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MEASUREMENTS AND STATE UPDATES IN QUANTUM FIELD THEORY

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Abstract

In recent years, a fully covariant framework was developed by Fewster and Verch to describe measurement processes in quantum field theory (QFT), resolving long-standing causality issues arising from trying to adapt standard non-relativistic rules to QFT. This framework is described in terms of the algebraic formulation of QFT (AQFT), which is a naturally well-suited approach for theories on curved spacetimes. While it is known that, in certain models, the class of non-selective measurements preserves the singularity structure associated with so-called Hadamard quantum states, the effect of selective measurements on these states is not yet fully established. This thesis presents a first approach to complete the picture. Techniques from microlocal analysis, distribution theory, and operator algebras are employed to study how the act of performing a selective measurement on a quantum field alters the singularity structure of its Hadamard states. It is shown that there are multiple ways to set up selective measurements, conditioned on local effects of the quantum Weyl algebra, that preserve the Hadamard condition of the preparation state. Conversely, analytical limits where the Hadamard condition is lost are also identified. Examples carrying relevant physical interpretations are presented alongside the development of these general results. Finally, an explicit model is chosen to carry out perturbative computations to analyze the extent to which a measurement process induces variations in the energy density of a specific QFT.

Contents

Introduction

A foundational aspect of quantum mechanics is its reformulation of the concepts of observables and states in a probabilistic framework, in contrast to the deterministic notions of classical mechanics. This feature of quantum theories led to a totally new understanding of the idea of physical measurement, culminating in a fully coherent mathematical formulation which provided practical rules to compute expectation values of quantum observables and, as a pure quantum novelty, the concept of post-measurement state [Neu32]. In the modern language, a quantum mechanical measurement is described in terms of effect operators modeling success/fail type of probabilistic tests [NC10]. In particular, this modeling is based on a system and probe paradigm, introducing the notions of state updates and quantum instruments [Bus+16]. This is actually an extremely natural way of modeling a physical experiment: a system, of which one wants to measure a physical property, interacts with a measuring apparatus (the probe) providing the actual experimental outcome that can be registered. The quantum measurement framework then describes procedures that infer information about the system for a given outcome of the probe, in terms of so-called induced system observables. This kind of mathematical modeling of quantum measurements serves as a rigorous and complete way of understanding physical experiments on quantum systems. The validity of the original version of this formulation, however, is limited to experimental regimes in which relativistic effects are irrelevant.

Quantum field theory (QFT) is the correct framework that allows one to incorporate the principles of special relativity into quantum mechanical systems. In this setting, it would be desirable to have an analogous theory of quantum measurements in order to rigorously describe physical experiments in QFT. The first natural guess would be to extend the rules of quantum measurements of non-relativistic quantum mechanics. This attempt, however, immediately generates inconsistencies with the causal structure of relativity, for if one tries to apply the standard rules to compute expectation values and state updates, even the most basic experiments would result in superluminal transfer of information [Sor93]. In recent developments [FV20], a QFT generalization of the quantum measurement framework was presented, which was shown to be free of such causal inconsistencies.

This new framework was set up in the algebraic formulation of QFT (AQFT). AQFT has been developed as an alternative, mathematically rigorous and axiomatic formulation for QFTs. Its key advantage resides in the fact that it ditches the need for an explicit Hilbert space on which the states and the observables of the QFT are represented. An immediate consequence is the absence of a preferred quantum state like the Minkowski vacuum. For this reason, this formulation is generally seen as the natural one for the study of QFTs on curved spacetimes, where the notion of vacuum is ill-defined. On curved spacetimes, the lack of such a state corresponds to the loss of a natural normal ordering procedure and, with it, a standard renormalization scheme for the expectation values of observables. Historically, this suggested the introduction of a class of states for which a generalization of such a procedure can be defined in a systematic and state-independent way also for curved spacetimes. This class of states is characterized by what is today known as the Hadamard condition [DB60; Chr78; KW91].

The aim of this thesis is to analyze the interplay between the quantum measurement framework of QFT and Hadamard states. This is a physical problem of great relevance due to the fact that one would ideally want to prepare a system in a physical (Hadamard) state and obtain an updated post-measurement state that is also physical after performing a measurement. This would provide a robust way to rigorously describe measurement schemes for physical phenomena on curved spacetimes of a genuine quantum field theoretical nature, such as the Unruh and Hawking effects [Unr76; Haw75]. In this thesis it is shown that there exist measurement schemes, in the sense of [FV20], that preserve the Hadamard condition of the initial quantum states in which the quantum fields are prepared for the experiment. This extends the case study of [Few25] from the class of purely non-selective measurements to more general selective ones. In addition to being a natural extension of the previous result, this also allows one to construct quantum measurements with physically meaningful descriptions and interpretations.

This thesis is organized as follows:

- **Chapter 1: Algebraic Quantum Field Theory**
  A review of the modern formulation of AQFT is given, starting from the physical observation that motivates the advantages of an algebraic description of physical systems. The causal structure of the general relativistic spacetime on which the quantum field lives is highlighted to show how it can be encoded in the algebras of quantum observables. The type of classical field theories considered in this work is then introduced. These theories are chosen for their broad physical relevance and for some properties that are extremely useful to define a systematic quantization procedure.

- **Chapter 2: Hadamard states and their Hilbert space representation**
  This chapter introduces the concept of Hadamard states. The original physical motivations for their definition are presented, together with the main applications in QFT. The singularity structure of Hadamard states is described both from a geometric and a microlocal perspective, highlighting the fundamental link between the two viewpoints, first established in [Rad96]. The Hadamard condition is finally translated into a more concrete Hilbert space language, which will turn out to be extremely useful for the study of quantum measurements.

- **Chapter 3: The AQFT measurement framework**
  The AQFT measurement framework is presented, together with the notion of state update in its selective and non-selective versions. An explicit example where the standard rules of quantum measurement fail is provided. Finally, a recent result from [Few25] is presented, showing how post-measurement states preserve the Hadamard condition of the original pre-measurement ones for non-selective measurements.

- **Chapter 4: Selective Hadamard state updates**
  This chapter contains the original results of this thesis. First, the groundwork for the study of selective measurements is set up. The main result of Chapter 3 is then extended to the significantly broader class of selective measurements and explicit constructions and examples are given. Finally, these results are employed in an explicit perturbative computation of the energy variation of a system that is undergoing a quantum measurement.

1 Algebraic Quantum Field Theory

1.1 Physical systems and their mathematical modeling

The modern approach to the subject of quantum mechanics is usually presented in terms of the first rigorous mathematical formulation of quantum theories, due to J. von Neumann [Neu32]. His work made extensive use of tools from functional analysis and operator theory establishing a solid ground on which the notions of quantum states, quantum observables and experimental measurement of a quantum system could be described in mathematical terms. In this framework the primary objects that are assigned to a quantum system are a Hilbert space and operators acting on it. In some sense, this introduces an explicit representation of states and observables, for example in terms of vectors and matrices in particularly simple cases. Subsequent works by I. Gel'fand, M. Naimark [GN43] and I.E. Segal [Seg47] showed that the introduction of such explicit objects was actually superfluous. The entire description of a quantum system can indeed be encoded in the algebra of its observables, without necessarily specifying how they act on some space. This point of view took its final form thanks to the work of R. Haag and D. Kastler [HK64] when it was applied to reformulate quantum field theory in terms of nets of observable algebras. This formulation of quantum field theory is known as algebraic quantum field theory (AQFT). Before reviewing the construction of a quantum field theory in this algebraic way, it is instructive trying to make sense of why physical systems are so well suited to such a description.

Any physical system is fully characterized by the set of experiments that can be performed on it. The notion of physical experiment is extremely primitive, but it can be appropriately stated as follows:

Definition 1.1.1 (*Experiment*)**.** An experiment on a physical system consists of the following subprocesses:

(i) Preparation of the system in a well defined condition which is called *physical state*.

(ii) Measurement of some quantity associated with the system which is called *observable*.

Denoting the set of all possible physical states $\mathcal{S}$ and all possible observables $\mathcal{O}$ associated with some physical system, it is natural to identify the set of all possible experiments on that system with $\mathcal{S} \times \mathcal{O}$.

The first logical step in the study of a physical system is to construct an appropriate mathematical model for it. Here appropriate means that it is expected that this mathematical model is able to replicate and predict experimental results. According to Definition 1.1.1 then, it is crucial that any one of these models is able to assign a mathematical object to the notion of physical state and observable. It is surprising that very few assumptions on the properties of experiments lead to a common structure underlying the mathematical description of physical systems. To better understand this, recall the standard mathematical description of classical and quantum systems.

Definition 1.1.2 (*Classical system*)**.** In the Hamiltonian formulation of classical mechanics, a classical system is associated with a symplectic manifold $(\mathcal{M}, \sigma)$ such that:

(i) Pure physical states of the system are described by points $s \in \mathcal{M}$.

(ii) Measurable quantities of the physical system are described by real phase functions, i.e. smooth real functions on the symplectic manifold $f \in C^\infty(\mathcal{M}; \mathbb{R})$.

Definition 1.1.3 (*Quantum system*)**.** A quantum system is associated with a Hilbert space $\mathcal{H}$ such that:

(i) Pure physical states of the system are described by rays in $\mathcal{H}$, i.e. equivalence classes of normalized vectors $\psi \in \mathcal{H}$ that differ by a phase.

(ii) Measurable quantities of the physical system are described by bounded selfadjoint operators $A^* = A \colon \mathcal{H} \to \mathcal{H}$.

It is clear that classical and quantum systems are generally described by different mathematical objects, however it is reasonable to expect that observables should obey some set of common rules as they describe physical quantities regardless of the context. Such rules are needed to define algebraic-like operations that are desirable in an experimental setting, like the composition of observables modeling a subsequent measurement of different physical quantities. These constraints give rise to a minimal structure of specific types of algebras.

Definition 1.1.4 (C^*-*algebra*)**.** An associative algebra is a vector space $\mathcal{A}$ with a product $\cdot \colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that $\forall A, B, C \in \mathcal{A}$, $\lambda \in \mathbb{C}$:

(i) $(A \cdot B) \cdot C = A \cdot (B \cdot C)$

(ii) $\lambda(A \cdot B) = (\lambda A) \cdot B = A \cdot (\lambda B)$

(iii) $(A + B) \cdot C = A \cdot C + B \cdot C,\ A \cdot (B + C) = A \cdot B + A \cdot C$

An associative algebra can be further characterized by the following properties:

- $\mathcal{A}$ is *unital* if $\exists \mathbb{1}_\mathcal{A} \in \mathcal{A}$ such that $\mathbb{1}_\mathcal{A} A = A \mathbb{1}_\mathcal{A} = A$.
- $\mathcal{A}$ is *normed* if there is a norm function $\| \cdot \| \colon \mathcal{A} \to \mathbb{R}$ such that $\|AB\| \le \|A\| \|B\|$. If $\mathcal{A}$ is unital, the unit is required to satisfy $\|\mathbb{1}_\mathcal{A}\| = 1$.
- $\mathcal{A}$ is *Banach* if it is complete with respect to its norm.
- $\mathcal{A}$ is a $*$-*algebra* if there is an involution $* \colon \mathcal{A} \to \mathcal{A}$ such that:
  (i) $(A + B)^* = A^* + B^*,\ (\lambda A)^* = \bar{\lambda} A^*$
  (ii) $(AB)^* = B^* A^*$
  (iii) $(A^*)^* = A$
- $\mathcal{A}$ is a C^*-*algebra* if it is a Banach $*$-algebra such that $\|A^* A\| = \|A\|^2$.

It is very easy to check that the space of smooth real functions on a symplectic manifold $C^\infty(\mathcal{M};\mathbb{R})$ and the space of bounded operators on a Hilbert space $\mathcal{B}(\mathcal{H})$ satisfy all the properties of a $*$-algebra (a C^* one in particular). This suggests that, in general, observables of a physical system can be mathematically modeled by a $*$-algebra. Additional structures on it will then result in different kinds of observables, but this description is completely general and independent even of the classical or quantum framework.

Remark. Notice that nothing has been said about whether or not a $*$-algebra is commutative. Clearly one core difference between a classical and a quantum system lies in the commutativity of the associated $*$-algebra.

This very brief review is meant to serve as a motivation for the introduction of the algebraic formulation of quantum mechanics and, in particular, of quantum field theory. A complete and detailed presentation of the standard mathematical formulation of quantum mechanics can be found for example in [Mor17].

1.2 Algebraic formulation and Locally Covariant Quantum Field Theories

The algebraic formulation of a quantum theory [FR19; Mor17] formalizes the ideas of Section 1.1. Its goal is to mathematically model a quantum system in terms of a unital $*$-algebra whose selfadjoint elements are to be interpreted as the system observables. As highlighted by Definition 1.1.1, these objects constitute only half of the full picture of a physical system. The physical states of the quantum system are mathematically modeled by linear functionals on the algebra associated to the system. The algebra, together with the set of physical states, fully reproduces the standard Hilbert space formulation, in the sense that it implements a way to compute expectation values of observables when the system is in a given physical state. Before diving into the algebraic formulation of quantum field theories, it is useful to review the main features of the algebraic formulation of quantum theories in general, including the non relativistic ones.

Definition 1.2.1 (*Algebraic quantum system*)**.** A quantum system is associated to a unital $*$-algebra $\mathfrak{A}$ called *algebra of observables*. The system is characterized by:

(i) Selfadjoint elements of $\mathfrak{A}$, i.e. $A = A^* \in \mathfrak{A}$ called *observables*.

(ii) Normalized positive linear functionals on its algebra of observables, i.e. $\omega\colon \mathfrak{A} \to \mathbb{C}$ such that $\omega(A^*A) \geq 0 \quad \forall A \in \mathfrak{A}$ and $\omega(\mathbb{1}_{\mathfrak{A}}) = 1$, called *states*.

The quantity $\omega(A)$ is interpreted as the *expectation value* of the observable A in the system state ω.

The relation between the two formulations can be appropriately formalized, it turns out that every physical state in the algebraic formulation gives rise to a full Hilbert space on which the expectation values of observables can be computed in the standard way. The link between abstract algebraic formulation and the concrete Hilbert space one is given by the following theorem due to Gel'fand-Naimark-Segal (GNS theorem).

Theorem 1.2.1. *Let $\mathfrak{A}$ be a unital $*$-algebra and ω a state on it, then there exists a quadruple $(\mathcal{H}_\omega, D_\omega, \pi_\omega, \Omega_\omega)$ consisting of a Hilbert space $\mathcal{H}_\omega$, a subspace $D_\omega \subset \mathcal{H}_\omega$ a linear map $\pi_\omega \colon \mathfrak{A} \to \mathcal{L}(D_\omega, \mathcal{H}_\omega)$ from the algebra to the space of linear operators acting on $\mathcal{H}_\omega$ with domain D_ω and $\Omega_\omega \in D_\omega, \|\Omega_\omega\|_{\mathcal{H}_\omega} = 1$ such that:*

(i) $\pi_\omega(\mathfrak{A})\Omega_\omega = D_\omega$ *and* D_ω *is dense in* $\mathcal{H}_\omega$*, making* Ω_ω *a cyclic vector.*

(ii) π_ω *is a* $*$*-algebra homomorphism, i.e.* $\pi_\omega(AB) = \pi_\omega(A)\pi_\omega(B), \pi_\omega(\mathbb{1}_\mathfrak{A}) = \mathbb{1}_{\mathcal{L}(D_\omega,\mathcal{H}_\omega)}$ *and* $\pi_\omega(A^*) = \pi_\omega(A)^*$ *on* D_ω *for any* $A, B \in \mathfrak{A}$*.*

(iii) $\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle$ *for any* $A \in \mathfrak{A}$*.*

If $\mathfrak{A}$ *is in particular a* C^**-algebra, then* $\mathrm{Ran}(\pi_\omega) \subset \mathcal{B}(\mathcal{H}_\omega)$ *and* $D_\omega = \mathcal{H}_\omega$*.*

Proof. See e.g. [Mor17, Th. 14.20]. □

What the GNS theorem is saying is that any algebraic state on a unital $*$-algebra gives rise to a representation of the algebra of observables on some Hilbert space. Most importantly, there is a unit vector in the Hilbert space that can be used to compute expectation values of observables in the original algebraic state. In more explicit terms, this means that observables are in general described by unbounded operators defined on a dense subspace of the Hilbert space. This is expected as most selfadjoint operators representing physical observables are necessarily unbounded in quantum mechanics. The GNS theorem gives additional constraints when $\mathfrak{A}$ is a C^*-algebra. The main result of having a unit vector to compute expectation values and a representation of the algebra of observables are obviously still there, as a C^*-algebra is a particular case of a $*$-algebra, however the representation π_ω takes now values in the bounded operators on $\mathcal{H}_\omega$. This is the main difference between choosing to work with a quantum algebra having a C^* structure. Bounded operators have more controlled behavior with respect to unbounded ones, so that they are easier to work with and can be employed in explicit calculations, on the other hand many physical observables are by nature represented by unbounded operators. This is reflected by the fact that it is possible to implement two main quantization procedures in the algebraic sense. Given a classical system it is possible to assign either a $*$-algebra or a C^*-algebra to it.

Definition 1.2.2 (*Folium of* ω)**.** Let ω be an algebraic state of a unital $*$-algebra $\mathfrak{A}$ and $(\mathcal{H}_\omega, D_\omega, \pi_\omega, \Omega_\omega)$ be its GNS representation. Let also ρ be a positive trace-class operator on $\mathcal{H}_\omega$ with unit trace (a density matrix), then the *folium* of the GNS representation is the set of states ω' on $\mathfrak{A}$ such that:

$$\omega'(A) = \mathrm{Tr}(\rho\pi_\omega(A)) \quad \forall A \in \mathfrak{A} \tag{1.2.1}$$

The folium of a GNS representation is essentially the set of states that can be reached by applying operators on the vector Ω_ω. One of the most commonly used representation in physics is the vacuum one associated to an appropriately defined vacuum state.

Turning now the attention to relativistic theories, the algebraic formulation of quantum field theory has the great advantage of being naturally suited to describe quantum fields on curved spacetimes. The following definitions recap the main features of the general relativistic description of spacetime [BGP08; Wal84]. A review of basic tools from differential geometry can be found in Appendix A.

Definition 1.2.3 (*Spacetime*)**.** A *spacetime* $(\mathcal{M}, g, \mathfrak{t}, \mathfrak{o})$, usually denoted just by $\mathcal{M}$, is a triple consisting of:

(i) A 4-dimensional smooth paracompact Lorentzian metric manifold $(\mathcal{M}, g)$ with finitely many components.

(ii) A time orientation $\mathfrak{t}$, i.e. a nowhere vanishing timelike 1-form on $\mathcal{M}$.

(iii) An orientation $\mathfrak{o}$, i.e. a nowhere vanishing 4-form on $\mathcal{M}$.

The metric and the time orientation endow $\mathcal{M}$ with a causal structure, given a subset $S \subset \mathcal{M}$ it is possible to define:

$$\begin{aligned} J^{\pm}(S) \coloneqq \{x \in \mathcal{M} \,|\exists \gamma \colon [a,b] \to \mathcal{M} \text{ future}(+)/\text{past}(-) \text{ directed causal curve} \\ \text{such that } \gamma(a) \in S, \gamma(b) = x\} \end{aligned} \tag{1.2.2}$$

The sets $J^{\pm}(S)$ are called *causal future*$(+)$*/past*$(-)$ *of* S.

Definition 1.2.4 (*Global hyperbolicity*)**.** A spacetime $\mathcal{M}$ is said to be *globally hyperbolic* if there are no closed causal curves in $\mathcal{M}$ and for any compact subset $K \subset \mathcal{M}$ the intersection $J^+(K) \cap J^-(K)$ is compact.

Definition 1.2.5 (*Causal sets*)**.** A subset $S \subset \mathcal{M}$ of a globally hyperbolic spacetime is said to be:

(i) *Causally convex* if it is equal to its *causal hull* $\mathrm{ch}(S) \coloneqq J^+(S) \cap J^-(S) = S$.

(ii) *Future*$(+)$*/past*$(-)$ *compact* if it is closed and $S \cap J^{\pm}(x)$ is compact $\forall x \in \mathcal{M}$.

(iii) *Spatially compact* if it is closed and $S \subset J^+(K) \cup J^-(K) \coloneqq J(K)$ for some compact $K \subset \mathcal{M}$.

(iv) A *region* if it is open, causally convex and has finitely many components. A region is by itself a globally hyperbolic spacetime.

A compact set $K \subset \mathcal{M}$ is always past/future/spatially compact but not vice versa. Fig. 1.2.1 qualitatively represents the causal future and past of a causally convex subset of Minkowski spacetime.

Definition 1.2.6 (*Cauchy surface*)**.** A smooth curve parametrized on an open interval $\gamma \colon (a,b) \to \mathcal{M}$ is said to be *inextendible* if there are no $p, q \in \mathcal{M}$ such that $\lim_{t\to a^+} \gamma(t) = p$ and $\lim_{t\to b^-} \gamma(t) = q$. A subset $S \subset \mathcal{M}$ is said to be a *Cauchy surface* if every inextendible timelike curve in $\mathcal{M}$ meets S in exactly one point.

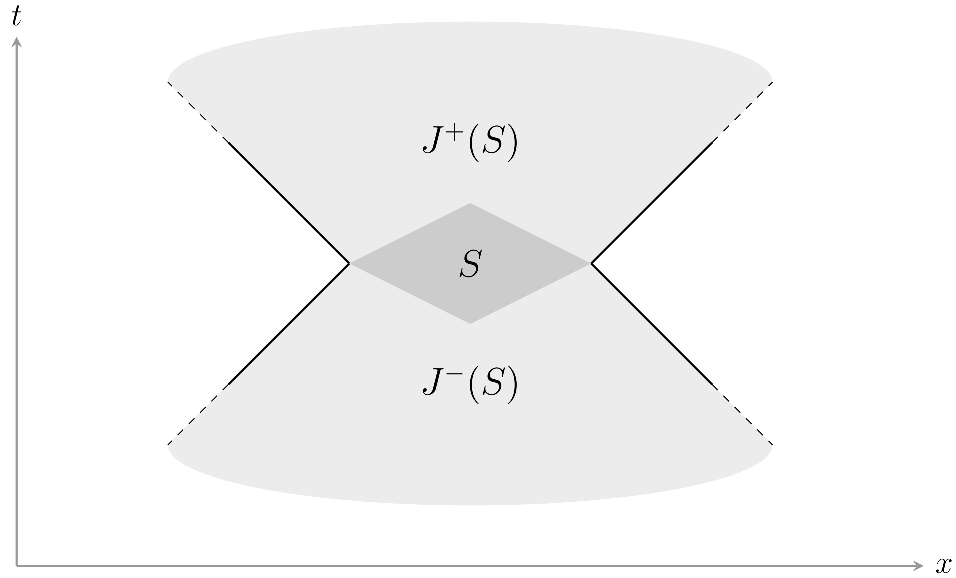

Figure 1.2.1: Causal future and causal past of a causally convex set S in Minkowski spacetime.

A quantum field is a particular type of quantum system, as such it can be associated to an algebra of observables. A proper assignment must evidently take into account the causal structure of the underlying spacetime in order for the quantum theory to be relativistically invariant. In these terms, one can intuitively think of a quantum field theory as some sort of rule that assigns an algebra of observables to a given spacetime. To formulate this idea in a clear and rigorous way it is possible to organize spacetimes and algebras into categories, the notion of assignment is then understood as a functor between categories. A collection of the main definitions from category theory can be found in Appendix B.

Definition 1.2.7 (**Loc** *category*)**.** The **Loc** *category* has globally hyperbolic spacetimes $(\mathcal{M}, g, \mathfrak{t}, \mathfrak{o})$ as objects and smooth isometric embeddings $\psi\colon (\mathcal{M}, g, \mathfrak{t}, \mathfrak{o}) \to (\mathcal{M}', g', \mathfrak{t}', \mathfrak{o}')$ such that $\psi(\mathcal{M}) \subset \mathcal{M}'$ is causally convex and the pullbacks of the orientations satisfy $\psi^*\mathfrak{t}' = \mathfrak{t}, \psi^*\mathfrak{o}' = \mathfrak{o}$ as morphisms. A **Loc** morphism $\psi\colon \mathcal{M} \to \mathcal{M}'$ is said to be *Cauchy* if $\psi(\mathcal{M})$ contains a Cauchy surface of $\mathcal{M}'$.

Definition 1.2.8 (**Alg** *category*)**.** The **Alg** *category* has unital $*$-algebras $\mathfrak{A}$ as objects and injective, unit preserving $*$-algebra homomorphisms as morphisms i.e. maps $\alpha\colon \mathfrak{A} \to \mathfrak{A}'$ such that $\alpha(AB) = \alpha(A)\alpha(B), \alpha(\mathbb{1}_{\mathfrak{A}}) = \mathbb{1}_{\mathfrak{A}'}$ and $\alpha(A^*) = \alpha(A)^*$ for all $A, B \in \mathfrak{A}$. Each **Alg** morphism defines a pullback on states as $(\alpha^*\omega)(A) := \omega(\alpha(A))$ for any $A \in \mathfrak{A}$ and ω state on $\mathfrak{A}'$.

Definition 1.2.9 (**C*-Alg** *category*)**.** The **C*-Alg** category is the subcategory of **Alg** whose objects are unital C^*-algebras and morphisms are **Alg** morphisms between C^*-algebras.

Definition 1.2.10 (*Locally covariant QFT*)**.** A *locally covariant quantum field theory* (LCQFT) is a functor $\mathscr{L}\colon$ **Loc** $\to$ **Alg**.

(i) A LCQFT is said to be *causal* or to satisfy *Einstein causality* if for every pair of

Loc morphisms $\psi_i \colon \mathcal{M}_i \to \mathcal{M}$ for $i = 1, 2$ such that $\psi_1(\mathcal{M}_1) \cap J(\psi_2(\mathcal{M}_2)) = \varnothing$ one has:

$$[\mathscr{L}(\psi_1)(A_1), \mathscr{L}(\psi_2)(A_2)] = 0 \quad \forall A_{1,2} \in \mathscr{L}(\mathcal{M}_{1,2}) \tag{1.2.3}$$

(ii) A LCQFT satisfies the *time-slice axiom* if $\mathscr{L}(\psi)$ is a **Alg** isomorphism for all Cauchy **Loc** morphisms $\psi \colon \mathcal{M} \to \mathcal{M}'$.

The definition of locally covariant quantum field theory perfectly incorporates the main features of relativity into the algebra of observables. The causality condition states that whenever two observables A_1, A_2 on spacetimes $\mathcal{M}_1, \mathcal{M}_2$ are interpreted as observables on a background spacetime $\mathcal{M}$ in which the embeddings of $\mathcal{M}_1$ and $\mathcal{M}_2$ are causally disconnected, then those two observables must commute, i.e. no information can travel faster than light. In particular, by Definition 1.2.5, a region is by itself a spacetime so that the causality condition of Definition 1.2.10 is interpreted as the non interference of measurements in causally disconnected regions of spacetime. On the other hand, the time-slice axiom states that the algebra of observables that is assigned on a Cauchy surface is the same as the one assigned to the whole spacetime. This is interpreted as the fact that assigning data on a Cauchy surface determines the evolution of that data on the entire spacetime.

Remark. The way that **C*-Alg** category has been defined makes sure that Definition 1.2.10 implements both a $*$-algebra and a C^*-algebra type of observables as discussed previously.

This formalization of quantum field theories automatically implements the concept of inequivalent theories.

Definition 1.2.11 (*Equivalent LCQFTs*)**.** Let $\mathscr{L}, \mathscr{L}' \colon$ **Loc** $\to$ **Alg** be two locally covariant quantum field theories. $\mathscr{L}$ and $\mathscr{L}'$ are said to be *equivalent theories* if there exists a natural transformation between them such that its component are isomorphisms. Equivalent theories are physically indistinguishable.

A practical example of inequivalent theories is given by two Klein-Gordon theories with different mass terms [BFV03].

In this work, locally covariant quantum field theories will be constructed in two main steps. The first one is to define a functor that assigns a certain class of classical field theories to a spacetime. The second one is to apply standard quantization functors that assign an algebra of observables to a classical field theory. Since the composition of two functors is again a functor, composing these two steps is enough to build a locally covariant quantum field theory. Appropriately defining such functors automatically results in a locally covariant quantum field theory that is both causal and satisfies the time-slice axiom. Further details about locally covariant quantum field theories can be found in the original paper [BFV03] and [FR16; FV15].

1.3 RFHGHO classical field theories

To introduce concrete models of quantum field theories, it is natural to start from an in-depth analysis of their classical counterpart. A classical field theory on a spacetime $\mathcal{M}$

is completely determined by a Lagrangian density $\mathcal{L}(\phi, \nabla_\mu \phi)$, which is a function of the field and its derivatives, specified in more details below. Euler-Lagrange equations can be employed to derive the equations of motion of the field, these are in general partial differential equations of the form $P\phi = 0$ where P is some linear partial differential operator. It is customary to identify a classical field theory with the partial differential operator appearing in its equations of motion.

Example 1.3.1. Some notable examples of classical field theories include:

(i) **Free complex Klein-Gordon field:** its equations of motion are $(\Box + m^2)\phi = 0$ where $\Box$ is the covariant wave operator on $\mathcal{M}$.

(ii) **Free Dirac field:** its equations of motion are $(\gamma^\mu \nabla_\mu + m)\psi = 0$ where γ^μ are the 4-dimensional gamma matrices.

(iii) **Complex Klein-Gordon field with pointlike source:** its equations of motion are $(\Box + m^2)\phi = \delta^4(x)$.

From these extremely basic and standard examples one can draw the following conclusions:

(i) Fields can take values in different vector spaces: the complex Klein-Gordon field is $\mathbb{C}$-valued, the Dirac field is spinor-valued.

(ii) In general, fields are not necessarily smooth functions on the spacetime manifold $\mathcal{M}$: the Klein-Gordon field with pointlike source is evidently a distribution.

It becomes clear how the natural mathematical setting to study the behavior of classical field theories is that of distributions on vector bundles. A direct consequence is that the study of the linear partial differential operator P will involve the theory of linear operators on the space of distributions on vector bundles, which is the topic of this section [Hör90; SV01; BGP08]. A brief review of the main facts about distributions on vector bundles can be found in Appendix A.

Definition 1.3.1 (*Hermitian vector bundle*)**.** Let $\mathcal{M}$ be a globally hyperbolic spacetime, a *Hermitian vector bundle* $(B, \langle\cdot,\cdot\rangle_x, \mathcal{C})$, usually denoted just by B, is a triple consisting of:

(i) A smooth finite rank complex vector bundle $\pi\colon B \to \mathcal{M}$.

(ii) A Hermitian bundle metric $\langle\cdot,\cdot\rangle_x$.

(iii) A fiberwise antilinear involution $\mathcal{C}$ such that $\langle \mathcal{C}u, \mathcal{C}v\rangle_x = \overline{\langle u, v\rangle}_x \quad \forall u, v \in \pi^{-1}(x)$ for any $x \in \mathcal{M}$.

The Hermitian bundle metric defines a sesquilinear pairing on the space of smooth sections of B:

$$\begin{aligned} \langle\cdot,\cdot\rangle\colon \Gamma^\infty(B) \times \Gamma^\infty(B) &\to \mathbb{C} \\ (f, g) &\mapsto \langle f, g\rangle := \int_{\mathcal{M}} \langle f(x), g(x)\rangle_x \, \mathrm{dV} \end{aligned} \tag{1.3.1}$$

for f, g with compactly intersecting supports, where dV is the metric induced volume form on $\mathcal{M}$. $\mathcal{C}$ naturally extends to an antilinear involution on $\Gamma^\infty(B)$.

Remark. From now on $\Gamma^\infty_{0/\mathrm{fc}/\mathrm{pc}/\mathrm{sc}}(B)$ will denote space of smooth sections of a Hermitian vector bundle B with compact/future compact/past compact/spatially compact support respectively. The absence of subscripts will correspond to a generic support type.

Definition 1.3.2 (*Adjoint, transpose and dual*)**.** Let B_1, B_2 be Hermitian vector bundles, with $\langle \cdot, \cdot \rangle_{1,2}$ being their respective sesquilinear pairings as in Definition 1.3.1, and $P\colon \Gamma^\infty(B_1) \to \Gamma^\infty(B_2)$ be a linear partial differential operator, the following operators can be defined:

(i) The *formal Hermitian adjoint* ${}^\dagger P\colon \Gamma^\infty(B_2) \to \Gamma^\infty(B_1)$ is the unique linear map such that $\langle f_1, {}^\dagger P f_2\rangle_1 = \langle P f_1, f_2\rangle_2 \quad \forall f_i \in \Gamma^\infty(B_i)$.

(ii) The *formal transpose* ${}^t P\colon \Gamma^\infty(B_2) \to \Gamma^\infty(B_1)$, defined as ${}^t P := \mathcal{C}_1 {}^\dagger P \mathcal{C}_2$. It is also possible to define the *formal conjugate* as $\overline{P} := \mathcal{C}_2 P \mathcal{C}_1$. The property $\langle f_1, {}^t P f_2\rangle_1 = \langle \overline{P} f_1, f_2\rangle_2 \quad \forall f_i \in \Gamma^\infty(B_i)$ is easily verified.

(iii) The *formal dual* ${}^\star P\colon \Gamma^\infty(B_2^*) \to \Gamma^\infty(B_1^*)$ is the unique linear map such that $[{}^\star P(F)](f_1) = F(P f_1) \quad \forall F \in \Gamma^\infty(B_2^*), f_1 \in \Gamma^\infty(B_1)$.

In the case where $B_1 = B_2$, the linear map P is said to be *formally Hermitian* if ${}^\dagger P = P$, *formally symmetric* if ${}^t P = P$ and *real* if $\overline{P} = P$. If P is real ${}^t P = {}^\dagger P$.

There is a particular class of classical field theories whose partial differential operator has certain useful properties related to the causal structure of the spacetime. These are called Green-hyperbolic operators [BGP08; Bär15], and the associated classical field theories turn out to be particularly well-suited to undergo a general quantization procedure. They will be the object of study in this work. The cases in Example 1.3.1 are, among others, all instances of Green-hyperbolic operators.

Definition 1.3.3 (*Green-hyperbolic operator*)**.** Let B be a Hermitian vector bundle over $\mathcal{M}$ and $P\colon \Gamma^\infty(B) \to \Gamma^\infty(B)$ be a linear partial differential operator. P is said to be *semi-Green-hyperbolic* if there are linear maps $E^\pm_P\colon \Gamma^\infty_0(B) \to \Gamma^\infty_{\mathrm{sc}}(B)$ such that $\forall f \in \Gamma^\infty_0(B)$:

(i) $P E^\pm_P f = f$

(ii) $E^\pm_P P f = f$

(iii) $\mathrm{supp}(E^\pm_P f) \subset J^\pm(\mathrm{supp}(f))$

The linear maps $E^\pm_P$ are called *advanced*$(-)$*/retarded*$(+)$ *Green operators for* P. If $P, {}^t P, {}^\dagger P, {}^\star P$ are all semi-Green-hyperbolic, then P is said to be *Green-hyperbolic*. The operator $E_P := E^-_P - E^+_P$ is called *causal propagator*.

What follows is a list of useful properties of Green operators and their associated causal propagator [Bär15].

Theorem 1.3.1. *Let B be a Hermitian vector bundle over $\mathcal{M}$ and $P\colon \Gamma^\infty_0(B) \to \Gamma^\infty_0(B)$ be a Green-hyperbolic operator, then:*

(i) The Green operators $E^{\pm}_P$ are unique and continuous.

(ii) $\overline{E^{\pm}_P} = E^{\pm}_{\overline{P}}$ and ${}^tE^{\pm}_P = E^{\mp}_{{}^tP}$.

(iii) If $\mathcal{M}$ is globally hyperbolic the following complex is exact:

$$\Gamma^\infty_0(B) \xrightarrow{P} \Gamma^\infty_0(B) \xrightarrow{E_P} \Gamma^\infty_{sc}(B) \xrightarrow{P} \Gamma^\infty_{sc}(B) \tag{1.3.2}$$

i.e. any composition of two maps gives 0 *and the range of each map is exactly the kernel of the one that follows.*

(iv) If $N \subset \mathcal{M}$ is a causally convex open set containing a Cauchy surface of $\mathcal{M}$, then any $f \in \Gamma^\infty_0(B)$ can be written as $f = h + Pg$ where $h \in \Gamma^\infty_0(B|_N), g \in \Gamma^\infty_0(B)$.

Proof. **(i)** Consider the advanced Green operator E^-_P, let $f \in \Gamma^\infty_{\mathrm{fc}}(B), x \in \mathcal{M}$ and $\chi \in C^\infty_0(\mathcal{M})$ with $\chi = 1$ in a neighborhood of $J^+(x) \cap \operatorname{supp}(f)$. Define the linear extension of E^-_P to $\Gamma^\infty_{\mathrm{fc}}(B)$ as:

$$\begin{aligned}\widetilde{E^-_P} \colon \Gamma^\infty_{\mathrm{fc}}(B) &\to \Gamma^\infty_{\mathrm{fc}}(B)\\ f(x) &\mapsto \widetilde{E^-_P}(f) \coloneqq [E^-_P(\chi f)](x)\end{aligned} \tag{1.3.3}$$

Since $J^+(y) \subset J^+(x) \quad \forall y \in J^+(x)$ the same cutoff function χ works both for x and y thus $\widetilde{E^-_P}(f) = [E^-_P(\chi f)](x)$ is a smooth section, because any point in $\mathcal{M}$ is contained in $J^+(x)$ for some $x \in \mathcal{M}$. Moreover:

$$\operatorname{supp}(\widetilde{E^-_P}(f)) \subset \bigcup_\chi \operatorname{supp}(E^-_P(\chi f)) \subset \bigcup_\chi J^-(\operatorname{supp}(\chi f)) \subset J^-(\operatorname{supp}(f)) \tag{1.3.4}$$

showing that $\widetilde{E^-_P}(f)$ has indeed range in $\Gamma^\infty_{\mathrm{fc}}(B)$. The linear extension also satisfies the relations in Definition 1.3.3:

$$[P\widetilde{E^-_P}(f)](x) = [\chi f](x) = f(x) \tag{1.3.5}$$

is immediate, however

$$\begin{aligned}[\widetilde{E^-_P}P(f)](x) &= [E^-_P(\chi P f)](x)\\ &= f(x) + [E^-_P([\chi, P]f)](x)\end{aligned} \tag{1.3.6}$$

The second term vanishes because it is not supported on x, this can be seen as follows:

$$\begin{aligned}\operatorname{supp}(E^-_P([\chi, P]f)) &\subset J^-(\operatorname{supp}([\chi, P]f))\\ &\subset J^-(\operatorname{supp}(f) \setminus J^+(x))\\ &\subset J^-(\operatorname{supp}(f)) \setminus \{x\}\end{aligned} \tag{1.3.7}$$

so that $[\widetilde{E^-_P}P(f)](x) = f(x)$. This shows that $\widetilde{E^-_P}$ is the inverse of the linear operator P on $\Gamma^\infty_{\mathrm{fc}}(B)$, thus it is unique. E^-_P, being a restriction of $\widetilde{E^-_P}$, is also unique. An analogous argument proves the uniqueness of E^+_P. Continuity follows from the fact that for any $K \subset \mathcal{M}$ future compact $J^-(J^-(K)) = J^-(K)$ so that $P|_{\Gamma^\infty_{J^-(K)}(B)}$ is a bijection with inverse $\left.\widetilde{E_P^-}\right|_{\Gamma^\infty_{J^-(K)}(B)}$ on a complete space with respect to the C^k norm (see Definition A.13) making them continuous. The continuous embedding of $\Gamma^\infty_0(B)$ into $\Gamma^\infty_{J^-(K)}(B)$ makes $\left.\widetilde{E_P^-}\right|_{\Gamma^\infty_0(B)} = E_P^-$ continuous, an analogous argument holds for E_P^+.

(ii) By definition:

$$\begin{aligned}\overline{P}\,\overline{E_P^\pm} f &= \mathcal{C}P\mathcal{C}\mathcal{C}E_P^\pm\mathcal{C}f \\ &= \mathcal{C}\mathcal{C}f = f\end{aligned} \tag{1.3.8}$$

Similarly $\overline{E_P^\pm}\,\overline{P} f = f$, moreover since $\operatorname{supp}(\mathcal{C}h) = \operatorname{supp}(h) \quad \forall h \in \Gamma^\infty_0(B)$:

$$\begin{aligned}\operatorname{supp}(\mathcal{C}E_P^\pm\mathcal{C}f) &= \operatorname{supp}(E_P^\pm\mathcal{C}f) \\ &\subset J^\pm(\operatorname{supp}(\mathcal{C}f)) \\ &= J^\pm(\operatorname{supp}(f))\end{aligned} \tag{1.3.9}$$

so that $\overline{E_P^\pm} = E_{\overline{P}}^\pm$. Consider then:

$$\begin{aligned}\langle {}^tP\,{}^tE_P^\pm f, g\rangle &= \langle \mathcal{C}\,{}^\dagger P\,{}^\dagger E_P^\pm \mathcal{C}f, g\rangle \\ &= \overline{\langle {}^\dagger P\,{}^\dagger E_P^\pm \mathcal{C}f, \mathcal{C}g\rangle} \\ &= \overline{\langle \mathcal{C}f, \mathcal{C}g\rangle} = \langle f, g\rangle\end{aligned} \tag{1.3.10}$$

Similarly ${}^tE_P^\pm\,{}^tPf = f$, moreover consider $g \in \Gamma^\infty_0(B)$ such that $\operatorname{supp}(g) \cap J^\mp(\operatorname{supp}(f)) = \varnothing$:

$$\begin{aligned}\langle {}^tE_P^\pm f, g\rangle &= \langle E_P^\pm(\mathcal{C}g), \mathcal{C}f\rangle \\ &= \int_{\mathcal{M}} \langle E_P^\pm(\mathcal{C}g)(x), \mathcal{C}f(x)\rangle_x \,\mathrm{dV}\end{aligned} \tag{1.3.11}$$

however, $\operatorname{supp}(\mathcal{C}f) \cap \operatorname{supp}(E_P^\pm\mathcal{C}g) = \operatorname{supp}(f) \cap J^\pm(\operatorname{supp}(g)) = \varnothing$ because by hypothesis $\operatorname{supp}(g) \cap J^\mp(\operatorname{supp}(f)) = \varnothing$. This means that:

$$\langle {}^tE_P^\pm f, g\rangle = \int_{\mathcal{M}} \langle {}^tE_P^\pm f(x), g(x)\rangle_x \,\mathrm{dV} = 0 \tag{1.3.12}$$

Since g is arbitrary $\operatorname{supp}({}^tE_P^\pm f) \subset J^\mp(\operatorname{supp}(f))$ so that ${}^tE_P^\pm = E_{{}^tP}^\mp$.

