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The phase-space distribution function for weakly flattened axisymmetric stellar systems

Graduation Thesis

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ABSTRACT

The astrophysical importance of galaxy modelling lies in the role it plays in understanding the structure, formation, and evolution of galaxies. Models can be constructed numerically, observationally (e.g. through the Jeans equations), or analytically. The analytical approach is based on the phase-space Distribution Function (DF), whose positivity is a necessary condition for the physical consistency of a galaxy model. This criterion allows physically unacceptable models to be discarded even when they provide a good fit to the observational data. Moreover, an analytical DF enables the computation of velocity moments of all orders, providing detailed predictions for stellar kinematics.

According to the Jeans theorem, the DF depends on the phase-space coordinates through the integrals of motion in the total gravitational potential. However, the relation between the DF and observable quantities is expressed by complicated integral equations, making its determination an inversion problem. To date, the only case for which a general inversion formula is available is spherical symmetry, through the Eddington formula. In axisymmetry, existing methods rely on analytical continuation into the complex plane, limiting their applicability to observational data.

In this thesis, we develop a novel inversion method for two-integral axisymmetric stellar systems by considering small departures from spherical symmetry. In this regime, density and gravitational potential admit homeoidal expansions, where the flattening parameter appears explicitly and the geometry separates naturally into spherical and non-spherical contributions. By exploiting this structure, we show that the inversion equations can be decomposed into terms that closely resemble the Eddington formula. This leads to original inversion formulae for the DF without introducing complex variables, providing a new framework for the analytical modelling of axisymmetric galaxies.

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CHAPTER 1

Introduction

Building dynamical models of galaxies is an important problem in astronomy, not only for stellar dynamics but also for the information we get from these models, regarding the formation, structure and evolution of these systems. From a methodological point of view, there exists a variety of approaches for the models construction: numerical, based on N-body simulations; phenomenological, such as by combining observations with the Jeans Equations; analytical, based on the phase-space distribution function (DF). This latter approach is the most fundamental but also the most difficult one.

The present thesis is focused on the last approach. The great advantage is that the analytical approach allows us to test the positivity of the DF, a crucial problem that is of difficult solution in other methods. Indeed, models obtained through numerical techniques or via the Jeans Equations may agree with observational data while still being physically unacceptable, if their associated DFs become negative in some point of phase space. The positivity condition for the DF is also known as consistency condition; useful presentations of consistency criteria and associated applications can be found in Ciotti and Pellegrini 1992, An and Evans 2006, Ciotti and Morganti 2010. For a given observed galaxy, more than one model may be consistent with the data; some of these models may still be unphysical. The analytical approach makes it possible to determine whether a galaxy model is physically acceptable and to discard those that are not.

Among all the approaches involving the DF, it is not a surprise that the vast majority of our knowledge comes from spherical systems: we know many remarkable results, like the Eddington inversion formula (Eddington 1916) and its generalizations, that allow to re-build the DF starting from given density and gravitational potential profiles. Spherically symmetric cases are quite well understood and widely used. Obviously, real galaxies can easily have shapes different from a sphere. Departing from spherical symmetry, the next natural generalization is the axial symmetry. The physical and mathematical passage from spherical to axial symmetry is way more complicated than one may imagine and introduces a series of problems that in the spherical case simply do not exist. The increasing difficulty is not only due to the increasing technical difficulties, but also to the fact that brand new problems arise in axial symmetry. For these reasons, the dynamics of axisymmetric systems is much more unexplored. An example of such new problems is that the equivalent Eddington inversion formula in axial symmetry needs the introduction of complex variables and relies on contour integrals (Hunter and Qian 1993), Laplace (Lynden-Bell 1962) and Laplace-Mellin (Dejonghe 1986) transforms, and in general need analytic extensions in the complex plane of density and gravitational potential. Unsurprisingly, axisymmetric systems are less understood and the general axisymmetric problem is still too much complicated.

In this work, we are going to use an already well-known philosophy: we do small steps beyond spherical symmetry and focus on axisymmetric systems with weak flattening, in hope that also the differences with respect to the spherical case will be small. Very often, in mathematics, something thought to be valid for "very small" reveals to be valid also for "small" and sometimes even for "not so small" values. There is not a general rule or certainty for this to happen, but it very often happens. The same can be said about those things thought to be valid only for "extremely big" values. An example is given by asymptotic expansions, which often work surprisingly well for values much smaller than "infinity". Moreover, in case of weak flattening the self-consistent density-potential pairs are much more manageable and the flattening parameter appears explicitly in their new expressions, the so called *homeoidal expansions*.

1.1 The astrophysical problem

For the following concepts, see also Binney and Tremaine 2008.

Whenever, for any instant in time, we can give the position $\mathbf{x}$ and the vector velocity $\mathbf{v}$ of a pointlike object, we have full knowledge of the entire trajectory that object makes, being it under the influence of some fundamental interaction or not. When such an interaction is gravity acting on celestial bodies (approximated as point-masses), we enter the realm of Celestial Mechanics. With this name we are more used to indicate that classical theoretical framework which allowed, along centuries, to predict the behaviour of planet orbits and the motions of minor Solar System objects, such as comets, asteroids, or even to understand some aspects of planetary rings. It is perfectly fine to see as point-masses even stars in binary systems, in open or globular clusters, up to stars constituting the stellar component of a galaxy. In these cases, Celestial Mechanics is renamed more properly as Stellar Dynamics. With the aid of mathematical techniques, such theories can explore completely and analytically any aspect of a 2-body problem, in which case each orbit is fully described in the 6-dimensional $\mathbb{R}^6$ *phase space*, where single points are identified (not by chance) as $(\mathbf{x}, \mathbf{v})$. However, when the point-masses interacting under mutual gravity are three, things get soon more difficult to handle, and only some strong assumptions can make it still possible to describe orbits analytically; the general 3-body problem has to be studied numerically. But even the numerical approach shows limits in the total number of objects that can be studied simultaneously: nowadays, most advanced numerical simulations of stellar motions on a galactic scale cannot include more than a few millions of particles, while real galaxies can contain a number of stars on the order of hundreds (if not a thousand) of billion. The lack of possibility to predict single orbits in systems made of an increasing number of point-masses seems discouraging: how is it possible to get some knowledge about stellar motions in a galaxy?

Unlike electromagnetic interaction, gravity cannot be shielded; each star in a galaxy is attracted by all the other members of the system. While they move inside the galaxy, stars continuously undergo many (small) deflections along their paths. In principle, each of these interactions could be able to change the motion of a star, although in reality these effects on orbits are soft and cumulative. In fact, it is possible to show that a star does not immediately feel the granular nature of the system it is moving in. That means, until a precise timescale has elapsed the stellar motion evolves as under the action of a smooth gravitational field, generated by a continuous density distribution $\rho(t, \mathbf{x})$. Of course, this is an approximation: what really happens is that the smooth description gives an ideal orbit from which the real one is allowed to deviate more and more in time. There will always be a time, defined for the specific stellar system, after which the cumulative kicks from many encounters have changed the star's orbit significantly; this timescale is the *2-body relaxation time* t_{2b}. After one t_{2b}, the star has lost memory of its initial conditions, its orbit is no longer consistent with the ideal «smooth» one, and it starts to feel significantly the discrete nature of the gravitational field of the system (Figure 1.1). As already mentioned, t_{2b} changes from one system to another, depending on the number of point-masses it contains, on its characteristic scale-length and on the typical observed behaviour of stars. For a spherical system with N point-masses all moving at characteristic speed v and having a global scale-length R , a good t_{2b} estimator is

$$t_{2b} \simeq \frac{N}{8 \ln N} \frac{R}{v}. \tag{1.1}$$

It is extremely useful to compare the estimated t_{2b} with the typical age of each particular type of stellar system, in order to understand its *degree of collisionality*, where with this term we indicate how much the discrete nature of the gravitational field has impacted on stellar orbits.

	N	R	v	t_{2b}	age/t_{2b}
open cluster	100	2 pc	0.5 km/s	10^7 yrs	≥ 1
globular cluster	10^5	4 pc	10 km/s	4×10^8 yrs	≥ 10
dwarf galaxy	10^9	1 kpc	50 km/s	10^{14} yrs	10^{-4}
elliptical galaxy	10^{12}	10 kpc	600 km/s	10^{17} yrs	10^{-7}

Table 1.1: Characteristic quantities for different types of stellar system. N is the number of point-masses, R the scale-length of the system, v the typical velocity of stars in that system, t_{2b} the 2-body relaxation time as estimated via (1.1). By comparing t_{2b} with the age of a system, its degree of collisionality can be evaluated. Open and globular clusters may be slightly collisional, while galaxies of any kind are almost perfectly collisionless systems.

As shown in Table 1.1, open and globular clusters we observe today in the Local Universe are starting to be influenced by the discrete nature of the distribution of stars; in other words, they are becoming *collisional* systems. Galaxies of any kind in the Local Universe, instead, show t_{2b} much larger than their ages, meaning that they are almost perfectly *collisionless* systems: every single star moves as under the influence of a mean gravitational field generated by a smooth mass distribution, rather than a collection of mass points. In galaxies of the Local Universe, stars do not experience significant encounters and their orbital energies are largely preserved.

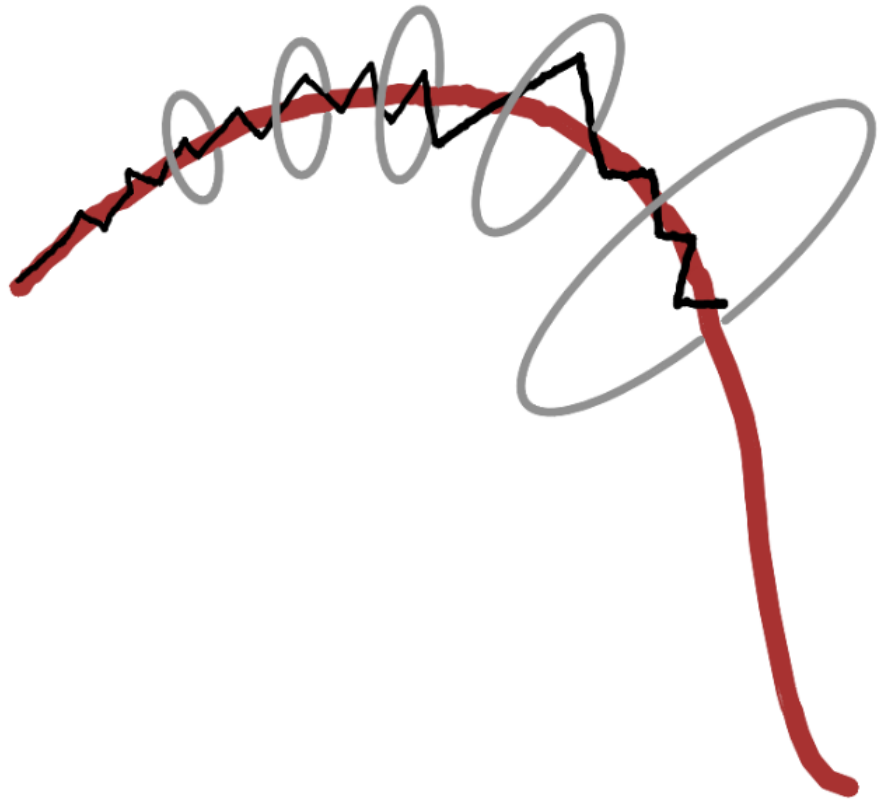


Figure 1.1: Intuitive, handmade sketch showing an ideal «smooth» orbit (red line), a possible corresponding true orbit of a star (black line) and the "uncertainties" about how much the true orbit can deviate from the smooth one, at different times (grey circumferences). Each grey circumference is centered in some point on the smooth orbit and perpendicular to it; its area is proportional to the maximum possible deviation the true orbit can have from the smooth orbit, at that instant in time. The set of circumferences describes a manifold centered on the smooth orbit and enclosing the volume in which the true orbit can exist. Notice how the uncertainty increases with time, but the true orbit does not necessarily deviate monotonically in time.

Collisionless systems allow for a smooth modelling, where the density and gravitational potential are continuous functions $\rho(t, \mathbf{x})$ and $\Phi(t, \mathbf{x})$ and where it can be usefully considered the existence of a crucial mathematical object in the phase space, the *distribution function* (DF) $f(t, \mathbf{x}, \mathbf{v})$. At any time t, the entire dynamical configuration of a stellar system is given if we can give its density distribution and the full local description of orbits, that means of the stellar velocity distribution in any point $\mathbf{x}$ of the system. In a collisionless system, we know everything about the internal dynamics if we can write its DF. We can immediately understand the importance of knowing the exact expression for f: by having this, it would be possible to know everything about the motions in a galaxy without actually solving any equation of motion for the single stars.

The reason for this can arise more clearly by understanding the meaning of f. The quantity $f(t, \mathbf{x}, \mathbf{v})\, d^3\mathbf{x}\, d^3\mathbf{v}$ gives the number (or mass) of stars which, inside a volume element centered on $\mathbf{x}$, move with vector velocities that (in the velocity space $\mathbb{R}^3$) have an ending point inside the closure of $d^3\mathbf{v}$ (see Figure 2.6). So, the DF can be seen as a density in the phase space. Like all those functions intended to be read as distributions, f is related to concepts such as average and dispersion. More precisely, all the dynamical quantities of the system are *moments* of f in the velocity space. Considering orders up to the second

$$\rho(t, \mathbf{x}) = \int d^3\mathbf{v}\ f(t, \mathbf{x}, \mathbf{v}) \qquad 0^{th} order \tag{1.2}$$

$$\overline{v_i}(t, \mathbf{x}) = \frac{\int d^3\mathbf{v}\ v_i\ f(t, \mathbf{x}, \mathbf{v})}{\rho(t, \mathbf{x})} \qquad 1^{st} order \tag{1.3}$$

$$\overline{v_i v_j}(t, \mathbf{x}) = \frac{\int d^3\mathbf{v}\ v_i v_j\ f(t, \mathbf{x}, \mathbf{v})}{\rho(t, \mathbf{x})} \qquad 2^{nd} order \tag{1.4}$$

$$\sigma_{ij}^2(t, \mathbf{x}) = \frac{1}{\rho(t, \mathbf{x})} \int d^3\mathbf{v}\ (v_i - \overline{v_i})(v_j - \overline{v_j})\ f(t, \mathbf{x}, \mathbf{v}) \qquad 2^{nd} order \tag{1.5}$$

where $i, j = 1, 2, 3$; with the overline " $-$ " we indicate weighted averages and with σ_{ij}^2 the *velocity dispersion tensor* (see Chapter 2 for a visual clarification about this object). These and all other moments of order up to infinity can be obtained if we can first write f, giving in this way the ultimate knowledge of the internal dynamics.

At this point, the existence of equations satisfied by f and its velocity moments is worth mentioning. First of all, in our study we consider systems which do not lose nor gain stars in their evolution. This is translated into a continuity equation involving f, since it is a density distribution in the phase space. Given the fact that we are using cartesian $(\mathbf{x}, \mathbf{v})$ as coordinates of the 6-dimensional phase space, we immediately get such an equation as

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi_T}{\partial x_i} \frac{\partial f}{\partial v_i} = 0 \tag{1.6}$$

where $i = 1, 2, 3$ (Einstein's rule for repeated indices has been used); Φ_T is the total gravitational potential of the system. This equation is better known as the *Collisionless Boltzmann Equation* (CBE). So far, no assumption has been made as to whether or not the potential Φ_T is only due to stars themselves or has further contributions from other sources, such as a Dark Matter (DM) halo, a central supermassive black hole or tidal fields by external objects. Nor we have discussed the possibility of stars having different masses. If Φ_T is only due to stars (all of the same mass) described by f, then *self-consistency* is fulfilled and it is expressed by Poisson Equation

$$lap(\Phi_T) = 4\pi G\rho = 4\pi G \int d^3\mathbf{v}\ f(t, \mathbf{x}, \mathbf{v}) \tag{1.7}$$

If again Φ_T is due only to stars but this time the system is made by stars of different masses, we consider, for each family of fixed mass m, self-consistency (Poisson equations) and then we sum all the potentials Φ_m to get Φ_T. In this way, we consider a f_m for each family of stars and the resulting DF is simply $f = \sum_m f_m$. Notice, however, that the CBE of each family still must include the total potential Φ_T and not just Φ_m (stars of each family are still under the action of the total gravity of the system).

CBE does not allow one to completely solve for f, and it represents just a constraint for it. Nevertheless, the usefulness of CBE is manifest if we take its velocity moments, obtaining the *Jeans equations* (JEs)

$$(0^{th} order) \qquad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho \overline{v_i}) = 0\,, \qquad \text{JE No.1} \tag{1.8}$$

$$(1^{st} order) \qquad \frac{\partial}{\partial t}(\rho \overline{v_j}) + \frac{\partial}{\partial x_i}(\rho\, \overline{v_i v_j}) + \rho \frac{\partial \Phi_T}{\partial x_j} = 0\,; \qquad \text{JE No.2} \tag{1.9}$$

again, $i, j = 1, 2, 3$. It can be shown, from its expression in (1.5), that the velocity dispersion tensor σ^2_{ij} can be generally expressed in a compact form

$$\sigma^2_{ij} = \overline{v_i v_j} - \overline{v_i}\, \overline{v_j}\ ; \tag{1.10}$$

by exploiting this and the JE No.1 (1.8), it's possible to rewrite the JE No.2 (1.9) in a more convenient way:

$$(1^{st} order) \qquad \rho \frac{\partial\, \overline{v_j}}{\partial t} + \rho \overline{v_i} \frac{\partial\, \overline{v_j}}{\partial x_i} = -\rho \frac{\partial \Phi_T}{\partial x_j} - \frac{\partial}{\partial x_i}(\rho \sigma^2_{ij}) \qquad \text{JE No.3} \tag{1.11}$$

which strongly reminds of the momentum equation of Hydrodynamics for an ideal fluid. As is evident by looking at them, JEs (1.8)-(1.11) are all equations involving the velocity moments of f, and are of crucial importance in modelling observed galaxies.

We have already pointed out that the CBE is not enough to solve completely for f. One may wonder whether the DF can be fully recovered by observing real galaxies: in general, observations provide local and luminosity-weighted average values of the velocity field (the moments of f in $\mathbf{v}$). Moments of *all* orders *together* completely define f. But, to date, observations cannot extract moments of order higher than second or third. The only way we can hopefully end up with analytic expressions or numerical results for f is via a mathematical reconstruction, supported by as many physical arguments as possible.

Unfortunately, building f is an extremely complicated problem. It all starts with the construction of models of observed galaxies based on the JEs; then, the focus goes for relating the observed quantities (de-projected density profile, streaming velocity and velocity dispersion fields) to the DF by using their mathematical definitions (1.2)-(1.5). This is the so called *ρ-to-f* approach, the logical steps of which are introduced in Chapter 2. It is due mentioning that in the ρ-to-f approach, in general, the underlying DF cannot be determined uniquely, and even when it is, its physical validity must be verified.

Our attention is dedicated entirely to what's next the JE modelling, the recovery of the DF, that is usually the least trivial step of the ρ-to$f$ approach. Usual approaches to this task involve integral inversion techniques. As will be clear in Chapter 2, the complete solution to this problem has been found only for spherical systems. For axisymmetric systems, integral inversion attempts have been carried out using complex analysis (e.g., Mellin integrals, contour integrals etc.). The main goal of this work is to show that an alternative to Mellin and philosophically similar methods exists, which actually does not rely on complex variables: after all, nature, at least in physical quantities like density and gravitational potential, should not be ascribed to complex quantities.

Our treatment will investigate exclusively stationary solutions, so all explicit time dependencies will vanish from now on. At the time being, even though the problem of obtaining analytic DF of stationary axisymmetric systems is still open, we know much about them (though, not so much): we know, for example, that their DFs can be written in terms of at least two integrals of motion (energy and axial component of the angular momentum), and how this generally translates in certain properties of the resulting density profiles, streaming motion fields and higher-order velocity moments; in other words, we at least know what to expect, qualitatively, from a DF that depends on the phase-space coordinates via such integrals of motion. The starting idea of this work is to see whether we can learn something more than this, in the special case of systems being *almost* spherical. The reader may question the motivation for adopting such a specific framework, considering (among other things) that real elliptical galaxies often exhibit shapes far from "almost spherical". A first point supporting our approach is that in this way we really start exploring what is beyond the spherical limit, and in doing this we can rely on several works about almost spherical systems, e.g. how to write the potential Φ, or how to re-arrange the JEs (Mancino et al. 2024). Secondly, what at first sight seems to be a very specific configuration turns out to be unexpectedly powerful for a larger variety of cases: from spherical harmonics expansion of the gravitational field, Φ always results being *rounder* than the density generating it; this means that an almost spherical potential can be associated to stellar systems that are more flattened. Quantitatively, it can be verified that an E1-like potential can be self-consistent with an E3-E4 density structure.

As already mentioned, stationary axisymmetric systems can be *two-integral*, in the sense that f can depend on the phase-space coordinates via the (relative) energy $\mathcal{E}$ and the axial component of the angular momentum J_z, so $f(\mathbf{x}, \mathbf{v}) = f(\mathcal{E}, J_z)$. Later on, it will be useful (actually, it will be necessary) to split f into its even and odd parts with respect to the argument J_z, because (as will be shown in Chapter 2 and Chapter 5) they determine completely, in turn, the density structure and the streaming motions of the system. These parts will be indicated as f_+ (*even*) and f_- (*odd*). We will first investigate f_+, with the same procedure being adaptable to f_-.

In addition to the consequences of the resulting f_+, it will be of great interest to lead another investigation regarding f_-. By writing the JEs in cylindrical coordinates

(R, φ, z), a degeneracy between ordered (streaming) and random motions along the azimuthal direction appears, which can possibly be broken by treating streaming motions as a "free parameter" of the problem. More precisely, a good phenomenological treatment of ordered motions is the *Satoh decomposition* (Satoh 1980), which expresses this velocity field as a combination of second-order moments of f in the velocity space. This, together with a wise use of the JEs for homeoidal-like density-potential pairs, can lead to a final expression for the odd part f_{-}.

CHAPTER 2

General Principles

The purpose of this Chapter is to get familiar with general properties of the DF and how these propagate to some of its velocity moments. By combining the Jeans theorem and the Noether theorem, the assumed symmetries of the system translate in the DF depending on the phase-space coordinates via known integrals of motion (IoM). In each of the two Sections of this Chapter, we exploit these theorems and write general results for the velocity moments of the DF. The two Sections differ in the assumed geometry for the density profile and are intended to go deeper into the particular cases of a spherically symmetric $\rho(r)$ (Section 2.1) and of an axisymmetric $\rho(R, z)$ (Section 2.2). It will be shown how in spherical symmetry the DF can be easily recovered with the Eddington formula, independently of the assumed internal orbital distribution. This formula will be crucial in the treatment of a weakly flattened, two-integral axisymmetric, stellar system.

Many of the derivations reported here refer to Ciotti 2021 (Chapter 12) and Binney and Tremaine 2008.

In collisionless stationary stellar systems the density profile ρ is the 0-th order moment of the DF f in the velocity space

$$\rho(\mathbf{x}) = \int_{\mathbb{R}^3} d^3\mathbf{v}\ f(\mathbf{x},\mathbf{v}) \tag{2.1}$$

Moreover, for these systems the Jeans theorem establishes that f depends on the phase-space coordinates only through the regular Integrals of Motion (IoM) of the total gravitational potential Φ_T. IoM can be found thanks to physical-mathematical arguments: if we are able to identify some relevant symmetries of the system, the Noether theorem associates each symmetry to an IoM. In particular, very familiar physical quantities (like energy, angular momentum or some of its components) are some of those IoM linked to symmetries. Finally, the *strong* Jeans theorem states that, in order to completely characterize the DF, a number of at most three IoM is sufficient. In brief, symmetries are able to unveil some (if not all) dependencies of f on the phase-space coordinates.

