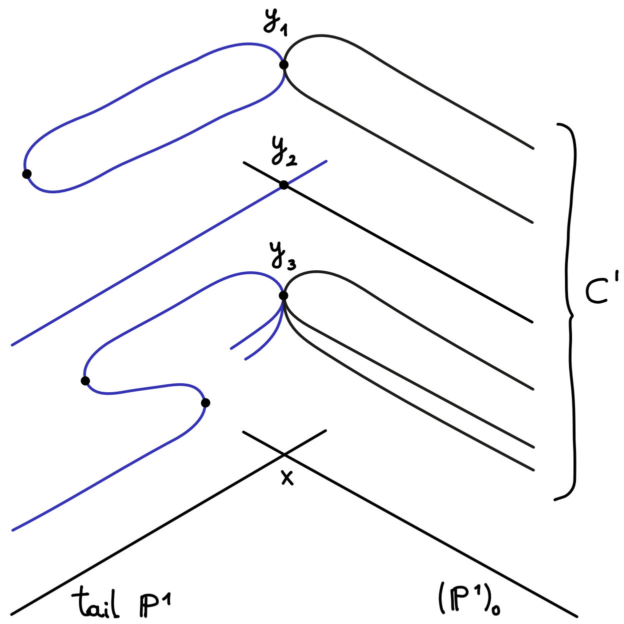


Figure 5.8: Example of Step 1 with $n_1 = 2$, $n_2 = 1$, and $n_3 = 3$.

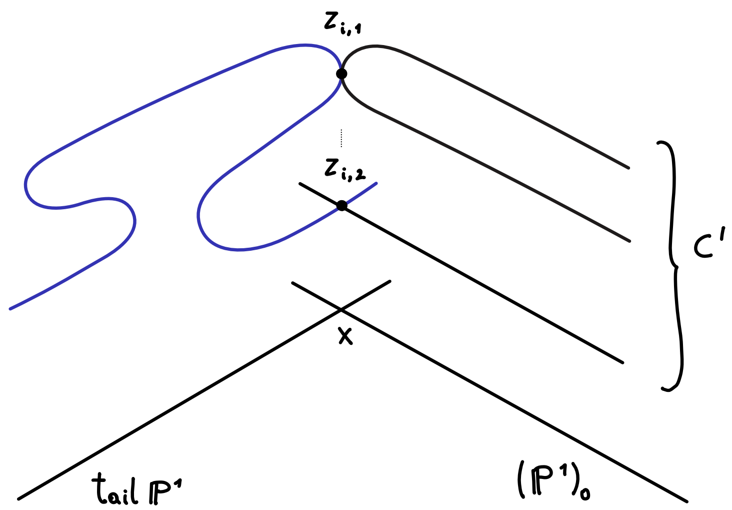


Figure 5.9: Example of Step 2 with $\ell_1 = 2$ and $\ell_2 = 1$.

we attach a copy B of $\mathbb{P}^1$ mapping isomorphically to $(\mathbb{P}^1)_0$. Finally, we attach to copy B further $\mathbb{P}^1$'s over the other tails. See Figure 5.11 below.

It is not difficult to see that the curve C'' resulting from the steps above, together with the map π to P defined starting from π', describe an admissible covering of degree k and genus $g = 2k - 1$. Furthermore, contracting the extra $\mathbb{P}^1$'s returns the original C, and this concludes the proof of part (a).

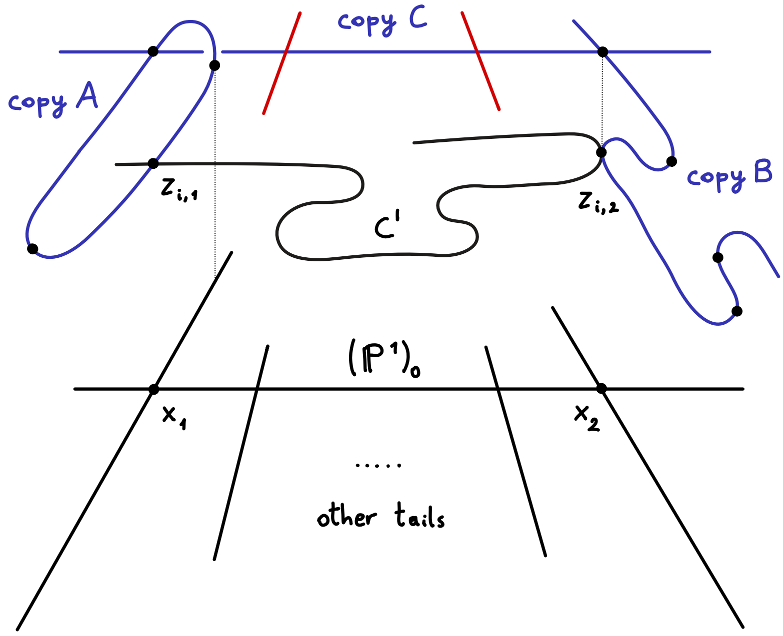


Figure 5.10: Example of Step 3 with $\ell_1 = 1$ and $\ell_2 = 2$.

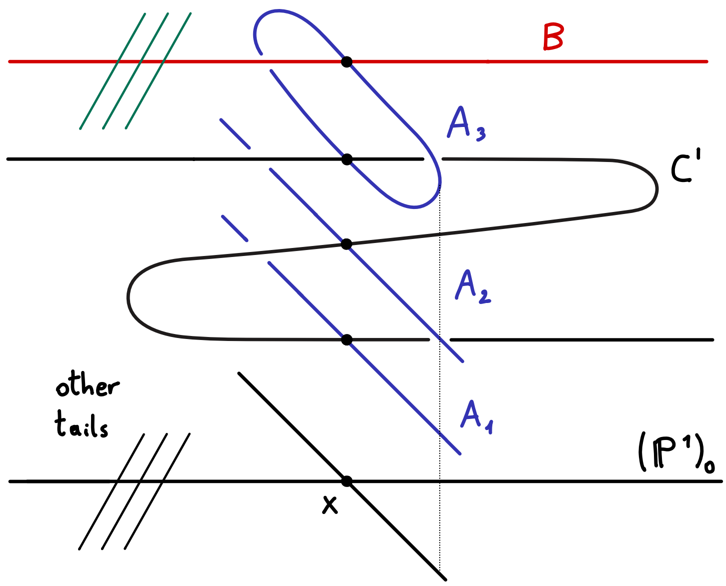


Figure 5.11: Example of Step 4. Note that A_3 maps to $(\mathbb{P}^1)_0$ with degree 2, and B serves to cover the other tails.

To prove part (b), denoting by $\overline{D}_k^* \subseteq \overline{M}_g$ the locus of curves satisfying conditions of parts (a) or (b), we show first that $\overline{D}_k^*$ is closed in the open subset $U \subseteq \overline{M}_g$

of curves with at most 2 irreducible components meeting at most once, i.e., having at most one node, implying that $\overline{D}_k \cap U \subseteq \overline{D}_k^*$. Secondly, we may show that points of $\overline{D}_k^*$ lift to $\overline{\mathrm{H}}_{g,k}$, hence $\overline{D}_k^* \subseteq \overline{D}_k$. Since $\overline{D}_k^* \subseteq U$, we get $\overline{D}_k^* = \overline{D}_k \cap U$.
Clearly, the locus of curves satisfying (b) is closed in the locus of curves $C_1 \cup C_2$ with $C_1 \cap C_2 = \{p\}$, thus $\overline{D}_k^* \subseteq \overline{M}_g$ is a constructible subset, i.e., a finite union of locally closed subsets. Let now $\mathcal{C} \to \operatorname{Spec} R$ be a family of stable curves over a DVR whose generic fibre C_η is in D_k (in particular, smooth) and whose central fibre C_0 is of the form $C_{0,1} \cup C_{0,2}$, $C_{0,1} \cap C_{0,2} = \{p\}$ where the $C_{0,i}$ are irreducible. Letting $\mathcal{F}_\eta$ be the invertible sheaf on C_η of degree k with $h^0(\mathcal{F}_\eta) \geq 2$, note that $\mathcal{F}_\eta$ extends to a torsion-free, reflexive, rank 1 sheaf on $\mathcal{C}$ in an infinite number of ways parametrised by an integer r. More precisely, if $\mathcal{F}$ is one of these extensions, then $\mathcal{F}(-rC_{0,1}) \simeq \mathcal{F}(rC_{0,2})$ are the others.
The completion $\widehat{\mathcal{O}}_{\mathcal{C},p}$ of the local ring at p is isomorphic to