(iii) Let $f \in \Gamma^\infty_0(B)$ such that $E_Pf = 0$, i.e. $f \in \ker(E_P)$, the goal is to show that f is of the form Pg for some $g \in \Gamma^\infty_0(B)$. Consider $g = E_P^+f = E_P^-f \in \Gamma^\infty_{\mathrm{sc}}(B)$ and $\operatorname{supp}(g) = \operatorname{supp}(E_P^-f) \cap \operatorname{supp}(E_P^+f) \subset J^-(\operatorname{supp}(f)) \cap J^+(\operatorname{supp}(f))$. Since $\mathcal{M}$ is globally hyperbolic $\operatorname{supp}(g)$ is compact meaning that $g \in \Gamma^\infty_0(B)$. Since $Pg = PE_P^+f = f$ one has $f \in P\Gamma^\infty_0(B)$ showing the exactness at $\Gamma^\infty_0(B)$. To show the exactness at $\Gamma^\infty_{\mathrm{sc}}(B)$ let $g \in \Gamma^\infty_{\mathrm{sc}}(B)$ such that $Pg = 0$, i.e. $g \in \ker(P)$, the goal is to show that g is of the form E_Pf for some $f \in \Gamma^\infty_0(B)$. Let $g \coloneqq g_- - g_+$ where $\operatorname{supp}(g_\pm) \subset J^\pm(K)$ for some compact subset $K \subset \mathcal{M}$. Then define $f \coloneqq Pg_- = Pg_+$ with $\operatorname{supp}(f) \subset J^+(K) \cap J^-(K)$ which is compact due to $\mathcal{M}$ being globally hyperbolic. Consider now an auxiliary $h \in \Gamma^\infty_0(B)$:

$$\begin{aligned}\langle h, E_P^-Pg_-\rangle &= \langle {}^\dagger P\,{}^\dagger E_P^- h, g_-\rangle \\ &= \overline{\langle \mathcal{C}\,{}^\dagger P\mathcal{C}\mathcal{C}\,{}^\dagger E_P^-\mathcal{C}\mathcal{C}h, \mathcal{C}g_-\rangle} \\ &= \overline{\langle {}^tPE_{{}^tP}^+\mathcal{C}h, \mathcal{C}g_-\rangle} \\ &= \langle h, g_-\rangle\end{aligned} \tag{1.3.13}$$

This shows that $E_P^- f = g_-$, similarly it can be shown $E_P^+ f = g_+$ thus $E_P f = E_P^- f - E_P^+ f = g_- - g_+ = g$ for $f \in \Gamma_0^\infty(B)$.

(iv) Consider an open set $U \subset N$ still containing the Cauchy surface and a real valued cutoff function $\chi \in C^\infty(\mathcal{M})$ such that $\chi = 0$ on a neighborhood of $\mathcal{M} \setminus J^-(U)$ and $\chi = 1$ on a neighborhood of $\mathcal{M} \setminus J^+(U)$. Notice that such a cutoff function can exist because U contains a Cauchy surface. Since χ is constant outside U and P is a partial differential operator one has $[P, \chi]E_P \colon \Gamma_0^\infty(B) \to \Gamma_0^\infty(B|_N)$ and $PE_P = 0$ on $\Gamma_0^\infty(B)$ due to (iii). Considering that f is compactly supported one can rewrite $f = \chi P(E_P^+ f) + (1-\chi)P(E_P^- f)$, then rearranging:

$$f - [P, \chi]E_P f = P(\chi E_P^+ f + (1-\chi)E_P^- f) \in P\Gamma_0^\infty(B) \tag{1.3.14}$$

This means that $f = [P, \chi]E_P f + P(\chi E_P^+ f + (1-\chi)E_P^- f)$ is decomposed in a section supported in N and a section which is in $P\Gamma_0^\infty(B)$. □

Definition 1.3.4 (*RFHGO*)**.** Let B be a Hermitian vector bundle over $\mathcal{M}$ and consider a Green-hyperbolic operator $P \colon \Gamma^\infty(B) \to \Gamma^\infty(B)$. If P is real and formally Hermitian, by Theorem 1.3.1 $\overline{E_P^\pm} = E_P^\pm = {}^t E_P^\mp$ and P is said to be a *real, formally Hermitian, Green-hyperbolic operator (RFHGHO)*. For any RFHGHO P it is possible to define an antisymmetric bilinear form on the space $E_P\Gamma_0^\infty(B)$:

$$\sigma_P(E_P f, E_P g) \coloneqq \langle \mathcal{C} f, E_P g \rangle \quad \forall f, g \in \Gamma_0^\infty(B) \tag{1.3.15}$$

satisfying $\sigma_P(\mathcal{C}E_P f, \mathcal{C}E_P g) = \overline{\sigma_P(E_P f, E_P g)}$.

The antisymmetric bilinear form of Definition 1.3.4 strongly suggests the possibility of associating some kind of symplectic space to any RFHGHO. This is a key property of these theories as symplectic spaces can be rigorously quantized following a standard procedure described in the next section [FV12]. In view of the formulation discussed in Section 1.2 and Definition 1.2.10, it is useful to organize the objects analyzed in this section into categories.

Definition 1.3.5 (**HVB** *category*)**.** The **HVB** *category* has Hermitian vector bundles $(B, \langle \cdot, \cdot \rangle_x, \mathcal{C})$ as objects and smooth vector bundle maps $\beta \colon (B, \langle \cdot, \cdot \rangle_x, \mathcal{C}) \to (B', \langle \cdot, \cdot \rangle'_x, \mathcal{C}')$ such that they cover a **Loc** morphism $\psi \colon \mathcal{M} \to \mathcal{M}'$ between the base manifolds, are fiberwise isometries and $\beta \circ \mathcal{C} = \mathcal{C}' \circ \beta$ as morphisms. A **HVB** morphism induces a pushforward map $\beta_* \colon \Gamma_0^\infty(B) \to \Gamma_0^\infty(B')$ acting as $\beta_* f = \beta \circ f \circ \psi^{-1}$ extended by zero.

Definition 1.3.6 (**GreenHyp** *category*)**.** Let P be a RFHGHO on a vector bundle B. The **GreenHyp** *category* has pairs (B, P) as objects and maps $S : (B, P) \to (B', P')$ such that there is an induced map on sections $S \colon \Gamma_0^\infty(B) \to \Gamma_0^\infty(B')$ satisfying $\mathcal{C}'S = S\mathcal{C}, SP\Gamma_0^\infty(B) \subset P'\Gamma_0^\infty(B')$ and $\sigma_{P'}(E_{P'}Sf, E_{P'}Sg) = \sigma_P(E_P f, E_P g) \quad \forall f, g \in \Gamma_0^\infty(B)$ as morphisms.

Definition 1.3.7 (**Sympl**$_\mathbb{K}$ *category*)**.** Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. A *complex symplectic vector space* is a triple $(V, \sigma, \mathcal{C})$ where V is a vector space over $\mathbb{C}, \sigma \colon V \times V \to \mathbb{C}$ is a symplectic form, i.e. a weakly non degenerate antisymmetric bilinear form on V and $\mathcal{C}$ is an antilinear involution on V such that $\sigma(\mathcal{C}v,\mathcal{C}u)=\overline{\sigma(u,v)}$. If V is a vector space over $\mathbb{R}$ and σ is $\mathbb{R}$-valued, the pair (V,σ) is a *real symplectic vector space*. The $\mathbf{Sympl}_{\mathbb{K}}$ *category* has $\mathbb{K}$ symplectic vector spaces as objects and linear maps $T\colon V\to V'$ such that $\sigma(u,v)=\sigma'(Tu,Tv)$ as morphisms. If $\mathbb{K}=\mathbb{C}$ it is also required that $\mathcal{C}'T=T\mathcal{C}$.

The properties required in the definition of the **GreenHyp** category were set up in such a way that it is now possible to systematically assign a symplectic vector space to a RFHGHO in a functorial way.

Theorem 1.3.2. *There is a well defined functor from* **GreenHyp** *to* $\mathbf{Sympl}_{\mathbb{C}}$. *It is defined on objects as:*

$$\begin{aligned}\mathscr{S}\colon\ \mathrm{ob}(\mathbf{GreenHyp})&\to\mathrm{ob}(\mathbf{Sympl}_{\mathbb{C}})\\(B,P)&\mapsto\mathscr{S}(B,P):=(E_P\Gamma_0^\infty(B),\sigma_P,\mathcal{C})\end{aligned}\tag{1.3.16}$$

It is defined on morphisms as:

$$\begin{aligned}\mathscr{S}\colon\ \mathrm{Hom}(\mathbf{GreenHyp})&\to\mathrm{Hom}(\mathbf{Sympl}_{\mathbb{C}})\\S&\mapsto\mathscr{S}(S)E_Pf:=E_{P'}Sf\end{aligned}\tag{1.3.17}$$

$S\in\mathrm{Hom}(\mathbf{GreenHyp})$ *is said to be Cauchy if* $\mathscr{S}(S)\in\mathrm{Hom}(\mathbf{Sympl}_{\mathbb{C}})$ *is an isomorphism. A pair of* **GreenHyp** *morphisms* $S,T\colon(B,P)\to(B',P')$ *are said to be equivalent if* $\mathscr{S}(S)=\mathscr{S}(T)$ *and they are denoted by* $S\sim T$.

Proof. By Definition 1.3.4 σ_P is a symplectic form on $E_P\Gamma_0^\infty(B)$, together with the antilinear involution $\mathcal{C}$ of B it makes $(E_P\Gamma_0^\infty(B),\sigma_P,\mathcal{C})\in\mathrm{ob}(\mathbf{Sympl}_{\mathbb{C}})$. The linear map $\mathscr{S}(S)\colon E_P\Gamma_0^\infty(B)\to E_{P'}\Gamma_0^\infty(B')$ is such that:

$$\sigma_{P'}(\mathscr{S}(S)E_Pf,\mathscr{S}(S)E_Pg)=\sigma_{P'}(E_{P'}Sf,E_{P'}Sg)=\sigma_P(E_Pf,E_Pg)\tag{1.3.18}$$

Since P is real it also holds that $\mathcal{C}'\mathscr{S}(S)E_Pf=\mathcal{C}'E_{P'}Sf=E_{P'}\mathcal{C}'Sf=E_{P'}S\mathcal{C}'f=\mathscr{S}(S)\mathcal{C}'f$ so that $\mathscr{S}(S)\in\mathrm{Hom}(\mathbf{Sympl}_{\mathbb{C}})$.

Let S,S' be two **GreenHyp** morphisms, then:

$$\mathscr{S}(S'\circ S)E_Pf=E_{P''}S'(Sf)=\mathscr{S}(S')E_{P'}Sf=\mathscr{S}(S')\circ\mathscr{S}(S)E_Pf\tag{1.3.19}$$

Moreover, $\mathscr{S}(\mathbb{1}_{(B,P)})E_Pf=E_Pf=\mathbb{1}_{E_P\Gamma_0^\infty(B)}E_Pf$ so that $\mathscr{S}$ is indeed a functor. □

The following theorem states a useful criterion to identify when a **GreenHyp** morphism is Cauchy.

Theorem 1.3.3. *Let* $S\colon(B,P)\to(B',P')$ *be a* **GreenHyp** *morphism and define the following maps:*

$$\begin{aligned}\hat{E}_P\colon\Gamma_0^\infty(B)/P\Gamma_0^\infty(B)&\to E_P\Gamma_0^\infty(B)\ \text{s.t. }\hat{E}_P[f]_P=E_Pf\\\hat{S}\colon\Gamma_0^\infty(B)/P\Gamma_0^\infty(B)&\to\Gamma_0^\infty(B')/P'\Gamma_0^\infty(B')\ \text{s.t. }\hat{S}[f]_P=[Sf]_{P'}\end{aligned}\tag{1.3.20}$$

then $\hat{E}_P$ *is an isomorphism, moreover* S *is Cauchy if and only if* $\hat{S}$ *is an isomorphism.*

Proof. The fact that $\hat{E}_P$ is an isomorphism directly follows from Theorem 1.3.1(iii). Consider now the following equivalence:

$$\hat{E}_{P'}\hat{S}[f]_P = \hat{E}_{P'}[Sf]_{P'} = E'_P Sf = \mathscr{S}(S)E_P f = \mathscr{S}(S)\hat{E}_P[f]_P \tag{1.3.21}$$

Since $\hat{E}_{P'}$ and $\hat{E}_P$ are isomorphisms, $\hat{S}$ is an isomorphism if and only if $\mathscr{S}(S)$ is an isomorphism i.e. S is Cauchy. □

In many cases it is useful to study **GreenHyp** morphisms between RFHGHOs acting on the same vector bundle. In particular, the existence of well behaved morphisms between RFHGHOs that agree on a portion of spacetime containing a Cauchy surface is guaranteed. This fact, stated in the next theorem, is a core result in the development of a measurement framework for quantum field theories.

Theorem 1.3.4. *Let $(B,P),(B,P') \in \mathrm{ob}(\mathbf{GreenHyp})$ be RFHGHOs on the same vector bundle such that there is a causally convex open subset $N \subset \mathcal{M}$ containing a Cauchy surface of $\mathcal{M}$ and $P'|_N = P|_N$. Then there is a Cauchy* **GreenHyp** *morphism $S\colon (B,P) \to (B,P')$ such that $\hat{S}[f]_P = [f]_{P'} \quad \forall f \in \Gamma_0^\infty(B|_N)$.*

Proof. Let $\mathfrak{i}_N\colon B|_N \to B$ be the inclusion bundle map and $\mathfrak{i}_{N*}\colon \Gamma_0^\infty(B|_N) \to \Gamma_0^\infty(B)$ its pushforward on smooth sections of B acting as the extension by zero of $\mathfrak{i}_{N*}f = \mathfrak{i}_N \circ f \circ \iota_N \quad \forall f \in \Gamma_0^\infty(B|_N)$, where ι_N is the natural inclusion map of N into $\mathcal{M}$. Notice that $\mathfrak{i}_{N*}\, P|_N\, \Gamma_0^\infty(B|_N) = P\mathfrak{i}_{N*}\Gamma_0^\infty(B) \subset P\Gamma_0^\infty(B)$. Moreover $P\mathfrak{i}_{N*} = \mathfrak{i}_{N*}\, P|_N$ and, since $\mathfrak{i}_{N*}$ is clearly real, one has $P|_N\, {}^t\mathfrak{i}_{N*} = {}^t\mathfrak{i}_{N*}P$ so that:

$$\begin{aligned} P|_N\, {}^t\mathfrak{i}_{N*}E_P^\pm \mathfrak{i}_{N*} &= {}^t\mathfrak{i}_{N*}PE_P^\pm\mathfrak{i}_{N*} = {}^t\mathfrak{i}_{N*}\mathfrak{i}_{N*} = \mathbb{1}_{\Gamma_0^\infty(B|_N)} \\ {}^t\mathfrak{i}_{N*}E_P^\pm\mathfrak{i}_{N*}\, P|_N &= {}^t\mathfrak{i}_{N*}E_P^\pm\, P|_N\, \mathfrak{i}_{N*} = {}^t\mathfrak{i}_{N*}\mathfrak{i}_{N*} = \mathbb{1}_{\Gamma_0^\infty(B|_N)} \end{aligned} \tag{1.3.22}$$

Moreover for any $f \in \Gamma_0^\infty(B|_N)$ one clearly has $\mathrm{supp}(\mathfrak{i}_{N*}f) \subset \mathrm{supp}(f)$ so that:

$$\mathrm{supp}({}^t\mathfrak{i}_{N*}E_P^\pm\mathfrak{i}_{N*}f) \subset \mathrm{supp}({}^t\mathfrak{i}_{N*}E_P^\pm f) \subset J^\pm(\mathrm{supp}(f)) \tag{1.3.23}$$

This shows that ${}^t\mathfrak{i}_{N*}E_P^\pm\mathfrak{i}_{N*}$ are Green operators for $P|_N$, however by Theorem 1.3.1(i) they are unique so that ${}^t\mathfrak{i}_{N*}E_P^\pm\mathfrak{i}_{N*} = E_{P|_N}^\pm$. With this property it is possible to verify:

$$\begin{aligned} \sigma_P(E_P\mathfrak{i}_{N*}f, E_P\mathfrak{i}_{N*}g) &= \langle \mathcal{C}\mathfrak{i}_{N*}f, E_P\mathfrak{i}_{N*}g\rangle \\ &= \langle \mathcal{C}f, {}^t\mathfrak{i}_{N*}E_P^\pm\mathfrak{i}_{N*}g\rangle \\ &= \sigma_{P|_N}(E_{P|_N}f, E_{P|_N}g) \end{aligned} \tag{1.3.24}$$

showing that $\mathfrak{i}_{N*}\colon (B|_N\,, P|_N) \to (B,P)$ is a **GreenHyp** morphism. The map $\hat{\mathfrak{i}}_{N*}$ as defined in Theorem 1.3.3 is surjective, indeed by Theorem 1.3.1(iv):

$$\exists h \in \Gamma_0^\infty(B|_N) \mid [f]_P = [\mathfrak{i}_{N*}h + Pg]_P = [\mathfrak{i}_{N*}h]_P = \hat{\mathfrak{i}}_{N*}[h]_{P|_N} \quad \forall f \in \Gamma_0^\infty(B) \tag{1.3.25}$$

It is also injective as a consequence of Theorem 1.3.1(iii) so that $\hat{\mathfrak{i}}_{N*}$ is an isomorphism, thus by Theorem 1.3.3 $\mathfrak{i}_{N*}$ is Cauchy. Clearly $\mathfrak{i}'_{N*}\colon (B|_N\,, P|_N) \to (B,P')$ is another Cauchy **GreenHyp** morphism. Consider now a function $\chi \in C^\infty(\mathcal{M})$ as in the proof of Theorem 1.3.1(iv), then the map:

$$\begin{aligned} L_N\colon \Gamma_0^\infty(B) &\to \Gamma_0^\infty(B|_N) \\ f &\mapsto L|_N\, f \coloneqq [P|_N\,, \chi]\hat{E}_{P|_N}\hat{\mathfrak{i}}_{N*}^{-1}[f]_P \end{aligned} \tag{1.3.26}$$

is a Cauchy **GreenHyp** morphism from (B, P) to $(B|_N, P|_N)$, this is checked in an analogous way to $\mathfrak{i}_{N*}$. The desired morphism is then $S \coloneqq \mathfrak{i}'_{N*} \circ L_N \colon \Gamma_0^\infty(B) \to \Gamma_0^\infty(B)$, indeed as a composition of Cauchy **GreenHyp** morphism it is a Cauchy **GreenHyp** morphism. Since $\hat{\mathfrak{i}}_{N*}$ is an isomorphism, let $[h]_P = \hat{\mathfrak{i}}_{N*}[f]_{P|_N} = [\mathfrak{i}_{N*}f]_P$, notice that:

$$\begin{aligned}
\langle g, {}^t\mathfrak{i}_{N*}E_P h\rangle &= \langle g, {}^t\mathfrak{i}_{N*}E_P\mathfrak{i}_{N*}f\rangle \\
&= \langle \mathcal{C}(\mathcal{C}g), {}^t\mathfrak{i}_{N*}E_P\mathfrak{i}_{N*}f\rangle \\
&= \sigma_P(E_P\mathfrak{i}_{N*}\mathcal{C}g, E_P\mathfrak{i}_{N*}f) \\
&= \sigma_{P|_N}(E_{P|_N}\mathcal{C}g, E_{P|_N}f) \\
&= \langle \mathcal{C}(\mathcal{C}g), E_{P|_N}f\rangle \\
&= \langle g, \hat{E}_{P|_N}[f]_{P|_N}\rangle \\
&= \langle g, \hat{E}_{P|_N}\hat{\mathfrak{i}}_{N*}^{-1}[h]_P\rangle
\end{aligned} \tag{1.3.27}$$

Then $S = \mathfrak{i}'_{N*}[P|_N, \chi]\, {}^t\mathfrak{i}_{N*}E_P$ but $[P|_N, \chi]$ vanishes outside N in any case so that one can rewrite $S = [P, \chi]E_P$. Finally, $P = P'$ in N and, following Theorem 1.3.1(iv) together with the support properties of the cutoff function χ, for any $f \in \Gamma_0^\infty(B|_N)$ it holds that:

$$\hat{S}[f]_P = [[P, \chi]E_P f]_{P'} = [f + Pg]_P = [f]_P = [f]_{P'} \tag{1.3.28}$$

□

In the beginning of this section it was mentioned how a classical field theory is identified with the partial differential operator appearing in its equations of motion. This statement can be made more precise in terms of a functor that assigns a RFHGHO to a spacetime.

Definition 1.3.8 (*Functorial equation of motion*)**.** A functor $\mathscr{B} \colon \mathbf{Loc} \to \mathbf{HVB}$ is called *functorial bundle.* A family of linear partial differential operators $\{P_\mathcal{M}\}, \mathcal{M} \in \mathbf{Loc}$ is said to be a *natural differential operator* on a functorial bundle $\mathscr{B}$ if $P_\mathcal{M} \colon \Gamma^\infty(\mathscr{B}(\mathcal{M})) \to \Gamma^\infty(\mathscr{B}(\mathcal{M}))$ satisfies $P_{\mathcal{M}'} \circ \mathscr{B}(\psi)_* = \mathscr{B}(\psi)_* \circ P_\mathcal{M}$ for all $\mathbf{Loc}$ morphisms $\psi \colon \mathcal{M} \to \mathcal{M}'$. A functor $\mathscr{E} \colon \mathbf{Loc} \to \mathbf{GreenHyp}$ is said to be a *functorial equation of motion* if $\mathscr{E}(\mathcal{M}) = (\mathscr{B}(\mathcal{M}), P_\mathcal{M})$ for some natural differential operator $\{P_\mathcal{M}\}$ on a functorial bundle $\mathscr{B}$ satisfying $\mathscr{E}(\psi) = \mathscr{B}(\psi)_*$ for any $\mathbf{Loc}$ morphism ψ.

This definition is essentially saying that it is possible to make an assignment:

$$\text{Spacetime } \mathcal{M} \to \text{Classical field theory } P\phi = 0,\ \phi \in \Gamma^\infty(B)$$

Once a functorial bundle and a natural differential operator have been specified, a functorial equation of motion takes a spacetime $\mathcal{M}$ and outputs a pair (B, P) of a vector bundle B and a linear partial differential operator P acting on its smooth sections $\phi \in \Gamma^\infty(B)$. The dynamics of the theory is described by the partial differential equation $P\phi = 0$. This is exactly the first step towards the construction of a locally covariant quantum field theory described at the end of Section 1.2.

Finally, as suggested by Example 1.3.1, it is useful to extend linear maps to act on distributions. It is also possible to define distributions from linear maps between smooth sections, this will come particularly handy in the study of the quantized theory.

Definition 1.3.9 (*2-point distribution of a linear operator*)**.** Any continuous linear map $T\colon \Gamma_0^\infty(B) \to \Gamma^\infty(B)$, where B is a vector bundle with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ fibers and bundle metric $\langle \cdot, \cdot \rangle_x$ on its fibers, uniquely defines a distributional section of the vector bundle $B \boxtimes B$ as:

$$\begin{aligned} \mathcal{D}'(B \boxtimes B) \ni T^{2\mathrm{pt}}\colon \Gamma_0^\infty(B^* \boxtimes B^*) &\to \mathbb{K} \\ (f^\sharp \otimes g^\sharp) &\mapsto T^{2\mathrm{pt}}(f^\sharp \otimes g^\sharp) \coloneqq \int_{\mathcal{M}} \langle \mathcal{C}f(x), Tg(x) \rangle_x \,\mathrm{dV} \end{aligned} \tag{1.3.29}$$

where $\sharp\colon B \to B^*$ is the canonical isomorphism induced by the bundle metric $[f^\sharp(g)](x) = \langle f, g \rangle_x \quad \forall x \in \mathcal{M}$.

Definition 1.3.10 (*Distributional kernel of a linear operator*)**.** Any continuous linear map $T\colon \Gamma_0^\infty(B) \to \Gamma^\infty(B)$, where B is a vector bundle with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ fibers, uniquely defines a distributional section of the vector bundle $B \boxtimes B^*$ as:

$$\begin{aligned} \mathcal{D}'(B \boxtimes B^*) \ni T^{\mathrm{knl}}\colon \Gamma_0^\infty(B^* \boxtimes B) &\to \mathbb{K} \\ (f \otimes g) &\mapsto T^{\mathrm{knl}}(f \otimes g) \coloneqq \int_{\mathcal{M}} f(Tg) \,\mathrm{dV} \end{aligned} \tag{1.3.30}$$

where $f \in \Gamma_0^\infty(B^*),\ g \in \Gamma_0^\infty(B)$ and the integral is well defined due to f, g being compactly supported. One formally writes:

$$T^{\mathrm{knl}}(f \otimes g) = \int_{\mathcal{M} \times \mathcal{M}} f(x) T(x, y) g(y) \,\mathrm{dV}_x \,\mathrm{dV}_y \tag{1.3.31}$$

for a so-called *Schwartz kernel* $T(x, y)$ associated to the linear map T [Hör90, Sec. 5.2].

Definition 1.3.11 (*Operator extension to distributions*)**.** Let $P\colon \Gamma^\infty(B) \to \Gamma^\infty(B)$ be a linear operator. Define its extension to distributional sections of B as:

$$\begin{aligned} P_{\mathcal{D}'(B)}\colon \mathcal{D}'(B) &\to \mathcal{D}'(B) \\ u(f) &\mapsto [P_{\mathcal{D}'(B)} u](f) \coloneqq u({}^\star P f) \quad \forall f \in \Gamma_0^\infty(B^*) \end{aligned} \tag{1.3.32}$$

Remark. All the fundamental properties relevant for this work are shared between T^{knl} and $T^{2\mathrm{pt}}$. This is due to the metric induced canonical isomorphisms for which $T^{\mathrm{knl}}(f \otimes g^\sharp) = T^{2\mathrm{pt}}(f^\sharp \otimes g^\sharp) \quad \forall f \in \Gamma_0^\infty(B_1), g \in \Gamma_0^\infty(B_2^*)$. The only main formal difference is that Definition 1.3.10 doesn't strictly require the metric induced isomorphism. When no confusion can arise the notation will be $T = T^{\mathrm{knl}} = T^{2\mathrm{pt}}$ and $P = P_{\mathcal{D}'(B)}$.

A comprehensive treatment of Green operators and their properties on globally hyperbolic spacetimes can be found in [Bär15; BGP08; Few25].

1.4 Quantization algebras

One of the main results of Section 1.3 was establishing a connection between RFHGHOs and symplectic vector space. This has been described via a functorial assignment in

Theorem 1.3.2. Symplectic vector spaces are particularly useful in the algebraic formulation of quantum theories. There are indeed a number of ways to assign a $*$-algebra to a symplectic vector space, again via functors. These are usually called quantization functors and the two quantization functors used in this work are described in Appendix B. In Section 1.2 it was discussed how in the algebraic formulation of quantum field theory it is desirable to work with $*$-algebras in certain situations and with C^*-algebras in some others, both construction will be presented in this section.

Theorem 1.4.1. *There is a well defined functor from* **GreenHyp** *to* **Alg**. *It is defined on objects as:*

$$\begin{aligned} \mathscr{A}\colon \ \mathrm{ob}(\mathbf{GreenHyp}) &\to \mathrm{ob}(\mathbf{Alg}) \\ (B,P) &\mapsto \mathscr{A}(P) \end{aligned} \tag{1.4.1}$$

where $\mathscr{A}(P)$ is the unital $*$-algebra generated by finite products and linear combinations of symbols $\Phi_P(f)$, such that for any $f,g \in \Gamma_0^\infty(B)$ they satisfy:

(i) $\Phi_P(\alpha f + \beta g) = \alpha \Phi_P(f) + \beta \Phi_P(g) \quad \forall \alpha, \beta \in \mathbb{C}$.

(ii) $\Phi_P(f)^* = \Phi_P(\mathcal{C} f)$.

(iii) $\Phi_P(Pf) = 0$.

(iv) $[\Phi_P(f), \Phi_P(g)] = i\sigma_P(E_P f, E_P g)\mathbb{1}_{\mathscr{A}(P)}$.

It is defined on morphisms as:

$$\begin{aligned} \mathscr{A}\colon \ \mathrm{Hom}(\mathbf{GreenHyp}) &\to \mathrm{Hom}(\mathbf{Alg}) \\ S &\mapsto \mathscr{A}(S) \end{aligned} \tag{1.4.2}$$

where for any $S\colon (B,P) \to (B',P')$ the **Alg** *morphism $\mathscr{A}(S)\colon \mathscr{A}(P) \to \mathscr{A}(P')$ acts as $\mathscr{A}(S)\Phi_P(f) = \Phi_{P'}(Sf) \quad \forall f \in \Gamma_0^\infty(B)$.*

Proof. The functor $\mathscr{A}$ is none other than the composition $\mathscr{Q} \circ \mathscr{S}$ of the quantization functor $\mathscr{Q}$ defined in Definition B.6 and the functor $\mathscr{S}$ defined in Theorem 1.3.2. In particular for each $f \in \Gamma_0^\infty(B)$ one can set $\Phi_P(f) \coloneqq (0, E_P f, 0, \dots) \in \bigoplus_{n\in\mathbb{N}_0} (E_P \Gamma_0^\infty(B))^{\odot n}$ from which all the properties of the symbols $\Phi_P(f)$ follow. $\square$

To get a sense of what the algebra elements $\Phi_P(f)$ are representing, it is worth taking a step back to the standard canonical quantization of a classical field theory. Given a classical field theory on Minkowski spacetime with field content $\phi(t,\vec{x})$, canonical quantization promotes the field to an operator $\hat{\phi}(t,\vec{x})$ acting on the Hilbert space of the quantum theory and imposes canonical commutation relations with the conjugate momentum $\hat{\pi}(t,\vec{y})$ as:

$$[\hat{\phi}(t,\vec{x}), \hat{\pi}(t,\vec{y})] = i\delta^3(\vec{x} - \vec{y}) \tag{1.4.3}$$

This relation shows that $\phi(t,\vec{x})$ and $\hat{\pi}(t,\vec{y})$ cannot be understood as operator-valued fields, but rather as operator-valued distributions, thus generally ill-defined at specific points. This apparent technical problem just reflects the physically well understood fact that measurements at a pointlike scale would require infinite energy. What this means is that, for fixed $x \in \mathcal{M}$, the expression $\hat{\phi}(x)$ cannot represent an operator, i.e. a physical observable, which, a posteriori, was actually to be expected. The algebra elements $\Phi_P(f)$ can be understood as the smearing of the operator-valued distribution $\hat{\phi}(x)$ against a test section $f \in \Gamma_0^\infty(B)$ supported in the spacetime region where the measurement is taking place, in other words:

$$\Phi_P(f) = \int_{\mathcal{M}} \hat{\phi}(x) f(x) \, \mathrm{dV} \tag{1.4.4}$$

The above expression has to be understood in some representation of the algebra $\mathscr{A}(P)$ acting on the Hilbert space of the theory. A popular instance of this fact is given by Minkowski spacetime where one takes the GNS representation induced by an appropriately defined Minkowski vacuum state, this fully recovers the usual construction of the Hilbert space of the canonical quantization scheme. In this sense the abstract algebra elements $\Phi_P(f)$ are said to be the field operators of the quantum theory and are regarded as the true observables of it [Haa96].

As previously mentioned, it is sometimes useful to have a full C^* structure on the algebra of observables. This can be achieved employing a different quantization functor.

Theorem 1.4.2. *There is a well defined functor from* **GreenHyp** *to* **C*-Alg**. *It is defined on objects as:*

$$\begin{aligned} \mathscr{A}_W \colon \ \mathrm{ob}(\mathbf{GreenHyp}) &\to \mathrm{ob}(\mathbf{C^*\text{-}Alg}) \\ (B, P) &\mapsto \mathscr{A}_W(P) \end{aligned} \tag{1.4.5}$$

where $\mathscr{A}_W(P)$ is the unital C^*-algebra given by the unique C^* completion of the algebra generated by finite products and linear combinations of symbols $W_P(f)$, such that for any real $f, g \in \Gamma_0^\infty(B)$ they satisfy:

(i) $W_P(f)^* = W_P(-f)$.

(ii) $W_P(0) = \mathbb{1}_{\mathscr{A}_W(P)}$.

(iii) $W_P(f)W_P(g) = e^{-\frac{i}{2}\sigma_P(E_P f, E_P g)} W_P(f+g)$.

It is defined on morphisms as:

$$\begin{aligned} \mathscr{A}_W \colon \ \mathrm{Hom}(\mathbf{GreenHyp}) &\to \mathrm{Hom}(\mathbf{C^*\text{-}Alg}) \\ S &\mapsto \mathscr{A}_W(S) \end{aligned} \tag{1.4.6}$$

where for any $S \colon (B, P) \to (B', P')$ the **C*-Alg** *morphism $\mathscr{A}_W(S) \colon \mathscr{A}_W(P) \to \mathscr{A}_W(P')$ acts as $\mathscr{A}_W(S)W_P(f) = W_{P'}(Sf)$. $\mathscr{A}_W(P)$ is called Weyl C^*-algebra.*

Proof. In a totally analogous way to Theorem 1.4.1, the functor $\mathscr{A}_W$ is again the composition $\mathscr{W} \circ \mathscr{S}$ where $\mathscr{W}$ is the Weyl quantization functor described in Definition B.8. All of the $W_P(f)$ properties are immediate. $\square$

Remark. From now on the antisymmetric bilinear form $\sigma_P(E_P f, E_P g)$ of Definition 1.3.4 will be denoted as $E_P(f,g) \quad \forall f,g \in \Gamma_0^\infty(B)$. This notation, though a bit redundant with the one of causal propagator, will be particularly useful as:

$$E_P(f,g) = \langle \mathcal{C}f, E_P g\rangle = \int_{\mathcal{M}} \langle \mathcal{C}f(x), E_P g(x)\rangle_x \,\mathrm{dV} \quad \forall f,g \in \Gamma_0^\infty(B) \tag{1.4.7}$$

In this way $E_P(f,g)$ can be understood as a distribution in the sense of Definition 1.3.9 as $E^{2\mathrm{pt}}(f^\sharp \otimes g^\sharp)$.

Theorem 1.4.3. *Consider the functor $\mathscr{A}$ in Theorem 1.4.1, then:*

(i) If $S\colon (B,P) \to (B',P')$ is a Cauchy **GreenHyp** *morphism, then $\mathscr{A}(S)\colon \mathscr{A}(P) \to \mathscr{A}(P')$ and $\mathscr{A}_W(S)\colon \mathscr{A}_W(P) \to \mathscr{A}_W(P')$ are an* **Alg** *and* **C*-Alg** *isomorphism respectively.*

(ii) For any pair P, P' of RFHGHOs on Hermitian vector bundles B, B' over the same base manifold, define the RFHGHO $P \oplus P'$ acting on $B \oplus B'$ acting as:

$$(P \oplus P')(f \oplus f') \coloneqq \begin{pmatrix} P & 0 \\ 0 & P' \end{pmatrix} \begin{pmatrix} f \\ f' \end{pmatrix} = (Pf \oplus P'f') \tag{1.4.8}$$

Then $\mathscr{A}(P \oplus P') \cong \mathscr{A}(P) \otimes \mathscr{A}(P')$ under the correspondence:

$$\Phi_{P\oplus P'}(f \oplus f') = \Phi_P(f) \otimes \mathbb{1}_{\mathscr{A}(P')} + \mathbb{1}_{\mathscr{A}(P)} \otimes \Phi_{P'}(f') \tag{1.4.9}$$

for any $f \in \Gamma_0^\infty(B), f' \in \Gamma_0^\infty(B')$.

Proof. **(i)** By Theorem 1.3.3, if S is Cauchy then $\hat{S}$ is an isomorphism. This means that $\mathscr{A}(P)\Phi_P(f) = \Phi'_P(Sf) = 0 \implies Sf = 0 \implies [f]_P = [0]_P \implies f = Pg$ so that $\Phi_P(f) = 0$. Similarly, consider any $\Phi_{P'}(g)$, since $\hat{S}$ is an isomorphism it is possible to find f such that $Sf = g + P'h$, then $\mathscr{A}(P)\Phi_P(f) = \Phi_{P'}(Sf) = \Phi_{P'}(g + P'h) = \Phi_{P'}(g)$. $\mathscr{A}_W(S)$ is proved analogously. These relations on the generators extend to the whole algebras by linearity.

(ii) By definition of the quantization functor and of the operator $P \oplus P'$ one has $\mathscr{A}(P \oplus P') = \bigoplus_{n\in\mathbb{N}_0} (E_P\Gamma_0^\infty(B) \oplus E_{P'}(B'))^{\odot n}$. This is a symmetric tensor algebra of a direct sum. The isomorphism with the tensor product of the individual graded algebras is proved in e.g. [Bou98, Ch. 3, Sec. 6]. □

All the functors defined can be now combined in the way proposed by Definition 1.2.10 to properly set up a locally covariant quantum field theory.

Theorem 1.4.4. *Let $\mathscr{E}\colon$* **Loc** $\to$ **GreenHyp** *be a functorial equation of motion. Then the functors $\mathscr{L} \coloneqq \mathscr{A} \circ \mathscr{E}$ and $\mathscr{L}_W \coloneqq \mathscr{A}_W \circ \mathscr{E}$ are locally covariant quantum field theories that satisfy both Einstein causality and the time-slice axiom.*

Proof. Clearly $\mathscr{L}\colon$ **Loc** $\to$ **Alg** so it is by definition a locally covariant quantum field theory. An algebra $\mathscr{L}(\mathcal{M})$ is generated by $\Phi_{P_\mathcal{M}}(f_\mathcal{M})$ $f_\mathcal{M} \in \Gamma_0^\infty(\mathscr{B}(\mathcal{M}))$ where $\mathscr{B}(\mathcal{M})$ is the Hermitian vector bundle over $\mathcal{M}$ assigned by $\mathscr{E}$ and $P_\mathcal{M}$ acts on its sections. Then consider a pair of **Loc** morphisms $\psi_i \colon \mathcal{M}_i \to \mathcal{M}$ for $i = 1, 2$ with causally disjoint image, Einstein causality is satisfied when:

$$\begin{aligned}
0 &= [\mathscr{L}(\psi_1)(\Phi_{P_{\mathcal{M}_1}}(f_{\mathcal{M}_1})), \mathscr{L}(\psi_2)(\Phi_{P_{\mathcal{M}_2}}(f_{\mathcal{M}_2}))] \\
&= [\mathscr{A}(\mathscr{E}(\psi_1))(\Phi_{P_{\mathcal{M}_1}}(f_{\mathcal{M}_1})), \mathscr{A}(\mathscr{E}(\psi_2))(\Phi_{P_{\mathcal{M}_2}}(f_{\mathcal{M}_2}))] \\
&= [\mathscr{A}(\mathscr{B}(\psi_1)_* f_{\mathcal{M}_1}), \mathscr{A}(\mathscr{B}(\psi_2)_* f_{\mathcal{M}_2})] \\
&= [\Phi_{P_{\mathcal{M}}}(\mathscr{B}(\psi_1)_* f_{\mathcal{M}_1}), \Phi_{P_{\mathcal{M}}}(\mathscr{B}(\psi_2)_* f_{\mathcal{M}_2})] \\
&= iE_{P_{\mathcal{M}}}(\mathscr{B}(\psi_1)_* f_{\mathcal{M}_1}, \mathscr{B}(\psi_2)_* f_{\mathcal{M}_2}) \mathbb{1}_{\mathscr{A}(P)}
\end{aligned} \tag{1.4.10}$$

The last line indeed vanishes due to the fact that $\mathscr{B}(\psi_1)_* f_{\mathcal{M}_1}$ and $\mathscr{B}(\psi_2)_* f_{\mathcal{M}_2}$ have disjoint support. In the proof of Theorem 1.3.4 it was shown that the immersion bundle map for an immersion containing a Cauchy surface gave rise to a Cauchy **GreenHyp** morphism. This means that $\mathscr{E}(\psi)$ is a Cauchy **GreenHyp** morphism whenever ψ is a Cauchy **Loc** morphism. By Theorem 1.4.3 it follows that $\mathscr{L}(\psi)$ is an **Alg** isomorphism so that $\mathscr{A}(\psi)\mathscr{A}(\mathcal{M}) = \mathscr{A}(\mathcal{M}')$ for any Cauchy **Loc** morphism $\psi \colon \mathcal{M} \to \mathcal{M}'$, showing that the time-slice axiom is satisfied.

