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On Alexander polynomial and Grid Homology for knots in the 3-sphere and in lens spaces

Tesi di Laurea Magistrale in Topologia Geometrica

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Contents

Introduction

The aim of this thesis is to analyze two different knot invariants: Grid Homology and the Alexander polynomial, for knots in the 3-sphere and in lens spaces, and to study their correlations in S^3.

Heegaard Floer Homology, defined by Ozsváth and Szabó [23], is a powerful invariant of closed, oriented three-manifolds. In 2003, Ozsváth and Szabó [22] and, independently, Rasmussen [26], defined Knot Floer Homology, providing an invariant for links in three-manifolds. This is the homology of a chain complex whose generators can be read off from a Heegaard diagram of the three-manifold, and whose differential counts pseudo-holomorphic disks.
In 2006, Manolescu, Ozsváth, and Sarkar [20] introduced a combinatorial method to compute Knot Floer Homology by reading the generators and the differentials from a grid diagram associated to the link in S^3; the resulting homology is called Grid Homology. Grid diagrams are simple combinatorial presentations of a link in S^3 defined by Brunn in the 19^{th} century [4] and widely studied by Cromwell [10] and Dynnikov [14]. In 2008, Baker, Grigsby, and Hedden [2] extended these results to lens spaces $L(p,q)$, providing a combinatorial description of Knot Floer Homology in these manifolds.
Lens spaces $L(p,q)$ are the 3-manifolds admitting a genus-1 Heegaard decomposition; that is, they are obtained by considering two solid tori V_1, V_2 and gluing the meridian α_2 of V_2 to a $p\beta_1 + q\alpha_1$ curve of V_1, which is a simple curve going p times around the longitude β_1 and q times around the meridian α_1 of V_1.
Knot Floer Homology has many important applications in knot theory due to its relations with the Alexander polynomial, the Seifert genus of a knot, and the Berge conjecture [3].
The relation between Grid Homology and the Alexander polynomial, exposed in [24], is well known for knots in S^3; indeed, it has been proven that the graded Euler characteristic of the grid chain complex associated to a knot is equal to its Alexander polynomial. However, in lens spaces, this relation is still unknown; one of the main goals of this thesis is to investigate these relations in lens spaces.

After recalling the basic definitions about the manifolds S^3 and $L(p,q)$ and about links in Chapter 1, we define grid diagrams in both S^3 and $L(p,q)$ in Chapter 2. In S^3, they can be seen as $n \times n$ grids equipped with two special markings, X and O, under the convention that each row and column of the grid must contain exactly one X and one O marking. If we connect these markings by vertical and horizontal segments, adopting the convention that vertical segments overcross horizontal ones, we can associate a link to each grid diagram.
In order to define the homology, we need to consider toroidal grid diagrams, which are grid diagrams viewed on a torus; they can be obtained by identifying the top edge with the bottom edge, and the left edge with the right edge of the grid diagram. This construction allows us to define grid diagrams in lens spaces as well, by considering the Heegaard decomposition of $L(p,q)$. Here, the diagrams appear as an $n \times pn$ grid (i.e., as p copies, called boxes, of an $n \times n$ grid glued together), where we identify the left edge with the right one, and the bottom edge of each box with the top edge of a box on the right, shifted by a factor of q due to the structure of $L(p,q)$. As before, we follow the convention that each row and column contains exactly one X and one O marking with respect to the identifications, and that each vertical segment overcrosses the horizontal ones to reconstruct the link. These diagrams are defined up to three *grid moves*: *commutation*, *stabilization*, and *translation*, which allow us to obtain equivalence classes of links and study their homology up to ambient isotopy. From the grid diagram of a link, we can compute its fundamental group in both S^3 and $L(p,q)$.
In Chapter 3, we analyze different definitions of the Alexander polynomial in S^3. Starting from a purely algebraic construction of the presentation of a module, we deduce a presentation of the Alexander matrix. Once this matrix is obtained, the Alexander polynomial can be defined as the greatest common divisor of the determinants of the maximal-dimension minors of the matrix. Given a knot, we follow two different approaches to obtain this presentation: the first one, more geometrically flavored, consists in building the Seifert matrix from the linking numbers of the generators of the homology group of a Seifert surface of the knot; the second one, more algebraic, consists in building the Alexander-Fox matrix by applying the Fox derivative over the group ring of the fundamental group of the knot complement. We also briefly recall the construction of the Alexander polynomial via Skein relations and prove that the Alexander polynomial is uniquely determined by the Skein relation and its value on the unknot.
We can also compute the Alexander polynomial of a knot from its grid diagram by defining a matrix whose elements are powers of a variable t, where the exponents are calculated using the winding number of the knot around the corresponding point in the grid diagram, and subsequently normalizing by the determinant of this matrix.

In $L(p, q)$, due to the presence of homology torsion, we can define the twisted Alexander polynomial of a knot alongside the classical one. This is due to the fact that the first homology group of the knot complement has torsion. If this torsion group is $\mathbb{Z}_d$, we can define as many twisted Alexander polynomials as there are d-th primitive roots of unity.
In Chapter 4, we present three different flavors of grid homology. We start by defining a grid state as an n-tuple of points in the grid such that each vertical and horizontal line contains exactly one point of the n-tuple. These grid states are the generators of the chain complexes, which are defined over a module $\mathcal{R}$. For knots in S^3, these complexes are equipped with two gradings: the Maslov and the Alexander grading, both of which can be read directly from the grid diagram. In $L(p, q)$, however, we need to define a third grading, called the Spin^c grading, which is related to the underlying structure of the lens space.
The differential required to define grid homology counts specific subregions of the grid called *rectangles*, and the different versions of grid homology differ in how these rectangles are counted. For the simplest version, *fully blocked grid homology*, we consider the field $\mathbb{Z}_2$ as the module $\mathcal{R}$ and we restrict our count to rectangles that do not contain any X or O markings. For the second version, *unblocked grid homology*, we take the module $\mathcal{R} = \mathbb{Z}_2[V_1, \ldots, V_n]$ and allow rectangles to contain O-markings, keeping track of which specific marking is contained by numbering them. For the third version, *simply blocked grid homology*, we consider the module $\mathcal{R} = \mathbb{Z}_2[V_1, \ldots, V_{n-1}]$ and allow rectangles to contain O-markings.
Next, we establish some relationships between these three flavors to assist in calculations, and we outline properties of the different homologies, such as the ability of *simply blocked grid homology* to distinguish a knot from its mirror image. The main result of this chapter is that fully blocked grid homology is not a knot invariant, whereas the other two versions are; this is proven by showing their invariance under grid moves.
For knots in lens spaces, the differentials and the grid homologies are defined in an analogous way, but it can be shown that we can only count rectangles between grid states that share the same Spin^c degree.
In the final section of the chapter, we investigate the relationship between the Alexander polynomial and the different versions of grid homology.
In S^3, it is a well-established result that the graded Euler characteristic of the grid homology of a knot, expressed as a Laurent polynomial, is equal to the symmetrized Alexander polynomial of the knot.
In lens spaces, however, this relationship remains an open question. Nonetheless, a work by Hedden, Wang, and Yang [17] on $(1, 1)$-knots provides an algorithm to compute Knot Floer Homology from a presentation of the knot group. This may offer a promising path toward discovering new correlations in this setting, given

that the twisted Alexander polynomial can also be computed from a presentation of the knot group.
Finally, we provide an appendix containing basic results about groups, modules, and bigraded modules, which may assist the reader throughout the thesis.

Chapter 1

Preliminaries

In this chapter we recall some basic definition from [15, 16, 18, 19] regarding the spaces and the objects that we are going to analyse.

1.1 Smooth manifolds and Heegaard splittings

This section is dedicated to defining the mathematical contexts and spaces within which we will operate, we start by giving the definition of smooth, orientable manifold and of Heegaard splitting and later we will focus on the two main manifold studied in this thesis: S^3 and $L(p,q)$.

Definition 1.1.1. Let M be a topological space and $n \in \mathbb{N}$. An ***n*-dimensional chart for *X*** at a point $x \in M$ is a homeomorphism $\Phi : U \longrightarrow V$ where U is an open neighbourhood of x and

1. V is an open subset of $\mathbb{R}^n$ or
2. V is an open subset of the upper half-space of $\mathbb{R}^n$ and $\Phi(x)$ lies in $\partial\mathbb{R}^n$.

In the former case we say that Φ is a chart of type (1) and in the latter case we say that Φ is a chart of type (2).
We say M is an ***n*-dimensional topological manifold** if M is second-countable, Hausdorff and for all $x \in M$ there exists an n-dimensional chart $\Phi : U \longrightarrow V$ at x.
An ***n*-dimensional atlas** for M is a collection of n-dimensional charts such that the domains of the charts cover all of M.

Sometimes will be useful to work with smooth manifolds, so we extend the previous definition.

Definition 1.1.2. Let M be a topological manifold and $n \in \mathbb{N}$, we say that the two n-dimensional charts $\Phi_1 : U_1 \longrightarrow V_1$ and $\Phi_2 : U_2 \longrightarrow V_2$ are **smoothly compatible** if the transition map

$$\Phi_2 \circ \Phi_1^{-1} : \Phi_1(U_1 \cap U_2) \longrightarrow \Phi_2(U_1 \cap U_2)$$

is a diffeomorphism.
An n-dimensional atlas of a topological manifold for M is called **smooth** if all charts in the atlas are smoothly compatible to one another. We call an atlas $\mathcal{A}$ **maximal** if every chart that is smoothly compatible with all the charts in $\mathcal{A}$ is already contained in $\mathcal{A}$.
We define a **smooth manifold** to be the pair $(M, \mathcal{A})$ where M is a topological manifold and $\mathcal{A}$ a maximal smooth atlas for M

To avoid notational overload we will drop the $\mathcal{A}$ from the definition of a smooth manifold.

Definition 1.1.3. Let M be a smooth manifold, we say that a point $x \in M$ is a **boundary point** if x does not admit a chart of type (1) in the smooth structure. We denote $\boldsymbol{\partial M}$ the set of all boundary points of M and refer to it as the boundary of M.
We say that a smooth manifold M is **closed** if it is compact with $\partial M = \emptyset$.

Definition 1.1.4. Let M be an n-dimensional smooth manifold and $n, k \in \mathbb{N}$.
We say that a subset $N \subseteq M$ is a ***k*-dimensional smooth submanifold** if given any $x \in N$ one of the following holds:

(α) there exists a chart $\Phi\colon U \to V$ of type (1) in the smooth structure for M at x such that

$$\Phi(U \cap N) = V \cap \{(0, \ldots, 0, x_1, \ldots, x_k) \in \mathbb{R}^n \mid x_i \in \mathbb{R}\},$$

(β) or there exists a chart $\Phi\colon U \to V$ of type (2) in the smooth structure for M at x such that $\Phi(x)$ lies in the upper half-space of $\partial\mathbb{R}^n$ and

$$\Phi(U \cap N) = V \cap \{(0, \ldots, 0, x_1, \ldots, x_k) \in \mathbb{R}^n \mid x_k \geq 0\},$$

(γ) or there exists a chart $\Phi\colon U \to V$ of type (1) in the smooth structure for M at x such that $\Phi(x)$ lies in the upper half-space of $\partial\mathbb{R}^n$ and

$$\Phi(U \cap N) = V \cap \{(0, \ldots, 0, x_1, \ldots, x_k) \in \mathbb{R}^n \mid x_k \geq 0\}.$$

We refer to such charts as **submanifold charts** of N. Any collection of submanifold charts of N that covers N is called a **submanifold atlas**.

We need to recall also the notion of orientability. For the aim of this thesis we will use the following one (see [16, Sec. 3.3] for another equivalent definition that sounds more intuitive).

Definition 1.1.5. We say that a closed, connected, n-manifold M is **orientable** if $H_n(M) \cong \mathbb{Z}$.
We say that a connected, n-manifold M with $\partial M \neq \emptyset$ is **orientable** if the homology of M relative to its boundary is $H_n(M, \partial M) \cong \mathbb{Z}$.

Since $\mathbb{Z}$ has two possible generators, namely $+1$ and -1, we can define two possible isomorphism from $H_n(M) \to \mathbb{Z}$. The choice of one of this generators is a choice of an **orientation** for the n-manifold.

In particular we will work with two different 3-manifolds: the sphere S^3 and the lens spaces $L(p, q)$.
As usual we define the *3-sphere* as the subset of $\mathbb{R}^4$

$$S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\} \tag{1.1}$$

We can also consider the 3-sphere to be the boundary of the 4-dimensional ball B^4 defined as

$$B^4 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 1\}.$$

Particularly, denote with B^2 the disc and let $S^1 = \partial B^2$ be the circle.
We introduce the *connected sum* operation for manifolds with or without boundary as a tool to join two separate manifolds.

Definition 1.1.6. Let M, M' be two connected non-empty n-dimensional smooth manifolds. We pick two smooth embeddings $\phi : B^n \longrightarrow M \smallsetminus \partial M$ and $\phi' : B^n \longrightarrow M' \smallsetminus \partial M'$. We define the **connected sum** of M and M' as

$$M \# M' := (M \smallsetminus \phi(B^n)) \sqcup (M' \smallsetminus \phi'(B^n))/\sim$$

where $\phi(x) \sim \phi'(x)$ for all $x \in \partial B^n \cong S^{n-1}$.
If M, M' are oriented we require that one of ϕ and ϕ' is orientation-preserving and the other orientation-reversing.

Definition 1.1.7. Let $n \in \mathbb{N}_{\geq 2}$ and M, M' be two oriented n-dimensional smooth manifolds with non-empty boundary. Let $\phi : B^{n-1} \longrightarrow \partial M$ and $\phi' : B^{n-1} \longrightarrow \partial M'$ be two smooth embeddings, where ϕ is orientation-preserving and ϕ' is orientation-reversing. We define the **boundary connected sum** of M and M' to be the smooth manifold

$$M \#_b M' = (M \sqcup M')/\sim$$

where $\phi(x) \sim \phi'(x)$ for all $x \in \partial B^{n-1}$.

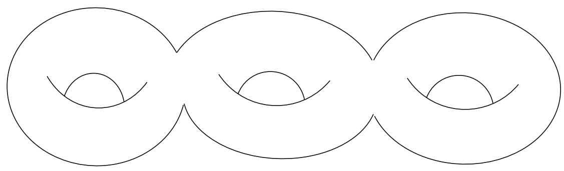


Figure 1.1: Example of an genus-3 handlebody H_3.

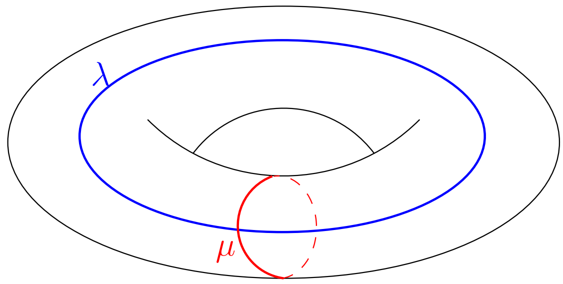


Figure 1.2: Torus with its meridian μ and its longitude λ.

Definition 1.1.8. Consider $g \in \mathbb{N}$, we define a **genus g handlebody** (Figure 1.1) as the oriented, connected, compact 3-manifold H_g with boundary, obtained as the boundary connected sum of g copies of the solid torus $S^1 \times B^2$.

Note that the boundary of a handlebody of genus g is a closed connected orientable surface of genus g.

Definition 1.1.9. Let M be a closed oriented 3-dimensional smooth manifold.

1. A **splitting of M** consists of a triple (X, Y, F) where X and Y are compact 3-dimensional submanifolds of M with $M = X \cup Y$ and such that their intersection is $F = X \cap Y = \partial X = \partial Y$. Given such a splitting we endow F with the orientation coming from $F = \partial X$.

2. A splitting (X, Y, F) of M is called **Heegaard splitting** if X and Y are both diffeomorphic handlebodies with the same genus. The surface F is called **Heegaard surface**. We define the **genus of the the Heegaard splitting** to be the genus of the handlebodies.

We consider, until the end of the thesis, p and q to be two coprime integer number such that $0 \leq q < p$.
We set the **torus** to be $\mathbb{T} = S^1 \times S^1 = \partial(S^1 \times B^2)$.

Definition 1.1.10. Consider the solid torus $V = S^1 \times B^2$, we define a **meridian** of the solid torus to be a simple closed curve $\mu \subset \partial V$ which bounds a disc in V, called *meridional disc*, i.e. $\mu = \{P\} \times \partial B^2$. We define a **longitude** of the solid

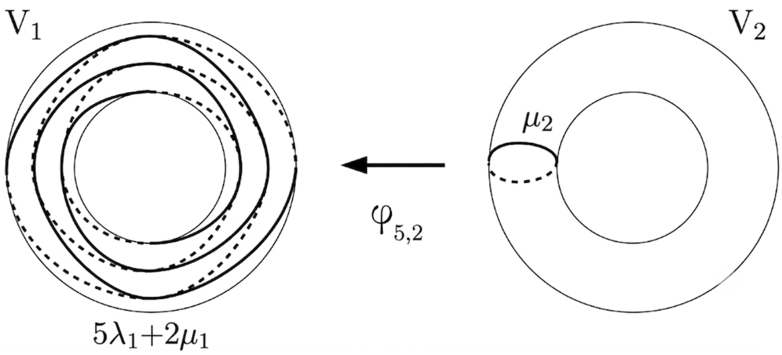


Figure 1.3: Gluing of V_1, V_2 sending μ_2 to $5\lambda_1 + 2\mu_1$.

torus to be a simple closed curve $\lambda \subset \partial V$ which intersects μ only once and is non trivial in the homology of the solid torus, i.e. $\lambda = S^1 \times \{Q\}$.

On the torus $\mathbb{T}$ we define $p\lambda + q\mu$ to be the closed curve that winds p times around the longitude λ and q times around the meridian μ. Choosing p and q to be coprime ensures that the curve is closed.

Definition 1.1.11. Consider two copies V_1, V_2 of a solid torus $V = S^1 \times B^2$. We define the **lens spaces $\boldsymbol{L(p,q)}$** to be the manifold admitting a genus-one Heegaard splitting $L(p,q) = V_1 \cup_{\varphi_{p,q}} V_2$ where we glue the two solid tori via the homeomorphism of their boundaries $\varphi_{p,q} : \partial V_2 \longrightarrow \partial V_1$ that sends the meridian $\mu_2 \subset V_2$ to the curve $p\lambda_1 + q\mu_1 \subset V_1$ as shown in Figure 1.3.

Remark 1. The definition of lens space is well posed, that is, the homeomorphism type of the manifold depends only on the curve $p\lambda_1 + q\mu_1$ chosen [25, Th. 10.2].

We briefly recall two results about the fundamental group and the first homology group of lens spaces:

Proposition 1.1. *Let $p, q \in \mathbb{Z}$ be two coprime integers such that $0 \leq q < p$ and consider the lens spaces $L(p,q)$. We have the following results:*

1. $\pi_1(L(p,q)) \cong \mathbb{Z}_p$
2. $H_1(L(p,q)) \cong \mathbb{Z}_p$.

This result is straightforward from the construction of the lens spaces via Heegaard splitting and the Van Kampen theorem. Furthermore, this construction allows us to see that the generator is the longitude λ_1 of V_1 (in the notation of Definition 1.1.11). As consequence, due to the Hurewicz theorem, also the $H_1(L(p,q))$ will be generated by λ_1.

Remark 2. We can also give a construction of the 3-sphere via Heegaard splitting, gluing two solid tori V_1, V_2 along their boundary with the homomorphism φ sending the meridian $\mu_2 \in V_2$ to the longitude $\lambda_1 \in V_1$.

It is possible to give an equivalent definition of lens spaces as quotients of S^3 under the action of a group isomorphic to $\mathbb{Z}_p$.
More precisely, consider S^3 as the unit vectors in $\mathbb{C}^2$, the lens space $L(p,q)$ is the quotient of S^3 under the action of the group $G_{p,q} \cong \mathbb{Z}_p$ generated by the application $\rho_{p,q} : S^3 \longrightarrow S^3$ defined by

$$\rho_{p,q}(z_1, z_2) = (z_1 e^{\frac{2i\pi}{p}}, z_2 e^{\frac{2i\pi q}{p}}).$$

Clearly, since the action is free and properly discontinuous, the quotient map $P : S^3 \longrightarrow S^3/G_{p,q}$ is the universal covering map. To show the equivalence with the other definition of lens spaces we address the reader to [19].

1.2 Knots and links

We now give the general definition of a link in a 3-manifold.

Definition 1.2.1. A **link** in a closed 3-dimensional manifold M is a pair (M, L) where L is a submanifold of M diffeomorphic to the disjoined union of $m \geq 1$ copies of S^1. This copies are called *components* of the link and denoted as L_i for $i \geq 1$.
A **knot** is a link with one component.
A link $L \subset M$ is called **trivial** if its components bound embedded pairwise disjoint 2-discs $B^2_1, \dots, B^2_m$ in M.
The trivial knot is called **unknot** and denoted as $\mathcal{O}$.
In Figure 1.4 is given an example of some knots and a link.

Remark 3. Along this thesis we will work mainly with links in S^3 and $L(p,q)$. Sometimes is useful, for links in S^3 to consider them as links in $\mathbb{R}^3$. This is always possible because, as said above, since if we look at S^3 as the Alexandroff compactification of $\mathbb{R}^3$, we can always find two disjoint open set, one containing the point to the infinity and the other containing the link. So we can remove this point and obtain a link in $\mathbb{R}^3$.

Definition 1.2.2. l Given a link in $\mathbb{R}^3 \subset S^3$, one can consider its reflected image along a plane which does not intersect it (such reflection is an orientation reversing homeomorphism of S^3). This is called the **mirror image** of the link.

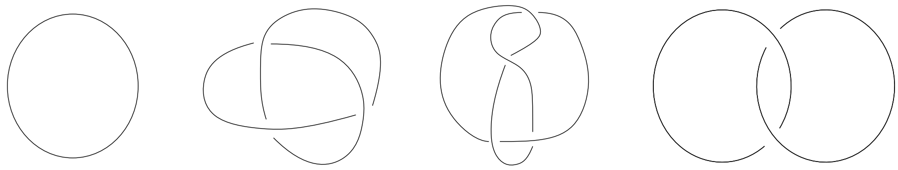


Figure 1.4: From left to right: the unknot, the trefoil knot, the figure-8 knot and the Hopf link.

It is useful to define a way to state whether two links are equal, i.e. i can deform one into the other.

Definition 1.2.3. Two m-components links $L = L_1 \sqcup \cdots \sqcup L_m$ and $\tilde{L} = \tilde{L}_1 \sqcup \cdots \sqcup \tilde{L}_m$ are **equivalent** if there exist a smooth isotopy from L to $\tilde{L}$, i.e. if there exist a smooth map

$$\begin{aligned} F : L \times [0,1] &\longrightarrow M \\ (z,t) &\mapsto F(z,t) \end{aligned}$$

such that holds:

1. For each $t \in [0,1]$ the map $F_t : L \longrightarrow M$ is a smooth embedding.
2. We have F_0=id and for each $i \in \{1, \dots, m\}$ we have $F_1(L_i) = \tilde{L}_i$

If $L, \tilde{L}$ are oriented links, then we also demand that the isotopy is orientation preserving, i.e. the map $F_1 \circ F_0^{-1} : L \longrightarrow \tilde{L}$ is orientation preserving.

Remark 4. It might be unexpected but not all the knots are equivalent to their mirror image, this class of knots is called *chiral knots*.
A simple example is the trefoil knot (Figure 1.4), which we will we will not be equivalent to its mirror image (see Example 14).

Definition 1.2.4. Given a knot K, we define the **knot exterior** to be the compact manifold (with boundary) $X := M \setminus U(K)$, where $U(K)$ is an open tubular neighbourhood of the knot, that is the image of a function $\varphi : S^1 \times B^2 \longrightarrow M$ which is a diffeomorphism restricted to the image, and such that $\varphi(S^1 \times \{(0,0)\}) = K$.

Remark 5. In a similar way can be defined the tubular neighbourhood $U(L)$ of a link $L \in M$, with the attention that the tubular neighbourhoods of the different components of the link are pairwise disjoint.

Given a link $L \subset S^3$ and a tubular neighbourhood $U(L)$, we define the **meridian of the component $\boldsymbol{L_i}$** to be the meridian, as in Definition 1.1.10, of the closure of the open tubular neighbourhood of L_i.

Remark 6. The exterior of a knot in S^3 is a solid torus. It can be easily seen considering the genus 1 Heegaard splitting of S^3, described in Definition 1.1.9; indeed an open tubular neighbourhood of the knot is a solid torus, knotted in S^3, following the knot. Removing this torus from the Heegaard splitting leaves us with a solid torus.

Definition 1.2.5. Given a link $L \subset M$, we define the **link group** as the $\pi_1(X) = \pi_1(M \smallsetminus U(L))$. When L is a knot we call it the **knot group**.

Throughout this thesis, when studying the link group, we will sometimes denote it explicitly as $\pi_1(M \setminus L)$ rather than $\pi_1(X)$. This is done to emphasize the ambient space in which the link is considered and, with a slight abuse of notation, to omit the tubular neighbourhood of the link from the definition.
In Section 2.4 we will show different ways to calculate the knot group for links in S^3 and in $L(p, q)$.

We define now an important surface which will be helpful in the next chapters to compute the Alexander polynomial of a link.

Definition 1.2.6. A **Seifert surface** S for an oriented link $L \subset S^3$ is a connected, compact, oriented 2-dimensional submanifold of S^3 such that $\partial S = L$.

Each link admits a Seifert surface, but we delay a sketch of the proof of this fact together with some properties of the Seifert surfaces in Section 3.1.2 after the definition of the grid diagrams. Moreover, it is not difficult to prove that each link admits infinitely many Seifert surfaces, so it is reasonable to set its genus as follows:

Definition 1.2.7. We define the **genus** g of a link $L \subset S^3$ to be the least possible genus of a Seifert surface for L.

Definition 1.2.8. Given two oriented knots K_1, K_2 in S^3 (oriented), let S be the Seifert surface of K_1 and assume that K_2 meets S transversally. At any intersection point assign $+1$ if the orientation given by S and K_2 agrees with the one of S^3 and -1 otherwise. The **linking number** of K_2 with respect to K_1 is the algebraic sum of all those ± 1 and is denoted by $lk(K_1, K_2)$.

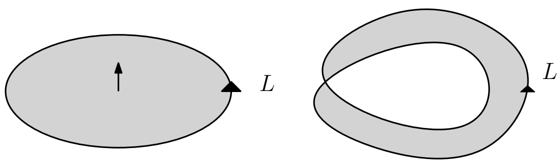


Figure 1.5: Two example of surfaces having the unknot as border. Only the left one is a Seifert surface since the right one is a Moebius strip.

Remark 7. This definition does not depend on the chosen Seifert surface, indeed if S' is another Seifert surface we can reverse its orientation and join them together obtaining the closed surface $\Sigma = S \cup S'$. Since $H_2(S^3) = 0$, then the algebraic intersection between K_2 and Σ is zero (i.e. K_2 enters Σ the same amount of times it exits). So the intersection with S cancel that with S', hence the algebraic intersection of K_2 and S is the same of K_2 and S'.

We consider now a knot in lens spaces and define a property which will be used in Section 4.2.2 to compute the gradings of the grid homology in $L(p, q)$ of the knot K.

Definition 1.2.9. Consider a knot $K \subset L(p, q)$, the **lift** $\widetilde{K}$ of K is the counter-image $P^{-1}(K)$ in S^3 under the quotient map P defined in page 12.

Chapter 2

Grid diagrams

In this chapter, we introduce the concept of grid diagrams, which provide a convenient combinatorial framework to represent knots and links in the 3-sphere S^3 and in lens spaces $L(p,q)$.The core idea is to define a grid on the plane as the intersection of two collections of curves, α and β, and then construct a toroidal grid by identifying the boundaries of the grid in an appropriate manner. Depending on the chosen identification, we obtain either a twisted toroidal grid in $L(p,q)$ or a standard toroidal grid in S^3. By equipping a toroidal grid with a suitable set of markings $\mathbb{X}$ and $\mathbb{O}$, we obtain a (twisted) toroidal grid diagram. Cutting this diagram along a specific choice of α and β curves yields a planar realization of the (twisted) toroidal grid diagram. We begin in Section 2.1 with the construction of toroidal grid diagrams in lens spaces, following [8, 9], and subsequently deduce the construction on the 3-sphere in Section 2.2 as a special case where $p = 1$ and $q = 0$, following [24]. Afterward, we introduce the notion of grid moves for both S^3 and $L(p,q)$, which allow us to transition between equivalent (twisted) toroidal grid diagrams representing the same link. In Section 2.4, we show how to compute the fundamental group of the link exterior from its planar realization in both spaces, following [24] and [7], respectively. Finally, Section 2.5 presents several propositions regarding the homology of links and their exterior in S^3 and $L(p,q)$.

Definition 2.0.1. Given $n, m \in \mathbb{N}$, following Figure 2.1, we define a $\boldsymbol{n \times m}$ **planar grid** G to be an $n \times m$ grid on the plane $\mathbb{R}^2$ obtained by the intersection of n horizontal segments $\tilde{\alpha}_i = (tn, i)$ and m vertical segments $\tilde{\beta}_j = (j, tm)$ with $i \in \{0, \dots, n\}, j \in \{0, \dots, m\}$ and $t \in [0,1]$. The numbers n, m are called **grid numbers** or **dimensions** of the grid.
We call **columns** and **rows** the vertical and horizontal regions of the planar grid, bounded by two consecutive $\tilde{\alpha}_i$ or $\tilde{\beta}_j$ segments respectively, thus the result of this intersection will be a rectangle with n rows and m columns of small squares.

As convention we refer to the left-most column and the bottom-most row as "first".

When only one of the grid number n or m is specified we imply that $n = m$. In Figure 2.1 is shown a 3×12 planar grid, with the segments $\tilde{\alpha}_i$ and $\tilde{\beta}_j$.

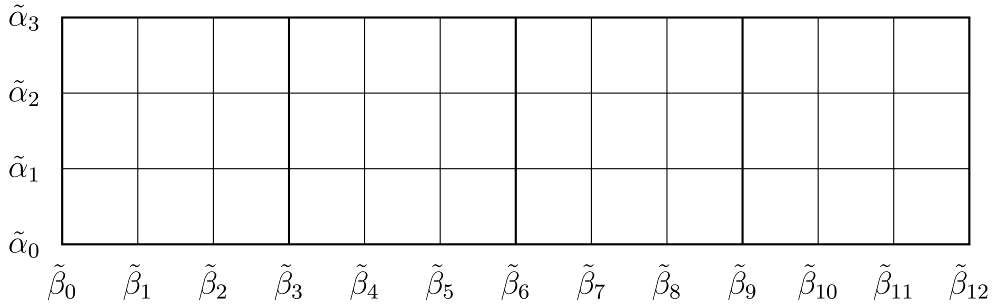


Figure 2.1: An example of a 3×12 planar grid G.

Definition 2.0.2. Given a $n \times m$ planar grid G, if it is possible to identify $\tilde{\alpha}_0$ with $\tilde{\alpha}_n$ and $\tilde{\beta}_0$ with $\tilde{\beta}_m$ so that the grid becomes a torus $\mathbb{T}$ and the set $\tilde{\alpha} = \{\tilde{\alpha}_i\}_{i=0}^n$ (resp. $\tilde{\beta} = \{\tilde{\beta}_i\}_{i=0}^m$) becomes a collection of n disjoint parallel curves on $\mathbb{T}$, we call the resulting grid **toroidal grid**.
We denote by $\alpha = \{\alpha_i\}_{i=1}^n$ (resp. $\beta = \{\beta_i\}_{i=1}^n$) the (collection of) n disjoint curves on $\mathbb{T}$.
As for the planar grid we call **columns** and **rows** the n vertical and horizontal regions of the toroidal grid - each homeomorphic to an annulus - bounded by two consecutive α_i or β_j curves respectively.
As before, the integers n, m are called **grid numbers** or **dimensions** of the toroidal grid.

Fore ease of illustration, we will draw toroidal grid as planar grid, minding the identification of the segments and, with an abuse of notation when the identification rule is clear, we will denote a toroidal grid with G, as for the planar one.
In Figure 2.2 is shown an example of toroidal grid of the planar grid of Figure 2.1, with the curves α_1 and β_1 obtained after the identification, where $\tilde{\alpha}_0$ has been identified with $\tilde{\alpha}_3$ with a horizontal shift such that $\tilde{\beta}_0$ will be joined to $\tilde{\beta}_3, \tilde{\beta}_6, \tilde{\beta}_9, \tilde{\beta}_{12}$ in order to form a single curve.

A toroidal grid inherits a little extra structure from the planar. Using the orientation of the plane determined by the oriented x and y-axes, there are induced orientations on the horizontal and vertical circles: explicitly, the torus is expressed as a product of two circles $\mathbb{T} = S^1 \times S^1$, where $S^1 \times \{p\}$ is a horizontal circle

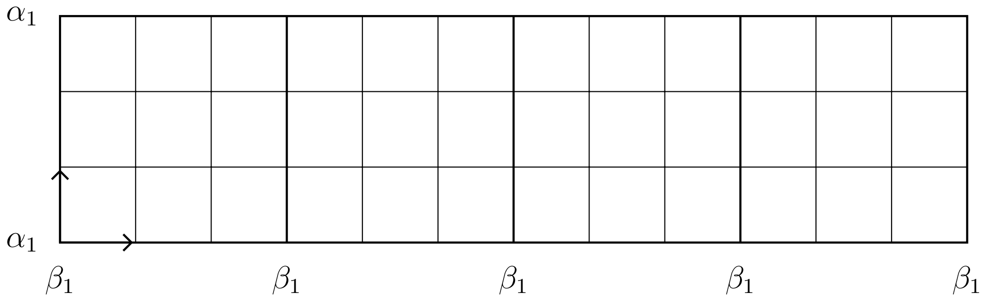


Figure 2.2: An example of a 3×12 toroidal grid G with its orientation.

and $\{p\} \times S^1$ is a vertical circle and both the horizontal and vertical circles are oriented. At each point in the torus, there are four preferred directions, which we think of as North, South, East, and West; more formally, "North" refers to the oriented tangent vector of the circles $\{p\} \times S^1$; "South" to the opposite direction; "East" refers to the positive tangent vector of the circles $S^1 \times \{p\}$; and "West" to its opposite. Correspondingly, each of the squares in the toroidal grid has a northern edge, an eastern edge, a southern edge, a western edge and similarly it has a northern, eastern, southern and western corner.

2.1 Grid diagrams in lens spaces

As seen in the preceding chapter, both S^3 and $L(p,q)$, can be constructed via a genus-one Heegaard splitting, gluing the boundary of two solid tori, with as Heegaard surface a torus $\mathbb{T}$.
When working with links in S^3 and $L(p,q)$, instead of the planar grid, can be useful to consider the toroidal grid, defined as before. Depending on the ambient space we are working with, we will consider two different type of identification obtaining different toroidal grid.

Definition 2.1.1. Consider $n, p, q \in \mathbb{N}$ such that $0 < n$, $0 \leq q < p$ and $(p,q) = 1$, as in the definition of lens spaces. Following Figure 2.1, we consider the toroidal grid G obtained by the $n \times pn$ planar grid by identifying $\tilde{\beta}_0$ to $\tilde{\beta}_{pn}$ and then $\tilde{\alpha}_0$ to $\tilde{\alpha}_n$ as follows:

$$\begin{aligned}
&\tilde{\alpha}_n \ni (s,n) \sim (s - qn \pmod{pn}, 0) \in \tilde{\alpha}_0 \qquad \text{with } s \in [0, pn] \\
&\tilde{\beta}_{pn} \ni (pn, t) \sim (0, t) \in \tilde{\beta}_0 \qquad \text{with } t \in [0, n].
\end{aligned}$$

Note that the condition $(p,q) = 1$ guarantees that after the identifications the planar grid becomes a toroidal grid consisting of n horizontal and n vertical circles,

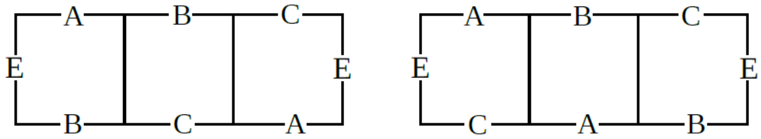


Figure 2.3: Top-bottom identifications for grids in $L(3,1)$ on the left, and $L(3,2)$ on the right.

$\alpha = \{\alpha_i\}_{i=1}^n$ and $\beta = \{\beta_i\}_{i=1}^n$, which bounds n rows and n columns each consisting of pn small squares.

In Figure 2.3 are shown two possible identification, depending on the value of q, of the planar grid done to obtain a toroidal grid, where two segments labelled with the same letter will be identified.

In order to use toroidal grids to describe links we need to introduce a marking on the toroidal grid.
We mark n of these small squares with the symbol X and n the symbol O according to the following rules:

- Each row and each column has exactly square marked with an X and a single square marked with an O.
- No square is marked with both an X and an O.

We denote the set of squares marked with an X by $\mathbb{X}$ and the set of squares marked with an O by $\mathbb{O}$. We will call the toroidal grid together with the choice of the marking a **twisted toroidal grid diagram** with **parameters** n, p, q and denote it $\mathbb{G}$. The integer n will be called **dimension** of the twisted toroidal grid diagram.
We define a **planar realization** of the twisted toroidal grid diagram to be the $n \times pn$ planar grid, together with the $\mathbb{X}$ and $\mathbb{O}$ markings, obtained after cutting the torus along α_i and β_j. The p squares of height/length n, obtained by cutting the torus along α_i and β_j are called **boxes**.
Note that cutting along the α_i and β_j we can obtain n^2 different planar realizations.
We will often forget about the distinction between twisted toroidal grid diagrams and their planar realizations, according to the convention "draw on the plane, think on the torus", and call them simply *grid diagrams* $\mathbb{G}$.
In Figure 2.4 is shown a planar realization of dimension 3 in $L(4,1)$, with the representation of circle α_1 and β_1 after the cutting.

Each twisted toroidal grid diagram encode a link L in $L(p,q)$ that could be reconstructed as follows: recall the Heegaard splitting of $L(p,q)$ as $V_1 \cup_{\varphi_{p,q}} V_2$ and

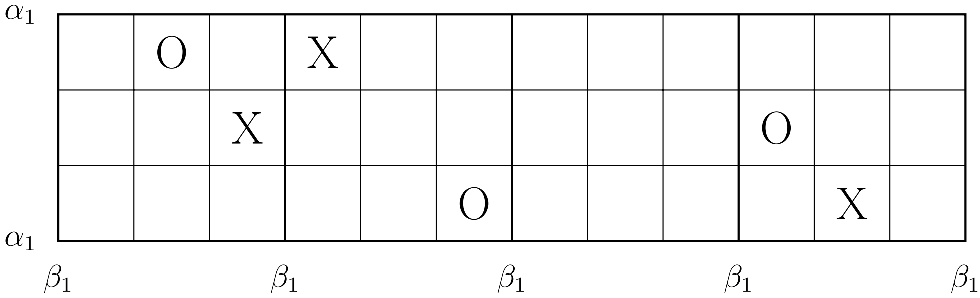


Figure 2.4: An example of a planar realization of dimension 3 in $L(4,1)$ with the sets of markings $\mathbb{X}$ and $\mathbb{O}$.

consider the torus $\mathbb{T}$ as an Heegaard surface for $L(p,q)$, so that the grid is a subset of $L(p,q)$; then, join with a segment each O to the X which lies in the same row, and each X to the O lying in the same column (keeping in mind the twisted identification) with the convention that vertical segments overcross the horizontal one, that is, pushing the horizontal segment into V_1 and the vertical one into V_2. We can assign an orientation to the link L by setting that each vertical segments goes from its X to its O marking and each horizontal segment goes from its O marking to its X marking, as shown in Figure 2.5.