$$\mathbb{C}[\![x, y, t]\!]/(xy - t^n)$$

for some n, where t is the uniformizer of R and $(x,t),(y,t)$ are the ideals corresponding to $C_{0,1}, C_{0,2}$ respectively. The group $\operatorname{Pic}(\operatorname{Spec} \widehat{\mathcal{O}}_{\mathcal{C},p} \setminus \{\mathfrak{m}\})$, where $\mathfrak{m} = t\widehat{\mathcal{O}}_{\mathcal{C},p}$, is given by the restrictions to $\operatorname{Spec} \widehat{\mathcal{O}}_{\mathcal{C},p} \setminus \{\mathfrak{m}\}$ of the ideal sheaves (x, t^r) for $0 \leq r \leq n-1$; in particular, it is cyclic of order n. If the restriction of $\mathcal{F}$ to $\operatorname{Spec} \widehat{\mathcal{O}}_{\mathcal{C},p} \setminus \{\mathfrak{m}\}$ is isomorphic to (x, t^r), then $\mathcal{F}(rC_{0,1})$ is invertible at p. Possibly after replacing $\mathcal{F}$ by this, we may assume that $\mathcal{F}$ is itself invertible at p, hence all the sheaves

$$\ldots, \mathcal{F}(-nC_{0,1}),\ \mathcal{F},\ \mathcal{F}(nC_{0,1}),\ \mathcal{F}(2nC_{0,1}), \ldots$$

are invertible at p as well.
Restricting $\mathcal{F}$ to C_0, we get a pair of torsion-free, rank 1 sheaves $\mathcal{F}_1$ on $C_{0,1}$ and $\mathcal{F}_2$ on $C_{0,2}$, invertible at p, where they are isomorphic. Restricting the twists of $\mathcal{F}$ above, we get pairs $\mathcal{F}_1(rp), \mathcal{F}_2(-rp)$ which glue to a sheaf $\mathcal{F}^{(r)}$ on C_0. Since h^0 is upper-semicontinuous, we also get $h^0(\mathcal{F}^{(r)}) \geq 2$.
Finally, fixing $r_1 > r_2$ such that

$$\begin{aligned} h^0(\mathcal{F}_1(rp)) \geq 2 \qquad & \text{if} \quad r \geq r_1, \quad \text{and} \\ h^0(\mathcal{F}_1(rp)) \geq 1 \qquad & \text{if} \quad r \geq r_2, \end{aligned}$$

then

$$\begin{aligned} h^0(\mathcal{F}^{(r_1)}) \geq 2 &\implies h^0(\mathcal{F}_2(-r_1p)) \geq 1, \quad \text{and} \\ h^0(\mathcal{F}^{(r_2)}) \geq 2 &\implies h^0(\mathcal{F}_2(-r_2p)) \geq 2. \end{aligned}$$

Therefore, if we let $\mathcal{F}_1' = \mathcal{F}_1(r_1p)$, $\mathcal{F}_2' = \mathcal{F}_2(-r_2p)$, and $\ell = r_1 - r_2$, we find that $\mathcal{F}_1', \mathcal{F}_2'$ and ℓ satisfy the conditions of (b), namely $[C_0] \in \overline{D}_k^*$.

It remains to show that points of $\overline{D}_k^*$ lift to $\overline{\mathrm{H}}_{g,k}$. Start with $C = C_1 \cup C_2$, $\mathcal{F}_1$, $\mathcal{F}_2$, and ℓ satisfying (b); in particular $C_1 \cap C_2 = \{p\}$. Take 2 sections of $\mathcal{F}_1$, one of them in $\mathcal{F}_1(-\ell p)$, spanning a subsheaf $\mathcal{F}_1'$, and take 2 sections of $\mathcal{F}_2$, one of them in $\mathcal{F}_2(-\ell p)$, spanning a subsheaf $\mathcal{F}_2'$. Their ratios define morphisms

$$\begin{aligned} \pi_1 \colon C_1' \to \mathbb{P}^1 \quad &\text{of degree } k_1, \quad \text{with ramification index } \ell_1 \text{ at } p,\\ \pi_2 \colon C_2' \to \mathbb{P}^1 \quad &\text{of degree } k_2, \quad \text{with ramification index } \ell_2 \text{ at } p, \end{aligned}$$

where C_i' is the normalization of C_i. To do this, we perform constructions similar to those used before.
If $\ell_1 = \ell_2$, we can simply join them into an admissible covering by identifying p and possibly adding tail $\mathbb{P}^1$'s to make it admissible.

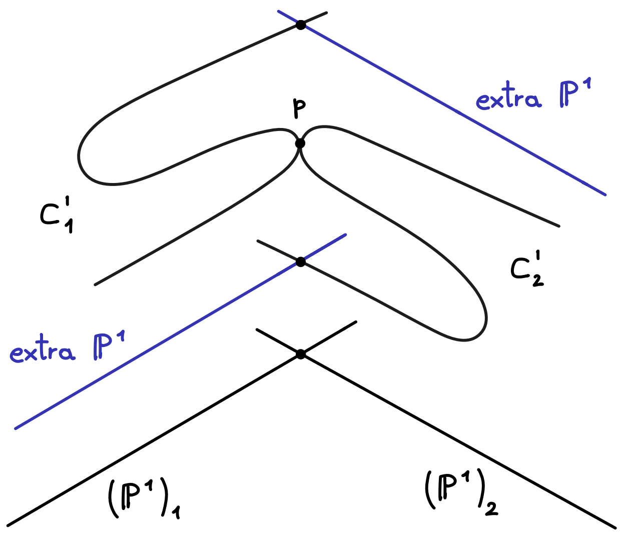


Figure 5.12: Example with $\ell_1 = \ell_2 = 2$ where we add extra $\mathbb{P}^1$'s to make the covering admissible.

Otherwise, we need other $\mathbb{P}^1$'s to join them. More specifically, we add to the base genus 0 curve a *link* $\mathbb{P}^1$, and we add a link $\mathbb{P}^1$ to the covering having degree ℓ_1 over the link $\mathbb{P}^1$ at the base with ramification index ℓ_1 (resp., ℓ_2) at the point $p \in C_1'$ (resp., $p \in C_2'$). Now we need to fill the covering with further $\mathbb{P}^1$'s at multiple branch points of π_1 and π_2, and with link $\mathbb{P}^1$'s to join the two points of C_1' (resp., C_2') over nodes of C_1 (resp., C_2). Finally, we need to increase the degree of the obtained covering up to k as in Step 4. In general, we also add $k_1 - \ell_1$ copies of $\mathbb{P}^1$ to C_1', and $k_2 - \ell_2$ copies to C_2' to extend the other points over the nodes at the base across the corresponding components.

The main question to be discussed is that k is large enough to add all these needed extra curves. In other words, if Z_i denotes the set of nodes of C_i where $\mathcal{F}_i'$

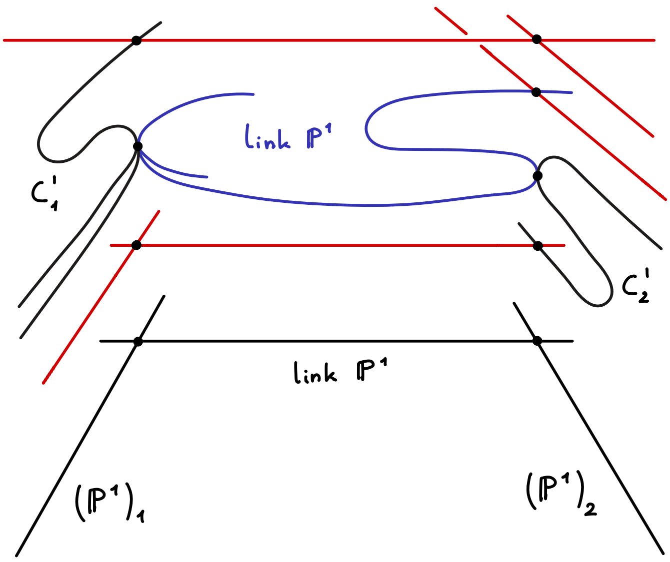


Figure 5.13: Example with $\ell_1 = 3$, $\ell_2 = 2$. Here we must add a link $\mathbb{P}^1$ (in blue) and extra $\mathbb{P}^1$'s (in red) to make the covering admissible.

is not invertible, to perform the constructions above, we need

$$(k_1 - \ell_1)(k_2 - \ell_2) + \max(\ell_1, \ell_2) + \#Z_1 + \#Z_2 \leq k.$$

As computed in the beginning of the proof, we have

$$k_i + \#Z_i = \chi(\mathcal{F}_i') + p_a(C_i) - 1;$$

moreover, since one of the sections of $\mathcal{F}_i$ defining π_i vanishes at p to order at least ℓ, it follows that

$$\begin{aligned} \ell &\leq \ell_i + \operatorname{length}((\mathcal{F}_i)_p/(\mathcal{F}_i')_p) \\ &\leq \ell_i + \chi(\mathcal{F}_i) - \chi(\mathcal{F}_i'). \end{aligned}$$

If $\ell_1 \geq \ell_2$, then

$$\begin{aligned} (k_1 - \ell_1)(k_2 - \ell_2) + \max(\ell_1, \ell_2) + \#Z_1 + \#Z_2 &= \chi(\mathcal{F}_1') + \chi(\mathcal{F}_2') - \ell_2 + p_a(C) - 2 \\ &\leq \chi(\mathcal{F}_1) + \chi(\mathcal{F}_2) - \ell + 2k - 3 \\ &= (3 - k + \ell) - \ell + 2k - 3 \\ &= k. \end{aligned}$$

This concludes the proof. $\square$

Recall the boundary divisors $\delta_0, \dots, \delta_{\lfloor g/2 \rfloor}$ defined in Definition 5.3.1 and the corresponding divisors $\Delta_i = \pi_* \delta_i$ on $\overline{M}_g$, where $\pi\colon \overline{\mathcal{M}}_g \to \overline{M}_g$ is the coarse moduli map. Further restricting to more specific classes of curves, we obtain the following corollaries.