The $\mathscr{L}_W$ case is proved in the exact same way using the Weyl relation Theorem 1.4.2(iii). □

Remark. Theorem 1.4.1 and Theorem 1.4.2 can be used to construct the local algebras of a certain region $\mathcal{O} \subset \mathcal{M}$ by taking the generators $\Phi_P(f)$ and $W_P(f)$ respectively with sections f supported in that region only, i.e. $\mathrm{supp}(f) \subset \mathcal{O}$. Theorem 1.4.4 formalizes this construction, which corresponds to the local algebras $\mathscr{L}(\mathcal{O})$ and $\mathscr{L}_W(\mathcal{O})$.

Now that a rigorous way of defining an algebra of observables of a quantum field theory has been established, the next step is to study its states and, consequently, the expectation values of the observables themselves. Recall from Definition 1.2.1 that a state on an algebra of observables is a normalized positive linear functional. In these terms the set of states on an algebra is a subset of its algebraic dual space. For this reason, given a $(B, P) \in$ **GreenHyp**, the set of states on $\mathscr{A}(P)$ is denoted as $\mathscr{A}(P)^*_{+,1}$, where the subscripts indicate positivity and normalization. The most important quantities of any quantum field theory are its correlation functions. Physically, these are expectation values of products of observables which, in the algebraic language, are computed by acting on the observables with a state functional. It has been argued previously that the field operators are regarded as the true observables of the quantum theory, thus the following definition is natural.

Definition 1.4.1 (*n-point function*)**.** Let $\omega \in \mathscr{A}(P)^*_{+,1}$ be a state on the algebra of observables $\mathscr{A}(P)$. Assuming that ω is regular enough that its action on the generators of the algebra is a distribution, the *distributional n-point function* of ω is defined as follows:

$$\begin{aligned}
\mathcal{D}'(B^{\boxtimes n}) \ni \mathfrak{W}^{(n)} \colon &\Gamma_0^\infty((B^*)^{\otimes n}) \to \mathbb{C} \\
&(f_1^\sharp \otimes \cdots \otimes f_n^\sharp) \mapsto \mathfrak{W}^{(n)}(f_1^\sharp \otimes \cdots \otimes f_n^\sharp) \coloneqq \omega(\Phi_P(f_1) \cdots \Phi_P(f_n))
\end{aligned} \tag{1.4.11}$$

The 2-point function is denoted just as $\mathfrak{W}(f^\sharp \otimes g^\sharp)$ due to its particular relevance.

The defining properties of the quantization algebras in Theorem 1.4.1, are basically encoding the canonical commutation relations (CCR) and the equations of motion of the quantum system into the algebra of observables. This is reflected by key properties of the 2-point function.

Proposition 1.4.5. *Let E_P^{2pt} be the 2-point distribution defined by the causal propagator $E_P\colon \Gamma_0^\infty(B) \to \Gamma_{sc}^\infty(B)$ as in Definition 1.3.9 and ${}^t\mathfrak{W}(f^\sharp \otimes g^\sharp) = \mathfrak{W}(g^\sharp \otimes f^\sharp)$, then:*

$$\mathfrak{W} - {}^t\mathfrak{W} = iE_P^{2pt} \tag{1.4.12}$$

The 2-point function can then be split into the sum of a symmetric and an antisymmetric part:

$$\mathfrak{W}(f^\sharp \otimes g^\sharp) := \mu^\omega(f^\sharp \otimes g^\sharp) + \frac{i}{2}E_P^{2pt}(f^\sharp \otimes g^\sharp) \tag{1.4.13}$$

Moreover, both $\mathfrak{W}$ and E_P^{2pt} are P-bisolutions when the action of P is extended to distributions as described by Definition 1.3.11. This means that:

$$(P \otimes \mathbb{1})\mathfrak{W} = (\mathbb{1} \otimes P)\mathfrak{W} = 0 \qquad (P \otimes \mathbb{1})E_P^{2pt} = (\mathbb{1} \otimes P)E_P^{2pt} = 0 \tag{1.4.14}$$

Proof. For any $f, g \in \Gamma_0^\infty(B)$, due to linearity one has:

$$\begin{aligned}
\mathfrak{W}(f^\sharp \otimes g^\sharp) - {}^t\mathfrak{W}(f^\sharp \otimes g^\sharp) &= \mathfrak{W}(f^\sharp \otimes g^\sharp - g^\sharp \otimes f^\sharp) \\
&= \omega([\Phi_P(f), \Phi_P(g)]) \\
&= iE_P(f,g)\omega(\mathbb{1}_{\mathscr{A}(P)}) \\
&= iE_P^{\mathrm{2pt}}(f^\sharp \otimes g^\sharp)
\end{aligned} \tag{1.4.15}$$

The decomposition is now automatic if one defines:

$$\mu^\omega(f^\sharp \otimes g^\sharp) := \frac{1}{2}(\mathfrak{W}(f^\sharp \otimes g^\sharp) + \mathfrak{W}(g^\sharp \otimes f^\sharp)) \tag{1.4.16}$$

Since P is formally Hermitian one can use the definition of the canonical isomorphism $\sharp$ to derive that ${}^\star P f^\sharp(h) = f^\sharp(Ph) = \langle f, Ph\rangle = \langle Pf, h\rangle = (Pf)^\sharp h$. This means that:

$$\begin{aligned}
(P \otimes \mathbb{1}_{\mathcal{D}'(B)})\mathfrak{W}(f^\sharp \otimes g^\sharp) &= \mathfrak{W}({}^\star P f^\sharp \otimes g^\sharp) \\
&= \omega(\Phi_P(Pf)\Phi_P(g)) = 0
\end{aligned} \tag{1.4.17}$$

The other equalities are derived in a similar way. □

Remark. For a lighter notation the arguments of distributions on $B^{\boxtimes n}$ will be denoted without the canonical isomorphism $\sharp$ from now on, for example the 2-point function $\mathfrak{W}$ acting on $f^\sharp \otimes g^\sharp$ becomes $\mathfrak{W}(f \otimes g)$.

Among the many different states on the algebra of observables, there is a class of so-called quasifree states that are in some sense completely determined by their 2-point function. This class of states have particularly good properties both in their algebraic form and in their Hilbert space representation through the GNS construction.

Definition 1.4.2 (*Quasifree state*)**.** An algebraic state $\omega \in \mathscr{A}(P)^*_{+,1}$ is said to be *quasifree* if its n-point function satisfies the following properties for any $f_i \in \Gamma_0^\infty(B), n \in \mathbb{N}_0$:

(i) $\omega(\Phi_P(f_1)\cdots\Phi_P(f_{2n+1}))=0$.

(ii) $\omega(\Phi_P(f_1)\cdots\Phi_P(f_{2n}))=\sum_{\text{Partitions}}\omega(\Phi_P(f_{i_1})\Phi_P(f_{i_2}))\cdots\omega(\Phi_P(f_{i_{2n-1}})\Phi_P(f_{i_{2n}}))$.

where the *partitions* of $2n$ elements are the possible subdivisions of the set $\{1,\ldots,2n\}$ into n disjoint pairs $\{i_1,i_2\},\ldots,\{i_{2n-1},i_{2n}\}$ with $i_{2k-1}<i_{2k}, k=1,\ldots n$.

An algebraic state $\omega\in\mathscr{A}_W(P)^*_{+,1}$ is said to be *quasifree* if there exists a symmetric bilinear form μ^ω on the symplectic space $(E_P\Gamma_0^\infty(B),\sigma_P,\mathcal{C})$ such that:

$$\omega(W_P(f))=e^{-\frac{1}{2}\mu^\omega(f,f)},\ f\in\Gamma_0^\infty(B) \tag{1.4.18}$$

The definition of quasifree state shows that any n-point function of such a state is fully determined by the value of the 2-point function via a procedure resembling the Wick contractions in quantum field theory on flat spacetimes. Moreover, the notation for the symmetric form in the Weyl algebra case suggests a relation with the 2-point function, i.e. a relation between $\mathscr{A}_W(P)$ and $\mathscr{A}(P)$. To better see this, notice the following equivalence using the Baker-Campbell-Hausdorff formula:

$$e^{i\Phi_P(f)}e^{i\Phi_P(g)}=e^{i\Phi_P(f)+i\Phi_P(g)-\frac{i}{2}E_P(f,g)\mathbb{1}_{\mathscr{A}(P)}}=e^{-\frac{i}{2}E_P(f,g)}e^{i\Phi_P(f+g)} \tag{1.4.19}$$

The abstract algebra $\mathscr{A}(P)$ has no topology on it, this means that the above expression has to be understood only as an equivalence between formal power series in $f,g\in\Gamma_0^\infty(B)$. This is due to the lack of a notion of convergence so that one cannot say that the power series of $e^{i\Phi_P(f)}$ converges to an element of $\mathscr{A}_W(P)$. It seems however that the formal operator $e^{i\Phi_P(f)}$ recreates all of the properties of the generators $W_P(f)$ of the Weyl C^*-algebra $\mathscr{A}_W(P)$. This formal identification becomes exact when the operators are represented on a GNS Hilbert space associated to a GNS representation induced by a quasifree state.

Theorem 1.4.6. *Let $\omega\in\mathscr{A}_W(P)^*_{+,1}$ be a quasifree state and $(\mathcal{H}_\omega,\pi_\omega,\Omega_\omega)$ its associated GNS construction. Then there is an induced representation of the unital $*$-algebra $\mathscr{A}(P)$ on $\mathcal{H}_\omega$ given by:*

$$\pi_\omega(\Phi_P(f))=-i\left.\frac{d}{dt}\pi_\omega(W_P(tf))\right|_{t=0} \tag{1.4.20}$$

where $\pi_\omega(\Phi_P(f))$ is defined on a dense domain $D_\omega(\pi_\omega(\Phi_P(f)))\subset\mathcal{H}_\omega$. This relation can also be written in its exponential form:

$$\pi_\omega(W_P(f))=e^{i\pi_\omega(\Phi_P(f))} \tag{1.4.21}$$

Proof. See e.g. [DG13, Sec. 8.1, 8.2, Prop. 17.3]. □

This result is based on the fact that quasifree states are, in some appropriate sense, well behaved enough that the induced GNS representation of the Weyl generators $W_P(f)$ creates strongly continuous 1-parameter groups of unitary operators. The rest of the statement is then an application of Stone's theorem, which also assures that the field operators $\pi_\omega(\Phi_P(f))$ are selfadjoint. The key takeaway is that the formal relation between the Weyl generators and the exponential of the field generators, suggested by Eq. (1.4.19), is exact in the Hilbert space formulation. This establishes a fundamental link between the two quantization schemes so that one can switch between the two of them based on the kind of computation that needs to be done.

Example 1.4.1 (*Quasifree states in Minkowski spacetime*)**.** Consider a real Klein-Gordon theory of mass m on Minkowski spacetime. The *Minkowski vacuum* state ω_0 is a quasifree state with 2-point function given by:

$$\mathfrak{W}_0(f \otimes g) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\hat{f}(-k)\hat{g}(k)}{2E_k} \qquad E_k := \sqrt{|k|^2 + m^2} \tag{1.4.22}$$

The *thermal state of inverse temperature* $\beta = \frac{1}{k_B T}$ is a quasifree state with 2-point function given by:

$$\mathfrak{W}_\beta(f \otimes g) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{2E_k} \left(\frac{\hat{f}(-k)\hat{g}(k)}{1 - e^{-\beta E_k}} + \frac{\hat{f}(k)\hat{g}(-k)}{e^{\beta E_k} - 1} \right) \tag{1.4.23}$$

The examples above are discussed in detail in [FR19], an in-depth analysis of general properties of quasifree states can be found in e.g. [Bru+15; DG13].

2 Hadamard states and their Hilbert space representation

2.1 Physical motivation and definition of Hadamard state

In classical field theories it is always possible to define a stress energy tensor for sufficiently regular field configurations. This quantity describes the energy density and the flow of momentum of the field across spacetime. For the prototypical Klein-Gordon classical field theory of mass m it has the following expression in terms of the field $\phi(x)$:

$$T_{\mu\nu}(x) = \nabla_\mu\phi(x)\nabla_\nu\phi(x) - \frac{1}{2}g_{\mu\nu}(x)(\nabla_\rho\phi(x)\nabla^\rho\phi(x) + m^2\phi^2(x)) \tag{2.1.1}$$

The stress energy tensor plays a crucial role even beyond the mere characterization of the field configuration, as it directly enters the Einstein field equations for general relativity:

$$G_{\mu\nu}(x) = \kappa T_{\mu\nu}(x) \tag{2.1.2}$$

where $G_{\mu\nu}$ is the Einstein tensor and κ is a proportionality constant. The importance of the stress energy tensor at classical level makes it especially desirable to figure out its quantum version. Upon canonical quantization the stress energy tensor gets promoted to an operator in the usual fashion. Among other applications, the expectation value of the stress energy tensor of a quantum field enters the semiclassical version of Eq. (2.1.2):

$$G_{\mu\nu}(x) = \kappa\langle\hat{T}_{\mu\nu}(x)\rangle_\omega \tag{2.1.3}$$

where $\langle\hat{T}_{\mu\nu}(x)\rangle_\omega$ is the expectation value of the quantum version of the stress energy tensor for a quantum field in the state ω. It is immediate to see that the distributional nature of the quantum fields makes it so that the naive quantization of Eq. (2.1.1) is ill-defined:

$$\hat{T}_{\mu\nu}(x) = \nabla_\mu\hat{\phi}(x)\nabla_\nu\hat{\phi}(x) - \frac{1}{2}g_{\mu\nu}(x)(\nabla_\rho\hat{\phi}(x)\nabla^\rho\hat{\phi}(x) + m^2\hat{\phi}^2(x)) \tag{2.1.4}$$

This prescription would indeed expect one to multiply distributions at the same spacetime point, which is in general an undefined operation [Hör90]. To overcome this problem it is possible to follow a procedure called point splitting regularization [Chr76]. The core strategy is to redefine $T_{\mu\nu}$ as a function of two separate spacetime points $x, y \in \mathcal{M}$ and then take the limit as $y \to x$. In the quantized version, this procedure would look like the following expression:

$$\begin{aligned}\hat{T}_{\mu\nu}(x) = \lim_{y\to x} g_\nu{}^{\nu'}(x,y)\Big[&\nabla^{(x)}_\mu\hat{\phi}(x)\nabla^{(y)}_{\nu'}\hat{\phi}(y) \\ &- \frac{1}{2}g_{\mu\nu'}(x,y)\left(g^{\rho\sigma'}(x,y)\nabla^{(x)}_\rho\hat{\phi}(x)\nabla^{(y)}_{\sigma'}\hat{\phi}(y) + m^2\hat{\phi}(x)\hat{\phi}(y)\right)\Big] \\ \coloneqq \lim_{y\to x} g_\nu{}^{\nu'}(x,y)&\mathcal{D}^{(x,y)}_{\mu\nu'}\hat{\phi}(x)\hat{\phi}(y)\end{aligned} \tag{2.1.5}$$

Here $g_{\mu\nu}(x,y)$ is a smooth bi-covector field agreeing with the metric on the diagonal $x=y$ used to correctly parallel transport vectors at different points [Chr76]. The differential operator $\mathcal{D}^{(x,y)}_{\mu\nu}$ is defined as:

$$\mathcal{D}^{(x,y)}_{\mu\nu} \coloneqq g_{\nu}{}^{\nu'}(x,y)\left[\nabla^{(x)}_{\mu}\nabla^{(y)}_{\nu'} - \frac{1}{2}g_{\mu\nu'}(x,y)\left(g^{\rho\sigma'}(x,y)\nabla^{(x)}_{\rho}\nabla^{(y)}_{\sigma'} + m^2\right)\right] \tag{2.1.6}$$

Finally, taking the expectation value in the state ω of both side yields:

$$\langle \hat{T}_{\mu\nu}(x)\rangle_{\omega} = \lim_{y\to x}\mathcal{D}^{(x,y)}_{\mu\nu}\langle\hat{\phi}(x)\hat{\phi}(y)\rangle_{\omega} \tag{2.1.7}$$

The expectation value in the state ω of this limit is divergent unless some subtraction procedure is specified, as it will be explained shortly. In light of Eq. (1.4.4) and Definition 1.3.10, notice that the right hand side contains none other than the Schwartz kernel of the 2-point function:

$$\mathfrak{W}(f\otimes g) = \omega(\Phi_P(f)\Phi_P(g)) = \int_{\mathcal{M}\times\mathcal{M}} f(x)\omega(\hat{\phi}(x)\hat{\phi}(y))g(y)\,\mathrm{dV}_x\,\mathrm{dV}_y \tag{2.1.8}$$

Remark. The operator in Eq. (2.1.6) is specific to the model of field theory, however it will always be a second order partial differential operator.

The kernel $\omega(\hat{\phi}(x)\hat{\phi}(y)) = \langle\hat{\phi}(x)\hat{\phi}(y)\rangle_{\omega} \coloneqq W(x,y)$ contains information about the correlation of the quantum field at different spacetime points $x,y\in\mathcal{M}$. As $y\to x$ the field becomes, in some appropriate sense, infinitely correlated with itself reflecting the undefined distributional multiplication at the same point.

The problem of infinite expectation values of observables is easily solved in Minkowski spacetime. Flat spacetime has a unique Poincaré invariant vacuum state ω_0 and the associated 2-point function $W_0(x,y)$ has a well known singularity structure as $y\to x$. To define finite physical quantities it is sufficient to subtract the vacuum contribution from the 2-point function of the state ω_0, this procedure is known as normal ordering. In curved spacetime the notion of vacuum state is not present due to the lack of global Poincaré symmetry, thus normal ordering is not a viable option to get finite results when taking expectation values of observables without the choice of a reference state. The flat spacetime case still suggests that one should look at the singularity structure of the 2-point function to define an appropriate renormalization scheme for observables such as the stress energy tensor. As stated in Theorem 1.4.3, the 2-point function is a bisolution of the differential equation determined by the operator P, the theory of this kind of linear partial differential equations was developed by J. Hadamard in [Had23]. In his work he introduced a prescription to find solutions to such differential equations based on their singularity structure in the limit $y\to x$, in terms of so-called Hadamard elementary solutions. Hadamard states are particular quantum states whose 2-point function has a specific type of singularity structure in the limit $y\to x$.

Definition 2.1.1 (*Hadamard state*)**.** A state $\omega\in\mathscr{A}(P)^{*}_{+,1}$ is said to be *Hadamard* if the kernel of its 2-point function has the following form:

$$W(x,y) = \frac{1}{4\pi^2}\left(\frac{u(x,y)}{\sigma(x,y)} + v(x,y)\log\left(\frac{\sigma(x,y)}{l^2}\right)\right) + w(x,y) \coloneqq H(x,y) + w(x,y) \tag{2.1.9}$$

where $\sigma(x, y)$ is the squared geodesic distance between x and y while u, v are smooth functions on $\mathcal{M} \times \mathcal{M}$ independent of ω and fully determined by the geometry of the spacetime via the *Hadamard recursion relations*. l is a positive length scale. The function w is a smooth function on $\mathcal{M} \times \mathcal{M}$ that depends on the state ω.

The results of the theory of PDEs developed by Hadamard was applied in [DB60] to determine the form of Green operators in terms of geometric constraint given by general relativity, resulting in the structure of the Hadamard 2-point function above. The singular structure in the limit $y \to x$ of Eq. (2.1.9), sometimes called Hadamard parametrix, is evident as the geodesic distance $\sigma(x, y)$ goes to zero. The length scale appearing in Definition 2.1.1 is naturally present in a massive field theory by m^2, however the definition is ambiguous in the massless case, especially in a conformally invariant theory where no absolute length scale is present. This matter is discussed in [Wal78; Chr78].

Remark. Definition 2.1.1 has some ambiguities even in the massive case:

(i) The Hadamard recursion relations allow to define the functions v, w in terms of power series in $\sigma(x, y)$:

$$v(x, y) = \sum_{n=0}^{\infty} v_n(x, y)\sigma^n(x, y) \qquad w(x, y) = \sum_{n=0}^{\infty} w_n(x, y)\sigma^n(x, y) \tag{2.1.10}$$

These series, however, have in principle little to vanishing convergence radii.

(ii) In general spacetimes, there can be many different geodesics connecting x and y, so that the geodesic distance can be only defined within a geodesically convex normal neighborhood of x, y, i.e. an open set with only one geodesic connecting any pair of points in it.

(iii) The restriction of Definition 2.1.1 to geodesically convex normal neighborhoods still doesn't solve the ambiguity in certain cases where there is no natural unique normal neighborhood of x, y.

These and other technical problems have been addressed in [KW91; Mor21] by modifying Definition 2.1.1 via some $i\varepsilon$ prescription that slightly deforms the geodesic distance, which is however suppressed in the form reported above. This technically ambiguous form of the definition contains nonetheless all the relevant details on the singularity structure needed for this work, especially in view of its reformulation in terms of microlocal analysis that will be given in Section 2.2. An example of the geodesic distance ambiguity is reported in Fig. 2.1.1.

The reason why Hadamard states are so convenient to solve the problem described in the beginning of this section, is that the singular part of their 2-point function is completely state independent and determined by the geometry of the spacetime. In order to understand the logic behind this, it is useful to compare the analogous construction in Minkowski spacetime.

In flat spacetime, normal ordering defines a renormalized version of the stress energy tensor as follows:

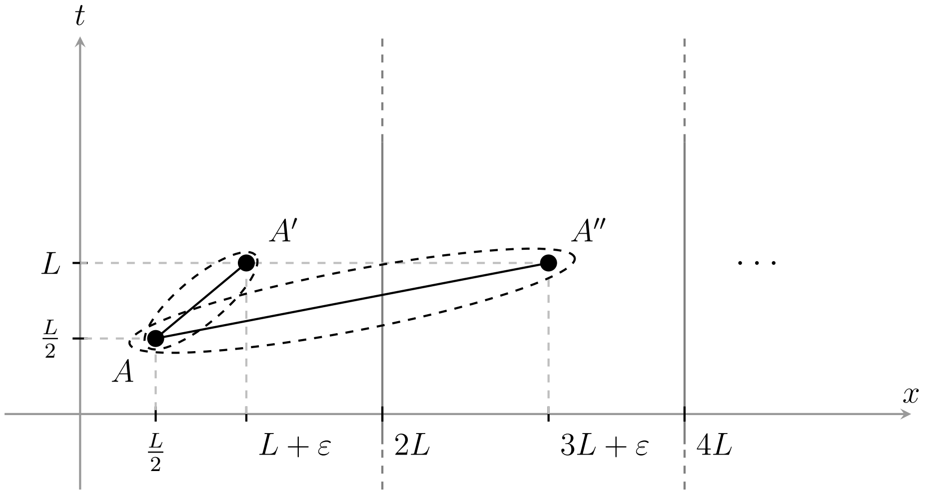

Figure 2.1.1: This example was considered in [Mor21] to show the ambiguity in the definition of the Hadamard parametrix. Consider a (1 + 1)-dimensional Minkowski spacetime wrapped into a cylinder via the identification $(t, x) \sim (t, x + 2L)$. In this way $A' = A''$ and it is possible to find different values for $\sigma(A, A')$ due to the possibility of identifying distinct normal neighborhoods.

$$:\hat{T}_{\mu\nu}: = \hat{T}_{\mu\nu} - \langle \hat{T}_{\mu\nu} \rangle_{\omega_0} \tag{2.1.11}$$

where ω_0 is the unique Minkowski vacuum state. In curved spacetime, the role of Minkowski vacuum is played by Hadamard states. It is possible to renormalize the stress energy tensor $\hat{T}_{\mu\nu}$ by regularizing the divergence with the point splitting procedure of Eq. (2.1.7) and subtracting the state independent singular term of the Hadamard parametrix of $W(x, y)$:

$$\langle \hat{T}^{\mathrm{ren}}_{\mu\nu} \rangle_\omega := \lim_{y \to x} \mathcal{D}^{(x,y)}_{\mu\nu} (W(x, y) - H(x, y)) = \lim_{y \to x} \mathcal{D}^{(x,y)}_{\mu\nu} w(x, y) \tag{2.1.12}$$

This procedure is fully covariant as $H(x, y)$ is a purely geometric quantity, making Eq. (2.1.12) a well defined renormalization scheme for the stress energy tensor in curved spacetime. This scheme is consistent with the axiomatic approach for the definition of a renormalized stress energy tensor of [Wal77].

A key result in the study of Hadamard states is that the condition in Eq. (2.1.9) is preserved under the dynamics of the theory [FSW78]. This means that if the 2-point function of a certain state has the singularity structure of Hadamard form in an open neighborhood of a Cauchy surface, then it has it everywhere in the spacetime. This property of Hadamard state implies that a state that is Hadamard on a Minkowski-like region of spacetime containing a Cauchy surface, will retain its Hadamard form regardless of the curvature of the spacetime in the future of that Cauchy surface. This fact is crucial in physical applications in order to give meaning to the expectation value of the renormalized stress energy tensor across the whole spacetime.

It is also worth noting that this point splitting renormalization scheme can be generalized in a locally covariant way to define arbitrary Wick polynomials $:\phi^k:$ on arbitrary spacetimes [HW01]. This is a crucial development in the construction of interacting theories, which are generally studied in the context of perturbative algebraic quantum field theory (pAQFT) [Rej16].

All these considerations suggest that Hadamard states are the most natural candidates for the definition of a class of physical or physically relevant states for a quantum field theory on curved spacetime, possibly even making Definition 2.1.1 the definition of such class of states.

2.2 The microlocal form of the Hadamard condition

In the previous section it was discussed how Hadamard states arise by requiring that certain bisolutions to the equations of motion of the field theory have a particular singularity structure. This highlights the fact that the singularities of the 2-point function of a Hadamard states are tightly related to the partial differential operator P defining the PDE that the 2-point function itself solves. In the last part of Section 1.3 it was shown how solutions to PDEs defined by operators like P are, in general, distributions. As such, they can have singularities, and it is reasonable to expect that they will be organized in a predictable way that is imposed by P. This point of view already starts to suggest that the singularity structure of the 2-point function of a Hadamard state might be described in terms of the singularity structure of a distribution with constraints given by the differential operator P. These ideas were first proposed by M.J. Radzikowski in [Rad96] for scalar fields, like the free Klein-Gordon theory, and later extended to Dirac fields [Koe95; Kra00; Hol01] and fields that are sections of general vector bundles [SV01]. This approach to the definition of the Hadamard condition is given in the context of microlocal analysis, that is the study of distributions and the propagation of their singularities via the so-called wavefront set. Microlocal analysis is a powerful theory that also allows one to rigorously manipulate distributions. Their singularity structure makes even the most basic operations generally ill-defined, unless one is careful to combine them in appropriate ways [Hör90]. The definition and interpretation of the wavefront set of a distribution can be found in Appendix A.

There is a class of partial differential operators, called normally hyperbolic, that is naturally suited to the analysis of the Hadamard condition. This is due to the particular behavior of the wavefront set under the action of a differential operator. This behavior is related to the form of higher derivative terms of the differential operator, which is encoded in its principal symbol.

Definition 2.2.1 (*Principal symbol*)**.** Let $P\colon \Gamma^\infty(B) \to \Gamma^\infty(B)$ be a partial differential operator of order m and $\alpha = (\alpha_1, \dots, \alpha_4)$ a multi-index. In local coordinates P takes the form:

$$P = \sum_{|\alpha|\leq m} A_\alpha(x) \frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1} \cdots (\partial x^4)^{\alpha_4}} := \sum_{|\alpha|\leq m} A_\alpha(x) D^\alpha \tag{2.2.1}$$

where $A_\alpha(x)$ are smooth bundle endomorphisms. Let $(x, k) \in T^*\mathcal{M}$, in local coordinates $k = (k_1, \dots, k_4)$, then the *principal symbol* of the differential operator P is defined as:

$$P_m(x, k) = \sum_{|\alpha|=m} A_\alpha(x) k_1^{\alpha_1} \cdots k_4^{\alpha_4} := \sum_{|\alpha|=m} A_\alpha(x) k^\alpha \tag{2.2.2}$$

Definition 2.2.2 (*Normally hyperbolic operator*)**.** A second order partial differential operator on a rank r vector bundle over a globally hyperbolic spacetime $P\colon \Gamma^\infty(B) \to \Gamma^\infty(B)$ is said to be *normally hyperbolic* if its principal symbol is given by the metric. This means that, in local coordinates (x^μ), and local trivialization of the bundle $f(x) = (f^1(x), \dots, f^r(x))$, it takes the form:

$$[Pf]^a(x) = g^{\mu\nu}(x)\partial_\mu\partial_\nu f^a(x) + [A^\nu]^a{}_b(x)\partial_\nu f^b + B^a{}_b(x) f^b \tag{2.2.3}$$

Specifically, its principal symbol is $P_2(x,k) = g^{\mu\nu}(x)k_\mu k_\nu \mathbb{1}_B$.

Remark. This kind of differential operators are also-called wave operators because their highest order term reproduces the D'Alembert operator. This also explains the name wavefront set, as that is the set encoding the geometric propagation of the singularities of solutions to the wave equation $P\phi = 0$.

The normally hyperbolic condition on P constrains the cones in the cotangent bundle along which the singularities propagate. In particular the constraint is determined by the principal symbol.

Definition 2.2.3 (*Characteristic set*)**.** Let $P\colon \Gamma^\infty(B) \to \Gamma^\infty(B)$ be a partial differential operator of order m, the *characteristic set of* P is defined as the subset of the cotangent bundle $T^*\mathcal{M}$ where its principal symbol is singular:

$$\mathrm{Char}(P) \coloneqq \{(x,k) \in \dot{T}^*\mathcal{M} \mid \det(P_m(x,k)) = 0\} \tag{2.2.4}$$

where $\dot{T}^*\mathcal{M}$ is the cotangent bundle with the zero section removed.

The following is a key result in the reformulation of the Hadamard condition in terms of microlocal analysis techniques.

Theorem 2.2.1. *Let $P\colon \Gamma^\infty(B) \to \Gamma^\infty(B)$ be a partial differential operator of order m with smooth coefficients and $u \in \mathcal{D}'(B,V)$ be a V-valued distribution on B, then:*

$$\mathrm{WF}(u) \subset \mathrm{Char}(P) \cup \mathrm{WF}(Pu) \tag{2.2.5}$$

where Pu is to be understood as the action of P on distributional sections of B as in Definition 1.3.11.

Proof. See e.g. [Hör90, Th. 8.3.1] □

Normally hyperbolic RFHGHOs define partial differential equations for which E_P and $\mathfrak{W}$ are bisolutions as in Proposition 1.4.5. Moreover, the characteristic set of a normally hyperbolic operators consists only of null covectors, as the inverse metric $g^{\mu\nu}(x)$ is non degenerate. By Theorem 2.2.1, it is immediate that the wavefront set of E_P and $\mathfrak{W}$ must be contained in the subset of the cotangent bundle of null covectors. Taking a look back at Definition 2.1.1, it seems like there is a connection emerging: the singularities of the Hadamard parametrix also arise along null geodesics connecting x with y (for which the geodesic distance $\sigma(x,y) = 0$). It turns out that this connection can be exactly formalized by the following statements.

Theorem 2.2.2. *Let P be a normally hyperbolic RFHGHO, define:*

$$\mathcal{N}^{\pm} := \{(x,k) \in \dot{T}^*\mathcal{M} \mid g(k,k) = 0,\ k \text{ is future}(+)/\text{past}(-) \text{ directed}\} \tag{2.2.6}$$

and define the equivalence relation $(x,k) \sim (y,l)$ if and only if there is a null geodesic γ connecting x and y such that the vectors $k^\sharp, l^\sharp$ are the tangent vectors of the curve at x, y respectively and are related by parallel transport along γ. Then:

$$\begin{aligned} \mathrm{WF}(E_P) &= \{(x,k;y,-l) \in \mathcal{N} \times \mathcal{N} \mid (x,k) \sim (y,l)\} \\ \mathrm{WF}(E_P^{\pm}) &= \{(x,k;y,-l) \in \mathrm{WF}(E_P) \mid x \in J^{\pm}(y)\} \cup \mathrm{WF}(\mathbb{1}) \end{aligned} \tag{2.2.7}$$

where $\mathbb{1}$ is the distribution induced by the identity map as in Definition 1.3.10, i.e. the Dirac delta, and $\mathcal{N} := \mathcal{N}^+ \cup \mathcal{N}^-$.

Proof. See e.g. [San10, Th. A.5]. □

Theorem 2.2.3. *Let P be a normally hyperbolic RFHGHO and $\omega \in \mathscr{A}(P)^*_{+,1}$ be a Hadamard state. The Hadamard condition is equivalent to the following microlocal condition on the wavefront set of the 2-point function associated to the state ω:*

$$\mathrm{WF}(\mathfrak{W}) = \{(x,k;y,-l) \in \mathcal{N}^+ \times \mathcal{N}^- \mid (x,k) \sim (y,l)\} \tag{2.2.8}$$

Proof. See e.g. [SV01, Th. 5.8]. □

Notice how this condition is in perfect accordance with Definition 2.1.1. If a pair of points $x, y \in \mathcal{M}$ are singular for the 2-point function, then they are joined by a null geodesic, i.e. their geodesic distance vanishes, which is exactly the limit in which the Hadamard parametrix is singular.

The wavefront set condition for the 2-point function of a Hadamard state reflects the one of the causal propagator, the additional constraint is given by restricting the singularities to propagate along positive frequencies. What this means is that a singularity at x propagates along a null geodesic in the direction $k^\sharp$, which by Theorem 2.2.3 is future directed, i.e. a positive frequency. This propagation is represented in Fig. 2.2.1.

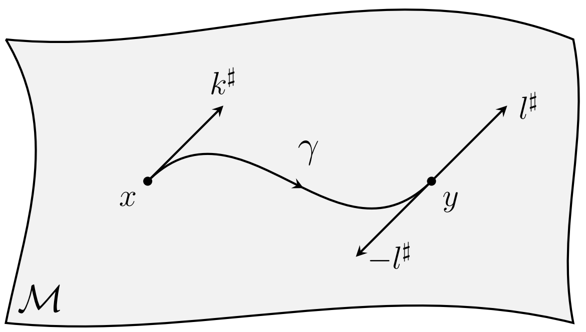

Figure 2.2.1: The singularity propagates from x to y along a null geodesic γ. The directions of the singularities are related by parallel transport, the wavefront set then contains the pair $k, -l$ and specifically selects k to be future pointing.

Remark. This discussion heavily relies on the fact that P is normally hyperbolic, meaning that its principal symbol vanishes exactly on the directions of propagation of the singularities. The microlocal form of the Hadamard condition can be consistently extended to non normally hyperbolic RFHGHOs. The idea behind this extension is to reformulate the discussion in terms of generalized conical neighborhoods $\mathcal{V}^{\pm}$ along which the singularities are allowed to propagate. This leads to the definition of $\mathcal{V}^{+}$-Hadamard states, which reduce to the standard $\mathcal{N}^{+}$-Hadamard states in the normally hyperbolic case. Most of the usual nice properties of Hadamard states are recovered in this generalized setting. A complete and detailed description of this construction can be found in [Few25].

2.3 Hadamard condition on Hilbert space vectors

In Section 1.2 the GNS theorem was presented as a way to systematically represent the algebraic observables and states of a quantum system in the more familiar context of linear operators acting on a Hilbert space and vectors therein. The necessity of working with both kinds of observable algebras introduced in Section 1.4 will be made clear throughout Chapter 3 and Chapter 4 in the context of the measurement framework of quantum field theories. For this reason, the preferred GNS representations of the quantum algebras will be those associated to quasifree states.

Remark. In order to make the notation lighter, from now on the GNS representation map π_ω will be suppressed. The distinction between an algebra element A and its representative $\pi_\omega(A)$ will be clear from the context.

For a quasifree state, the exponential expression:

$$W_P(f) = e^{i\Phi_P(f)} \tag{2.3.1}$$

is well defined as the exponentiation of a selfadjoint operator into a unitary one, both acting on the Hilbert space of the theory $\mathcal{H}_\omega$, where ω is an algebraic quasifree state, as stated by Theorem 1.4.6. It is then desirable to translate the microlocal Hadamard condition in the Hilbert space setting.

Notice first that the 2-point function can be expressed also in terms of state on the Weyl C^*-algebra thanks to the possibility of differentiating the Weyl generators.

Definition 2.3.1 (2*-point function in quasifree representations*)**.** Let $\omega \in \mathscr{A}_W(P)^*_{+,1}$ be a quasifree algebraic state on the Weyl C^*-algebra and Ω_ω the Hilbert space and GNS vector of the representation associated to ω. Then the 2-point function of ω is defined to reproduce the standard 2-point function as follows:

$$\begin{aligned}
\mathfrak{W}(f \otimes g) &= \langle \Omega_\omega, \Phi_P(f)\Phi_P(g)\Omega_\omega\rangle \\
&= -\frac{\partial^2}{\partial s \partial t}\langle \Omega_\omega, W_P(sf)W_P(tg)\Omega_\omega\rangle\Big|_{s=t=0} \\
&= -\frac{\partial^2}{\partial s \partial t}\omega(W_P(sf)W_P(tg))\Big|_{s=t=0}
\end{aligned} \tag{2.3.2}$$

The Hadamard condition can be then equivalently formulated for quasifree states on the Weyl C^*-algebra as follows.

Definition 2.3.2 (*Hadamard state on the Weyl C^*-algebra*)**.** A state $\omega \in \mathscr{A}_W(P)^*_{+,1}$ is Hadamard if and only if:

$$\begin{aligned} \mathrm{WF}(\mathfrak{W}) &= \mathrm{WF}\left(-\frac{\partial^2}{\partial s \partial t}\omega(W_P(s\cdot)W_P(t\cdot))\bigg|_{s=t=0}\right) \\ &= \{(x,k;y,-l) \in \mathcal{N}^+ \times \mathcal{N}^- \mid (x,k) \sim (y,l)\} \end{aligned} \tag{2.3.3}$$

where the distribution is understood as the extension of Eq. (2.3.2) by $\mathbb{C}$-linearity.

A natural question is whether or not quasifree Hadamard states on the unital $*$-algebra $\mathscr{A}(P)$ and on the Weyl C^*-algebra $\mathscr{A}_W(P)$ are equivalent. The following result shows that there is indeed a natural one-to-one correspondence.

Proposition 2.3.1. *Let $\alpha \in \mathscr{A}_W(P)^*_{+,1}$ and $\beta \in \mathscr{A}(P)^*_{+,1}$ be quasifree (Definition 1.4.2) Hadamard (Definition 2.3.2, Theorem 2.2.3) states, then α defines a quasifree Hadamard state $\tilde{\alpha}$ on $\mathscr{A}(P)$ and β defines a quasifree Hadamard state $\tilde{\beta}$ on $\mathscr{A}_W(P)$.*

Proof. First, α defines a state on the unital $*$-algebra $\mathscr{A}(P)$ acting on the algebra in the GNS representation induced by α as follows:

$$\tilde{\alpha}(\Phi_P(f_1)\cdots\Phi_P(f_n)) := (-i)^n \left.\frac{\partial^n}{\partial t_1 \cdots \partial t_n}\langle \Omega_\alpha, W_P(t_1 f_1)\cdots W_P(t_n f_n)\Omega_\alpha\rangle\right|_{t_i=0} \tag{2.3.4}$$

where the index $i = 1,\ldots,n$ labels the different generators of the algebras. This defines a quasifree state on $\mathscr{A}(P)$, as stated in [DG13, Prop. 17.8]. β defines a state on the Weyl algebra $\mathscr{A}_W(P)$ acting on its generators as follows:

$$\tilde{\beta}(W_P(f)) := e^{-\frac{1}{2}\mu^\beta(f\otimes f)} \tag{2.3.5}$$

where $\mu^\beta(f\otimes f)$ is the symmetric part of the 2-point function associated to β, understanding now $\mu^\beta(f\otimes f) = \mu^{\tilde{\beta}}(f,f)$ of Definition 1.4.2. This indeed defines a quasifree state on the Weyl C^*-algebra satisfying positivity and normalization because [Bru+15, Prop. 5.2.9]:

$$|E_P(f,g)| \leq 2\sqrt{\mu^\beta(f\otimes f)}\sqrt{\mu^\beta(g\otimes g)} \tag{2.3.6}$$

This equivalence is stated in e.g. [DG13, Prop. 17.5].

$\tilde{\alpha}$ is Hadamard if $\mathrm{WF}(\tilde{\alpha}(\Phi_P(\cdot)\Phi_P(\cdot)))$ satisfies the microlocal Hadamard condition, but:

$$\tilde{\alpha}(\Phi_P(f)\Phi_P(g)) = -\frac{\partial^2}{\partial s\partial t}\alpha(W_P(sf)W_P(tg))|_{s=t=0} \tag{2.3.7}$$

which satisfies the microlocal Hadamard condition because α is Hadamard [FV03, Eq. 4.3]. Similarly, $\tilde{\beta}$ is Hadamard if $\mathrm{WF}(-\frac{\partial^2}{\partial s\partial t}\tilde{\beta}(W_P(s\cdot)W_P(t\cdot))|_{s=t=0})$ satisfies the microlocal Hadamard condition, however:

$$-\frac{\partial^2}{\partial s\partial t}\tilde{\beta}(W_P(sf)W_P(tg))|_{s=t=0} = \beta(\Phi_P(f)\Phi_P(g)) \tag{2.3.8}$$

Thus $\tilde{\beta}$ is Hadamard on β being Hadamard. ∎

Remark. This result is particularly useful because it allows one to translate results on the unital $*$-algebra $\mathscr{A}(P)$ to the Weyl C^*-algebra $\mathscr{A}_W(P)$, which are frequently used together in the context of quantum measurements in quantum field theory, as will be clear in Chapter 3 and Chapter 4. This shows why quasifree states are naturally preferred for this discussion. The set of algebraic quasifree states that satisfy the Hadamard condition, either on a unital $*$-algebra or on a Weyl C^*-algebra, is denoted as $\mathcal{S}_{qH}$.

