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# Small-scale structure of cosmic superstrings with time-varying tension

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**Abstract**

This thesis project investigates the small-scale structure of cosmic superstrings with time variable tension. Building on the framework of Polchinski and Rocha for constant tension strings, we extend this by allowing the tension to vary as $\mu(t) \sim t^{-m}$. The correlator constructed from the scalar product of left- or right-moving unit vectors at different points along the string obeys the same evolution equation as in the original framework, but with a modified scaling exponent $\chi'$. Crucially, this depends on $\kappa = 1 - mn/4$, related to the modified Hubble parameter entering the equations of motion, and on $\bar{\alpha} = 1 - 2\bar{v}^2$. Interestingly, $\chi'$ remains positive both when the variation of the tension dominates over the expansion of the Universe ($mn > 4$) and viceversa ($mn < 4$). This is due to the squared velocity $\bar{v}^2$ interpolating between values greater or smaller than $1/2$, thereby affecting the sign of $\bar{\alpha}$. Therefore, within the generalized framework considered, the time variation of the tension does not alter the smooth small-scale structure of the strings. We further investigate corrections to the squared velocity of the network and circular loops as proxies for subhorizon physics, to understand the effect of the varying tension at different scales. Notably, the Velocity-One-Scale model gives a correction to the network velocity proportional to $mn - 4$. Conversely, the formal and perturbative loop analyses show the corrections are extremely suppressed by the subhorizon nature of the system, appearing at order $\mathcal{O}((\mathcal{H}L)^2)$. Furthermore, the deviation of the correlations between left- and right-moving unit vectors from the original framework is shown to be suppressed by $\mathcal{O}(\ell/t)$ beyond leading order. Overall, the time-dependent tension affects the network velocity and loop amplitudes, but the leading smooth small-scale structure predicted by the Polchinski-Rocha framework remains valid.

# Contents

# Introduction

The electromagnetic, weak, and strong interactions are well described by the Standard Model (SM) of particle physics. Through the framework of quantum field theory (QFT), these interactions are encoded in the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ and several remarkable predictions have been made so far. Notably, the most famous one is the discovery of the Higgs boson in 2012 [1], the scalar particle associated with excitations of the Higgs field responsible for generating particles' masses through spontaneous symmetry breaking of gauge symmetries. This mechanism has a close analogue to condensed matter physics, specifically in the theory of superconductivity [2]. Indeed, analogies between particle physics and condensed matter physics appear in a large number of areas. This is mainly related to the properties of QFT at non zero temperatures. In this case, phase transitions can occur in a similar way as condensed matter systems undergo phase transitions and one is able to relate field theory quantities, such as the generating functional and the effective action, to condensed matter quantities, for instance the partition function and Gibbs free energy [3].

Proceeding along this direction, one is naturally led to the notion of topological defects in the condensed matter literature. These are associated with symmetry breaking mechanisms leading to non-trivial degenerate vacuum states. They have been directly observed in certain materials: one-dimensional defects, or *lines*, appear in Type II superconductors as magnetic flux lines or quantized vortex-lines in superfluid Helium-4 [4]. Another common example is the presence of domain structures in ferromagnetic materials. Here the minimum energy configuration is achieved by domain walls of different magnetization. It is then natural to wonder whether such defects have a counterpart in field theory models and the answer turns out to be positive. Topological defects can equivalently arise in field theory during symmetry breaking phase transitions, where the vacuum states are stochastically distributed in different regions of spacetime. A notable example is that of monopoles, associated to a symmetry breaking process which terminates with a residual $U(1)$ symmetry. These are expected to be produced from any Grand Unification Theory (GUT) which recovers the electromagnetic component, $U_{em}(1)$, of the SM. Furthermore, these objects carry a non-zero amount of energy, and therefore have several implications both at the phenomenological and cosmological level. Indeed, the existence of monopoles from any GUT leads to an overclosed universe, where the energy density of monopoles exceeds the other forms of energy in the universe. This is another one of the problems solved by the theoretical inflationary epoch in the beginning of the universe [5], together with the usually mentioned problems of horizon and flatness.

Another interesting class of topological defect that can arise during symmetry breaking phase transitions is that of *cosmic strings*. These are one-dimensional defects produced due to the vacuum manifold being not simply connected and they are characterized by a tension $\mu$ which is proportional to the symmetry breaking parameter $\eta$ of the underlying field theory, $\mu \sim \eta^2$. Out of the various defects, cosmic strings are topologically stable and the study of their dynamical evolution in terms of cosmic string network reveals the achievement of a scaling regime, where the string energy density tracks that of the universe [4]. Moreover, they have a distinguished gravitational field that could lead to specific lensing effects which may be observed. Interestingly, these objects were among the promising candidates for the seed of structure formation. However, in light of the Cosmic Microwave Background (CMB) experiments, their dimensionless tension $G_N\mu$ with $G_N$ Newton constant, is far too constrained, $G_N\mu \lesssim 10^{-7}$ [6].

Despite the great success of the SM, it lacks the description of the gravitational interaction. Although General Relativity can be treated as an effective quantum field theory at energies below the Planck scale, its quantization is non-renormalizable and therefore does not provide a predictive ultraviolet complete theory of quantum gravity. A possible unified description of the SM and gravity is provided by Superstring Theory. This is an entirely different framework to that of QFT since it postulates the fundamental objects of our universe are one-dimensional, the strings, instead of zero dimensional point particles. Among the many controversial results in string theory, the most mind-bending notion is that for the quantum theory to be consistent, it is required to live in a $D = 10$ dimensional spacetime, whereas our observed universe is four-dimensional. This makes it necessary to develop mechanisms by which one can extract $4D$ physical quantities from the higher dimensional theory. This is the concept of *string compactification*, where the six extra dimensions are compactified to reduce the $10D$ theory to a $4D$ effective one. Imposing certain conditions on the $4D$ theory, namely on the amount of supersymmetries, selects a specific geometry for the extra-dimensions which is usually taken to be a *Calabi-Yau* manifold. For instance, compactification of Type II string theory on a Calabi-Yau threefold leads to an effective four-dimensional theory with $\mathcal{N} = 2$ supersymmetry, while further ingredients such as orientifold projections and fluxes are typically introduced in phenomenological models to obtain $\mathcal{N} = 1$. The main consequence of a compactification is the dependence of $4D$ quantities on geometrical data of the extra dimensional space. This is often summarized by saying that string theory has no free dimensionless parameters. The $4D$ couplings, masses, mixing angles and other values are entirely derived from the compactification data. Namely, the biggest obstacle in practically computing these values is the presence of a large number of scalar fields arising from the compactification which are massless at tree level, the so-called *moduli fields*. This leads to the problem of *moduli stabilization*, the necessity of devising a mechanism for which the moduli fields develop a non-trivial potential and therefore gain a mass term.

Potential observational evidence of superstring theory could come from cosmic superstrings. These are in some aspects analogous to string-like topological defects arising in field theory models, albeit their tension does not depend on the symmetry breaking parameter, rather on the compactification data as well as the moduli fields. Cosmic superstrings have been investigated since [7], where it was argued that stretching over cosmological distances leads to a dimensionless tension larger than the observational constraints. However, development in string compactification made it possible to devise models in which the tension was systematically lowered, mainly due to warping effects and large volume of the extra dimensions. This opened up a large class of scenarios featuring the production of cosmic superstrings with interesting phenomenological properties. As mentioned before, the peculiar characteristic of cosmic superstrings is the dependence of the tension on moduli fields, and, since these evolve dynamically, it inherits a time evolution. This constitutes the motivation for this thesis project where cosmic strings with time dependent tension are studied. In particular, the small-scale structure of these objects, that is on scales below the cosmological horizon, represents an interesting window for observation due to the difference with respect to ordinary field theory strings. This is because the small structure can retain an imprint of the microscopic and cosmological evolution of the network. Therefore, if cosmic superstrings with time-dependent tension generate correlations different from those of ordinary field-theory strings, these could in principle lead to distinct observational signatures.

The study of the small scale structure of cosmic string networks with constant tension was quantitatively presented in [8]. In this article, the authors introduced a framework based on the evolution of the scalar product of left- and right-moving unit vectors evaluated at different points along the string. By studying the evolution in time of the correlation between these vectors, one is able to characterize the network of cosmic strings through the *fractal dimension*. This quantity was shown to approach unity in the small scale limit, namely when the ratio between the string segment length $\ell$ and the cosmological horizon size $d_H \sim t$ approaches zero. This signals the strings become smooth at this scale. In this thesis, I will generalize this model to the case of a time varying tension and elaborate on the implications. From a physical point of view, allowing the tension to vary modifies the damping terms in the microscopic equations of motion. Therefore, one might expect this to affect the balance between cosmological stretching and the development of small-scale irregularities. In particular, if the variation of the tension dominates against the expansion, small-scale irregularities could survive or grow more efficiently than in the constant-tension case, which would be represented by a breakdown of the original constant tension framework. A central goal of this thesis is to investigate what happens when the tension is allowed to vary in time, whether the smoothing predicted by the original framework is preserved or not.

The thesis is outlined as follows: in the first chapter, I will review the origin of topological defects in field theory, later focusing on the particular case of cosmic strings and exploring their dynamical evolution. The classification based on homotopy theory will be explained and the model used to capture the dynamics of string networks will be described. In the second chapter, I will present an overview of superstring theory, describing the problem of moduli stabilization, and outlining one of the most studied scenarios to solve this. This will motivate the study of exponential potentials in cosmology, which will be the topic of the third chapter. There, I will discuss solutions for cosmological backgrounds with exponential potential, and I will explore non-standard evolutionary periods such as kination and tracker epochs. All of this will serve as motivation for the time dependence of cosmic string tension. The fourth chapter will then be dedicated to the small-scale structure of these topological defects. I will first review the framework used with constant tension and generalize it to the case of time-varying tension. This will motivate the study of the squared velocity of both long string networks and circular loops, to understand the effect of the time varying tension on different scales. Finally, I will examine the correlation between left- and right-moving unit vectors as a consistency check of the generalized framework.

# Chapter 1

# Topological defects and cosmic strings

In this chapter, I will present the mechanisms underlying the formation of topological defects with an emphasis on cosmic strings. This requires the discussion of topological defects in field theory and cosmological phase transitions, as well as finite-temperature field theories and spontaneous symmetry breaking. I will also describe the classification of such defects through the use of homotopy theory. In addition, I will discuss the microscopic evolution of cosmic strings, as well as the formation and macroscopic evolution of string networks via the Velocity-One-Scale model. Finally, I will briefly explore the connection between field theory strings and cosmic superstrings arising from superstring theories.

## 1.1 Topological defects in field theory

Topological defects are stable field configurations associated with spontaneous symmetry breaking (SSB) of a given field theory whose set of vacua states is not trivial [4]. Vacuum states are energetically equivalent and there may be regions of spacetime where the field settles in one particular vacuum configuration different from that of another region of spacetime. Continuity in the field requires it to leave the manifold of vacuum states, the so-called *vacuum manifold*, thereby creating an energetic field configuration which is typically denoted as a *topological defect*. As the name suggests, the nature of defects is profoundly connected to the topology of the vacuum manifold. This will be made more clear by the use of homotopy theory, as I shall discuss later. For now, let us begin by reviewing some examples of SSB.

Consider the following Lagrangian for a real scalar field $\phi$ in two spacetime dimensions:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi), \qquad V(\phi) = \frac{\lambda}{4}(\phi^2 - \eta^2)^2 \tag{1.1.1}$$

which enjoys a discrete $\mathbb{Z}_2$ symmetry represented by the transformation $\phi \to -\phi$ that leaves the Lagrangian invariant. The minima of the potential do not possess this same symmetry. Indeed, these are found by:

$$\frac{\partial V}{\partial \phi} = 0 \implies \phi = \pm\eta \tag{1.1.2}$$

while the solution $\phi = 0$ corresponds to a maximum. If the field $\phi(x)$ were to approach $+\eta$ as $x \to \infty$ and $\phi(x) \to -\eta$ as $x \to -\infty$, then the continuity of the field $\phi(x)$ demands it to pass through the value $\phi = 0$. However, this does not correspond to either minimum of the potential and therefore constitutes an energetic field configuration since $V(\phi) \neq 0$ when $\phi = 0$. These field configurations are actually the limits as $x \to \pm\infty$ of an analytic solution to the field equations resulting from the above Lagrangian, typically known as the $\phi^4$-kink. To recover it, instead of solving the second order partial differential equation, a clever method relies on using these as boundary conditions of the energy functional, which for this two dimensional theory is [9]:

$$\begin{aligned}
E &= \int dx \left[\frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\partial_x\phi)^2 + V(\phi)\right] \\
&= \int dx \left[\frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}\left(\partial_x\phi - \sqrt{2V(\phi)}\right)^2 + \partial_x\phi\,\sqrt{2V(\phi)}\right] \\
&= \int dx \left[\frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}\left(\partial_x\phi - \sqrt{2V(\phi)}\right)^2\right] + \int_{\phi(-\infty)}^{\phi(+\infty)} d\phi'\,\sqrt{2V(\phi')}.
\end{aligned}$$

This is minimized for:

$$\partial_t\phi = 0, \qquad\qquad \partial_x\phi - \sqrt{2V(\phi)} = 0. \tag{1.1.3}$$

The first equation implies the solution is time independent while the second one is a simple first order partial differential equation that can be solved using the potential of (1.1.1) giving:

$$\partial_x\phi = \sqrt{\frac{\lambda}{2}}(\eta^2 - \phi^2) \implies \phi(x) = \eta\tanh\left(\eta\sqrt{\frac{\lambda}{2}}\,x\right). \tag{1.1.4}$$

Indeed, this is a time-independent and non dissipative solution of finite energy. The stability of such solution can be understood in terms of topological conservation laws, that are related to the particular topology of the vacuum manifold. In this case, such manifold is disconnected, hence the formation of the defect. There exists a current $j^\mu = \epsilon^{\mu\nu}\partial_\nu\phi$, which is conserved as $\partial_\mu j^\mu = 0$ by antisymmetry of the tensor $\epsilon^{\mu\nu}$. Using Noether's theorem, it follows the existence of the conserved charge:

$$Q = \int dx j^0 = \int dx\,\partial_x\phi = \phi\big|_{x=\infty} - \phi\big|_{x=-\infty} \tag{1.1.5}$$

which is zero in the symmetric state and non-zero in the presence of the $\phi^4-$kink with $\phi$ in different vacua at $x = \pm\infty$. Since this is a conserved quantity, it follows that the kink configuration cannot continuously relax into the vacuum, since it belongs to a different topological sector. If the model were 4$-$dimensional, then the kink solution could extend for every $y$ and $z$, leading to the notion of *domain walls*.

Domain walls are typically associated with the breaking of a discrete symmetry, where the vacuum manifold $\mathcal{M}$ involves disconnected components [4]. Physically, domain walls appear at the boundaries between regions of spacetime where the field $\phi$ lies in different vacuum states. When the curvature radius of the wall is greater than its thickness, the solution is precisely that of the $\phi^4-$kink and its width is approximately given by $\delta \sim 1/(\sqrt{\lambda}\eta)$ with a vacuum energy of $\rho \sim \lambda\eta^4$. Thus, its surface energy density will be $\zeta \sim \rho\delta \sim \sqrt{\lambda}\eta^3$ which, unless the symmetry breaking scale $\eta$ is very small, it is a very large quantity and can drastically impact the homogeneity of the universe. It is worth noting that the origin of these defects has been pointed out by using the field $\phi$ to map points at spatial infinity in physical space to non-trivial points in the vacuum manifold, $\phi = \pm\eta$ in this case. This mapping will be at the heart of the use of homotopy to classify topological defects.

Let us now discuss the SSB of a global symmetry. Consider the following Lagrangian for a complex scalar field $\phi$:

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - V(\phi), \qquad V(\phi) = m^2\phi^*\phi + \lambda(\phi^*\phi)^2 \tag{1.1.6}$$

where $\lambda > 0$. This theory enjoys a $U(1)$ global symmetry, as can be seen by applying the transformation:

$$\phi \to \phi' = e^{iq\alpha}\phi, \qquad \phi^* \to \phi'^* = e^{-iq\alpha}\phi^* \tag{1.1.7}$$

with $q$ and $\alpha$ constants. If $m^2 > 0$, the potential has a single minimum at $\phi = 0$, instead if $m^2 \equiv -\mu^2 < 0$, there is an entire set of degenerate minima of the so called Mexican hat potential. As before, the set of minima do not enjoy the same symmetry as the Lagrangian. Indeed, the vacuum can be found by minimizing the potential, which leads to the solutions $|\phi_0|^2 = \mu^2/(2\lambda) \equiv v^2/2$, but the phase of the field $\phi$ is left untouched. Interestingly, radial excitations of the field require energy, they are massive, while angular excitations cost no potential energy and will correspond to massless particles. Hence, the vacuum configuration is not unique and can be parametrized with polar coordinates as $\phi_0 = |\phi_0|e^{i\theta}$, to exploit the symmetry of the theory. Clearly, this isn't invariant under global $U(1)$ transformations anymore, since the phase will change as $\theta \to \theta + q\alpha$, suggesting the vacuum of the theory spontaneously breaks the symmetry of the full Lagrangian. This is an instance of SSB, albeit in the case of a global symmetry. The vacuum can be seen as a circle of radius $v/\sqrt{2}$ in the complex space $(\mathrm{Re}(\phi_0), \mathrm{Im}(\phi_0))$. Since all vacuum states correspond to energetically equivalent configurations, one is free to choose the real part to be $v/\sqrt{2}$ and the imaginary part to vanish, i.e. $|\phi_0| = v/\sqrt{2}$ and $\theta_0 = 0$. In polar coordinates, one can rewrite the field as $\phi(x) = \rho(x)\exp(i\theta(x)/v)$, where $\rho(x) = \rho_0 + \tilde{\rho}(x) = (v + \tilde{\rho}(x))/\sqrt{2}$ and $\theta(x) = \theta_0 + \tilde{\theta}(x) = \tilde{\theta}(x)$. Thus:

$$\phi(x) = \frac{1}{\sqrt{2}}(\tilde{\rho} + v)\, e^{i\tilde{\theta}(x)/v} \tag{1.1.8}$$

This can be substituted in the original Lagrangian to yield:

$$\mathcal{L}_{kin} = \partial_\mu\phi^*\partial^\mu\phi = \frac{1}{2}(\partial_\mu\tilde{\rho})^2 + \frac{1}{2}(\partial_\mu\tilde{\theta})^2 + \frac{1}{v}\tilde{\rho}(\partial_\mu\tilde{\theta})^2 + \frac{1}{2v^2}\tilde{\rho}^2(\partial_\mu\tilde{\theta})^2 \tag{1.1.9}$$

$$V(\phi^*\phi) = \frac{1}{2}(2v^2\lambda)\tilde{\rho}^2 + \lambda v\tilde{\rho}^3 + \frac{\lambda}{4}\tilde{\rho}^4 + constants \tag{1.1.10}$$

Interestingly, this describes a massive scalar field $\tilde{\rho}(x)$ with mass given by $m^2_{\tilde{\rho}} = 2\lambda v^2$ which couples *derivatively* to a massless scalar field $\tilde{\theta}(x)$. This massless field is known as

*Goldstone boson*, and it associated to the breaking of the global $U(1)$, while the derivative couplings are a consequence of the residual symmetry of the vacua states, which is precisely a shift symmetry in the phase $\theta$. This result can be generalized into what is known as the *Goldstone theorem*, which states that whenever a continuous symmetry of the Lagrangian is not a symmetry of the vacuum, then massless particles known as *Goldstone bosons* appear in the spectrum of the theory.

Of more interest is the SSB of a gauge symmetry, which leads to the Higgs mechanism, the foundation of the SM. Consider the previous example but when the $U(1)$ symmetry is promoted to a local one, meaning $\alpha = \alpha(x)$ now. The Lagrangian would not be invariant anymore because of the derivative terms that act precisely on the function $\alpha(x)$. The symmetry is preserved at the price of introducing a generalization of the derivative operator, the so-called covariant derivative:

$$D_\mu = \partial_\mu + iqA_\mu \tag{1.1.11}$$

where $q$ is a constant and $A_\mu(x)$ is a vector field (gauge field) for consistency of index structure. For the theory to be invariant under local $U(1)$, the following is required:

$$(D_\mu\phi) \to (D_\mu\phi)' = e^{iq\alpha(x)}(D_\mu\phi) \qquad \text{when} \qquad \phi \to \phi' = e^{iq\alpha(x)}\phi \tag{1.1.12}$$

which implies the transformation law for the gauge field:

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu\alpha \tag{1.1.13}$$

This is the standard gauge transformation for the 4-potential in the covariant formulation of electromagnetism. The introduction of this gauge field requires one to add its kinetic terms in the Lagrangian to describe its propagation. This leads to the scalar QED Lagrangian [10]:

$$\mathcal{L} = (D_\mu\phi)(D^\mu\phi)^* - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - m^2\phi^*\phi \tag{1.1.14}$$

which can be expanded to yield:

$$\mathcal{L} = \partial_\mu\phi\,\partial^\mu\phi^* - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - m^2\phi^*\phi - iqA_\mu J^\mu + q^2A_\mu A^\mu\phi\phi^* \tag{1.1.15}$$

where $J^\mu$ is the conserved current associated with the global $U(1)$ symmetry, $J^\mu = \phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*$ with these conventions. The result is rather remarkable as it generates an interaction between the gauge field and the scalar field simply by promoting a global symmetry to a local one. This undergoes the name of *gauge principle* and it is at the foundations of gauge theories used to described the electroweak and strong interactions in the SM. It is also worth mentioning that if the field $\phi$ were to acquire a non-zero Vacuum Expectation Value (VEV), the last term would constitute a mass term for the field $A_\mu$, which would otherwise be forbidden by the requirement of gauge invariance. This is the essence of the Higgs mechanism. The procedure is then similar to the global case, one substitutes the exponential field parametrization of (1.1.8) into the above Lagrangian with the Mexican hat potential to see excitations. For sake of recognizing the Higgs field, denote $\tilde{\rho}$ as $H(x)$ and $\tilde{\theta} = \theta$. By expanding the covariant derivative terms, three terms can be noticed to yield:

$$\frac{1}{2}(\partial_\mu\theta)^2 + \frac{q^2v^2}{2}A_\mu A^\mu + qv\,\partial_\mu\theta\,A^\mu = \frac{q^2v^2}{2}\left(A_\mu + \frac{1}{qv}\partial_\mu\theta\right)\left(A^\mu + \frac{1}{qv}\partial^\mu\theta\right) \tag{1.1.16}$$

where in the round brackets one recognizes the typical form of gauge transformations for the vector potential as in (1.1.13). Indeed, by defining the function $\alpha(x) = -\theta(x)/(qv)$ one has:

$$A'_\mu = A_\mu - \partial_\mu \alpha = A_\mu + \frac{1}{qv}\partial_\mu \theta \tag{1.1.17}$$

so that (1.1.16) can be written as $\frac{1}{2}m_A^2 A'_\mu A'^\mu$ with $m_A^2 = q^2 v^2$. A mass term for the gauge field has appeared and the goldstone boson $\theta$ has been *eaten up* by the massless field $A_\mu$, generating the correct numbers of degrees of freedom for a massive spin 1 field. Choosing the function $\alpha(x)$ as above corresponds to the *unitary gauge*, where the Goldstone boson does not appear explicitly. In the end, one is left with the following [11]:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu H)^2 - \frac{1}{2}(2\lambda v^2)H^2 + \frac{m_A^2}{2}A'_\mu A'^\mu + q^2 v H A'_\mu A'^\mu + \frac{q^2}{2}H^2 A'_\mu A'^\mu - \lambda v H^3 - \frac{\lambda}{4}H^4. \tag{1.1.18}$$

corresponding to the theory of a massive scalar field $H(x)$ with mass $m_H^2 = 2\lambda v^2$ interacting with a massive gauge field.

There are also instances of SSB of non-abelian symmetries, i.e. those described by non-abelian groups. In this case, one can simply check whether the vacuum state(s) is invariant under the full symmetry group of the Lagrangian, $G$, by computing the action of group elements, $g \in G$, on the vacuum state $\phi_0$. In many cases of interest, the symmetry groups are Lie groups, for which the elements can be represented as $\exp(i\theta_a T^a)$, where $T_a$ are the generators of the group and $\theta_a$ the parameters of the transformation which identify the specific group element. Expanding around small parameters:

$$\phi'_0 = g\,\phi_0 \approx (1 + i\theta_a T^a)\phi_0 \implies \delta\phi_0 = i\theta_a T^a \phi_0 \tag{1.1.19}$$

It can be concluded that the vacuum is symmetric if the action of the generators of the group $G$ on it give zero. Then, the elements of $G$ corresponding to transformations which leave $\phi_0$ unchanged form a group $H$, called the unbroken subgroup or the little group of $G$ with respect to $\phi_0$ [4]. Furthermore, one has $g\phi_0 = g'\phi_0$ if and only if $g' = g\,h$ with $h \in H$. This is exactly the statement that the vacuum states, which are typically represented as points in the vacuum manifold $\mathcal{M}$, are in $1-1$ correspondence with the coset $G/H$. Hence one can write:

$$\mathcal{M} = G/H \tag{1.1.20}$$

which will be referred to as the vacuum manifold. The topology of this manifold dictates whether and what defects can arise, as well as their properties.

Consider the example of SSB of the global $U(1)$. As emphasized there, the vacuum state is not unique. Rather, the possible vacua are parametrized by a phase $\theta$ or an angle while the modulus of the field is fixed, $|\phi_0| = v/\sqrt{2}$. Therefore, the vacuum manifold is $\mathcal{M} = S^1$. Also, the field $\phi$ could traverse a closed path $L$ in physical space such that the phase of the field develops a non-trivial winding, $\Delta\theta = 2\pi$. For example, there may be distinct regions of spacetime where the field has a different phase, see Fig. 1.1. However, since the field is single valued, there must exist a point in field space where $\phi = 0$, effectively leaving the vacuum manifold and constituting an energetic configuration. In two spatial dimensions, this type of defect is referred to as a *vortex* and its location may be defined as the location of such point in physical space. If the model is extended to three spatial dimensions, this point would becomes a $1-$dimensional object, a vortex line or a *string*.

Since the broken symmetry is a global one, this is an instance of *global strings* solution, while in the presence of the SSB of a gauge symmetry one refers to *gauge (local) strings*. The existence of these defects relies on the possibility of wrapping the phase around a loop in the vacuum manifold $\mathcal{M}$. Therefore, these defects may arise if and only if $\mathcal{M}$ contains incontractible closed paths or loops. This is precisely encoded in the requirement of non trivial fundamental homotopy group of $\mathcal{M}$. In addition, to keep the analogy with domain walls, it is worth noting that the field $\phi$ is still mapping points in physical space to the vacuum manifold, in this case loops in spacetime and loops in $\mathcal{M}$.

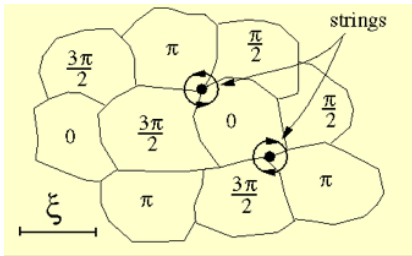


Figure 1.1: Formation of strings due to a non-trivial winding of the phase of the field $\phi$ around a closed path in physical space. Image taken from [12].

Some properties of cosmic strings or vortex lines follow from simple considerations. Consider the abelian Higgs model described by (1.1.14) with the different convention $D_\mu = \partial_\mu - ieA_\mu$. This amounts only to a convention choice for the sign of the gauge coupling ($q \to -e$) and has no physical effect. The requirement of finite energy far from the core imposes $D_\mu\phi \approx 0$ and $F^{\mu\nu} \approx 0$. Thus:

$$\partial_\mu\phi \approx ieA_\mu\phi \implies A_\mu \approx \frac{1}{ie}\partial_\mu(\ln\phi) \tag{1.1.21}$$

In the Lorentz gauge, $\partial_\mu A^\mu = 0$, the Higgs field takes the same form as the global string at large distances from the core, $\phi \approx v/\sqrt{2}\, e^{in\theta}$, with $n$ the winding number of the string. Therefore:

$$\partial_\mu(\ln\phi) = in\partial_\mu\theta \implies A_\mu \simeq \frac{n}{e}\partial_\mu\theta \tag{1.1.22}$$

Integrating along a closed path surrounding the string and using Stokes' theorem reveals the presence of a magnetic flux $\Phi_B$ flowing along the string [4]:

$$\Phi_B = \int \mathbf{B}\cdot d\mathbf{S} = \oint \mathbf{A}\cdot d\mathbf{l} = \frac{n}{e}\oint \partial_i\theta = \frac{2\pi n}{e} \tag{1.1.23}$$

stemming from the fact that the phase of $\phi$ changes by $2\pi$ when traversing a circle. This is analogous to the quantized flux lines in superconductors, once again highlighting the analogies between condensed matter defects and field theory defects. With the potential of the abelian Higgs model, the relevant mass scales in the system are $m_\phi = \sqrt{\lambda}\eta$ and $m_A = e\eta$, the masses of the Higgs field and gauge field, respectively. These determine two length scale via their Compton wavelength:

$$\delta_\phi \sim \frac{1}{\sqrt{\lambda}\eta} \qquad\qquad \delta_A \sim \frac{1}{e\eta} \tag{1.1.24}$$

For $m_A > m_\phi$, the scalar core is wider than the magnetic flux tube, thus the string has an inner false vacuum ($\phi = 0$) core and thinner magnetic flux tube with linear energy densities given by:

$$\mu_\phi \sim V(\phi = 0)\,\delta_\phi^2 \sim \frac{\lambda\eta^4}{\lambda\eta^2} \sim \eta^2 \qquad\qquad \mu_A \sim B^2\delta_A^2 \sim (\Phi_B/\delta_A^2)^2\delta_A^2 \sim n^2\eta^2 \tag{1.1.25}$$

Overall, the linear energy density (tension) of the string is $\mu \sim \eta^2$, dependent on the symmetry breaking parameter $\eta$. This parameter will be linked to the temperature of the Universe when discussing cosmological phase transitions in section 1.3. For now, it is interesting to note that some string models have $\eta \sim 10^{16}$GeV, hinting at the possible cosmological significance of such defects [4].

These statements can be made more mathematically precise by searching for cylindrically symmetric solutions to the field equations descending from the Lagrangians in (1.1.6) and (1.1.14), which will lead to global or gauge string solutions. In a slightly different notation, the potential can be written as:

$$V(|\phi|^2) = \frac{\lambda}{4}\left(|\phi|^2 - \eta^2\right)^2 \tag{1.1.26}$$

The Euler-Lagrange equations of motion descending from the local $U(1)$ model then are:

$$(\partial_\mu - ieA_\mu)(\partial^\mu - ieA^\mu)\phi + \frac{\lambda}{2}\phi(\bar{\phi}\phi - \eta^2) = 0, \qquad \partial_\mu F^{\mu\nu} = j^\nu$$

where

$$j^\nu = 2e\,\mathrm{Im}\left[\bar{\phi}(\partial^\nu - ieA^\nu)\phi\right]. \tag{1.1.27}$$

is the conserved current associated to the original $U(1)$ global symmetry. Working in Lorentz gauge, it is also convenient to perform the rescalings: $\phi \to \eta^{-1}\phi$, $A^\mu \to \eta^{-1}A^\mu$ and $x \to \eta x$. The additional rescaling $x \to ex$ reveals the only relevant parameter in the system is the ratio $\beta = m_\phi^2/m_A^2 \simeq \lambda/e^2$. Following [4], the existence of string-like solutions lying along the z-axis requires one to search for cylindrically symmetric solution $(\phi_s, \mathbf{A}_s)$ through the use of the following ansatz:

$$\phi(r,\theta) = e^{in\theta}f(r), \qquad A_r = 0, \qquad A_\theta = \frac{n}{er}\alpha(r). \tag{1.1.28}$$

with $n$ the vortex winding number. Consistency conditions at the vortex center $\phi = 0$ require $f(0) = \alpha(0) = 0$ and $f(\infty) = 1, \alpha(\infty) = 1$. The solution will depend only on the polar coordinates $(r,\theta)$ and the field equations can be then written as:

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{n^2 f}{r^2}(\alpha - 1)^2 - \frac{\lambda}{2}f(f^2 - 1) = 0, \tag{1.1.29}$$

$$\frac{d^2\alpha}{dr^2} - \frac{1}{r}\frac{d\alpha}{dr} - 2e^2 f^2(\alpha - 1) = 0. \tag{1.1.30}$$

Explicit solutions to these equations are not known and for the purpose of this thesis is sufficient to observe that numerical simulations find the vortex configuration to have an energy per unit length:

$$\mu = 2\pi\eta^2 g(\beta) \tag{1.1.31}$$

where the function $g(\beta)$ is found to approach 1.

As should be clear from the discussion above, the details of the field theory responsible for the production of the topological defects determine the precise core structure. However, when considering the cosmological and phenomenological implications of defects, these details become mostly irrelevant as one is interested in the macroscopic spacetime propagation and possible interaction mechanisms. In general, several types of strings and defects can be theoretically realized stemming from larger and more complicated symmetry groups than $U(1)$. The question then is how to understand whether such a theory admits stable vortex solutions or other types of topological defects from its vacuum manifold. This question will be addressed in the next section through the homotopy classification of defects.

## 1.2 Homotopy classification

In order to determine what defects may appear in a theory with a symmetry group $G$, one needs to examine the topology of the vacuum manifold $\mathcal{M}$, the set of minima of the potential. The classification of defects in terms of topology of the vacuum manifold was first established by Kibble [13] and it relies on the use of homotopy theory. From the examples of the previous section, it is clear that domain walls appear if $\mathcal{M}$ has disconnected components, while strings or vortices form if there are uncontractable loops in $\mathcal{M}$, i.e. if the vacuum manifold is not simply connected. These properties are precisely captured by homotopy theory which classifies mappings from the $n-$dimensional sphere into $\mathcal{M}$ through the $n^{\text{th}}$ homotopy group $\pi_n(\mathcal{M})$. A non-trivial first homotopy group or fundamental group indicates that $\mathcal{M}$ is not simply connected, hence string solutions exist. Generally, the dimensionality of the defect dictates the relevant homotopy group to use, as shown in Table 1.1 below.

Table 1.1: Classification of topological defects with the appropriate homotopy group $\pi_n(\mathcal{M})$.

| Topological defect | Dimension | Classification |
|---|---|---|
| Domain walls | 2 | $\pi_0(\mathcal{M})$ |
| Strings | 1 | $\pi_1(\mathcal{M})$ |
| Monopoles | 0 | $\pi_2(\mathcal{M})$ |
| Textures | – | $\pi_3(\mathcal{M})$ |

Textures and monopoles are other defects that have not been discussed here. Monopoles correspond to non-trivial $\pi_2(\mathcal{M})$. If produced abundantly at an early Universe phase transition, they can overclose it. This is the monopole problem, potentially solved by the theory of inflation.

To be more precise about these statements, it is necessary to review and introduce some important mathematical definitions and theorems following [14].

**Definition 1.2.1.** A topological space $X$ is connected if it cannot be written as $X = X_1 \cup X_2$, where $X_1$ and $X_2$ are open and $X_1 \cap X_2 = \emptyset$. Otherwise, $X$ is disconnected.

**Definition 1.2.2.** A loop in a topological space $X$ is a continuous map $f : [0, 1] \to X$ such that $f(0) = f(1)$. If any loop in $X$ can be continuously shrunk to a point, $X$ is simply connected.

To give a few examples, the 1-dimensional sphere $S^1$ is not simply connected, because a loop winding once around the circle cannot be continuously contracted to a point while remaining on $S^1$. $S^n$, $n \geq 2$ is simply connected, while the $n-$dimensional torus

$T^n$ is not simply connected. Now, the purpose of topology is to classify space [14], and two figures are said to be equivalent in topology if one can be deformed into the other via a *continuous* deformation. This naturally suggests the identification of an equivalence class between these figures. More precisely, this continuous deformation is called a *homeomorphism*.

**Definition 1.2.3.** Let $X_1$ and $X_2$ be topological spaces. A map $f : X_1 \to X_2$ is a homeomorphism if it is continuous and has inverse $f^{-1} : X_2 \to X_1$ which is also continuous.

If such a map exists, then $X_1$ is homeomorphic to $X_2$. It is important to stress the continuity requirement, as it implies this deformation happens without any tearing or pasting of such spaces. Consider the image in Fig. 1.2.
Any loop in $Y$ can be continuously shrunk to a point, while this is not the case for the loop $\alpha$ in $X$ due to the presence of the hole. One says that $\alpha$ is *homotopic* to $\beta$ if $\alpha$ can be obtained from $\beta$ by a continuous deformation. It is clear then that each loop in $Y$ is homotopic to a point. Since this resembles an equivalence relation, it suggests the introduction of an equivalence class which is called *homotopy class*. There is only one homotopy class in $Y$, since all loops can be shrunk to a point, while in $X$ each class is characterized by the number of times $n \in \mathbb{Z}$ the loop goes around the hole. Since $\mathbb{Z}$ is additive, going around $n$ times and then $m$ is equivalent to going around $n+m$ times. This already points at the fact the set of homotopy classes have a group structure.

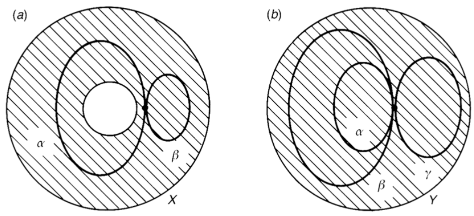



Figure 1.2: Disc with hole $(a)$ and without $(b)$. Image taken from [14].

To see this in detail, consider the following definition of paths and loops:

**Definition 1.2.4.** Let $X$ be a topological space and $I = [0, 1]$. A path is a continuous map $\alpha : I \to X$ with initial point $x_0$ and final point $x_1$ if $\alpha(0) = x_0$ and $\alpha(1) = x_1$. If $\alpha(0) = \alpha(1)$, the path is a loop with base point $x_0$.

The set of loops in a topological space $X$ can be endowed with group structure as follows. Define the product between two paths $\alpha, \beta : I \to X$, such that $\alpha(1) = \beta(0)$ as:

$$(\alpha * \beta)(s) = \begin{cases} \alpha(2s), & 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases} \tag{1.2.1}$$

with $(\alpha*\beta)(s)$ is also a path. Since the goal is to endow the set of homotopy classes with a group structure, it is necessary to identify loops that can be continuously deformed into one another.

**Definition 1.2.5.** Two loops $\alpha, \beta : I \to X$ with base point $x_0$ are said homotopic, $\alpha \sim \beta$ if there exists a continuous map $F : I \times I \to X$ such that:

$$F(s,0) = \alpha(s),\ F(s,1) = \beta(s),\ \forall s \in I \qquad F(0,t) = F(1,t) = x_0,\ \forall t \in I \tag{1.2.2}$$

$F(s,t)$ is called a *homotopy* between $\alpha$ and $\beta$.

This homotopy can be seen as an intermediate family of paths on the interval $s, t \in I$. Then, $[\alpha]$ will denote the equivalent class of loops homotopic to $\alpha$. We are now ready to see the full group structure.

**Definition 1.2.6.** Let X be a topological space. The set of homotopy classes of loops at $x_0 \in X$ is denoted as $\pi_1(X, x_0)$, the fundamental group of $X$ and $x_0$.

The product of homotopy classes $[\alpha], [\beta]$ is defined as $[\alpha * \beta]$ and satisfies the following:

$$1)\quad ([\alpha] * [\beta]) * [\gamma] = [\alpha] * ([\beta] * [\gamma]) \tag{1.2.3}$$

$$2)\quad [\alpha] * [cx] = [\alpha] \tag{1.2.4}$$

$$3)\quad [\alpha] * [\alpha^{-1}] = [cx] \tag{1.2.5}$$

where $[cx]$ is the identity element, provided by the homotopy class of all loops contractible to the point $x$. The inverse corresponds to the same loop traversed in the opposite direction. Therefore, $\pi_1(X, x_0)$ is a group. Typically the base point is omitted because on a connected space one can define a $1-1$ homeomorphism that preserves the group structure, i.e. an isomorphism, between $\pi_1(X, x_0)$ and the fundamental group at any other base point $y$. This involves simply adding a path from $x_0$ to $y$ and its inverse. Therefore, one typically denotes the fundamental group as $\pi_1(X)$. Let us give some examples for clarity.

- The first fundamental group of a contractible space, such as $\mathbb{R}^n$, is trivial $\pi_1(\mathbb{R}^n) = \mathbb{I}$. This follows from the very definition of the group. If an (arcwise) connected space has trivial fundamental group, then it is simply connected by definition.

