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PHYSICS

Black Hole Complementarity and the Island Paradigm

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Abstract

The black-hole information paradox highlights the tension between unitarity and semiclassical gravity. The island paradigm resolves this by reproducing the Page curve. This thesis argues the algebraic structure of the island paradigm realises Bohr's complementarity in quantum gravity.

We evaluate the paradox and the island formula in Jackiw-Teitelboim gravity. In a fixed spacetime, local exterior algebras are type III_1 factors. Including the observer's clock (the ADM Hamiltonian) converts this into a type II_∞ crossed product with a trace. We construct dressed observables for asymptotic and infalling observers. Since infalling observables are generated by a modular shift from the asymptotic clock, they form non-commuting subalgebras, suggesting complementarity.

Finally, we introduce a relational framework using quantum reference frames. Gravitational entropy becomes observer-dependent: the asymptotic entropy features an island, while the infalling one does not. Therefore, the horizon, interior, and island emerge as relational structures.

Contents

Chapter 1

Introduction

The discovery that black holes radiate [1] is one of the deepest results of twentieth-century theoretical physics. It unites quantum mechanics, general relativity, and thermodynamics in a single phenomenon. Yet it also confronts us with a profound puzzle: if a black hole formed from a pure quantum state evaporates completely into thermal radiation, the final state is mixed. Unitarity, the fundamental principle that quantum evolution preserves information, appears to be violated. This is the black-hole information problem.

For nearly fifty years the problem has driven a search for a consistent quantum theory of gravity. The tension is not merely technical; it forces us to re-examine the very concepts of space, time, and observables. In the last few years a remarkable resolution has emerged. Using the gravitational path integral, several groups showed that the von Neumann entropy of Hawking radiation follows the Page curve [26]—the hallmark of unitary evolution—if one allows the entanglement wedge of the radiation to include a region inside the black hole, an island [3, 4, 5]. The island formula,

$$S(\rho_R) = \min_I \left\{ \frac{\text{Area}(\partial I)}{4G_N} + S_{\text{matter}}(R \cup I) \right\}, \tag{1.1}$$

successfully reproduces the Page curve and restores unitarity, at least at the semiclassical level.

The island formula is now well established, but its conceptual implications are still being digested. What does the appearance of an island behind the horizon tell us about the nature of the black-hole interior and the horizon itself? Is the island a physical membrane, or is it a book-keeping device that signals a deeper relational structure?

This thesis reconstructs and synthesises the route from the black-hole information paradox to the island paradigm and its algebraic/relational reading. The interpretive contribution is the claim that the crossed-product structure suggests an algebraic, observer-dependent version of complementarity, which we argue is consistent with—and extends—the early black-hole complementarity program. The complementarity claim is presented as suggestive, not as a proven analogue of $[x, p] = i\hbar$.

This thesis argues that the island paradigm can be understood as a realisation of Bohr's complementarity in quantum gravity. More precisely, we show that the observables associated with the black hole's interior and exterior are non-commuting, suggesting a structural parallel with position and momentum in ordinary quantum mechanics. We interpret this algebraic non-commutation as a structural realisation of black-hole complementarity, providing a mathematical framework for the early proposals of Susskind, Thorlacius, and Uglum.

The argument proceeds in two main thrusts. First, we analyse the algebraic structure of quantum fields on a black-hole background. In a fixed spacetime, the local algebras of the exterior and interior are type III_1 factors, which have no trace and no density matrices. Gravity changes this: the inclusion of the asymptotic observer's clock—the ADM Hamiltonian—converts the exterior algebra into a type II_∞ crossed product [17, 18, 19]. Physical observables must be dressed to this clock. An infalling observer uses a different clock (her proper time), and her algebra is obtained from the exterior algebra by a half-sided modular translation. We discuss the non-commutation of dressed exterior and interior observables, which suggests a structural parallel with the non-commutation of complementary observables in ordinary quantum mechanics. We interpret this as an algebraic statement of complementarity.

Second, we embed this algebraic complementarity into a broader philosophical framework. Drawing on the relational quantum mechanics of Rovelli [20] and the modern theory of quantum reference frames (QRFs) [23], we interpret the two observers as QRFs whose algebras are complementary. The island appears as a relational structure: it is the piece of the interior that the asymptotic observer must "borrow" from the infalling observer's algebra to purify the radiation. Gravitational entropy becomes observer-dependent, and the Page curve is the asymptotic observer's perspective on a fundamentally relational process. The non-commutation of sub-algebras is interpreted as a structural realisation of Bohr's complementarity.

This thesis is therefore situated at the intersection of algebraic quantum field theory, holography, and the foundations of quantum mechanics. Its central thesis is that the island paradigm can be understood as a realisation of Bohr's complementarity, with the algebraic QFT of type II von Neumann algebras and the theory of quantum reference frames providing a precise mathematical framework for this interpretation. The black-hole interior, long thought to be a region of spacetime, emerges instead as a relational concept, defined only with respect to the observer's clock.

The rest of the thesis is structured as follows.

- **Chapter 2** traces the historical development of the black-hole information problem, from Hawking's original calculation through the Page curve, the AMPS firewall, and the early complementarity proposals, to the modern island formula.

- **Chapter 3** reviews Bohr's original formulation of complementarity and examines the first black-hole complementarity program of Susskind, Thorlacius, and Uglum, together with the philosophical criticisms that it lacked precise mathematical realisation.

- **Chapter 4** presents the island formula in detail using the Jackiw–Teitelboim (JT) gravity model, and shows how it yields the Page curve for the eternal two-sided black hole.

- **Chapter 5** contains the algebraic core of the thesis. It introduces local von Neumann algebras, reviews the type III_1 property, constructs the crossed product, derives the dressing of observables, and discusses the non-commutativity of interior and exterior dressed operators, which we interpret as a structural realisation of complementarity.

- **Chapter 6** develops the relational interpretation. It connects Bohr's complementarity to quantum reference frames, shows that the black-hole observers can be interpreted as QRFs, and establishes the observer-dependence of gravitational entropy. The horizon, interior, and island are interpreted as relational structures.

- **Chapter 7** summarises the main results and outlines directions for future research, including the application of the relational framework to cosmology.

By the end of this work, the reader will see that the resolution of the black-hole information problem does not require a modification of quantum mechanics, but rather a refinement of our concepts of space and time. The island is not a thing, but a relation.

Chapter 2

The Black Hole Information Problem

A black hole formed from the collapse of a pure quantum state and subsequently evaporating via Hawking radiation appears to violate unitarity—the cornerstone of quantum mechanics. This chapter states the paradox in its simplest form, introduces the Page curve as the quantitative benchmark of unitary evolution, and explains why semiclassical gravity fails to reproduce it. We conclude that resolving the paradox requires a radical revision of the concepts of spacetime and observables, which the remainder of this thesis will explore through the algebraic and relational framework.

2.1 The Hawking Effect and the Information Paradox

In 1975, Stephen Hawking demonstrated that black holes are not truly black [1]. By quantising a free scalar field on the background of a collapsing star, he showed that the black hole emits particles with a thermal spectrum (see appendix A for the derivation of Hawking effect). This result—now known as Hawking radiation - is one of the cornerstones of modern theoretical physics, but it also sparked a crisis: if the radiation is exactly thermal, it carries no information about the initial quantum state of the collapsing matter. A pure state would evolve into a mixed state, violating unitarity. This is the **black-hole information paradox**.

2.1.1 The physical picture

The intuitive mechanism of Hawking radiation is pair creation near the horizon. In the vacuum of quantum field theory, virtual particle-antiparticle pairs are constantly created and annihilated. Near the event horizon, one member of a pair can fall into the black hole while the other escapes to infinity. The energy for this process is supplied by the gravitational field; as a result, the black hole loses mass and shrinks. An observer at infinity detects a steady flux of particles with a thermal distribution.

This heuristic picture, while not a replacement for a rigorous calculation, captures the essential physics: the horizon provides the physical boundary that allows one particle to be trapped while its partner is radiated away. The escaping particles are uncorrelated and carry no information about the interior—at least in the semiclassical approximation.

2.1.2 Hawking temperature and thermal spectrum

The rigorous derivation of appendix A and [1] shows that the radiation from a Schwarzschild black hole of mass M is exactly thermal, with a temperature

$$T_H = \frac{\hbar c^3}{8\pi G M k_B} = \frac{\hbar \kappa}{2\pi c k_B} \tag{2.1}$$

where k_B is the Boltzmann constant and k is the surface gravity. This is the Hawking temperature. The corresponding spectrum for bosonic particles is the Planck distribution

$$N(\omega) = \frac{1}{e^{\hbar\omega/k_B T_H} - 1} \tag{2.2}$$

and for fermions the appropriate Fermi-Dirac distribution. The temperature is inversely proportional to the mass: smaller black holes are hotter, and as the black hole radiates it heats up further, leading to a runaway evaporation.

The emission of energy causes the black hole to lose mass. Integrating the Stefan–Boltzmann law gives the lifetime of a black hole of initial mass M,

$$\tau = \frac{t_p}{3\beta}\left(\frac{M}{m_p}\right)^3, \tag{2.3}$$

where m_p and t_p are Planck mass and Planck time. So that a solar-mass black hole would live far longer than the age of the universe, while microscopic black holes evaporate almost instantly. The evaporation process is depicted in the Penrose diagram of Fig. 2.1.

2.1.3 The information paradox

The thermal nature of Hawking radiation leads directly to the paradox. In quantum mechanics, a pure state evolves unitarily to another pure state. If a black hole forms from a pure state (e.g., a collapsing star with a well-defined wavefunction) and then evaporates completely, the final state is a thermal bath of Hawking quanta—a mixed state, described by a density matrix rather than a single state vector. The von Neumann entropy of the radiation starts at zero (before evaporation) and ends at a value comparable to the initial Bekenstein–Hawking entropy $S_{\text{BH}} = A/4G_N$, which is huge. Information about the initial state appears to be lost, in direct conflict with the unitarity of quantum theory.

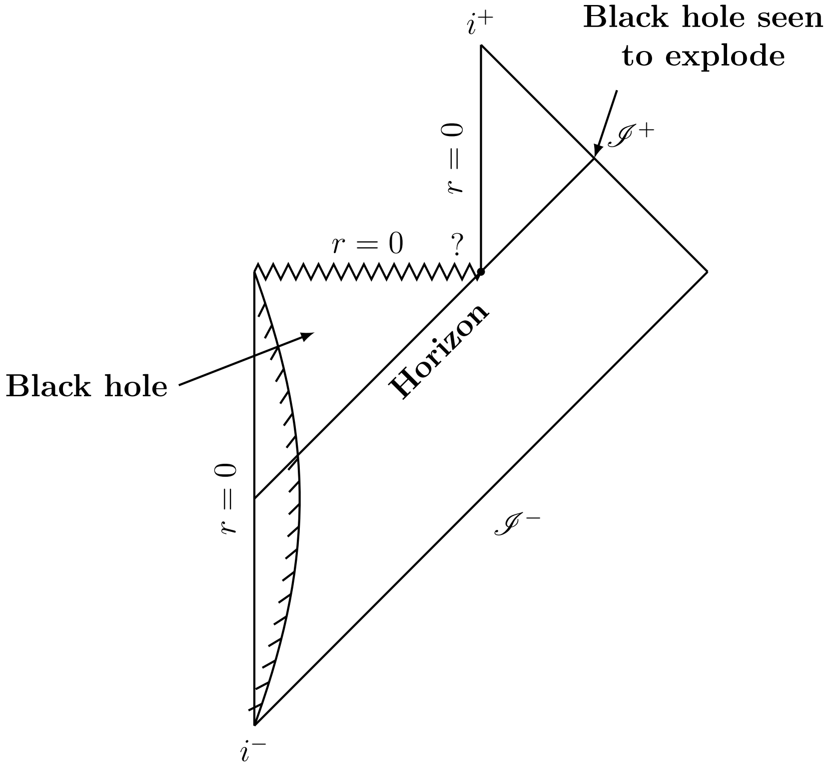

Figure 2.1: The Penrose diagram for an evaporating black hole

Hawking himself initially concluded that quantum gravity must be non-unitary, and that information is permanently destroyed in black-hole evaporation [2]. This radical proposal was met with widespread scepticism. The paradox, therefore, is not merely a technical loophole; it signals a deep tension between general relativity and quantum mechanics. Resolving it requires a consistent theory of quantum gravity.

In the decades that followed, a consensus emerged that unitarity must be preserved, but the mechanism by which information escapes the black hole remained elusive. A crucial step in quantifying the problem was provided by Don Page in 1993, who showed that if the evaporation is unitary, the entanglement entropy of the radiation must follow a characteristic curve—now called the Page curve—which we discuss in the next section.

2.2 The Page Curve and the Need for Unitarity

Hawking's result—that the radiation from a black hole is exactly thermal—implies a monotonically increasing entropy for the radiation, ending in a mixed state when the black hole has completely evaporated. This is in direct conflict with the unitarity of quantum mechanics. In 1993, Don Page sharpened the paradox by showing that if the evaporation is unitary, the entanglement entropy of the Hawking radiation must follow a characteristic, symmetric curve. This curve, now called the **Page curve**, provides the quantitative benchmark that any resolution of the information problem must reproduce.

2.2.1 The Page theorem for random pure states

Consider a bipartite quantum system with Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $\dim \mathcal{H}_A = d_A$ and $\dim \mathcal{H}_B = d_B$, and assume without loss of generality that $d_A \leq d_B$. Page [26, 27] proved that the average von Neumann entropy of subsystem A, when the total system is in a uniformly random pure state (Haar measure), is

$$\langle S_A \rangle \approx \ln d_A - \frac{d_A}{2d_B} \qquad \text{for integer } d_A, d_B. \tag{2.4}$$

For $d_A \ll d_B$, this reduces to $S_A \approx \ln d_A$, while for $d_A \approx d_B$ the entropy reaches a maximum near $S_A \approx \frac{1}{2}\ln(d_A d_B)$ and then decreases symmetrically. The resulting curve, shown in Fig. 2.2, rises almost linearly with $\ln d_A$, turns over when the subsystem comprises about half the total degrees of freedom, and finally falls back to zero. This is the **Page curve**.

The physical origin of the turnover is simple: when the subsystem A is small, it is highly entangled with the complementary system B, and its entropy is nearly maximal. As A grows to become larger than B, it effectively contains almost the entire system, so its entropy is now limited by the dimension of B and must decrease. The curve is symmetric because the average entropy of A and B are equal in a pure state, $\langle S_A \rangle = \langle S_B \rangle$.

2.2.2 The Page curve for an evaporating black hole

For a black hole, we identify subsystem A with the Hawking radiation and subsystem B with the remaining black hole. The total Hilbert space dimension is set by the Bekenstein–Hawking entropy,

$$D = e^{S_{\mathrm{BH}}}, \qquad S_{\mathrm{BH}} = \frac{A}{4G_N}. \tag{2.5}$$

Initially the black hole has entropy $S_{\mathrm{BH}}(0)$ and the radiation is empty ($d_R = 1$). As the black hole evaporates, d_R grows while d_{BH} shrinks, but the product $d_R\, d_{BH} = e^{S_{\mathrm{BH}}(0)}$ remains constant if unitarity holds.

At early times, $d_R \ll d_{BH}$, and the radiation entropy increases, closely tracking the thermal entropy produced by Hawking. At the **Page time** t_{Page}, the black hole has lost roughly half its initial entropy, $S_{\mathrm{BH}}(t_{\mathrm{Page}}) \approx S_{\mathrm{BH}}(0)/2$, and the radiation subsystem becomes the larger of the two. From that moment on, the entanglement entropy of the radiation must *decrease*, eventually returning to zero when the black hole has completely evaporated. For a Schwarzschild black hole in four dimensions, the Page time is

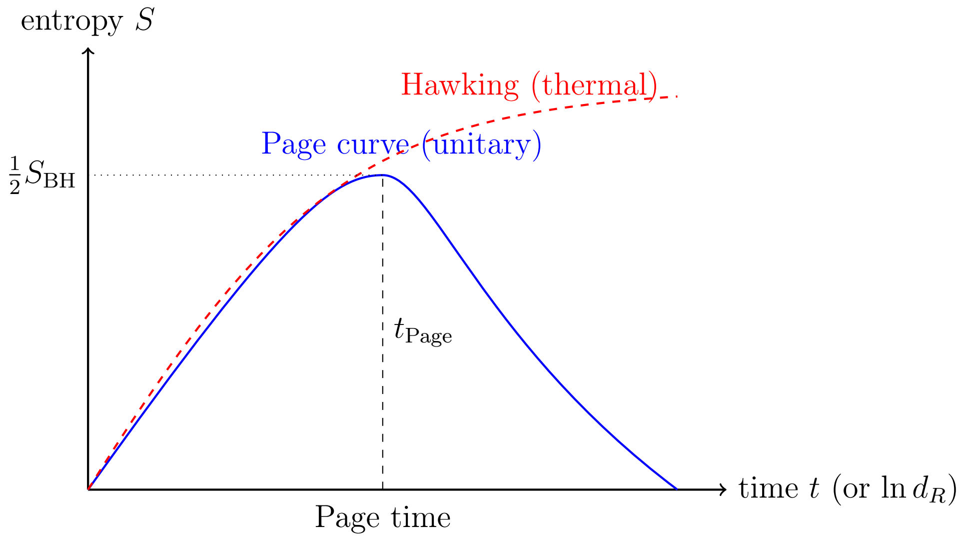

Figure 2.2: The Page curve (blue) for a unitary evaporation, compared with Hawking's ever-increasing entropy (red, dashed). The Page time t_{Page} marks the moment when the radiation entropy reaches its maximum and begins to decrease, returning to zero when the black hole has evaporated. The maximum entropy is approximately half the initial Bekenstein–Hawking entropy S_{BH}.

approximately

$$t_{\mathrm{Page}} \sim M^3 \sim \tau_{\mathrm{evap}}, \tag{2.6}$$

where τ_{evap} is the full evaporation lifetime.

2.2.3 The conflict with Hawking's calculation

Hawking's semiclassical computation yields a radiation entropy that grows monotonically and never turns over (red dashed curve in Fig. 2.2). Even after the black hole has completely evaporated, the radiation remains in a mixed state. This contradicts the unitarity of quantum mechanics, which demands that a pure initial state evolve to a pure final state. The Page curve therefore makes the information paradox quantitative: any consistent theory of quantum gravity must explain how the radiation entropy can follow the Page curve, turning over at the Page time and returning to zero.

2.3 The Failure of Semiclassical Physics

Semiclassical gravity—quantum field theory on a fixed classical black-hole background—is an approximation that treats spacetime as externally given. Hawking's calculation is performed entirely within this framework. The resulting thermal radiation arises from the infinite redshift at the horizon, which effectively erases any initial correlations.

The failure of semiclassical physics is rooted in two related shortcomings:

1. **The information paradox as an entanglement problem:** The Hawking process creates entangled pairs across the horizon. As long as the black hole remains macroscopic, the interior partner modes are inaccessible to the asymptotic observer, and the radiation appears mixed. To restore unitarity, these interior modes must be transferred to the radiation in some way, contradicting the local factorisation of the Hilbert space assumed in semiclassical reasoning.