For technical reasons, at least for systems of finite total mass (this is our case, as discussed in Chapter 3), or when Φ_T can be set equal to zero at infinity, it can be convenient to work with relative energies and potentials: the relative potential is defined as $\Psi \equiv -\Phi$, while the relative energy per unit mass as $\mathcal{E} \equiv -E = \Psi_T - \frac{\|\mathbf{v}\|^2}{2}$. Thus, the total relative potential defines a family of isopotential surfaces enclosing more and more volume of space as the value of Ψ_T decreases. From now on, when not specified, we will refer equivalently to "energy" or "relative energy" and to "potential" or "relative potential", with this not being confusing since the difference is only due to a minus sign.

It is useful to dive into the consequences, on the velocity moments, of the different dependence scenarios, when the symmetries of the system are given.

2.1 The spherical case

Let's consider a stationary spherical system. This means that $\Phi_T = \Phi_T(r)$ and the system is symmetric under traslations in time, so $\partial\Phi_T/\partial t = 0$. The Noether theorem associates stationarity to the energy conservation along each single orbit, meaning that E (energy per unit mass) is an IoM; moreover, if the system is spherically symmetric in all its physical properties, then another possible IoM is the norm of the specific angular momentum J, again along each single orbit.

Let's first focus on the case in which the energy is enough to characterize the DF, so $f = f(E)$. In spherical coordinates (r, θ, φ), the definitions (1.3)-(1.5) for the velocity moments of f give the following results:

$$\overline{v_r}(r) = \frac{1}{\rho(r)} \int dv_\theta \int dv_\varphi \int_{-\infty}^{+\infty} dv_r \; v_r \; f(\frac{1}{2}(v_r^2 + v_\theta^2 + v_\varphi^2) + \Phi_T(r)) = 0\,, \tag{2.2}$$

$$\overline{v_\theta}(r) = \overline{v_\varphi}(r) = \overline{v_r}(r) = 0\,, \tag{2.3}$$

$$\overline{v_r v_\theta}(r) = \frac{1}{\rho(r)} \int dv_\varphi \int dv_\theta \; v_\theta \int_{-\infty}^{+\infty} dv_r \; v_r \; f(\frac{1}{2}(v_r^2 + v_\theta^2 + v_\varphi^2) + \Phi_T(r)) = 0\,, \tag{2.4}$$

$$\overline{v_r v_\varphi}(r) = \overline{v_\theta v_\varphi}(r) = \overline{v_r v_\theta}(r) = 0\,, \tag{2.5}$$

$$\overline{v_r^2}(r) = \frac{1}{\rho(r)} \int dv_\varphi \int dv_\theta \int_{-\infty}^{+\infty} dv_r \; v_r^2 \; f(\frac{1}{2}(v_r^2 + v_\theta^2 + v_\varphi^2) + \Phi_T(r)) \neq 0\,, \tag{2.6}$$

$$\overline{v_\theta^2}(r) \neq 0 \quad , \quad \overline{v_\varphi^2}(r) \neq 0\,, \tag{2.7}$$

$$\sigma_{ij}^2(r) = \overline{v_i v_j}(r) - \overline{v_i}(r)\,\overline{v_j}(r) = \begin{cases} 0 & i \neq j \\ \left.\begin{array}{l} \sigma_r^2(r) = \overline{v_r^2}(r) \\ \sigma_\theta^2(r) = \overline{v_\theta^2}(r) \\ \sigma_\varphi^2(r) = \overline{v_\varphi^2}(r) \end{array}\right\} & i = j \end{cases} . \tag{2.8}$$

The results (2.2)-(2.5) are zero because they are integrating an odd function over an interval centered on the symmetry point (zero in these cases). Result (2.3) tells that the system is not expanding nor contracting ($\overline{v_r} = 0$) and that net rotation is absent either in φ-direction or in θ-direction ($\overline{v_\varphi} = \overline{v_\theta} = 0$); single stars can have non-null motions in all directions around a point $\mathbf{x}$, but the *net* (or *streaming*) motions are all zero. From (2.8) we see that the velocity dispersion tensor is diagonalized in the spherical coordinates. Moreover, it's possible to notice how the non-mixed, second order, moments (2.6)-(2.7) are all equal, because f depends in the same way on each of the three velocity components $(v_r, v_\theta, v_\varphi)$ and the integrals involving each of them are always performed in the interval $[-\infty, +\infty]$, so that mathematically there is no difference in using one component instead of another. This implies

$$\overline{v_r^2}(r) = \overline{v_\theta^2}(r) = \overline{v_\varphi^2}(r)\,, \tag{2.9}$$

and from (2.8) we get $\sigma_r^2(r) = \sigma_\theta^2(r) = \sigma_\varphi^2(r) \equiv \sigma^2(r)$, allowing the velocity dispersion tensor to be completely determined by just one function $\sigma(r)$ and to write its corresponding matrix (at a fixed r) as

$$\sigma_{ij}^2 = \begin{pmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{pmatrix} . \tag{2.10}$$

The DF $f = f(E)$ is called *ergodic* and the consequent velocity dispersion tensor is *isotropic*, meaning that its properties do not vary from a radial direction to another in the velocity space, starting from its origin. There is a very convenient way of visualizing σ^2_{ij} that makes use of the *velocity ellipsoid* (VE), which at any point $\mathbf{x}$ is defined as the ellipsoid centered in $\mathbf{x}$ having the diagonalizing coordinate axes as principal (geometrical) axes and the associated components σ_{11}, σ_{22} and σ_{33} as semi-axis lengths. Some VEs of a spherical system with ergodic DF are represented in Figure 2.1.

Now, let's see what happens if, besides the orbital specific energy E, the DF depends also on J, so $f = f(E, J)$. As for the ergodic case, by using the definitions (1.3)-(1.5) in spherical coordinates (r, θ, φ) we obtain:

$$\overline{v_r}(r) = \frac{1}{\rho(r)} \int dv_\varphi \int dv_\theta \int_{-\infty}^{+\infty} dv_r \; v_r \; f(\frac{1}{2}(v_r^2 + v_\theta^2 + v_\varphi^2) + \Phi_T(r)\,,\, r\left|\sqrt{v_\theta^2 + v_\varphi^2}\right|) = 0\,, \tag{2.11}$$

$$\overline{v_\theta}(r) = \overline{v_\varphi}(r) = \overline{v_r}(r) = 0\,, \tag{2.12}$$

$$\overline{v_r v_\theta}(r) = \overline{v_r v_\varphi}(r) = \overline{v_\varphi v_\theta}(r) = 0\,, \tag{2.13}$$

$$\overline{v_r^2}(r) \neq 0 \quad , \quad \overline{v_\theta^2}(r) = \overline{v_\varphi^2}(r) \neq 0\,, \tag{2.14}$$

$$\sigma^2_{ij}(r) = \overline{v_i v_j}(r) - \overline{v_i}(r)\overline{v_j}(r) = \begin{cases} 0 & i \neq j \\ \left.\begin{array}{l} \sigma_r^2(r) = \overline{v_r^2}(r) \\ \sigma_\theta^2(r) = \sigma_\varphi^2(r) = \overline{v_\theta^2}(r) \end{array}\right\} & i = j \end{cases} . \tag{2.15}$$

Again, there are no net rotations nor net radial motions in the system. At any point $\mathbf{x}$, the velocity dispersion tensor σ^2_{ij} is still diagonalized in the local spherical coordinates, but now it depends on two numbers, σ_r and σ_θ, which are not necessarily equal:

$$\sigma^2_{ij} = \begin{pmatrix} \sigma_r^2 & 0 & 0 \\ 0 & \sigma_\theta^2 & 0 \\ 0 & 0 & \sigma_\theta^2 \end{pmatrix} . \tag{2.16}$$

There can be *anisotropy* between the radial direction and the tangential plane. This can be quantified by the *anisotropy parameter*

$$\beta(r) \equiv 1 - \frac{\sigma_t^2(r)}{2\sigma_r^2(r)} = 1 - \frac{\sigma_\theta^2(r) + \sigma_\varphi^2(r)}{2\sigma_r^2(r)}\,, \tag{2.17}$$

which can have the extremal values of $-\infty$ (all orbits are locally circular), zero (orbits are locally isotropic) or one (all orbits are locally radial). When $\beta > 0$ orbits are *radially biased*, or *radially anisotropic*, because the radial velocity dispersion is higher than the tangential one; they represent an intermediate way between isotropic and purely radial orbits. When $\beta < 0$ orbits are *tangentially biased*, or *tangentially anisotropic*, and they are in an intermediate way between being circular and being isotropic. All the kinds of isotropy/anisotropy for a spherical system are represented in Figure 2.2.

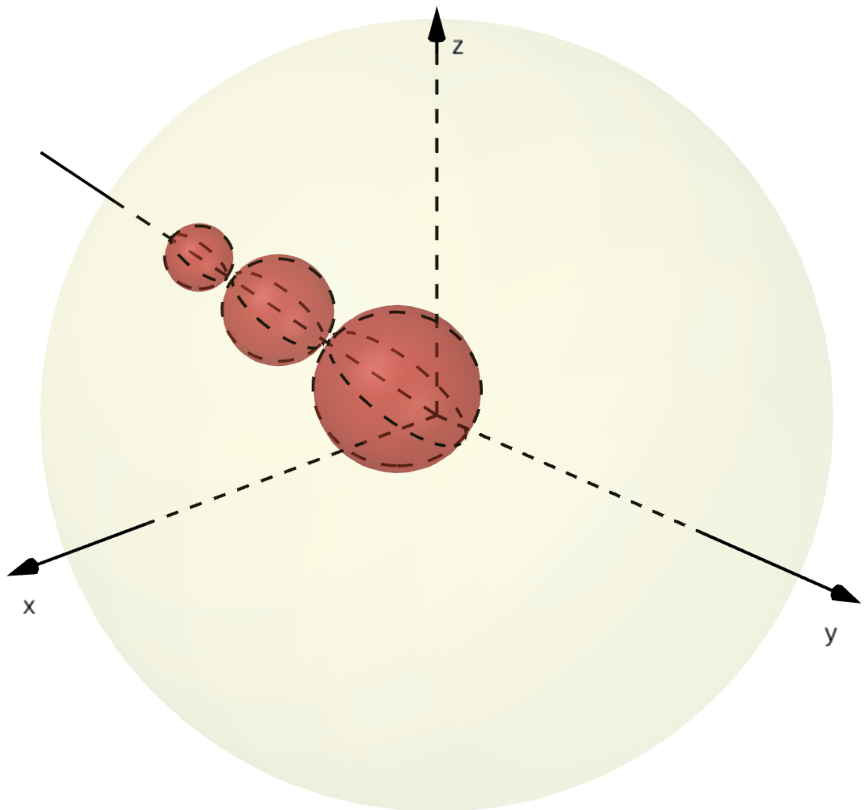
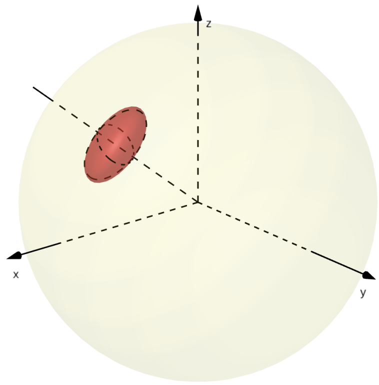
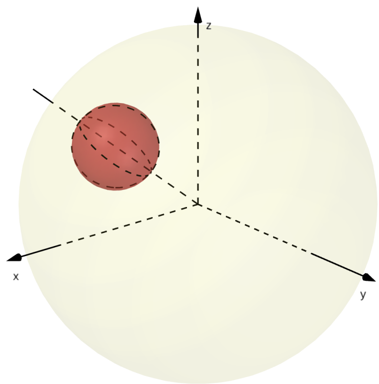


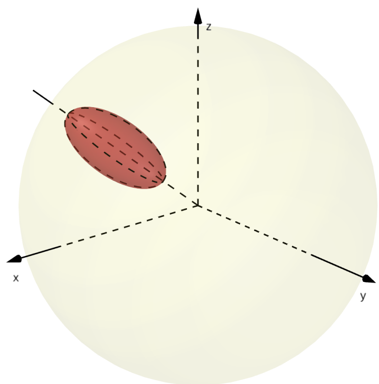
Figure 2.1: Visualization of the VEs (red) in a spherical system (yellow) having an ergodic DF. Each VE is centered in a point in the configuration space and represents the velocity dispersion at that point, giving an idea of how orbits are distributed in its neighborhood. The black line indicates a radial direction; along this, three VEs are considered, at different distances from the center of the system. Because the velocity dispersion tensor is everywhere isotropic, all VEs are spheres. Usually, we expect the velocity dispersion to be higher where the gravitational potential is stronger, so in this example the size of a VE decreases from the center to the outer regions.



(a) $-\infty < \beta < 0$



(b) $\beta = 0$



(c) $0 < \beta < 1$

Figure 2.2: Visualization of all the possible types of VEs (red) of a spherical system (yellow) having a $f = f(E, J)$. At any point $\mathbf{x}$ having distance r from the center, the value of $\beta(r)$ suggests the shape of the VE centered at $\mathbf{x}$. In all cases, the VE at $\mathbf{x}$ is aligned with the local spherical coordinates axes $(\hat{\mathbf{e}}_r(\mathbf{x}), \hat{\mathbf{e}}_\theta(\mathbf{x}), \hat{\mathbf{e}}_\varphi(\mathbf{x}))$. **(a)**: The anisotropy parameter is negative, meaning that the velocity dispersion is higher in the tangential plane (θ, φ); since $\sigma_\theta^2 = \sigma_\varphi^2 > \sigma_r^2$, the VE at $\mathbf{x}$ is oblate. **(b)**: $\beta = 0$, so $\sigma_r^2 = \sigma_\theta^2 = \sigma_\varphi^2$; the VE at $\mathbf{x}$ is a sphere. **(c)**: β is positive, so $\sigma_r^2 > \sigma_\theta^2 = \sigma_\varphi^2$; the VE at $\mathbf{x}$ is prolate with major axis along the radial direction.

As we will now see, for spherically symmetric systems it is always possible to obtain an expression for the DF, based on the assumed density profile. This makes spherical systems the most valid objects in order to get direct information on the DF. In showing this, we are first going to re-write the relation (2.1) between $\rho(r)$ and f, in both ergodic and anisotropic cases. Then, by using the general Abel inversion technique (see Eq. (A.4)), it will be possible to invert this relation and express the DF as an integral function involving the density.

Let's first consider a spherical system with only isotropic orbits. The DF is ergodic and depends only on the energy, so $f = f(\mathcal{E})$. The consistency of our stellar system (which has only one stellar component, with stars all of the same mass) can be expressed as a "truncation" leaving only non-negative relative energies:

$$f = h(\mathcal{E})\theta(\mathcal{E}) \geq 0\,, \tag{2.18}$$

where $h(\mathcal{E})$ is some regular function and $\theta(\mathcal{E})$ is the Heaviside step function in Eq. (A.3). The relation (2.1) between ρ and f becomes

$$\rho(r) = \int_0^\infty dv\ 4\pi v^2 h(\mathcal{E})\theta(\mathcal{E})\,. \tag{2.19}$$

Given $\mathcal{E} = \Psi_T - \frac{v^2}{2}$, it is easy to change the integration variable from v to $\mathcal{E}$, obtaining

$$\rho(r) = \int_{-\infty}^{\Psi_T} d\mathcal{E}\ 4\pi\sqrt{2(\Psi_T - \mathcal{E})}\ h(\mathcal{E})\theta(\mathcal{E})\,; \tag{2.20}$$

the Heaviside step function is null for negative $\mathcal{E}$, so we are left only with

$$\rho(r) = 4\pi \int_0^{\Psi_T} d\mathcal{E}\ \sqrt{2(\Psi_T - \mathcal{E})}\ h(\mathcal{E})\,. \tag{2.21}$$

An important note: for spherical systems, $\Psi(r)$ is always a monotonic function of radius r; this implies that ρ can be expressed equivalently in r or in Ψ. This turns out to be useful, because now we can see ρ as an integral function of Ψ_T and we re-arrange (2.21) in

$$\frac{1}{\sqrt{8}\pi}\rho(\Psi_T) = 2\int_0^{\Psi_T} d\mathcal{E}\ \sqrt{\Psi_T - \mathcal{E}}\ h(\mathcal{E})\,. \tag{2.22}$$

By deriving this equation with respect to Ψ_T, we get this *Abel integral equation*:

$$\frac{1}{\sqrt{8}\pi}\frac{d\rho(\Psi_T)}{d\Psi_T} = \int_0^{\Psi_T} d\mathcal{E}\ \frac{h(\mathcal{E})}{\sqrt{\Psi_T - \mathcal{E}}}\,, \tag{2.23}$$

which is suitable for the *Abel inversion technique* enunciated in Section A.2 of the Appendix. The inversion of (2.23) is the *Eddington formula* for the DF of a spherical isotropic system

$$h(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2}\frac{d}{d\mathcal{E}}\int_0^{\mathcal{E}} d\Psi_T\ \frac{d\rho}{d\Psi_T}\ \frac{1}{\sqrt{\mathcal{E} - \Psi_T}}\,. \tag{2.24}$$

We can now deal with a spherical anisotropic system, the DF of which must depend on two IoM, as shown previously in this Section. Now $f = f(\mathcal{E}, J)$ and we let our

VEs have whatever ellipsoidal shape; in particular, the anisotropy parameter $\beta = \beta(r)$ can vary as a function of radius r. Because we are introducing the norm J of the angular momentum, now the integration of (2.1) over the velocity space cannot simply be performed along concentric spherical shells as we did in the ergodic case (we were allowed to do this because of isotropy in the velocity space). In fact, by using spherical coordinates (r, θ, φ) for the configuration space and $(v_r, v_\theta, v_\varphi)$ for the velocity space, we have that $J = r\ v_T = r\sqrt{v_\theta^2 + v_\varphi^2}$. The DF now depends, in the second argument, only on the tangential component of velocity v_T, with this being re-written as $v_T = r\ v\ sin\,\beta$ once we decompose a velocity vector as in Figure 2.3. In other words, f is not anymore spherically symmetric in the velocity space because now it depends, besides v, on the colatitude β.

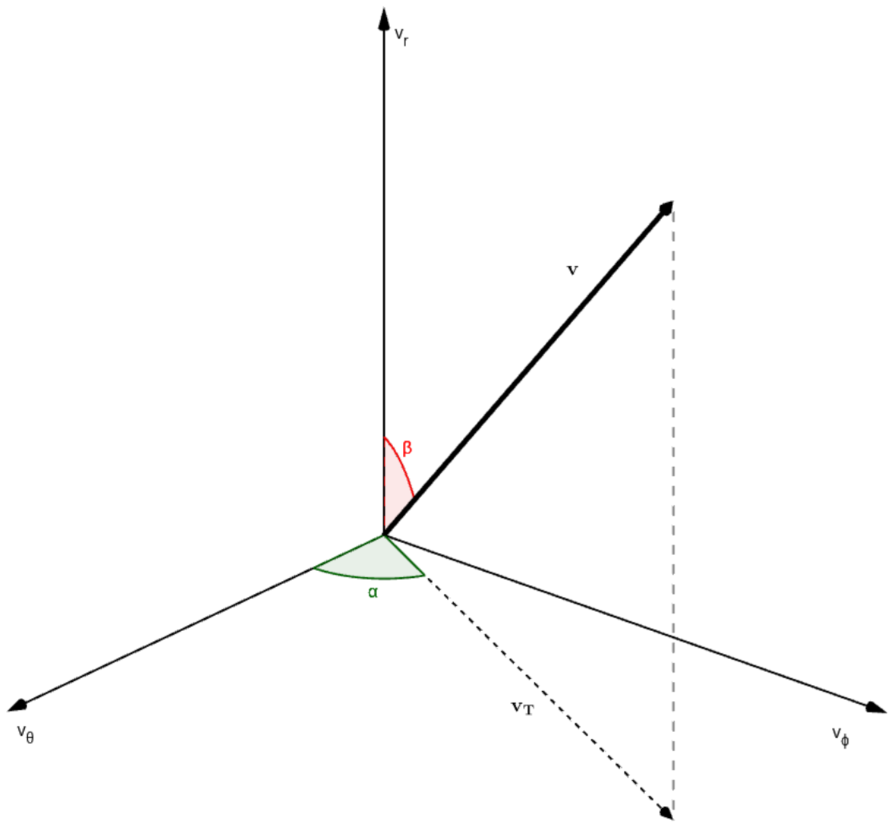


Figure 2.3: Decomposition of the velocity vector $\mathbf{v}$ adopted in the integration of the DF for spherically symmetric systems with two integrals of motion. The colatitude β is measured from $\mathbf{v_r}$, while the longitude α from $\mathbf{v}_\theta$ in the tangential plane $(\mathbf{v}_\theta, \mathbf{v}_\varphi)$.

With such decomposition, the volume element is $d^3\mathbf{v} = v^2 dv\ sin\,\beta\ d\beta\ d\alpha$ (for reasons analogous to the ones discussed in detail in Section 2.2), and the relation (2.1) between ρ and f becomes

$$\rho(r) = \int_0^{2\pi} d\alpha \int_0^{\pi} d\beta\ sin\,\beta \int_0^{\infty} dv\ v^2\, h(\Psi_T - \frac{v^2}{2}\,,\, r\,v\,sin\,\beta)\,. \qquad (2.25)$$

In order to make some progress with (2.25) we are forced to loose generality, by doing some further assumption on the orbital properties of the system and/or on the form of the DF. As long as possible, we are interested more in not touching the DF, so the only illustrative example of spherical anisotropic system we are introducing here is the one known as the Osipkov-Merritt system (Merritt 1985; Osipkov 1979), which can be recovered under the hypothesis that $f = f(\mathcal{E}, J) = f(Q)$, with

$$Q \equiv \mathcal{E} - \frac{J^2}{2r_a^2}\,, \tag{2.26}$$

$$f = h(Q)\theta(Q) \geq 0\,, \tag{2.27}$$

where r_a is the *anisotropy radius*; it represents the scale-radius inside which orbits are almost isotropic and beyond which orbits start to be significantly radially-biased. Actually, this is exactly what we are implicitly assuming, about the distribution of orbits, when we introduce the particular expression (2.26): indeed, the resulting anisotropy parameter $\beta(r)$ is

$$\beta(r) = \frac{r^2}{r^2 + r_a^2}\,. \tag{2.28}$$

This means that, in Osipkov-Merritt systems, the velocity dispersion tensor is isotropic at the center independently of the particular value of r_a. Moreover, when $r_a \to \infty$, $\beta(r) = 0 \quad \forall r \geq 0$; the system is globally isotropic, and we recover $Q = \mathcal{E}$. For r_a fixed, the velocity dispersion tensor becomes more and more radially anisotropic at $r \to \infty$, while inside r_a the velocity dispersion tensor becomes more and more isotropic. This is the reason why r_a is called the anisotropy radius. r_a is the only free parameter of the orbital configuration.