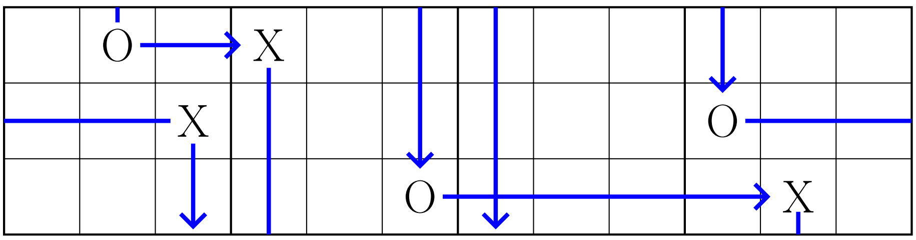


Figure 2.5: An example of a knot K in $L(4,1)$ with grid number 3 and an orientation induced by its X and O markings.

Remark 8. There are two possible ways to connect each X to the corresponding O marking in the same row/column, but the isotopy class of the resulting link does not depend upon the possible choice, indeed the two links given by this different choices for each row/column are isotopic. This follows from the fact that the two possible arcs connecting two markings on the same row/column are related by a slide along a meridional disc of the Heegaard splitting of $L(p,q)$. In other word, the two arcs form a decomposition of the boundary of a meridional disc in the Heegaard splitting.

Theorem 2.1. *Every link in $L(p,q)$ can be represented by a grid diagram.*

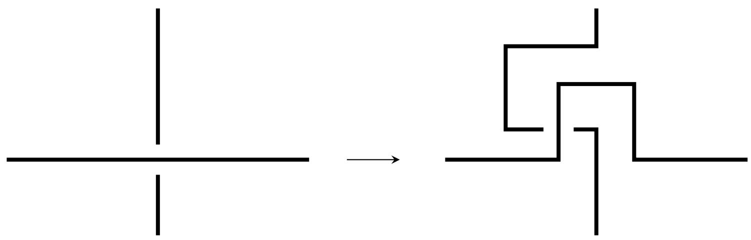


Figure 2.6: Local modification applied to an horizontal crossing.

Proof. Approximate the link L by a PL-embedding with the property that the projection over the Heegaard surface $\mathbb{T}$ admits only horizontal and vertical segments and double points (see [5]). At each crossing for which the horizontal segment is an overcrossing, apply the modification indicated in Figure 2.6. Finally, move the link into general position, so that different horizontal (or vertical) segments are not collinear. Mark the turns by X's and O's. In particular, if L is oriented, choose the marking so that vertical segments point from X to O, while horizontal segments point from O to X. The result is a grid diagram representing L. For all the details see [2]. □

2.2 Grid diagrams in S^3

As we have seen in Section 1.1, we can consider the 3-sphere S^3 as a particular case of lens spaces, with parameter $p = 1$ and $q = 0$; therefore, the notion of toroidal grid diagram in the 3-sphere arises as a particular case of Definition 2.1.1, nevertheless, let's state it explicitly.

Definition 2.2.1. Consider a planar grid of dimension n as in Definition 2.0.1. We consider the toroidal grid obtained by identifying $\tilde{\beta}_0$ to $\tilde{\beta}_n$ and $\tilde{\alpha}_0$ to $\tilde{\alpha}_n$ as follows:

$$\tilde{\alpha}_n \ni (s, n) \sim (s, 0) \in \tilde{\alpha}_0 \qquad \text{with } s \in [0, n].$$
$$\tilde{\beta}_n \ni (n, t) \sim (0, t) \in \tilde{\beta}_0 \qquad \text{with } t \in [0, n]$$

After the identifications, the planar grid is a toroidal grid consisting of n horizontal and n vertical circles, $\alpha = \{\alpha_i\}_{i=1}^n$ and $\beta = \{\beta_i\}_{i=1}^n$, which bounds n rows and n columns each consisting of n small squares.

As before, in order to encode a link in the grid diagram, we mark n of these small squares with the symbol X and n the symbol O according to the following rules:

- Each row and each column has exactly square marked with an X and a single square marked with an O.
- No square is marked with both an X and an O.

We denote the set of squares marked with an X by $\mathbb{X}$ and the set of squares marked with an O by $\mathbb{O}$. We will call the toroidal grid together with the choice of the marking a **toroidal grid diagram** of **dimension** n and denote it $\mathbb{G}$.
We define a **planar realization** of the toroidal grid diagram to be the $n \times pn$ planar grid, together with the $\mathbb{X}$ and $\mathbb{O}$ markings, obtained after cutting the torus along α_i and β_j. Note that cutting along the α_i and β_j we can obtain n^2 different planar realizations.
As before, with the same convention, we will forget about the distinction between the toroidal grid diagram and its planar realizations and, when the context is clear, call them *grid diagrams* $\mathbb{G}$.

As before a toroidal grid diagram encode a link $L \subset S^3$ that could be reconstructed as in the case of lens spaces, by joining with a segment each O to the X which lies in the same row, and each X to the O lying in the same column, with the convention that vertical segments overcrosses the horizontal one. In particular remark (8) is still valid.
Especially we can assign an orientation to the link L by setting that each vertical segment goes from its X to its O marking and each horizontal segment goes from its O marking to its X marking.
Moreover Theorem 2.1 for link in lens spaces remains valid, so each oriented link in the 3-sphere could be represented via grid diagrams.

In Figure 2.7 is shown an example of the trefoil knot in grid diagram of dimension 5. This representation is not unique and, as we will see in the Section 2.3, we can obtain equivalent grid diagrams representing the same knot which will differ by grid moves.
Sometimes, when working with planar realization of toroidal grid diagrams in S^3, can be useful to describe them using permutation $\sigma_{\mathbb{O}}$ and $\sigma_{\mathbb{X}}$ of S_n (i.e. the symmetric group of n elements) as follow: if there is an O marking in the intersection of the i^{th} column and the j^{th} row, then $\sigma_{\mathbb{O}}(i) = j$. We will indicate this permutation as the n-tuple $\sigma_{\mathbb{O}} = (\sigma_{\mathbb{O}}(1), \ldots, \sigma_{\mathbb{O}}(n))$. The permutation $\sigma_{\mathbb{X}}$ is defined analogously using the X markings in place of the O markings.

Remark 9. The orientation of the link in S^3 or in $L(p, q)$, induced by the grid, can be reversed by switching the roles of the $\mathbb{X}$ a the $\mathbb{O}$ and obtaining an equivalent link with the opposite orientation.
Furthermore, for links in S^3, we can obtain the same result by performing a reflection along the diagonal of the grid. The resulting knot will have an orientation

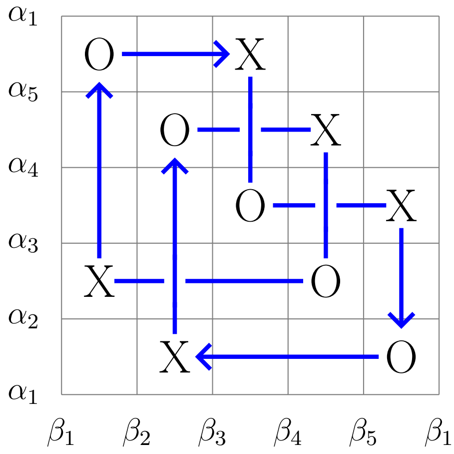


Figure 2.7: Grid diagram for the trefoil knot with its orientation induced by the markings.

not consistent with the one defined above, indeed vertical segments will go from the O markings to the X markings and horizontal segments vice-versa.

Remark 10. Given a grid diagram $\mathbb{G}$ in S^3 and a link L encoded by the grid diagram, recalling the definition of mirror image (see Definition 1.2.2), we can translate it terms of grid diagrams. The mirror image $m(L)$ of L will be obtained by a reflection of $\mathbb{G}$ along the horizontal symmetry axis, indeed, by reflecting, we will commute each overcrossing in an undercrossing and vice-versa.

Usually when studying links in S^3, it is common to use a different notion of diagram (see [24, pg.14]) used to represent links in Figure 1.4.
Given a link in $\mathbb{R}^3$ (we have seen in Remark 3 that we can consider links in S^3 as links in $\mathbb{R}^3$), consider $P : \mathbb{R}^3 \longrightarrow T$ to be the orthogonal projection to an oriented plane $T \subset \mathbb{R}^2$. For a generic choice of T the projection P restricted to L is an immersion with finitely many double points. At these double points, called crossings, we illustrate the strand passing under as an interrupted curve segment. The strand passing under is called undercrossing, while the strand passing over is called overcrossing. If L is oriented, the orientation is specified by placing an arrow on the diagram tangent to each component of L. The resulting diagram is called **regular diagram**.
Sometimes it is useful to transition from one diagram to the other. To do so, we observe that the planar realization of a grid diagram is essentially a PL-realization (piecewise-linear) of a regular diagram. Indeed, we simply need to straighten the strands of the regular diagram so that only vertical and horizontal segments appear, apply local modifications as in Figure 2.6 to ensure that vertical strands always overcross horizontal ones, and place the X and O markings at the corners

according to the convention.
Conversely, consider a planar grid diagram of a link L. By viewing the plane as $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$, we can push the vertical strands up into the upper half-space and the horizontal strands down into the lower half-space. At each marking, we insert a vertical segment (along the z-axis) to connect the horizontal and vertical strands, thereby obtaining a link embedded in $\mathbb{R}^3$. Finally, by smoothing the strands to ensure a well-defined projection, we define $\mathcal{D} = \mathcal{D}(\mathbb{G})$ to be the resulting regular diagram projected onto the plane.

2.3 Grid Moves

In this section we deal with the problem of understanding whether two grid diagram represent equivalent link or not. To do so we introduce the notion of grid moves, which represents a set of moves of the markings in the grid which will modify the presentation of the grid, preserving the equivalence class of the link represented. We state the moves for links in S^3 and then analyse how to extend them to the case of links in $L(p, q)$.

In this section we will call grid diagram a planar realization of a toroidal grid diagram for link in S^3.
We define the *commutation move*, which will allow us to switch two particular rows or columns.

Definition 2.3.1. Each column in a grid diagram determines a closed interval of real numbers that connects the height of its O-marking with the height of its X-marking. Consider a pair of consecutive columns in a grid diagram $\mathbb{G}$ (i.e. they are divided by the same β_j). Suppose that the two intervals associated to the consecutive columns are either disjoint, or one is contained in the interior of the other, as shown in Figure 2.8. Interchanging these two columns gives rise to a new grid diagram $\mathbb{G}'$. We say that the two grid diagrams $\mathbb{G}$ and $\mathbb{G}'$ differ by a **column commutation**. A **row commutation** is defined analogously, using rows in place of columns.

We now state the second grid move, which is the *stabilization move*. Differently from the commutation moves, this increases the dimension of the grid by one, adjoining a new row and a new column. There are 4 possible type of stabilization moves depending on where we place the new X and O markings, as shown in Figure 2.9. There is an inverse of this move called *destabilization move*, which will decrease the grid number by one, deleting a particular row and column.

Definition 2.3.2. Suppose that $\mathbb{G}$ is an $n \times n$ grid diagram. A grid diagram $\mathbb{G}'$ is called a **stabilization** of $\mathbb{G}$ if it is an $(n+1) \times (n+1)$ grid diagram obtained by

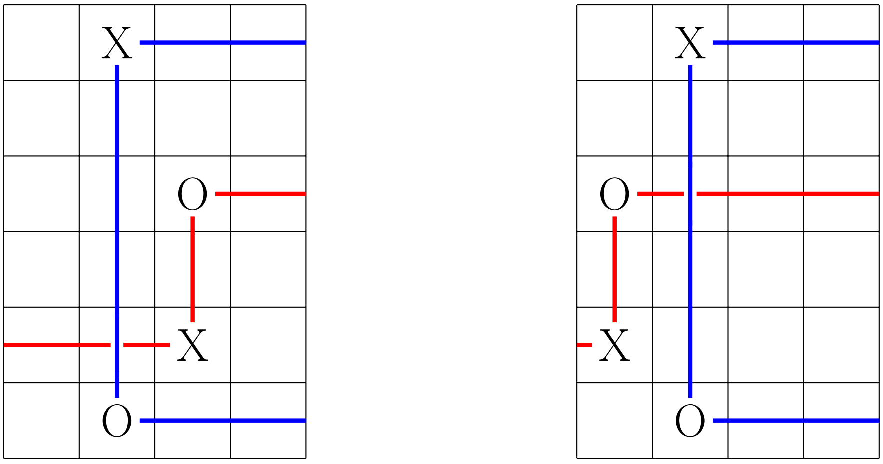


Figure 2.8: Commutation moves of concatenated columns.

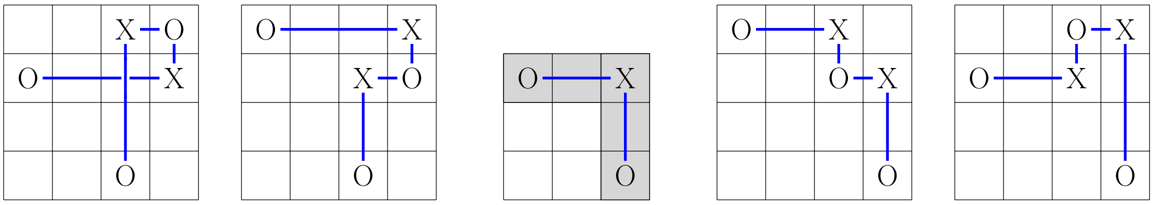


Figure 2.9: The four possible stabilization move where we chose the marking X in the grid of dimension 3 and split its the left-most column and the top-most row, shaded in grey.

splitting a row and column of $\mathbb{G}$ in two as follows. Choose some marked square in $\mathbb{G}$, and erase the markings in that square and in the others marked squares in its row and in its column. Now, split the row and column in two (i.e. add a new horizontal and a new vertical line). There are four ways to insert markings in the two new columns and rows in the $(n+1) \times (n+1)$ grid to obtain a grid diagram. Let $\mathbb{G}'$ be any of these four new grid diagrams as in Figure 2.9. The inverse of a stabilization is called a **destabilization**.

With the definition of these two moves is possible to prove the following theorem:

Theorem 2.2. *Two (planar) grid diagrams represent equivalent links in $\mathbb{R}^3$ if and only if there exists a finite sequence of commutation, stabilization and destabilization moves that transform one into the other.*

The proof of this theorem is due to Cromwell in [10] and is quite technical we leave the reference for the reader. An alternative version can be found in Appendix

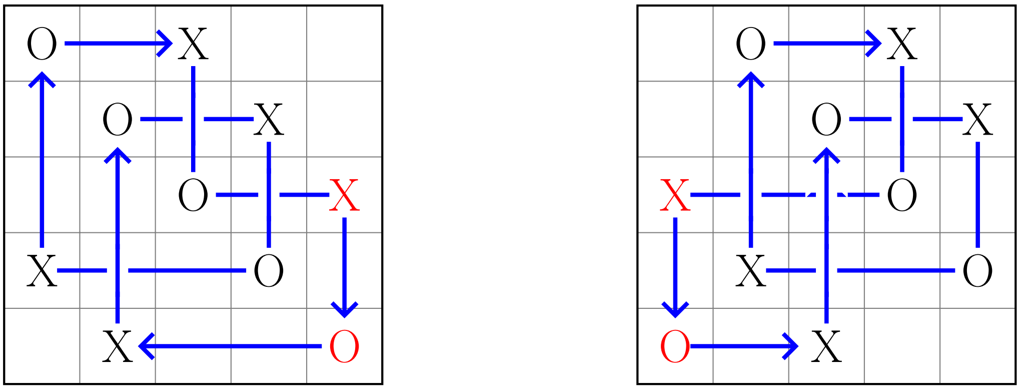


Figure 2.10: Horizontal translation of the trefoil in S^3.

B.4 of [24].
Grid moves have a natural adaptation to toroidal grid diagrams, where we say that two toroidal grid diagrams differ by a particular grid move if their planar realization differ for that grid move. However, if we want to have an equivalence result for toroidal grid diagrams we need to consider another move, accounting for the relation between different planar realization associated to the same toroidal grid diagram.

Definition 2.3.3. Given two planar realizations $\mathbb{G}, \mathbb{G}'$, we say that they differ by a **translation** if one can be obtained from the other by a cyclic permutation of the rows or of the columns.

More explicitly, in a planar realization, a translation corresponds to a horizontal or vertical cyclic shift of all markings. Any marking leaving the grid through one side reappears on the opposite side according to the toroidal identifications, as shown in Figure 2.10. Clearly, two different planar realization of a toroidal grid diagram can be connected by a sequence of translations.

Definition 2.3.4. We call commutations, (de)stabilizations, translation **grid moves**, collectively.

For links in lens spaces we can define the grid moves in the same way as for a link in S^3 since a twisted toroidal grid diagram is a special case of a toroidal grid diagram. The only difference is in the translation, where we have to respect the identifications of the grid diagram, given as in Figure 2.3. In Figure 2.11 is shown an example of translation of a knot in $L(3,1)$, where we coloured the marking to keep track of their movement.

Figure 2.11: Upward vertical translation of a knot in a grid of dimension 2 in $L(3,1)$.

Theorem 2.3. *Two toroidal grid diagrams represent equivalent links in $L(p,q)$ (including the case of S^3) if and only if there is a finite sequence of grid moves that transforms one into the other.*

As before, the proof is quite technical and can be found in [2, 10].

We can define an additional move for grid diagrams in S^3, called *switch*, which will be a particular type of commutation move.

Definition 2.3.5. Consider two consecutive columns in the planar realization such that the X marking in one of the columns occurs in the same row as the O marking of the other column. We call these columns **special**.
If $\mathbb{G}'$ is obtained from $\mathbb{G}$ interchanging a pair of special columns, we say that the grid diagrams are related by a switch. An analogous definition can be given form special rows.

Remark 11. Two grid diagrams related by a switch represent equivalent link, indeed each switch can be expressed as a sequence of commutations, stabilizations and destabilizations.

A further move can be defined but it differs from the other moves since the resulting diagram will not represent equivalent links as stated in Proposition 2.4. Consider a planar realization of a toroidal grid diagram for link in S^3.

Definition 2.3.6. Fix two consecutive columns (or rows) in a grid diagram $\mathbb{G}$, and let $\mathbb{G}'$ be obtained by interchanging those two columns (or rows). Suppose that the interiors of their corresponding intervals intersect non-trivially, but neither is contained in the other. We say that the grid diagrams $\mathbb{G}$ and $\mathbb{G}'$ are related by a **cross-commutation**. In Figure 2.12 we switched the second column with the third column whose interval intersect non trivially.

The next proposition shows that cross commutation moves correspond to crossing changes for a digram of a link, defined as in page 23.

Definition 2.3.7. Given a link $L \subset S^3$ and its regular diagram, we consider one of its crossing. Its preimage, throughout the regular projection, are two strand in

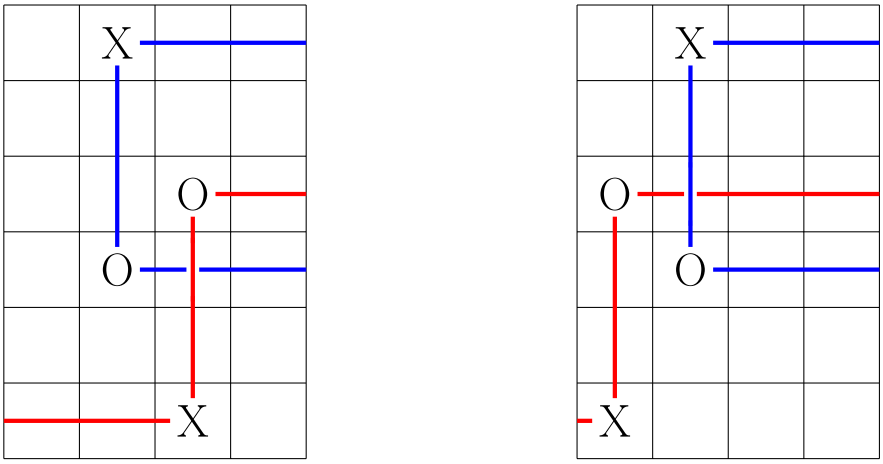


Figure 2.12: Cross-commutation moves of columns.

the 3-sphere, inside a ball B^3. We suppose that, for one instant, we can slide one strand through the other, obtaining another link L'. We say that L and L' are related by a **crossing change**.

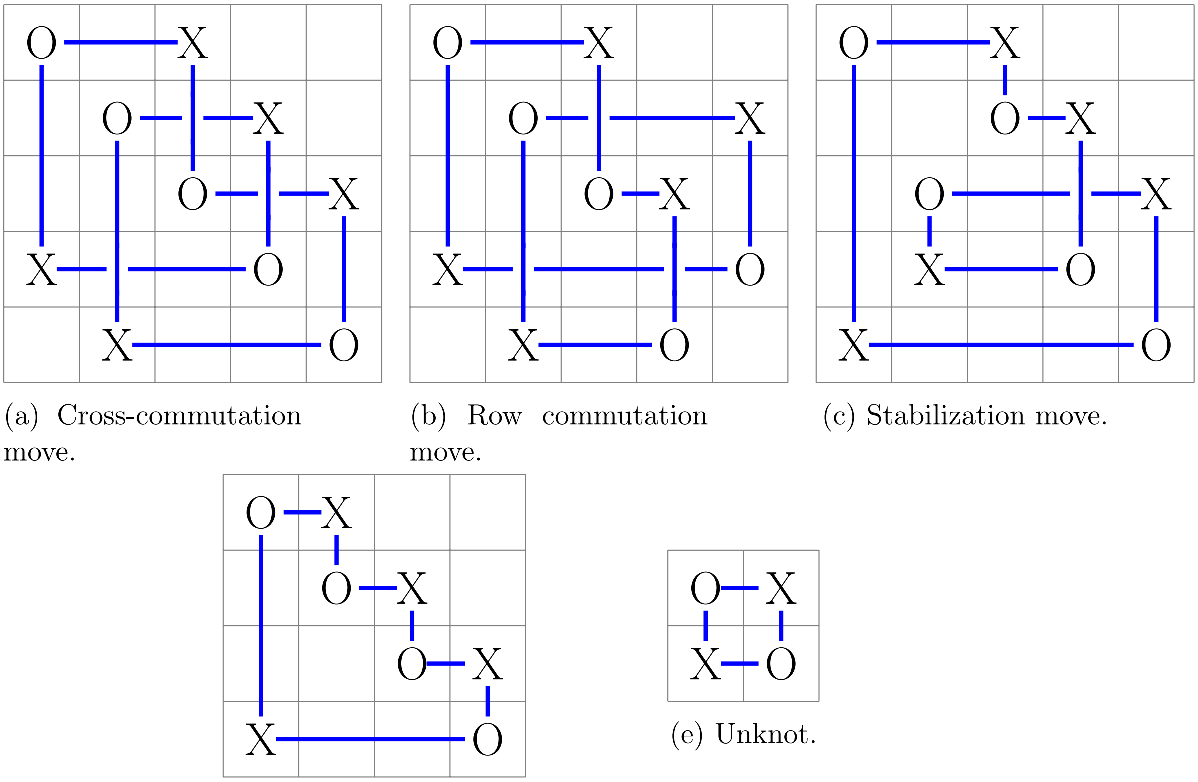
The two links related by a crossing change are no more equivalent, and each link can be related to the unknot after a finite number of crossing change. The minimal number of crossing changes required to unknot L is called *unknotting number*.

Proposition 2.4. *If $\mathbb{G}$ and $\mathbb{G}'$ are two grid diagrams that are related by a cross-commutation, then their associated oriented links L and L' are related by a crossing change.*

In Figure 2.13 we show an example of this proposition, in the case of the trefoil knot, in order to convince the reader. We performed a cross-commutation moves between the third and the fourth row, in the first two diagrams. In the following diagrams, a commutation between the first and the second rows and four stabilization moves have been performed. As we can see, after the cross-commutation move, the trefoil knot becomes equivalent to the unknot, and the reader can try themself to see that we obtain the same result performing a crossing change in S^3. We leave as reference Proposition (3.1.13 of [24]).

2.4 Fundamental Group

We will describe how to compute a presentation (see Definition A.1.2) for the link group (see Definition 1.2.5) in S^3 and in $L(p,q)$, starting from a grid diagram



(a) Cross-commutation move.

(b) Row commutation move.

(c) Stabilization move.

(d) Stabilization move.

(e) Unknot.

Figure 2.13: Evolution of the grid diagram through cross-commutations and grid moves.

for the link.
Let L be a link in S^3. Following Figure 2.14, we begin by producing a presentation for $\pi_1(S^3 \smallsetminus L)$ described by a planar realization $\mathbb{G}$ of dimension n. Fix a point P in $\mathbb{R}^3$ above the grid diagram:

- To each vertical segment C_i in the grid diagram (which connects the O and X markings within a given column) we associate a loop x_i based at P. This loop starts from P, enters the grid through the column immediately to the left of C_i, winds around C_i, and returns to P by leaving the grid through the column immediately to its right, without intersecting any other column or segment C_j. The set $\{x_i\}_{i=1}^n$ then forms the set of generators for $\pi_1(X)$.
- the relations $\{r_1, \ldots, r_{n-1}\}$ correspond to the horizontal lines α; the relator r_i is the product of the generators corresponding to those vertical segments of the link in the diagram which meet the i^{th} horizontal line, in the order they are encountered, from left to right.

Theorem 2.5. *The presentation*

$$\langle x_1, \ldots, x_n \mid r_1, \ldots, r_{n-1} \rangle$$

described as above, is a presentation of the link group $\pi_1(S^3 \smallsetminus L)$ of L.

Proof. The result follows from the Van Kampen theorem for a suitable decomposition of the link complement into two open subsets. Consider the grid $\mathbb{G}$ and assume that the link is moved with isotopy into the following position. In the usual coordinates $(x, y, z) \in \mathbb{R}^3$ (with the convention that the grid lies in the plane $\{z = 0\}$), we assume that the horizontal segments of the grid presenting L lies in the plane $\{z = 0\}$ while the vertical segments are in the plane $\{z = 1\}$. Over the markings $\mathbb{X}, \mathbb{O}$ these segments are joined by segments parallel to the z-axis. The resulting is a PL representative of L.
Take $X_1 = \{(x, y, z) \in \mathbb{R}^3 \smallsetminus L \mid z > 0\}$ and $X_2 = \{(x, y, z) \in \mathbb{R}^3 \smallsetminus L \mid z < 1\}$, decomposing the knot complement into two path-connected open subset, in such a way that $X_1 \cap X_2$ is also path connected. Fix a basepoint x_0 on the plane $\{z = 1/2\}$. By choosing a convenient generator of the free group $\pi_1(X_1, x_0)$ and $\pi_1(X_2, x_0)$, the Van Kampen theorem provides the desired presentation of the link group $\pi_1(\mathbb{R}^3 \smallsetminus L, x_0)$, furthermore, due to Remark 3, we have $\pi_1(S^3 \smallsetminus L) \cong \pi_1(\mathbb{R}^3 \smallsetminus L)$. $\square$

Example 1. Let's now compute the the fundamental group for the trefoil knot K in Figure 2.14. We have five generators x_1, x_2, x_3, x_4, x_5, as shown in figure, and four relators r_1, r_2, r_3, r_4, one for each α_i for $i = 1, \ldots, 4$ of the grid diagram; in

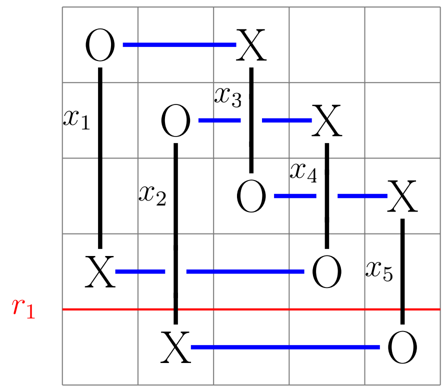


Figure 2.14: Computing a presentation of the fundamental group of the trefoil knot where we highlighted the relator $r_1 = x_2 x_5$.

the figure α_1, corresponding to the relator r_1 is in red. So the fundamental group is

$$\pi_1(S^3 \setminus K) = \langle x_1, \dots, x_5 \mid r_1, \dots, r_4 \rangle$$

where

$$\begin{aligned} r_1 &: x_2 x_5 = 1 \\ r_2 &: x_1 x_2 x_4 x_5 = 1 \\ r_3 &: x_1 x_2 x_3 x_4 = 1 \\ r_4 &: x_1 x_3 = 1. \end{aligned}$$

Using Titze transformations (see Definition A.1.3) over the relations r_1, r_4 we can delete generators x_3, x_5 and obtain two relations

$$\begin{aligned} &r_2 : x_1 x_2 x_4 x_2^{-1} = 1 \longrightarrow x_4 = x_2^{-1} x_1^{-1} x_2 \\ &r_3 : x_1 x_2 x_1^{-1} x_2^{-1} x_1^{-1} x_2 = 1 \end{aligned}$$

so we end up having only two generators x_1, x_2 and one relation

$$r_3 : x_2 x_1 x_2 = x_1 x_2 x_1.$$

Consequently the fundamental group is

$$\pi_1(S^3 \setminus K) = \langle x_1, x_2 \mid x_2 x_1 x_2 = x_1 x_2 x_1 \rangle. \tag{2.1}$$

Let's consider now a link $L \subset L(p, q)$ and let $\mathbb{G}$ be a grid diagram with grid number n associated. We denote with $B_1, \dots, B_p$ the boxes of the diagram (numbered from left to right). Each box is divided by the β-curves into n annuli: denote

with B_{ij} the i^{th} annuli (numbered from left to right) in the j^{th} box, for $i = 1, \dots, n$ and $j = 1, \dots, p$.
Fix a point P inside the solid torus of the Heegaard splitting having β curves as meridians. For each $i = 1, \dots, n$ let j_i be the first index such that B_{ij_i} contains a vertical segment of the link connecting an X marking to the corresponding O marking in the same column. Denote this vertical segment with C_i and associate to it a generator x_i starting from P, going around C_i once and coming back to P; we require that x_i intersect the diagram only in B_{ij_i} and we choose its orientation such that it enters to the left of C_i (with respect to the orientation of α_1). Moreover we associate a generator f to β_1: it start from P, goes around the arc of β_1 separating B_1 and B_2; we orient it such that it enters into the diagram in B_{p1} and comes out in B_{12}.
There are n relators, one for each α curve which are

$$r_h = \prod_{j=1}^{p} x_1^{\epsilon_{1j}} \dots x_n^{\epsilon_{nj}} f$$

where $\epsilon_{ij} = 1$ if α_h intersect an arc of L contained in B_{ij} and zero otherwise, for $h = 1, \dots, n$.

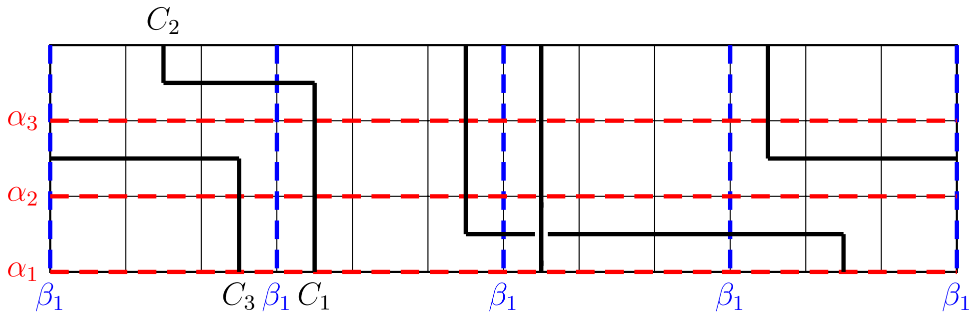


Figure 2.15: An example of the construction of generators and the relations for a knot K in $L(4,1)$ represented with a grid having grid number 3.

Theorem 2.6. *Let L be a link in $L(p,q)$ and let $\mathbb{G}$ be a grid diagram of grid number n representing L. Then*

$$\pi_1(L(p,q) \smallsetminus L, P) = \langle x_1, \dots, x_n, f \mid r_1, \dots, r_n \rangle$$

where x_i and r_j are defined as before.

Proof. This proof generalizes the one given for the link group in S^3.
Consider the Heegaard splitting of $L(p,q)$ as union of two solid tori V_1, V_2, whose

intersection surface is $\mathbb{T}$, as in Definition 1.1.11. The link L is obtained from the union of curves immersed in $\mathbb{T}$ by pushing the n horizontal arcs into V_1 and the n vertical ones into V_2. We have that $\pi_1(V_2 \smallsetminus L, P)$ is the free group generated by $x_1, \ldots, x_n, f$, moreover $L(p,q) \smallsetminus L$ is homotopic to the space obtained by gluing a disc along each α_h and so a presentation for $\pi_1(L(p,q) \smallsetminus L, P)$ is obtained by adding to $\pi_1(V_2 \smallsetminus L, P)$ the corresponding relations which are exactly r_h for $h = 1, \ldots, n$. □

Example 2. Let's compute the fundamental group of the knot K in Figure 2.15. As shown in the figure, we have three generators x_1, x_2, x_3 associated to the vertical strand of the by the knot, and a generator f due to the structure of the space. We have three relations r_1, r_2, r_3, one for each α_i.

$$\begin{aligned}
r_1 &: x_3 f x_1 f x_1 f x_2 f = 1 \\
r_2 &: x_3 f x_1 x_3 f x_1 f^2 = 1 \\
r_3 &: f x_1 x_3 f x_1 f x_1 f = 1.
\end{aligned}$$

From the first relation we can write $x_2 = f^{-1}x_1^{-1}f^{-1}x_1^{-1}f^{-1}x_3^{-1}f^{-1}$ and from the third relation we can write $x_3 = x_1^{-1}f^{-2}x_1^{-1}f^{-1}x_1^{-1}f^{-1}$, so we can delate both generators. Substituting in r_2 the value of x_3 and defining a new generator $g := f^{-1}x_1^{-1}$, we obtain a new relation $r : fgf^{-1}g^2f^{-1}gf = 1$. Consequently the fundamental group of the exterior of the knot is

$$\pi_1(L(p,q) \smallsetminus K) = \langle f, g \mid fgf^{-1}g^2f^{-1}gf = 1\rangle.$$

So far in this section we gave algorithm to compute the fundamental group of a link exterior, starting from its grid diagram. As seen in Definition 2.2, for links in the 3-sphere, we can define a regular diagram, therefore we want to expose an algorithm to obtain a presentation for the link group, based on the regular diagram. To do so we need to set a characterization for the crossings of a regular diagram.

Definition 2.4.1. Given a diagram for a link $L \subset S^3$, we define a crossing to be **positive** if you need to rotate the overcrossing segment counter-clockwise in order to overlap with the undercrossing segment, and **negative** if you need to rotate it clockwise to overlap it with the undercrossing segment as in Figure 2.16.
We define a positive crossing a to have **crossing sign** $Cr(a) = +1$, while a negative crossing b to have crossing sign $Cr(b) = -1$.

Given a regular diagram for $L \subset S^3$, following [18, p.110-111]) we define the **Wirtinger presentation** of the link group as follows:

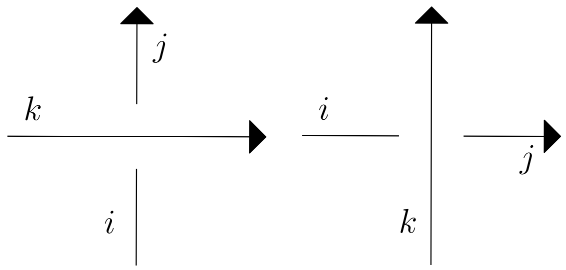


Figure 2.16: A positive crossing (left) and a negative crossing (right) for an oriented link L, with its segments labelled as in the Wirtinger presentation.

- Select an orientation of the link and label each segment of the regular diagram (where a segment is defined to be the arc between two undercrossings). Corresponding to each segment i take a generator g_i of the fundamental group, considering a loop starting from a point $P \in S^3 \smallsetminus L$, going around the i^{th} segment following the right-hand rule and without encircling other segments.

- For each crossing c_h define a relator r_{c_h} as follow: suppose that the crossing c_h is labelled by segments i, j, k (where k is the overcrossing segments while i, j are the two segments of the undercrossing in this order, according to the link orientation as in Figure 2.16) and with the corresponding generators g_i, g_j, g_k. The relator will be

$$\begin{aligned} r_{c_h} &: g_i g_k g_j^{-1} g_k^{-1} = 1 \quad \text{if the crossing is } \textit{positive} \\ r_{c_h} &: g_i g_k^{-1} g_j^{-1} g_k = 1 \quad \text{if the crossing is } \textit{negative}. \end{aligned}$$

This procedure will generate a presentation $\langle g_1, \dots, g_n, \mid r_1, \dots, r_m \rangle$ for the link exterior.

2.5 First homology group

The aim of this thesis, is to build an invariant called Grid Homology. One of the reasons why it has been defined is because the first homology group of a link, i.e. the first homology group of its exterior, does not give us much information about the link as shown in the following propositions.

Proposition 2.7. *Let $K \subset S^3$ be an oriented knot, with $U = U(K)$ an open tubular neighbourhood of K, and let $X = S^3 \smallsetminus U$ be its exterior. Then $H_1(X) \cong \mathbb{Z}$, generated by the class the meridian μ of U. If C is an oriented simple closed* *curve in X, then the homology class $[C] \in H_1(X)$ is $lk(C, K)$. Furthermore, $H_3(X) = H_2(X) = 0$.*

Proof. Note that, denoted $\overline{U}$ the closure of U, $X \cap \overline{U} = \partial X = \partial \overline{U} = \mathbb{T}$ and $X \cup U = S^3$. Consider the following part of the Mayer-Vietoris exact sequence associated with the above decomposition of S^3:

$$\begin{aligned} &\longrightarrow H_3(X) \oplus H_3(\overline{U}) \longrightarrow H_3(S^3) \xrightarrow{\delta_3} \\ \xrightarrow{\delta_3} H_2(X \cap \overline{U}) &\longrightarrow H_2(X) \oplus H_2(\overline{U}) \longrightarrow H_2(S^3) \xrightarrow{\delta_2} \\ \xrightarrow{\delta_2} H_1(X \cap \overline{U}) &\longrightarrow H_1(X) \oplus H_1(\overline{U}) \longrightarrow H_1(S^3) \xrightarrow{\delta_1} \end{aligned}$$

Now $H_3(X) \oplus H_3(\overline{U}) = 0$, since any connected smooth 3-manifold with non-empty boundary retracts to some 2-dimensional subspace, hence it has zero 3-dimensional homology. The homology of $X \cap \overline{U} \cong \mathbb{T}$, $\overline{U} \cong S^1 \times B^2$, S^3 are known, so we only need to investigate $H_2(X), H_1(X)$ and the sequence now results

$$\begin{aligned} &\longrightarrow 0 \oplus 0 \longrightarrow \mathbb{Z} \xrightarrow{\delta_3} \\ \xrightarrow{\delta_3} \mathbb{Z} &\longrightarrow H_2(X) \oplus 0 \longrightarrow 0 \xrightarrow{\delta_2} \\ \xrightarrow{\delta_2} \mathbb{Z} \oplus \mathbb{Z} &\longrightarrow H_1(X) \oplus \mathbb{Z} \longrightarrow 0 \xrightarrow{\delta_1} \end{aligned}$$

A generator of $H_3(S^3)$ is represented by the chain consisting of sum of all the 3-simplexes of S^3, coherently oriented. This pulls back to the sum of the 3-simplexes in X plus those of $\overline{U}$, and is mapped by the boundary map to the sum of the 2-simplexes in ∂X plus those in $\partial \overline{U}$, which pulls back to the sum of, coherently oriented, 2-simplexes of $X \cap \overline{U}$; this represent a generator of $X \cap \overline{U}$, so the map δ_3 is an isomorphism. Since $H_2(S^3) = 0$, the exactness of the sequence implies that $H_2(X) = 0$.
As $H_2(S^3) = H_1(S^3) = 0$, the map from $H_1(X \cap \overline{U})$ to $H_1(X) \oplus H_1(\overline{U})$ is an isomorphism, so $H_1(X) = \mathbb{Z}$.
This isomorphism $H_1(X \cap \overline{U}) \longrightarrow H_1(X) \oplus H_1(\overline{U})$ is induced by the inclusion maps of $X \cap \overline{U}$ into each of X and $\overline{U}$. Suppose that μ is a non-separating, simple closed curve in $X \cap \overline{U}$ that bounds a disc in the solid torus $\overline{U}$, oriented so that μ encircles K with the right-hand rule, then μ represents an indivisible element (i.e is not multiple a of any another element by a non-unit integer) of $H_1(X \cap \overline{U})$, and of course $[\mu] = 0$ in $H_1(\overline{U})$.
Thus under the above isomorphism $[\mu] \mapsto (1, 0) \in \mathbb{Z} \oplus \mathbb{Z} = H_1(X) \oplus H_1(\overline{U})$, for the image must still be indivisible, and this can be taken to define the choice of identification of $H_1(X)$ with $\mathbb{Z}$.
By the definition of linking number (see Definition 1.2.8) we can see that $[C]$ is homologous in $H_1(X)$ to $lk(C, K)[\mu]$. □

Now we can describe the first homology group of the link exterior.