2. **The absence of back-reaction on the algebra of observables:** In a fixed background, the algebras of the exterior and interior are independent and commute. Gravitational back-reaction, however, forces us to include the observer's clock as a physical degree of freedom, modifying the algebra of gauge-invariant observables. Semiclassical physics ignores this modification.

The firewall argument of AMPS [30] sharpened this failure: the three assumptions of unitarity, the equivalence principle, and effective field theory are mutually incompatible. To preserve unitarity and a smooth horizon, one must abandon the naive factorisation of the Hilbert space and accept that the interior is not an independent region but is encoded in the radiation in a nontrivial way.

Chapter 3

Bohr's Complementarity and the Early Black Hole Complementarity Program

The black-hole information paradox is not merely a conflict between two theories; it is a clash between two deeply held principles about the nature of physical reality. To understand why the modern island paradigm resolves the paradox in a way that is both mathematically precise and conceptually satisfying, we must first return to the foundational insight that ultimately inspired it: Bohr's complementarity. This chapter recalls Bohr's original formulation of complementarity in quantum mechanics and then traces how it was adapted to the black-hole context in the early 1990s by Susskind, Thorlacius, and Uglum. We will see that, while the early program correctly identified the need for an observer-dependent description, it lacked the algebraic machinery—specifically, the identification of non-commuting observables—that would make complementarity a rigorous structural feature of the theory. The remainder of this thesis will offer an interpretive framework, drawing on the algebraic crossed-product construction, in which the island paradigm can be understood as a realisation of Bohr's complementarity in quantum gravity.

3.1 Bohr's Complementarity

In the 1920s, Niels Bohr articulated a radical new framework for understanding quantum phenomena. The central tenet was that the classical concepts used to describe an experiment—position, momentum, wave, particle—cannot be applied simultaneously without contradiction. Instead, the description of a quantum system depends irreducibly on the experimental arrangement, and arrangements that are mutually exclusive give rise to complementary descriptions of the same underlying reality [31, 32].

The canonical example is the wave-particle duality of light. An experiment that measures interference (e.g., a Young's double-slit setup) forces us to describe light as a wave; an experiment that measures photon counts (e.g., the photoelectric effect) forces us to describe light as a stream of particles. The two experimental arrangements are incompatible: one cannot perform both simultaneously. Correspondingly, the mathematical representation of these complementary descriptions involves observables that do not commute. The most familiar case is position and momentum, whose commutator

$$[x, p] = i\hbar \tag{3.1}$$

encodes the impossibility of simultaneously assigning sharp values to both. Complementarity is therefore not a philosophical preference but a structural property of the algebra of observables in quantum mechanics.

Bohr emphasised that the "measuring agency"—the experimental apparatus—is an indispensable part of the description. One cannot separate the object from the agency that observes it. This insight had profound consequences for the interpretation of quantum states: a state is not an absolute attribute of a system but is defined only *relative to* a specified experimental context. As we shall see, this relational spirit is the thread that connects Bohr's complementarity directly to the modern treatment of black-hole observers as quantum reference frames.

3.2 The Membrane Paradigm and the Stretched Horizon

The first step towards a gravitational analogue of complementarity was the realisation that a black-hole event horizon behaves, in many respects, like a physical membrane. In the 1970s and 1980s, the membrane paradigm was developed by Thorne, Price, Macdonald, and others [33], which showed that an observer who remains outside the black hole can model the horizon as a two-dimensional surface endowed with electrical conductivity, viscosity, and thermodynamic properties. This was not a statement about the intrinsic nature of the horizon, but a prescription that correctly reproduces all classical observable consequences of the black-hole exterior.

't Hooft pushed this idea further by suggesting that the black-hole horizon might play a crucial role in restoring unitarity. He argued that infalling matter is thermalised at the horizon, which then acts as a source for the Hawking radiation. In this picture, information that falls into the black hole is not lost but stored on the horizon, eventually returning to the exterior via subtle correlations in the radiation.

Building on these ideas, Susskind, Thorlacius, and Uglum [34] introduced the concept of the stretched horizon. This is a hot, Planck-thick membrane located roughly one Planck length above the event horizon, which absorbs, scrambles, and re-emits information. For an asymptotic observer, the stretched horizon is the physical surface that encodes the unitarity of the evaporation process. Crucially, an infalling observer encounters nothing special at the stretched horizon; the equivalence principle holds, and spacetime remains smooth.

3.3 Black-Hole Complementarity

The stretched horizon picture leads directly to the principle of **black-hole complementarity** [34]. According to this principle, the descriptions of the black hole by an exterior observer and by an infalling observer are *complementary* in Bohr's sense. The exterior observer sees a unitary process: information falls onto the stretched horizon, is scrambled, and eventually returns to the asymptotic region through the Hawking radiation. The infalling observer, on the other hand, experiences a perfectly smooth horizon and a classical interior, in accordance with the equivalence principle.

These two descriptions are mutually exclusive because no single observer can simultaneously be outside the black hole collecting radiation and inside crossing the horizon. Just as one cannot perform an experiment that simultaneously reveals the wave and particle nature of an electron, one cannot construct an apparatus that simultaneously measures the exterior unitarity and the interior smoothness. Susskind et al. explicitly acknowledged Bohr's influence, proposing that black-hole complementarity should be understood as the gravitational analogue of Bohr's complementarity.

The early complementarity proposal was an important conceptual breakthrough. It shifted the discussion of the information paradox from an absolute statement ("information is lost") to an observer-dependent one ("information is in the radiation for the exterior observer, and in the interior for the infalling observer"). This relational view anticipated many of the ideas that would later be made precise by the algebraic QFT and quantum reference frame frameworks.

3.4 Towards an Algebraic Formulation

Despite its conceptual appeal, black-hole complementarity as formulated in 1993 remained qualitative. It did not identify a specific algebraic mechanism that would enforce the mutual exclusivity of the two descriptions. The proposal relied on the statement that the two observers cannot share their information, but it did not derive this from the structural properties of the theory. As a result, critics such as Bokulich [35] and van Dongen–de Haro [36] pointed out that the exterior and interior descriptions could, in the eternal black-hole spacetime, be consistently combined into a single global picture. There was, at that stage, no clear algebraic obstruction to doing so.

This lack of a precise mathematical formulation made the early complementarity program vulnerable to the AMPS firewall argument [30], which used monogamy of entanglement to expose an apparent contradiction between unitarity, the equivalence principle, and effective field theory. The firewall crisis highlighted that a purely epistemological complementarity was insufficient; a structural, algebraic realisation was required.

Subsequent developments in gravitational physics have begun to address this gap. In particular, the study of gravitationally dressed observables in [37, 38] revealed that observables dressed to different reference frames do not, in general, commute. More recently, the algebraic approach based on the crossed product of von Neumann algebras [14, 17, 18, 19] has provided a systematic framework in which the observer's clock is explicitly included as a physical degree of freedom. It is becoming clear that the missing algebraic ingredient was not necessarily a specific pair of non-commuting observables, but rather the fact that the entire families of observables accessible to different observers form non-commuting subalgebras.

The present thesis builds on these developments to propose an explicit interpretive link between the island paradigm and Bohr's complementarity. We will review how the gauge-invariant observables accessible to an asymptotic observer and to an infalling observer belong to two distinct, unitarily related type II_∞ algebras. Because the infalling algebra is generated by a modular translation built from the asymptotic clock, these two subalgebras do not mutually commute. It is important to note that constructing a genuinely new, purely gravitational obstruction to global descriptions remains an open problem, as the non-commutation presented here ultimately reduces to ordinary timelike non-commutation. Therefore, rather than serving as a definitive refutation of Bokulich and van Dongen–de Haro, this algebraic framework should be viewed as a strong interpretive proposal. This non-commutation provides a robust mathematical framework for complementarity wherein the island formula emerges as a physical consequence: the island is the piece of the interior that the asymptotic observer must "borrow" from the infalling observer's algebra to restore unitarity. In this sense, the algebraic and relational framework developed in the chapters that follow offers a natural home for the core ideas of the early black-hole complementarity program.

Chapter 4

The Island Paradigm – A Modern Synthesis

4.1 Introduction: Why a Concrete Model?

In Chapter 2 we saw that the black-hole information paradox reduces to a quantitative conflict: unitarity demands that the entropy of the Hawking radiation follow the Page curve, whereas semiclassical gravity predicts an ever-increasing thermal entropy. The island formula 1.1 was proposed in the literature as the resolution, but its physical content remains obscured if it is merely stated as an abstract extremisation principle. To understand *why* the island appears, *how* it restores unitarity, and *what* it implies about the nature of the black-hole interior, we must work in a concrete, analytically tractable model.

This chapter presents the island paradigm in the simplest possible setting: Jackiw -Teitelboim (JT) gravity in two dimensions, coupled to a flat CFT bath. JT gravity is an ideal toy model for several reasons. First, it is exactly solvable: the dilaton field ϕ forces the geometry to be locally AdS_2, so the gravitational dynamics reduces to a boundary Schwarzian mode. Second, the area term in the generalised entropy is simply the value of the dilaton at the quantum extremal surface, making the extremisation problem tractable. Third, when coupled to a large-c conformal field theory, the matter entanglement entropy can be computed using standard CFT techniques, yielding explicit analytic expressions for the radiation entropy. Finally, the model captures all the essential features of the information paradox - a thermal atmosphere, an eternal (or evaporating) horizon, and a non-trivial Page curve - while remaining under full computational control.

Our goal in this chapter is to derive the Page curve for the eternal two-sided JT black hole. Since the black hole is kept in thermal equilibrium with the baths, the Page curve in this setup does not decrease to zero as in a fully evaporating black hole. Instead, the radiation entropy grows at early times and then saturates at a constant value of order twice the Bekenstein–Hawking entropy of the two black holes. This plateau is the equilibrium analogue of the Page curve. The discussion follows the semiclassical island formula and the replica-wormhole derivation of the fine-grained entropy [5], the explicit eternal-JT calculation reviewed in [8], and the quantum extremal surface perspective developed in [4]. We also briefly comment on the broader modern interpretation of islands, including the possibility that islands can extend outside the horizon in more realistic higher-dimensional settings [6]. By the end of this chapter, the reader will have seen the island formula at work in a fully explicit, analytically solvable model, laying the groundwork for the deeper algebraic and relational interpretation developed in Chapters 5 and 6.

4.2 JT Gravity Coupled to a Bath

JT gravity is a two-dimensional theory of dilaton gravity. The metric is locally fixed to be AdS_2, while the nontrivial gravitational degree of freedom is the dilaton ϕ. In Lorentzian signature, a convenient form of the JT action is

$$I_{\rm JT} = -\frac{1}{16\pi G_N}\left[\int_{\mathcal{M}} d^2x\sqrt{-g}\,\phi\,(R+2) + 2\int_{\partial\mathcal{M}} du\,\phi_b K\right] + I_{\rm matter}. \tag{4.1}$$

The equation of motion obtained by varying the dilaton imposes

$$R = -2, \tag{4.2}$$

so the geometry is locally AdS_2. The dilaton plays the role of transverse area in higher-dimensional black holes. Consequently, the Bekenstein–Hawking entropy of a JT black hole is proportional to the value of the dilaton at the horizon.

We take the eternal two-sided black hole and attach a nongravitating half-line bath to each asymptotic boundary. The CFT propagates through the AdS–bath interface with transparent boundary conditions. The gravity region is therefore dynamical, while the baths are ordinary nongravitating quantum systems in which the radiation can be collected.

A useful form of the AdS_2 metric is

$$ds^2 = -\frac{4\,dx^+dx^-}{(x^+-x^-)^2}. \tag{4.3}$$

The right exterior region is covered by Schwarzschild-like coordinates

$$y^\pm = t \pm r_*, \tag{4.4}$$

where t is the boundary time. These are related to the Poincare coordinates by

$$x^\pm = \frac{\beta}{\pi}\tanh\left(\frac{\pi y^\pm}{\beta}\right), \tag{4.5}$$

where β is the inverse temperature of the black hole. It is also convenient to introduce Kruskal-like coordinates $w^{\pm}$. In the right bath,

$$w^{\pm} = \pm e^{\pm 2\pi y^{\pm}/\beta}. \tag{4.6}$$

In these coordinates, the eternal black-hole dilaton takes the form

$$\phi(w^+, w^-) = \phi_0 + \frac{2\pi\phi_r}{\beta}\frac{1 - w^+w^-}{1 + w^+w^-}. \tag{4.7}$$

The horizon is at $w^- = 0$ on the right side. Therefore the thermal Bekenstein–Hawking entropy of one black hole is

$$S_{\mathrm{BH}}^{(\beta)} = \frac{\phi_h}{4G_N} = \frac{1}{4G_N}\left(\phi_0 + \frac{2\pi\phi_r}{\beta}\right), \tag{4.8}$$

where ϕ_h denotes the value of the dilaton at the horizon.

The matter sector is a large-c CFT defined on the full glued spacetime. The initial state is chosen to be the thermofield double state at the same temperature as the black hole,

$$|\Psi_{\mathrm{TFD}}\rangle = \frac{1}{\sqrt{Z(\beta)}}\sum_n e^{-\beta E_n/2}\,|E_n\rangle_L \otimes |E_n\rangle_R\,. \tag{4.9}$$

Because the baths have the same temperature as the black hole and the boundary conditions are transparent, there is no net energy loss. The black holes are in thermal equilibrium. However, they are not in *entanglement equilibrium*: Hawking modes continually enter the baths, and the matter entropy of the bath regions grows with boundary time.

The complete spacetime consists of the black-hole interior and exteriors (described by JT gravity) and the two flat baths attached at the left and right boundaries. A Penrose diagram of the eternal black hole with the baths is shown in Fig. 4.1. The matter CFT lives on the entire diagram.

With the full geometry established, we now turn to the quantum extremal surface prescription and the computation of the entropy of the Hawking radiation.

4.3 The Quantum Extremal Surface Prescription

The semiclassical entropy of Hawking radiation cannot be computed by a naive partial trace over the exterior region alone. As the black hole evaporates, correlations build up between the radiation and the interior, and a consistent accounting of the von Neumann entropy must include the possibility that some interior degrees of freedom purify the radiation. The quantum extremal surface (QES) prescription and the associated island formula, provides the precise rule for including these contributions. In this section we formulate the prescription in general terms and then specialise it to the JT gravity model that we will solve in the following sections.

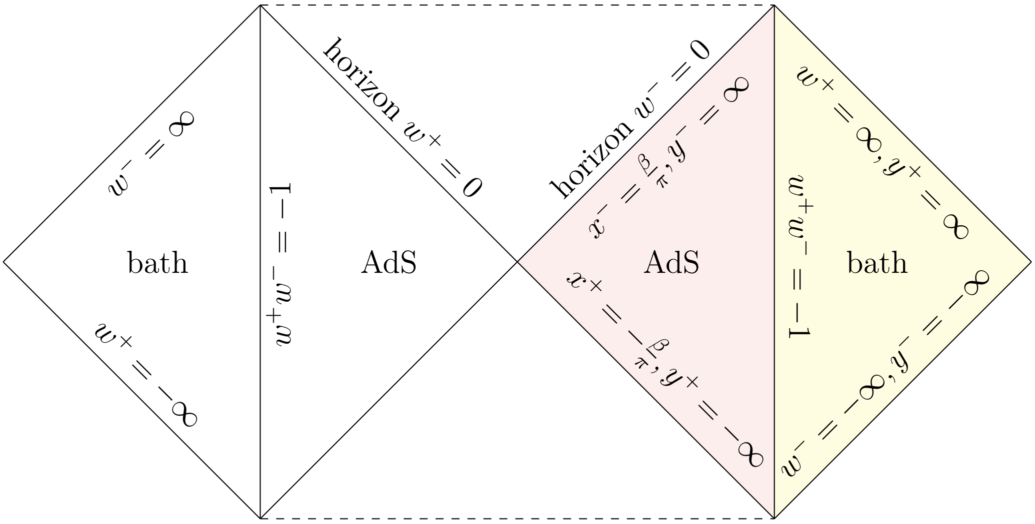

Figure 4.1: The coordinates of the eternal black hole pair along with their half-Minkowski space bath regions. The pink region is part of the AdS geometry outside the right black hole. The yellow region is the right bath region. The right Schwarzschild coordinates $y^\pm$ cover the pink and yellow regions. The global coordinates $w^\pm$ cover all regions on both the left and the right.

4.3.1 The Ryu-Takayanagi and HRT Prescriptions

In semiclassical theory, the Bekenstein–Hawking entropy of the black hole is by the **generalised entropy** [42, 43]:

$$S_{\text{gen}} = \frac{\text{A}}{4G_N} + S_{\text{matter}}. \tag{4.10}$$

Where A is the area of the event horizon and S_{matter} is the entanglement entropy of the quantum fields in the black hole exterior.

In the holographic context, Ryu and Takayanagi [44] showed that, in the classical limit ($N \to \infty$, $\lambda \to \infty$), the entanglement entropy of a subregion A in a static holographic CFT is given by the Ryu-Takayanagi (RT) formula:

$$S_{RT}(A) = \min_{\gamma_A} \frac{\text{Area}(\gamma_A)}{4G_N} \tag{4.11}$$

where γ_A is a codimension-2 static surface in the bulk homologous to A (i.e., $\partial\gamma_A = \partial A$). The homology constraint is crucial; it ensures that the bulk region bounded by $A \cup \gamma_A$ is a valid geometric entity.

For time-dependent spacetimes, Hubeny, Rangamani, and Takayanagi (HRT) generalized this to the extremal surface prescription. The surface $\mathcal{X}_A$ must be a saddle point of the area functional. If multiple extremal surfaces exist, the one with the minimal area is the correct holographic dual. The entanglement wedge $\mathcal{E}(A)$ is then defined as the domain of dependence of the homology hypersurface Σ_A, where $\partial\Sigma_A = A \cup \mathcal{X}_A$.

4.3.2 The Quantum Extremal Surface (QES)

The classical HRT formula is insufficient to describe phenomena where bulk quantum effects are significant, such as black hole evaporation. The semi-classical correction, first introduced by Faulkner, Lewkowycz, and Maldacena (FLM) [45] and later refined by Engelhardt and Wall [46], replaces the area functional with the generalized entropy functional:

$$S_{\text{gen}}[\gamma; A] = \frac{\text{Area}(\gamma)}{4G_N} + S_{\text{bulk}}(\Sigma_\gamma) \tag{4.12}$$

Here, $S_{\text{bulk}}(\Sigma_\gamma)$ is the von Neumann entropy of the bulk quantum fields on the partial Cauchy slice Σ_γ bounded by A and γ.