The Q variable makes computations possible:

$$\begin{aligned} Q &= \Psi_T - \frac{v^2}{2} - \frac{r^2 v^2 sin^2\beta}{2r_a^2} \\ &= \Psi_T - \frac{v^2}{2}\left(1 + \frac{r^2 sin^2\beta}{r_a^2}\right)\,; \end{aligned}$$

$$\begin{aligned} \rho(r) &= 2\pi \int_0^\pi d\beta\ sin\,\beta \int_0^\infty dv\ v^2\ h(Q) \\ &= 2\pi \int_0^\pi d\beta\ sin\,\beta \int_{-\infty}^{\Psi_T} dQ\ \frac{\sqrt{2(\Psi_T - Q)}}{\left(1 + \left(\frac{r}{r_a}\right)^2 sin^2\beta\right)^{\frac{3}{2}}}\ h(Q)\,; \end{aligned}$$

by performing the integration in β and considering $Q \geq 0$ from (2.27), we get

$$\left(1 + \left(\frac{r}{r_a}\right)^2\right)\rho(r) = 4\pi \int_0^{\Psi_T} dQ\ \sqrt{2(\Psi_T - Q)}\ h(Q)$$

$$\rho_Q(r) = 4\pi \int_0^{\Psi_T} dQ\ \sqrt{2(\Psi_T - Q)}\ h(Q)\,, \tag{2.29}$$

which is very similar to the ρ-f relation (2.21) valid for spherical isotropic systems, having now the variable Q taking the place of $\mathcal{E}$. For simplicity, we have defined

$$\rho_Q(r) \equiv \left(1 + \left(\frac{r}{r_a}\right)^2\right)\rho(r)\,. \tag{2.30}$$

Analogously to the isotropic case, by deriving (2.29) with respect to Ψ_T we get an Abel integral equation suitable for the Abel inversion, finally giving

$$h(Q) = \frac{1}{\sqrt{8}\pi^2}\frac{d}{dQ}\int_0^Q d\Psi_T\ \frac{d\rho_Q}{d\Psi_T}\ \frac{1}{\sqrt{Q-\Psi_T}}\,, \tag{2.31}$$

which is the Eddington formula for the DF of a spherical system with anisotropic orbits described by (2.28). The Osipkov-Merritt system comes out by a very specific combination of the two original arguments of f. If we considered another combination of $\mathcal{E}$ and J, in principle, a new anisotropic spherical system would have been obtained. The advantage of Osipkov-Merritt systems, besides their analytical ductility, is that their orbital configuration is predicted by some galaxy formation scenarios.

Summarizing, for any spherical system an expression for the DF can be obtained, regardless the distribution of orbits. Whenever the DF cannot be computed analytically, this can be done numerically. The Eddington formula (2.24) will turn out to be useful starting from Chapter 4 on, where we'll exploit the homeoidal expansions to obtain the formulae for some "pieces" of the DF in weakly flattened, two-integral axisymmetric, stellar systems.

2.2 The two-integral axisymmetric case

The system we are going to consider for the rest of this work is stationary, which means that Φ_T is independent of time ($\partial\Phi_T/\partial t = 0$), so a first integral of motion we can identify is the specific energy E (i.e., E is conserved along the orbit of each star). Geometrically, we let our system be axisymmetric, so $\Phi_T = \Phi_T(R, z)$ and this introduces another integral of motion, J_z (i.e., also the axial component J_z of the angular momentum is conserved along the orbit of each star). In general, axisymmetric systems can have a DF which depends on a third IoM I_3, but, unless we know how to write I_3, we can't say anything a priori about the consequent properties of the velocity moments of $f(E, J_z, I_3)$. The only case in which we are guaranteed to obtain analytical results is when $f = f(E, J_z)$, so it is interesting to see what properties its velocity moments will have. By using cylindrical coordinates (R, φ, z), definitions (1.3)-(1.5) give:

$$\overline{v_R}(R,z) = \frac{1}{\rho(R,z)} \int dv_z \int dv_\varphi \int_{-\infty}^{+\infty} dv_R \; v_R \; f(\frac{1}{2}(v_R^2 + v_z^2 + v_\varphi^2) + \Phi_T(R,z)\,,\; Rv_\varphi) = 0\,, \tag{2.32}$$

$$\overline{v_z}(R,z) = 0\,, \tag{2.33}$$

$$\overline{v_\varphi}(R,z) = \frac{1}{\rho(R,z)} \int dv_z \int dv_R \int_{-\infty}^{+\infty} dv_\varphi \; v_\varphi \; f(\frac{1}{2}(v_R^2 + v_z^2 + v_\varphi^2) + \Phi_T(R,z)\,,\; Rv_\varphi)\,, \tag{2.34}$$

$$\overline{v_R v_z}(R,z) = \overline{v_z v_\varphi}(R,z) = \overline{v_R v_\varphi}(R,z) = 0\,, \tag{2.35}$$

$$\overline{v_R^2}(R,z) = \frac{1}{\rho(R,z)} \int dv_z \int dv_\varphi \int_{-\infty}^{+\infty} dv_R \; v_R^2 \; f(\frac{1}{2}(v_R^2 + v_z^2 + v_\varphi^2) + \Phi_T(R,z)\,,\; Rv_\varphi)\,, \tag{2.36}$$

$$\overline{v_z^2}(R,z) = \overline{v_R^2}(R,z)\,, \tag{2.37}$$

$$\sigma_{ij}^2(R,z) = \overline{v_i v_j}(R,z) - \overline{v_i}(R,z)\overline{v_j}(R,z) = \begin{cases} 0 & i \neq j \\ \left.\begin{array}{l} \sigma_R^2(R,z) = \overline{v_R^2}(R,z) \\ \sigma_\varphi^2(R,z) = \overline{v_\varphi^2}(R,z) - \overline{v_\varphi}^2(R,z) \\ \sigma_z^2(R,z) = \sigma_R^2(R,z) \end{array}\right\} & i = j \end{cases}\,. \tag{2.38}$$

The system is now allowed to rotate, around the z-axis, since $\overline{v_\varphi}$ can be non-zero; whether there is rotation or not depends on the parity of the DF with respect to its second argument, that is something we cannot know a priori. At any point $\mathbf{x}$, σ_{ij}^2 is diagonalized in the local cylindrical coordinates and the corresponding matrix is

$$\sigma_{ij}^2 = \begin{pmatrix} \sigma_R^2 & 0 & 0 \\ 0 & \sigma_\varphi^2 & 0 \\ 0 & 0 & \sigma_R^2 \end{pmatrix}\,. \tag{2.39}$$

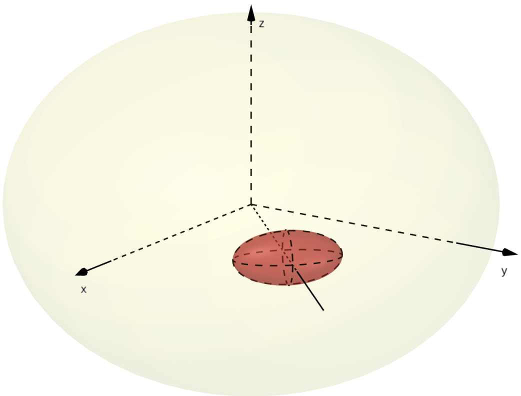
On the meridional plane (R, z) there is isotropy, and the velocity dispersion tensor can be anisotropic if the azimuthal dispersion is different than the meridional dispersion. We have already introduced the anisotropy parameter β for spherical systems in (2.17). Similarly, parameters quantifying anisotropy can be defined for axisymmetric systems;

in the most general case we can define two parameters

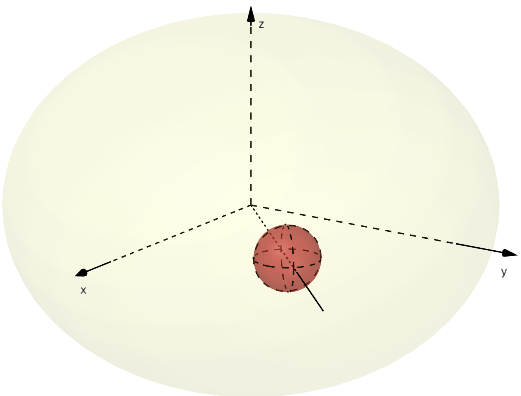
$$\beta_z(R,z) \equiv 1 - \frac{\sigma_z^2(R,z)}{\sigma_R^2(R,z)}\,, \tag{2.40}$$

$$\gamma(R,z) \equiv 1 - \frac{\sigma_\varphi^2(R,z)}{\sigma_R^2(R,z)}\,; \tag{2.41}$$

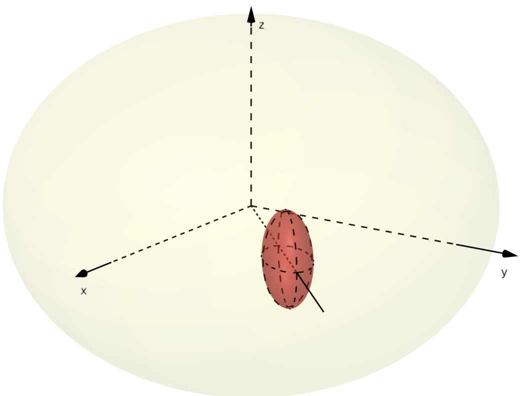
in our case, for the result (2.38), $\beta_z(R,z) = 0$. By giving β_z and γ, we fully characterize σ_{ij}^2 and so the size and shape of the VE, which is always aligned with the local cylindrical coordinates basis (Figure 2.4).



(a) $\beta_z = 0\,,\ -\infty < \gamma < 0$



(b) $\beta_z = 0\,,\ \gamma = 0$



(c) $\beta_z = 0\,,\ 0 < \gamma < 1$

Figure 2.4: Visualization of all the possible types of VEs (red) of an axisymmetric system (yellow) having a $f = f(E, J_z)$. At any point $\mathbf{x}$, the values of β_z and γ suggest the shape of the VE centered at $\mathbf{x}$. In all cases, the VE at $\mathbf{x}$ is aligned with the local cylindrical coordinates axes $(\hat{\mathbf{e}}_R(\mathbf{x}), \hat{\mathbf{e}}_\varphi(\mathbf{x}), \hat{\mathbf{e}}_z(\mathbf{x}))$. **(a)**: The azimuthal anisotropy parameter γ is negative while $\beta_z = 0$, meaning that the velocity dispersion is higher in the azimuthal direction and isotropic on the meridional plane; the VE at $\mathbf{x}$ is prolate, with major axis along the φ-direction. **(b)**: $\gamma = 0$ and $\beta_z = 0$, so $\sigma_R^2 = \sigma_z^2 = \sigma_\varphi^2$; the VE at $\mathbf{x}$ is a sphere. **(c)**: $\beta_z = 0$ while γ is positive, so $\sigma_R^2 = \sigma_z^2 > \sigma_\varphi^2$; the VE at $\mathbf{x}$ is oblate, with minor axis along the φ-direction.

Similarly to the spherical case, the consistency of the one-component axisymmetric stellar system writes

$$f = h(\mathcal{E}, J_z)\, \theta(\mathcal{E}) \geq 0 \tag{2.42}$$

where $h(\mathcal{E}, J_z)$ is some regular function and $\theta(\mathcal{E})$ is the Heaviside step function in Eq. (A.3). For the truncation above letting only $\mathcal{E} \geq 0$, the velocity sections satisfying this condition are $\Omega_{\mathbf{x}} = \left\{ \mathbf{v} :\ 0 \leq \|\mathbf{v}\| \leq \sqrt{2\Psi_T(\mathbf{x})} \right\}$, spheres of radius $\sqrt{2\Psi_T(\mathbf{x})}$. This gives useful geometric information for rewriting the integral (2.1) in a way which is possible to handle, that is going to rearrange the relation between ρ, the total potential Ψ_T and the DF. The natural coordinates in configuration space are the cylindrical ones (R, φ, z) and the associated velocity components are (v_R, v_φ, v_z). As a direct consequence of the geometry of $\Omega_{\mathbf{x}}$, the natural coordinates for integration in velocity space are the spherical ones; considering the colatitude $\lambda \in [0, \pi]$, the longitude $\mu \in [0, 2\pi]$ and the radial coordinate $v = \|\mathbf{v}\| \in \left[0, \sqrt{2\Psi_T}\right]$ (see Figure 2.5), the velocity components write as

$$v_R = v\ sin\, \lambda\ sin\, \mu\,, \quad v_z = v\ sin\, \lambda\ cos\, \mu\,, \quad v_\varphi = v\ cos\, \lambda \tag{2.43}$$

In this way $v^2 = v_R^2 + v_z^2 + v_\varphi^2 = v_m^2 + v_\varphi^2$, with the meridional velocity $v_m = v\ sin\, \lambda$; since $\mathcal{E} = \Psi_T - \frac{v^2}{2}$ and $J_z = R\, v_\varphi = R\, v\, cos\, \lambda$, the DF $f(\mathcal{E}, J_z)$ is independent of the longitude μ. Moreover, the volume element $d^3\mathbf{v}$ in spherical coordinates is $d^3\mathbf{v} = v^2\, sin\, \lambda\, d\lambda\, d\mu\, dv$ (see Figure 2.6). With all this in mind, the integral (2.1) becomes

$$\rho(R, \Psi_T) = \int_0^{2\pi} d\mu \int_0^{\pi} \int_0^{\sqrt{2\Psi_T}} d\lambda\, dv\ v^2 sin\lambda\ h(\Psi_T - \frac{v^2}{2}, R\, v\, cos\lambda) \tag{2.44}$$

Since we are working in a fixed (arbitrary) position $\mathbf{x}$, in this integral the potential $\Psi_T(\mathbf{x})$ is a fixed value with respect to the integration variables (λ, v); let the integration in $d\lambda$ be the innermost one. After integration in $d\mu$, we are left with

$$\rho(R, \Psi_T) = 2\pi \int_0^{\sqrt{2\Psi_T}} dv\ v^2 \int_0^{\pi} d\lambda\, sin\lambda\ h(\Psi_T - \frac{v^2}{2}, R\, v\, cos\lambda) \tag{2.45}$$

Focusing just on the λ-integration, the parity of h becomes of crucial importance in the outcome of the whole integration. We can notice this in a more clear way by using the change of variable $x \equiv cos\lambda$, suggested by the second argument of h and by the $d\lambda\, sin\lambda$ appearing in the same integral. Now $dx = -sin\lambda\, d\lambda$ and the integral (2.45) becomes

$$\rho(R, \Psi_T) = 2\pi \int_0^{\sqrt{2\Psi_T}} dv\ v^2 \int_{-1}^{1} dx\ h(\Psi_T - \frac{v^2}{2}, R\, v\, x) \tag{2.46}$$

We indicate as $h_\pm(\mathcal{E}, J_z)$ the even (odd) component of the DF with respect to the second argument (J_z):

$$h_\pm(\mathcal{E}, J_z) = \frac{h(\mathcal{E}, J_z)\ \pm\ h(\mathcal{E}, -J_z)}{2} \tag{2.47}$$

A direct look at the x-integral in (2.46) makes it evident that only the even part of the DF gives a non-zero contribution to the density. Reducing the integration over x as

$\int_{-1}^{1} dx\ h_+ = 2\int_0^1 dx\ h_+$ and going back from x to λ variable, we have

$$\rho(R,\Psi_T) = 4\pi \int_0^{\sqrt{2\Psi_T}} dv\ v^2 \int_0^{\frac{\pi}{2}} d\lambda\ sin\lambda\ h_+(\Psi_T - \frac{v^2}{2}, R\,v\,cos\lambda) \tag{2.48}$$

Finally, going back to the $(\mathcal{E}, J_z)$ variables allows to read the relation (2.48) between density, cylindrical radius, gravitational potential and DF in a more direct and clean way, leading to the fundamental reference relation to work on in the next Chapters. This last change of variables comes simply by the definitions of $\mathcal{E} = \Psi_T - \frac{v^2}{2}$ and $J_z = R\,v\,cos\lambda$, giving respectively (considering the order of integration: v is fixed while integrating over λ)

$$v = \sqrt{2(\Psi_T - \mathcal{E})} \qquad , \qquad dv = -\frac{d\mathcal{E}}{\sqrt{2(\Psi_T - \mathcal{E})}} \tag{2.49}$$

$$cos\lambda = \frac{J_z}{R\,v} \qquad , \qquad sin\lambda\, d\lambda = -\frac{dJ_z}{R\,\sqrt{2(\Psi_T - \mathcal{E})}} \tag{2.50}$$

with

$$v \in \left[0, \sqrt{2\Psi_T}\right] \to\ \mathcal{E} \in [0, \Psi_T] \tag{2.51}$$

$$\lambda \in \left[0, \frac{\pi}{2}\right] \to\ J_z \in \left[0, R\,\sqrt{2(\Psi_T - \mathcal{E})}\right] \tag{2.52}$$

Putting all this together in (2.48) leads to the remarkable iterated integral relation

$$\rho(R,\Psi_T) = \frac{4\pi}{R}\int_0^{\Psi_T} d\mathcal{E} \int_0^{R\sqrt{2(\Psi_T-\mathcal{E})}} dJ_z\ h_+(\mathcal{E}, J_z)\,. \tag{2.53}$$

Similar computations lead to another fundamental relation:

$$\rho\overline{v_\varphi}(R,\Psi_T) = \frac{4\pi}{R^2}\int_0^{\Psi_T} d\mathcal{E} \int_0^{R\sqrt{2(\Psi_T-\mathcal{E})}} dJ_z\ h_-(\mathcal{E}, J_z)\,J_z\,. \tag{2.54}$$

These are the ground base on which we will construct the entire investigation of a possible formula for the DF of a weakly flattened stellar system (Chapters 4 and 5).

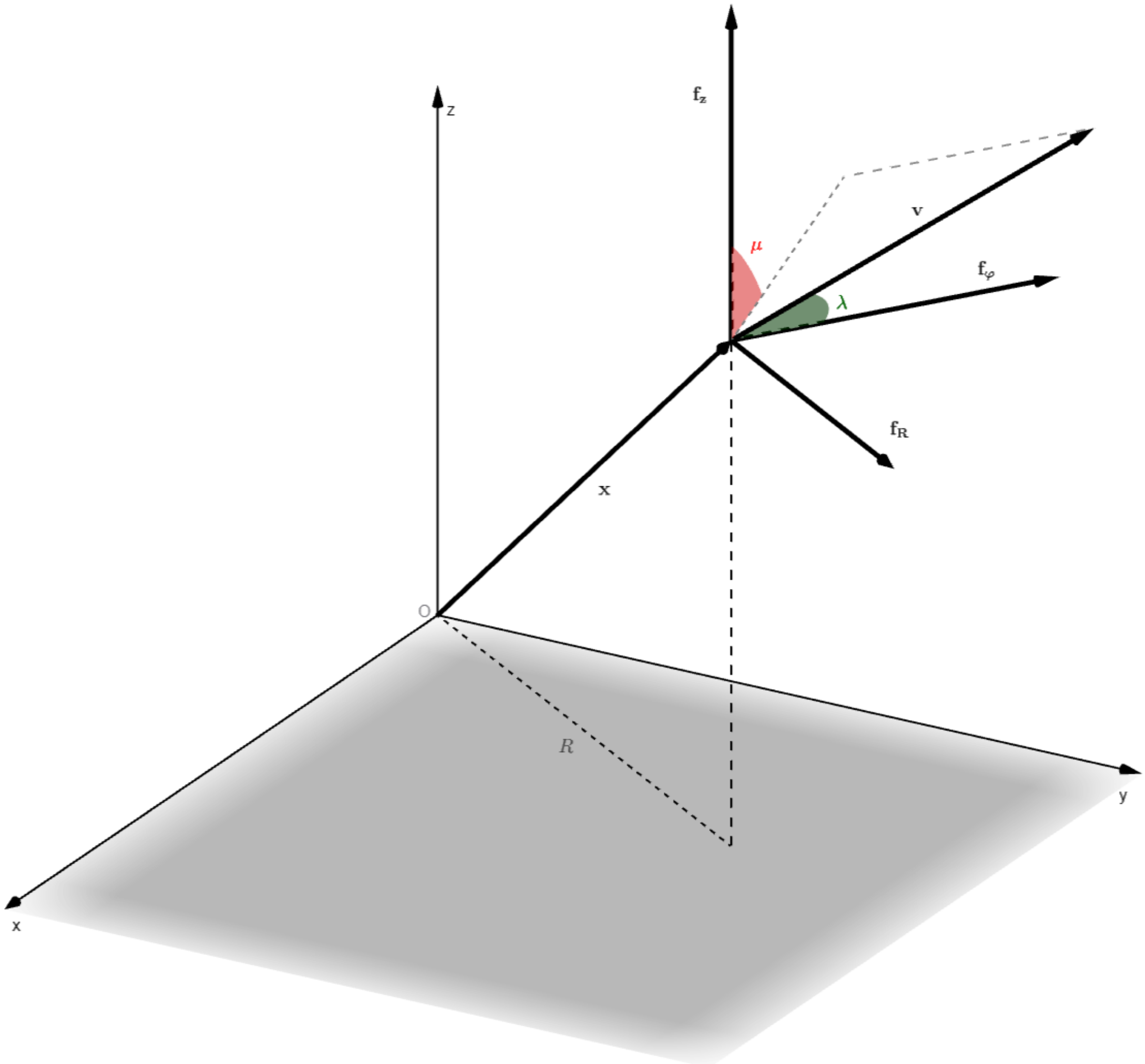


Figure 2.5: Decomposition of the velocity vector $\mathbf{v}$ adopted in the integration of the DF for cylindrically symmetric systems with two integrals of motion (Section 2.2). Due to the consistency condition and the consequent geometry of the velocity sections $\Omega_{\mathbf{x}}$, the natural coordinates are the spherical ones. The colatitude λ is measured from $\mathbf{f}_\varphi$, while the longitude μ from $\mathbf{f_z} = \mathbf{e_z}$ in the plane $(\mathbf{f_z}, \mathbf{f_R})$.

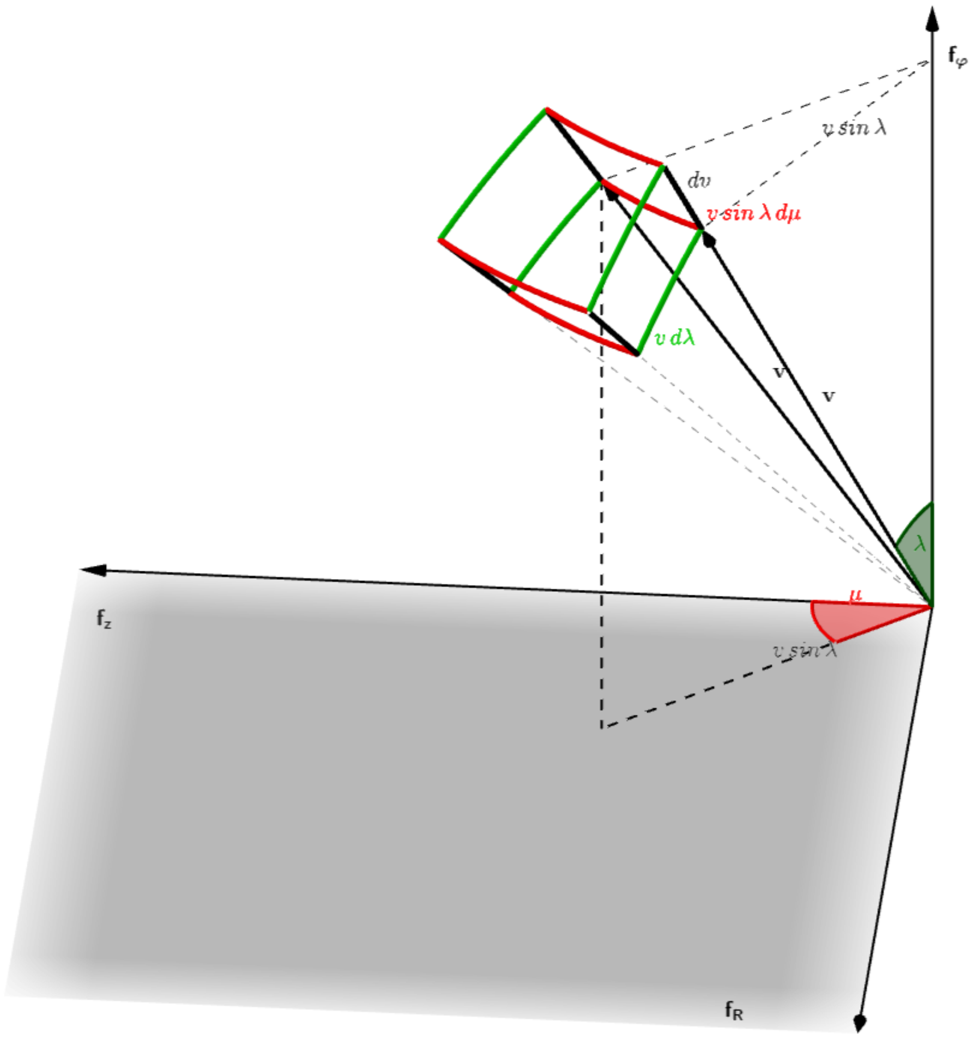


Figure 2.6: Volume element in the velocity space. When the colatitude, longitude and v variations are infinitesimal, the measure of the volume element is $d^3\mathbf{v} = v\,d\lambda \times v\,sin\,\lambda\,d\mu \times dv = v^2\,sin\,\lambda\,d\lambda\,d\mu\,dv$. In this image, all these variations of the spherical coordinates are purposely enhanced for visual clarity. The grey plane is the analogous, in the velocity space, of the meridional plane (R, z); here lies the meridional velocity $v_m = v\,sin\,\lambda$.