The reformulation of the microlocal Hadamard condition on Hilbert spaces is essentially given by the following characterization.

Theorem 2.3.2. *Let $\omega \in \mathscr{A}_W(P)^*_{+,1}$ be a quasifree state on the Weyl C^*-algebra associated to P and $\mathcal{H}$ be the GNS Hilbert space associated to some state on $\mathscr{A}_W(P)$, which is not necessarily ω. If there exists a vector $\psi \in \mathcal{H}$ such that $\omega(A) = \langle \psi, A\psi \rangle$ for any $A \in \mathscr{A}_W(P)$ and for which $W_P(tf)\psi$ is differentiable for any $f \in \Gamma_0^\infty(B)$, define the $\mathcal{H}$-valued distribution:*

$$\begin{aligned} \Phi_P(\cdot)\psi \colon \Gamma_0^\infty(B) &\to \mathcal{H} \\ f &\mapsto \Phi_P(f)\psi := -i\, \frac{d}{dt} W_P(tf)\psi \Big|_{t=0} \end{aligned} \tag{2.3.9}$$

Then ω is Hadamard if and only if $\mathrm{WF}(\Phi_P(\cdot)\psi) \subset \mathcal{N}^-$.

Proof. See e.g. [SVW02, Prop. 6.1], [FV03, Th. 4.2]. □

Theorem 2.3.2 essentially translates the Hadamard condition of a state ω from a statement about the wavefront set of a $\mathbb{C}$-valued bidistribution on the vector bundle B (which is $\mathfrak{W}$), to a statement about the wavefront set of a $\mathcal{H}$-valued distribution on the vector bundle B (which is $\Phi_P(\cdot)\psi$).

Remark. The field operator $\Phi_P(f)$ in Theorem 2.3.2 is defined on the dense domain $D(\Phi_P(f)) \subset \mathcal{H}$ for which its definition exists as a strong limit:

$$\left\| \frac{d}{dt} W_P(tf)\psi \Big|_{t=0} - \left(\frac{W_P(tf)\psi - \psi}{t} \right) \right\| \xrightarrow{t \to 0} 0 \tag{2.3.10}$$

Then the hypotheses of theorem are evidently satisfied in the GNS representation induced by ω itself, taking the GNS vector $\psi = \Omega_\omega$, due to Theorem 1.4.6. It is however convenient to allow for other representations and, consequently, other vectors.

Definition 2.3.3 (*Hadamard set*)**.** Let $\omega \in \mathcal{S}_{qH}$ be a quasifree Hadamard state on $\mathscr{A}_W(P), (\mathcal{H}_\omega, \Omega_\omega)$ its associated GNS Hilbert space and vector and $D_\omega(\Phi_P(f)) \subset \mathcal{H}_\omega$ the dense domain of definition of $-i\, \frac{d}{dt} W_P(tf)\big|_{t=0}$. Define the *Hadamard set of* ω as:

$$\mathrm{Had}(\omega) := \left\{ \psi \in \bigcap_{f \in \Gamma_0^\infty(B)} D_\omega(\Phi_P(f)) \mid \mathrm{WF}(\Phi_P(\cdot)\psi) \subset \mathcal{N}^- \right\} \tag{2.3.11}$$

By Theorem 2.3.2, every $\psi \in \mathrm{Had}(\omega)$ defines an algebraic Hadamard state ω_ψ defined as the state whose GNS Hilbert space and vector is the pair $(\mathcal{H}_\omega, \psi)$, i.e. $\omega_\psi \in \mathcal{S}_{qH}$.

A particularly useful property of the Hadamard set is that it is invariant under the action of finite linear combinations of Weyl generators. What this means is that if $\psi \in \mathrm{Had}(\omega)$ for some $\omega \in \mathcal{S}_{qH}$, then $W_P(f)\psi \in \mathrm{Had}(\omega)$ meaning that $W_P(f)\psi$ again gives rise to an Hadamard state. This is formalized by the next results [FV03].

Lemma 2.3.3. *Let $u, v \in \mathcal{D}'(B; \mathcal{H})$ be Hilbert space valued distributional sections of a vector bundle B. If $\|u(f)\| \leq \|v(f)\|$ for any test section $f \in \Gamma_0^\infty(B^*)$, then* $\mathrm{WF}(u) \subset \mathrm{WF}(v)$.

Proof. If $(y, l) \in \dot{T}^*\mathcal{M}$ is a regular directed point for v, then:

$$\sup_{k\in\mathcal{V}}(1 + \|k\|^n)\|u(\chi e^{ix^\mu k(\partial_\mu)})\|_V \leq \sup_{k\in\mathcal{V}}(1 + \|k\|^n)\|v(\chi e^{ix^\mu k(\partial_\mu)})\|_V < \infty \;\forall n \in \mathbb{N} \tag{2.3.12}$$

showing that (y, l) is regular directed also for u proving the statement. □

Theorem 2.3.4. *Let $\omega \in \mathcal{S}_{qH}$ be a quasifree Hadamard state, then* $\mathrm{Had}(\omega)$ *is invariant under the action of the algebra finitely generated by Weyl operators $W_P(f)$, $f \in \Gamma_0^\infty(B)$.*

Proof. The goal is to show that $\mathrm{WF}(\Phi_P(\cdot)W_P(f)\psi) \subset \mathcal{N}^-$ as the wavefront set of linear combinations of distributions is contained in the union of the individual wavefront sets (see e.g. [Hör90, Sec. 8.2]). Then, using $\Phi_P(\cdot)W_P(f)\psi = W_P(f)\Phi_P(\cdot)\psi + [\Phi_P(\cdot), W_P(f)]\psi$:

$$\mathrm{WF}(\Phi_P(\cdot)W_P(f)\psi) \subset \mathrm{WF}(W_P(f)\Phi_P(\cdot)\psi) \cup \mathrm{WF}([\Phi_P(\cdot), W_P(f)]\psi) \tag{2.3.13}$$

Weyl generators are unitary so that $\|W_P(f)\Phi_P(g)\psi\| = \|\Phi_P(g)\psi\|$, by Lemma 2.3.3 the first term satisfies:

$$\mathrm{WF}(W_P(f)\Phi_P(\cdot)\psi) \subset \mathrm{WF}(\Phi_P(\cdot)\psi) \subset \mathcal{N}^- \tag{2.3.14}$$

For the second term, recall the definition of the field operator in terms of differentiation of Weyl operators and let $\psi \in \mathrm{Had}(\omega)$:

$$\Phi_P(g)\psi = -i \lim_{t\to 0} \frac{W_P(tg)\psi - \psi}{t} \tag{2.3.15}$$

Then the combination $\Phi_P(g)W_P(f)$ can be rewritten using the properties of the Weyl generators in Theorem 1.4.2:

$$\begin{aligned}\Phi_P(g)W_P(f)\psi &= -i \lim_{t\to 0} \frac{W_P(tg) - \mathbb{1}}{t} W(f)\psi \\ &= -iW_P(f) \lim_{t\to 0} \left(\frac{e^{-itE_P(g,f)} - 1}{t} + \frac{W_P(tg) - \mathbb{1}}{t} \right) \psi \\ &= W_P(f)(-E_P(g, f)\psi + \Phi_P(g)\psi)\end{aligned} \tag{2.3.16}$$

This means that:

$$[\Phi_P(g), W_P(f)]\psi = -E_P(g, f)W_P(f)\psi \tag{2.3.17}$$

Again, since Weyl generators are unitary, $\|[\Phi_P(g), W_P(f)]\psi\| = |E_P(g, f)|\|\psi\|$. The integral representation of $E_P(g, f)\psi$ is given in terms of an integration of $g(x)$ multiplied by $E_P f(x)$, which is smooth, thus $\mathrm{WF}(E_P(\cdot, f)\psi) = \varnothing$. Again, by Lemma 2.3.3:

$$\mathrm{WF}([\Phi_P(\cdot), W_P(f)]\psi) \subset \mathrm{WF}(E_P(\cdot, f)\psi) = \varnothing \tag{2.3.18}$$

proving the original statement. □

Theorem 2.3.4 will play a key role in determining how certain types of measurements in quantum field theory influence the properties of Hadamard states.

3 The AQFT measurement framework

3.1 Causal measurements in Quantum Field Theory

Non relativistic quantum mechanics has precise and complete prescriptions that describe the process of measuring a quantum system in a given state. Consider a non relativistic quantum system with associated Hilbert space $\mathcal{H}$ and a density matrix ρ representing a general quantum state. The action of performing a measurement on the system is modeled by the following postulates (see e.g. [NC10]).

Definition 3.1.1 (*Quantum measurement*)**.** A *quantum measurement* is described by a collection of *measurement operators* $\{M_m\} \subset \mathcal{B}(\mathcal{H})$ labeled by the possible outcomes of the experiment. The measurement operators satisfy the *completeness equation*:

$$\sum_m M_m^* M_m = \mathbb{1} \tag{3.1.1}$$

The probability that the result m occurs is given by:

$$p(m) = \mathrm{Tr}[\rho M_m^* M_m] \tag{3.1.2}$$

The state of the system after a measurement that returns the outcome m is given by:

$$\rho_m = \frac{M_m \rho M_m^*}{\mathrm{Tr}[\rho M_m^* M_m]} \tag{3.1.3}$$

The completeness equation represents the fact that probabilities sum to one:

$$\sum_m p(m) = \sum_m \mathrm{Tr}[\rho M_m^* M_m] = \mathrm{Tr}[\rho \sum_m M_m^* M_m] = \mathrm{Tr}[\rho] = 1 \tag{3.1.4}$$

If one is only interested in computing probabilities of the outcomes of some measurement, Definition 3.1.1 can be recast in the formalism of POVMs.

Definition 3.1.2 (*POVM*)**.** Let $\{E_m\} \subset \mathcal{B}(\mathcal{H})$ be a collection of operators labeled by a discrete set of indices. The collection $\{E_m\}$ is said to be a *positive operator-valued measure* (POVM) if $\mathrm{Tr}[\rho E_m] \in [0, 1]$ for any m, density matrix ρ on $\mathcal{H}$ and $\sum_m E_m = \mathbb{1}$. The operators E_m are also-called *effects*.

It is immediate to see that the set $\{E_m\} = \{M_m^* M_m\}$, subject to Definition 3.1.2, defines a POVM. Probabilities can then be computed as $p(m) = \mathrm{Tr}[\rho E_m]$.

Example 3.1.1 (*Expectation value*)**.** A standard example of quantum measurement is the measurement of an observable of the quantum system in a state ρ. The observable is described by a selfadjoint operator $A \in \mathcal{B}(\mathcal{H})$. Assuming that A has a discrete spectrum, the spectral theorem for selfadjoint operators provides a POVM such that:

$$A = \sum_m \lambda_m E_m \tag{3.1.5}$$

where the values $\{\lambda_m\}$ constitute the spectrum of A. The expectation value of the observable A in the state ρ is then computed in the usual statistical sense:

$$\langle A\rangle_\rho = \sum_m \lambda_m p(m) = \sum_m \lambda_m \operatorname{Tr}[\rho E_m] = \operatorname{Tr}[\rho A] \tag{3.1.6}$$

recovering the standard expression. In particular, the set $\{E_m\}$ is a projection-valued measure (PVM), which is a special case of POVM for which the operators E_m are orthogonal projectors.

Remark. The apparatus describing quantum measurements and the computation of expectation values in quantum mechanics has been summarized in the case where the outcomes of an experiment take a discrete number of values. This has been done only for the sake of a cleaner notation, since it is already enough to provide examples in which the non relativistic measurement description clashes with fundamental properties of quantum field theory, which will be the main topic of this section. This machinery, however, can be fully extended to the general case of continuous experimental outcomes. The labels m are replaced by intervals Δ of the Borel σ-algebra of $\mathbb{R}$, encoding the fact that, in the continuous case, the physical instrument has a finite precision and $p(\Delta)$ is interpreted as the probability of the outcome to be in some interval Δ of the real line. This generalization also makes sense of the measure terminology for a POVM, as the spectral theorem for selfadjoint operators defines so-called spectral measures, which are measures on the Borel σ-algebra of $\mathbb{R}$ with values in the set of effects on $\mathcal{H}$. For a detailed treatment of this topic, see e.g. [Mor17].

In Section 1.2 the algebraic formulation of quantum field theory was given in terms of an assignment $\mathcal{O} \to \mathscr{A}(\mathcal{O})$ of an algebra to each region of the spacetime $\mathcal{O} \subset \mathcal{M}$. The algebra $\mathscr{A}(\mathcal{O})$ contains operators that can be interpreted to be localized in the spacetime region $\mathcal{O}$, in particular the selfadjoint ones represent localized observables. The first natural guess for a theory of quantum measurement in the relativistic setting would be to extend the rules of non relativistic quantum mechanics to work for measurement operators $M_m \in \mathscr{A}(\mathcal{O})$ localized in certain regions of spacetime, defining in this way the concept of local quantum measurement. This naive extension, however, ends up failing even in the most basic instance of quantum measurement. This issue was first highlighted in [Sor93] by setting up a theoretical experiment which, assuming the validity of the postulates in Definition 3.1.1, would lead to the possibility of transmitting information at superluminal speed. In other words, the rules of non relativistic quantum mechanical measurements are not compatible with the causal structure of a relativistic quantum field theory. This experiment, which will be now presented, is known as Sorkin's impossible measurement.

Consider three spacetime regions $\mathcal{O}_A, \mathcal{O}_B, \mathcal{O}_C$ in which Alice, Bob and Charlie, respectively, perform their quantum field theoretical measurements. The regions are located in such a way that the following relations hold:

(i) $\mathcal{O}_C \cap J^-(\mathcal{O}_B) = \varnothing, \mathcal{O}_B \cap J^-(\mathcal{O}_A) = \varnothing$

(ii) $\mathcal{O}_C \cap J^+(\mathcal{O}_B) \neq \varnothing, \mathcal{O}_B \cap J^+(\mathcal{O}_A) \neq \varnothing$

(iii) $J(\mathcal{O}_A) \cap \mathcal{O}_C = \varnothing$

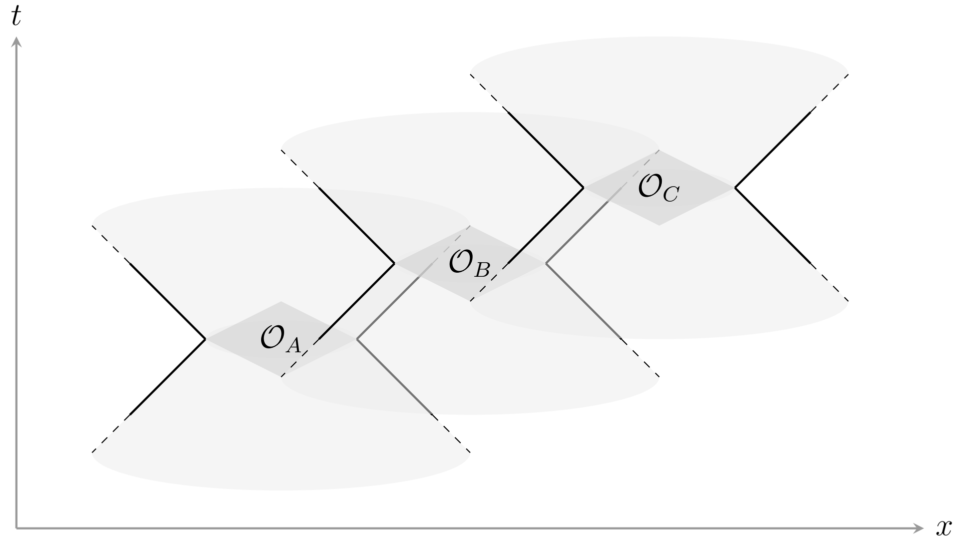

Figure 3.1.1: The setup of the measurements regions in Minkowski spacetime. Alice's and Charlie's regions are evidently causally disconnected. The shaded cones represent the causal future and past of each region as usual.

A pictorial representation of the relative position of the regions is given in Fig. 3.1.1.

Alice and Bob set up the most basic quantum measurement possible, which is described by a single unitary measurement operator. Notice that any unitary operator U satisfies the completeness equation of Definition 3.1.1 $U^*U = \mathbb{1}$. Since the measurement is now in the context of quantum field theory, these unitary operators belong to the local algebras of the regions $\mathcal{O}_A$ and $\mathcal{O}_B$. Following Theorem 1.4.2 and Theorem 1.4.6 the two measurement operators can be built by taking $f_A, f_B \in \Gamma_0^\infty(B)$ with $\mathrm{supp}(f_A) \subset \mathcal{O}_A$ and $\mathrm{supp}(f_B) \subset \mathcal{O}_B$ and defining $U_A := e^{i\Phi_P(f_A)}$ and $U_B := e^{i[\Phi_P(f_B)]^2}$ [FV24]. Charlie performs the quantum measurement of the selfadjoint operator $\Phi_P(f_C)$ localized in its region for $f_C \in \Gamma_0^\infty(B)$ with $\mathrm{supp}(f_C) \subset \mathcal{O}_C$. Now suppose that Alice chooses whether or not to perform the measurement, while Bob certainly performs it. If ρ is the density matrix representing a quasifree state ω of the quantum field and one assumes that the prescriptions of Definition 3.1.1 remain valid in quantum field theory, Charlie performs his measurement in one of the following states:

$$\rho_{AB} = U_B U_A \rho U_A^* U_B^*, \quad \rho_B = U_B \rho U_B^* \tag{3.1.7}$$

Now, the expectation value of Charlie's quantum measurement can be either one of the following:

$$\langle \Phi_P(f_C) \rangle_{AB} = \mathrm{Tr}[\rho_{AB} \Phi_P(f_C)], \quad \langle \Phi_P(f_C) \rangle_B = \mathrm{Tr}[\rho_B \Phi_P(f_C)] \tag{3.1.8}$$

Using the cyclic property of the trace, the expectation values can be rewritten as:

$$\langle \Phi_P(f_C) \rangle_{AB} = \mathrm{Tr}[\rho U_A^* U_B^* \Phi_P(f_C) U_B U_A], \quad \langle \Phi_P(f_C) \rangle_B = \mathrm{Tr}[\rho U_B^* \Phi_P(f_C) U_B] \tag{3.1.9}$$

Using BCH formula and the defining property of the field operators presented in Theorem 1.4.1 $[\Phi_P(g), \Phi_P(h)] = iE_P(g,h)\mathbb{1}$, one can write:

$$
\begin{aligned}
U_B^* e^{i\Phi_P(tf_C)} U_B &= e^{-i[\Phi_P(f_B)]^2} e^{i\Phi_P(tf_C)} e^{i[\Phi_P(f_B)]^2} \\
&= e^{i\Phi_P(tf_C)+2iE_P(f_B,tf_C)\Phi_P(f_B)} \\
&= e^{i\Phi_P(tf_C+2tE_P(f_B,f_C)f_B)}
\end{aligned}
\tag{3.1.10}
$$

Then using the Weyl relation in its exponential form one can rearrange the expression $e^{i\Phi_P(g)}e^{i\Phi_P(h)} = e^{-\frac{i}{2}E_P(g,h)\mathbb{1}}e^{i\Phi_P(g+h)} = e^{-iE_P(g,h)\mathbb{1}}e^{i\Phi_P(h)}e^{i\Phi_P(g)}$, so that:

$$
\begin{aligned}
U_A^* U_B^* e^{i\Phi_P(tf_C)} U_B U_A &= e^{i\Phi_P(-f_A)} e^{i\Phi_P(tf_C+2tE_P(f_B,f_C)f_B)} e^{i\Phi_P(f_A)} \\
&= e^{-iE_P[-f_A,tf_C+2tE_P(f_B,f_C)f_B]\mathbb{1}} U_B^* e^{i\Phi_P(tf_C)} U_B \\
&= e^{2itE_P(f_A,f_B)E_P(f_B,f_C)\mathbb{1}} U_B^* e^{i\Phi_P(tf_C)} U_B
\end{aligned}
\tag{3.1.11}
$$

where $E_P(f_A, f_C) = 0$ because $\mathrm{supp}(f_A)$ and $\mathrm{supp}(f_C)$ are causally disjoint. Then differentiating in $t = 0$ yields:

$$
\begin{aligned}
U_A^* U_B^* \Phi_P(f_C) U_B U_A &= -i \frac{d}{dt} \left(U_A^* U_B^* e^{i\Phi_P(tf_C)} U_B U_A \right)\Big|_{t=0} \\
&= U_B^* \Phi_P(f_C) U_B + 2E_P(f_A, f_B)E_P(f_B, f_C)\mathbb{1}
\end{aligned}
\tag{3.1.12}
$$

This means that:

$$
\langle \Phi_P(f_C) \rangle_{AB} = \langle \Phi_P(f_C) \rangle_B + 2E_P(f_A, f_B)E_P(f_B, f_C) \tag{3.1.13}
$$

In general $2E_P(f_A, f_B)E_P(f_B, f_C) \neq 0$ due to how the measurement regions were set up. This means that Charlie can perform a quantum measurement in his local region of spacetime $\mathcal{O}_C$ and determine whether or not Alice has performed her experiment in the region $\mathcal{O}_A$ even though their regions are causally disconnected. This kind of set up could be, in principle, implemented to transmit information in a superluminal way from $\mathcal{O}_A$ to $\mathcal{O}_B$, suggesting that the non relativistic framework of quantum measurement cannot be straightforwardly adapted to quantum field theory.

3.2 Systems and probes for QFT measurements

In recent years C.J. Fewster and R. Verch proposed a new framework to model local measurements in QFT [FV20]. Mimicking Definition 3.1.1, this framework comes with prescriptions to compute expectation values of observables and rules to predict the post-measurement state update but, unlike the non relativistic one, it can be shown to be free of causal inconsistencies such as Sorkin's impossible measurement. The main idea is that in physics an experiment is carried out by means of a measurement instrument interacting with the system that one wants to measure. This approach was already in use for non relativistic theories and can be seen as an operational axiomatic description of the quantum measurements of Section 3.1, see e.g. [Bus+16]. The system and the instrument (probe) are then modeled as QFTs that interact in a region of spacetime and are free outside that interaction zone. The AQFT formulation of this framework has the great advantages of being both model-independent and applicable to curved spacetimes. The latter is of utmost importance for purely quantum field theoretical phenomena like the

Unruh [Unr76] and Hawking [Haw75] effects, where a solid theoretical description often lacks a complementary treatment of their measuring processes.

Definition 3.2.1 (*LCQFTs combinations*)**.** Let $\mathscr{L}_A$ and $\mathscr{L}_B$ be LCQFTs as in Definition 1.2.10. The *uncoupled combination* of $\mathscr{L}_A$ and $\mathscr{L}_B$ is the LCQFT whose local algebras of observables are defined as:

$$\mathscr{L}_U(\mathcal{O}) \coloneqq \mathscr{L}_A(\mathcal{O}) \otimes \mathscr{L}_B(\mathcal{O}) \tag{3.2.1}$$

for all regions $\mathcal{O} \subset \mathcal{M}$.

Let $K \subset \mathcal{M}$ be a compact subset of the spacetime manifold. A LCQFT $\mathscr{L}_C$ is a *coupled combination* of $\mathscr{L}_A$ and $\mathscr{L}_B$ with *coupling zone* K if for each $L \subset \mathcal{M} \setminus \mathrm{ch}(K)$ causally convex subset outside the causal hull of K there is an **Alg** isomorphism:

$$\chi_L \colon \mathscr{L}_A(L) \otimes \mathscr{L}_B(L) \to \mathscr{L}_C(L) \tag{3.2.2}$$

which is compatible with the locality structure of the LCQFTs, i.e. for each $L' \subset L$ the following diagram commutes:

$$\begin{tikzcd}
\mathscr{L}_A(L') \otimes \mathscr{L}_B(L') \arrow[r, "\mathscr{L}_A(\iota_{L;L'})\otimes\mathscr{L}_B(\iota_{L;L'})"] \arrow[d, "\chi'_L"'] & \mathscr{L}_A(L) \otimes \mathscr{L}_B(L) \arrow[d, "\chi_L"] \\
\mathscr{L}_C(L') \arrow[r, "\mathscr{L}_C(\iota_{L;L'})"'] & \mathscr{L}_C(L)
\end{tikzcd} \tag{3.2.3}$$

where $\iota_{L;L'}$ is the inclusion **Loc** morphism of the region L' into the region L.

Let's unpack Definition 3.2.1. The uncoupled combination of a pair of QFTs is just a QFT whose local algebras are non interacting products of the individual ones. This means that a local operator of $\mathscr{L}_U$ that can be written as a tensor product of operators from $\mathscr{L}_A$ and $\mathscr{L}_B$ is acted on by tensor products of states as follows:

$$\upsilon(U) = (\omega \otimes \sigma)(A \otimes B) = \omega(A)\sigma(B) \tag{3.2.4}$$

where $U \in \mathscr{L}_U(\mathcal{O}), \upsilon \in \mathscr{L}_U(\mathcal{O})^*_{+,1}$ and similarly for A, B and ω, σ. In particular, $\mathscr{L}_U$ satisfies Einstein causality and the time-slice axiom if the original pair does.

The coupled combination is basically saying that the QFT $\mathscr{L}_C$ is equivalent to the uncoupled combination $\mathscr{L}_U$ in the sense of Definition 1.2.11, when it is restricted to regions strictly outside the coupling zone. This definition formalizes the fact that the interaction needed to measure a QFT happens in a compact spacetime zone and, outside that zone, the QFTs are completely independent.

The goal is now to identify the coupled combination $\mathscr{L}_C$ with the uncoupled $\mathscr{L}_U$ at early times (before the interaction) and at late times (after the interaction). To do that consider the following lemma.

Lemma 3.2.1. *Let $\mathcal{M}^{\pm} \coloneqq \mathcal{M} \setminus J^{\mp}(K)$ be the natural in/out regions of the coupling zone K. The* **Alg** *morphisms*

$$\mathscr{L}_A(\iota_{\mathcal{M};\mathcal{M}^{\pm}}) \coloneqq \alpha^{\pm},\ \mathscr{L}_B(\iota_{\mathcal{M};\mathcal{M}^{\pm}}) \coloneqq \beta^{\pm},\ \mathscr{L}_C(\iota_{\mathcal{M};\mathcal{M}^{\pm}}) \coloneqq \gamma^{\pm} \tag{3.2.5}$$

are **Alg** *isomorphisms.*

Proof. The sets $\mathcal{M}^{\pm}$ contain a Cauchy surface because K is compact. For this reason the maps $\iota_{\mathcal{M}^{\pm}}$ are Cauchy **Loc** morphism. By the time-slice axiom, all of the **Alg** morphisms in the lemma are isomorphisms. □

Fig. 3.2.1 displays an example of the spacetime regions considered up to this point.

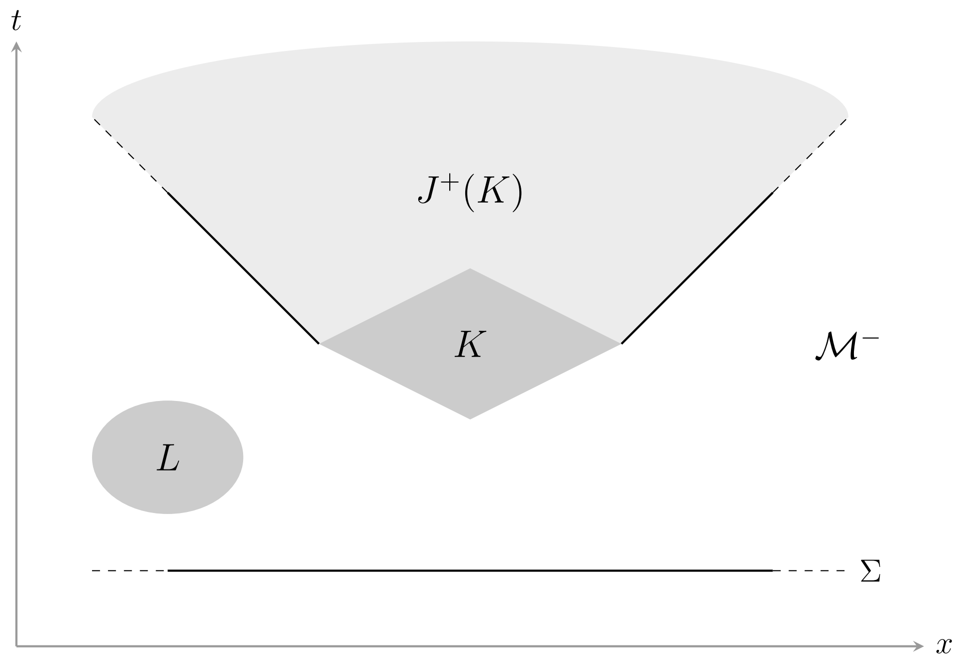

Figure 3.2.1: In Minkowski spacetime it is immediate to se that $\mathcal{M}^-$ contains a Cauchy surface Σ of constant time. The set L does not intersect the causal hull of the coupling zone K.

The following definition brings the fundamental object necessary to model the measurement interaction. Its algebraic and model independent definition makes it possible to develop the theory of QFT measurements at a very general level.

Definition 3.2.2 (*Response maps and scattering morphism*)**.** Let $\chi^{\pm} := \chi_{\mathcal{M}^{\pm}}$ be the **Alg** isomorphisms of Definition 3.2.1 associated to the in/out regions. The **Alg** isomorphisms:

$$\tau^{\pm} := \gamma^{\pm} \circ \chi^{\pm} \circ (\alpha^{\pm} \otimes \beta^{\pm})^{-1} \colon \mathscr{L}_A(\mathcal{M}) \otimes \mathscr{L}_B(\mathcal{M}) \to \mathscr{L}_C(\mathcal{M}) \tag{3.2.6}$$

identifying the uncoupled system at early$(-)$/late$(+)$ time with the coupled combination are called *advanced$(-)$/retarded$(+)$ response maps.*

The composition $\Theta := (\tau^-)^{-1} \circ \tau^+$ is an automorphism of the algebra $\mathscr{L}_A(\mathcal{M}) \otimes \mathscr{L}_B(\mathcal{M})$ called *scattering morphism.*

Remark. The scattering morphism evolves the free observables of $\mathscr{L}_A(\mathcal{M}) \otimes \mathscr{L}_B(\mathcal{M})$ after the interaction backwards in time, through the coupling zone into the free region before the interaction. This can be seen as the familiar action of the S-matrix in the standard scattering formalism, hence the name scattering morphism. Here before/after is interpreted as causal past/future, which in relativity is not strictly related to actual time.

The action of the response maps can be better seen as the following chain:

$$\mathscr{L}_A(\mathcal{M}) \otimes \mathscr{L}_B(\mathcal{M}) \xrightarrow{(\alpha^\pm \otimes \beta^\pm)^{-1}} \mathscr{L}_A(\mathcal{M}^\pm) \otimes \mathscr{L}_B(\mathcal{M}^\pm) \xrightarrow{\chi^\pm} \mathscr{L}_C(\mathcal{M}^\pm) \xrightarrow{\gamma^\pm} \mathscr{L}_C(\mathcal{M}) \quad (3.2.7)$$

which by Lemma 3.2.1 and Definition 3.2.1 is a combination of isomorphisms. Fig. 3.2.2 represents the action of the response maps taking the uncoupled observables into the coupled combination algebra.

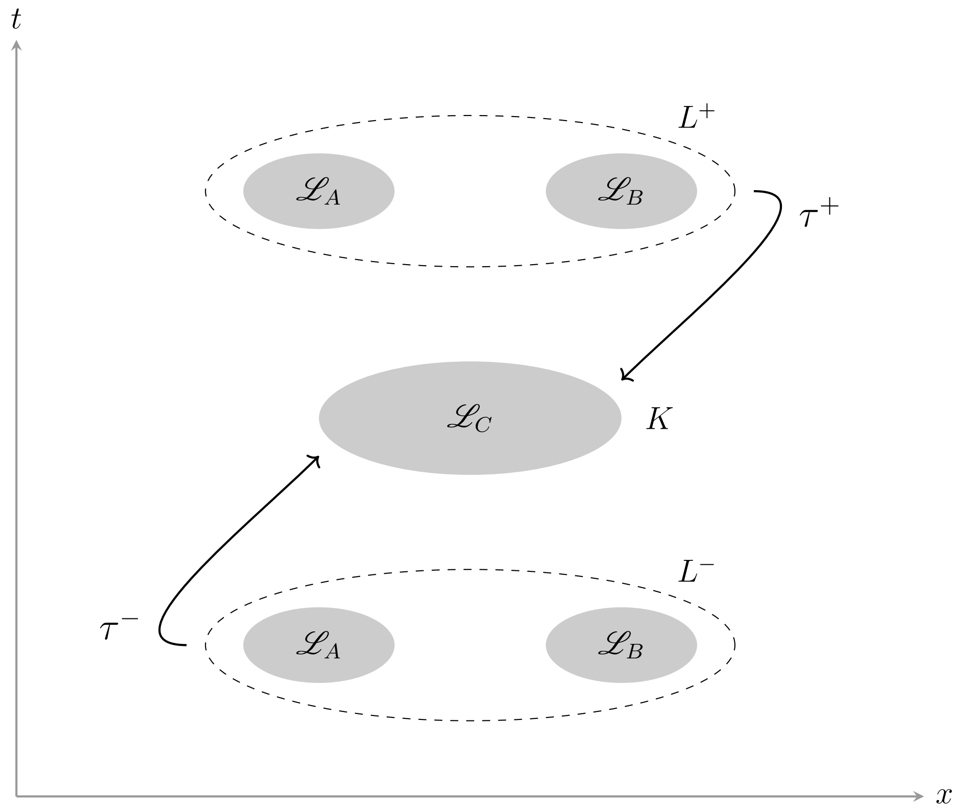

Figure 3.2.2: The response map $\tau^\pm$ are isomorphisms from the uncoupled quantum algebras localized in $L^\pm \subset \mathcal{M}^\pm$ to the coupled quantum algebra localized in K. Outside K the uncoupled and the coupled algebras are isomorphic by definition via the χ_L maps.

In a QFT experiment, the system and probe are prepared at early times (before the interaction) in states ω and σ respectively, while the measurement of observables happens at late times (after the interaction). It is useful to consider effects, which are interpreted as success/fail tests, so that $\omega(A) \coloneqq p(A;\omega)$ is interpreted as the probability of success of the test A on the system in the state ω. On a pair of QFTs, an effect A of the system and an effect B on the probe create an effect $A \otimes B$ on the uncoupled combination. On the state $\omega \otimes \sigma$ this evaluates to:

$$(\omega \otimes \sigma)(A \otimes B) = \omega(A)\sigma(B) = p(A;\omega)p(B;\sigma) = p(A\&B;\omega \otimes \sigma) \qquad (3.2.8)$$

which is clearly interpreted as the probability of success of the joint test $A\&B$ on the states ω and σ. The link between effects (POVMs) and observables (selfadjoint operators) has been already mentioned in Example 3.1.1, for further details see e.g. [Bus+16, Ch. 9].

In the language of QFT measurements just described, the experimental setup consists of the following objects:

(i) An uncorrelated state at early times $(\omega \otimes \sigma) \in (\mathscr{L}_A(\mathcal{M}) \otimes \mathscr{L}_B(\mathcal{M}))^*_{+,1}$.

(ii) An uncorrelated effect at late times $(A \otimes B) \in \mathscr{L}_A(\mathcal{M}) \otimes \mathscr{L}_B(\mathcal{M})$.

Being able to read the outcome of a probe measurement means that it is desirable to derive an expression for the following conditioned probability:

$$p(A \mid B; \omega \otimes \sigma) = \frac{p(A\&B; \omega \otimes \sigma)}{p(\mathbb{1}_A\&B; \omega \otimes \sigma)} \tag{3.2.9}$$

This is interpreted as the probability of success of the system effect A, which is what one wants to infer, conditioned on the success of the probe's effect B, which is what one actually measures. The denominator acts as the normalization factor given by the probability of success of B, which in terms of the uncoupled combination figures as the effect $\mathbb{1} \otimes B$.

To compute $p(A\&B; \omega \otimes \sigma)$ it is needed to identify late-time observables with early time states. This is where the scattering morphism plays its main role so that:

$$p(A\&B; \omega \otimes \sigma) = (\omega \otimes \sigma)(\Theta(A \otimes B)) \tag{3.2.10}$$

This equation can be understood visually from Fig. 3.2.2 and from the chain Eq. (3.2.7) composed as:

$$(A\otimes B) \in \mathscr{L}_A(\mathcal{M})\otimes\mathscr{L}_B(\mathcal{M}) \xrightarrow{\tau^+} \mathscr{L}_C(\mathcal{M}) \xrightarrow{(\tau^-)^{-1}} \mathscr{L}_A(\mathcal{M})\otimes\mathscr{L}_B(\mathcal{M}) \ni \Theta(A\otimes B) \tag{3.2.11}$$

Eq. (3.2.9) can be rewritten as a σ and B dependent functional on the local algebras of the system. This defines a state that can be used to directly compute the conditioned probabilities, called updated state. In some sense, this takes the role of the state update rule of Definition 3.1.1.

Definition 3.2.3 (*State update*)**.** Let $\mathscr{L}_A$ and $\mathscr{L}_B$ be the system and probe QFTs, respectively prepared in states ω and σ at early time, and Θ be the scattering morphism modeling the interaction. The *selective state update* of ω is the state of the system QFT that reproduces the conditional probability for the joint test of a probe effect B with any system effect A, conditioned on the success of B. It takes the following form:

$$\omega^{\text{sel}}(A) := \frac{(\omega \otimes \sigma)(\Theta(A \otimes B))}{(\omega \otimes \sigma)(\Theta(\mathbb{1}_A \otimes B))} \tag{3.2.12}$$

If the conditioning effect of the probe is the identity, the new state is called *non-selective state update* and takes the following form:

$$\omega^{\text{n.s.}}(A) := (\omega \otimes \sigma)(\Theta(A \otimes \mathbb{1}_B)) \tag{3.2.13}$$

The induced measurement of the system observable A is said to be either a *selective measurement* or *non-selective measurement* accordingly.

To see how this framework solves the problem presented in Section 3.1, it is needed to introduce the concept of multiple measurements, make an assumption of so-called causal factorization and state the following key properties of the scattering morphism [FV20].

Theorem 3.2.2. *Let $\mathscr{L}_A$ and $\mathscr{L}_B$ be a system and a probe QFT coupled in a compact set $K \subset \mathcal{M}$, $\mathscr{L}_U$ their uncoupled combination and Θ the scattering morphism of Definition 3.2.2.*

(i) If L is a causally convex open subset of the causal complement $K^\perp = \mathcal{M}^+ \cap \mathcal{M}^-$ of K, then the scattering morphism acts trivially on its local uncoupled combination algebra, i.e. $\Theta \mathscr{L}_U(L) = \mathscr{L}_U(L)$.

(ii) If $L^\pm$ are causally convex open subsets of $\mathcal{M}^\pm$ such that $L^+ \subset D(L^-)$, where $D(L^-)$ is the Cauchy development of L^-, then $\Theta \mathscr{L}_U(L^+) \subset \mathscr{L}_U(L^-)$.