A Quantum Extremal Surface (QES) is a surface γ that extremizes S_{gen}. The entanglement entropy of the boundary region A is given by the QES with the minimal generalized entropy:

$$S(A) = \min_{\gamma \in \text{QES}} S_{\text{gen}}[\gamma; A] \tag{4.13}$$

This prescription is transformative. The inclusion of the bulk entropy term S_{bulk} allows the location of the extremal surface to shift dynamically based on the entanglement structure of the bulk fields. This shift is the mechanism behind the Page curve transition: as a black hole evaporates, the bulk entropy of the interior modes grows, eventually causing the minimal QES to jump from a trivial surface to a surface near the horizon, abruptly changing the entanglement wedge.

4.3.3 The Island Formula

Let R be the radiation region in the two baths. The island formula states that the fine-grained entropy of R is

$$\boxed{S(R) = \min_I \left[\operatorname*{Ext}_{\partial I} \left\{ \frac{\phi(\partial I)}{4G_N} + S_{\text{matter}}(R \cup I) \right\} \right].} \tag{4.14}$$

Here I is a possible island region in the gravitational spacetime, ∂I is its boundary, and $S_{\text{matter}}(R \cup I)$ is the ordinary von Neumann entropy of the quantum fields on the fixed semiclassical background. In JT gravity, the "area" term is replaced by the dilaton evaluated at the quantum extremal surface (QES).

For a two-sided black hole, the island has two endpoints, one on the right side and one on the left side. If the right QES is denoted by p_1 with coordinates $w_1^{\pm}$, the left QES is its reflected partner p_3. The radiation region has two endpoints p_2 and p_4, one on each bath boundary (fig. 4.3). By symmetry, one may work on the right side and then double the result.

The island formula contains two competing saddles:

1. The **no-island saddle**, $I = \emptyset$, which reproduces Hawking's semiclassical result.
2. The **island saddle**, $I \neq \emptyset$, in which the entropy is computed from $R \cup I$ and includes the dilaton contribution at ∂I.

The physical entropy is the smaller of these two answers.

4.4 Entropy without an Island

We first compute the entropy of the radiation region without including an island. The relevant radiation region consists of the two bath intervals outside the left and right black holes at boundary time t. Since the full CFT state is pure, this entropy can equivalently be computed as the entropy of the complementary interval D crossing the AdS region as a subset of a complete Cauchy slice across the spacetime as shown in figure 4.2 [8].

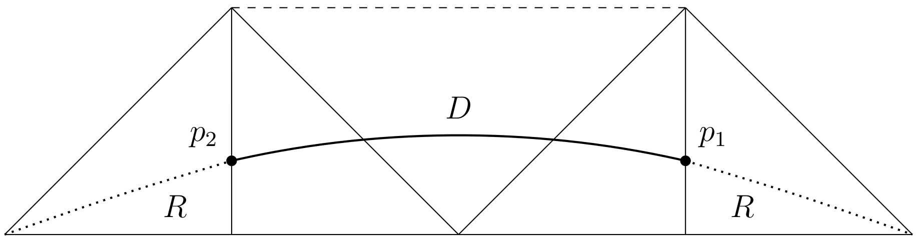

Figure 4.2: The calculation of the entropy of the baths region to the AdS region involves calculating a CFT entanglement entropy for an interval D across the AdS region between boundary points at a given time t.

For a two-dimensional CFT in the vacuum coordinates $w^\pm$, the entropy of an interval with endpoints p_1 and p_2 is

$$S_{\text{CFT}} = \frac{c}{6} \log \left[\frac{-(w_1^+ - w_2^+)(w_1^- - w_2^-)}{\Omega_1 \Omega_2} \right], \tag{4.15}$$

where Ω_i are the conformal factors relating the physical metric to the flat metric in the $w^\pm$ coordinates. In the bath,

$$\Omega^{-2} = \frac{\beta^2}{(2\pi)^2 w^+ w^-}. \tag{4.16}$$

Taking the two endpoints to lie at boundary time t, the radiation entropy in the no-island saddle is

$$S_{\varnothing}(R(t)) = \frac{c}{3}\log\left[\frac{\beta}{\pi\epsilon}\cosh\left(\frac{2\pi t}{\beta}\right)\right], \tag{4.17}$$

where ϵ is a UV cutoff. The cutoff-dependent contribution is a time-independent constant. It does not affect the late-time growth rate of the entropy, and it only shifts the Page time by a cutoff-dependent constant amount. For completeness, we keep this constant explicitly. At late times $t \gg \beta$, using

$$\cosh\left(\frac{2\pi t}{\beta}\right) \simeq \frac{1}{2}e^{2\pi t/\beta},$$

we obtain

$$S_{\varnothing}(R(t)) = \frac{2\pi c}{3\beta}t + \frac{c}{3}\log\left(\frac{\beta}{2\pi\epsilon}\right) + \cdots. \tag{4.18}$$

Thus the physically important part of the no-island entropy is its linear growth,

$$S_{\varnothing}(R(t)) = \frac{2\pi c}{3\beta}t + \text{constant} + \cdots. \tag{4.19}$$

Thus the no-island entropy grows linearly forever. This is precisely the Hawking result: the radiation appears to become more and more entangled with degrees of freedom behind the horizon. However, this cannot be the complete fine-grained entropy, because the two black holes have only a finite entropy of order $2S_{\rm BH}^{(\beta)}$.

4.5 Entropy with an Island

We now include an island bounded by two QES points. By symmetry, the right QES is at $p_1 = (w_1^+, w_1^-)$ and the left QES is its reflected partner at $p_3 = (w_3^-, w_1^+)$ as shown in figure 4.3. The right bath endpoint p_2 is at

$$w_2^+ = e^{2\pi t/\beta}, \qquad w_2^- = -e^{-2\pi t/\beta}. \tag{4.20}$$

and the left bath endpoint p_4 is at

$$w_4^{\mp} = \pm e^{\pm 2\pi t/\beta} \tag{4.21}$$

At late times, the two-sided four-point CFT entropy factorizes into equal left and right contributions. The generalized entropy is then

$$S_{\rm gen}(w_1^{\pm}) = 2\left[\frac{\phi(w_1^{\pm})}{4G_N} + \frac{c}{6}\log\left(\frac{-w_{12}^2}{\Omega_1\Omega_2}\right)\right], \tag{4.22}$$

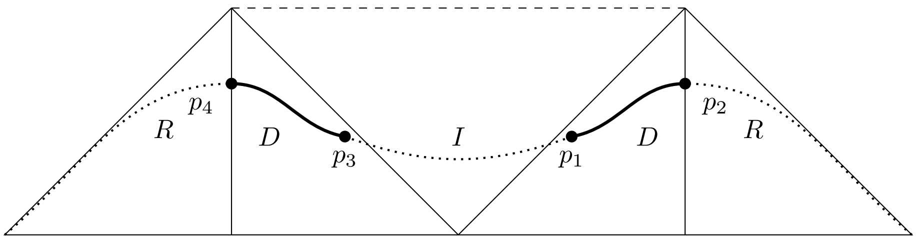

Figure 4.3: The generalized entropy for points p_2 and p_4 at the boundaries with QES's at points p_1 and p_3 in the bulk involves calculating the entropy of disjoint intervals.

where

$$w_{12}^2 = (w_1^+ - w_2^+)(w_1^- - w_2^-). \tag{4.23}$$

The conformal factors are

$$\Omega_1^{-2} = \frac{4}{(1 + w_1^+ w_1^-)^2}, \qquad \Omega_2^{-2} = \frac{\beta^2}{(2\pi)^2}. \tag{4.24}$$

It is useful to define the dimensionless parameter

$$k = \frac{G_N c}{3\phi_r}. \tag{4.25}$$

The semiclassical limit corresponds to $k \ll 1$. In the high-temperature regime $\beta k \ll 1$, the generalized entropy can be written as

$$S_{\text{gen}}(w_1^\pm) = \frac{\phi_0}{2G_N} + \frac{c}{3} F(w_1^\pm) + \frac{c}{3} \log\left(\frac{\beta}{\pi\epsilon}\right), \tag{4.26}$$

where

$$F(w_1^\pm) = \frac{\pi}{\beta k} \frac{1 - w_1^+ w_1^-}{1 + w_1^+ w_1^-} + \log\left[\frac{(e^{2\pi t/\beta} - w_1^+)(w_1^- + e^{-2\pi t/\beta})}{1 + w_1^+ w_1^-}\right]. \tag{4.27}$$

The QES is obtained by extremizing:

$$\frac{\partial S_{\text{gen}}}{\partial w_1^+} = 0, \qquad \frac{\partial S_{\text{gen}}}{\partial w_1^-} = 0. \tag{4.28}$$

In the limit $\beta k \ll 1$, the solution is simple:

$$w_1^\pm = -\frac{\beta k}{2\pi} \frac{1}{w_2^\mp} = \pm \frac{\beta k}{2\pi} e^{\pm 2\pi t/\beta}. \tag{4.29}$$

Thus

$$w_1^+ = \frac{\beta k}{2\pi} e^{2\pi t/\beta}, \qquad w_1^- = -\frac{\beta k}{2\pi} e^{-2\pi t/\beta}. \tag{4.30}$$

Since $w_1^- < 0$ is small, the right QES lies just outside the right future horizon in these coordinates. The factor $\beta k/(2\pi)$ also shows that the QES is displaced relative to the boundary time. Writing

$$w_1^+ = e^{2\pi(t-\Delta t_s)/\beta}, \tag{4.31}$$

one obtains

$$\Delta t_s = \frac{\beta}{2\pi}\log\left(\frac{2\pi}{\beta k}\right). \tag{4.32}$$

This time delay is the scrambling time scale. It is the same logarithmic time scale expected from the Hayden–Preskill interpretation of information recovery from an old black hole.

Substituting the QES solution back into the generalized entropy gives, to leading order,

$$S_I(R(t)) = \frac{\phi_0}{2G_N} + \frac{\pi c}{3\beta k} + \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\right). \tag{4.33}$$

Using $k = G_N c/(3\phi_r)$, the second term becomes

$$\frac{\pi c}{3\beta k} = \frac{\pi\phi_r}{\beta G_N}. \tag{4.34}$$

Therefore,

$$S_I(R(t)) = 2S_{\rm BH}^{(\beta)} + \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\right), \tag{4.35}$$

where the last term is the CFT cutoff-dependent contribution, which may be absorbed into a renormalized definition of the black-hole entropy. Hence, up to such renormalization,

$$S_I(R(t)) \simeq 2S_{\rm BH}^{(\beta)}. \tag{4.36}$$

The important point is that the island entropy is approximately constant in time. It is therefore subdominant at early times but becomes dominant at late times.

4.6 The Page Curve

The fine-grained radiation entropy is obtained by minimizing over the no-island and island saddles:

$$S(R(t)) = \min\left\{S_\varnothing(R(t)), S_I(R(t))\right\}. \tag{4.37}$$

Using the results above,

$$S(R(t)) = \min\left\{\frac{2\pi c}{3\beta}\,t,\; 2S_{\rm BH}^{(\beta)}\right\}, \tag{4.38}$$

At early times,

$$S_{\varnothing}(R(t)) < S_I(R(t)), \tag{4.39}$$

so the no-island saddle dominates and the radiation entropy grows. At late times,

$$S_{\varnothing}(R(t)) > S_I(R(t)), \tag{4.40}$$

so the island saddle dominates and the entropy saturates. The Page time is defined by the equality

$$S_{\varnothing}(R(t_{\mathrm{Page}})) = S_I(R(t_{\mathrm{Page}})). \tag{4.41}$$

Using the late-time approximation, this gives

$$\frac{2\pi c}{3\beta} t_{\mathrm{Page}} \simeq 2S_{\mathrm{BH}}^{(\beta)}. \tag{4.42}$$

Thus

$$t_{\mathrm{Page}} \simeq \frac{3\beta}{\pi c} S_{\mathrm{BH}}^{(\beta)}. \tag{4.43}$$

The resulting entropy curve is

$$S(R(t)) \simeq \begin{cases} \dfrac{2\pi c}{3\beta} t, & t < t_{\mathrm{Page}}, \\[2ex] 2S_{\mathrm{BH}}^{(\beta)}, & t > t_{\mathrm{Page}}. \end{cases} \tag{4.44}$$

This is the Page curve for the eternal two-sided JT black hole coupled to baths. The entropy rises linearly at early times, as Hawking's calculation predicts, but it does not grow forever. At the Page time, the dominant saddle changes from the no-island saddle to the island saddle. After this transition, the entropy is bounded by the black-hole entropy as shown in the figure 4.4.

Because the black hole is eternal and remains in thermal equilibrium with the bath, the entropy does not decrease to zero. Instead, it reaches a plateau. This plateau is the equilibrium version of the Page curve. In a fully evaporating black hole, the black-hole entropy itself decreases with time, and the island saddle tracks this decrease, producing the familiar rise-and-fall Page curve.

4.7 Replica Wormholes and the Origin of the Island Rule

The island formula may look mysterious from the Lorentzian semiclassical viewpoint because it says that a region I behind the horizon contributes to the entropy of a distant

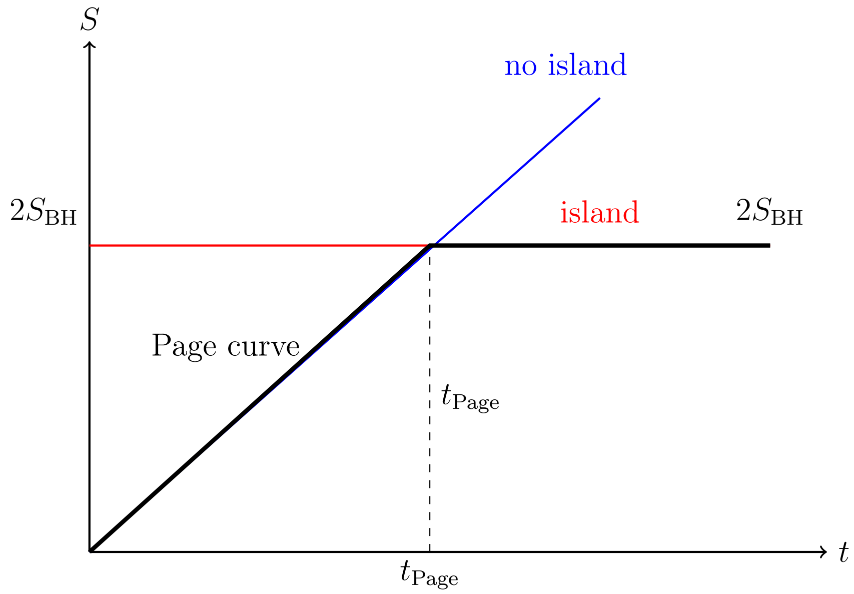

Figure 4.4: The Page curve (thick black) obtained by taking the minimum of the two saddle entropies. The no-island entropy (blue) grows linearly with time. The island entropy (red) is constant, given by twice the Bekenstein–Hawking entropy. The Page time t_{Page} is the moment when the two curves cross.

radiation region R. The gravitational path-integral explanation is provided by replica wormholes [49, 5, 6, 8, 29].

To compute the von Neumann entropy, one begins with the replica trick:

$$S(R) = -\,\partial_n \log \mathrm{Tr}\, \rho_R^n\big|_{n=1}\,. \tag{4.45}$$

In a nongravitational QFT, the n replicas are sewn together only along the radiation region R. In a gravitational theory, however, the geometry itself is dynamical. Therefore the gravitational path integral for $\mathrm{Tr}\, \rho_R^n$ must include all allowed saddle geometries consistent with the replica boundary conditions.

There is an obvious disconnected saddle: n copies of the original black-hole geometry. This saddle gives the Hawking result, namely the no-island entropy. But there are also connected saddles in which the different replicas are joined through the gravitational region. These are the replica wormholes. In the limit $n \to 1$, the fixed points of the replica symmetry become quantum extremal surfaces, and the replica-wormhole saddle reduces precisely to the island contribution in the generalized entropy.

Thus the island rule is not an extra assumption added by hand. It is the effective $n \to 1$ description of new semiclassical saddles in the gravitational replica path integral. The Page transition occurs when the replica-wormhole saddle becomes dominant over the disconnected Hawking saddle.

4.8 Physical Interpretation

The Page transition has a sharp interpretation in terms of entanglement wedges. Before the Page time, the radiation region R has no island. The Hawking radiation is described semiclassically as being entangled with interior partner modes. The entropy of R therefore grows as more Hawking pairs are produced.

After the Page time, the island I is included in the entanglement wedge of the radiation. The entropy is no longer the entropy of R alone, but of $R \cup I$. The island contains the interior partner modes that purify the late Hawking radiation. In this sense, the island degrees of freedom are secretly encoded in the radiation system. The semiclassical geometry still contains an interior, but the fine-grained gravitational entropy assigns part of that interior to the radiation.

This is the essential resolution of the entropy paradox in the island paradigm. Hawking's calculation is correct as a calculation of matter entropy on a fixed background, but it is not the correct fine-grained entropy in a theory with dynamical gravity. The correct entropy includes the possibility of islands and is computed by a generalized entropy saddle.

4.9 Modern Synthesis

The island paradigm has three interconnected lessons.

First, the fine-grained entropy of radiation in gravity is not a purely local QFT quantity. Even if the radiation region lies in a nongravitating bath, the state was prepared by a gravitational path integral, and the entropy must be computed using gravitational rules. This is why an island in the gravitational region can contribute to the entropy of a distant nongravitating radiation system.

Second, the Page curve can be obtained within semiclassical gravity. The calculation does not require explicit microscopic knowledge of the black-hole Hilbert space. The microscopic unitarity is reflected semiclassically through the dominance of replica-wormhole saddles and their $n \to 1$ island limit.

Third, the island transition changes the entanglement wedge. After the Page time, interior operators in the island are reconstructible from the radiation, at least within the appropriate code subspace. This provides a geometric realization of the idea that the late radiation contains information about the black-hole interior.

The eternal JT setup is special because it is analytically tractable and remains in thermal equilibrium. Nevertheless, it captures the essential mechanism. More general situations, including evaporating one-sided black holes, shockwave perturbations, and higher-dimensional black holes, preserve the same basic structure: the competition between a Hawking saddle and an island saddle. In more dynamical settings, the island location can move in time, the Page time can shift, and the QES may lie inside or outside the horizon depending on the state. Recent work [6] has even shown that, in appropriate higher-dimensional situations, islands can extend parametrically outside the horizon, emphasizing that the island paradigm is not merely a Planck-scale correction at the stretched horizon.

In this chapter we derived the Page curve for the eternal two-sided black hole in JT gravity using the island formula. The main steps were:

1. The no-island saddle gives the Hawking entropy
$$S_{\varnothing}(R(t)) = \frac{c}{3}\log\left[\frac{\beta}{\pi\epsilon}\cosh\left(\frac{2\pi t}{\beta}\right)\right], \tag{4.46}$$
which grows linearly at late times.