- $\pi_1(S^1) = \mathbb{Z}$ since as discussed before, winding the $S^1$ $n$ times cannot be continuously deformed into a situation with $m$ encirclings. Therefore the group is not trivial.

- Generalizing for $n-$dimensional spheres: $\pi_n(S^n) = \mathbb{Z}$ for the same reasoning.

To see why loops in topological spaces are relevant consider the abelian Higgs model with vacuum manifold $\mathcal{M} = S^1$. The presence of a potential string or line defect can be detected by encircling this with a closed path $\gamma(t)$ in physical space, or simply put an $S^1$. Then the field $\phi$ maps points along $\gamma(t)$ into $\mathcal{M}$, effectively $\phi : S^1 \to \mathcal{M}$. The values of $\phi$ along $\gamma$ are mapped into a path $g(t)$ in $\mathcal{M}$, $g(t) = \phi(\gamma(t))$. Hence we are dealing with loops in topological spaces, that are characterized by the first homotopy group of the vacuum manifold. This is why homotopy theory plays a crucial role. The possible mappings from $S^1$ to $U(1)$ are classified by $\pi_1(U(1)) \simeq \mathbb{Z}$, therefore we can assign an integer to the line defect, which is precisely the string winding number, i.e. the number of times the image of $S^1$ wraps around the $U(1)$. In a similar fashion, the existence of monopole defects can be detected by surrounding them with an $S^2$. The field then represents a mapping from $S^2$ to $\mathcal{M}$, which are classified by the second homotopy group, $\pi_2(\mathcal{M})$.

The computation of the fundamental group is simplified if the vacuum manifold can be expressed in terms of compact Lie groups, since in this case the vacuum manifold can be written as $\mathcal{M} = G/H$. It is well known that any compact Lie group $G$ can be embedded in a larger simply connected group $\tilde{G}$, ($\pi_1(\tilde{G}) \cong \mathbb{I}$), known as the *universal* *covering group* of $G$. For example, $SO(3)$ is not simply connected but has universal covering group $SU(2)$. By enlarging the symmetry group $G$, it follows that the unbroken subgroup $H$ must also be enlarged by $\tilde{H}$ containing any element of $\tilde{G}$ that leaves the VEV of $\phi$ invariant. However, these enlargements cancel out in the vacuum manifold $G/H$. Moreover, the subgroup $H$ can have several disconnected components $H_i$ and one of them $H_0$ connected to the identity. Then, the disjoint connected components of $H$ are the cosets of the subgroup $H_0$, $H/H_0$. These form a group called the quotient group typically denoted as $\pi_0(H)$ as it corresponds to the equivalence class of mappings from $S^0$ into $H$. Then, the fundamental theorem holds.

**Theorem 1.2.7.** *Let $G$ be a connected and simply connected Lie group, with subgroup $H$ having a component $H_0$ connected to the identity. The quotient group $\pi_0(H) \equiv H/H_0$ is isomorphic to the fundamental group $\pi_1(G/H)$:*

$$\pi_1(G/H) \cong \pi_0(H) \tag{1.2.6}$$

Therefore, the problem of understanding the formation of defects is translated from the computation of the fundamental group of the entire vacuum manifold to simply determining whether the unbroken subgroup $H$ is connected or not. Clearly, the nature of $H$ depends on details of the SSB process, but let us give a few examples of application of this powerful theorem:

- The existence of string solutions in both global and local $U(1)$ models can be confirmed by using its universal covering. Indeed, using directly $\pi_1(U(1)/I) = \pi_0(I)$ would lead to a wrong result because $U(1)$ is connected but not simply connected. Instead, the universal covering is $\mathbb{R}$ modded out by $\mathbb{Z}$, corresponding to $\theta \to \theta + 2\pi n$, with $n \in \mathbb{Z}$. Using the theorem above leads to $\pi_1(\mathbb{R}/\mathbb{Z}) = \pi_0(\mathbb{Z}) = \mathbb{Z}$, where the last equality follows from $\mathbb{Z}$ being a discrete group, so every point is its own connected component.

- SSB of $G = SU(2) \xrightarrow{3} U(1) = H$. This breaking occurs via a triplet scalar field in the adjoint representation of $SU(2)$ [4]. Since $SU(2)$ is connected and simply connected, using the theorem gives $\pi_1(SU(2)/U(1)) = \pi_0(U(1)) = \mathbb{I}$ since $U(1)$ is connected. Therefore, no strings solutions exists. However, $\mathcal{M} \simeq SU(2)/U(1) \simeq S^2$ and $\pi_2(S^2) = \mathbb{Z}$, so this breaking produces monopoles.

- In Grand Unified Theories, one eventually needs to recover $U(1)_{em}$ as a factor of the unbroken subgroup $H$, $G \to U(1) \times ....$ Therefore, $\pi_2(G/U(1)) = \pi_1(U(1)) = \mathbb{Z}$, these models necessarily include (magnetic) monopoles.

Before closing this section, it is worth noting that the first fundamental theorem can be generalized and it is found to be the lower component of exact homotopy sequences depicted as:

$$\begin{aligned} \cdots \longrightarrow \pi_n(G) \longrightarrow \pi_n(G/H) \longrightarrow \pi_{n-1}(H) \longrightarrow \pi_{n-1}(G) \longrightarrow \\ \cdots \longrightarrow \pi_1(G) \longrightarrow \pi_1(G/H) \longrightarrow \pi_0(H) \end{aligned}$$

where at each step the image of the homomorphisms is the kernel of the next homomorphism.

This concludes the classification of topological defects based on homotopy theory and allows one to understand more easily whether and what type of defects can arise in a given SSB process. Now, I would like to turn the attention towards the physical aspects of these defects, especially their formation, which is likely to have happened in the early universe, a fascinating playground for particle physics.

## 1.3 Cosmological phase transitions

There are many reasons to believe the Universe underwent several phase transitions in its early times. If cosmic strings do exist, they must have been formed during the first fraction of a second after the Big Bang [2], where the state of the Universe was a hot plasma of radiation. To understand potential observable effects today, one must first study how these objects were formed in such an environment and their propagation up until now. Since the early universe was rather hot, it is necessary to consider finite-temperature field theories.

The chain of symmetry breaking possibly started from a Grand Unified Theory (GUT) symmetry group $G$, such as $SU(5)$ or $SO(10)$ to name the most famous examples, which broke into the vacuum subgroup $H$ and eventually recovered the SM gauge group schematically as:

$$G \to H \to \cdots \to SU(3)_C \times SU(2)_L \times U(1)_Y \to SU(3)_C \times U(1)_{\text{em}} \tag{1.3.1}$$

These phase transitions were caused by the temperature drop as a result of the expansion of the Universe, since generally $T \sim 1/a(t)$ [15], where $a(t)$ is the scale factor in an FLRW cosmology. The critical temperature, $T_c$, of each transition dictates the symmetry breaking scale and, as mentioned in section 1.1, this is tightly connected to all the parameters of the field theory generating topological defects, especially the tension of strings, $\mu$. These symmetry breaking phase transitions can be described as SSB processes, albeit one needs to consider thermal correction to the potential. Moreover, this also makes clear the statements about different regions of spacetime with different vacuum states. Indeed, because of causality, disconnected regions of spacetime can have different choices of vacuum values.

Another type of corrections to be taken into consideration are the quantum ones. The SSB examples explored in section 1.1 relied on the tree-level classical potential. However, quantum effects (loops) can affect the potential and actually influence the occurrence of SSB. These quantum corrections arise due to self interactions of the quantum field $\phi$, as well as interactions with other fields. The fully quantum corrected potential for $\phi$, typically referred to as the *effective potential* $V_{eff}$ can be evaluated using perturbation theory by expanding in powers of small coupling constants as:

$$V_{eff}(\phi) = V(\phi) + \sum_n V_n(\phi) \tag{1.3.2}$$

where $V(\phi)$ is the classical tree level potential and $V_n(\phi)$ represents the contribution of $n-$loops Feynman diagrams. The non-abelian generalization of the abelian Higgs model is:

$$\mathcal{L} = \frac{1}{2} D_\mu \phi_i D^\mu \phi_i - V(\phi) - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}. \tag{1.3.3}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e f_{abc} A^b_\mu A^c_\nu. \tag{1.3.4}$$

is the non-abelian gauge field strength and:

$$D_\mu \phi = \left(\partial_\mu - ieA^a_\mu T^a\right)\phi. \tag{1.3.5}$$

is the non-abelian covariant derivative, with $T^a$ generators of the non-abelian gauge group. One can also add fermions to the theory by introducing:

$$\mathcal{L}_F = i\bar{\psi}\gamma^\mu D_\mu \psi - \bar{\psi} Y_i \psi \phi_i \tag{1.3.6}$$

where the interaction between the Higgs and fermion fields is of the Yukawa type with Yukawa coupling matrix $Y_i$. For this general model, the one-loop contribution to $V_{eff}$ is known as the Coleman-Weinberg potential [16]:

$$V_1(\phi) = \frac{1}{64\pi^2}\left[\mathrm{Tr}\left(\mu^4 \ln\frac{\mu^2}{\sigma^2}\right) + 3\,\mathrm{Tr}\left(M^4 \ln\frac{M^2}{\sigma^2}\right) - 4\,\mathrm{Tr}\left(m^4 \ln\frac{m^2}{\sigma^2}\right)\right]. \tag{1.3.7}$$

where $\sigma$ is the renormalization scale and $\mu$, $M$, and $m$ are the scalar, vector and fermions mass matrices respectively. It is useful to perform a treatment in analogy with condensed matter phase transitions, where symmetries that are spontaneously broken can be restored at sufficiently high temperatures. In this picture, the expectation value of the Higgs field in the abelian $U(1)$ and $\mathbb{Z}_2$ models have condensed matter analogues corresponding to superfluids and superconductivity. Thus, the VEV of the Higgs field can be thought of as describing a Bose condensate of Higgs particles. In thermodynamics, phase transitions are investigated by looking at the Helmholtz free energy of the system:

$$F = E - TS \tag{1.3.8}$$

which becomes a non trivial function of $\phi$ in our case. The equilibrium value of $\phi$ is found precisely by minimizing this function and its VEV will generally be temperature dependent. At low temperatures, the minima of F will correspond to minima of the energy $E$, as the second term is negligible. Instead, at higher temperatures the second term becomes relevant as the entropy also increases. Indeed, if $\phi$ decreases, the masses of the particles in the theory will also decrease, increasing the overall phase space and thus the number of microstates available. This results in an entropy increase. Therefore, it is reasonable to expect a tendency of the Higgs field to decrease with temperature and eventually approach $\phi = 0$ at a critical temperature $T_c$. To describe these types of symmetry breaking processes in finite temperature field theories, one has to evaluate the Free energy of (1.3.8) in terms of $\phi$ and $T$. The formalisms was first developed by [17] as well as [18], and it relies on the the fact that the Free energy density $F(\phi, T)/\mathcal{V}$ has the same diagrammatic expansions as the effective potential $V_{eff}(\phi)$ provided one replaces the standard Green's functions with finite temperature Green's functions. This naturally leads to the definition of:

$$V_{eff}(\phi, T) = F(\phi, T)/\mathcal{V} \tag{1.3.9}$$

as the finite-temperature effective potential, where $\mathcal{V}$ is the three dimensional volume of space of the system.

Let us derive the one-loop temperature dependent contribution to $V_{eff}(\phi,T)$. To lowest order in the coupling constants, thermal particles are not interacting, thus one can write:

$$V_{\text{eff}}(\phi,T) = V(\phi) + \sum_n F_n(\phi,T) \tag{1.3.10}$$

where $V(\phi)$ is the tree level or zero temperature potential and each $F_n$ is given by:

$$F_n = \pm T \int \frac{d^3k}{(2\pi)^3} \ln\left(1 \mp \exp(-\varepsilon_k/T)\right), \tag{1.3.11}$$

with $\varepsilon_k = \left(k^2 + m_n^2\right)^{1/2}$. When the temperature is much smaller than all particle masses, $T \ll m(T)$, $F_n$ is exponentially small and can be neglected. Contrary, when $T \gg m(T)$ the contribution depends on the statistical nature of the particles. In full generality one can write:

$$V_{\text{eff}}(\phi,T) = V(\phi) + \frac{1}{24}\mathcal{M}^2 T^2 - \frac{\pi^2 g_*}{90} T^4, \tag{1.3.12}$$

where

$$g_* = g_b + \frac{7}{8} g_f, \qquad \mathcal{M}^2 = \sum_b m_n^2 + \frac{1}{2}\sum_f m_n^2. \tag{1.3.13}$$

with $g_b$ and $g_f$ the number of bosonic and fermionic spins states of the theory. Since the $T^4$ term is independent of $\phi$, it does not affect the symmetry breaking. Instead, the interesting term is the $T^2$ correction. For the general non-abelian Higgs model of (1.3.3), the coefficient $\mathcal{M}^2$ is given by:

$$\mathcal{M}^2 = \operatorname{Tr}\mu^2 + 3\operatorname{Tr}M^2 + \frac{1}{2}\operatorname{Tr}\left(\gamma^0 m \gamma^0 m\right) \tag{1.3.14}$$

Consider the abelian Higgs model with $q = e$. There is a complex scalar $\phi$ and massless gauge field $A_\mu$, so one has $g_* = 4$. Assuming $\lambda \gg e^4$, the quantum corrections to the Higgs potential can be neglected, leaving only the thermal ones. Moreover, after SSB, the coefficient $\mathcal{M}^2$ reads:

$$\left.\begin{aligned} m_b &= \sqrt{2}e|\phi| \\ m_\phi &= \sqrt{\lambda}|\phi| \end{aligned}\right\} \qquad \mathcal{M}^2 = 2(\lambda + 3e^2)|\phi|^2 \tag{1.3.15}$$

which leads to the following finite temperature effective potential:

$$V_{\text{eff}}(\phi,T) = V(\phi) + \frac{\lambda + 3e^2}{12} T^2 |\phi|^2 - \frac{2\pi^2}{45} T^4 \tag{1.3.16}$$

As can be seen, there is an additional positive contribution proportional to $|\phi|^2$ affecting the minima and maxima of the potential, since these are dictated by its second derivative. Interestingly, the potential can be written as [4]:

$$V_{eff}(\phi,T) = \frac{\lambda}{4}(|\phi|^2 - \eta^2)^2 + \frac{\lambda + 3e^2}{12} T^2 |\phi|^2 = m^2(T)|\phi|^2 + \frac{\lambda}{4}|\phi|^4 \tag{1.3.17}$$

where we have defined the temperature dependent mass

$$m^2(T) = \frac{\lambda + 3e^2}{12} T^2 - \frac{\lambda}{2}\eta^2 \tag{1.3.18}$$

This is the effective mass of the field $\phi$ in the symmetric state $|\phi| = 0$ and it equals zero at the critical temperature $T_c$, hence:

$$T_c = \eta\sqrt{\frac{6\lambda}{\lambda + 3e^2}} \qquad \eta^2 = \frac{\lambda + 3e^2}{6\lambda}T_c^2 \tag{1.3.19}$$

which puts on firm grounds previous statements about the symmetry breaking parameter $\eta$ being related to the critical temperature. Furthermore, the fixed points of the potential are:

$$\frac{\partial V_{eff}}{\partial|\phi|} = |\phi|\left[\lambda(|\phi|^2 - \eta^2) + \frac{\lambda + 3e^2}{6}T^2\right] = 0 \implies \begin{cases} |\phi| = 0, \\ |\phi| = \sqrt{\dfrac{\lambda + 3e^2}{6\lambda}(T_c^2 - T^2)} \equiv |\phi|^* \end{cases} \tag{1.3.20}$$

and, using relations (1.3.19), the second derivative of the potential at these points can be written as:

$$\begin{cases} \left.\dfrac{\partial^2 V_{eff}}{\partial|\phi|^2}\right|_{|\phi|=0} = \dfrac{\lambda + 3e^2}{6}(T^2 - T_c^2) \\ \left.\dfrac{\partial^2 V_{eff}}{\partial|\phi|^2}\right|_{|\phi|=|\phi|^*} = \dfrac{\lambda + 3e^2}{2}(T_c^2 - T^2) \end{cases} \tag{1.3.21}$$

This clearly shows that for $T > T_c$, $|\phi| = 0$ becomes a minimum of the potential while the other fixed point is imaginary. Instead, as the temperature drops below $T_c$, $m^2(T) < 0$, the symmetric state becomes unstable and the field develops a non-zero vacuum expectation value where $|\phi| = 0$ corresponds to a maximum and the minima then satisfy $|\phi| = |\phi|^*$ with arbitrary phase, corresponding to the usual vacuum degeneracy. The description presented so far is that of a *second order* phase transition, characterized by a continuous growth of the order parameter, $|\phi|$ here, as the temperature decreases below $T_c$. See Fig. 1.3 for a visual representation.

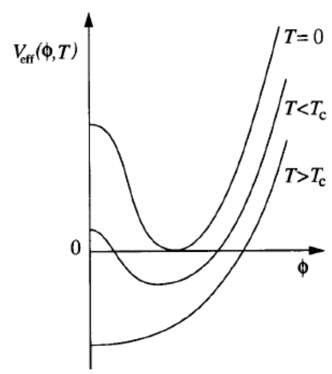


Figure 1.3: Evolution of the effective potential in a second order phase transition [4].

The situation is not quite the same for a *first order* phase transition. Such a transition can be realized in the abelian Higgs model of (1.1.14) by considering symmetry breaking solely induced by quantum corrections (1.3.7). For instance, when $|\phi| \ll T/e$ the finite temperature effective potential is:

$$V_{\text{eff}}(\phi, T) = m^2(T)|\phi|^2 + \frac{3e^4}{16\pi^2}|\phi|^4 \ln\left(\frac{|\phi|^2}{\sigma^2}\right), \tag{1.3.22}$$

with effective mass:

$$m^2(T) = \mu_0^2 + \frac{1}{4}e^2T^2. \tag{1.3.23}$$

For very large temperatures, this potential is dominated by the $e^2T^2$ term, and there is a single minimum at $\phi = 0$. The variation of the potential with temperature is illustrated in Fig. 1.4. As $T$ becomes closer to $T_c$ but still $T > T_c$, the symmetry breaking minimum appears, but the symmetric state, the so called *false vacuum* is still energetically favored. Only after $T < T_c$, this becomes higher and therefore constitutes a metastable configuration. One cannot use the formula of (1.3.12) to evaluate $T_c$ analytically, since it is not valid in this regime. The remarkable feature of a first order phase transition is that the symmetric state remains metastable below the critical temperature. In the presence of strong supercooling, the decay of such metastable state requires quantum tunnelling through the potential barrier. In this instance, spherical bubbles of the true SSB vacuum appear and the phase transition is completed when these collide or coalesce. Further details on this matter are irrelevant for the purpose of the formation of string network, which will be discussed in the next section together with the macroscopic model used to understand its evolution.

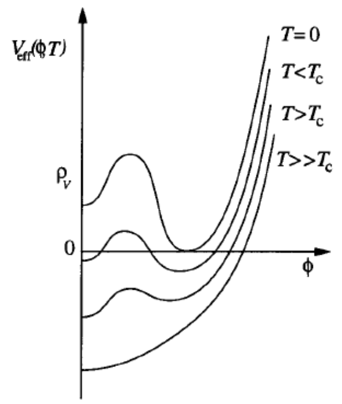


Figure 1.4: Evolution of the effective potential in a first order phase transition[4].

# 1.4 Cosmic strings formation and evolution

In this section, the focus will be shifted to the evolution of strings both in their field theory description as well as the cosmological formation of a network of these defects, hence the name cosmic strings. The microscopic equations of motion will be derived and will serve as the basis for the development of the macroscopic model used to study the evolution of the network.

## 1.4.1 Microscopic dynamics

The microscopic dynamics of strings can be precisely studied when the curvature radius is greater than the thickness, in which case these objects can be treated as one-dimensional. A one dimensional objects spans a two-dimensional surface or *worldsheet* $\Sigma$ as it propagates through spacetime. Coordinates on the worldsheet are arbitrary parameters $\zeta^a = (\zeta^0, \zeta^1)$, where $\zeta^0$ is chosen to be timelike while $\zeta^1$ spacelike and label points along the string. The worldsheet is a surface embedded in 4-dimensional spacetime, therefore one can write:

$$x^\mu = x^\mu(\zeta^a) \tag{1.4.1}$$

Given a spacetime metric $g_{\mu\nu}$, the worldsheet metric is defined as the pullback:

$$\gamma_{ab} = \frac{\partial x^\mu(\zeta)}{\partial \zeta^a}\frac{\partial x^\nu(\zeta)}{\partial \zeta^b}\, g_{\mu\nu} \tag{1.4.2}$$

We now want to look for an action principle governing the motion of strings in spacetime of the form:

$$S = \int d^2\zeta \, \sqrt{-\gamma} \, \mathcal{L} \tag{1.4.3}$$

where $\gamma$ is the determinant of the worldsheet metric $\gamma_{ab}$. The Lagrangian should be invariant under general spacetime coordinate transformations $x^\mu \to \tilde{x}^\mu(x^\nu)$ and under worldsheet reparametrization $\zeta^a \to \tilde{\zeta}^a(\zeta^b)$. Having noted this, the only physical quantities that can enter in the Lagrangian are the string tension and geometrical quantities such as worldsheet intrinsic and extrinsic curvature and derivatives. The string four velocity cannot enter as one can Lorentz boost to the rest frame. Since the mass dimension of $\mathcal{L}$ must be 2 one has:

$$\mathcal{L} = -\mu + \alpha K + \beta \frac{K^2}{\mu} + \dots \tag{1.4.4}$$

with $\alpha$ and $\beta$ numerical coefficients. For the string to be really one dimensional, the curvature radius has to be much larger than the thickness, $R \gg \delta$ and recalling $\delta \sim \eta^{-1}$, $\mu \sim \eta^2$, one has $\delta \sim \mu^{-1/2}$. Therefore:

$$K \sim 1/R^2 \ll 1/\delta^2 \sim \mu \tag{1.4.5}$$

and one can neglect the curvature corrections to the Lagrangian leading to the Nambu-Goto action for a string:

$$S = -\mu \int d^2\zeta \, \sqrt{-\gamma} \tag{1.4.6}$$

Note that the tension is assumed to be constant here. This action is proportional to the area swept by the string in spacetime, and the least action principle determines its dynamics by minimizing such area, similarly to the case of a particle's worldline. Proceeding as usual, one performs the variation of the action with respect to the spacetime coordinates to recover the equations of motion finding:

$$\frac{1}{\sqrt{-\gamma}} \partial_a \left( \sqrt{-\gamma} \, \gamma^{ab} g_{\rho\sigma} x^\sigma_{,b} \right) - \frac{1}{2} \partial_\rho g_{\lambda\sigma} \gamma^{ab} x^\lambda_{,a} x^\sigma_{,b} = 0 \tag{1.4.7}$$

Manipulating this expression one can turn it into:

$$\frac{1}{\sqrt{-\gamma}} \partial_a \left( \sqrt{-\gamma} \, \gamma^{ab} x^\sigma_{,b} \right) + \gamma^{ab} \left[ \frac{1}{2} g^{\sigma\rho} (\partial_\nu g_{\mu\rho} + \partial_\mu g_{\nu\rho} - \partial_\rho g_{\mu\nu}) \right] \partial_a x^\mu \partial_b x^\nu = 0 \tag{1.4.8}$$

where we recognize in the square brackets the Christoffel symbols $\Gamma^\sigma_{\mu\nu}$ for the metric $g_{\mu\nu}$. Therefore the equations of motion can be written in a compact form as:

$$x^{\sigma;a}_{,a} + \Gamma^\sigma_{\mu\nu} \gamma^{ab} \partial_a x^\mu \partial_b x^\nu = 0 \tag{1.4.9}$$

Let us consider some relevant spacetimes for future discussion.

### Minkowski spacetime

In this flat spacetime, $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^\mu_{\nu\rho} = 0$, so the equations of motion are greatly simplified. Indeed:

$$\partial_a \left( \sqrt{-\gamma} \, \gamma^{ab} x^\sigma_{,b} \right) = 0 \tag{1.4.10}$$

Now, we can use the gauge redundancy represented by the reparameterization invariance of the worldsheet coordinates to set:

$$\left.\begin{aligned} \gamma_{01} &= 0 \\ \gamma_{11} + \gamma_{00} &= 0 \end{aligned}\right\} \implies \begin{aligned} \dot{x} \cdot x' &= 0 \\ \dot{x}^2 + x'^{\,2} &= 0 \end{aligned} \tag{1.4.11}$$

where $\dot{x}^\mu = \partial x^\mu / \partial \zeta^0$ and $x^{\mu\,\prime} = \partial x^\mu / \partial \zeta^1$. This is known as the *conformal gauge* since the worldsheet metric is conformally the Minkowski one:

$$\gamma_{ab} = \sqrt{-\gamma}\,\eta_{ab} \qquad\qquad \gamma^{ab} = \frac{\eta^{ab}}{\sqrt{-\gamma}} \tag{1.4.12}$$

In this gauge, the equations of motion are:

$$\ddot{x}^\mu - x^{\mu\,\prime\prime} = 0 \tag{1.4.13}$$

simply the two dimensional wave equation. There is a residual gauge freedom in reparameterization of the kind $\dot{\zeta}^1 = \zeta^{0\,\prime}$ and $\zeta^{1\,\prime} = \dot{\zeta}^0$ which can be removed by identifying worldsheet time with physical time $t = x^0 = \zeta^0$. With this choice, one can write the string trajectory as a $3-$vector $\mathbf{x}(t, \zeta^1)$ where the gauge fixing conditions and equations of motion become:

$$\left.\begin{aligned} \dot{x} \cdot x' &= 0 \\ \dot{x}^2 + x'^{\,2} &= 0 \\ \ddot{x}^\mu - x^{\mu\,\prime\prime} &= 0 \end{aligned}\right\} \implies \begin{aligned} \dot{\mathbf{x}} \cdot \mathbf{x}' &= 0 \\ \dot{\mathbf{x}}^2 + \mathbf{x}'^{\,2} &= 1 \\ \ddot{\mathbf{x}} - \mathbf{x}'' &= 0 \end{aligned} \tag{1.4.14}$$

The first condition tells us that $\dot{\mathbf{x}}$ is orthogonal to the string tangent vector $\mathbf{x}'$ and thus constitutes the string velocity. The second condition can be written as $d\zeta^1 = (1 - \dot{\mathbf{x}}^2)^{-1/2} |d\mathbf{x}| \equiv dE/\mu$, with $E$ the energy of the string. Thus, $\zeta^1$ is proportional to the string energy:

$$E = \mu \int d\zeta^1 \tag{1.4.15}$$

The fact that $E$ is the string energy can be inferred from the 00 component of the energy momentum tensor, which can be found by varying the action with respect to the spacetime metric:

$$\sqrt{-g}\, T^{\mu\nu}(x) = \mu \int d^2\zeta \sqrt{-\gamma}\, \gamma^{ab} \partial_a X^\mu \partial_b X^\nu \delta^{(4)}(x - X). \tag{1.4.16}$$

In this spacetime and with this gauge choice, it becomes:

$$T^{\mu\nu}(\mathbf{x}, t) = \mu \int d\zeta^1 \, \left(\dot{x}^\mu \dot{x}^\nu - x'^\mu x'^\nu\right) \delta^{(3)}\big(\mathbf{x} - \mathbf{x}(\zeta^1, t)\big)\,. \tag{1.4.17}$$

Hence the total energy of the string is:

$$E = \int d^3x\, T^0_0(x) = \mu \int d\zeta^1 \int d^3x\, \delta^{(3)}(\mathbf{x} - \mathbf{x}(\zeta^1, t)) = \mu \int d\zeta^1 \tag{1.4.18}$$

Instead, the equations of motion tell us the string acceleration is inversely proportional to the local curvature radius $R^{-1}$, so the string tends to straighten. Furthermore, a relevant aspect of strings in Minkowski spacetime regards the Root Mean Squared (RMS) velocity of segments on an oscillating loop. The loop configuration is required to be periodic in $\zeta^1$, hence $\mathbf{x}(t,\zeta^1) = \mathbf{x}(t,\zeta^1+L)$, with $L$ the invariant length of the loop [4]. The dynamical equations also impose periodicity in the time coordinate $t$, of period $T$. Then, the RMS velocity of segments on the loop can be defined as $\bar{v} = \langle v^2\rangle^{1/2}$, with

$$\langle v^2\rangle = \int_0^T \frac{dt}{T}\int_0^L \frac{d\zeta^1}{L}\dot{\mathbf{x}}^2 \tag{1.4.19}$$

Integrating by parts leads to:

$$\int_\Sigma d^2\zeta\, \dot{\mathbf{x}}\cdot\dot{\mathbf{x}} = \int_\Sigma d^2\zeta \frac{d}{dt}(\mathbf{x}\cdot\dot{\mathbf{x}}) - \int_\Sigma d^2\zeta\, \mathbf{x}\cdot\ddot{\mathbf{x}} \tag{1.4.20}$$

The total derivative term can be neglected and the second term can be written using the equations of motion and the gauge fixing condition as:

$$\int d^2\zeta\, \mathbf{x}\cdot\ddot{\mathbf{x}} = \int d^2\zeta\, \mathbf{x}\cdot\mathbf{x}'' = -\int d^2\zeta\, \mathbf{x}'\cdot\mathbf{x}' = -\int d^2\zeta\,(1-\dot{\mathbf{x}}^2) \tag{1.4.21}$$

Therefore,

$$2\int_\Sigma d^2\zeta\, \dot{\mathbf{x}}\cdot\dot{\mathbf{x}} = \int d^2\zeta = 1 \implies \bar{v}^2 = \int_\Sigma d^2\zeta\, \dot{\mathbf{x}}\cdot\dot{\mathbf{x}} = \frac{1}{2} \tag{1.4.22}$$

where the integration is over one period. It can be concluded that the RMS velocity of segments on a loop is $\bar{v} = 1/\sqrt{2}$.

### Expanding Universe

For a general curved spacetime, the second gauge choice in (1.4.11) is not convenient and instead it is useful to replace it with $\zeta^0 = t$. Then, the string equations of motion can be written as:

$$\frac{\partial}{\partial t}\left[\frac{\dot{x}^\mu x'^2}{\sqrt{-\gamma}}\right] + \frac{\partial}{\partial\zeta^1}\left[\frac{x'^\mu \dot{x}^2}{\sqrt{-\gamma}}\right] + \frac{1}{\sqrt{-\gamma}}\Gamma^\mu_{\nu\sigma}\left(x'^2\dot{x}^\nu\dot{x}^\sigma + \dot{x}^2 x'^\nu x'^\sigma\right) = 0. \tag{1.4.23}$$

In an expanding FLRW universe, the line element reads:

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2 = a^2(\tau)(-d\tau^2 + d\mathbf{x}^2) \tag{1.4.24}$$

where conformal time $\tau$, $dt = a\,d\tau$ has been introduced. The gauge fixing conditions are $\dot{\mathbf{x}}\cdot\mathbf{x}' = 0$ and the conformal time is identified with the worldsheet time, $\zeta^0 = \tau$. The Christoffel symbols are given by:

$$\Gamma^0_{00} = \frac{\dot{a}}{a} \qquad \Gamma^0_{ij} = \frac{\dot{a}}{a}\delta_{ij} \qquad \Gamma^0_{0i} = \Gamma^i_{00} = \Gamma^i_{jk} = 0 \qquad \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j \tag{1.4.25}$$

and, by defining the linear energy density parameter $\varepsilon$ as:

$$\varepsilon \equiv \sqrt{\frac{\mathbf{x}'^2}{1-\dot{\mathbf{x}}^2}}, \tag{1.4.26}$$

the equations of motion are found to be:

$$\ddot{\mathbf{x}} + 2\mathcal{H}(1-\dot{\mathbf{x}}^2)\dot{\mathbf{x}} - \frac{1}{\varepsilon}\left(\frac{\mathbf{x}'}{\varepsilon}\right)' = 0 \qquad \dot{\varepsilon} + 2\frac{\dot{a}}{a}\varepsilon\dot{\mathbf{x}}^2 = 0 \tag{1.4.27}$$

where dots denote derivative with respect to conformal time, $\mathcal{H}$ is the Hubble parameter in conformal time, and scalar products are with respect to the Euclidean metric. The physical energy of the string is given by:

$$E = \mu a(\tau) \int \mathrm{d}\zeta^1 \varepsilon \tag{1.4.28}$$

and one can investigate its time dependence by differentiating it with respect to conformal time:

$$\dot{E} = \frac{\dot{a}}{a} E - 2\dot{a}\mu \int d\zeta^1 \varepsilon \dot{\mathbf{x}}^2 = \mathcal{H}\left(1 - 2\bar{v}^2\right) E \tag{1.4.29}$$

At this point, one can understand the physical effect of the expansion of the Universe. In flat spacetime, $\bar{v}^2 = 1/2$, implying energy is conserved for loops. Instead, the expansion of the universe leads to a damping term in the equations of motion proportional to the Hubble parameter. This damping will typically lead to a lower squared velocity than the Minkowskian value, suggesting a loss of energy due to velocity redshifting [4]. This effects becomes negligible for loops of invariant length much smaller than the horizon, $L \ll d_H$, since the damping term is proportional to $(L/d_H)^2$.

### Perturbations

It is instructive to consider perturbations to the dynamics of a straight and static string in an expanding universe. Its trajectory will be described by $\mathbf{x}(\zeta^1) = \mathbf{c}\zeta^1$ and the expansion of the universe simply stretches it. Small perturbations to this solution can be represented as:

$$\mathbf{x}(\zeta^1, \tau) = \mathbf{c}\zeta^1 + \delta\mathbf{x}(\zeta^1, \tau) \tag{1.4.30}$$

Using this into the equations of motion (1.4.27) and expanding to first order in $\delta\mathbf{x}$ leads to:

$$\delta\ddot{\mathbf{x}} + \frac{2\alpha}{\tau}\,\delta\dot{\mathbf{x}} - \delta\mathbf{x}'' = 0, \qquad \mathbf{c}\cdot\delta\dot{\mathbf{x}} = 0 \tag{1.4.31}$$

where a power law expansion $a(\tau) \sim \tau^\alpha$ has been assumed. The general solution to this equation can be found by going to Fourier space and it is a superposition of Bessel functions:

$$\delta\mathbf{x} = \mathbf{A}\tau^{-\nu} J_\nu(k\tau)\, e^{ik\zeta} \tag{1.4.32}$$

with $\nu = \alpha - 1/2$. Physical quantities are obtained by multiplying the comoving ones with the scale factor $a(\tau)$, thus the physical wavelength associated to a momentum $k$ mode is $\lambda = 2\pi a(\tau)/k$. As standard practice in the treatment of the evolution of perturbations in cosmology, two regimes are identified based on the ratio between the horizon $d_H \sim \tau$ and $\lambda$. Equivalently, $k\tau \ll 1$ corresponds to superhorizon modes while $k\tau \gg 1$ subhorizon modes. In the first regime, the time dependent factor approaches a constant value, therefore these long wavelength modes do not evolve in comoving coordinates, they are conformally stretched by the expansion of the universe. Instead, subhorizon modes oscillate and their comoving amplitude is damped by the expansion, implying irregularities on scales smaller than the horizon are washed out by the expansion of the universe.

### 1.4.2 Network formation

Having understood cosmological phase transitions and the formation of defects in field theory, we are ready to combine the two aspects and discuss the formation of cosmic strings in a cosmological setting. Since the relevant scales are much larger than the string thickness, these objects will be considered effectively as one-dimensional and the details of the specific field theory giving rise to them are mostly irrelevant. However, for sake of clarity it is useful to keep in mind the global or local cosmic strings associated to the SSB of $U(1)$, which predicts a tension $\mu \sim \eta^2$ as in (1.1.31). Furthermore, we will consider only thermal phase transitions induced by the cooling of the Universe in a standard radiation dominated FLRW cosmology.

Initially, the Universe was very hot, $T \gg T_c$, so the system was in the symmetric state without any strings. When the temperature falls below the critical value, the field acquires a non-zero VEV and SSB occurs. The vacuum manifold of the considered model will be the usual circle, and so the field will tend to one of the vacuum states along such circle. The choice of minimum, of the phase in this case, will depend on random fluctuations in the field $\phi$ which can vary across regions of spacetime. If we denote as $\xi$ the correlation length beyond which these fluctuations are uncorrelated, due to causality it must be $\xi < d_H$, with $d_H$ the cosmological horizon. The transition can be first or second order based on the underlying field theory details:

1. In the case of a first order phase transition, bubbles nucleate where the phase of the field inside each bubble is more or less a random variable. When bubbles coalesce, the value of $\phi$ across the different regions will interpolate and, if the net phase change along the mutual boundary is $2\pi$ a string can be trapped alongside it (recall Fig. 1.1). Several of these processes can happen and the result is a network configuration of cosmic strings with a characteristic scale given by the separation of bubble nucleation centers and the probability of trapping a string at a boundary.

2. In the event of second order phase transition no bubble nucleates. Instead, the field $\phi$ tends to move away from the symmetric state at the same time everywhere. For a certain amount of time, thermal fluctuations can be sufficiently high as to move the field again up to the symmetric state. Only after the temperature reaches the *Ginzburg temperature*, $T_G$, these higher order corrections can be neglected and the phase of the field $\phi$ freezes in. Then the outcome is the same, strings can be trapped where the phase of field changes by $2\pi$ and the scale of the network is proportional to the correlation length at $T_G$ as well as the probability of string trapping.

In both situations, the outcome is a random network of cosmic strings, either infinite or loops, with a certain characteristic length $\xi < d_H$, where $d_H \sim t$ for power law cosmologies. After the SSB originating the strings, their tension becomes $\mu \sim \eta^2$. Furthermore, the environment surrounding the strings is a hot plasma of particles and radiation, therefore their evolution is subject to a lot of friction, which causes a damping effect. This effect can be approximated analytically introducing a friction scale $l_f$. However, for our purposes it is sufficient to note that after $T < T_*$, where $T_* \sim T_c^2/M_{pl}$, the damping becomes negligible. For example, for $T_c \sim 10^{15}$ GeV, $T_* \sim 10^{12}$ GeV which corresponds to very early times, $10^{-31}$ s.

The question then is what happens to the characteristic scale $\xi$? First of all, this can be defined more precisely as the average length of string in a volume of size $\xi^3$. Basically, this is saying the string density is:

$$\rho_s = \frac{\mu}{\xi^2} \tag{1.4.33}$$

Since causality requires $\xi$ to never become greater than $t$, only two possible outcomes are available. Either (1) $\xi/t$ decreases or (2) $\xi/t$ is constant. This second option is known as the scaling regime of the network, where $\xi$ is a fixed fraction of cosmic time $t$. In a radiation dominated background $a \sim t^{1/2}$ and $\rho \sim a^{-4}$, therefore $\rho \sim t^{-2}$. Moreover, the comoving horizon is:

$$d_H^{com} = \int \frac{dt}{a(t)} = 2t^{1/2} \implies d_H = a(t) d_H^{com} = 2t \tag{1.4.34}$$

The relevant scales in the system are $\xi$ and the physical horizon $d_H$, whose ratio can be defined as:

$$\gamma = \frac{2t}{\xi} \tag{1.4.35}$$

and it grows in regime (1) while remains constant in regime (2). From the Friedman equation we have:

$$\rho_{tot} = \frac{3H^2}{8\pi G_N} = \frac{3}{32\pi G_N} \frac{1}{t^2} \tag{1.4.36}$$

Therefore, the ratio of string density to the universe energy density is:

$$\frac{\rho_s}{\rho_{tot}} = \frac{\mu}{\xi^2} \frac{32\pi G_N t^2}{3} = \frac{8\pi}{3} (G_N \mu) \gamma^2 \tag{1.4.37}$$

Since in regime (1) $\gamma$ increases in time, it would lead to a Universe dominated by string energy density, which is not what observations suggest. Instead, in the second regime, the strings track the overall energy density of the universe. The combination $G_N\mu$ is typically referred to as the *dimensionless string tension* and it is the quantity constrained by experiments. This scaling regime is actually a result of the underlying dynamical equations governing the string network which physically arises due to a balance between the expansion of the universe and the intersection of strings, known as *intercommutation*, which results in the production of loops that oscillate and radiate energy. This macroscopic behavior of the network is best understood by averaging the string equations of motion (1.4.27) to obtain evolution equations for both the characteristic scale of the network and the average velocity.