At this point, we know how the arguments of f lead to the properties of its velocity-moments, that constitute the density profile and the internal orbital distribution of a stellar system. This is known as the *f-to-ρ approach*. One difficulty with the f-to-ρ approach is the lack of control on the resulting dynamical quantities and their profiles: it is always possible to assume a very specific expression for f but we are not guaranteed that the consequent velocity-moments turn out to agree with observed properties of real galaxies, even though these results are certainly physically acceptable equilibrium models (not necessarily stable), once f is well-defined (meaning that it is positive definite over the whole phase space). All the information learned from the f-to-ρ approach can possibly be useful in solving the opposite problem, that is to obtain an exact form of the DF of a system given its density profile, gravitational potential and some velocity moments of f. The so called *ρ-to-f approach* is the problem of determining the DF starting from a *dynamical model*, that is given as

$$\textit{dynamical model} = \left(\rho\,, \quad \begin{matrix} \textit{internal} \\ \textit{orbital} \\ \textit{distribution} \end{matrix} \right).$$

This method is based on the Jeans Equations (JEs) and is very useful for its ability to relate observationally accessible quantities (e.g. streaming velocity and velocity dispersion fields) to other fundamental but not directly accessible quantities (e.g. the total potential and the total density distribution in a galaxy). The logical steps of the ρ-to$f$ approach can be summarized as follows:

(1) Guided by theoretical/numerical/observational indications, a specific form for the density ρ (or for the projected density Σ) of the component of interest (e.g. a particular stellar population, a dark matter halo etc.) is assumed.

(2) For the density component of interest, the JEs are solved, either analytically or numerically. Solutions are then projected in the "plane of the sky". The closure of the JEs is in general obtained by imposing physically motivated ansatz, such as with the aim of reproducing the features of observed (projected) velocity fields.

(3) If possible, the underlying DF is recovered and its positivity tested. In case of somewhere negative f, the DF cannot be accepted and the model obtained so far must be discarded, even if it is successful at reproducing observations.

The last step (the recovery of the DF) is the least trivial of the ρ-to-f approach, and is the topic of this entire work. Such problem is in general a technically difficult *inverse problem*, and even when it is doable, unicity and positivity of the recovered DF are not guaranteed. In the few cases in which the DF can be obtained analytically, this is generally expressed in integral form. In case of spherically symmetric systems, as seen in Section 2.1, the integral form of the DF is quite easy to obtain, and the knowledge of the density profile is sufficient to completely determine f (analytically or numerically). For axisymmetric systems, instead, the recovery of the DF is not as affordable as in spherical systems. To date, the methods aimed to recover the DF of axisymmetric systems rely on classical integral transforms (Laplace transforms, Stieltjes transforms, Laplace-Mellin transforms) and/or on analytic continuations, in the complex plane, of quantities such as $\rho = \rho(R, \Psi_T)$; others are based on the theorem of residues for functions of complex variables. All of these tools are strictly inherited form *complex* analysis (something which did not happen for spherical systems), and are not meant to be directly applied to observational data. The purpose of this work is to explore the possibility of a new method based on mathematical arguments which do not involve complex analysis. The logic of this new possible method is shown in Chapter 4; for now, we only anticipate that a priority for this new approach is to investigate what general properties of f can be learned by perturbing spherical symmetry with a small flattening giving a slightly axisymmetric system, in the hope of obtaining inversion formulae similar to the well known ones of spherical systems.

CHAPTER 3

Homeoidal expansions

The general two-integral axisymmetric case shows several technical difficulties. An example of this is in the fundamental relations (2.53)-(2.54), that have the completely unknown even and odd parts of the DF as integrands and the axisymmetric potential $\Psi_T(R, z)$ as a limit of integration. Just the problem of determining $\Psi_T(R, z)$ given $\rho(R, z)$ is not trivial from a technical point of view. In order to avoid additional difficulties, we can exploit the existence of special expressions for the self-consistent pair (ρ, Ψ_T) in case the axisymmetric system is weakly flattened. These useful formulae are known as *homeoidal expansions*. The advantage of using them are multiple: not only they allow to overcome technical difficulties, but we also have under control the flattening parameter η, which appears in a linear and explicit form. The essential goal of this thesis is to understand the effects of the flattening, and this Chapter provides the basic tools needed to achieve it.

3.1 The gravitational field of ellipsoids: general results

An ellipsoid is the geometric locus of points in space obeying the equation

$$\sum_{i=1}^{3} \frac{x_i^2}{a_i^2} = m^2 , \qquad a_1 \geq a_2 \geq a_3 \tag{3.1}$$

with m being the linear size of the ellipsoid and a_1, a_2 and a_3 being the major, intermediate and minor semi-axes of the ellipsoid. We will often consider $x_1 = x$, $x_2 = y$, $x_3 = z$. Ellipsoids with different values of m but same values of (a_1, a_2, a_3) are said to be similar. The region of space $m_1 \leq \mathcal{S}(\mathbf{x}) \leq m_2$ between two concentric, coaxial, similar ellipsoids having linear sizes m_1 and m_2 (Figure 3.1) is called *homeoid* (of finite thickness), and can be described as

$$\mathcal{S}^2(\mathbf{x}) = \sum_{i=1}^{3} \frac{x_i^2}{a_i^2} \tag{3.2}$$

Considering the ellipsoidal surface $m = 1$ and adding a parameter τ to each denominator of (3.1) we obtain the following equation for (unitary) *confocal quadrics* (Figures 3.2 - 3.4 and Figure 3.5)

$$\sum_{i=1}^{3} \frac{x_i^2}{a_i^2 + \tau} = 1 \tag{3.3}$$

The gravitational potential in a point $\mathbf{x}$ generated by an ellipsoidal density distribution $\rho(m)$ is given by

$$\Phi(\mathbf{x}) = -G\,\pi\, a_1 a_2 a_3 \int_0^\infty d\tau\, \frac{\Delta\Psi(\mathbf{x},\tau)}{\sqrt{\Delta(\tau)}} \tag{3.4}$$

also known as the *Chandrasekhar formula*, where

$$\Delta\Psi(\mathbf{x},\tau) \equiv 2 \int_{m(\mathbf{x},\tau)}^{\infty} \rho(t)\, t\, dt , \qquad m^2(\mathbf{x},\tau) \equiv \sum_{i=1}^{3} \frac{x_i^2}{a_i^2 + \tau} \tag{3.5}$$

$$\Delta(\tau) \equiv (a_1^2 + \tau)(a_2^2 + \tau)(a_3^2 + \tau) \tag{3.6}$$

The limit for small flattenings of the Chandrasekhar formula (3.4) leads to a set of formulae known as the *homeoidal expansions* for the density distribution $\rho(m)$ and the related potential $\Phi = -\Psi$.

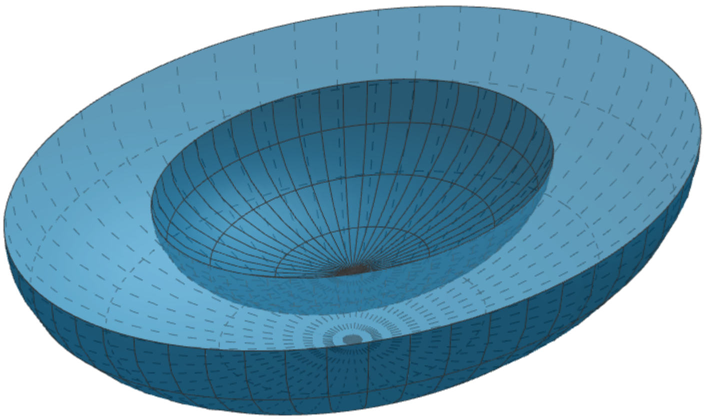


Figure 3.1: Finite-thickness homeoid, the region of space between two concentric, co-axial, similar ellipsoids. The homeoid in this picture has been represented as half of its volume, for visual clarity.

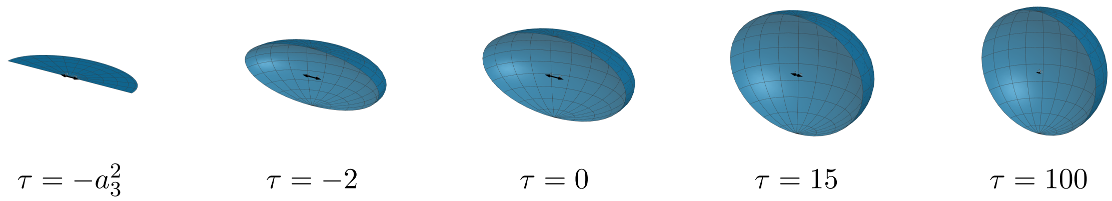


Figure 3.2: Confocal ellipsoids having same m, same $a_1 = 4$, $a_2 = 3$, $a_3 = 2$ but different values of $\tau \geq -a_3^2$. The black double-arrow runs between the two foci. As the black double-arrow changes in size with respect to the size of each ellipsoid, it can be visualized how the ellipsoids become bigger and rounder by increasing τ.

Figure 3.3: Confocal elliptic one-sheeted hyperboloids having same m, same $a_1 = 4$, $a_2 = 3$, $a_3 = 2$ but different values of $-a_2^2 \leq \tau \leq -a_3^2$.

Figure 3.4: Confocal elliptic two-sheeted hyperboloids having same m, same $a_1 = 4$, $a_2 = 3$, $a_3 = 2$ but different values of $-a_1^2 \leq \tau \leq -a_2^2$.

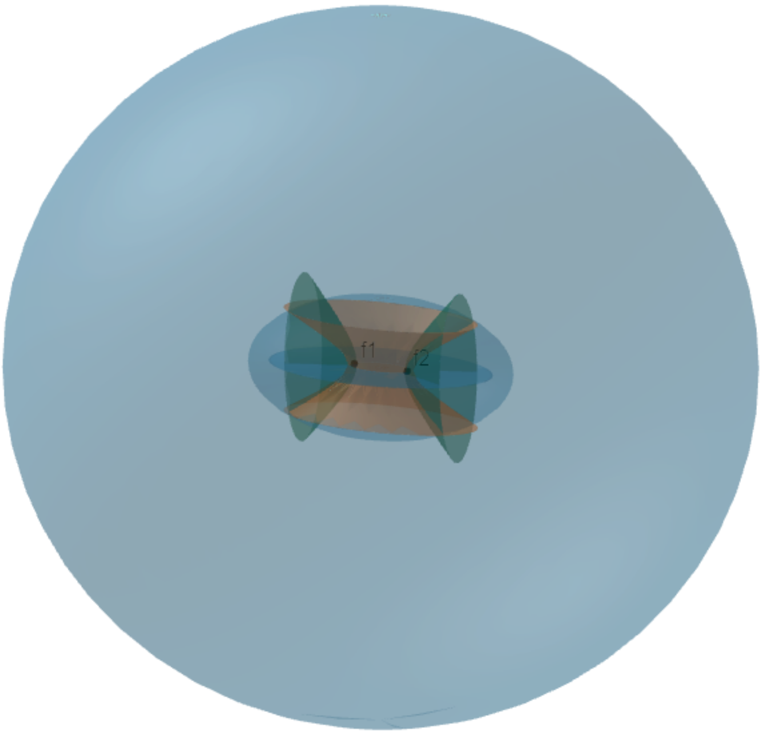


Figure 3.5: All the three families of confocal quadrics (ellipsoids in blue, elliptic one-sheeted hyperboloids in orange and elliptic two-sheeted hyperboloids in green) represented together in the same frame. The two foci f_1 and f_2 are also shown. All the quadrics have same m, same $a_1 = 4$, $a_2 = 3$, $a_3 = 2$ but different values of $\tau \geq -a_1^2$.

3.2 Homeoidal expansions for oblate ellipsoidal systems

Our stellar system is ellipsoidal oblate, with an infinitesimal flattening in the minor axis, which in an ellipsoid of the form

$$m^2 = \frac{x^2}{a_1^2} + \frac{y^2}{a_1^2} + \frac{z^2}{a_3^2} \tag{3.7}$$

coincides with the z-axis. However, the method leading to homeoidal expansions (*homeoidal expansion technique*) does not consider at the outset an ellipsoidal object, but a spherically symmetric density distribution, written as

$$\rho(r) = \rho_0 \tilde{\rho}(s) \ , \qquad s \equiv \frac{r}{a_1} \tag{3.8}$$

where ρ_0 is a density normalization constant, $\tilde{\rho}(s)$ is the dimensionless density profile, and a_1 is a length scale (for now, completely generic and not forced to be related to the major semiaxis a_1 of (3.7)). Associated to the density profile (3.8) is a mass profile

$$M(r) = 4\pi \int_0^r l^2 \rho(l)\, dl \tag{3.9}$$

We impose a finite and constant total mass M_{tot} to our stellar system. This choice comes from a mathematical and physical argument: in case of infinite total mass, the integration extrema in the integral (2.53) would be problematic (Ψ_T and $\mathcal{E}$ would be able to assume all possible values without any limitations, etc.), together with the fact that we could not freely assume $\mathbf{x_0} = \infty$, $\Psi_T(\infty) = 0$ anymore (as was already assumed also in (3.4)). By using $l = a_1 s$ and (3.8) in (3.9), M_{tot} writes

$$M_{tot} = 4\pi a_1^3 \rho_0 \int_0^\infty \tilde{\rho}(s) s^2 ds \tag{3.10}$$

Without loss of generality, we set

$$\int_0^\infty \tilde{\rho}(s) s^2 ds = 1 \quad \Longrightarrow \quad \rho_0 = \frac{M_{tot}}{4\pi a_1^3} \tag{3.11}$$

We now transform the spherical density (3.8) into an *oblate* ellipsoidal density of the same total mass with the substitution

$$s \mapsto m = \sqrt{\frac{x^2}{a_1^2} + \frac{y^2}{a_1^2} + \frac{z^2}{a_3^2}} = \sqrt{\frac{R^2}{a_1^2} + \frac{z^2}{q_z^2 a_1^2}} \ , \qquad \rho_0 \mapsto \rho_n \tag{3.12}$$

where $a_1 \geq a_3$ and where we have implicitly introduced the ellipticity (or flattening) η as

$$\frac{a_3}{a_1} \equiv q_z = 1 - \eta \ , \qquad 0 \leq \eta \leq 1 \ ; \tag{3.13}$$

in the spherical limit $q_z = 1$ and $\eta = 0$. There are many ways to apply some flattening along the z-direction while conserving the total mass M of the system: for example, we could reduce q_z while increasing the major semiaxis a_1 so that the density in the center of the system ($\rho(0)$) is conserved, or so that the average value $\langle\rho\rangle$ is conserved; or we could keep the major semiaxis a_1 constant while reducing the minor semiaxis a_3 only, or even expand a_1 while fixing a_3, etc.. In the last two examples, the values of $\rho(0)$ and $\langle\rho\rangle$ are forced to change due to the conservation of the total mass. Fortunately, this cannot in any way affect qualitatively the investigation about the DF: how the density profile re-arranges in the flattening can, if anything, change some coefficients, but not how the DF will come out to depend on its arguments ($\mathcal{E}, J_z$). We choose to keep the equatorial size of the system fixed ($a_1 = const$) while reducing a_3; in this way we accept ending up with larger values of ρ in each point internal to the ellipsoidal surface. From now on, the length scale in (3.8) is chosen to be equal to the major semiaxis a_1 of the ellipsoid. Now, the density is stratified over concentric, coaxial, similar ellipsoids and is expressed as

$$\rho(\mathbf{x}) = \rho_n \tilde{\rho}(m) \tag{3.14}$$

Again, we want the total mass to be conserved in the transformation from a spherical to an ellipsoidal system, independently of the value of η; this means that

$$M_{tot} = \begin{cases} 4\pi \displaystyle\int_0^\infty r^2 \rho(r)\, dr & spherical \\ \\ \displaystyle\int \rho(\mathbf{x})\, dV & ellipsoidal \end{cases} \tag{3.15}$$

The volume element dV in (oblate) ellipsoidal geometry can be given either by computing the determinant of the jacobian of the transormation of coordinates from ellipsoidal to cartesian ones, or by a simple geometric analogy argument. The starting spherical system can be seen as an ellipsoid like (3.1) having all the semiaxes of the same value a_1; in the following, the mathematical equation of a sphere of radius $r = a_1 m$, the spherical-to-cartesian transformation of coordinates and the consequent volume element expression are listed

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_1^2} + \frac{z^2}{a_1^2} = m^2 \quad \Longleftrightarrow \quad x^2 + y^2 + z^2 = (a_1 m)^2 = r^2 \tag{3.16}$$

$$\begin{cases} x = r\, sin\theta\, cos\varphi \\ \\ y = r\, sin\theta\, sin\varphi \\ \\ z = r\, cos\theta \end{cases} \tag{3.17}$$

$$dV = dx\, dy\, dz = r^2 sin\theta\, dr\, d\theta\, d\varphi \tag{3.18}$$

where the colatitude $\theta \in [0, \pi]$ is measured from the positive z-axis and the longitude $\varphi \in [0, 2\pi]$ from the positive x-axis.

The oblate ellipsoidal analogous of (3.16)-(3.18) are

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_1^2} + \frac{z^2}{q_z^2 a_1^2} = m^2 \quad \Longleftrightarrow \quad x^2 + y^2 + \frac{z^2}{q_z^2} = (a_1 m)^2 = r^2 \tag{3.19}$$

$$\begin{cases} x = r\, sin\theta\, cos\varphi \\ y = r\, sin\theta\, sin\varphi \\ z = q_z\, r\, cos\theta \end{cases} \tag{3.20}$$

$$dV = dx\, dy\, dz = q_z\, r^2 sin\theta\, dr\, d\theta\, d\varphi \tag{3.21}$$

and it can be noticed that the determination of dV in the oblate ellipsoidal case is immediate once we realize that the only cartesian differential to be modified after the system is flattened is $dz_{ell} = q_z\, dz_{sph}$, so $dV_{ell} = q_z\, dV_{sph}$. There is an analogy between spherical and oblate ellipsoidal geometries, in which the latter can be seen as a "modified" spherical geometry depending on a factor q_z; this allows to treat any integration stratified over similar oblates as we do for the integration over spherical shells. Therefore, (3.15) writes

$$M_{tot} = \begin{cases} 4\pi \int_0^\infty r^2 \rho(r)\, dr & spherical \\ q_z\, 4\pi \int_0^\infty a_1^2 m^2 \rho(m)\, a_1\, dm & ellipsoidal \end{cases} \tag{3.22}$$

By equating the two expressions and remembering that now the scale factor in (3.8) is the major semiaxis a_1, we have

$$4\pi a_1^3\, \rho_0 \int_0^\infty s^2 \tilde{\rho}(s)\, ds = q_z\, 4\pi a_1^3\, \rho_n \int_0^\infty m^2 \tilde{\rho}(m)\, dm \tag{3.23}$$

The normalization of the dimensionless density profile $\tilde{\rho}$ must be conserved independently of the flattening, so the two integral terms above are algebraically equal. We are left with

$$\rho_n = \frac{\rho_0}{q_z} = \frac{\rho_0}{(1-\eta)} \tag{3.24}$$

We now have everything to re-write the new oblate ellipsoidal density distribution

$$\rho(\mathbf{x}) = \frac{\rho_0}{(1-\eta)}\, \tilde{\rho}(m) \tag{3.25}$$

Before starting the series expansion of (3.25) for small flattening η, notice that in addition to the term $(1-\eta)^{-1}$ also the term $\tilde{\rho}(m)$ depends on η:

$$m^2 = \frac{R^2}{a_1^2} + \frac{z^2}{q_z^2 a_1^2} = \tilde{R}^2 + \tilde{z}^2\, (1-\eta)^{-2} \quad \Longrightarrow \quad m = \sqrt{\tilde{R}^2 + \tilde{z}^2\, (1-\eta)^{-2}} \tag{3.26}$$

so $m = m(\eta)$ and consequently $\tilde{\rho}(m) = \tilde{\rho}(m(\eta))$. We have used the " ˜ " notation for dimensionless coordinates $\tilde{R} \equiv \frac{R}{a_1}$ and $\tilde{z} \equiv \frac{z}{a_1}$; in case of future possible need, same holds for $\tilde{x} \equiv \frac{x}{a_1}$, $\tilde{y} \equiv \frac{y}{a_1}$ and $\tilde{\mathbf{x}} \equiv (\tilde{x}, \tilde{y}, \tilde{z}) = (\tilde{R}, \varphi, \tilde{z})$. A basic Taylor expansion of (3.25) for the flattening $\eta \to 0$ leads directly to the homeoidal expansion of the density profile. At the first order in η:

$$\frac{\tilde{\rho}(m)}{1-\eta} \sim (1+\eta)\tilde{\rho}(s) + \eta \tilde{z}^2 \frac{1}{s} \frac{d\tilde{\rho}(s)}{ds} \tag{3.27}$$

For the density at the right-hand side to be physically acceptable, the flattening η must satisfy some criterion in order to ensure the positivity of the expansion, which is threatened by the general negativity of $\frac{d\tilde{\rho}(s)}{ds}$ valid for any reasonable stellar system. Let's impose the positivity of the expansion (3.27)

$$(1+\eta)\,\tilde{\rho}(s) + \eta\tilde{z}^2\frac{1}{s}\frac{d\tilde{\rho}(s)}{ds} \geq 0$$

$$(1+\eta)\tilde{\rho}(s) + \eta\tilde{z}^2\frac{1}{s}\left(-\left|\frac{d\tilde{\rho}(s)}{ds}\right|\right) \geq 0$$

$$(1+\eta) - \eta\frac{s^2 cos^2\theta}{s}\left|\frac{d\tilde{\rho}(s)}{ds}\right|\frac{1}{\tilde{\rho}(s)} \geq 0$$

$$(1+\eta) - \eta cos^2\theta\left|\frac{dln\tilde{\rho}(s)}{dlns}\right| \geq 0$$

$$\frac{\eta cos^2\theta}{(1+\eta)}\left|\frac{dln\tilde{\rho}(s)}{dlns}\right| \leq 1$$

$$\sup_{\theta\in[0,\pi]}\left(\frac{\eta cos^2\theta}{(1+\eta)}\right)\sup_{s\geq 0}\left(\left|\frac{dln\tilde{\rho}(s)}{dlns}\right|\right) \leq 1$$

$$\frac{\eta}{(1+\eta)}A_M \leq 1$$

$$(A_M - 1)\,\eta \leq 1 \tag{3.28}$$

where we have used $\tilde{z}^2 = s^2cos^2\theta$, $\tilde{\rho}(s) \geq 0\ \forall s$, and defined $A_M \equiv \sup_{s\geq 0}\left(\left|\frac{dln\tilde{\rho}(s)}{dlns}\right|\right)$.

Two remarks are in order: first, the term A_M is known once we know $\tilde{\rho}(s)$; second, the homeoidal expansion for the density at the first order in η is made of a spherical term plus a non-spherical one, due to the multiplication of a spherically symmetric object by $\tilde{z}^2$.