2. The island saddle has two QES endpoints near the horizons. Extremizing the generalized entropy gives
$$w_1^{\pm} = \pm\frac{\beta k}{2\pi}e^{\pm 2\pi t/\beta}, \qquad k = \frac{G_N c}{3\phi_r}. \tag{4.47}$$

3. Substitution back into the generalized entropy gives the constant island answer
$$S_I(R(t)) \simeq 2S_{\rm BH}^{(\beta)}. \tag{4.48}$$

4. The physical entropy is the minimum of the two saddles:
$$S(R(t)) = \min\left\{S_{\varnothing}(R(t)), S_I(R(t))\right\}. \tag{4.49}$$

Thus the radiation entropy initially follows Hawking's calculation but saturates at the Page time:
$$t_{\rm Page} \simeq \frac{3\beta}{\pi c}S_{\rm BH}^{(\beta)}. \tag{4.50}$$
The saturation is caused by the dominance of the island saddle, whose path-integral origin is the replica wormhole. The island paradigm therefore provides a semiclassical derivation of the Page curve and a modern geometric account of how black-hole radiation can encode information about the interior.

In the following chapters we will embed this result into a deeper algebraic and relational framework, demonstrating that the island paradigm can be understood as a realisation of Bohr's complementarity in quantum gravity..

Chapter 5

Non-Commutativity of Interior and Exterior Observables

5.1 Introduction: The Need for an Algebraic Formulation

In Chapter 4 we saw that the island formula successfully reproduces the Page-like curve in the eternal two-sided JT black hole and restores unitarity to black-hole evaporation, at least at the semiclassical level. The inclusion of an island behind the horizon in the entanglement wedge of the radiation is a profound realisation: the black-hole interior is not merely a causally disconnected region; its degrees of freedom are essential for the purification of the Hawking radiation. Yet the computational success of the island formula leaves several conceptual questions unanswered. Why does the interior participate in the entropy of the radiation? What is the precise mathematical mechanism that allows the interior to be "reconstructed" from the outside? And how does this resolution relate to the long-standing idea of black-hole complementarity?

These questions point to a fundamental tension in the semiclassical description of black holes. In quantum field theory on a fixed curved background, the algebras of observables in the exterior and interior are independent and commute with each other by microcausality. The information paradox arises precisely because this factorisation forces the Hawking radiation to be exactly thermal, with no correlations between the interior and exterior modes. To restore unitarity, this factorisation must break down. Gravitational back-reaction, which is ignored in the semiclassical approximation, must modify the algebraic structure of the theory in a way that entangles the interior with the exterior.

The modern algebraic approach to gravity provides a promising mathematical framework for such a modification. When the observer's clock—the ADM Hamiltonian or, more generally, the generator of time translations—is treated as a physical operator rather than a classical parameter, the algebra of observables is deformed. The local algebras of quantum fields, which in a fixed background are type III_1 factors with no trace and no density matrices, are converted into type II_∞ **crossed products** [17, 18, 19]. This transition endows the algebra with a well-defined trace and a finite von Neumann entropy, and it forces physical observables to be gravitationally dressed to the observer's clock. The half-sided modular translation of Leutheusser and Liu [14] further allows one to move operators from the exterior into the interior, revealing that the interior is not an independent region but a relational construct generated by the action of the clock.

In this chapter we will review the crossed-product construction and show how the gauge-invariant observables accessible to an asymptotic observer (Alice) and those accessible to an infalling observer (Bob) belong to two distinct, non-commuting type II_∞ algebras. The non-commutation of interior and exterior dressed operators—which we compute in §5.5—suggests a structural parallel with the non-commutation of complementary observables in ordinary quantum mechanics. We interpret this algebraic structure as a realisation of black-hole complementarity. This provides a mathematically robust framework for complementarity, extending the early proposals reviewed in Chapter 3. We emphasise that this is an interpretive proposal rather than a definitive proof; constructing a genuinely new algebraic obstruction to global descriptions remains an open question.

The chapter is organised as follows. §5.2 reviews the algebraic structure of quantum field theory on a fixed background, establishing the type III_1 property of local algebras and the Reeh–Schlieder theorem. §5.3 introduces the crossed product and shows how gravity changes the algebra of the exterior to type II_∞. §5.4 constructs the dressed observables for the asymptotic and infalling observers and shows that their algebras are unitarily related but distinct. §5.5 discusses the non-commutation of interior and exterior dressed operators, presenting a structural argument that suggests an algebraic form of complementarity. A fully rigorous derivation of the commutator is left as an open question. Finally, §5.6 synthesises these results to propose an algebraic interpretation of complementarity, linking it back to Bohr and to the island paradigm.

5.2 Local Algebras in Quantum Field Theory

The semiclassical description of a black hole relies on quantum field theory on a fixed curved background. To understand why this description fails to give a unitary Page curve, and why the resolution—the island formula—requires a radical reformulation of the observables, we must revisit the algebraic structure of relativistic quantum field theory. In this section we recall the essential results that force local algebras of observables to be of type III, and we show that the exterior of an eternal black hole is precisely such a type III_1 algebra. Detailed definitions of von Neumann algebra types and the Tomita–Takesaki modular theory are collected in Appendix B; here we present only the physical aspects needed for the black-hole exterior.

We follow the clear exposition of Witten [11] and the classic monograph of Haag [9], emphasising only those aspects that will be necessary for the crossed-product construction in §5.3.

5.2.1 The Net of Observables and the Reeh–Schlieder Theorem

In algebraic quantum field theory, to every causally complete open region $\mathcal{O}$ of spacetime one associates a von Neumann algebra $\mathcal{A}(\mathcal{O})$ of bounded operators. The self-adjoint elements of $\mathcal{A}(\mathcal{O})$ represent physical observables that can be measured within $\mathcal{O}$. The collection of these algebras forms a **net** satisfying the following axioms:

- **Isotony:** $\mathcal{O}_1 \subset \mathcal{O}_2 \;\Rightarrow\; \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$.
- **Locality (microcausality):** If $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated, then $[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0$.
- **Poincaré covariance:** A unitary representation of the Poincaré group acts geometrically on the net.

The global Hilbert space $\mathcal{H}$ carries a vacuum state $|\Omega\rangle$. The **Reeh–Schlieder theorem** states that for any open region $\mathcal{O}$, the vacuum is both cyclic and separating for $\mathcal{A}(\mathcal{O})$. Cyclicity means that the set $\{\, A|\Omega\rangle \mid A \in \mathcal{A}(\mathcal{O}) \,\}$ is dense in $\mathcal{H}$; the separating property means that if $A|\Omega\rangle = 0$ for some $A \in \mathcal{A}(\mathcal{O})$, then $A = 0$. Physically, this reflects the fact that the vacuum is highly entangled across all scales: one cannot isolate a local subsystem with a finite number of degrees of freedom. In particular, a local algebra cannot contain a finite-dimensional factor of type I and possesses no pure states.

5.2.2 Modular Theory and the Rindler Wedge

The Tomita–Takesaki theory (see Appendix B.3) associates to any von Neumann algebra $\mathcal{M}$ with a cyclic and separating vector Ω a **modular operator** Δ and a **modular conjugation** J. The modular flow $\sigma_t(A) = \Delta^{it} A \Delta^{-it}$ is an automorphism of $\mathcal{M}$, and J maps $\mathcal{M}$ to its commutant $\mathcal{M}'$.

For the right Rindler wedge in Minkowski space, $x > |t|$, the Bisognano–Wichmann theorem [12] provides the explicit identification

$$\Delta = e^{-2\pi K}, \tag{5.1}$$

where K is the generator of boosts in the x–t plane. The modular flow σ_t therefore acts geometrically, boosting the wedge by rapidity $2\pi t$. This is the prototypical example of modular theory in quantum field theory and will serve as our bridge to black-hole physics.

5.2.3 Type III$_1$ Algebras and the Absence of a Trace

A von Neumann algebra is classified by its type (see Appendix B.2). Why local algebras are type III$_1$? Witten [11] gives an elegant argument that relies on the Reeh–Schlieder theorem and the nature of the modular flow. For the algebra of a wedge, the Bisognano–Wichmann theorem gives $\Delta = e^{-2\pi K}$, and the boost generator K has a continuous spectrum ranging over all of $\mathbb{R}$. Hence the spectrum of Δ is the whole $\mathbb{R}^+$, with no eigenvalue equal to 1 (except the vacuum). The absence of an eigenvalue 1 is directly tied to the fact that the vacuum is separating; if K had a zero eigenvector in the algebra, there would be a local charge that annihilates the vacuum, contradicting Reeh–Schlieder. Consequently, there are no minimal projectors and no trace. The algebra is type III$_1$.

Local algebras in relativistic quantum field theory are type III$_1$ factors. Physically, this means:

- There are no density matrices in the conventional sense. No normal state can be represented by a density operator $\rho \in \mathcal{M}$ because that would require a trace.
- The von Neumann entropy is not defined without additional structure. The semi-classical Hawking calculation attempts to assign an entropy using a naive partial trace that does not exist in the strict type III framework. Any finite result requires an implicit regulator—which, as we will see, corresponds to the missing gravitational back-reaction.
- The absence of a trace is a direct consequence of the Reeh–Schlieder entanglement: the vacuum is so entangled that the algebra "does not fit" into a finite-dimensional or trace-admitting structure.

5.2.4 The Black-Hole Exterior as a Type III$_1$ Algebra

The right exterior of an eternal black hole (fig.5.1) is a wedge region analogous to the Rindler wedge. Let $\mathcal{A}_R$ be the von Neumann algebra generated by smeared fields in this region. The Hartle–Hawking state $|\Psi_{\mathrm{HH}}\rangle$ is cyclic and separating for $\mathcal{A}_R$ [14]. By symmetry, the modular Hamiltonian H_{mod} of $\mathcal{A}_R$ with respect to $|\Psi_{\mathrm{HH}}\rangle$ is the generator of the Killing boost that fixes the bifurcation surface. In asymptotically AdS spacetimes, this is precisely the ADM Hamiltonian H (up to a constant shift), so

$$\Delta = e^{-2\pi H}. \tag{5.2}$$

Consequently, $\mathcal{A}_R$ is a type III$_1$ factor, exactly like the Rindler wedge in flat space. It possesses no trace, no density matrices, and no well-defined von Neumann entropy. This is the algebraic root of the information paradox: the semiclassical Hawking calculation assumes a tensor-product structure and a trace that do not exist in the strict type III framework.

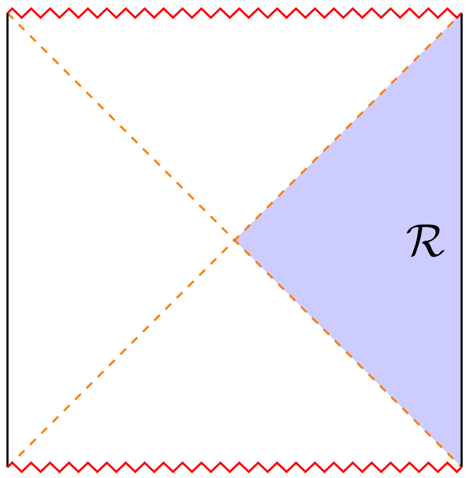

Figure 5.1: The right exterior wedge $\mathcal{R}$ of the eternal two sided black hole (shaded).

To resolve the paradox, gravity must modify the algebra. As we will see in the next section, the inclusion of the observer's clock—the ADM Hamiltonian as a physical operator—converts $\mathcal{A}_R$ into a type II_∞ crossed product $\mathcal{A}_R \rtimes_\sigma \mathbb{R}$, which does admit a trace and a well-defined entropy.

5.3 The Crossed Product and Gravitational Back-reaction

In the previous section we established that the right exterior algebra $\mathcal{A}_R$ of an eternal black hole is a type III_1 von Neumann factor. Such an algebra possesses no trace, no density matrices, and no well-defined von Neumann entropy. This algebraic fact is the mathematical origin of the information paradox: the semiclassical Hawking calculation implicitly assumes a factorisation of the Hilbert space and a trace that do not exist in the strict type III framework. The island formula of chapter 4, on the other hand, produced a finite entropy that followed the Page curve. How can a type III algebra give a finite entropy? The resolution lies in the back-reaction of gravity, which forces us to enlarge the algebra by including the observer's clock. Algebraically, this enlargement is the **crossed product** of $\mathcal{A}_R$ by its modular automorphism group. The resulting algebra is type II_∞ and possesses a natural semi-finite trace, from which the Bekenstein–Hawking entropy emerges directly.

5.3.1 The Crossed Product: General Construction

Motivation: Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and let $\Omega \in \mathcal{H}$ be a cyclic and separating vector for $\mathcal{M}$. The Tomita–Takesaki theory then provides a modular operator Δ and a modular automorphism group

$$\sigma_t(A) = \Delta^{it} A \Delta^{-it}, \qquad A \in \mathcal{M}. \tag{5.3}$$

In a quantum field theory on a fixed spacetime, Δ is not an element of the local algebra $\mathcal{M}$; it is an external object derived from the global state. Consequently, the time evolution it generates is not a gauge symmetry but a physical boost, and the algebra $\mathcal{M}$ is type III_1, with no intrinsic notion of a Hamiltonian.

In a gravitational theory, however, the symmetry generated by Δ is *gauge*. The time translation that shifts the boundary of a black-hole spacetime is a diffeomorphism, and the corresponding charge—the ADM Hamiltonian H—becomes a physical observable that must be included in the algebra of any asymptotic observer. Conversely, an observer with a clock can measure H, and the operators she uses to describe the black-hole exterior must commute with the generator of her time translations. The minimal algebra that contains both the original local fields and the charge H is the **crossed product** of $\mathcal{M}$ by the modular group.

Definition: The crossed product $\mathcal{M} \rtimes_\sigma \mathbb{R}$ is the von Neumann algebra generated by

1. all operators $A \in \mathcal{M}$,
2. a one-parameter family of unitaries $u(s) = e^{isX}$ $(s \in \mathbb{R})$, representing the group of modular automorphisms, i.e.

$$u(s)\, A\, u(-s) = \sigma_s(A), \tag{5.4}$$

The canonical way to realise this algebra is on the enlarged Hilbert space

$$\widetilde{\mathcal{H}} = \mathcal{H} \otimes L^2(\mathbb{R}, dx), \tag{5.5}$$

where X acts as multiplication by x on the second factor, and a local operator $A \in \mathcal{M}$ is represented as

$$\pi(A) = \int_{-\infty}^{\infty} \sigma_{-x}(A) \otimes |x\rangle\langle x|\, dx. \tag{5.6}$$

One verifies that this representation is faithful and that the commutant of $\pi(\mathcal{M})$ is $\pi(\mathcal{M}') \otimes 1$, so the crossed product inherits all the physical information of the original algebra while also containing the 'clock' X.

Trace and type: The crossed product $\mathcal{M} \rtimes_\sigma \mathbb{R}$ is type II_∞. Its semi-finite faithful normal trace is given by

$$\tau(\widehat{A}) = \langle \Omega \otimes \psi_0 |\, \widehat{A}\, | \Omega \otimes \psi_0 \rangle, \tag{5.7}$$

where $\psi_0 \in L^2(\mathbb{R})$ is the vector such that $\langle\psi_0|e^{itX}|\psi_0\rangle = e^{-t}$ (for instance the exponential distribution $|\psi_0(x)|^2 = e^{2\pi x}$ for $x \leq 0$, appropriately normalized). In the fibered representation, this trace takes the form

$$\tau(\widehat{A}) = \langle\Omega| \int_{-\infty}^{\infty} e^{-2\pi x}\, \widehat{A}(x)\, dx |\Omega\rangle, \tag{5.8}$$

where $\widehat{A}(x)$ is the operator acting on $\mathcal{H}$ obtained by restricting $\widehat{A}$ to the fiber at $X = x$. The factor $e^{-2\pi x}$ weighs heavily negative x, i.e. low energies, so that τ is semi-finite but not finite—exactly what one expects for a system with a continuous energy spectrum.

A caveat is in order regarding the use of this trace for entropy computations. The trace τ is semifinite (type II_∞), which means it fixes the entropy only up to an additive constant. To obtain a finite entropy that matches the Page curve, a renormalisation step is required. A full treatment of the II_∞ versus II_1 distinction, and its implications for the interpretation of gravitational entropy, is an important direction for future work.

Importantly, the trace allows us to compute von Neumann entropies once the renormalisation is specified. For a state $\Phi \in \widetilde{\mathcal{H}}$ that is cyclic and separating for the crossed product, the entropy of a subalgebra can be defined using the relative modular operator; after renormalisation, the result is finite and matches the generalised entropy formula of Chapter 4. This is the algebraic reason why the island prescription works.

5.3.2 The Black Hole Exterior with Gravity

Identification of the modular Hamiltonian: For the right exterior of an eternal black hole, the Hartle–Hawking state $|\Psi_{\mathrm{HH}}\rangle$ is cyclic and separating for $\mathcal{A}_R$. The modular Hamiltonian derived from this state is, by the Bisognano–Wichmann theorem in curved spacetime, the generator of the horizon Killing boost:

$$H_{\mathrm{mod}} = \frac{1}{2\pi}\, \widehat{K}, \tag{5.9}$$

where $\widehat{K}$ is the conserved charge associated with the horizon-generating Killing vector [11][13]. In an asymptotically flat (or AdS) black hole, this charge is precisely the ADM Hamiltonian H—the total energy of the spacetime. Thus we have the identification

$$\Delta = e^{-2\pi H}. \tag{5.10}$$

Gauging the time translation: An asymptotic observer measures time along the boundary. Her clock is the ADM Hamiltonian H, which generates translations of the boundary time t. Diffeomorphism invariance demands that physical observables commute with the total Hamiltonian constraint; in the quantum theory, this means that the algebra of observables accessible to the asymptotic observer must be invariant under the adjoint action of e^{itH}. The local algebra $\mathcal{A}_R$ alone is not invariant because its elements depend on a choice of time slicing. The minimal invariant extension is obtained by adjoining the unitary group generated by H itself, giving the crossed product [17]

$$\mathcal{A}_R^{\mathrm{G}} = \mathcal{A}_R \rtimes_\sigma \mathbb{R}, \qquad \sigma_t(\cdot) = e^{iHt} \cdot e^{-iHt}. \tag{5.11}$$

Thus $\mathcal{A}_R^{\mathrm{G}}$ is the gauge-invariant algebra of the black-hole exterior including the asymptotic clock.

Trace and Bekenstein–Hawking entropy: The trace on $\mathcal{A}_R^{\mathrm{G}}$ defined by (5.8) now reads

$$\tau(\widehat{A}) = \mathrm{Tr}_{\mathcal{H}_R}\big(e^{-\beta H}\,\widehat{A}\,\big), \qquad \beta = 2\pi, \tag{5.12}$$

where $\mathcal{H}_R$ is the physical Hilbert space of the right exterior (including perturbative gravity). It should be noted that because the boost Hamiltonian of the eternal two-sided black hole is unbounded from below, this trace is strictly well-defined only when working in the single-sided (asymptotic observer) setting where the energy is bounded below. Assuming this restriction, the thermal density matrix $e^{-\beta H}$ is precisely the one that appears in the Gibbons–Hawking path integral. Its entropy

$$S = -\,\mathrm{Tr}\big(\rho \log \rho\big), \qquad \rho = \frac{e^{-\beta H}}{Z}, \tag{5.13}$$

reproduces the Bekenstein–Hawking area law $S = A/4G_N$. Moreover, for a subregion R of the boundary (or of the bath), the trace allows the definition of a relative entropy that coincides with the generalised entropy $S_{\mathrm{gen}} = \frac{A(\partial I)}{4G_N} + S_{\mathrm{matter}}$. In this sense, the island formula is the semi-classical limit of the algebraic entropy defined on the type II_∞ algebra.