### 1.4.3 Velocity One Scale model

The Velocity-One Scale (VOS) model [19] describes the macroscopic evolution of cosmic strings networks in terms of the string velocity and a single scale, denoted as the correlation length $\xi$. This is the same length introduced in section 1.4.2 and physically it can be interpreted as the average length of string found in a volume $\xi^3$ of space. The VOS model is not meant to resolve the microscopic structure of the network, but rather to provide an effective, averaged description of its large-scale evolution. Its foundation lies in the fact that cosmic string networks are statistical systems, for which a small number of macroscopic quantities can capture the relevant dynamics. The original one-scale model was introduced by Kibble [20] as well as Albrecht and Turok a few years later [21]. This was refined by including the RMS velocity as an additional dynamical variable, leading to the velocity-dependent one-scale model developed by Martins and Shellard [22]. This framework has been extensively compared with numerical simulations [23] and remains the standard analytic description of the large-scale evolution of cosmic string networks.

Let us rename this as $\xi = L$ as typical in the literature. The equations governing the VOS model are averaged versions of the microscopic equations of motion (1.4.27). These describe the evolution in time of the string density $\rho_s$, or equivalently the correlation length, and the RMS velocity. Consider the time evolution of the string energy (1.4.29), which can be written as $\rho \equiv \rho_s = E/a^3$, leading to:

$$\dot{\rho} = \frac{\dot{E}}{a^3} - 3\mathcal{H}\rho \tag{1.4.38}$$

Therefore:

$$\frac{d\rho}{d\tau} = \frac{\mathcal{H}E}{a^3}(1 - 2\bar{v}^2) - 3\mathcal{H}\rho \implies a\frac{d\rho}{dt} = aH\rho(1 - 2\bar{v}^2) - 3aH\rho \tag{1.4.39}$$

where $H$ is the cosmic time Hubble parameter, $aH = \mathcal{H}$. Thus:

$$\frac{d\rho}{dt} + 2H(1 + \bar{v}^2)\rho = 0 \tag{1.4.40}$$

At this point, one has to take into account the rate of loop production from long string intersections. This was first described by Kibble [20] as:

$$\left(\frac{d\rho}{dt}\right)_{\text{to loops}} = \rho\frac{v}{L}\int w\left(\frac{\ell}{L}\right)\frac{\ell}{L}\frac{d\ell}{L} = \tilde{c}\,v\frac{\rho}{L} \tag{1.4.41}$$

where $w(\ell/L)$ is a scale invariant function giving the probability of creating loops with length in between $\ell$ and $\ell + d\ell$, while the chopping coefficient $\tilde{c}$ is assumed to be constant. This contribution is to be added to the density evolution equation. Moreover, recalling (1.4.33) and assuming a constant tension:

$$\frac{d\rho}{dt} = -2\frac{\mu}{L^3}\frac{dL}{dt} \tag{1.4.42}$$

So that the density evolution equation can be turned into an equation for the evolution of the correlation length $L$:

$$2\frac{dL}{dt} = 2HL(1 + \bar{v}^2) + \tilde{c}\bar{v} \tag{1.4.43}$$

This is one of the core equations of the VOS model. An equation for the evolution of the velocity can be derived in a similar fashion by differentiating (1.4.19):

$$2v\frac{dv}{d\tau} = \frac{1}{\int d\zeta\varepsilon}\left[-2\mathcal{H}\int d\zeta\,\varepsilon\dot{\mathbf{x}}^4 - 4\mathcal{H}\int d\zeta\varepsilon(1 - \dot{\mathbf{x}}^2)\dot{\mathbf{x}}^2\right] + 2\mathcal{H}(\bar{v}^2)^2 + I \tag{1.4.44}$$

where

$$I \equiv \left[2\int d\zeta\varepsilon\dot{\mathbf{x}}\frac{1}{\varepsilon}\frac{d}{d\zeta}\left(\frac{1}{\varepsilon}\frac{d}{d\zeta}\mathbf{x}\right)\right] \Big/ \left[\int d\zeta\varepsilon\right] \tag{1.4.45}$$

Up to second order terms, we can take $\langle \dot{\mathbf{x}}^4 \rangle = \langle \dot{\mathbf{x}}^2 \rangle^2$. To evaluate $I$, one introduces the curvature radius $R$ as:

$$\frac{a(\tau)}{R}\,\hat{\mathbf{u}} = \frac{d^2\mathbf{x}}{ds^2}\,, \tag{1.4.46}$$

where $\hat{\mathbf{u}}$ is the unit curvature vector along the string and $s$ is the physical length along the string related to $\zeta$ by $ds = |\mathbf{x}'|\,d\zeta = \sqrt{1-\dot{\mathbf{x}}^2}\,\epsilon\,d\zeta$. Furthermore, the momentum parameter $k$ may be introduced as:

$$\Big\langle (1-\dot{\mathbf{x}}^2)\,(\dot{\mathbf{x}}\cdot\hat{\mathbf{u}})\Big\rangle \equiv k\,v\,(1-v^2)\,. \tag{1.4.47}$$

Exchanging the $\zeta$ derivatives with $s$ gives:

$$I = \frac{2akv}{R}(1-v^2) \tag{1.4.48}$$

Therefore,

$$\frac{dv}{dt} = (1-v^2)\left(\frac{k(v)}{R} - 2Hv\right) \tag{1.4.49}$$

which is the other fundamental equation of the VOS model. Technically one should also take into account the production of loops of physical length $\ell$. Without going into much details, the equation governing this can be found to be:

$$\frac{d\ell}{dt} = (1-2v_\ell^2)H\ell - \Gamma' G\mu. \tag{1.4.50}$$

with $v_\ell$ the loop velocity and $\Gamma'$ a numerical coefficient. This model has been validated through numerical simulations and it accurately describes the large scale structure of cosmic string networks. Discrepancies seem to be related to the presence of non-trivial small-scale structure such as kinks and wiggles on scales below $L$. Further extensions of the model take into account friction due to the motion of strings in a background of radiation fluid or by considering a time dependent tension. This latter case will be investigated in section 4.2.2.

This concludes the discussion of cosmic string evolution, both at the microscopic and network level. Next, a few words will be dedicated to the string theory view of field theory strings and how these objects can arise in such a context.

## 1.5 Cosmic superstrings

Superstring theory replaces the point particle description of QFT with 1-dimensional extended objects, the superstrings, whose vibrational modes correspond to different particle states. As one dimensional objects, their propagation is studied by means analogous to those of cosmic strings. Indeed, these two "categories" of strings were regarded as distinct in the beginning, mainly due to cosmic strings being on horizon scales while superstrings are extremely small objects. However, under certain circumstances, superstrings can also grow to cosmological size and play the same role of cosmic strings. Interestingly, observations of cosmic superstrings could provide a direct evidence in support of superstring theory or M-theory [24].

It has been shown by [25] that 10-dimensional Type IIB superstring theory contains extended objects of various dimensions embedded in spacetime, known as *D-branes*. A $p-$dimensional brane ($Dp$-brane) spans a $p+1$ dimensional worldvolume as it propagates in spacetime. Thus, a D1$-$brane is a string, as it sweeps a two dimensional surface or worldsheet. These strings are denoted as D-strings, while fundamental string theory strings are referred to as F-strings. The possible existence of cosmic size superstrings was investigated in [7] where it was argued the stability of such objects was not possible because of several reasons. Firstly, the tension of superstrings is of the order of $M_{pl}^2$, leading to a dimensionless string tension $G\mu$ far larger than the constraints at the time. Secondly, it was argued that any inflationary period would have diluted the number of any pre-existing strings, and so only those produced afterwards could have been observationally relevant. The situation drastically changed with the development of inflationary models in string theory, such as [26, 27, 28, 29], based on the idea of brane-antibrane inflation and warping of the extra dimensions.

These inflationary models rely on a particular type of bulk geometry which is that of a Calabi-Yau cone, resembling a throat region in the extra dimensions. A D3-brane is attracted to a $\overline{D3}-$brane placed at the bottom of this throat similar to how opposite charged particles attract. The radial distance between these branes plays the role of the inflaton field in $4D$. Notably, the throat constitutes a highly warped region, meaning $4-$dimensional spacetime is scaled by a factor dependent on the position of the $D3$-brane in the internal space [24]:

$$ds^2 = e^{2A(y_\perp)}\eta_{\mu\nu}dx^\mu dx^\nu + ds_\perp^2 \tag{1.5.1}$$

where $y_\perp$ are the internal space coordinates. The existence of such solution in Type IIB was first found by [30]. In the throat region, redshift effects are very important, achieving a maximum at the bottom of the throat where the warp factor reaches its minimum value, $e^{2A_0} \ll 1$. The string scale as measured by a ten-dimensional observer is close to the $4D$ Planck scale, however, for a four-dimensional observer, there is an exponential suppression precisely due to the warping:

$$\mu_{eff} \sim e^{2A_0}\mu_F \sim \frac{e^{2A_0}}{2\pi\alpha'} \tag{1.5.2}$$

Therefore, the dimensionless string tension can be made compatible with constraints thanks to warping. One-dimensional objects are expected to be produced in this region, and the natural candidates are $F$- and $D$-strings.

In addition, another mechanism to reduce the dimensionless string tension involves compactifications with large extra dimensions. Notably, a moduli stabilization scenario in this setting has been constructed in [31] and it is known as the Large Volume Scenario (LVS). To see why this is helpful, recall that performing a dimensional reduction of the $10-$dimensional Einstein-Hilbert action of superstring theory leads to a relation between the $10D$ and $4D$ Newton constants $G_{N,10}$ and $G_{N,4}$ which schematically is:

$$G_{N,10} \sim V_6\, G_{N,4} \tag{1.5.3}$$

where $V_6$ is the volume of the internal $6D$ space. This implies the $4D$ Planck scale is derived from the higher dimensional one. For Type II theories, $G_{N,10} \sim g_s^2 l_s^8$, with $l_s$ the string length. Therefore, the four dimensional Planck scale can be estimated as:

$$l_{pl}^2 = G_{N,4} \sim g_s^2 \left( \frac{l_s^6}{V_6} \right) l_s^2 \tag{1.5.4}$$

Thus, when $V_6 \gg l_s^6$, the four-dimensional Planck length is much smaller than the string length, equivalently the four-dimensional Planck mass is much larger than the string mass, provided that $g_s < 1$. In turn, this implies the fundamental string tension is:

$$\mu \sim 1/l_s^2 \ll 1/l_{pl}^2 \tag{1.5.5}$$

which could bring the dimensionless tension back into the observational bounds. More generally, in string theoretic inflationary models involving $Dp$-brane annihilation, $D$-strings can form. This is because the reheating process in such setting is described by a tachyon field, and these $D-$strings correspond to topological defects of this field. This is closely analogous to the mechanism outlined in section 1.1. $F-$strings can also form in a $S$-dual view, and in some instances these can combine to form bound states of $p$ $F-$ and $q$ $D-$strings, denoted as $(p, q)$ strings. It is also necessary to point out that models of inflation in string theory without production of defects are available. However, they can be formed and therefore one should investigate the potential stability and resulting observational signatures. This lies outside the scope of this section and will not be discussed any further.

Although cosmic superstrings can behave macroscopically like ordinary field-theory cosmic strings, there are important differences between the two. Field theory strings arise as solitonic defects after the spontaneous breaking of a gauge or global symmetry, and their tension is set by the corresponding symmetry breaking scale. Cosmic superstrings, instead, originate as fundamental F-strings, D-strings, or their $(p, q)$ bound states in string compactifications. Their effective four-dimensional tension is therefore controlled not only by a symmetry breaking scale, but also by string-theoretic data such as the string scale, the string coupling, warping, internal volumes, and background fluxes. This makes it possible for cosmic superstrings to have tensions compatible with observational bounds, for example through warped throats or large compact dimensions. Moreover, while ordinary field-theory gauge strings typically intercommute with probability close to unity, cosmic superstrings can have significantly smaller reconnection probabilities and may form bound states and three-string junctions [24]. These features provide possible ways of distinguishing a cosmic superstring network from an ordinary field-theory cosmic string network.

The important point for the present thesis is that string compactifications naturally provide mechanisms through which macroscopic cosmic superstrings can arise and behave, at large scales, similarly to field-theory cosmic strings, although their effective four-dimensional tension depends on the compactification. To investigate this connection further, and ultimately to discuss the possibility of a time-dependent string tension, it is necessary to review some basic aspects of superstring theory and string compactification. This will be the subject of the next chapter.

# Chapter 2

# String compactification and moduli stabilization

The existence of cosmic strings in a string theoretic context motivates the discussion of superstring theory. For this reason, in this chapter I will provide a general overview of superstring theory, focusing on the basic ingredients of the theory and the properties needed to address the issue of moduli stabilization. This naturally leads to the description of the Large Volume Scenario, where the existence of an exponential potential governing moduli dynamics will be presented. Kaluza-Klein dimensional reduction will also be discussed, considering its important role in determining the kinetic terms for scalar fields in the lower dimensional actions and the dynamics of the Planck scales in different dimensions.

## 2.1 Overview of superstring theory

String theory postulates the existence of one-dimensional objects as fundamental entities, the strings. These sweep out a two-dimensional surface, the *worldsheet*, as they propagate in spacetime. As usual, the evolution in spacetime of a physical object is best determined via the least action principle, which requires one to have an action for said object. Through simple arguments, one can generalize the action for a zero-dimensional object to the case of a $n$-dimensional one, simply by requiring the action to minimize its world volume, which in the case of a particle is one-dimensional, typically called the worldline. This leads to the same Nambu-Goto action of (1.4.6). However, the equations of motion coming from that action are highly non linear. It is therefore typical to introduce the intrinsic worldsheet metric $h_{ab}$, which is independent of the spacetime coordinates. The new action principle can then be written as [32]:

$$S_P = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}\, h^{ab}\partial_a X^M(\sigma)\partial_b X^N(\sigma)\eta_{MN}\,. \tag{2.1.1}$$

known as the *Polyakov action*, describing the propagation of a bosonic string in $D$-dimensional Minkowski spacetime. Here $X^M$, with $M = 0, \ldots, D-1$ are spacetime coordinates while $\sigma^a$, with $a = 0, 1$ are worldsheet coordinates. Also, $T_s = (2\pi\alpha')^{-1}$ is the string tension. Consistency conditions of the quantum theory, namely the absence of the Weyl anomaly, require it to live in $D = 26$ spacetime dimensions. However, this theory fails at describing our universe mainly for two reasons: it has a tachyon in the spectrum, signaling vacuum instability, and lacks spacetime fermions. This is where superstring theory comes to the rescue. By imposing $\mathcal{N} = (1,1)$ worldsheet supersymmetry, the tachyon can be effectively removed from the spectrum and spacetime fermions appear in specific sectors. In conformal gauge, the worldsheet action is [33]:

$$S = S_P + S_F = -\frac{1}{4\pi\alpha'}\int d^2\sigma \left(\partial^a X^M \partial_a X_M - i\bar{\psi}^M \gamma^a \partial_a \psi_M\right). \tag{2.1.2}$$

where $\gamma^a$ are 2-dimensional Dirac matrices obeying the usual Clifford algebra and $\psi_M$ is a Dirac spinor on the worldsheet which transforms as a vector under spacetime Lorentz transformations. In terms of the components:

$$\psi^M = \begin{bmatrix} \psi_+^M \\ \psi_-^M \end{bmatrix} \tag{2.1.3}$$

the fermionic part of action (2.1.2) is:

$$S_F = \frac{i}{2\pi\alpha'}\int d^2\sigma \left(\psi_-^M \partial_+ \psi_-^N + \psi_+^M \partial_- \psi_+^N\right)\eta_{MN}, \tag{2.1.4}$$

where $\partial_\pm \equiv \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$, with $\tau \equiv \sigma^0$ and $\sigma \equiv \sigma^1$. A careful treatment of boundary terms in the above action leads to possible choices which divide the solutions into sectors. Namely, this reduces to choosing periodic or antiperiodic boundary conditions for the fermionic fields. For the closed string, this is encoded in:

$$\psi_+^\mu(\sigma) = \pm\psi_+^\mu(\sigma + l), \qquad \psi_-^\mu(\sigma) = \pm\psi_-^\mu(\sigma + l). \tag{2.1.5}$$

where $l$ is the string length. The $+$ sign is historically denoted as "Ramond sector" (R), while the $-$ sign is the "Neveu-Schwarz sector" (NS). Therefore, one has generally four independent contributions: R-R, NS-NS, R-NS, and NS-R. This choice determines whether the mode expansions are half-integer or integer. These solutions are all independent and one can combine the different sectors to yield additional degrees of freedom. While quantum constraints will remove some of these extra solutions, the final sectors are found by requiring a well defined worldsheet Conformal Field Theory (CFT) [34]. This is implemented by the so-called "GSO projections" that give only two inequivalent combinations of sectors if one demands supersymmetry in spacetime, which is usually required to have computational control over the effective field theory and the advantageous features of supersymmetry breaking [35]. These inequivalent combinations are Type IIA and Type IIB superstring theory [32]:

$$\text{IIA:} \quad (\text{NS}_+, \text{NS}_-),\ (\text{R}_+, \text{R}_-),\ (\text{NS}_+, \text{R}_-),\ (\text{R}_+, \text{NS}_-), \tag{2.1.6}$$

$$\text{IIB:} \quad (\text{NS}_+, \text{NS}_+),\ (\text{R}_+, \text{R}_+),\ (\text{NS}_+, \text{R}_+),\ (\text{R}_+, \text{NS}_+). \tag{2.1.7}$$

where the signs label the two possible GSO projections, related to the worldsheet fermion number. The critical number of spacetime dimensions is found by requiring the consistency of the quantum worldsheet theory. In particular, anomaly cancellation on the worldsheet implies that superstring theory must live in a ten-dimensional spacetime. Notably, the ten-dimensional spacetime effective actions describing the low-energy excitations of the superstring can actually be equivalently described by 10-dimensional $\mathcal{N} = 2$ supergravity theories involving bosonic and fermionic contributions. The former arise from string states in NS-NS and R-R sectors, while the latter come from R-NS and

NS-R sectors.

Let us now discuss the low energy structure of $10D$ Type IIB supergravity. This is phenomenologically favorable compared with Type IIA since it is chiral and the SM is a chiral theory. For this reason, several research lines in the field of string phenomenology and string cosmology take Type IIB as the starting point. The $(\mathrm{NS}_+, \mathrm{NS}_+)$ sector contains the massless states associated with the graviton $G_{MN}$, the dilaton field $\Phi$, and the antisymmetric Kalb-Ramond field $B_{MN}$. Instead, the R-R sectors contain a zero-form (axion) $C_0$, a two-form $C_2$, and a four-form $C_4$. These forms are generalization of the electromagnetic field and as such have an associated field strength given by $F_p = dC_{p-1}$, with $d$ the exterior derivative operator. Overall, the bosonic action governing the dynamics of these fields has three contributions:

$$S_{\mathrm{IIB}} = S_{\mathrm{NS}} + S_{\mathrm{R}}^{(\mathrm{IIB})} + S_{\mathrm{CS}}^{(\mathrm{IIB})}, \tag{2.1.8}$$

with $S_{\mathrm{CS}}^{(\mathrm{IIB})}$ the *Chern-Simons* action which encodes topological interactions among forms. Explicitly, these terms are:

$$S_{\mathrm{NS}} = \frac{1}{2\kappa^2}\int d^{10}X\sqrt{-G}\, e^{-2\Phi}\left(R + 4(\partial\Phi)^2 - \frac{1}{2}|H_3|^2\right), \tag{2.1.9}$$

$$S_{\mathrm{R}}^{(\mathrm{IIB})} = -\frac{1}{4\kappa^2}\int d^{10}X\sqrt{-G}\left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right), \tag{2.1.10}$$

$$S_{\mathrm{CS}}^{(\mathrm{IIB})} = -\frac{1}{4\kappa^2}\int C_4\wedge H_3\wedge F_3, \tag{2.1.11}$$

where $\tilde{F}_3 = F_3 - C_0\wedge H_3$, and $\tilde{F}_5 = F_5 - \frac{1}{2}C_2\wedge H_3 + \frac{1}{2}B_2\wedge F_3$. Here $\kappa^2$ is Newton constant in 10D and it can be related to the string tension by comparing the worldsheet and supergravity calculations of graviton scattering amplitudes. It is worth noting the dilaton factor in front of the Ricci scalar in the NS action. This signals one is working in *string frame*, where the standard gravitational action (Einstein-Hilbert) is not manifest. This frame is used to compare results of string perturbations theory with supergravity calculations [36]. However, eventually one would like to recover Einstein theory of gravity, which has no dilaton dependent prefactor in the action. Therefore, one can perform a Weyl rescaling of the metric to go in *Einstein frame* (denoted by the subscript $E$):

$$G_{E,MN} \equiv e^{-\Phi/2}G_{MN} \tag{2.1.12}$$

Moreover, in Type IIB it is useful to define the combinations:

$$G_3 \equiv F_3 - \tau H_3, \qquad \tau \equiv C_0 + ie^{-\Phi} \tag{2.1.13}$$

where $\tau$ is usually referred to as the *axio-dilaton*, as it contains the axion ($C_0$) and the dilaton. This leads to the following Einstein-frame bosonic action for Type IIB superstring theory:

$$S_{\mathrm{IIB}} = \frac{1}{2\kappa^2}\int d^{10}X\sqrt{-G_E}\left[R_E - \frac{|\partial\tau|^2}{2\,(\mathrm{Im}\tau)^2} - \frac{|G_3|^2}{2\mathrm{Im}\tau} - \frac{|\tilde{F}_5|^2}{4}\right] - \frac{i}{8\kappa^2}\int\frac{C_4\wedge G_3\wedge\bar{G}_3}{\mathrm{Im}\tau} \tag{2.1.14}$$

### 2.1.1 Compactification and energy scales

As mentioned in the previous section, perturbative superstring theory contains a massless spin-2 excitation and it is free of the tachyonic instability of the bosonic string after the GSO projection. Quantum consistency further requires anomaly cancellation and fixes the critical dimension to $D = 10$. Therefore, one is forced to work in a 10-dimensional spacetime, $X_{10}$. However, the observed number of spacetime dimensions is $D = 4$, therefore it is necessary to study *string compactifications* to make contact with the real world. The basic idea is that the six extra dimensions (ED) need to be made very small, compactified. The concept of ED was first introduced by Kaluza and Klein [37] as they explored general relativity in five dimensions. For superstring theory, compactification can be summarized as:

$$X_{10} \to \mathbb{R}^{1,3} \times Y_6 \tag{2.1.15}$$

where $Y_6$ is a six-dimensional compact space. The volume of these extra dimensions already determines a mass scale, known as the Kaluza-Klein (KK) scale $M_{KK}$ given by:

$$\mathrm{Vol}(Y_6)^{-1/6} \equiv M_{KK} \tag{2.1.16}$$

which is typically larger than the Large Hadron Collider (LHC) energy scale, $E_{LHC} \sim 13\,\mathrm{TeV}$. In addition, the string length, $l_s = 2\pi\sqrt{\alpha'}$, introduces another mass scale:

$$M_s = \frac{1}{l_s} = \left(2\pi\sqrt{\alpha'}\right)^{-1} \tag{2.1.17}$$

Furthermore, as any theory of quantum gravity, the Planck scale $M_P$ must also play a role. Indeed, these three energy scales are not independent as one can see by performing the dimensional reduction of the Einstein-Hilbert (EH) term of the action (2.1.9), which can be schematically written as [35]:

$$S_{10D} \supset M_s^8 \int d^{10}X \, \sqrt{-G_{10D}} \, e^{-2\Phi} R_{10D} \tag{2.1.18}$$

where the power of the string tension is chosen to have a dimensionless action. After the decomposition (2.1.15), this becomes:

$$S_{10D} \supset M_s^8 \left(\int d^6y\sqrt{g_{6D}}\right) \int d^4x \, e^{-2\Phi} R_{4D} + \dots \tag{2.1.19}$$

with $y^m$ , $m = 4, \dots 9$ the ED coordinates. The integral over these yields the volume of the extra dimensions, which can be written in terms of the dimensionless volume $\mathcal{V}$: $\mathrm{Vol}(Y_6) = \mathcal{V} l_s^6 = \mathcal{V} M_s^{-6}$. Moreover, in the vacuum, $\Phi \to \langle\Phi\rangle = \ln g_s$, using the standard relation between the VEV of the dilaton and string coupling $g_s$ [36]. Overall, this makes it possible to recover the standard $4D$ EH action provided that one identifies the $4D$ Planck mass with:

$$S_{4D} \sim \frac{M_{P,4}^2}{2} \int d^4x \, \sqrt{-g_{4D}} R_{4D} \qquad\qquad M_{P,4}^2 \simeq M_s^2 \mathcal{V} e^{-2\langle\Phi\rangle} \tag{2.1.20}$$

In turn, this implies a relation between the string scale and Planck scale [36]:

$$M_s \simeq \frac{g_s M_P}{\sqrt{\mathcal{V}}} \tag{2.1.21}$$

modulo some factors of $2\pi$. Using this result, the relation between the KK scale and the Planck scale can easily be found by:

$$M_{KK} = \mathrm{Vol}(Y_6)^{-1/6} = \frac{M_s}{\mathcal{V}^{1/6}} \simeq \frac{g_s M_P}{\mathcal{V}^{2/3}} \tag{2.1.22}$$

Note how these expressions depend on the string coupling and the dimensionless volume of the ED. For perturbation theory to hold, one typically requires $g_s \ll 1$, so that we are in the weakly coupled regime and the above action can be trusted. Moreover, there exist scenarios, such as the Large Volume Scenario (LVS) [38], where the ED volume is actually large, therefore the combination of $g_s$ and $\mathcal{V}$ can be small and the string scale may be lower than the Planck scale, potentially leading to observational signatures of string theory at not so incredible high energies. This is simply a taste of the power of dimensional reduction, which will be discussed more in detail in the following section.

This hierarchy of scales also clarifies the effective description at different energies. The basic picture of string compactification is the following: at energies $E \gtrsim M_s$ the full $10D$ superstring theory is the relevant theory, one is able to resolve the individual strings. At energies $M_s > E > M_{KK}$, an observer would see only the massless excitations of strings and the effective theory is a $10D$ $\mathcal{N} = 2$ supergravity theory. Lowering again the energy scale, one goes below the KK scale, $E \ll M_{KK}$, so the extra dimensions are compactified. In this regime, the effective theory is a $4D$ supergravity theory with a number of supersymmetry which depends on the geometry of the extra dimensional compact space. The most studied geometry for the ED space is the *Calabi-Yau* (CY) one, a particular manifold which is Ricci flat and possesses $SU(3)$ holonomy. This condition follows from the requirement of a covariantly constant spinor, which implies a certain number of supersymmetries in the $4D$ effective theory. When compactifying on this type of Calabi-Yau manifolds, $\mathcal{N} = 2$ supersymmetry in $10D$ becomes $\mathcal{N} = 2$ supersymmetry in $4D$ [39]. In practice, one does not want to break entirely supersymmetry when reducing to $4D$, since this allows for a larger computational control and has several nice features related to its breaking. Furthermore, one should also obtain a chiral $4D$ theory if one wishes to describe the real world. However, $\mathcal{N} = 2$ supersymmetric theories are not chiral. The situation is solved by performing an *orientifold projection*, which is essentially an identification of points in the ED space. This has the effect of cutting in half the degrees of freedom and therefore one is left with $4D\,, \mathcal{N} = 1$ supergravity theory as low energy effective description in spacetime [33].

This motivates a brief review of the structure of four-dimensional $\mathcal{N} = 1$ supergravity, which will provide the effective framework for the discussion of moduli stabilization. Supergravity is the local version of supersymmetry and can be realized by gauging global supersymmetric theories. As seen in section 1.1, when promoting a global symmetry to a local one, it is necessary to introduce a gauge field to make the derivative operator transform nicely under the specific gauge transformation. In this case, one is gauging a fermionic symmetry and as such, its gauge field will be a fermion not a boson. The gauge field of local supersymmetry is found to be a spin 3/2 particle, called the *gravitino*. The bosonic field content of a general four-dimensional $\mathcal{N} = 1$ supergravity theory are the metric tensor $g_{\mu\nu}$, gauge potentials $A^a_\mu$ and complex scalar fields $\phi^i$ [36]. As typical of supersymmetric theories, the kinetic terms as well as gauge fermion or boson interactions are encoded in a real analytic function of the fields $K(\phi_i, \bar{\phi}^{\bar{i}})$, known as the *Kähler potential*. Instead, the scalar masses and possible Yukawa-like interactions are encoded in an holomorphic function of the fields $W(\phi^i)$, known as the *superpotential*. In absence of gauge interactions, the scalar fields Lagrangian is [36]:

$$\mathcal{L}_\Phi = -K_{i\bar{j}}\,\partial^\mu\phi^i\partial_\mu\bar{\phi}^{\bar{j}} - V_F, \tag{2.1.23}$$

where $K_{i\bar{j}} = \partial_i\partial_{\bar{j}}K$ is the Kähler metric, the metric in field space. Instead, the potential is the well-known F-term scalar potential of $\mathcal{N}=1$ supergravity:

$$V_F(\phi^i, \bar{\phi}^{\bar{i}}) = e^{K/M_{\rm pl}^2}\left[K^{i\bar{j}}D_iW\overline{D_jW} - \frac{3}{M_{\rm pl}^2}|W|^2\right], \tag{2.1.24}$$

where $D_iW = \partial_iW + \frac{1}{M_{\rm pl}^2}(\partial_iK)W$ is the Kähler covariant derivative. The fundamental objective of string compactification is to determine the Kähler potential and superpotential in terms of compactification data. This would allow the computation of the low energy four-dimensional effective theory describing the dynamics of $4D$ fields. However, when performing a dimensional reduction one faces the issue related to the presence of several scalar fields typically associated with deformation of the ED space. These are called *moduli fields* and constitute a major challenge in achieving a realistic description of our Universe in superstring theory compactifications. Before delving into some of the details related to this problem, let us thoroughly discuss the procedure of dimensional reduction, as it is very important to understand the physics in four dimensions.

## 2.2 Dimensional reduction

Dimensional reduction of the full superstring theory action (2.1.8) is a very difficult task and, for the purpose of this thesis, it is sufficient to present the detailed mechanism for a simpler case. Consider Einstein gravity in a $D$-dimensional spacetime. This is described by the following action:

$$S = \frac{M_{P,D}^{D-2}}{2}\int \mathrm{d}^Dx\sqrt{-\hat{G}}\,\hat{R} \tag{2.2.1}$$

where $M_{P,D}$, $\hat{G}$, and $\hat{R}$ are the $D$-dimensional Planck mass, determinant of the $D$-dimensional metric tensor, and $D$-dimensional Ricci scalar, respectively. In the following, hats will denote higher dimensional objects. In light of (2.1.15), the goal is to obtain an equation relating the higher dimensional Ricci scalar to the lower dimensional one. This is done by implementing an adequate ansatz for the higher dimensional metric tensor, exploiting the symmetries of the problem. This is typically suggested by the structure of the compactification manifold. For example, consider the case of a $(D = d+1)$-dimensional theory reduced to a $d$-dimensional one by compactifying a single extra dimension along a circle $S^1$. It is immediate to understand that there will be a $U(1)$ symmetry in the dimensionally reduced theory. This is because translations along the extra dimension correspond to phase rotations, and such rotations constitute a $U(1)$ group. This suggests the introduction of a $U(1)$ gauge field $A_\mu$ in the metric ansatz. Recall the Ricci scalar definition in terms of the Ricci tensor:

$$\hat{R} = \hat{G}^{MN}\hat{R}_{MN} \tag{2.2.2}$$

Capital latin indices are $D$-dimensional, while lower case latin indices will refer to the extra dimensions, and lower case Greek indices to the non compact dimensions. In terms of the Christoffel symbols, the Ricci tensor is given by:

$$\hat{R}_{MN} = \partial_P \hat{\Gamma}^P_{MN} - \partial_N \hat{\Gamma}^P_{MP} + \hat{\Gamma}^P_{PQ} \hat{\Gamma}^Q_{MN} - \hat{\Gamma}^P_{NQ} \hat{\Gamma}^Q_{MP} \tag{2.2.3}$$

The computation of this then requires one to evaluate the Christoffel symbols associated with the higher dimensional metric tensor. This is greatly facilitated by the use of the Kaluza-Klein ansatz for the $D$-dimensional metric which, for the case of a single extra dimension is [40]:

$$\hat{ds}^2 = e^{2\alpha\phi(x)} g_{\mu\nu} dx^\mu dx^\nu + e^{2\beta\phi(x)} \left(dz + A_\mu dx^\mu\right)^2 \tag{2.2.4}$$

Equivalently, in matrix form:

$$\hat{G}_{MN} = \begin{bmatrix} e^{2\alpha\phi(x)} g_{\mu\nu} + e^{2\beta\phi(x)} A_\mu A_\nu & e^{2\beta\phi(x)} A_\mu \\ e^{2\beta\phi(x)} A_\nu & e^{2\beta\phi(x)} \end{bmatrix} \tag{2.2.5}$$

where $\phi(x)$ is a scalar field describing the fluctuations of the proper size of the extra dimensions, typically called the *radion* [41]. For reductions on a higher-dimensional compact space, one would in general obtain several scalar moduli and, when the internal space has continuous isometries, several gauge fields $A^m_\mu$.

A motivation for this ansatz relies simply on the index structure of the metric components. In the $D = d + 1$ case considered, the $D$-dimensional metric will include a $d \times d$-dimensional tensor, two $d$-vectors, and a scalar. Now, the diffeomorphism invariance of the higher dimensional gravity theory corresponds to invariances of the various components of the metric tensor under certain transformation rules. In general, this can be written as [40]:

$$\delta \hat{x}^M = -\hat{\xi}^M \qquad \delta \hat{G}_{MN} = \hat{\xi}^P \partial_P \hat{G}_{MN} + \hat{G}_{PN} \partial_M \hat{\xi}^P + \hat{G}_{MP} \partial_N \hat{\xi}^P \tag{2.2.6}$$

The KK ansatz won't be preserved by such transformations. Indeed, the general form of transformations leaving the KK ansatz invariant are with $\hat{\xi}^\mu = \xi^\mu(x^\nu)$ and $\hat{\xi}^z = \lambda(x^\mu)$, where the $D$-dimensional coordinates split as $\hat{x}^M = (x^\mu, z)$. These local transformations correspond to the general coordinate invariance of the $d$- dimensional theory and a $U(1)$ local transformation. For instance, the $zz$-component transforms as:

$$\delta \hat{G}_{zz} = \xi^\sigma \partial_\sigma \hat{G}_{zz} \tag{2.2.7}$$

which implies, through the KK ansatz:

$$\delta \phi = \xi^\rho \partial_\rho \phi \tag{2.2.8}$$

This is the transformation of a scalar field, $\phi$, under $d-$dimensional general coordinate transformations parametrized by $\xi^\mu$, while it remains untouched by the $U(1)$ transformation parametrized by $\lambda$. Instead, the $\mu z$-component transforms as:

$$\delta \hat{G}_{\mu z} = \xi^\rho \, \partial_\rho \hat{G}_{\mu z} + \hat{G}_{\rho z} \, \partial_\mu \xi^\rho \implies \delta A_\mu = \xi^\rho \, \partial_\rho A_\mu + A_\rho \, \partial_\mu \xi^\rho + \partial_\mu \lambda \, . \tag{2.2.9}$$

The first two terms represent the transformation of a covector under the diffeomorphism while the third one is the typical transformation of a $U(1)$ gauge field (see (1.1.13)). This tells us that the field $A_\mu$ can be though of as the connection associated with translations along the compact circle. Finally, the $\mu\nu-$component yields:

$$\begin{aligned}\delta\hat{G}_{\mu\nu} =&\hat{\xi}^\rho\,\partial_\rho\hat{G}_{\mu\nu}+\hat{G}_{\rho\nu}\,\partial_\mu\hat{\xi}^\rho+\hat{G}_{\mu\rho}\,\partial_\nu\hat{\xi}^\rho+\hat{G}_{z\nu}\,\partial_\mu\hat{\xi}^z+\hat{G}_{\mu z}\,\partial_\nu\hat{\xi}^z\\ \Longrightarrow \delta g_{\mu\nu} =& \,\xi^\rho\,\partial_\rho g_{\mu\nu}+g_{\rho\nu}\,\partial_\mu\xi^\rho+g_{\mu\rho}\,\partial_\nu\xi^\rho\end{aligned} \tag{2.2.10}$$

This is the standard transformation of the metric tensor components under diffeomorphisms, showing indeed the $d$-dimensional theory possesses general coordinate invariance.