Proof. **(i)** For any $X \subset Y \subset \mathcal{M}$ define the morphisms:

$$\mathscr{L}_A(\iota_{Y;X}) \coloneqq \alpha_{Y;X}, \; \mathscr{L}_B(\iota_{Y;X}) \coloneqq \beta_{Y;X}, \; \mathscr{L}_C(\iota_{Y;X}) \coloneqq \gamma_{Y;X} \tag{3.2.14}$$

Notice that with this notation $\alpha^\pm = \alpha_{\mathcal{M};\mathcal{M}^\pm}$, and similarly for β, γ. Using the definition of response maps and of coupled combination, the following holds:

$$\begin{aligned} \tau^\pm \circ (\alpha_{\mathcal{M};L} \otimes \beta_{\mathcal{M};L}) &= \gamma^\pm \circ \chi^\pm \circ (\alpha^\pm \otimes \beta^\pm)^{-1} \circ (\alpha_{\mathcal{M};L} \otimes \beta_{\mathcal{M};L}) \\ &= \gamma^\pm \circ \chi^\pm \circ (\alpha_{\mathcal{M}^\pm;L} \otimes \beta_{\mathcal{M}^\pm;L}) \\ &= \gamma^\pm \circ \gamma_{\mathcal{M}^\pm;L} \circ \chi_L \\ &= \gamma_{\mathcal{M};L} \circ \chi_L \end{aligned} \tag{3.2.15}$$

Then, since $L \subset \mathcal{M}^+ \cap \mathcal{M}^-$, it follows that the morphism χ_L is the same for both τ^+ and τ^-:

$$\tau^+ \circ (\alpha_{\mathcal{M};L} \otimes \beta_{\mathcal{M};L}) = \tau^- \circ (\alpha_{\mathcal{M};L} \otimes \beta_{\mathcal{M};L}) \tag{3.2.16}$$

Then the scattering morphism acts as follows:

$$\begin{aligned} \Theta(\alpha_{\mathcal{M};L} \otimes \beta_{\mathcal{M};L}) &= (\tau^-)^{-1} \circ \tau^+ \circ (\alpha_{\mathcal{M};L} \otimes \beta_{\mathcal{M};L}) \\ &= (\alpha_{\mathcal{M};L} \otimes \beta_{\mathcal{M};L}) \end{aligned} \tag{3.2.17}$$

This means that any $U = (A \otimes B) \in \mathscr{L}_U(L)$ uncoupled observable localized in L, included in the global algebra $\mathscr{L}_U(\mathcal{M})$ via the inclusion map $(\alpha_{\mathcal{M};L} \otimes \beta_{\mathcal{M};L})$, is invariant under the action of the scattering morphism:

$$\Theta \mathscr{L}_U(\mathcal{M}) = \Theta(\alpha_{\mathcal{M};L} \otimes \beta_{\mathcal{M};L}) \mathscr{L}_U(L) = (\alpha_{\mathcal{M};L} \otimes \beta_{\mathcal{M};L}) \mathscr{L}_U(L) \tag{3.2.18}$$

(ii) Eq. (3.2.15) can be applied for $L = L^\pm$ giving:

$$\tau^+ \circ (\alpha_{\mathcal{M};L^+} \otimes \beta_{\mathcal{M};L^+}) = \gamma_{\mathcal{M};L^+} \circ \chi_{L^+}, \quad (\tau^-)^{-1} \circ (\gamma_{\mathcal{M};L^-}) = (\alpha_{\mathcal{M};L^-} \otimes \beta_{\mathcal{M};L^-}) \circ \chi_{L^-}^{-1} \tag{3.2.19}$$

By the time-slice axiom, since $L^+ \subset D(L^-)$ there is an embedding of local coupled algebras $\iota_C \colon \mathscr{L}_C(L^+) \hookrightarrow \mathscr{L}_C(L^-)$ such that $\gamma_{\mathcal{M};L^+} = \gamma_{\mathcal{M};L^-} \circ \iota_C$, then:

$$\begin{aligned} \Theta(\alpha_{\mathcal{M};L^+} \otimes \beta_{\mathcal{M};L^+}) &= (\tau^-)^{-1} \circ \tau^+ \circ (\alpha_{\mathcal{M}:L^+} \otimes \beta_{\mathcal{M};L^+}) \\ &= (\tau^-)^{-1} \circ \gamma_{\mathcal{M};L^+} \circ \chi_{L^+} \\ &= (\alpha_{\mathcal{M};L^-} \otimes \beta_{\mathcal{M};L^-}) \circ \chi_{L^-}^{-1} \circ \iota_C \circ \chi_{L^+} \\ &\coloneqq (\alpha_{\mathcal{M};L^-} \otimes \beta_{\mathcal{M};L^-}) \circ \delta \end{aligned} \tag{3.2.20}$$

where $\delta\colon \mathscr{L}_A(L^+) \otimes \mathscr{L}_B(L^+) \hookrightarrow \mathscr{L}_A(L^-) \otimes \mathscr{L}_B(L^-)$ is an embedding of the uncoupled combination localized at L^+ into the uncoupled combination localized at L^- existing thanks to the hypothesis $L^+ \subset D(L^-)$. Then as for the first part of the theorem the following holds:

$$\Theta\mathscr{L}_U(\mathcal{M}) = \Theta(\alpha_{\mathcal{M};L^+} \otimes \beta_{\mathcal{M};L^+})\mathscr{L}_U(L^+) \subset (\alpha_{\mathcal{M};L^-} \otimes \beta_{\mathcal{M};L^-})\mathscr{L}_U(L^-) \tag{3.2.21}$$

□

Remark. The first point of Theorem 3.2.2 says that the scattering morphism acts trivially on observables that are localized in regions of spacetime causally disconnected from the interacting region, as one would expect. The second point is essentially stating that the observables of a post interaction region L^+ that is fully determined by a pre interaction region L^- via Cauchy development, are necessarily scattered back to observables localized in the domain of determinacy of L^+, which is L^-.

Multiple successive measurements can be modeled in a covariant way.

Definition 3.2.4 (*Causally ordered sets*)**.** Let K_1, K_2 be compact sets, if $K_1 \cap J^+(K_2) = \varnothing$, the pair is said to be causally orderable and it is denoted as $K_1 \lhd K_2$.

Definition 3.2.5 (*Causal factorization*)**.** Consider a pair of causally ordered compact sets $K_1 \lhd K_2$, a system QFT $\mathscr{L}_A$ and a pair of probe QFTs $\mathscr{L}_{B_j}, j = 1, 2$. Let $\mathscr{L}_{C_j}$ be the coupled combinations of $\mathscr{L}_A$ and $\mathscr{L}_{B_j}$ with coupling zone K_j, and associated scattering morphisms Θ_j. Consider also $\mathscr{L}_{B_1} \otimes \mathscr{L}_{B_2}$ as a single probe theory and let $\mathscr{L}_C$ be a coupled combination of $\mathscr{L}_A$ with the combined probe theory and associated scattering morphism Θ. The theory satisfies *causal factorization*, if the scattering morphism factorizes as follows:

$$\Theta = (\Theta_1 \otimes_3 \mathbb{1}) \circ (\Theta_2 \otimes_2 \mathbb{1}) \tag{3.2.22}$$

The tensor product subscript indicates the slot on which the second factor acts:

$$\begin{aligned} (\Theta_2 \otimes_2 \mathbb{1})(A \otimes (B_1 \otimes B_2)) &= (\Theta_2(A \otimes B_2) \otimes_2 B_1) \\ (\Theta_1 \otimes_3 \mathbb{1})(A \otimes (B_1 \otimes B_2)) &= (\Theta_1(A \otimes B_1) \otimes_3 B_2) \end{aligned} \tag{3.2.23}$$

Here and in the following discussion, the identities are understood to act on their respective spaces where no subscript is present. Using the same notation of Section 3.1, in this framework Sorkin's impossible measurement is modeled by assigning QFTs $\mathscr{L}_A, \mathscr{L}_B$ to Alice and Bob respectively, both acting as probe theories for a third system QFT $\mathscr{L}_S$, not to be confused with the A, B system/probe of the general discussion above. Alice and Bob perform non-selective measurements by coupling their probes in zones $K_A \subset \mathcal{O}_A, K_B \subset \mathcal{O}_B$. Assume that Alice and Bob prepare their probes in states σ_A, σ_B respectively, at early times, while the system QFT is prepared in a state ω, also at early time. Picking now a system observable localized in $\mathcal{O}_C$, i.e. $C \in \mathscr{L}_S(\mathcal{O}_C)$, the problem reduces to comparing its expectation value on the non-selective state update conditioned on Bob's measurement alone vs Bob's and Alice's measurements:

$$\omega_B^{\text{n.s.}}(C) \stackrel{?}{=} \omega_{AB}^{\text{n.s.}}(C) \tag{3.2.24}$$

The first member is easy and just evaluates to:

$$\omega_B^{\text{n.s.}}(C) = (\omega \otimes \sigma_B)(\Theta_B(C \otimes \mathbb{1})) \tag{3.2.25}$$

where Θ_B is the scattering morphism relative to Bob's measurement. The second member reads as:

$$\omega_{AB}^{\text{n.s.}}(C) = (\omega \otimes \sigma_A \otimes \sigma_B)(\Theta(C \otimes \mathbb{1} \otimes \mathbb{1})) \tag{3.2.26}$$

Assume causal factorization so that the scattering morphism relative to the full double probe reads as:

$$\Theta = (\Theta_A \otimes_3 \mathbb{1}) \circ (\Theta_B \otimes_2 \mathbb{1}) \tag{3.2.27}$$

Consider the first action:

$$(\Theta_B \otimes_2 \mathbb{1})(C \otimes \mathbb{1} \otimes \mathbb{1}) = \Theta_B(C \otimes \mathbb{1}) \otimes \mathbb{1} \tag{3.2.28}$$

In [BFR21, Lemma 3, 4] it is shown that in this setup it is always possible to find a region $\mathcal{N} \subset \mathcal{M}$ such that $\mathcal{N} \cap J(\mathcal{O}_A) = \varnothing$ and $\mathcal{O}_C \subset D(\mathcal{N})$, see Fig. 3.2.3 for an illustration.

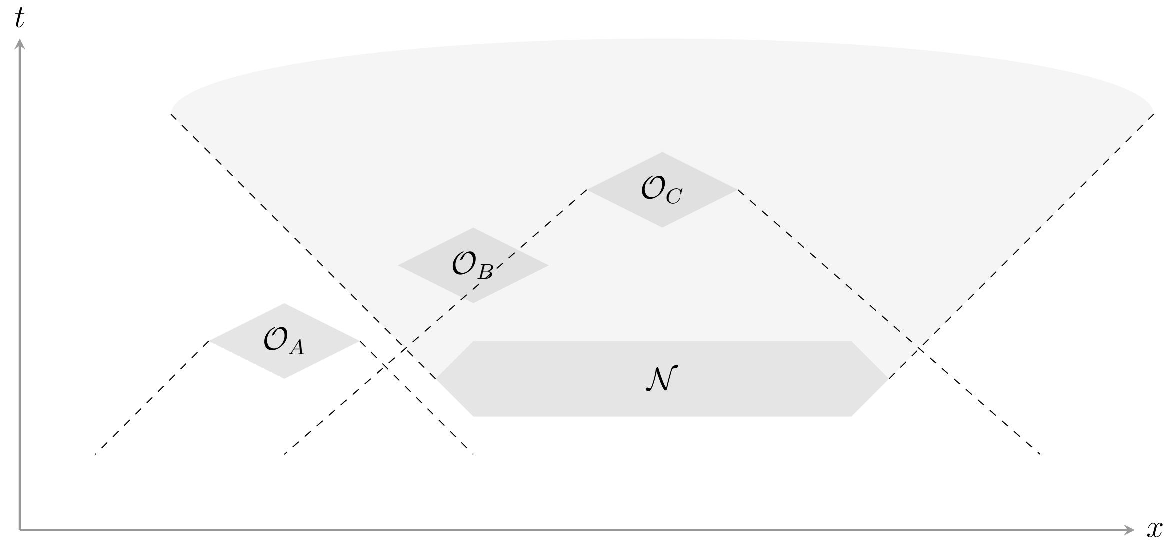

Figure 3.2.3: The region $\mathcal{N}$ is causally disconnected from Alice's local measurement in $\mathcal{O}_A$, while also serving as a domain of determinacy for the region $\mathcal{O}_C$.

By Theorem 3.2.2, it follows that $\Theta_B(C \otimes \mathbb{1})$ is in the uncoupled combination of $\mathscr{L}_S$ and $\mathscr{L}_B$ localized in $\mathcal{N}$. But then, since $\mathcal{N}$ is causally disjoint from $\mathcal{O}_A$, it follows again from Theorem 3.2.2 that $(\Theta_A \otimes_3 \mathbb{1})$ acts trivially on $(\Theta_B \otimes_2 \mathbb{1})(C \otimes \mathbb{1} \otimes \mathbb{1})$ so that:

$$\begin{aligned} \omega_{AB}^{\text{n.s.}}(C) &= (\omega \otimes \sigma_A \otimes \sigma_B)(\Theta(C \otimes \mathbb{1} \otimes \mathbb{1})) \\ &= (\omega \otimes \sigma_A \otimes \sigma_B)((\Theta_B \otimes_2 \mathbb{1})(C \otimes \mathbb{1} \otimes \mathbb{1})) \\ &= (\omega \otimes \sigma_B)(\Theta_B(C \otimes \mathbb{1})) \\ &= \omega_B^{\text{n.s.}}(C) \end{aligned} \tag{3.2.29}$$

showing that the expectation value of an observable localized in a region causally disconnected from Alice's region cannot tell anything about whether or not Alice performed her measurement.

3.3 Hadamard states in the measurement framework

The measurement framework presented in Section 3.2 completely characterizes the process of a quantum measurement, using the modern paradigm of systems and probes, in the context of QFT. On one hand, it has the great advantage of being fully model independent as no mention of a specific type of QFT for either the system or the probe has been made and no assumptions on the interaction were required. On the other hand, the technical difficulties of scattering theory have not magically vanished, they are present in this framework in the guise of response maps, scattering morphisms and **Alg** isomorphisms between the uncoupled and coupled combinations. Once explicit calculations are needed, it is possible to select particular models and interactions to approach the problem with perturbation theory. An example of this will be given in Chapter 4. Even without this level of specialization, refining some assumptions about the system and the probe is enough to provide a further characterization of the measurement scheme and its fundamental objects. In particular, the RFHGHO theories of Chapter 1 are the perfect playground to study the interaction between the measurement framework of Chapter 3 and the Hadamard condition of Chapter 2.

The first thing to notice is that Theorem 1.3.4 can be interpreted as a result about theories that agree outside a compact subset of the spacetime manifold. Definition 3.2.1 needs maps χ_L that identify two (quantum) theories outside a compact subset of the spacetime manifold, which is the causal hull of the coupling zone. Even though the first case concerns classical field theories, while the second one is about quantum local algebras, Section 1.4 presented a quantization scheme in terms of functors, meaning that the classical morphisms of Theorem 1.3.4 can be eventually quantized in hope that they can be useful for a quantum measurement setup.

Theorem 3.3.1. *Let $(B,P),(B,Q) \in$ **GreenHyp** be RFHGHOs on a spacetime $\mathcal{M}$ that agree outside a compact set $K \subset \mathcal{M}$. Then there are Cauchy **GreenHyp** morphisms $\tau^{\pm}_{QP}\colon (B,P) \to (B,Q)$ and $\tau^{\pm}_{PQ}\colon (B,Q) \to (B,P)$ such that $\tau^{\pm}_{QP}\Gamma^\infty_0(B) \subset \Gamma^\infty_0(B|_{\mathcal{M}^\pm})$ and $\hat{\tau}^{\pm}_{QP}[f]_P = [f]_Q$, and analogous properties for $\tau^{\pm}_{PQ}$. Then $\theta \coloneqq \tau^-_{PQ} \circ \tau^+_{QP}$ is a **GreenHyp** endomorphism of (B,P) and can be chosen to take the following form:*

$$\theta f = f - (Q-P)E^-_Q f, \;\; f \in \Gamma^\infty_0(B|_{\mathcal{M}^+}) \tag{3.3.1}$$

In the spirit of Section 3.2, the maps $\tau^\pm$ are called response maps and the map θ is called scattering morphism for Q relative to P with coupling zone K.

Proof. The response maps are the Cauchy **GreenHyp** morphisms defined in Theorem 1.3.4. Choosing them as proposed in the proof of the theorem and exploiting the fact that P and Q are injective, the specific form of θ follows. See e.g. [Few25, Th. 5.21] for details. □

The Cauchy **GreenHyp** morphisms that play the role of response maps can be characterized by their action on the sections of the bundles they act on. This characterization is useful to define a notion of regularity which, in a sense, preserves the singularity structure of distributions.

Definition 3.3.1 (*Regular morphism*)**.** A linear map $S\colon \Gamma_0^\infty(B) \to \Gamma_0^\infty(B')$ is said to be *regular* if:

(i) S is continuous and has continuous formal transpose ${}^tS\colon \Gamma_{\mathrm{sc}}^\infty(B') \to \Gamma^\infty(B)$.

(ii) For any compact $K \subset \mathcal{M}$ there is a compact $K' \subset \mathcal{M}'$ such that $S\Gamma_K^\infty(B) \subset \Gamma_{K'}^\infty(B')$.

(iii) $\mathrm{WF}(S^{\mathrm{knl}}) \cap \Big((0_{T^*\mathcal{M}'} \times \dot{T}^*\mathcal{M}) \cup (\dot{T}^*\mathcal{M}' \times 0_{T^*\mathcal{M}}) \Big) = \varnothing.$

This means that there can be sufficiently regular morphisms that do not disturb the singularity structure of a distribution. This is particularly relevant for the measurement framework. One would like for the scattering morphism to satisfy some kind of regularity condition, so that the measurement process does not introduce new singularities in the states of the system.

The following two results are all that is needed to study the behavior of Hadamard states under non-selective state updates [Few25]. They will play an important role also in the case of selective state updates, which will be the topic of Chapter 4.

Theorem 3.3.2. *Let $(B,P),(B,Q) \in$ **GreenHyp** be RFHGHOs on a spacetime $\mathcal{M}$ that agree outside a compact set $K \subset \mathcal{M}$ and θ be the scattering morphism of Q relative to P with coupling zone K. Define the **Alg** scattering automorphism:*

$$\Theta := \mathscr{A}(\theta)\colon \mathscr{A}(P) \to \mathscr{A}(P) \tag{3.3.2}$$

Then its pullback $\Theta^*\colon \mathscr{A}(P)^*_{+,1} \to \mathscr{A}(P)^*_{+,1}$ is an isomorphism on the set of Hadamard states on $\mathscr{A}(P)$.

Proof. [Few25, Lemma 5.15] shows that the response maps of Theorem 3.3.1 are regular. [Few25, Lemma 5.16] shows that, for any regular **GreenHyp** morphism S, the pullback of the **Alg** morphism $\mathscr{A}(S)$ maps Hadamard states to Hadamard states. Since $\theta = \tau^-_{PQ} \circ \tau^+_{QP}$, then $\Theta = \mathscr{A}(\tau^-_{PQ}) \circ \mathscr{A}(\tau^+_{QP})$ pulls back to a composition of Hadamard preserving morphisms. As this holds also for $\Theta^{-1} = \mathscr{A}(\tau^+_{PQ}) \circ \mathscr{A}(\tau^-_{QP})$, the map Θ^* is indeed an isomorphism on the set of Hadamard states. □

Theorem 3.3.3. *Let $(B_P,P),(B_Q,Q) \in$ **GreenHyp** be normally hyperbolic RFHGHOs on a spacetime $\mathcal{M}$ and define the RFHGHO $(P \oplus Q, B_P \oplus B_Q)$ as in Theorem 1.4.3. Let also $\omega \in \mathscr{A}(P)^*_{+,1}$ and $\sigma \in \mathscr{A}(Q)^*_{+,1}$ be Hadamard states. Then $\omega \otimes \sigma$ is a Hadamard state on the algebra $\mathscr{A}(P \oplus Q) \cong \mathscr{A}(P) \otimes \mathscr{A}(Q)$.*

Moreover, if α is a Hadamard state on $\mathscr{A}(P \oplus Q)$, the partial traces $\alpha_P(A) = \alpha(A \otimes \mathbb{1}_{\mathscr{A}(Q)})$ and $\alpha_Q(B) = \alpha(\mathbb{1}_{\mathscr{A}(P)} \otimes B)$ are Hadamard states on $\mathscr{A}(P)$ and $\mathscr{A}(Q)$ respectively.

Proof. Consider the 2-point function associated to α_P:

$$\begin{aligned}
\mathfrak{W}_{\alpha_P}(f \otimes g) &= \alpha_P(\Phi_P(f)\Phi_P(g)) \\
&= \alpha((\Phi_P(f) \otimes \mathbb{1}_{\mathscr{A}(Q)})(\Phi_P(g) \otimes \mathbb{1}_{\mathscr{A}(Q)})) \\
&= \alpha(\Phi_{P\oplus Q}(f \oplus 0)\Phi_{P\oplus Q}(g \oplus 0)) \\
&= \mathfrak{W}_\alpha((f \oplus 0) \otimes (g \oplus 0))
\end{aligned} \tag{3.3.3}$$

Since α is Hadamard, the following holds:

$$\mathrm{WF}(\mathfrak{W}_{\alpha_P}) \subset \mathrm{WF}(\mathfrak{W}_{\alpha}) \subset \mathcal{N}^+ \times \mathcal{N}^- \tag{3.3.4}$$

Now, by Proposition 1.4.5:

$$\begin{aligned}\mathrm{WF}(\mathfrak{W}_{\alpha_P}) &= \mathrm{WF}(iE_P + {}^t\mathfrak{W}_{\alpha_P}) \\ &\subset \big(\mathrm{WF}(E_P) \cup (\mathcal{N}^- \times \mathcal{N}^+)\big) \cap \mathrm{WF}(\mathfrak{W}_{\alpha_P}) \\ &\subset \mathrm{WF}(E_P) \cap (\mathcal{N}^+ \times \mathcal{N}^-)\end{aligned} \tag{3.3.5}$$

where $\mathrm{WF}({}^t\mathfrak{W}_{\alpha_P})$ is computed using the rule of composition of $\mathfrak{W}_{\alpha_P}$ with the map $\phi(x, y) = (y, x)$, see e.g. [Hör90, Th. 8.2.4]. Conversely:

$$\begin{aligned}\mathrm{WF}(E_P) \cap (\mathcal{N}^+ \times \mathcal{N}^-) &\subset \mathrm{WF}(-i(\mathfrak{W}_{\alpha_P} - {}^t\mathfrak{W}_{\alpha_P})) \cap (\mathcal{N}^+ \times \mathcal{N}^-) \\ &\subset \big(\mathrm{WF}(\mathfrak{W}_{\alpha_P}) \cup \mathrm{WF}({}^t\mathfrak{W}_{\alpha_P})\big) \cap (\mathcal{N}^+ \times \mathcal{N}^-) \\ &\subset \mathrm{WF}(\mathfrak{W}_{\alpha_P})\end{aligned} \tag{3.3.6}$$

Thus $\mathrm{WF}(\mathfrak{W}_{\alpha_P}) = \mathrm{WF}(E_P) \cap (\mathcal{N}^+ \times \mathcal{N}^-) = \{(x, k; y, -l) \in \mathcal{N}^+ \times \mathcal{N}^- \mid (x, k) \sim (y, l)\}$ showing that α_P is Hadamard. The proof for the state α_Q follows analogous computations.

This computation shows in general that if $\mathfrak{W}$ is the 2-point distribution of some algebraic state, then the condition $\mathrm{WF}(\mathfrak{W}) \subset \mathcal{N}^+ \times \mathcal{N}^-$ implies that the state is Hadamard. Consider then the identification $\Phi_{P\oplus Q}(f \oplus g) = \Phi_P(f) \otimes \mathbb{1}_{\mathscr{A}(Q)} + \mathbb{1}_{\mathscr{A}(P)} \otimes \Phi_Q(g)$ given in Theorem 1.4.3, the 2-point function associated to $\omega \otimes \sigma$ takes the form:

$$\mathfrak{W}_{\omega\otimes\sigma}((f\oplus g)\otimes(f'\oplus g')) = \mathfrak{W}_{\omega}(f\otimes f') + \mathfrak{W}_{\sigma}(g\otimes g') + \mathfrak{W}_{\omega}(f)\mathfrak{W}_{\sigma}(g') + \mathfrak{W}_{\omega}(f')\mathfrak{W}_{\sigma}(g) \tag{3.3.7}$$

The 1-point functions are smooth [Few25, Corollary 5.10] so that:

$$\mathrm{WF}(\mathfrak{W}_{\omega\otimes\sigma}) \subset \mathrm{WF}(\mathfrak{W}_{\omega}) \cup \mathrm{WF}(\mathfrak{W}_{\sigma}) \subset \mathcal{N}^+ \times \mathcal{N}^- \tag{3.3.8}$$

and $\omega \otimes \sigma$ is Hadamard. □

The main result of this section is presented in the following theorem [Few25]:

Theorem 3.3.4. *Let $(B_P, P), (B_Q, Q), (B_P \oplus B_Q, T) \in$ **GreenHyp** be normally hyperbolic RFHGHOs such that $P \oplus Q$ agrees with T outside a compact set $K \subset \mathcal{M}$. Let $\omega \in \mathscr{A}(P)^*_{+,1}, \sigma \in \mathscr{A}(Q)^*_{+,1}$ be Hadamard states. Assume that $\mathscr{A}(P)$ and $\mathscr{A}(Q)$ are the system and probe QFTs respectively, with $\mathscr{A}(T)$ being their coupled combination with coupling zone K and scattering morphism $\Theta = \mathscr{A}(\theta)$ as defined in Theorem 3.3.1. Then the non-selective state update $\omega^{n.s.} \in \mathscr{A}(P)^*_{+,1}$ of ω is a Hadamard state.*

Proof. By Definition 3.2.3, the non-selective state update of ω is defined to act on the system observables as:

$$\omega^{\text{n.s.}}(A) = (\omega \otimes \sigma)(\Theta(A \otimes \mathbb{1}_{\mathscr{A}(Q)})) = [\Theta^*(\omega \otimes \sigma)](A \otimes \mathbb{1}_{\mathscr{A}(Q)}) \tag{3.3.9}$$

By Theorem 3.3.2, $\Theta^*(\omega \otimes \sigma)$ is a Hadamard state on the uncoupled combination algebra $\mathscr{A}(P) \otimes \mathscr{A}(Q)$. Finally, evaluating it on $A \otimes \mathbb{1}_{\mathscr{A}(Q)}$ amounts to taking the partial trace which, by Theorem 3.3.3, gives a Hadamard state. □

The preservation of the Hadamard condition for non-selective measurements heavily relies on the fact that this kind of state update essentially boils down to taking partial traces, for which Theorem 3.3.3 guarantees the preservation of the Hadamard property. Selective state updates are instead obtained (up to a normalization factor), by replacing the identity $\mathbb{1}_{\mathscr{A}(Q)}$ with an arbitrary effect.

Another key point is that the results of this section revolved around the unital $*$-algebra $\mathscr{A}(P)$ only. This is enough to model non-selective measurements, however, it is too restrictive if one considers selective ones. This is because $\mathscr{A}(P)$ is generated by finite linear combinations and products of the generators $\Phi_P(f)$, which are unbounded, making the only possible effects multiples of the identity:

$$\omega(A) \in [0,1] \quad \forall \omega \in \mathscr{A}(P)^*_{+,1} \iff A = \lambda \mathbb{1}_{\mathscr{A}(P)},\ \lambda \in [0,1] \tag{3.3.10}$$

In order to construct non trivial effects, the introduction of the Weyl C^*-algebra is needed. This approach will be adopted in Chapter 4.

4 Selective Hadamard state updates

4.1 Finitely generated Weyl effects

This chapter contains the original results of this thesis. The main goal is to extend the conclusions of Chapter 3 to the case of selective state updates. In this first section, the focus is on identifying the problems that the selective case poses and on providing a first example of how they can be treated. This turns out to already give a first meaningful result on the preservation of the Hadamard condition for selective measurements. In the second section, the analysis is extended to more complex classes of selective updates, also providing physically relevant examples of selective measurements that preserve the Hadamard condition. Finally, the last section shifts its focus to the analysis of a specific model in perturbation theory, providing an explicit result on the energy variation of a system undergoing a selective measurement.

The analysis of selective state updates requires the introduction of effect operators. These are used as the conditioning effects of the probe QFT in the measurement framework of Section 3.2. Effects were briefly described in Definition 3.1.2 and used to model success/fail tests. The following is an equivalent definition given in a more algebraic setting.

Definition 4.1.1 (*Effect*)**.** An *effect* of a C^*-algebra $\mathfrak{A}$ is an element $F \in \mathfrak{A}$ such that $F \geq 0$ and $\mathbb{1}_{\mathfrak{A}} - F \geq 0$, where:

$$F \geq 0 \iff F = \sum_{\alpha} F_\alpha^* F_\alpha,\ \{F_\alpha\} \subset \mathfrak{A} \iff \omega(F) \geq 0 \quad \forall \omega \in \mathfrak{A}^*_{+,1} \tag{4.1.1}$$

Notice that in a Hilbert space representation the condition $\mathrm{Tr}[\rho F] \in [0,1]$ is reproduced by the property that $\omega_\rho(F) \geq 0, \omega_\rho(\mathbb{1} - F) \geq 0$ where ω_ρ is the algebraic state induced by ρ, defined as $\omega_\rho(F) \coloneqq \mathrm{Tr}[\rho F]$. The middle equivalent positivity condition also shows that effects are always selfadjoint.

As mentioned in the end of Section 3.3, the unital $*$-algebra quantization is not enough to model non trivial conditioning effects, for this reason the introduction of the Weyl C^*-algebra quantization is needed. Since the main focus of this work is on Hadamard states, it is desirable to be able to work with 2-point functions, which are basically defined as expectation values of the $*$-algebra generators. This shows that it is necessary to work simultaneously with both kind of algebras, for this reason only quasifree states will be considered. The connection that this class of states provides between $\mathscr{A}(P)$ and $\mathscr{A}_W(P)$ has been extensively discussed in Section 2.3.

In this chapter, the notation follows that of Chapter 3 so that indices P, Q indicate the normally hyperbolic RFHGHOs labeling a system and a probe QFT respectively, unless stated otherwise.

Proposition 4.1.1. *Let $S[\mathscr{A}_W(Q)]$ be the set of finite linear combinations and products of generators $W_Q(f)$ of the Weyl C^*-algebra $\mathscr{A}_W(Q)$. Let $T \in S[\mathscr{A}_W(Q)]$, then for any $\beta \in [1, \infty)$, the operator*

$$F_Q \coloneqq \frac{T^*T}{\beta \|T^*T\|} \in \mathscr{A}_W(Q) \tag{4.1.2}$$

is an effect in the Weyl C^*-algebra of the probe.

Proof. For any state $\sigma \in \mathscr{A}_W(Q)^*_{+,1}$ one has:

$$\sigma(F_Q) = \frac{\sigma(T^*T)}{\beta\|T^*T\|} \in [0,1], \quad \sigma(\mathbb{1} - F_Q) = 1 - \frac{\sigma(T^*T)}{\beta\|T^*T\|} \in [0,1] \tag{4.1.3}$$

□

The following lemma provides a key tool in the analysis to come.

Lemma 4.1.2. *Let $\omega \in \mathscr{A}_W(P)^*_{+,1}$, $\sigma \in \mathscr{A}_W(Q)^*_{+,1}$ be quasifree Hadamard states on the system and probe Weyl C^*-algebras and α be a quasifree Hadamard state on the Weyl C^*-algebra $\mathscr{A}_W(P \oplus Q)$. Let also Θ be the scattering morphism of Theorem 3.3.4, the action of the scattering morphism on Weyl algebra states can be interpreted in terms of Proposition 2.3.1:*

$$\Theta^*_W \widetilde{(\omega \otimes \sigma)} = \Theta^*(\tilde{\omega} \otimes \tilde{\sigma}) \tag{4.1.4}$$

Then the following hold:

(i) $\Theta^*_W(\omega \otimes \sigma)$ is a quasifree Hadamard state on the uncoupled combination algebra $\mathscr{A}_W(P) \otimes \mathscr{A}_W(Q)$, where the tensor product is taken to be the unique C^* one inducing the isomorphism $\mathscr{A}_W(P \oplus Q) \cong \mathscr{A}_W(P) \otimes \mathscr{A}_W(Q)$.

(ii) The partial trace $\alpha_P(\cdot) \coloneqq \alpha(\cdot \otimes \mathbb{1}_Q)$ is a quasifree Hadamard state on $\mathscr{A}_W(P)$.

Proof. **(i)** The state $\omega \otimes \sigma$ is a Hadamard state on $\mathscr{A}_W(P \oplus Q)$ by the first part of Theorem 3.3.3, translated in terms of Weyl algebras following Proposition 2.3.1. It is also quasifree, with associated symmetric bilinear form:

$$\mu^{\omega\otimes\sigma}(f_1 \oplus f_2, g_1 \oplus g_2) = \mu^{\omega}(f_1, g_1) + \mu^{\sigma}(f_2, g_2) \tag{4.1.5}$$

where the notation follows Definition 1.4.2. Θ^*_W preserves the quasifree property as shown in e.g. [HR15, Proposition 29.3-2] and the Hadamard property by Theorem 3.3.2, again translating the result between unital $*$-algebras and Weyl C^*-algebras following Proposition 2.3.1. The algebra isomorphism and the choice of tensor product is discussed in [FJR22, Sec. 6.1].

(ii) Using the defining properties of the Weyl generators in Theorem 1.4.2 one has:

$$\begin{aligned}
\alpha_P(W_P(sf)W_P(tg)) &= \alpha(W_P(sf)W_P(tg) \otimes \mathbb{1}_Q) \\
&= e^{-\frac{i}{2}stE_P(f,g)}\alpha(W_{P\oplus Q}((sf + tg) \oplus 0)) \\
&= e^{-\frac{i}{2}stE_P(f,g)}e^{+\frac{i}{2}E_{P\oplus Q}((sf\oplus 0),(tg\oplus 0))} \\
&\quad \times \alpha(W_{P\oplus Q}(sf \oplus 0)W_{P\oplus Q}(tg \oplus 0)) \\
&= \alpha(W_{P\oplus Q}(s(f \oplus 0))W_{P\oplus Q}(t(g \oplus 0)))
\end{aligned} \tag{4.1.6}$$

Since α is Hadamard by hypothesis, the partial trace α_P is Hadamard as well. The quasifree property is immediate from an analogous computation, showing that:

$$\alpha_P(W_P(f)) = \alpha(W_{P\oplus Q}(f \oplus 0)) = e^{-\frac{1}{2}\mu^{\alpha}(f\oplus 0, f\oplus 0)} \tag{4.1.7}$$

□

To avoid heavy notation, the scattering morphism will be denoted by Θ in both cases, suppressing the subscript W when it is acting on a Weyl C^*-algebra. The main result of this section is an extension of Theorem 3.3.4 to selective measurements conditioned on the class of effects of Proposition 4.1.1. Recall that $\mathcal{S}_{\mathrm{qH}}$ is the set of quasifree Hadamard state on a certain algebra.

Proposition 4.1.3. *Let $\omega \in \mathscr{A}_W(P)^*_{+,1}$, $\sigma \in \mathscr{A}_W(Q)^*_{+,1}$ be quasifree Hadamard states on the system and probe Weyl C^*-algebras and $F_Q \in \mathscr{A}_W(Q)$ be an effect as in Proposition 4.1.1, then the selectively updated state $\omega^{sel} \in \mathscr{A}_W(P)^*_{+,1}$ conditioned on F_Q is Hadamard.*

Proof. By Lemma 4.1.2, the state $\alpha := \Theta^*(\omega \otimes \sigma) \in \mathcal{S}_{\mathrm{qH}} \subset \mathscr{A}_W(P \oplus Q)^*_{+,1}$. For any $A \in \mathscr{A}_W(P)$, redefining the effect by incorporating the constants into T such that $\|T\| \leq 1$ and $F_Q = T^*T$, the following holds in the GNS representation of α:

$$\begin{aligned} \alpha(A \otimes F_Q) &= \langle \Omega_\alpha, (A \otimes F_Q)\Omega_\alpha \rangle \\ &= \langle \Omega_\alpha, (A \otimes \mathbb{1})\,(\mathbb{1} \otimes T^*)\,(\mathbb{1} \otimes T)\,\Omega_\alpha \rangle \\ &= \langle (\mathbb{1} \otimes T)\,\Omega_\alpha, (A \otimes \mathbb{1})\,(\mathbb{1} \otimes T)\,\Omega_\alpha \rangle \end{aligned} \tag{4.1.8}$$

Since $\mathscr{A}_W(P \oplus Q) \cong \mathscr{A}_W(P) \otimes \mathscr{A}_W(Q)$ in the sense that:

$$\mathscr{A}_W(P \oplus Q)(f \oplus g) = \mathscr{A}_W(P)(f) \otimes \mathscr{A}_W(Q)(g) \tag{4.1.9}$$

one can choose $f = 0$ so that $\mathscr{A}_W(P \oplus Q)(0 \oplus g) = \mathbb{1} \otimes \mathscr{A}_W(Q)(g)$ and $(\mathbb{1} \otimes T)$ can be indeed seen as a finite linear combination of finite products of $\mathscr{A}_W(P \oplus Q)$ generators. Then, by Theorem 2.3.4, the state $\psi_\alpha^{[T]}$ defined as follows is in the Hadamard set of α:

$$\psi_\alpha^{[T]} := (\mathbb{1} \otimes T)\,\Omega_\alpha \in \mathrm{Had}(\alpha) \tag{4.1.10}$$

Following Definition 2.3.3, one can rewrite:

$$\alpha(A \otimes F_Q) = \langle \psi_\alpha^{[T]}, (A \otimes \mathbb{1})\psi_\alpha^{[T]} \rangle = \omega_{\psi_\alpha^{[T]}}(A \otimes \mathbb{1}) \tag{4.1.11}$$

The state $\omega_{\psi_\alpha^{[T]}} \in \mathscr{A}_W(P \oplus Q)^*_{+,1}$ is Hadamard because $\psi_\alpha^{[T]} \in \mathrm{Had}(\alpha)$ and one recognizes that:

$$\omega^{sel}(A) = \frac{\alpha(A \otimes F_Q)}{\alpha(\mathbb{1} \otimes F_Q)} = \frac{1}{E_0}\omega_{\psi_\alpha^{[T]}}(A \otimes \mathbb{1}) \quad \forall A \in \mathscr{A}_W(P) \tag{4.1.12}$$

where $E_0 = \alpha(\mathbb{1} \otimes F_Q)$ is the normalization factor of the updated state. Finally, by Lemma 4.1.2 it follows that ω^{sel} is Hadamard as well, it being proportional to the partial trace of the Hadamard state $\omega_{\psi_\alpha^{[T]}}$. □

This approach can be implemented to model an actual selective measurement with a clear physical interpretation.

Example 4.1.1 (*Squared cosine effect*)**.** The easiest example of effect one can define, following the prescription of Proposition 4.1.1, is given by taking $T = \mathbb{1} + \lambda W_Q(h)$. Recalling the exponential form of the Weyl generators, one obtains $T^*T = (1 + |\lambda|^2)\mathbb{1} +$

$2\mathrm{Re}(\lambda W(h))$. Moreover, $\|T^*T\| = \|T\|^2 \leq (1+|\lambda|)^2$, and taking $\lambda \in \mathbb{R}^+$ the normalization factor can be chosen as $\beta = (1+\lambda)^2$:

$$F_Q := \frac{(1+\lambda^2)\mathbb{1} + 2\lambda\cos(\Phi_Q(h))}{(1+\lambda)^2} = \frac{(1-\lambda)^2\mathbb{1} + 4\lambda\cos^2\left(\frac{\Phi_Q(h)}{2}\right)}{(1+\lambda)^2} \tag{4.1.13}$$

This gives a random combination of the trivially true effect, represented by $\mathbb{1}$, together with the $\cos^2\left(\frac{\Phi_Q(h)}{2}\right)$ effect, with a λ dependent weight. The operational meaning of the $\cos^2\left(\frac{\Phi_Q(h)}{2}\right)$ effect can be understood by considering the following set of measurement operators in the sense of Definition 3.1.1, now not necessarily labeled by experimental outcomes:

$$\left\{M_1 = \cos\frac{\Phi_Q(h)}{2}, M_2 = \sin\frac{\Phi_Q(h)}{2}\right\} \tag{4.1.14}$$

Notice that $M_1^*M_1 + M_2^*M_2 = \mathbb{1}$, so that they are indeed measurement operators. This set can be interpreted as the one associated to the binary question: "*Is the value of the quantum field* $\Phi_Q(h) \in [-\frac{\pi}{2}, +\frac{\pi}{2}] \mod 2\pi$?". The expectation value of the effect $\cos^2\left(\frac{\Phi_Q(h)}{2}\right) = M_1^*M_1$ on a given state represents the probability that the answer to the question is a yes. The complementary effect $\sin^2\left(\frac{\Phi_Q(h)}{2}\right) = M_2^*M_2$ models instead the probability that the answer to the question is a no.

Example 4.1.1 shows that by setting up a quantum measurement, in the sense of Section 3.2, one can observe the probability for the probe field to assume a certain value and infer probability distributions for arbitrary effects of the system. Proposition 4.1.3 shows that this can be done all while preserving the Hadamard condition of the original states.

The main drawback of Proposition 4.1.3 is that it is valid only for particularly simple conditioning effects, i.e. the finitely generated ones. A Weyl C^*-algebra has however a much richer structure, providing additional ways to construct effect operators. The question of how some of these other classes of effects influence Hadamard states will be the topic of the next section.

4.2 General Weyl effects and Gaussian measurements

The goal of this section is to present different ways to construct effects on a Weyl C^*-algebra, which are not finitely generated like the ones in Section 4.1. Once their definition is in place, it is shown that these effects preserve the Hadamard condition in the Hilbert space sense of Theorem 2.3.2. Finally, these results are applied to the case where the effect operators are employed in the quantum measurement scheme.