Proposition 2.8. *Let $L \subset S^3$ be an oriented link of m components and let X be its exterior. Then $H_1(X) \cong \bigoplus_m \mathbb{Z}$ generated by the homology classes of the meridians $\{\mu_i\}$ of the components of L.*

Proof. The proof is an adaptation of the previous proposition. Now $\overline{U} = \overline{U(L)}$ is a disjoint union of m solid tori. Furthermore $H_1(\overline{U} \cap X) = \oplus_{2m}\mathbb{Z}$ and $H_1(\overline{U}) = \oplus_m\mathbb{Z}$ and we have the isomorphism $H_1(\overline{U} \cap X) \longrightarrow H_1(\overline{U}) \oplus H_1(X)$, so $H_1(X) = \oplus_m\mathbb{Z}$. As before the generators will be the meridians of the tubular neighbourhood of each component. $\square$

As a consequence, if we consider a link $L \subset S^3$ with $L_1, \dots, L_m$ components of L, we can define the linking number of an oriented simple closed curve $C \subset S^3 \smallsetminus L$ to be $lk(C, L) = \sum_i lk(C, L_i)$, where $lk(C, L)$ is the image of $[C] \in H_1(S^3 \smallsetminus L)$.

When we consider links in $L(p, q)$ the first homology group of the link exterior becomes quite more interesting and, especially for links, gives us some more informations about the link.

Lemma 2.9. *The homology class of a knot $K \subset L(p, q)$ can be read directly from the grid diagram, more precisely, is the sum of the signed number of intersection of the knot with a meridian α_i of the torus, with the orientations convention for links established at page 19*

$$H_1(L(p, q)) \ni [K] = \#\{\alpha_i \cap K\} \pmod p$$

Proof. Considered the construction of the lens spaces of Definition 1.1.11, we have seen in Proposition 1.1, the group $H_1(L(p, q))$ is generated by the longitude λ_1 of V_1. From its definition, the longitude intersects each meridian α_i in exactly one point. Thus the homology class of the knot is the number intersection, counted with respect to the orientation, of the knot K with the meridian α_i, which corresponds to the number of times K runs along the longitude. $\square$

Lemma 2.10. *Let L be a link in $L(p, q)$, with components $L_1, \dots, L_m$. For each $j = 1, \dots, m$, let $\delta_j = [L_j] \in \mathbb{Z}_p = H_1(L(p, q))$. Then*

$$H_1(L(p, q) \smallsetminus L) \cong \mathbb{Z}^m \oplus \mathbb{Z}_d$$

where $d = gcd(\delta_1, \dots, \delta_m, p)$.

The proof of this Lemma may be find in [6, Cor. 5].

Chapter 3

Alexander Polynomial

3.1 Alexander polynomial in S^3

The idea of Alexander [1] was to build an invariant for knots. Since the first homology group of the knot complement, being always $\mathbb{Z}$, is quite a weak invariant, he build *the* infinite cyclic cover $\tilde{X}_\infty$ of the knot complement and calculated the first homology group of this covering, which turned out to be a very strong invariant for knots. Seeing $H_1(\tilde{X}_\infty)$ as a $\mathbb{Z}[t, t^{-1}]$-module allow us to use the theory of presentations of module, in order to study this invariant.
In the following section we will explore three different approaches to compute the Alexander polynomial.

3.1.1 Infinite cyclic covering

We want to find a presentation for this module as defined in Appendix A.2. To do so, we need to have a better understanding of this covering (for the theory of coverings we leave as reference [16, Ch. 1]).
Let S be the Seifert surface of an oriented link $L = L_1 \sqcup \cdots \sqcup L_m \subset S^3$, that is $\partial S = L$ (see Definition 1.2.6). Let U be tubular neighbourhood of L and $X = S^3 \smallsetminus U$ (see Definition 1.2.4), then $X \cap S$ is S with a (collar) neighbourhood of ∂S removed, thus $X \cap S$ is just a copy of S, that, to simplify notation, we will continue to call S. This S has a regular neighbourhood $S \times [-1, 1]$ in X, with S identified with $S \times \{0\}$ and the notation chosen so that the meridian μ (see Definition 1.1.10 of every component of L enters the neighbourhood in $S_- := S \times \{-1\}$ and leaves it at $S_+ := S \times \{1\}$.
Let now Y to be the space X cut along S; this means that Y is $X \smallsetminus S$, compactified with two copies S_+, S_- of S replacing the removed copy of S.
We can recover X from Y by gluing S_+ and S_- together, thus $X = Y/\phi$, where $\phi : S_- \longrightarrow S \longrightarrow S_+$ is the natural homeomorphism. Let $\{Y_i \mid i \in \mathbb{Z}\}$ be spaces

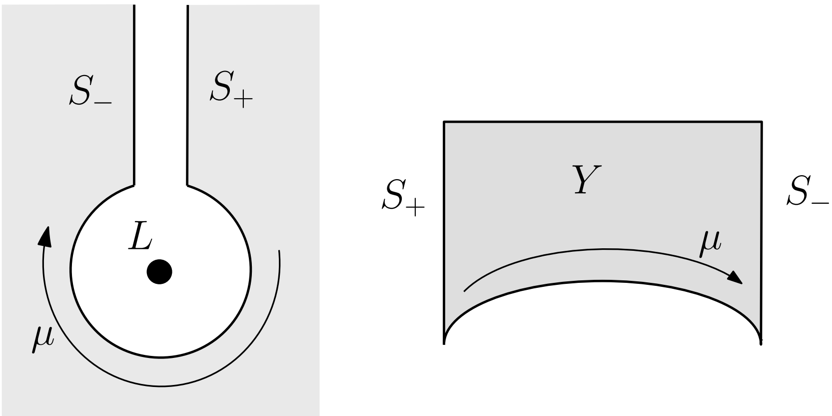


Figure 3.1: Construction of the infinite cyclic covering $\tilde{X_\infty}$ with L the link and μ a meridian of X.

homeomorphic to Y, and let $h_i : Y \longrightarrow Y_i$ be a homeomorphism. We define $\tilde{X}_\infty$ be the space formed from the disjoint union of all the Y_i, by identifying $h_i S_-$ with $h_{i+1}S_+$ with the homeomorphism $h_{i+1}\phi h_i^{-1} : Y_i \longrightarrow Y_{i+1}$
By construction $\tilde{X}_\infty$ is a covering of X. Furthermore, we claim that it is the covering associated to $\operatorname{Ker}\psi$, where $\psi : \pi_1(X) \longrightarrow \mathbb{Z}$, is the map defined as

$$\psi([\gamma]) = \sum_{i=1}^{n} \psi_i([\gamma]) = \sum_{i=1}^{n} lk(\gamma, L_i) \tag{3.1}$$

where $\psi_i : \pi_1(X) \longrightarrow H_1(X) \longrightarrow H_1(S^3 \smallsetminus L_i) \cong \mathbb{Z}$ are surjective homomorphisms induced by the Hurewicz map and the inclusion $S^3 \smallsetminus L \hookrightarrow S^3 \smallsetminus L_i$, and the isomorphism is given by to Proposition 2.8. By the classification theorem of covering spaces, we need to show that Ker $\psi = \pi_1(\tilde{X}_\infty, \tilde{x}_0)$.
An element γ of Ker ψ is a loop in $\pi_1(X)$ with linking number 0 with the link. Its lifting to $\tilde{X}_\infty$, is a path from $\tilde{x}_0$ to another $\tilde{x}_1$ not necessarily in the same sheet of the covering. Each time the path $\tilde{\gamma}$ passes through S going in another sheet, its linking number increases or decreases by one. In order to have linking number 0 its final point $\tilde{x}_1$ must lay in the same sheet as $\tilde{x}_0$, so $\tilde{\gamma}$ is a loop in $\tilde{X}_\infty$.
Conversely each loop in $\pi_1(\tilde{X}_\infty)$ has linking number 0 with the knot, for the same reasoning.

Since Ker ψ is a normal subgroup of $\pi_1(x)$, the covering is regular and so the group of deck transformations $\operatorname{Aut}(\tilde{X}_\infty)$, is

$$\operatorname{Aut}(\tilde{X}_\infty) = \frac{\pi_1(X)}{\text{Ker } \psi} = \langle t \rangle \cong \mathbb{Z}$$

and acts transitively on the fiber over each point.
This action, defined as $t_{|_{Y_i}} = h_{i+1}h_i^{-1}$, can be thought as a translation of *"one*

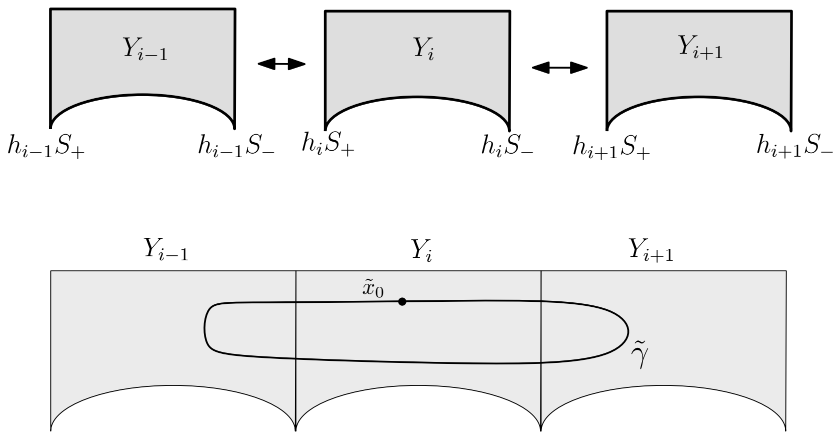


Figure 3.2: The copies of Y glued together by h_i to form $\tilde{X}_\infty$ (top).
A lift of a loop $\gamma \in \pi_1(X, x_0)$ with $lk(\gamma, L) = 0$.

sheet to the right" in $\tilde{X}_\infty$. Consequently the group ring $\mathbb{Z}\langle t\rangle$ induces an action over $H_1(\tilde{X}_\infty)$. By definition of group ring, we can see $\mathbb{Z}\langle t\rangle$ as the ring $\mathbb{Z}[t, t^{-1}]$ of Laurent polynomials in t and this means that we can see $H_1(\tilde{X}_\infty)$ as a module over $\mathbb{Z}[t, t^{-1}]$ called **the Alexander module** of the oriented link.
We are now ready to define the Alexander polynomial.

Definition 3.1.1. The $\boldsymbol{r^{th}}$**Alexander ideal** of an oriented link L is the r^{th} elementary ideal of the $\mathbb{Z}[t, t^{-1}]$-module $H_1(\tilde{X}_\infty)$. (Definition A.2.10)
The $\boldsymbol{r^{th}}$**Alexander polynomial** of L is a generator of the smallest principal ideal of $\mathbb{Z}[t, t^{-1}]$ that contains the r^{th} Alexander ideal.
The *first* Alexander polynomial is called **the Alexander polynomial** and is written $\Delta_L(t)$.

Since $\mathbb{Z}[t, t^{-1}]$ is a g.c.d domain (see Definition A.1.8), we can apply Lemma A.3 and A.4 which imply that the r-th Alexander polynomial as the greatest common divisor of the determinants of all the $(m - r + 1) \times (m - r + 1)$ minors of the Alexander matrices A.
Furthermore, due to Lemma A.3, the units of $\mathbb{Z}[t, t^{-1}]$ are elements of the form $\pm t^{\pm k}$, $k \in \mathbb{Z}$. If f, g are polynomials in $\mathbb{Z}[t, t^{-1}]$ we set $\boldsymbol{f} \doteq \boldsymbol{g}$ if $f = \pm g t^{\pm k}$.
As consequence, the Alexander polynomial of L is unique up to multiplication by unit of $\mathbb{Z}[t, t^{-1}]$.

In order to compute this Alexander polynomial, we need to find a way to obtain a presentation for the Alexander module $H_1(\tilde{X}_\infty)$. There are two principal way to obtain this presentation: the first one, more geometrically flavoured, passes trough the Seifert matrix associated to the link (see Section 3.1.2); the second one, more algebraic, uses Fox calculus and a presentation of $\pi_1(X)$ (see Section 3.1.3).

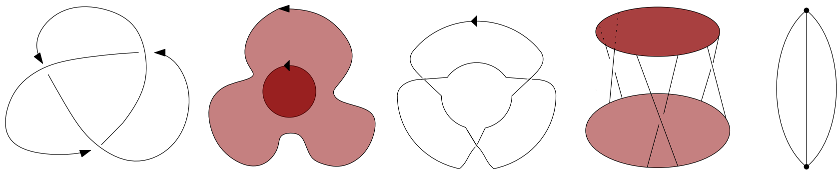


Figure 3.3: Seifert Algorithm for the trefoil knot and its Seifert graph.

3.1.2 Seifert Matrix

One of the possible ways to compute a presentation matrix for the Alexander module is by computing the so called *Seifert matrix*, which arises from the Seifert surface of the link. We start by recalling the Seifert surfaces introduced in Definition 1.2.6 and give some results which will be useful in the computations.
We start by proving that every oriented link in S^3 admits a Seifert surface, through a procedure known as *Seifert Algorithm*.

Theorem 3.1. *Any oriented link in S^3 ha a Seifert surface.*

We present here a sketch of the proof. For more details see [15, Prop. 94.1]

Proof. Take an oriented regular diagram D of L and modify each crossing, locally, resolving the crossing as in Figure 3.3. We obtained a regular diagram $\hat{D}$ which is the same ad D except for a small neighbourhood of each crossing, where the crossing has been removed accordingly with the orientation. Thus $\hat{D}$ is the disjoint union of oriented simple closed curves in S^2 and it is the boundary of some disjoint discs (they may be nested). Join this discs together with half-twisted strips at each crossing following the orientation given by the discs: this gives rise to an oriented surface with L as boundary. If it is not connected, connect the components removing a small disc and inserting a thin tube. □

In Definition 1.2.7 we set the genus of a link to be the minimal genus of its Seifert surfaces. From its definition, the genus of a surface might be rather difficult to find, so we introduce the following notion in order to solve this problem for Seifert surfaces.

Definition 3.1.2. We define the **Seifert graph G_S** of a Seifert surface, to be the graph obtained collapsing each disc found in the Seifert algorithm to a point and each twisted band to a segment. In this manner we obtain a graph to which the Seifert surface deformation retracts, as shown (on the right) of Figure 3.3.

Proposition 3.2. *The genus g of a link with m components can be computed from its Seifert graph, indeed $g = \frac{n-d+2-m}{2}$ where n is the number of half-twisted strips and d the number of discs in the Seifert algorithm.*

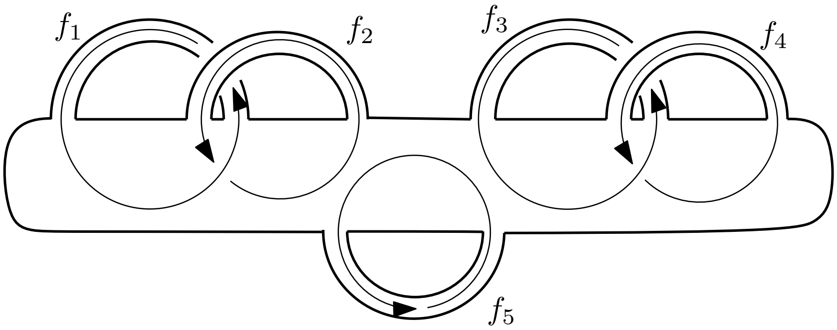


Figure 3.4: Seifert surface of genus 2 of a link L with 2 components with the generators of its first homology group.

Proof. We can compute the Euler characteristic of the Seifert graph G_S, which is $\chi(G_S) = d - n$, where $d = \#$ of vertices, $n = \#$ of edges. Conversely, the Euler characteristic of a closed surface of genus g with boundary with m boundary components is $\chi = 2 - 2g - m$. Since the Euler characteristic is an homotopy invariant and that the Seifert surface deformation retracts onto G_S we can combine the two equations and find $d - n = 2 - 2g - m$, so $g = \frac{n-d+2-m}{2}$. $\square$

We state some classical results about the homology of surfaces, which will lay the foundation to define the Seifert matrix. Unless stated otherwise, all homology groups in this chapter are assumed to have integer coefficients, and the coefficient group will be dropped from the notation for simplicity.
Let S be the Seifert surface of genus g of an oriented link $L \subset S^3$ with n components. By the theory of surfaces we have the following proposition:

Proposition 3.3. *$H_1(S) \cong \mathbb{Z}^{2g+n-1}$, generated by $\{[f_i]\}$, where f_i are oriented simple closed curves depicted in Figure 3.4.*

Proposition 3.4. *It holds $H_1(S^3 \smallsetminus S) \cong H_1(S) \cong \mathbb{Z}^{2g+n-1}$. Moreover, there exist an unique bilinear form*

$$\beta : H_1(S^3 \smallsetminus S) \times H_1(S) \longrightarrow \mathbb{Z}$$

such that

$$\beta([c], [d]) = lk(c, d)$$

whenever c and d are simple oriented closed curves in $S^3 \smallsetminus S$ and S respectively and $lk(c, d)$ is the linking number of Definition 1.2.8

Remark 12. Proposition 3.3 and 3.4 are true in general for all orientable, compact, connected surfaces F of genus g with n boundary components.

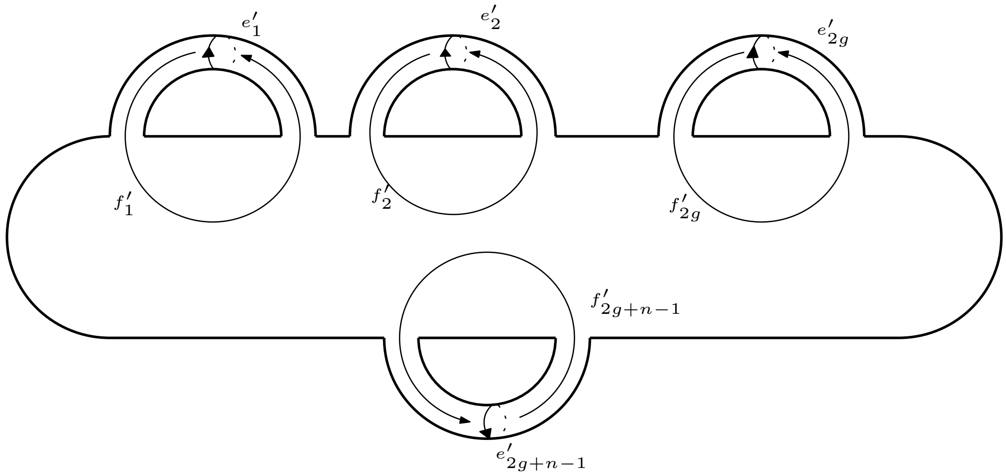


Figure 3.5: Knotted Handlebody V with its generators.

Proof. Consider $V := \overline{U(S)}$ be a closed tubular neighbourhood of S and consider $V' := \overline{S^3 \smallsetminus U(S)}$. By representing S as a disc with knotted bands, as in Figure 3.4, we can see V as a (knotted) handlebody of genus $2g+n-1$ with each 1-handle coming from each band of S. So $V \cap V' = \partial V = \partial V' \cong \Sigma_{2g+n-1}$, with Σ_{2g+n-1} a closed, connected, orientable surface of genus $2g+n-1$.
Following Figure 3.5, as basis for $H_1(\partial V)$, we can take classes of loops $f'_1, \dots, f'_{2g+n-1}$, $e_1, \dots, e_{2g+n-1}$, so that f_i is homologous to f'_i in $H_1(V)$ and e_j bounds a disc in V and such that e_j intersect f_j transversally in one point and is disjoint from f_k for $k \neq j$. This leads us to $H_1(\partial V) \cong \mathbb{Z}^{2g+n-1} \oplus \mathbb{Z}^{2g+n-1}$.
Since $\{e_j\}$ are nullhomologus in V we obtain $H_1(V) \cong H_1(S) \cong \mathbb{Z}^{2g+n-1}$ thanks to Proposition 3.3 and, since $S^3 \smallsetminus V$ is a deformation retract of $S^3 \smallsetminus S$, we have $H_1(V') \cong H_1(S^3 \smallsetminus S)$.
We can apply the Mayer-Vietoris sequence, using $V \cup V' = S^3$

$$0 = H_2(S^3) \longrightarrow H_1(V \cap V') \xrightarrow{\phi} H_1(V) \oplus H_1(V') \longrightarrow H_1(S^3) = 0$$

Our choice of bases gives $\phi([f'_i]) = ([f_i], x_i)$ and $\phi([e_i]) = (0, y_i)$, for some elements $x_i, y_i \in H_i(V')$.
Observing the sequence, we deduce that ϕ is an isomorphism and that the $[e_j]$ are a basis of $H_1(V')$, thus we have

$$\mathbb{Z}^{2g+n-1} \oplus \mathbb{Z}^{2g+n-1} \cong H_1(\partial V) \cong H_1(V) \oplus H_1(V') \cong \mathbb{Z}^{2g+n-1} \oplus H_1(S^3 \smallsetminus S)$$

so $H_1(S^3 \smallsetminus S) \cong \mathbb{Z}^{2g+n-1} \cong H_1(S)$.
Fix and orientation for e_j and f_i so that $lk(e_j, f_i) = \delta_{ij}$. Now define

$$\beta\left(\sum_{i=1}^{t} a_j[e_j], \sum_{i=1}^{t} b_i[f_i]\right) = \sum_{i=1}^{t} a_i b_i$$

with $t = 2g + n - 1$. Clearly β is a $\mathbb{Z}$-valued bilinear form, we just have to check its behaviour on simple closed curve c and d.
Let $[c] = \sum_{k=1}^{t} a_k[e_k]$ and $[d] = \sum_{h=1}^{t} b_h[f_h]$. Since the homology class of a simple closed curve $[c]$ in $H_1(S^3 \setminus f_i)$ is its linking number with f_i, as shown in Proposition 2.7 we have

$$lk(c, [f_i]) = [c] = \left[\sum_{k=1}^{t} a_k[e_k]\right] = \sum_{k=1}^{t} a_k\, lk(e_k, f_i) = a_i$$

Thus in $H_1(S^3 \setminus c)$ it holds

$$lk(c, d) = [d] = \left[\sum_{h=1}^{t} b_h[f_h]\right] = \sum_{h=1}^{t} b_h\, lk(f_h, c) = \sum_{h=1}^{t} b_h a_h = \beta([c], [d]).$$

□

We want to use β to define a bilinear form, called Seifert form, on $H_1(S)$. Since S is oriented $V \cong S \times [-1, 1]$, hence we have two embeddings $i^{\pm} : S \longrightarrow S^3 \setminus S$ given by $i^{\pm}(x) = (x, \pm 1)$.
Consider the induced map in homology $i_*^{\pm} : H_1(S) \longrightarrow H_1(S^3 \setminus S)$ and denote the image of $x \in H_1(S)$ by $x^{\pm} := i_*^{\pm}(x)$

Definition 3.1.3. The **Seifert form** $\alpha : H_1(S) \times H_1(S) \longrightarrow \mathbb{Z}$ is a $\mathbb{Z}$-valued bilinear form defined as

$$\alpha(x, y) = \beta(x^-, y) = \beta(x, y^+).$$

Remark 13. If c, d are simple closed curves in S then $\alpha([c], [d]) = lk(c^-, d) = lk(c, d^+)$.

Taking $\{[f_i]\}$ as basis for $H_1(S)$, defined as before, α is represented by the **Seifert matrix A**, where

$$A_{ij} = \alpha([f_i], [f_j]) = lk\,(f_i^-, f_j) = lk\,(f_i, f_j^+)$$

Remark 14. An immediate consequence is that in $H_1(S^3 \setminus S)$, with $\{[e_i]\}$ as basis, $[f_i^-] = \sum_j A_{ij}[e_j]$ and $[f_j^+] = \sum_i A_{ij}[e_i]$.
Indeed

$$[f_i^-] = \sum_j lk\,(f_i^-, e_j)[e_j] = \sum_j \alpha([f_i], [ej])[e_j] = \sum_j A_{ij}[e_j].$$

Theorem 3.5. *Let L be an oriented link and S a Seifert surface for L. A presentation matrix for the Alexander module of L is $tA - A^T$, where A is the Seifert matrix associated to the Seifert form α on $H_1(S)$ with respect to a fixed basis of $H_1(S)$.*

Proof. Let $Y' := \cup_i Y_{2i+1}$ and $Y'' := \cup_i Y_{2i}$ be disjoint union of countably many copies of Y and such that $\tilde{X}_\infty = Y' \cup Y''$ and $Y' \cap Y''$ is countably many copies of S. We want to investigate the homology of $\tilde{X}_\infty$ using the Mayer-Vitoris exact sequence over the chain complexes in the following way

$$0 \longrightarrow C_n(Y' \cap Y'') \xrightarrow{\alpha_n} C_n(Y') \oplus C_n(Y'') \xrightarrow{\beta_n} C_n(\tilde{X}_\infty) \longrightarrow 0$$

where C_n is the n^{th} chain group.
Note that, a priori, the chain groups of Y' and Y'' are not modules over $\mathbb{Z}[t, t^{-1}]$ since t maps Y_i into Y_{i+1}, so it interchanges Y' and Y''; however, each term in the above sequence is a module over $\mathbb{Z}[t, t^{-1}]$.
To achieve an exact sequence of homology modules we need that $\beta_n \alpha_n = 0$, and this is true if we define

$$\begin{aligned} \beta_n(a, b) &= a + b \quad \forall\, a \in C_n(Y'), \forall\, b \in C_n(Y'') \\ \alpha_n(x) &= (-x, x) \quad \forall x \in C_n(Y_{i-1} \cap Y_i) \end{aligned}$$

This short exact sequence of chain complexes of modules over $\mathbb{Z}[t, t^{-1}]$, induces a long exact sequence of homology modules

$$\begin{aligned} &\longrightarrow H_1(Y' \cap Y'') \xrightarrow{\alpha_1} H_1(Y') \oplus H_1(Y'') \xrightarrow{\beta_1} H_1(\tilde{X}_\infty) \xrightarrow{\delta_*} \\ &\xrightarrow{\delta_*} H_0(Y' \cap Y'') \xrightarrow{\alpha_0} H_0(Y') \oplus H_0(Y'') \xrightarrow{\beta_0} H_0(\tilde{X}_\infty) \longrightarrow 0 \end{aligned}$$

Since $\tilde{X}_\infty$ is path connected we obtain $H_0(\tilde{X}_\infty) = \mathbb{Z}$. Analogously S is the Seifert surface of L, so it is connected by definition, therefore $H_0(S) = \mathbb{Z}$. But $Y' \cup Y''$ is countably many copies of S, each moved to the next by the homeomorphism t, thus $H_0(Y' \cap Y'')$ consists of one copy of $\mathbb{Z}$ for every power of t, so it can be identified, as a module, with $\mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}} H_0(S)$ with a generator of $H_0(Y_0 \cap Y_1)$ corresponding to $1 \otimes 1$. Similarly $H_0(Y') \oplus H_0(Y'')$ is the direct sum of countably many copies of $H_0(Y)$, so this may be identified with $\mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}} H_0(Y)$, with a generator of $H_0(Y_0)$ corresponding to $1 \otimes 1$.
As consequence $\mathrm{Im}(\alpha_0) = (t-1)\mathbb{Z}[t, t^{-1}]$, indeed $\alpha_0(1 \otimes 1) = -(1 \otimes 1) + (t \otimes 1) = (t-1) \otimes 1$, where the term $t \otimes 1$ appears since $\mathrm{Im}(\alpha_0) \subset H_0(Y_0) \oplus H_0(Y_1)$. This implies that on $H_0(Y' \cap Y'')$, the map α_0 is injective, since $(t-1)$ is not a zero divisor in $\mathbb{Z}[t, t^{-1}]$ and, furthermore, β_0 is surjective since

$$\mathrm{Im}(\beta_0) \cong \frac{\mathbb{Z}[t, t^{-1}]}{\mathrm{Ker}(\beta_0)} \cong \frac{\mathbb{Z}[t, t^{-1}]}{\mathrm{Im}(\alpha_0)} \cong \frac{\mathbb{Z}[t, t^{-1}]}{(t-1)\mathbb{Z}[t, t^{-1}]} \cong \mathbb{Z}$$

Since α_0 is injective we have that also β_1 is a surjective map, indeed $\mathrm{Im}(\delta_*) = \mathrm{Ker}(\alpha_0) = 0$, so $\mathrm{Im}(\beta_1) = \mathrm{Ker}(\delta_*) = H_1(\tilde{X}_\infty)$.

Now apply the same reasoning to H_1.
The module $H_1(Y' \cap Y'')$ can be identified with $\mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}} H_1(S)$ so that $x \in H_1(Y_0 \cap Y_1)$ correspond to $1 \otimes x$. Similarly $H_1(Y') \oplus H_1(Y'')$ can be identified with $\mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}} H_1(Y_0)$, so that $y \in H_1(Y_0)$ corresponds to $1 \otimes y$.
As result, as modules, $H_1(Y' \cap Y'')$ has base $\{1 \otimes [f_i]\}$ and $H_1(Y') \oplus H_1(Y'')$ has base $\{1 \otimes [e_i]\}$, where f_i and e_i are the simple closed curves used in Proposition 3.4 and depicted in Figure 3.5.
In order to express elements of $\mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}} H_1(S)$ in terms of elements of $\mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}} H_1(Y)$, we need to use the induced function $i_*^{\pm} : H_1(S) \longrightarrow H_1(Y)$. Now from the definition of α_1 we have:

$$\begin{aligned}
\alpha_1(1 \otimes [f_i]) &= -(1 \otimes i_*^-([f_i])) + (t \otimes i_*^+([f_i])) = \\
&= -(1 \otimes [f_i^-]) + (t \otimes [f_i^+]) = \\
&= -(1 \otimes \sum_j A_{ij}[e_j]) + (t \otimes \sum_j A_{ji}[e_j]) \\
&= \sum_j (-A_{ij}(1 \otimes [e_j]) + A_{ji}(t \otimes [e_j]))
\end{aligned}$$

where the first equality allows us to express $[f_i] \in H_1(S)$ as the sum of an element $i_*^-([f_i])$ of $H_1(Y_0)$ and an element $i_*^+([f_i])$ of $H_1(Y_1)$, the third equality is due to *Remark* 14 and A is the Seifert matrix, with respect to the given bases.
Hence, with respect to the module bases $\{1 \otimes [f_i]\}$ and $\{1 \otimes [e_i]\}$, the map α_1 is represented by matrix $tA - A^T$ and, since β_1 is surjective, is a presentation matrix for $H_1(\tilde{X}_\infty)$.
For higher degrees, since $H_k(\tilde{X}_\infty) = 0$ for $k \geq 2$, the map β_k is the zero map and α_k is an isomorphism. $\square$

Since Seifert surfaces are not unique, neither will be their associated Seifert matrices, so we need to establish whether two Seifert matrices are equivalent. To do so we need the definition of *S-equivalence*

Definition 3.1.4. Let A be a square matrix over $\mathbb{Z}$. An **elementary enlargement** of A is the matrix B of the form

$$B = \begin{pmatrix} A & \xi & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} A & 0 & 0 \\ \eta^T & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

for some column ξ or some row η^T.
Two square matrices A and B are called **S-equivalent** if they are related by a sequence of elementary enlargements, elementary reductions and unimodular congruences (this last being a relation of the form $B = P^T A P$ where $\det P = \pm 1$).

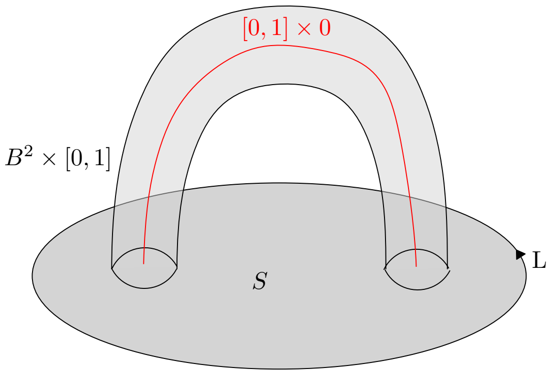


Figure 3.6: A surgery along the arc $[0,1]\times 0$ performed on the Seifert surface of the unknot.

Definition 3.1.5. Suppose that S is a Seifert surface for an oriented link $L \subset S^3$. Suppose there is a solid cylinder, $[0,1]\times B^2 \subset S^3$, such that $([0,1]\times B^2)\cap S = \{0,1\}\times B^2$ and that it is on the side of S near $\{0,1\}\times B^2$, with respect to the orientation of S. Let $S' = (S \setminus \{0,1\}\times B^2)\cup[0,1]\times\partial B^2)$ as shown in Figure 3.6. Then S' is said to be obtained from S by a **surgery along the arc** $[0,1]\times 0$.

Proposition 3.6. *Suppose S_1 and S_2 to be Seifert surfaces for an oriented link $L \subset S^3$. Then there is a sequence of Seifert surfaces $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$ with $\Sigma_1 = S_1$ and $\Sigma_n = S_2$, such that, for each i, either Σ_i is obtained from Σ_{i-1} or Σ_{i-1} is obtained from Σ_i, by surgery along an arc embedded in S^3, or they are related by an isotopy of S^3.*

We refer the reader to [18, Th. 8.4] for the proof.

Proposition 3.7. *Let A and B be Seifert matrices for an oriented link L, then A is S-equivalent to B.*

Proof. Suppose that A is a $n\times n$ matrix corresponding to a Seifert surface S, with respect to some base of $H_1(S)$. Changing the basis, changes A to a matrix of the form P^TAP, where P is a unimodular base-change matrix. Thus it suffices to check what happens when the Seifert surface is changed, that is, by Proposition 3.6, to check (with respect to any base) the effect of a surgery along the arc (Definition 3.1.5).
Suppose that S is changed to S' by a surgery along the arc. A base for $H_1(S')$ can be chosen to be the homology classes of curves f_i that constitute a base for $H_1(S)$ together with the classes of a curve f_{n+1} that goes once over the solid cylinder defining the surgery and of a curve f_{n+2} around the middle of the cylinder (that is $f_{n+1} = \{1/2\}\times\partial B^2$).

Then, because f_{n+2} bounds a disc $\{1/2\} \times B^2$ that is disjoint form $\bigcup_{i \leq n} f_i$, we have that $lk(f_{n+2}^{\pm}, f_i) = 0$ for all $i \neq n+1$. Further, as f_{n+1} meets this disc at one point in its boundary, choosing orientations carefully gives either $lk(f_{n+1}^{+}, f_{n+2}) = 0$ and $lk(f_{n+1}^{-}, f_{n+2}) = 0$ or $lk(f_{n+1}^{+}, f_{n+2}) = 1$ and $lk(f_{n+1}^{-}, f_{n+2}) = 0$.
In the first case the new Seifert matrix is of the form

$$\begin{pmatrix} A & \xi & 0 \\ ? & ? & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{which is congruent to} \quad \begin{pmatrix} A & \xi & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The second case leads to a matrix of the form

$$\begin{pmatrix} A & 0 & 0 \\ \eta^T & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

□

It follows from this theorem that any invariant well-defined on S-equivalence classes of square matrices of integers, gives at once an invariant of oriented links. Following Definition 3.1.1 we can compute the Alexander polynomial of a given link $L \subset S^3$ using the Seifert matrix. The Alexander module $H_1(\tilde{X}_\infty)$ has a square presentation matrix, since the Seifert matrix is squared, thus the first elementary ideal is principal, and *the* Alexander polynomial of L is given by

$$\Delta_L(t) \doteq \det(tA - A^T). \tag{3.2}$$

We state now some properties of the Alexander polynomial:

Proposition 3.8. *Let L be a link, then:*

1. $\Delta_L(t) \doteq \Delta_L(t^{-1})$
2. *If K is a knot, then* $\Delta_K(1) = \pm 1$
3. *If L is a link, with at least two components, then* $\Delta_L(1) = 0$
4. $\Delta_L(t) \doteq \Delta_{m(L)}(t)$, *with $m(L)$ the mirror image of L*
5. $\Delta_L(t) \doteq \Delta_{-L}(t)$, *with $-L$ the link L with its orientation reversed*

Proof. (1) Using the properties of the determinant and the transpose we have

$$\begin{aligned} \Delta_L(t^{-1}) &= \det(t^{-1}A - A^T) = (-t^{-1})^k \det(-A + tA^T) = \\ &= (-t^{-1})^k \det((-A + tA^T)^T) = (-t^{-1})^k \det(tA - A^T) \doteq \Delta_L(t) \end{aligned}$$

with k the order of A.
(2) If K is a knot then any Seifert surface has just one boundary component and, by considering the core of the bands, we can take a basis $([f_1], \ldots, [f_{2g}])$ of $H_1(S)$ such that the f_i are simple closed curves where f_{2j-1} and f_{2j} intersect once and there are no intersections between other couples of curves. If A is the associated Seifert matrix $(A - A^T)_{ij} = lk(f_i^-, f_j) - lk(f_i^+, f_j)$ and this is the algebraic intersection of f_i and f_j of S, so

$$A - A^T = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & B_g \end{pmatrix}$$

where $B_i = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. As consequence $\Delta_K(1) = \det(A - A^T) = \pm 1$.
(3) If L has at least two components, by taking as before, the core of the bands, we have a basis $([f_1], \ldots, [f_{2g}], [f_{2g+1}], \ldots, [f_{2g+n-1}])$, so that the columns of $A - A^T$ corresponding to $[f_{2g+1}], \ldots, [f_{2g+n-1}]$ are zero, since, if $i > 2g$, then f_i has no intersection with the other curves.
(4) Let $r : S^3 \longrightarrow S^3$ be a reflection so that $r(L) = m(L)$. If S is a Seifert surface for L then $\bar{S} = r(S)$ is a Seifert surface for $m(L)$. Moreover, if A is the Seifert matrix associated to a basis $\mathcal{B}$ of $H_1(S)$, then $-A$ is a Seifert matrix of $\bar{S}$ associated to the basis $r(\mathcal{B})$. So $\Delta_{m(L)} = \pm \Delta_L$.
(5) If S is an oriented Seifert surface for L, then $-S$ (i.e. S with the opposite orientation) is an oriented Seifert surface for $-L$. So if A is a Seifert matrix for L, a Seifert matrix for $-L$ is A^T, since i^- exchanges with i^+.

□

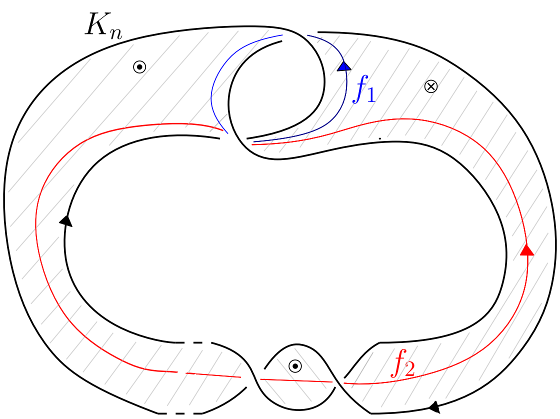


Figure 3.7: Twisted knot K_n.