5.3.3 The Origin of Non-Commutativity

A crucial feature of the crossed product is the appearance of the element X, the generator of the modular unitaries. From (5.4) we find the commutator of X with an element of the original algebra:

$$[X, A] = i\,\frac{d}{dt}\sigma_t(A)\Big|_{t=0}. \tag{5.14}$$

In the gravitational interpretation, X is the ADM Hamiltonian H, and the right-hand side is the time derivative of the local field with respect to the Killing time. Therefore, the physical clock H does *not* commute with the local fields of the exterior; it generates their evolution. This is, of course, the familiar statement that the Hamiltonian is the generator of time translations.

However, the interior of the black hole is accessed from the exterior via the half-sided modular translation

$$U(s) = \Delta^{-is/2\pi} = e^{isH}, \tag{5.15}$$

which is precisely the unitary that shifts operators into the interior. Because $U(s)$ is built from H, any interior operator $A_{\text{int}} = U(s)A_{\text{ext}}U(s)^\dagger$ will necessarily fail to commute with H, and hence with the exterior operators that are dressed to the boundary clock. Quantitatively, this suggests the relation

$$[A_{\text{ext}}, A_{\text{int}}] \propto [\partial_t \widehat{A}, \widehat{A}], \tag{5.16}$$

indicating that interior and exterior observables are non-commuting. As we will discuss in §5.5, this non-commutation suggests an algebraic, observer-dependent reading of complementarity.

In summary, we have reviewed how the inclusion of gravitational back-reaction—through the necessity of dressing observables to a clock—converts the type III_1 algebra of the black-hole exterior into a type II_∞ crossed product. This algebraic transition is what makes the island formula possible and, at the same time, suggests that interior and exterior observables are non-commuting, as we discuss further in §5.4.

5.4 Clock Dressing and Two Families of Observers

The crossed-product algebra $\mathcal{A}_R^{\text{G}} = \mathcal{A}_R \rtimes_\sigma \mathbb{R}$ constructed in §5.3 provides the gauge-invariant observables accessible to an asymptotic observer. But a black hole can also be probed by an infalling observer who crosses the horizon. In a diffeomorphism-invariant quantum theory, every physical observable must be dressed to a material reference system—a clock. The asymptotic observer dresses her operators to the boundary time, while the infalling observer dresses them to her proper time along a geodesic. In this section we construct the two families of dressed observables, discuss how they generate two distinct type II_∞ algebras, and explore the algebraic structure underlying their non-commutation, which we interpret as central to black-hole complementarity.

5.4.1 The Asymptotic Observer: Dressing to the Boundary Clock

Alice's clock and algebra: Consider an eternal AdS black hole. The asymptotic observer (Alice) lives at the AdS boundary, where she measures the boundary time t. Her physical clock is the ADM Hamiltonian H, which generates translations in t. The gauge-invariant algebra she can access was shown in §5.3.2 to be the crossed product

$$\mathcal{A}_A \equiv \mathcal{A}_R^{\text{G}} = \mathcal{A}_R \rtimes_\sigma \mathbb{R}, \tag{5.17}$$

with modular automorphism $\sigma_t(\cdot) = e^{iHt} \cdot e^{-iHt}$. This is a type II_∞ factor, equipped with the trace $\tau_A(\cdot) = \text{Tr}_{\mathcal{H}_R}(e^{-2\pi H} \cdot)$.

Dressing a local field: Let $\phi(t)$ be a matter field operator in the right exterior, expressed as a function of the boundary time t on the fixed background. On its own,

$\phi(t)$ is an element of the type III_1 algebra $\mathcal{A}_R$ and is not gauge-invariant. To obtain a physical observable, we must **dress** it to Alice's clock. The dressing is implemented by the integral representation of the crossed product:

$$\widehat{\phi}(t) = \int_{-\infty}^{\infty} \phi(t-u) \otimes |u\rangle\langle u| \, du \, , \tag{5.18}$$

where $|u\rangle$ are the eigenstates of the boundary clock operator X (the generator of H), and the tensor product acts on $\mathcal{H}_R \otimes L^2(\mathbb{R}, du)$. This is the formal dressing to the modular clock (following the CLPW construction), meaning invariance under the modular flow generated by H; it should be distinguished from the gravitational/diffeomorphism dressing of Donnelly and Giddings, which implements invariance under spatial diffeomorphisms. Equivalently, one may write

$$\widehat{\phi}(t) = \int_{-\infty}^{\infty} e^{i(t-u)H} \, \phi(0) \, e^{-i(t-u)H} \otimes |u\rangle\langle u| \, du. \tag{5.19}$$

The physical meaning is transparent: when the clock reads u, the field operator is $\phi(t-u)$, i.e. the field evaluated at a time shifted by the clock reading. The integral over u smears the field against the clock states, ensuring that the total operator is invariant under simultaneous translation of the boundary time and the clock variable. All such dressed operators belong to $\mathcal{A}_A$ and reproduce the correct Hartle–Hawking correlation functions.

5.4.2 The Infalling Observer: Dressing to a Geodesic Clock

Bob's worldline: Now consider an infalling observer (Bob) who falls freely into the black hole along a radial geodesic. Bob carries a clock that measures his proper time τ. His worldline penetrates the horizon and eventually reaches the singularity; we will only need the segment that enters the interior. The question is: what is the gauge-invariant algebra of observables accessible to Bob?

Half-sided modular translation: The crucial tool is the half-sided modular translation introduced by Leutheusser and Liu [14][15]. Starting from the asymptotic clock H, define the unitary

$$U(s_0) = e^{is_0 H}, \qquad s_0 > 0. \tag{5.20}$$

For a sufficiently large s_0, conjugation by $U(s_0)$ maps an operator $A \in \mathcal{A}_R$ into the black-hole interior[16]:

$$A_{\text{int}} = U(s_0) \, A \, U(s_0)^\dagger \; \in \; \mathcal{A}_R'. \tag{5.21}$$

Geometrically, $U(s_0)$ boosts the right exterior wedge across the bifurcation surface, translating it along the horizon by a null shift proportional to s_0, thereby placing the operator behind the horizon. In the language of quantum reference frames, $U(s_0)$ is the transformation that relates Alice's frame to Bob's frame [23].

Bob's algebra: Let $\mathcal{A}_B^{\text{bulk}} = U(s_0)\mathcal{A}_R U(s_0)^\dagger$ be the algebra of fields in Bob's causal diamond (including the interior). Bob's gauge-invariant algebra is then the crossed product of this bulk algebra with respect to his own modular flow:

$$\mathcal{A}_B = \mathcal{A}_B^{\text{bulk}} \rtimes_{\sigma_B} \mathbb{R}, \tag{5.22}$$

where σ_B is the modular automorphism for Bob's natural vacuum state (the Minkowski-like vacuum near his geodesic). Because $U(s_0)$ is unitary, $\mathcal{A}_B$ is unitarily equivalent to Alice's algebra $\mathcal{A}_A$:

$$\mathcal{A}_B = U(s_0)\,\mathcal{A}_A\,U(s_0)^\dagger. \tag{5.23}$$

However, the equivalence does *not* preserve the trace: Bob's trace τ_B is related to Alice's trace τ_A by the adjoint action of $U(s_0)$,

$$\tau_B(\cdot) = \tau_A\big(U(s_0)\cdot U(s_0)^\dagger\big) = \tau_A\big(e^{is_0H}\cdot e^{-is_0H}\big), \tag{5.24}$$

and because H does not commute with the operators in $\mathcal{A}_A$, the two traces are different. Consequently, the von Neumann entropies computed by Alice and Bob for the same physical system will differ—a point we shall develop in Chapter 6.

Infalling dressed operators: A gauge-invariant observable for Bob is obtained by dressing a bulk field to his proper time. Using the unitary $U(s_0)$, we can construct it directly from Alice's dressed operators:

$$\widehat{\phi}_{\text{in}}(\tau) = U(s_0)\,\widehat{\phi}(t(\tau))\,U(s_0)^\dagger\ , \tag{5.25}$$

where $t(\tau)$ is the boundary time coordinate along Bob's geodesic. This operator belongs to $\mathcal{A}_B$ and represents the field as measured by Bob at his proper time τ inside the black hole.

5.5 Non-Commutativity of Interior and Exterior Observables

In §5.4 we constructed two families of gauge-invariant dressed observables: the asymptotic family $\widehat{\phi}(t)$, built from matter fields dressed to the boundary clock, and the infalling family $\widehat{\phi}_{\text{in}}(\tau)$, obtained from the asymptotic operators by the half-sided modular translation $U(s_0) = e^{is_0H}$. Both families are elements of type II_∞ algebras, and their descriptions of the black hole are equally valid. The central question of black-hole complementarity is whether these two descriptions are mutually compatible—i.e. whether their observables can be assigned simultaneous sharp values.

In this section we explore this question and argue that the two descriptions suggest an algebraic form of complementarity. The argument proceeds as follows. We first give an intuitive detector argument (§5.5.1), then we present a structural algebraic argument (§5.5.2) that shows why the two observer algebras do not commute. We interpret this non-commutation as a structural realisation of Bohr's complementarity in quantum gravity. We emphasise that this is a heuristic, structural observation; a fully rigorous treatment of the commutator, and its status as an algebraic obstruction, is left as an open question.

5.5.1 Detector-Based Intuition

Place two Unruh–DeWitt detectors near an eternal black hole (Fig. 5.2). Detector D_A is static outside the horizon, following the orbit of the Killing time t, and couples to the exterior dressed field $\widehat{\phi}(t)$. Detector D_B is freely falling, crosses the horizon, and couples to the infalling dressed field $\widehat{\phi}_{\text{in}}(\tau)$ along its worldline. Both detectors are idealised point-like systems that click when the local field strength exceeds a threshold.

Let P_A and P_B be the projection operators onto the clicked states of the two detectors. In a classical world one could ask for the joint probability $P(\text{click}_A \wedge \text{click}_B)$; quantum mechanically, this probability is well-defined only if $[P_A, P_B] = 0$. Because P_A is a function of $\widehat{\phi}(t)$ and P_B a function of $\widehat{\phi}_{\text{in}}(\tau)$, the condition $[P_A, P_B] = 0$ requires that the dressed fields themselves commute.

To see that they cannot, note that detector A measures the field at boundary time t dressed to the asymptotic clock H, while detector B measures the field dressed to the infalling clock, which is related to the asymptotic clock by the half-sided modular translation $U(s_0)$. The dressing operations are incompatible; the asymptotic dressing uses H as its reference, whereas the infalling dressing uses a clock that is itself built from H (through $U(s_0)$) and therefore does not commute with the exterior fields. Consequently, the two dressed fields cannot be simultaneously diagonalised, and the joint click distribution fails to factorise:

$$P(\text{click}_A \wedge \text{click}_B) \neq P(\text{click}_A)\, P(\text{click}_B). \tag{5.26}$$

This failure is the operational signature of non-commuting observables, and it reflects the physical impossibility of an omniscient observer who simultaneously reads both clocks. The next subsection explore this structural point further.

5.5.2 Algebraic Non-Commutativity

The non-commutation follows from a purely structural algebraic argument. Recall from §5.4 that

$$\mathcal{A}_A = \mathcal{A}_R \rtimes_\sigma \mathbb{R}, \qquad \mathcal{A}_B = U(s_0)\, \mathcal{A}_A\, U(s_0)^\dagger, \tag{5.27}$$

with $U(s_0) = e^{is_0 H}$. The modular automorphism σ_t is inner on the crossed product: $\sigma_t(A) = e^{iHt} A e^{-iHt}$. Because H does not commute with the original algebra $\mathcal{A}_R$, the

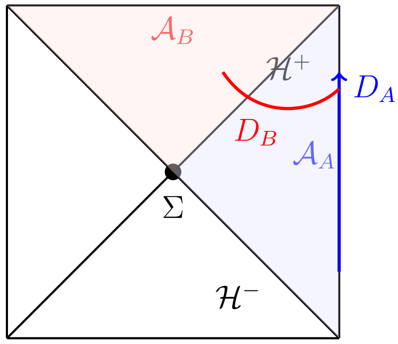

Figure 5.2: Penrose diagram of an eternal AdS black hole. The vertical boundaries are timelike AdS infinity. Detector D_A (blue) remains on the right boundary and couples to the asymptotic dressed field $\widehat{\phi}(t)$. Detector D_B (red) falls from the boundary and crosses the future horizon $\mathcal{H}^+$ into the interior; it couples to the infalling dressed field $\widehat{\phi}_{\text{in}}(\tau)$. The shaded regions indicate the algebras accessible to each observer: $\mathcal{A}_A$ (blue, right exterior) and $\mathcal{A}_B$ (red, black-hole interior). The non-factorisability of their joint click probabilities reflects the non-commutation of these two algebras.

unitary $U(s_0)$ is not an element of $\mathcal{A}'_A$ (the commutant of $\mathcal{A}_A$). Consequently, $\mathcal{A}_B$ is not a subalgebra of $(\mathcal{A}_A)'$; instead, $\mathcal{A}_B$ contains operators that fail to commute with elements of $\mathcal{A}_A$.

To see the failure explicitly, take any $A \in \mathcal{A}_R$ that is not invariant under the modular flow (i.e. $[H, A] \neq 0$). Its dressed version $\widehat{A} \in \mathcal{A}_A$ and the corresponding infalling operator $\widehat{A}_{\text{in}} = U(s_0)\widehat{A}U(s_0)^\dagger \in \mathcal{A}_B$ satisfy

$$\begin{aligned} [\widehat{A}, \widehat{A}_{\text{in}}] &= \widehat{A}U(s_0)\widehat{A}U(s_0)^\dagger - U(s_0)\widehat{A}U(s_0)^\dagger\widehat{A} \\ &= U(s_0)\big[U(s_0)^\dagger\widehat{A}U(s_0), \widehat{A}\big]U(s_0)^\dagger \\ &= U(s_0)\big[\sigma_{-s_0}(\widehat{A}), \widehat{A}\big]U(s_0)^\dagger. \end{aligned} \tag{5.28}$$

Since $\sigma_{-s_0}(\widehat{A})$ differs from $\widehat{A}$ by a time derivative,

$$\sigma_{-s_0}(\widehat{A}) = \widehat{A} - s_0\,\partial_t\widehat{A} + \mathcal{O}(s_0^2), \tag{5.29}$$

the commutator is non-zero and proportional to $[\partial_t\widehat{A}, \widehat{A}]$. Thus, non-commutation is an algebraic consequence of the crossed product and the modular translation: the two observer algebras are related by a modular translation and so do not generally commute. We emphasise that this is a structural statement about non-commuting subalgebras; the commutator $[\partial_t\widehat{A}, \widehat{A}]$ is operator-valued and is not a central analogue of $[x, p] = i\hbar$.

This algebraic argument suggests that the non-commutation is not merely a perturbative artifact but a structural feature of the type II_∞ description of the black hole. Any two observers whose clocks are related by a half-sided modular translation will possess algebras that do not mutually commute. We interpret this as the algebraic signature of complementarity. A fully rigorous treatment—including a precise characterisation of the commutator and its status as an algebraic obstruction—is left for future work.

5.6 Complementarity Formulated Algebraically

We can now summarize the algebraic picture developed in this chapter. The purpose of the preceding discussion has not been to derive a new theorem establishing black-hole complementarity. Rather, it has been to show that the crossed-product framework provides a natural language in which observer-dependent descriptions can be formulated algebraically. Bohr's original insight was that mutually exclusive experimental arrangements correspond to observables represented by non-commuting operators. Position and momentum, or spin components along different axes, are the elementary examples. In the black-hole context:

> **Algebraic complementarity:** *The gauge-invariant observables accessible to an asymptotic observer (Alice) and those accessible to an infalling observer (Bob) are described by two distinct type II_∞ algebras $\mathcal{A}_A$ and $\mathcal{A}_B$, related by a half-sided modular translation. These algebras do not commute; their elements generally fail to commute. Hence, no single observer can assign simultaneous sharp values to both the exterior dressed field and the interior dressed field. The descriptions suggest an algebraic form of complementarity.*

A caveat is in order: the existence of non-commuting algebras is a necessary structural ingredient for complementarity in this context, but it is not by itself sufficient to establish Bohr-style complementarity. A complete realisation would require a detailed analysis of the experimental arrangements and the precise sense in which they are mutually exclusive. Our aim here is more modest: to show that the algebraic framework is consistent with a structural realisation of complementarity.

This formulation suggests a way in which the long-standing tension between unitarity and the equivalence principle might be understood. Unitarity is a property of the algebra $\mathcal{A}_A$: the trace τ_A gives a unitary Page curve for the radiation. The equivalence principle is a property of the algebra $\mathcal{A}_B$: the infalling observer sees a locally Minkowskian vacuum and no drama at the horizon. The two algebras do not commute, so the corresponding descriptions are complementary—both are correct, but they cannot be combined into a single, observer-independent picture.

The firewall paradox of AMPS [30] is cited in this context as the sharpest statement of the tension. However, a full confrontation with the AMPS argument—which would require demonstrating that the monogamy of entanglement constraints are satisfied in this framework—is beyond the scope of this thesis. We leave this important test for future work.

The island formula can now be interpreted in this algebraic framework. The island I can be understood as an element of $\mathcal{A}_A$: it is the piece of the interior that the asymptotic observer must include to compute the entropy of her radiation. For Bob, who possesses the algebra $\mathcal{A}_B$, the same degrees of freedom are not an "island" but simply part of his local causal diamond. The very existence of the island is *observer-dependent*—a relational feature defined with respect to the observer's clock. This relational interpretation of black-hole physics is the subject of the next chapter.

Chapter 6

A Relational Interpretation of Black Hole Physics

6.1 Introduction: From Algebraic Complementarity to a Relational Interpretation

Chapter 5 presented a series of algebraic results that reshape our understanding of the black-hole interior. The exterior algebra of an eternal black hole, initially a type III_1 factor with no trace and no well-defined entropy, becomes a type II_∞ crossed product upon the inclusion of the observer's clock—the ADM Hamiltonian (5.11). Gauge-invariant observables were constructed by dressing bulk fields to the asymptotic boundary clock (Alice) and to the infalling proper time (Bob) (5.18)–(5.25). Because the infalling observables are generated by a half-sided modular translation built from the asymptotic clock, we observed that these two families of dressed observables are non-commuting, suggesting a structural parallel with the non-commutation of complementary observables in quantum mechanics.

These results are not merely technical curiosities. They invite a fundamental rethinking of what we mean by the black-hole interior and the horizon. If the gauge-invariant observables accessible to an asymptotic observer and to an infalling observer fail to commute, then the corresponding descriptions of the black hole are mutually exclusive—they cannot be combined into a single, observer-independent picture. This mirrors the signature of complementarity in Bohr's original sense. The question that arises is: does the island paradigm, which successfully reproduces the Page curve, provide the physical manifestation of this algebraic complementarity?