Going back to our original goal, a direct computation of the Ricci tensor and Ricci scalar from the KK ansatz leads to an equation of the kind:

$$\hat{R}\propto R+\tilde{R}+\dots \tag{2.2.11}$$

with possible exponential factors of the constants $\alpha$ and $\beta$ appearing in some combination. $\tilde{R}$ is the Ricci scalar of the extra dimensions and the dots include contributions that are Lorentz scalars built out of covariant derivatives of the radion, or of the gauge field $A_\mu$, namely their kinetic terms will appear in the lower dimensional action. Thus, the constants $\alpha$ and $\beta$ can be fixed by requiring canonical kinetic terms for the field $\phi$ and the standard EH term to appear in the dimensionally reduced action. This last condition can be readily derived by considering powers of the metric tensor entering the action. For simplicity, one can compute this in the vacuum where $A_\mu = 0$ because of Lorentz invariance and consider a $D$-dimensional theory reduced to a $d$-dimensional one by compactifying $n$ of the $D$ dimensions. The line element then reads:

$$\hat{ds}^2 = e^{2\alpha\phi(x)}g_{\mu\nu}(x)dx^\mu dx^\nu + e^{2\beta\phi(x)}\gamma_{mn}(y)dy^m dy^n \tag{2.2.12}$$

where $\gamma_{mn}$ is the metric of the $n$ extra dimensions and $y^m$, $m = 0,\dots,n-1$ the respective coordinates. In matrix form this looks like:

$$\hat{G}_{MN} = \begin{bmatrix} e^{2\alpha\phi(x)}g_{\mu\nu} & 0\\ 0 & e^{2\beta\phi(x)}\gamma_{mn}\end{bmatrix} \tag{2.2.13}$$

The upper left block is $d\times d$, while the lower right block is $n\times n$. The determinant of the metric then will be:

$$\hat{G} = \det G_{MN} = e^{2d\alpha\phi}\,\det g_{\mu\nu}\,e^{2n\beta\phi}\,\det\gamma_{mn} \tag{2.2.14}$$

Thus

$$\sqrt{-\hat{G}} = \sqrt{-g}\sqrt{\gamma}e^{(\alpha d+\beta n)\phi} \tag{2.2.15}$$

Moreover, the relevant contribution from the D-dimensional Ricci in the EH action is proportional to the $d\times d$ component of the inverse metric tensor, that is $e^{-2\alpha\phi}$. Therefore:

$$\sqrt{-\hat{G}}\hat{R}\propto\sqrt{-g}\sqrt{\gamma}\,e^{(\alpha(d-2)+\beta n)\phi}(R+\tilde{R}+\dots), \tag{2.2.16}$$

and the standard EH term in $d$-dimensions is recovered provided that the argument of the exponential vanishes, that is:

$$\alpha(d-2)+\beta n = 0 \tag{2.2.17}$$

In $d = 4$ and a single extra dimension, $n = 1$, and this gives:

$$\alpha = -\frac{\beta}{2} \tag{2.2.18}$$

while in the case of $n = 6$ and $d = 4$, this is $\alpha = -3\beta$. The additional relation between $\alpha$ and $\beta$ requires some computations. We shall assume the field $\phi$ and $g_{\mu\nu}$ to be functions only of the $d$-dimensional coordinates, and $\gamma_{mn}$ to be a function of the extra dimensional coordinates only. This is the zero mode expansion in KK reduction. The Christoffel symbols are defined as:

$$\hat{\Gamma}^P_{MN} = \frac{1}{2}\hat{G}^{PT}(\partial_M \hat{G}_{NT} + \partial_N \hat{G}_{MT} - \partial_T \hat{G}_{MN}) \tag{2.2.19}$$

Since there are 2 possible types of indices and 3 available slots, there are in total 6 types of Christoffel symbols. Direct computations give:

$$\begin{aligned}
\hat{\Gamma}^\mu_{\nu\sigma} &= \Gamma^\mu_{\nu\sigma} + \alpha(\delta^\mu_\sigma \partial_\nu \phi + \delta^\mu_\nu \partial_\sigma \phi - \partial_\lambda \phi g^{\mu\lambda} g_{\sigma\nu}) & \qquad \hat{\Gamma}^m_{nq} &= \tilde{\Gamma}^m_{nq} \\
\hat{\Gamma}^\mu_{mq} &= -\beta\, e^{2\phi(\beta-\alpha)}\partial^\mu \phi\, g_{mq} & \hat{\Gamma}^m_{\mu\nu} &= 0 \\
\hat{\Gamma}^\mu_{\nu q} &= 0 & \hat{\Gamma}^m_{\mu q} &= \beta\, \partial_\mu \phi\, \delta^m_q
\end{aligned}$$

where $\tilde{\Gamma}^m_{nq}$ are the Christoffel symbols associated with the metric $\gamma_{mn}$. Two of these vanish due to the assumptions above. Since with $A_\mu = 0$ the metric is block diagonal, the Ricci scalar receives only two contributions:

$$\hat{R} = \hat{G}^{MN}\hat{R}_{MN} = \hat{G}^{\mu\nu}\hat{R}_{\mu\nu} + \hat{G}^{mn}\hat{R}_{mn} \tag{2.2.20}$$

The summation over all possible indices in the expression for $\hat{R}_{\mu\nu}$ and $\hat{R}_{mn}$ involves products of the Christoffel symbols computed before. Some intermediate steps are reported in Appendix A. In the end, the Ricci scalar is found to be:

$$\hat{R} = e^{-2\alpha\phi}R(g_{\mu\nu}) + e^{-2\beta\phi}\tilde{R}(\gamma_{mn}) - e^{-2\alpha\phi}\nabla^2\phi\,[2\alpha(d-1) + 2n\beta] + e^{-2\alpha\phi}K(\alpha,\beta)(\partial\phi)^2$$

where

$$K(\alpha,\beta) \equiv n\alpha\beta(2-d) - n^2\beta^2 + (d-2)(1-d)\alpha^2 + (2-d)n\alpha\beta - n\beta^2 \tag{2.2.21}$$

and with $R(g_{\mu\nu})$ and $\tilde{R}(\gamma_{mn})$ the Ricci scalars for the metrics $g_{\mu\nu}$ and $\gamma_{mn}$, respectively. Canonical kinetic terms for $\phi$ in the $d$-dimensional theory are obtained by requiring $K(\alpha,\beta) = -\frac{1}{2}$:

$$2(d-2)n\alpha\beta + n\beta^2(n+1) + (d-2)(d-1)\alpha^2 = \frac{1}{2} \tag{2.2.22}$$

which, using the Einstein frame condition (2.2.17) to write $\alpha$ in terms of $\beta$, yields:

$$\beta^2 = \frac{d-2}{2n(d+n-2)} \tag{2.2.23}$$

Therefore, we also have:

$$\alpha^2 = \frac{n}{2(d+n-2)(d-2)} \tag{2.2.24}$$

In the particular case relevant for superstring theory, $d = 4$ and $n = 6$, the above equations yield:

$$\alpha = -\frac{\sqrt{3}}{4} \qquad\qquad \beta = \frac{1}{4\sqrt{3}} \tag{2.2.25}$$

Instead, for $n = 1$ and general $d$, these reduce to:

$$\alpha^2 = \frac{1}{2(d-1)(d-2)} \qquad\qquad \beta^2 = \frac{d-2}{2(d-1)} \tag{2.2.26}$$

which agree with the results of [40]. The relation between the higher and lower dimensional Ricci scalars considering also the contribution of the determinant of the metric is:

$$\sqrt{-\hat{G}}\,\hat{R} = \sqrt{-g}\sqrt{\gamma}\left[R_4 + e^{-2\phi/\sqrt{3}}\tilde{R}_6 - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{\sqrt{3}}{2}\nabla^2\phi\right] \tag{2.2.27}$$

where we have considered $d = 4$, $n = 6$. Considering the $(D = 10)$-dimensional gravity action, the last term in the previous equation is a total derivative and therefore can be neglected, giving:

$$S = \frac{M_{P,10}^8}{2}\int \mathrm{d}^{10}x\sqrt{-\hat{G}}\,\hat{R} = \frac{M_{P,10}^8}{2}\int \mathrm{d}^6y\sqrt{\gamma}\int \mathrm{d}^4x\sqrt{-g}\,\left[R_4 - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + e^{-2\phi/\sqrt{3}}\tilde{R}_6\right]$$

Moreover, assuming the extra dimensions to be a Ricci flat manifold as in the case of CY, $\tilde{R}_{mn} = 0 \implies \tilde{R}_6 = 0$, thus:

$$S = \frac{M_{P,10}^8}{2}\int \mathrm{d}^{10}x\sqrt{-\hat{G}}\,\hat{R} = \mathrm{Vol}(Y_6)\frac{M_{P,10}^8}{2}\int \mathrm{d}^4x\sqrt{-g}\,\left[R_4 - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi\right] \tag{2.2.28}$$

where $\mathrm{Vol}(Y_6) = \int \mathrm{d}^6y\sqrt{\gamma}$ since the metric has Euclidean signature. In string theory, this is usually written as $\mathrm{Vol}(Y_6) = \mathcal{V}l_s^6 = \mathcal{V}\left(2\pi\sqrt{\alpha'}\right)^6$. At this point, the coefficient in front of the integral can be set equal to the 4-dimensional Planck mass squared, as the standard EH term. This results in a relation between the higher and lower dimensional Planck scales:

$$S = \frac{M_{P,4}^2}{2}\int \mathrm{d}^4x\sqrt{-g}\,\left[R_4 - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi\right] \qquad M_{P,4}^2 = (2\pi)^6(\alpha')^3\mathcal{V}M_{P,10}^8 \tag{2.2.29}$$

Notice that in the case of non Ricci-flat compactifications, $\tilde{R}_6$ contributes as a potential energy exponential in the scalar field $\phi$: positive internal curvature contributes with a negative potential, $V \propto -e^{-2\phi/\sqrt{3}}$, leading to smaller compactification volume. Instead, a negative internal curvature gives a positive potential term $V \propto +e^{-2\phi/\sqrt{3}}$, undermining the stability of the compactification [36]. Remarkably, we recognize the derivatives of $\phi$ as standard kinetic terms for a four-dimensional scalar field without a scalar potential. This is an instance of a modulus, which corresponds to massless deformations of the $10D$-solution. In string theory, this kinetic term will be derived from a particular Kähler potential entering the action (2.1.23).

So far these calculations assumed zero background gauge fields, $A_\mu = 0$, but clearly this isn't the most general setting. Indeed one can lift this assumption and recover the standard Maxwell Lagrangian $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ as additional term to (2.2.27). The strategy to recover this result is the same as before, only now the Christoffel symbols will change because the metric will contain the gauge field. For simplicity, let us consider the case of a single extra dimension, $n = 1$, compactified on a $S^1$ and assume the field $\phi$ to be constant, not dependent on the coordinates of the non-compact dimensions. The goal is to reduce a $(D = d+1)$-dimensional gravity theory to a $d$-dimensional one starting from the metric ansatz:

$$\hat{G}_{MN} = \begin{bmatrix} e^{2\alpha\phi}g_{\mu\nu} + e^{2\beta\phi}A_\mu A_\nu & e^{2\beta\phi}A_\mu \\ e^{2\beta\phi(x)}A_\nu & e^{2\beta\phi} \end{bmatrix} = \begin{bmatrix} \hat{G}_{\mu\nu} & \hat{G}_{z\mu} \\ \hat{G}_{\nu z} & \hat{G}_{zz} \end{bmatrix} \tag{2.2.30}$$

This contains a $d \times d$ -block, a $1 \times 1$-block, and two $d \times 1$ blocks, where $z$ is the index referring to the single extra dimension. The inverse metric is:

$$\hat{G}^{MN} = \begin{bmatrix} e^{-2\alpha\phi}g^{\mu\nu} & -e^{-2\alpha\phi}A^\mu \\ -e^{-2\alpha\phi}A^\nu & e^{-2\beta\phi} + e^{-2\alpha\phi}A_\rho A_\rho \end{bmatrix} = \begin{bmatrix} \hat{G}^{\mu\nu} & \hat{G}^{\mu z} \\ \hat{G}^{z\nu} & \hat{G}^{zz} \end{bmatrix} \tag{2.2.31}$$

The Christoffel symbols associated with this metric to quadratic order in $A_\mu$ are given in Appendix A. Contrary to the previous case, now there are three contributions to the Ricci tensor since the metric is not diagonal and therefore $\hat{R}_{\mu z}$ is non zero:

$$\hat{R} = \hat{G}^{MN}\hat{R}_{MN} = \hat{G}^{\mu\nu}\hat{R}_{\mu\nu} + \hat{G}^{zz}\hat{R}_{zz} + 2\hat{G}^{\mu z}\hat{R}_{\mu z} \tag{2.2.32}$$

We report here the final result and refer to Appendix A for the detailed derivation:

$$\hat{R} = e^{-2\alpha\phi}\left[R - \frac{1}{4}e^{2\phi(\beta-\alpha)}F_{\mu\nu}F^{\mu\nu}\right] + \mathcal{O}(A^3) \tag{2.2.33}$$

Considering also the contribution from the determinant of the metric tensor, in Einstein frame we have:

$$\sqrt{-\hat{G}}\hat{R} = \sqrt{-g}\left[R - \frac{1}{4}e^{-2\phi\alpha(d-1)}F_{\mu\nu}F^{\mu\nu}\right] \tag{2.2.34}$$

Now, we can add the kinetic terms for the field $\phi$ using the solutions for $\alpha$ and $\beta$ previously found specifying $n = 1$, arriving at the final identity:

$$\sqrt{-\hat{G}}\hat{R} = \sqrt{-g}\left[R - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{4}e^{-2\phi\alpha(d-1)}F_{\mu\nu}F^{\mu\nu}\right] \tag{2.2.35}$$

In the $d = 4$ case, this becomes:

$$\sqrt{-\hat{G}}\hat{R} = \sqrt{-g}\left[R - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{4}e^{-6\phi\alpha}F_{\mu\nu}F^{\mu\nu}\right] \tag{2.2.36}$$

with

$$\alpha = -(2\sqrt{3})^{-1}, \qquad \beta = 1/\sqrt{3}. \tag{2.2.37}$$

With the appropriate Planck mass term in front, this would be a Lagrangian describing the Einstein-Maxwell system if $\phi$ could be set to zero or constant. The general formulae found here agree with those in [40] for the case $n = 1$ and general $d$. In other treatments, one may use a different parameterization of the metric tensor and obtain the following dimensionally reduced action from five-dimensional gravity [42]:

$$S_{\mathrm{EH}} = \frac{2\pi\sqrt{\alpha'}\,\sigma_0^{1/3}M_{P,5}^3}{2}\int d^4x\sqrt{-g}\left(R_4 - \frac{1}{4}\sigma F_{\mu\nu}F^{\mu\nu} - \frac{1}{6}\frac{\partial_\mu\sigma\,\partial^\mu\sigma}{\sigma^2}\right) + \dots \tag{2.2.38}$$

where $\sigma$ parametrizes the geometrical modulus encoded in exponential form above. Indeed, requiring canonical kinetic terms for $\sigma$ leads to:

$$\frac{1}{6}\frac{(\partial_\mu\sigma)^2}{\sigma^2} = \frac{1}{2}(\partial_\mu\phi)^2 \implies \phi = \frac{1}{\sqrt{3}}\ln\sigma \tag{2.2.39}$$

This is precisely what one finds using the above derivation. Indeed, setting the $F_{\mu\nu}F^{\mu\nu}$ terms equal yields:

$$\sigma = e^{\sqrt{3}\phi} = e^{-6\alpha\phi} \implies \alpha = -\frac{1}{2\sqrt{3}} \tag{2.2.40}$$

which is exactly the value for $\alpha$ in (2.2.37). Once again, it is important to stress that the four-dimensional Planck scale in Einstein frame is not a fundamental input of the higher-dimensional theory, but a derived quantity fixed by the background value of the compactification volume, or equivalently by the VEV of the corresponding geometric modulus. In a similar fashion, dimensional reduction of the full superstring theory action on a Calabi-Yau manifold leads to several four-dimensional scalar fields, as well as vector and tensor fields, whose properties are determined by the geometry of the internal space.

As a final note, it is important to note the logarithmic relation between the canonically normalized scalar field and the geometrical modulus, $\phi \propto \ln\sigma$. This relation will play a central role in later chapters. If the modulus evolves in time during a cosmological epoch, then the compactification volume, and therefore any string tension depending on it, acquires a time dependence. This will be the origin of the time dependence of the tension of cosmic superstrings, which will be further discussed in the next chapter.

## 2.3 Type IIB compactifications and no-scale structure

Compactification of Type IIB superstring theory on a Calabi-Yau manifold is clearly more involved than the examples explored in the previous section. However, the main ideas are carried over. Here, I shall review the basic geometrical ingredients relevant for the discussion of the resulting $4D$ spacetime theory which is described by a particular Kähler potential and superpotential. Further details on the geometry and the complete derivation of the $\mathcal{N}=1$ effective action can be found in [43] and [44], respectively.

The first step is to understand the field content resulting from a CY compactification with $\mathcal{N}=2$ supersymmetry. To do so, let us begin by considering the $10D$ metric written as:

$$G_{MN}dX^M dX^N = \eta_{\mu\nu}dx^\mu dx^\nu + g_{mn}dy^m dy^n, \tag{2.3.1}$$

with $g_{mn}$ the Ricci flat metric of the ED compact space $Y_6$. Compactifications of Type IIB on this background yield a $4D\,\mathcal{N}=2$ supergravity theory. The geometric moduli fields are associated with two types of metric deformation that preserve the overall CY condition:

- *Kähler moduli*: these are associated with deformations of the Kähler form $J = ig_{i\bar{j}}\, dz^i \wedge d\bar{z}^{\bar{j}}$, where $z^i, \bar{z}^{\bar{j}}$, $i, \bar{j} = 1, \dots, 3$ are the complex coordinates on $Y_6$. This form controls the area of two-cycles in the CY. It can be written in terms of components by introducing a set of harmonic $(1,1)$-forms $\omega^I$, with $I = 1, \dots, h^{1,1}$ as:
$$J = t^I(x)\omega_I \tag{2.3.2}$$
The $t^I(x)$ constitute $h^{1,1}$ four-dimensional scalar fields. Physically, these correspond to deformation in the size of $Y_6$. For example, the volume $\mathcal{V}$ is a function of Kähler moduli.

- *Complex structure moduli*: these are associated with deformations of the complex structure on $Y_6$. In a similar way, one can introduce a basis of harmonic $(1,2)$-forms $\chi^A$, $A = 1, \dots, h^{1,2}$, to write these deformations:
$$\delta g_{ij} = \frac{i}{\|\Omega\|^2}\, \zeta^A(x)\, (\chi_A)_{i\bar{j}\bar{k}}\, \Omega^{\bar{j}\bar{k}}{}_j. \tag{2.3.3}$$
where $\Omega$ is the holomorphic $(3,0)$ form of $Y_6$. The $\zeta^A(x)$ constitute $h^{1,2}$ four-dimensional scalar fields, the complex structure moduli. Physically, these describe deformation in the shape of $Y_6$.

By expanding the various $p$-forms arising from the NS-NS and R-R sectors, one has additional scalars. Overall, these can be packaged into multiplets of $4D$, $\mathcal{N}=2$ supersymmetry. Namely, one has $4 \times h^{1,1}$ scalars furnishing the bosonic content of $h^{1,1}$ hypermultiplets, while the $h^{1,2}$ complex structure moduli are found in $\mathcal{N}=2$ vector multiplets. There is also the universal hypermultiplet containing the dilaton, axion $C_0$ and additional axionic degrees of freedom descending from $B_2$ and $C_2$ [36].

The reduction to $\mathcal{N}=1$ supersymmetry requires one to perform an orientifold involution. An orientifold action is a symmetry that includes the worldsheet orientation reversal $\Omega_{ws}$ and basically identifies points in the ED space resulting in a reduction of the degree of freedom. The loci of points left invariant by such geometrical transformation are denoted as *orientifold planes*. These are non-dynamical extended objects with negative tension. Interestingly, these are needed for internal consistency of the theory. Indeed, the ED space is compact and therefore the total charge must vanish. In Type IIB there exists positive tension objects known as $Dp$-branes, which have a positive charge. Although antibranes could exists, these manifestly break spacetime supersymmetry. Therefore, orientifold planes are preferred. In type IIB, these have three or seven spatial dimensions, hence are denoted as $O3/O7$-planes. Under this action, the bases used to decompose the Kähler form and the complex structure form split into bases for even and odd eigenspaces: $\omega^I \to (\omega^i, \tilde{\omega}^\alpha)$ where $i = 1, \dots, h^{1,1}_+$ and $\alpha = 1, \dots, h^{1,1}_-$. The resulting orientifolded Kähler form can be written as:
$$J = t^i(x)\omega_i \tag{2.3.4}$$
where the $t^i(x)$ constitute $h^{1,1}_+$ four-dimensional real scalar fields. Similarly, the complex structure moduli invariant under the orientifold are $\zeta^a$ with $a = 1, \dots, h^{1,2}_-$. As before, it

is useful to assemble the resulting four-dimensional scalar fields into irreducible representations of $\mathcal{N} = 1$ supersymmetry. The details of such procedure lie outside the scope of this thesis so I will present only the final result: the two form scalars descending from $B_2$ and $C_2$ are $b^\alpha(x)$ and $c^\alpha(x)$, respectively, while the scalars associated with the four-form $C_4$ are $\theta^i(x)$. The good Kähler coordinates are schematically given by

$$T_i = \tau_i + i\theta_i \tag{2.3.5}$$

known as the complexified four cycles volumes as well as a combination of the axiodilaton $\tau$ (see (2.1.13)) and two-form scalars

$$G_\alpha \equiv c_\alpha - \tau b_\alpha. \tag{2.3.6}$$

The bottom line is that a compactification of Type IIB superstring theory on CY orientifolds leads to a total of $h^{1,1}_+ + h^{1,1}_- + h^{1,2}_- + 1 = h^{1,1} + h^{1,2}_- + 1$ complex scalar fields in the four-dimensional effective theory.

These scalar fields are massless and therefore can lead to important cosmological and phenomenological problems that string theory has to face. Namely, the quantum fluctuations of moduli fields during inflation could contribute to the primordial density perturbations [36], they could spoil the energy density of the universe, overclose it, and decay to some dark radiation, to name a few scenarios. It is thus a major challenge in the field of string cosmology to realize a concrete model in which all moduli fields are stabilized, meaning they have a positive mass squared. This is the problem of *moduli stabilization*. In this context, the best understood string theory solutions are Imaginary Self-Dual (ISD) [30]. In these backgrounds, one allows non-trivial three-form fluxes $G_3$ wrapping cycles of the internal CY space, together with localized sources such as $D$-branes and $O$-planes. The ten-dimensional equations of motion and the five-form Bianchi identity impose non-trivial consistency conditions on these ingredients. In particular, for supersymmetric localized sources, the compactification admits solutions in which the three-form flux is imaginary self-dual, $\star_6 G_3 = iG_3$, and the warp factor is related to the four-form potential. The detailed ten-dimensional derivation will not be reviewed here since the important consequence for the present discussion is that such fluxes generate a scalar potential for the axiodilaton and complex structure moduli. Indeed, the ten-dimensional action of Type IIB (2.1.8) contains the following term:

$$V_{\text{flux}} = \frac{1}{2\kappa^2} \int d^{10}X \sqrt{-G_E} \left[ -\frac{|G_3|^2}{2\,\text{Im}(\tau)} \right] \tag{2.3.7}$$

which includes the axiodilaton through the denominator and the definition of $G_3$ ((2.1.13)) as well as complex structure moduli through metric contractions. Due to the absence of any derivatives, this can be interpreted as a scalar potential for these fields. Thus, at the classical level, $\tau$ and $\zeta^a$ receive masses at leading order in $\alpha'$ for suitable choices of fluxes.

Furthermore, the geometrical data of the four-dimensional effective theory of an ISD compactification can be derived from a specific Kähler potential and superpotential of $\mathcal{N} = 1$ supergravity. At leading order in $\alpha'$ and $g_s$ one has:

$$K_0 = -2\ln\mathcal{V} - \ln[-i(\tau - \bar{\tau})] - \ln\left(-i\int \Omega \wedge \bar{\Omega}\right) \tag{2.3.8}$$

where the volume $\mathcal{V}$ and $\Omega$ depend on the Kähler moduli $T_i$ and complex structure moduli $\zeta^a$, respectively. Instead, the correct superpotential is known as the Gukov-Vafa-Witten (GVW) superpotential [45]:

$$W_0 = \frac{c}{\alpha'} \int G_3 \wedge \Omega\,, \tag{2.3.9}$$

where $c$ is a constant. As mentioned before, the superpotential encodes the mass terms for supersymmetric theories and, given the dependence of $G_3$ on $\tau$ and of $\Omega$ on $\zeta_a$, it follows that these develop a non-trivial scalar potential. In particular, this potential has the typical structure of $\mathcal{N} = 1$ supergravity as in (2.1.24):

$$V_F = e^{K_0} \left[ K_0^{I\bar{J}} D_I W_0 \overline{D_J W_0} - 3|W_0|^2 \right] \tag{2.3.10}$$

with $I, J$ taking values over all moduli fields $T_i, \zeta^a, G_\alpha$ and $\tau$. This scalar potential has very interesting features which descend from the *no-scale* property of (2.3.8). To see what this is and implies, consider the case of a single Kähler modulus denoted as $T$, where one can write the Kähler potential as:

$$K_0 = -3 \log(T + \overline{T}). \tag{2.3.11}$$

provided that the CY volume scales as $\mathcal{V} \sim (T + \overline{T})^{3/2}$. The computation of $V_F$ requires one to evaluate the derivatives:

$$K_T = \partial_T K = -\frac{3}{T + \overline{T}} = \partial_{\overline{T}} K \qquad\qquad K_{T\overline{T}} = \frac{3}{(T + \overline{T})^2} = K_{\overline{T}T} \tag{2.3.12}$$

and, since there is a single Kähler modulus, the inverse Kähler metric is simply

$$K^{T\overline{T}} = \frac{(T + \overline{T})^2}{3} = K^{\overline{T}T} \tag{2.3.13}$$

Therefore, the Kähler covariant derivative is:

$$D_T W_0 = \partial_T W_0 + K_T W_0 = K_T W_0 = \frac{-3W_0}{T + \overline{T}} \tag{2.3.14}$$

where $\partial_T W_0 = 0$ since the GVW superpotential is independent of the Kähler moduli, modulo non-perturbative corrections. Therefore:

$$\begin{aligned} V &= e^{K_0} \left[ K_0^{i\bar{j}} D_i W_0 D_{\bar{j}} \overline{W}_0 - 3 \left| W_0^2 \right| \right] = e^{K_0} \left[ K_0^{T\bar{T}} D_T W_0 D_{\bar{T}} \overline{W_0} - 3 \left| W_0 \right|^2 \right] = \\ &= \frac{1}{(T + \bar{T})^3} \left[ 3 \frac{(T + \bar{T})^2}{(T + \bar{T})^2} \left| W_0 \right|^2 - 3 \left| W_0 \right|^2 \right] = 0 \end{aligned}$$

This is a consequence of both $W_0$ being independent of $T$ and the particular form of the Kähler potential which satisfies:

$$K^{T\bar{T}} K_T K_{\bar{T}} = 3 \tag{2.3.15}$$

which is called *no-scale* property. In general, this can be written as:

$$\sum_{I,J=T_i} K_0^{I\bar{J}} \partial_I K_0 \partial_{\bar{J}} K_0 = 3 \tag{2.3.16}$$

for a generic number of Kähler moduli $T_i$. The implications of this are large [31]. First of all, this tree level Kähler potential cancels off the negative contribution to the scalar potential, leaving:

$$V_{no\text{-}scale} = e^{K_{tree}} \left[ K^{M\bar{N}} D_M W_{tree} D_{\bar{N}} \overline{W}_{tree} \right] \tag{2.3.17}$$

with $M, N$ running over the $h^{2,1}$ complex structure moduli and one dilaton. Therefore, it is semipositive definite. The value of the potential at the minimum is zero, corresponding to a zero cosmological constant. Moreover, supersymmetry breaking occurs when $D_M W \neq 0$ along the given moduli directions. This potential has $D_M W_{tree} = 0$ for complex structure moduli and dilaton, meaning supersymmetry is preserved in those directions. Instead, $D_{T_i} W_{tree} \neq 0$ for the Kähler moduli. Therefore, supersymmetry is broken in these directions. Thus, the potential features a minimum with zero cosmological constant and broken supersymmetry. However, this also means that at tree level the Kähler moduli are not stabilized, since they do not appear in any potential. Therefore, it is necessary to consider possible perturbative corrections to $K$ or non perturbative corrections to $W$, which is precisely what the KKLT and LVS moduli stabilization approaches do.

## 2.4 Moduli stabilization: KKLT and LVS

The aforementioned no-scale structure of the Kähler potential is a tree-level property of Type IIB compactifications and, albeit useful to obtain a broken supersymmetric vacuum with zero cosmological constant, it leaves unstabilized all the Kähler moduli. This is not acceptable if one wishes to derive realistic consequences from such string theoretic models. Therefore, perturbative and/or non-perturbative corrections must be important and can break the no-scale structure of the effective actions, thereby destabilizing the moduli and alter the vacuum energy. Indeed, supersymmetric renormalization theorems prove the Kähler potential is corrected order by order in perturbation theory, while the superpotential receives only non-perturbative corrections. The most famous perturbative correction to the Kähler potential comes from $(\alpha')^3$ curvature corrections to the ten-dimensional action. In $4D$ this is [36]:

$$K = -2 \ln \left[ \mathcal{V} + \frac{\xi}{2 g_s^{3/2}} \right] \qquad\qquad \xi \equiv - \frac{\chi(Y_6)\zeta(3)}{2(2\pi)^3} \tag{2.4.1}$$

where $\chi(Y_6)$ is the Euler characteristic of the compact $6D$ space. Unless $\chi = 0$, this Kähler potential does not satisfy the no-scale property. Other types of corrections, such as $g_s$ corrections from spacetime loops, can occur but these are not relevant for the present discussion. Instead, because of holomorphicity and shift symmetry, the superpotential receives no $\alpha'$ corrections. The reasoning is the following: the imaginary part $\theta_i$ of the Kähler moduli in (2.3.5) enjoys a shift symmetry, $\theta_i \to \theta_i + const$, to all orders in perturbation theory as a result of higher dimensional gauge invariance. Since the superpotential must be holomorphic, it cannot depend on any combination of the kind $T_i + \overline{T}_i$, but only on polynomials in $T_i$. However, any polynomial form in $T_i$ is not invariant under the shift symmetry and therefore one concludes the superpotential cannot depend on the Kähler moduli at the perturbative level, perhaps only non-perturbatively. Moreover, since $\alpha'$ corrections change the magnitude of $T_i$ but $W$ is independent of them, $W$ does not receive any perturbative $\alpha'$ correction.

Non-perturbative corrections to $W$ come mainly from two effects:

- Gaugino condensation: compactifications in which a stack of $N_c$ $D7$-branes wraps a four-cycle $\Sigma_4$ satisfying certain topological properties yield a four-dimensional effective theory which is an $SU(N_c)$ $\mathcal{N}=1$ Yang-Mills theory. At low energies, a non-perturbative superpotential is generated from gaugino condensation:

$$W_{gaugino} = A\, e^{-aT} \tag{2.4.2}$$

  where $a = 2\pi/N_c$, $T$ is related to the volume $\mathcal{V}_4$ of the four-cycle $\Sigma_4$, and $A$ is independent of the Kähler moduli but can depend on complex structure moduli as well as the axiodilaton.

- Euclidean $D3$-branes instantons: wrapping the four-cycle with Euclidean $D3$-branes instead of spacetime-filling $D7$-branes leads to a superpotential of the form:

$$W_{ED3} = Ae^{-aT} \tag{2.4.3}$$

  where $a = 2\pi$ and $A$ depends on the moduli as the previous case.

Overall, the general non-perturbative corrections to $W$ can be written as:

$$W_{np} = \sum_i A_i e^{-a_i T_i} \tag{2.4.4}$$

Moduli stabilization is a tough problem because as we have shown before, the tree level results for $K$ and $W$ do not stabilize the Kähler moduli and one is forced to compute corrections to these. The problem is that doing so is hard because when corrections are important, they cannot be computed, and when they are computable, they are not important [46]. This represents the *Dine-Seiberg problem* [47]. The main conclusion one derives from theoretical understanding of the possible corrections is that string theory vacua arise as a competition among terms at different perturbative order or between perturbative and non perturbative corrections. Indeed, the KKLT scenario [48] analyses the competition between the classical tree-level superpotential (2.3.9) and the non-perturbative one (2.4.4). Instead, the LVS construction compares the leading $\alpha'$ correction to $K$ and the non-perturbative correction to $W$. Under particular circumstances one can argue the higher order corrections are negligible and therefore trust the results. I will discuss only the LVS scenario since it is one of most compelling setting for Type IIB and also because it directly leads to an exponential potential for the volume modulus, which will lead to important consequences for the cosmic string tension.

### 2.4.1 The Large Volume Scenario

In this section, I will describe the basic ingredients and steps presented in [38] to realize a moduli stabilization scenario in the large volume limit of the Calabi-Yau manifold. The reader should keep in mind the motivation for this entire section: it should serve as a foundation for the time-dependence of cosmic string tension. This will depend heavily on the final form of the potential recovered in the LVS and therefore it is reasonable to dedicate a few pages to its description.

The setup involves the Kähler potential with the $\alpha'$ correction and the non-perturbative corrected superpotential outlined in the previous section. The tree level superpotential involves the complex structure moduli and axiodilaton which develop a non-trivial mass, $m_{flux} \sim \alpha'/R^3$, with $R$ the radius of $Y_6$ [48]. Then, one integrates out these massive fields to focus on the effective field theory for the Kähler moduli which will have a mass $m \ll m_{flux}$. After this, the Kähler potential and superpotential become [49]:

$$K = K_{\text{cs}} - 2\ln\left[\mathcal{V} + \frac{\xi}{2g_s^{3/2}}\right] \tag{2.4.5}$$

$$W = W_0 + \sum_n A_n e^{-a_n T_n}. \tag{2.4.6}$$

with $K_{cs}$ the part of $K$ containing the complex structure moduli and axiodilaton. Substituting these expressions in (2.1.24) leads to the following scalar potential [38]:

$$\begin{aligned} V = e^K \Bigg[ & G^{T_j \bar{T}_k}\left(a_j a_k A_j \bar{A}_k e^{-a_j T_j - a_k \bar{T}_k} - a_j A_j e^{-a_j T_j} \overline{W}\, \partial_{\bar{T}_k} K - a_k \bar{A}_k e^{-a_k \bar{T}_k} W \partial_{T_j} K\right) \\ & + 3\xi \frac{(\xi^2 + 7\xi\mathcal{V} + \mathcal{V}^2)}{(\mathcal{V} - \xi)(2\mathcal{V} + \xi)^2} |W|^2 \Bigg] \equiv V_{\text{np1}} + V_{\text{np2}} + V_{\alpha'} \end{aligned} \tag{2.4.7}$$

By requiring small values of $W_0$, usually $W_0 \ll 10^{-4}$, a class of minima can be found and corresponds to the KKLT [48] solutions. This allows the non-perturbative contribution to the superpotential to balance the flux term and stabilize the Kähler modulus in a controlled regime. Such vacua correspond to a specific region of the flux landscape, and their existence must be checked in explicit constructions. However, small values of $W_0$ are not the most common in the flux landscape. Therefore, one should search for large volume minima with larger values of $W_0$. Indeed, this is what the LVS does. It is useful to present some relations between the overall CY volume $\mathcal{V}$ and the Kähler moduli:

$$\tau_i = \partial_{t_i}\mathcal{V} = \frac{1}{2}k_{ijk}t^j t^k \qquad\qquad \mathcal{V} = \int_{Y_6} J^3 = \frac{1}{6}k_{ijk}t^i t^j t^k \tag{2.4.8}$$

Note that because of the relation between the $\tau_i$ and $t_i$, the volume will depend on the real part of the complexified four-cycle moduli $\tau_i = \operatorname{Re} T_i$. In addition, in the large volume limit one has:

$$e^K \sim \frac{e^{K_{cs}}}{\mathcal{V}^2} + \mathcal{O}\left(\frac{1}{\mathcal{V}^3}\right) \tag{2.4.9}$$

which is essential to evaluate each of the three terms appearing in the potential in this limit. The $\alpha'$ correction is the simplest to analyze and it leads to:

$$V_{\alpha'} \sim +\frac{3\xi}{16\mathcal{V}^3} e^{K_{\text{cs}}} |W|^2 + \mathcal{O}\left(\frac{1}{\mathcal{V}^4}\right) \tag{2.4.10}$$

which is always positive and depends only on the overall volume. The situation is not as simple for $V_{np1}$ and $V_{np2}$ since these depend on the Kähler moduli explicitly. The decompactification limit $\mathcal{V} \to \infty$, is taken in the region of moduli space where all $\tau_i \to \infty$

except one, denoted by $\tau_s$. This limit has to be well defined and this $\tau_s$ must appear in the non-perturbative correction to $W$. In this regime then one can recover:

$$V_{\rm np1} \sim \frac{(-k_{ssk}t^k)a_s^2|A_s|^2 e^{-2a_s\tau_s}e^{K_{\rm cs}}}{\mathcal{V}} + \mathcal{O}\left(\frac{e^{-2a_s\tau_s}}{\mathcal{V}^2}\right) \tag{2.4.11}$$

In a similar way, using some technicalities which can be found in the original LVS paper [38], one finds the following expression for $V_{np2}$:

$$V_{\rm np2} \sim -\frac{a_s\tau_s e^{-a_s\tau_s}}{\mathcal{V}^2}|A_sW_0|e^{K_{\rm cs}} + \mathcal{O}\left(\frac{e^{-a_s\tau_s}}{\mathcal{V}^3}\right). \tag{2.4.12}$$

Combining these three expression yields the total potential:

$$V \sim \left[\frac{1}{\mathcal{V}}a_s^2|A_s|^2(-k_{ssk}t^k)e^{-2a_s\tau_s} - \frac{1}{\mathcal{V}^2}a_s\tau_s e^{-a_s\tau_s}|A_sW| + \frac{\xi}{\mathcal{V}^3}|W|^2\right] \tag{2.4.13}$$

where factors of $e^{K_{cs}}$ are absorbed into $A$ and $W$. Furthermore, there exists a large volume limit in which the potential approaches zero from below. This is given by:

$$\mathcal{V}\to\infty \qquad \text{with} \qquad a_s\tau_s = \ln\mathcal{V} \tag{2.4.14}$$

where the potential becomes:

$$V \sim \left[a_s^2A_s^2\frac{(-k_{ssj}t^j)}{\mathcal{V}^3} - |A_sW_0|\frac{(a_sk_{sjk}t^jt^k)}{\mathcal{V}^3} + \frac{\xi}{\mathcal{V}^3}|W_0|^2\right] + \mathcal{O}\left(\frac{1}{\mathcal{V}^4}\right) \tag{2.4.15}$$

where $W \to W_0$ since in this limit the non-perturbative corrections are subleading. The important question then regards which one of these terms is dominant in the large volume limit, as all the terms naively scale like $1/\mathcal{V}^3$. However, the numerator of the second term is proportional to $\tau_s$, thus overall it scales like $\ln(\mathcal{V})/\mathcal{V}^3$. Instead, the other terms schematically behave as $(t_i)^2$ or $t_i$ and, since $t^kt^k \sim \ln\mathcal{V}$, it follows that $t^i \sim \sqrt{\ln\mathcal{V}}$. Hence, the first and third term schematically scale as $\sqrt{\ln\mathcal{V}}/\mathcal{V}^3$ and $1/\mathcal{V}^3$, respectively. Therefore, in the large volume limit with $a_s\tau_s = \ln\mathcal{V}$ the second term dominates and the potential is approximately:

$$V \sim -e^{K_{\rm cs}}|A_sW_0|\frac{\ln\mathcal{V}}{\mathcal{V}^3} \tag{2.4.16}$$

which approaches zero from below. Given that at smaller volumes the other terms dominate and both are positive, the potential must have a local Anti-de Sitter (AdS) minimum along the direction in Kähler moduli space where the volume changes. It can be argued that this is a minimum in the other directions of moduli space so that the full potential does exhibit an AdS minimum with non-vanishing F-terms, therefore supersymmetry is broken. As usual, it is then necessary to perform an *uplift* to de Sitter (dS) spacetime, to be able to derive some interesting cosmology from such construction.

The important point for the present thesis is that the relevant dynamical variable is not directly the geometrical volume $\mathcal{V}$, but its canonically normalized scalar field appearing in the four-dimensional effective action. As already anticipated in the dimensional reduction section 2.2, the canonically normalized field has a logarithmic dependence on the geometrical modulus. Indeed, the four-dimensional supergravity action involves the following kinetic terms for a single Kähler modulus $T = \tau + i\theta$:

$$\mathcal{L}_{\rm kin} = -K_{T\bar{T}}\partial_\mu T\partial^\mu\bar{T} = -\frac{3}{4\tau^2}\partial_\mu\tau\,\partial^\mu\tau. \tag{2.4.17}$$

where we have fixed the axionic component $\theta$. The canonically normalized volume modulus $\Phi$ is then found by requiring:

$$\frac{1}{2}(\partial\Phi)^2 = \frac{3}{4\tau^2}(\partial\tau)^2 \qquad \Longrightarrow \qquad \Phi = \sqrt{\frac{3}{2}}M_P \ln\tau \tag{2.4.18}$$

Furthermore, geometrical constructions supporting LVS, the so-called *swiss-cheese* manifolds, typically have the property $\mathcal{V} \sim \tau^{3/2}$ for large volumes. Thus:

$$\Phi = M_P\sqrt{\frac{2}{3}}\ln\mathcal{V} \qquad \Longrightarrow \qquad \mathcal{V} \simeq e^{\sqrt{3/2}\,\Phi/M_P}. \tag{2.4.19}$$

This logarithmic canonical normalization is of crucial importance for the link between the LVS potential and the time evolution of the cosmic string tension. As a result, inverse powers of the compactification volume become exponential functions of the canonical field. For example:

$$\mathcal{V}^{-3} = \exp\left(-3\sqrt{\frac{3}{2}}\frac{\Phi}{M_P}\right) = \exp\left(-\sqrt{\frac{27}{2}}\frac{\Phi}{M_P}\right). \tag{2.4.20}$$

Thus the leading large-volume behaviour of the LVS potential can be approximated, along the runaway direction and away from the minimum, by a steep exponential potential

$$V(\Phi) \simeq V_0 \exp\left(-\lambda\frac{\Phi}{M_{\mathrm{P}}}\right), \qquad \lambda = \sqrt{\frac{27}{2}}. \tag{2.4.21}$$

The main point is that the LVS provides a concrete string-theoretic origin for exponential-type potentials for geometric moduli. Since the effective tension of four-dimensional cosmic superstrings can depend on the compactification volume, which is the case for $F$-strings $\mu = \mu(\mathcal{V})$, the cosmological evolution of the volume modulus induces a time dependence of the effective four-dimensional string tension. The next chapter will therefore study how rolling scalar fields with exponential potentials can lead to power-law tension evolution of the form $\mu(t) \propto t^{-m}$.

# Chapter 3

# Rolling moduli and time-dependent tension

In this chapter, I shall take the final result of the previous one to motivate the study of exponential potentials in cosmology. For this reason, I shall briefly describe the dynamics of scalar fields in an expanding universe. Then, the use of exponential potentials will reveal the existence of non-standard evolutionary periods such as kination and tracker epochs which will constitute interesting scenarios. Remarkably, if the scalar field is taken to be the canonically normalized volume modulus $\Phi$, a time evolution of the geometrical volume $\mathcal{V}$ can be found and therefore it will naturally imply a time dependence for cosmic superstring tension.

## 3.1 Scalar fields in an expanding universe

In an expanding FLRW background, the metric is given by:

$$ds^2 = -dt^2 + a(t)^2 \left[ \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right] \tag{3.1.1}$$

where $k = -1, 0, +1$ is the curvature constant and specifies the topology of the Universe. The dynamics of the scale factor $a(t)$ is determined by the Einstein equations:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu} \tag{3.1.2}$$

with $G_N$ the four-dimensional Newton constant. In most cases, due to its homogeneity and isotropy, one considers the universe to be filled with barotropic perfect fluids characterized by the proper energy density $\rho$ and pressure $p$. In this case, the energy momentum tensor on the r.h.s. of (3.1.2) becomes:

$$T^\mu{}_\nu = \text{diag}(-\rho, p, p, p), \tag{3.1.3}$$

and, plugging the metric (3.1.1) in (3.1.2) with the energy momentum tensor above, yields the Friedmann and Raychaudhuri equations:

$$H^2 = \frac{8\pi G_N}{3}\rho - \frac{k}{a^2}, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}(\rho + 3p). \tag{3.1.4}$$

where $H \equiv \dot{a}/a$ is the cosmic time Hubble parameter. These equations are to be paired with energy-momentum conservation, which follows from the Bianchi identity $\nabla_\mu T^{\mu\nu}=0$:

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{3.1.5}$$

Observations [50] indicate the universe is very close to being spatially flat, therefore we shall consider $k = 0$ henceforth. Furthermore, the equation of state of such barotropic fluids is taken to be $p = w\rho$, where $w$ is the equation of state parameter, a constant value. Different fluids have different values of $w$. The notable examples are $w = 1/3$ for radiation, $w = 0$ for standard matter, and $w = -1$ for vacuum energy or cosmological constant. Using the equation of state, one is able to derive the dependence of $\rho$ on the scale factor through (3.1.5) and the behavior of $a(t)$ with respect to cosmic time by (3.1.4).