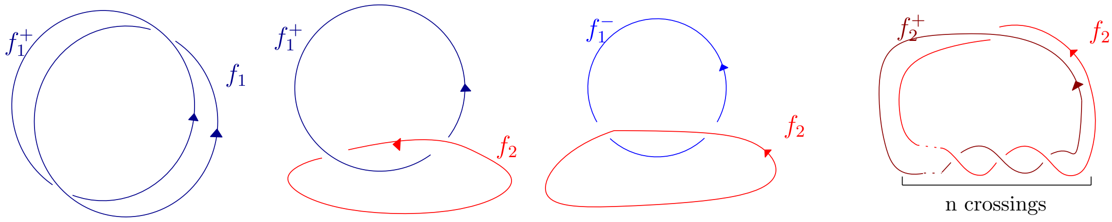


Figure 3.8: Linking between $f_i^{\pm}, f_j^{\pm}$.

Example 3. We define K_n with $n \in \mathbb{Z}$ to be the class of knots obtained by linking together the two endings of the unknot, after performing n half twists. In particular, the resulting diagram, will have $n+2$ crossing with n crossings in the lower part, as shown in Figure 3.7. If $n > 0$ the twists will be clockwise while, if $n < 0$, they will be counter-clockwise, with respect to the knot orientation.
For example, $K_0 = \mathcal{O}$ is the unknot, K_1 is the trefoil knot, K_2 is the figure-8 knot, shown in Figure 1.4.
Let's calculate the Alexander polynomial for a general K_n knot, with n odd, in order to obtain a Seifert surface as in Figure 3.7. Consider the Seifert surface S of the knot, thanks to Proposition 3.2 the surface has genus 1 and, since K_n is a knot, it has 1 boundary component, so $H_1(S)$ has $2g+n-1=2$ generators, f_1, f_2 represented by oriented simple closed curves as in Figure 3.7.
As seen before

$$A_{ij} = \alpha(f_i, f_j) = lk(f_i^-, f_j) = lk(f_i, f_j^+)$$

where $f^{\pm}$ is the curve f lifted in $S^3 \setminus S$, accordingly to the orientation of S.
It is possible to follow Figure 3.8 to see the crossings between $f_i, f_j^{\pm}$.
$A_{1,1} = lk(f_1, f_1^+) = 1$, since f_1^+ intersects the Seifert surface of f_1 only once and the orientations agrees.
$A_{1,2} = lk(f_1^-, f_2) = 0$, since, due to the orientation of the Seifert surface the curve f_1 is lifted above f_2.
$A_{2,1} = lk(f_2, f_1^+) = -1$.
$A_{2,2} = lk(f_2, f_2^+) = \frac{n+1}{2}$.
We obtained the matrix A

$$\begin{pmatrix} 1 & 0 \\ -1 & \frac{n+1}{2} \end{pmatrix}$$

it follows that

$$tA - A^T = \begin{pmatrix} t-1 & 1 \\ -t & \frac{n+1}{2}(t-1) \end{pmatrix}$$

so

$$\Delta_{K_n}(t) = \det(tA - A^T) = \frac{n+1}{2}t^2 - nt + \frac{n+1}{2}$$

Using this formula we obtain $\Delta_{K_1}(t) \doteq t^2 - t + 1$.
If n is even we need a different Seifert surface and we obtain

$$\Delta_{K_n}(t) = -\frac{n}{2}t^2 + (n+1)t - \frac{n}{2}.$$

In particular, we obtain $\Delta_{\mathcal{O}}(t) = t \doteq 1$ and $\Delta_{K_2}(t) = -t^2 + 3t - 1$.

3.1.3 Alexander-Fox Matrix

We want now to give another way to compute a presentation matrix for the Alexander module, using the knot group. This method is due to Fox [12] and can be generalized to arbitrary finite presentable group. We use the description given by Lickorish in [18] in order to derive this construction from the knot group. For simplicity we expose the construction only for knot, but it can be generalized to links with m components.
Fix X to be the exterior of a knot $K \subset S^3$ and $G = \pi_1(X)$. It is known that $H_1(X) = G/G'$ with $G' = \{aba^{-1}b^{-1} \mid a, b \in G\}$ the commutator subgroup of G.
The infinite cyclic cover $\tilde{X}_\infty$ has its fundamental group equal to G', indeed a priori $\pi_1(\tilde{X}_\infty) = \text{Ker } \psi$ but, recalling that an element in Ker ψ is a loop in $[\gamma] \in \pi_1(X)$ with $lk(\gamma, K) = 0$, we have that each element of the commutator is contained in Ker ψ; indeed, if $\tau \in G'$, then $\exists\, a, b \in G$ such that $\tau = aba^{-1}b^{-1}$. Since the linking number is additive we have that $lk(\tau, K) = lk(aba^{-1}b^{-1}, K) = lk(a, K) + lk(b, K) + lk(a^{-1}, K) + lk(b^{-1}, K) = lk(a, K) + lk(b, K) - lk(a, K) - lk(b, K) = 0$.
On the other hand, each class $[\gamma] \in \text{Ker}\, \psi$ is trivial in the abelianization of $\pi_1(X)$, so $G' = \pi_1(\tilde{X}_\infty)$.
Now $H_1(\tilde{X}_\infty)$ is the abelianization of $\pi_1(\tilde{X}_\infty)$, so it is G'/G'', with G'' the commutator subgroup of G'. The conjugation gives an action of G on G and this passes to quotients, giving an action of G/G' on G'/G'', which is the action of $H_1(X)$ over $H_1(\tilde{X}_\infty)$ given by $\mathbb{Z}\langle t\rangle$, so we can represent $H_1(\tilde{X}_\infty)$ as a $\mathbb{Z}[t, t^{-1}]$-module.
The module $H_1(\tilde{X}_\infty)$ can thus be defined directly in terms of G and using Fox free differential calculus, we can compute the Alexander polynomial by calculating a presentation matrix as follows:
We consider the group homomorphism $\alpha : G \longrightarrow G/G' = \langle t\rangle$ and its extension to the group rings (see Definition A.1.4 $\alpha : \mathbb{Z}G \longrightarrow \mathbb{Z}H_1(X) \cong Z[t, t^{-1}]$.
Consider a presentation of $G = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m\rangle$ and define F to be the free group generated by $x_1, \ldots, x_n$. The quotient map from F to G induces a homomorphism of group rings $\phi : \mathbb{Z}F \longrightarrow \mathbb{Z}G$.
We can define a *derivative* as a function $\frac{\partial}{\partial x_j} : \mathbb{Z}F \longrightarrow \mathbb{Z}F$ to be the linear extension

of the map defined over elements of F by:

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \qquad \frac{\partial x_i^{-1}}{\partial x_j} = -\delta_{ij} x_i^{-1}$$
$$\frac{\partial(uv)}{\partial x_j} = \frac{\partial u}{\partial x_j} + u\frac{\partial v}{\partial x_j}$$

for $u, v = \sum_i a_i x_i^{k_i} \in \mathbb{Z}F$, $a_i, k_i \in \mathbb{Z}$.
We define the **Alexander-Fox matrix** to be the matrix $A = (a_{ij})_{i,j} \in M_{m,n}(\mathbb{Z}[t, t^{-1}])$ where

$$a_{ij} = \alpha\phi\left(\frac{\partial r_i}{\partial x_j}\right). \tag{3.3}$$

In order to show that this is indeed the presentation matrix of $H_1(\tilde{X}_\infty)$ instead of working directly on X, we consider a space P, described below, with $\pi_1(P) = G$ and let $\tilde{P}$ be the cover of P corresponding to G'. The above construction of the Alexander module that depend only on the fundamental group, shows that $H_1(\tilde{P})$, regarded as a $\mathbb{Z}[t, t^{-1}]$-module, is equivalent to the Alexander module of K.
Take as P a complex consisting of one 0-cell V, n oriented 1-cells labelled $x_1, \dots, x_n$, having all their end points identified with V to form n loops and m oriented 2-cells $c_1, \dots, c_m$, with each ∂c_i glued to the 1-cells according to the relator r_i. The lifts of this cells to $\tilde{P}$ gives us a cell structure on $\tilde{P}$ and can be used to investigate its homology. Indeed let $\tilde{V}$ be a chosen lift of the point V, let $\tilde{x}_i$ be the lift of x_i starting at $\tilde{V}$ and let $\tilde{c}_i$ be the lift of c_i whose boundary is the lift of r_i beginning at $\tilde{V}$. The space $\tilde{P}$ is just the union of all translates of these cells under the action of $\langle t \rangle$.
As consequence, the chain complex of $\mathbb{Z}[t, t^{-1}]$-modules for $\tilde{P}$

$$C_2(\tilde{P}) \xrightarrow{d_2} C_1(\tilde{P}) \xrightarrow{d_1} C_0(\tilde{P})$$

has each $C_i(\tilde{P})$ freely generated as a module by the i-cells in $\tilde{P}$.
In this chain complex, the boundary map d_2 sends $\tilde{c}_i$ to the lift of r_i beginning at $\tilde{V}$ regarded as an element of $C_1(\tilde{P})$. Any occurrence of x_j in r_i contributes some $\langle t \rangle$-translate of $\tilde{x}_j$ to $d_2(\tilde{c}_i)$, indeed, if $r_i = w_1 x_j w_2$ this occurrence of x_j contributes to $d_2(\tilde{c}_i)$ the lift of x_j that begins at the final point of the lift of w_1 which starts at $\tilde{V}$; thus the contribution is $\alpha(w_1)$ acting on $\tilde{x}_j$. If $r_i = w_1 x_j^{-1} w_2$, this occurrence of x_j^{-1} contributes $-\alpha(v_1 x_j^{-1})\tilde{x}_j$. The $\tilde{x}_j$ term in $d_2(\tilde{c}_i)$ is thus the sum, over all occurrences of x_j^{-1} and x_j in r_i of these contribution, which means

$$d_2(\tilde{c}_i) = \sum_j \alpha\phi\left(\frac{\partial r_i}{\partial x_j}\right)\tilde{x}_j$$

where ϕ and $\frac{\partial}{\partial x_i}$ are defined as before.
Thus the transpose of $\alpha\phi\left(\frac{\partial r_i}{\partial x_j}\right)$ is a matrix representing d_2, so it is a presentation matrix for the module $C_1(\tilde{P})/d_2(C_2(\tilde{P}))$.
We have the following short exact sequence of modules

$$0 \longrightarrow \frac{\text{Ker } d_1}{\text{Im } d_2} \hookrightarrow \frac{C_1(\tilde{P})}{\text{Im } d_2} = \frac{C_1(\tilde{P})}{\text{Ker } d_1} \xrightarrow{d_1} \text{Im } d_1 \longrightarrow 0$$
$$0 \longrightarrow H_1(\tilde{P}) \hookrightarrow \frac{C_1(\tilde{P})}{d_2(C_2(\tilde{P}))} \xrightarrow{d_1} d_1(C_1(\tilde{P})) \longrightarrow 0$$

The boundary map d_1 is determined by $d_1(\tilde{x}_j) = t^{a_j}\tilde{V} - \tilde{V} = (t^{a_j} - 1)\tilde{V}$, where $t^{a_j}\tilde{V} = \alpha\phi(x_j)$, for some $a_j \in \mathbb{Z}$. Thus $d_1(C_1(\tilde{P}))$ is $\mathcal{I}\hat{V}$ where $\mathcal{I}$ is an ideal of $\mathbb{Z}[t,t^{-1}]$ generated by $\{(t^{a_j} - 1) \mid j = 1,\dots,n\}$.
Observing that $(t-1) = (t^a - 1) - t(t^{a-1} - 1)$ we get that $\mathcal{I} = (t-1)$ is a principal ideal. Then $d_1(C_1(\tilde{P}))$ is a free rank-one module generated by the element $(t-1)\hat{V}$ (it has a bases of cardinality one containing the only element $(t-1)$ which is not a zero divisor in $\mathbb{Z}[t,t^{-1}]$) and there is an isomorphism $\psi : \mathbb{Z}[t,t^{-1}] \longrightarrow \mathcal{I}$.
As $d_1(C_1(\tilde{P}))$ is free, the sequence splits and we get

$$C_1(t\tilde{P})/d_2(C_2(\tilde{P})) \cong H_1(\tilde{P}) \oplus \mathbb{Z}[t,t^{-1}]$$

To conclude is enough to observe that if A is a presentation matrix for a module M over a ring R, a presentation matrix for $M \oplus R$ is A with an extra row of zeros, but this new matrix will have the same non-zero minors of A. The transpose of the $m \times n$ matrix $\alpha\phi\left(\frac{\partial r_i}{\partial x_j}\right)$ presents $H_1(\tilde{P}) \oplus \mathbb{Z}[t,t^{-1}]$, so the first elementary ideal of $\mathbb{Z}[t,t^{-1}]$ for the module $H_1(\tilde{P})$ is generated by the $(n-1)\times(n-1)$ minors of the matrix, and this is the ideal generating the Alexander polynomial $\Delta_K(t)$.

Remark 15. In [12] is shown that is possible to define an Alexander-Fox matrix in a more general general setting, taking a finitely presentable group G over a ring of integers R and defining a derivative $\partial/\partial x_i$ over its generators.

Remark 16. Due to the definition of the Fox derivative, if we have a relation of type $r_i : w = u$, with w, u words in the generators, we can compute $a_{i,j}$ $\forall j$ as follow.
Since the abelianizer α sends each generator to t we have the following:

$$\begin{aligned} a_{i,j} = \alpha\phi\left(\frac{r_i}{x_j}\right) &= \alpha\phi\left(\frac{wu^{-1}}{x_j}\right) = \alpha\phi\left(\frac{\partial w}{\partial x_j} - wu^{-1}\frac{\partial u}{\partial x_j}\right) \\ &= \alpha\phi\left(\frac{\partial w}{\partial x_j}\right) - tt^{-1}\alpha\phi\left(\frac{\partial u}{\partial x_j}\right) = \alpha\phi\left(\frac{\partial(w-u)}{\partial x_j}\right). \end{aligned}$$

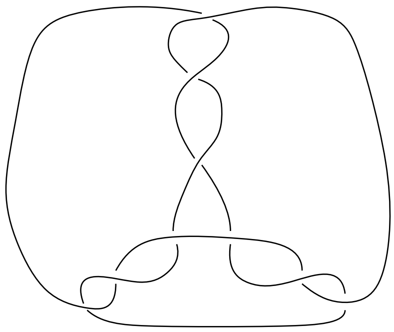


Figure 3.9

Example 4. Given the trefoil knot, we compute its Alexander polynomial calculating its Alexander-Fox matrix.
Consider a presentation of the trefoil group $G = \langle x, y \mid xyx = yxy \rangle$, as obtained in Example 1. Since we have two generators and one relator, the resulting matrix will be $A \in M_{1,2}(\mathbb{Z}[t, t^{-1}])$. Using Remark 16 we obtain the following:

$$\frac{\partial r_1}{\partial x} = \frac{\partial(xyx - yxy)}{\partial x} = 1 + xy - y$$
$$\frac{\partial r_1}{\partial y} = \frac{\partial(xyx - yxy)}{\partial y} = x - 1 - yx$$

we apply $\alpha\phi$ to the derivation which abelianize and sends $x \mapsto t$ and $y \mapsto t$

$$a_{1,1} = \alpha\phi\left(\frac{\partial r_1}{\partial x}\right) = \alpha\phi(1 + xy - y) = 1 + t^2 - t$$
$$a_{1,2} = \alpha\phi\left(\frac{\partial r_1}{\partial 2}\right) = \alpha\phi(x - 1 - yx) = t - 1 - t^2$$

and we obtained the matrix

$$A = \begin{pmatrix} t^2 - t + 1 & -t^2 + t - 1 \end{pmatrix}$$

taking the generator of the ideal generated by the minors of order one, we get that the Alexander polynomial is $\Delta_K(t) \doteq t^2 - t + 1$.

Example 5. Consider the Wirtinger presentation of the knot in Figure 3.9, which can be simplified $G = \langle x, y, z \mid r_1, r_2 \rangle$ where $r_1 : y^{-1}xyx^{-1}y = x^{-1}zx^{-1}zxz^{-1}x$ and $r_2 : x^{-1}zxz^{-1}x = y^{-1}zyz^{-1}y$.

We start by calculating the derivative of the relators:

$$
\begin{aligned}
\frac{\partial r_1}{\partial x} &= y^{-1}(1+xy(-x^{-1})) - (-x^{-1}+x^{-1}z(-x^{-1}+x^{-1}z(1+xz^{-1}))) = \\
&= y^{-1} - y^{-1}xyx^{-1} + x^{-1} + x^{-1}zx^{-1} - x^{-1}zx^{-1}z - x^{-1}zx^{-1}zxz^{-1} = \\
\frac{\partial r_1}{\partial y} &= -y^{-1} + y^{-1}x(1+yx^{-1}) = -y^{-1} + y^{-1}x + y^{-1}xyx^{-1} \\
\frac{\partial r_3}{\partial z} &= 0 - x^{-1}(1+zx^{-1}(1+zx(-z^{-1}))) = -x^{-1} - x^{-1}zx^{-1} + x^{-1}zx^{-1}zxz^{-1} \\
\frac{\partial r_2}{\partial x} &= -x^{-1} + x^{-1}z + x^{-1}zxz^{-1} \\
\frac{\partial r_2}{\partial y} &= y^{-1} - y^{-1}z - y^{-1}zyz^{-1} \\
\frac{\partial r_2}{\partial z} &= x^{-1} - x^{-1}zxz^{-1} - y^{-1} + y^{-1}zyz^{-1}
\end{aligned}
$$

applying α and ϕ we obtain

$$
\begin{aligned}
a_{1,1} &= \alpha\phi(\frac{\partial r_1}{\partial x}) = t^{-1} - 1 + t^{-1} +^{-1} -1 - 1 = 3t^{-1} - 3 \\
a_{1,2} &= -t^{-1} + 2 \\
a_{1,3} &= -2t^{-1} + 1 \\
a_{2,1} &= -t^{-1} + 2 \\
a_{2,2} &= t^{-1} - 2 \\
a_{2,3} &= 0
\end{aligned}
$$

Consequently

$$
A = \begin{pmatrix} 3t^{-1}-3 & -t^{-1}+2 & -2t^{-1}+1 \\ -t^{-1}+2 & t^{-1}-2 & 0 \end{pmatrix} \sim \begin{pmatrix} 3-3t & -1+2t & 0 \\ -1+2t & 1-2t & 0 \end{pmatrix} \sim \begin{pmatrix} 2-t & 0 & 0 \\ 0 & 1-2t & 0 \end{pmatrix}
$$

where we performed moves (3) of Definition A.2.9 in order to simplify the matrix. Considering the ideal generated by the determinant of the minors of order 2 we obtain $I = ((2-t)(1-2t))$, which are distinct irreducibles, so the generator of I is $2t^2 - 5t + 2$. Thus we obtain $\Delta_K(t) \doteq 2t^2 - 5t + 2$.

3.1.4 Alexander-Conway polynomial and Skein relation

Since the Alexander polynomial of a link L defined above is unique up to units of $\mathbb{Z}[t, t^{-1}]$, it is natural investigate if there is a way to normalize it and

obtain an unique polynomial associated to L without ambiguity of sign or units. For knot, Proposition 3.8 ensure that is possible to chose a representative of the Alexander polynomial such that $\Delta_K(1) = 1$ and $\Delta_K(t) = \Delta_K(t^{-1})$ as follow: Since it is defined up to units we can find $e_1 \in \mathbb{Z}$ such that $\Delta_K(t) \doteq \Delta_K(t) \cdot t^{e_1} = a_0 + a_1 t + \cdots + a_n t^n$, where, due to (1) of Proposition 3.8 $a_{n-r} = \pm a_r$ with the same choice of sign for all r. If n is odd, we have an even number of terms, so $\Delta_K(1) = 2a_0 + 2a_1 + \cdots + 2a_{(n+1)/2} \neq 1$ contradicting (2) from the proposition. So n is even and $a_{n-r} = a_r$ otherwise $a_{n/2} = -a_{n/2}$ and $\Delta_K(1) = a_{n/2} = 0$.
Thus we can write $\Delta_K(t) = b_0 + b_1(t + t^{-1}) + b_2(t^2 + t^{-2}) + \ldots$ which is the unique representative satisfying $\Delta_K(1) = 1$ and $\Delta_K(t) = \Delta_K(t^{-1})$. This representative is called the **Alexander-Conway polynomial** and is denoted $\nabla_K(t)$.
Fixed A to be a Seifert matrix for a knot, it is easy to check that the Alexander-Conway polynomial is $\nabla_K(t) = \det(t^{1/2}A - t^{-1/2}A^T)$, since it satisfies the above equalities (see proof of Proposition 3.8) and, due to the fact that n is even, it is equivalent to $\Delta_K(t)$.
For a link with at least two components, since $\Delta_L(1) = 0$, a different reasoning is needed, but it is still possible to demonstrate that $\nabla_L(t) = \det(t^{1/2}A - t^{-1/2}A^T) \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$ does not depend on the Seifert matrix A as follows.

Proposition 3.9. *The Alexander-Conway polynomial is well defined invariant of the oriented link $L \subset S^3$.*

Proof. It is sufficient to check the invariance of the Alexander-Conway polynomial when A changes by S-equivalence thanks to Proposition 3.7.
Firstly note that

$$\det(t^{\frac{1}{2}} P^T A P - t^{-\frac{1}{2}} P^T A^T P) = (\det P)^2 \det(t^{\frac{1}{2}} A - t^{-\frac{1}{2}} A^T)$$

so that the Alexander-Conway polynomial $\nabla_L(t)$ is invariant under unimodular congruence. If now

$$B = \begin{pmatrix} A & \xi & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

then

$$(t^{\frac{1}{2}} B - t^{-\frac{1}{2}} B^T) = \begin{pmatrix} t^{\frac{1}{2}} A - t^{-\frac{1}{2}} A^T & t^{\frac{1}{2}} \xi & 0 \\ -t^{-\frac{1}{2}} \xi^T & 0 & t^{\frac{1}{2}} \\ 0 & -t^{-\frac{1}{2}} & 0 \end{pmatrix}$$

which has the same determinant as $(t^{\frac{1}{2}} A - t^{-\frac{1}{2}} A^T)$. Similarly for the other types of elementary enlargements.

□

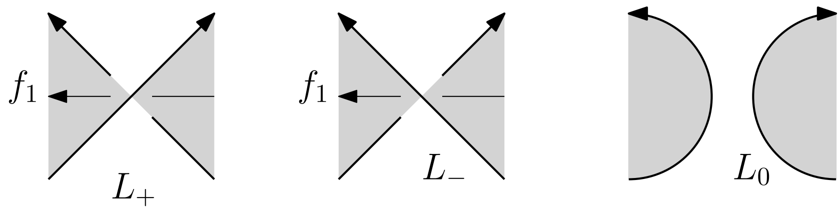


Figure 3.10: The loop f_1 going through the added band.

Definition 3.1.6. Let L be a link. Define the **skein triple** to be the triplet L_+, L_-, L_0 obtained by changing one crossing of L and leaving the rest of the link unchanged, as in Figure 3.11.

Theorem 3.10. *For oriented links L, the Alexander-Conway polynomial $\nabla_L(t) \in \mathbb{Z}[t^{-1/2}, t^{1/2}]$ is characterised by:*

1. $\nabla_{\mathcal{O}}(t) = 1$
2. $\nabla_{L_+}(t) - \nabla_{L_-}(t) = (t^{1/2} - t^{-1/2})\nabla_{L_0}(t)$ ***(Skein relation)***

Proof. The first statement is a direct consequence of the construction of the Alexander-Conway polynomial. Since $\Delta_{\mathcal{O}}(t) = 1$ then $\nabla_{\mathcal{O}}(t) = 1$.
Let now S_0 be a Seifert surface for L_0, constructed via the Seifert algorithm, we can construct Seifert surfaces for L_+ ans L_- by adding bands at specific crossings. If $\{[f_2], \dots, [f_n]\}$ is a basis for $H_1(S_0)$, we obtain a bases for $H_1(S_\pm)$ by adding the class of a loop $[f_1]$ going through the added band as in Figure 3.10.
Computing the Seifert matrices A_0 and $A_\pm$ associated to S_0 and $S_\pm$ with respect to bases $\{[f_2], \dots, [f_n]\}$ and $\{[f_1], [f_2], \dots, [f_n]\}$ we have

$$A_+ = \begin{pmatrix} n & \mathbf{a} \\ \mathbf{b} & A_0 \end{pmatrix} \quad A_- = \begin{pmatrix} n-1 & \mathbf{a} \\ \mathbf{b} & A_0 \end{pmatrix}$$

for $n \in \mathbb{Z}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^{n-1}$.
Now

$$\nabla_{L_+} = \det(t^{\frac{1}{2}} A_+ - t^{-\frac{1}{2}} A_+^T) = \det \begin{pmatrix} n(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) & t^{\frac{1}{2}}\mathbf{a} - t^{-\frac{1}{2}}\mathbf{b}^T \\ t^{\frac{1}{2}}\mathbf{b} - t^{-\frac{1}{2}}\mathbf{a}^T & t^{\frac{1}{2}} A_0 - t^{-\frac{1}{2}} A_0^T \end{pmatrix}$$
$$\nabla_{L_-} = \det(t^{\frac{1}{2}} A_- - t^{-\frac{1}{2}} A_-^T) = \det \begin{pmatrix} (n-1)(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) & t^{\frac{1}{2}}\mathbf{a} - t^{-\frac{1}{2}}\mathbf{b}^T \\ t^{\frac{1}{2}}\mathbf{b} - t^{-\frac{1}{2}}\mathbf{a}^T & t^{\frac{1}{2}} A_0 - t^{-\frac{1}{2}} A_0^T \end{pmatrix}$$

so using the multilinearity of the determinant

$$\begin{aligned} \nabla_{L_+} + \nabla_{L_-} &= \det \begin{pmatrix} t^{\frac{1}{2}} - t^{-\frac{1}{2}} & 0 \\ 0 & t^{\frac{1}{2}} A_0 - t^{-\frac{1}{2}} A_0^T \end{pmatrix} \\ &= (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \det(t^{\frac{1}{2}} A_0 - t^{-\frac{1}{2}} A_0^T) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \nabla_{L_0}. \end{aligned}$$

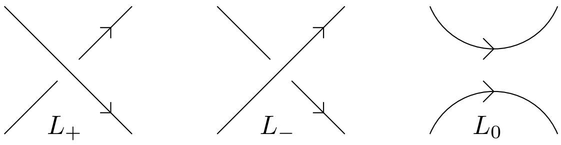


Figure 3.11: Local crossing changes.

□

Corollary 3.11. *Considered L to be the trivial link with k components, then $\nabla_L(t) = 0$, for $k \geq 2$.*

Proof. By induction suppose that $L = \mathcal{O} \sqcup \mathcal{O}$ has two components. Consider now a knot $K = \mathcal{O}$ and twist it by an half twist. We obtain an equivalent knot K', and, since ∇ is a knot invariant,$\nabla_{K'} = \nabla_K$
We compute the crossing changing obtaining $K_+ = K_- = K'$ and $K_0 = L$. Using the Skein relation we have $(t^{1/2} - t^{-1/2})\nabla_{K_0} = \nabla_{K+} - \nabla_{K_-} = \nabla_K - \nabla_K = 0$ so $\nabla_L(t) = \nabla_{K_0}(t) = 0$. Inductively, it follows for $L = \mathcal{O} \sqcup \cdots \sqcup \mathcal{O}$. □

Theorem 3.12. *Any knot invariant satisfying the Skein relation and having value 1 on the unknot is the Alexander-Conway polynomial.*

Since the Alexander-Conway polynomial satisfies the skein relation, it is enough to show that it can be computed using only its value on the unknot, thus it is completely determined by the Skein relation and its value on the unknot.

Definition 3.1.7. A **resolving tree** for a link L is a binary tree diagram with L at the root of the tree, trivial links as terminal leafs, and such that at each stage of the tree, one crossing is isolated and changed, according to the Skein relation as in Figure 3.11, so that every triplet (parent, left child, right child) is of the form (L_+, L_-, L_0) or (L_-, L_+, L_0). Each regular diagram may be replaced by an isotopic regular diagram before applying the Skein relation.

Given a link L it is always possible to build a resolving tree since at each step we are performing a crossing change which leaves constant or decreases the number of crossings. Since each link has a finite unknotting number (see Page 28), in a finite number of step we will obtain a finite number of trivial link.

We give a sketch of proof of Theorem 3.12 using the resolving tree to define an algorithm to calculate ∇_L as follows:
First create a resolving tree for L. The terminal nodes will be trivial links. By assumption $\nabla_{\mathcal{O}} = 1$ and $\nabla_L = 0$ if L is a split links by Corollary 3.11. Using Skein relation, ∇_L is a linear combination of ∇ applied to each terminal node. For a complete proof see [11, Pg. 167].

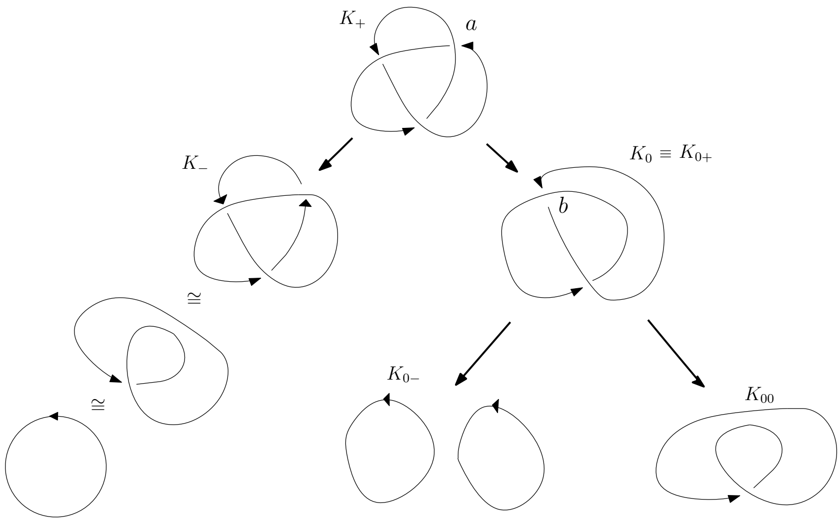


Figure 3.12: Resolving tree for the trefoil knot.

Example 6. We want to compute the Alexander polynomial of the trefoil knot K using the Skein relation. In order to do so we need to create a resolving tree for K as in Figure 3.12. We choose the crossing labelled with a, which is a positive crossing K_+ accordingly with Figure 3.11 so we need to create K_-, K_0.
The knot K_- is the unknot $\mathcal{O}$ so its Alexander polynomial is 1. The knot K_0 is the Hopf link. We label its crossing and we continue to create our resolving tree considering the crossing b which is the positive crossing $K_{0+} = K_0$ so we create crossings K_{0-}, K_{00}. The first one is a two component trivial link, so its Alexander polynomial is zero, the second one is the unknot.
We can now use the Skein relation to compute the Alexander polynomial of the trefoil.

$$\begin{aligned}
\nabla_K(t) = \nabla_{K_+}(t) &= \nabla_{K_-}(t) + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})\nabla_{K_0}(t) = \\
&= 1 + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})(\nabla_{K_{0-}} + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})\nabla_{K_{00}}) = \\
&= 1 + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})(0 + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})) = \\
&= 1 + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})^2 = \\
&= t^{-1} + t - 1.
\end{aligned}$$

3.1.5 Grid Diagram

So far we described many ways to compute the Alexander polynomial of a link starting from a covering or from a presentation of the fundamental group. All this description seems to be take distance from the description of links given in Chapter

2 which uses grid diagrams. We want now to describe a way to compute the Alexander polynomial starting from is grid diagram. To do so we will investigate how the links winds around each point of the grid and we will build a matrix with this information called *grid matrix*.

Definition 3.1.8. Let γ be a closed, oriented, piecewise linear curve in the plane, and let $p \in \mathbb{R}^2 \setminus \gamma$. The **winding number** $w_\gamma(p)$ of γ around p is defined as follows. Choose a ray ρ, starting from p and extending to infinity, that intersects γ transversally. The winding number $w_\gamma(p)$ is the sum of the signs of all crossings between ρ and γ (see Definition 2.4.1).

Note that since ρ is in general position, every intersection is a transverse crossing, meaning γ locally passes from one side of ρ to the other, which ensures that each crossing sign is well-defined.

Remark 17. The winding number is independent of the choice of the transversal ray ρ. Furthermore, by the general position assumption, we can always ensure that ρ does not pass through any vertices or self-intersection (double) points of γ

We now consider as grid diagram a planar realization $\mathbb{G}$ with grid number n, representing a link L and we associate a matrix to the grid $\mathbb{G}$ as follows:

Definition 3.1.9. Form an $n \times n$ matrix whose $(i, j)^{th}$ entry is obtained by rasing the formal variable t to the power given by (-1)-times the winding number of the regular diagram given by $\mathbb{G}$, around the $(j-1, n-i)^{th}$ lattice point with $1 \leq i, j \leq n$. We call this matrix the **grid matrix** and denote it with $\mathbf{M}(\mathbb{G})$

Consider the function $\det(\mathbf{M}(\mathbb{G}))$ associated to the diagram. This is not a link invariant, however, with a suitable normalization, we can obtain the Alexander-Conway polynomial of the link.
We start by considering a function $\mathrm{a}(\mathbb{G})$ associated to the grid: for each O and X-marking consider the four corners of the squares in the grid occupied by the marking and sum up the winding numbers in these corners. By summing these contributions for all O's and X's and dividing the result by 8, we get the number $\mathrm{a}(\mathbb{G})$ associated to the grid. Finally let $\epsilon(\mathbb{G}) \in \{\pm 1\}$ be the sign of the permutation that connects $\sigma_{\mathbb{O}}$ (see Page 22) and $\tau = (n, n-1, \ldots, 1)$ (which is the permutation sending $i \mapsto \tau(i) = n - i + 1$ for $i = 1, \ldots, n$).

Definition 3.1.10. Consider a grid diagram $\mathbb{G}$ of dimension n. We define the function

$$D_{\mathbb{G}}(t) = \epsilon(\mathbb{G}) \cdot \det(\mathbf{M}(\mathbb{G})) \cdot (t^{-\frac{1}{2}} - t^{\frac{1}{2}})^{1-n} t^{a(\mathbb{G})} \tag{3.4}$$

Theorem 3.13. *Let $\mathbb{G}$ be a grid diagram that represents L. Then, the function $D_{\mathbb{G}}(t)$, is a link invariant and it coincides with the Alexander-Conway polynomial $\nabla_L(t)$ of the link L.*

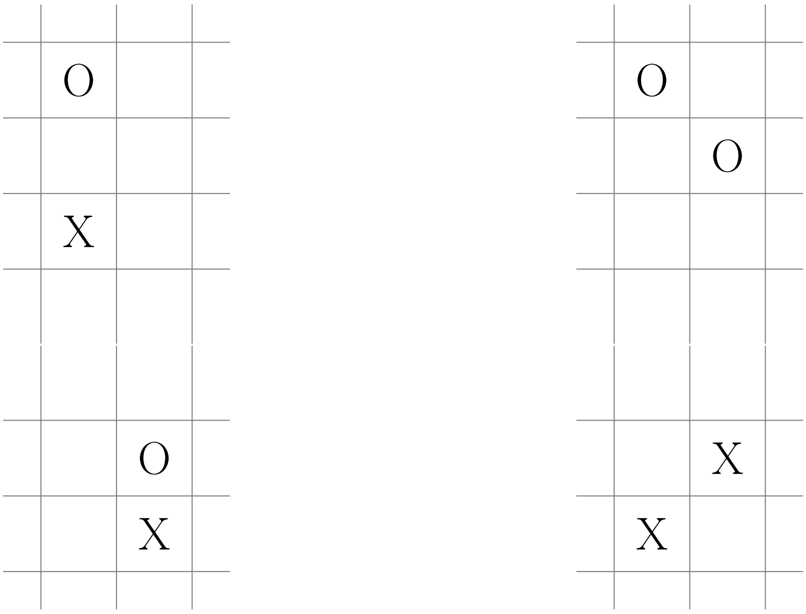


Figure 3.13: Commutation moves, on the left the y-projections of the X, O intervals are disjoint, while on the right the projections are nested.

In order to prove this theorem we need to prove that the function $D_{\mathbb{G}}(t)$ is invariant under commutation and stabilization moves, then thanks to Theorem 2.2 we obtain the first part of our thesis.

Proposition 3.14. *The function $D_{\mathbb{G}}(t)$ is invariant under commutation moves.*

Proof. Supposed that the grid $\mathbb{G}'$ is derived from $\mathbb{G}$ by commuting the i^{th} and the $(i+1)^{st}$ columns. Then the matrices $M(\mathbb{G})$ and $M(\mathbb{G}')$ will differ only in the $(i+1)^{st}$ column.
We distinguish two cases, depending on whether the two intervals we want to commute are project disjointly to the y-axis, or one projects inside the other as in Figure 3.13.
In the first case we subtract the i^{th} column from the $(i+1)^{st}$ in $M(\mathbb{G})$ and the $(i+2)^{nd}$ from the $(i+1)^{st}$ in $M(\mathbb{G}')$. The resulting matrix will differ only in the sign of the $(i+1)^{st}$ column, hence their determinant will differ of a sign. Since the size of the grid does not changes, quantity $a(\mathbb{G})$ is preserved, i.e. $a(\mathbb{G}) = a(\mathbb{G}')$, and $\epsilon(\mathbb{G}) = \epsilon(\mathbb{G}')$ we get that $D_{\mathbb{G}}(t)$ is invariant under commutation moves.
In the second case we consider nested commutation of columns and we need to distinguish four further subcases, depending on the relative position of the O and the X markings. We give details only for the case in Figure 3.13 where the O-markings are over the X-markings in both columns, the other can be obtained

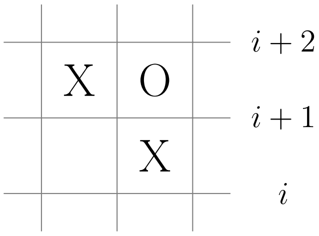


Figure 3.14: Stabilization move.

similarly.
As before we need to subtract one column from the other in each matrix, the difference is the choice of the column, which depends on the position of the markings. In general we need to subtract form the $(i+1)^{st}$ column the corresponding on the shorter O-X interval (that is the column where the two markings are closer to each other). After performing the subtraction in both matrices we realize that the $(i+1)^{st}$ column differ not only by a sign, but also by a multiple of t. It can be shown that this difference is compensated with the difference of the terms $a(\mathbb{G})$ and $a(\mathbb{G}')$, while the sign is absorbed by the change of ϵ. This conclude the invariance of $D_{\mathbb{G}}(t)$ under commutation of columns. A similar argument verifies the result for the row commutation. □

Proposition 3.15. *The function $D_{\mathbb{G}}(t)$ is invariant under stabilization moves.*

Proof. Let's consider the stabilization shown in Figure 3.14. In the matrix of the stabilized diagram we subtract the $(i+2)^{nd}$ row from the $(i+1)^{st}$ row, then we get a matrix which has a single non-zero element in this row. The determinant of the minor corresponding to this element is, up to sign, the determinant of the matrix before the stabilization move. The sign change is compensated by $\epsilon(\mathbb{G})$ while the t-power by the change of size of the grid and of the quantity $a(\mathbb{G})$. This shows that $D_{\mathbb{G}}(t)$ is invariant under stabilization/de-stabilization moves. The other cases of stabilization behave similarly. □

To show that it coincides with the symmetrized Alexander polynomial we will use the fact that $D_{\mathbb{G}}(t)$ satisfies the Skein relations.