Proposition 4.2.1. *Let R be a normally hyperbolic RFHGHO and $\Omega_\rho \in \mathrm{Had}(\rho)$ for some algebraic quasifree Hadamard state $\rho \in \mathcal{S}_{qH} \subset \mathscr{A}_W(R)^*_{+,1}$. Define the operator $T := \sum_{n=1}^\infty c_n W_R(h_n)$ where $\{c_n\} \in \ell^1$ and $\{h_n\} \subset \Gamma^\infty_K(B_R)$ is a sequence of smooth sections uniformly bounded and supported on a compact set $K \subset \mathcal{M}$. Then $T \in \mathscr{A}_W(R)$ and $TD(\Phi_R(g)) \subset D(\Phi_R(g))$.*

Proof. The definition of T assures that the series converges in operator norm topology:

$$\|T\| = \|\sum_{n=1}^{\infty} c_n W_R(h_n)\| \leq \sum_{n=1}^{\infty} \|c_n W_R(h_n)\| \leq \sum_{n=1}^{\infty} |c_n| < \infty \tag{4.2.1}$$

as $W_R(h_n)$ is unitary so that its norm is 1. Define the graph of $\Phi(g)$ as:

$$\begin{aligned}\Gamma(\Phi_R(g)) &\coloneqq \{(\Omega_\rho, \Phi_R(g)\Omega_\rho) \mid \Omega_\rho \in D(\Phi_R(g))\} \\ &\subset D(\Phi_R(g)) \times \mathrm{Ran}(\Phi_R(g)) \subset \mathcal{H}_\rho \times \mathcal{H}_\rho\end{aligned} \tag{4.2.2}$$

with the standard product norm topology. By Theorem 1.4.6, $\Phi_R(g)$ is selfadjoint, thus closed. This means that convergent sequences in its graph have limit in the graph itself. Let $T_k \coloneqq \sum_{n=1}^{k} c_n W_R(h_n)$, then T_k is a finite linear combination of Weyl generators meaning that, by Theorem 2.3.4, $T_k\Omega_\rho \in D(\Phi_R(g))$ for $\Omega_\rho \in D(\Phi_R(g))$. Let $(T_k\Omega_\rho, \Phi_R(g)T_k\Omega_\rho) \coloneqq G_k \in \Gamma(\Phi_R(g))$ and $k < l$, it follows that:

$$\begin{aligned}\|G_l - G_k\| &= \left\| \left(\sum_{n=k+1}^{l} c_n W_R(h_n)\Omega_\rho, \sum_{n=k+1}^{l} c_n \Phi_R(g) W_R(h_n)\Omega_\rho \right) \right\| \\ &= \left\| \left(\sum_{n=k+1}^{l} c_n W_R(h_n)\Omega_\rho \right) \right\| + \left\| \left(\sum_{n=k+1}^{l} c_n \Phi_R(g) W_R(h_n)\Omega_\rho \right) \right\| \\ &\leq \sum_{n=k+1}^{l} |c_n| + \sum_{n=k+1}^{l} \|c_n \Phi_R(g) W_R(h_n)\Omega_\rho\|\end{aligned} \tag{4.2.3}$$

Consider then the following:

$$\begin{aligned}\Phi_R(g)W_R(h_n)\Omega_\rho &= -i \lim_{t\to 0} \frac{W_R(tg) - \mathbb{1}}{t} W_R(h_n)\Omega_\rho \\ &= -iW_R(h_n) \lim_{t\to 0} \left(\frac{e^{-itE_R(g,h_n)} - 1}{t} \mathbb{1} + \frac{W_R(tg) - \mathbb{1}}{t} \right) \Omega_\rho \\ &= W_R(h_n) \left(-E_R(g, h_n)\Omega_\rho + \Phi_R(g)\Omega_\rho \right)\end{aligned} \tag{4.2.4}$$

Then the following estimate holds:

$$\begin{aligned}\|c_n \Phi_R(g) W_R(h_n)\Omega_\rho\| &\leq |c_n| \|\Phi_R(g) W_R(h_n)\Omega_\rho\| \\ &= |c_n| \|W_R(h_n) \left(-E_R(g, h_n)\Omega_\rho + \Phi_R(g)\Omega_\rho \right)\| \\ &\leq |c_n| \left(|E_R(g, h_n)| + \|\Phi_R(g)\Omega_\rho\| \right)\end{aligned} \tag{4.2.5}$$

Putting all together:

$$\|G_l - G_k\| \leq \sum_{n=k+1}^{l} |c_n| \left(|E_R(g, h_n)| + \|\Phi_R(g)\Omega_\rho\| + 1 \right) \tag{4.2.6}$$

Since $\Omega_\rho \in D(\Phi(g))$, the second and third terms in parentheses are bounded constants. For the first term, recall the integral representation of $E_R(g, h_n)$ and let H be the constant bounding the sequence $\{h_n\}$ so that:

$$|E_R(g, h_n)| \leq \mathrm{Vol}(K) H \sup_{x\in K} |[E_R g](x)| \coloneqq C \tag{4.2.7}$$

where $\sup_{x\in K} |[E_R g](x)|$ is finite because $E_R g$ is smooth in a compact K. Since the sequence $\{(C + \|\Phi_R(g)\Omega_\rho\| + 1)|c_n|\} \in \ell^1$, its partial sums are Cauchy, this gives the following result:

$$\|(T_l\Omega_\rho, \Phi_R(g)T_l\Omega_\rho) - (T_k\Omega_\rho, \Phi_R(g)T_k\Omega_\rho)\| \xrightarrow{l,k\to\infty} 0 \tag{4.2.8}$$

This means that, under the stated hypotheses, $\{(T_k\Omega_\rho, \Phi_R(g)T_k\Omega_\rho)\}$ is a Cauchy sequence in $\Gamma(\Phi_R(g))$. The graph of $\Phi_R(g)$ is a Hilbert space, thus complete. It follows that:

$$(T_k\Omega_\rho, \Phi_R(g)T_k\Omega_\rho) \xrightarrow{k\to\infty} (T\Omega_\rho, \Phi_R(g)T\Omega_\rho) \in \Gamma(\Phi_R(g)) \implies T\Omega_\rho \in D(\Phi_R(g)) \tag{4.2.9}$$

□

The hypotheses of Proposition 4.2.1 are not particularly strict. Despite that, there are simple ways to break the conclusion, like in the following example.

Example 4.2.1 (*Unbounded sequence*)**.** Let $\{c_n\} = \{\frac{1}{n^2}\} \in \ell^1$ and $h_n(x) = n^3 h(x)$ for some $h \in \Gamma^\infty_K(B_R)$. Notice that this sequence does not respect the hypotheses of Proposition 4.2.1, so convergence is not guaranteed. The series defining $\Phi_R(g)T\Omega_\rho$ converges to a vector in $\mathcal{H}_\rho$ only if $\|c_n\Phi_R(g)W_R(h_n)\Omega_\rho\| \to 0$ as $n \to \infty$, the reverse triangle inequality implies in fact:

$$\begin{aligned}
\|c_n\Phi_R(g)W_R(h_n)\Omega_\rho\| &= \|c_nW_R(h_n)\Phi_R(g)\Omega_\rho - c_nE_R(g,h_n)W_R(h_n)\Omega_\rho\| \\
&\geq |\|c_nW_R(h_n)\Phi_R(g)\Omega_\rho\| - \|c_nE_R(g,h_n)W_R(h_n)\Omega_\rho\|| \\
&= \left|\left\|\frac{1}{n^2}W(n^3h)\Phi_R(g)\Omega_\rho\right\| - \left\|\frac{n^3}{n^2}E_R(g,h)W_R(n^3h)\Omega_\rho\right\|\right| \\
&\overset{n\to\infty}{\sim} n|E_R(g,h)| \xrightarrow{n\to\infty} \infty
\end{aligned} \tag{4.2.10}$$

Under some more additional constraints, the operators of Proposition 4.2.1 preserve the Hilbert space Hadamard condition.

Proposition 4.2.2. *Let $T = \sum_{n=1}^\infty c_nW_R(h_n)$ as in Proposition 4.2.1, $\Omega_\rho \in \mathrm{Had}(\rho)$ for some $\rho \in \mathcal{S}_{qH} \subset \mathscr{A}_W(R)^*_{+,1}$. If $\{|c_n|\|h_n\|_{C^k(K)}\} \in \ell^1 \quad \forall k \in \mathbb{N}$, then $T\Omega_\rho \in \mathrm{Had}(\rho)$.*

Proof. For $T\Omega_\rho$ to be Hadamard, the wavefront set of $\Phi_R(\cdot)T\Omega_\rho$ needs to satisfy Theorem 2.3.2. Notice that:

$$\begin{aligned}
\mathrm{WF}(\Phi_R(\cdot)T\Omega_\rho) &= \mathrm{WF}(T\Phi(\cdot)\Omega_\rho + [\Phi(\cdot),T]\Omega_\rho) \\
&\subset \mathrm{WF}(T\Phi(\cdot)\Omega_\rho) \cup \mathrm{WF}([\Phi(\cdot),T]\Omega_\rho)
\end{aligned} \tag{4.2.11}$$

By Lemma 2.3.3, for the first term it holds that:

$$\mathrm{WF}(T\Phi_R(\cdot)\Omega_\rho) \subset \mathrm{WF}(\Phi_R(\cdot)\Omega_\rho) \subset \mathcal{N}^- \tag{4.2.12}$$

For the second term:

$$\begin{aligned}
[\Phi_R(g),T]\Omega_\rho &= -\sum_{n=1}^\infty c_nE_R(g,h_n)W_R(h_n)\Omega_\rho \\
&= \int_{\mathrm{supp}(g)} g(x)\left\{-\sum_{n=1}^\infty c_n[E_Rh_n](x)W_R(h_n)\Omega_\rho\right\} \mathrm{dV} \\
&= \int_{\mathrm{supp}(g)} g(x)(H(x)\Omega_\rho)\,\mathrm{dV}
\end{aligned} \tag{4.2.13}$$

The goal now is to prove that $H(x)\Omega_\rho$ is a smooth Hilbert space valued function. Notice that the definition of the Hilbert space valued function $H(x)\Omega_\rho$ is well posed only when the series converges, and this is guaranteed by Proposition 4.2.1. Consider a compact set $K_g \supset \mathrm{supp}(g)$, then $\|\cdot\|_{C^k(K_g)}$ is a family of seminorms on $E_R\Gamma_0^\infty(B_R) \subset \Gamma_{\mathrm{sc}}^\infty(B_R)$. This family is also countable because $\mathcal{M}$ is second countable, so a countable sequence of compacts $\{K_g\}$ eventually covers all of the spacetime. The family of $C^k(K)$ seminorms on $\Gamma_K^\infty(B_R)$ is clearly directed. Since E_R is a continuous linear map, for any m and K_g there are a k and a $C > 0$ such that [RS81, Ch. 5, Th. V.2]:

$$\|E_R h_n\|_{C^m(K_g)} \leq C\|h_n\|_{C^k(K)} \tag{4.2.14}$$

The derivatives of $E_R h_n$ along arbitrary directions $v_\mu, \|v_\mu\| = 1$ are bounded in K_g by $\|v_\mu \nabla^\mu [E_R h_n](x)\| \leq \|\nabla[E_R h_n](x)\| \leq \|E_R h_n\|_{C^1(K_g)}$ where all the norms are understood as in Definition A.13. Let now $\sum_{n=1}^{k} c_n v_\mu \nabla^\mu [E_R h_n](x) W_R(h_n)\Omega_\rho \coloneqq D_k$ and $k < l$, then for some $p \in \mathbb{N}$:

$$\begin{aligned}
\|D_l - D_k\| &= \left\| \sum_{n=k+1}^{l} c_n v_\mu \nabla^\mu [E_R h_n](x) W_R(h_n)\Omega_\rho \right\| \\
&\leq \sum_{n=k+1}^{l} |c_n| \|E_R h_n\|_{C^1(K_g)} \\
&\leq C \sum_{n=k+1}^{l} |c_n| \|h_n\|_{C^p(K)} \xrightarrow{l,k\to\infty} 0
\end{aligned} \tag{4.2.15}$$

The partial sums are then Cauchy so that the series:

$$\sum_{n=1}^{\infty} c_n v_\mu \nabla^\mu [E_R h_n](x) W_R(h_n)\Omega_\rho \tag{4.2.16}$$

converges uniformly, it being a Cauchy sequence in a complete Hilbert space, to a function $v_\mu \nabla^\mu H(x)\Omega_\rho$ which is the derivative of $H(x)\Omega_\rho$ [Rud76, Th. 7.17]. Notice that the bound on the partial sums is independent of K_g so that the convergence of $v_\mu \nabla^\mu H(x)$ is independent of the test section g. Iterating this process one has that any mixed directional derivative along arbitrary directions $\nabla^{(k)} H(x)(n^1, \ldots, n^k)\Omega_\rho$ exists thanks to the sequence $\|h_n\|_{C^k(K)}$ satisfying the hypothesis. This makes $H(x)\Omega_\rho$ a smooth Hilbert space valued function on $\mathcal{M}$. Then the distribution $[\Phi_R(\cdot), B]\Omega_\rho \coloneqq u(\cdot)\Omega_\rho$ has the property that there exists a smooth function $H(x)\Omega_\rho$ for which:

$$u(g)\Omega_\rho = \int_{\mathcal{M}} g(x) H(x)\Omega_\rho \, \mathrm{dV} \quad \forall x \in \mathcal{M} \tag{4.2.17}$$

Consequently this distribution has empty wavefront set and:

$$\mathrm{WF}(\Phi_R(\cdot) B\Omega_\rho) \subset \mathrm{WF}(T\Phi_R(\cdot)\Omega_\rho) \cup \mathrm{WF}([\Phi_R(\cdot), B]\Omega_\rho) \subset \mathcal{N}^- \tag{4.2.18}$$

□

Remark. It is fairly easy to find sequences of smooth sections satisfying the hypotheses above. For any fixed smooth section $h \in \Gamma_0^\infty(B_R)$ take $K = \mathrm{supp}(h)$ and a sequence $\{t_n\} \in \ell^1$. Then the sequence of sections $\{h_n\} := \{t_n h\}$ is such that:

$$\|h_n\|_{C^k(K)} = |t_n| \|h\|_{C^k(K)} = a_k |t_n| \tag{4.2.19}$$

where a_k is a finite constant dependent on k since h is smooth and compactly supported, then:

$$|c_n| \|h_n\|_{C^k(K)} = a_k |c_n| |t_n| \in \ell^1 \ \forall k \in \mathbb{N} \tag{4.2.20}$$

satisfying the hypothesis in Proposition 4.2.2.

Moreover, the request for the sequence $\{h_n\}$ to be compactly supported on a common set $K \subset \mathcal{M}$ is reasonable from a physical point of view. The sections h_n are used to define an operator that will be employed as a conditioning effect in a quantum measurement, i.e. a probe observable. Experimentally, the measurement of the probe happens in a compact region of spacetime, meaning that the effect operator should belong to the local algebra of some compact set $K \subset \mathcal{M}$, which is the same set where the sections h_n are supported.

Proposition 4.2.2 showed that it is possible to construct operators (not necessarily effects) in the Weyl C^*-algebra that are not finitely generated, but still preserve the Hadamard condition. Most importantly, this result can be used to build more complex effects preserving the Hadamard condition, following a procedure that extends Proposition 4.1.1 as follows.

Proposition 4.2.3. *Let $T = \sum_{n=1}^\infty c_n W_R(h_n) \notin S[\mathscr{A}_W(R)]$ such that it satisfies the hypotheses of Proposition 4.2.2 and $\beta \in [1, \infty)$. Then the operator:*

$$F_Q = \frac{T^* T}{\beta \|T^* T\|} \in \mathscr{A}_W(R) \tag{4.2.21}$$

is an effect preserving the Hadamard condition, i.e. if $\Omega_\rho \in \mathrm{Had}(\rho)$ for some $\rho \in \mathcal{S}_{qH}$, then $F_Q \Omega_\rho \in \mathrm{Had}(\rho)$.

Proof. Notice that $T^* = \sum_{n=1}^\infty c_n^* W_R(-h_n)$. Since $|c_n^*| = |c_n|$ and $|-h_n| = |h_n|$, the hypotheses of Proposition 4.2.2 remain valid for T^*. Then one has $T\Omega_\rho \in \mathrm{Had}(\rho)$ and $T^*(T\Omega_\rho) \in \mathrm{Had}(\rho)$. Since F_Q is a positive scalar multiple of T^*T, also $F_Q \Omega_\rho \in \mathrm{Had}(\rho)$. The normalization factor $\beta \|T^* T\|$ guarantees that F_Q is an effect operator in the same way of Proposition 4.1.1. $\square$

The following is an example of non finitely generated effect that preserves the Hadamard condition, which can be given a physical interpretation.

Example 4.2.2 (*Characteristic set effects*)**.** Consider the characteristic function of a generic interval of the real line $\chi_{[a,b]}(x)$ for an interval such that $a > -\pi,\ b < \pi$ and the sequence of smooth functions:

$$f_\epsilon(x) = \sum_{n \in \mathbb{Z}} e^{-\epsilon n^2} e^{inx} \xrightarrow{\epsilon \to 0} \frac{1}{2\pi} \delta(x) \tag{4.2.22}$$

Their convolution $F_\epsilon(x) = (\chi_{[a,b]} * f_\epsilon)(x)$ is a smooth function converging to $\chi_{[a,b]}$ in the limit $\epsilon \to 0$. The function can be rewritten via its Fourier series:

$$\begin{aligned} F_\epsilon(x) = \sum_{n\in\mathbb{Z}} \widehat{\chi_{[a,b]} * f_\epsilon}(n) e^{inx} &= \sum_{n\in\mathbb{Z}} \widehat{\chi_{[a,b]}}(n) \widehat{f}_\epsilon(n) e^{inx} \\ &= \frac{b-a}{2\pi} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{ie^{-\epsilon n^2}}{2\pi n}(e^{-inb} - e^{-ina}) e^{inx} \end{aligned} \tag{4.2.23}$$

This can be used to define the following operator for any $h \in \Gamma^\infty_K(B_R)$:

$$F_\epsilon(\Phi_R(h)) = \frac{b-a}{2\pi}\mathbb{1} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{ie^{-\epsilon n^2}}{2\pi n}(e^{-inb} - e^{-ina}) W_R(nh) \tag{4.2.24}$$

Simply splitting the sum over $\mathbb{Z}$ into a pair of separate sums over $\mathbb{N}$ it is possible to show that the conditions of the hypothesis in Proposition 4.2.2 are satisfied:

$$\sum_{n\in\mathbb{N}} \left| \frac{ie^{-\epsilon n^2}}{2\pi n}(e^{-inb} - e^{-ina}) \right| n\|h\|_{C^k(K)} \leq C \sum_{n\in\mathbb{N}} e^{-\epsilon n^2} < \infty \tag{4.2.25}$$

This means that the operator $F_\epsilon(\Phi_R(h))$ approximates with arbitrary precision $\chi_{[a,b]}(\Phi_R(h))$ and preserves the Hadamard condition for any fixed value of ϵ. Since $F_\epsilon(\Phi_R(h))$ is already a positive operator bounded between 0 and 1, it represent an effect by itself and can be interpreted as the unsharp measurement of whether or not the value of $\Phi_R(h)$ lies in $[a,b]$ mod 2π.

An alternative way to construct non finitely generated effects that preserve the Hadamard condition is suggested by the fact that the Weyl generators are unitary operators. For such operators on a Hilbert space, one can employ the continuous functional calculus version of the spectral theorem for unitary operators.

Proposition 4.2.4. *Let $f(e^{it}) \in C^1(\mathbb{T})$ be a continuously differentiable function on the unit circle in the complex plane, then the operator $f(W_R(h))$, in the sense of the continuous functional calculus, preserves the Hadamard condition, i.e. if $\Omega_\rho \in \mathrm{Had}(\rho)$ for some $\rho \in \mathcal{S}_{qH} \subset \mathscr{A}_W(R)^*_{+,1}$, then $f(W_R(h))\Omega_\rho \in \mathrm{Had}(\rho)$.*

Proof. The spectral theorem for unitary operators in its continuous calculus form is expressed in terms of the existence of the following algebra homomorphism for any unitary operator $U \in \mathcal{B}(\mathcal{H})$, see e.g. [Arc25, Th. 7.6]:

$$\begin{aligned} \Lambda \ &: \ C(\mathbb{T}) \to \mathcal{B}(\mathcal{H}) \\ f(e^{it}) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{int} &\mapsto \Lambda(f) := \sum_{n\in\mathbb{Z}} \hat{f}(n) U^n \end{aligned} \tag{4.2.26}$$

A fundamental property of this homomorphism is that the unitary operator function's norm satisfies $\|\Lambda(f)\| := \|f(U)\| \leq \|f\|_{L^\infty}$. Then as usual one wants to check the following wavefront set:

$$\mathrm{WF}(\Phi_R(\cdot) f(W_R(h))\Omega_\rho) \subset \mathrm{WF}(f(W_R(h))\Phi_R(\cdot)\Omega_\rho) \cup \mathrm{WF}([\Phi_R(\cdot), f(W_R(h))]\Omega_\rho) \tag{4.2.27}$$

For the first term $\|f(W_R(h))\Phi_R(g)\Omega_\rho\| \leq \|f\|_{L^\infty}\|\Phi_R(g)\Omega_\rho\| \quad \forall g \in \Gamma_0^\infty(B_R)$ so that:

$$\mathrm{WF}(f(W_R(h))\Phi_R(\cdot)\Omega_\rho) \subset \mathrm{WF}(\Phi_R(\cdot)\Omega_\rho) \subset \mathcal{N}^- \tag{4.2.28}$$

and no new singularities appear. For the second one the following holds:

$$\begin{aligned}
\|[\Phi_R(g), f(W_R(h))]\Omega_\rho\| &= \left\| \sum_{n\in\mathbb{Z}} \hat{f}(n) E_R(g, nh) W_R(nh)\Omega_\rho \right\| \\
&= |E_R(g,h)| \left\| \sum_{n\in\mathbb{Z}} n\hat{f}(n) W_R(nh)\Omega_\rho \right\| \\
&= |E_R(g,h)| \|f'(W_R(h))\Omega_\rho\| \\
&\leq |E_R(g,h)| \|f'\|_{L^\infty}
\end{aligned} \tag{4.2.29}$$

These manipulations are consistent for C^1 functions, using an approximation argument, without the need to restrict to C^2 for uniform convergence of the Fourier series as argued in [Arc25, Th. 7.6]. Because f is a C^1 function on the torus, its derivative is a continuous function on the torus, thus bounded. Since $\mathrm{WF}(E_R(\cdot, h)) = \varnothing$, also $\mathrm{WF}([\Phi_R(\cdot), f(W_R(h))]\Omega_\rho) = \varnothing$. This proves that $f(W_R(h))\Omega_\rho$ is Hadamard. □

The methods developed in this section can be employed to study a particular collection of effects, which constitutes a family of measurement operators in the form of a POVM (see Definition 3.1.2). This POVM models a so-called Gaussian modulated measurement, see e.g. [MN25] for a detailed discussion.

Example 4.2.3 (*Gaussian modulated measurement*)**.** Let $h \in \Gamma_0^\infty(B_R)$ and $\epsilon > 0$, we define the POVM as the following set of effects in the probe algebra:

$$\{E_m^{h,\epsilon}\} \text{ such that } E_m^{h,\epsilon} := \frac{1}{\sqrt{2\pi}\epsilon} e^{-\frac{(\Phi_R(h)-m)^2}{2\epsilon^2}} \tag{4.2.30}$$

These effects can be shown to preserve the Hadamard condition. To do that consider the function:

$$H(x) = \frac{1}{\sqrt{2\pi}\epsilon} e^{-\frac{(x-m)^2}{2\epsilon^2}} = \int_{\mathbb{R}} \tilde{H}(t) e^{itx}\, \mathrm{d}t \tag{4.2.31}$$

Its Fourier transform can be evaluated explicitly:

$$\tilde{H}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}\epsilon} e^{-\frac{(x-m)^2}{2\epsilon^2}} e^{-itx}\, \mathrm{d}x = \frac{1}{2\pi} e^{-itm - \frac{1}{2}\epsilon^2 t^2} \tag{4.2.32}$$

Since the smeared field operator $\Phi_R(h)$ is selfadjoint, one can make sense of $H(\Phi_R(h))$ in terms of the spectral theorem for selfadjoint operators (see e.g. [RS81, Sec. 8.3]):

$$\begin{aligned}
H(\Phi_R(h)) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itm - \frac{1}{2}\epsilon^2 t^2} e^{it\Phi_R(h)}\, \mathrm{d}t \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itm - \frac{1}{2}\epsilon^2 t^2} W_R(th)\, \mathrm{d}t
\end{aligned} \tag{4.2.33}$$

where the integral is understood in terms of a Bochner integral on the Hilbert space when acting on vectors $H(\Phi_R(f))\Omega_\rho$ since $\|e^{-itm-\frac{1}{2}\epsilon^2 t^2} W_R(tf)\Omega_\rho\| \in L^1(\mathbb{R})$, which is a sufficient condition for a Hilbert space valued function to be Bochner integrable (see e.g. [DU77, Sec. 2.2, Th. 2]). Then the rest of the proof goes as the one for Proposition 4.2.2, to preserve the Hadamard condition one needs that $\mathrm{WF}(\Phi_R(\cdot)H(\Phi_R(h))\Omega) \subset \mathcal{N}^-$ for $\Omega_\rho \in \mathrm{Had}(\rho), \rho \in \mathcal{S}_{qH} \subset \mathscr{A}_W(R)^*_{+,1}$, which is equivalently stated using commutators as:

$$\mathrm{WF}(H(\Phi_R(h))\Phi_R(\cdot)\Omega_\rho) \cup \mathrm{WF}([\Phi_R(\cdot), H(\Phi_R(h))]\Omega_\rho) \subset \mathcal{N}^- \tag{4.2.34}$$

By the spectral theorem for selfadjoint operators $\|H(\Phi_R(h))\| \leq \|H\|_{L^\infty} < \infty$, so that by Lemma 2.3.3 the first term satisfies:

$$\mathrm{WF}(H(\Phi_R(h))\Phi_R(\cdot)\Omega_\rho) \subset \mathrm{WF}(\Phi_R(\cdot)\Omega_\rho) \subset \mathcal{N}^- \tag{4.2.35}$$

The second term can be rewritten as follows:

$$\begin{aligned}[\Phi_R(g), H(\Phi_R(h))]\Omega_\rho &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itm-\frac{1}{2}\epsilon^2t^2} E_R(g, th) W_R(th)\Omega_\rho \,\mathrm{d}t \\ &= \int_{\mathcal{M}} g(x)\left[\frac{1}{2\pi}\int_{\mathbb{R}} e^{-itm-\frac{1}{2}\epsilon^2t^2} t[E_R h](x) W_R(th)\Omega_\rho \,\mathrm{d}t\right] \mathrm{dV} \\ &= \int_{\mathcal{M}} g(x)[E_R h](x) A\Omega_\rho \,\mathrm{dV}\end{aligned} \tag{4.2.36}$$

where $A\Omega_\rho := \frac{1}{2\pi}\int_{\mathbb{R}} te^{-itm-\frac{1}{2}\epsilon^2t^2} W_R(th)\Omega_\rho \,\mathrm{d}t$. This is a well defined Bochner integral because the Hilbert space valued function of the real variable t that is being integrated satisfies $\|te^{-itm-\frac{1}{2}\epsilon^2t^2} W_R(th)\Omega_\rho\| \in L^1(\mathbb{R})$. Since $E_R h$ is a smooth section of B_R, then the distribution $[\Phi_R(\cdot), H(\Phi_R(h))]\Omega_\rho$ is smooth and has empty wavefront set, thus Eq. (4.2.34) is proven.

The two constructions of Proposition 4.2.3 and Proposition 4.2.4 presented in this section give effects that preserve the Hadamard condition of a single state on a general Weyl C^*-algebra algebra. The following result shows how the selective state update of a system Hadamard state, conditioned on probe effects of those types, is still Hadamard.

Proposition 4.2.5. *Let $\omega \in \mathscr{A}_W(P)^*_{+,1}$, $\sigma \in \mathscr{A}_W(Q)^*_{+,1}$ be quasifree Hadamard states on the system and probe Weyl C^*-algebras and consider the following effects:*

(i) $F_1 = \frac{T^*T}{\beta\|T^*T\|}$ *with $\beta \in [1, \infty)$ and $T = \sum_{n=1}^{\infty} c_n W_Q(h_n)$ as in Proposition 4.2.2.*

(ii) $F_2 = f(W_Q(h))$ *with $f \in C(\mathbb{T})$ such that $f > 0$ and $\|f\|_{L^\infty} \leq 1$ as in Proposition 4.2.4.*

Then the selectively updated states $\omega_i^{sel}(A) = [\Theta^*(\omega \otimes \sigma)](A \otimes F_i)$ with $i = 1, 2$ and $A \in \mathscr{A}_W(P)$ are Hadamard.

Proof. **(i)** The proof goes basically as the one for Proposition 4.1.3, one only needs to show that:

$$\left(\mathbb{1}_P \otimes \frac{T}{\sqrt{\beta\|T^*T\|}}\right)\Omega_\alpha \in \mathrm{Had}(\alpha) \tag{4.2.37}$$

Up to a constant factor it holds that:

$$\mathbb{1}_P \otimes T = \sum_n c_n\left(W_P(0) \otimes W_Q(h_n)\right) = \sum_n c_n\left(W_{P\oplus Q}(0 \oplus h_n)\right) \tag{4.2.38}$$

Then it follows that:

$$\|0 \oplus h_n\|_{C^k(K\times K)} = \|0\|_{C^k(K)} + \|h_n\|_{C^k(K)} = \|h_n\|_{C^k(K)} \tag{4.2.39}$$

The hypotheses of Proposition 4.2.2 are satisfied by $\mathbb{1}_P \otimes T$, here with $R = P \oplus Q$, proving the result.

(ii) If $f > 0$ then $\sqrt{f} \in C^1(\mathbb{T})$, so that $\sqrt{f}(W_Q(h))$ is well defined in the sense of Proposition 4.2.4. Then, like in the previous point, one needs to show that:

$$\left(\mathbb{1}_P \otimes (\sqrt{f})(W_Q(h))\right)\Omega_\alpha \in \mathrm{Had}(\alpha) \tag{4.2.40}$$

A direct computation gives:

$$\begin{aligned}\left(\mathbb{1} \otimes \sqrt{f}(W_Q(h))\right)\Omega_\alpha &= \sum_{n\in\mathbb{Z}} \widehat{(\sqrt{f})}(n) W_{P\oplus Q}(n(0\oplus h))\Omega_\alpha \\ &= \sqrt{f}(W_{P\oplus Q}(0 \oplus h))\Omega_\alpha\end{aligned} \tag{4.2.41}$$

which is in $\mathrm{Had}(\alpha)$ by Proposition 4.2.4, again with $R = P \oplus Q$. □

The results of this section show how it is possible to construct non finitely generated operators in the Weyl C^*-algebra in such a way that they preserve the Hadamard condition. This has been also proven to be sufficient to guarantee that one can build effect operators that preserve the Hadamard condition of the system states after a selective measurement. This discussion has also been shown to be applicable to relevant situations by providing examples of effect and measurement operators that are physically meaningful from an operational perspective.

4.3 Stress energy tensor update

Section 2.1 presented the original motivation for the introduction of Hadamard states, showing how they can be used to appropriately define a finite expectation value for the quantum version of the stress energy tensor. The stress energy tensor of a quantum field contains crucial information about the energy density of the system, for this reason it is desirable that physical states can model a finite value of such energy density. In the framework of QFT measurements, the results obtained in Section 3.3 and in Chapter 4 so far, showed that a quantum field theoretical experiment can be set up in such a way that measuring a physical observable of a quantum field in a Hadamard state does not disturb the system too much, so that it preserves the Hadamard condition. In these situations, it is possible to define a post-measurement stress energy tensor, so that one can compare it with the pre-measurement one to gauge the scale of the energy variation in the system, which is induced by the act of performing a measurement. The goal of this section is to study this phenomenon by looking at how the expression $\langle \hat{T}^{\mathrm{ren}}_{\mu\nu}\rangle_\omega$ changes when ω is updated to some ω^{sel}. To do that, the first step will be to get an explicit expression for the 2-point function $\mathfrak{W}^{\mathrm{sel}}$ of a post-measurement selectively updated state and compute the difference $\Delta\mathfrak{W}^{\mathrm{sel}} = \mathfrak{W}^{\mathrm{sel}} - \mathfrak{W}$ between that and the 2-point function $\mathfrak{W}$ of the pre-measurement state. Once this is achieved, the computation is taken further using perturbation theory on a particular interacting model of choice. The variation of the expectation value of the stress energy tensor follows from Eq. (2.1.12) by acting on the final result with the differential operator $\mathcal{D}^{(x,y)}_{\mu\nu}$ in the limit $y \to x$.

Starting with the first objective, consider the usual system and probe QFT pair, prepared at early times in quasifree Hadamard states ω, σ respectively. Take now a Weyl effect $F_Q = b(W_Q(h)) = \sum_{n\in\mathbb{Z}} \hat{b}(n) W_Q(nh)$ written in terms of its Fourier series in the continuous calculus sense of Proposition 4.2.4 and recall the definition of $\alpha := \Theta^*(\omega \otimes \sigma) \in \mathcal{S}_{qH}$:

$$\begin{aligned} \omega^{\mathrm{sel}}(W_P(f)) &\propto \alpha(W_P(f) \otimes F_Q) \\ &= \sum_{n\in\mathbb{Z}} \hat{b}(n) \alpha(W_{P\oplus Q}(f \oplus nh)) \\ &= \sum_{n\in\mathbb{Z}} \hat{b}(n) e^{-\frac{1}{2}\mu^\alpha(f\oplus nh, f\oplus nh)} \end{aligned} \tag{4.3.1}$$

because α is a quasifree state and as such it has an associated symmetric bilinear form μ^α (Definition 1.4.2). Then the updated 2-point function looks as follows:

$$\begin{aligned} \mathfrak{W}^{\mathrm{sel}}(f \otimes g) &= \omega^{sel}(\Phi_P(f)\Phi_P(g)) \\ &= -\left.\frac{\partial^2}{\partial s \partial t} \omega^{sel}(W_P(sf)W_P(tg))\right|_{s=t=0} \\ &= -\left.\frac{\partial^2}{\partial s \partial t} e^{-\frac{i}{2} st E_P(f,g)} \omega^{sel}(W_P(sf+tg))\right|_{s=t=0} \\ &= -\left.\frac{1}{E_0}\frac{\partial^2}{\partial s \partial t} e^{-\frac{i}{2} st E_P(f,g)} \sum_{n\in\mathbb{Z}} \hat{b}(n) e^{-\frac{1}{2}\mu^\alpha((sf+tg)\oplus nh, (sf+tg)\oplus nh)}\right|_{s=t=0} \end{aligned} \tag{4.3.2}$$

where $E_0 := \alpha(\mathbb{1}_P \otimes F_Q)$ is the normalization factor of the selective state update ω^{sel}. Let $e^{-\frac{i}{2} st E_P(f,g)} := A(s,t)$ and $\sum_{n\in\mathbb{Z}} \hat{b}(n) e^{-\frac{1}{2}\mu^\alpha((sf+tg)\oplus nh, (sf+tg)\oplus nh)} := B(s,t)$, then:

$$-\left.\frac{\partial^2}{\partial s \partial t} A(s,t)B(s,t)\right|_{s=t=0} = -\left.\left[(\partial_s\partial_t A(s,t))B(s,t) + A(s,t)(\partial_s\partial_t B(s,t))\right]\right|_{s=t=0} \tag{4.3.3}$$

where the terms with only one derivative acting on either A or B vanish at $s = t = 0$. The term $-\left.(\partial_s\partial_t A(s,t))B(s,t)\right|_{s=t=0}$ can be computed as follows:

$$\begin{aligned} \frac{i}{2} E_P(f,g) \sum_{n\in\mathbb{Z}} \hat{b}(n) e^{-\frac{1}{2}\mu^\alpha(0\oplus nh, 0\oplus nh)} &= \frac{i}{2} E_P(f,g) \sum_{n\in\mathbb{Z}} \hat{b}(n) \alpha(W_{P\oplus Q}(0 \oplus nh)) \\ &= \frac{i}{2} E_P(f,g) \sum_{n\in\mathbb{Z}} \hat{b}(n) \alpha(\mathbb{1}_P \otimes W_Q(nh)) \\ &= \frac{i}{2} E_P(f,g) \alpha(\mathbb{1}_P \otimes F_Q) \\ &= \frac{i}{2} E_P(f,g) E_0 \end{aligned} \tag{4.3.4}$$

Let $M(s,t) := \mu^\alpha((sf+tg) \oplus nh, (sf+tg) \oplus nh)$ so that, using $A(0,0) = 1$, one can rewrite $-\left.A(s,t)(\partial_s\partial_t B(s,t))\right|_{s=t=0}$ as:

$$-\sum_{n\in\mathbb{Z}}\hat{b}(n)\left[-\frac{1}{2}\partial_s\partial_t M(s,t)+\left(-\frac{1}{2}\partial_s M(s,t)\right)\left(-\frac{1}{2}\partial_t M(s,t)\right)\right]e^{-\frac{1}{2}M(s,t)}\Bigg|_{s=t=0} \tag{4.3.5}$$

The symmetric bilinear form $M(s,t)$ can be expanded as follows:

$$\begin{aligned}M(s,t) = s^2\mu^\alpha(f\oplus 0, f\oplus 0) + 2st\mu^\alpha(f\oplus 0, g\oplus 0) + t^2\mu^\alpha(g\oplus 0, g\oplus 0)+\\ + 2ns\mu^\alpha(f\oplus 0, 0\oplus h) + 2nt\mu^\alpha(g\oplus 0, 0\oplus h) + n^2\mu^\alpha(0\oplus h, 0\oplus h)\end{aligned} \tag{4.3.6}$$

This simplifies the last expression to:

$$-\sum_{n\in\mathbb{Z}}\hat{b}(n)\left[-\mu^\alpha(f\oplus 0, g\oplus 0) + n^2\mu^\alpha(f\oplus 0, 0\oplus h)\mu^\alpha(g\oplus 0, 0\oplus h)\right]e^{-\frac{1}{2}\mu^\alpha(0\oplus nh, 0\oplus nh)} \tag{4.3.7}$$

Defining the operator $\tilde{F}_Q = \sum_{n\in\mathbb{Z}} n^2\hat{b}(n)W_Q(nh)$, the Fourier sums can be rewritten once again using the normalization factor for the selective state update conditioned on $\tilde{F}_Q$, i.e. $\alpha(\mathbb{1}_P\otimes\tilde{F}_Q) := E_1$. Moreover, $\sum_{n\in\mathbb{Z}}\hat{b}(n)e^{-\frac{1}{2}\mu^\alpha(0\oplus nh, 0\oplus nh)} = E_0$. Then collecting all together:

$$\begin{aligned}\omega^{sel}(\Phi(f)\Phi(g)) = \mu^\alpha(f\oplus 0, g\oplus 0) + \frac{i}{2}E_P(f,g)\\ - \mu^\alpha(f\oplus 0, 0\oplus h)\mu^\alpha(g\oplus 0, 0\oplus h)\frac{E_1}{E_0}\end{aligned} \tag{4.3.8}$$

The difference between the original system 2-point function in the state ω and the updated one in the state ω^{sel} looks then as follows:

$$\begin{aligned}\mathfrak{W}^{\mathrm{sel}}(f\otimes g) - \mathfrak{W}(f\otimes g) &= \mu^\alpha(f\oplus 0, g\oplus 0) + \frac{i}{2}E_P(f,g)\\ &\quad - \mu^\alpha(f\oplus 0, 0\oplus h)\mu^\alpha(g\oplus 0, 0\oplus h)\frac{E_1}{E_0}\\ &\quad - \left(\mu^\omega(f,g) + \frac{i}{2}E_P(f,g)\right)\\ &= \mu^\alpha(f\oplus 0, g\oplus 0) - \mu^\omega(f,g)\\ &\quad - \mu^\alpha(f\oplus 0, 0\oplus h)\mu^\alpha(g\oplus 0, 0\oplus h)\frac{E_1}{E_0}\end{aligned} \tag{4.3.9}$$

where $\frac{E_1}{E_0}$ is a factor dependent on the choice of the effect F_Q. Notice how the final expression is symmetric, this is expected because the antisymmetric part of the 2-point function is state independent and cancels out when computing the difference between 2-point functions of any state. It would be desirable to find a more explicit form for expressions like $\mu^\alpha(a\oplus c, b\oplus d)$, where a, b, c, d are smooth sections of their appropriate bundles. Recall that the symmetric form μ^α is actually identified with the symmetric part of the 2-point function associated to the quasifree state α (Proposition 2.3.1). For a quasifree state the 2-point function can be decomposed in the state dependent symmetric part and the state independent antisymmetric part:

$$\alpha(\Phi_{P\oplus Q}(a \oplus c)\Phi_{P\oplus Q}(b \oplus d)) = \mu^{\alpha}(a \oplus c, b \oplus d) + \frac{i}{2}E_{P\oplus Q}(a \oplus c, b \oplus d) \tag{4.3.10}$$

so that in general:

$$\mu^{\alpha}(a \oplus c, b \oplus d) = \frac{1}{2}\left(\alpha(\Phi_{P\oplus Q}(a \oplus c)\Phi_{P\oplus Q}(b \oplus d)) + (a \oplus c \leftrightarrow b \oplus d)\right) \tag{4.3.11}$$

Since $\alpha(\cdot) = \Theta^*(\omega \otimes \sigma)(\cdot) = (\omega \otimes \sigma)(\Theta\, \cdot)$, it is useful to consider the action of the scattering morphism $\Theta(\Phi_{P\oplus Q}(a \oplus c)) = \Phi_{P\oplus Q}(\theta(a \oplus c))$ where:

$$\theta(a \oplus c) = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \theta_{11}a + \theta_{12}c \\ \theta_{21}a + \theta_{22}c \end{pmatrix} \coloneqq \begin{pmatrix} T_1(a;c) \\ T_2(c;a) \end{pmatrix} \tag{4.3.12}$$

Then the full 2-point function $\alpha(\Phi_{P\oplus Q}(a \oplus c)\Phi_{P\oplus Q}(b \oplus d)) \coloneqq \mathfrak{a}((a \oplus c) \otimes (b \oplus d))$ can be expanded as follows using Eq. (1.4.9):

$$\begin{aligned} \mathfrak{a}((a \oplus c) \otimes (b \oplus d)) &= (\omega \otimes \sigma)\left(\Phi_{P\oplus Q}(T_1(a;c) \oplus T_2(c;a))\Phi_{P\oplus Q}(T_1(b;d) \oplus T_2(d;b))\right) \\ &= (\omega \otimes \sigma)[\Phi_P(T_1(a;c))\Phi_P(T_1(b;d)) \otimes \mathbb{1}_Q \\ &\quad + \mathbb{1}_P \otimes \Phi_Q(T_2(c;a))\Phi_Q(T_2(d;b))] + Z \\ &= \mathfrak{W}(T_1(a;c) \otimes T_1(b;d)) + \mathfrak{S}(T_2(c;a) \otimes T_2(d;b)) + Z \end{aligned} \tag{4.3.13}$$

where Z is a term containing products of 1-point functions of the states ω and σ, which vanish, and $\mathfrak{S}$ is the 2-point function of the probe state σ. Again the 2-point functions of the system state ω and probe state σ can be split into their symmetric and antisymmetric terms:

$$\begin{aligned} \mathfrak{W}(a \otimes b) &= \mu^{\omega}(a,b) + \frac{i}{2}E_P(a,b) \\ \mathfrak{S}(c \otimes d) &= \mu^{\sigma}(c,d) + \frac{i}{2}E_Q(c,d) \end{aligned} \tag{4.3.14}$$

Then $\mu^{\alpha}(a \oplus c, b \oplus d)$ is just the symmetrized part of $\mathfrak{a}((a \oplus c) \otimes (b \oplus d))$:

$$\mu^{\alpha}(a \oplus c, b \oplus d) = \mu^{\omega}(T_1(a;c), T_1(b;d)) + \mu^{\sigma}(T_2(c;a), T_2(d;b)) \tag{4.3.15}$$

The difference of the original 2-point function and the updated 2-point function can be thus rewritten as:

$$\begin{aligned} \Delta\mathfrak{W}^{sel}(f \otimes g) &= \mu^{\omega}(T_1(f;0), T_1(g;0)) + \mu^{\sigma}(T_2(0;f), T_2(0;g)) - \mu^{\omega}(f,g) \\ &\quad - (\mu^{\omega}(T_1(f;0), T_1(0;h)) + \mu^{\sigma}(T_2(0;f), T_2(h;0))) \\ &\quad \times (\mu^{\omega}(T_1(g;0), T_1(0;h)) + \mu^{\sigma}(T_2(0;g), T_2(h;0)))\frac{E_1}{E_0} \end{aligned} \tag{4.3.16}$$

This is the full form of the difference between the updated and the original 2-point functions, it depends on:

(i) The scattering morphism (i.e. the model of interaction between the system and probe) through the transformations $T_{1,2}$ and the normalization factors $E_{0,1}$.