Having established this background, the evolution of a scalar field $\varphi(x)$ in a spatially flat FLRW background is determined by the following action principle:

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)\right] \tag{3.1.6}$$

with scalar potential $V(\varphi)$ and $M_P$ the four-dimensional Planck mass. Assuming the scalar field to depend only on cosmic time $\varphi(t)$, varying the action with respect to it yields the equation of motion:

$$\ddot{\varphi} + 3H\dot{\varphi} + V_{,\varphi} = 0 \tag{3.1.7}$$

where $V_{,\varphi} \equiv dV/d\varphi$. Moreover, the energy momentum tensor for such scalar field is:

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\left(\frac{1}{2}\partial_\sigma\varphi\partial^\sigma\varphi + V(\varphi)\right) \tag{3.1.8}$$

and, using (3.1.3), the energy density and pressure of the scalar field are found to be:

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi), \qquad p_\varphi = \frac{1}{2}\dot{\varphi}^2 - V(\varphi). \tag{3.1.9}$$

which in turn can be substituted in (3.1.4) to give:

$$H^2 = \frac{1}{3M_P^2}\left(\frac{\dot{\varphi}^2}{2} + V(\varphi)\right), \qquad \frac{\ddot{a}}{a} = -\frac{1}{3M_P^2}\left(\dot{\varphi}^2 - V(\varphi)\right). \tag{3.1.10}$$

This generic scalar field $\varphi$ can be taken to be one of the moduli descending from string compactifications. In particular, this can take the role of the canonically normalized volume modulus, $\Phi$, in which case it inherits a natural time dependence. However, it is necessary also to specify a given potential if one wants to describe the dynamics of such field when this is not negligible. Given that the LVS model prescribes exponential potentials for the canonically normalized volume modulus, it is natural to study the general behaviour of scalar fields in background cosmology with such form of potentials. This shall be the topic of the next section.

# 3.2 Exponential potentials and scaling solutions

Exponential potentials in cosmology are not usually attractive because they can be very steep, as is the nature of the exponential function, and do not constitute a great playground for inflationary scenarios. However, as mentioned in the previous section, non perturbative effects are naturally exponential in form and moduli fields develop exponential potentials. Furthermore, in section 2.2 it was pointed out that the curvature of the compact internal space could contribute to a positive or negative potential energy for the scalar field associated to the extra dimension and this potential was precisely exponential. It is therefore interesting to study the consequence of exponential potentials in flat FLRW cosmology.

We shall follow [51] and consider a scalar field with potential:

$$V(\varphi) = V_0 \exp\left(-\lambda \frac{\varphi}{M_P}\right) \tag{3.2.1}$$

where $\lambda$ is constant. Consider also a perfect fluid in the background, with equation of state of the form:

$$p_\gamma = (\gamma - 1)\rho_\gamma \tag{3.2.2}$$

with $0 \leq \gamma \leq 2$. The reason for this choice will be made clear throughout the section. The system then evolves according to the equations:

$$\dot{H} = -\frac{1}{2M_P^2}\left(\rho_\gamma + p_\gamma + \dot{\varphi}^2\right) = -\frac{1}{2M_P^2}\left(\gamma\rho_\gamma + \dot{\varphi}^2\right) \tag{3.2.3}$$

$$\dot{\rho}_\gamma = -3H(\rho_\gamma + p_\gamma) = -3H\gamma\rho_\gamma \tag{3.2.4}$$

$$\ddot{\varphi} = -3H\dot{\varphi} - V_{,\varphi} \tag{3.2.5}$$

with the constraint:

$$H^2 = \frac{1}{3M_P^2}\left(\rho_\gamma + \rho_\varphi\right) = \frac{1}{3M_P^2}\left(\rho_\gamma + \frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right) \tag{3.2.6}$$

This system of equations can be converted into a plane autonomous system with the following change of variable:

$$x = \frac{\dot{\varphi}}{M_P}\frac{1}{\sqrt{6}H} \qquad\qquad y = \sqrt{\frac{V}{3}}\frac{1}{M_P H} \tag{3.2.7}$$

Then, the constraint equation (3.2.6) becomes:

$$\frac{\rho_\gamma}{3H^2M_P^2} + x^2 + y^2 = 1 \implies \Omega_\gamma + \Omega_\varphi = 1 \tag{3.2.8}$$

where the density parameters ($\Omega_i \equiv \frac{\rho_i}{3H^2M_P^2}$) have been introduced. The one associated to the field $\varphi$ is precisely given by:

$$\Omega_\varphi = x^2 + y^2 \tag{3.2.9}$$

and it is bounded between 0 and 1. Therefore, the evolution of the dynamical system is described by trajectories within a unit disc. In addition, the variables $x$ and $y$ encode the fractional energy densities of kinetic and potential energy, $\Omega_k = x^2$ and $\Omega_p = y^2$, respectively. With this change of variable, the whole system can be casted as (setting $M_P = 1$) [51]:

$$\begin{cases} x'(N) = -3x - \dfrac{V'(\varphi)}{V(\varphi)}\sqrt{\dfrac{3}{2}}\,y^2 + \dfrac{3}{2}x\left[2x^2 + \gamma(1 - x^2 - y^2)\right], \\ y'(N) = \dfrac{V_{,\varphi}(\varphi)}{V(\varphi)}\sqrt{\dfrac{3}{2}}\,xy + \dfrac{3}{2}y\left[2x^2 + \gamma(1 - x^2 - y^2)\right], \\ H'(N) = -\dfrac{3}{2}H\left[2x^2 + \gamma(1 - x^2 - y^2)\right], \\ \varphi'(N) = \sqrt{6}\,x. \end{cases} \tag{3.2.10}$$

with $N = \log a$ the number of e-foldings as new time variable and where primes denote derivatives with respect to $N$. For the potential in consideration, $V_{,\varphi}/V = -\lambda/M_P$ and the field $\varphi$ has an effective equation of state parameter $\gamma_\varphi$ given by [51]:

$$\gamma_\varphi = \frac{\rho_\varphi + p_\varphi}{\rho_\varphi} = \frac{\dot{\varphi}^2}{V(\varphi) + \dot{\varphi}^2/2} = \frac{2x^2}{y^2 + x^2} \tag{3.2.11}$$

The dynamical system analysis in the original paper shows there are five fixed points $x' = 0 = y'$ characterized by different values or ranges of $\gamma$ and $\lambda$. If these occur at finite values of $x$ and $y$, the solution to the dynamical system corresponds to a barotropic equation of state for $\varphi$ and scale factor evolution given by $a \sim t^p$, with $p = 2/(3\gamma_\varphi)$. Summarizing the results, one has two fixed points $(x, y) = (\pm 1, 0)$ where the energy density of the scalar field is dominated by its kinetic contribution, since $\Omega_p = 0$. From (3.2.11), the effective equation of state parameter is $\gamma_\varphi = 2$. These solutions are unstable and dominate at early times. This is a period denoted as *kination*, where the kinetic energy of the scalar field is dominating with respect to the potential one, exactly the opposite case of slow-roll inflation. In addition, there is a fluid dominated solution $(x, y) = (0, 0)$, $\Omega_\varphi = 0$, which is found to be unstable for all values of $\gamma > 0$ and $\lambda$. However, the solution of interest here is the late time scaling solution, in which the scalar field and the background fluid have a fixed ratio of energy densities. Indeed, the fixed point conditions, $x' = 0 = y'$, imply the scalar field has an effective equation of state parameter equal to the fluid one, $\gamma_\varphi = \gamma$. At the same time, solving the system of equations gives:

$$x = \sqrt{\frac{3}{2}}\frac{\gamma}{\lambda} \tag{3.2.12}$$

Therefore, this fixed point is characterized by:

$$\Omega_k = x^2 = \frac{3}{2}\frac{\gamma^2}{\lambda^2}, \qquad \Omega_p = y^2 = \frac{3(2-\gamma)\gamma}{2\lambda^2}, \qquad \Omega_\gamma = 1 - x^2 - y^2 = 1 - \frac{3\gamma}{\lambda^2} \tag{3.2.13}$$

and this period of evolution is denoted as *attractor tracker solution*. Equivalently, the fixed point is located at:

$$(x, y) = \left(\sqrt{\frac{3}{2}}\frac{\gamma}{\lambda}, \sqrt{\frac{3(2-\gamma)\gamma}{2\lambda^2}}\right) \tag{3.2.14}$$

The attractor feature represents a convergence to the same solution for various initial conditions. Instead, the tracker characterization refers to the fixed ratio of energy densities of scalar field and fluid. In this regime, reintroducing $M_P$, the last two equation of system (3.2.10) are:

$$\begin{cases} H'(N) = -\dfrac{3}{2}H\gamma \\ \varphi'(N) = \dfrac{3\gamma}{\lambda}M_P \end{cases} \Longrightarrow \begin{cases} H(t) = H_0\left(\dfrac{a(t)}{a_0}\right)^{-3\gamma/2} \\ \varphi(a) = \varphi_0 + \dfrac{3\gamma}{\lambda}M_P \ln\left(\dfrac{a(t)}{a_0}\right) \end{cases} \tag{3.2.15}$$

where $t_0$ denotes the start of the tracker with scale factor $a(t_0) \equiv a_0$ and $\varphi(t_0) \equiv \varphi_0$. Given that $a(t) \sim t^{2/(3\gamma)}$, the evolution in time of $\varphi$ is:

$$\varphi(t) = \varphi_0 + \frac{2M_P}{\lambda}\ln\left(\frac{t}{t_0}\right) \tag{3.2.16}$$

where the dependence on $\gamma$ drops out. Therefore, for both radiation and matter trackers ($\gamma_r = 4/3, \gamma_m = 1$), the solution is the same.

One might wonder whether such cosmological scenarios are physically relevant or merely a mathematical possibility allowed by the theory. To answer this, I shall dedicate a few words to the *overshoot problem* in string cosmology [52]. Standard cosmology is widely believed to have started with an initial period of inflation, featuring large energy scales $\Lambda_{inf} \sim (10^{15}\,\text{GeV})^4$ [5], where the universe underwent an exponential expansion. The current state of the Universe however exhibits an electroweak symmetry breaking at scales $\Lambda_{EW} \sim 100\,\text{GeV}$, which remains a great mystery to explain. This is because it seems highly unstable to quantum corrections unless one severely fine tunes the parameters or includes additional symmetries, such as supersymmetry. If string theory is the theory describing our world, and thus our universe, it must provide a description of it from the early times to today. However, many string theory models exhibit a potential energy which vanishes in the decompactification limit described by:

$$\mathcal{V} \to \infty \qquad\qquad g_s \to 0 \tag{3.2.17}$$

Therefore, there is a typical runaway behavior at the heart of all cosmological scenarios arising from string theory which must be addressed. As a consequence of moduli stabilization, the immediate post-inflationary scenarios typically involve *rolling moduli* with energies far larger than those of particle physics models. The overshoot problem is then why and how did the universe end up in its current vacuum rather than the decompactification limit ?[53] The current state of the universe is a vacuum of such potential and therefore it is surrounded by a barrier to the decompactification limit which in scale is like a nanometer to the height of mount Everest [54]. As such, it seems unlikely that the evolution winds up in the minimum and does not *overshoot* to $V \to 0$ at $\mathcal{V} \to \infty$.

Two obvious solutions or better escamotages to the problem above consist in either a drastically low energy scale of inflation, $\Lambda_{inf} \sim (1\text{TeV})^4$ or extremely high scales of post-inflationary minimum. In both cases, the two potential "barriers" are comparable in height and therefore the overshoot problem is not present at all. Of course, this completely bypasses the problem and does not compel the search for new physics. Instead, the fixed points described above provide the basic dynamical ingredients of the standard resolution of this problem in string cosmology. Immediately after inflation, the modulus may enter a kination regime, in which its energy density is dominated by kinetic energy and, as will be shown in the next section, redshifts as $\rho_{\rm kin} \propto a^{-6}$. Any radiation or matter component redshifts more slowly ($\rho_r \sim a^{-4}$, $\rho_m \sim a^{-3}$) and can eventually catch up with the kinetic energy, increasing Hubble friction and driving the system towards the tracker solution. If this happens before the field crosses the barrier of the minimum, the modulus can be guided into the stabilized vacuum rather than overshooting it.

However, for the purposes of this thesis the main role of kination is slightly different. During this epoch, the volume modulus can be shown to evolve logarithmically in cosmic time. Since $\Phi \propto \ln \mathcal{V}$, this implies a power-law evolution of the compactification volume and hence of any cosmic superstring tension depending on it. This will now be shown explicitly.

## 3.3 Kination and power-law tension evolution

As briefly mentioned in the previous section, after inflation the inflaton field starts rolling down the potential and achieves quite a large amount of kinetic energy compared to its potential energy. Therefore, the universe enters a phase dominated by the kinetic energy of the rolling field, denoted as *kination*. In pure kination environment, $V(\varphi) = 0$, the equation of motion for such a scalar field are:

$$\ddot{\varphi} + 3H\dot{\varphi} = 0 \tag{3.3.1}$$

$$H^2 = \frac{\dot{\varphi}^2}{6M_P^2} \tag{3.3.2}$$

Defining $y = \dot{\varphi}$, from the second equation one has $H = \frac{y}{\sqrt{6}M_P}$, where the positive sing has been chosen so that the evolution is towards larger volumes. Substituting it into the first gives:

$$\dot{y} + \sqrt{\frac{3}{2}}\frac{y^2}{M_P} = 0 \tag{3.3.3}$$

which can be integrated once to yield:

$$\dot{\varphi} = \sqrt{\frac{2}{3}}\frac{M_P}{t} \tag{3.3.4}$$

and twice to get:

$$\varphi(t) = \sqrt{\frac{2}{3}}M_P \ln t + D \tag{3.3.5}$$

One of the integration constants can be removed by formally requiring $t \to 0$ to represent a singularity in the energy density. The other one is fixed by initial condition $\varphi(t_0) = \varphi_0$ and yields:

$$\varphi(t) = \sqrt{\frac{2}{3}}M_P \ln\left(\frac{t}{t_0}\right) + \varphi_0 \tag{3.3.6}$$

Using (3.3.2), the evolution of the scale factor during such phase is found to be:

$$\frac{\dot{a}}{a} \sim \frac{1}{3t} \implies a \sim t^{1/3} \tag{3.3.7}$$

which in turn corresponds to an energy density scaling like:

$$\frac{1}{t^2} \sim H^2 = \frac{\rho}{3} \implies \rho \sim t^{-2} \sim a^{-6} \tag{3.3.8}$$

as was mentioned in the previous section. This shows kinetic energy redshifts faster than matter and radiation, therefore it dilutes much more quickly. Consequently, any matter or radiation sources present during the onset of the kination phase will eventually come to dominate the energy density and contribute to Hubble friction. In turn, this will cause the rolling field to slow down and potentially stop its evolution. After this phase, the system then enters the tracker solutions outlined in the previous section. The reason why $\gamma$ was taken in between the values of 0 and 2 in (3.2.2) can now be made clear. With that equation of state, the energy density of the fluid scales as $\rho_\gamma \sim a^{-3\gamma}$ and, if one wishes for this additional fluid to catch up with the kinetic energy of the scalar field, its energy density cannot dilute any faster than $a^{-6}$, therefore $\gamma$ is bounded by zero and two.

The important point is the following: if the scalar field $\varphi(t)$ is taken to be the canonically normalized volume modulus $\Phi(t)$ of (2.4.19), it inherits a logarithmic dependence on cosmic time. If we compare this with the dependence of $\Phi$ on the geometrical volume of the underlying CY manifold, arising from canonical normalization, for a field excursion $\Delta\Phi = \Phi - \Phi_0$, it leads to the following relation:

$$\sqrt{\frac{2}{3}} M_P \ln\left(\frac{t}{t_0}\right) = \Delta\varphi = \Delta\Phi = \sqrt{\frac{2}{3}} M_P \ln\left(\frac{\mathcal{V}}{\mathcal{V}_0}\right) \implies \mathcal{V}(t) = \mathcal{V}_0\left(\frac{t}{t_0}\right) \tag{3.3.9}$$

This tells us that in a kination epoch descending from string cosmology the evolution of the internal compactification volume is *linear* in cosmic time. To understand the consequence for cosmic strings, recall that cosmic superstrings can arise as F-strings whose tension is the fundamental string tension, $\mu = M_s^2$, with $M_s$ given by (2.1.21). Then, using the dependence of the geometrical CY volume on time just found, one has:

$$\mu = M_s^2 \sim \frac{M_P^2}{\mathcal{V}} \sim \frac{M_P^2}{\mathcal{V}_0}\left(\frac{t}{t_0}\right)^{-1} \tag{3.3.10}$$

In terms of the dimensionless string tension, $G_N\mu = \mu/M_P^2$, this is:

$$G_N\mu \sim t^{-1} \tag{3.3.11}$$

For the case of the LVS potential (2.4.21), the time dependence of $\mathcal{V}$ implies:

$$V_{LVS} \sim 1/\mathcal{V}^3 \sim t^{-3} \tag{3.3.12}$$

during such kination phase. This also follows from using the logarithmic evolution of the field $\Phi$ directly into the LVS potential:

$$V_{LVS} = V_0 \exp\left(-\sqrt{\frac{27}{2}}\sqrt{\frac{2}{3}}\ln t\right) = V_0 \exp(-3\ln t) \implies V_{LVS} \sim t^{-3} \tag{3.3.13}$$

A similar analysis can lead to a different time dependence of the tension for background tracker solutions. Indeed, recalling (3.2.16) and letting $\varphi$ be the canonically normalized volume modulus, one has:

$$\Delta\varphi = \left(\frac{2}{3}\right)^{3/2} M_P \ln\left(\frac{t}{t_0}\right) \qquad\qquad \Delta\Phi = \sqrt{\frac{2}{3}} M_P \ln\left(\frac{\mathcal{V}}{\mathcal{V}_0}\right) \tag{3.3.14}$$

Setting $\Delta\varphi = \Delta\Phi$ implies:

$$\mathcal{V} = \mathcal{V}_0 \left(\frac{t}{t_0}\right)^{2/3} \tag{3.3.15}$$

This represents a non-linear evolution of the geometrical CY volume as function of cosmic time. As a consequence, the tension of fundamental $F$-strings scales as:

$$\mu(t) \sim \frac{M_P^2}{\mathcal{V}_0}\left(\frac{t}{t_0}\right)^{-2/3} \implies G_N\mu \sim t^{-2/3} \tag{3.3.16}$$

It should be kept in mind that the value of the power is not a universal property of tracker solutions. It follows from the specific LVS slope, $\lambda = \sqrt{27/2}$, of the volume modulus potential. Different exponential slopes, or different rolling moduli, would lead to different powers.

These examples provide a concrete illustration of why string theory naturally involves cosmic superstrings whose tension exhibits a power law time dependence, $\mu \sim t^{-m}$. This is a very natural phenomenon descending mainly from two aspects. Firstly, as pointed out in section 2.3, the four-dimensional effective theory coming from compactification of Type IIB superstring theory can be described in terms of the Kähler potential in (2.3.8). This naturally leads to canonically normalized scalar fields that are logarithmic in the geometrical moduli (see (2.4.19)). Consequently, if the cosmological evolution of the canonical field is itself logarithmic in time, as happens during kination and scaling regimes, then the geometrical modulus evolves as a power of time. It is therefore the combination of canonical normalization of scalar fields and cosmological evolutions that makes this possible.

The exponential form of the potential is crucial for the existence of scaling or tracker solutions, since it makes $V_{,\Phi}/V$ constant. However, during a pure kination epoch the potential energy is subdominant and the logarithmic evolution of the field follows instead from kinetic domination. In this sense, the LVS exponential potential motivates the presence of a steep runaway direction, while the power-law evolution of the compactification volume during kination follows from the combination of kinetic domination and the logarithmic canonical normalization of the volume modulus.

One might then wonder whether cosmic superstrings have particular properties during such kination phase or how it affects them. The beautiful thing is that this period of evolution not only provides a clear way to see the time dependence of the tension but also constitutes an ideal environment for cosmic superstrings to grow, as will be clear when discussing the microscopic equations of motion governing string with time dependent tension. This will be done in the next chapter, together with the analysis of their small-scale structure.

# Chapter 4

# Small-scale structure of cosmic strings

The main objective of this chapter is to investigate the small scale structure of cosmic strings with a time varying tension, building on the model developed by Polchinski and Rocha (PR) [8] for constant tension strings in an expanding universe, under appropriate assumptions. I will begin by reviewing the original model, emphasizing key assumptions and results. Afterwards, I will study the microscopic and macroscopic dynamics of cosmic strings with time varying tension using the generic dependence $\mu \sim t^{-m}$, as motivated by the previous chapter. This requires a review of the modified equations of motion and the generalized VOS model, which clarify how the dynamics differs from the constant tension case. Finally, I will present an original extension of the PR framework to time-varying tension. The purpose of this analysis is to determine whether the modified microscopic dynamics can change the scaling of the small-scale correlator or whether the PR result remains valid once the new terms are consistently taken into account.

## 4.1 Constant tension framework

The framework proposed by PR to analyze the small scale structure of cosmic strings relies on the correlation of either left- or right- moving unit vectors at nearby points along the string. In this first section, I shall give an overview of this framework, reproducing their original calculations, emphasizing the key physical assumptions at its foundation, and explaining the physical mechanism playing a role.

As outlined in section 1.4.2, after symmetry breaking phase transitions powered by the expansion of the universe, a network of cosmic strings may be formed. Its evolution is mainly governed by three processes:

1. Expansion of the Universe: this results in a conformal stretch of superhorizon modes ($k\tau \ll 1$), while subhorizon modes ($k\tau \gg 1$) straighten with the expansion, meaning irregularities are smoothed by the expansion of the universe.

2. Gravitational radiation: cosmic strings produce a gravitational field, because of their non zero energy momentum tensor. However, this is significant only for length scales below a certain $\ell_{GW} \propto (G\mu)^p d_H$ where $p$ is a positive exponent,

and given that the dimensionless tension is very small, it can be neglected in first approximation.

3. Intercommutation events: these refer to intersection of long strings that generate loops and play an important role in reaching a scaling solution for the network.

Consider a segment of long string with length $\ell$ such that $\ell_{GW} < \ell \ll d_H$. For a network in the scaling regime, the probability of intercommutation events can be shown to be proportional to $\ell/d_H$ [2]. Therefore, taking $\ell \ll d_H$ allows for these processes to be neglected, and stretching due to the expansion of the universe is the only relevant process, neglecting the formation of loops of size $\ell$ or smaller. As the network scales, the ratio between $\ell$ and $d_H$ will decrease as cosmic time advances, while it will grow approaching 1 evolving backwards in time. The logic of [8] is to use simulations to study the non-linear dynamics when $\ell/d_H \sim 1$ and then evolve the results in time using the standard equations of motion derived from the Nambu-Goto action as shown in section 1.4.1.

Let us recall these equations, which in conformal gauge can be written as:

$$\ddot{\mathbf{x}} + 2\frac{\dot{a}}{a}(1-\dot{\mathbf{x}}^2)\dot{\mathbf{x}} - \frac{1}{\varepsilon}\left(\frac{\mathbf{x}'}{\varepsilon}\right)' = 0 \qquad\qquad \dot{\varepsilon} + 2\frac{\dot{a}}{a}\varepsilon\dot{\mathbf{x}}^2 = 0 \tag{4.1.1}$$

Then, the energy of a string segment of coordinate length $\sigma$ (previously denoted as $\zeta^1$) is (1.4.28):

$$E = \mu a(\tau)\int \mathrm{d}\sigma\varepsilon = \mu a(\tau)\varepsilon(\tau)\sigma \tag{4.1.2}$$

taking $\varepsilon$ to be $\sigma$-independent, as allowed by residual gauge transformations of the world-sheet coordinate $\sigma$. Then, on dimensional grounds, the physical length of this segment is defined as:

$$\ell \equiv \frac{E}{\mu} = a(\tau)\sigma\varepsilon(\tau) \tag{4.1.3}$$

The RMS velocity $\bar{v}^2$ instead is given by (1.4.19):

$$\bar{v}^2 = \frac{\int \mathrm{d}\sigma\,\varepsilon\,\dot{\mathbf{x}}^2}{\int \mathrm{d}\sigma\,\varepsilon} \tag{4.1.4}$$

Since the equation of motion for $\varepsilon$ implies its variation time-scale is the Hubble time, one can substitute $\dot{\mathbf{x}}^2 \to \bar{v}^2$, which leads to

$$\varepsilon \propto a^{-2\bar{v}^2} \tag{4.1.5}$$

This relation will be important later on. In the FLRW universe, the evolution of the scale factor in cosmic and conformal time can be written as:

$$a \sim t^{\nu} \sim \tau^{\nu'} \tag{4.1.6}$$

where $\nu = \frac{1}{2}, \nu' = 1, \bar{v}^2 \approx 0.41$ in radiation domination, and $\nu = \frac{2}{3}, \nu' = 2, \bar{v}^2 \approx 0.35$ in matter domination. The values of $\bar{v}^2$ are taken from simulations [55]. Interestingly, the worldsheet line element can be written as:

$$ds^2 = \gamma_{ab}dx^a dx^b = \gamma_{00}d\tau^2 + \gamma_{11}d\sigma^2 \tag{4.1.7}$$

with

$$\gamma_{00} = a^2(1-\dot{\mathbf{x}}^2) \qquad\qquad \gamma_{11} = -a^2\mathbf{x}'^2 \tag{4.1.8}$$

Implying

$$ds^2 = a^2\left[(1-\dot{\mathbf{x}}^2)d\tau^2 - \mathbf{x}'^2 d\sigma^2\right] = a^2(1-\dot{\mathbf{x}}^2)\left[d\tau^2 - \varepsilon^2 d\sigma^2\right] \tag{4.1.9}$$

This naturally suggests the introduction of left and right movers to describe the propagation of signals along the string, as these are associated to null directions on the worldsheet $d\sigma = \pm d\tau/\varepsilon$. The left and right moving unit vectors are thus defined as $\mathbf{p}_\pm \equiv \dot{\mathbf{x}} \pm \frac{1}{\varepsilon}\mathbf{x}'$ and the equations of motion can be casted into equations describing the action of left/right moving operators $\hat{L}_\pm = (\partial_\tau \pm \frac{\partial_\sigma}{\varepsilon})$ on the right/left moving unit vectors by proceeding as follows. First, evaluate the derivative of $\mathbf{p}_+$ with respect to (conformal) time and $\sigma$:

$$\dot{\mathbf{p}}_+ = \ddot{\mathbf{x}} + \frac{1}{\varepsilon}\partial_\tau \mathbf{x}' - \frac{\dot{\varepsilon}}{\varepsilon^2}\mathbf{x}' \qquad\qquad \mathbf{p}'_+ = \partial_\sigma \dot{\mathbf{x}} + \left(\frac{\mathbf{x}'}{\varepsilon}\right)' \tag{4.1.10}$$

Hence:

$$\hat{L}_-\mathbf{p}_+ = \dot{\mathbf{p}}_+ - \frac{1}{\varepsilon}\mathbf{p}'_+ = \ddot{\mathbf{x}} - \frac{\dot{\varepsilon}}{\varepsilon^2}\mathbf{x}' - \frac{1}{\varepsilon}\left(\frac{\mathbf{x}'}{\varepsilon}\right)' \tag{4.1.11}$$

Now, using the definition of the unit vectors, some useful relations are:

$$\begin{aligned} \dot{\mathbf{x}} &= \frac{1}{2}(\mathbf{p}_+ + \mathbf{p}_-) & \ddot{\mathbf{x}} &= \frac{1}{2}(\dot{\mathbf{p}}_+ + \dot{\mathbf{p}}_-) \\ \frac{1}{\varepsilon}\mathbf{x}' &= \frac{1}{2}(\mathbf{p}_+ - \mathbf{p}_-) & (1-\dot{\mathbf{x}}^2) &= \frac{1}{2}(1 - \mathbf{p}_+\cdot\mathbf{p}_-) \end{aligned} \tag{4.1.12}$$

Therefore,

$$\frac{\dot{\varepsilon}}{\varepsilon} = -\frac{\dot{a}}{a}(1 + \mathbf{p}_+\cdot\mathbf{p}_-) \tag{4.1.13}$$

Using both the original equations of motion in (4.1.11) and writing everything in terms of the unit vector gives:

$$\dot{\mathbf{p}}_+ - \frac{1}{\varepsilon}\mathbf{p}'_+ = -\frac{\dot{a}}{a}\left[\mathbf{p}_- - (\mathbf{p}_+\cdot\mathbf{p}_-)\,\mathbf{p}_+\right] \tag{4.1.14}$$

A similar equation for $\mathbf{p}_-$ can be derived. In a compact representation, both of these are:

$$\dot{\mathbf{p}}_\pm \mp \frac{1}{\varepsilon}\mathbf{p}'_\pm = -\frac{\dot{a}}{a}\left[\mathbf{p}_\mp - (\mathbf{p}_+\cdot\mathbf{p}_-)\,\mathbf{p}_\pm\right]. \tag{4.1.15}$$

The crucial point is that left and right movers are not independent of each other as one has in flat Minkowski spacetime (with constant tension), but they are coupled by the expansion of the universe, represented by the conformal time Hubble parameter $\mathcal{H} = \dot{a}/a$.

With the aim of gaining insights about the correlations of these unit vectors at nearby points along the string, it is useful to consider the evolution in time of the product $\mathbf{p}_+(\tau,\sigma)\cdot\mathbf{p}_+(\tau,\sigma')$, since it tells us whether these unit vectors remain parallel or not in time. To do this, it is convenient to introduce the variable $s$, constant along the left moving curves $\hat{L}_- s = \dot{s} - \frac{1}{\varepsilon}s' = 0$, in replacement of $\sigma = \sigma(s,\tau)$ now. This leads to:

$$\partial_\tau \mathbf{p}_+(s,\tau) = \frac{\partial \mathbf{p}_+}{\partial \tau}\frac{\partial \tau}{\partial \tau}\Big|_\sigma + \frac{\partial \mathbf{p}_+}{\partial \sigma}\frac{\partial \sigma}{\partial \tau} = \dot{\mathbf{p}}_+ - \frac{1}{\varepsilon}\mathbf{p}'_+$$

Therefore,

$$\partial_\tau\big(\mathbf{p}_+(s,\tau)\cdot\mathbf{p}_+(s',\tau)\big) = -\frac{\dot a}{a}\Big(\mathbf{p}_-(s,\tau)\cdot\mathbf{p}_+(s',\tau)+\mathbf{p}_+(s,\tau)\cdot\mathbf{p}_-(s',\tau)+ \\ \alpha(s,\tau)\,\mathbf{p}_+(s,\tau)\cdot\mathbf{p}_+(s',\tau)+\alpha(s',\tau)\,\mathbf{p}_+(s,\tau)\cdot\mathbf{p}_+(s',\tau)\Big)$$

with $\alpha\equiv-\mathbf{p}_+\cdot\mathbf{p}_- = 1-2v^2$. In the case of perfect smooth structure, the map $s\to\mathbf{p}_+(s,\tau)$ would constitute a smooth curve on the unit sphere. Thus, one could expand $\mathbf{p}_+(s',\tau)$ as:

$$\mathbf{p}_+(s',\tau)=\mathbf{p}_+(s,\tau)+(\partial_s\mathbf{p}_+)\big|_s(s'-s)+\mathcal{O}\left((s'-s)^2\right) \tag{4.1.16}$$

Taking the scalar product with $\mathbf{p}_+(s,\tau)$ gives:

$$\mathbf{p}_+(s,\tau)\cdot\mathbf{p}_+(s',\tau)=1+\mathcal{O}\big((s'-s)^2\big) \tag{4.1.17}$$

since $2\mathbf{p}_+\partial_s\mathbf{p}_+=\partial_s(\mathbf{p}_+\cdot\mathbf{p}_+)=0$. This implies:

$$1-\mathbf{p}_+(s,\tau)\cdot\mathbf{p}_+(s',\tau)=\mathcal{O}\big((s'-s)^2\big) \tag{4.1.18}$$

Equivalently, one can argue that terms of order $\mathcal{O}(s'-s)$ drop because the product is even under $s\leftrightarrow s'$. Interested in deviation from this regular regime, the scenario investigated in [8] neglects terms of higher order in $(s'-s)$. This is because higher powers would signal an approach to unity of the scalar product even faster than this idealized case. Therefore, possible irregularities appear to be linked to $(s'-s)$ to some power smaller than two. Moreover, the time scale of variation of left (right) moving vectors along right (left) characteristics is of the Hubble time scale, as by the equations of motion. Therefore, the products between them can be evaluated at the intersection point of the curves passing through $(s,\tau)$ and $(s',\tau)$:

$$\mathbf{p}_+(s,\tau)\cdot\mathbf{p}_-(s',\tau)=-\alpha(s,\tau-\delta)+\mathcal{O}(s'-s) \tag{4.1.19}$$

For example, for $\tau$ such that $\varepsilon(\tau)=1$, this is given by:

$$\sigma^*-\tau^*=\text{const}=s-\tau \implies \sigma^*=s-(\tau-\tau^*) \tag{4.1.20}$$
$$\sigma+\tau^*=\text{const}=s'+\tau \implies \sigma=s'+(\tau-\tau^*) \tag{4.1.21}$$

Imposing $\sigma=\sigma^*$ gives $\tau^*-\tau=(s'-s)/2\equiv\delta$. Then:

$$\partial_\tau\big(\mathbf{p}_+(s,\tau)\cdot\mathbf{p}_+(s',\tau)\big)=\frac{\dot a}{a}\Big(\alpha(s',\tau-\delta)+\alpha(s,\tau+\delta)-\alpha(s,\tau)\,\mathbf{p}_+(s,\tau)\cdot\mathbf{p}_+(s',\tau) \\ -\alpha(s',\tau)\,\mathbf{p}_+(s,\tau)\cdot\mathbf{p}_+(s',\tau)\Big)+\mathcal{O}\big((s'-s)^2\big).$$

Integrating over scales of the Hubble time, the $\delta$ in the arguments of $\alpha$ becomes negligible, as this is of the order $s'-s\approx\sigma'-\sigma$, meaning it is roughly the coordinate distance between two points along the string, and this is by assumption much smaller than $d_H\sim t$. The *correlator* can then be defined as:

$$h_{++}(s,s',\tau)=1-\mathbf{p}_+(s,\tau)\cdot\mathbf{p}_+(s',\tau) \tag{4.1.22}$$

representing the fact that when the left moving vectors at $s$ and $s'$ are parallel, meaning perfect smooth small scale structure, this vanishes. Therefore, the presence of small scale structure is encoded in deviations of $h_{++}$ from zero, or equivalently of $\mathbf{p}_+(s,\tau)\cdot\mathbf{p}_+(s',\tau)$ from one. This correlator obeys the equation:

$$\partial_\tau h_{++}(s,s',\tau) = -\frac{\dot{a}}{a}\, h_{++}(s,s',\tau)\left[\alpha(s',\tau)+\alpha(s,\tau)+O\big((s'-s)^2\big)\right] \tag{4.1.23}$$

whose solution is:

$$h_{++}(s,s',\tau_1) = h_{++}(s,s',\tau_0)\exp\left\{-\nu'\int_{\tau_0}^{\tau_1}\frac{d\tau}{\tau}\left[\alpha(s',\tau)+\alpha(s,\tau)+O\big((s'-s)^2\big)\right]\right\}.$$

When considering an ensemble of segments, the differences in the exponents will average out and so the $\alpha$ can be replaced by the RMS value $\bar{\alpha} = 1-2\bar{v}^2$. This leads to the solution:

$$\big\langle h_{++}(s,s',\tau_1)\big\rangle \approx \big\langle h_{++}(s,s',\tau_0)\big\rangle\left(\frac{\tau_1}{\tau_0}\right)^{-2\nu'\bar{\alpha}} \tag{4.1.24}$$

Or more generally, considering a translational invariant ensemble, since the physics should not depend on the specific points along the string:

$$\big\langle h_{++}(\sigma-\sigma',\tau)\big\rangle \approx \frac{f(\sigma-\sigma')}{\tau^{2\nu'\bar{\alpha}}} \propto a^{-2\bar{\alpha}} \tag{4.1.25}$$

Now, to determine the specific form of the function $f(\sigma-\sigma')$ one can look at the ratio between the segment length and the horizon size:

$$\frac{\ell}{d_H} \propto \frac{a\varepsilon(\sigma-\sigma')}{t} \propto a^{1-2\bar{v}^2-1/\nu}(\sigma-\sigma') \propto \frac{\sigma-\sigma'}{\tau^{\,1+2\nu'\bar{v}^{\,2}}} \tag{4.1.26}$$

where the dependence of $\varepsilon$ on $a$ is the one in (4.1.5). This can be used as an initial condition, setting the correlator $h_{++}=h_0$, where $h_0$ is time independent, when $\sigma-\sigma' = x_0\tau^{\,1+2\nu'\bar{v}^{\,2}}$, for some constant $x_0$ determined by simulations. Using a power law ansatz $f(\Delta\sigma) = C(\Delta\sigma)^{2\chi}$, as motivated by the expected time dependence of the correlator in (4.1.25) gives:

$$h_0\tau^{2\nu'\bar{\alpha}} = f(\Delta\sigma = x_0\tau^{1+2\nu'\bar{v}^2}) = C\tau^{(1+2\nu'\bar{v}^2)2\chi}(x_0)^{2\chi} \tag{4.1.27}$$

The constant $C$ can be set to $h_0/(x_0^{2\chi})$ to remove the coefficient, and then matching powers of $\tau$ yields:

$$2\chi(1+2\nu'\bar{v}^2) = 2\nu'\bar{\alpha} \implies \chi = \frac{\nu'\bar{\alpha}}{1+2\nu'\bar{v}^2} \tag{4.1.28}$$

Therefore:

$$f(\Delta\sigma) = \frac{h_0}{x_0^{2\chi}}(\Delta\sigma)^{2\chi} \tag{4.1.29}$$

and the correlator can be written as:

$$\big\langle h_{++}(\sigma-\sigma',\tau)\big\rangle \approx \frac{h_0}{x_0^{2\chi}}\frac{(\sigma-\sigma')^{2\chi}}{\tau^{2\nu'\bar{\alpha}}} = A\left(\frac{\sigma-\sigma'}{\tau^{1+2\nu'\bar{v}^2}}\right)^{2\chi} \tag{4.1.30}$$

In terms of physical time $t$ and $\ell$, this is given by:

$$\big\langle h_{++}(\sigma-\sigma',\tau)\big\rangle \approx A\left(\frac{\ell}{t}\right)^{2\chi} \tag{4.1.31}$$

Equivalently, in terms of scalar products between right or left unit vectors, which are related by $\sigma$ parity, this is:

$$\mathbf{p}_+(s,\tau)\cdot\mathbf{p}_+(s',\tau)=\mathbf{p}_-(s,\tau)\cdot\mathbf{p}_-(s',\tau)\approx 1-A(\ell/t)^{2\chi} \tag{4.1.32}$$

The above result can be used to determine the normalized spatial and temporal correlators defined as:

$$\begin{aligned}\mathrm{corr}_x(l,t)&\equiv\frac{\langle\mathbf{x}'(\sigma,\tau)\cdot\mathbf{x}'(\sigma',\tau)\rangle}{\langle\mathbf{x}'(\sigma,\tau)\cdot\mathbf{x}'(\sigma,\tau)\rangle}\approx 1-\frac{A}{2(1-\bar{v}^2)}\left(\frac{l}{t}\right)^{2\chi}\\ \mathrm{corr}_t(l,t)&\equiv\frac{\langle\dot{\mathbf{x}}(\sigma,\tau)\cdot\dot{\mathbf{x}}(\sigma',\tau)\rangle}{\langle\dot{\mathbf{x}}(\sigma,\tau)\cdot\dot{\mathbf{x}}(\sigma,\tau)\rangle}\approx 1-\frac{A}{2\bar{v}^2}\left(\frac{l}{t}\right)^{2\chi}\end{aligned} \tag{4.1.33}$$

representing the correlation between velocities and spatial tangent vectors at different points along the string. Intuitively, if these approach 1, the tangent vectors are parallel, meaning the string is straight, and similarly, the velocities are parallel, so there is a sort of coherent motion. It must be emphasized that the scaling behavior of these correlators is a consequence of the initial assumption in the stretching regime, that the expansion of the universe is the only relevant process.