Definition 3.1.11. Let (L_+, L_-, L_0) be and oriented skein triple (see Definition 3.1.6). A **grid realization of the oriented skein triple**, consists of four grid diagrams $\mathbb{G}_+, \mathbb{G}_-, \mathbb{G}_0, \mathbb{G}_0'$ representing the links L_+, L_-, L_0 and L_0 respectively. These diagrams are further related as follows: $\mathbb{G}_+$ and $\mathbb{G}_-$ differ by a cross-commutation, $\mathbb{G}_0$ and $\mathbb{G}_0'$ differ by a commutation and $\mathbb{G}_+$ and $\mathbb{G}_0$ differ in the placement of their X-markings, as shown in Figure 3.15

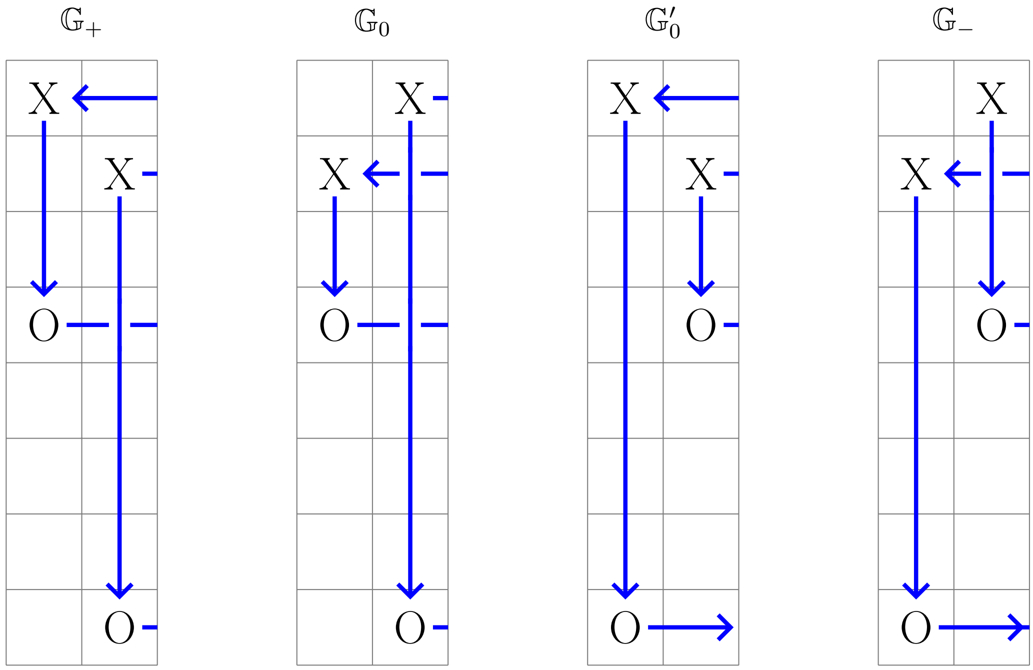


Figure 3.15: The four diagrams of the Skein relation.

Lemma 3.16. *Any oriented skein triple has a grid realization.*

Proof. Given the diagrams (D_+, D_-, D_0) given by the oriented skein triple , defined as in Page 23, we can approximate the diagrams by horizontal and vertical segments as in the proof of Theorem 2.1, with the additional property that in the small disk where the diagrams differ, the approximation is as given by $\mathbb{G}_+, \mathbb{G}_-, \mathbb{G}'_0$ of Figure 3.15, while outside the disk the approximations are identical. Applying a commutation move on $\mathbb{G}'_0$ and we get $\mathbb{G}_0$. $\square$

Proposition 3.17. *The invariant $D_L(t)$ satisfies the skein relation, that is, for an oriented skein triple (L_+, L_-, L_0) we have*

$$D_{L_+}(t) - D_{L_-}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})D_{L_0}(t)$$

Proof. Let (L_+, L_-, L_0) be an oriented skein triple, and let $\mathbb{G}_+, \mathbb{G}_-, \mathbb{G}_0, \mathbb{G}'_0$ be its grid realization. These grid diagrams agree in the placements of their $\mathbb{X}$ and $\mathbb{O}$-markings in all but two consecutive columns, which we think of as left-most. In these two columns of $\mathbb{G}_+$ we either move the two X-markings (transforming $\mathbb{G}_+$ to $\mathbb{G}_0$) or the two O-markings (transforming $\mathbb{G}'_0$ to $\mathbb{G}_+$) or both (realizing a cross-commutation, transforming $\mathbb{G}_+$ to $\mathbb{G}_-$).

Now, the four associated grid matrices differ only in their second columns and we

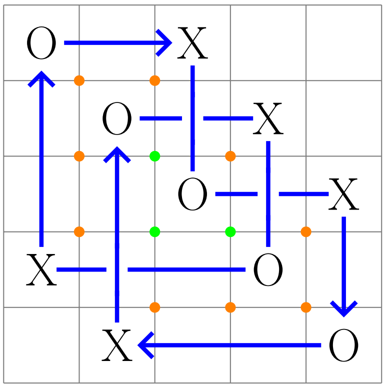


Figure 3.16: Grid diagram for the trefoil knot with orientation, where green points have winding number -2, while orange points have winding number -1.

have the relation

$$\det(\mathbf{M}(\mathbb{G}_+)) + \det(\mathbf{M}(\mathbb{G}_-)) = \det(\mathbf{M}(\mathbb{G}_0)) + \det(\mathbf{M}(\mathbb{G}_0'))$$

It's straightforward to verify that

$$\begin{aligned} a(\mathbb{G}_-) &= a(\mathbb{G}_+) = a(\mathbb{G}_0) + \frac{1}{2} = a(\mathbb{G}_0') + \frac{1}{2} \\ \epsilon(\mathbb{G}_+) &= -\epsilon(\mathbb{G}_-) = \epsilon(\mathbb{G}_0) = -\epsilon(\mathbb{G}_0'). \end{aligned}$$

Combining these with Equation (3.4) we obtain the skein relation for $D_L(t)$. $\square$

Proof Theorem 3.13. The Alexander-Conway polynomial is characterized by the skein relation and by its normalization for the unknot. Since $D_L(t)$ satisfies this skein relation and $D_{\mathcal{O}}(t) = 1$ the result follows. $\square$

Example 7. Consider the trefoil knot and its grid diagram with the orientation as in Figure 3.16. We can compute the winding matrix defining $m_{i,j} = t^{(-1)w_\gamma(j-1,n-i)}$ where γ is the a regular projection of the trefoil encoded by the grid diagram. Thus we obtain

$$\mathbf{M}(\mathbb{G}) = \begin{pmatrix} 1 & t & t & 1 & 1 \\ 1 & t & t^2 & t & 1 \\ 1 & t & t^2 & t^2 & t \\ 1 & 1 & t & t & t \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Using a calculator we can compute $\det(\mathbf{M}(\mathbb{G}))$.

$$\det(\mathbf{M}(\mathbb{G})) = (t^6 - 5t^5 + 11t^4 - 14t^3 + 11t^2 - 5t + 1)$$

Considering the points in Figure 3.16 we calculate $a(\mathbb{G})$. For each marking we sum the winding number of its corners, so the top left X marking contributes $+1$ to the sum, while the X marking in the middle contributes $2+2+2+1=7$. Summing up for all the markings and dividing the result by 8 we obtain $a(\mathbb{G})=-3$.
Since the permutation $\sigma_{\mathbb{O}}=(5,4,3,2,1)=(n,n-1,\dots,1)$ we get that $\epsilon(\mathbb{G})=1$.
Combining this results we obtain that:

$$\begin{aligned}\nabla_K(t) &= 1\cdot(t^6-5t^5+11t^4-14t^3+11t^2-5t+1)\cdot(t^{-1/2}-t^{1/2})^{-4}t^{-3}\\ &= (t-1)^4(t^2-t+1)t^2(t-1)^{-4}t^{-3}\\ &= t^{-1}-1+t\end{aligned}$$

3.2 Twisted Alexander Polynomial

While the previous section focused exclusively on links in the 3-sphere, we now aim to extend the definition of the Alexander polynomial to links in lens spaces. A priori, one could generalize Fox's description of the Alexander polynomial by starting from a presentation of the fundamental group of the link exterior in $L(p,q)$. However, since the first homology group of $L(p,q)$ has non-trivial torsion (as shown in Lemma 1.1), such an approach would result in a significant loss of information. To overcome this issue, the idea is to introduce a new invariants, known as the *twisted Alexander polynomials*. This polynomials account for the torsion of $L(p,q)$ while recovering the classical Alexander polynomial when applied to links in S^3. In what follows, we provide a general construction of the twisted Alexander polynomials for an arbitrary finitely generated group G, and subsequently specialize its behaviour to the case of links in lens spaces.
Considered a finitely generated group $G=\langle x_1,\dots,x_m \mid r_1,\dots,r_n\rangle$ and denote with $H=G/G'$ its abelianization and let $N=H/\mathrm{Tors}(H)$. Consider the Alexander-Fox matrix A, defined as in Equation (3.3), associated to the presentation of G. Moreover let $E(G)$ be the first elementary ideal of G, which is the ideal of $\mathbb{Z}[H]$, generated by the $(m-1)$-minors of A.
For each homomorphism $\sigma:\mathrm{Tors(H)}\longrightarrow\mathbb{C}^*=\mathbb{C}\setminus\{0\}$ we can define a **twisted Alexander polynomial** $\Delta^\sigma(G)$ of G as follow:
Fix a splitting $H=\mathrm{Tors(H)}\times N$ and consider the ring homomorphism (that we still denote with σ):

$$\begin{aligned}\sigma\colon \mathbb{Z}[H] &\longrightarrow \mathbb{C}[N]\\ (f,g) &\longmapsto \sigma(f)g\end{aligned}$$

where $f\in\mathrm{Tors(H)}$, $g\in N, \sigma(f)\in\mathbb{C}^*$.
Since the ring $\mathbb{C}[N]$ is a unique factorization domain, we define $\Delta^\sigma(G)=\gcd(\sigma(E(G)))$.

This is an element of $\mathbb{C}[N]$ defined up to multiplication by elements of N and non-zero complex numbers.
If $\Delta(G)$ denote the classic Alexander polynomial of a knot we have $\Delta^1(G) = \alpha\Delta(G)$ with $\alpha \in \mathbb{C}^*$.
Let's consider now a knot $K \subset L(p, q)$, then the **σ-twisted Alexander polynomial** of K is $\Delta_K^\sigma = \Delta^\sigma(\pi_1(L(p, q) \smallsetminus K))$. Since in this case Tors(H)=$\mathbb{Z}_d$ then $\sigma(\text{Tors}(H))$ is contained in the cyclic group generated by ζ, where ζ is a d^{th} primitive root of the unity.
When $\mathbb{Z}[\zeta]$ is a principal ideal domain (PID), in order to define Δ_K^σ we consider the restriction $\sigma : \mathbb{Z}[H] \longrightarrow \mathbb{Z}[\zeta][N]$. Here $\Delta_K^\sigma \in \mathbb{Z}[\zeta][N]$ is defined up to multiplication by $\zeta^h g$, with $g \in N$.
We have the following proposition:

Proposition 3.18. [21] *If ζ is a d^{th} primitive root of unity, then the ring $\mathbb{Z}[\zeta]$ is a PID if and only id $d \cong 2$ mod 4, or $d \in \{1, 3, 4, 5, 7, 8, 9, 11, 12, 13, 15, 16, 17, 19, 20, 21, 24, 25, 27, 28, 32, 35, 36, 40, 44, 45, 48, 60, 84\}$.*

Example 8. We show how to compute the twisted Alexander polynomial of the knot $K_1 \subset L(4, 1)$ in Figure 3.17.
We can compute its fundamental groups using Theorem 2.6.
If we call $X = L(4, 1) \smallsetminus K_1$ we have

$$\pi_1(X) = \langle v_1, \ldots, v_5, f \mid f^4 = 1, v_2v_5f^4 = 1, v_2v_2v_4v_5f^4 = 1, v_1v_2v_3v_4f^4 = 1, v_1v_3f^4 = 1\rangle$$

and with some calculation we obtain

$$\pi_1(X) = \langle v_1, , v_2, f \mid f^4 = 1, v_1v_2v_1 = v_2v_1v_2\rangle. \tag{3.5}$$

We need now to compute $H_1(X)$ in order to obtain its Torsion subgroup. So abelianizing $\pi_1(X)$ we obtain

$$H_1(X) = \langle v_1, f \mid f^4 = 1\rangle \cong \mathbb{Z} \oplus \mathbb{Z}_4.$$

We compute the Alexander-Matrix of $\pi_1(X)$ with respect to $H_1(X) \cong \mathbb{Z}\langle v_1\rangle \oplus \mathbb{Z}_4\langle f\rangle$, mapping $v_1 \mapsto t$ and $f \mapsto s$

$$A = \begin{pmatrix} 1 + t^2 - t & t - 1 - t^2 & 0 \\ 0 & 0 & 1 + s + s^2 + s^3 \end{pmatrix}. \tag{3.6}$$

So the first elementary ideal is $E(\pi_1(X)) = ((1 + t^2 - t)(1 + s + s^2 + s^3))$.
Since $\text{Tors}(H_1(X) \cong \mathbb{Z}_4$, for each homomorphism $\sigma : \mathbb{Z}_4 \longrightarrow \mathbb{C}^*$, that is for every 4^{th} primitive root of the unity, we can compute a twisted Alexander polynomial

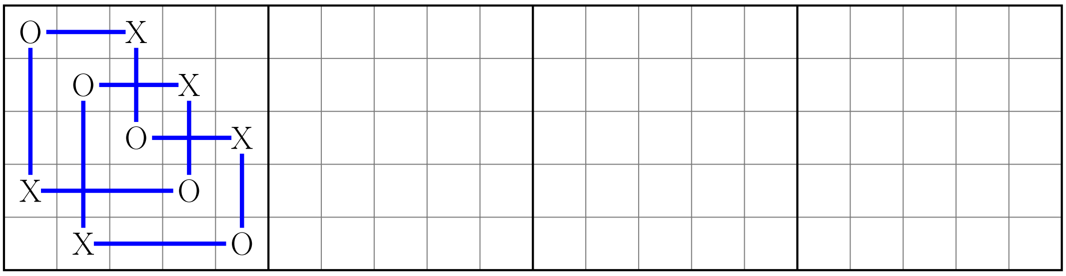


Figure 3.17: Grid diagram of $K_1 \subset L(4,1)$.

$\Delta^{\sigma}_{K_1}(t)$ which means, to evaluate $E(\pi_1(X))$ in $1, -1, i, -i$.

$$\begin{aligned}
&\Delta^{1}_{K_1}(t) = 4(t^2 - t + 1) \doteq \Delta_{K_1}(t)\\
&\Delta^{-1}_{K_1}(t) = 0(t^2 - t + 1) = 0 \quad \text{since} \quad (1 + s + s^2 + s^3)_{|-1} = 0\\
&\Delta^{i}_{K_1}(t) = 0\\
&\Delta^{-i}_{K_1}(t) = 0.
\end{aligned}$$

We can observe that the the only non zero twisted Alexander polynomial obtained is when $\sigma = 1$, that is the Alexander polynomial. This is due to the fact that K_1 is a local knot.
A knot is called **local** if it is contained in a ball embedded in $L(p, q)$.
We now enunciate some important properties of the twisted Alexander polynomials

Proposition 3.19. *Let K be a local knot in $L(p, q)$. Then $\Delta^{\sigma}_{K} = 0$ if $\sigma \neq 1$, and $\Delta_K = p \cdot \Delta_{\tilde{K}}$ otherwise, where $\tilde{K}$ is the knot K considered as a knot in S^3.*

Proof. The fundamental group of K can be presented with the relations of Wirtinger type and the lens relation $f^p = 1$ (see [7]). Therefore the column in the Alexander-Fox matrix A corresponding to the Fox derivative of the lens relation is everywhere zero except for the entry corresponding to the f-derivative, which is $1 + f + f^2 + \cdots + f^{p-1}$. Moreover, removing the column and the row corresponding to this entry we obtain the Alexander-Fox matrix of $\tilde{K}$. So the statement follows by observing that in the case of Δ_K, the generator f is sent to 1, while if $\sigma \neq 1$, the generator f is sent in a k-th root of the unity, where k divides p, so $\sigma(1 + f + f^2 + \cdots + f^{p-1}) = 0$. □

As a consequence, a knot with a non trivial twisted Alexander polynomial cannot be local.

Proposition 3.20. [27] *Let K be a knot in lens space, then:*

1. $\Delta^{\sigma}_{K}(t) = \Delta^{\sigma}_{K}(t^{-1})$

2. $\Delta_K(1) \equiv |\mathrm{Tors}(H_1(L(p,q) \smallsetminus K))| \pmod p$.

Example 9. We produce now the calculation of a non local knot $K_2 \subset L(4,1)$ (see Figure 3.18 and call $Y = L(4,1) \smallsetminus K_2$. From its grid diagram we obtain:

$$\begin{aligned}\pi_1(Y) = \langle v_1, \dots, v_5, f \mid\ & v_1 f v_2 f v_4 v_5 f^2 = 1,\ v_1 f v_1 f v_4 v_5 f^2 = 1,\ v_1 f v_1 f v_4 f v_4 f = 1, \\ & f v_1 f v_3 v_4 f v_4 f = 1,\ f v_1 f v_2 v_3 f v_4 f = 1\rangle \\ = \langle v_1, v_4, f \mid\ & v_1 f v_1 f v_4 f v_4 f = 1,\ f v_4 f^2 v_1 f v_1 = v_4 f v_4 f^2 v_1 f\rangle \\ = \langle v, w, f \mid\ & v f v f w f w f = 1,\ f w f^2 v f v = w f w f^2 v f\rangle.\end{aligned}$$

Abelianizing we obtain

$$H_1(Y) = \langle v, w, f \mid f^4 v^2 w^2 = 1,\ f^4 v^2 w = f^4 w^2 v\rangle = \langle v, f \mid f^4 v^4 = 1\rangle = \langle x, f \mid x^4 = 1\rangle$$

so we can write $H_1(Y) \cong \mathbb{Z}\langle f\rangle \oplus \mathbb{Z}_4\langle x\rangle$ and compute the Alexander-Fox matrix with respect to this basis, mapping $f \mapsto t$ and $x \mapsto u$.

$$A = \begin{pmatrix} 1+u & u^2(1+u) & ut^{-1} + u^2t^{-1} + u^3t^{-1} + t^{-1} \\ ut^2 + u^2t^2 - u^2t & t-1-u & 1+u+ut+u^2t-ut^{-1}-u^2t^{-1}-u^2-u^3 \end{pmatrix}.$$

In order to calculate the twisted Alexander polynomial we need to compute the determinants of all the 2×2 minors E_{ij} of A, where E_{ij} is the minor obtained combining the columns i, j

$$\begin{aligned}E_{1,2} &= t(2+2u) - t^2(u^3+2+u) - (1+u^2+2u) \\ E_{1,3} &= -t(u^3+u+2) - (u^3-u^2-3u-1) - t^{-1}(2u^2+u+u^3) \\ E_{2,3} &= t(u^3+u+2) + (u^3-u^2-3u-1) + t^{-1}(2u^2+u^3+u) = -E_{1,3}\end{aligned}$$

As before, for each homomorphism $\sigma : \mathbb{Z}_4 \longrightarrow \mathbb{C}^*$ we can compute a twisted Alexander polynomial, calculating $E^u_{i,j}$, this means, evaluating the minors $E_{i,j}$ in

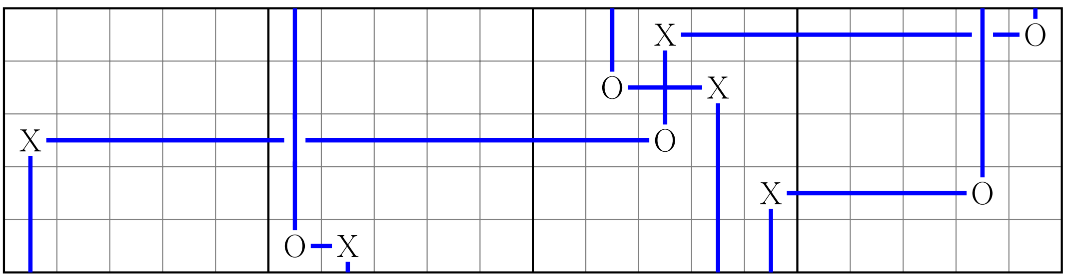


Figure 3.18: Grid diagram of $K_2 \subset L(4,1)$.

$u = 1, -1, i, -i$, and taking the g.c.d$(E^u_{i,j})$ for each i, j, u.
For $u = 1$

$$E^1_{1,2} = 4t^2 - 4t + 4 \quad E^1_{1,3} = -4t - 4t^{-1} + 4 \doteq 4t^2 - 4t + 4 \quad E^1_{2,3} = 4t^2 - 4t + 4$$

so $\Delta^1_{K_2}(t) = \text{g.c.d}(E^1_{1,2}, E^1_{1,3}, E^1_{2,3}) \doteq 4(t^2 - t + 1)$.
For $u = -1$

$$E^{-1}_{1,2} = 0 \quad E^{-1}_{1,3} = 0 \quad E^{-1}_{2,3} = 0$$

so $\Delta^{-1}_{K_2}(t) = 0$.
For $u = i$

$$E^i_{1,2} = -2(t-1)(t-i) \quad E^i_{1,3} = 2(t-i)^2 \quad E^i_{2,3} = -2(t-i)^2$$

so $\Delta^i_{K_2}(t) = \text{g.c.d}(E^i_{1,2}, E^i_{1,3}, E^i_{2,3}) \doteq 2(t-i)$.
For $u = -i$

$$E^{-i}_{1,2} = -2(t-1)(t-i) \quad E^{-i}_{1,3} = 2(t-i)^2 \quad E^{-i}_{2,3} = -2(t-i)^2$$

so $\Delta^{-i}_{K_2}(t) = \text{g.c.d}(E^{-i}_{1,2}, E^{-i}_{1,3}, E^{-i}_{2,3}) \doteq (2(t-i))$.

As we can notice, K_1 and K_2 have the same Alexander polynomial but K_2 has non trivial twisted Alexander polynomial for $\sigma = i, -i$, thus the knots are not equivalent in $L(4, 1)$. This confirm the observation made in the introduction of this section and the necessity to introduce a twisted version of the Alexander polynomials, to distinguish knots in lens spaces.

Chapter 4

Grid Homology

In this chapter we will describe the construction for the different flavours of the grid homology. At the beginning we will define the generators of the grid complex and their bigradings. We'll then explain how to calculate the differential and the three different flavours of grid homology $\widetilde{GC}, GC^{-}, \widehat{GC}$ for both S^3 and $L(p,q)$. From now to the end of the thesis we are going to work with homologies with $\mathbb{Z}_2$ coefficients and we will consider only knots.

The definition of a grid diagram is briefly recalled from Section 2.2. Given a planar grid and a suitable identification of its segments, a grid diagram consists of a diagram on the torus, defined by the sets of curves α, β and equipped with a set of markings. Unless stated otherwise, the term grid diagram throughout this chapter will refer to a toroidal grid diagram (or twisted toroidal grid diagram for knots in lens spaces).

4.1 Generators

4.1.1 Grid states in S^3

Definition 4.1.1. A **grid state x** for a grid diagram $\mathbb{G}$ with grid number n, representing a knot $K \subset S^3$, is a one-to-one correspondence between the horizontal circles β and vertical circles α.
This corresponds to choosing n points $\{x_1, \ldots, x_n\}$ on the torus in $\alpha \cap \beta$ such that there is exactly one in each α and β curve as shown in Figure 4.1.
The set of grid states for $\mathbb{G}$ is denoted $S(\mathbb{G})$.

As for the sets $\mathbb{X}$ and $\mathbb{O}$, we can represent a grid state as permutation of n elements. If $\mathbf{x} = \{x_1, \ldots, x_n\}$, then $\sigma = \sigma_{\mathbf{x}}$ is specified by the property that $x_i = \alpha_{\sigma(i)} \cap \beta_i$.

The correspondence between grid states and permutations is non canonical but it depends on a numbering of the horizontal and vertical circles. When illustrating the diagrams and the states, we use planar realizations of the grid diagrams.

Remark 18. Due to the boundary identifications of the toroidal grid, a point of the grid state on the bottom horizontal circle α_1 can also be viewed as a point on the top horizontal circle. Analogously, points on the rightmost vertical circle correspond to points on the leftmost vertical circle β_1.
As a convention throughout this thesis, when illustrating grid states and computing functions over them, we consider these points as lying exclusively on the bottom circle and on the leftmost circle of the toroidal grid, as shown in Figure 4.1, where the alternative positions of the points are shaded in gray.

4.1.2 Rectangles in S^3

The chain complexes associated to a grid diagram are generated by grid states, and their differentials counts rectangles connecting states. The various versions of the grid complex differ in how they count rectangles.

Fix two grid states $\mathbf{x}, \mathbf{y} \in \mathbf{S}(\mathbb{G})$ and an embedded rectangle r in the torus, whose boundary lies in the union of the horizontal and vertical circles, satisfying the following relationship. The sets $\mathbf{x}$ and $\mathbf{y}$ overlap in $n-2$ points in the torus and the four points left out are the four corners of r. The rectangle r inherits an orientation from the torus and induce an orientation on its boundary ∂r. The oriented boundary of r consists of four oriented segments, two of which are vertical and two of which are horizontal. The rectangle r *goes from* $\mathbf{x}$ *to* $\mathbf{y}$ if the horizontal segments in ∂r point from the components of $\mathbf{x}$ to the components of $\mathbf{y}$, while the vertical segments point from the components of $\mathbf{y}$ to the components of $\mathbf{x}$.
For $\mathbf{x}, \mathbf{y} \in \mathbf{S}(\mathbb{G})$ let $\mathrm{Rect}(\mathbf{x},\mathbf{y})$ denote the set of rectangles from $\mathbf{x}$ to $\mathbf{y}$. This set is either empty, if $\mathbf{x}$ and $\mathbf{y}$ do not coincide in $n-2$ points, or it consists of exactly two rectangles since, as shown in Figure 4.1, due to the identifications of the boundary of the grid, the four corners define a complementary rectangle.
Label the four corners of any rectangle as northeast, southeast, northwest, and southwest as at Page 18. In particular if r is a rectangle from $\mathbf{x}$ to $\mathbf{y}$ then the northeast and southwest corners of r are elements of $\mathbf{x}$, while the northwest and southeast corners of r are elements of $\mathbf{y}$.
We denote $\mathrm{Int}(r)$ the interior of the rectangle r and we define a rectangle to be **empty** if:

$$\mathbf{x} \cap \mathrm{Int}(r) = \mathbf{y} \cap \mathrm{Int}(r) = \emptyset$$

that is it does not contain any element of $\mathbf{x}$ or $\mathbf{y}$ in its interior The set of empty rectangles from $\mathbf{x}$ to $\mathbf{y}$ is denoted $\mathrm{Rect}^\circ(\mathbf{x},\mathbf{y})$.
Given a rectangle $r_1 \in \mathrm{Rect}(\mathbf{x},\mathbf{y})$, we will think of it as a geometric subset of the torus, together with the initial and terminal grid states $\mathbf{x}$,$\mathbf{y}$. Thus if $\mathbf{x} \neq \mathbf{x}'$, a rectangle from $\mathbf{x}$ to $\mathbf{y}$ is always thought as off different from a rectangle from $\mathbf{x}'$ to $\mathbf{y}'$, even if their underlying polygons in the torus are the same. We call this underlying polygon the **support** of the rectangle.

Remark 19. Since we associated to each grid state a permutation, we can give a definition of rectangle in terms of permutations. Given $\mathbf{x},\mathbf{y} \in S(\mathbb{G})$, they define a rectangle if their permutations differ of a transposition, indeed set $\sigma_{\mathbf{x}}$ and $\sigma_{\mathbf{y}}$ to be the two associated permutation, saying that they differ of a transposition τ means that $\sigma_{\mathbf{x}} = \tau\sigma_{\mathbf{x}}$ (or vice-versa), thus the points of the grid states are all equivalent, except for two points which are switched in the grid, forming a rectangle.
In particular any two grid states $\mathbf{z}$ and $\mathbf{w}$ are connected by a sequence of grid states $\mathbf{z} = \mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k = \mathbf{w}$ and rectangles $r_i \in \mathrm{Rect}(\mathbf{z}_i, \mathbf{z}_{i+1})$. This follows from the fact that any permutation $\sigma_{\mathbf{z}} \in S_n$ can be written as product of permutations.

Definition 4.1.2. Taken $\mathbf{x}$,$\mathbf{y}$,$\mathbf{z} \in S(\mathbb{G})$, $r_1 \in \mathrm{Rect}(\mathbf{x},\mathbf{y})$ and $r_2 \in \mathrm{Rect}(\mathbf{y},\mathbf{z})$, we can consider the concatenation $r_1 * r_2$ which we call a **polygon** connecting $\mathbf{x}$ to $\mathbf{z}$ through $\mathbf{y}$. Its corners will be the union of the corners of r_1, r_2 and the boundary the union of the boundaries of r_1, r_2. If the two rectangles share a side (meaning that a side of r_1 coincide or is contained in a side of r_2, or vice-versa), we remove this side from the boundary of $r_1 * r_2$. Thus the boundary inherits an orientation from r_1 and r_2, which is the union of the orientations.
We are going to denote $\mathrm{Poly}(\mathbf{x},\mathbf{z})$ the set of polygons connecting $\mathbf{x}$ to $\mathbf{z}$, and $\mathrm{Poly}^\circ(\mathbf{x},\mathbf{z})$ the set of the empty ones.
Polygons can be in turn connected, considering a concatenation between two polygons defined as for the rectangles .
If a polygon coincide with a row or a column of the grid we denote in **thin annulus**.

4.1.3 Grid states in $L(p,q)$

The definition of grid state in this more general situation, does not change with respect to the S^3 case.

Definition 4.1.3. Given a grid diagram $\mathbb{G}$ of dimension n representing a knot $K \subset L(p,q)$, a **grid state** is a one-to-one correspondence between α and β curves. This corresponds to choosing n points in $\alpha \cap \beta$ such that there is exactly one in each α and β curve.
The set of grid states is denoted $S(\mathbb{G})$.

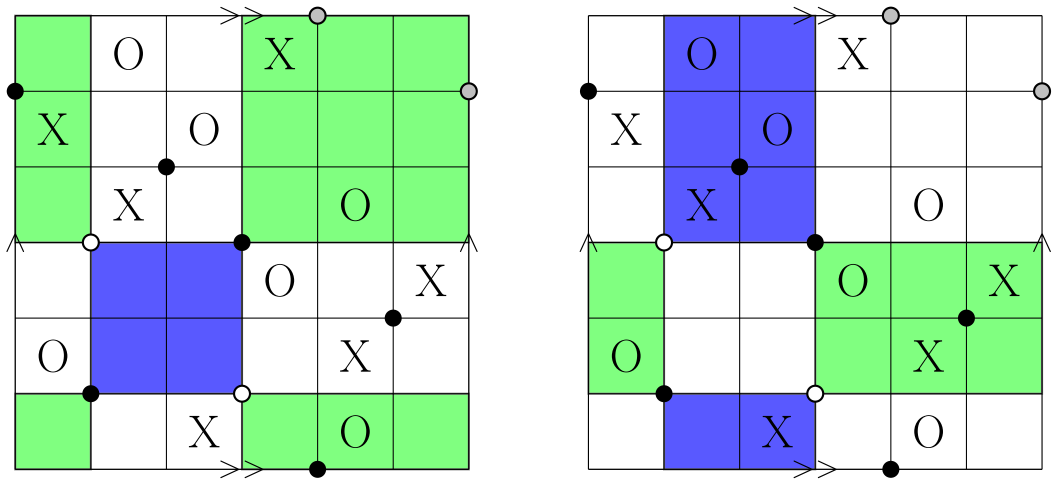


Figure 4.1: Two grid state $\mathbf{x}$ (black dots) and $\mathbf{y}$ (white dots) and the rectangles connecting them. In the left picture we see the two rectangles going from $\mathbf{x}$ to $\mathbf{y}$ while in the right picture we see the two rectangles going from $\mathbf{y}$ to $\mathbf{x}$. We highlighted only the points of the grid state where $\mathbf{x}$ and $\mathbf{y}$ differ.
In grey the alternative positions of the points in the leftmost vertical and bottom horizontal circle.

There is a bijection between $S(\mathbb{G})$ and $S_n \times \mathbb{Z}_p^n$ which can defined as follows: since we fixed a cyclic labelling of the α and β curves it makes sense to speak of the m-th intersection between two curves, with $0 \le m \le p-1$; so if the l-th component of a generator lies on the m-th intersection of α_l and β_j then the permutation $\sigma \in S_n$ associated will be such that $\sigma(l) = j$ and the l-th component of $\mathbb{Z}_p^n$ will be m as shown in Figure 4.2. If $\mathbf{x} \in S(\mathbb{G})$, we can thus write $\mathbf{x} = (\sigma_x, (x_1^p, \dots, x_n^p))$. We will refer to σ_x as the *permutation of the generator* $\mathbf{x}$ and to $(x_1^p, \dots, x_n^p)$ as its *p-coordinates*.

Remark 20. This bijection implies that $\#S(\mathbb{G}) = n!p^n$ for a grid of dimension n in $L(p, q)$.

4.1.4 Rectangles in $L(p, q)$

As for grid states in S^3 we the same definition for rectangles in $L(p, q)$. The main difference is due to the twisted identification of the grid which complicates the search of the rectangles. Given $\mathbf{x}$,$\mathbf{y}$$\in S(\mathbb{G})$ we continue to denote $\text{Rect}(\mathbf{x}, \mathbf{y})$ the set of the rectangles from $\mathbf{x}$ to $\mathbf{y}$ and $\text{Rect}^\circ(\mathbf{x}, \mathbf{y})$ the set of empty rectangles, defined as at Page 71.

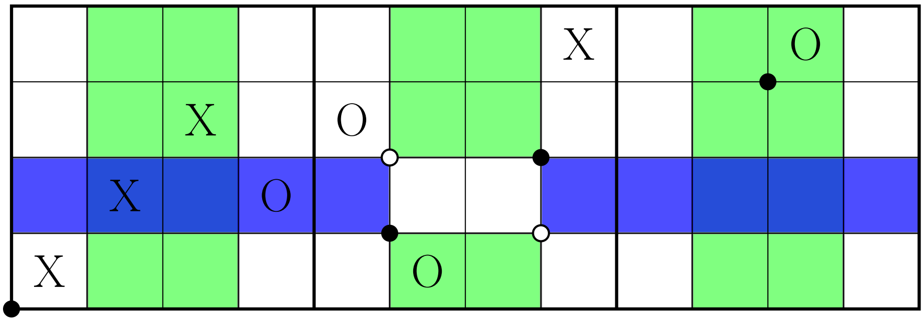


Figure 4.2: Grid states and rectangles from **x**=((2,3),(0,1,1,2)) (black dots) to **y**=((1,3,2),(0,1,1,2)) (white dots) for a grid of dimension 4 in L(3,1). Only the blue rectangle is empty.

4.2 Bigradings

The grid complexes $\widetilde{GC}, GC^-, \widehat{GC}$ (that we will define in the next section) are equipped with two grading which can be obtained directly by the grid states, thanks to the combinatorial description of the grid homology.

4.2.1 Maslov and Alexander grading in S^3

If we consider a grid diagram $\mathbb{G}$ as a grid $[0,n) \times [0,n)$ in $\mathbb{R}^2$, a grid state $\mathbf{x} \in S(\mathbb{G})$ can be seen as a collection of points in with integer coordinates and the set $\mathbb{O} = \{O_i\}_{i=1}^n$ can be thought as a collection of points in the plane with half integer coordinates (i.e. we think at our O-markings as points in the centre of its square).

Definition 4.2.1. Let A and B denote two finite set of points in $\mathbb{R}^2$. We define the function $I(A,B)$ which counts the number of pairs

$$(a,b) = ((a_1,a_2),(b_1,b_2)) \subset A \times B$$

such that $a_i < b_i$ for $i = 1,2$.

We state a useful property, which will simplify our calculations of the gradings.

Definition 4.2.2. We say that a point on a grid diagram $x = (x_1, x_2)$ has a **weight** $w(x) = c - d$, where c is the number of $X = (a,b)$ marking such that $a \geq x_1$ and $b \geq x_2$, and d is the number of X marking such that $x_1 > a$ and $x_2 > b$.

Proposition 4.1. *Take a planar grid of dimension n for a knot in S^3, a generator* $\mathbf{x}$ *in* $S(\mathbb{G})$ *and denote with J the set $\mathbb{O}$ or $\mathbb{X}$, then*

$$I(\mathbf{x}, J) - I(J, \mathbf{x}) = n.$$

Proof. We prove the lemma for the X markings. It can be done also for the O markings with a symmetric definition of weight.
Say that the $x_1, x_2, \ldots, x_n$ are the points that make up the generator $\mathbf{x}$. Then:

$$\begin{aligned} I(\mathbf{x}, X) - I(X, \mathbf{x}) &= w(x_1) + w(x_2) + \cdots + w(x_n) \\ &= n + (n-2) + (n-4) + \cdots + (n - 2(n-1)) \\ &= n^2 - 2\sum_{i=1}^{n-1} i \\ &= n^2 - 2\frac{n(n-1)}{2} = n. \end{aligned}$$

□

Remark 21. We will use this function to define the bigradings. Given a grid diagram G, we want to observe the behaviour of I under a cyclic translation of the grid. We analyse the case of a vertical translation of the top row R, noting that the argument can be iterated for any other row or column translation.
We denote by A_1, B_1 the sets of points in G before the translation, and by A_2, B_2 the corresponding sets after the translation. By definition, for any two generic sets of points A and B, $I(A, B)$ is the sum of the characteristic function $\chi(a, b)$ over all possible pairs $(a, b) \in A \times B$, where $\chi(a, b) = 1$ if $x_a < x_b$ and $y_a < y_b$, and zero otherwise. Depending on the position of the initial points $a \in A_1$ and $b \in B_1$ within the grid, we distinguish four cases under the row translation:

1. If $a, b \in G \setminus R$, their relative position is unchanged; hence, these pairs yield the same contribution to both $I(A_1, B_1)$ and $I(A_2, B_2)$.

2. If $a, b \in R$, they are translated together, preserving both their relative horizontal and vertical positions; hence, these pairs yield the same contribution to both $I(A_1, B_1)$ and $I(A_2, B_2)$.

3. If $a \in R$ and $b \in G \setminus R$, then a is moved from the top row to the bottom row. Consequently, the vertical condition $y_a < y_b$ (which was always false for A_1, B_1) becomes always true for A_2, B_2. This case adds a contribution of 1 to $I(A_2, B_2)$ for each pair where $x_a < x_b$.

4. If $a \in G \setminus R$ and $b \in R$, then b is moved to the bottom row. Thus, the vertical condition $y_a < y_b$ (which was always true for A_1, B_1) becomes always false for A_2, B_2. This case subtracts a contribution of 1 from $I(A_2, B_2)$ for each pair where $x_a < x_b$.

When considering the global sum over all possible pairs, cases (1) and (2) yield no net change. When summing over all possible pairs, the terms depending on the

specific horizontal positions of the points in cases (3) and (4) cancel each other out.
Let $N_R(A_1) = |A_1 \cap R|$ and $N_R(B_1) = |B_1 \cap R|$ denote the number of points of A_1 and B_1 in the row R, respectively. Assuming the standard grid closure where the total cardinalities satisfy $|A_1| = |B_1|$, the value of I for the translated grid is:

$$I(A_2, B_2) = I(A_1, B_1) + |A_1| \cdot \big(N_R(A_1) - N_R(B_1)\big). \tag{4.1}$$

Definition 4.2.3. The **Maslov grading** is a function

$$M_{\mathbb{O}} : S(\mathbb{G}) \longrightarrow \mathbb{Z}$$

defined as

$$M_{\mathbb{O}}(\mathbf{x}) = I(\mathbf{x}, \mathbf{x}) - I(\mathbf{x}, \mathbb{O}) - I(\mathbb{O}, \mathbf{x}) + I(\mathbb{O}, \mathbb{O}) + 1. \tag{4.2}$$

where the function I is computed on a planar realization of $\mathbb{G}$.

Similarly we can define the function $M_{\mathbb{X}}$ using the X-markings instead of the O-markings. If not specified we will use the function $M_{\mathbb{O}}$ and drop the $\mathbb{O}$ from the notation.