(ii) The conditioning Weyl effect through the normalization factors $E_{0,1}$ and the section $h(x)$.

It is now possible to start working on the second objective of this section, that is taking Eq. (4.3.16) and expand it in perturbation theory by choosing a specific interaction model. To do that, consider a measurement that happens via a quadratic interaction of the system and probe, which are taken to be two scalar fields Φ_P and Φ_Q:

$$\mathcal{L}_{int} = -\lambda\rho(x)\phi(x)\varphi(x),\ \Phi_P(f) = \int_{\mathcal{M}} \phi(x)f(x)\,\mathrm{dV},\ \Phi_Q(g) = \int_{\mathcal{M}} \varphi(x)g(x)\,\mathrm{dV} \tag{4.3.17}$$

Here $\rho(x)$ is compactly supported within the interaction region. Then $T = P \oplus Q + \lambda R$ is a RFHGHO coinciding with $P \oplus Q$ outside the interacting region when R is represented as:

$$R = \begin{pmatrix} R_{PP} & R_{PQ} \\ R_{QP} & R_{QQ} \end{pmatrix} = \begin{pmatrix} 0 & \rho(x) \\ \rho(x) & 0 \end{pmatrix} \tag{4.3.18}$$

Then the full equations of motion, in accordance with the interaction Lagrangian, look like:

$$T\begin{pmatrix} \phi(x) \\ \varphi(x) \end{pmatrix} = P\phi(x) + Q\varphi(x) + \lambda\rho(x)(\phi(x) + \varphi(x)) = 0 \tag{4.3.19}$$

Following [FV20, Sec. 5.3], a Born expansion can be used to estimate at second order in λ the terms $T_1(a;c)$ and $T_2(a;c)$:

$$\begin{aligned}\begin{pmatrix} T_1(a;c) \\ T_2(c;a) \end{pmatrix} &= \begin{pmatrix} a \\ c \end{pmatrix} - \lambda\begin{pmatrix} 0 & \rho \\ \rho & 0 \end{pmatrix}\begin{pmatrix} E_P^- & 0 \\ 0 & E_Q^- \end{pmatrix}\begin{pmatrix} a \\ c \end{pmatrix} \\ &\quad + \lambda^2\begin{pmatrix} 0 & \rho \\ \rho & 0 \end{pmatrix}\begin{pmatrix} E_P^- & 0 \\ 0 & E_Q^- \end{pmatrix}\begin{pmatrix} 0 & \rho \\ \rho & 0 \end{pmatrix}\begin{pmatrix} E_P^- & 0 \\ 0 & E_Q^- \end{pmatrix}\begin{pmatrix} a \\ c \end{pmatrix} + o(\lambda^3) \\ &= \begin{pmatrix} a - \lambda\rho E_Q^- c + \lambda^2\rho E_Q^-\rho E_P^- a \\ c - \lambda\rho E_P^- a + \lambda^2\rho E_P^-\rho E_Q^- c \end{pmatrix} + o(\lambda^3)\end{aligned} \tag{4.3.20}$$

With this one can rewrite Eq. (4.3.16) at second order in λ. To do that, it is possible to expand individually the two type of terms in its expression, letting $\rho E_{P,Q}^- := A_{P,Q}$ this looks like:

$$\begin{aligned}\mu^\alpha(f \oplus 0, g \oplus 0) &= \mu^\omega(T_1(f;0), T_1(g;0)) + \mu^\sigma(T_2(0;f), T_2(0;g)) \\ &= \mu^\omega(f,g) + \lambda^2[\mu^\omega(A_Q A_P f, g) + \mu^\omega(f, A_Q A_P g) \\ &\quad + \mu^\sigma(A_P f, A_P g)] + o(\lambda^3)\end{aligned} \tag{4.3.21}$$

$$\mu^{\alpha}(f \oplus 0, 0 \oplus h)\mu^{\alpha}(g \oplus 0, 0 \oplus h) = \lambda^2 \left[\mu^{\omega}(f, A_Q h) + \mu^{\sigma}(A_P f, h)\right][f \leftrightarrow g] + o(\lambda^3) \quad (4.3.22)$$

Moreover, the term multiplying the factor $\frac{E_1}{E_0}$ cannot be ignored in general, as there are effects F_Q for which its expansion in λ has non vanishing zero-th order. Terms of its expansion proportional to any power of λ are irrelevant here as $\frac{E_1}{E_0}$ only appears multiplied by a term already of order $o(\lambda^2)$. To better see this, consider a series expansion in λ:

$$\frac{E_1}{E_0}[\lambda] = \frac{E_1}{E_0}[0] + o(\lambda) \tag{4.3.23}$$

At $\lambda = 0$ there is no interaction, this means that $\alpha[\lambda = 0] = \omega \otimes \sigma$ and the leading order vanishes if and only if:

$$(\omega \otimes \sigma)(\mathbb{1}_P \otimes \tilde{F}_Q) = \sigma(\tilde{F}_Q) = 0 \tag{4.3.24}$$

Take for example the function $b(x) = \frac{2}{3} + \frac{\cos(x)}{3}$, satisfying the hypothesis in Proposition 4.2.4. Its Fourier expansion has non vanishing coefficients $\hat{b}(0) = \frac{2}{3}$ and $\hat{b}(\pm 1) = \frac{1}{6}$, then:

$$\sigma(\tilde{F}_Q) = \sum_{n \in \mathbb{Z}} n^2 \hat{b}(n) e^{-\frac{1}{2}n^2 \mu^{\sigma}(h,h)} = \frac{1}{3} e^{-\frac{1}{2}\mu^{\sigma}(h,h)} \neq 0 \tag{4.3.25}$$

so that $\frac{E_1}{E_0}[0] = \sigma(\tilde{F}_Q) \coloneqq k_0 \neq 0$. Assembling all together:

$$\begin{aligned} \Delta\mathfrak{W}^{sel}(f \otimes g) = \lambda^2 \big\{ &\mu^{\omega}(A_Q A_P f, g) + \mu^{\omega}(f, A_Q A_P g) + \mu^{\sigma}(A_P f, A_P g) \\ &- \left[\mu^{\omega}(f, A_Q h) + \mu^{\sigma}(A_P f, h)\right][f \leftrightarrow g]\, k_0 \big\} + o(\lambda^3) \end{aligned} \tag{4.3.26}$$

This shows that the 2-point function of the selective state update conditioned on some effect F_Q where the measurement scheme is set up via a weak coupling of the fields, doesn't receive any correction at first order in the coupling w.r.t. the old 2-point function.

Recall the following integral representations:

$$\begin{aligned} [E_P^{\pm} f](x) &= \int_{\mathcal{M}_y} E_P^{\pm}(x, y) f(y)\, \mathrm{dV}_y = \int_{\mathcal{M}_y} E_P^{\mp}(y, x) f(y)\, \mathrm{dV}_y \\ \mu^{\omega}(f, g) &= \int_{\mathcal{M}_x \times \mathcal{M}_y} f(x) \mu^{\omega}(x, y) g(y)\, \mathrm{dV}_x\, \mathrm{dV}_y \end{aligned} \tag{4.3.27}$$

and the analogous ones for $[E_Q^{\pm} f](x)$ and $\mu^{\sigma}(f, g)$. Now, denoting $\mathrm{dV}_x = \mathrm{d}x$ and omitting the domain of integration for a cleaner notation, notice that:

$$
\begin{aligned}
\mu^\omega(A_Q A_P f, g) &= \int \mathrm{d}z\, \mathrm{d}y\, (A_Q A_P f)(z)\mu^\omega(z,y)g(y) \\
&= \int \mathrm{d}z\, \mathrm{d}y\, \rho(z) \int \mathrm{d}x\, \mathrm{d}w\, E^-_Q(z,w)\rho(w)E^-_P(w,x)f(x)\mu^\omega(z,y)g(y) \\
&= \int \mathrm{d}x\, \mathrm{d}y\, f(x) \left(\int \mathrm{d}z\, \mathrm{d}w\, E^+_P(x,w)\rho(w)E^+_Q(w,z)\rho(z)\mu^\omega(z,y) \right) g(y) \\
&\coloneqq \int \mathrm{d}x\, \mathrm{d}y\, f(x)K_1(x,y)g(y)
\end{aligned}
\tag{4.3.28}
$$

Here the following integral kernel has been defined:

$$
K_1(x,y) \coloneqq \int \mathrm{d}z\, \mathrm{d}w\, E^+_P(x,w)\rho(w)E^+_Q(w,z)\rho(z)\mu^\omega(z,y) \tag{4.3.29}
$$

and the antisymmetry of $E^-_P(w,x)$ and $E^-_Q(z,w)$ has been used to rearrange the integral. Similarly:

$$
\begin{aligned}
\mu^\omega(f, A_Q A_P g) &= \int \mathrm{d}x\, \mathrm{d}y\, f(x)K_2(x,y)g(y) \\
\mu^\sigma(A_P f, A_P g) &= \int \mathrm{d}x\, \mathrm{d}y\, f(x)K_3(x,y)g(y) \\
[\mu^\omega(f, A_Q h) + \mu^\sigma(A_P f, h)]\, [f \leftrightarrow g]\, k_0 &= \int \mathrm{d}x\, \mathrm{d}y\, f(x)K_4(x,y)g(y)
\end{aligned}
\tag{4.3.30}
$$

The integral kernels K_2, K_3 are defined as:

$$
\begin{aligned}
K_2(x,y) &\coloneqq \int \mathrm{d}z\, \mathrm{d}w\, \mu^\omega(x,w)\rho(w)E^-_Q(w,z)\rho(z)E^-_P(z,y) \\
K_3(x,y) &\coloneqq \int \mathrm{d}z\, \mathrm{d}w\, E^+_P(x,w)\rho(w)\mu^\sigma(w,z)\rho(z)E^-_P(z,y)
\end{aligned}
\tag{4.3.31}
$$

The kernel $K_4(x,y) = k_0 t(x)t(y)$ instead correctly factors into x, y dependent functions defined as:

$$
t(x) \coloneqq \int \mathrm{d}z\, \mathrm{d}w\, [\mu^\omega(x,z)\rho(z)E^-_Q(z,w)h(w) + E^+_P(x,z)\rho(z)\mu^\sigma(z,w)h(w)] \tag{4.3.32}
$$

This is expected as the original term was a product of an f dependent and a g dependent part. The variation of the 2-point function can now finally be rewritten in terms of a new integral kernel:

$$
\begin{aligned}
\Delta\mathfrak{W}^{\mathrm{sel}}(f\otimes g) &= \lambda^2\int \mathrm{d}x\,\mathrm{d}y\, f(x)\Delta W^{\mathrm{sel}}_{(2)}(x,y)g(y)+o(\lambda^3)\\
\Delta W^{\mathrm{sel}}_{(2)}(x,y) &= K_1(x,y)+K_2(x,y)+K_3(x,y)-K_4(x,y)\\
&= \int \mathrm{d}z\,\mathrm{d}w\,\Big\{E^+_P(x,w)\rho(w)E^+_Q(w,z)\rho(z)\mu^\omega(z,y)\\
&\quad+\mu^\omega(x,w)\rho(w)E^-_Q(w,z)\rho(w)E^-_P(z,y)\\
&\quad+E^+_P(x,w)\rho(w)\mu^\sigma(w,z)\rho(z)E^-_P(z,y)\Big\}\\
&\quad-k_0t(x)t(y)
\end{aligned}
\tag{4.3.33}
$$

This is the final explicit form of Eq. (4.3.16) at second order in perturbation theory, for the specific interaction model considered. It is now interesting to analyze more in depth this result and, based on the discussion of Section 2.1, act on $\Delta W^{\mathrm{sel}}_{(2)}(x,y)$ with the differential operator $\mathcal{D}^{(x,y)}_{\mu\nu}$.

Notice first the expression for the function $t(x)$ in Eq. (4.3.32). Because both h and ρ are compactly supported smooth sections, any combination $\rho E^-_Q h$ is also a compactly supported smooth section by the properties of the advanced/retarded Green operators. Since ω and σ are Hadamard, the wavefront set of μ^ω,μ^σ cannot contain vectors of the type $(x,k,y,0)$, because the Hadamard condition would imply $k=0$, but $(x,0;y,0)\notin \dot{T}^*(\mathcal{M}\times\mathcal{M})$. Then both $\int\mu^\omega(x,w)\rho(z)E^-_Q(z,w)h(w)\,\mathrm{d}w$ and $\int E^+_P(x,z)\rho(z)\mu^\sigma(z,w)h(w)\,\mathrm{d}w$ are smooth sections [Hör90, Th. 8.2.12]. Thus, K_4 is just a product of two terms, each comprising a sum of smooth sections, i.e. a smooth section with $\mathrm{WF}(K_4)=\varnothing$. Moreover, since $t(x)$ is a smooth section, there are no distributional singularities creating problems when taking the following limit:

$$
\lim_{y\to x}\mathcal{D}^{(x,y)}_{\mu\nu}K_4(x,y)\coloneqq k_0T_4[t](x)
\tag{4.3.34}
$$

Here $T_4[t](x)$ is a classical stress energy tensor of a fictitious classical field configuration $t(x)$ defined by Eq. (4.3.32). The term classical refers to the fact that $t(x)$ is smooth, unlike the distributional quantum field $\hat{\phi}(x)$ of Eq. (2.1.4).

Notice now that taking $h=0$ means that $F_Q=\sum_{n\in\mathbb{Z}}\hat{b}(n)W(nh)=K\mathbb{1}$. The selective state update in this case is:

$$
\omega^{\mathrm{sel}}(A)=\frac{\alpha(A\otimes K\mathbb{1})}{\alpha(\mathbb{1}\otimes K\mathbb{1})}=\alpha(A\otimes\mathbb{1})=\omega^{n.s.}(A)
\tag{4.3.35}
$$

This shows that $h=0$ directly reduces to the non-selective state update. It is also evident from the perturbative expansion that in this case $K_4=0$ so that:

$$
\Delta\mathfrak{W}^{n.s.}(f\otimes g)=\lambda^2\big\{\mu^\omega(A_QA_Pf,g)+\mu^\omega(f,A_QA_Pg)+\mu^\sigma(A_Pf,A_Pg)\big\}+o(\lambda^3)
\tag{4.3.36}
$$

To interpret this result physically, consider the formal power series expansion of the stress energy tensor in the non-selectively updated state:

$$
\langle\hat{T}^{\mathrm{reg}}_{\mu\nu}\rangle_{\omega^{n.s.}}=\langle\hat{T}^{\mathrm{reg}}_{\mu\nu}\rangle_{\omega^{\mathrm{n.s.}}}\Big|_{\lambda=0}+\lambda\,\frac{d}{d\lambda}\langle\hat{T}^{\mathrm{reg}}_{\mu\nu}\rangle_{\omega^{n.s.}}\Big|_{\lambda=0}+\lambda^2\,\frac{d^2}{d\lambda^2}\langle\hat{T}^{\mathrm{reg}}_{\mu\nu}\rangle_{\omega^{n.s.}}\Big|_{\lambda=0}+o(\lambda^3)
\tag{4.3.37}
$$

Clearly $\omega^{\text{n.s.}}|_{\lambda=0} = \omega$ so that:

$$\lim_{y\to x} \mathcal{D}^{(x,y)}_{\mu\nu} \Delta W^{\text{sel}}_{(2)}[h=0] = \lambda^2 \frac{d^2}{d\lambda^2} \langle \hat{T}^{\text{reg}}_{\mu\nu} \rangle_{\omega^{n.s.}} \bigg|_{\lambda=0} + o(\lambda^3) \tag{4.3.38}$$

The main result here is that the action of the differential operator $\mathcal{D}^{(x,y)}_{\mu\nu}$ on (the second order kernel of) Eq. (4.3.36) should be interpreted as the second order correction to the expectation value of the regularized stress energy tensor of the non-selectively updated state $\omega^{n.s.}$ w.r.t. the original state ω. On top of that, Eq. (4.3.34) shows how smoothly deviating from the non-selective to the selective state update by choosing a non vanishing $h(x)$, adds a contribution to the second order correction which is proportional to a classical stress energy tensor of a field configuration $t(x)$ as in Eq. (4.3.32).

As a final consistency check, the methods developed in this section can be used to prove that this specific interaction modeling the measurement preserves the wavefront set of the original 2-point function, so that the updated state is still Hadamard as expected.

Proposition 4.3.1. *Let $\mathfrak{W}^{n.s.}$ be the 2-point function of the non-selective state update in the measurement scheme proposed in this chapter, then* $\mathrm{WF}(\mathfrak{W}^{n.s.}) \subset \mathcal{N}^+ \times \mathcal{N}^-$ *order by order in perturbation theory.*

Proof. The 2-point distribution can be rewritten as:

$$\begin{aligned} \mathfrak{W}^{\text{n.s.}}(f\otimes g) &= \alpha(\Phi_P(f)\Phi_P(g)\otimes \mathbb{1}_Q) \\ &= \alpha(\Phi_{P\oplus Q}(f\oplus 0)\Phi_{P\oplus Q}(g\oplus 0)) \\ &= \mathfrak{a}((f\oplus 0)\otimes(g\oplus 0)) \\ &= \mathfrak{W}(T_1(f;0),T_1(g;0)) + \mathfrak{S}(T_2(0;f),T_2(0;g)) + Z \end{aligned} \tag{4.3.39}$$

where Z denotes again vanishing 1-point terms. Notice that $T_1(f;0) = \theta^\lambda_{11} f$, $T_2(0;f) = \theta^\lambda_{21} f$ where the λ dependence is explicitly shown to keep track of the orders in the expansion. In this way the two terms can be rewritten as the scattering morphism operator acting on a distribution:

$$\begin{aligned} \mathfrak{W}(T_1(f;0),T_1(g;0)) &= [\mathfrak{W}\circ(\theta^\lambda_{11}\otimes\theta^\lambda_{11})](f\otimes g) \\ \mathfrak{S}(T_2(0;f),T_2(0;g)) &= [\mathfrak{S}\circ(\theta^\lambda_{21}\otimes\theta^\lambda_{21})](f\otimes g) \end{aligned} \tag{4.3.40}$$

The map θ^λ has the expansion:

$$\theta^\lambda = \mathbb{1} - \lambda \begin{pmatrix} 0 & \rho E^-_Q \\ \rho E^-_P & 0 \end{pmatrix} + \lambda^2 \begin{pmatrix} \rho E^-_Q \rho E^-_P & 0 \\ 0 & \rho E^-_P \rho E^-_Q \end{pmatrix} + o(\lambda^3) \tag{4.3.41}$$

At order zero in λ the hypothesis is checked trivially as the result is just $\mathfrak{W}(f\otimes g)$. The next contribution is at second order in λ and one needs to check terms of the form:

$$\begin{gathered} \mathfrak{W}\circ(\rho E^-_Q \rho E^-_P \otimes \mathbb{1}) \\ \mathfrak{W}\circ(\mathbb{1}\otimes \rho E^-_P \rho E^-_Q) \\ \mathfrak{S}\circ(\rho E^-_P \otimes \rho E^-_P) \end{gathered} \tag{4.3.42}$$

Let now $\mathfrak{D}$ be a distribution such that $\mathrm{WF}(\mathfrak{D}) \subset \mathcal{N}^+ \times \mathcal{N}^-$, from [Few25, Lemma 5.14] the following composition of wavefront sets holds:

$$\mathrm{WF}(\mathfrak{D}\circ(\mathbb{1}\otimes\rho E^-_P)) \subset (\mathrm{WF}'({}^t\mathbb{1})\times \mathrm{WF}'({}^t\rho E^-_P))\circ(\mathcal{N}^+\times\mathcal{N}^-) \tag{4.3.43}$$

with:

$$\begin{aligned}
\mathrm{WF}'({}^t\rho E_P^-) &= \{(x,k;y,l) \in \dot{T}^*(\mathcal{M}\times\mathcal{M}) \mid (y,-l;x,k) \ \in \mathrm{WF}(\rho E_P^-)\} \\
&\subset \{(x,k;y,l) \in \mathcal{N}\times\mathcal{N} \mid (y,-l) \sim (x,-k),\ y \in J^-(x)\cap \operatorname{supp}(\rho)\} \\
\mathrm{WF}'({}^t\mathbb{1}) &= \{(x,k;y,l) \in \dot{T}^*(\mathcal{M}\times\mathcal{M}) \mid (y,-l;x,k) \ \in \mathrm{WF}(\mathbb{1})\} \\
&= \{(x,k;y,l) \in \dot{T}^*(\mathcal{M}\times\mathcal{M}) \mid x=y,\ l=k\}
\end{aligned} \tag{4.3.44}$$

This holds if $\mathcal{N}^+\times\mathcal{N}^-$ does not intersect $(0_{T^*\mathcal{M}}\times\dot{T}^*\mathcal{M})\cup(\dot{T}^*\mathcal{M}\times 0_{T^*\mathcal{M}})$, which is verified given the definition of $\mathcal{N}^\pm$, and because $\mathbb{1}$ and E_P^- are regular morphisms in the sense of Definition 3.3.1. The composition of wavefront sets takes the following form:

$$\begin{aligned}
\mathrm{WF}(\mathfrak{D}\circ(\mathbb{1}\otimes\rho E_P^-)) \subset \{&(x,k;y,l)\in\dot{T}^*(\mathcal{M}\times\mathcal{M}) \mid \exists(w,m;z,n)\in\mathcal{N}^+\times\mathcal{N}^-, \\
&((x,k;w,m),(y,l;z,n)) \in (\mathrm{WF}'(\mathbb{1})\times\mathrm{WF}'({}^t\rho E_P^-))\}
\end{aligned} \tag{4.3.45}$$

This means that $m\in\mathcal{N}^+ \implies k\in\mathcal{N}^+$, $n\in\mathcal{N}^- \implies l\in\mathcal{N}^-$, so that:

$$\mathrm{WF}(\mathfrak{D}\circ(\mathbb{1}\otimes\rho E_P^-)) \subset \mathcal{N}^+\times\mathcal{N}^- \tag{4.3.46}$$

Similarly:

$$\mathrm{WF}(\mathfrak{D}\circ(\rho E_P^-\otimes\mathbb{1})) \subset \mathcal{N}^+\times\mathcal{N}^- \tag{4.3.47}$$

All of the terms in the expansion are compositions of $\mathfrak{W}$ and $\mathfrak{S}$ with sequences of $(\mathbb{1}\otimes\rho E^\pm_{P,Q})$ and $(\rho E^\pm_{P,Q}\otimes\mathbb{1})$. Both $\mathfrak{W}$ and $\mathfrak{S}$ satisfy the defining condition of $\mathfrak{D}$ because they are 2-point functions of Hadamard states, this means that:

$$\mathrm{WF}(\mathfrak{W}^{\text{n.s.}}) \subset \mathcal{N}^+\times\mathcal{N}^- \tag{4.3.48}$$

□

The fact that $\mathfrak{W}^{\text{n.s.}}$ satisfies the Hadamard condition then follows from the same argument used in the proof of Theorem 3.3.3, where it was shown that:

$$\mathrm{WF}(\mathfrak{W}) \subset \mathcal{N}^+\times\mathcal{N}^- \implies \mathrm{WF}(\mathfrak{W}) = \{(x,k;y,-l)\in\mathcal{N}^+\times\mathcal{N}^- \mid (x,k)\sim(y,l)\} \tag{4.3.49}$$

This result explicitly checks the consistency of Theorem 3.3.4 in the specific measurement scheme presented in this section, confirming that a non-selective measurement preserves the Hadamard property.

Conclusions

The original results of this thesis, presented in Chapter 4, showed that it is possible to set up a measurement scheme, in the sense discussed in Chapter 3, in such a way that the selective state update of a Hadamard state is still Hadamard. This was done by presenting several different ways to construct effect operators of the probe Weyl C^*-algebra that are, in some appropriate sense, sufficiently regular, so that they do not introduce new singularities in the 2-point function of the updated state. Some examples with physically meaningful interpretations were also presented such as Example 4.1.1 and Example 4.2.2, both conditioning the measurement on certain ranges of values of the probe field. Finally, a specific model in perturbation theory provided an example of explicit computations representing the energy variation of a system that is subjected to a quantum measurement, showing how the structure of the measurement scheme affects the energy variation itself. A final consistency check also concluded that, in the specific measurement scheme presented in Section 4.3, the preservation of the Hadamard condition holds order by order in perturbation theory.

This work presented sufficient conditions to set up a conditioning effect for a QFT measurement that preserves the Hadamard condition. A further step in this direction would be to better understand the physical implications of these conditions, while also trying to sharpen them. Additional methods to construct such effects might also be uncovered, complementing the spectral theorem and Weyl generators series shown here.

Finally, the perturbative model showed how a selective measurement comes with a possibly negative contribution to the variation of the system energy after the measurement. This could suggest a possible link with the theory of Quantum Energy Inequalities (QEIs) [Few12]. This theory aims to put constraints on the negative local energy fluctuation of a quantum field, and, in particular, it constrains the energy density of Hadamard states. The main results of this work show that QEIs should apply to post-measurement states, which could eventually suggest the existence of thresholds for which measurement schemes are physically realizable.

Appendix

A Distributions on vector bundles

This appendix contains the main definitions and results on the theory of vector bundles on manifolds and distributions on them. The content reported here is meant to be a concise summary of the tools used in the main work, for a more exhaustive treatment with detailed proofs see e.g. [Tu11; Tu17; Lee03; BGP08; Hör90].

Definition A.1 (*Topological manifold*). A second countable, Hausdorff topological space $\mathcal{M}$ is called a *topological manifold* of dimension n (or n-*manifold*) if it is locally homeomorphic to $\mathbb{R}^n$, i.e. $\forall p \in \mathcal{M}$ there are an open neighborhood U of p and a homeomorphism $\phi\colon U \to \mathbb{R}^n$. The pair (U, ϕ) is called *chart* of $\mathcal{M}$. A collection of charts $\{(U_\alpha, \phi_\alpha)\}$ such that $\{U_\alpha\}$ is an open cover of $\mathcal{M}$ is called an *atlas* of $\mathcal{M}$.

Definition A.2 (*Smooth manifold*). Let $\mathcal{M}$ be a topological n-manifold and $\{(U_\alpha, \phi_\alpha)\}$ an atlas of $\mathcal{M}$. If:

$$\phi_\beta \circ \phi_\alpha^{-1}\colon \mathbb{R}^n \supset \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n \quad \forall \alpha, \beta \tag{A.1}$$

is smooth in the $\mathbb{R}^n$ sense, the atlas is called *smooth*. An atlas is called *maximal* if there is no other atlas of $\mathcal{M}$ containing all of its charts. A topological n-manifold together with a maximal smooth atlas is called a *smooth manifold*.

Definition A.3 (*Smooth map between manifolds*). Let $\mathcal{N}$ and $\mathcal{M}$ be two smooth manifolds of dimension n and m respectively. A (topologically) continuous map $F : \mathcal{N} \to \mathcal{M}$ is said to be *smooth* at a point $p \in \mathcal{N}$ if there are charts (U, ϕ) and (V, ψ) of $\mathcal{N}$ and $\mathcal{M}$ respectively with $p \in U$ and $F(p) \in V$ such that:

$$\psi \circ F \circ \phi^{-1}\colon \mathbb{R}^n \supset \phi(F^{-1}(V) \cap U) \to \mathbb{R}^m \tag{A.2}$$

is smooth at $p \in \mathcal{N}$ in the $\mathbb{R}^n$ sense. If F is smooth at every point of $\mathcal{N}$, then F is called a *smooth map*. The set of all smooth maps between $\mathcal{N}$ and $\mathcal{M}$ is denoted as $C^\infty(\mathcal{N}, \mathcal{M})$, if $\mathcal{M} = \mathbb{R}$ one usually writes $C^\infty(\mathcal{N})$.

Definition A.4 (*Tangent space*). Let $\mathcal{M}$ be a smooth manifold and $p \in \mathcal{M}$. A linear map $v_p\colon C^\infty(\mathcal{M}) \to \mathbb{R}$ is called a *derivation* at p if it satisfies the Leibniz rule:

$$v_p(fg) = v_p(f)g(p) + f(p)v_p(g) \quad \forall f, g \in C^\infty(\mathcal{M}) \tag{A.3}$$

The set of all derivations at $p \in \mathcal{M}$ is a vector space over $\mathbb{R}$ called the *tangent space* of $\mathcal{M}$ at p and is denoted as $T_p\mathcal{M}$. If $\mathcal{M}$ has dimension n, then as a vector space $\dim(T_p\mathcal{M}) = n$.

Definition A.5 (*Differential of a smooth function*). Let $F\colon \mathcal{N} \to \mathcal{M}$ be a smooth map between manifolds. At each point $p \in \mathcal{N}$, F induces a linear map between tangent spaces called the *differential* at p:

$$\begin{aligned} F_{*,p}\colon T_p\mathcal{N} &\to T_{F(p)}\mathcal{M} \\ v_p &\mapsto F_{*,p}(v_p) \end{aligned} \tag{A.4}$$

where $F_{*,p}(v_p)f := v_p(f \circ F) \quad \forall f \in C^\infty(\mathcal{M})$.

With these definitions it is possible to introduce the concept of vector bundle over a smooth manifold.

Definition A.6 (*Vector Bundle*)**.** Let $E, \mathcal{M}$ be two smooth manifolds and $\pi\colon E \to \mathcal{M}$ a smooth surjection between manifolds. Let also $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, then the triple $(E, \mathcal{M}, \pi)$ is called a *smooth vector bundle* of rank r if:

(i) $\forall p \in \mathcal{M}$, the set $E_p \coloneqq \pi^{-1}(p)$ is a vector space over $\mathbb{K}$ of dimension r.

(ii) $\forall p \in \mathcal{M}$, there is an open neighborhood U of p and a diffeomorphism $\phi_U\colon \pi^{-1}(U) \to U \times \mathbb{K}^r$ that restricts to a fiber preserving vector space isomorphism $\phi_U|_p : E_p \to \{p\} \times \mathbb{K}^r$, i.e. $\pi_1 \circ \phi_U|_p = \pi$ where $\pi_1\colon U \times \mathbb{K}^r \to U$ is the projection on the first factor.

We call E the *total space*, $\mathcal{M}$ the *base space* and E_p the *fiber above* p of the vector bundle. The map ϕ_U is called *local trivialization* of the *trivializing open subset* U.

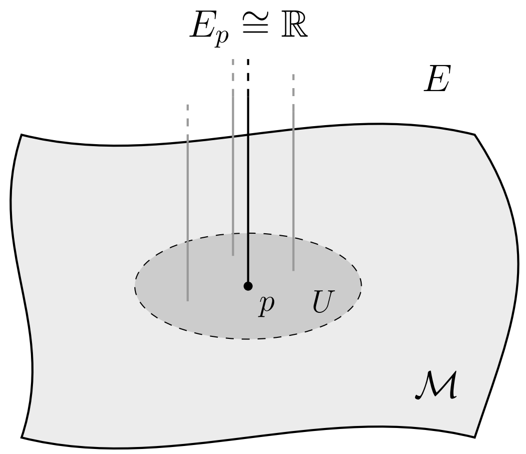

Figure A.1: Pictorial representation of a line bundle over a smooth manifold. To each point $p \in \mathcal{M}$ there is an associated fiber E_p isomorphic to the real line $\mathbb{R}$.

Remark. It is customary to refer to a vector bundle both as its total space E and its projection map $\pi\colon E \to \mathcal{M}$ and calling it a vector bundle over $\mathcal{M}$. See Fig. A.1 for a pictorial representation of a vector bundle.

A vector bundle E is basically a smooth manifold that locally looks like its base with a copy of $\mathbb{K}^r$ attached to each point. Among the many examples of vector bundles, the tangent bundle of a smooth manifold is of crucial importance.

Example A.1 (*Tangent bundle*)**.** Let $\mathcal{M}$ be a smooth manifold and E a vector bundle over $\mathcal{M}$ such that $E_p = T_p\mathcal{M}$, the tangent space of $\mathcal{M}$ at p. Then E is called the *tangent bundle* of $\mathcal{M}$ and is denoted as $T\mathcal{M}$.

This construction of the tangent bundle is basically reversed with respect to the usual one where one defines $T\mathcal{M}$ as the disjoint union of the tangent spaces $T_p\mathcal{M}$ at each point p of $\mathcal{M}$ and then builds a manifold structure on it.

Definition A.7 (*Smooth maps between vector bundles*)**.** Let $\pi_E \colon E \to \mathcal{M}$ and $\pi_F \colon F \to \mathcal{M}$ be vector bundle over the same base manifold $\mathcal{M}$. A smooth map between manifolds $\beta \colon E \to F$ is said to be a *smooth bundle map* if it is fiber preserving, that is $\pi_F \circ \beta = \pi_E$.

There are 3 ways of constructing new vector bundles from old ones that are of particular importance:

Definition A.8 (*Composition of vector bundles*)**.** Let E, F be a vector bundles with base manifolds $\mathcal{M}, \mathcal{N}$ respectively, then:

(i) If $\mathcal{M} = \mathcal{N}$, the *tensor product bundle* $E \otimes F$ is the vector bundle with base manifold $\mathcal{M}$ such that the fiber above $p \in \mathcal{M}$ is the tensor product of the fibers of E and F above p, i.e. $(E \otimes F)_p = E_p \otimes F_p$.

(ii) If $\mathcal{M} = \mathcal{N}$, the *direct sum bundle* $E \oplus F$ is the vector bundle with base manifold $\mathcal{M}$ such that the fiber above $p \in \mathcal{M}$ is the direct sum of the fibers of E and F above p, i.e. $(E \oplus F)_p = E_p \oplus F_q$.

(iii) The *external tensor product bundle* $E \boxtimes F$ is the vector bundle with base manifold $\mathcal{M} \times \mathcal{N}$ such that the fiber above $(p, q) \in \mathcal{M} \times \mathcal{N}$ is the tensor product of the fibers of E and F above p and q respectively, i.e. $(E \boxtimes F)_{(p,q)} = E_p \otimes F_q$.

Relevant objects related to smooth manifolds are vector fields. A vector field can be seen as an assignment of a tangent vector to each point of the manifold, i.e. a choice of an element of each fiber of the tangent bundle. This is formalized for a general vector bundle by the following definition.

Definition A.9 (*Section of a vector bundle*)**.** Let $\pi : E \to \mathcal{M}$ be a smooth vector bundle. A *section* of E over an open set U is a smooth map $f : U \to E$ such that $\pi \circ f = \mathrm{id}_{\mathcal{M}}$. The section is called *smooth* if the map f is smooth as a map between manifolds.The set of all smooth sections of E is denoted as $\Gamma^\infty(E)$ and it has the structure of a $\mathbb{K}$-vector space.

Vector bundles can be further characterized by the introduction of a metric and a connection on them. This characterization has a key role for the tangent bundle.

Definition A.10 (*Bundle metric*)**.** Let E be a vector bundle over $\mathcal{M}$, a *bundle metric* on E is an assignment $\mathcal{M} \ni p \to \langle \cdot, \cdot \rangle_p$ where $\langle \cdot, \cdot \rangle_p$ is a non degenerate inner product on the fiber E_p. This assignment is required to be smooth in the following sense: for any two smooth sections $s_1, s_2 \in \Gamma(E)$, the map $\mathcal{M} \ni p \to \langle s_1(p), s_2(p) \rangle_p \in \mathbb{R}$ is a smooth function on $\mathcal{M}$. The bundle metric $\langle \cdot, \cdot \rangle_p$ can be:

- *Riemannian* if the inner product is pointwise positive definite.
- *Hermitian* if the fibers are $\mathbb{C}$-vector spaces and the inner product is pointwise Hermitian.
- *Lorentzian* if the inner product has pointwise signature $(+, -, \dots, -)$.

Proposition A.1. *Every smooth vector bundle E with real (complex) fibers over a manifold $\mathcal{M}$ has a Riemannian (Hermitian) bundle metric.*

Proof. See e.g. [Tu17]. □

Definition A.11 (*Metric manifold*)**.** A smooth manifold $\mathcal{M}$ together with a Riemannian bundle metric on its tangent bundle $T\mathcal{M}$ is called a metric manifold.

It is often useful to be able to compare elements of different fibers of a vector bundle. There is no natural way to do that with the tools presented so far because vectors in different fibers belong to different vector spaces. It is possible to introduce the concept of connection on a vector bundle to solve this problem. Again the introduction of a connection on the tangent bundle of a metric manifold is of particular importance.

Definition A.12 (*Connection on a vector bundle*)**.** Let E be a vector bundle over a smooth manifold $\mathcal{M}$. A *connection* on E is a map:

$$\nabla \colon \Gamma^\infty(T\mathcal{M}) \times \Gamma^\infty(E) \to \Gamma^\infty(E) \tag{A.5}$$

such that $\forall X \in \Gamma^\infty(T\mathcal{M})$ and $\forall f \in \Gamma^\infty(E)$:

(i) $\nabla(X, f) \coloneqq \nabla_X f$ is $C^\infty(\mathcal{M})$-linear in X and $\mathbb{R}$-linear in s.

(ii) If $g \in C^\infty(\mathcal{M})$, then $\nabla_X(gf) = X(g)f + g\nabla_X f$.

If E is a metric bundle with bundle metric $\langle\cdot,\cdot\rangle$, the connection ∇ is called *metric connection* or *compatible with the metric* if it satisfies the following compatibility condition:

$$X\langle f_1, f_2\rangle = \langle \nabla_X f_1, f_2\rangle + \langle f_1, \nabla_X f_2\rangle \quad \forall X \in \Gamma^\infty(T\mathcal{M}), \quad \forall f_1, f_2 \in \Gamma^\infty(E) \tag{A.6}$$

Example A.2 (*Levi-Civita connection*)**.** Let $E = T\mathcal{M}$, in this case it is possible to define the notion of torsion of a connection ∇ as the map $T : \Gamma^\infty(T\mathcal{M}) \times \Gamma^\infty(T\mathcal{M}) \to \Gamma^\infty(T\mathcal{M})$ defined as:

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \in \Gamma^\infty(T\mathcal{M}) \tag{A.7}$$

where $[X, Y]$ is the Lie bracket of the vector fields X and Y. If $T(X, Y) = 0 \quad \forall X, Y \in \Gamma^\infty(T\mathcal{M})$, the connection ∇ is said to be *torsion-free*. A torsion-free metric connection on $T\mathcal{M}$ is called *Levi-Civita connection*.