To characterize a network of cosmic strings, a useful quantity is the fractal dimension, defined as the dimensionality of fractal object, i.e. those without integer dimension [55]. Typically, this is found by understanding the dependence of a network property on a certain scale in the system. In this case, one can use the RMS spatial distance between points along the segment and the length of the segment itself. The definition of the RMS spatial distance is:

$$\begin{aligned}\langle r^2(l,t)\rangle=&\langle\mathbf{x}'\cdot\mathbf{x}'\rangle\int_0^l dl'\int_0^l dl''\,\mathrm{corr}_x(l'-l'',t)\approx\\ \approx&(1-\bar{v}^2)\,l^2\left[1-\frac{A(l/t)^{2\chi}}{(2\chi+1)(2\chi+2)(1-\bar{v}^2)}\right]\equiv(1-\bar{v}^2)\,l^2\left[1-D\left(\frac{l}{t}\right)^{2\chi}\right]\end{aligned}$$

The fractal dimension is then defined as

$$d_f=\frac{2d\ln\ell}{d\ln\langle r^2(\ell,t)\rangle} \tag{4.1.34}$$

The direct computation starts from evaluating $\ln\langle r^2(l,t)\rangle$:

$$\ln\langle r^2(l,t)\rangle=\ln(1-\bar{v}^2)+2\ln\ell+\ln\left[1-D\left(\frac{l}{t}\right)^{2\chi}\right] \tag{4.1.35}$$

Differentiating with respect to $\ln\ell$ gives:

$$\frac{d\ln\langle r^2(l,t)\rangle}{d\ln\ell}=2-\frac{2\chi D(\ell/t)^{2\chi}}{1-D(\ell/t)^{2\chi}}\approx 2-2\chi D(\ell/t)^{2\chi}+\mathcal{O}((\ell/t)^{4\chi}) \tag{4.1.36}$$

Therefore:

$$\frac{d\ln\ell}{d\ln\langle r^2(l,t)\rangle}\approx\frac{1}{2}(1+\chi D(\ell/t)^{2\chi})+\mathcal{O}((\ell/t)^{4\chi}) \tag{4.1.37}$$

Hence:

$$d_f = 1 + \frac{A\chi(l/t)^{2\chi}}{(2\chi+1)(2\chi+2)(1-\bar{v}^2)} + \mathcal{O}((\ell/t)^{4\chi}) \tag{4.1.38}$$

At small scales, $\ell \ll d_H \sim t$, the fractal dimension approaches one, since the exponent $\chi$ is positive ($\chi_r \sim 0.10$ and $\chi_m \sim 0.25$). Thus, the strings approach smoothness in this limit. The form of the two point function found above has been compared with simulations in [8] and a good agreement was found in regimes where the model was actually not supposed to perfectly hold. Instead, there is a significance departure of numerical and analytical results at the small-scale limit, where the model should work. This has been subject of work in the past years where additional effects were considered, such as kink production [56] and small-scale wiggles propagating along the string [57]. However, all these studies assume a constant string tension and, as discussed in the previous chapter, this assumption need not hold in string compactifications during epochs in which the relevant moduli are still evolving. Therefore, it is natural to wonder how this model changes when the tension is allowed to vary in time. This will be the discussion of the upcoming sections, but I shall first provide the description of microscopic and macroscopic string dynamics with a time-dependent tension.

# 4.2 Time-varying tension dynamics

Consider a time-varying tension of the form $\mu(t) \sim t^{-m}$, with $m \geq 0$ a real parameter. As shown in section 3.3, $m$ is generally an order 1 number, $m = 1$ for kination and $m = 2/3$ for tracker epoch with the LVS potential. Allowing such a time dependence will naturally modify the equations of motion (1.4.7) describing the microscopic evolution of the strings, as well as the macroscopic evolution governed by the VOS model equations (1.4.40), (1.4.43), and (1.4.49). For this reason, this section will include the derivation of the modified equations of motion and proper generalization of the VOS model in this time-varying tension scenario.

## 4.2.1 Equations of motion

The action governing the dynamics of a string with non-constant tension can be simply obtained from the Nambu-Goto action (1.4.6). The generalization simply requires $\mu \to \mu(x)$, such that:

$$S = -\int d^2\zeta\, \mu\left(x^\nu(\zeta)\right) \sqrt{-\gamma} \tag{4.2.1}$$

where the tension is allowed to generally depend on spacetime coordinates for now. I will later restrict the dependence on time. Using the least action principle under a variation of the spacetime coordinates yields:

$$\delta S = \delta S_1 + \delta S_2 = \int d^2\zeta\, \mu\left(x^\nu(\zeta)\right) \delta\left(\sqrt{-\gamma}\right) - \int d^2\zeta\, \left(\delta\mu\left(x^\nu(\zeta)\right)\right) \sqrt{-\gamma} \tag{4.2.2}$$

The first terms gives the usual equations of motion (1.4.7), while the second term appears due to the varying tension. Indeed, since $\delta\mu(x^\nu) = \delta x^\nu \partial_\nu \mu$, it follows that:

$$\delta S_2 = -\int d^2\zeta \sqrt{-\gamma}\, \delta x^\nu \partial_\nu \mu \tag{4.2.3}$$

Therefore, the final form of the equations of motion is [58]:

$$\frac{1}{\mu(x)\sqrt{-\gamma}}\partial_a\left(\mu(x)\sqrt{-\gamma}\,\gamma^{ab}g_{\rho\sigma}x^\sigma_{,b}\right)-\frac{1}{2}\partial_\rho g_{\lambda\sigma}\gamma^{ab}x^\lambda_{,a}x^\sigma_{,b}-\frac{\partial_\rho\mu(x)}{\mu(x)}=0 \tag{4.2.4}$$

which can be manipulated to make the Christoffel symbols associated with the spacetime metric appear:

$$\frac{1}{\mu(x)\sqrt{-\gamma}}\partial_a\left(\mu(x)\sqrt{-\gamma}\,\gamma^{ab}\partial_b x^\sigma\right)+\Gamma^\sigma_{\mu\nu}\gamma^{ab}\partial_a x^\mu\partial_b x^\nu-\frac{\partial^\sigma\mu(x)}{\mu(x)}=0 \tag{4.2.5}$$

Now the dependence of the tension becomes only on physical time $x^0$. As in the constant tension case, to obtain a closed form it is necessary to specify the spacetime metric. We shall distinguish the cases of Minkowski spacetime and expanding flat FLRW universe.

### Minkowski spacetime

In this instance $g_{\mu\nu}=\eta_{\mu\nu}$ and so $\Gamma^\mu_{\rho\sigma}=0$. Following the same steps as the constant tension case leads to:

$$\ddot{\mathbf{x}}+\frac{\dot\mu}{\mu}(1-\dot{\mathbf{x}}^2)\dot{\mathbf{x}}-\frac{1}{\varepsilon}\left(\frac{\mathbf{x}'}{\varepsilon}\right)'=0 \qquad\qquad \dot\varepsilon+\frac{\dot\mu}{\mu}\varepsilon\dot{\mathbf{x}}^2=0 \tag{4.2.6}$$

where it is clear that the variation of the tension acts in a way similar to the scale factor in the case of an expanding universe. It is interesting to write these in terms of left and right moving unit vectors, for comparison with the PR framework. This gives:

$$\dot{\mathbf{p}}_\pm\mp\frac{1}{\varepsilon}\mathbf{p}'_\pm=-\frac{\dot\mu}{2\mu}\left[\mathbf{p}_\mp-\left(\mathbf{p}_+\cdot\mathbf{p}_-\right)\mathbf{p}_\pm\right]. \tag{4.2.7}$$

where now there is no distinction between cosmic and conformal time, but the physical time is still identified with worldsheet time $\zeta^0$. Notably, the left and right moving unit vectors are not independent, the coupling is precisely due to the time variation of the tension. In a constant tension scenario, the RHS of the above equation is zero and left- and right- movers evolve independently. As it will be clear in a moment, this effect will add up to the expansion of the universe.

### FLRW spacetime

Considering the Christoffel symbols associated with the flat FLRW metric and using the relation between $\varepsilon$ and $\gamma$ one can find the $\rho=i$ and $\rho=0$ equations of motion to be:

$$\ddot{\mathbf{x}}+\left(2\frac{\dot a}{a}+\frac{\dot\mu}{\mu}\right)(1-\dot{\mathbf{x}}^2)\dot{\mathbf{x}}-\frac{1}{\varepsilon}\left(\frac{\mathbf{x}'}{\varepsilon}\right)'=0 \qquad \dot\varepsilon+\left(2\frac{\dot a}{a}+\frac{\dot\mu}{\mu}\right)\varepsilon\,\dot{\mathbf{x}}^2=0 \tag{4.2.8}$$

where dots denote derivatives with respect to conformal time. At the microscopic level, these equations of motion are equivalent to those with constant tension but with a modified Hubble parameter:

$$2\tilde{\mathcal{H}}=2\mathcal{H}+\frac{\dot\mu}{\mu} \tag{4.2.9}$$

As pointed out in the previous paragraph, the variation of the tension enters the equations in the same way as the expansion of the universe. However, its sign can be positive or negative, and therefore constitute an enhancement of the usual Hubble damping or act as an anti-damping effect. This will turn out to be of great importance in understanding the small-scale structure of the strings. For completeness, let us write these in terms of left and right moving unit vectors, where it should be clear that the only modification regards the Hubble parameter. Indeed, the equations are:

$$\dot{\mathbf{p}}_{\pm} \mp \frac{1}{\epsilon}\mathbf{p}'_{\pm} = -\tilde{\mathcal{H}}\left[\mathbf{p}_{\mp} - \left(\mathbf{p}_{+}\cdot\mathbf{p}_{-}\right)\mathbf{p}_{\pm}\right]. \tag{4.2.10}$$

which highlights how both the expansion of the Universe and the variation of the tension couple the evolution of left and right moving unit vectors.

The variation of the tension leads to interesting phenomenology which has been explored in [42, 59, 60, 61, 62]. A significant case to investigate is the one where the interplay between the tension and the expansion of the universe balance each other. From the structure of the equations, this can happen if the modified Hubble parameter term vanishes. This condition can be treated as a differential equation giving the necessary dependence of $\mu$ on $a$:

$$2\frac{da}{d\tau}\frac{1}{a} + \frac{d\mu}{d\tau}\frac{1}{\mu} = 0 \implies \mu \sim a^{-2} \tag{4.2.11}$$

In such instance, the equations reduce to the constant tension scenario in Minkowski spacetime. For future reference, it is useful to translate this solution in a relation between the power of the tension $m$ and of the scale factor. This latter typically depends on cosmic time as $a(t) \sim t^{2/n}$. Then, the above condition is:

$$-m = -4/n \implies mn = 4 \tag{4.2.12}$$

This will be identified as the *threshold value* of $mn$. Values of $mn > 4$ indicate the variation of the tension dominates against the expansion, while for $mn < 4$ it is the opposite. For example, volume-modulus kination ($m = 1$, $n = 6$) falls into the $mn > 4$ category, while matter and radiation trackers ($m = 2/3$, $n = 3, 4$) are cases of $mn < 4$.

To illustrate another phenomenological effect and for later purposes, let us consider the case of a circular loop. This is one of the most studied configurations as it ensures analytical control most of the times. The configuration requires $\mathbf{x}(\tau,\zeta) = \mathbf{x}(\tau,\zeta + 2\pi L)$, which in turn allows for the loop trajectory to be written as $\mathbf{x}(\tau,\zeta) = R(\tau)L\hat{u}(\zeta)$, with $L$ a dimensionful length, $R(\tau)$ a dimensionless function which describes the time evolution of the loop amplitude, and $\hat{u}(\zeta)$ the unit vector describing the spatial form of the string. For simplicity, consider a loop along the $z$-plane, so that $\hat{u} = (\cos(\zeta/L), \sin(\zeta/L), 0)$. The string trajectory is thus:

$$\mathbf{x}(\tau,\zeta) = R(\tau)L(\cos(\zeta/L), \sin(\zeta/L), 0) \tag{4.2.13}$$

Using this ansatz, some useful relations are:

$$\varepsilon = \frac{R}{\sqrt{1-\dot{R}^2L^2}}, \qquad \frac{\mathbf{x}'}{\varepsilon} = L\frac{\partial\hat{u}}{\partial\zeta}\sqrt{1-\dot{R}^2L^2} \implies \frac{1}{\varepsilon}\left(\frac{\mathbf{x}'}{\varepsilon}\right)' = \frac{-\hat{u}}{RL}(1-\dot{R}^2L^2)$$

Therefore, upon substitution of (4.2.13) in the spatial equation of motion, one has:

$$\ddot{R}L\hat{u} + 2\tilde{\mathcal{H}}(1-\dot{R}^2L^2)\dot{R}L\hat{u} + \frac{\hat{u}}{RL}(1-\dot{R}^2L^2) = 0 \tag{4.2.14}$$

Focusing on the components:

$$\ddot{R} + 2\tilde{\mathcal{H}}(1 - \dot{R}^2 L^2)\dot{R} + \frac{(1 - \dot{R}^2 L^2)}{RL^2} = 0 \tag{4.2.15}$$

while the $\varepsilon$ equations reads:

$$\frac{\dot{\varepsilon}}{\varepsilon} = -2\tilde{\mathcal{H}} L^2 \dot{R}^2(\tau) \tag{4.2.16}$$

Note that these differ from the equations in [59] because we are working in conformal time while that paper uses cosmic time. Of course, these are related by $dt = a(t)d\tau$, so they are equivalent. The important thing to remember is that comoving scales are related to physical ones by factors of $a(t)$. For instance, the physical and comoving radii of the loop are given by:

$$R_{phys}(\tau) = a(\tau)R_{com}(\tau) \qquad \text{with} \qquad R_{com}(\tau) = R(\tau)L \tag{4.2.17}$$

In particular, the definition of $\varepsilon$ can be used to infer its meaning with regards to the loop amplitude. Indeed, an oscillating loop has zero velocity at the point of maximum amplitude and $\dot{\mathbf{x}}(\tau, \zeta) = \dot{R}(\tau)L\hat{u}$. Therefore, at such point:

$$\varepsilon_\tau = \left.\frac{R}{\sqrt{1 - \dot{R}^2 L^2}}\right|_{max} = R_{max} \implies \varepsilon_t = aR_{max} \tag{4.2.18}$$

In cosmic time, $\varepsilon_t$ represents the maximum physical radius of the loop [59]. In the constant tension case, the r.h.s of (4.2.16) is $-\mathcal{H}$ since fast oscillating loops have $\langle L^2 \dot{R}^2(\tau)\rangle \simeq 1/2$. This implies $\varepsilon_\tau \sim a^{-1}(\tau)$, therefore the physical radius of the loop is $a\varepsilon_\tau \sim 1$ and the maximum amplitude of the loop is constant in physical coordinates, thus decreasing in comoving ones. The condition for them to grow in comoving coordinates is $\dot{\varepsilon}/\varepsilon > 0$ [59], which requires a negative modified Hubble parameter, since $L^2 \dot{R}^2(\tau) > 0$. Assuming $a(t) \sim t^{2/n}$, it follows that $a(\tau) \sim \tau^{2/(n-2)}$, therefore we can write:

$$2\tilde{\mathcal{H}} = \left(\frac{4}{n-2} - \frac{mn}{n-2}\right)\frac{1}{\tau} = \frac{4}{n-2}\left(1 - \frac{mn}{4}\right)\frac{1}{\tau} \equiv \frac{2\nu'\kappa}{\tau} \tag{4.2.19}$$

Equivalently, in cosmic time this is:

$$2\tilde{H} = \left(\frac{4}{nt} - \frac{m}{t}\right) = \frac{4}{n}\left(1 - \frac{mn}{4}\right)\frac{1}{t} \equiv \frac{2\kappa\nu}{t} \tag{4.2.20}$$

where $\nu$ and $\nu'$ are the conventions used in [8] for the evolution of the scale factor in cosmic and conformal time, respectively, while $\kappa$ was defined as:

$$\kappa \equiv 1 - mn/4 \tag{4.2.21}$$

It is thus clear that a negative modified Hubble parameter is achieved for $\kappa < 0$, or equivalently for $mn > 4$ ($n$ is greater than 2 in all cosmologically relevant evolutionary periods). The best conditions for loops to grow is achieved for smallest values of $\mathcal{H}$, thus a slowly varying scale factor, and rapidly varying tension. The first condition suggests a kination environment, where the scale factor evolution is slower than in matter and radiation domination, $a(t)_{kin} \sim t^{1/3}$. Consequently, the time variation of the tension in a volume modulus kination environment is $\mu(t) \sim t^{-1}$, as shown in section 3.3. In this setting, $m = 1$ and $n = 6$, therefore $mn > 4$, providing the conditions for loops to grow.

However, the growth mechanism does not appear to be restricted to a purely kinating background. Indeed, the dynamical system analysis carried out in [60], with the typical exponential potential of section 3.2, shows that besides the kination fixed point, this condition can also be realized in other regions of phase space, in particular in modulus-dominated and scaling fixed points. This is especially relevant because scaling solutions can be stable, unlike kination, and therefore may support a longer growth epoch. For fundamental strings this can occur when the volume modulus rolls in a suitable power-law potential.

### 4.2.2 VOS model with time-varying tension

Let us consider the effects of the varying tension on the macroscopic evolution of cosmic strings network. As outlined in section 1.4.3, this is well captured by the VOS model, whose fundamental equations determine the time evolution of the characteristic length, density, and velocity of the network. With similar steps, one can generalize those equations to the case of a varying tension. First of all, the energy of a string with varying tension is simply given by:

$$E(\tau) = \mu(\tau)a(\tau)\int d\zeta^1 \varepsilon \tag{4.2.22}$$

Differentiating this with respect to conformal time leads to:

$$\dot{E} = \mathcal{H}E(1-2\bar{v}^2) + \frac{\dot{\mu}}{\mu}E(1-\bar{v}^2) \tag{4.2.23}$$

Then, using $\rho \sim E/a^3$, this becomes an equation for the string energy density:

$$\frac{d\rho}{dt} = -2H(1+\bar{v}^2)\rho + \frac{1}{\mu}\frac{d\mu}{dt}(1-\bar{v}^2)\rho \tag{4.2.24}$$

Considering loop chopping processes for long strings, this can be casted into an evolution equation for $L$ (not to be confused with the $L$ in 4.2.13), defined by $\rho = \mu(t)/L^2$. This implies:

$$\frac{d\rho}{dt} = \frac{d\mu}{dt}\frac{1}{L^2} - \frac{2\mu}{L^3}\frac{dL}{dt} \tag{4.2.25}$$

which leads to

$$2\frac{dL}{dt} = 2HL(1+\bar{v}^2) + \tilde{c}\bar{v} + \frac{L}{\mu}\frac{d\mu}{dt}\bar{v}^2 \tag{4.2.26}$$

Instead, the new velocity evolution equation simply involves the replacement of $2H$ with $2\tilde{H} = 2H + \frac{1}{\mu}\frac{d\mu}{dt}$:

$$\frac{d\bar{v}}{dt} = (1-\bar{v}^2)\left[\frac{k(\bar{v})}{L} - \left(2H + \frac{1}{\mu}\frac{d\mu}{dt}\right)\bar{v}\right] \tag{4.2.27}$$

where the curvature terms provides an acceleration, while the expansion of the universe and variation of the tension act as friction, but now the dominant contribution depends on the sign of the modified Hubble parameter. Overall, the fundamental equations of the VOS model with time-varying tension are (4.2.24), (4.2.26), and (4.2.27). Some comments on the physical impact of these equations are in order. First of all, the existence of fixed points requires:

$$\frac{k(\bar{v})}{L} = \left(2H + \frac{1}{\mu}\frac{d\mu}{dt}\right)\bar{v} \tag{4.2.28}$$

explicitly showing the sign of $k(\bar{v})$ corresponds to the one of the modified Hubble parameter, and it is thus negative for $mn > 4$ and positive when $mn < 4$. Furthermore, from the explicit phenomenological form of the momentum parameter, $k(\bar{v})$ is positive for $\bar{v}^2 < 1/2$, vanishes at $\bar{v}^2 = 1/2$, and becomes negative for $\bar{v}^2 > 1/2$. Therefore, in the tension-dominated regime $mn > 4$, where a non-trivial fixed point requires $k(\bar{v}) < 0$, the fixed point cannot lie on the branch $\bar{v}^2 \leq 1/2$. This point will be clarified further in section 5.1. Indeed, analysis of small perturbations of a straight string with time varying tension suggests a sign flip of the momentum parameter when $mn$ crosses the threshold value [42]. This points towards the existence of a scaling fixed point with $\bar{v}^2 > 1/2$ and $mn > 4$. This last statement regarding the velocity and the value of $mn$ will be made precise in several ways as it will turn out to be important in determining the scaling behavior of the correlator in the small-scale structure context.

Another notable effect of time varying tension is that scaling solutions will not be characterized by a string energy density which is a constant fraction of the Universe one anymore. Indeed, scaling solutions have a constant ratio $L/t = \xi$, and the tension behaves as $\mu(t) = \mu_0(t_0^m/t^m)$, with $t_0$ some initial time. Therefore, string energy density will be:

$$\rho = \frac{\mu}{L^2} = \frac{\mu_0 t_0^m}{\xi^2 t^{2+m}} \tag{4.2.29}$$

and, in analogy to (1.4.37), the same ratio can be written as:

$$\frac{\rho_s}{\rho_{tot}} = \frac{\mu_0 t_0^m}{\xi^2 t^{2+m}} \frac{32\pi G_N t^2}{3} = \frac{32\pi}{3\xi^2}(G_N\mu_0)\left(\frac{t_0}{t}\right)^m \tag{4.2.30}$$

This ratio decreases in time for $m > 0$, remains constant for $m = 0$, and grows for $m < 0$. Having understood the modifications to the microscopic string dynamics and network evolution brought by the time-varying tension, the natural question regards whether the small-scale structure of the strings in this setting features a different behavior than the constant tension case. This leads to a generalization of the PR framework discussed in section 4.1, which will be presented in the next section.

### 4.2.3 Extension of PR framework

In what follows, I extend the analysis of Polchinski and Rocha to the case of a time-varying tension. To the best of my knowledge, the PR small-scale framework has so far been formulated for strings with constant tension, so this extension represents one of the original contributions of this thesis. As in the original framework, the starting point is to consider the equations of motion written in terms of left and right moving unit vectors (4.2.10), which we report here for clarity:

$$\dot{\mathbf{p}}_\pm \mp \frac{1}{\epsilon}\mathbf{p}'_\pm = -\tilde{\mathcal{H}}\left[\mathbf{p}_\mp - \left(\mathbf{p}_+ \cdot \mathbf{p}_-\right)\mathbf{p}_\pm\right]. \tag{4.2.31}$$

Everything proceeds as in the constant tension case, provided that the original assumption remain valid. Gravitational backreaction can be neglected by restricting to segment lengths $\ell$ larger than the length at which these effects become relevant, which is schematically given by $\ell_{\rm GW} \sim (G\mu)^p d_H$, with $p > 0$. Since the dimensionless time-varying tension of cosmic (super)strings is bounded by observation to small values, this is indeed negligible. In addition, intercommutation events are also subleading for sufficiently short segments: as in the constant-tension case, the probability per Hubble time for a segment of physical length $\ell$ to participate in a long-string intercommutation is suppressed by $\ell/d_H$, up to the reconnection probability. Thus, in the intermediate range

$$\ell_{\rm GW}(t) \ll \ell \ll d_H(t),$$

stretching remains the dominant effect. Then, one can replace $\dot{\mathbf{x}}^2$ with $\bar{v}^2$ in the equation for $\varepsilon$. This is justified provided the coefficient multiplying $\dot{\mathbf{x}}^2$ in the equation for $\varepsilon$ varies only on cosmological time scales, while the local oscillations of the strings occur on much shorter time scales. In the present case this coefficient is the modified Hubble parameter $\tilde{\mathcal{H}} = \mathcal{H} + \dot{\mu}/(2\mu)$. For a power law tension evolution $\mu \sim t^{-m}$, with $m = O(1)$, both $\mathcal{H}$ and $\dot{\mu}/\mu$ vary on the Hubble time scale. Therefore the same averaging assumption used in the constant tension PR framework can be applied and one can then write:

$$\frac{\dot{\varepsilon}}{\varepsilon} = -2\tilde{\mathcal{H}}\bar{v}^2 = -\frac{2\kappa\nu'}{\tau}\bar{v}^2 \tag{4.2.32}$$

This implies the dependence on conformal time of $\varepsilon$ is

$$\varepsilon = \varepsilon_0 \tau^{-2\kappa\nu'\bar{v}^2} \sim a^{-2\kappa\bar{v}^2}$$

and so similarly to the constant tension case, the ratio between the segment length and horizon size is:

$$\frac{\ell}{d_H} \propto \frac{a\varepsilon(\sigma-\sigma')}{t} \propto a^{1-2\kappa\bar{v}^2-1/\nu}(\sigma-\sigma') \propto \frac{\sigma-\sigma'}{\tau^{1+2\kappa\nu'\bar{v}^2}} \tag{4.2.33}$$

The correlator $h_{++}$ then will satisfy the same equation as before, just with the modified Hubble parameter, leading to an additional factor of $\kappa$ in the exponential. The structure of the solution is also analogous, only the exponent $\chi$ is replaced by $\chi'$:

$$\big\langle h_{++}(\sigma-\sigma',\tau)\big\rangle \approx \tilde{A}\left(\frac{\ell}{t}\right)^{2\chi'} \qquad \text{with} \quad \chi' = \frac{\kappa\nu'\bar{\alpha}}{1+2\kappa\nu'\bar{v}^2} \tag{4.2.34}$$

where $\tilde{A}$ is in principle a different constant than the original one, but could still be inferred from simulations. Everything boils down to understanding the sign of $\chi'$, which depends on the sign of $\kappa$, hence on the vale of the product $mn$. As mentioned before, this determines the existence of scaling fixed points investigated in [42], but it is also linked to the value of $\bar{v}^2$, which is found to be greater or smaller than $1/2$ when $\kappa < 0$ and $\kappa > 0$, respectively. Therefore, the sign of $\bar{\alpha} = 1 - 2\bar{v}^2$ in the numerator will also be affected. In particular, the results of [42] can be summarized as:

$$\begin{cases} mn < 4 \implies k(v) > 0, v^2 < 1/2 \\ mn > 4 \implies k(v) < 0, v^2 > 1/2 \end{cases} \tag{4.2.35}$$

Thus, going back to the sign of (4.2.34), we have:

$$\begin{cases} mn > 4 \implies \kappa < 0 \\ \bar{v}^2 > 1/2 \implies \bar{\alpha} < 0 \end{cases} \qquad \begin{cases} mn < 4 \implies \kappa > 0 \\ \bar{v}^2 < 1/2 \implies \bar{\alpha} > 0 \end{cases} \tag{4.2.36}$$

However, to understand the sign of $\chi'$ one also needs to look at its denominator:

$$1 + 2\kappa\nu'\bar{v}^2 > 0 \implies mn < 4 + \frac{2}{\nu'\bar{v}^2} = 4 + \frac{n-2}{\bar{v}^2} \tag{4.2.37}$$

Therefore, when $mn > 4$ the denominator is not guaranteed to be negative, this happens only if $mn > 4 + \frac{2}{\nu'\bar{v}^2}$. In this case, the numerator $\kappa\nu'\bar{\alpha}$ is positive since both $\kappa$ and $\bar{\alpha}$ are negative, making the exponent $\chi'$ negative, which is something that could not have happened in the constant tension case. Thus, there are three relevant regimes:

$$\begin{cases} mn < 4 \implies \chi' > 0 \\ 4 < mn < 4 + \dfrac{2}{\nu'\bar{v}^2} \implies \chi' > 0 \\ mn > 4 + \dfrac{2}{\nu'\bar{v}^2} \implies \chi' < 0 \end{cases}$$

When $\chi' > 0$, the results are the same as in the constant tension case. However, when $\chi' < 0$, the correlators in (4.1.33), that are the same in the varying tension case just with $\chi \to \chi'$, diverge at small scales $\ell \ll t$, signaling the framework is breaking down. Indeed, the negative sign of the denominator implies the ratio $\ell/d_H$ increases in time, contrary to the assumption $\ell \ll d_H$ of the stretching regime. This would correspond to string segments being superhorizon, which is not the case we are looking for to discuss the small-scale structure. Therefore, this parameter range lies outside the domain of validity of the present small-scale extension and the sign of the new exponent is fully determined by the numerator. Strikingly, this seems to prefer an overall positive sign as a result of a cancellation between the sign $\kappa$ and $\bar{\alpha}$.

To better understand the sign of this exponent, it is natural to investigate more deeply the deviation of the RMS velocity from the Minkowskian value of 1/2. However, this also raises an important question regarding the development of correlations based on the scale of the system in consideration. Indeed, the entire framework focuses on long strings network properties, and analyses the correlations in this setting. The question is then whether these correlations play an important role also beyond the network scale, for instance for deep subhorizon physics, and whether the implications are the same. Therefore, this motivates the comparison between network dynamics and subhorizon physics, which shall be described in the following chapter, taking the circular loop as a proxy for subhorizon physics.

# Chapter 5

# Subhorizon dynamics and consistency of PR approximation

In this chapter, insights about the squared velocity are derived from several physical settings which should be carefully distinguished. It should be kept in mind that the RMS velocity relevant for the sign of the modified exponent is a network averaged quantity, arising after integration over Hubble times and considering an ensemble of string segments. This naturally suggests a treatment starting from the VOS model, whose evolution equations are precisely designed to describe the averaged dynamics of the long-string network. At the same time, the dependence of the exponent on the velocity raises a broader question. If a time-dependent tension can modify the network averaged velocity and hence the development of correlations on long strings, one may ask whether analogous effects also appear in local, deep-subhorizon dynamics. This motivates a comparison between two physically distinct regimes. First, we consider a network of long strings, for which the VOS model is found to provide the appropriate macroscopic averaged description of the velocity shift induced by a time-dependent tension. Second, we study a circular loop as a controlled proxy for deep-subhorizon physics. The purpose of the latter calculation is not to replace the network-averaged velocity entering the correlator, but to test whether the same time dependence can generate relevant local corrections. Our results will show the loop dynamics does receive relevant corrections which, however, are Hubble-scale suppressed and do not lead to a robust deviation from the Minkowskian value of the squared velocity.

## 5.1 VOS approach

The velocity equation in the varying-tension VOS model, (4.2.27), involves a fundamental quantity which is the momentum parameter, $k(\bar{v})$. This actually includes effects of small scale structure along the strings, albeit one does not capture their origin in this macroscopic description. In the literature, the following phenomenological expression for the momentum parameter has been widely used [22]:

$$k(\bar{v}) = \frac{2\sqrt{2}}{\pi}(1-\bar{v}^2)\left(1+2\sqrt{2}\,\bar{v}^3\right)\frac{1-8\bar{v}^6}{1+8\bar{v}^6} \tag{5.1.1}$$

which has been inferred from the study of helicoidal strings, together with other analytical considerations, and it reproduces the correct relativistic and non relativistic limits of

$k(\bar{v})$. In particular, it vanishes in the Minkowskian case of $\bar{v}^2 = 1/2$. Instead, deviations from this value determine the sign of this important quantity. Since the other terms are positive for $\bar{v}^2 \in [0, 1]$, the sign of $k(\bar{v})$ is solely determined by $1 - 8\bar{v}^6$, which is negative for $\bar{v}^2 > 1/2$ and positive for $\bar{v}^2 < 1/2$, as anticipated in section 4.2.2. For the present purposes, it is sufficient to take this parameter as encoding the possible small scale structure and elaborate on it.

Assuming the RMS squared velocity can be written as $\bar{v}^2 = 1/2 + \delta$, with $\delta$ a small correction whose sign is to be determined, one can simplify the expression for $k(\bar{v})$ at linear order in $\delta$. Indeed:

$$\left.\begin{aligned} \bar{v}^3 &\simeq \frac{1}{2\sqrt{2}}(1+3\delta) + \mathcal{O}(\delta^2) \\ \bar{v}^6 &\simeq \frac{1}{8}(1+6\delta) + \mathcal{O}(\delta^2) \end{aligned}\right\} \quad \Longrightarrow \quad k(\bar{v}) \simeq -\frac{6\sqrt{2}}{\pi}\delta + \mathcal{O}(\delta^2) \tag{5.1.2}$$

Next, consider the equations governing the evolution in time of the velocity (4.2.27) and the characteristic length (4.2.26) in the varying tension VOS model. These can be re-casted as the autonomous system [42]:

$$\begin{cases} \dfrac{d\xi}{dz} = \xi\left[\left(\dfrac{2}{n} - 1\right) + \dfrac{v^2}{2}\left(\dfrac{4}{n} - m\right)\right] + \dfrac{\tilde{c}(v)v}{2}, \\ \dfrac{dv}{dz} = (1 - v^2)\left[\dfrac{k(v)}{\xi} - \left(\dfrac{4}{n} - m\right)v\right]. \end{cases} \tag{5.1.3}$$

via the definitions $z = \log t$ and $L = \xi(t)t$. The existence of scaling fixed points with $\xi$ and $v$ constants leads to the solutions:

$$\bar{\xi} = \sqrt{\frac{n}{8}\frac{k(\bar{v})\left(k(\bar{v}) + \tilde{c}\right)}{\left(1 - \frac{mn}{4}\right)\left(1 - \frac{2}{n}\right)}}, \qquad \bar{v} = \sqrt{\frac{n}{2}\frac{k(\bar{v})}{k(\bar{v}) + \tilde{c}}\frac{1 - \frac{2}{n}}{1 - \frac{mn}{4}}}. \tag{5.1.4}$$

Squaring the equation for $\bar{v}$ and substituting the linear order expression for the momentum parameter leads to:

$$\bar{v}^2 = \frac{n-2}{2(1 - \frac{mn}{4})}\frac{1}{1 - \frac{\tilde{c}\pi}{6\sqrt{2}\delta}} \simeq -\frac{n-2}{2(1 - \frac{mn}{4})}\frac{6\sqrt{2}\delta}{\tilde{c}\pi} \tag{5.1.5}$$

where the last step is justified provided that $|\delta| < \tilde{c}\pi/(6\sqrt{2}) \sim 0.085$, assuming $\tilde{c} = 0.23$ as inferred from numerical simulations [23]. This does not make the physical effect irrelevant, as it corresponds to a change of order 10% in the velocity with respect to the Minkowskian value of 1/2, far from negligible. Moreover, if we define $\Delta = mn - 4$, the r.h.s is of order $\mathcal{O}(\delta/\Delta)$, while the l.h.s is finite, $\bar{v}^2 \in [0, 1]$. This suggests $\delta \sim \mathcal{O}(\Delta)$, hence we can write $\bar{v}^2 = 1/2$ on the l.h.s, leading to:

$$\frac{1}{2} \simeq \frac{12\sqrt{2}(n-2)}{\tilde{c}\pi}\frac{\delta}{\varepsilon} \quad \Longrightarrow \quad \delta \simeq \frac{\tilde{c}\pi}{24\sqrt{2}(n-2)}(mn - 4) \tag{5.1.6}$$

Therefore:

$$\bar{v}^2 = \frac{1}{2} + \delta \simeq \frac{1}{2} + \frac{\tilde{c}\pi}{24\sqrt{2}(n-2)}(mn - 4) \tag{5.1.7}$$

Since in cosmological settings $n > 2$, the sign of this correction is solely determined by the difference of $mn$ from 4. Specifically, when the tension is dominating with respect to the Hubble expansion, $mn > 4$, the correction is positive, $\delta > 0$, leading to a negative $\bar{\alpha}$. Meanwhile, $\kappa = 1 - mn/4 < 0$, making the overall exponent $\chi' > 0$. Equivalently, the numerator in this exponent is effectively of order $\Delta^2$, hence independent on the sign of $mn - 4$.

Furthermore, to gain physical intuition, it is interesting to relate this correction to the correlation between left and right moving unit vectors along the string. Recall that $\bar{\alpha} = 1 - 2\bar{v}^2 = -\langle \mathbf{p}_+ \cdot \mathbf{p}_- \rangle$, which implies, using the above equation for $\bar{v}^2$:

$$\langle \mathbf{p}_+ \cdot \mathbf{p}_- \rangle = 2\delta = \frac{\tilde{c}\pi}{12\sqrt{2}(n-2)}(mn-4) \tag{5.1.8}$$

Interestingly, this suggests that the interplay between the tension and Hubble expansion directly influences the correlation between left and right moving unit vectors along the string. In the constant tension case, $\bar{\alpha}$ could only be positive, so only negative correlation was allowed. Instead, when the tension is allowed to vary and dominates against the expansion, $mn > 4$, the correlation can become positive, implying left and right movers behave similarly and potentially justifying a smooth structure of the strings. The potential implications of this observations are consolidated in section 5.4.

# 5.2 Circular loop: formal solution

As mentioned in section 4.2, the circular loop is a particular solution to the string equations of motion which typically allows for greater analytical treatment than other more complicated solutions. To this end, I shall consider this as a representative for the study of the role of time varying tension in the context of subhorizon dynamics. I will present original calculations regarding the squared velocity of the loop, using this as a diagnostic quantity where to investigate possible corrections due to the variation of the tension. However, it should be kept in mind that this is different than the network averaged velocity appearing in the PR scaling exponent. The purpose of this analysis is therefore to test whether a time-dependent tension can generate significant correlations or whether such effects are parametrically suppressed in the deep-subhorizon regime.

A first approach consists of formally solving the equation for $\varepsilon$ (4.2.8) and expanding for subhorizon modes. Following [42], for $\mu \sim t^{-m}$ and $a \sim t^{2/n}$, the formal solution is:

$$\varepsilon(\tau) = \varepsilon(\tau_0) \exp\left[-q \int_{\tau_0}^{\tau} \frac{d\tau}{\tau} \dot{\mathbf{x}}^2\right]. \tag{5.2.1}$$

where $q = (4/n - m)/(1 - 2/n) = 2(1 - mn/4)\nu' = 2\kappa\nu'$, using the previous definition of $\kappa$. Because of causality, $\dot{\mathbf{x}}^2 \in [0, 1]$, implying no poles for $\varepsilon(\tau)$. However, the very definition of $\varepsilon$ leads to poles when the string velocity approaches unity, $\dot{\mathbf{x}}^2 = 1$, i.e. the string moves at the speed of light. Focusing on the circular loop, this suggests looking for solutions with:

$$\dot{R}(\tau)L = -\sin(f(\tau/L)) \implies \dot{\mathbf{x}}^2 = \sin^2(f(\tau/L)) \tag{5.2.2}$$

which through (5.2.1) implies:

$$R(\tau) = \exp\left[-q\int_{\tau_0}^{\tau}\frac{d\bar{\tau}}{\bar{\tau}}\sin^2\left(f\left(\frac{\bar{\tau}}{L}\right)\right)\right]\cos\left(f\left(\frac{\tau}{L}\right)\right). \tag{5.2.3}$$

Consistency conditions of these equations leads to the following differential equation for the function $f(x)$, with $x \equiv \tau/L$:

$$2x^2 f''(x) - qxf'(x)\left(1 - 3\cos(2f(x))\right) - q\left(1 + q\sin^2(f(x))\right)\sin(2f(x)) = 0. \tag{5.2.4}$$

where a prime denotes derivatives with respect to $x$. This equation was first presented in [42] and the details of the derivation can be found in Appendix B. In the limit where $f(x) \gg 1$, the phase of the trigonometric functions is rapidly oscillating, therefore those can be replaced by their averages over a period, namely $\langle\cos(2f)\rangle \sim \langle\sin(2f)\rangle \sim 0$, such that the equation is:

$$2x^2 f''(x) - qxf'(x) = 0 \tag{5.2.5}$$

whose solutions can be found with a standard power law ansatz, $f(x) \sim x^p$, which gives $p = 0$ and $p = 1 + q/2$, for $q \neq -2$. Thus:

$$f(x) = A_1 + A_2 x^{1+q/2} \tag{5.2.6}$$

Instead, when $f(x) \ll 1$, $\sin(2f) \simeq 2f$, $\cos(2f) \simeq 1$, and $\sin^2(f) \simeq 0$, the averaged equation is:

$$x^2 f''(x) + qxf'(x) - qf(x) = 0 \tag{5.2.7}$$

whose solution is, for $q \neq -1$:

$$f(x) = A_1 x + A_2 x^{-q} \tag{5.2.8}$$

The subhorizon regime is $x \gg 1$ and in such regime the loop is expected to oscillate many times over cosmological timescales, so it should have a rapidly varying phase. As it will be clear later on, $1 + q/2 > 0$ for the allowed values of $q$, such that the condition just described is satisfied by the large $f(x)$ solution. Therefore, $f(x) \gg 1$ is self-consistent and corresponds to the subhorizon oscillatory regime. Note that this also reproduces the correct Minkowski behavior in the constant tension case ($q = 0$) [42]. Moreover, the integration constants can be set by requiring $f(x_0) = 0$ and $f'(x_0) = 1$.