Corollary 5.6.9. *Let* $\operatorname{Int}\Delta_0$ *be the locus of irreducible curves with a single node as in Figure 5.14a, and let* $\operatorname{Int}\Delta_{0,1}$ *be the locus of reducible curves with a single node, whose irreducible components are a smooth curve* C_1 *of genus* $g-1$ *and* $\mathbb{P}^1$ *with* $0, \infty$ *identified, intersecting at a point* p *as in Figure 5.14b.*

(a) (b)

Figure 5.14

(1) Writing curves in $\operatorname{Int}\Delta_0$ *as the quotient of a smooth genus* $g-1$ *curve* C' *by identifying two distinct points* p_1 *and* p_2*, we have*

$$(\operatorname{Int}\Delta_0) \cap \overline{D}_k = \left\{ (C', p_1, p_2) \;\middle|\; \begin{array}{l} \exists \text{ a line bundle } L \text{ on } C' \text{ of degree } k \\ s.t.\ h^0(L) \geq 2,\ h^0(L(-p_1-p_2)) \geq 1 \end{array} \right\},$$

and the map $(\operatorname{Int}\Delta_0) \cap \overline{D}_k \ni (C', p_1, p_2) \mapsto C' \in M_{g-1}$ *is surjective.*

(2) Describing curves in $\operatorname{Int}\Delta_{0,1}$ *by the pair* (C_1, p)*, we have*

$$(\operatorname{Int}\Delta_{0,1}) \cap \overline{D}_k = \left\{ (C_1, p) \;\middle|\; \begin{array}{l} \exists \text{ a line bundle } L \text{ on } C_1 \text{ of degree } k \\ s.t.\ h^0(L) \geq 2,\ h^0(L(-2p)) \geq 1 \end{array} \right\}.$$

Let $S_{k,\ell,g} \subseteq M_{g,1}$ *be the subset of pointed curves* (C,p) *possessing a line bundle* L *of degree* k *with* $h^0(L) \geq 2$ *and* $h^0(L(-\ell p)) \geq 1$*. For* $1 \leq g_1 \leq k-1$ *and* $g_2 = g - g_1$*, let* $\operatorname{Int}\Delta_{g_1}$ *be the locus of curves* $C = C_1 \cup C_2$ *where* C_i *is smooth of genus* g_i *and* $C_1 \cap C_2 = \{p\}$*.*

(3) Observing that $\operatorname{Int}\Delta_{g_1} \simeq M_{g_1,1} \times M_{g_2,1}$*, we have*

$$\begin{aligned} (\operatorname{Int}\Delta_{g_1}) \cap \overline{D}_k = &\bigcup_{\frac{g_1+1}{2} \leq k_1 \leq \min\{k, g_1\}} (S_{k_1, 2k_1-g_1, g_1} \times M_{g_2,1}) \\ \cup &\bigcup_{\frac{g_2+1}{2} \leq k_2 \leq \min\{k, g_2\}} (M_{g_1,1} \times S_{k_2, 2k_2-g_2, g_2}). \end{aligned}$$

Proof. Item (1) follows from part (a) of Theorem 5.6.8 recalling that on smooth curves being torsion-free is equivalent to being locally free [SP, Tag 0CC4]. Indeed, if $\mathcal{F}$ is a line bundle, we have nothing to prove. Otherwise, over the node p we get $\mathcal{F}_p \simeq \mathfrak{m}_p$, and such a sheaf will be the image of a line bundle M on C' of degree $k-1$ such that $h^0(M) \geq 2$. Letting $L = M(p_1)$, we conclude the proof.

For item (2) we use part (b) of the theorem with C_2 the other irreducible component. Such an $\mathcal{F}_2$ exists if and only if

$$\ell = 1,\ \chi(\mathcal{F}_2) \geq 2 \quad \text{or} \quad \chi(\mathcal{F}_2) \geq \ell \geq 2.$$

Therefore $\mathcal{F}_1$ is a line bundle on C_1 with $h^0(\mathcal{F}_1) \geq 2$ satisfying respectively either

$$\deg \mathcal{F}_1 \leq k-1, \quad h^0(\mathcal{F}_1(-p)) \geq 1$$

or

$$\deg \mathcal{F}_1 = k, \quad h^0(\mathcal{F}_1(-2p)) \geq 1.$$

In any case, there will be a line bundle L such that

$$\deg L = k, \quad h^0(L) \geq 2, \quad \text{and} \quad h^0(L(-2p)) \geq 1.$$

Finally, applying again (b), we prove (3). Since the C_i are smooth, the $\mathcal{F}_i$ are line bundles. Letting $k_i = \deg \mathcal{F}_i$, by Riemann-Roch we get

$$(k_1 + 1 - g_1) + (k_2 + 1 - g_2) = \chi(\mathcal{F}_1) + \chi(\mathcal{F}_2) = 3 - k + \ell,$$

hence

$$k_1 + k_2 = 1 - k + \ell + g = k + \ell.$$

Moreover, rephrasing part (b) of the theorem yields

$$(\operatorname{Int} \Delta_{g_1}) \cap \overline{D}_k = \bigcup_{k_1+k_2=k+\ell} (S_{k_1,\ell,g_1} \times S_{k_2,\ell,g_2}). \tag{$\star$}$$

However, this can be simplified observing that

- $2k - \ell - 1 \geq g \implies S_{k,\ell,g} = M_{g,1}$ (see [GH80]), and
- $S_{k,\ell,g} \subseteq S_{k+1,\ell+1,g}$,

since given L of degree k putting (C,p) in $S_{k,\ell,g}$, then $L(p)$ puts (C,p) in $S_{k+1,\ell+1,g}$. Now if $k_1 + k_2 = 1 - k + \ell + g = k + \ell$, necessarily either $2k_1 - \ell - 1 \geq g_1$ or $2k_2 - \ell - 1 \geq g_2$, otherwise $g_i \geq 2k_i - \ell$ would yield

$$2k - 1 = g_1 + g_2 \geq 2k_1 + 2k_2 - 2\ell = 2k,$$

which is a contradiction.
As a consequence, in the union in $(\star)$, one of the 2 factors is always $M_{g_1,1}$ or $M_{g_2,1}$. Furthermore, when $2k_1-\ell-1>g_1$, if we replace k_2 by k_2+1 and ℓ by $\ell+1$, we get

$$\begin{aligned} S_{k_1,\ell,g_1}\times S_{k_2,\ell,g_2} &= M_{g_1,1}\times S_{k_2,\ell,g_2}\\ &= M_{g_1,1}\times S_{k_2+1,\ell,g_2+1}\\ &= S_{k_1,\ell+1,g_1}\times S_{k_2+1,\ell,g_2+1}. \end{aligned}$$

Therefore, in $(\star)$ it suffices to only consider terms with

$$\begin{aligned} 2k_1-\ell-1=g_1,\quad 2k_2-\ell=g_2,\quad \text{or with}\\ 2k_1-\ell=g_1,\quad 2k_2-\ell-1=g_2. \end{aligned}$$

□

Note that for $g_1=1,2$, item (3) of Corollary 5.6.9 gives

$$\begin{aligned} &(\operatorname{Int}\Delta_1)\cap\overline{D}_k = M_{1,1}\times S_{k,2,g-1},\\ &(\operatorname{Int}\Delta_2\cap\overline{D}_k = (S_{k,2,g-1}\times M_{g-2,1})\cup(M_{2,1}\times S_{k,3,g-2})\cup(M_{2,1}\times S_{k-1,1,g-2}). \end{aligned}$$

Combining this with item (2) proves the following.

Corollary 5.6.10. *Letting Δ_1' be the locus of curves $C=C_1\cup C_2$ with C_1 smooth of genus $g-1$ and C_2 smooth or nodal of genus 1 such that $C_1\cap C_2=\{p\}$, that is $\Delta_1'=\operatorname{Int}\Delta_{0,1}\cup\operatorname{Int}\Delta_1$, we get*

$$\Delta_1'\cap\overline{D}_k=\overline{M}_{1,1}\times S_{k,2,g-1}.$$

Intersection multiplicities

In the previous section we have computed some set-theoretic intersections of the divisor $\overline{D}_k$ with the boundary of $\overline{M}_g$. Though in Section 5.5 we have defined the intersection product between classes only over smooth varieties, this can be generalised over any scheme of finite type over a field k, compatibly with the smooth case; see [Ful84, §17]. Actually, intersection theory has been further generalised on DM stacks; see [Vis89]. However, we will always consider intersections outside the singular locus of $\overline{M}_g$.