To show that the Maslov grading is well-defined, we recall that any two planar realizations of the same toroidal grid diagram are related by a sequence of translation moves, i.e. cyclic permutations of rows or columns.
As shown in Remark 21, under a cyclic translation of a row R, the value of I shifts by $|A_1| \cdot \big(N_R(A_1) - N_R(B_1)\big)$. Since we are evaluating this function on grid states and markings, we have $N_R(A_1) = N_R(B_1) = 1$, because every row and column contains exactly one point of the grid state, one X marking, and one O marking. Consequently, the error term vanishes, yielding $I(A_2, B_2) = I(A_1, B_1)$.
Furthermore, since any cyclic permutation can be decomposed into a product of such elementary translations, the permutation τ does not affect the value of I. Applying this invariance to each term of Equation 4.2, we conclude that M_O (and analogously M_X) is well-defined and independent of the chosen planar realization.

Definition 4.2.4. The **Alexander grading** is a function

$$A : S(\mathbb{G}) \longrightarrow \mathbb{Z}$$

defined as

$$A(\mathbf{x}) = \frac{1}{2}[I(\mathbb{O}, \mathbb{O}) - I(\mathbb{X}, \mathbb{X}) + 2I(\mathbb{X}, \mathbf{x}) - 2I(\mathbb{O}, \mathbf{x})] + \frac{1-n}{2} \tag{4.3}$$

The same observations made for the Maslov function, show that the Alexander function is well defined and does not depend of the planar realization chosen.

Remark 22. Usually the Alexander function is defined as

$$A(\mathbf{x}) = \frac{1}{2}(M_{\mathbb{O}}(\mathbf{x}) - M_{\mathbb{X}}(\mathbf{x})) + \frac{1-n}{2} \tag{4.4}$$

Using Proposition 4.1 we can obtain the Equation (4.4) by trivial calculations.

From both the formulation it is not straightforward at all that the Alexander grading is integer valued, the following proposition ensures this fact.

Proposition 4.2. *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are two grid states that can be connected by some rectangle* $r \in \mathrm{Rect}(\mathbf{x}, \mathbf{y})$*, then*

$$M_{\mathbb{O}}(\mathbf{x}) - M_{\mathbb{O}}(\mathbf{y}) = 1 - 2\#(r \cap \mathbb{O}) + 2\#(\mathbf{x} \cap \mathrm{Int}(r)). \tag{4.5}$$

$$A(\mathbf{x}) - A(\mathbf{y}) = \#(r \cap \mathbb{X}) - \#(r \cap \mathbb{O}). \tag{4.6}$$

Furthermore $M_{\mathbb{O}}$ *and* A *are integer valued.*

Proof. Taken A, B finite subset of $\mathbb{R}^2$ we define the function

$$J(A, B) = \frac{I(A, B) + I(B, A)}{2} \tag{4.7}$$

where $I(A, B)$ is defined as in (4.2.1).
From trivial substitution we can obtain that

$$M_{\mathbb{O}}(\mathbf{x}) = J(\mathbf{x}, \mathbf{x}) - 2J(\mathbf{x}, \mathbb{O}) + J(\mathbb{O}, \mathbb{O}) + 1.$$

To prove the Equation (4.5) we split our proof in three cases, depending on the position of the rectangle $r \in \mathrm{Rect}(\mathbf{x}, \mathbf{y})$.
We start by the case where the rectangle r is contained in the planar realization used to define M. Label the southwest, northeast, northwest and southeast corners of r by x_1, x_2, y_1, y_2 respectively. Clearly

$$\mathbf{x} = \{x_1, x_2\} \cup \mathbf{p} \text{ and } \mathbf{y} = \{y_1, y_2\} \cup \mathbf{p}.$$

where $\mathbf{p} = \mathbf{x} \cap \mathbf{y}$. It's easy to see that

$$\begin{aligned} J(\mathbf{x}, \mathbf{x}) - J(\mathbf{y}, \mathbf{y}) &= \#\{(x_1, x_2)\} + \#\{x \in \mathbf{p} \mid x > x_1\} + \#\{x \in \mathbf{p} \mid x < x_1\} \\ &+ \#\{x \in \mathbf{p} \mid x > x_2\} + \#\{x \in \mathbf{p} \mid x < x_2\} - \#\{x \in \mathbf{p} \mid x > y_1\} \\ &- \#\{x \in \mathbf{p} \mid x < y_1\} - \#\{x \in \mathbf{p} \mid x > y_2\} - \#\{x \in \mathbf{p} \mid x < y_2\} \\ &= 1 + 2\#\{x \in \mathbf{p} \mid x_1 < x < x_2\} = 1 + 2\#(\mathbf{x} \cap Int(r)). \end{aligned}$$

where we count $\#\{(x_1, x_2)\} = 1$ since $x_1 < x_2$. Similarly,

$$\begin{aligned} 2J(\mathbf{x}, \mathbb{O}) - 2J(\mathbf{y}, \mathbb{O}) &= \#\{O_i \in \mathbb{O} \mid O_i > x_1\} + \#\{O_i \in \mathbb{O} \mid O_i < x_1\} + \#\{O_i \in \mathbb{O} \mid O_i > x_2\} \\ &+ \#\{O_i \in \mathbb{O} \mid O_i < x_2\} - \#\{O_i \in \mathbb{O} \mid O_i > y_1\} - \#\{O_i \in \mathbb{O} \mid O_i < y_1\} \\ &- \#\{O_i \in \mathbb{O} \mid O_i > y_2\} - \#\{O_i \in \mathbb{O} \mid O_i < y_2\} \\ &= 2\#\{O_i \in \mathbb{O} \mid x_1 < O_i < x_2\} = 2\#(\mathbb{O} \cap r). \end{aligned}$$

The above two equations end the first case.
Next we suppose that r satisfies Equation (4.5) and we suppose $r' \in \mathrm{Rect}(\mathbf{y}, \mathbf{x})$ is the rectangle with the property that r and r' meet along both horizontal edges, so in particular they have the same width v. Then, since every column contains an O and every β_i contains a component of $\mathbf{x}$, it follows that

$$\#(r' \cap \mathbb{O}) + \#(r \cap \mathbb{O}) = v$$
$$\#(\mathbf{x} \cap Int(r')) + \#(\mathbf{x} \cap Int(r)) = v - 1.$$

These two equations, together with Equation (4.5) for r shows that

$$M(\mathbf{y}) - M(\mathbf{x}) = 1 - 2\#(r' \cap \mathbb{O}) + 2\#(\mathbf{x} \cap Int(r')) \tag{4.8}$$

which is the analogue of Equation (4.5) for r'.
In the same way we can prove the case when r and r' meet along both vertical edges.
It follows that if Equation (4.5) holds for every $r \in \mathrm{Rect}(\mathbf{x}, \mathbf{y})$ then it holds for every rectangle in $\mathrm{Rect}(\mathbf{x}, \mathbf{y}) \cup \mathrm{Rect}(\mathbf{y}, \mathbf{x})$. It's easy to see that at least one of the four rectangle in $\mathrm{Rect}(\mathbf{x}, \mathbf{y}) \cup \mathrm{Rect}(\mathbf{y}, \mathbf{x})$ is contained in the fundamental domain, for which we already proved Equation (4.5), hence this completes the first part of our proof.
To prove Equation 4.6, if $r \in \mathrm{Rect}(\mathbf{x}, \mathbf{y})$ is any rectangle connecting two grid states $\mathbf{x}$ and $\mathbf{y}$, by Equation (4.5) follows that

$$M_{\mathbb{O}}(\mathbf{x}) - M_{\mathbb{O}}(\mathbf{y}) = 1 - 2\#(r \cap \mathbb{O}) + 2\#(\mathbf{x} \cap Int(r)).$$
$$M_{\mathbb{X}}(\mathbf{x}) - M_{\mathbb{X}}(\mathbf{y}) = 1 - 2\#(r \cap \mathbb{O}) + 2\#(\mathbf{x} \cap Int(r)).$$

Taking the difference of these two equations and and applying Equation (4.4), we conclude that Equation (4.6) holds.
We are left to show that the Alexander and the Maslov function are integer valued. Clearly M is integer valued from its definition since is the sum of integer numbers. This implies that A takes values in $\frac{1}{2}\mathbb{Z}$. In view of Equation (4.6) and Remark 19, it suffices to show that there is one grid state $\mathbf{x}$ for which $A(\mathbf{x})$ is integral, thereafter any other value of $A(\mathbf{y})$ will be calculated recursively.
We consider the grid state $\mathbf{x}^{NWO}$ whose point are all in north-west corner of the squares marked with an O and show that $M_{\mathbb{X}}(\mathbf{x}^{NWO}) \equiv n - 1 \mod 2$.
To this end we find a sequence of grid states $\mathbf{x}_i \in S(\mathbb{G})$ for $i = \{1, \dots, n\}$, with the following properties:

- $\mathbf{x}_1 = \mathbf{x}^{NWX}$ is the grid state whose point are the north-west corners of the squares marked with an X
- $\mathbf{x}_n = \mathbf{x}^{NWO}$

- there is a (not necessarily empty) rectangle connecting $\mathbf{x}_i$ to $\mathbf{x}_{i+1}$.

This sequence always exists, since the permutation connecting $\mathbf{x}^{NWX}$ to $\mathbf{x}^{NWO}$ is a cycle of length n (since the grid represents a knot), and it can be written as a product of $n-1$ transpositions.
Furthermore, $M_{\mathbb{X}}(\mathbf{x}_1) = 0$, indeed it has all the points of the grid state in the north-west corner of the squares marked with an X, but, due to Remark 18, one of the points of $\mathbf{x}_1$ will be in the first row. Since we have n points in the grid state, we suppose that this point is in the circle β_q, this will lead to the following equalities

$$\begin{aligned} I(\mathbf{x}_1, \mathbf{x}_1) &= I(\mathbb{X}, \mathbb{X}) + n - 1 - q - q \\ I(\mathbf{x}_1, \mathbb{X}) &= I(\mathbb{X}, \mathbb{X}) + n - 1 - q + 1 \\ I(\mathbb{X}, \mathbf{x}_1) &= I(\mathbb{X}, \mathbb{X}) - q \end{aligned}$$

Thus

$$\begin{aligned} M_{\mathbb{X}}(\mathbf{x}_1) &= I(\mathbf{x}_1, \mathbf{x}_1) - I(\mathbf{x}_1, \mathbb{X}) - I(\mathbb{X}, \mathbf{x}_1) + I(\mathbb{X}, \mathbb{X}) + 1 = \\ &= n - 1 - 2q - 1 + q + q + 1 = 0 \end{aligned}$$

As consequence, the thesis follows from the reduction $\mod 2$ of Equation (4.5). ☐

Furthermore we have the following relation

Lemma 4.3. *Given a planar realization $\mathbb{G}$ and a grid state $\mathbf{x} \in S(\mathbb{G})$, the sign of the permutation that connects $\mathbf{x}$ with $\mathbf{x}^{NWO}$ is $(-1)^{M(\mathbf{x})}$, where $\mathbf{x}^{NWO}$ is the grid state composed by elements in the northwestern point of each O-marking.*

Proof. Recalling Remark 19 and using a reduction mod 2 of Equation (4.5), we see that every time we have a rectangle between two grid states the parity of the Maslov degree changes and this can be read in terms of signs of permutations because we can associate to each grid state a permutation of S_n and because we have a rectangle between two grid states only when they differ of a transposition. We can always find a permutation which connect $\mathbf{x} \in S(\mathbb{G})$ with $\mathbf{x}^{NWO}$ and this will be the product of a certain number of transposition and every transposition produces a change in the parity of the Maslov degree. ☐

We introduce a notation which will be useful during the computations. Given a grid state $\mathbf{x}$, to denote its bigrading (m, a), where $m = M(\mathbf{x})$ and $a = A(\mathbf{x})$, we write $\mathbf{x} \sim (m, a)$.

There is a more geometric way to describe the Alexander grading that uses the definition of winding number for a knot projection in terms of grid diagram. As seen in the last chapter we can use the winding number of the knot projection to compute the Alexander polynomial of the knot, this gives us the first insight of possible connection between the Alexander polynomial and the grid homology.

Definition 4.2.5. Let $\mathbb{G}$ be a planar realization of a knot K and let $\mathcal{D} = \mathcal{D}(\mathbb{G})$ be the projection of the corresponding knot in the plane, and let p be any point not in $\mathcal{D}$. Let $w_{\mathcal{D}}(p)$ denote the winding number of $\mathcal{D}$ around p. Then we have the following formula for the **Alexander grading**

$$A(\mathbf{x}) = -\sum_{x\in\mathbf{x}} w_{\mathcal{D}}(x) + \frac{1}{8}\sum_{j=1}^{8n} w_{\mathcal{D}}(p_j) + \frac{1-n}{2} = -\sum_{x\in\mathbf{x}} w_{\mathcal{D}}(x) + a(\mathbb{G}) + \frac{1-n}{2}. \quad (4.9)$$

Where $p_1, \ldots, p_{8n}$ are the $8n$ lattice point on the grid (counted with multiplicity) corresponding to the 4 corners of each of the $2n$ squares of the grid marked with and O or an X, $a(\mathbb{G})$ is defined as in Page 59 and $x \in \mathbf{x}$ are the points of the grid state $\mathbf{x}$ in the grid diagram.

In order to verify the equivalence of the two different definitions for the Alexander grading we want to recover Definition 4.2.5 from Definition 4.2.4. Firstly we need to show how to express the winding number $w_{\mathcal{D}}(p)$ in terms of $\mathbb{X}$ and $\mathbb{O}$ in the grid.

Lemma 4.4. *Let $\mathbb{G}$ be a planar grid diagram of a knot K, let $\mathcal{D} = \mathcal{D}(\mathbb{G})$ be the corresponding knot projection in the plane, and let p be any point not on $\mathcal{D}$.Then the winding number $w_{\mathcal{D}}(p)$ is computed by*

$$w_{\mathcal{D}}(p) = J(p, \mathbb{O} - \mathbb{X}).$$

Proof. If $(x, y) = p \notin \mathcal{D}$ then $I(p, \mathbb{O} - \mathbb{X})$ is the intersection number of the ray ρ_+, from p to $(+\infty, y)$,with $\mathcal{D}$: the vertical arc connecting some O with X contributes $+1$ if O lies in this upper right quadrant and the X does not, and in contributes -1 if the X lies in this upper right quadrant and the O does not, and it contributes 0 otherwise, i.e.

$$\#(\rho_+ \cap \mathcal{D}) = I(p, \mathbb{O} - \mathbb{X})$$

Similarly the intersection of ρ_- from p to $(-\infty, y)$ with $\mathcal{D}$ is

$$\#(\rho_- \cap \mathcal{D}) = I(\mathbb{O} - \mathbb{X}, p)$$

Since the winding number does not depend on the choice of the ray ρ that we choose, $w_{\mathcal{D}}(p) = \#(\rho_+ \cap \mathcal{D}) = \#(\rho_- \cap \mathcal{D})$. Averaging the two equation above we obtain our thesis. $\square$

Summing over all the components $x \in \mathbf{x}$ we obtain

$$-\sum_{x \in \mathbf{x}} w_{\mathcal{D}}(x) = -J(\mathbf{x}, \mathbb{O} - \mathbb{X}). \tag{4.10}$$

Lemma 4.5. *Let $\mathbb{G}$ be a planar grid diagram of a knot K, let $\mathcal{D} = \mathcal{D}(\mathbb{G})$ be the corresponding knot projection in the plane, and $w_{\mathcal{D}}(p)$ be the winding number around p, then*

$$\frac{1}{2} J(\mathbb{O} + \mathbb{X}, \mathbb{O} - \mathbb{X}) = \frac{1}{8} \sum_{i=1}^{8n} w_{\mathcal{D}}(p_i). \tag{4.11}$$

Proof. We first need to verify that given any small square in the grid, whose centre z is marked with an O or an X, if $z_1, \ldots, z_4$ denote its four corner points in the plane, then

$$J(z, \mathbb{O} - \mathbb{X}) = \frac{1}{4} J(z_1 + z_2 + z_3 + z_4, \mathbb{O} - \mathbb{X}) + \begin{cases} -\frac{1}{4} & \text{if } z \text{ is marked with an } O \\ \frac{1}{4} & \text{if } z \text{ is marked with an } X. \end{cases}$$

Suppose for definiteness that z is marked with an O. Then, for any $O' \in \mathbb{O} \setminus \{O\}$,

$$J(z, O') = \frac{1}{4} J(z_1 + z_2 + z_3 + z_4, O'). \tag{4.12}$$

Also, for any X-marking not in the same row or column as z,

$$J(z, X) = \frac{1}{4} J(z_1 + z_2 + z_3 + z_4, X). \tag{4.13}$$

The correction term of $-\frac{1}{4}$ comes from the pairing of the X-markings in the same row and column as z with the formal sum $z_1 + \cdots + z_4$, combined with the pairing of the O-marking on z with $z_1 + \cdots + z_4$.
A similar reasoning gives the same result when z is marked with an X.
Summing up over all O and X-marked squares and using the previous lemma we conclude. □

Combining the two lemmas we obtain:

$$\begin{aligned} A(\mathbf{x}) &= \frac{1}{2}(M_{\mathbb{O}}(\mathbf{x}) - M_{\mathbb{X}}(\mathbf{x})) - \left(\frac{n-1}{2}\right) \\ &= -J(\mathbf{x}, \mathbb{O} - \mathbb{X}) + \frac{1}{2}(J(\mathbb{O}, \mathbb{O}) - J(\mathbb{X}, \mathbb{X})) - \left(\frac{n-1}{2}\right) \\ &= -\sum_{x \in \mathbf{x}} w_{\mathcal{D}}(x) + \frac{1}{8} \sum_{i=1}^{8n} w_{\mathcal{D}}(p_i) - \left(\frac{n-1}{2}\right). \\ &= -\sum_{x \in \mathbf{x}} w_{\mathcal{D}}(x) + a(\mathbb{G}) - \left(\frac{n-1}{2}\right). \end{aligned}$$

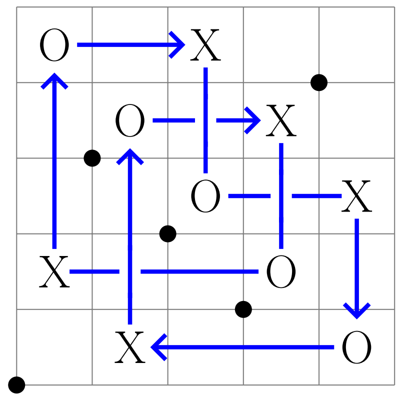


Figure 4.3: A grid state $\mathbf{x}$ for the trefoil.

Example 10. Given the grid diagram in Figure 4.3 we want to calculate Maslov and Alexander grading of its grid state.
Starting with $M(\mathbf{x})$:

$$\begin{aligned} I(\mathbf{x}, \mathbf{x}) &= 7 \\ I(\mathbf{x}, \mathbb{O}) &= 8 \\ I(\mathbb{O}, \mathbf{x}) &= 3 \\ I(\mathbb{O}, \mathbb{O}) &= 0. \end{aligned}$$

So $M_{\mathbb{O}}(\mathbf{x}) = 7 - 8 - 3 + 1 = -3$ To compute $A(\mathbf{x})$ we need to knot the following values:

$$\begin{aligned} I(\mathbb{X}, \mathbb{X}) &= 6 \\ I(\mathbb{X}, \mathbf{x}) &= 7. \end{aligned}$$

So $A(\mathbf{x}) = \frac{1}{2}[-6+14-6]+\frac{1-5}{2} = -1$ and the grid state $\mathbf{x} = (0, 3, 2, 1, 4) \sim (-3, -1)$. To check this result we can compute the Alexander grading using the winding number. From Equation 4.9 we need to compute

$$a(\mathbb{G}) = \frac{1}{8}\sum_{j=1}^{40} w_{\mathcal{D}}(p_j) = \frac{1}{8}(-24) = -3$$

and

$$-\sum_{x \in \mathbf{x}} w_{\mathcal{D}}(x) = -(0 - 1 - 2 - 1) = 4$$

so $A(\mathbf{x}) = 4 - 3 + \frac{1-5}{2} = -1$.

4.2.2 Maslov, Alexander and Spin^c grading in $L(p, q)$

In lens spaces, besides the Maslov and the Alexander grading, we can define another grading, the Spin^c, due to the structure of $L(p, q)$, with value in $\mathbb{Z}_p$. The Maslov and the Alexander grading instead will be defined similarly as in S^3 but will take values in $\mathbb{Q}$.
In order to define the Maslov and the Alexander gradings we need to consider the *lifting* of the grid diagrams to S^3, with respect to the universal covering (see Definition 1.2.9), which consists on a $pn \times pn$ grid as in Figure 4.4. For more details on the lifting of grid diagrams we refer the reader to [19].
With this purpose we denote $X(p, n)$ the set of n-tuples of points contained in the $n \times pn$ rectangle in $\mathbb{R}^2$ whose bottom vertices are $(0, 0)$ and $(pn, 0)$ and $Y(p, n)$ the set of pn-tuples of points contained in the $pn \times pn$ rectangle in $\mathbb{R}^2$ with the same bottom vertices as $X(p, n)$. We then define

$$C_{p,q} : X(p, n) \longrightarrow Y(p, n)$$

as the function sending the n-tuple $\{(c_i, b_i)\}_{i=1,\dots,n}$ to the pn-tuple

$$\{(c_i + nqk(\text{mod } np), b_i + nk)\}^{i=1,\dots,n}_{k=0,\dots,p-1}.$$

In order to avoid notational overloads, we denote $C_{p,q}(q) = \tilde{q}$, with q an n-tuple of points.
Geometrically, we are computing the Maslov and the Alexander grading on the lift of the grid in S^3, minus a correctional factor, but we will give a completely combinatorial description of the gradings so, morally, we can forget about the fact that we are considering the lift.

Definition 4.2.6. The **Maslov grading** is a function

$$M : S(\mathbb{G}) \longrightarrow \mathbb{Q}$$

defined as

$$M(\mathbf{x}) = \frac{1}{p}[I(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}) - I(\tilde{\mathbf{x}}, \tilde{\mathbb{O}}) - I(\tilde{\mathbb{O}}, \tilde{\mathbf{x}}) + I(\tilde{\mathbb{O}}, \tilde{\mathbb{O}})] + d(p, q, q-1) + 1 \tag{4.14}$$

Where $d(p, q, q-1)$ is known as the **correction term** of $L(p, q)$ associated to the $(q-1)^{th}$ Spin^c structure and can be computed as follows:

- $d(1, 0, 0) = 0$
- $d(p, q, i) = \left(\frac{pq-(2i+1-p-q)^2}{4pq}\right) - d(q, r, j)$ where $r \equiv_q p$ and $j \equiv_q i$

Definition 4.2.7. The **Alexander grading** is a function

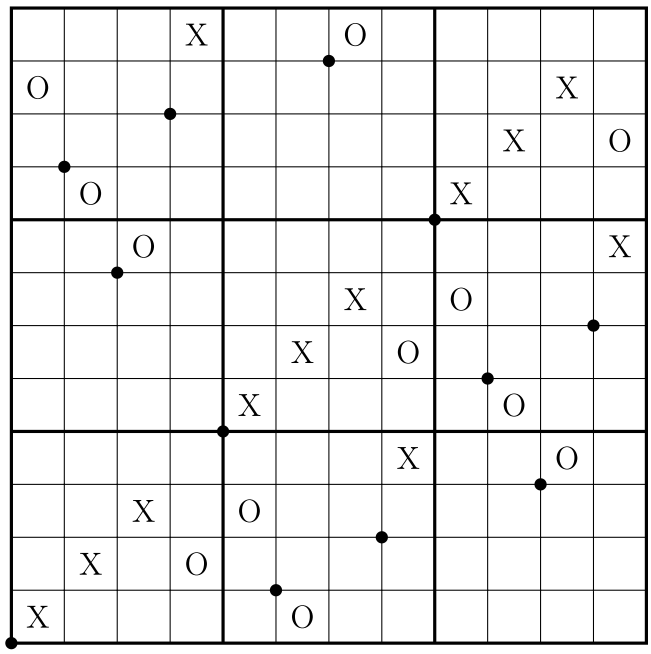


Figure 4.4: Lifting of the grid diagram in Figure 4.2.

$$A : S(\mathbb{G}) \longrightarrow \mathbb{Q}$$

defined as

$$A(\mathbf{x}) = \frac{1}{2p}(I(\tilde{\mathbb{O}}, \tilde{\mathbb{O}}) - I(\tilde{\mathbb{X}}, \tilde{\mathbb{X}}) + 2I(\tilde{\mathbb{X}}, \tilde{\mathbf{x}}) - 2I(\tilde{\mathbb{O}}, \tilde{\mathbf{x}})) + \frac{1-n}{2} \tag{4.15}$$

Once again, using Proposition 4.1, we can obtain from this equation an alternative formulation for the Alexander grading (as seen for S^3).

Let's now consider $(x_1^{\mathbb{O}}, \ldots, x_n^{\mathbb{O}})$ the p-coordinates of the generator $\mathbf{x}_{\mathbb{O}}$ whose components are in the lower left vertex of the squares which contain a O-marking.

Definition 4.2.8. The **Spin^c degree** of $\mathbf{x} = (\sigma_x, (x_1, \ldots, x_n)) \in S(\mathbb{G})$ is a function:

$$S : S_n \times \mathbb{Z}_p^n \longrightarrow \mathbb{Z}_p$$

defined as

$$S(\mathbf{x}) = q - 1 + \sum_{i=1}^{n}(x_i - x_i^{\mathbb{O}}) \mod p \tag{4.16}$$

Remark 23. We can observe that the Maslov grading and the Spin^c degree depends only on the position of the O-markings, while the Alexander grading depends on the placement of both markings.

Example 11. Consider the grid diagram of dimension $n = 4$ of a knot in $L(3,1)$ as in Figure 4.2 and its lifting in Figure 4.4 and compute the three gradings.
We start by the Maslov grading.

$$\begin{aligned}
&I(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}) = 33\\
&I(\tilde{\mathbf{x}}, \tilde{\mathbb{O}}) = 40\\
&I(\tilde{\mathbb{O}}, \tilde{\mathbf{x}}) = 28\\
&I(\tilde{\mathbb{O}}, \tilde{\mathbb{O}})] = 29\\
&d(3,1,0) = \left(\frac{3-(1-3-1)^2}{12}\right) + d(1,0,0) = -\frac{1}{2}.
\end{aligned}$$

So

$$M(\mathbf{x}) = \frac{1}{3}(33 - 40 - 28 + 29) - \frac{1}{2} + 1 = -\frac{3}{2}.$$

For the Alexander grading we need to calculate $I(\tilde{\mathbb{X}}, \tilde{\mathbf{x}})$ and $I(\tilde{\mathbb{X}}, \tilde{\mathbb{X}})$.

$$\begin{aligned}
&I(\tilde{\mathbb{X}}, \tilde{\mathbf{x}}) = 38\\
&I(\tilde{\mathbb{X}}, \tilde{\mathbb{X}}) = 52.
\end{aligned}$$

So

$$A(\mathbf{x}) = \frac{1}{6}(29 - 52 + 76 - 56) - \frac{1-4}{2} = -2.$$

As stated in Figure 4.2 the generator $\mathbf{x} = ((2,3),(0,1,1,2)$.
Consider the generator $\mathbf{x}_{\mathbb{O}} = (0,1,3,2),(1,0,1,2))$ whose elements are in the south-west corners of the O-markings.

$$S(\mathbf{x}) = 0 + \sum_{i=1}^{4}(x_i + x_i^{\mathbb{O}}) = (0-1) + (1-0) + (1-1) + (2-2) = 0$$

Thus the generator $\mathbf{x} = ((2,3),(0,1,1,2)) \sim (-\frac{3}{2}, -2, 0)$.

4.3 Homologies in S^3

In this section we are going to define the grid complexes $\widetilde{GC}, GC^-, \widehat{GC}$ and their differentials $\tilde{\partial}$, ∂^- and $\hat{\partial}$ in order to obtain the grid homologies.
We recall that we are going to work with homologies with $\mathbb{Z}_2$ coefficients. Since we are working with coefficients in a field, the grid complexes and the homology groups will be vector spaces.

4.3.1 Fully blocked grid homology

Definition 4.3.1. The **fully blocked grid chain complex** associated to the grid diagram $\mathbb{G}$ is the chain complex $\widetilde{CG}(\mathbb{G})$, whose underlying vector space over $\mathbb{Z}_2$ has basis corresponding to the set of grid states $S(\mathbb{G})$, and whose differential is specified by

$$\tilde{\partial}_{\mathbb{O},\mathbb{X}}(\mathbf{x}) = \sum_{\mathbf{y} \in S(\mathbb{G})} \#\{r \in \text{Rect}^\circ(\mathbf{x}, \mathbf{y}) \mid r \cap \mathbb{O} = r \cap \mathbb{X} = \emptyset\} \cdot \mathbf{y}. \tag{4.17}$$

where $\#\{A\}$ denotes the number of elements in the set A, modulo 2.
The subscript $\tilde{\partial}_{\mathbb{O},\mathbb{X}}$ indicates the fact that the differential counts rectangles disjoint from $\mathbb{O}$ and $\mathbb{X}$. For notational simplification we will denote $\tilde{\partial}_{\mathbb{O},\mathbb{X}} = \tilde{\partial}$

Lemma 4.6. *The operator $\tilde{\partial} : \widetilde{GC}(\mathbb{G}) \longrightarrow \widetilde{GC}(\mathbb{G})$ is a differential, i.e. it satisfies $\tilde{\partial} \circ \tilde{\partial} = 0$.*

Proof. For grid stats $\mathbf{x}$ and $\mathbf{z}$ fix $\psi \in \text{Poly}(\mathbf{x}, \mathbf{z})$, defined as in Definition 4.1.2, and let $N(\psi)$ denote the number of ways we can decompose ψ as a composite of two empty rectangle $r_1 * r_2$.
Observe that if $\psi = r_1 * r_2$ for some $r_1 \in \text{Rect}^\circ(\mathbf{x}, \mathbf{y})$ and $r_2 \in \text{Rect}^\circ(\mathbf{y}, \mathbf{z})$ then:

- $\psi \cap \mathbb{X}$ is empty $\iff$ $r_i \cap \mathbb{X} = \emptyset$ per i=1,2.
- $\psi \cap \mathbb{O}$ is empty $\iff$ $r_i \cap \mathbb{O} = \emptyset$ per i=1,2.

It follows that for any $\mathbf{x} \in S(\mathbb{G})$,

$$\tilde{\partial} \circ \tilde{\partial}(\mathbf{x}) = \sum_{\mathbf{z} \in S(\mathbb{G})} \sum_{\{\psi \in \text{Poly}^\circ(\mathbf{x},\mathbf{z}) \mid \psi \cap \mathbb{X} = \psi \cap \mathbb{O} = \emptyset\}} N(\psi) \cdot \mathbf{z}.$$

We will show that there are always an even number of ways to connect $\mathbf{x}$ to $\mathbf{z}$ with two empty rectangles in order, and since $\widetilde{GC}(\mathbb{G})$ is defined over $\mathbb{Z}_2$, this will complete the proof.
There are five possible configurations of the polygons, according to the quantity $Q = \#(\mathbf{x} \setminus (\mathbf{x} \cap \mathbf{z})) \in \{0, \dots, 4\}$.

- $Q = 0$. Here the two rectangles shares all 4 vertices and will create an annulus using the identifications of the borders of the grid so $N(\psi) = 1$,but either way it will contain an O or an X marking so it does not contribute to the differential.
- $Q = 1$. This is not possible since the two rectangles should be sharing 3 vertices out of 4.

- $Q = 2$. This is not possible since they should be sharing a side, but there would not exist a grid state $\mathbf{y}$ connecting $\mathbf{x}$ to $\mathbf{z}$ since it would have two of its points in the same α or β circle.

- $Q = 3$. In this case the two rectangles share one vertex and we obtain an L-shaped polygon. Here we can first count, following Figure 4.5, the bottom left rectangle (shaded in blue), arriving at the intermediate generator $\mathbf{y}$ (in grey), before counting the right rectangle (in green), ending in $\mathbf{z}$. Or, as in the right of the figure, we can first count the top right rectangle (shaded in blue), giving the intermediate generator $\mathbf{y}$ (in white), before counting the bottom rectangle (in green), again ending in $\mathbf{z}$. So $N(\psi) = 2$

- $Q = 4$. In this case the two rectangles do not share any vertex and they can either be disjoint or intersect each other as shown in Figure 4.6. There are two possible ways to count the rectangles: either counting r_1 first then r_2 or vice versa, so $N(\psi) = 2$.

Either of the last two cases, since we are working with $\mathbb{Z}_2$ coefficients, will do not contribute to the differential. $\square$

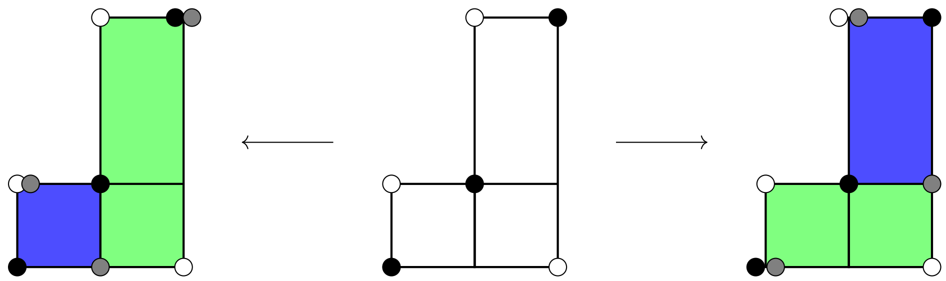


Figure 4.5: Case Q=3, where we split the polygon connecting $\mathbf{x}$ (black) to $\mathbf{z}$ (white) in the two possible ways, using the grid state $\mathbf{y}$ (gray).

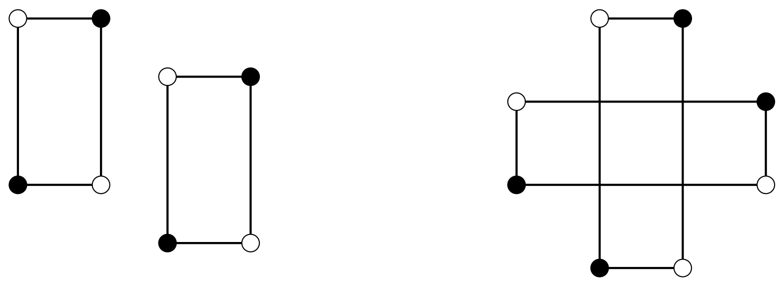


Figure 4.6: Two possible cases for Q=4, when the rectangles do not share any vertex.

Lemma 4.7. *The differential $\tilde{\partial}(\mathbf{x})$ is homogeneous of degree $(-1, 0)$.*

Proof. If $\mathbf{y}$ appears in $\tilde{\partial}(\mathbf{x})$ then there is a rectangle $r \in \mathrm{Rect}^\circ(\mathbf{x}, \mathbf{y})$ with $r \cap \mathbb{X} = r \cap \mathbb{O} = \emptyset$. By Equations (4.5),

$$M_{\mathbb{O}}(\mathbf{y}) = M_{\mathbb{O}}(x) - 1 + 2\#(r \cap \mathbb{O}) + 2\#(\mathbf{x} \cap Int(r)) = M_{\mathbb{O}}(\mathbf{x}) - 1.$$

Similarly by Equation (4.6)

$$A(\mathbf{y}) = A(\mathbf{x}) - \#(r \cap \mathbb{X}) + \#(r \cap \mathbb{O}) = A(\mathbf{x}).$$

So the differential drops the Maslov grading by one and preserves the Alexander grading. $\square$

The Maslov and Alexander functions on $S(\mathbb{G})$ induce two gradings on $\widetilde{GC(\mathbb{G})}$: we define $\widetilde{GC_m}(\mathbb{G}, a)$ to be the $\mathbb{Z}_2$-vector space generated by grid states $\mathbf{x}$ with $\mathrm{M}(\mathbf{x}){=}m$ and $\mathrm{A}(\mathbf{x}){=}a$.
It follows from the lemma that the restriction of $\tilde{\partial}$ to $\widetilde{GC}_m(\mathbb{G}, a)$ maps into $\widetilde{GC}_{m-1}(\mathbb{G}, a)$. Thus the bigrading descends to a bigrading on the homology groups of $\widetilde{GC}(\mathbb{G})$. Explicitly, letting

$$\widetilde{GH}_m(\mathbb{G}, a) = \frac{\mathrm{Ker}(\tilde{\partial}) \cap \widetilde{GC}_m(\mathbb{G}, a)}{\mathrm{Im}(\tilde{\partial}) \cap \widetilde{GC}_m(\mathbb{G}, a)} \tag{4.18}$$

then

$$\widetilde{GH}(\mathbb{G}) = \bigoplus_{m,a \in \mathbb{Z}} \widetilde{GH}_m(\mathbb{G}, a).$$

Definition 4.3.2. The **fully blocked grid homology** of $\mathbb{G}$, denoted $\widetilde{GH}(\mathbb{G})$, is the homology oh the chain complex $(\widetilde{GC}(\mathbb{G}), \tilde{\partial}_{\mathbb{O},\mathbb{X}})$, thought as a bigraded vector space.

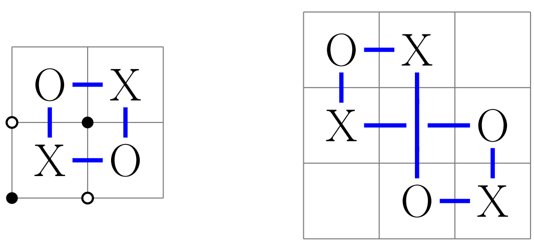
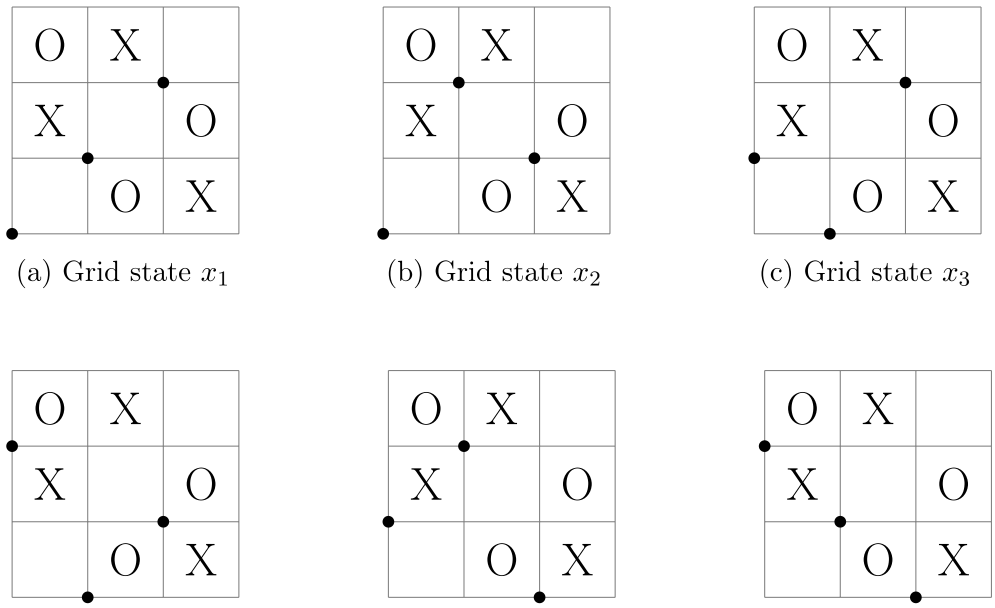


Figure 4.7: Grid diagram for the unknot, with grid number 2 (left) and 3 (right).



(a) Grid state x_1 (b) Grid state x_2 (c) Grid state x_3

(d) Grid state x_4 (e) Grid state x_5 (f) Grid state x_6

Figure 4.8: Grid states for the grid diagram of the unknot with dimension 3.