To investigate possible corrections induced by the time variation of the tension, we will now use the semi-analytic solution reviewed above to perform new calculations not found in the current literature. In particular, to compute the average squared velocity of the loop from $x_0$ to $x$ one has to evaluate:

$$\left\langle v^2\right\rangle_{[x_0,x]} = \frac{1}{x - x_0}\int_{x_0}^{x} d\bar{x}\, R'(\bar{x})^2. \tag{5.2.9}$$

To do so, first note that (5.2.3) can be written as $R(x) = A(x)\cos f(x)$, where $A(x) \equiv \exp\left(-q\int\frac{dx}{x}\sin^2(f(x))\right)$. Then:

$$R'(x) = -A(x)\left[f'(x)\sin f(x) + \frac{q}{x}\sin^2 f(x)\cos f(x)\right] \tag{5.2.10}$$

which implies, omitting the dependence of $A$ on $x$ for simplicity:

$$\begin{aligned} R'(x)^2 =& A^2 f'(x)^2 \sin^2 f(x) + A^2 \frac{2q}{x} f'(x) \sin^3 f(x) \cos f(x) + A^2 \frac{q^2}{x^2} \sin^4 f(x) \cos^2 f(x) \\ \equiv& T_1 + T_2 + T_3 \end{aligned} \tag{5.2.11}$$

We are interested in the subhorizon regime, where $f(x) \sim A_2 x^\alpha$ with integration constant $A_2 = \frac{1}{\alpha} x_0^{-q/2}$ and $\alpha \equiv 1 + q/2$, while $A_1$ si irrelevant since it enters only in the arguments of the trigonometric functions and thus can be neglected via a constant shift of $x$. For fast oscillations, one can approximate $A(x)$ as:

$$A(x) \sim \exp\left(-\frac{q}{2} \int_{x_0}^{x} \frac{d\bar{x}}{\bar{x}}\right) = \left(\frac{x}{x_0}\right)^{-\frac{q}{2}} \tag{5.2.12}$$

It is also convenient to define $y \equiv A_2 x^\alpha$, which leads to:

$$y_0 = A_2 x_0^\alpha = \frac{x_0}{\alpha} \qquad\qquad dx = \frac{1}{\alpha} A_2^{-\frac{1}{\alpha}} y^{\frac{1}{\alpha}-1} dy \tag{5.2.13}$$

Each term in (5.2.11) has to be averaged over the interval $[x_0, x]$. Taking the dominant terms for the trigonometric functions one has:

$$\sin^2 y \sim \frac{1}{2}, \qquad \sin^3 y \cos y \sim 0, \qquad \sin^4 y \cos^2 y \sim \frac{1}{16} \tag{5.2.14}$$

The first term is:

$$\langle T_1 \rangle_{[x_0,x]} = \frac{1}{x - x_0} \int_{x_0}^{x} d\bar{x}\, A(\bar{x})^2 f'(\bar{x})^2 \sin^2 f(\bar{x}) = \frac{1}{2} \tag{5.2.15}$$

which correctly reproduces the Minkowskian result. Instead, the second term vanishes:

$$\langle T_2 \rangle_{[x_0,x]} = \frac{1}{x - x_0} \int_{x_0}^{x} d\bar{x}\, \frac{2q}{\alpha} \frac{1}{y} \sin^3 y \cos y \longrightarrow 0 \tag{5.2.16}$$

Finally, the third term is:

$$\langle T_3 \rangle_{[x_0,x]} = \frac{1}{x - x_0} \int_{x_0}^{x} d\bar{x}\, \frac{q^2}{\alpha^2} \frac{1}{y^2} \sin^4 y \cos^2 y = \frac{q^2}{16\alpha^2(1-2\alpha)} \frac{y^{1/\alpha-2} - y_0^{1/\alpha-2}}{y^{1/\alpha} - y_0^{1/\alpha}}. \tag{5.2.17}$$

Overall, the velocity is:

$$\left\langle v^2 \right\rangle_{[x_0,x]} = \langle T_1 \rangle_{[x_0,x]} + \langle T_2 \rangle_{[x_0,x]} + \langle T_3 \rangle_{[x_0,x]} \simeq \frac{1}{2} + \frac{q^2}{16\alpha^2(1-2\alpha)} F_\alpha(y, y_0) \tag{5.2.18}$$

where $F_\alpha(y, y_0)$ was defined as:

$$F_\alpha(y, y_0) = \frac{y^{1/\alpha-2} - y_0^{1/\alpha-2}}{y^{1/\alpha} - y_0^{1/\alpha}}. \tag{5.2.19}$$

At first sight, the sign of this prefactor appears to change when $1 - 2\alpha$ interpolates between $\alpha > 1/2$ and $\alpha < 1/2$, or equivalently, when $q > -1$ and $q < -1$. However, this interpretation is misleading, since the fraction dependent on the initial and final phases, $F_\alpha(y, y_0)$, changes sign at the same point. For $y > y_0$ and $\alpha > 0$, the denominator is positive, while the numerator has the sign of $1/\alpha - 2 = (1 - 2\alpha)/\alpha$. Therefore, $\text{sign}(F_\alpha) = 1 - 2\alpha$, compensating the apparent sign change of the prefactor. The full $\mathcal{O}(q^2)$ contribution is thus non-negative, consistently with the positivity of the original integrand in (5.2.17). As a direct consequence of this result, if one were to insert this in the numerator of the modified exponent $\chi'$, this would be proportional to $\Delta^3$, not of order $\Delta^2$ as the macroscopic VOS argument suggested, with $\Delta = mn - 4$. However, the important observation to be made regards the size of this correction. The first non zero contribution arises from $T_3$ which is of order $1/x^2$, meaning of order of the loop dimensionful length $L$ squared in Hubble units, $\mathcal{O}((\mathcal{H}L)^2)$. This highlights the correction is extremely suppressed due to the subhorizon nature of the loop, independently of the detailed sign discussion.

However, one could argue the result is not completely correct due to the replacement of the trigonometric functions with their average contributions (5.2.14), which may not be well justified. Indeed, the typical form of the integrals entering the average squared velocity is:

$$\frac{1}{x - x_0} \int_{x_0}^{x} d\bar{x}\, p(\bar{x}) T(f(\bar{x})) \tag{5.2.20}$$

with $T(f(x))$ a rapidly oscillating function of $f(x)$ and $p(x)$ a slowly varying prefactor. When averaging over a short period of time, because of its slow variation, one can take $p(x)$ to be roughly constant. Instead, the rapidly varying contribution has to be properly averaged:

$$\langle p(x) T(f(x)) \rangle_{loc} \approx p(x) \langle T(f(x)) \rangle_{loc} \tag{5.2.21}$$

which is what has been assumed throughout the derivation. However, if many periods occur from $x_0$ to $x$, then the slow variation of $p(x)$ adds up period after period, leading to a non-constant contribution. Therefore, the cumulative average is the relevant quantity. For $\langle T_1 \rangle$ and $\langle T_3 \rangle$ this is not a significant effect, since the trigonometric functions in those terms have a non-zero average. Specifically, one is neglecting a term proportional to $\int dy\, y^{1/\alpha - 1} \cos(2y)$ and $\int dy\, y^{1/\alpha - 3} \cos(ny)$, with $n = 2, 4, 6$, respectively for $T_1$ and $T_3$. However, since over a single period these trigonometric functions have zero average, their corrections are subleading. Instead, the situation is different for $T_2$, whose trigonometric component has zero average. A more detailed treatment can be done by performing an integration by parts:

$$\langle T_2 \rangle \supset \int_{y_0}^{y} d\bar{y}\, \bar{y}^{1/\alpha - 2} \sin^3 \bar{y} \cos \bar{y} = \frac{1}{4} \left[ \bar{y}^{1/\alpha - 2} \sin^4 \bar{y} \Big|_{y_0}^{y} - \left( \frac{1}{\alpha} - 2 \right) \int_{y_0}^{y} d\bar{y}\, \bar{y}^{1/\alpha - 3} \sin^4 \bar{y} \right] \tag{5.2.22}$$

The boundary term depends on the specific initial and final configurations of the loop trajectory, therefore it can be neglected as it is phase dependent. The second term instead contains physical information, precisely the cumulative weight of the prefactor. Notably, $\sin^4 y$ has a non-zero average, 3/8 to be precise, hence:

$$\langle T_2 \rangle = -\frac{3q A_2^{-1/\alpha}}{16\alpha^2 (x - x_0)} \int_{y_0}^{y} d\bar{y}\, \bar{y}^{1/\alpha - 3} = -\frac{3q}{16\alpha^2} F_\alpha(y, y_0) \tag{5.2.23}$$

Therefore, the RMS velocity is:

$$\left\langle v^2 \right\rangle_{[x_0, x]} \simeq \frac{1}{2} + \frac{1}{16\alpha^2} \left[ -3q + \frac{q^2}{(1 - 2\alpha)} \right] F_\alpha(y, y_0) \tag{5.2.24}$$

This can be written as $\bar{v}^2 = 1/2 + \delta(q) + \delta(q^2)$. In general, the sign of the correction is controlled by

$$\text{sign}\left[\left(-3q + \frac{q^2}{(1-2\alpha)}\right) F_\alpha\right] = \text{sign}\left(-3q + \frac{q^2}{(1-2\alpha)}\right)\text{sign}(F_\alpha) \tag{5.2.25}$$

In particular, one can easily see that $\text{sign}(F_\alpha) = -\text{sign}(1+q)$ and the term in square brackets can be written as:

$$-3q + \frac{q^2}{1-2\alpha} = -\frac{q(3+4q)}{1+q} \tag{5.2.26}$$

Overall, the sign of the correction is then given by $\text{sign}(q(3+4q))$. At this point, before investigating all possible ranges of $q$ available, it should be noted that the lowest value for $q$ is obtained for volume modulus kination, where $q = -1/2$ and so $\alpha = 3/4$. Therefore, $(3+4q)$ cannot be negative, leading to the entirety of the sign dictated by $\text{sign}(q)$. The interesting result is that with this careful treatment of the oscillatory functions in $T_2$ one is able to generate a correction, which would be missed by a purely local averaging prescription. This is the first place where the calculation captures a genuinely cumulative effect of the time-dependent background on the subhorizon loop dynamics. Naively one would think it is at first order as of (5.2.11), however, this is found to contribute much the same way as $T_3$, thus effectively behaving as a second order correction. Once again, this marks the Hubble scale suppression of the deep subhorizon loops and the distinction between the network scale dynamics and this last regime.

### 5.2.1 Residual oscillatory correction

Despite the outcome, the careful treatment of the oscillating functions highlights the importance of treating those adequately. As a direct consequence, one might then ask whether the expression for $f(x)$ is truly reliable, since it was obtained precisely by averaging the trigonometric functions in (5.2.4). Motivated by such argument, we have considered a novel approach where the phase function $f(x)$ can be written as $f(x) = F(x)+\xi(x)$, with $F(x)$ the leading order averaged phase function, while $\xi(x)$ describes small deviations to the averaged phase generated by the oscillatory terms previously discarded in the equation. The purpose of this correction is to test whether the discarded oscillatory terms can generate sizeable local effects in the deep-subhorizon loop dynamics and to improve the solution found in the literature.

Substituting this expression for $f(x)$ into the original equation leads to a differential equation for the oscillatory correction $\xi(x)$. Indeed, using:

$$\cos(F(x) + \xi(x)) \approx \cos(F(x)) - \xi(x)\sin(F(x)) \tag{5.2.27}$$
$$\sin(F(x) + \xi(x)) \approx \sin(F(x)) + \xi(x)\cos(F(x)) \tag{5.2.28}$$

the equation can be expanded as (omitting the dependence on $x$ of the functions):

$$\begin{aligned} 0 =& 2x^2F'' - qxF'(1 - 3\cos 2F) - q\left(1 + q\sin^2 F\right)\sin 2F \\ &+ 2x^2\xi'' - qx\xi'(1 - 3\cos 2F) - 2q\xi\left(1 + q\sin^2 F\right)\cos 2F \\ &- 6qx\xi F'\sin 2F - 4q^2\xi\sin^2 F\cos^2 F \end{aligned}$$

up to order $\mathcal{O}(\xi^2)$ corrections. In a more compact notation, we can write this as $E[F] + L_F[\xi] = 0$, where:

$$E[F] = 2x^2F'' - qxF'(1 - 3\cos 2F) - q\left(1 + q\sin^2 F\right)\sin 2F \tag{5.2.29}$$

and

$$L_F[\xi] = 2x^2\xi'' - qx\xi'(1 - 3\cos 2F) - 2q\xi\left(1 + q\sin^2 F\right)\cos 2F - 6qx\xi F'\sin 2F - 4q^2\xi\sin^2 F\cos^2 F$$

Here, $E[F]$ represents the original equation for the leading phase $F(x)$, while the other terms depend on $\xi$ and its derivatives. Since $F(x) = A_1 + A_2x^\alpha$ is a solution of the original equation with the averaged trigonometric functions, $E_{avg}[F] = 2x^2F'' - qxF' = 0$, we can use this inside the equation to evaluate the oscillatory corrections previously neglected, which will be denotes as $E_{osc}[F]$. In turn, these will constitute a source term of the differential equation describing the evolution of $\xi$. Thus, the residual oscillatory terms are:

$$E_{osc}[F] = 3qxF'\cos(2F) - q\left(1 + q\sin^2 F\right)\sin(2F) \tag{5.2.30}$$

Since $F' = \alpha F/x$, the first term is proportional to $F\cos(2F)$, hence of order $\mathcal{O}(F)$, while the others are purely oscillatory, of order $\mathcal{O}(1)$. Therefore, in the regime of interest where $F(x) \gg 1$, the residual oscillatory term can be approximated as:

$$E_{osc}[F] \approx 3q\alpha F\cos(2F) \tag{5.2.31}$$

Instead, the $\xi$ component of the equation can be written as:

$$L_F[\xi] = L_{avg}[\xi] + L_{osc}[\xi] \tag{5.2.32}$$

in the same spirit as $E[F]$. $L_{avg}[\xi]$ represents the equation with trigonometric functions approximated by their averages for large $F(x)$, while $L_{osc}[\xi]$ is the residual oscillatory term. However, in the large phase regime, one can approximate $L_F[\xi] \simeq L_{avg}[\xi]$ and later check that the oscillatory terms are indeed subleading. This is consistent with our intent, solving for the correction $\xi(x)$ generated by the oscillatory terms arising from the solution to the avearged equation, $F(x)$. Therefore, the equation is:

$$L_{avg}[\xi] = -E_{osc}[F] \implies 2x^2\xi'' - qx\xi' = -3q\alpha F\cos(2F) \tag{5.2.33}$$

At this point, it is convenient to switch from $x$ derivatives to derivatives with respect to $F$. This means $d/dx = F'd/dF$, thus:

$$2x^2F'^2\frac{d^2\xi}{dF^2} + 2x^2F''\frac{d\xi}{dF} - qxF'\frac{d\xi}{dF} = -3q\alpha F\cos(2F) \tag{5.2.34}$$

Recalling $F' = \alpha F/x$ and $F'' = \alpha(\alpha-1)F/x^2$, this leads to:

$$2\alpha^2F^2\frac{d^2\xi}{dF^2} + 2\alpha(\alpha-1)F\frac{d\xi}{dF} - q\alpha F\frac{d\xi}{dF} = -3q\alpha F\cos(2F) \tag{5.2.35}$$

However, $q = 2(\alpha - 1)$ and the terms linear in $d\xi/dF$ cancel out, leaving the final equation:

$$2\alpha^2F^2\frac{d^2\xi}{dF^2} = -3q\alpha F\cos(2F) \implies \frac{d^2\xi}{dF^2} = -\frac{3q}{2\alpha}\frac{\cos 2F}{F} \tag{5.2.36}$$

The structure of the source suggests looking for an ansatz of the kind $\xi(F) = A\cos(2F)/F$, which upon substitution gives:

$$-4A\frac{\cos(2F)}{F} + \mathcal{O}(F^{-2}) = -\frac{3q}{2\alpha}\frac{\cos 2F}{F} \tag{5.2.37}$$

where terms of order $F^{-2}$ have been neglected since we are in the regime $F(x) \gg 1$. It is then immediate to identify $A = 3q/(8\alpha)$ so that the first order particular solution for $\xi(F)$ is:

$$\xi(F) = \frac{3q}{8\alpha}\frac{\cos(2F)}{F} + \mathcal{O}(F^{-2}) \tag{5.2.38}$$

whose $x$ dependence is found by using the averaged form of $F(x)$. The solution just found is consistently small in the large phase regime, since $\xi(x)/F \sim 1/F^2$. Nevertheless, its effect is to remove the oscillatory terms neglected in the derivation of the solution to the averaged equation. This is numerically checked and quantified in Appendix C.

The implications for the velocity can be found by taking (5.2.10) and using $f(x) = F(x) + \xi(x)$ to explicitly see where the oscillatory correction $\xi(x)$ enters. The amplitude can be expressed as:

$$\begin{aligned} A(x) &= \exp\left(-q\int\frac{dx}{x}\sin^2 f(x)\right) \approx \\ &\approx \exp\left(-q\int\frac{dx}{x}\sin^2 F(x)\right)\exp\left(-q\int\frac{dx}{x}\xi\sin(2F)\right) \equiv A_0(x)e^{\delta A(x)} \end{aligned}$$

where $\delta A \equiv -q\int\frac{dx}{x}\xi\sin(2F)$ and $A_0(x) \equiv \exp\left(-q\int\frac{dx}{x}\sin^2 F(x)\right)$. In addition, let us denote the square bracket term in (5.2.10) as $B(x)$, that is:

$$B(x) = \left[f'(x)\sin f(x) + \frac{q}{x}\sin^2 f(x)\cos f(x)\right] \tag{5.2.39}$$

With $f(x) = F(x) + \xi(x)$, this becomes:

$$\begin{aligned} B(x) =&(F' + \xi')(\sin F + \xi\cos F) + \frac{q}{x}(\sin^2 F + \xi\sin 2F)(\cos F - \xi\sin F) = \\ =&F'\sin F + \frac{q}{x}\sin^2 F\cos F + \xi'\sin F + \xi F'\cos F + \frac{q}{x}\xi\sin F\left(2 - 3\sin^2 F\right) \\ \equiv&B_0(x) + \delta B(x) \end{aligned}$$

where $\delta B(x)$ includes all terms with $\xi$ while the remaining ones are in $B_0(x)$. This notation allows for an easier expansion for $R'^2$ to first order in $\delta A$ and $\delta B$. First of all, since $\delta A \sim 1/F$, this is small in the large $x$ regime, thus one can expand $A(x) \simeq A_0(x)(1 + \delta A(x))$. Overall, one has:

$$\begin{aligned} R'^2 = A^2B^2 =&A_0^2B_0^2 + 2A_0^2B_0\delta B + 2A_0^2B_0^2\delta A + \mathcal{O}(\delta^2) \\ =&R_0'^2 + \delta R_B'^2 + \delta R_A'^2 + \mathcal{O}(\delta^2) \end{aligned} \tag{5.2.40}$$

Both $\delta R_B'^2$ and $\delta R_A'^2$ involve several terms and, to avoid computing the average of each one of them, we can try to determine which one is the dominant contribution. This can be done by using:

$$\xi \sim \frac{q}{\alpha F} \qquad\qquad \xi' \sim -\frac{q}{\alpha x} \qquad\qquad F' = \frac{\alpha F}{x} \tag{5.2.41}$$

Then, $B_0$ can be estimated as:

$$B_0(x) = \frac{\alpha F}{x}\sin F + \frac{q}{x}\sin^2 F\cos F \approx \frac{\alpha F}{x}\sin F \tag{5.2.42}$$

for large $F(x)$. Instead, $\delta B$ can be approximated as:

$$\begin{aligned}\delta B =& -\frac{3q}{4x}\sin 2F\sin F + \frac{3q}{8x}\cos 2F\cos F + \frac{3q^2}{8\alpha x}\frac{\cos 2F}{F}\sin F\left(2-3\sin^2 F\right)\\ \approx& \frac{3q}{8x}\left(\cos 2F\cos F - 2\sin 2F\sin F\right) = \frac{3q}{8x}\left(\cos F - 6\sin^2 F\cos F\right)\end{aligned}$$

Overall, the full $\delta R_B'^2$ correction is found to be:

$$\delta R_B'^2(x) = \frac{3\alpha q}{4x^2}A_0^2 F\left(\sin F\cos F - 6\sin^3\cos F\right) \tag{5.2.43}$$

Instead, the amplitude correction can be estimated via integration by parts. Indeed, one can write $\delta A$ as:

$$\delta A = -q\int\frac{dx}{x}\xi\sin 2F = \frac{-3q^2}{16\alpha}\int\frac{dx}{x}\frac{\sin 4F}{F} \tag{5.2.44}$$

Moreover, using a substitution $u = F(x)$, such that $du = F'dx$, the integral becomes:

$$I \equiv \int\frac{dx}{x}\frac{\sin 4F}{F} = \int du\, g(u)\sin 4u \tag{5.2.45}$$

with $g(u) = 1/(\alpha u^2)$. Performing an integration by parts leads to:

$$I = -\frac{1}{4}g(u)\cos 4u + \frac{1}{4}\int du\frac{dg}{du}\cos 4u \tag{5.2.46}$$

However, $dg/du \sim 1/F^3$, while the first term is proportional to $1/F^2$. Therefore, the dominant contribution in the large $F(x)$ limit is given by $I \sim -\cos(4F)/(4\alpha F^2)$. The full $\delta A$ correction is then:

$$\delta A \approx \frac{3q^2}{64\alpha^2}\frac{\cos 4F}{F^2} \tag{5.2.47}$$

Overall, the correction this brings at the level of $R'^2$ is:

$$\delta R_A'^2 = \frac{3q^2}{32x^2}A_0^2\cos 4F\sin^2 F \tag{5.2.48}$$

With the aim of comparing these corrections with the leading order value of $R'^2$, it is necessary to compute also $R_0'^2$, which is:

$$R_0'^2 = \frac{\alpha^2 A_0^2}{x^2}F^2\sin^2 F \tag{5.2.49}$$

Taking the ratios of $\delta R_B'^2$ and $\delta R_A'^2$ with $R_0'^2$ highlights a clear hierarchy between the two corrections:

$$\frac{\delta R_A'^2}{R_0'^2} \sim \frac{q^2}{\alpha^2 F^2} \qquad\qquad \frac{\delta R_B'^2}{R_0'^2} \sim \frac{q}{\alpha F} \tag{5.2.50}$$

Therefore, the amplitude correction is suppressed by an additional power of $F$ compared with the one from $\delta B$. This is because $\delta A$ involves only $\xi$, which is already suppressed by $F$, while $\delta B$ depends on $\xi'$, which does not contain any powers of $F$.

Having understood the dominant contribution from the oscillatory phase correction $\xi(x)$, one can proceed to evaluate the impact this has on the value of $v^2$. The procedure is analogous to the one outlined before, it is just necessary to compute $\langle \delta R_B'^2 \rangle$ over a range $[x_0, x]$. It is convenient to split (5.2.43) into two contributions, namely $\delta R_B'^2 = B_1 + B_2$ with

$$B_1(y) = \frac{3q}{4\alpha}\frac{\cos y \sin y}{y} \qquad\qquad B_2(y) = -\frac{9q}{2\alpha}\frac{\sin^3 y \cos y}{y} \tag{5.2.51}$$

where the variable $y$ has been introduced as before, $y = A_2 x^\alpha$. Both these terms contain trigonometric functions with zero average, and would thus give a null contribution to the velocity in this naive approach. However, as seen before, it is necessary to treat the $y$-dependent prefactors with care in such instances. The trick here lies in recognizing that $\cos y \sin y = \frac{1}{2}\frac{d}{dy}\sin^2 y$ and $\sin^3 y \cos y = \frac{1}{4}\frac{d}{dy}\sin^4 y$, such that the correction brought by $B_1(y)$ can be computed as:

$$\begin{aligned}\langle B_1 \rangle =& \frac{3qA_2^{-1/\alpha}}{8\alpha^2(x-x_0)}\int_{y_0}^{y} d\bar{y}\,\bar{y}^{1/\alpha-2}\frac{d}{d\bar{y}}\sin^2\bar{y} = \\ =& \frac{3qA_2^{-1/\alpha}}{8\alpha^2(x-x_0)}\left[\bar{y}^{1/\alpha-2}\sin^2\bar{y}\Big|_{y_0}^{y} - \left(\frac{1}{\alpha}-2\right)\int_{y_0}^{y} d\bar{y}\,\bar{y}^{1/\alpha-3}\sin^2\bar{y}\right]\end{aligned}$$

The first term in square brackets is a phase-dependent boundary term which can be neglected. Instead, the second term has a non-zero average trigonometric function inside the integral. Therefore, this gives:

$$\langle B_1 \rangle = -\frac{3q}{16\alpha^2}F_\alpha(y, y_0) \tag{5.2.52}$$

with $F_\alpha(y, y_0)$ as in (5.2.19). On the other hand, the full computation of $\langle B_2 \rangle$ is not required, since one can notice it has a similar structure to the original $T_2$ term. Indeed, $B_2 = -\frac{9}{4}T_2$, therefore its contribution when integrated is:

$$\langle B_2 \rangle = -\frac{9}{4}\langle T_2 \rangle = \frac{27q}{64\alpha^2}F_\alpha(y, y_0) \tag{5.2.53}$$

Overall, the correction to $\bar{v}^2$ brought by $\delta B$ is:

$$\langle \delta R_B'^2 \rangle = -\frac{3q}{16\alpha^2}F_\alpha(y, y_0) + \frac{27q}{64\alpha^2}F_\alpha(y, y_0) = \frac{15q}{64\alpha^2}F_\alpha(y, y_0) \tag{5.2.54}$$

Combining this with the previously computed correction in (5.2.24), the squared velocity reads:

$$\langle v^2 \rangle_{[x_0,x]} \simeq \frac{1}{2} + \frac{1}{16\alpha^2}\left[-3q + \frac{q^2}{(1-2\alpha)} + \frac{15q}{4}\right]F_\alpha(y, y_0) \tag{5.2.55}$$

Therefore, the effect of the phase correction is a contribution to the squared velocity which is of the same order as the other corrections previously found. Given that the sign dependence is highly sensitive on the numerical coefficients, this result should not be fully interpreted as giving a robust argument for a sign flip of the correction. Indeed, additional contributions could potentially arise from higher order corrections to the phase function $f(x)$. Some of these terms may be in phase with with the leading order $T_1$ and not average out over the oscillation period. Consequently, the final form of the squared velocity could be modified. This reinforces the fact that the numerical sign of the velocity correction should not be interpreted as a robust prediction of the present approximation. Instead, the main result is that the corrected solution significantly improves the previous one used in the literature and consistently shows the discrepancy between subhorizon dynamics and macroscopic networks physics. The time variation of the tension does generate oscillatory corrections which are shown to be important. However, at the level of the corrections considered, these do not induced a significant deviation of the squared velocity to the Minkowskian value of 1/2.

Before moving on to the next section, it is worth pointing out that the oscillatory terms neglected in obtaining (5.2.33) can now be estimated using the results of (5.2.41) in the full operator $L_{osc}[\xi]$. This gives:

$$L_{osc}[\xi] \simeq -\frac{3q^2}{\alpha}\cos 2F - \frac{2q^2}{\alpha F}(1+\sin^2 F)\cos 2F - 6q^2\sin 2F - \frac{4q^2}{\alpha F}\sin^2 F\cos 2F \quad (5.2.56)$$

which clearly shows that $L_{avg}[\xi] = -E_{osc}[F] \sim -F\cos 2F$ is the dominant contribution since the terms in the above equation are either purely oscillatory or the product of oscillatory functions and $1/F$, which are subleading in the large $F$ regime. This validates our result. However, to completely verify this, we checked the analytic phase correction derived above against the numerical solution of the full phase equation. Since the perturbative correction is defined only up to homogeneous solutions, corresponding to redefinitions of the integration constants of the leading-order solution $F(x)$, these modes are first fitted and subtracted. The remaining residual is then found to be controlled throughout the oscillatory regime. The details of this check, together with the error behavior, are reported in Appendix C.

In addition, one might wonder whether the amplitude correction, $\delta A$, we neglected can significantly impact the correction to the squared velocity. This can readily be checked by noting that $\delta R_A'^2$ can be written as:

$$\delta R_A'^2 = \frac{3q^2}{32x^2}A_0^2\cos 4F\sin^2 F = \frac{3q^2}{32x^2}A_0^2\left(\sin^2 F - 8\sin^4 F\cos^2 F\right) \quad (5.2.57)$$

Since these trigonometric functions have a non zero average, they can be directly replaced by said values when integrating. However, $8\langle\sin^4 F\cos^2 F\rangle = 1/2$ and $\langle\sin^2\rangle = 1/2$, therefore this exactly cancel out and leave a zero effect on the squared velocity.

## 5.3 Circular loop: perturbative approach

To further investigate whether the variation of the tension generates interesting effects in terms of correlations and deviations from the 1/2 value of the circular loop velocity, in this section we explore another approach based on a perturbative treatment of the spatial equation of motion describing the loop, instead of starting from the $\varepsilon$ equation as in the previous section.

First of all, recall the circular loop solution (4.2.13) and the microscopic equations of motion related to it, namely (4.2.15) and (4.2.16). Defining $x \equiv \tau/L$, one can exchange derivatives with respect to $\tau$ with derivatives with respect to $x$, $\frac{d}{d\tau} = \frac{1}{L}\frac{d}{dx}$, which will be denoted by a prime. In Minkowski this satisfies $R(\tau) = \cos(\tau/L)$. For this ansatz, $\dot{\mathbf{x}} = \dot{R}L\hat{u}$, where $\hat{u}$ is the unit vector $\hat{u} = (\cos(\sigma/L), \sin(\sigma/L), 0)$, and thus $v^2 = \dot{R}^2L^2 = R'^2$. A first, albeit naive, approach would simply assume small perturbations around the circular loop solution in Minkowski, writing $R(x) = R_0(x) + \delta R(x)$, with $R_0(x)$ the Minkowski solution and $\delta R(x)$ smaller than $R_0(x)$. Hence, to first order in the perturbation the velocity is:

$$v^2 = R'^2 = R_0'^2 + 2R_0'\delta R' = \sin^2 x - 2\delta R' \sin x \tag{5.3.1}$$

Since this is periodic over $\tau/L \to \tau/L + 2\pi$, we can average over $x$ ranging from 0 to $2\pi$ to get:

$$\bar{v}^2 = \frac{1}{2\pi}\int_0^{2\pi} dx \sin^2 x - \frac{1}{\pi}\int_0^{2\pi} dx\, \sin x \delta R' = \frac{1}{2} - \frac{1}{\pi}\int_0^{2\pi} dx\, \sin x \delta R' \tag{5.3.2}$$

It is thus necessary to compute $\delta R'$ to evaluate the correction. To do this, one needs to consider the spatial equation of motion written in terms of derivatives with respect to $x$:

$$R'' + \frac{2\kappa\nu'}{x}(1 - R'^2)R' + \frac{1 - R'^2}{R} = 0 \tag{5.3.3}$$

where the conformal time modified Hubble parameter is $2\tilde{\mathcal{H}} = 2\kappa\nu'/\tau$. Indeed, $R(x) = R_0(x) = \cos(x)$ is a solution to this equation with $\tilde{\mathcal{H}} = 0$:

$$R_0'' + \frac{1 - {R_0}'^2}{R_0} = 0 \tag{5.3.4}$$

Substituting the small perturbation to the Minkowski solution:

$$R_0'' + \delta R'' + \frac{2\kappa\nu'}{x}(1 - {R_0}'^2)R_0' + \frac{1 - {R_0}'^2 - 2R_0'\delta R'}{R_0 + \delta R} = 0 \tag{5.3.5}$$

where in the Hubble friction term we replaced $R$ with $R_0$ since in the subhorizon regime, $x \gg 1$, the term $1/x$ is already small. The third term can be written as:

$$\frac{1 - {R_0}'^2 - 2R_0'\delta R'}{R_0 + \delta R} \approx \frac{1 - {R_0}'^2 - 2R_0'\delta R'}{R_0}\left(1 - \frac{\delta R}{R_0}\right) = \frac{1 - {R_0}'^2 - 2R_0'\delta R'}{R_0} - \frac{\left(1 - {R_0}'^2\right)}{R_0^2}\delta R$$

Thus, (5.3.5) becomes:

$$R_0'' + \frac{1 - {R_0}'^2}{R_0} - \frac{2R_0'\delta R'}{R_0} - \frac{\left(1 - {R_0}'^2\right)}{R_0^2}\delta R + \delta R'' + \frac{2\kappa\nu'}{x}(1 - {R_0}'^2)R_0' = 0 \tag{5.3.6}$$

The first two terms are zero by (5.3.4), while the others can be simplified by using $R_0' = -\sin(x)$, leading to:

$$\delta R'' + 2\tan x\, \delta R' - \delta R = \frac{2\kappa\nu'}{x} \sin x \cos^2 x \tag{5.3.7}$$

This second order inhomogeneous differential equation with variable coefficient and source term $f(x) = \frac{2\kappa\nu'}{x}\sin x\cos^2 x$ can be solved as follows. First, focus on the homogeneous equation:

$$\delta R'' + 2\tan x\,\delta R' - \delta R = 0 \tag{5.3.8}$$

A solution can be found by slightly perturbing the Minkowski solution $\cos(x+\delta x) \approx \cos x - \delta x \sin x$. Indeed, using $y_1(x) = \sin x$ gives:

$$-\sin x + 2\frac{\sin x}{\cos x}\cos x - \sin x = 0 \tag{5.3.9}$$

Hence $y_1(x) = C_1 \sin x$ is a solution. A second solution can be found by defining new variables to reduce the order of the equation. Indeed, let $y_2(x) = \alpha(x)y_1(x)$, and let $v = \alpha'(x)$. Substituting in the equation gives a first order differential equation for $v$:

$$v' + \left(\frac{2}{\tan x} + 2\tan x\right)v = 0 \tag{5.3.10}$$

whose solution is $v(x) = \alpha'(x) = \cos^2 x/\sin^2 x$. Therefore, $\alpha(x) = x + \cot x$ and the second solution is $y_2(x) = C_2(x\sin x + \cos x)$. Overall, the two linearly independent solutions to the homogeneous equation are:

$$y_{hom}(x) = C_1\sin x + C_2(x\sin x + \cos x) \tag{5.3.11}$$

The particular solution can be found by using general techniques of differential equations. Namely, given $y_1$ and $y_2$ solutions of the homogeneous equation and source term $f(x)$, the general formula for the particular solution is:

$$y_p(x) = -y_1(x)\int dx\,\frac{f(x)y_2(x)}{W(y_1,y_2)} + y_2(x)\int dx\,\frac{f(x)y_1(x)}{W(y_1,y_2)} = y_{p1} + y_{p2} \tag{5.3.12}$$

where $W(y_1, y_2)$ is the Wronskian. This can be computed as:

$$W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -\cos^2 x \tag{5.3.13}$$

Leading to:

$$y_{p1}(x) = 2\kappa\nu'\sin x\left[\frac{x}{2} - \frac{\sin 2x}{4} + \frac{1}{2}\mathrm{Si}(2x)\right] \tag{5.3.14}$$

$$y_{p2}(x) = -2\kappa\nu'(x\sin x + \cos x)\left(\frac{1}{2}\ln x - \frac{1}{2}\mathrm{Ci}(2x)\right) \tag{5.3.15}$$

These complicated expressions come from the $1/x$ in the source. To analytically compute the correction proportional of $v^2$, which is what interests us, we can consider averaging over a single oscillation period, meaning we take $x = x_* + \delta x$, with $x_*$ a large fixed value and $\delta x$ small. Therefore $1/x \approx 1/x_*$ constant. In this case, the source term $f(x) = \frac{2\kappa\nu'}{x_*}\sin x\cos^2 x$ becomes periodic in $x$. Now the particular solution is nicer ($\kappa\nu'/x_* \equiv C$):

$$y_{p1} = C\sin x\left[\frac{x^2}{2} - \frac{x\sin 2x}{4} - \frac{3}{4}\cos 2x + \frac{3}{4}\right] \qquad y_{p2}(x) = -C(x\sin x + \cos x)\left(x - \frac{\sin 2x}{2}\right)$$

Adding these together leaves:

$$y_p(x) = C\left(-\frac{x^2}{2}\sin x - x\cos x + \frac{1}{2}\sin^3 x + \sin x\right) \tag{5.3.16}$$

and thus the full solution to (5.3.7) is:

$$\delta R(x) = C_1 \sin x + C_2(x\sin x + \cos x) + C\left(-\frac{x^2}{2}\sin x - x\cos x + \frac{1}{2}\sin^3 x + \sin x\right)$$

At this point, a few remarks are in order. First of all, the perturbative expansion is valid provided the correction $\delta R$ remains small compared with the background solution $R_0(x)$, away from the collapse points of the unperturbed loop. In addition, the final solution contains terms that are proportional to $x$ and therefore grow in time, undermining the validity of the perturbative expansion. Indeed, the homogeneous equation already admits a solution proportional to $x\sin x + \cos x$. However, this corresponds to an infinitesimal change in the amplitude, and hence frequency, of the exact Minkowski circular loop solution. Indeed, the general solution in Minkowski can be written as $R_0(x) = \mathcal{A}\cos\left(\frac{x}{\mathcal{A}} + \delta\right)$ and differentiating this with respect to $\mathcal{A}$ at $\mathcal{A} = 1$ gives:

$$\frac{\partial}{\partial \mathcal{A}}\left[\mathcal{A}\cos\frac{x}{\mathcal{A}}\right]\Big|_{\mathcal{A}=1} = \cos x + x\sin x \tag{5.3.17}$$

Instead, the other homogeneous solution $\sin x$ can be absorbed into a constant shift of the frequency, since $\cos(x+\delta) \approx \cos x - \delta\sin x$. The truly problematic *secular* terms appearing in $\delta R(x)$ come from the particular solution generated by the forcing function. These should not be removed by imposing ad hoc initial conditions, but by introducing a more adequate form of perturbative ansatz, such as a strained phase or slow amplitude variable. Let us work this out in details.