In order to compute the class of $\overline{D}_k$, we must refine the results above, describing not only set-theoretical intersections, but also their "multiplicities". This requires the two following results, which are consequences of the discussion in [GH80].

Theorem A. *For all $g \geq 1$ and all d such that*

$$g/2 + 1 \leq d \leq g + 1,$$

let

$$a(d, g) = (2d - g - 1)\frac{g!}{d!(g - d + 1)!}.$$

Then for almost all pairs (C, p), where C is a curve of genus g and $p \in C$, there is a finite number $a(d, g)$ of line bundles L on C of degree d such that

$$h^0(L) \geq 2, \quad h^0(L(-(2d - g - 1)p)) \geq 1.$$

Moreover, each L with $h^0(L) = 2$ is globally generated, and $H^0(L)$ defines a branched covering $\pi\colon C \to \mathbb{P}^1$ of degree d which is simply branched except for one branch point with ramification index $(2d - g - 1)$ at p.

Theorem B. *For all $g \geq 1$ and all d such that*

$$g/2 + 1 \leq d \leq g,$$

let

$$b(d, g) = (2d - g - 1)(2d - g)(2d - g + 1)\frac{g!}{d!(g - d)!}.$$

Then for almost all curves C of genus g, there is a finite number $b(d, g)$ of pairs (L, p), where L is a line bundle on C of degree d and $p \in C$, such that

$$h^0(L) \geq 2, \quad h^0(L(-(2d - g)p)) \geq 1.$$

Moreover, each L with $h^0(L) = 2$ is globally generated, and $H^0(L)$ defines a branched covering $\pi\colon C \to \mathbb{P}^1$ of degree d which is simply branched except for one branch point with ramification index $(2d - g)$ at p, when $2d - g \geq 3$.

Proof. See [HM82, Thms A, B]. □

Using these theorems, we can now compute the multiplicities of the intersection product $\operatorname{Int}(\Delta_i) \cdot \overline{D}_k$.

Theorem 5.6.11. *In the notation of Corollary 5.6.9,*

(1) Δ_0 and $\overline{D}_k$ intersect generically transversely, more specifically

$$\operatorname{Int} \Delta_0 \cdot \overline{D}_k = \left\{ (C', p_1, p_2) \;\middle|\; \begin{array}{l} \exists \text{ a line bundle } L \text{ on } C' \text{ of degree } k \\ \text{s.t. } h^0(L) \geq 2,\ h^0(L(-p_1 - p_2)) \geq 1 \end{array} \right\}$$

with multiplicity one;

(2) if $g_1 \geq 3$, we have

$$\begin{aligned}(\operatorname{Int}\Delta_{g_1})\cdot\overline{D}_k = &\sum_{\frac{g_1+1}{2}\leq k_1\leq \min\{k,g_1\}} a(k_1+g_2-k+1,g_2)\,(S_{k_1,2k_1-g_1,g_1}\times M_{g_2,1})\\ &+\sum_{\frac{g_2+1}{2}\leq k_2\leq \min\{k,g_2\}} a(k_2+g_1-k+1,g_1)\,(M_{g_1,1}\times S_{k_2,2k_2-g_2,g_2})\,.\end{aligned}$$

Proof. Let us first prove (1). We claim that in the set-theoretic intersection $(\operatorname{Int}\Delta_0)\cap\overline{D}_k$ there is an open dense consisting of curves $C = C'/(p_1\sim p_2)$ for which

- C' has no automorphisms;
- there is a unique line bundle L on C' of degree k such that $h^0(L) = 2$ and $h^0(L(-p_1-p_2))\geq 1$;
- L is globally generated; and
- if $\pi\colon C'\to\mathbb{P}^1$ is the covering defined by $H^0(L)$, then π is simply branched.

Indeed, by Theorem B, for almost all C' of genus $g-1 = 2k-2$, there are $b(k,2k-2)$ pairs (L,q) such that $\deg L = k$, $h^0(L)\geq 2$, and $h^0(L(-2q))\geq 1$. Each line bundle L_j defines a covering $\pi_j\colon C'\to\mathbb{P}^1$, hence a curve

$$\begin{tikzcd} \Gamma_j \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C' \arrow[d, "\pi_j"] \\ C' \arrow[r, "\pi_j"] & \mathbb{P}^1. \end{tikzcd}$$

Note that Γ_j is a curve in $C'\times_{\mathbb{C}} C'$, and almost all $(p_1,p_2)\in C'\times_{\mathbb{C}} C'$ are on a unique Γ_j if they are on any of them, hence the claim follows.

Furthermore, such a curve $C = C'/(p_1\sim p_2)$ has $[C]$ in the image of a unique point $\pi\colon C''\to P$ of $\overline{\mathrm{H}}_{g,k}$, depicted in Figure 5.15. More precisely, if $b = 3g-1$ the symmetric group $\mathfrak{S}_b$ acts on $\overline{\mathrm{H}}_{g,k}$ permuting the labelling of the b branching points and the admissible covering is the unique point of $\overline{\mathrm{H}}_{g,k}/\mathfrak{S}_b$ over $[C]$. Note that this covering has an automorphism ϕ of order 2 given by

$$\begin{aligned}\phi|_{C'} &= \mathrm{id},\\ \phi|_T &= \text{autom. fixing } q \text{ and interchanging } x_1,\ x_2,\\ \phi|_A &= \text{autom. fixing } p_1,\ p_2 \text{ and interchanging the two branching points.}\end{aligned}$$

However, C'' has no automorphisms over P fixing all the branching points, thus the point $z\in\overline{\mathrm{H}}_{g,k}$ representing $\pi\colon C''\to P$ is smooth. But z is fixed by the transposition $\sigma \in \mathfrak{S}_b$ interchanging x_1 and x_2. More generally, σ fixes the smooth divisor $Z \subseteq \overline{\mathrm{H}}_{g,k}$ of all admissible coverings like $\pi\colon C'' \to P$ but where $x_3, \ldots, x_b$ are allowed to vary, and C' varies accordingly.

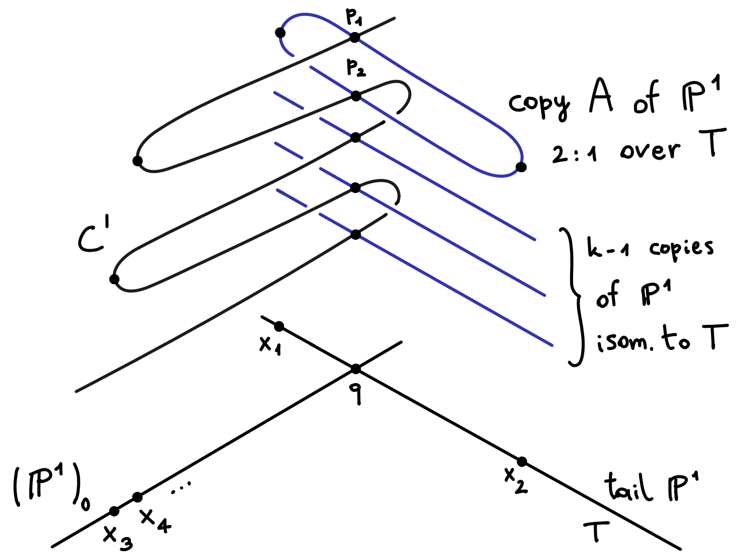


Figure 5.15

Consider now the universal family

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\Psi} & \mathcal{P} \\
& \searrow \quad \swarrow & \\
& \overline{\mathrm{H}}_{g,k}. &
\end{array}
$$

Near z we have local coordinates $t_1, \ldots, t_{b-3}$ on $\overline{\mathrm{H}}_{g,k}$ such that at all points of $\pi^{-1}(q)$, the universal family is given by

$$xy = t_1,$$

and $t_1 = 0$ is the local equation of Z. Moreover

$$
\begin{aligned}
\sigma^*(t_1) &= -t_1 \quad \text{and} \\
\sigma^*(t_i) &= t_i \quad \text{for} \quad 2 \le i \le b-3,
\end{aligned}
$$

hence $t_1^2, t_2, \ldots, t_{b-3}$ are local coordinates on $\overline{\mathrm{H}}_{g,k}/\langle\sigma\rangle$ near the image of z.