Proposition A.2. *Every smooth vector bundle E over a manifold $\mathcal{M}$ has a connection. In particular, if E is a metric bundle, it has a metric connection. Moreover if $E = T\mathcal{M}$ is the tangent bundle of a metric manifold, it has a unique Levi-Civita connection.*

Proof. See e.g. [Tu17]. □

This proposition assures that it is possible to introduce a connection on any vector bundle. The uniqueness property is exclusive to the torsion free Levi-Civita connection on the tangent bundle, however it is useful to be able to introduce auxiliary connections on other vector bundles. This, together with the parallel result for the bundle metric, comes particularly handy when trying to define the norm of the covariant derivative of a bundle vector field, which can be done as follows:

Definition A.13 (*C^k-norm of a section of a vector bundle*)**.** Let E be a smooth vector bundle with base manifold $\mathcal{M}$. Equip E and the cotangent bundle $T^*\mathcal{M}$ with connections $\nabla^E, \nabla^{T^*\mathcal{M}}$ and let $f \in \Gamma^\infty(E)$. There is a natural induced connection on the tensor product bundle $(T^*\mathcal{M})^{\otimes k} \otimes E$:

$$\nabla^k \colon \Gamma^\infty(T\mathcal{M}) \times \Gamma^\infty(\underbrace{T^*\mathcal{M} \otimes \cdots \otimes T^*\mathcal{M}}_{k-\text{times}} \otimes E) \to \Gamma^\infty(\underbrace{T^*\mathcal{M} \otimes \cdots \otimes T^*\mathcal{M}}_{k-\text{times}} \otimes E) \tag{A.8}$$

It is defined by imposing the Leibniz rule on all of his factors:

$$\begin{aligned} \nabla^k_X(\omega_1 \otimes \cdots \otimes \omega_k \otimes f) = {} & \nabla^{T^*\mathcal{M}}_X \omega_1 \otimes \cdots \otimes \omega_k \otimes f \\ & + \cdots \\ & + \omega_1 \otimes \cdots \otimes \nabla^{T^*\mathcal{M}}_X \omega_k \otimes f \\ & + \omega_1 \otimes \cdots \otimes \omega_k \otimes \nabla^E_X s \end{aligned} \tag{A.9}$$

The *k-th covariant derivative* of a smooth section of the vector bundle E is the section $\nabla^k s \in \Gamma^\infty((T^*\mathcal{M})^{\otimes k} \otimes E)$. Choosing a Riemannian (Hermitian) bundle metric on E and on $T^*\mathcal{M}$ there is a naturally induced bundle metric on $(T^*\mathcal{M})^{\otimes k} \otimes E$. For any $K \subset \mathcal{M}$, it is possible to define the *C^k-norm* of f as:

$$\|f\|_{C^k(K)} \coloneqq \max_{j=0,\ldots,k} \sup_{x \in K} \|\nabla^j f(x)\|_{(T^*\mathcal{M})^{\otimes j} \otimes E} \tag{A.10}$$

Remark. It is common to work with compactly supported sections of a vector bundle. In the case of a compact set $K \subset \mathcal{M}$, different choices of connections and metrics on E and $T^*\mathcal{M}$ lead to equivalent C^k-norms [BGP08]. This means that there is no need to specify the choice as by Proposition A.1 and Proposition A.2 it is guaranteed that they exist.

It is also particularly useful to introduce the concept of integration on a smooth manifold. The natural objects to integrate on a manifold are differential forms, however they require the manifold to be orientable which might not always be the case. To solve this, one introduces the concept of densities. It is remarkable that on a metric manifold there is a natural metric induced density.

Definition A.14 (*k-covector*)**.** Let $\mathcal{M}$ be a smooth manifold and $p \in \mathcal{M}$. A *k-covector* $\omega_p \in T^*_p\mathcal{M}^{\otimes k}$ is an *alternating k-tensor*:

$$\omega_p \colon \underbrace{T_p\mathcal{M} \otimes \cdots \otimes T_p\mathcal{M}}_{k-\text{times}} \to \mathbb{R} \tag{A.11}$$

where alternating means that for every element of the permutation group of k elements $\sigma \in S_k$ one has:

$$(\sigma\omega_p)(v_1, \ldots, v_k) \coloneqq \omega_p(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \operatorname{sgn}(\sigma)\omega_p(v_1, \ldots, v_k) \tag{A.12}$$

Definition A.15 (*Wedge product and k-forms*)**.** Let $\mathcal{M}$ be a smooth manifold of dimension $n, p \in \mathcal{M}$ and $\alpha_p \in T_p^*\mathcal{M}^{\otimes m}, \beta_p \in T_p^*\mathcal{M}^{\otimes l}$ be a m-covector and a l-covector respectively. Then:

$$T_p^*\mathcal{M}^{\otimes(m+l)} \ni \alpha_p \wedge \beta_p := \frac{1}{m!l!} \sum_{\sigma \in S_{m+l}} \operatorname{sgn}(\sigma)\sigma(\alpha_p \otimes \beta_p) \tag{A.13}$$

is a $m + l$-covector. For this reason the vector space of dimension $\binom{n}{k}$ of k-covectors at $p \in \mathcal{M}$ is denoted as $\bigwedge^k T_p^*\mathcal{M}$.

Let $\bigwedge^k T^*\mathcal{M}$ be the vector bundle over $\mathcal{M}$ whose fiber above $p \in \mathcal{M}$ is the vector space $\bigwedge^k T_p^*\mathcal{M}$ of k-covectors at p, then a *differential k-form* ω is a section of $\bigwedge^k T^*\mathcal{M}$.

Definition A.16 (*Pullback of a k-form*)**.** Let $F \colon \mathcal{N} \to \mathcal{M}$ be a smooth map between manifolds $\omega \in \Gamma^\infty(\bigwedge^k T^*\mathcal{M})$ be a smooth k-form on $\mathcal{M}$. The *pullback* of ω by the smooth map F is the k-form on $\mathcal{N}$ defined as follows on each point $p \in \mathcal{N}$:

$$(F^*\omega)_p(v_1, \ldots, v_k) := (\omega)_{F(p)}(F_{*,p}v_1, \ldots, F_{*,p}v_k) \quad \forall v_i \in T_p\mathcal{M} \tag{A.14}$$

where $F_{*,p}$ is the differential of F at $p \in \mathcal{N}$.

Definition A.17 (*Density*)**.** Let $\mathcal{M}$ be a smooth manifold of dimension $n, p \in \mathcal{M}$. A *density* μ on $T_p\mathcal{M}$ is a map:

$$\mu \colon T_p\mathcal{M}^{\otimes n} \to \mathbb{R} \tag{A.15}$$

such that $\forall A : T_p\mathcal{M} \to T_p\mathcal{M}$ it satisfies $\mu(Av_1, \ldots, Av_n) = |\det(A)|\mu(v_1, \ldots, v_n)$. The set of all densities on $T_p^*\mathcal{M}$ is denoted as $\Omega(T_p\mathcal{M})$.

Proposition A.3. *Let $\mathcal{M}$ be a smooth manifold of dimension n, then:*

(i) *$\Omega(T_p\mathcal{M})$ is a 1-dimensional vector space spanned by $|\omega|$, for all $0 \neq \omega \in \bigwedge^n T_p^*\mathcal{M}$ where $|\omega|(v_1, \ldots, v_n) := |\omega(v_1, \ldots, v_n)|$.*

(ii) *$\Omega(\mathcal{M})$ is the line bundle over $\mathcal{M}$ whose fiber above $p \in \mathcal{M}$ is $\Omega(T_p\mathcal{M})$ called density bundle of $\mathcal{M}$. A smooth section of $\Omega(\mathcal{M})$ is called a smooth density on $\mathcal{M}$.*

(iii) *If $\mathcal{M}$ is a metric manifold, there is a unique metric induced smooth density on $\mathcal{M}$ called volume density* dV.

Given a chart $(U, \phi = x^i)$ of $\mathcal{M}$ the volume form takes a standard expression which allows to define its integral:

$$\begin{aligned} \int_U \mathrm{dV} &= \int_{\phi(U)} (\phi^{-1})^* \,\mathrm{dV} \\ &= \int_{\phi(U)} \sqrt{|g|}|\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n| := \int_{\phi(U)} \sqrt{|g|}\mathrm{d}x^1 \cdots \mathrm{d}x^n \end{aligned} \tag{A.16}$$

where $(\phi^{-1})^*\mathrm{dV}$ is the pullback of the smooth density dV acting pointwise as Definition A.16 and $g(x) := \det[g_{\mu\nu}](x)$ is the determinant of the metric tensor field in the chart coordinates.

For any compactly supported continuous function $f \in C_c^\infty(\mathcal{M})$ it is then possible to define the integral of f over $\mathcal{M}$ as $\int_\mathcal{M} f\,\mathrm{dV}$ recalling that:

$$\int_{\phi(U)} (\phi^{-1})^*(f\,\mathrm{dV}) = \int_{\phi(U)} (\phi^{-1})^* f (\phi^{-1})^*\,\mathrm{dV} = \int_{\phi(U)} (f \circ \phi^{-1})(\phi^{-1})^*\,\mathrm{dV} \tag{A.17}$$

Proof. See e.g. [Lee03]. □

This concludes the summary of the necessary notions on vector bundles. Having described the concepts of sections, their differentiation via connections and their integration via volume densities, it is now possible to introduce the theory of distributions.

Definition A.18 (*Space of test sections*)**.** Let E be a smooth vector bundle over a smooth manifold $\mathcal{M}$. Endow E and $T^*\mathcal{M}$ with auxiliary connections and Riemannian (possibly Hermitian) bundle metrics to define the C^k-norm as in Definition A.13. The space $\Gamma_0^\infty(E)$ of compactly supported smooth sections of E is called *space of test sections of E*. A sequence of test sections $\{f_n\}_n \subset \Gamma_0^\infty(E)$ is said to converge to a test section $f \in \Gamma_0^\infty(E)$ if:

(i) There is a compact set $K \subset \mathcal{M}$ such that $\mathrm{supp}(f_n) \subset K \quad \forall n$.

(ii) $\|f - f_n\|_{C^k(K)} \xrightarrow{n\to\infty} 0 \quad \forall k \in \mathbb{N}$.

Definition A.19 (*Distribution on a vector bundle*)**.** Let E be a smooth vector bundle over a smooth manifold $\mathcal{M}, E^*$ the dual vector bundle whose fibers above $p \in \mathcal{M}$ are $E_p^* = (E_p)^*$ and V a vector space over a field $\mathbb{K}$. A $\mathbb{K}$-linear map:

$$u\colon \Gamma_0^\infty(E^*) \to V \tag{A.18}$$

is called a *V-valued distribution on E* or *distributional section of E* if it is continuous in the sense that:

$$V \ni u(f_n) \xrightarrow{n\to\infty} u(f) \in V \quad \forall \Gamma_0^\infty(E^*) \ni f_n \xrightarrow{n\to\infty} f \in \Gamma_0^\infty(E^*) \tag{A.19}$$

The space of V-valued distributions on E is denoted as $\mathcal{D}'(E, V)$.

Remark. The space of V-valued distributional sections of E is essentially the topological dual of $\Gamma_0^\infty(E^*)$ with respect to the weak-$*$ topology, i.e. the weakest topology on $\Gamma_0^\infty(E^*)$ making the V-valued functionals continuous. In this sense it is possible to define a V-valued distribution on the manifold $\mathcal{M}$ itself as $u \in C_0^\infty(\mathcal{M}, V)^* \coloneqq \mathcal{D}'(\mathcal{M})$, i.e. an element of the dual space of the compactly supported smooth functions on $\mathcal{M}$.

Example A.3 (*Delta distribution*)**.** Let $\mathcal{M} = \mathbb{R}$ and $E = \mathbb{R} \times \mathbb{R}$. Then the space of test sections of E^* is just the space of compactly supported real smooth functions $C_0^\infty(\mathbb{R})$. Let $V = \mathbb{R}$, then the *delta distribution* δ_x is the $\mathbb{R}$-valued distribution on E such that:

$$\delta_x(f) \coloneqq f(x)\, x \in \mathbb{R} \tag{A.20}$$

There is a standard way to represent smooth sections as distributions.

Proposition A.4. *Let E be a smooth vector bundle over a smooth metric manifold $\mathcal{M}$ with $\mathbb{K}$ fibers. There is a canonical embedding $\iota : \Gamma^\infty(E) \hookrightarrow \mathcal{D}'(E,\mathbb{K})$. Usually one denotes $\mathcal{D}'(E,\mathbb{K}) = \mathcal{D}'(E)$.*

Proof. Since $\mathcal{M}$ is a metric manifold there is a natural metric induced volume form that allows one to integrate functions on the manifold. Let $f \in \Gamma^\infty(E)$ and $g \in \Gamma_0^\infty(E^*)$. Then $\iota(f) \in \mathcal{D}'(E,\mathbb{K})$ is a distribution defined as follows:

$$[\iota(f)](g) := \int_{\mathcal{M}} g(f)\,\mathrm{dV} \tag{A.21}$$

Since g is a test section of E^*, the quantity $g(f)$ is a compactly supported $\mathbb{K}$ function on $\mathcal{M}$ for which integration with respect to dV is well defined. □

Definition A.20 (*Support of a distribution*)**.** Let $u \in \mathcal{D}'(E,V)$ where E is a vector bundle over $\mathcal{M}$. The support of the distribution u is defined to be the set:

$$\mathrm{supp}(u) := \{p \in \mathcal{M} \mid \forall U \ni p\, \exists f \in \Gamma_0^\infty(E) \text{ s.t. } \mathrm{supp}(f) \subset U,\, u(f) \neq 0\} \tag{A.22}$$

Definition A.21 (*Restriction of a distribution*)**.** Let $\Omega \subset \mathcal{M}$ be an open subset of a manifold $\mathcal{M}$ and E be a vector bundle. Define the vector bundle $E|_\Omega$ as the vector bundle over Ω whose fiber above $p \in \Omega$ coincides with the fiber above p of E. The *restriction of a distributional section u of E to the open subset* Ω is the distributional section $u|_\Omega$ of $E|_\Omega$ such that:

$$u|_\Omega(f) = u(\tilde{f}) \quad \forall f \in \Gamma_0^\infty(E^*|_\Omega),\ \tilde{f}(q) := \begin{cases} f(q) & \text{if } q \in \Omega \\ 0 & \text{if } q \in \mathcal{M} \setminus \Omega \end{cases} \tag{A.23}$$

Definition A.22 (*Singular support of a distribution*)**.** Let $u \in \mathcal{D}'(E,V)$, the *singular support of* u is defined to be the set:

$$\mathrm{singsupp}(u) := \{p \in \mathrm{supp}(u) \mid \nexists U \ni p \text{ s.t. } u|_U \in \Gamma_0^\infty(E|_U)\} \tag{A.24}$$

i.e. the singular support of u is the set of points in its support for which there is no neighborhood such that its restriction to it coincides with a smooth section.

The notion of singularities of a distribution, in the sense of the points near which it can't be described as a smooth section, can be sharpened with respect to Definition A.22. It is indeed possible to study not only the position of singularities, i.e. the singular support, but also their direction in momentum space.

Definition A.23 (*Regular directed point of a distribution*)**.** Let $u \in \mathcal{D}'(\mathcal{M},V)$ be a V-valued distribution on a manifold $\mathcal{M}$ and $(y,l) \in \dot{T}^*\mathcal{M}$ be an element of the cotangent bundle with its zero section removed. Let $(U, \phi = x^\mu)$ be a chart of $\mathcal{M}$ covering a neighborhood of $y \in \mathcal{M}$ and $\mathcal{V} \subset \dot{T}^*\mathcal{M}$ be an open cone. Then (y,l) is said to be a *regular directed point of* u if there is a smooth function $\chi \in C_0^\infty(U)$ such that $\chi(y) \neq 0$ and:

$$\sup_{k \in \mathcal{V}} (1 + \|k\|^n)\|u(\chi e^{ix^\mu k(\partial_\mu)})\|_V < \infty \quad \forall n \in \mathbb{N} \tag{A.25}$$

where $\|\cdot\|_V$ is a norm on the vector space V. That is, the formal Fourier transform of the distribution is fast decreasing in any direction k in a conical neighborhood $\mathcal{V}$ of $l \in \dot{T}_y^*\mathcal{M}$.

Remark. The definition of regular directed point can be understood by explicitly writing down the distribution:

$$u(\chi e^{ix^\mu k(\partial_\mu)}) = \int_U u(x)\chi(x)e^{ix^\mu k(\partial_\mu)}\,\mathrm{dV} \tag{A.26}$$

This can be seen as a local Fourier transform of $u(x)$ where the function $\chi(x)$ localizes the integral around $y \in U$ and the function $e^{ik(x^\mu \partial_\mu)}$ gives the Fourier transform evaluated in the direction $k \in \mathcal{V}$. The condition for (y, l) to be a regular directed point can be then interpreted as the requirement for the Fourier transform of the distribution u to be fast decreasing in a conical neighborhood of l. A non regular directed point (y, l) signals the presence of a singularity of the distribution in $y \in U$ along the direction $l \in T_y^*\mathcal{M}$.

Definition A.24 (*Wavefront set*)**.** Let $u \in \mathcal{D}'(\mathcal{M}, V)$ be a V-valued distribution on a manifold $\mathcal{M}$. The *wavefront set* of u is the set:

$$\mathrm{WF}(u) \coloneqq \{(y, l) \in \dot{T}^*\mathcal{M} \mid (y, l) \text{ is not a regular directed point of } u\} \tag{A.27}$$

If $u \in \mathcal{D}'(E, V)$ is a V-valued distribution on a smooth vector bundle E, let $\{f_i\} \subset \Gamma_0^\infty(E^*)$ be a collection of test sections forming a local basis of E^*, then the wavefront set of u can be defined as:

$$\mathrm{WF}(u) \coloneqq \bigcup_i \mathrm{WF}(\tilde{u}_i) \text{ s.t. } \mathcal{D}'(\mathcal{M}, V) \ni \tilde{u}_i \colon g \mapsto \tilde{u}_i(g) \coloneqq u(gf_i) \tag{A.28}$$

As one might expect it can be proven that $(y, l) \in \mathrm{WF}(u) \implies y \in \mathrm{singsupp}(u)$ (see e.g. [Hör90] and [SV01] for a translation from distributions on $\mathbb{R}^n$ to distributions on vector bundles). This means that the wavefront set adds directional information to the singular support of a distribution which, by itself, provides only positional information about the location of its singularities.

B Category theory and quantization functors

Section 1.2 outlines a general framework in which category theory plays a central role in the quantization of classical field theories by assigning an algebra of observables to define the quantized theory. The main concepts used in this work are the defining properties of category and functor, which are reported in this appendix. A general introduction to the theory of categories can be found in [Lei25]. The tools described here are then used to define the so-called quantization functors. Details of their construction and the proof of their functorial nature can be found in [FV12].

Definition B.1 (*Category*)**.** A *category* is a pair $(\mathrm{ob}(\mathbf{A}), \mathrm{Hom}(\mathbf{A}))$ such that:

(i) $\mathrm{ob}(\mathbf{A})$ is a collection of *objects*.

(ii) $\forall A, B \in \mathrm{ob}(\mathbf{A})$ there is a collection $\mathbf{A}(A, B) \subset \mathrm{Hom}(\mathbf{A})$ of *morphisms* from A to B.

(iii) $\forall A, B, C \in \mathrm{ob}(\mathbf{A})$ there is a map called *composition*:

$$\begin{aligned} \circ \colon \mathbf{A}(A, B) \times \mathbf{A}(B, C) &\to \mathbf{A}(A, C) \\ (f, g) &\mapsto g \circ f \end{aligned} \tag{B.1}$$

(iv) $\forall A \in \mathrm{ob}(\mathbf{A})$ there is a morphism $\mathbb{1}_{\mathbf{A}(A,A)}$ called *identity on* A.

(v) $\forall A, B, C, D \in \mathrm{ob}(\mathbf{A})$ and $f \in \mathbf{A}(A, B), g \in \mathbf{A}(B, C), h \in \mathbf{A}(C, D)$ one has $(h \circ g) \circ f = h \circ (g \circ f)$ and $f \circ \mathbb{1}_A = \mathbb{1}_B \circ f$.

Remark. In a similar fashion to the vector bundle case, it is customary to refer to a category $(\mathrm{ob}(\mathbf{A}), \mathrm{Hom}(\mathbf{A}))$ just as $\mathbf{A}$ so that $A \in \mathbf{A}$ means $A \in \mathrm{ob}(\mathbf{A})$ and $f\colon A \to B$ means $f \in \mathbf{A}(A, B)$.

Definition B.2 (*Subcategory*)**.** Let $\mathbf{A}$ be a category. A *subcategory* of $\mathbf{A}$ is a pair $(\mathrm{ob}(\mathbf{S}), \mathrm{Hom}(\mathbf{S}))$ where $\mathrm{ob}(\mathbf{S})$ is a subclass of $\mathrm{ob}(\mathbf{A})$ and $\mathrm{Hom}(\mathbf{S})$ is a subclass of $\mathrm{Hom}(\mathbf{A})$ that is closed under composition and identities.

In a sense, a category generalizes the concept of sets and functions between them. In particular, a category could be the family of sets with some structure on them and functions which preserve that particular structure.

Example B.1 (**Top** *category*)**.** The pair of topological spaces (X, τ) as objects and continuous maps $f\colon X \to Y$ as morphisms is a category called **Top**, with the composition map being the standard composition of functions and the identity morphisms $\mathbb{1}_X$ being the standard identity functions on the topological spaces (X, τ).

Following this intuition, it is clear that it is possible to further characterize the morphisms of a category:

Definition B.3 (*Isomorphism*)**.** A morphism $f\colon A \to B$ of a category $\mathbf{A}$ is called *isomorphism* if there exists a morphism $g\colon B \to A$ such that $g \circ f = \mathbb{1}_A$ and $f \circ g = \mathbb{1}_B$. The morphism g is called the *inverse of* f and it is usually denoted as $g = f^{-1}$. The objects A and B are called *isomorphic* and this is usually denoted as $A \cong B$.

Example B.2 (**Top** *isomorphisms*)**.** Let **Top** be the category of topological spaces. The isomorphisms of **Top** are the homeomorphisms between topological spaces.

The notion of map can be applied to categories themselves.

Definition B.4 (*Functor*)**.** Let $\mathbf{A}$ and $\mathbf{B}$ be categories. A *Functor* $\mathscr{F}\colon \mathbf{A} \to \mathbf{B}$ is a pair of maps, both denoted by $\mathscr{F}$:

(i) $\mathscr{F}\colon\ \mathrm{ob}(\mathbf{A}) \to \mathrm{ob}(\mathbf{B})$

(ii) $\mathscr{F}\colon\ \mathrm{Hom}(\mathbf{A}) \supset \mathbf{A}(A, A') \to \mathbf{B}(\mathscr{F}(A), \mathscr{F}(A')) \subset \mathrm{Hom}(\mathbf{B}) \quad \forall A, A' \in \mathbf{A}$

They satisfy the properties:

(i) $\mathscr{F}(f' \circ f) = \mathscr{F}(f') \circ \mathscr{F}(f) \quad \forall f \in \mathbf{A}(A, A'), f' \in \mathbf{A}(A', A'')$ and $A, A', A'' \in \mathbf{A}$.

(ii) $\mathscr{F}(\mathbb{1}_A) = \mathbb{1}_{\mathscr{F}(A)} \quad \forall A \in \mathbf{A}$.

Example B.3 (**Cat** *category*)**.** The pair of categories $\mathbf{A}$ as objects and functors $\mathscr{F}\colon \mathbf{A} \to \mathbf{B}$ as morphisms is a category called **Cat** with the composition map being the composition of $\mathscr{F}$ and $\mathscr{F}'$ on objects and morphisms of $\mathbf{A}$ and $\mathbf{B}$. This category gives a natural notion of composition of functors.

It is also useful to define maps between functor themselves. If one considers a pair of functors between the same categories, one can introduce the notion of natural transformation as a map between the images of the functors.

Definition B.5 (*Natural transformation*)**.** Let $\mathscr{F}, \mathscr{G}\colon \mathbf{A} \to \mathbf{B}$ be a pair of functors between the same categories. A *natural transformation* between the functors $\mathscr{F}$ and $\mathscr{G}$ is a family of $\mathbf{B}$ morphisms $\{\beta_A\colon \mathscr{F}(A) \to \mathscr{G}(A)\}_{A\in\mathbf{A}} \subset \mathrm{Hom}(\mathbf{B})$ such that for every $\mathbf{A}$ morphism $\mathrm{Hom}(\mathbf{A}) \ni f\colon A \to A'$ the diagram commutes:

$$\begin{tikzcd} \mathscr{F}(A) \arrow[r, "\mathscr{F}(f)"] \arrow[d, "\beta_A"'] & \mathscr{F}(A') \arrow[d, "\beta_{A'}"] \\ \mathscr{G}(A) \arrow[r, "\mathscr{G}(f)"'] & \mathscr{G}(A') \end{tikzcd} \tag{B.2}$$

The morphisms β_A are called the *components* of the natural transformation $\beta\colon \mathscr{F} \to \mathscr{G}$.

The categories $\mathbf{Sympl}_{\mathbb{K}}$, **Alg** and $\mathbf{C^*}$**-Alg** introduced in Section 1.3 can be linked by so-called quantization functors. The motivation for this name can be understood following the discussion in Section 1.1. Classical physical systems are modelled by appropriate symplectic spaces, quantum physical systems are modelled instead by appropriate observable algebras. In this sense this functor associate to a classical system its corresponding quantum version.

Definition B.6 (*Quantization functor*)**.** Let $(V, \sigma, \mathcal{C}), (V', \sigma', \mathcal{C}') \in \mathbf{Sympl}_{\mathbb{C}}$. Let also $f\colon (V, \sigma, \mathcal{C}) \mapsto (V', \sigma', \mathcal{C}')$. The *quantization functor* $\mathscr{Q}\colon \mathbf{Sympl}_{\mathbb{C}} \to \mathbf{Alg}$ is defined as follows:

(i) $\mathscr{Q}(V, \sigma, \mathcal{C}) = \mathcal{A} \in \mathbf{Alg}$ is the algebra whose vector space is $\mathscr{Q}(V) := \bigoplus_{n\in\mathbb{N}_0} V^{\odot n}$ where $V^{\odot n}$ is the symmetrized n-th tensor product of the vector space V with itself. The algebra product is defined $\forall u^{\odot m} \in V^{\odot m}$ and $v^{\odot n} \in V^{\odot n}$ as:

$$u^{\odot m} \cdot v^{\odot n} := \sum_{r=0}^{\min\{m,n\}} \left(\frac{i\sigma(u,v)}{2}\right)^r \frac{m!n!}{r!(m-r)!(n-r)!} S\left(u^{\otimes(m-r)} \otimes v^{\otimes(n-r)}\right) \tag{B.3}$$

(ii) $\mathscr{Q}(f) \in \mathrm{Hom}(\mathbf{Alg})$ is the **Alg** morphism defined as:

$$\mathscr{Q}(f) := \bigoplus_{n\in\mathbb{N}_0} f^{\odot n} \tag{B.4}$$

where $f^{\odot n}(v_1 \odot \cdots \odot v_n) := f(v_1) \odot \cdots \odot f(v_n)$.

Proposition B.1. *The map $\mathscr{Q}\colon \mathbf{Sympl}_{\mathbb{C}} \to \mathbf{Alg}$ defined in Definition B.6 is a well defined functor.*

Proof. See e.g. [FV12]. □

Definition B.7 (*Weyl relation*)**.** Let $(V, \sigma, \mathcal{C}) \in \mathbf{Sympl}_{\mathbb{C}}$ and $u, v \in V$. Define:

$$W(\lambda u) := \bigoplus_{n \in \mathbb{N}_0} \frac{(i\lambda)^n}{n!} u^{\odot n} \tag{B.5}$$

The algebra product of $\mathscr{Q}(V, \sigma, \mathcal{C}) \in \mathbf{Alg}$ can be written as:

$$W(\lambda u) W(\mu v) = e^{-\frac{i}{2}\sigma(\lambda u, \mu v)} W(\lambda u + \mu v) \tag{B.6}$$

understood as an equivalence between formal power series in λ and μ. Eq. (B.6) is known as *Weyl relation*.

The definition of the Weyl relation suggests another way to assign an algebra to a symplectic space.

Definition B.8 (*Weyl quantization functor*)**.** Let $(V, \sigma, \mathcal{C}), (V', \sigma', \mathcal{C}') \in \mathbf{Sympl}_{\mathbb{R}}$. Let also $f\colon (V, \sigma, \mathcal{C}) \mapsto (V', \sigma', \mathcal{C}')$. The *Weyl quantization functor* $\mathscr{W}\colon \mathbf{Sympl}_{\mathbb{R}} \to \mathbf{C^*\text{-}Alg}$ is defined as follows:

(i) $\mathscr{W}(V, \sigma, \mathcal{C}) \in \mathbf{C^*\text{-}Alg}$ is the unique completion to a C^*-algebra of the algebra generated by operators $\{W(u) \mid u \in V\}$ obeying $W(0_V) = \mathbb{1}_{\mathscr{W}(V,\sigma,\mathcal{C})}, W(u)^* = W(-u)$ and the Weyl relation Eq. (B.6).

(ii) $\mathscr{W}(f) = \alpha_f \in \mathrm{Hom}(\mathbf{C^*\text{-}Alg})$ is the $\mathbf{C^*\text{-}Alg}$ morphism defined as $\alpha_f(W(u)) = W(f(u))$.

Proposition B.2. *The map* $\mathscr{W}\colon \mathbf{Sympl}_{\mathbb{R}} \to \mathbf{C^*\text{-}Alg}$ *defined in Definition B.8 is a well defined functor which can also be extended to* $\mathbf{Sympl}_{\mathbb{C}}$ *via the composition with the restriction functor* $\mathscr{R}\colon \mathbf{Sympl}_{\mathbb{C}} \to \mathbf{Sympl}_{\mathbb{R}}$.

Proof. The functorial properties of the map $\mathscr{W}$ are immediately verified. It is a standard result that the algebra generated by the collection of $W(u)$ can be given a unique C^* norm so that the completion of $\mathscr{W}(V)$ with respect to the C^* induced metric is a C^*-algebra. For further details of this construction see e.g. [Mor17; FV12]. □

References

[Arc25] N. Arcozzi. *Functional Analysis and Operator Theory*. 2025. arXiv: 2505.05526 [math.FA].

[BGP08] C. Baer, N. Ginoux, and F. Pfaeffle. *Wave Equations on Lorentzian Manifolds and Quantization*. 2008. arXiv: 0806.1036 [math.DG].

[Bär15] C. Bär. "Green-Hyperbolic Operators on Globally Hyperbolic Spacetimes". *Communications in Mathematical Physics* 333 (2015), pp. 1585–1615. DOI: 10.1007/s00220-014-2097-7.

[BFR21] H. Bostelmann, C. J. Fewster, and M. H. Ruep. "Impossible measurements require impossible apparatus". *Phys. Rev. D* 103 (2021), p. 025017. DOI: 10.1103/PhysRevD.103.025017.

[Bou98] N. Bourbaki. *Algebra I: Chapters 1-3*. Springer-Verlag, 1998. ISBN: 978-3-540-64243-5.

[BFV03] R. Brunetti, K. Fredenhagen, and R. Verch. "The generally covariant locality principle: a new paradigm for local quantum physics". *Commun. Math. Phys.* 237 (2003), pp. 31–68. DOI: https://doi.org/10.1007/s00220-003-0815-7.

[Bru+15] R. Brunetti et al., eds. *Advances in Algebraic Quantum Field Theory*. Springer, 2015. DOI: 10.1007/978-3-319-21353-8.

[Bus+16] P. Busch et al. *Quantum Measurement*. Springer International Publishing, 2016. DOI: 10.1007/978-3-319-43389-9.

[Chr76] S. M. Christensen. "Vacuum expectation value of the stress tensor in an arbitrary curved background: The covariant point-separation method". *Phys. Rev. D* 14 (1976), pp. 2490–2501. DOI: 10.1103/PhysRevD.14.2490.

[Chr78] S. M. Christensen. "Regularization, renormalization, and covariant geodesic point separation". *Phys. Rev. D* 17 (1978), pp. 946–963. DOI: 10.1103/PhysRevD.17.946.

[DG13] J. Dereziński and C. Gérard. *Mathematics of Quantization and Quantum Fields*. Cambridge University Press, 2013. DOI: https://doi.org/10.1017/9781009290876.

[DB60] B. S. DeWitt and R. W. Brehme. "Radiation damping in a gravitational field". *Annals of Physics* 9 (1960), pp. 220–259. DOI: https://doi.org/10.1016/0003-4916(60)90030-0.

[DU77] J. Diestel and J. J. Uhl. *Vector Measures*. American Mathematical Society, 1977. URL: https://books.google.co.uk/books?id=fEKZAwAAQBAJ.

[Few12] C. J. Fewster. *Lectures on quantum energy inequalities*. 2012. arXiv: 1208.5399 [gr-qc].

[Few25] C. J. Fewster. *Hadamard states for decomposable Green-hyperbolic operators*. 2025. arXiv: 2503.12537 [math-ph].

[FJR22] C. J. Fewster, I. Jubb, and M. H. Ruep. "Asymptotic Measurement Schemes for Every Observable of a Quantum Field Theory". *Annales Henri Poincaré* 24 (2022), pp. 1137–1184. DOI: 10.1007/s00023-022-01239-0.

[FR19] C. J. Fewster and K. Rejzner. *Algebraic Quantum Field Theory – an introduction*. 2019. arXiv: 1904.04051 [hep-th].

[FV03] C. J. Fewster and R. Verch. “Stability of Quantum Systems at Three Scales: Passivity, Quantum Weak Energy Inequalities and the Microlocal Spectrum Condition”. *Communications in Mathematical Physics* 240 (2003), pp. 329–375. DOI: 10.1007/s00220-003-0884-7.

[FV12] C. J. Fewster and R. Verch. “Dynamical Locality of the Free Scalar Field”. *Annales Henri Poincaré* 13 (2012), pp. 1675–1709. DOI: 10.1007/s00023-012-0166-z.

[FV15] C. J. Fewster and R. Verch. *Algebraic quantum field theory in curved spacetimes*. 2015. arXiv: 1504.00586 [math-ph].

[FV20] C. J. Fewster and R. Verch. “Quantum Fields and Local Measurements”. *Communications in Mathematical Physics* 378 (2020), pp. 851–889. DOI: 10.1007/s00220-020-03800-6.

[FV24] C. J. Fewster and R. Verch. *Measurement in Quantum Field Theory*. 2024. arXiv: 2304.13356 [math-ph].

[FR16] K. Fredenhagen and K. Rejzner. “Quantum field theory on curved spacetimes: Axiomatic framework and examples”. *Journal of Mathematical Physics* 57 (2016), p. 031101. eprint: https://pubs.aip.org/aip/jmp/article-pdf/doi/10.1063/1.4939955/14803718/031101_1_online.pdf.

[FSW78] S. A. Fulling, M. Sweeny, and R. M. Wald. “Singularity structure of the two-point function in quantum field theory in curved spacetime”. *Communications in Mathematical Physics* 63 (1978), pp. 257–264. DOI: 10.1007/BF01196934.

[GN43] I. Gelfand and M. Neumark. “On the imbedding of normed rings into the ring of operators in Hilbert space”. *Rec. Math. [Mat. Sbornik] N.S.* 12(54) (1943), pp. 197–217. URL: http://mi.mathnet.ru/eng/sm6155.

[Haa96] R. Haag. *Local Quantum Physics*. Springer Berlin Heidelberg, 1996. DOI: 10.1007/978-3-642-61458-3.

[HK64] R. Haag and D. Kastler. “An Algebraic Approach to Quantum Field Theory”. *Journal of Mathematical Physics* 5 (1964), pp. 848–861. DOI: 10.1063/1.1704187.

[Had23] J. Hadamard. *Lectures on Cauchy’s problem in linear partial differential equations*. Yale University Press, 1923.

[Haw75] S. W. Hawking. “Particle creation by black holes”. *Communications in Mathematical Physics* 43 (1975), pp. 199–220. DOI: 10.1007/BF02345020.

[Hol01] S. Hollands. “The Hadamard Condition for Dirac Fields and Adiabatic States on Robertson–Walker Spacetimes”. *Communications in Mathematical Physics* 216 (2001), pp. 635–661. DOI: 10.1007/s002200000350.

[HW01] S. Hollands and R. M. Wald. “Local Wick Polynomials and Time Ordered Products of Quantum Fields in Curved Spacetime”. *Communications in Mathematical Physics* 223 (2001), pp. 289–326. DOI: 10.1007/s002200100540.

[HR15] R. Honegger and A. Rieckers. *Photons in Fock Space and Beyond*. World Scientific Publishing Company, 2015. DOI: 10.1142/9251.

[Hör90] L. Hörmander. *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*. 2nd. Springer-Verlag, 1990. ISBN: 978-3-540-00662-6.

[KW91] B. S. Kay and R. M. Wald. "Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on spacetimes with a bifurcate killing horizon". *Physics Reports* 207 (1991), pp. 49–136. DOI: `https://doi.org/10.1016/0370-1573(91)90015-E`.

[Koe95] M. Koehler. *The stress energy tensor of a locally supersymmetric quantum field on a curved spacetime*. 1995. arXiv: `gr-qc/9505014 [gr-qc]`.

[Kra00] K. Kratzert. "Singularity structure of the two point function of the free Dirac field on a globally hyperbolic spacetime". *Annalen der Physik* 512 (2000), pp. 475–498. DOI: `https://doi.org/10.1002/andp.20005120605`.

[Lee03] J. M. Lee. *Introduction to Smooth Manifolds*. Vol. 218. Springer-Verlag New York, 2003. DOI: `10.1007/978-0-387-21752-9`.

[Lei25] T. Leinster. *Basic Category Theory*. 2025. arXiv: `1612.09375 [math.CT]`.

[MN25] J. Mandrysch and M. Navascués. "Quantum field measurements in the Fewster–Verch framework". *Letters in Mathematical Physics* 115 (2025), p. 115. DOI: `10.1007/s11005-025-02001-3`.

[Mor17] V. Moretti. *Spectral Theory and Quantum Mechanics*. 2nd ed. Springer Nature, 2017. DOI: `10.1007/978-3-319-70706-8`.

[Mor21] V. Moretti. "On the global Hadamard parametrix in QFT and the signed squared geodesic distance defined in domains larger than convex normal neighbourhoods". *Letters in Mathematical Physics* 111 (2021), p. 130. DOI: `10.1007/s11005-021-01464-4`.

[Neu32] J. von Neumann. *Mathematische Grundlagen der Quantenmechanik*. Springer, 1932. URL: `http://eudml.org/doc/203794`.

[NC10] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information: 10th Anniversary Edition*. Cambridge University Press, 2010. DOI: `https://doi.org/10.1017/CBO9780511976667`.

[Rad96] M. J. Radzikowski. "Micro-local approach to the Hadamard condition in quantum field theory on curved space-time". *Communications in Mathematical Physics* 179 (1996), pp. 529–553. DOI: `10.1007/BF02100096`.

[RS81] M. Reed and B. Simon. *I: Functional Analysis*. Academic Press, 1981. URL: `https://books.google.co.uk/books?id=rpFTTjxOYpsC`.

[Rej16] K. Rejzner. *Perturbative algebraic quantum field theory*. en. 1st ed. Springer International Publishing, 2016. DOI: `https://doi.org/10.1007/978-3-319-25901-7`.

[Rud76] W. Rudin. *Principles of Mathematical Analysis*. McGraw-Hill, 1976. URL: `https://books.google.co.uk/books?id=kwqzPAAACAAJ`.

[SV01] H. Sahlmann and R. Verch. "Microlocal spectrum condition and Hadamard form for vector-valued quantum fields in curved spacetime". *Reviews in Mathematical Physics* 13 (2001), pp. 1203–1246. DOI: `https://doi.org/10.1142/S0129055X01001010`.

[San10] K. Sanders. "The locally covariant Dirac field". *Reviews in Mathematical Physics* 22 (2010), pp. 381–430. DOI: 10.1142/S0129055X10003990.

[Seg47] I. E. Segal. "Postulates for General Quantum Mechanics". *Annals of Mathematics* 48 (1947), p. 930. URL: https://api.semanticscholar.org/CorpusID:123583952.

[Sor93] R. D. Sorkin. *Impossible Measurements on Quantum Fields.* 1993. arXiv: gr-qc/9302018 [gr-qc].

[SVW02] A. Strohmaier, R. Verch, and M. Wollenberg. "Microlocal analysis of quantum fields on curved space–times: Analytic wave front sets and Reeh–Schlieder theorems". *Journal of Mathematical Physics* 43 (2002), pp. 5514–5530. DOI: 10.1063/1.1506381.

[Tu11] L. W. Tu. *An Introduction to Manifolds.* Springer New York, NY, 2011. DOI: https://doi.org/10.1007/978-1-4419-7400-6.

[Tu17] L. W. Tu. *Differential Geometry.* Springer Cham, 2017. DOI: https://doi.org/10.1007/978-3-319-55084-8.

[Unr76] W. G. Unruh. "Notes on black-hole evaporation". *Phys. Rev. D* 14 (1976), pp. 870–892. DOI: 10.1103/PhysRevD.14.870.

[Wal77] R. M. Wald. "The back reaction effect in particle creation in curved spacetime". *Communications in Mathematical Physics* 54 (1977), pp. 1–19. DOI: 10.1007/BF01609833.

[Wal78] R. M. Wald. "Trace anomaly of a conformally invariant quantum field in curved spacetime". *Phys. Rev. D* 17 (1978), pp. 1477–1484. DOI: 10.1103/PhysRevD.17.1477.

[Wal84] R. M. Wald. *General Relativity.* Chicago Univ. Pr., 1984. DOI: 10.7208/chicago/9780226870373.001.0001.