Example 12. Let $\mathcal{O}$ denote the unknot. We can compute the fully blocked grid homology for the unknot for a grid of dimension 2 and 3.
If $n = 2$ we have 2 generators, black and white as in Figure 4.7, in bidegree $(0, 0)$ and $(-1, -1)$ respectively and we obtain:

$$\widetilde{GH}_0(\mathbb{G}, 0) = \mathbb{Z}_2 \qquad \widetilde{GH}_{-1}(\mathbb{G}, -1) = \mathbb{Z}_2$$

If $n = 3$ we have 6 generators in bigradings:

$$\begin{aligned} x_1 &= (0, -1) & x_2 &= (-1, -1) \\ x_3 &= (-1, -1) & x_4 &= (-2, -2) \\ x_5 &= (0, 0) & x_6 &= (-1, -1) \end{aligned}$$

with differentials $\tilde{\partial}(x_i)$:

$$\begin{aligned} \tilde{\partial}(x_1) &= x_2 + x_3 + x_6 \\ \tilde{\partial}(x_i) &= 0, \forall i \in \{2, \ldots, 6\}. \end{aligned}$$

Thus we obtain:

$$\begin{aligned}
\widetilde{GH}_0(\mathbb{G},0) &= \mathbb{Z}_2\langle x_5\rangle \\
\widetilde{GH}_0(\mathbb{G},-1) &= 0 \\
\widetilde{GH}_{-1}(\mathbb{G},-1) &= \mathbb{Z}_2\langle x_3\rangle \oplus \mathbb{Z}_2\langle x_6\rangle \\
\widetilde{GH}_{-2}(\mathbb{G},-2) &= \mathbb{Z}_2\langle x_4\rangle.
\end{aligned}$$

This shows that the grid homology $\widetilde{GH(\mathbb{G})}$ is not a knot invariant and its total dimension depends on the grid number of the presentation.

Theorem 4.8. *If $\mathbb{G}$ is a grid diagram with grid number n representing a knot K, then the normalized dimension $\dim_{\mathbb{Z}_2}(\widetilde{GH(\mathbb{G})})/2^{n-1}$ is an integer-valued knot invariant and it is independent of the chosen grid presentation for K.*

We will prove this theorem in Page 102, after the proof of the invariance of GH^- and $\widehat{GH}$.

4.3.2 Unblocked grid homology

We want to enrich the grid complex to a bigraded chain complex over the ring $\mathcal{R} = \mathbb{Z}_2[V_1,\dots,V_n]$. Unlike the fully blocked grid homology, the unblocked grid homology is equipped with a differential $\partial^-_{\mathbb{X}}$ counting empty rectangles that may contain the O-markings but not the X-markings.
With this purpose it is useful to enumerate the set $\mathbb{O} = \{O_i\}_{i=1}^n$ in order to put the O-markings in one-to-one correspondence with the generators V_i of the polynomial algebra $\mathcal{R}$.

Definition 4.3.3. The **unblocked grid complex** $GC^-(\mathbb{G})$ is the free module over $\mathcal{R}$ generated by $S(\mathbb{G})$, equipped with the $\mathcal{R}$-module endomorphism whose value on any $\mathbf{x} \in S(\mathbb{G})$ is given by

$$\partial^-_{\mathbb{X}}(\mathbf{x}) = \sum_{\mathbf{y}\in S(\mathbb{G})} \sum_{\{r\in \mathrm{Rect}^\circ(\mathbf{x},\mathbf{y}) | r\cap\mathbb{X}=\emptyset\}} V_1^{O_1(r)} \dots V_n^{O_n(r)} \cdot \mathbf{y} \tag{4.19}$$

where $O_i(r)$ count the multiplicity of the marking O_i in the rectangle r and is defined to be either 0 or 1.
To avoid notational overload we will denote $\partial^-_{\mathbb{X}} = \partial^-$.

The elements $V_1^{k_1}\dots V_n^{k_n}\cdot \mathbf{x}$ where $\mathbf{x} \in S(\mathbb{G})$ and $k_1,\dots,k_n$ are arbitrary non negative integers, form a basis for the $\mathbb{Z}_2$-vector space $GC^-(\mathbb{G})$. We extend the

Maslov and the Alexander grading to this basis by

$$M(V_1^{k_1}\dots V_n^{k_n}\cdot \mathbf{x}) = M(\mathbf{x}) - 2k_1 - \dots - 2k_n \tag{4.20}$$

$$A(V_1^{k_1}\dots V_n^{k_n}\cdot \mathbf{x}) = A(\mathbf{x}) - k_1 - \dots - k_n. \tag{4.21}$$

These extensions equip $GC^-(\mathbb{G})$ with a bigrading: let $GC^-_m(\mathbb{G},a)$ denote the vector subspace spanned by basis vectors $V_1^{k_1}\dots V_n^{k_n}\cdot\mathbf{x}$ with $M(V_1^{k_1}\dots V_n^{k_n}\cdot\mathbf{x}) = m$ and $A(V_1^{k_1}\dots V_n^{k_n}\cdot \mathbf{x}) = a$.
It follows from its definition that the complex $CG^-(\mathbb{G})$ generalizes $\widetilde{GC}(\mathbb{G})$, indeed

$$\frac{GC^-(\mathbb{G})}{V_1 = \dots = V_n = 0} \cong \widetilde{GC}(\mathbb{G}).$$

Theorem 4.9. *The object $(GC^-(\mathbb{G}), \partial^-)$ is a bigraded chain complex over the ring $\mathcal{R}$=$\mathbb{Z}_2[V_1,\dots,V_n]$, in the sense of Definition A.3.2.*

Lemma 4.10. *The operator $\partial^- : GC^-(\mathbb{G}) \longrightarrow GC^-(\mathbb{G})$ satisfies $\partial^-\circ\partial^- = 0$.*

Proof. The proof follows step by step the proof of Lemma 4.6. Here we consider polygons which may contain O-markings so we observe that if $\phi = r_1 * r_2$ for some $r_1 \in \mathrm{Rect}(\mathbf{x},\mathbf{y})$ and $r_2 \in \mathrm{Rect}(\mathbf{y},\mathbf{z})$ then:

- $\phi \cap \mathbb{X}$ is empty $\iff$ $r_i \cap \mathbb{X} = \emptyset$ per i=1,2.
- the multiplicities of ϕ, r_1, r_2 at any $O_i \in \mathbb{O}$ are related by

$$O_i(\phi) = O_i(r_1) + O_i(r_2).$$

It follows that for any $\mathbf{x} \in S(\mathbb{G})$,

$$\partial^-\circ\partial^-(\mathbf{x}) = \sum_{\mathbf{z}\in S(\mathbb{G})}\ \sum_{\{\phi\in \mathrm{Poly}(\mathbf{x},\mathbf{z}) | \phi\cap\mathbb{X}=\emptyset\}} N(\phi)\cdot V_1^{O_1(\phi)}\cdots V_n^{O_n(\phi)}\cdot \mathbf{z}.$$

With the same cases and considerations of Lemma 4.6 we conclude our proof. □

Lemma 4.11. *The differential ∂^- is homogeneous of degree $(-1,0)$.*

Proof. If $V_1^{k_1}\dots V_n^{k_n}\cdot\mathbf{y}$ appears in $\partial^-(\mathbf{x})$, then there is a rectangle $r \in \mathrm{Rect}^\circ(\mathbf{x},\mathbf{y})$ with $r\cap\mathbb{X} = \emptyset$ and $O_i(r) = k_i$ for $i = 1,\dots,n$. By Equations (4.5) and (4.20)

$$M(V_1^{k_1}\dots V_n^{k_n}\cdot\mathbf{y}) = M(\mathbf{y}) - 2\#(r\cap\mathbb{O}) = M(\mathbf{x}) - 1.$$

Similarly by Equations (4.6) and (4.21)

$$A(V_1^{k_1}\dots V_n^{k_n}\cdot\mathbf{y}) = A(\mathbf{y}) - \#(r\cap\mathbb{O}) = A(\mathbf{x}) - \#(r\cap\mathbb{X}) = A(\mathbf{x}).$$

□

Proof Theorem 4.9. Equations (4.21) and (4.20) ensure that multiplication by V_i is an homogeneous map of degree $(-2,-1)$, i.e. $GC^-(\mathbb{G})$ is a bigraded module over $\mathbb{Z}_2[V_1,\ldots,V_n]$. The operator ∂^- is defined to be an $\mathcal{R}$-module homomorphism, Lemma 4.11 ensures it is homogeneous of degree $(-1,0)$ and Lemma 4.10 ensure that it is a differential ending our proof. $\square$

Lemma 4.12. *For any pair of integers $i,j \in \{1,\ldots,n\}$ multiplication by V_i is chain homotopic to multiplication by V_j, when thought of as homogeneous maps from $GC(\mathbb{G})$ to itself of degree $(-2,-1)$.*

Proof. Variables V_i, V_j are called *consecutive* if there is a square marked by X in the same row as O_i and in the same column as O_j. Suppose that V_i, V_j are consecutive and let X_i denote the X-marking in the same row as O_i and in the same column ad O_j. Define a corresponding homotopy operator that counts rectangles that contain X_i in their interior

$$\mathcal{H}_i(\mathbf{x}) = \mathcal{H}_{X_i}(\mathbf{x}) = \sum_{\mathbf{y}\in S(\mathbb{G})} \sum_{\{r\in \mathrm{Rect}^\circ(\mathbf{x},\mathbf{y}) | Int(r)\cap\mathbb{X}=X_i\}} V_1^{O_1(r)}\ldots V_n^{O_n(r)}\cdot \mathbf{y}.$$

It follows from Equations (4.5) and (4.6) that $\mathcal{H}_i$ is homogeneous of degree $(-1,-1)$. We can adapt the proof of Lemma 4.6 to count decomposition of domains ϕ with $N(\phi) > 0$ and which contain X_i, and no other $X \in \mathbb{X}$, with multiplicity one in their interior. In addition to the types of pairs appearing in cases 1 and 2, there are two thin annuli that contribute to $\partial^- \circ \mathcal{H}_i + \mathcal{H}_i \circ \partial^-$, and those are the two annuli through X_i. The contribution of this two annuli are the multiplication by V_i and by V_j. Thus we obtain

$$\partial^- \circ \mathcal{H}_i + \mathcal{H}_i \circ \partial^- = V_i - V_j$$

which is the definition of chain homotopy.
For some generic V_i, V_j, since K is a knot, there is a sequence of consecutive variables $V_i = V_{n_1},\ldots,V_{n_m} = V_j$, such that V_{n_k} and $V_{n_{k+1}}$ are consecutive; adding the chain homotopies, we deduce that V_i is homotopic to V_j. $\square$

Definition 4.3.4. Fix some $i \in \{1,\ldots,n\}$. The **unblocked grid homology** of $\mathbb{G}$, denoted $GH^-(\mathbb{G})$, is the homology of $(GH^-(\mathbb{G}), \partial^-_{\mathbb{X}})$, viewed ad a bigraded module over $\mathbb{Z}_2[U]$, where the action of U is induced by multiplication by V_i.

Lemma 4.12 shows that the grid homology groups, thought of as bigraded modules over $F[U]$, are independent of the choice of i.

4.3.3 Simply blocked grid homology

We now want to describe the third type ho grid homology, the $\widehat{GH}(\mathbb{G})$.

Definition 4.3.5. Fix some $i \in \{1, \ldots, n\}$. The quotient complex $GC^-(\mathbb{G})/V_i$ is called the **simply blocked grid complex**, and it is denoted $\widehat{GC}(\mathbb{G})$.
The **simply blocked grid homology** of $\mathbb{G}$, denoted as $\widehat{GH}(\mathbb{G})$, is the bigraded vector space obtained as the homology of $\widehat{GH}(\mathbb{G}) = (GC^-(\mathbb{G})/V_i, \partial^-_{\mathbb{X}})$.

Explicitly, $GC^-(\mathbb{G})/V_n$ is the bigraded $\mathbb{Z}_2$-vector space provided with basis $V_1^{k_1} \ldots V_{n-1}^{k_{n-1}} \cdot \mathbf{x}$ where $k_1, \ldots, k_{n-1}$ are arbitrary non negative integers and $\mathbf{x} \in S(\mathbb{G})$, equipped with the differential $\hat{\partial}_{\mathbb{X},O_n}$ specified by $\hat{\partial}_{\mathbb{X},O_n} \circ V_j = V_j \circ \hat{\partial}_{\mathbb{X},O_n}$ for $j = 1, ..., n-1$, where

$$\hat{\partial}_{\mathbb{X},O_n}(\mathbf{x}) = \sum_{\mathbf{y} \in S(\mathbb{G})} \sum_{\{r \in \mathrm{Rect}^\circ(\mathbf{x},\mathbf{y}) | r \cap \mathbb{X} = \emptyset, O_n(r) = 0\}} V_1^{O_1(r)} \ldots V_{n-1}^{O_{n-1}(r)} \cdot \mathbf{y} \tag{4.22}$$

where $O_i(r)$ count the multiplicity of the marking O_i in the rectangle r and is defined to be either 0 or 1.
In order to avoid notational overload we will denote $\hat{\partial}_{\mathbb{X},O_n}$ as $\hat{\partial}$.

We are now ready to prove the main theorem, which ensures us that the unblocked grid homology and the simply blocked grid homology are knots invariants.

Theorem 4.13. *The homologies $\widehat{GH}(\mathbb{G})$ and $GH^-(\mathbb{G})$, the former thought of as a bigraded $\mathbb{Z}_2$-vector space and the latter thought of as a bigraded $\mathbb{Z}_2[U]$-module, depend on the grid $\mathbb{G}$ only through its underlying (unoriented) knot K.*

In view of Theorem 2.2, to prove this theorem is sufficient to show the invariance of the homologies under commutation and stabilization moves (see Definition 2.3.4). Furthermore we will prove that orientation changes of the knot, obtained performing a reflection of the grid diagram along its diagonal (Remark 9), will not affect the homology of the knot, thus $\widehat{GH}(\mathbb{G})$ and $GH^-(\mathbb{G})$, depend on the grid $\mathbb{G}$ only through its underlying (unoriented) knot K.

Proposition 4.14. *If $\mathbb{G}$ and $\mathbb{G}'$ are two grid diagrams which differ by a commutation move, we have an isomorphism of bigraded vector spaces $\widehat{GH}(\mathbb{G}) \cong \widehat{GH}(\mathbb{G}')$ and an isomorphism of bigraded $\mathbb{Z}_2[U]$-module $GH^-(\mathbb{G}) \cong GH^-(\mathbb{G}')$.*

In order to prove this proposition we need some preliminary construction.
It is useful to view both diagrams on one torus: regard all the X and O-markings as fixed for both $\mathbb{G}$ and $\mathbb{G}'$ and draw two vertical circles curved, as shown in Figure 4.9. Let $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_n\}$ and $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_n\}$ denote the horizontal and

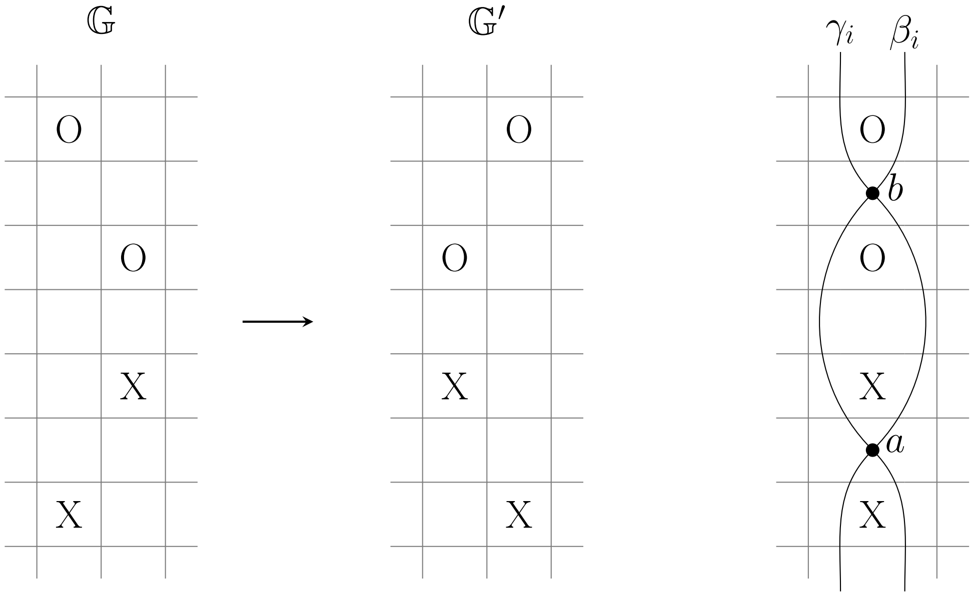


Figure 4.9: The two grid diagrams related by a commutation moves.

vertical circles for $\mathbb{G}$. The set of horizontal circles for $\mathbb{G}'$ is $\boldsymbol{\alpha}$ while the set of vertical circles is $\boldsymbol{\gamma} = \{\beta_1, \ldots, \beta_{i-1}, \gamma_i, \beta_{i+1}, \ldots, \beta_n\}$. Draw the vertical circles so that β_i and γ_i meet perpendicularly in two points, denoted a and b, both of which miss the horizontal circles. Distinguish a, b as follows: the complement of $\beta_i \cup \gamma_i$ in the grid torus contains two bigons, which intersect in $\{a, b\}$, one of this bigons has a portion of β_i on its western boundary and a portion of γ_i on its eastern boundary. For this bigons, the intersection point a occurs at its southern tip.

Definition 4.3.6. Fix grid states $\mathbf{x} \in S(\mathbb{G})$ and $\mathbf{y}' \in S(\mathbb{G}')$. An embedded disc p in the torus whose boundary is the union of five arcs, each of which lies in some $\alpha_j, \beta_j, \gamma_j$ is called a **pentagon from $\mathbf{x}$ to $\mathbf{y}'$** if it satisfies the following:

- Four of the corners of p are in $\mathbf{x} \cup \mathbf{y}'$
- Each corner point x of p is an intersection of two of the curves taken from $\{\alpha_j, \beta_j, \gamma_j\}_{j=1}^n$ and a small disc centred at x is divided into four quadrants by these two curves. The pentagon p contains exactly one of the four quadrants.
- Letting $\partial_\alpha p$ denote the portion of the boundary of p in $\alpha_1 \cup \cdots \cup \alpha_n$, then

$$\partial(\partial_\alpha p) = \mathbf{y}' - \mathbf{x}.$$

The set of such pentagons is denoted $\mathrm{Pent}(\mathbf{x}, \mathbf{y}')$.

The conditions on the pentagon ensure that its fifth corner is the point a.
We can define analogously $\mathrm{Pent}(\mathbf{y}', \mathbf{x})$ replacing the last condition with $\partial(\partial_\alpha p) = \mathbf{x} - \mathbf{y}'$. For such pentagons, the fifth corner is the point b.
We define the set of **empty pentagons** $\mathrm{Pent}^\circ(\mathbf{x}, \mathbf{y}')$ analogously as for rectangles.

Definition 4.3.7. Fix grid sates $\mathbf{x}, \mathbf{y} \in S(\mathbb{G})$. An embedded disc h is the torus, whose boundary is the union of the α_j, β_j and γ_i is called an **hexagon from x to y** if it satisfies the following:

- At any of the six corner points x of h, the hexagon contains exactly one of the four quadrants determined by the two intersecting curves at x.
- Four of the corner points of h are in $\mathbf{x} \cup \mathbf{y}$ and the two other corners are a and b
- $\partial(\partial_\alpha h) = \mathbf{y} - \mathbf{x}$.

Denote the set of hexagons from $\mathbf{x}$ to $\mathbf{y}$ by $\mathrm{Hex}(\mathbf{x}, \mathbf{y})$ and the set of **empty hexagons** by $\mathrm{Hex}^\circ(\mathbf{x}, \mathbf{y})$.

Definition 4.3.8. Fix grid sates $\mathbf{x}, \mathbf{y} \in S(\mathbb{G})$. An embedded disc t is the torus, whose boundary is the union of the α_j, β_j and γ_i is called an **triangle from x to y** if it satisfies the following:

- Two corners of t are in $\mathbf{x} \cup \mathbf{y}$.
- Each corner point is the intersection of two curves taken from $\{\alpha_j, \beta_j, \gamma_j\}_{j=1}^n$.
- $\partial(\partial_\alpha t) = \mathbf{y} - \mathbf{x}$.

The vertex not in $\mathbf{x} \cup \mathbf{y}$ will be called **initial point** of the triangle.

Proposition 4.15. *The $\mathbb{Z}_2[V_1, \ldots, V_n]$-module map $P : GC^-(\mathbb{G}) \longrightarrow GC^-(\mathbb{G}')$ defined as*

$$P(\mathbf{x}) = \sum_{\mathbf{y}' \in S(\mathbb{G}')} \sum_{\{p \in \mathrm{Pent}^\circ(\mathbf{x}, \mathbf{y}') | p \cap \mathbb{X} = \emptyset\}} V_1^{O_1(p)} \cdots V_n^{O_n(p)} \cdot \mathbf{y}' \tag{4.23}$$

is a bigraded map.

Proof. There is a one-to-one correspondence

$$I : S(\mathbb{G}') \longrightarrow S(\mathbb{G})$$

sending a grid state $\mathbf{x}' \in S(\mathbb{G}')$ to the unique grid state $\mathbf{x} = I(\mathbf{x}')$ that agrees with $\mathbf{x}'$ in all but one component (the one on γ_i). There is a canonical small triangle

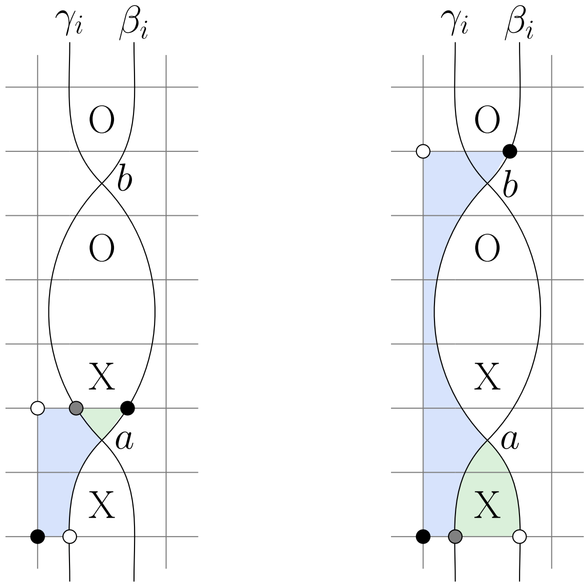


Figure 4.10: On the left, the pentagon (blue) can be pre-composed with a triangle (green) to get a rectangle; on the right, the pentagon (blue) can be post-composed with a triangle (green) to get a rectangle.

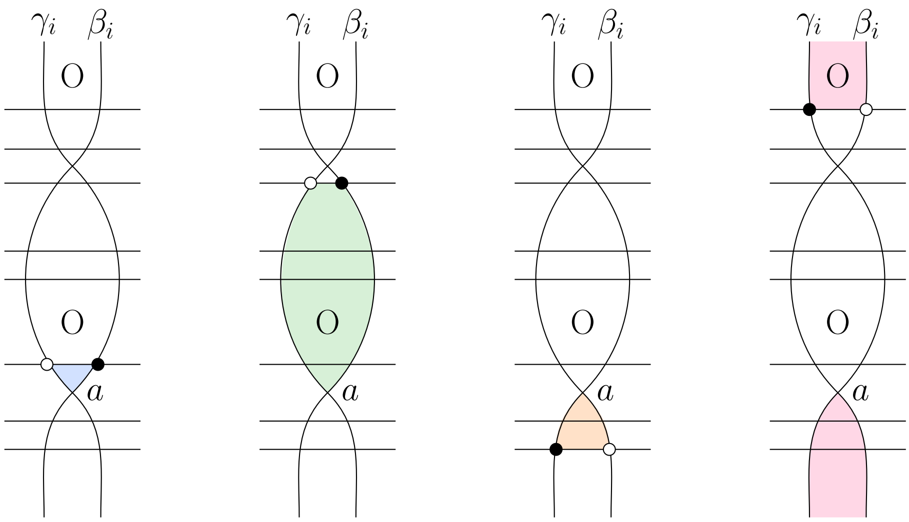


Figure 4.11: Four possible small triangle with initial vertex a.

from $\mathbf{x}'$ to $\mathbf{x}$ denoted $t_{\mathbf{x}}$. These triangles can be composed with pentagons, to obtain rectangle, via a juxtaposition denoted $*$, as for polygons.
If $p \in \text{Pent}(\mathbf{x},\mathbf{y}')$ then at least one of the following holds: $t_{\mathbf{x}} * p \in \text{Rect}(\mathbf{x}', \mathbf{y}')$ or $p * t_{\mathbf{x}} \in \text{Rect}(\mathbf{x}, \mathbf{y})$. See Figure 4.10.
We claim that if $t_{\mathbf{x}}$ is a triangle from $\mathbf{x}'$ to $\mathbf{x} = I(\mathbf{x}')$ then

$$M(\mathbf{x}) - M(\mathbf{x}') = -1 + 2\#(t_{\mathbf{x}} \cap \mathbb{O}). \tag{4.24}$$

To see this, choose a planar realization so that the triangle is one of the four possible triangle with initial vertex a showed in Figure 4.11. In all four cases $J(\mathbf{x}, \mathbf{x}) - J(\mathbf{x}', \mathbf{x}') = 0$, $J(\mathbb{O}, \mathbb{O}) - J(\mathbb{O}', \mathbb{O}') = -1$, while $J(\mathbb{O}, \mathbf{x}) - J(\mathbb{O}', \mathbf{x}')$ can be 0 or -1. All four cases leads to Equation (4.24).
Consider $p \in \text{Pent}(\mathbf{x},\mathbf{y}')$, let $\mathbf{y} = I(\mathbf{y}')$ and denote $r = t_{\mathbf{y}} * p \in \text{Rect}(\mathbf{x}, \mathbf{y})$, defined as for polygons in Definition 4.1.2. From Equation (4.5) we obtain that

$$M(\mathbf{x}) - M(\mathbf{y}) = 1 - 2\#(p \cap \mathbb{O}) - 2\#(t_{\mathbf{y}} \cap \mathbb{O}) + 2\#(\mathbf{x} \cap \text{Int}(p)) + 2\#(\mathbf{x} \cap \text{Int}(t_{\mathbf{y}})).$$

Combining we (4.24) we obtain that

$$M(\mathbf{x}) - M(\mathbf{y}') = M(\mathbf{x}) - M(\mathbf{y}) + M(\mathbf{y}) - M(\mathbf{y}') = -2\#(p \cap \mathbb{O}) + 2\#(\mathbf{x} \cap \text{Int}(p)).$$

since that $\mathbf{y} \cap \text{Int}(t_{\mathbf{x}}) = \emptyset$ for any triangle $t_{\mathbf{x}}$ and any grid state $\mathbf{y}$. This implies that P preserves Maslov grading since we can express $M(\mathbf{y}')$ as $M(\mathbf{x}) + \text{cost}$.
Switching the roles of $\mathbb{X}$ and $\mathbb{O}$ and using Equation (4.6) we see that for any $p \in \text{Pent}(\mathbf{x},\mathbf{y}')$, then

$$A(\mathbf{x}) - A(\mathbf{y}') = \#(p \cap \mathbb{X}) - \#(p \cap \mathbb{O}).$$

Since all the pentagons counted in P have $p \cap \mathbb{X} = \emptyset$, it follows that P preserves Alexander grading. □

Lemma 4.16. *The map P defined in Equation (4.23) is a chain map.*

Proof. The idea is the same of Lemma 4.6 and 4.10, we construct an operators which analyse how pentagons and rectangles interact. For further details we refer the reader to Lemma 5.1.4 in [24]. □

Proposition 4.17. *The function $H : GC^-(\mathbb{G}) \longrightarrow GC^-(\mathbb{G})$ is a $\mathbb{Z}_2[V_1, \ldots, V_n]$-module homomorphism defined as*

$$H(\mathbf{x}) = \sum_{\mathbf{y} \in S(\mathbb{G})} \sum_{\{h \in \text{Hex}^\circ(\mathbf{x},\mathbf{y}) | h \cap \mathbb{X} = \emptyset\}} V_1^{O_1(h)} \cdots V_n^{O_n(h)} \cdot \mathbf{y} \tag{4.25}$$

and it provides a homotopy from the bigraded chain map $P' \circ P$ to the identity map on $GC^-(\mathbb{G})$, where $P' : GC^-(G') \longrightarrow GC^-(G')$ is an analogous function counting empty pentagons.

We define $H' : GC^-(G') \longrightarrow GC^-(G')$ counting empty hexagons.

Proof. We first show that H is homogeneous of degree $(1,0)$. Suppose that $V_1^{k_1} \cdots V_n^{k_n} \cdot \mathbf{y}$ occurs with non-zero coefficient in $H(\mathbf{x})$. The corresponding hexagon $h \in \mathrm{Hex}^\circ(\mathbf{x}, \mathbf{y})$ can be naturally extended to a rectangle $r \in \mathrm{Rect}(\mathbf{x}, \mathbf{y})$ that differs from h by addition of a bigon next to β_i containing exactly one X and exactly one O. Combining with Equation (4.5) and (4.6) we obtain

$$\begin{aligned} M(\mathbf{x}) - M(V_1^{O_1(h)} \cdots V_n^{O_n(h)} \cdot \mathbf{y}) &= -1 + 2\#(\mathbf{x} \cap \mathrm{Int}(h)) \\ A(\mathbf{x}) - A(V_1^{O_1(h)} \cdots V_n^{O_n(h)} \cdot \mathbf{y}) &= \#(h \cap \mathbb{X}) \end{aligned}$$

and, since h is empty and $h \cap \mathbb{X} = \emptyset$, the claim follows.
The proof of the identity

$$\partial_{\mathbb{X}}^- \circ H + H \circ \partial_{\mathbb{X}}^- = \mathrm{Id} + P' \circ P \tag{4.26}$$

is done analysing how empty rectangles and empty hexagons can interact. Specifically, for $\psi \in \pi(\mathbf{x}, \mathbf{z})$ let $N(\psi)$ denote the number of ways of decomposing ψ as either:

1. $\psi = r * h$, where r is an empty rectangle and h an empty hexagon
2. $\psi = h * r$, where r is an empty rectangle and h an empty hexagon
3. $\psi = p * p'$, where p is an empty pentagon from $\mathbb{G}$ to $\mathbb{G}'$ and p' an empty pentagon from $\mathbb{G}'$ to $\mathbb{G}$.

Clearly

$$(\partial_{\mathbb{X}}^- \circ H + H \circ \partial_{\mathbb{X}}^- + P' \circ P)(\mathbf{x}) = \sum_{\mathbf{z} \in S(\mathbb{G})} \sum_{\{\psi \in \pi(\mathbf{x}, \mathbf{z}) | \psi \cap \mathbb{X} = \emptyset\}} N(\psi) \cdot V_1^{O_1(\psi)} \cdots V_n^{O_n(\psi)} \cdot \mathbf{z}.$$

There are three cases of $\psi \in \pi(\mathbf{x}, \mathbf{z})$ with $N(\psi) > 0$:

1. $\mathbf{x} \smallsetminus (\mathbf{x} \cap \mathbf{z})$ consists of 4 elements. In this case $N(\psi) = 2$ and the two decompositions are as $r_1 * h_1$ and $h_2 * r_2$, where r_1, r_2 are rectangle from $\mathbb{G}$, while h_1, h_2 are hexagons for $\mathbb{G}$ with the same support (see Page 71).

2. $\mathbf{x} \smallsetminus (\mathbf{x} \cap \mathbf{z})$ consists of 3 elements. In this case, ψ has eight corners. Either seven of these corners are 90° and one is 270°, or five are 90° and three are 270°. In the first case, cutting at the 270° corner in two different directions gives the two decompositions of ψ. In the second case, the three corners with 270° include both a and b. Again $N(\psi) = 2$: at one of the three corners there is a choice of two different directions for cutting, while at the other corners, the direction for cutting is uniquely determined, as shown in Figure 4.12.

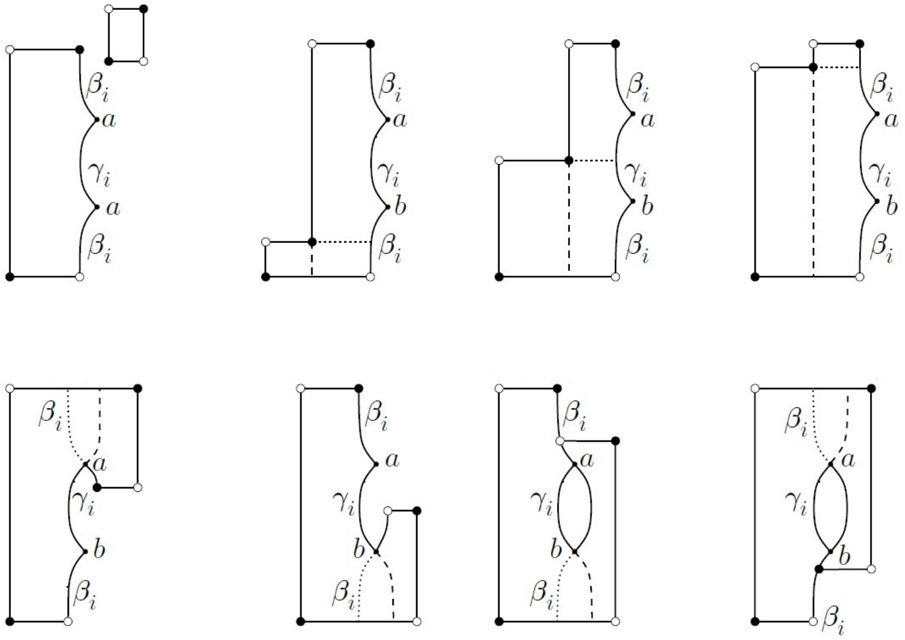


Figure 4.12: The top left image is the case 1 of Proposition 4.17. The other seven images are the possible composition of rectangles and hexagons of case 2 of Proposition 4.17.

3. **x**=**z**. In this case, ψ is supported inside an annulus between β_i and one of the other consecutive vertical circles. In this case $N(\psi) = 1$, and ψ could have decompositions of any of the three types (that is, rectangle-hexagon, hexagon-rectangles or pentagon-pentagon), depending on the exact placement of **x** on β_{i-1}, β_i and β_{i+1}, as in Figure 4.13. Hence, it contributes the identity map, verifying the Equation (4.26)

□

Proof Proposition 4.14. Suppose that $\mathbb{G}$ and $\mathbb{G}'$ differ by a column commutation. The map $P : GC^-(\mathbb{G}) \longrightarrow GC^-(\mathbb{G}')$ is a chain map by Lemma 4.16 that respect gradings. Applying Proposition 4.17 twice, for $P' \circ P$ and H first and then for $P \circ P'$ and H', it is a bigraded chain homotopy equivalence.
Furthermore, a bigraded chain homotopy equivalence between $GC^-(\mathbb{G})$ and $GC^-(\mathbb{G}')$ induces a bigraded chain homotopy equivalence on the $V_1 = 0$ specialization $\widehat{GC}(\mathbb{G}) \longrightarrow \widehat{GC}(\mathbb{G}')$.
Since a bigraded chain homotopy equivalence induces isomorphism on homology in

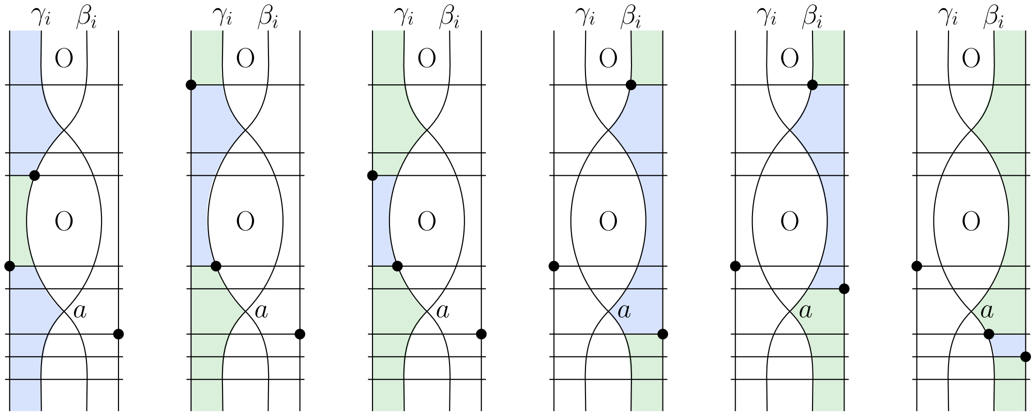


Figure 4.13: Possible decomposition in rectangle-hexagon (Figure 1,4), hexagon-rectangle (Figure 3,6) or pentagon-pentagon (Figure 2,5), of the annulus of Case (3) of Proposition 4.17.

$$\begin{array}{c|c} X & O \\ \hline & X \end{array} \qquad \begin{array}{c|c} X_1 & O_1 \\ \hline & X_2 \end{array}$$

Figure 4.14: Stabilization move and the markings used in Proposition 4.18.

both cases, the result is proved for column commutations. For row commutations the proof is the same rotating the diagram by 90°. $\square$

For the stabilization invariance we will investigate the effect of stabilization moves on $\widetilde{GH}$ and deduce from it the invariance for $\widehat{GH}$. The invariance of GH^- is slightly more complicated and requires more algebraic construction so we leave the reference to the reader (Section 5.2.3 of [24]).

Proposition 4.18. *Suppose that G' is a stabilization of $\mathbb{G}$, then there is an isomorphism of bigraded vector spaces*

$$\widetilde{GH}(\mathbb{G}') \cong \widetilde{GH}(\mathbb{G}) \oplus \widetilde{GH}(\mathbb{G})[\![1,1]\!].$$

where $\widetilde{GH}(\mathbb{G})[\![1,1]\!]$ is the shift of $\widetilde{GH}(\mathbb{G})$ according to Definition A.3.3.

In order to prove this proposition we establish some notation. Assume that $\mathbb{G}'$ is obtained from $\mathbb{G}$ by a stabilization as shown in Figure 4.14 and number the markings such that O_1 is the new marking while O_2 is the marking in the row below O_1, and X_1, X_2 as in Figure 4.14. Denote c the intersection point of the new

horizontal and vertical circle.
Decompose the set of grid states of $S(\mathbb{G}')$ as the disjoint union $I(G')\cup N(\mathbb{G}')$, where $I(\mathbb{G}')$ is the set of grid states $\mathbf{x} \in S(\mathbb{G}')$ such that $c \in \mathbf{x}$. This decomposition splits the vector space $\widetilde{GH}(\mathbb{G}') \cong \widetilde{I} \oplus \widetilde{N}$, where $\widetilde{I}, \widetilde{N}$ are the spans of the grid states $I(\mathbb{G}'), N(\mathbb{G}')$ respectively. Note that $\widetilde{N}$ is a subcomplex: any rectangle $\mathrm{Rect}(\mathbf{x}, \mathbf{y})$ with $\mathbf{x} \in N(\mathbb{G}')$ and $\mathbf{y} \in I(\mathbb{G}')$ must contain one of X_1 or X_2.
With respect to this splitting the map $\tilde{\partial}$ can be written in the matrix form as

$$\tilde{\partial} = \begin{pmatrix} \tilde{\partial}_I^I & 0 \\ \tilde{\partial}_I^N & \tilde{\partial}_N^N \end{pmatrix}$$

so that $\tilde{\partial}_I^N : (\widetilde{I}, \tilde{\partial}_I^I) \longrightarrow (\widetilde{N}, \tilde{\partial}_N^N)$ is a chain map.
Proposition 4.18 will follow from this statements:

- The homology of the chain complex $(\widetilde{I}, \tilde{\partial}_I^I)$ is isomorphic to $\widetilde{GH}(\mathbb{G})[\![1,1]\!]$ as a bigraded vector space.
- The homology of the chain complex $(\widetilde{N}, \tilde{\partial}_N^N)$ is isomorphic to $\widetilde{GH}(\mathbb{G})$ as a bigraded vector space.
- The chain map $\tilde{\partial}_I^N$ induces the zero map in homology.