In this chapter we propose that the answer is yes. We argue that the island formula, the Page curve, and the very notion of the black-hole interior can be naturally understood within a relational interpretation of black-hole physics. In this interpretation, the horizon and the interior are not absolute geometric entities; they are relational features that emerge from the choice of the observer's quantum reference frame. The island, which appears in the asymptotic observer's entropy calculation but is absent for the infalling observer, is the concrete expression of this observer-dependence.

Our approach is deeply rooted in the relational quantum mechanics of Rovelli [20] and the modern formalism of quantum reference frames (QRFs) [23, 21, 22]. In these frameworks, physical facts—including the very notion of a subsystem—are defined only relative to a physical clock. The crossed-product construction of Chapter 5 behaves precisely as the algebra of a QRF for time translations, and the half-sided modular translation $U(s_0) = e^{is_0H}$ acts as a QRF transformation linking Alice's and Bob's perspectives. The observer-dependence of gravitational entropy, recently established by De Vuyst et al. [24], is a natural consequence of this relational structure.

6.2 Relational Quantum Mechanics and Quantum Reference Frames

The algebraic complementarity discussed in Chapter 5 invites a radical revision of the concepts of space, time, and observables in quantum gravity. A natural framework for understanding this revision is the relational interpretation of quantum mechanics, pioneered by Rovelli [20], together with its modern mathematical realisation in the theory of quantum reference frames (QRFs) [23, 21, 22]. We adopt this interpretive framework for the relational reading developed in this chapter.

6.2.1 Rovelli's relational quantum mechanics

Relational quantum mechanics (RQM) is one of several interpretations of quantum mechanics that address the role of the observer. It is not uncontroversial; critics have questioned its treatment of probability, the consistency of its notion of 'relative states', and its implications for the reality of the wavefunction. Nevertheless, we adopt RQM here as a useful interpretive framework because its emphasis on observer-dependent descriptions aligns naturally with the algebraic structures arising from the crossed-product construction.

In relational quantum mechanics (RQM), the state of a quantum system is not an absolute attribute but is defined only *relative to* another physical system that serves as the observer [20]. There is no God's-eye view of the universe; every physical fact—the outcome of a measurement, the value of an observable—is a statement about the relation between two systems. Different observers may assign different states to the same system, and these assignments are related by unitary transformations that encode the relative information each observer possesses.

The canonical example is a spin-1/2 particle and a Stern–Gerlach apparatus. The particle's spin state "up" is not a property of the particle alone but of the particle *relative to the apparatus orientation.* Another apparatus with a different orientation will assign a different state. No single, absolute description encompasses both; instead, the descriptions are complementary, related by a rotation.

For a closed universe—such as the black-hole plus radiation system—there is no external classical observer. All observing systems are inside the universe, and their descriptions are necessarily relational. This insight is the starting point for a relational interpretation of black-hole physics.

We emphasise that this is an interpretive choice. The algebraic structures of Chapter 5 are mathematically well-defined regardless of one's preferred interpretation of quantum mechanics. The relational reading is a way of making sense of these structures, not a necessary consequence of them.

6.2.2 Quantum reference frames

The modern formalism of quantum reference frames (QRFs) provides a mathematical framework for RQM. A QRF is a quantum system that serves as the origin of coordinates for describing other systems. The fundamental Hilbert space $\mathcal{H}_{\text{kin}}$ of the universe contains all degrees of freedom, but it is subject to constraints that express the gauge symmetries (e.g., diffeomorphism invariance, translational invariance). Physical states lie in the perspective-neutral Hilbert space $\mathcal{H}_{\text{phys}}$, defined as the subspace annihilated by the constraints [22].

Choosing a specific QRF corresponds to *gauge-fixing* one or more of these constraints. This breaks the redundancy and yields a reduced Hilbert space $\mathcal{H}_A$ that contains the states of the remaining systems *relative to* the chosen QRF. The observables that are well-defined in this frame are those that commute with the constraints when dressed to the QRF; they form a von Neumann algebra $\mathcal{A}_A$ [21].

Transformations between different QRFs are represented by unitary operators on $\mathcal{H}_{\text{phys}}$. If A and B are two QRFs, the transformation $U_{A\to B}$ is a unitary that intertwines their respective algebras, translating a physical description relative to frame A into a description relative to frame B. This structure is central to our interpretive proposal, as it provides the exact mathematical language needed to identify the half-sided modular translation $U(s_0)$ behaves as the QRF transformation linking the asymptotic and infalling observers.

6.2.3 The Page–Wootters formalism and the crossed product

A particularly important class of QRFs is provided by the Page–Wootters (PW) formalism [47], which treats time itself as a relational observable. One introduces a clock system with Hilbert space $L^2(\mathbb{R})$ and a Hamiltonian H_{clock} that generates its evolution.

The total Hamiltonian constraint is

$$\mathbf{H} = H_{\text{sys}} + H_{\text{clock}} \approx 0, \tag{6.1}$$

where H_{sys} is the Hamiltonian of the system of interest. Physical states are annihilated by $\mathbf{H}$, and the algebra of observables that are gauge-invariant with respect to this constraint—i.e. that commute with $\mathbf{H}$—is precisely the crossed product [24, 25]

$$\mathcal{A}_{\text{phys}} = \mathcal{A}_{\text{sys}} \rtimes_{\alpha} \mathbb{R}, \tag{6.2}$$

where $\alpha_t(\cdot) = e^{iH_{\text{sys}}t} \cdot e^{-iH_{\text{sys}}t}$ is the modular flow generated by the system Hamiltonian.

This construction is precisely the one we encountered in Chapter 5. The gravitational algebra of the black-hole exterior, $\mathcal{A}_A = \mathcal{A}_R \rtimes_\sigma \mathbb{R}$, is the PW algebra for the clock being the ADM Hamiltonian [24]. The equivalence "PW = CLPW" (Chandrasekaran–Longo–Penington–Witten) is the statement that the crossed-product algebra of gravitational observables *is* the algebra of a quantum reference frame for time translations[24, 25, 19]. The observer's clock—the ADM Hamiltonian—is not an external classical parameter but a quantum operator, and the gauge-invariant observables are those dressed to this clock.

6.2.4 QRFs in the black-hole context

In the black-hole setting we have two natural QRFs:

- **Alice (asymptotic observer)**: her clock is the ADM Hamiltonian H, and her algebra is the crossed product $\mathcal{A}_A = \mathcal{A}_R \rtimes_\sigma \mathbb{R}$.

- **Bob (infalling observer)**: his clock is his proper time τ, and his algebra is obtained from Alice's by the half-sided modular translation $U(s_0) = e^{is_0 H}$: $\mathcal{A}_B = U(s_0)\mathcal{A}_A U(s_0)^\dagger$.

The unitary $U(s_0)$ acts as a QRF transformation relating the two frames. Because $U(s_0)$ is constructed from the very same Hamiltonian that generates Alice's modular flow, these two observer-dependent algebras do not mutually commute [14]. This structural non-commutation is the algebraic signature of the fact that the two QRFs are complementary—their descriptions cannot be trivially combined into a single absolute picture.

In the following sections, we will trace this complementarity back to its roots in Bohr's philosophy, proposing that the black-hole interior and the island formula are physical manifestations of the relational structure that naturally emerges within this algebraic framework.

6.3 The Black Hole as a Quantum Reference Frame

The quantum reference frame formalism introduced in §6.2 provides a natural mathematical language for interpreting the two observers of black-hole complementarity. In this section, we identify Alice (the asymptotic observer) and Bob (the infalling observer) as two distinct QRFs, and we re-interpret the algebraic results of Chapter 5 in light of this identification.

6.3.1 Alice and Bob as quantum reference frames

In the black-hole setting, the perspective-neutral Hilbert space $\mathcal{H}_{\text{phys}}$ contains all degrees of freedom—the bulk fields, the metric, and the physical clocks. Choosing a specific clock corresponds to a gauge-fixing of the Hamiltonian constraint, yielding a reduced description relative to that clock.

- **Alice (asymptotic observer)** uses the ADM Hamiltonian H as her clock. The gauge-invariant observables accessible to her are the dressed operators constructed in §5.4. Their algebra is the crossed product

$$\mathcal{A}_A = \mathcal{A}_R \rtimes_\sigma \mathbb{R}, \tag{6.3}$$

  where $\sigma_t(\cdot) = e^{iHt} \cdot e^{-iHt}$ is the modular flow of the exterior algebra $\mathcal{A}_R$. This is precisely the algebra of a quantum reference frame for time translations [24].

- **Bob (infalling observer)** falls across the horizon along a radial geodesic and uses his proper time τ as his clock. His gauge-invariant observables are obtained from Alice's by the half-sided modular translation $U(s_0) = e^{is_0 H}$:

$$\mathcal{A}_B = U(s_0)\, \mathcal{A}_A\, U(s_0)^\dagger, \tag{6.4}$$

  where $s_0 > 0$ is chosen large enough that the translated operators lie behind the horizon. $\mathcal{A}_B$ is a distinct type II_∞ factor, unitarily equivalent to $\mathcal{A}_A$ but with a different trace.

The two algebras represent the perspectives of two different QRFs on the same physical system. The physical reference in Alice's construction is the boundary clock; Bob's reference is his proper time. The relational transformation between these two perspectives is governed by the modular flow of the asymptotic algebra.

6.3.2 The half-sided modular translation as a QRF transformation

In the general QRF formalism, a change of clock is implemented by a unitary operator on the perspective-neutral Hilbert space [24, 22]. For two clocks that are both functions of the same Hamiltonian constraint, this unitary is given by the Connes cocycle between the two clock states.

In the black-hole case, the unitary $U(s_0)$ that relates Alice's and Bob's algebras can be interpreted as such a QRF transformation. To see this, note that Bob's clock τ is the proper time along an infalling geodesic. In the semiclassical limit, the relation between τ and the boundary time t is given by the boost generated by the modular Hamiltonian H; advancing τ corresponds to evolving the boundary time by a shift proportional to s_0 [15]. Consequently, the algebra of Bob's observables is obtained from Alice's by the adjoint action of $U(s_0)$:

$$\mathcal{A}_B = U(s_0)\,\mathcal{A}_A\,U(s_0)^\dagger, \qquad U(s_0) = e^{is_0 H}. \tag{6.5}$$

While constructed from the asymptotic clock Hamiltonian, the physical role of $U(s_0)$ in this context is to act as a relational map that connects the two frames, translating the observer's perspective from the boundary into the interior.

6.3.3 Non-commutation as QRF complementarity

In §5.5 we observed that the dressed observables of Alice and Bob do not commute, suggesting a structural parallel with the non-commutation of complementary observables. From the QRF perspective, this structural non-commutation is the algebraic signature of the fact that the two QRFs are complementary. Two QRFs are compatible if their algebras commute; they are complementary if they do not. Conceptually, this mirrors the relationship between position and momentum QRFs: the algebra of position measurements and the algebra of momentum measurements do not commute, reflecting the impossibility of a simultaneous sharp specification of both. For the black hole, the asymptotic QRF (Alice) and the infalling QRF (Bob) are complementary in a structurally similar sense. No single QRF can contain both the exterior dressed field $\widehat{\phi}(t)$ and the interior dressed field $\widehat{\phi}_{\text{in}}(\tau)$ as commuting observables.

This complementarity can be understood as a structural feature of the QRF formalism and the crossed-product construction, rather than a purely philosophical assertion. The next section will trace this structure back to Bohr's original complementarity and show that the island paradigm is its physical expression.

6.4 Observer-Dependent Entropy

The algebraic complementarity discussed in Chapter 5 and re-interpreted in the language of quantum reference frames in Sections 6.2–6.3 has a direct physical consequence: the von Neumann entropy of the Hawking radiation is *observer-dependent*. In this section we argue that the island formula—and the Page curve it yields—can be understood as the entropy measured by the asymptotic observer Alice, while the infalling observer Bob computes a different entropy that does not contain an island contribution. The difference between the two entropies is precisely the relative entropy of the two clocks, as recently proven by De Vuyst et al. [24]. This develops the relational interpretation of black-hole physics.

6.4.1 Alice's entropy: the island formula

For Alice, the gauge-invariant algebra is the crossed product $\mathcal{A}_A = \mathcal{A}_R \rtimes_\sigma \mathbb{R}$ constructed in §5.3. The semi-finite trace on this type II_∞ algebra is

$$\tau_A(\cdot) = \mathrm{Tr}_{\mathcal{H}_R}\big(e^{-2\pi H}\,\cdot\,\big), \tag{6.6}$$

where H is the ADM Hamiltonian. The von Neumann entropy of the radiation region R is obtained from this trace and, as we reviewed in Chapter 4, is given by the island formula

$$S_A(R) = \min_I \left\{ \frac{\phi(\partial I)}{4G_N} + S_{\text{matter}}(R \cup I) \right\}, \tag{6.7}$$

where I is the entanglement island behind the horizon. This entropy follows the Page curve for the eternal two-sided black hole, rising linearly during the Hawking evaporation and plateauing at the Bekenstein–Hawking entropy $2S_{\text{BH}}$ after the Page time.

6.4.2 Bob's entropy: no island

Bob's algebra is obtained from Alice's by the half-sided modular translation $U(s_0) = e^{is_0 H}$:

$$\mathcal{A}_B = U(s_0)\,\mathcal{A}_A\,U(s_0)^\dagger, \tag{6.8}$$

and its trace is $\tau_B(\cdot) = \tau_A(U(s_0)\,\cdot\,U(s_0)^\dagger)$. Because H does not commute with the local operators in $\mathcal{A}_R$, τ_B differs from τ_A. Following the framework developed in recent literature [24, 25], one expects that Bob's entropy, evaluated with respect to τ_B, does not require an island. Because his causal diamond naturally includes the interior, the von Neumann entropy of the matter fields he can access should simply be the bulk entanglement entropy in his wedge,

$$S_B(R') = S_{\text{matter}}(R'), \tag{6.9}$$

where R' denotes the region accessible to Bob. Consequently, one expects this entropy to remain small and not exhibit the Page transition. A rigorous, independent computation of Bob's entropy in the crossed-product framework is left for future work.

6.4.3 The entropy difference as relative entropy of clocks

The difference between Alice's and Bob's entropies is not arbitrary; it is determined by the algebraic relation between the two traces. The Connes cocycle $(D\tau_A : D\tau_B)_t$ encodes the relative entropy between the two clock states. For any state ρ that is normal on both algebras, De Vuyst et al. [24] [25] show that

$$S_A(\rho) - S_B(\rho) = S_{\text{rel}}\big(\rho^A_{\text{clock}} \,\|\, \rho^B_{\text{clock}}\big), \tag{6.10}$$

where ρ^A_{clock} and ρ^B_{clock} are the reduced density matrices of the two clocks in the state ρ. As established by De Vuyst et al., in the semiclassical limit, the island area term $\frac{\phi(\partial I)}{4G_N}$ corresponds precisely to this relative entropy.

Explicitly, for the eternal black-hole in JT gravity, one finds [24][25]

$$S_A(R) - S_B(R') = \frac{\phi_0 + \bar{\phi}}{4G_N} + \mathcal{O}(G_N^0), \tag{6.11}$$

where ϕ_0 is the constant background dilaton and $\bar{\phi}$ is the renormalised boundary value, which is exactly the Bekenstein–Hawking area term. This foundational result, supports the interpretation of the island as the physical manifestation of the observer-dependence of gravitational entropy.

6.4.4 Interpretation: gravitational entropy is observer-dependent

This relationship between the island area and the relative clock entropy serves as the central pillar of our relational interpretation. It suggests that gravitational entropy is not an absolute property of the black hole, but rather a relational quantity that depends on the observer's quantum reference frame. Alice, whose clock is the ADM Hamiltonian, must include an island to purify the radiation; the island's area term contributes to her entropy. Bob, whose clock is his proper time, already includes the interior in his algebra; he needs no island, and his entropy is purely matter entanglement.

This observer-dependence offers a natural explanation for the long-standing puzzle of why the island appears only in the asymptotic description. In this framework, the island is not a physical membrane, but a relational entropy contribution that emerges from the incompatibility of the two clocks. If the two clocks were to coincide—i.e., in a classical limit where the relative entropy between the clock states vanishes—this geometric discrepancy would disappear, and the two entropies would agree. In the full quantum theory, however, the clock must be treated as a quantum system, rendering the gravitational entropy genuinely observer-dependent.

6.5 The Horizon and Interior as Relational Structures

The algebraic complementarity and observer-dependent entropy established in the preceding sections invite a profound revision of the concepts of horizon and interior in black-hole physics. If the algebra of observables depends on the observer's quantum reference frame, then the geometric objects that are defined by those algebras—the horizon as a boundary, the interior as a region—must also be interpreted relationally. In this section we develop this relational interpretation.

6.5.1 The horizon as a QRF boundary

In a diffeomorphism-invariant quantum theory, the notion of a "region of spacetime" is not fundamental. It is derived from the set of observables that an observer can access with a given clock. For the asymptotic observer Alice, the gauge-invariant algebra is the crossed product $\mathcal{A}_A = \mathcal{A}_R \rtimes_\sigma \mathbb{R}$, which is generated by the local fields in the right exterior wedge $\mathcal{R}$ dressed to the ADM Hamiltonian H (§5.3). The future horizon $\mathcal{H}^+$ is the null boundary of the causal domain of this algebra: it is the surface beyond which the fields can no longer be represented as operators in $\mathcal{A}_A$ without leaving the algebra.

For the infalling observer Bob, the algebra is $\mathcal{A}_B = U(s_0)\mathcal{A}_A U(s_0)^\dagger$, where $U(s_0) = e^{is_0 H}$ is the half-sided modular translation [14]. In Bob's frame, the horizon is not a boundary; it is a smooth patch of locally flat spacetime inside his causal diamond. The same null surface $\mathcal{H}^+$ that is the edge of Alice's observable universe is an ordinary region for Bob.

Thus the horizon can be understood not as an absolute geometric entity but as the boundary of the region that a particular quantum reference frame can probe. In the perspective-neutral language of §6.2, the horizon can be seen as the gauge-dependent shadow of the Hamiltonian constraint, viewed differently by different clocks. This relational definition of the horizon provides a natural framework to address the tension between the equivalence principle (which requires a smooth horizon) and unitarity (which requires an exterior description): the two demands refer to two different QRFs, whose associated algebras are complementary.

6.5.2 The interior from the half-sided modular translation

The interior is not an independent region of spacetime that pre-exists as the quantum state; it emerges from the exterior algebra via the action of the observer's clock. The fundamental mathematical tool here is the half-sided modular translation $U(s_0) = e^{is_0 H}$, introduced by Leutheusser and Liu [14, 15]. In the Kruskal coordinates (U, V) of the eternal black hole, this unitary acts geometrically as a boost:

$$U(s_0): \ (U, V) \longmapsto \big(U\, e^{-s_0},\ V\, e^{s_0}\big). \tag{6.12}$$

For a sufficiently large s_0, a point in the right exterior $(U < 0, V > 0)$ is pushed across the bifurcation surface into the interior $(U > 0, V > 0)$. Consequently, for any operator $A \in \mathcal{A}_R$, the conjugated operator $A_{\text{int}} = U(s_0)AU(s_0)^\dagger$ is localised behind the horizon.