Analytically, we can now consider the local curve $\gamma\colon \Delta_s \to \overline{\mathrm{H}}_{g,k}/\langle\sigma\rangle$ given by

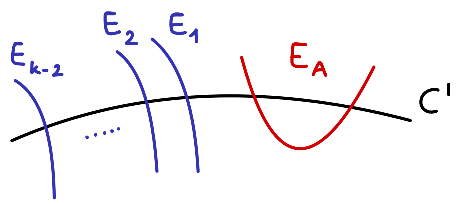
$t_1^2 = s$ and $t_2 = \cdots = t_{b-3} = 0$. We thus have the diagram

$$\begin{tikzcd}
\overline{\mathrm{H}}_{g,k}/\langle\sigma\rangle \arrow[r, two heads, "\lambda"] & \overline{D}_k \arrow[r, hook] & \overline{M}_g \\
\Delta_s \arrow[u, hook, "\gamma"'] & & \operatorname{Int}\Delta_0, \arrow[u, hook]
\end{tikzcd}$$

and we claim that the curve $\lambda \circ \gamma \colon \Delta_s \to \overline{M}_g$ is generically transverse to $\operatorname{Int}\Delta_0$ at $[C]$. This will imply that the divisor $\lambda^*(\operatorname{Int}\Delta_0)$ on $\overline{\mathrm{H}}_{g,k}/\langle\sigma\rangle$ is Δ_s with multiplicity one, hence $\overline{D}_k$ and $\operatorname{Int}\Delta_0$ intersect transversely.
To prove the claim, consider the restriction of $\mathcal{C}$ to $t_2, \ldots, t_{b-3} = 0$, This gives a family of nodal curves

$$\mathcal{C}_1 \to \Delta_{t_1}$$

whose fibres at $t_1 \neq 0$ are smooth. Moreover, the central fibre at $t_1 = 0$ is



and $\mathcal{C}_1$ is a smooth surface as its local equation is $xy = t_1$ and Δ_{t_1} is a smooth curve. Since any smooth fibre $F_{t_1} = (\mathcal{C}_1)_{t_1}$ is a deformation of the central one $F_0 = (\mathcal{C}_1)_0$, for any component Γ of F_0 we get

$$F_0 \cdot \Gamma = F_t \cdot \Gamma = 0,$$

hence for $\Gamma \neq C'$,

$$0 = F_0 \cdot \Gamma = C' \cdot \Gamma + \Gamma^2.$$

In particular,

$$(E_A)^2 = -2, \quad \text{and} \quad (E_1)^2 = \cdots = (E_{k-2})^2 = -1,$$

and contracting these curves as in the proof of Stable Reduction (Theorem 5.1.1), yields a family of stable curves

$$\mathcal{C}_2 \to \Delta_{t_1}.$$

This new family has the same fibres as the previous over $t_1 \neq 0$, and the central fibre is now C. However, observe that contracting E_A produces an ordinary surface double point, namely $\mathcal{C}_2$ has local equation $xy = t_1^2$ at the image of E_A.

Since $\mathcal{C}_2 \to \Delta_{t_1}$ is the pullback to Δ_{t_1} of the family over Δ_s corresponding to $\lambda \circ \gamma \colon \Delta_s \to \overline{M}_g$, it is induced by a family

$$\mathcal{C}_3 \to \Delta_s,$$

where $\mathcal{C}_3$ has local equation $xy = s$ at the image of E_A, hence it is smooth. This last family then gives a curve on $\overline{M}_g$ transverse to the nodal locus Δ_0.

Let us focus now on the proof of (2). Since the discussion for the two sums is similar, we will take a sufficiently general point $[C] \in S_{k_1,2k_1-g_1,g_1} \times \mathcal{M}_{g_2,1}$ and compute the intersection multiplicity at it. Consider $C = C_1 \cup C_2$, where C_1 has genus g_1 and C_2 has genus g_2, with $C_1 \cap C_2 = \{p\}$. By Theorem B, for almost all such C_1, there are $b(k_1, g_1)$ pairs (L_1, q) with $\deg L_1 = k_1$, $h^0(L_1) \geq 2$, and $h^0(L_1(-(2k_1 - g_1)q)) \geq 1$. Since for almost all C_1 the points q are distinct, for almost all $(C_1, p) \in S_{k_1,2k_1-g_1,g_1}$, there is a unique L_1 of degree k_1 with $h^0(L_1) \geq 2$, and $h^0(L_1(-(2k_1 - g_1)p)) \geq 1$. Actually, for this L_1, we have $h^0(L_1) = 2$, hence it defines a covering

$$\pi_1 \colon C_1 \to \mathbb{P}^1$$

which is simply branched except for one branch point with ramification index $(2k_1 - g_1)$ at p.
On the other hand, since (C_2, p) is a general point of $\mathcal{M}_{g_2,1}$, by Theorem A for all k_2 it has exactly $a(k_2, g_2)$ line bundles L_2 with $\deg L_2 = k_2$, $h^0(L_2) \geq 2$, and $h^0(L_2(-(2k_2 - g_2 - 1)p)) \geq 1$. Actually, for all of these, we have $h^0(L_2) = 2$, hence $H^0(L)$ defines a covering

$$\pi_2 \colon C_2 \to \mathbb{P}^1$$

which is simply branched except for one branch point with ramification index $(2k_2 - g_2)$ at p. In particular, there are no line bundles L_2 of degree k_2 with $h^0(L_2) \geq 2$, and $h^0(L_2(-(2k_2 - g_2)p)) \geq 1$.

Letting P be the union of two copies of $\mathbb{P}^1$ glued at a node, it follows that the only ways to lift C to a point of $\overline{\mathrm{H}}_{g,k}$ are to use those admissible coverings $\pi \colon C' \to P$ as in Figure 5.16 below defined by L_1 and L_2 where both have ramification index $\ell = 2k_1 - g_1 = 2k_2 - g_2 - 1$ at p. More precisely, the degree of π is k and over $(\mathbb{P}^1)_1$ we have C_1 of degree k_1 and $k - k_1$ copies of $\mathbb{P}^1$ mapping isomorphically to $(\mathbb{P}^1)_1$; similarly, over $(\mathbb{P}^1)_2$ we have C_2 of degree k_2 and $k - k_2$ copies of $\mathbb{P}^1$ mapping isomorphically to $(\mathbb{P}^1)_2$.
We then necessarily choose k_2 such that

$$2k_1 - g_1 = 2k_2 - g_2 - 1,$$

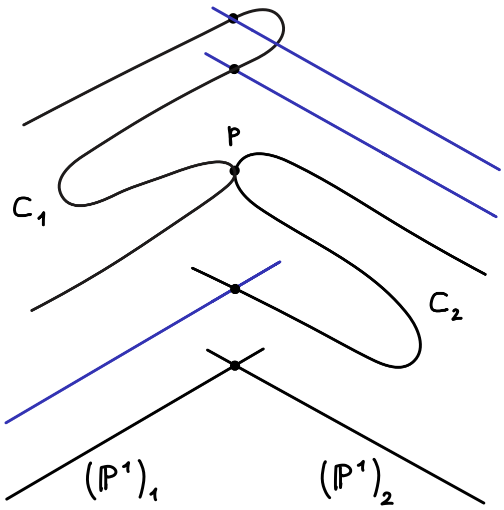


Figure 5.16: Example of admissible covering with $\ell = 2$, $k_1 = 4$, $k_2 = 3$, and $k = 5$.

that is

$$\begin{aligned} k_2 &= k_1 + \frac{g_2 + 1 - g_1}{2} \\ &= k_1 + g_2 + \frac{1 - g}{2} \\ &= k_1 + g_2 - k + 1. \end{aligned}$$

While L_1 is unique, we have $a(k_2, g_2)$ choices for L_2, hence there are $a(k_2, g_2)$ points of $\overline{\mathrm{H}}_{g,k}/\mathfrak{S}_b$ over the admissible covering of C just described. In other words, $\overline{D}_k$ has $a(k_2, g_2)$ branches at $[C]$. It remains to prove that $\operatorname{Int}\Delta_{g_1}$ meets each of these branches transversely.

For $g_1 \geq 3$, we argue similarly to (1). In this case, the covering $\pi\colon C'' \to P$ has no automorphisms, hence $\overline{\mathrm{H}}_{g,k}$ is smooth at the point z representing π, and also there is no $\sigma \in \mathfrak{S}_b$ fixing z. If we embed this covering in a one-dimensional family of admissible coverings as in (1), we get

$$\begin{tikzcd} \mathcal{C}_1 \arrow[rr, "\Psi"] \arrow[dr] & & \mathcal{P} \arrow[dl] \\ & \Delta_1 & \end{tikzcd}$$

which is described, locally near p, as

$$\begin{aligned} \mathcal{C}_1 :& \qquad xy = t, \\ \mathcal{P} :& \qquad uv = t^\ell, \\ \Psi :& \qquad u = x^\ell, \quad v = y^\ell. \end{aligned}$$

In particular, $\mathcal{C}_1$ is a smooth surface, and all the extra rational tails in the central fibre $(\mathcal{C}_1)_0$ are curves $E \subseteq \mathcal{C}_1$ with $E^2 = -1$. Contracting these, we get a family of stable curves

$$\mathcal{C}_2 \to \Delta_t$$

with $\mathcal{C}_2$ still smooth, and the central fibre is now C. This last family then gives a curve on $\overline{M}_g$ transverse to the nodal locus Δ_{g_1} at $[C]$. □

Remark. Item (2) of the theorem above actually holds also for $g_1 = 1, 2$ if the intersection is taken, not in $\overline{M}_g$, but in the miniversal deformation space of a curve in $\operatorname{Int}\Delta_{g_1}$ [HM82, Thm 6]. This slight change is necessary since $\operatorname{Int}\Delta_1$ and $\operatorname{Int}\Delta_2$ meet the singular locus of $\overline{M}_g$, hence we cannot work with the usual intersection product.