The most general form of perturbative solution introduces a slowly varying amplitude and rapid phase. Indeed, the homogeneous solution proportional to $x\sin x + \cos x$ suggests the perturbation has a direct effect on the amplitude. However, since this effect is generated by the Hubble damping term, it is small over a single period. Therefore, one should allow the amplitude to slowly change over a single period, by introducing a slow time variable $X = \varepsilon x$, with $\varepsilon$ a small parameter. In the present problem, the naturally small parameter to be identified with $\varepsilon$ is $1/x_*$ and derivatives with respect to $x$ will be given by $\frac{d}{dx} = \varepsilon\frac{d}{dX}$. For what concerns the phase, we can write $x \to \psi(x, X)$, such that the frequency of oscillation is given by $\omega(X) = d\psi/dx = \omega_0(X) + \varepsilon\omega_1(X) + \mathcal{O}(\varepsilon^2)$. Overall, the starting point for this perturbative treatment is:

$$R(\psi, X) = A(X)\cos(\psi(x, X)) \tag{5.3.18}$$

with $X = \varepsilon x$ and $\psi$. Similarly, one can expand $R(\psi, X)$ in power series of $\varepsilon$ as $R(\psi, X) = R_0(\psi, X) + \varepsilon R_1(\psi, X) + \mathcal{O}(\varepsilon^2)$. The next step consists of writing the equation of motion in terms of derivatives with respect to $\psi$ and $X$ treated as independent variables. This means:

$$\frac{dR(\psi, X)}{dx} = \frac{\partial\psi}{\partial x}\frac{\partial R}{\partial\psi} + \frac{\partial X}{\partial x}\frac{\partial R}{\partial X} = \omega(X)R_\psi + \varepsilon R_X \tag{5.3.19}$$

$$\frac{d^2R(\psi, X)}{dx^2} = \omega(X)^2 R_{\psi\psi} + 2\varepsilon\omega(X)R_{X\psi} + \varepsilon(\partial_X\omega)R_\psi + \mathcal{O}(\varepsilon^2) \tag{5.3.20}$$

Using the $\varepsilon$ expansion for both $\omega$ and $R$ leads to:

$$\begin{aligned} R' =& \omega_0(X)R_0(X)_\psi + \varepsilon\left[\omega_1(X)R_0(X)_\psi + \omega_0(X)R_1(X)_\psi + R_0(X)_X\right] + \mathcal{O}(\varepsilon^2) \\ R'^2 =& \omega_0^2(X)(\partial_\psi R_0)^2 + 2\omega_0\varepsilon R_0(X)_\psi\left[\omega_1(X)R_0(X)_\psi + \omega_0 R_1(X)_\psi + R_0(X)_X\right] + \mathcal{O}(\varepsilon^2) \end{aligned}$$

while the second derivative reads:

$$R'' = \omega_0^2 R_{0,\psi\psi} + \varepsilon\left[2\omega_0\omega_1 R_{0,\psi\psi} + \omega_{0,X}R_{0,\psi} + \omega_0^2 R_{1,\psi\psi} + 2\omega_0 R_{0,\psi X}\right] + \mathcal{O}(\varepsilon^2) \tag{5.3.21}$$

Therefore, the order $\varepsilon^0$ equation is:

$$\omega_0^2 R_{0,\psi\psi} + \frac{1-\omega_0^2 R_{0,\psi}^2}{R_0} = 0 \tag{5.3.22}$$

whose solution is the Minkowski circular loop $R_0(\psi, X) = A(X)\cos(\psi)$ with amplitude $A(X)$ and oscillation frequency $\omega_0(X) = 1/A(X)$. Indeed:

$$\frac{1}{A^2}(-A\cos\psi) + \frac{1 - A^{-2}A^2\sin^2\psi}{A\cos\psi} = 0 \tag{5.3.23}$$

At this perturbative order, $A(X)$ is not fixed yet, but it is promoted to a slowly varying amplitude. Its evolution will be determined precisely by the condition to remove secular terms. Instead, at order $\varepsilon^1$ the equation is found to be:

$$R_{1,\psi\psi} + 2R_{1,\psi}\tan\psi - R_1 = 2A^2\omega_1\sec\psi - A_X A\sin\psi + qA^2\cos^2\psi\sin\psi \tag{5.3.24}$$

where $q \equiv 2\kappa\nu'$ as in the previous section. The structure of the equation on the l.h.s. is the same as the naive perturbative approach, one recovers the same differential operator:

$$\hat{L} \equiv \frac{\partial^2}{\partial\psi^2} + 2\tan\psi\frac{\partial}{\partial\psi} - 1 \tag{5.3.25}$$

acting on $R_1(\psi, X)$ whereas the source term on the r.h.s, $S(\psi)$, has three contributions coming from different physical aspects. The first contains the shift correction to the frequency of oscillation. The second term comes from the slow variation of the amplitude. Finally, the term proportional to $q$ originates due to the modified Hubble parameter, hence it encapsulates the variation of the tension as well. The original perturbative approach had $\omega_1 = 0$ and $A = 1$, so $A_X = 0$, removing important physical degrees of freedom which are crucial to remove the dangerous secular terms.

The solutions to the homogeneous equation are the same as (5.3.11) with $x \to \psi$, corresponding to a constant phase shift and amplitude variation. Instead, the particular solution has to be evaluated from the new source term $S(\psi)$. However, a simplification can be made by noting that the differential operator $\hat{L}$ acting on a cosine function gives: $\hat{L}(\cos\psi) = -2\sec\psi$, which is the functional form of the first term on the r.h.s. of the above equation. Therefore, a particular solution is $R_{1,p} = -A^2\omega_1\cos(\psi)$. However, this has the same structure as the zero-th order solution, hence it can be absorbed into it via a redefinition of the leading order amplitude. Moreover, since we are interested in the condition to remove secular terms, this source term is irrelevant and we can set $\omega_1 = 0$ for simplicity. Then, the source is $S(\psi) = -AA_X\sin\psi + qA^2\sin\psi\cos^2\psi$ and the correspondent particular solution can be found with the method outlined before:

$$y_p(\psi) = -y_1(\psi)\int d\psi\,\frac{S(\psi)y_2(\psi)}{W(y_1,y_2)} + y_2(\psi)\int d\psi\,\frac{S(\psi)y_1(\psi)}{W(y_1,y_2)} = y_{p1}(\psi) + y_{p2}(\psi) \tag{5.3.26}$$

The Wronskian is unchanged, $W(y_1, y_2) = -\cos^2\psi$, while the two terms making up the particular solution after integration are:

$$
\begin{aligned}
y_{p1}(\psi) =& AA_X \left[-\psi\sin\psi\tan\psi + \frac{\psi^2}{2}\sin\psi\right] + qA^2\left[\frac{\psi^2}{4}\sin\psi - \frac{\psi}{2}\sin^2\psi\cos\psi + \frac{3}{4}\sin^3\psi\right] \\
yp_2(\psi) =& (\psi\sin\psi + \cos\psi)\left[AA_X\tan\psi - \left(AA_X + \frac{qA^2}{2}\right)\psi + \frac{qA^2}{2}\sin\psi\cos\psi\right]
\end{aligned}
$$

After adding up these contributions, one is left with:

$$
y_p(\psi) = -\left(AA_X + \frac{qA^2}{2}\right)\left(\frac{\psi^2}{2}\sin\psi + \psi\cos\psi\right) + \left(AA_X + \frac{qA^2}{2}\right)\sin\psi + \frac{qA^2}{4}\sin^3\psi
$$

Secular terms appear in the same functional form as in the naive perturbative approach (5.3.16). However, these can be removed by requiring the amplitude dependent term multiplying them to vanish, that is:

$$
AA_X + \frac{qA^2}{2} = 0 \implies A_X = -\frac{qA}{2} \tag{5.3.27}
$$

With this condition on the amplitude evolution, the particular solution is:

$$
R_{1,p}(\psi) = \frac{qA^2}{4}\sin^3\psi + \mathcal{O}(\varepsilon^2) \tag{5.3.28}
$$

and can be interpreted as the full solution to the equation to first order in $\varepsilon$ since the homogeneous ones do not described physically relevant effects. The squared velocity can now be computed using the same definition as before but integrating over a period from $\psi_*$ to $\psi_* + 2\pi$:

$$
\begin{aligned}
\bar{v}^2 &= \frac{1}{2\pi}\int_{\psi_*}^{\psi_*+2\pi} d\psi\, R'^2 \\
&= \frac{1}{2\pi}\int_{\psi_*}^{\psi_*+2\pi} d\psi\ \left[\omega_0^2(\partial_\psi R_0)^2 + 2\omega_0\varepsilon R_{0,\psi}\left(\omega_0 R_{1,\psi} + R_{0,X}\right)\right] + \mathcal{O}(\varepsilon^2)
\end{aligned} \tag{5.3.29}
$$

Using our results, this is found to be:

$$
R'^2 = \sin^2\psi - \frac{3q\varepsilon A}{2}\sin^3\psi\cos\psi + q\varepsilon A\sin\psi\cos\psi + \mathcal{O}(\varepsilon^2) \tag{5.3.30}
$$

which upon integration yields:

$$
\bar{v}^2 = \frac{1}{2} + \mathcal{O}(\varepsilon^2) \tag{5.3.31}
$$

This reproduces the expected leading order result of the squared velocity for circular loops but fails at capturing the correction due to the expansion of the universe and time-varying tension. This is a consequence of the first order correction being proportional to odd trigonometric functions, $\sin^3\psi\cos\psi$ and $\sin\psi\cos\psi$ in our case. Therefore, this perturbative approach further consolidates the conclusions of the previous section, highlighting that although the variation of the tension introduces extra terms in the equations, the effect is highly suppressed by the subhorizon nature of the loop. While in the previous treatment analytical results were available, this perturbative approach manifestly shows the correction appears at order $\varepsilon^2$, consistent with the previous result of it being of order $\mathcal{O}((\mathcal{H}L)^2)$. Nonetheless, the removal of secular terms implies a noteworthy behavior of the loop amplitude, as of (5.3.27). Recalling that $q = 2\kappa\nu'$, the amplitude grows for negative values of $\kappa$, thus when $mn > 4$, while it decreases for $mn < 4$, in comoving coordinates. This is consistent with the discussion of section 4.2. Furthermore, the solution to this equation leads to the same functional form of the amplitude used in section 5.2. Indeed, one can write:

$$\frac{dA}{dX} = \frac{1}{\varepsilon}\frac{dA}{dx} = -\frac{qA}{2} \tag{5.3.32}$$

and, if one recovers $\varepsilon \sim 1/x$, this becomes:

$$\frac{dA}{dx} \simeq -\frac{qA}{2x} \implies A(x) \sim x^{-q/2} \tag{5.3.33}$$

exactly as (5.2.12). Overall, the solution to the perturbed loop equation consistently reproduces the correct behavior of the amplitude, highlighting the fact that the variation of the tension and expansion of the universe significantly impact the evolution in time of loop amplitudes. At the same time, this approach puts on firm grounds previous statements about the correction being of order $1/x^2$, heavily suppressed due to the deep subhorizon nature of the loop.

It is useful to compare this conclusion with the role of loops in the original PR framework. There, the scaling form of the small-scale correlator was first derived under the assumption that stretching was the only dominant effect. Loop production was included only afterwards, as a perturbation to this leading picture. However, this perturbative treatment led to a divergent production rate for small loops, indicating that loops cannot be regarded as a completely harmless correction and that a more complicated fragmentation process is required [8]. The present analysis in the varying-tension case strengthens this even more. While the perturbative circular loop calculation does not generate a first order correction to the averaged squared velocity, it does show that the comoving amplitude of a loop grows when the tension dominates against the expansion of the universe. This does not directly modify the loop production calculation in the PR framework, which concerns the self-intersections of long strings and the resulting small-scale structure. Nevertheless, it suggests that loops are potentially more dynamically relevant in the varying-tension case. When produced, or if already present in the initial configuration, they need not remain negligible, since their comoving size can increase.

For this reason, a complete extension of the PR framework to time-dependent tension should eventually include the coupled evolution of the long-string correlator and the loop population. This lies beyond the scope of the present thesis, but the growth of loop amplitudes found here provides a useful indication that the loop sector should not be ignored.

This concludes the section dedicated to the velocity correction as probe of the scale of the effect induced by the variation of the tension. The key takeaway is that that the corrections induced by the time-varying tension are deeply suppressed in the subhorizon regime. The following and last section will be dedicated to another possible quantity on which the time varying tension could have a relevant effect, that is the correlation between left and right moving unit vectors.

## 5.4 Left-right correlations beyond the PR approximation

As demonstrated by the results in the previous sections, the effect of the time variation of the tension on the squared velocity of the strings significantly depends on the scale in consideration. The macroscopic VOS approach suggests this quantity interpolates between values greater and smaller than 1/2 when the variation of the tension dominates against the expansion and vice versa. On the other hand, the investigation of the circular loop as proxy for subhorizon physics, with a formal and perturbative approach, showed the correction appears to be significantly suppressed, of order $\mathcal{O}((\mathcal{H}L)^2)$. Overall, this raises a question about the validity of the constant tension framework outlined in section 4.1, as the velocity was taken to be smaller than 1/2 throughout the entire process. Furthermore, as supported by the macroscopic VOS arguments, the correlation between left and right moving unit vectors receives interesting contributions from the variation of the tension. This suggests a potential intricate structure of the correlator between left and right moving unit vectors at different points along the strings. This is a departure from the constant tension approach, where this last correlator was treated as a fixed quantity based on the argument of slow evolution of right (left) moving unit vectors along left (right) curves. Overall, this motivates the study of a left-right correlator in the same spirit of (4.1.22):

$$h_{+-}(s, s', \tau) \equiv 1 - \mathbf{p}_+(s, \tau) \cdot \mathbf{p}_-(s', \tau) \tag{5.4.1}$$

which could be investigated in a similar way by computing the time derivative of the scalar product $\mathbf{p}_+(s, \tau) \cdot \mathbf{p}_-(s', \tau)$. However, the original framework introduced the variable $s$, constant along left moving curves, to allow one to compute the evolution of $\mathbf{p}_+$ much easily. This is because the $\partial_\tau \mathbf{p}_+(s, \tau)$ yields exactly the left moving operator. The problem is that $\partial_\tau \mathbf{p}_-(s, \tau) = \dot{\mathbf{p}}_- - \frac{1}{\varepsilon}\mathbf{p}'_-$, which is not the right moving operator, the one evolving $\mathbf{p}_-$ in (4.2.10). To overcome this, we can introduce another variable constant along the $\mathbf{p}_-$ characteristics, such that the time evolution correspond to the equation of motion operator. Subsequently, one can focus on the situation where the left and right moving curves intersect, and consider this as the first order approximation of the correlator. This more general analysis will show that the PR assumption $\langle \mathbf{p}_- \cdot \mathbf{p}_+ \rangle = -\alpha(s, \tau - \delta)$, consisting of treating this scalar product as evaluated at the intersection point, is the first order result while the correction is proportional to the ratio between string segment length $\ell$ and horizon size $d_H$.

Let us call this variable $v$, whose evolution is given by $\dot{v} + \frac{1}{\varepsilon}v' = 0$. Then, it is necessary to denote the dependence of $\sigma$ on the two variables $s$ and $v$. This is important because:

$$\frac{d\sigma_s(\tau)}{d\tau} = -\frac{1}{\varepsilon} \qquad\qquad \frac{d\sigma_v(\tau)}{d\tau} = +\frac{1}{\varepsilon} \tag{5.4.2}$$

which is crucial to avoid the aforementioned problem. Derivatives with respect to sigma will be denoted with a prime and implicitly $\mathbf{p}_-$ is differentiated with respect to $\sigma_v$ while $\mathbf{p}_+$ with respect to $\sigma_s$. At this point, one can consider the left-right scalar product

$M(\tau) \equiv \mathbf{p}_+(s,\tau)\cdot\mathbf{p}_-(v,\tau)$ and take the time derivative of it:

$$\begin{aligned}\partial_\tau M(\tau) =&(\partial_\tau \mathbf{p}_+(s,\tau))\cdot\mathbf{p}_-(v,\tau)+\mathbf{p}_+(s,\tau)\cdot(\partial_\tau\mathbf{p}_-(v,\tau))\\ =&-\tilde{\mathcal{H}}\left[P^{sv}_{--}(\tau)+P^{sv}_{++}(\tau)\right]-\tilde{\mathcal{H}}A^{sv}(\tau)M(\tau)\end{aligned}$$

where

$$P^{sv}_{--}(\tau)=\mathbf{p}_-(s,\tau)\cdot\mathbf{p}_-(v,\tau) \qquad P^{sv}_{++}(\tau)=\mathbf{p}_+(s,\tau)\cdot\mathbf{p}_+(v,\tau) \tag{5.4.3}$$

and

$$A^{sv}(\tau)=-\mathbf{p}_+(v,\tau)\cdot\mathbf{p}_-(v,\tau)-\mathbf{p}_+(s,\tau)\cdot\mathbf{p}_-(s,\tau)\equiv\alpha(v,\tau)+\alpha(s,\tau) \tag{5.4.4}$$

Then, the above is a first order linear differential equation for $M(\tau)$ whose solution can be found with the method of the integrating factor. In this case, the latter is:

$$I(\tau_2,\tau_1)=\int_{\tau_1}^{\tau_2}d\bar{\tau}\tilde{\mathcal{H}}A^{sv}(\bar{\tau}) \tag{5.4.5}$$

and the general solution reads:

$$M(\tau)=e^{-I(\tau,\tau_*)}M(\tau_*)-\int_{\tau_*}^{\tau}d\bar{\tau}\,\tilde{H}(\bar{\tau})\left[P^{sv}_{++}(\bar{\tau})+P^{sv}_{--}(\bar{\tau})\right]e^{-I(\tau,\bar{\tau})} \tag{5.4.6}$$

where $\tau_*$ denotes the time at which the two curves intersect. This can be computed by using the two characteristic equations, $\sigma_s(\tau)=s-\tau/\varepsilon$, $\sigma_v(\tau)=v+\tau/\varepsilon$, and imposing $\sigma_s(\tau_*)=\sigma_v(\tau_*)$. This leads to $\tau_*=\varepsilon_*(s-v)/2$, assuming $\varepsilon$ to be constant at the value $\varepsilon(\tau_*)\equiv\varepsilon_*$. At such instant, the solution becomes $M(\tau_*)=-\alpha(\tau_*)$ as in the PR model. Interested in deviations of the scalar product from this value, we can approximate the other terms in the equation at this time. This means:

$$A(s,v,\tau_*)=2\alpha_* \qquad P^{sv}_{--}(\tau_*)=P^{sv}_{++}(\tau_*)=1 \tag{5.4.7}$$

Thus, the differential equation at $\tau_*$ reads:

$$\partial_\tau M\big|_{\tau_*}\simeq-2\tilde{H}_*\left(1-\alpha_*^2\right). \tag{5.4.8}$$

The Hubble parameter is given by $\tilde{\mathcal{H}}_*=\kappa\nu'/\tau_*$, such that the solution is logarithmic in time:

$$M(\tau)=M(\tau_*)-2\kappa\nu'(1-\alpha_*^2)\ln(\tau/\tau_*) \tag{5.4.9}$$

The deviation we are looking for is $\delta M(\tau)\equiv M(\tau)-M(\tau_*)$, given by:

$$\delta M(\tau)=-2\kappa\nu'(1-\alpha_*^2)\ln(\tau/\tau_*) \tag{5.4.10}$$

For $\tau$ close to the crossing time $\tau_*$, meaning $|\tau-\tau_*|/\tau_*\ll1$, one can expand the natural log as:

$$\ln(\tau/\tau_*)=\ln\left(1+\frac{\tau-\tau_*}{\tau_*}\right)\simeq\frac{\tau-\tau_*}{\tau_*} \tag{5.4.11}$$

such that the final expression for $\delta M(\tau)$ is:

$$\delta M(\tau)=-2\tilde{H}_*\left(1-\alpha_*^2\right)(\tau-\tau_*)+\mathcal{O}\left(\left(\frac{\tau-\tau_*}{\tau_*}\right)^2\right) \tag{5.4.12}$$

In particular, one can write this in terms of physical quantities of interest, namely the length of string segment $\ell$ and the horizon size $d_H \sim t$. Indeed, the physical distance between the points that coincided at $\tau_*$ can be written as:

$$\Delta\sigma(\tau) = \sigma_v(\tau) - \sigma_s(\tau) = \frac{2}{\varepsilon_*}(\tau - \tau_*) \tag{5.4.13}$$

Moreover, noting the conformal time modified Hubble parameter $\tilde{\mathcal{H}} = a\tilde{H} = a\kappa\nu/t$, the correction $\delta M(\tau)$ becomes:

$$\delta M(\tau) \simeq -(1-\alpha_*^2)\frac{\kappa\nu}{t_*}(a\varepsilon_*\Delta\sigma) \tag{5.4.14}$$

and recalling (4.1.3), the round bracket term $(a\varepsilon_*\Delta\sigma)$ is precisely the string segment length between the two points that evolved along their characteristics from $\tau_*$. Finally, this deviation is:

$$\delta M(\tau) \simeq -\kappa\nu(1-\alpha_*^2)\left(\frac{\ell}{t_*}\right) \tag{5.4.15}$$

which clearly shows it is of order of the ratio between the segment length and the horizon size, at the intersection time. However, since we are performing a local expansion near this crossing time, one can equivalently replace $t_* \to t$ at leading order, effectively leading to:

$$\delta M(\tau) \simeq \mathcal{O}\left(\frac{\ell}{t}\right) \tag{5.4.16}$$

Since in the stretching regime one considers $\ell \ll d_H \sim t$, and $\ell/t \to 0$ corresponds to the small-scale structure limit, $\delta M(\tau) \to 0$ in this regime. This highlights the fact that although the variation of the tension enters the correlation between left and right moving unit vectors, the induced correction is suppressed in the small-scale regime and therefore it does not modify the leading order PR assumption. This can be considered as further evidence in support of the original framework.

# Chapter 6

# Conclusion

This thesis set out to investigate the small-scale structure of cosmic strings with a time dependent tension. This line of research has received a lot of attention in the recent years and it is widely motivated by string theory inspired models where analogues of cosmic strings can be realized. After showing the mechanism underlying the formation of these object in a field theoretic context, we reviewed in detail the microscopic equations of motion and the macroscopic VOS model that capture the evolution of these defects on different scales. This scale separation is what ultimately leads to our original results.

The framework proposed by Polchinski and Rocha to investigate the structure of constant tension string segments of length $\ell$ below the horizon size has been thoroughly reviewed and examined, highlighting the positivity of the scaling exponent $\chi$, consequence of the network averaged velocity $\bar{v}^2$ being smaller than the Minkowskian value of 1/2 in an expanding universe. Afterwards, this has been extended by allowing the tension to vary in time. The notable effect is that the parameter $\kappa = 1 - mn/4$, related to the Hubble parameter by $2\tilde{\mathcal{H}} = 2\kappa\nu'/\tau$, has been shown to enter the modified scaling exponent:

$$\chi' = \frac{\kappa\bar{\alpha}\nu'}{1 + 2\kappa\nu'\bar{v}^2} \tag{6.1}$$

both at the numerator and denominator. However, the sign of this exponent has been found to be positive in both regimes of rapidly varying tension and slowly varying tension. This has been shown to be a consequence of the sign of $\bar{\alpha} = 1 - 2\bar{v}^2$ being identical to the one of $\kappa$, due to $\bar{v}^2$ interpolating between values greater and smaller than 1/2 when $\kappa$ was smaller or greater than zero, respectively. The conclusion is therefore that the correlator $h_{++}(\sigma - \sigma', \tau)$, whose evolution is:

$$\big\langle h_{++}(\sigma - \sigma', \tau)\big\rangle \approx \tilde{A}\left(\frac{\ell}{t}\right)^{2\chi'} \tag{6.2}$$

vanishes in the small-scale limit $\ell/t \to 0$. Therefore, the scalar product between left-left or right-right moving unit vectors evaluated at different points along the string approaches 1 in this same limit, signaling a regular small-scale structure and motion of the string. In other words, it can be conclusively stated that, within the assumptions of the generalized stretching regime, the fractal dimension of a cosmic string network with a time varying tension approaches unity at small-scale, so that the strings are smooth.

The crucial role of the squared velocity motivated the understanding of the effect of a time varying tension on the deviation of this quantity from the Minkowskian value of

1/2. This has been investigated in two systems with different characteristic scales. On the one hand, the VOS approach has been used to compute this deviation at the level of the averaged network velocity. The result is the following:

$$\bar{v}^2 = \frac{1}{2} + \delta \simeq \frac{1}{2} + \frac{\tilde{c}\pi}{24\sqrt{2}(n-2)}(mn-4) \tag{6.3}$$

clearly demonstrating the flip of the sign of $\bar{\alpha}$ when $\kappa$ also changes sign. On the other hand, the circular loop solution has been thoroughly studied as a representative for subhorizon physics, with the aim of understanding whether the time-variation of the tension had a significant effect in this regime. The formal solution approach based on the phase function $f(x)$ was found to be able to generate analytical corrections to the squared velocity with a careful treatment of oscillatory functions. The novel introduction of the phase correction $\xi(x)$ lead to the solution:

$$f(x) = F(x) + \xi(x) = F(x) + \frac{3q}{8\alpha}\frac{\cos 2F}{F} + \mathcal{O}(F^{-2}) \tag{6.4}$$

which has been numerically shown to improve the control of residual oscillatory corrections previously neglected in the search for an averaged solution $F(x) = A_1 + A_2 x^\alpha$ and to lower the discrepancy with the full numerical solution. The main conclusion from this approach is that the size of the correction induced by the time variation of the tension are highly suppressed in the subhorizon regime. Both the formal loop solution and perturbative treatment reproduce the leading order result $v^2 = 1/2$ with corrections displayed at order $\mathcal{O}((\mathcal{H}L)^2)$. In addition, the perturbative approach allowed us to recover the evolution of the loop amplitude following from the removal of secular terms in the particular solution. Therefore, this perturbative approach validates the conclusion about the order of the possible effect on subhorizon dynamics, since we recover $v^2 = 1/2 + \mathcal{O}(\varepsilon^2)$, and demonstrates the main effect of the time varying tension is on the amplitude, not on the velocity. This is found to grow when the tension varies faster than the scale factor and vice versa, agreeing with the literature.

As a final consistency check, we have discussed the impact of the time varying tension on the correlation between left and right movers, since this is closely connected with $\bar{\alpha}$ and thus $v^2$. Our analysis showed the original assumption, in which the mixed scalar product is evaluated at the intersection point of the corresponding characteristics, remains valid at leading order. The corrections are found to be suppressed by $\mathcal{O}(\ell/t)$ and therefore vanish in the small scale limit. Overall, within the generalized stretching regime considered in this thesis, the time-dependent tension does not alter the smooth small-scale structure predicted by the PR framework. The possible effects of this are found at the level of the network averaged squared velocity and in the evolution of loop amplitudes, while subhorizon corrections to the loop velocity and left-right correlator remain suppressed. A complete treatment involving both the evolution of long string networks and loop population constitutes a potential future research direction which is left for future work.

# Appendix A

# Details of dimensional reduction

Here we present some of the details regarding the calculations leading to the final results in section 2.2.

For the scalar field kinetic terms, the Ricci tensor with both $d-$dimensional indices is found to be:

$$\begin{aligned}\hat{R}_{\mu\nu} =& R_{\mu\nu}(g_{\mu\nu}) - \alpha g_{\mu\nu}\nabla^2\phi - ((d-2)\alpha^2 + n\alpha\beta)g_{\mu\nu}(\partial\phi)^2 + \\ & \left((d-2)\alpha^2 + 2n\alpha\beta - n\beta^2\right)\partial_\mu\phi\partial_\nu\phi - ((d-2)\alpha + n\beta)\nabla_\mu\nabla_\nu\phi\end{aligned}$$

where covariant derivatives have been used to group terms:

$$\nabla_\mu\nabla_\nu\phi = \partial_\mu\partial_\nu\phi - \Gamma^\sigma_{\mu\nu}\partial_\sigma\phi \qquad \nabla^2\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi \tag{A.1}$$

Note that the Ricci tensor for the lower dimensional metric $g_{\mu\nu}$ has appeared. As mentioned before, the only possible contributions from the index structure involve derivatives of the field $\phi$ or the metric tensor. Instead, the Ricci tensor with both $n$-dimensional indices is:

$$\hat{R}_{mn} = \tilde{R}_{mn}(\gamma_{mn}) - \beta\, e^{2\phi(\beta-\alpha)}\gamma_{mn}\nabla^2\phi + \left(\alpha\beta(2-d) - n\beta^2\right)(\partial\phi)^2\gamma_{mn} \tag{A.2}$$

where $\tilde{R}_{mn}$ is the Ricci tensor associated with $\gamma_{mn}$. Combining these results in (2.2.20) gives:

$$\hat{R} = e^{-2\alpha\phi}R(g_{\mu\nu}) + e^{-2\beta\phi}\tilde{R}(\gamma_{mn}) - e^{-2\alpha\phi}\nabla^2\phi\,[2\alpha(d-1) + 2n\beta] + +e^{-2\alpha\phi}K(\alpha,\beta)(\partial\phi)^2$$

where

$$K(\alpha,\beta) \equiv n\alpha\beta(2-d) - n^2\beta^2 + (d-2)(1-d)\alpha^2 + (2-d)n\alpha\beta - n\beta^2 \tag{A.3}$$

For the Maxwell term, the Christoffel symbols associated with the KK metric tensor to quadratic order in $A_\mu$ are:

$$\begin{aligned}
&\hat{\Gamma}^\mu_{\nu\sigma} = \Gamma^\mu_{\nu\sigma} + \frac{1}{2}e^{2\phi(\beta-\alpha)}\left(F_\sigma{}^\mu A_\nu + F_\nu{}^\mu A_\sigma\right) && \hat{\Gamma}^z_{zz} = 0 \\
&\hat{\Gamma}^\mu_{zz} = 0 && \hat{\Gamma}^z_{\mu\nu} = \frac{1}{2}\left(\nabla_\mu A_\nu + \nabla_\nu A_\mu\right) + \mathcal{O}(A^3) \\
&\hat{\Gamma}^\mu_{z\sigma} = \frac{1}{2}e^{2\phi(\beta-\alpha)}F_\sigma{}^\mu && \hat{\Gamma}^z_{\mu z} = -\frac{1}{2}e^{2\phi(\beta-\alpha)}F_{\mu\lambda}A^\lambda
\end{aligned}$$

The $dd$-contribution to the Ricci tensor is the more involved to compute and here we present some of the intermediate steps:

$$\hat{R}_{\mu\nu} = R_{\mu\nu} + \frac{1}{2}k^2 \left[ \left( \nabla_\lambda (F_\nu{}^\lambda A_\mu) - \frac{1}{2} F_\mu{}^\lambda (\nabla_\lambda A_\nu + \nabla_\nu A_\lambda) \right) + (\mu \leftrightarrow \nu) \right] \tag{A.4}$$

where $k^2 \equiv e^{2\phi(\beta-\alpha)}$ and the covariant derivatives are of the kind:

$$\nabla_\sigma (F_\mu{}^\sigma A_\nu) = \partial_\sigma (F_\mu{}^\sigma A_\nu) - \Gamma^\lambda_{\mu\sigma} F_\lambda{}^\sigma A_\nu - \Gamma^\lambda_{\nu\sigma} F_\mu{}^\sigma A_\lambda + \Gamma^\sigma_{\lambda\sigma} F_\mu{}^\lambda A_\nu \tag{A.5}$$

Note that in the case of a symmetric connection, one has:

$$\nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu} \tag{A.6}$$

Grouping terms in an appropriate way yields the following result:

$$\hat{R}_{\mu\nu} = R_{\mu\nu} + \frac{1}{2}k^2 \left[ (\nabla_\sigma F_\nu{}^\sigma) A_\mu + (\nabla_\sigma F_\mu{}^\sigma) A_\nu \right] - \frac{1}{2} k^2 F_{\mu\sigma} F_\nu{}^\sigma \tag{A.7}$$

The $zz-$ and $z\mu-$components, instead, can be easily computed to find:

$$\hat{R}_{zz} = \frac{1}{4} k^2 F_{\lambda\mu} F^{\lambda\mu} \qquad\qquad \hat{R}_{z\mu} = \frac{1}{2} k^2 \nabla_\nu F_\mu{}^\nu \tag{A.8}$$

Overall, performing the contraction with the inverse metric as in (2.2.32) leads to the important result (2.2.33):

$$\hat{R} = e^{-2\alpha\phi} \left[ R - \frac{1}{4} e^{2\phi(\beta-\alpha)} F_{\mu\nu} F^{\mu\nu} \right] + \mathcal{O}(A^3) \tag{A.9}$$

# Appendix B

# Master equation for $f(x)$

The discussion of section 5.2 heavily relies on equation (5.2.4) first derived in [42]. Because of its importance, the details of its derivation are presented in this appendix. By differentiating (5.2.3) with respect to $x$ and matching it with $R'(x) = -\sin(f(x))$, one has:

$$-\sin f(x) = -\left[f' \sin f(x) + \frac{q}{x}\cos f(x)\sin^2 f(x)\right]\exp\left[-q\int \frac{d\bar{x}}{\bar{x}}\sin^2\left(f(\bar{x})\right)\right] \tag{B.1}$$

which implies:

$$\exp\left[q\int \frac{d\bar{x}}{\bar{x}}\sin^2\left(f(\bar{x})\right)\right] = f' + \frac{q}{x}\cos f(x)\sin f(x) \tag{B.2}$$

Therefore, $R(x)$ can be written as:

$$R(x) = \frac{\cos f(x)}{f' + \frac{q}{x}\cos f(x)\sin f(x)} \tag{B.3}$$

Repeating the procedure again leads to a single differential equation relating $f$ and its derivatives, the master equation we are after. Indeed:

$$-\sin f = R' = \frac{-f'\sin f}{f' + \frac{q}{x}\cos f\sin f} - \frac{\cos f\left[f'' - \frac{q}{x^2}\cos f\sin f + \frac{q}{x}f'\left(1-2\sin^2 f\right)\right]}{\left(f' + \frac{q}{x}\cos f(x)\sin f(x)\right)^2} \tag{B.4}$$

Rearranging terms, this becomes:

$$0 = \frac{\cos f}{\sin f}f'' - \frac{3qf'}{x}\sin f\cos f - \frac{q}{x^2}\cos^2 f - \frac{q^2}{x^2}\cos^2 f\sin^2 f + \frac{q}{x}f'\frac{\cos f}{\sin f} \tag{B.5}$$

To isolate the $f''$ term and since $2\sin f\cos f = \sin 2f$, it is useful to multiply by $2x^2\sin f/\cos f$, which leads to:

$$0 = 2x^2 f'' + 2qxf'\left(1-3\sin^2 f\right) - q(1+q\sin^2 f)\sin 2f \tag{B.6}$$

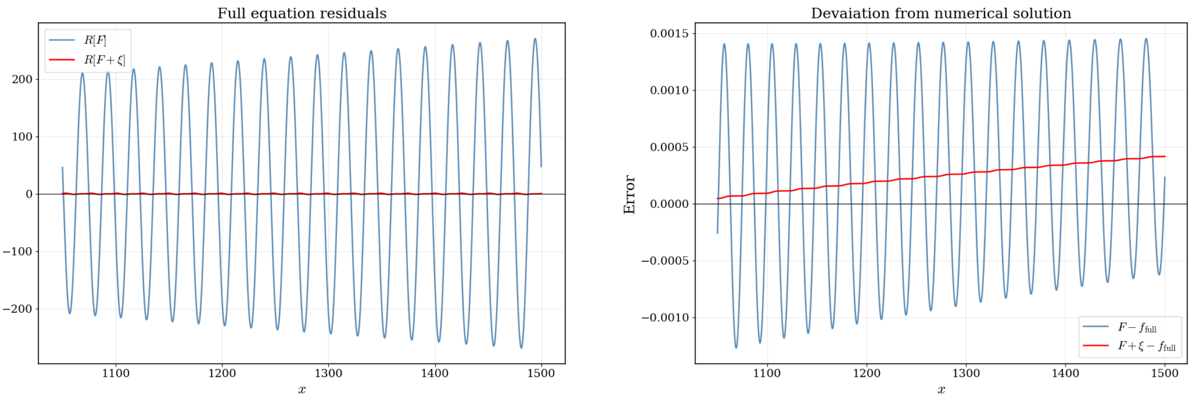
Finally, by writing $\sin^2 f = (1-\cos 2f)/2$ one recovers the final equation presented in [42] with the correct presence of $f'(x)$ in the second term:

$$0 = 2x^2 f'' - qxf'\left(1-3\cos 2f\right) - q(1+q\sin^2 f)\sin 2f \tag{B.7}$$

# Appendix C

# Numerical check of phase correction

The numerical checks of the accuracy and consistency of the phase correction $\xi(x)$ compared to the averaged solution $F(x) = A_1 + A_2 x^\alpha$ are presented in this appendix. A useful way to compare the two approaches is to evaluate the residual terms using the averaged solution $F(x)$ and $F(x)+\xi(x)$. These are given by $R[F]$ and $R[F+\xi]$, obtained by substituting the averaged solution $F(x)$ or $F(x)+\xi(x)$ into the master phase equation, respectively. According to the decomposition used in the main text, $R[F] \simeq E_{osc}[F]$ while $R[F+\xi] \simeq E_{osc}[F] + L_{avg}[\xi] + L_{osc}[\xi] \simeq L_{osc}[\xi]$, where the equality sign is not used as this right hand expression contains also subleading terms on top of the oscillatory ones. Therefore, these residual terms measure the size of the omitted oscillatory and subleading terms.




(a) Residual terms of the full equation.

(b) Deviation from the full numerical simulation.

Figure C.1: Numerical consistency check for the kination benchmark with $q = -1/2$ over $x \in [1000, 1500]$. The first panel shows the corresponding residuals in the full equation for the averaged solution $F(x)$ with and without the phase correction $\xi(x)$. The second panel compares the error before and after including the oscillatory correction $\xi$.

The plot in Fig.C.1a shows the original averaged solution neglected oscillatory terms interpolating between roughly $-200$ and $+200$, while after the introduction of the phase correction these are under much better control, oscillating between $-1$ and $+1$ values. This is consistent with the equation determining $\xi(x)$ it self, equation (5.2.33), which is exactly the condition to remove the large oscillatory terms remaining in the full equation when one substitutes the averaged $F(x)$. Quantitatively, this can be confirmed by looking at the RMS residual ratio, defined as:

$$\mathcal{R}_{\text{RMS}} = \frac{\sqrt{\frac{1}{N}\sum_{i=1}^{N}\left[\text{Res}(F+\xi)(x_i)\right]^2}}{\sqrt{\frac{1}{N}\sum_{i=1}^{N}\left[\text{Res}(F)(x_i)\right]^2}} \tag{C.1}$$

where $N$ is the number of sampled points in the interval $x \in [1000, 1500]$, which in the numerical implementation used here is $N = 3001$. For the present case, this is found to be $\mathcal{R}_{RMS} \simeq 0.0038$, while it is $\mathcal{R}_{RMS} \simeq 0.0033$ for a larger window, namely for $x \in [1000, 2000]$. Since these values are smaller than one, they indicate the correction reduces the residual terms significantly over longer ranges of $x$ as well.

Another argument can be based on the difference between the two solutions, $F(x)$ and $F(x) + \xi(x)$, and the true numerical solution to the master phase equation. The result is shown in Fig.C.1b and it highlights the averaged initial solution was already consistent with the numerical one, although the oscillations were larger than when the phase correction is taken into account. The effect of such phase is to significantly decrease the discrepancy as further supported by the RMS error ratio defined as:

$$\mathcal{E}_{\text{RMS}} = \frac{\sqrt{\frac{1}{N}\sum_{i=1}^{N}\left[F(x_i)+\xi(x_i)-f_{\text{full}}(x_i)\right]^2}}{\sqrt{\frac{1}{N}\sum_{i=1}^{N}\left[F(x_i)-f_{\text{full}}(x_i)\right]^2}} \tag{C.2}$$

which is found to be $\mathcal{E}_{RMS} \simeq 0.30$ with the current window, while it grows to $\mathcal{E}_{RMS} \simeq 0.54$ with $x \in [1000, 2000]$. This growth can be seen by directly looking at the graph, clearly showing a larger discrepancy of the phase corrected solution as $x$ increases. However, this deviation does not correspond to a physical inefficiency of the correction, but merely to the homogeneous terms of the full $\xi(x)$ solution. Indeed, (5.2.33) is a second order inhomogeneous differential equation, and the solution we used is the particular one. The two independent homogeneous solutions correspond to $\xi(x) = B_1 + B_2 x^\alpha$, which is exactly the functional form of the averaged solution $F(x)$, as they satisfy the same equation.

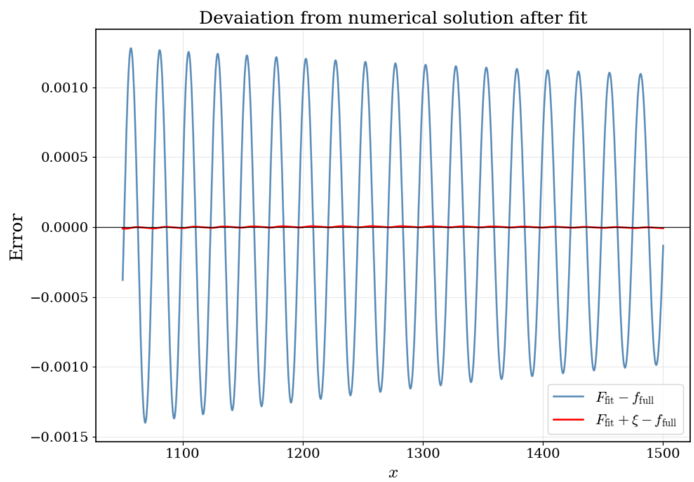



Figure C.2: Deviation from the numerical solution after refitting the integration constants.

Therefore, these homogeneous solutions correspond to a small shift of the integration constants of $F(x)$. This suggests performing a fit of the integration constants of $F(x)$

before comparing it with the full numerical solution. The result of this procedure is shown in Fig.C.2. This result clearly shows that a slow drift between $F(x) + \xi(x)$ and the numerical solution should not be interpreted as a failure of the oscillatory correction itself, but rather as a small mismatch in the choice of integration constants, which can be removed once we allow these constants to be fitted against the numerical solution. After this fit, the RMS error ratio is found to be $\mathcal{E}_{RMS} \simeq 0.0092$, which demonstrates once again the phase correction improves the solution by almost two orders of magnitude.

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