With the dimensionless coordinates, also other previously introduced quantities can be easily normalized or can be simply written in terms of them:

$$m^2(\mathbf{x},\tau) = \frac{R^2}{a_1^2+\tau} + \frac{z^2}{(1-\eta)^2a_1^2+\tau} \;\rightarrow\; m^2(\tilde{\mathbf{x}},\tilde{\tau}) = \frac{\tilde{R}^2}{1+\tilde{\tau}} + \frac{\tilde{z}^2}{(1-\eta)^2+\tilde{\tau}}\ , \tag{3.29}$$

$$\widetilde{\Delta\Psi}(m(\tilde{\mathbf{x}},\tilde{\tau})) \equiv \frac{\Delta\Psi(m(\tilde{\mathbf{x}},\tilde{\tau}))}{\rho_n} = 2\int_{m(\tilde{\mathbf{x}},\tilde{\tau})}^{\infty}\tilde{\rho}(t)\,t\,dt\ , \tag{3.30}$$

$$\tilde{\Delta}(\tilde{\tau}) \equiv \frac{\Delta(\tau)}{a_1^6} = (1+\tilde{\tau})^2((1-\eta)^2+\tilde{\tau})\ , \tag{3.31}$$

with $\tilde{\tau} \equiv \frac{\tau}{a_1^2}$ and where we have put (3.25) in (3.5). We normalize also the relative potential as $\Psi(\mathbf{x}) = \Psi_n\tilde{\Psi}(\mathbf{x})$ and determine its normalization Ψ_n by using the Chandrasekhar formula (3.4) for an oblate ellipsoid with dimensionless quantities

$$\Psi_n \tilde{\Psi}(\mathbf{x}) = G\,\pi\, a_1^2 a_3 \int_0^\infty a_1^2\, d\tilde{\tau}\; \frac{\rho_n \widetilde{\Delta\Psi}(m(\mathbf{x},\tilde{\tau}))}{\sqrt{a_1^6 \tilde{\Delta}(\tilde{\tau})}}$$

$$\Psi_n \tilde{\Psi}(\mathbf{x}) = G\,\pi\,(1-\eta)\, a_1^5 \frac{\rho_0}{(1-\eta)} \int_0^\infty d\tilde{\tau}\; \frac{\widetilde{\Delta\Psi}(m(\mathbf{x},\tilde{\tau}))}{a_1^3\sqrt{\tilde{\Delta}(\tilde{\tau})}}$$

$$\Psi_n \tilde{\Psi}(\mathbf{x}) = G\,\pi\, a_1^2 \frac{M_{tot}}{4\pi a_1^3} \int_0^\infty d\tilde{\tau}\; \frac{\widetilde{\Delta\Psi}(m(\mathbf{x},\tilde{\tau}))}{\sqrt{\tilde{\Delta}(\tilde{\tau})}}$$

Let's set

$$\Psi_n = \frac{GM_{tot}}{a_1} \implies \tilde{\Psi}(\mathbf{x}) = \frac{1}{4}\int_0^\infty d\tilde{\tau}\; \frac{\widetilde{\Delta\Psi}(m(\mathbf{x},\tilde{\tau}))}{\sqrt{\tilde{\Delta}(\tilde{\tau})}}\ . \tag{3.32}$$

For $\eta \to 0$

$$\tilde{\Psi}(\mathbf{x}) \sim I_0(s) + \eta I_1(s) + \eta \tilde{z}^2 I_2(s) \tag{3.33}$$

where

$$\begin{cases} I_0(s) = \dfrac{1}{s}\displaystyle\int_0^s \tilde{\rho}(t) t^2 dt \ + \int_s^\infty \tilde{\rho}(t) t\, dt\ , \\ I_1(s) = \dfrac{1}{3s^3}\displaystyle\int_0^s \tilde{\rho}(t) t^4 dt \ + \dfrac{1}{3}\int_s^\infty \tilde{\rho}(t) t\, dt\ , \\ I_2(s) = -\dfrac{1}{s^5}\displaystyle\int_0^s \tilde{\rho}(t) t^4 dt\ . \end{cases} \tag{3.34}$$

The homeoidal expansions (3.27) and (3.33) describe *exact*, *nonspherical* density-potential pairs obeying the Poisson equation.

In Section 2.2, we have used the total potential Ψ_T because in general the DF depends on the integrals of motion in the *total* potential itself. The homeoidal expansions instead refer to a self-consistent density(ρ)-potential(Ψ) pair, which satisfies the Poisson equation. For now, our (isolated) stellar system has only one component which is of stellar nature (no externally imposed gravity from, e.g., dark matter components, external tidal fields, central black hole, etc.). So, only in this very simplified frame, $\Psi = \Psi_T$, and we omit possible external sources of gravity. For this reason, we are allowed to put both ρ and Ψ_T in the integral (2.53) using respectively (3.27) and (3.33). In precence of an externally imposed potential Ψ_{ext}, we would have $\Psi_T = \Psi + \Psi_{ext}$ with known Ψ_{ext}, therefore the homeoidal expansion for the total potential Ψ_T would be obtained by applying (3.33) to Ψ only and then summing this expansion to Ψ_{ext}.

It is possible to rewrite the homeoidal expansions (3.27) and (3.33) in terms of the other linear cylindrical coordinate $\tilde{R}$, instead of using $\tilde{z}$. This will turn out to be extremely useful in Chapter 4, where we start to ask some requirements to the DF by performing a "matching" with the terms of the homeoidal expansion for the density, which will take the place of $\rho(R, \Psi_T)$ at the left-hand side of the integral (2.53). The change of formalism simply relies on $\tilde{z}^2 = s^2 - \tilde{R}^2$, and gives the following homeoidal expansions:

$$\frac{\tilde{\rho}(m)}{(1-\eta)} \sim \tilde{\rho}_0(s) + \eta\, \tilde{\rho}_1(s) + \eta\, \tilde{R}^2 \tilde{\rho}_2(s)\ , \tag{3.35}$$

where

$$\begin{cases} \tilde{\rho}_0(s) \equiv \tilde{\rho}(s)\ , \\ \tilde{\rho}_1(s) \equiv \tilde{\rho}(s) + s\dfrac{d\tilde{\rho}(s)}{ds}\ , \\ \tilde{\rho}_2(s) \equiv -\dfrac{1}{s}\dfrac{d\tilde{\rho}(s)}{ds}\ ; \end{cases} \tag{3.36}$$

$$\tilde{\Psi}(\mathbf{x}) \sim \tilde{\Psi}_0(s) + \eta\, \tilde{\Psi}_1(s) + \eta\, \tilde{R}^2 \tilde{\Psi}_2(s)\ , \tag{3.37}$$

where

$$\begin{cases} \tilde{\Psi}_0(s) \equiv I_0(s) = \dfrac{1}{s}\displaystyle\int_0^s \tilde{\rho}(t)t^2 dt + \int_s^\infty \tilde{\rho}(t)t\, dt\ , \\ \tilde{\Psi}_1(s) \equiv I_1(s) + s^2 I_2(s) = -\dfrac{2}{3}\dfrac{1}{s^3}\displaystyle\int_0^s \tilde{\rho}(t)t^4 dt\ + \dfrac{1}{3}\int_s^\infty \tilde{\rho}(t)t\, dt\ , \\ \tilde{\Psi}_2(s) \equiv -I_2(s) = \dfrac{1}{s^5}\displaystyle\int_0^s \tilde{\rho}(t)t^4 dt\ . \end{cases} \tag{3.38}$$

From a conceptual point of view, we will assume to have an object for which the expressions (3.35) and (3.37) are known. Not only they describe what happens to density and gravitational potential when the flattening of the system is small, but they also describe a self-consistent density-potential pair independently of the value of η.

In the remaining of this thesis, we will work with the density-potential pair that, in full generality, we write as

$$\rho(R,z) = \rho_0(r) + \eta\rho_1(r) + \eta R^2 \rho_2(r)\,, \tag{3.39}$$

$$\Psi(R,z) = \Psi_0(r) + \eta\Psi_1(r) + \eta R^2 \Psi_2(r)\,. \tag{3.40}$$

The crucial question will be: upon inserting these expressions in the fundamental integrals (2.53)-(2.54) of Chapter 2, what form must h_+ and h_- take?

CHAPTER 4

General considerations about the inversion

Our purpose is to understand how h_+ and h_- are made, exploiting (2.53) and (2.54) which are *exact*. In addition, we are provided with the homeoidal expressions obtained in Chapter 3 as expansions for small flattening; however, if one computes the laplacian of $\Psi_{homeoidal}$ the result is nothing else than $\rho_{homeoidal}$ and the homeoidal expansions are by themselves an *exact* pair. In this Chapter we show that general information about the DF can be unveiled if one imposes that (2.53) and (2.54) are satisfied by the exact homeoidal-like density-potential pair. Besides the technical kernel of the mathematical problem, a geometrical study of the integration domain is performed, useful to understand how integrations behave with flattening regardless of the DF. Given the similarity of (2.53) and (2.54), without loss of generality we focus on the even part h_+ because, as apparent, once the problem is solved for the even part the treatment applies unchanged also to the odd part h_-.

4.1 Setting the stage

We are now in position to set the stage for our problem, that for clarity we repeat. Suppose we have a density-potential pair given by the two Eq.s (3.39)-(3.40), that we repeat here:

$$\rho(R,z) = \rho_0(r) + \eta\rho_1(r) + \eta R^2\rho_2(r)\,, \tag{4.1}$$

$$\Psi(R,z) = \Psi_0(r) + \eta\Psi_1(r) + \eta R^2\Psi_2(r)\,. \tag{4.2}$$

These two expressions are *exact* in that they satisfy rigorously the Poisson equation; moreover, the parameter η measures deviations from spherical symmetry. For small η, the pair also gives an accurate representation of ellipsoidally stratified systems weakly deviating from spherical shape. As well known, the JEs for this density-potential pair can be formally solved under the assumption of a supporting two-integral DF (Mancino et al. 2024). We also know from Chapter 2 that density and streaming velocity field are associated with the even and odd parts of the DF, respectively. This can be seen in Eq.s (2.53)-(2.54) that we repeat as well:

$$\rho(R,\Psi) = \frac{4\pi}{R}\int_0^{\Psi} d\mathcal{E}\int_0^{R\sqrt{2(\Psi-\mathcal{E})}} dJ_z\ h_+(\mathcal{E},J_z)\,, \tag{4.3}$$

$$\rho\overline{v_\varphi}(R,\Psi) = \frac{4\pi}{R^2}\int_0^{\Psi} d\mathcal{E}\int_0^{R\sqrt{2(\Psi-\mathcal{E})}} dJ_z\ h_-(\mathcal{E},J_z)\,J_z\,. \tag{4.4}$$

The focus of this thesis is to discover what we can say about h_+ and h_-. As the problem of inversion of the two integrals is formally the same, in the following Sections we just work on the even part of the DF.

We now explain why the problem, even in the limit $\eta \to 0$, is by far non-trivial.

$$\rho_0(r)+\eta\rho_1(r)+\eta R^2\rho_2(r) = \frac{4\pi}{R}\int_0^{\Psi_0(r)+\eta\Psi_1(r)+\eta R^2\Psi_2(r)} d\mathcal{E}\int_0^{R\sqrt{2[\Psi_0(r)+\eta\Psi_1(r)+\eta R^2\Psi_2(r)-\mathcal{E}]}} dJ_z\ h_+(\mathcal{E},J_z)\,. \tag{4.5}$$

This identity must be rigorously verified when performing the two-dimensional integration of the unknown function $h_+(\mathcal{E},J_z)$. So, important difficulties can be immediately spotted.

The first comes from the exact linear dependence of the density on η, that must be obtained respecting that *all non-linearities on the right-hand-side must cancel out exactly*; this is going to be the first result of this thesis, about how h_+ must be written so that all non-linearities vanish upon two-dimensional integration.

The second difficulty is that there must appear a dependence ηR^2 in some of the resulting terms.

Third point, all the resulting functional terms must be spherically symmetric (they must depend only on r)

Last but not least, when $\eta = 0$ everything must reduce to the spherical case, giving back an Eddington-like formula (Chapter 2).

It should be emphasized that the linear structure depicted above appears, at the right-hand side of (4.5), in the two upper limits of integration, and in one of them (the innermost one) in a non-linear way. This makes evident why the structure of h_+ function must be highly non-trivial.

At the moment, we don't know if and how the DF changes with the flattening η, but a dependence of h_+ on η is reasonable, given the amount of limiting conditions and the physical meaning of "distribution function" of a flattened system. It will be shown how this simple but almost necessary assumption makes the agreement with all the requirements possible. So, from now on, $h_+ = h_+(\mathcal{E}, J_z; \eta)$. In the next Sections we face the technical problem and find mathematical formulae allowing for the necessary non-linear cancellations. The already known inversion formulae of Sec 2.1 turn out to be useful and allow us to learn something general about the DF.

4.2 The $\eta = 0$ term: recovering the spherical limit

Theorem 1. *In the spherical limit $\eta = 0$, the function $h_+(\mathcal{E}, J_z)$ reduces to $h_+(\mathcal{E})$, i.e. the spherical limit is isotropic and the inversion formula for (4.3) reduces to the Eddington inversion. Mathematically, $\eta = 0$ cancels the dependence of h_+ on J_z.*

Proof. Taking $\eta = 0$ in (4.5), we obtain

$$\rho_0(r) = \frac{4\pi}{R} \int_0^{\Psi_0(r)} d\mathcal{E} \int_0^{R\sqrt{2[\Psi_0(r)-\mathcal{E}]}} dJ_z \; h_{+0}(\mathcal{E}, J_z)\,, \tag{4.6}$$

where $h_{+0}(\mathcal{E}, J_z) \equiv h_+(\mathcal{E}, J_z; \eta = 0)$.

Here, one of the necessary requirements of Section 4.1 already comes in play. The factor behind the integral contains a R^{-1} (a *cylindrical* radius), which must be canceled out by assuming some property of $h_{+0}(\mathcal{E}, J_z)$, in order to match the *spherical* symmetry of the term $\rho_0(r)$ appearing at the left-hand side. The most direct and simple way this can happen is by assuming that h_{+0} does not depend on J_z, so that R^{-1} behind the integral immediately cancels out with the integration in dJ_z, that leaves a $R\sqrt{2(\Psi_0 - \mathcal{E})}$.

If $\Psi_0(r)$ is a monotonic and invertible function, $r = r(\Psi_0)$ can be found and ρ_0 can be expressed as a function of Ψ_0 instead of r:

$$\rho_0\left[\Psi_0(r)\right] = 4\pi \int_0^{\Psi_0} d\mathcal{E} \; h_{+0}(\mathcal{E}) \sqrt{2(\Psi_0 - \mathcal{E})}$$

$$\frac{d\rho_0}{d\Psi_0}\left[\Psi_0(r)\right] = 4\pi \int_0^{\Psi_0} d\mathcal{E} \; \frac{h_{+0}(\mathcal{E})}{\sqrt{2(\Psi_0 - \mathcal{E})}} \qquad \textit{Abel}$$

$$\boxed{h_{+0}(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2} \frac{d}{d\mathcal{E}} \int_0^{\mathcal{E}} d\Psi_0 \; \frac{d\rho_0}{d\Psi_0} \frac{1}{\sqrt{\mathcal{E} - \Psi_0}}\,;} \tag{4.7}$$

here with "*Abel*" we indicate that after that passage an Abel inversion (see Eq. (A.4)) is performed. The **Theorem 1** is proved. □

4.3 Higher order η terms

Theorem 2. *The function $h_+(\mathcal{E}, J_z; \eta)$ cannot be linear in η, because otherwise, once integrated in (4.3), it would give non-linear terms in η. $h_+(\mathcal{E}, J_z; \eta)$ must contain all orders of η.*

Proof. By contradiction, suppose $h_+ = h_+(\mathcal{E}; \eta)$ (ergodic), and that $h_+(\mathcal{E}; \eta) = h_{+0}(\mathcal{E}) + \eta h_{+1}(\mathcal{E})$ (linear in η). Let us also consider $\rho = \rho_0(r) + \eta\rho_1(r)$ and $\Psi = \Psi_0(r) + \eta\Psi_1(r)$. Under these assumptions, (4.3) becomes:

$$\rho_0(r)+\eta\rho_1(r) = 4\pi \int_0^{\Psi} d\mathcal{E}\ h_{+0}(\mathcal{E})\sqrt{2\left(\Psi_0 + \eta\Psi_1 - \mathcal{E}\right)}+\eta 4\pi \int_0^{\Psi} d\mathcal{E}\ h_{+1}(\mathcal{E})\sqrt{2\left(\Psi_0 + \eta\Psi_1 - \mathcal{E}\right)}\,. \tag{4.8}$$

Whatever the arbitrary functions $h_{+0}(\mathcal{E})$ and $h_{+1}(\mathcal{E})$ may be, it is evident that the integral terms yield non-linear contributions in the parameter η. □

In order to gain further insights on this problem, we perform a similar analysis for a linearly-perturbed spherical model. In other words, in this Section we limit ourselves to consider only the spherically symmetric terms of the homeoidal-like expressions (4.1)-(4.2) and only isotropic orbits (meaning that DFs are ergodic); the reason for the isotropy condition has arisen in the previous Section and will be repeated at the end of this Section as well. Then, based on the completely *general* results obtained in this way, in Section 4.4 we perform the same method by inserting also the cylindrical terms, finally considering the full homeoidal-like expressions (4.1)-(4.2).

In the following passages, we introduce the spherical terms $\eta\rho_1(r)$ and $\eta\Psi_1(r)$, and see whether something new around the DF can be learnt. Again, results dealing with the DF are going to be highlighted in boxes.

$$\rho(r) = \rho_0(r) + \eta\,\rho_1(r), \qquad \Psi(r) = \Psi_0(r) + \eta\,\Psi_1(r), \qquad h_+ = h_+(\mathcal{E};\eta)$$

$$\rho\left[\Psi(r)\right] = 4\pi \int_0^{\Psi} d\mathcal{E}\ h_+(\mathcal{E};\eta)\,\sqrt{2(\Psi - \mathcal{E})}$$

$$\rho_0 + \eta\,\rho_1 = 4\pi \int_0^{\Psi_0+\eta\,\Psi_1} d\mathcal{E}\ h_+(\mathcal{E};\eta)\,\sqrt{2(\Psi_0 + \eta\,\Psi_1 - \mathcal{E})}$$

Both left-hand side and right-hand side are functions of η; the equation can be seen as

$$\mathcal{F}(\eta) = \mathcal{G}(\eta) \qquad \forall\eta$$

$$\mathcal{F}(0) = \mathcal{G}(0)$$

$$\rho_0 = 4\pi \int_0^{\Psi_0} d\mathcal{E}\ h_+(\mathcal{E};0)\,\sqrt{2(\Psi_0 - \mathcal{E})}\,, \tag{4.9}$$

which is identical to the case of Section 4.2, when we determined h_{+0}. So $h_+(\mathcal{E};0) = h_{+0}(\mathcal{E})$ and its expression is (4.7).

In case η is small, looking for the DF means looking for a possible "homeoidal" expansion for the DF; in this framework h_{+0} can be seen as the 0-order term of the following Taylor expansion for $\eta \to 0$

$$h_+(\mathcal{E};\eta) = \sum_{n=0}^{\infty} \eta^n\, h_{+n}(\mathcal{E}) \ , \quad h_{+n}(\mathcal{E}) \equiv \left. \frac{\partial^n h_+(\mathcal{E};\eta)}{n!\,\partial\eta^n}\right|_{\eta=0} . \tag{4.10}$$

The assumption of h_+ being a series like (4.10) comes from the necessity of including the case of a small flattening, but can be kept for non-small flattening as well, just forgetting the interpretation of a Taylor expansion and by considering the series as fully generic. The success of this representation of $h_+(\mathcal{E};\eta)$, as will become apparent below, will provide evidence for the validity of **Theorem 2**.

The integral equation becomes

$$\rho_0 + \eta\,\rho_1 = 4\pi \int_0^{\Psi_0+\eta\,\Psi_1} d\mathcal{E}\, \left(\sum_{n=0}^{\infty} \eta^n\, h_{+n}(\mathcal{E})\right) \sqrt{2(\Psi_0+\eta\,\Psi_1-\mathcal{E})}\,. \tag{4.11}$$

Again, it is possible to interpret this equation as the equality between two functions of η; the advantage of this consideration is that not only the two functions are equal regardless of η, but also their derivatives are always equal, regardless of the order of derivation:

$$\begin{aligned} \mathcal{F}(\eta) &= \mathcal{G}(\eta) && \forall\eta \\ \frac{d^i}{d\eta^i}\mathcal{F}(\eta) &= \frac{d^i}{d\eta^i}\mathcal{G}(\eta) && \forall\eta\,,\ \forall i \end{aligned}$$

For $i = 1$ (see (A.5) in the Appendix for the derivative in η of the right-hand side, which is an integral function depending on the parameter η in both a limit of integration and the integrand)

$$\rho_1 = 4\pi \int_0^{\Psi(\eta)} d\mathcal{E}\, \left[(h_{+1}(\mathcal{E}) + 2\eta\, h_{+2}(\mathcal{E}) + ...)\, \sqrt{2(\Psi(\eta)-\mathcal{E})} + \frac{\sum_{n=0}^{\infty} \eta^n\, h_{+n}(\mathcal{E})}{\sqrt{2(\Psi(\eta)-\mathcal{E})}} \Psi_1 \right]$$

and again considering both sides as functions of η only, they are equal also for $\eta = 0$

$$\rho_1 = 4\pi \int_0^{\Psi_0} d\mathcal{E}\, \left[h_{+1}(\mathcal{E})\, \sqrt{2(\Psi_0-\mathcal{E})} + \frac{h_{+0}(\mathcal{E})}{\sqrt{2(\Psi_0-\mathcal{E})}} \Psi_1 \right]$$

$$\rho_1 = 4\pi \int_0^{\Psi_0} d\mathcal{E}\ h_{+1}(\mathcal{E}) \sqrt{2(\Psi_0-\mathcal{E})}\ + 4\pi \Psi_1 F_1(\Psi_0)\ , \quad F_1(\Psi_0) \equiv \int_0^{\Psi_0} d\mathcal{E}\, \frac{h_{+0}(\mathcal{E})}{\sqrt{2(\Psi_0-\mathcal{E})}}$$

$$\rho_1(r) - 4\pi\Psi_1(r) F_1[\Psi_0(r)] = 4\pi \int_0^{\Psi_0} d\mathcal{E}\ h_{+1}(\mathcal{E}) \sqrt{2(\Psi_0-\mathcal{E})}\,.$$

We define the term (known once ρ is determined by observation/modelling)

$$H_1(r) \equiv \rho_1(r) - 4\pi\Psi_1(r)F_1[\Psi_0(r)] \; . \tag{4.12}$$

If the function $\Psi_0(r)$ is monotonic and we are able to invert it, finding $r = r(\Psi_0)$, we can change the independent variable from r to Ψ_0. Note that after this change of variable

$$H_1(\Psi_0) = \rho_1(\Psi_0) - 4\pi\Psi_1(\Psi_0)F_1(\Psi_0) \, . \tag{4.13}$$

By following Eq. (A.6), we go on with the computations

$$\begin{aligned}
H_1(\Psi_0) =& 4\pi \int_0^{\Psi_0} d\mathcal{E} \; h_{+1}(\mathcal{E})\sqrt{2(\Psi_0 - \mathcal{E})} \\
\frac{dH_1(\Psi_0)}{d\Psi_0} =& 4\pi \int_0^{\Psi_0} d\mathcal{E} \; \frac{h_{+1}(\mathcal{E})}{\sqrt{2(\Psi_0 - \mathcal{E})}} \\
\frac{\sqrt{2}}{4\pi}\frac{dH_1}{d\Psi_0} =& \int_0^{\Psi_0} d\mathcal{E} \; \frac{h_{+1}(\mathcal{E})}{\sqrt{\Psi_0 - \mathcal{E}}} \qquad \mathit{Abel}
\end{aligned}$$

$$\boxed{h_{+1}(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2}\frac{d}{d\mathcal{E}}\int_0^{\mathcal{E}} d\Psi_0 \; \frac{dH_1}{d\Psi_0}\frac{1}{\sqrt{\mathcal{E} - \Psi_0}}} \tag{4.14}$$