Test curves

Recall that in Corollary 5.6.5 we have proven the relation

$$\overline{D}_k = a\lambda - \sum_{i=0}^{k-1} b_i\delta_i$$

In this last section, we finally compute these coefficients. We will determine them by using the technique of restricting this relation to suitable families of stable curves.

First, we describe the general method. Let D be a divisor on $\overline{M}_g$, and assume

$$D = a\lambda - b_0\delta_0 - \cdots - b_{k-1}\delta_{k-1}$$

in the Chow ring. For a smooth projective curve S, let

$$f\colon \mathcal{C} \to S$$

be a family of stable curves of genus g, and denote by

$$\gamma\colon S \to \overline{M}_g$$

the corresponding morphism. If $\gamma(S) \nsubseteq \operatorname{Supp}(D)$, then we may define the divisor γ^*D on S in a canonical way. In fact, for every $s \in S$, let

$$\pi_s\colon \mathcal{D}_s \to \Delta^{3g-3} := \operatorname{Spec}\mathbb{C}[\![t_1, \dots, t_{3g-3}]\!]$$

be the miniversal deformation of the fibre $\mathcal{C}_s$ of f over s. Then there exists a neighbourhood U of s such that γ factors as

$$U \xrightarrow{\gamma_1} \Delta^{3g-3} \xrightarrow{\gamma_2} \overline{M}_g.$$

Furthermore, $\gamma_2^*(D)$ is a divisor on Δ^{3g-3}, hence a Cartier one as Δ^{3g-3} is smooth, and then we can take its pullback via γ_1, defining

$$\gamma^*(D) \coloneqq \gamma_1^*(\gamma_2^* D).$$

This is equivalent to considering the pullback $\pi^*\mathcal{O}_{\overline{M}_g}(D)$ on $\overline{\mathcal{M}}_g$, where $\pi\colon \overline{\mathcal{M}}_g \to \overline{M}_g$ is the coarse moduli map, and then we get a line bundle $L = (\pi^*\mathcal{O}_{\overline{M}_g}(D))_S$ on S. Finally, since S is smooth, the map

$$\begin{aligned} \mathrm{Cl}(S) &\to \mathrm{Pic}(S) \\ [D] &\mapsto \mathcal{O}(D) \end{aligned}$$

is an isomorphism, hence $L = \mathcal{O}_S(D')$ for some divisor D' on S, unique up to linear equivalence; actually, the normalization axiom in Theorem 5.5.3 implies that $D' = c_1(L)$ in the Chow group, thus we get $\gamma^*(D) = c_1(L)$.
In particular,

$$\gamma^*\lambda = c_1(\det f_*\omega_{\mathcal{C}/S}) = c_1(f_*\omega_{\mathcal{C}/S}),$$

and if $\gamma(S) \not\subseteq \Delta_i$, we get

$$\gamma^*\delta_i = \begin{cases} \frac{1}{2}\gamma^*\Delta_1, & \text{if} \quad i = 1 \\ \gamma^*\Delta_i, & \text{if} \quad i \neq 1 \end{cases},$$

otherwise γ factors through the clutching morphism ξ_i defined in Section 5.3 and we get

$$\gamma^*\delta_i = c_1((\mathcal{N}_{\xi_i})_S)$$

where $\mathcal{N}_{\xi_i}$ is the normal sheaf to the clutching morphism. More precisely, such a family $F = (\mathcal{C} \to S)$ in Δ_i comes from an object

- of $\overline{\mathcal{M}}_{g-1,2}$ if $i = 0$, or
- of $\overline{\mathcal{M}}_{g-i,1} \times_{\mathbb{C}} \overline{\mathcal{M}}_{i,1}$ otherwise.

In the first case, F is obtained from a family $(\mathcal{C}' \to S, \sigma_1, \sigma_2) \in \overline{\mathcal{M}}_{g-1,2}$ by glueing its two sections, and it holds

$$(\mathcal{N}_{\xi_0})_S = \sigma_1^*\Omega^\vee_{\mathcal{C}'/S} \otimes \sigma_2^*\Omega^\vee_{\mathcal{C}'/S} = \mathcal{N}_{\sigma_1} \otimes \mathcal{N}_{\sigma_2},$$

where the last equality is due to [SP, Tag 0474].
In the second case, we obtain F by glueing two families $(\mathcal{C}_1' \to S, \sigma_1) \in \overline{\mathcal{M}}_{i,1}$ and $(\mathcal{C}_2' \to S, \sigma_2) \in \overline{\mathcal{M}}_{g-i,1}$ along their sections, and it holds

$$(\mathcal{N}_{\xi_i})_S = \sigma_1^*\Omega^\vee_{\mathcal{C}_1'/S} \otimes \sigma_2^*\Omega^\vee_{\mathcal{C}_2'/S} = \mathcal{N}_{\sigma_1} \otimes \mathcal{N}_{\sigma_2}.$$

We refer to [ACG11, §XIII.3] for the proof of the statements above.

In the following, we will compute the degree $\deg_S D := \deg \gamma^* D$ of the pullback to S for three different families of curves and for the divisors λ, δ_i, and $\overline{D}_k$.

First family. Choose smooth curves C_1 and C_2 of genera $\alpha \leq k-1$ and $2k-1-\alpha$ respectively. Choose then $p \in C_2$ a general point, and let $S_2 = C_1 \times \{p\} \subseteq C_1 \times C_2$. Finally, denote by S_1 the diagonal of $C_1 \times C_1$, and let

$$\pi \colon T \to C_1$$

be the curve obtained by identifying S_1 and S_2 in $(C_1 \times C_1) \cup (C_1 \times C_2)$ over C_1. In other words, $T \to C_1$ is the family whose fibre over $q \in C_1$ is the nodal curve obtained by identifying $q \in C_1$ with $p \in C_2$ as depicted below.

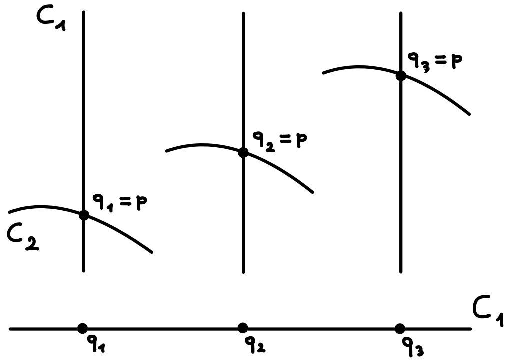


Figure 5.17: First test family.

Now we may compute the degree of λ and δ_i over S. Recall that the stalk of $\pi_* \omega_\pi$ at a point $q \in C_1$ is simply given by the global sections of the sheaf ω_{T_q} over the fibre T_q. The structure of the dualizing bundle of a nodal curve (see Section 2.2) and the Residue Theorem [GH78, §2.1] imply that these sections are exactly given canonically by $H^0(C_1, \omega_{C_1}) \oplus H^0(C_2, \omega_{C_2})$. Since this is independent of the point q, $\pi_*\omega_\pi$ is the trivial bundle $(H^0(C_1, \omega_{C_1}) \oplus H^0(C_2, \omega_{C_2})) \otimes \mathcal{O}_{C_1}$, hence $\deg_{C_1} \lambda = 0$. Clearly, $\deg_{C_1} \delta_i = 0$ for $i \neq \alpha$, and

$$\deg_{C_1} \delta_\alpha = \deg(\mathcal{N}_{\xi_\alpha})_{C_1} = \deg(\sigma_1^* \Omega^\vee_{(C_1 \times C_1)/C_1} \otimes \sigma_2^* \Omega^\vee_{(C_1 \times C_2)/C_1})$$

where $\sigma_i \colon C_1 \to C_1 \times C_i$ is the section defining S_i. Recalling that for $i = 1, 2$

$$\Omega_{(C_1 \times C_i)/C_1} \simeq p_2^* \Omega_{C_i},$$

we get

$$\sigma_1^* \Omega^\vee_{(C_1 \times C_1)/C_1} \otimes \sigma_2^* \Omega^\vee_{(C_1 \times C_2)/C_1} \simeq T_{C_1} \otimes \mathcal{O}_{C_1} \simeq T_{C_1}$$

since $p_2 \circ \sigma_1 = \mathrm{id}_{C_1}$ and $p_2 \circ \sigma_2$ is the constant morphism in $p \in C_2$. Finally,

$$\begin{aligned}\deg_{C_1} \delta_\alpha &= \deg(\mathcal{N}_{\xi_\alpha})_{C_1} \\ &= \deg T_{C_1} \\ &= 2 - 2\alpha.\end{aligned}$$