The algebra of interior observables is thus unitarily equivalent to the exterior algebra, but it is a different type II_∞ factor:

$$\mathcal{A}_B = U(s_0)\,\mathcal{A}_A\,U(s_0)^\dagger, \tag{6.13}$$

as first explicitly constructed by Krishnan and Mohan [16]. The interior can be understood as the *relational image* of the exterior under a large modular boost. No new degrees of freedom are created; the interior is simply the exterior as seen from Bob's clock. This provides a concrete algebraic expression of the idea that "the interior is encoded in the radiation" [59].

In the QRF framework of §6.2, the unitary $U(s_0)$ acts as a clock-change transformation. Alice's clock is the ADM Hamiltonian H; Bob's clock is his proper time τ, which, as noted earlier, can be modeled via the boost generated by H. The interior exists only relative to Bob's QRF; for Alice, the same degrees of freedom are encoded in the radiation via the island mechanism.

6.5.3 The island as a relational structure

The relational interpretation of the horizon and interior provides a natural explanation for the island formula of Chapter 4. Alice, whose algebra $\mathcal{A}_A$ is restricted to the right exterior, cannot directly access the interior. To purify the Hawking radiation, she must include in her entropy calculation an "island" I behind the horizon. The island is precisely the piece of the interior that her clock forces her to reconstruct. In the language of §6.4, the island's area term $\frac{\phi(\partial I)}{4G_N}$ is the relative entropy between Alice's clock state and Bob's clock state. Bob, whose algebra $\mathcal{A}_B$ already contains the interior, needs no island; his entropy is purely the matter entanglement in his causal diamond.

Thus the island can be understood not as a physical membrane but as a relational book-keeping device that accounts for the observer-dependence of the algebra. Its presence or absence can be seen as depending on the choice of quantum reference frame. This offers an interpretation of the apparent paradox that the interior seems to be simultaneously "inside" the black hole (for Bob) and "outside" (for Alice): the two descriptions refer to two complementary QRFs, and their respective islands can be understood as the shadows cast by the interior onto the exterior algebra.

In summary, the horizon can be understood as the boundary of a QRF, the interior as a relational boost of the exterior, and the island as the observer-dependent entropy contribution that emerges from the incompatibility of the two clocks. Together, these ideas develop the relational interpretation of black-hole physics.

6.6 Complementarity and the Island Paradigm

The technical results of Chapters 5 and 6 converge on a single, profound conclusion: the algebraic structure of the island paradigm offers a natural physical realisation of complementarity in quantum gravity. In this section, we synthesise the algebraic non-commutation, the QRF interpretation, and the observer-dependent entropy into a coherent philosophical framework, and we argue that this framework provides a robust way to address the long-standing tension between unitarity and the equivalence principle.

6.6.1 Three Senses of Complementarity

To understand how this framework functions, it is crucial to carefully distinguish between three distinct senses of complementarity:

1. **Bohr's empirical complementarity:** The fundamental epistemological lesson that mutually exclusive experimental arrangements cannot be comprehended within a single picture [32].
2. **Canonical non-commuting observables:** The mathematical realisation of this principle in ordinary quantum mechanics via specific pairs of operators, such as position and momentum ($[x, p] = i\hbar$).
3. **Non-commuting subalgebras:** The broader algebraic property where entire families of observables (e.g., the algebras accessible to different observers) fail to mutually commute.

Bohr insisted that "the empirical data obtained under different experimental conditions cannot be comprehended within a single picture" [32]. In the algebraic approach to quantum gravity, we do not rely on finding a specific canonical commutator (sense 2), but rather on the structural property of the observer algebras (sense 3) to implement Bohr's original empirical vision (sense 1).

6.6.2 Black-hole complementarity: the first attempt

In 1993, Susskind, Thorlacius, and Uglum [34] proposed that the black-hole information problem could be addressed by extending Bohr's complementarity to the geometry of the horizon. They argued that the experiences of an asymptotic observer and an infalling observer are complementary: the former sees a unitary evaporation via a stretched horizon membrane, while the latter sees a smooth, featureless horizon in accordance with the equivalence principle. The two descriptions should not be combined into a single global picture.

Despite its conceptual appeal, early black-hole complementarity was heavily criticised for lacking a precise mathematical formulation. The AMPS firewall argument [30] sharpened this tension by showing that the postulates of unitarity, the equivalence principle, and effective field theory cannot all hold simultaneously in a global description. For a detailed philosophical analysis of the structure of black holes and the implicit assumptions underlying the AMPS paradox, we refer the reader to the work of Cinti and Sanchioni [50, 51]. Ultimately, the firewall crisis demonstrated that a purely epistemological complementarity was insufficient; a structural, algebraic realisation was required.

6.6.3 Algebraic complementarity: an interpretive framework

The modern algebraic framework provides a natural way to address this impasse. As we reviewed in Chapter 5, the inclusion of gravitational back-reaction forces us to dress observables to physical clocks. The asymptotic observer Alice uses the ADM Hamiltonian H as her clock, generating the type II_∞ algebra $\mathcal{A}_A = \mathcal{A}_R \rtimes_\sigma \mathbb{R}$. The infalling observer Bob uses a different clock—his proper time along a geodesic—and his algebra is $\mathcal{A}_B = U(s_0)\mathcal{A}_A U(s_0)^\dagger$, with $U(s_0) = e^{is_0 H}$ representing the half-sided modular translation.

Crucially, because Bob's algebra is generated by a modular shift built from Alice's clock, these two subalgebras do not mutually commute. This structural non-commutation acts as an algebraic obstruction to combining the two descriptions. No single observer can simultaneously assign sharp values to both Alice's and Bob's dressed observables; the two experimental arrangements—asymptotic measurement vs. infalling measurement—are mutually exclusive in Bohr's strict sense. The algebra of observables in quantum gravity is observer-dependent, and the failure of commutativity between these different observer algebras serves as the mathematical expression of complementarity.

6.6.4 The Island Paradigm as a Realisation of Complementarity

The island formula of Chapter 4 can be understood as the physical manifestation of this algebraic complementarity. Alice, whose algebra $\mathcal{A}_A$ is restricted to the right exterior, computes the entropy of the Hawking radiation and finds the Page curve,

$$S_A(R) = \min_I \left\{ \frac{\phi(\partial I)}{4G_N} + S_{\text{matter}}(R \cup I) \right\}, \tag{6.14}$$

which includes an island I behind the horizon. Bob, whose algebra $\mathcal{A}_B$ contains the interior, has no need for an island; his entropy is expected to be simply the bulk matter entanglement,

$$S_B(R') = S_{\text{matter}}(R'), \tag{6.15}$$

where R' denotes the region accessible to Bob. The difference between the two entropies is the relative entropy of the two clock states, a result due to De Vuyst et al. [24]:

$$S_A(R) - S_B(R') = S_{\text{rel}}(\rho^A_{\text{clock}} \| \rho^B_{\text{clock}}) = \frac{\phi_0 + \bar{\phi}}{4G_N} + \mathcal{O}(G_N^0). \tag{6.16}$$

Thus, the island can be understood not as a physical membrane but as a relational bookkeeping device that accounts for the observer-dependence of the algebra. Alice sees a unitary Page curve because her algebra includes the island as the interior reconstructed from the outside; Bob sees a smooth horizon because his algebra already contains the interior.

This observer-dependent algebraic picture suggests a natural way the firewall tension might be avoided, which we do not develop rigorously here [54]. Because the modes appearing in the firewall argument are dressed to incompatible clocks, the monogamy of entanglement constraints may be bypassed. However, a definitive quantitative test of this framework against the AMPS argument, the firewall Apologia, and the Harlow-Hayden computational complexity bounds remains an open challenge.

In this sense, the island paradigm can be understood as extending the programme of black-hole complementarity. Bohr taught us that complementary experimental arrangements correspond to non-commuting observables. The algebraic framework shows that the asymptotic and infalling descriptions correspond to non-commuting subalgebras. The island formula is the physical consequence: the entropy of Hawking radiation is an observer-dependent quantity, and the Page curve is the signature of unitarity in the asymptotic frame. The black-hole information problem is thus reframed not by modifying quantum mechanics, but by recognising that the very notion of "where the information is" is fundamentally observer-dependent.

Chapter 7

Conclusions

The black-hole information problem has been the sharpest test of our ideas about quantum gravity for nearly five decades. This thesis has argued that the recent island paradigm, which successfully reproduces the Page curve and restores unitarity, is not merely a computational technique but can be understood as the physical manifestation of a deeper principle: Bohr's complementarity, now made mathematically precise by the algebraic quantum field theory of type II von Neumann algebras and the theory of quantum reference frames.

The original contribution of this thesis lies in synthesizing the results from the existing literature—the island formula, the crossed-product construction of von Neumann algebras, and the quantum reference frame formalism—and offering an interpretive framework: the crossed-product structure suggests an algebraic, observer-dependent reading of complementarity. Several crucial mathematical and physical questions remain open: the rigorous formulation of an operator-valued commutator for the dressed fields, the extension of these algebraic structures from eternal to dynamically evaporating black holes, and a definitive, quantitative test of this framework against the AMPS firewall paradox. These remain important open challenges for the field.

We began by tracing the historical arc of the paradox—from Hawking's thermal radiation and the Page curve, through the early black-hole complementarity proposal of Susskind, Thorlacius, and Uglum, to the discovery of the island formula. Chapter 4 provided a detailed derivation of the island formula in the Jackiw–Teitelboim model of gravity, demonstrating how the quantum extremal surface prescription yields the Page curve for the eternal two-sided black hole. The island, a region behind the horizon that must be included in the entanglement wedge of the radiation, was seen to be essential for purity.

The central technical component of this thesis is the algebraic construction showing that the gauge-invariant observables accessible to an asymptotic observer and those accessible to an infalling observer form two distinct, non-commuting type II_∞ algebras. In a fixed background, the exterior and interior algebras are type III_1 factors with no trace and no well-defined entropy. The inclusion of the observer's clock—the ADM Hamiltonian—converts the exterior algebra into a type II_∞ crossed product $\mathcal{A}_A = \mathcal{A}_R \rtimes_\sigma \mathbb{R}$, which possesses a faithful normal semi-finite trace. Physical observables must be dressed to this clock. The infalling observer's algebra $\mathcal{A}_B$ is obtained from $\mathcal{A}_A$ by the half-sided modular translation $U(s_0) = e^{is_0 H}$, which can be interpreted as a quantum reference frame transformation. Because the infalling observables are generated by a modular shift built from the asymptotic clock, we observed that these two families of dressed observables belong to non-commuting subalgebras. We interpret this non-commutation as a structural realisation of black-hole complementarity, providing a mathematical framework for the early proposals.

Building on this algebraic foundation, we developed a relational interpretation of black-hole physics. The asymptotic observer (Alice) and the infalling observer (Bob) were identified as distinct quantum reference frames, whose clocks are the ADM Hamiltonian and the infalling proper time, respectively. The half-sided modular translation can be interpreted as the QRF transformation connecting their descriptions. Their algebras do not commute, signalling that the two frames are complementary in Bohr's sense: the experimental arrangements are mutually exclusive, and no single observer can simultaneously assign sharp values to both the exterior and the interior dressed observables.

A direct physical consequence of this observer-dependence is that gravitational entropy is not absolute. Alice's entropy follows the Page curve and includes an island contribution, whereas Bob's entropy is expected to be purely matter entanglement and contains no island. The difference between the two entropies is the relative entropy of the two clock states, a result due to De Vuyst et al. The horizon, the interior, and the island itself can be understood not as fundamental geometric entities but as relational structures that emerge from the choice of the observer's quantum clock.

Taken together, these results suggest that the island paradigm can be understood as a realisation of Bohr's complementarity in quantum gravity, extending the early black-hole complementarity program. The early program was visionary but lacked the mathematical machinery to make the mutual exclusivity of the two descriptions precise. The algebraic framework, centred on the crossed product and the type III to type II transition, provides a mathematical framework for this complementarity. The island can be understood as the physical expression of this complementarity: it is the shadow cast by the interior onto the exterior algebra. Within this interpretive framework, the black-hole information paradox is thus reframed not by modifying quantum mechanics, but by recognising that the very concepts of "region," "interior," and "entropy" are observer-dependent.

Several directions for future research open up from this work.

- **De Sitter space and cosmology.** The crossed-product construction applies equally to the static patch of de Sitter space. A relational interpretation of the Gibbons–Hawking radiation would reveal observer-dependent entropies and possibly an island formula for cosmological horizons.

- **General quantum reference frame calculus.** The half-sided modular translation used here is specific to the black-hole background. Developing a systematic calculus of QRF transformations for gravitational subregions would provide a powerful tool for understanding holography and emergent spacetime.

- **Information-theoretic formulations.** The complementarity can also be captured in finite-dimensional quantum error-correction models using the Petz recovery map. Expanding this connection could lead to a fully operational understanding of the black-hole interior as a quantum code.

- **AMPS and the firewall.** The algebraic framework presented here offers a structural framework for complementarity, but a full confrontation with the AMPS paradox—demonstrating that monogamy of entanglement constraints are satisfied—remains an open question. This is an essential test for the framework developed in this thesis.

In conclusion, this thesis has argued that the island formula, the Page curve, and a possible resolution of the black-hole information problem can be understood through a single profound idea: complementarity, rooted in Bohr's insight and now equipped with the full mathematical armoury of modern theoretical physics. The black-hole interior is not a place; it is a relation. The horizon is not a barrier; it is a boundary between complementary descriptions. And the island is not a membrane; it is the observer-dependent signature of a relational universe. These insights, we believe, will play an essential role in the ongoing quest for a complete theory of quantum gravity.

Appendix A

Hawking Effect

A.1 Hawking Radiation

One of the most profound discoveries in theoretical physics is that black holes are not truly "black". In 1974, Stephen Hawking demonstrated that quantum mechanical effects near the event horizon lead to the spontaneous emission of radiation from black holes. Classically, once matter crosses the event horizon, it cannot escape; the escape velocity exceeds the speed of light. However, when quantum field theory is properly formulated in curved spacetime, this picture fundamentally changes. The presence of the event horizon introduces a causal boundary that alters how quantum fields are decomposed, and different observers—a static observer at infinity versus a freely infalling observer—perceive the vacuum state differently. This difference in vacuum definition, combined with quantum uncertainty near the horizon, leads to what appears as thermal radiation to an external observer: a continuous flux of particles being spontaneously created and emitted from the black hole.

The physical mechanism underlying this effect can be understood as follows. According to the uncertainty principle, virtual particle-antiparticle pairs constantly fluctuate in and out of existence in the quantum vacuum. Near the event horizon, if one member of such a pair crosses the event horizon while the other remains outside, the pair can no longer annihilate. From the perspective of an external observer, this manifests as the emission of real particles carrying away energy and causing the black hole to gradually evaporate. To make this intuitive picture precise, we quantize a scalar field on the black hole background using the Kruskal-Szekeres coordinates, which provide a clean description covering both interior and exterior regions. The key insight is to construct two different bases of modes—the Boulware basis, and the Unruh basis. These bases are related by a Bogoliubov transformation that reveals the thermal nature of the radiation: what one observer considers a vacuum contains thermal particles when viewed in the other observer's frame. The calculation proceeds as follows:

Let's consider maximal analytic extension of Schwarschild metric.

$$ds^2 = \frac{32m^3}{r} \exp \frac{-r}{2m} dU dV - r^2 d\Omega^2 \tag{A.1}$$

We will now consider a massless scalar field and quantize it in this particular manifold. So the classical field equation is

$$\widehat{\Box} f = 0 \tag{A.2}$$

Because the background is spherically symmetric we can expand the field f as,

$$f = \sum_{l,m} \frac{f(r,t)}{r} Y_l^m(\theta, \phi) \tag{A.3}$$

Where Y_l^m are spherical harmonics. So equation (A.2) becomes,

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^{*2}} + V_l(r) \right) f_l(r,t) = 0 \tag{A.4}$$

Where $V_l(r)$ is the potential given by,

$$V_l(r) = \left(1 - \frac{2m}{r}\right) \left(\frac{2m}{r^3} - \frac{l(l+1)}{r^2} \right) \tag{A.5}$$

The full wave equation, including the angular momentum barrier and gravitational potential, is difficult to solve exactly. By setting $l = 0$, we remove the angular term, simplifying the radial equation. By setting $V_l(r) = 0$, we ignore the effects of the gravitational potential, leaving a simple wave equation.

In terms of the null coordinates, equation (A.2) becomes,

$$\frac{\partial}{\partial v} \frac{\partial}{\partial u} f(v,u) = 0 \tag{A.6}$$

The general solution is the arbitrary function of u and v,

$$f = F(u) + G(v) \tag{A.7}$$

How do we choose F(u) and G(v)? I^+ and I^- are assymptotically flat regions, so choose modes such that at $r \to \infty$, modes approach usual Minkowski modes.

$$f_\omega = \{u_R \ , \ u_L \ , \ u_I \ , \ c.c\} \tag{A.8}$$

With

$$u_R = \begin{cases} A\frac{1}{r} e^{-i\omega u} & \text{if r > 2m} \\ 0 & \text{if r < 2m} \end{cases} \tag{A.9}$$

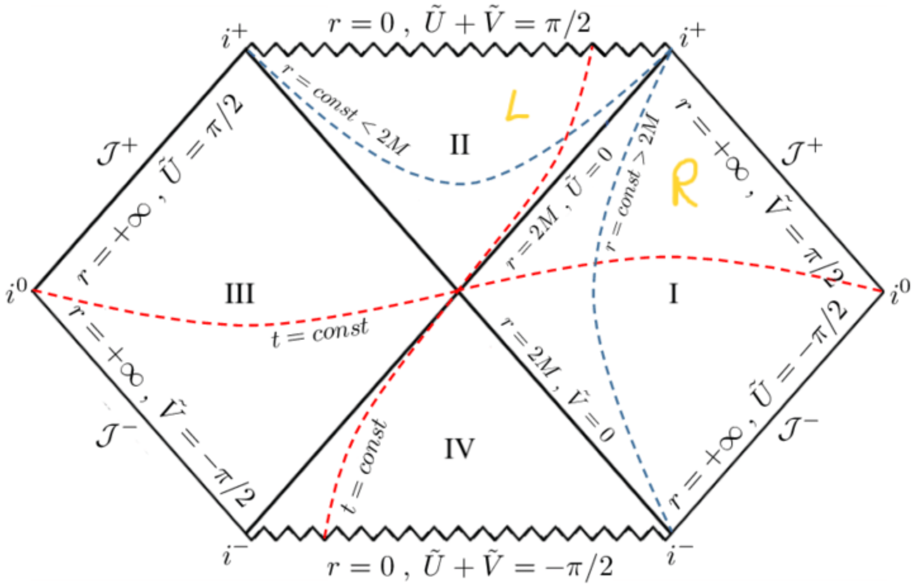

Figure A.1: Penrose diagram of Kruskal extension of Schwarzschild metric

$$u_L = \begin{cases} 0 & \text{if r} > \text{2m} \\ A\frac{1}{r}e^{-i\omega u} & \text{if r} < \text{2m} \end{cases} \tag{A.10}$$

$$u_I = A\frac{1}{r}e^{-i\omega v} \quad A = \frac{1}{4\pi\sqrt{\omega}} \tag{A.11}$$

Where

- u_R : Outgoing mode for R region
- u_L : Outgoing mode for L region
- u_I : Ingoing mode for both L and R

This choice of the basis is called the Boulware basis which is normalized. The vacuum state associated to these modes is called the Boulware vacuum. Boulware vacuum approaches the Minkowski vacuum asymptotically.

$$|B\rangle \xrightarrow{\overset{+}{\mathcal{I}^-}} |0_M\rangle \tag{A.12}$$

Outgoing Boulware modes u_R are positive frequency and positive norm with respect to Schwarzschild time t and outgoing Boulware modes u_L are negative frequency and positive norm with respect to Schwarzschild time t.