For $i = 2$

$$\begin{aligned}
0 =& 4\pi \left\{ \int_0^{\Psi(\eta)} d\mathcal{E} \left[(2\,h_{+2} + 6\eta\,h_{+3} + ...)\sqrt{2(\Psi(\eta) - \mathcal{E})} + \frac{(h_{+1} + 2\eta h_{+2} + 3\eta^2 h_{+3} + ...)}{\sqrt{2(\Psi(\eta) - \mathcal{E})}}\Psi_1 \right] + \right. \\
&+ \int_0^{\Psi(\eta)} d\mathcal{E} \; \Psi_1 \left[\frac{(h_{+1} + 2\eta h_{+2} + 3\eta^2 h_{+3} + ...)}{\sqrt{2(\Psi(\eta) - \mathcal{E})}} - \frac{\sum_{n=0}^{\infty} \eta^n h_{+n}}{(2(\Psi(\eta) - \mathcal{E})\,)^{\frac{3}{2}}}\Psi_1 \right] + \\
&\left. + \; \Psi_1^2 \left. \frac{\sum_{n=0}^{\infty} \eta^n h_{+n}}{\sqrt{2(\Psi(\eta) - \mathcal{E})}} \right|_{\mathcal{E}=\Psi} \right\} ;
\end{aligned}$$

the last term is divergent, unless $\sum_{n=0}^{\infty} \eta^n h_{+n}(\mathcal{E})$ goes to zero faster than $\sqrt{2(\Psi(\eta) - \mathcal{E})}$ for $\mathcal{E} \to \Psi(\eta)$ (more rigorously, the series is an infinitesimal of higher order than $\sqrt{2(\Psi(\eta) - \mathcal{E})}$ for $\mathcal{E} \to \Psi(\eta)$). In such case that term vanishes, and we are left with

$$\begin{aligned}
0 =& 4\pi \left\{ \int_0^{\Psi(\eta)} d\mathcal{E} \left[(2\,h_{+2} + 6\eta\,h_{+3} + ...)\sqrt{2(\Psi(\eta) - \mathcal{E})} + \frac{(h_{+1} + 2\eta h_{+2} + 3\eta^2 h_{+3} + ...)}{\sqrt{2(\Psi(\eta) - \mathcal{E})}}\Psi_1 \right] + \right. \\
&\left. + \int_0^{\Psi(\eta)} d\mathcal{E} \; \Psi_1 \left[\frac{(h_{+1} + 2\eta\,h_{+2} + 3\eta^2\,h_{+3} + ...)}{\sqrt{2(\Psi(\eta) - \mathcal{E})}} - \frac{\sum_{n=0}^{\infty} \eta^n h_{+n}}{(2(\Psi(\eta) - \mathcal{E}))^{\frac{3}{2}}}\Psi_1 \right] \right\} .
\end{aligned}$$

For the same argument as above, the equation is valid also for $\eta = 0$, in which case it becomes:

$$0 = 4\pi \left\{ \int_0^{\Psi_0} d\mathcal{E}\; 2\, h_{+2} \sqrt{2(\Psi_0-\mathcal{E})} + 2\int_0^{\Psi_0} d\mathcal{E}\; \Psi_1 \frac{h_{+1}}{\sqrt{2(\Psi_0-\mathcal{E})}} - \int_0^{\Psi_0} d\mathcal{E}\; \Psi_1^2 \frac{h_{+0}}{(2(\Psi_0-\mathcal{E}))^{\frac{3}{2}}} \right\} . \tag{4.15}$$

Let's re-write (4.15) and define some new quantities in order to clean the notation

$$-2\Psi_1 \int_0^{\Psi_0} d\mathcal{E}\, \frac{h_{+1}(\mathcal{E})}{\sqrt{2(\Psi_0-\mathcal{E})}} + \Psi_1^2 \int_0^{\Psi_0} d\mathcal{E}\, \frac{h_{+0}(\mathcal{E})}{(2(\Psi_0-\mathcal{E}))^{\frac{3}{2}}} = 2\int_0^{\Psi_0} d\mathcal{E}\; h_{+2}(\mathcal{E})\sqrt{2(\Psi_0-\mathcal{E})}$$

$$\Psi_1(\Psi_0)\, A_2(\Psi_0) + \Psi_1^2(\Psi_0)\, B_2(\Psi_0) = 2\int_0^{\Psi_0} d\mathcal{E}\; h_{+2}(\mathcal{E})\sqrt{2(\Psi_0-\mathcal{E})}$$

$$A_2(\Psi_0) \equiv -\,2\int_0^{\Psi_0} d\mathcal{E}\, \frac{h_{+1}(\mathcal{E})}{\sqrt{2(\Psi_0-\mathcal{E})}} , \tag{4.16}$$

$$B_2(\Psi_0) \equiv \int_0^{\Psi_0} d\mathcal{E}\, \frac{h_{+0}(\mathcal{E})}{(2(\Psi_0-\mathcal{E}))^{\frac{3}{2}}} . \tag{4.17}$$

Let us define the full known term

$$H_2(\Psi_0) \equiv \Psi_1(\Psi_0)\, A_2(\Psi_0) + \Psi_1^2(\Psi_0)\, B_2(\Psi_0) \tag{4.18}$$

so the new equation is

$$H_2(\Psi_0) = 2\int_0^{\Psi_0} d\mathcal{E}\; h_{+2}(\mathcal{E})\sqrt{2(\Psi_0-\mathcal{E})}$$

$$\frac{1}{\sqrt{2}} \frac{dH_2(\Psi_0)}{d\Psi_0} = \int_0^{\Psi_0} d\mathcal{E}\, \frac{h_{+2}(\mathcal{E})}{\sqrt{\Psi_0-\mathcal{E}}} \qquad \textit{Abel}$$

$$\boxed{h_{+2}(\mathcal{E}) = \frac{1}{\sqrt{2}\,\pi} \frac{d}{d\mathcal{E}} \int_0^{\mathcal{E}} d\Psi_0\, \frac{dH_2}{d\Psi_0} \frac{1}{\sqrt{\mathcal{E}-\Psi_0}}} \tag{4.19}$$

It is possible to proceed by taking derivatives of orders higher than two and, possibly, to obtain the other terms of the expansion (4.10).

Here we want to remind why, so far, we have focused on a isotropic $h_+(\mathcal{E};\eta)$. This was suggested by the "geometrical" structure of the integral (4.5) considering *spherical* terms only: the factor behind the integral contains a R^{-1} (a *cylindrical* radius), which must be canceled out by assuming some property of h_+, in order to match the *spherical* symmetry of the terms on the left-hand side. The most direct and simple way this can happen is by assuming that h_+ does not depend on J_z, so that R^{-1} behind the integral immediately cancels out with the integration in dJ_z, that leaves a $R\sqrt{2(\Psi_T-\mathcal{E})}$. The isotropy is not an arbitrary assumption, but a necessary consequence of the spherical geometry at the left-hand side, given by the adopted density being only spherical.

Summarizing, we have provided a new method able to give the constructive formulae for all terms of the ergodic even part of the DF, with this having the structure of a generic series $h_+(\mathcal{E};\eta) = \sum_{n=0}^{\infty} \eta^n h_{+n}(\mathcal{E})$, interpretable as a Taylor expansion in case of small η. Moreover, we obtained the following *theorem about the inversion of spherically perturbed systems*:

Theorem 3. *Given a spherically perturbed system with density $\rho(r) = \rho_0(r) + \eta\rho_1(r)$ and associated potential $\Psi(r) = \Psi_0(r) + \eta\Psi_1(r)$, the even part of the DF of the system is a series $\sum_{n=0}^{\infty} \eta^n h_{+n}(\mathcal{E})$.*

Although this new mathematical method is powerful in determining the $h_{+n}(\mathcal{E})$ terms, these functions must be such that the right-hand side of (4.5) always matches the geometrical structure of the left-hand side. For this reason, the following theorem must be satisfied:

Theorem 4. *In the function $h_+(\mathcal{E}, J_z;\eta)$, all terms of order $n \geq 2$ in η are such that, when the integrations in (4.5) are performed, their resulting integrals cancel out. The full integration results in a linear structure on the parameter η.*

4.4 Including cylindrical terms

Theorem 5. *The η-linear-order term of $h_+(\mathcal{E}, J_z;\eta)$ is*

$$h_{+,lin}(\mathcal{E}, J_z) \equiv h_{+10}(\mathcal{E}) + h_{+11}(\mathcal{E})J_z^2\,, \tag{4.20}$$

where:

$$h_{+10}(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2}\frac{d}{d\mathcal{E}}\int_0^{\mathcal{E}} d\Psi_0\, \frac{dH_{10}}{d\Psi_0}\frac{1}{\sqrt{\mathcal{E}-\Psi_0}}\,, \tag{4.21}$$

$$h_{+11}(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2}\frac{d}{d\mathcal{E}}\int_0^{\mathcal{E}} d\Psi_0\, \frac{d^2H_{11}}{d\Psi_0^2}\frac{1}{\sqrt{\mathcal{E}-\Psi_0}}\,, \tag{4.22}$$

given

$$H_{10}(\Psi_0) \equiv \rho_1(\Psi_0) - 4\pi\Psi_1(\Psi_0)\int_0^{\Psi_0} d\mathcal{E}\, \frac{h_{+00}(\mathcal{E})}{\sqrt{2(\Psi_0-\mathcal{E})}}\,, \tag{4.23}$$

$$H_{11}(\Psi_0) \equiv \rho_2(\Psi_0) - 4\pi\Psi_2(\Psi_0)\int_0^{\Psi_0} d\mathcal{E}\, \frac{h_{+00}(\mathcal{E})}{\sqrt{2(\Psi_0-\mathcal{E})}}\,, \tag{4.24}$$

and with $h_{+00}(\mathcal{E})$ being the zero-order term of $h_+(\mathcal{E}, J_z;\eta)$.

Proof. The lesson learnt in the previous Section is that a series of the form $\sum_{n=0}^{\infty} \eta^n h_{+n}(\mathcal{E})$ can successfully account for a spherical density like $\rho(r) = \rho_0(r) + \eta\rho_1(r)$. This is extremely useful, because we have just discovered that piece of a possible DF able to explain two out of the three terms composing the homeoidal expansion for the density profile.

Only one term of the density is left, that is $\eta R^2\rho_2(r)$. Our aim is now to find a possible (if not the only possible) missing piece of the DF such that the entire expression of h_+ in (4.3) can cover successfully the (non-spherical) density $\rho_0(r)+\eta\rho_1(r)+\eta R^2\rho_2(r)$. Keeping in mind that the series (4.10) can explain all the spherically symmetric terms of the density, the most simple scenario in which a "ηR^2" term can appear due to the DF is given by the following *ansatz*:

$$\boxed{h_+(\mathcal{E},J_z;\eta)=\sum_{n=0}^{\infty}\eta^n\sum_{k=0}^{n}h_{+nk}(\mathcal{E})J_z^{2k}\,.} \tag{4.25}$$

The reader should remind the meaning of the subscript "+", which indicates that what we are considering is the even part of the DF *with respect to the argument* J_z, and that whenever a function of $\mathcal{E}$ only has the subscript "+" this should be interpreted just as a component function for the whole $h_+(\mathcal{E},J_z;\eta)$. In case h_+ is the one in (4.25), the integral equation (4.3) becomes

$$\begin{aligned}\rho_0+\eta\rho_1+\eta R^2\rho_2 =&\frac{4\pi}{R}\int_0^{\Psi(\eta)}d\mathcal{E}\int_0^{R\sqrt{2[\Psi(\eta)-\mathcal{E}]}}dJ_z\ \{h_{+00}(\mathcal{E})+\\ &+\eta\left[h_{+10}(\mathcal{E})+h_{+11}(\mathcal{E})J_z^2\right]+\\ &+\eta^2\left[h_{+20}(\mathcal{E})+h_{+21}(\mathcal{E})J_z^2+h_{+22}(\mathcal{E})J_z^4\right]+...\}\ .\end{aligned} \tag{4.26}$$

We remind that now $\Psi(\eta)=\Psi_0(r)+\eta\Psi_1(r)+\eta R^2\Psi_2(r)$. The same mathematical argument supporting the reasoning in Sec. 4.3 is useful here as well: by interpreting both the left-hand side and the right-hand side of (4.26) as functions of η only, such functions must be equal also for $\eta=0$, obtaining:

$$\rho_0(r)=4\pi\int_0^{\Psi_0}d\mathcal{E}\ h_{+00}(\mathcal{E})\sqrt{2(\Psi_0-\mathcal{E})}\,;$$

again, if $\Psi_0(r)$ is a monotonic function and can be inverted, finding $r=r(\Psi_0)$, we can derive both members with respect to Ψ_0, obtain an integral identity suitable for the Abel inversion technique and finally get

$$\boxed{h_{+00}(\mathcal{E})=\frac{1}{\sqrt{8}\pi^2}\frac{d}{d\mathcal{E}}\int_0^{\mathcal{E}}d\Psi_0\ \frac{d\rho_0}{d\Psi_0}\frac{1}{\sqrt{\mathcal{E}-\Psi_0}}\,.} \tag{4.27}$$

Taking the first η-derivative of (4.26) leads to

$$\begin{aligned}\rho_1 + R^2\rho_2 =& 4\pi \left\{ \int_0^{\Psi(\eta)} d\mathcal{E}\, \frac{h_{+00}(\mathcal{E})}{\sqrt{2\left[\Psi(\eta)-\mathcal{E}\right]}} \left(\Psi_1 + R^2\Psi_2\right) + \right. \\ & + \int_0^{\Psi(\eta)} d\mathcal{E}\, h_{+10}(\mathcal{E}) \sqrt{2\left[\Psi(\eta)-\mathcal{E}\right]} + \eta \frac{\partial}{\partial\eta} \int_0^{\Psi(\eta)} d\mathcal{E} ... + \\ & + \int_0^{\Psi(\eta)} d\mathcal{E}\, h_{+11}(\mathcal{E}) \frac{R^2}{3} \left(2\left[\Psi(\eta)-\mathcal{E}\right]\right)^{3/2} + \eta \frac{\partial}{\partial\eta} \int_0^{\Psi(\eta)} d\mathcal{E} ... + \\ & \left. + \frac{2}{R}\eta ... + \frac{1}{R}\eta^2 \frac{\partial}{\partial\eta} ... \right\} ,\end{aligned} \tag{4.28}$$

which for $\eta = 0$ becomes

$$\begin{aligned}\rho_1 + R^2\rho_2 =& 4\pi \left\{ \int_0^{\Psi_0} d\mathcal{E}\, \frac{h_{+00}(\mathcal{E})}{\sqrt{2\left(\Psi_0-\mathcal{E}\right)}} \left(\Psi_1 + R^2\Psi_2\right) + \right. \\ & + \int_0^{\Psi_0} d\mathcal{E}\, h_{+10}(\mathcal{E}) \sqrt{2\left(\Psi_0-\mathcal{E}\right)} + \\ & \left. + \int_0^{\Psi_0} d\mathcal{E}\, h_{+11}(\mathcal{E}) \frac{R^2}{3} \left(2\left[\Psi_0-\mathcal{E}\right]\right)^{3/2} \right\} .\end{aligned}$$

Defining the (known) terms

$$A_1(\Psi_0) \equiv \int_0^{\Psi_0} d\mathcal{E}\, \frac{h_{+00}(\mathcal{E})}{\sqrt{2\left(\Psi_0-\mathcal{E}\right)}} , \tag{4.29}$$

$$H_{10}(\Psi_0) \equiv \rho_1(\Psi_0) - 4\pi\Psi_1(\Psi_0)A_1(\Psi_0) , \tag{4.30}$$

$$H_{11}(\Psi_0) \equiv \rho_2(\Psi_0) - 4\pi\Psi_2(\Psi_0)A_1(\Psi_0) , \tag{4.31}$$

and isolating the unknown terms at the right-hand side:

$$H_{10}(\Psi_0) + R^2 H_{11}(\Psi_0) = 4\pi \int_0^{\Psi_0} d\mathcal{E}\, h_{+10}(\mathcal{E}) \sqrt{2\left(\Psi_0-\mathcal{E}\right)} + \frac{4\pi}{3} R^2 \int_0^{\Psi_0} d\mathcal{E}\, h_{+11}(\mathcal{E}) \left(2\left[\Psi_0-\mathcal{E}\right]\right)^{3/2} .$$

We can univocally associate the terms in this way (A.5):

$$H_{10}(\Psi_0) = 4\pi \int_0^{\Psi_0} d\mathcal{E}\, h_{+10}(\mathcal{E}) \sqrt{2\left(\Psi_0-\mathcal{E}\right)} ; \tag{4.32}$$

$$H_{11}(\Psi_0) = \frac{4\pi}{3} \int_0^{\Psi_0} d\mathcal{E}\, h_{+11}(\mathcal{E}) \left(2\left[\Psi_0-\mathcal{E}\right]\right)^{3/2} . \tag{4.33}$$

Let's first focus on (4.32).

$$\frac{1}{4\pi} H_{10}(\Psi_0) = \int_0^{\Psi_0} d\mathcal{E}\, h_{+10}(\mathcal{E}) \sqrt{2(\Psi_0 - \mathcal{E})}$$

$$\frac{1}{\sqrt{8}\,\pi} \frac{dH_{10}(\Psi_0)}{d\Psi_0} = \int_0^{\Psi_0} d\mathcal{E}\, \frac{h_{+10}(\mathcal{E})}{\sqrt{\Psi_0 - \mathcal{E}}} \qquad \textit{Abel}$$

$$\boxed{h_{+10}(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2} \frac{d}{d\mathcal{E}} \int_0^{\mathcal{E}} d\Psi_0\, \frac{dH_{10}}{d\Psi_0} \frac{1}{\sqrt{\mathcal{E} - \Psi_0}}\,.} \tag{4.34}$$

Now consider (4.33).

$$\frac{3}{4\pi} H_{11}(\Psi_0) = \int_0^{\Psi_0} d\mathcal{E}\, h_{+11}(\mathcal{E}) \left(2\left[\Psi_0 - \mathcal{E}\right]\right)^{3/2}$$

$$\frac{1}{4\pi} \frac{dH_{11}(\Psi_0)}{d\Psi_0} = \int_0^{\Psi_0} d\mathcal{E}\, h_{+11}(\mathcal{E}) \sqrt{2(\Psi_0 - \mathcal{E})}$$

$$\frac{1}{\sqrt{8}\,\pi} \frac{d^2 H_{11}(\Psi_0)}{d\Psi_0^2} = \int_0^{\Psi_0} d\mathcal{E}\, \frac{h_{+11}(\mathcal{E})}{\sqrt{\Psi_0 - \mathcal{E}}} \qquad \textit{Abel}$$

$$\boxed{h_{+11}(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2} \frac{d}{d\mathcal{E}} \int_0^{\mathcal{E}} d\Psi_0\, \frac{d^2 H_{11}}{d\Psi_0^2} \frac{1}{\sqrt{\mathcal{E} - \Psi_0}}\,.} \tag{4.35}$$

□

The remaining unknown terms of $h_+(\mathcal{E}, J_z; \eta)$ are the second- and higher-η-order terms of the series (4.25). If one wants to obtain the solution for the n-th term of the series, it is enough to apply $d^n/d\eta^n$ to Eq. (4.26) and then proceed with the same logic shown for the first derivative case. As a demonstration of the effectiveness of the method, we are going to compute all the second-order terms.

The second η-derivative of (4.26) is (for convenience, compute it as the first η-

derivative of (4.28)):

$$
\begin{aligned}
0 =& 4\pi \left\{ \frac{d\Psi(\eta)}{d\eta} \left(\Psi_1 + R^2\Psi_2\right) \frac{h_{+00}(\mathcal{E})}{\sqrt{2\left[\Psi(\eta) - \mathcal{E}\right]}}\Bigg|_{\mathcal{E}=\Psi(\eta)} - \int_0^{\Psi(\eta)} d\mathcal{E}\, \frac{h_{+00}(\mathcal{E})}{\left(2\left[\Psi(\eta) - \mathcal{E}\right]\right)^{3/2}} \left(\Psi_1 + R^2\Psi_2\right)^2 + \right. \\
& + 2\int_0^{\Psi(\eta)} d\mathcal{E}\, \frac{h_{+10}(\mathcal{E})}{\sqrt{2\left[\Psi(\eta) - \mathcal{E}\right]}} \left(\Psi_1 + R^2\Psi_2\right) + \eta \frac{\partial^2}{\partial\eta^2} \int_0^{\Psi(\eta)} d\mathcal{E} ... + \\
& + 2\int_0^{\Psi(\eta)} d\mathcal{E}\, h_{+11}(\mathcal{E}) R^2 \sqrt{2\left[\Psi(\eta) - \mathcal{E}\right]} \left(\Psi_1 + R^2\Psi_2\right) + \eta \frac{\partial^2}{\partial\eta^2} \int_0^{\Psi(\eta)} d\mathcal{E} ... + \\
& + \frac{2}{R} \int_0^{\Psi(\eta)} d\mathcal{E} \int_0^{R\sqrt{2[\Psi(\eta)-\mathcal{E}]}} dJ_z \left[h_{+20}(\mathcal{E}) + h_{+21}(\mathcal{E}) J_z^2 + h_{+22}(\mathcal{E}) J_z^4\right] + \\
& \left. + \frac{2}{R}\eta \frac{\partial}{\partial\eta} ... + \frac{2}{R}\eta \frac{\partial}{\partial\eta} ... + \frac{1}{R}\eta^2 \frac{\partial^2}{\partial\eta^2} ... + ... \right\} ;
\end{aligned}
\tag{4.36}
$$

a necessary condition for the present derivative to be convergent is that $h_{+00}(\mathcal{E})$ is an infinitesimal of higher order than $\sqrt{2\left[\Psi - \mathcal{E}\right]}$, for $\mathcal{E} \to \Psi$. If this is the case, the first term vanishes.