Regarding $\overline{D}_k$, by Theorem 5.6.8 and Corollary 5.6.9, set-theoretically

$$\mathrm{Supp}(\gamma^* \overline{D}_k) = \left\{ q \in C_1 \;\middle|\; \begin{array}{l} \text{For some } 0 \le i \le (\alpha-1)/2, \\ \exists \text{ a line bundle } L \text{ on } C_1 \text{ of degree } \alpha - i \\ \text{with } h^0(L) \ge 2,\ h^0(L(-(\alpha-2i)q)) \ge 1 \end{array} \right\},$$

which by Theorem B consists in $b(\alpha - i, \alpha)$ points. Moreover, Theorem A implies that the multiplicity of each of these intersection points is $a(k-i, 2k-1-\alpha)$. Putting it all together, we obtain

$$\begin{aligned}\deg_{C_1} \overline{D}_k &= \sum_{i=0}^{(\alpha-1)/2} a(k-i, 2k-1-\alpha)\, b(\alpha-i, \alpha) \\ &= \sum_{i=0}^{\alpha/2} (\alpha-2i-1)(\alpha-2i)^2(\alpha-2i+1) \frac{\alpha!\,(2k-1-\alpha)!}{(\alpha-i)!\,i!\,(k-i)!\,(k-\alpha+i)!} \\ &= \frac{1}{2} \sum_{i=0}^{\alpha} (\alpha-2i-1)(\alpha-2i)^2(\alpha-2i+1) \frac{\alpha!\,(2k-1-\alpha)!}{(\alpha-i)!\,i!\,(k-i)!\,(k-\alpha+i)!},\end{aligned}$$

where the last equality comes from the fact that the sum is unaffected by the substitution $i \mapsto \alpha - i$. Writing $(\alpha-2i-1)(\alpha-2i)^2(\alpha-2i+1)$ as

$$\begin{aligned}&[(k-i) - (k-\alpha+i)] \cdot [(\alpha-i)(\alpha-i-1)(\alpha-i-2) - 3(\alpha-i)(\alpha-i-1)i \\ &+ 3(\alpha-i)i(i-1) - i(i-1)(i-2) + 3(\alpha-i)(\alpha-i-1) - 3i(i-1)],\end{aligned}$$

the sum becomes

$$\begin{aligned}
\frac{\alpha(\alpha-1)(\alpha-2)}{2} \sum_{i=0}^{\alpha} \Bigg[& \binom{\alpha-3}{i}\binom{2k-1-\alpha}{k-i-1} - 3\binom{\alpha-3}{i-1}\binom{2k-1-\alpha}{k-i-1} \\
& + 3\binom{\alpha-3}{i-2}\binom{2k-1-\alpha}{k-i-1} - \binom{\alpha-3}{i-3}\binom{2k-1-\alpha}{k-i-1} \\
& - \binom{\alpha-3}{i}\binom{2k-1-\alpha}{k-i} + 3\binom{\alpha-3}{i-1}\binom{2k-1-\alpha}{k-i} \\
& -3\binom{\alpha-3}{i-2}\binom{2k-1-\alpha}{k-i} - \binom{\alpha-3}{i-3}\binom{2k-1-\alpha}{k-i}\Bigg] \\
+\frac{3\alpha(\alpha-1)}{2} \sum_{i=0}^{\alpha} \Bigg[& \binom{\alpha-2}{i}\binom{2k-1-\alpha}{k-i-1} - \binom{\alpha-2}{i-2}\binom{2k-1-\alpha}{k-i-1} \\
& -\binom{\alpha-2}{i}\binom{2k-1-\alpha}{k-i} + \binom{\alpha-2}{i-2}\binom{2k-1-\alpha}{k-i}\Bigg],
\end{aligned}$$

hence

$$\begin{aligned}
\deg_{C_1} \overline{D}_k = & \frac{\alpha(\alpha-1)(\alpha-2)}{2}\left[-6\binom{2k-4}{k-2} + 8\binom{2k-4}{k-3} - 2\binom{2k-4}{k-4}\right] \\
& + \frac{3\alpha(\alpha-1)}{2}\left[\binom{2k-3}{k-2} - \binom{2k-3}{k-3}\right] \\
= & -6\alpha(\alpha-1)(\alpha-2)\frac{(2k-4)!}{k!\,(k-2)!} + 6\alpha(\alpha-1)\frac{(2k-3)!}{k!\,(k-2)!} \\
= & \, 6\alpha(\alpha-1)(2k-1-\alpha)\frac{(2k-4)!}{k!\,(k-2)!}.
\end{aligned}$$

On the other hand,

$$\deg_{C_1} \overline{D}_k = b_\alpha(\deg_{C_1} \delta_\alpha) = b_\alpha(2-2\alpha),$$

thus, for $\alpha \geq 2$,

$$b_\alpha = \frac{3\alpha(2k-1-\alpha)}{k} \cdot \frac{(2k-4)!}{(k-1)!\,(k-2)!}. \tag{5.7}$$

Similarly, if we choose to vary the point $p \in C_2$ and fix $q \in C_1$, we obtain a family $\pi\colon T \to C_2$ with

$$\begin{aligned}
\deg_{C_2} \lambda &= 0 \\
\deg_{C_2} \delta_i &= 0 \quad \text{for} \quad i \neq \alpha, \\
\deg_{C_2} \delta_\alpha &= -2(2k-2-\alpha).
\end{aligned}$$

We may compute again

$$\begin{aligned}\deg_{C_2} \overline{D}_k &= \sum_{i=0}^{(\alpha+1)/2} b(k-i, 2k-1-\alpha)\, a(\alpha-i+1, \alpha)\\ &= \sum_{i=0}^{(\alpha+1)/2} \frac{\alpha!\,(2k-1-\alpha)!}{(\alpha-i+1)!\,i!\,(k-i)!\,(k-\alpha+i-1)!}\\ &\qquad \cdot (\alpha-2i)(\alpha-2i+1)^2(\alpha-2i+2),\end{aligned}$$

and setting $\beta = \alpha+1$, this becomes

$$\begin{aligned}&\sum_{i=0}^{\beta/2} \frac{(\beta-1)!\,(2k-\beta)!}{(\beta-i)!\,i!\,(k-i)!\,(k-\alpha+i)!}(\beta-2i-1)(\beta-2i)^2(\beta-2i+1)\\ &= \frac{2k-\beta}{\beta}\cdot 6\beta(\beta-1)(2k-\beta-1)\frac{(2k-4)!}{k!\,(k-2)!}\\ &= 6\alpha(2k-\alpha-1)(2k-\alpha-2)\frac{(2k-4)!}{k!\,(k-2)!}.\end{aligned}$$

We then conclude that

$$b_\alpha = \frac{\deg_{C_2} \overline{D}_k}{2(2k-2-\alpha)} = \frac{3\alpha(2k-1-\alpha)}{k}\cdot\frac{(2k-4)!}{(k-1)!\,(k-2)!},$$

which holds, unlike (5.7), also for $\alpha = 1$, yielding

$$b_1 = \frac{6(k-1)}{k}\cdot\frac{(2k-4)!}{(k-1)!\,(k-2)!}. \tag{5.8}$$

Second family. We consider a family consisting of a fixed smooth curve C_2 of genus $g-1$, plus a variable elliptic curve E attached to a fixed point of C_2. More precisely, let $(\pi_1\colon X \to B, \sigma_1)$ be family of stable curves of genus 1, where B is a proper curve and X a smooth surface, and let also $S_1 \coloneqq \sigma_1(B) \subseteq X$ be the image of the section. If $p \in C_2$ is a general point, set $S_2 = B \times \{p\} \subseteq B \times C_2$, and define

$$\pi\colon U \to B$$

as the curve over B obtained by identifying $S_1 \subseteq X$ and $S_2 \subseteq B \times C_2$ in the disjoint union $X \cup (B \times C_2)$.