In this basis we can expand $\widehat{\phi}$ as

$$\widehat{\phi} = \sum_{\omega}(a^R_\omega u_R + a^L_\omega u_L + a^I_\omega u_I + h.c) \tag{A.13}$$

and define Boulwarw vacuum as

$$|B\rangle : \begin{cases} a^I_\omega \, |B\rangle = 0 \\ a^R_\omega \, |B\rangle = 0 \quad \forall \ \omega \\ a^L_\omega \, |B\rangle = 0 \end{cases} \tag{A.14}$$

We can also define,

$a^{I\dagger}_\omega \, |B\rangle$: 1 particle state on $\mathcal{I}^-$

$a^{R\dagger}_\omega \, |B\rangle$: 1 particle state on $\mathcal{I}^+$

$a^{L\dagger}_\omega \, |B\rangle$: This is called 1 partner state, which has a negative Killing energy.

This is just a mathematical construction. To see what it physically mean we need to consider the expectation value of covariantized energy-momentum tensor operator in the Boulware vacuum.

$$\hat{T}_{\mu\nu}(\hat{\phi}) = \partial_\mu\hat{\phi}\partial_\nu\hat{\phi} \ - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\hat{\phi}\partial_\beta\hat{\phi} \tag{A.15}$$

The epectation value of $\hat{T}_{\mu\nu}$ diverges but upon renormalization one finds,

$$\langle B| \, \hat{T}_{\mu\nu} \, |B\rangle \xrightarrow{r\to\infty} 0 \quad \text{as} \quad \frac{m^2}{r^6} \tag{A.16}$$

$$\langle B| \, \hat{T}_{\mu\nu} \, |B\rangle \xrightarrow{r\to 2m} \big(1 - \frac{2m}{r}\big)^{-2} \to \text{geometrical divergence} \tag{A.17}$$

In general $\langle B| \, \hat{T}_{\mu\nu} \, |B\rangle = f(g_{\mu\nu})$ which is UV finite, but if the metric is singular then $\langle B| \, \hat{T}_{\mu\nu} \, |B\rangle$ will have geometrical divergence. Indeed using (v,u) we have divergence on r = 2m.

Since we neglected the back scattering, u and v states do not mix. So the Boulware states can be composed of u and v sectors,

$$|B\rangle = |B\rangle_u \otimes |B\rangle_v \tag{A.18}$$

From $\langle B| \, \hat{T}_{\mu\nu} \, |B\rangle \xrightarrow{H}$ divergence, Boulware modes are singular on horizon H^+ and H^-. Physically, Boulware vacuum describes vacuum polarization outside the static star.

Since in curved spacetime we can have any number of modes, so, can we have modes which is regular on future and past horizon?

Unruh proposed a new basis. He considered ingoing modes same as Boulware ingoing and outgoing modes constructed out of Kruskal U,

$$u^I_\omega = \frac{1}{4\pi\sqrt{\omega}}\frac{1}{r}e^{-i\omega v} \qquad u_k(\omega_k) = \frac{1}{4\pi\sqrt{\omega}}\frac{1}{r}e^{-i\omega_k U} \tag{A.19}$$

Where $U = \pm 4m\, e^{u/4m}$ and ω_k is different than Boulware ω . This forms the Unruh basis,

$$U : \{u_k, u^I_\omega, c.c\} \tag{A.20}$$

In this basis we can expand field as,

$$\hat{\phi} = \sum_{\omega_k}(a_{\omega_k} + h.c) + \sum_{\omega}(a^I_\omega u^I_\omega + h.c) \tag{A.21}$$

and define the Unruh vacuum as

$$|U\rangle \quad \text{s.t.} \quad \begin{cases} a_{\omega_k}|U\rangle = 0 & \forall\omega_k \\ a^I_\omega |U\rangle = 0 & \forall\omega \end{cases} \tag{A.22}$$

Since the ingoing modes are same as Boulware, we have,

$$|U\rangle \xrightarrow{I^-} |0_M\rangle \tag{A.23}$$

But,

$$|U\rangle \not\xrightarrow{I^+} |0_M\rangle \tag{A.24}$$

So, the state is Minkowski in past infinity but not in the future infinity. That means we have no particles in the past null infinity in the vacuum but we will find flux of particle in the future null infinity. With this construction the expectation value of covariantized energy - momentum is

$$\langle U|\, T_{\mu\nu}\, |U\rangle \xrightarrow{H^+} regular \tag{A.25}$$

$$\langle U|\, T_{\mu\nu}\, |U\rangle \xrightarrow{H^-} \big(1 - \frac{2m}{r}\big)^{-2} \to diverge \tag{A.26}$$

Unruh states are regular on future horizon but not on the past horizon. However this divergence on the past horizon is an artifact without any physicsal significance since there is no past horizon in the gravitational collapse forming black hole.

Is there any physical system in which we can apply this construction? To see this recall in the case of the gravitational collapse of a star, near the horizon i.e u $\to \infty$ we found independent of the collapse model that

$$\text{Outgoing mode} \approx \lim_{u\to\infty} e^{-i\omega F(u)} = e^{-i\omega U} = \text{Kruskal mode} \tag{A.27}$$

So as u $\to \infty$, $|in\rangle \sim |U\rangle$. At late time u $\to \infty$, the maximal analytic extension of Schwarzschild spacetime with respect to Unruh modes mimic real gravitational collapse. Physically, Unruh construction corresponds to late time near horizon behaviour of a real collapsing star.

What is the relation between $|U\rangle$ and $|B\rangle$? Since ingoing modes of both the construction is the same we will just focus on outgoing modes. So there is a Bogoliubov transformation between $u_k \to u_{L,R}$.

$$u_k(\omega_k) = \int_0^{+\infty} d\omega(\alpha^L_{\omega_k,\omega} u^L_\omega + \beta^L_{\omega_k,\omega} u^{*L}_\omega) + \int_0^{+\infty} d\omega(\alpha^R_{\omega_k,\omega} u^R_\omega + \beta^R_{\omega_k,\omega} u^{*R}_\omega) \tag{A.28}$$

The Bogoliubov coefficients can be expressed as a scalar product,

$$\alpha^R_{\omega_k,\omega} = (u_k(\omega_k), u^R_\omega) \qquad \beta^R_{\omega_k,\omega} = -(u_k(\omega_k), u^{*R}_\omega) \tag{A.29}$$

A similar relation holds for the L region. One can compute these coefficients by choosing H^- as a Cauchy surface.

$$\alpha^R_{\omega_k,\omega} = \frac{1}{2\pi k}\sqrt{\frac{\omega}{\omega_k}}(\frac{-i\omega_k}{k})^{i\omega/k}\Gamma(\frac{-i\omega}{k}) \tag{A.30}$$

$$\beta^R_{\omega_k,\omega} = \frac{1}{2\pi k}\sqrt{\frac{\omega}{\omega_k}}(\frac{-i\omega_k}{k})^{-i\omega/k}\Gamma(\frac{i\omega}{k}) \tag{A.31}$$

$$\alpha^L_{\omega_k,\omega} = \frac{1}{2\pi k}\sqrt{\frac{\omega}{\omega_k}}(\frac{i\omega_k}{k})^{-i\omega/k}\Gamma(\frac{i\omega}{k}) \tag{A.32}$$

$$\alpha^L_{\omega_k,\omega} = \frac{1}{2\pi k}\sqrt{\frac{\omega}{\omega_k}}(\frac{i\omega_k}{k})^{i\omega/k}\Gamma(\frac{-i\omega}{k}) \tag{A.33}$$

Where Γ is a Euler gamma function and the k in the denominator is the surface gravity $k = 1/4$m. Notice that,

$$\begin{aligned}\alpha^R_{\omega_k,\omega} &= \frac{1}{2\pi k}\sqrt{\frac{\omega}{\omega_k}}\Gamma\left(\frac{-i\omega}{k}\right)(-i)^{i\omega/k}\left(\frac{\omega_k}{k}\right)^{i\omega/k} \\ &= \frac{1}{2\pi k}\sqrt{\frac{\omega}{\omega_k}}\Gamma\left(\frac{-i\omega}{k}\right)\left(\frac{\omega_k}{k}\right)^{i\omega/k} e^{\pi\omega/2k}\end{aligned} \tag{A.34}$$

Similarly,

$$\beta^R_{\omega_k,\omega} = \frac{1}{2\pi k}\sqrt{\frac{\omega}{\omega_k}}\Gamma(\frac{i\omega}{k})(\frac{\omega_k}{k})^{-i\omega/k}e^{-\pi\omega/2k} \tag{A.35}$$

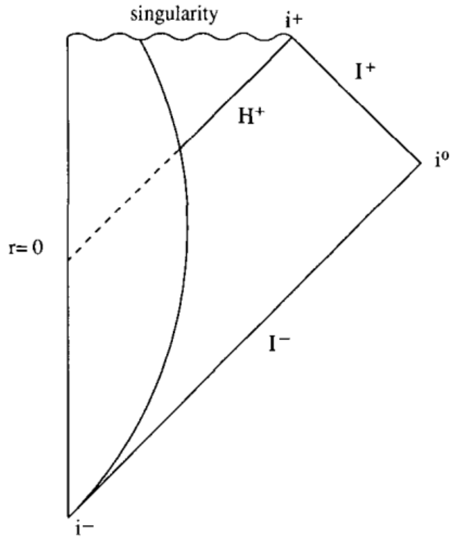

Figure A.2: Penrose diagram of a Schwarzschild black hole formed by gravitational collapse.

Taking modulo square we find,

$$|\alpha^R_{\omega_k,\omega}|^2 = e^{2\pi\omega/k}|\beta^R_{\omega_k,\omega}|^2 \tag{A.36}$$

We know that the Bugoliubov coefficients are not independent but holds,

$$\sum_{\omega_k} \alpha^{*R}_{\omega_k,\omega}\alpha^R_{\omega_k,\omega'} - \beta^R_{\omega_k,\omega'}\beta^{*R}_{\omega_k,\omega} = \delta_{\omega,\omega'} \tag{A.37}$$

For $\omega = \omega'$,

$$\sum_{\omega_k}(|\alpha^R_{\omega_k,\omega}|^2 - |\beta^R_{\omega_k,\omega}|^2) = 1 \tag{A.38}$$

Substituting equation (A.36) into equation (A.38) we get,

$$\sum_{\omega_k}|\beta^R_{\omega_k,\omega}|^2 = \frac{1}{e^{8\pi m\omega} - 1} \tag{A.39}$$

Lets go back to the gravitational collapse. The state $|in\rangle$ corresponds to no particles on I^-. In Heisenberg picture, $|in\rangle$ is also the final state on I^+. But the operator evolves, so on I^+ observer measures a flux of particles given by $\langle in|\hat{N}_\omega|in\rangle$. Where $\hat{N}_\omega = a^{R\dagger}_\omega a^R_\omega$ is the number of particles on I^+.

So we are interested in $\langle in|\hat{N}_\omega|in\rangle$. We will use the fact that at late times $u \to +\infty$, $|in\rangle \to |U\rangle$.

$$\lim_{u\to\infty}\langle in|a^{R\dagger}_\omega a^R_\omega|in\rangle = \langle U|a^{R\dagger}_\omega a^R_\omega|U\rangle \tag{A.40}$$

$\hat{N}_\omega$ is in terms of ladder operators of the Boulware states and $|U\rangle$ is in terms of Unruh vacuum. We can relate these two with Bogoliubov transformation,

$$a_\omega^R = \sum_{\omega_k} \alpha_{\omega_k,\omega}^R a_{\omega_k} + \beta_{\omega_k,\omega}^{R*} a_{\omega_k}^\dagger \tag{A.41}$$

Substituting equation (A.41) into equation (A.40) we find,

$$\begin{aligned} \lim_{u\to\infty} \langle in| a_\omega^{R\dagger} a_\omega^R |in\rangle &= \langle U| a_\omega^{R\dagger} a_\omega^R |U\rangle \\ &= \sum_{\omega_k} |\beta_{\omega_k,\omega}|^2 \\ &= \frac{1}{e^{8\pi m\omega} - 1} \\ &= \frac{1}{e^{\hbar\omega/k_B T_H} - 1} \end{aligned} \tag{A.42}$$

Where,

$$T_H = \frac{\hbar c^3}{8\pi G m k_B} = \frac{\hbar\kappa}{2\pi c k_B} \tag{A.43}$$

When we include effective potential V_{eff} which causes back scattering so some modes will be transmitted and others will get reflected with probability T and R such that, $|T|^2 + |R|^2 = 1$. In this case we will get,

$$\begin{aligned} \lim_{u\to\infty} \langle in| a_\omega^{R\dagger} a_\omega^R |in\rangle &= \langle U| a_\omega^{R\dagger} a_\omega^R |U\rangle \\ &= \sum_{\omega_k} |T|^2 |\beta_{\omega_k,\omega}|^2 \\ &= \frac{|T|^2}{e^{\hbar\omega/k_B T_H} - 1} \\ &= \frac{\Gamma(\omega)}{e^{\hbar\omega/k_B T_H} - 1} \end{aligned} \tag{A.44}$$

Where, $\Gamma(\omega)$ is a grey body factor.

Hence we can see that the black hole emits a thermal radiation with temperature T_H.

Appendix B

Von Neumann Algebras and Tomita–Takesaki Theory

This appendix collects the essential definitions and results from the theory of von Neumann algebras that are used in the algebraic treatment of black-hole physics in Chapter 5. Detailed expositions can be found in the standard textbooks [9, 10, 11]. We assume familiarity with basic operator algebra concepts (bounded operators, Hilbert spaces, C^*-algebras).

B.1 Von Neumann Algebras and Factors

Let $\mathcal{H}$ be a complex Hilbert space. The algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$ is a $*$-algebra with the operator norm. A **von Neumann algebra** $\mathcal{M}$ is a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed in the weak operator topology, or equivalently (by von Neumann's bicommutant theorem) satisfies $\mathcal{M}'' = \mathcal{M}$, where the prime denotes the commutant:

$$\mathcal{M}' = \{\, x \in \mathcal{B}(\mathcal{H}) \mid xy = yx \;\; \forall y \in \mathcal{M} \,\}.$$

A von Neumann algebra is called a **factor** if its centre is trivial, i.e. $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}\,\mathbb{I}$.

B.2 Classification of Factors

Factors are classified by the structure of their projections and by the existence of a trace. Let $\mathcal{M}$ be a factor.

- **Type I:** $\mathcal{M}$ is isomorphic to $\mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$. It has minimal (non-zero) projections. Finite type I_n factors ($\dim \mathcal{K} = n$) have the ordinary trace.
- **Type II:** $\mathcal{M}$ has no minimal projections but admits a **faithful normal semi-finite trace** τ. If $\tau(\mathbb{I})$ is finite, the factor is type II_1; if $\tau(\mathbb{I})$ is infinite, it is type II_∞. Type II factors are "continuous" matrix algebras.
- **Type III:** $\mathcal{M}$ admits no faithful normal semi-finite trace. All non-zero projections are infinite and equivalent to the identity. The spectrum of the modular operator (see below) provides a finer classification into type III_0, III_λ ($0 < \lambda < 1$), and III_1.

In quantum field theory, local algebras are typically type III_1 factors [9]. They have no trace, no density matrices in the usual sense, and no pure states. This property is central to the information paradox.

B.3 Tomita–Takesaki Theory

Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and let $\Omega \in \mathcal{H}$ be a **cyclic and separating** vector for $\mathcal{M}$ (i.e. $\mathcal{M}\Omega$ is dense in $\mathcal{H}$, and $A\Omega = 0$ implies $A = 0$ for $A \in \mathcal{M}$). The **Tomita–Takesaki theorem** states that the map

$$S_0 : \mathcal{M}\Omega \to \mathcal{M}\Omega, \qquad S_0(A\Omega) = A^\dagger\Omega, \tag{B.1}$$

extends to a closed, anti-linear operator S on $\mathcal{H}$. The polar decomposition of S defines two canonical objects:

$$S = J\Delta^{1/2}, \tag{B.2}$$

where

- Δ is a positive, self-adjoint operator (the **modular operator**), and
- J is an anti-unitary involution (the **modular conjugation**), $J^2 = \mathbb{I}$.

The fundamental theorem then states:

1. $\Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}$ for all $t \in \mathbb{R}$ (the modular automorphism group $\sigma_t(A) = \Delta^{it}A\Delta^{-it}$ leaves $\mathcal{M}$ invariant).
2. $J\mathcal{M}J = \mathcal{M}'$, i.e. the modular conjugation exchanges the algebra with its commutant.

The vector Ω satisfies the **KMS condition** at inverse temperature $\beta = 1$ with respect to the modular flow parameter t: for any $A, B \in \mathcal{M}$,

$$\langle\Omega| A\, \sigma_t(B)\, \Omega\rangle = \langle\Omega| \sigma_{-t-i}(A)\, B\, \Omega\rangle, \tag{B.3}$$

where the function $t \mapsto \langle\Omega| A\, \sigma_t(B)\, \Omega\rangle$ extends to an analytic function on the strip $0 < \operatorname{Im} t < 1$.

Physically, we can interpret this flow as time evolution by defining a dimensionless modular Hamiltonian $K_{\text{mod}} = -\ln \Delta$. The KMS condition (B.3) implies that the state $\langle\Omega| \cdot |\Omega\rangle$ is exactly thermal with respect to K_{mod} at inverse temperature $\beta = 1$. To connect this to a physical Hamiltonian H with units of energy, we must perform a constant rescaling. If we define $K_{\text{mod}} = 2\pi H$, the modular operator takes the form $\Delta = e^{-2\pi H}$. Consequently, the state is thermal with respect to the physical time evolution generated by H at the rescaled inverse temperature $\beta = 2\pi$.

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