The identity (4.36) for $\eta = 0$ becomes:

$$
\begin{aligned}
0 =& 4\pi \left\{ - \int_0^{\Psi_0} d\mathcal{E}\, \frac{h_{+00}(\mathcal{E})}{\left(2\left[\Psi_0 - \mathcal{E}\right]\right)^{3/2}} \left(\Psi_1 + R^2\Psi_2\right)^2 + \right. \\
& + 2\int_0^{\Psi_0} d\mathcal{E}\, \frac{h_{+10}(\mathcal{E})}{\sqrt{2(\Psi_0 - \mathcal{E})}} \left(\Psi_1 + R^2\Psi_2\right) + \\
& + 2\int_0^{\Psi_0} d\mathcal{E}\, h_{+11}(\mathcal{E}) \sqrt{2\left(\Psi_0 - \mathcal{E}\right)}\, R^2 \left(\Psi_1 + R^2\Psi_2\right) + \\
& + 2\int_0^{\Psi_0} d\mathcal{E}\, h_{+20}(\mathcal{E}) \sqrt{2(\Psi_0 - \mathcal{E})} + \frac{2}{3} \int_0^{\Psi_0} d\mathcal{E}\, h_{+21}(\mathcal{E}) \left(2\left[\Psi_0 - \mathcal{E}\right]\right)^{3/2} R^2 + \\
& \left. + \frac{2}{5} \int_0^{\Psi_0} d\mathcal{E}\, h_{+22}(\mathcal{E}) \left(2\left[\Psi_0 - \mathcal{E}\right]\right)^{5/2} R^4 \right\}
\end{aligned}
$$

Let's define the (known) terms

$$A_2(\Psi_0) \equiv -\int_0^{\Psi_0} d\mathcal{E}\, \frac{h_{+00}(\mathcal{E})}{\left(2\left[\Psi_0 - \mathcal{E}\right]\right)^{3/2}}\,, \tag{4.37}$$

$$B_{20}(\Psi_0) \equiv 2\int_0^{\Psi_0} d\mathcal{E}\, \frac{h_{+10}(\mathcal{E})}{\sqrt{2(\Psi_0 - \mathcal{E})}}\,, \tag{4.38}$$

$$B_{21}(\Psi_0) \equiv 2\int_0^{\Psi_0} d\mathcal{E}\, h_{+11}(\mathcal{E})\sqrt{2(\Psi_0 - \mathcal{E})}\,, \tag{4.39}$$

$$H_{20}(\Psi_0) \equiv -4\pi A_2 \Psi_1^2 - 4\pi B_{20}\Psi_1\,, \tag{4.40}$$

$$H_{21}(\Psi_0) \equiv -8\pi A_2 \Psi_1 \Psi_2 - 4\pi B_{20}\Psi_2 - 4\pi B_{21}\Psi_1\,, \tag{4.41}$$

$$H_{22}(\Psi_0) \equiv -4\pi A_2 \Psi_2^2 - 4\pi B_{21}\Psi_2\,, \tag{4.42}$$

where $\Psi_1 = \Psi_1(\Psi_0)$ and $\Psi_2 = \Psi_2(\Psi_0)$. Isolating all known terms at left-hand side:

$$\begin{aligned} H_{20}(\Psi_0) + H_{21}(\Psi_0)R^2 + H_{22}(\Psi_0)R^4 =& 8\pi \int_0^{\Psi_0} d\mathcal{E}\, h_{+20}(\mathcal{E})\sqrt{2(\Psi_0 - \mathcal{E})}+ \\ &+ \frac{8\pi}{3}\int_0^{\Psi_0} d\mathcal{E}\, h_{+21}(\mathcal{E})\left(2\left[\Psi_0 - \mathcal{E}\right]\right)^{3/2} R^2 + \\ &+ \frac{8\pi}{5}\int_0^{\Psi_0} d\mathcal{E}\, h_{+22}(\mathcal{E})\left(2\left[\Psi_0 - \mathcal{E}\right]\right)^{5/2} R^4\,. \end{aligned}$$

We can equate terms multiplied by the same order of R (A.5):

$$H_{20}(\Psi_0) = 8\pi \int_0^{\Psi_0} d\mathcal{E}\, h_{+20}(\mathcal{E})\sqrt{2(\Psi_0 - \mathcal{E})}\,; \tag{4.43}$$

$$H_{21}(\Psi_0) = \frac{8\pi}{3}\int_0^{\Psi_0} d\mathcal{E}\, h_{+21}(\mathcal{E})\left(2\left[\Psi_0 - \mathcal{E}\right]\right)^{3/2}\,; \tag{4.44}$$

$$H_{22}(\Psi_0) = \frac{8\pi}{5}\int_0^{\Psi_0} d\mathcal{E}\, h_{+22}(\mathcal{E})\left(2\left[\Psi_0 - \mathcal{E}\right]\right)^{5/2}\,. \tag{4.45}$$

Taking the derivative with respect to Ψ_0 once for Eq. (4.43), twice for Eq. (4.44) and three times for Eq. (4.45), we obtain identities suitable for Abel inversion, leading to the following results:

$$\boxed{h_{+20}(\mathcal{E}) = \frac{\sqrt{2}}{8\pi^2}\frac{d}{d\mathcal{E}}\int_0^{\mathcal{E}} d\Psi_0\, \frac{dH_{20}}{d\Psi_0}\frac{1}{\sqrt{\mathcal{E} - \Psi_0}}\,;} \tag{4.46}$$

$$\boxed{h_{+21}(\mathcal{E}) = \frac{\sqrt{2}}{8\pi^2}\frac{d}{d\mathcal{E}}\int_0^{\mathcal{E}} d\Psi_0\, \frac{d^2H_{21}}{d\Psi_0^2}\frac{1}{\sqrt{\mathcal{E} - \Psi_0}}\,;} \tag{4.47}$$

$$\boxed{h_{+22}(\mathcal{E}) = \frac{\sqrt{2}}{24\pi^2}\frac{d}{d\mathcal{E}}\int_0^{\mathcal{E}} d\Psi_0 \frac{d^3 H_{22}}{d\Psi_0^3}\frac{1}{\sqrt{\mathcal{E}-\Psi_0}}\,.} \tag{4.48}$$

We discovered that the even part of the DF can be written as in (4.25). Its terms with order $n \geq 2$ in η necessarily satisfy the **Theorem 4** even with the introduction of cylindrical terms, for the same "geometric-matching" reasons seen in Section 4.3.

4.5 Geometrical study of the integration domain

For a complete understanding of the integral (2.53), we can also study it geometrically, in the context of an infinitesimally flattened stellar system. More precisely, we can study the integration domain of (2.53) in the $(\mathcal{E}, J_z)$ plane, in particular how the boundary of this domain varies with η considering the homeoidal expansion of the potential being put in substitution to Ψ_T. In fact, h_+ is not the only object that varies with η. We are going to show that also the geometry of the integration domain is strictly affected by it. It will be clarified how the integration domain (more precisely, its boundary) changes for different values of η, and how the domain can even be formally expanded when the flattening is infinitesimal.
Let's start by using an alternative form of (2.53):

$$\rho(R;\eta) = \frac{4\pi}{R}\iint_{\Omega(\eta)} d\mathcal{E}\, dJ_z\; h_+(\mathcal{E}, J_z) \tag{4.49}$$

where with $\Omega(\eta)$ we indicate the integration domain as a set of points in the $(\mathcal{E}, J_z)$ plane. If we manage to write an expansion of $\Omega(\eta)$ for small η, we would know how the domain contributes with terms in η in the consequent split form of the integral. This investigation starts once a point $(\tilde{R}, \tilde{z})$ and a model for $\tilde{\rho}(s)$ are chosen: this will allow to treat $I_0(s)$, $I_1(s)$ and $I_2(s)$ as fixed parameters of this geometrical study (we refer to their expressions in (3.34)). Since the expansion (3.33) of the potential is dimensionless, in the integral we use $\Psi_T = \Psi_n\tilde{\Psi}$ and $\mathcal{E} = \Psi_n\tilde{\mathcal{E}}$, where $\tilde{\mathcal{E}}$ is the dimensionless relative energy per unit mass; the normalization factor Ψ_n has been computed in Chapter 3 (see (3.32)). We expect the domain $\Omega(\eta)$ to vary with η because this one appears in the limit of integration of (2.53); from a geometrical point of view, in the plane $(\tilde{\mathcal{E}}, J_z)$ $\Omega(\eta)$ is delimited by

$$J_z(\mathcal{E}) = R\sqrt{2(\Psi_T - \mathcal{E})}$$

$$J_z(\tilde{\mathcal{E}};\eta) = R\sqrt{\Psi_n}\sqrt{2(\tilde{\Psi}_T - \tilde{\mathcal{E}})}$$

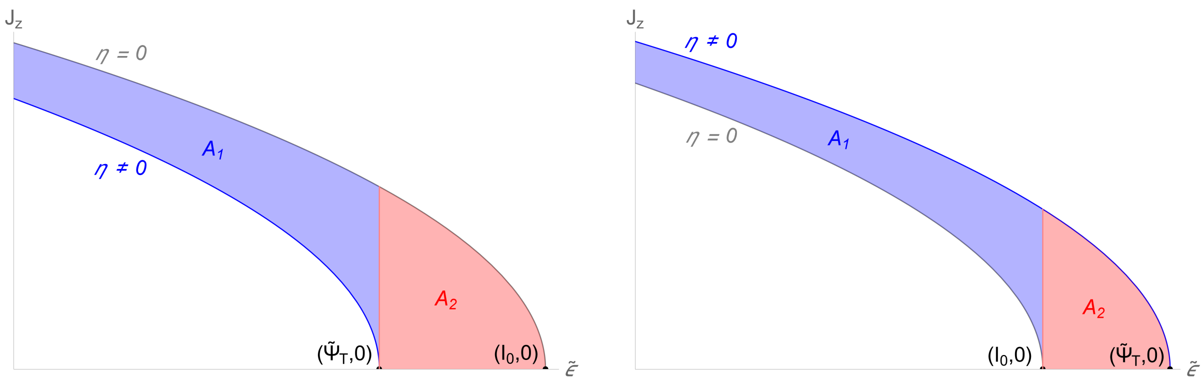
$$= R\sqrt{\Psi_n}\sqrt{2(I_0 + \eta(I_1 + \tilde{z}^2 I_2) - \tilde{\mathcal{E}})}$$

$$J_z(\tilde{\mathcal{E}};\eta) = R\sqrt{\Psi_n}\sqrt{2(I_0 + \eta A - \tilde{\mathcal{E}})} \tag{4.50}$$

where we have defined

$$A \equiv I_1 + \tilde{z}^2 I_2 \quad \begin{cases} < 0 & , \; I_1 < \tilde{z}^2 \, |I_2| \\ = 0 & , \; \tilde{z} = \pm\sqrt{-\dfrac{I_1}{I_2}} \\ > 0 & , \; I_1 > \tilde{z}^2 \, |I_2| \end{cases} \tag{4.51}$$

In fact, the sign of A depends on how I_1 and I_2 combine; as can be noticed in (3.34), $I_1(s) \geq 0 \quad \forall s$ and $I_2(s) \leq 0 \quad \forall s$. With the parameter A (constant in our geometrical study since we have fixed a point $(\tilde{R}, \tilde{z})$ and a model for $\tilde{\rho}(s)$), $\Psi_T = I_0 + \eta A$; this simplifies the formalism and the geometric intuition on where and how things are going to change in $(\tilde{\mathcal{E}}, J_z)$ for different values of η. In Figure 4.1 the boundary $\partial\Omega(\eta)$ for $\eta \neq 0$ is compared to $\partial\Omega(0)$, for both cases $A < 0$ (Figure 4.1a) and $A > 0$ (Figure 4.1b).



(a) $A < 0$ **(b)** $A > 0$

Figure 4.1: How integration domain varies in the plane $(\tilde{\mathcal{E}}, J_z)$ for $\eta \neq 0$ in two different model scenarios, distinguished by the sign of the parameter $A \equiv I_1(s) + \tilde{z}^2 I_2(s)$. The plots refer to an arbitrary point $(\tilde{R}, \tilde{z}) \neq (0, 0)$. When $A < 0$ (left panel) the domain gets restricted with increasing η, while it gets enlarged by η when $A > 0$ (right panel). The difference-area between the two curves $\eta = 0$ (grey curve) and $\eta \neq 0$ (blue curve) is divided in two sub-areas: $\mathcal{A}_1(\eta)$ (blue) is the area vertically delimited by the inferior curve and its projection on the superior curve; $\mathcal{A}_2(\eta)$ (red) is the remaining area, delimited by the superior curve and the $\tilde{\mathcal{E}}$-axis.

It can be seen that $\Omega(\eta)$ can be smaller or larger than the spherically symmetric case $\Omega(0)$, depending on the sign of A. In either cases, $\Omega(\eta)$ can be related to $\Omega(0)$ using the two areas $\mathcal{A}_1(\eta)$ and $\mathcal{A}_2(\eta)$ in Figure 4.1:

$$A < 0: \qquad \Omega(\eta) = \Omega(0) - \mathcal{A}_1(\eta) - \mathcal{A}_2(\eta) \tag{4.52}$$

$$A > 0: \qquad \Omega(\eta) = \Omega(0) + \mathcal{A}_1(\eta) + \mathcal{A}_2(\eta) \tag{4.53}$$

This has consequences on the development of the integral in the form (4.49)

$$\rho(R; \eta) = \frac{4\pi}{R} \iint_{\Omega(0) \mp \mathcal{A}_1(\eta) \mp \mathcal{A}_2(\eta)} d\mathcal{E} \, dJ_z \; h_+(\mathcal{E}, J_z) \tag{4.54}$$

To isolate the contribution of the integration domain from that of the DF h_+, in the following we study the *measures* of the domains $\Omega(\eta)$ and try to find a way of expanding $\Omega(\eta)$ for small values of η.
The measure $\mathcal{A}[\Omega]$ of a generic area Ω in $(\tilde{\mathcal{E}}, J_z)$ is

$$\mathcal{A}[\Omega] = \iint_\Omega d\tilde{\mathcal{E}}\, dJ_z \tag{4.55}$$

For simplicity, we are going to indicate $\Omega(\eta)$, $\Omega(0)$, $\mathcal{A}_1(\eta)$ and $\mathcal{A}_2(\eta)$ as their own measures.

Consider $A < 0$.

$$\begin{aligned}
\Omega(\eta) =& \Omega(0) - \mathcal{A}_1(\eta) - \mathcal{A}_2(\eta) \\
\Omega(\eta) =& \Omega(0) - R\sqrt{\Psi_n}\left\{\int_0^{\tilde{\Psi}_T}\left(\sqrt{2(I_0-\tilde{\mathcal{E}})} - \sqrt{2(I_0+\eta A-\tilde{\mathcal{E}})}\right) d\tilde{\mathcal{E}}\right\} + \\
& - R\sqrt{\Psi_n}\left\{\int_{\tilde{\Psi}_T}^{I_0}\sqrt{2(I_0-\tilde{\mathcal{E}})}\; d\tilde{\mathcal{E}}\right\} \\
=& \Omega(0) + R\sqrt{\Psi_n}\sqrt{2}\frac{2}{3}\left\{\eta^{\frac{3}{2}}(-A)^{\frac{3}{2}} - I_0^{\frac{3}{2}} + (I_0+\eta A)^{\frac{3}{2}}\right\} + \\
& - R\sqrt{\Psi_n}\sqrt{2}\frac{2}{3}\left\{\eta^{\frac{3}{2}}(-A)^{\frac{3}{2}}\right\} \\
\underset{\eta\to 0}{\sim}& \Omega(0) + R\sqrt{\Psi_n}\sqrt{2}\frac{2}{3}I_0^{\frac{3}{2}}(-1+1+\frac{3}{2}\eta\frac{A}{I_0} + o(\eta^2)) \\
\underset{\eta\to 0}{\sim}& \Omega(0) + \eta\, RA\sqrt{2I_0\Psi_n} + o(\eta^2)
\end{aligned}$$

Notice how the two sub-areas contribute with terms in powers of $\eta \to 0$: $\mathcal{A}_1(\eta)$ gives terms in η, $\eta^{\frac{3}{2}}$ and higher order terms $o(\eta^2)$; $\mathcal{A}_2(\eta)$ instead gives only $\eta^{\frac{3}{2}}$, which even simplifies *exactly* with the corresponding power term from $\mathcal{A}_1(\eta)$. This last fact means that $\mathcal{A}_2(\eta)$ contributes to the net summation of areas only through orders higher than $\frac{3}{2}$ in η.

For $A > 0$

$$\begin{aligned}
\Omega(\eta) =&\Omega(0) + \mathcal{A}_1(\eta) + \mathcal{A}_2(\eta) \\
\Omega(\eta) =&\Omega(0) + R\sqrt{\Psi_n}\left\{\int_0^{I_0}\left(\sqrt{2(I_0+\eta A-\tilde{\mathcal{E}})} - \sqrt{2(I_0-\tilde{\mathcal{E}})}\right)\,d\tilde{\mathcal{E}}\right\} + \\
&+ R\sqrt{\Psi_n}\left\{\int_{I_0}^{\tilde{\Psi}_T}\sqrt{2(I_0+\eta A-\tilde{\mathcal{E}})}\,d\tilde{\mathcal{E}}\right\} \\
=&\Omega(0) + R\sqrt{\Psi_n}\sqrt{2}\frac{2}{3}\left\{-\eta^{\frac{3}{2}}A^{\frac{3}{2}} + (I_0+\eta A)^{\frac{3}{2}} - I_0^{\frac{3}{2}}\right\} + \\
&+ R\sqrt{\Psi_n}\sqrt{2}\frac{2}{3}\left\{\eta^{\frac{3}{2}}A^{\frac{3}{2}}\right\} \\
\underset{\eta\to 0}{\sim}&\Omega(0) + R\sqrt{\Psi_n}\sqrt{2}\frac{2}{3}I_0^{\frac{3}{2}}(1+\frac{3}{2}\eta\frac{A}{I_0} + o(\eta^2) - 1) \\
\underset{\eta\to 0}{\sim}&\Omega(0) + \eta\,RA\sqrt{2I_0\Psi_n} + o(\eta^2)
\end{aligned}$$

The same things noticed right after the $A < 0$ computations are valid here as well. Not only are we able to write a "homeoidal" expansion for the measure of $\Omega(\eta)$, but we also get that this expansion is independent of the sign of parameter A. That means, $\forall A$, at the first order in $\eta \to 0$ the integration over the domain gives

$$\Omega(\eta) \sim \Omega(0) + \eta\,R(I_1 + \tilde{z}^2 I_2)\sqrt{2I_0\Psi_n} \tag{4.56}$$

or, using $\tilde{R}$ only

$$\Omega(\eta) \sim \Omega(0) + (a_1\sqrt{2I_0\Psi_n})\left\{\eta\tilde{R}\,(I_1 + s^2 I_2) - \eta\tilde{R}^3 I_2\right\} \tag{4.57}$$

Keeping in mind the terms at left-hand side of the integral, which consist in the first-order homeoidal expansion of the density, and again (for now) forgetting about the DF h_+, let's see if we can associate terms from both sides univocally. Mathematically, this means to write (4.49) without h_+ (set $h_+(\tilde{\mathcal{E}}, J_z) = 1$) and to consider homeoidal expansions of both density and measure of the integration domain $\Omega(\eta)$

$$\tilde{\rho}_0(s)+\eta\tilde{\rho}_1(s)+\eta\tilde{R}^2\tilde{\rho}_2(s) = \frac{4\pi}{a_1}\frac{1}{\tilde{R}}\left\{\Omega(0) + (a_1\sqrt{2I_0(s)\Psi_n})\left(\eta\tilde{R}\,(I_1(s) + s^2 I_2(s)\,) - \eta\tilde{R}^3 I_2(s)\right)\right\} \tag{4.58}$$

Going on with the computations:

$$\Omega(0) = \int_0^{I_0} d\tilde{\mathcal{E}}\,R\sqrt{\Psi_n}\sqrt{2(I_0-\tilde{\mathcal{E}})} = \tilde{R}\,a_1\sqrt{2\Psi_n}\frac{2}{3}I_0^{\frac{3}{2}}$$

$$
\begin{aligned}
\tilde{\rho}_0(s) + \eta\tilde{\rho}_1(s) + \eta\tilde{R}^2\tilde{\rho}_2(s) = & \frac{8\pi}{3}\sqrt{2\Psi_n}I_0^{\frac{3}{2}}(s) + \\
& + \eta\, 4\pi\sqrt{2\Psi_n I_0(s)}\,\big(I_1(s) + s^2 I_2(s)\big) + \\
& - \eta\tilde{R}^2 4\pi\sqrt{2\Psi_n I_0(s)}\; I_2(s)
\end{aligned}
$$

It is evident how single terms from both sides of the equality can be associated (see Section A.5). We obtain three equations:

$$\tilde{\rho}_0(s) = \frac{8\pi}{3}\sqrt{2\Psi_n}I_0^{\frac{3}{2}}(s) \ , \tag{4.59}$$

$$\tilde{\rho}_1(s) = 4\pi\sqrt{2\Psi_n I_0(s)}\,\big(I_1(s) + s^2 I_2(s)\big) \ , \tag{4.60}$$

$$\tilde{\rho}_2(s) = -4\pi\sqrt{2\Psi_n I_0(s)}\; I_2(s) \ . \tag{4.61}$$

CHAPTER 5

Streaming motions and the odd part of the DF

Having developed the new method for the even part of the DF, we now wish to show how it can be applied to gain insight into the odd part $h_-(\mathcal{E}, J_z; \eta)$. Similarly to what done in Chapter 4, the starting point is the integral identity (4.4). This time, however, the left-hand side is not immediately available: we must first find a way to express the product between density ρ and streaming velocity field $\overline{v_\varphi}$ so that the flattening η appears explicitly. At this end, we rely on the solutions of axisymmetric Jeans Equations (JEs) in the particular case of homeoidal density-potential pairs and in combination with the Satoh (1980) k-decomposition.

5.1 Setting a new stage

As already stressed, homeoidal density-potential pairs like in Eq.s (4.1)-(4.2) represent exact pairs, in that they are related via the Poisson equation. In these expressions, the flattening η along the minor axis appears explicitly and in linear form; moreover, they are written as a spherical part plus a non-spherical part, the latter being given by a spherical function multiplied by the square of the cylindrical radius R. This mathematical structure allows for manageable solutions of the JEs (Mancino et al. 2024), which are necessary in order to apply our new method to solve for at least one term of the odd $h_-(\mathcal{E}, J_z; \eta)$. We repeat here, for clarity, the starting integral to consider for our purposes:

$$\rho\overline{v_\varphi}(R, \Psi) = \frac{4\pi}{R^2} \int_0^\Psi d\mathcal{E} \int_0^{R\sqrt{2(\Psi-\mathcal{E})}} dJ_z \; h_-(\mathcal{E}, J_z)\, J_z \, . \tag{5.1}$$

An important remark is in order. The homeoidal expressions we have used so far are truncated density–potential pairs. They can be seen in two different ways: as the first-order expansion of the ellipsoidal parent galaxy model in the limit of small flattening or as an independent non-spherical system. In the first interpretation ("η-linear"), only linear terms in the flattening are retained in the solution of the Jeans equations; in the second interpretation ("η-quadratic"), the Jeans equations contain up to quadratic terms in η. We are going to report the η-quadratic solution and the logical steps leading to it; if one chooses to work in the η-linear regime, orders higher than one in η can be neglected.

The JEs for a two-integral axisymmetric system (Chapter 2) are written in this way:

$$\begin{cases} \dfrac{\partial \rho\sigma^2}{\partial z} = \rho \dfrac{\partial \Psi_T}{\partial z} \, , \\[2ex] \dfrac{\partial \rho\sigma^2}{\partial R} - \dfrac{\rho\Delta}{R} = \rho \dfrac{\partial \Psi_T}{\partial R} \, , \end{cases} \tag{5.2}$$

where $\sigma_R^2 = \sigma_z^2 \equiv \sigma^2$ and $\Delta \equiv \overline{v_\varphi^2} - \sigma^2$. In our case, $\Psi_T = \Psi$ given by (4.2).

The streaming velocity $\overline{v_\varphi}$ is a free parameter of the problem. A useful parametrization, allowing to split $\overline{v_\varphi^2}$ into its ordered ($\overline{v_\varphi}$) and dispersion (σ_φ) components, is the phenomenological Satoh (1980) k-decomposition:

$$\overline{v_\varphi} = k\sqrt{\Delta} \, , \qquad \sigma_\varphi^2 = \sigma^2 + (1 - k^2)\Delta \, ; \tag{5.3}$$

$k = 1$ is the case of the isotropic rotator, while for $k = 0$ no net rotation is present.

If one integrates first the "vertical" JE (first Eq. in (5.2)), at fixed R, and imposing the boundary condition of a vanishing "pressure" $\rho\sigma^2 = 0$ for $z \to \infty$, it is possible to obtain σ:

$$\rho\sigma^2 = -\int_z^\infty dz' \, \rho \frac{\partial \Psi}{\partial z'} = -\int_r^\infty dr' \, \rho \frac{\partial \Psi}{\partial r'} \, , \tag{5.4}$$

where $r' = \sqrt{R^2 + z'^2}$. Here, the first advantage of using homeoidal expressions (4.1)-(4.2) is evident: since the integration is performed at constant R, and the functions ρ_i and Ψ_i are spherically symmetric, the expression of $\rho\sigma^2$ consists just in the evaluation of a certain number of integrals over the spherical radius.