Let us compute the degrees on B. First of all, we get

$$\deg \delta_1 = \deg \sigma_1^* \Omega_{X/B}^\vee$$

where the computation is carried out like for the first family. Note that the image of the induced morphism $B \to \overline{M}_g$ intersects Δ_0 exactly for points $b \in B$ such that the fibre X_b is nodal. The Noether's formula (see Corollary 5.5.7) provides the relation

$$12\chi(X, \mathcal{O}_X) = K_X^2 + e(X),$$

and from the theory of elliptic fibrations (see [Bar+04]) we know that the class K_X is a multiple of the generic fibre, hence $K_X^2 = 0$. Note that $e(X)$ is exactly the number of nodal fibres of π_1, since smooth ones have topological Euler characteristic 0. On the other hand, the GRR formula (see Theorem 5.5.6) yields

$$\int_B \mathrm{ch}((\pi_1)_! \mathcal{O}_X) \cdot \mathrm{td}(B) = \int_X \mathrm{ch}(\mathcal{O}_X) \cdot \mathrm{td}(X) = \chi(X, \mathcal{O}_X),$$

where the last equality is the Noether's formula. The term on the left is

$$\int_B \mathrm{ch}((\pi_1)_! \mathcal{O}_X) \cdot \mathrm{td}(B) = \int_B (\mathrm{ch}(\mathcal{O}_B) - \mathrm{ch}(R^1(\pi_1)_* \mathcal{O}_X)) \cdot \mathrm{td}(B).$$

Moreover, by Grothendieck duality [Liu02, §6.4.3],

$$\mathcal{H}om_{\mathcal{O}_B}(\mathrm{R}^1(\pi_1)_* \mathcal{O}_X, \mathcal{O}_B) \simeq (\pi_1)_* \omega_{X/B},$$

and $\mathrm{R}^1(\pi_1)_* \mathcal{O}_X$ is a vector bundle [Har77, Cor. III.12.9], hence $\mathrm{R}^1(\pi_1)_* \mathcal{O}_X \simeq (\pi_1)_* \omega_{X/B}^\vee$, implying

$$\mathrm{ch}(R^1(\pi_1)_* \mathcal{O}_X) = 1 - \lambda$$

since the Hodge line bundle on B with respect to π is

$$\begin{aligned}
\det(\pi_* \omega_{U/B}) &= \det((\pi_1)_* \omega_{X/B} \oplus f_* \omega_{(B \times C_2)/B}) \\
&= \det((\pi_1)_* \omega_{X/B}) \otimes \det(f_* \omega_{(B \times C_2)/B}) \\
&= \det((\pi_1)_* \omega_{X/B}),
\end{aligned}$$

where $f : B \times C_2 \to B$ is the trivial family. Therefore, we get

$$\begin{aligned}
\chi(X, \mathcal{O}_X) &= \int_B (\mathrm{ch}(\mathcal{O}_B) - \mathrm{ch}(R^1(\pi_1)_* \mathcal{O}_X)) \cdot \mathrm{td}(B) \\
&= \int_B \lambda(1 + c_1(T_B)) \\
&= \deg \lambda,
\end{aligned}$$

yielding

$$12 \deg \lambda = e(X) = \deg \delta_0.$$

Finally, observe that $\sigma_1^*\Omega_{X/B} = \sigma_1^*\omega_{X/B}$ since σ_1 does not meet the nodal locus of fibres; moreover, we have the evaluation map

$$\phi\colon \pi^*(\pi_*\omega_{X/B}) \to \omega_{X/B},$$

and restricting it on fibres we get an isomorphism, as they have trivial dualizing bundle (genus 1), then ϕ itself is an isomorphism. This implies

$$\begin{aligned}\sigma_1^*\Omega_{X/B} &= \sigma_1^*\omega_{X/B}\\ &= \sigma_1^*(\pi^*\pi_*)\omega_{X/B}\\ &= \pi_*\Omega_{X/B},\end{aligned}$$

proving that $\deg\lambda = -\deg\delta_1$. Clearly, $\deg\delta_\alpha = 0$ for $\alpha \geq 2$.
On the other hand, by Corollary 5.6.10 it follows that the image of B in $\overline{M}_g$ corresponding to π is disjoint from $\overline{D}_k$, since a general C_2 possesses no line bundles L of degree $k-1$ having $h^0(L) = 2$, and there are only finitely many of degree k; also p, being general, will not be a branch point of the associated coverings $C \to \mathbb{P}^1$. Therefore, we conclude that

$$a(\deg\lambda) - b_0(\deg\delta_0) - b_1(\deg\delta_1) = 0,$$

implying

$$a - 12b_0 + b_1 = 0. \tag{5.9}$$

Third family. It follows the last construction. Let C be a general curve of genus $g-1 = 2k-2$, and set $p \in C$ a general point. Let S_1 and S_2 be the proper transforms of respectively the diagonal Δ and the section $C \times \{p\}$ in the blow-up $\widetilde{C\times C}$ of $C \times C$ at (p,p). Since S_1 and S_2 are disjoint, we may identify them in $\widetilde{C\times C}$ over C. This procedure yields a family

$$\pi\colon W \to C$$

of stable curves of genus g over C.
For this family, we clearly find

$$\begin{aligned}&\deg\delta_\alpha = 0 \quad \text{for} \quad \alpha > 1,\\ &\deg\delta_1 = 1, \quad \text{and}\\ &\deg\delta_0 = \deg(\mathcal{N}_{S_1/\widetilde{C\times C}}) + \deg(\mathcal{N}_{S_12/\widetilde{C\times C}})\end{aligned}$$

where

$$\begin{aligned}\deg(\mathcal{N}_{S_1/\widetilde{C\times C}}) &= \deg(\mathcal{N}_{\Delta/(C\times C)}) - 1, \quad \text{and}\\ \deg(\mathcal{N}_{S_2/\widetilde{C\times C}}) &= \deg(\mathcal{N}_{(C\times\{p\})/(C\times C)}) - 1\end{aligned}$$

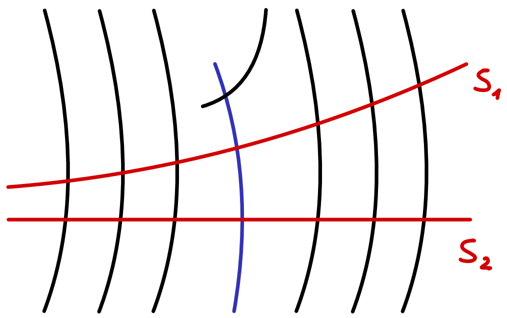


Figure 5.18: Surface $\widetilde{C \times C}$ with the exceptional divisor (in blue). To construct the third test family, we identify sections S_1 and S_2.

from [Ful84, Ex. 4.3.9]. We then find

$$\deg \delta_0 = (2 - 2(2k-2) - 1) - 1 = 4 - 4k.$$

Moreover, we have the short exact sequence

$$0 \to H^0(C, \omega_C) \otimes \mathcal{O}_C \to \pi_* \omega_{W/C} \to \mathcal{O}_C \to 0$$

where the last map takes the residues at S_1, implying

$$\deg \lambda = 0.$$

Note that since C and $p \in C$ are general, Theorem A states that C possesses exactly $\frac{(2k-2)!}{(k-1)!\,k!}$ line bundles L of degree k with $h^0(L) = 2$, each of which having a unique section zero at p, and none of these sections has a multiple zero. Therefore, there are $\frac{(2k-2)!}{(k-1)!\,k!}(k-1)$ points $q \in C$ such that $h^0(L(-p-q)) = 1$ for one of these line bundles. Furthermore, Theorem 5.6.11 ensures us that these points occur with multiplicity one in the divisor given by the pullback of $\overline{D}_k$ on C, hence

$$\deg \overline{D}_k = \frac{(2k-2)!}{(k-2)!\,k!},$$

and this gives us the last relation

$$4(k-1)b_0 - b_1 = \frac{(2k-2)!}{(k-2)!\,k!}, \tag{5.10}$$

We can finally prove the last result needed to conclude the proof of Theorem 5.6.1.

Theorem 5.6.12. *The divisor $\overline{D}_k \subseteq \overline{M}_g$ has class*

$$[\overline{D}_k] = c\left(6(k+1)\lambda - k\delta_0 - \sum_{i=1}^{k-1} 3i(2k-1-i)\delta_i\right)$$

in $A^\bullet(\overline{M}_g)$, where $c = \frac{(2k-4)!}{k!(k-2)!}$.

Proof. Combining the relations (5.8) and (5.10), we deduce

$$\begin{aligned}
b_0 &= \frac{1}{4(k-1)}\left(\frac{(2k-2)!}{k!\,(k-2)!} + \frac{6(k-1)}{k}\cdot\frac{(2k-4)!}{(k-1)!\,(k-2)!}\right)\\
&= \frac{1}{4}\cdot\frac{(2k-4)!}{k!\,(k-2)!}(2(2k-3)+6)\\
&= \frac{(2k-4)!}{(k-1)!\,(k-2)!},
\end{aligned}$$

and the relation (5.9) gives us

$$\begin{aligned}
a &= 12b_0 - b_1\\
&= \left(12 - \frac{6(k-1)}{k}\right)\frac{(2k-4)!}{(k-1)!\,(k-2)!}\\
&= 6(k+1)\frac{(2k-4)!}{k!\,(k-2)!}.
\end{aligned}$$

Summing up, we conclude that

$$\begin{aligned}
a &= 6(k+1)c,\\
b_0 &= kc, \quad \text{and}\\
b_\alpha &= 3\alpha(2k-\alpha-1)c \quad \text{for} \quad \alpha > 0.
\end{aligned}$$

□

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