Once a solution for the vertical JE is available, no further integration would be required to obtain Δ from the "radial" JE (the second Eq. in (5.2)):

$$\begin{aligned}\frac{\rho\Delta}{R} &= \frac{\partial\rho\sigma^2}{\partial R} - \rho\frac{\partial\Psi}{\partial R} \\ &= \int_z^\infty dz' \left(\frac{\partial\Psi}{\partial R}\frac{\partial\rho}{\partial z'} - \frac{\partial\Psi}{\partial z'}\frac{\partial\rho}{\partial R}\right) \equiv [\Psi,\rho] \;,\end{aligned} \tag{5.5}$$

where the last expression is in terms of a commutator between potential and density. The advantage of using the commutator for the specific case of homeoidal expressions is in the following property:

$$\begin{aligned}[f(R)u(r), g(R)v(r)] =& \frac{df(R)}{dR}g(R)\int_r^\infty dx\, u(x)\frac{dv(x)}{dx} + \\ & - f(R)\frac{dg(R)}{dR}\int_r^\infty dx\,\frac{du(x)}{dx}v(x)\end{aligned} \;. \tag{5.6}$$

Combining the Eq.s (4.1)-(4.2), (5.5) and (5.6), it can be shown that

$$\rho\Delta = 2\eta R^2\left[Z_{02}(r) + \eta Z_{12}(r) + \eta R^2 Z_{22}(r)\right] \;, \tag{5.7}$$

with

$$\begin{aligned}Z_{ij}(r) &\equiv H_{ij}(r) - \rho_i(r)\Psi_j(r)\,, \\ H_{ij}(r) &\equiv X_{ij}(r) + X_{ji}(r)\,, \\ X_{ij}(r) &\equiv -\int_r^\infty dt\,\rho_i(t)\frac{d\Psi_j(t)}{dt}\,.\end{aligned} \tag{5.8}$$

Let's focus now on the left-hand side of (5.1). Thanks to the Satoh decomposition (5.3),

$$\begin{aligned}\rho\overline{v_\varphi} &= \rho k\sqrt{\Delta} \\ &= k\sqrt{\rho\;(\rho\Delta)}\,;\end{aligned}$$

our purpose is just to show that our method works for the first term of h_-, so we take only the η-linear term of the expression (5.7). In this way, and by expanding $\rho^{1/2}$ for small η, we end up with:

$$\rho\overline{v_\varphi} = k\sqrt{2\rho_0 Z_{02}}\eta^{1/2}R + \eta^{3/2}R... + \eta^{3/2}R^3...\,. \tag{5.9}$$

By assuming that the first term of the above expression is the left-hand side of (5.1), this one becomes:

$$k\sqrt{2\rho_0 Z_{02}}\eta^{1/2}R = \frac{4\pi}{R^2}\int_0^{\Psi(\eta)} d\mathcal{E}\int_0^{R\sqrt{2[\Psi(\eta)-\mathcal{E}]}} dJ_z\, h_-(\mathcal{E},J_z;\eta)\,J_z\,. \tag{5.10}$$

5.2 The first term of the odd part

By looking at (5.10), in order to obtain at the right-hand side the analogous of $k\eta^{1/2}R\cdot$ "spherical", we make the *ansatz*

$$\boxed{h_-(\mathcal{E}, J_z; \eta) = k\eta^{1/2} J_z h_{-00}(\mathcal{E})\,,} \tag{5.11}$$

so we assume that *the DF contains the k Satoh parameter*. By inserting this in (5.10), we get

$$k\sqrt{2\rho_0 Z_{02}}\eta^{1/2}R = k\eta^{1/2}R\frac{4\pi}{3}\int_0^{\Psi(\eta)} d\mathcal{E}\, h_{-00}(\mathcal{E})\left(2\left[\Psi(\eta)-\mathcal{E}\right]\right)^{3/2}\,,$$

and we immediately see that the terms $k\eta^{1/2}R$ cancel out, with the remaining identity given by:

$$\sqrt{2\rho_0 Z_{02}} = \frac{4\pi}{3}\int_0^{\Psi(\eta)} d\mathcal{E}\, h_{-00}(\mathcal{E})\left(2\left[\Psi(\eta)-\mathcal{E}\right]\right)^{3/2}\,. \tag{5.12}$$

The identity above is valid $\forall\eta$, included $\eta = 0$, in which case:

$$A(\Psi_0) = \int_0^{\Psi_0} d\mathcal{E}\, h_{-00}(\mathcal{E})\left(2\left[\Psi_0-\mathcal{E}\right]\right)^{3/2}$$

$$\frac{\sqrt{2}}{3}\frac{d^2A}{d\Psi_0^2}(\Psi_0) = \int_0^{\Psi_0} d\mathcal{E}\,\frac{h_{-00}(\mathcal{E})}{\sqrt{\Psi_0-\mathcal{E}}} \qquad \textit{Abel}$$

$$\boxed{h_{-00}(\mathcal{E}) = \frac{\sqrt{2}}{3\pi}\frac{d}{d\mathcal{E}}\int_0^{\mathcal{E}} d\Psi_0\,\frac{d^2A}{d\Psi_0^2}\frac{1}{\sqrt{\mathcal{E}-\Psi_0}}\,,} \tag{5.13}$$

where we have supposed that $\Psi_0(r)$ is monotonic and invertible, so to get $r = r(\Psi_0)$; we also have defined

$$A(\Psi_0) \equiv \frac{3}{4\pi}\sqrt{2\rho_0 Z_{02}}\,. \tag{5.14}$$

CHAPTER 6

Discussion and Conclusions

6.1 The problem

The unknown answers around formation, structure and evolution of galaxies are strictly related to our ability of modelling them. However, there are many cases in which, given an observed galaxy, more than one dynamical model is possible for it. Among the techniques used as of today, the analytical approach is the only one allowing, in principle, to check the physical validity of a model, by testing the positivity of the phase-space Distribution Function (DF) of the system across the entire phase space. Whenever a model turns out to have a somewhere negative DF, this must be discarded as unphysical, even if it well fits the observational data for that galaxy. In order to apply these tests on models, their DFs must be well known mathematical objects.

Unsurprisingly, the vast majority of our knowledge about DFs comes from spherical systems, where remarkable inversion techniques (Eddington formula and its generalizations) allow to compute the DF of a model given its density and potential profiles. The problem arises when exiting the spherical symmetry, the next natural generalization of which being axial symmetry. The mathematical passage from spherical to axisymmetric models not only exhibits technical difficulties, but introduces new problems which were not existing in spherical symmetry. In fact, the axisymmetric equivalent of Eddington inversion formula requires the introduction of complex variables (as in contour integrals, Laplace-Mellin transforms, analytic extensions in the complex plane) in substitution of the real (physical) variables. It is not a surprise that the DF of axisymmetric systems is much less understood, and, whenever it can be obtained form these methods, its determination often requires an enormous effort.

In this thesis, we explore the axisymmetric scenario by considering small deviations from spherical symmetry. In this regime, the general expressions for self-consistent density-potential pairs become the homeoidal expansion formulae, which show the parameter η (representing the flattening of the system along the minor axis) in a linear fashion. It must be pointed out that, even though they are obtained in the regime of small η, the mathematical structure of the homeoidal expansions can be also valid for non-weakly flattened systems. Moreover, given the homeoidal-like expressions, the associated JEs are well known. These, together with the assumption of a two-integral DF

$f(\mathcal{E}, J_z)$ (where, for each stellar orbit, the relative specific energy $\mathcal{E}$ and the axial component of the specific angular momentum J_z are regular Integrals of Motion (IoM) of the total gravitational potential), allow to find new expressions for the DF. Our expectations were that the inversion formulae in the weak flattening regime have to be similar to the Eddington inversion inherited from spherical systems. In a two-integral axisymmetric system, density distribution is simply given by the even part of the DF in the argument J_z (indicated as $f_+(\mathcal{E}, J_z)$), while the odd part ($f_-(\mathcal{E}, J_z)$) completely determines the streaming velocity field. We first studied the even part, finding a new mathematical method able to invert its integral formulation, given the homeoidal density-potential pair. The same method is applicable to the odd part, given the same density-potential pair and the streaming velocity field, with the support of JEs and the phenomenological Satoh decomposition.

6.2 Our approach

In Chapter 2 we presented general dependence properties of the DF and how these translate to the dynamical features of the corresponding system, being it spherical or axisymmetric. Jeans theorem and Noether theorem associate symmetries of the system to its IoM, which become the new arguments of the DF. Depending on whether the DF is written in terms of one or two IoM, general properties of its velocity moments can be shown. In spherical symmetry, the DF can be easily recovered from the density profile thanks to the Eddington inversion formula, regardless of the internal orbital distribution.

Chapter 3 is dedicated to introducing the useful homeoidal expansions, obtained in the framework of weakly flattened stellar systems but which actually show a mathematical structure valid also for non-weakly flattened objects. Homeoidal expansions give a self-consistent density-potential pair, and overcome the difficulties related to determining the gravitational potential given a density profile. Moreover, they guarantee control on the flattening parameter η, that appears in a linear and explicit form. Since this thesis aims to understand the effects of flattening, these expressions will represent the basic tools required for this purpose.

Chapter 4 displays new general results about the even part of the DF, obtained by imposing that (2.53) is satisfied by the exact homeoidal density-potential pair. Once the method is verified to work well for the even part, the treatment is applied unchanged, in Chapter 5, to the odd part of the DF, with the support of the JEs (adapted to the homeoidal expressions) and of the phenomenological Satoh decomposition, which parametrizes the streaming velocity field starting from second-order moments of the DF in the velocity space.

6.3 Results

Given the even part of the DF (even in the argument J_z) $h_+(\mathcal{E}, J_z; \eta)$, we found that, in order for it to be compatible with the homeoidal density-potential pair as in (4.1)-(4.2), it must rigorously satisfy the properties stated in the following theorems.

Theorem 1. *In the spherical limit* $\eta = 0$*, the function* $h_+(\mathcal{E}, J_z)$ *reduces to* $h_+(\mathcal{E})$*, i.e. the spherical limit is isotropic and the inversion formula for (4.3) reduces to the Eddington inversion. Mathematically,* $\eta = 0$ *cancels the dependence of* h_+ *on* J_z*.*

Proof: see page 43

Theorem 2. *The function* $h_+(\mathcal{E}, J_z; \eta)$ *cannot be linear in* η*, because otherwise, once integrated in (4.3), it would give non-linear terms in* η*.* $h_+(\mathcal{E}, J_z; \eta)$ *must contain all orders of* η*.*

Proof: see page 44

Theorem 3. *Given a spherically perturbed system with density* $\rho(r) = \rho_0(r) + \eta\rho_1(r)$ *and associated potential* $\Psi(r) = \Psi_0(r) + \eta\Psi_1(r)$*, the even part of the DF of the system is a series* $\sum_{n=0}^{\infty} \eta^n h_{+n}(\mathcal{E})$*.*

Theorem 4. *In the function* $h_+(\mathcal{E}, J_z; \eta)$*, all terms of order* $n \geq 2$ *in* η *are such that, when the integrations in (4.5) are performed, their resulting integrals cancel out. The full integration results in a linear structure on the parameter* η*.*

Theorem 5. *The* η*-linear-order term of* $h_+(\mathcal{E}, J_z; \eta)$ *is*

$$h_{+,lin}(\mathcal{E}, J_z) \equiv h_{+10}(\mathcal{E}) + h_{+11}(\mathcal{E}) J_z^2 \,, \tag{4.20}$$

where:

$$h_{+10}(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2} \frac{d}{d\mathcal{E}} \int_0^{\mathcal{E}} d\Psi_0 \, \frac{dH_{10}}{d\Psi_0} \frac{1}{\sqrt{\mathcal{E} - \Psi_0}} \,, \tag{4.21}$$

$$h_{+11}(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2} \frac{d}{d\mathcal{E}} \int_0^{\mathcal{E}} d\Psi_0 \, \frac{d^2 H_{11}}{d\Psi_0^2} \frac{1}{\sqrt{\mathcal{E} - \Psi_0}} \,, \tag{4.22}$$

given

$$H_{10}(\Psi_0) \equiv \rho_1(\Psi_0) - 4\pi\Psi_1(\Psi_0) \int_0^{\Psi_0} d\mathcal{E} \, \frac{h_{+00}(\mathcal{E})}{\sqrt{2(\Psi_0 - \mathcal{E})}} \,, \tag{4.23}$$

$$H_{11}(\Psi_0) \equiv \rho_2(\Psi_0) - 4\pi\Psi_2(\Psi_0) \int_0^{\Psi_0} d\mathcal{E} \, \frac{h_{+00}(\mathcal{E})}{\sqrt{2(\Psi_0 - \mathcal{E})}} \,, \tag{4.24}$$

and with $h_{+00}(\mathcal{E})$ *being the zero-order term of* $h_+(\mathcal{E}, J_z; \eta)$*.*

Proof: see page 48

Putting all these theorems together, the mathematical structure of the even part of the DF is given by the following expression:

$$h_+(\mathcal{E}, J_z; \eta) = \sum_{n=0}^{\infty} \eta^n \sum_{k=0}^{n} h_{+nk}(\mathcal{E}) J_z^{2k} \, .$$

It is worth to point out that the functions $H_{10}(\Psi_0)$ and $H_{11}(\Psi_0)$ are known terms, once we know how to write the single terms of the homeoidal expressions (4.1)-(4.2), for example from observational/theoretical modelling.

All the functions constituting the $\mathcal{E}$-dependence of $h_+(\mathcal{E}, J_z; \eta)$ are ergodic, in the sense that they have $\mathcal{E}$ as the only argument, describing an isotropic behaviour. The anisotropic behaviour is introduced by the dependence on J_z, in the innermost series of the expression above. The full series $\sum_{n=0}^{\infty}$ can be interpreted, in the particular case of small flattening η, as a Taylor expansion of $h_+(\mathcal{E}, J_z; \eta)$. The corresponding series in the spherical case $(\sum_{n=0}^{\infty} \eta^n h_{+n}(\mathcal{E}))$ must be an infinitesimal of higher order than $\sqrt{2(\Psi - \mathcal{E})}$, for $\mathcal{E} \to \Psi$, where Ψ is the relative gravitational potential of the spherical homeoidal density-potential pair.

For what concerns the odd part, we have seen that our method works well in determining, at least, the term

$$\boxed{h_-(\mathcal{E}, J_z; \eta) = k\eta^{1/2} J_z h_{-00}(\mathcal{E}) \, ,} \tag{6.1}$$

with

$$\boxed{\begin{aligned} h_{-00}(\mathcal{E}) &= \frac{\sqrt{2}}{3\pi} \frac{d}{d\mathcal{E}} \int_0^{\mathcal{E}} d\Psi_0 \, \frac{d^2 A}{d\Psi_0^2} \frac{1}{\sqrt{\mathcal{E} - \Psi_0}} \, , \\ A(\Psi_0) &\equiv \frac{3}{4\pi} \sqrt{2\rho_0 Z_{02}} \, , \\ Z_{ij}(r) &\equiv H_{ij}(r) - \rho_i(r)\Psi_j(r) \, , \\ H_{ij}(r) &\equiv X_{ij}(r) + X_{ji}(r) \, , \\ X_{ij}(r) &\equiv - \int_r^{\infty} dt \, \rho_i(t) \frac{d\Psi_j(t)}{dt} \, . \end{aligned}} \tag{6.2}$$

This is responsible for the first term of the JEs solutions in case of homeoidal density-potential pairs. Notice that the *ansatz* (6.1) necessarily depends on $\sqrt{\eta}$ and contains the Satoh k parameter.

6.4 Future developments

In this thesis, we have found original inversion formulae for the DF of two-integral axisymmetric stellar systems, thereby introducing a novel inversion technique in a general sense. It will be important to study in a more accurate way the convergence and other mathematical/algebraic properties of the series appearing in the DF. Moreover, the more general odd part of the DF is still to be studied with this method. We also still have to understand how the new method behaves when applied to the known analytical models of galaxies, and/or to real astrophysical systems.

The technique developed in this thesis lays the groundwork for a deeper understanding of the phenomenological Satoh decomposition, since it can possibly give a hint on how this decomposition is encoded in the phase space, which is, to date, an unsolved problem. An interesting direction for future research would be to investigate whether the properties of Satoh decomposition can be derived from those of the DF (e.g. positivity condition, etc.). We expect, for example, that by requiring a consistent DF we can find limits on the k parameter. Our step is the first constructive one in this sense.

In conclusion, the present work may represent a first step towards a phase-space interpretation of the Satoh decomposition and, more generally, towards a deeper understanding of the dynamical structure of axisymmetric stellar systems.

Appendix

A.1 Special functions

Heaviside step function

$$\theta(x) = \begin{cases} 1 \ , & x \geq 0 \\ 0 \ , & x < 0 \end{cases} \tag{A.3}$$

A.2 Abel inversion

Let $g_1(x)$ be a sufficiently regular function defined over the interval $x_m \leq x \leq x_M$. For $0 < \alpha < 1$, the following identity holds:

$$f_1(x) = \int_{x_m}^{x} \frac{g_1(t)\, dt}{(x-t)^{\alpha}} \ , \quad g_1(x) = \frac{sin(\pi\alpha)}{\pi} \frac{d}{dx} \int_{x_m}^{x} \frac{f_1(t)\, dt}{(x-t)^{1-\alpha}} \tag{A.4}$$

A.3 Derivative of an integral function in different dependence scenarios

Let both the upper limit of the integral and the integrand depend on a parameter α; the derivative with respect to α is:

$$\frac{d}{d\alpha} \int_0^{B(\alpha)} dx\ q(x;\alpha) = \int_0^{B(\alpha)} dx\ \frac{\partial}{\partial\alpha} q(x;\alpha)\ + \frac{dB}{d\alpha} q\left(B(\alpha);\alpha\right) \tag{A.5}$$

Let the integrand depend on the variable B (which is the variable of the whole integral function); the derivative with respect to B is:

$$\frac{d}{dB} \int_0^{B} dx\ q(x,B) = \int_0^{B} dx\ \frac{\partial q}{\partial B}\ + q(B,B) \tag{A.6}$$

A.4 Test: the "amphibious" case

Consider the case $\rho_0 = \rho_1$. The expression for the density becomes $\rho(r) = (1+\eta)\rho_0(r)$. What about the potential?
First of all, (ρ_0, Ψ_0) and (ρ_1, Ψ_1) are exact density-potential pairs related by the Poisson equation. This can be proved in few passages, starting from the general Poisson equation:

$$\begin{aligned} lap(\Psi) &= -\, 4\pi G\rho \\ lap(\Psi_0 + \eta\Psi_1) &= -\, 4\pi G\, (\rho_0 + \eta\rho_1) \\ lap(\Psi_0) + 4\pi G\rho_0 &= -\, \eta\, (lap(\Psi_1) + 4\pi G\rho_1) \ ; \end{aligned}$$

this equality is valid $\forall\eta$ if and only if $lap(\Psi_1) + 4\pi G\rho_1 = 0$, from which it follows that also $lap(\Psi_0) + 4\pi G\rho_0 = 0$.
However, if $\rho_0 = \rho_1$, it is not trivial that also $\Psi_0 = \Psi_1$. In fact:

$$\begin{aligned} lap(\Psi_0 + \eta\Psi_1) &= -\, 4\pi G(1+\eta)\rho_0 \\ 0 &= -\, \eta\, (lap(\Psi_1) + 4\pi G\rho_0) \\ lap(\Psi_1) + 4\pi G\rho_0 &= 0 \\ lap(\Psi_1) &= lap(\Psi_0) \\ div\, grad(\Psi_1) &= div\, [grad(\Psi_0 + c) + \mathbf{f}] \\ \Psi_1(r) &= \Psi_0(r) + c\ , \end{aligned}$$

where c is an arbitrary constant and $\mathbf{f}$ is an arbitrary solenoidal vector field. For simplicity, we are going to consider only the case in which the arbitrary quantities are null. So $\Psi_1 = \Psi_0$ and the potential becomes $\Psi(r) = (1+\eta)\Psi_0$. We are interested in checking whether the factor $(1+\eta)$, which enters the integral in a non linear way, has some effect on the resulting expressions for h_{+0}, h_{+1} and h_{+2}.
The starting integral is now

$$(1+\eta)\rho_0 = 4\pi \int_0^{(1+\eta)\Psi_0} d\mathcal{E}\ h_+(\mathcal{E};\eta)\sqrt{2(\,(1+\eta)\Psi_0 - \mathcal{E})}$$

and again $h_+(\mathcal{E};\eta)$ is a series of the form given in (4.10). It can already be seen that by setting $\eta = 0$ we immediately obtain the same equation which led, in the more general case, to the expression for h_{+0}, so we end up with the same formula as (4.7).
By taking the first derivative of the equation above with respect to η and then setting $\eta = 0$ we get

$$\rho_0 = 4\pi \int_0^{\Psi_0} d\mathcal{E} \left[h_{+1}(\mathcal{E})\sqrt{2(\Psi_0 - \mathcal{E})} + \frac{h_{+0}(\mathcal{E})}{\sqrt{2(\Psi_0 - \mathcal{E})}}\Psi_0 \right] \ ,$$

which differs from the more general case only for ρ_0 in place of ρ_1 and Ψ_0 in place of Ψ_1. Therefore, instead of having H_1 as defined in (4.12) we use

$$H_{1,test} = \rho_0 - 4\pi\Psi_0 F_1$$

and obtain

$$h_{+1}(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2}\frac{d}{d\mathcal{E}}\int_0^{\mathcal{E}} d\Psi_0 \frac{dH_{1,test}}{d\Psi_0}\frac{1}{\sqrt{\mathcal{E}-\Psi_0}} \ .$$

Taking the second derivative of the starting integral with respect to η, we get

$$\begin{aligned}0 =& 4\pi \Bigg\{ \int_0^{(1+\eta)\Psi_0} d\mathcal{E} \left[(2h_{+2} + 6\eta h_{+3} + ...) \sqrt{2((1+\eta)\Psi_0 - \mathcal{E})} + \frac{(h_{+1} + 2\eta h_{+2} + ...)}{\sqrt{2((1+\eta)\Psi_0 - \mathcal{E})}}\Psi_0 \right] + \\ &+ \Psi_0 \int_0^{(1+\eta)\Psi_0} d\mathcal{E} \left[\frac{(h_{+1} + 2\eta h_{+2} + ...)}{\sqrt{2((1+\eta)\Psi_0 - \mathcal{E})}} - \frac{\sum_{n=0}^{\infty}\eta^n h_{+n}}{(2((1+\eta)\Psi_0 - \mathcal{E}))^{\frac{3}{2}}}\Psi_0 \right] + \\ &+ \Psi_0^2 \left. \frac{\sum_{n=0}^{\infty}\eta^n h_{+n}}{\sqrt{2((1+\eta)\Psi_0 - \mathcal{E})}} \right|_{\mathcal{E}=(1+\eta)\Psi_0} \Bigg\} \ ,\end{aligned}$$

which is the same as we obtained in the more general case, with the only difference that now Ψ_0 replaces Ψ_1. So, instead of H_2 defined in (4.18), we use

$$H_{2,test} = \Psi_0 A_2(\Psi_0) + \Psi_0^2 B_2(\Psi_0)$$

with $A_2(\Psi_0)$ and $B_2(\Psi_0)$ defined in (4.16)-(4.17), and finally get

$$h_{+2}(\mathcal{E}) = \frac{1}{\sqrt{2}\pi}\frac{d}{d\mathcal{E}}\int_0^{\mathcal{E}} d\Psi_0 \frac{dH_{2,test}}{d\Psi_0}\frac{1}{\sqrt{\mathcal{E}-\Psi_0}} \ .$$

In conclusion, all the general formulas (4.7), (4.14) and (4.19) giving the expressions for h_{+0}, h_{+1} and h_{+2} are valid also for the special case $\rho_0(r) = \rho_1(r)$, in which they can be used normally just by substituting Ψ_1 with Ψ_0.

A.5 Biunivocal correspondence of coefficients in a second-order linear algebraic equation

Consider a, b, c, d, e, f be arbitrary real numbers, and $\eta \in [0;1]$ a real variable for which the following equation is true

$$a + b\eta + c\eta^2 = d + e\eta + f\eta^2 \quad , \forall \eta \in [0;1] \ . \tag{A.7}$$

One may be temped to believe that the validity of this equation can be reached via some strange combinations of the real numbers a, b, c, d, e, f, which may not necessarily be correspondingly equal. However, the truth is that in order for Eq. (A.7) to be valid, $a = d \wedge b = e \wedge c = f$ is a necessary condition. Indeed, if the equation is valid $\forall \eta \in [0;1]$, then it must be valid also when $\eta = 0$; in this case, the equation reduces to

$a = d$. But both a and d are real *numbers*, and once their values are determined these must be preserved whatever η is. So after algebraic simplifications, Eq. (A.7) becomes $b + c\eta = e + f\eta$. Again, this latter equation is valid also for $\eta = 0$, leading to $b = e$ and finally to $c = f$.

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