We observe that there is a one-to-one correspondence between grid states in $S(\mathbb{G})$ and those in $I(\mathbb{G}')$, taking $\mathbf{x} \in S(\mathbb{G})$ to $\mathbf{x} \cup \{c\} \in I(\mathbb{G}')$. By trivial consideration it follows that

$$M(\mathbf{x}') = M(\mathbf{x}) - 1 \qquad \text{and} \qquad A(\mathbf{x}') = A(\mathbf{x}) - 1. \tag{4.27}$$

Lemma 4.19. *The map $\widetilde{e} : \widetilde{I} \to \widetilde{GH}(\mathbb{G})$ induced by the correspondence $\mathbf{x}\cup\{c\} \mapsto \mathbf{x}$ gives an isomorphism between $(\widetilde{I}, \tilde{\partial}_I^I)$ and $\widetilde{GH}(\mathbb{G})[\![1,1]\!]$.*

Proof. The map $\widetilde{e}$ is a bijection on the corresponding grid states, so it induces an isomorphism between the vector space $\widetilde{I}$ and $\widetilde{GC(\mathbb{G})}$. Empty rectangles disjoint from $\mathbb{X} \cup \mathbb{O}$ in $\mathbb{G}$ correspond to empty rectangles disjoint from $\mathbb{X}' \cup \mathbb{O}'$ in $\mathbb{G}'$, so $\widetilde{e}$ is an isomorphism of chain complexes and the grading shift is given by Equation (4.27). $\square$

We define two linear maps $\widetilde{H}_{X_2}^I : \widetilde{N} \longrightarrow \widetilde{I}$ whose value on $\mathbf{x} \in N(\mathbb{G}')$ is

$$\widetilde{H}_{X_2}^I(\mathbf{x}) = \sum_{\mathbf{y} \in I(\mathbb{G}')} \#\{r \in \mathrm{Rect}^\circ(\mathbf{x}, \mathbf{y}) \mid \mathrm{Int}(r) \cap \mathbb{X} = X_2, \mathrm{Int}(r) \cap \mathbb{O} = \emptyset) \cdot \mathbf{y}$$

and $\widetilde{H}_{O_1} : \widetilde{I} \longrightarrow \widetilde{N}$ whose value on $\mathbf{x} \in I(\mathbb{G}')$ is

$$\widetilde{H}_{O_1}(\mathbf{x}) = \sum_{\mathbf{y} \in N(\mathbb{G}')} \#\{r \in \mathrm{Rect}^\circ(\mathbf{x}, \mathbf{y}) \mid \mathrm{Int}(r) \cap \mathbb{X} = \emptyset, \mathrm{Int}(r) \cap \mathbb{O} = O_1) \cdot \mathbf{y}.$$

Lemma 4.20. *The map* $\widetilde{H}^I_{X_2}$ *drops both the Maslov and the Alexander gradings by one, while* $\widetilde{H}_{O_1}$ *increases both gradings by one. Moreover the maps* $\widetilde{H}_{O_1}$ *and* $\widetilde{H}^I_{X_2}$ *are chain maps and induce isomorphism on homology.*

Proof. From Equations (4.6) and (4.5) follows the grading changes. To obtain the chain map needed we observe that the composition $\widetilde{H}^I_{X_2} \circ \widetilde{H}_{O_1}$ on a generator $\mathbf{x} \in I(\mathbb{G}')$ counts juxtapositions $r_1 * r_2$ of pairs of empty rectangles, where r_1 starts at $\mathbf{x} \in I(\mathbb{G}')$ and contains O_1 while r_2 contains X_2 and ends in some state in $I(\mathbb{G}')$. By geometric reasoning, for any $\mathbf{x} \in I(\mathbb{G}')$, there exists a unique such juxtaposition, forming a vertical annulus of width one through O_1, X_2 and it follows that $\widetilde{H}^I_{X_2} \circ \widetilde{H}_{O_1} = \mathrm{Id}_{\widetilde{I}}$.
We can define the linear map $\widetilde{H}_{O_1,X_2} : \widetilde{N} \longrightarrow \widetilde{N}$ which counts rectangles containing both O_1 and X_2. It is easy to see that it is homogeneous of degree $(1,0)$ and using the preceding argument it an be shown that

$$\widetilde{H}^I_{O_1} \circ \widetilde{H}_{X_2} + \widetilde{H}_{O_1,X_2} \circ \tilde{\partial}^N_N + \tilde{\partial}^N_N \circ \widetilde{H}_{O_1,X_2} = \mathrm{Id}_{\widetilde{N}}$$

thus $\widetilde{H}^I_{O_1}$ and $\widetilde{H}_{X_2}$ are chain homotopy equivalences. □

Lemma 4.21. *The chain map* $\tilde{\partial}^N_I : (\widetilde{I}, \tilde{\partial}^I_I) \longrightarrow (\widetilde{N}, \tilde{\partial}^N_N)$ *induces the trivial map in homology.*

Proof. If $\mathbf{x},\mathbf{z} \in I(\mathbb{G}')$, $\mathbf{y} \in N(\mathbb{G}')$, $r_1 \in \mathrm{Rect}^\circ(\mathbf{x},\mathbf{y})$ and $r_2 \in \mathrm{Rect}^\circ(\mathbf{y},\mathbf{z})$ it is easy to see that $\mathbf{x} = \mathbf{z}$ and $r_1 * r_2$ is a thin annulus (see Definition 4.1.2). In particular one of r_1 or r_2 contains an O-marking. It follows that $\widetilde{H}_{X_2} \circ \tilde{\partial}^N_I = 0$ and since $\widetilde{H}_{X_2}$ induces an isomorphism in homology we conclude. □

Proof Proposition 4.18. We consider the short exact sequence

$$0 \longrightarrow \widetilde{N} \longrightarrow \widetilde{GC}(\mathbb{G}') \longrightarrow \widetilde{I} \longrightarrow 0$$

where $\widetilde{N}$ is a subcomplex of $\widetilde{GC}(\mathbb{G}')$ with quotient complex $\widetilde{I}$.
This induces a long exact sequence in homology with the connecting homomorphism induced by $\tilde{\partial}^N_I : \widetilde{I} \longrightarrow \widetilde{N}$ and, since $\tilde{\partial}^N_I$ vanishes on homology, the long exact sequence becomes the short exact sequence

$$0 \longrightarrow H(\widetilde{N}) \longrightarrow \widetilde{GH}(\mathbb{G}') \longrightarrow H(\widetilde{I}) \longrightarrow 0.$$

Replacing $H(\widetilde{I})$ and $H(\widetilde{N})$ with $\widetilde{GH}(\mathbb{G})[\![1,1]\!]$ $\widetilde{GH}(\mathbb{G})$ respectively, due to the previous lemmas, we obtain the short exact sequence

$$0 \longrightarrow \widetilde{GH}(\mathbb{G}) \longrightarrow \widetilde{GH}(\mathbb{G}') \longrightarrow \widetilde{GH}(\mathbb{G})[\![1,1]\!] \longrightarrow 0.$$

Since we are working on a field this concludes our proof.
For the other types of stabilizations, we can express them as a combination of commutations and switches (see Definition 2.3.5) so we have the invariance from Proposition 4.14 and Proposition 5.1.8 of [24] respectively. □

Proposition 4.22. *If the grid diagram G' is obtained performing a stabilization move over $\mathbb{G}$, then $\widehat{GH}(\mathbb{G}') \cong \widehat{GH}(\mathbb{G})$ and $GH^-(\mathbb{G}') \cong GH^-(\mathbb{G})$, i.e the simply blocked and the unblocked grid homology are invariant under stabilization moves.*

Proof. As explained above, we will prove the proposition only for $\widehat{GH}(\mathbb{G})$.
Recalling that if $\mathbb{G}$ has grid dimension n, then $\mathbb{G}'$ has grid dimension $n+1$, due to Proposition 4.18 we have that $\widetilde{GH}(\mathbb{G}') \cong \widetilde{GH}(\mathbb{G}) \oplus \widetilde{GH}(\mathbb{G})[\![1,1]\!]$. Applying Proposition 4.24 to $\mathbb{G}'$ we obtain

$$\widehat{GH}(\mathbb{G}')\otimes W^{\otimes n} \cong \widetilde{GH}(\mathbb{G}') \cong \widetilde{GH}(\mathbb{G})\oplus\widetilde{GH}(\mathbb{G})[\![1,1]\!] \cong \widehat{GH}(\mathbb{G})\otimes W^{\otimes n-1}\oplus(\widehat{GH}(\mathbb{G})\otimes W^{\otimes n-1})[\![1,1]\!]$$

but tensorizing $\widehat{GH}(\mathbb{G})$ with $W^{\otimes(n-1)}$ and summing it with itself shifted in bigrading of $(1,1)$ gives us exactly $\widehat{GH}(\mathbb{G}) \otimes W^{\otimes n}$, thus $\widehat{GH}(\mathbb{G}')$ and $\widehat{GH}(\mathbb{G})$ correspond in each bigrading so $\widehat{GH}(\mathbb{G}') \cong \widehat{GH}(\mathbb{G})$. □

Proposition 4.23. *The bigraded vector space $\widehat{GH}(K)$ and the bigraded $\mathbb{Z}_2[U]$-module $GH^-(K)$ are independent of the choice of orientation on K.*

Proof. If $\mathbb{G}$ represent K, then the grid diagram $\mathbb{G}'$ obtained by reflecting $\mathbb{G}$ across the diagonal represents $-K$. Reflection induces a one-to-one correspondence $\phi : S(\mathbb{G}) \longrightarrow S(\mathbb{G})$ and it preserves the fundamental domain $[0,n) \times [0,n)$. Given two sets P, Q in the fundamental domain, we have $J(P,Q) = J(\phi(P), \phi(Q))$, where J is defined as in Equation (4.7). From this follows that $M(\mathbf{x}) = M(\phi(\mathbf{x}))$ and $A(\mathbf{x}) = A(\phi(\mathbf{x}))$. Reflection also identifies empty rectangles of $\mathrm{Rect}^\circ(\mathbf{x},\mathbf{y})$ with empty rectangles of $\mathrm{Rect}^\circ(\phi(\mathbf{x}),\phi(\mathbf{y}))$, so it follows that ϕ extends to an isomorphism $GC^-(\mathbb{G}) \longrightarrow GC^-(\mathbb{G}')$ of bigraded complexes, which induces an isomorphism between homology groups. □

Proof Theorem 4.13. Fix a toroidal grid diagram $\mathbb{G}$, the vector space $\widehat{GH}(\mathbb{G})$ and the $\mathbb{Z}_2[U]$-module $GH^-(\mathbb{G})$ depends on a choice of V_i. Due to Lemma 4.12 and Proposition 4.27, respectively, $GH^-(\mathbb{G})$ and $\widehat{GH}(\mathbb{G})$ are independent of the choice of V_i and depend only on the toroidal grid diagram $\mathbb{G}$. Independence from the choice of the grid follows from Theorem 2.2, together with commutation invariance (Proposition 4.14), stabilization invariance (Proposition 4.22) and orientation invariance (Proposition 4.23). □

Proof Theorem 4.8. Using Proposition 4.14, Proposition 4.22 and Proposition 4.23, for the independence from the grid presentation and Proposition 4.26 for the dimension we obtain our thesis. □

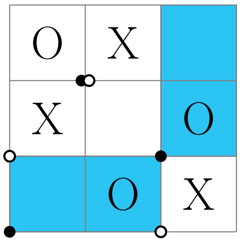


Figure 4.15: Rectangles between x_2 (black) and x_5 (white).

Sometimes, since $GH^-(\mathbb{G}), \widehat{GH}(\mathbb{G})$ are knot invariants, when we want to underline the knot rather than its grid diagram we will write $GH^-(K), \widehat{GH}(K)$.

Example 13. We compute the differentials ∂^- and $\hat{\partial}$ of the unknot $\mathcal{O}$ in Figure 4.8.

$$\begin{aligned}
\partial^-(x_1) &= x_2 + x_3 + x_6 & \hat{\partial}(x_1) &= x_2 + x_3 + x_6 & \\
\partial^-(x_2) &= (V_2 + V_3)x_5 & \hat{\partial}(x_2) &= V_2 x_5 & &\text{(As shown in Figure 4.15)} \\
\partial^-(x_3) &= (V_1 + V_2)x_5 & \hat{\partial}(x_3) &= V_2 x_5 + V_1 x_5 & \\
\partial^-(x_4) &= V_1 x_2 + V_3 x_3 + V_2 x_6 & \hat{\partial}(x_4) &= V_1 x_2 + V_2 x_6 & \\
\partial^-(x_5) &= 0 & \hat{\partial}(x_5) &= 0 & \\
\partial^-(x_6) &= (V_1 + V_3)x_5 & \hat{\partial}(x_6) &= V_1 x_5 &
\end{aligned}$$

Thus

$$GH^- = \frac{\langle x_5, V_1x_5, V_2x_5, V_3x_5\rangle}{\langle (V_1+V_2)x_5, (V_2+V_3)x_5\rangle} = \langle x_5, V_1x_5\rangle = \mathbb{Z}_2[U]\langle x_5\rangle.$$

$$\widehat{GH} = \frac{\langle x_5, V_1x_5, V_2x_5\rangle}{\langle V_1x_5, V_2x_5\rangle} = \langle x_5\rangle = \mathbb{Z}_2\langle x_5\rangle.$$

4.3.4 Properties and examples

We explore now some properties of the three flavours of Grid Homology defined in the preceding sections and we state ways to connect them.

Proposition 4.24. *Let $\mathbb{G}$ be a grid diagram representing a knot. Let W be the two dimensional bigraded vector space, with one generator in bigrading $(0,0)$ and the other one in bigrading $(-1,-1)$. Then, there is an isomorphism*

$$\widetilde{GH}(\mathbb{G}) \cong \widehat{GH}(\mathbb{G}) \otimes W^{\otimes(n-1)} \tag{4.28}$$

of bigraded vector spaces, where $\widehat{GH}(\mathbb{G}) = H(\frac{GH^-(\mathbb{G})}{V_i})$ for any i=1,...,n.

Proof. We will prove by induction on j that

$$H\left(\frac{GC^{-}(\mathbb{G})}{V_1=\cdots=V_j=0}\right)\cong H\left(\frac{GC^{-}(\mathbb{G})}{V_1=0}\right)\otimes W^{\otimes(j-1)}. \tag{4.29}$$

We interpret W^0 as a one-dimensional vector space in bigrading $(0,0)$, so that the isomorphism $W^{\oplus a}\otimes W\cong W^{\oplus(a+1)}$ holds for all $a\geq 0$. In the basic case where $j=1$, Equation (4.29) is a tautology.
For the inductive step, for $j>1$ consider the short exact sequence

$$0\longrightarrow\frac{GC^{-}(\mathbb{G})}{V_1=\cdots=V_{j-1}=0}\xrightarrow{\cdot V_j}\frac{GC^{-}(\mathbb{G})}{V_1=\cdots=V_{j-1}=0}\xrightarrow{\psi}\frac{GC^{-}(\mathbb{G})}{V_1=\cdots=V_j=0}\longrightarrow 0 \tag{4.30}$$

where $\cdot V_j$ is the multiplication by V_j.
Since V_j and V_1 are chain homotopic due to Lemma 4.12, this induces a chain homotopy between the chain map $\cdot V_j$ and the 0 map (Proposition A.12), so the long exact sequence induced on homology (see Lemma A.11), associated to the short exact sequence (4.30) becomes

$$0\longrightarrow H\left(\frac{GC^{-}(\mathbb{G})}{V_1=\cdots=V_{j-1}=0}\right)\to H\left(\frac{GC^{-}(\mathbb{G})}{V_1=\cdots=V_j=0}\right)\longrightarrow H\left(\frac{GC^{-}(\mathbb{G})}{V_1=\cdots=V_{j-1}=0}\right)\longrightarrow 0$$

where the second arrow preserves bigradings and the third one is homogeneous of degree $(1,1)$, due to Lemma A.11, since V_j was a bigraded module homomorphism of degree $(-2,-1)$ and the homomorphism ψ has degree $(-1,0)$. Thus, this sequence of vector spaces splits and induces isomorphisms

$$\begin{aligned}H\left(\frac{GC^{-}(\mathbb{G})}{V_1=\cdots=V_j=0}\right)&\cong H\left(\frac{GC^{-}(\mathbb{G})}{V_1=\cdots=V_{j-1}=0}\right)\oplus H\left(\frac{GC^{-}(\mathbb{G})}{V_1=\cdots=V_{j-1}=0}\right)\\&\cong H\left(\frac{GC^{-}(\mathbb{G})}{V_1=\cdots=V_{j-1}=0}\right)\otimes W\cong GH\otimes W^{\otimes(j-1)}\end{aligned}$$

where in the direct sum the two terms are in degrees (m,a) and $(m-1,a-1)$ so they are isomorphic to the tensor product with W thanks to Definition A.3.3, and where the second isomorphism follows from inductive hypothesis. This verifies Equation (4.29) for $j=1,\ldots,n$.
Due to Lemma 4.3.2, when $j=n$, Equation (4.29) gives the thesis for $i=1$. Numbering our formal variables differently, we obtain the thesis for arbitrary i. $\square$

Remark 24. Due to definition A.3.3, we can compute recursively the tensor product $\widehat{GH}(\mathbb{G})\otimes W^{\otimes(n-1)}$.

Proposition 4.25. *There is a long exact sequence relating $\widehat{GH}(\mathbb{G})$ and $GH^{-}(\mathbb{G})$:*

$$\to GH^{-}_{m+2}(\mathbb{G},a+1)\to GH^{-}_{m}(\mathbb{G},a)\to\widehat{GH}_m(\mathbb{G},a)\to GH^{-}_{m+1}(\mathbb{G},a+1)\to$$

Proof. Consider the short exact sequence

$$0 \to GC^-(\mathbb{G}) \xrightarrow{V_i} GC^-(\mathbb{G}) \to \widehat{GC}(\mathbb{G}) \to 0$$

of bigraded chain complexes of $\mathbb{Z}_2[V_i]$-modules, where the first map is homogeneous of degree $(-2,-1)$. The associated long exact sequence in homology (see Lemma A.11) gives the statement of the proposition. □

Proposition 4.26. *For a grid diagram $\mathbb{G}$ with grid number n, the vector space $\widehat{GH}(\mathbb{G})$ is finite dimensional and $2^{n-1} \cdot \dim_{\mathbb{Z}_2} \widehat{GH}(\mathbb{G}) = \dim_{\mathbb{Z}_2} \widetilde{GH}(\mathbb{G})$.*

Proof. The proof of this proposition follows directly from Proposition 4.24 since

$$\begin{aligned}\dim(\widetilde{GH}(\mathbb{G})) &= \dim(\widehat{GH}(\mathbb{G}) \otimes W^{\otimes(n-1)}) \\ &= \dim(\widehat{GH}(\mathbb{G})) \cdot \dim(W^{\otimes(n-1)}) = \dim(\widehat{GH}(\mathbb{G})) \cdot 2^{n-1}.\end{aligned}$$

□

Proposition 4.27. *The simply blocked grid homology*

$$\widehat{GH}(\mathbb{G}) = H(GC^-(\mathbb{G})/V_i)$$

is independent of the choice of i=1,...,n.

Proof. From Proposition 4.24 it follows that for *i,j*

$$H\left(\frac{GC^-(\mathbb{G})}{V_i}\right) \otimes W^{(n-1)} \cong H\left(\frac{GC^-(\mathbb{G})}{V_j}\right) \otimes W^{(n-1)} \tag{4.31}$$

as bigraded vector spaces.

A finite dimensional bigraded vector space Y is determined up to isomorphism by its Poincaré polynomial (see Definition A.3.8), since we are fixing the number of generators of each $Y_{m,a}$ in each bidegree (m,a).

Defining $Y_i = H\left(\frac{GC^-(\mathbb{G})}{V_i}\right)$, Equation (4.31) translates into

$$(1+q^{-1}t^{-1})^{n-1} \cdot P_{Y_i}(q,t) = (1+q^{-1}t^{-1})^{n-1} \cdot P_{Y_i}(q,t)$$

so $P_{Y_i} = P_{Y_j}$ hence $H\left(\frac{GC^-(\mathbb{G})}{V_i}\right) = H\left(\frac{GC^-(\mathbb{G})}{V_j}\right)$ as bigraded vector spaces. □

Proposition 4.28. *Take a grid diagram $\mathbb{G}$ with grid number n for a knot. Then, there is an isomorphism of bigraded $\mathbb{Z}_2[U]$-modules:*

$$H\left(\frac{GC^-(\mathbb{G})}{V_1 = \cdots = V_n}\right) \cong GH^-(\mathbb{G}) \otimes W^{\otimes(n-1)}.$$

Proof. For each $i = 1, \ldots, n$ we construct inductively a quasi-isomorphism of $\mathbb{Z}_2[V_i, \ldots, V_n]$-modules (see Definition A.3.5)

$$\Phi_i : GC^-(\mathbb{G}) \otimes W^{\otimes(i-1)} \longrightarrow \frac{GC^-(\mathbb{G})}{V_1 = \cdots = V_i}. \tag{4.32}$$

For $i = 1$ we have a tautology. For the inductive step, since $GC^-(\mathbb{G})$ is a free module over $\mathbb{Z}_2[V_1, \ldots, V_n]$, the quotient $\frac{GC^-(\mathbb{G})}{V_1=\cdots=V_i}$ is a free module over $\mathbb{Z}_2[V_i, \ldots, V_n]$, so Lemma A.13 gives a quasi-isomorphism

$$\frac{GC^-(\mathbb{G})}{V_1 = \cdots = V_i} \otimes W \longrightarrow \frac{GC^-(\mathbb{G})}{V_1 = \cdots = V_{i+1}}.$$

Precomposing with Φ_i, which holds for inductive hypothesis, tensored with W, gives the quasi-isomorphism Φ_{i+1} needed for Equation (4.32) for all $i = 1, \ldots, n$. Fixing $i = n$ gives the statement of the proposition. $\square$

Proposition 4.29. *If K is a knot, then for all $m, a \in \mathbb{Z}$, we have $\widehat{GH}_m(K, a) \cong \widehat{GH}_{m-2a}(K, -a)$.*

Proof. Let $\mathbb{G}_1$ be a grid diagram for K and let $\mathbb{G}_2$ be a diagram for K, but with opposite orientations (see Remark 9). There is an isomorphism of chain complexes between $\widetilde{GC}(\mathbb{G}_1)$ and $\widetilde{GC}(\mathbb{G}_2)$, induced by the natural identification of the grid states, which does not respect the bigradings, since they depend on the position of $\mathbb{X}$ and $\mathbb{O}$. Let M_i and A_i denote the Maslov and Alexander gradings calculated with respect to the diagram $\mathbb{G}_i$. From Equation (4.4) we obtain

$$\begin{aligned} M_1(\mathbf{x}) - M_2(\mathbf{x}) &= 2A_1(\mathbf{x}) + n - 1 \\ A_1(\mathbf{x}) - A_2(\mathbf{x}) &= 1 - n. \end{aligned}$$

It follows that

$$\widetilde{GH}_m(\mathbb{G}_1, a) \cong \widetilde{GH}_{m-2a+1-n}(\mathbb{G}_1, -a + 1 - n). \tag{4.33}$$

This symmetry, together with Proposition 4.24 implies the thesis, indeed, expressing the symmetries in terms of the Poincaré polynomials (see Definition A.3.8)

$$\widetilde{P}_{\mathbb{G}}(q,t) = \sum_{m,a} \dim \widetilde{GH}_m(\mathbb{G}, a) \cdot q^m t^a \quad \text{and} \quad \widehat{P}_{\mathbb{G}}(q,t) = \sum_{m,a} \dim \widehat{GH}_m(\mathbb{G}, a) \cdot q^m t^a$$

we can rewrite Equation (4.33) as

$$\widetilde{P}_{\mathbb{G}_1}(q,t) = (qt)^{1-n} \cdot \widetilde{P}_{\mathbb{G}_2}(q, q^{-2}t^{-1}).$$

Proposition 4.24, applied to $\mathbb{G}_1, \mathbb{G}_2$ gives us the relations

$$(1+q^{-1}t^{-1})^{n-1}\widehat{P}_{\mathbb{G}_1}(q,t) = \widetilde{P}_{\mathbb{G}_1}(q,t) \quad \text{and} \quad (1+q^{-1}t^{-1})^{n-1}\widehat{P}_{\mathbb{G}_2}(q,t) = \widetilde{P}_{\mathbb{G}_2}(q,t) \tag{4.34}$$

where the factor $(1+q^{-1}t^{-1})^{n-1}$ is due to the dimension of W with a generator in bigrading $(0,0)$ and one in bigrading $(-1,-1)$. Thus combining the previous equations we obtain $\widehat{P}_{\mathbb{G}_1}(q,t) = \widehat{P}_{\mathbb{G}_2}(q,q^{-2}t^{-1})$, which gives us the thesis. $\square$

Proposition 4.30. *If K is a knot and $m(K)$ is its mirror image, then for all $m, a \in \mathbb{Z}$, $\widehat{GH}_m(K,a) \cong \widehat{GH}_{2a-m}(m(K),a)$.*

Proof. Let $\mathbb{G}$ be a grid diagram with markings $\mathbb{O}, \mathbb{X}$ for K and let $\mathbb{G}^*$ be the diagram with markings $\mathbb{O}^*, \mathbb{X}^*$ obtained by reflecting $\mathbb{G}$ through a horizontal axis. The diagram $\mathbb{G}^*$ represents $m(K)$ (see Remark 10).
Reflection induces a bijection $x \mapsto x^*$ between grid states for $\mathbb{G}$ and those for $\mathbb{G}^*$, inducing a bijection between empty rectangles in $\mathrm{Rect}^\circ(\mathbf{x},\mathbf{y})$ and empty rectangles in $\mathrm{Rect}^\circ(\mathbf{y}^*,\mathbf{x}^*)$. Thus, reflection induces an isomorphism of chain complexes $\widetilde{GC}(\mathbb{G}^*) \cong \mathrm{Hom}(\widetilde{GC}(\mathbb{G}),\mathbb{Z}_2)$, sending $x^* \mapsto \delta_{x,z}$, with $\delta_{x,z} = 1$ if $x = z$ and 0 otherwise. Since $\mathbb{Z}_2$ is a field, the Universal Coefficient Theorem in cohomology (see [24, Th. A.5.2]) gives an isomorphism $\widetilde{GH}(\mathbb{G}) \cong \widetilde{GH}(\mathbb{G}^*)$ and by Proposition 4.24 we get that $\widehat{GH}(K) \cong \widehat{GH}(m(K))$.
It remain to keep track of the gradings in the above isomorphism. We verify that

$$M_{\mathbb{O}}(\mathbf{x}) + M_{\mathbb{O}^*}(\mathbf{x}^*) = 1-n. \tag{4.35}$$

Indeed, choosing $\mathbf{x} = \mathbf{x}^{NWO}$ (the grid state with all its components in the north-western corner of the O-markings) we have $M_{\mathbb{O}}(\mathbf{x}) = 0$ and $M_{\mathbb{O}^*}(\mathbf{x}^*) = 1-n$ since $\mathbf{x}^* = \mathbf{x}^{SWO}$. For arbitrary grid states $\mathbf{x}$,$\mathbf{y}$, if $r \in \mathrm{Rect}(\mathbf{x},\mathbf{y})$ then $r^* \in \mathrm{Rect}(\mathbf{y}^*,\mathbf{x}^*)$ so from Equation (4.5) we have $M_{\mathbb{O}}(\mathbf{x}) + M_{\mathbb{O}^*}(\mathbf{x}^*) = M_{\mathbb{O}}(\mathbf{y}) + M_{\mathbb{O}^*}(\mathbf{y}^*)$.
Since any two grid states can be connected by a sequence of rectangles, as seen in Remark 19, Equation (4.35) follows for all grid states.
It follows from Equation (4.35), that

$$A_{\mathbb{G}}(\mathbf{x}) + A_{\mathbb{G}^*}(\mathbf{x}^*) = 1-n$$

which, along with Equation (4.35) identifies $\widetilde{GH}_m(\mathbb{G},a) \cong \widetilde{GH}_{1-n-m}(\mathbb{G}^*, 1-n-a)$. Combining with Equation (4.33), we obtain the isomorphism

$$\widetilde{GH}_m(\mathbb{G},a) \cong \widetilde{GH}_{2a-m}(\mathbb{G}',a) \tag{4.36}$$

where $\mathbb{G}'$ is obtained from $\mathbb{G}^*$ by switching the roles of $\mathbb{X}$ and $\mathbb{O}$. Combining with Proposition 4.24 we obtain the thesis.

In particular from Equation (4.36) we have the relation between the Poincaré polynomials

$$\widetilde{P}_G(q,t) = \widetilde{P}_{\mathbb{G}'}(q^{-1}, q^2t^{-1})$$

which leads to

$$(1+q^{-1}t^{-1})^{n-1}\widehat{P}_{\mathbb{G}}(q,t) = \widetilde{P}_{\mathbb{G}}(q,t) \quad \text{and} \quad (1+q^{-1}t^{-1})^{n-1}\widehat{P}_{\mathbb{G}'}(q,t) = \widetilde{P}_{\mathbb{G}'}(q,t)$$

so $\widehat{P}_{\mathbb{G}}(q,t) = \widehat{P}_{\mathbb{G}'}(q^{-1}, q^2t^{-1})$, which gives the thesis.

□

We want now give a standard form to the $\mathbb{Z}_2[U]$-module $GH^-(\mathbb{G})$ and the bigraded vector space $\widehat{GH}(\mathbb{G})$. In order to do so we need to recall the definition of *torsion* of a module over a PID as in Theorem A.7.
When M is bigraded, the quotient $M/\mathrm{Tors}(M)$ inherits a bigrading. We define the **rank** $r = \mathrm{rk}(M)$ of M to be the integer such that $M/\mathrm{Tors}(M) \cong \mathbb{Z}_2[U]^r$.

Lemma 4.31. *For any knot K, the module $GH^-(K)$ has rank 1; more precisely*

$$GH^-(K)/\mathrm{Tors}(GH^-(K)) \cong \mathbb{Z}_2[U]$$

is supported in bigradings (m,a) satisfying $m - 2a = 0$.

This proposition implies that there are homogeneous, non-torsion elements in $GH^-(K)$ for any knot. We can now define our torsion:

Definition 4.3.9. For any knot K, $\boldsymbol{\tau(K)}$ is -1 times the maximal integer i for which there is a homogeneous, non-torsion element in $GH^-(K)$ with Alexander grading equal to i.

Lemma 4.32. *For each $m, a \in \mathbb{Z}$, the vector space $GH^-_m(K,a)$ is a finite dimensional and vanishes if either m ora is sufficiently large. Moreover, $GH^-(K)$ is a finitely generated bigraded $\mathbb{Z}_2[U]$-module.*

Proof. Fixed a grid diagram $\mathbb{G}$ representing K, the chain complex $GC^-(\mathbb{G})$ is, by construction, a finitely generated module over $\mathbb{Z}_2[V_1,\dots,V_n]$, that is bigraded. Since multiplication by V_i shifts both Maslov and Alexander gradings down, it follows that for each $m, a \in \mathbb{Z}$, the vector space $GC^-_m(\mathbb{G},a)$ is finite dimensional and vanishes if either m or a are sufficiently large.
The homology $GH^-(K)$ inherits these properties, in particular $GH^-(K)$ is finitely generated as a $\mathbb{Z}_2[V_1\dots,V_n]$-module, because $\mathbb{Z}_2[V_1\dots,V_n]$ is a Noetherian ring (see Proposition A.14 and Definition A.1.9), thus each submodule and quotient of submodules is finitely generated as module (see Theorem A.5). Since each V_i acts on $GH^-(K)$ the same way, due to Lemma 4.12, we conclude that $GH^-(K)$ is a finitely generated module over $\mathbb{Z}_2[U]$. □

Let $\mathbb{Z}_2[U]/U^n_{(m,a)}$ denote the bigraded cyclic torsion module whose generator g has bigrading given by (m,a) and satisfies the relation $U^n g = 0$ (while $U^{n-1}g \neq 0$); that is $\mathbb{Z}_2[U]/U^n_{(m,a)}$ is isomorphic to $\mathbb{Z}_2$ in bigradings $(m-2i, a-i)$ for every $i = 0, \dots, n-1$, and U times the non-zero element in bigrading $(m-2i, a-i)$ is the one in bigrading $(m-2i-2, a-i-1)$, provided that $0 \leq i \leq n-2$. Similarly, let $\mathbb{Z}_2[U]_{(m,a)}$ denote the free, rank one $\mathbb{Z}_2[U]$-module whose generator has bigrading (m,a).

Proposition 4.33. *For each knot K, we can find $k \geq 0$ and a set of triples of integers $\{(m_i, a_i, n_i)\}_{i=1}^k$ with $n_i > 0$, so that*

$$GH^-(K) \cong \left(\bigoplus_{i=1}^k \mathbb{Z}_2[U]/U^{n_i}_{(m_i,a_i)} \right) \oplus \mathbb{Z}_2[U]_{(-2\tau,-\tau)}$$

as $\mathbb{Z}_2[U]$-modules, where $\tau = \tau(K)$.
Similarly, we have

$$\widehat{GH}(K) \cong \left(\bigoplus_{i=1}^k \mathbb{Z}_{2(m_i,a_i)} \oplus \mathbb{Z}_{2(m_i-2n_i+1, a_i-n_i)} \right) \oplus \mathbb{Z}_{2(-2\tau,-\tau)}$$

as bigraded vector space.

Proof. According to Lemma 4.32, the module $GH^-(K)$ is finitely generated over $\mathbb{Z}_2[U]$. By the classification of finitely generated modules over a PID (Theorem A.7), the module $GH^-(K)$ splits as a direct sum of finitely many cyclic modules, which are of the form $\mathbb{Z}_2[U]/p$, for some polynomial $p \in \mathbb{Z}_2[U]$. The gradings force these summands to be either of the form $\mathbb{Z}_2[U]$ (i.e. free summands), or of the form $\mathbb{Z}_2[U]/U^n$ for some $n > 0$. The free summand in $GH^-(K)$ is given by $GH^-(K)/\mathrm{Tors} \cong \mathbb{Z}_2[U]_{(2a,a)}$, due to Lemma 4.31, so $a = -\tau(K)$. Thus we obtained the first equation. To obtain the statement for $\widehat{GH}(K)$ we use Proposition 4.25. $\square$

Example 14. We want to try to compute the three grid homology flavours for the trefoil knot K, oriented as in Figure 4.3 (the orientation does not effect the grid homology but we need to fix an orientation to compute the Alexander grading using the winding number). Since the grid has dimension $n = 5$ we knot that there are $5! = 120$ different grid states. Although we can try to enlist and calculate the differentials for all of them, using the proposition proved above, it is possible to compute the grid homology just by analysing the grid states with Alexander degrees $1, 0, -5$.
We begin by showing that the Alexander degree spans between 1 and -5, using

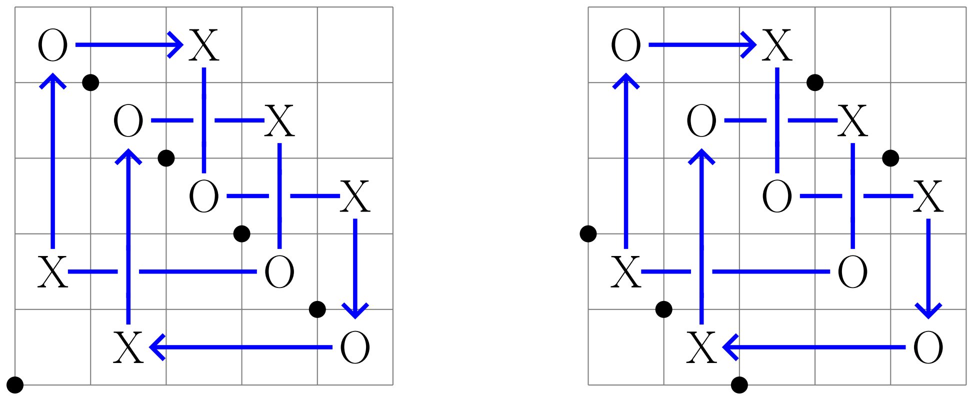
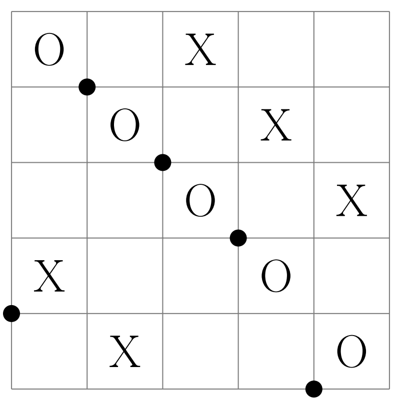


Figure 4.16: Grid states b (on the right) and d (on the left) with Alexander grading 1 and -5 respectively.

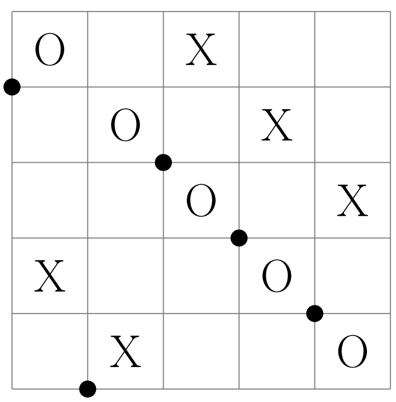
the winding number. As seen in Equation (4.9) we need to compute $a(\mathbb{G})$ and the winding number of the elements of the grid state. Due to Example 10 we already know that $a(\mathbb{G}) = -3$. Combining this results we obtain

$$A(\mathbf{x}) = -\sum_{x \in \mathbf{x}} w_{\mathcal{D}}(x) - 5 = -A'(\mathbf{x}) - 5 \tag{4.37}$$

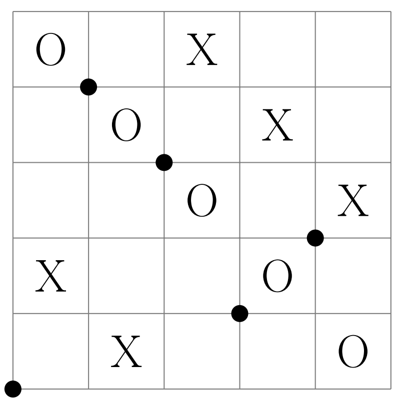
so we have that the maximum value of $A(\mathbf{x})$ is obtained minimizing $A'(\mathbf{x})$. Since the winding numbers of the points of the grid are ≤ 0, due to the orientation (see Figure 4.16), the minimal value for $A'(\mathbf{x})$ is obtained by the grid state on the left of Figure 4.16 and correspond to -6. Thus the maximal value for $A(\mathbf{x}) = 6 - 5 = 1$. Vice versa, the minimal value of $A(\mathbf{x})$ is obtained by the grid state on the right of Figure 4.16 and its Alexander grading $A(\mathbf{x}) = -5$. We denote this two grid state b, d respectively.
With a similar reasoning we can compute all the grid states for a fixed Alexander degree, but we restrict ourself to show the 5 grid states with Alexander grading 0. The idea to find them is to keep fixed two of the five points of the state in correspondence of the point of the grid with winding number -2. To obtain $A(\mathbf{x}) = 0$ we need now a point with winding number -1 and there are two possible choices for this point which allow us to take the remaining two points of the states with winding number 0. Then we can consider grid states with only one point of winding number -2 and 3 points with winding number -1; for each of them we find one grid state. These are the only possibilities since, due to the configuration of the points, there not exists a grid state with all the points with winding number -1. These grid states with $A(\mathbf{x}) = 0$ are shown in Figure 4.17 and denominated $c_1, \dots, c_5$.
We want now to compute the Maslov grading of $b, c_1, \dots, c_5, d$. Applying Equation



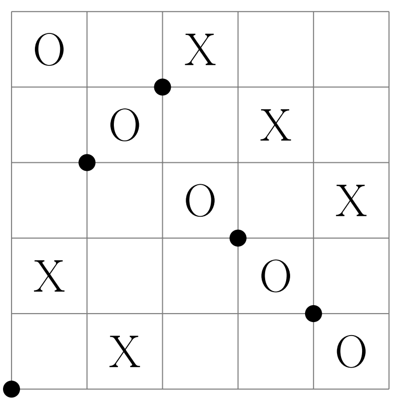
(a) Grid state c_1



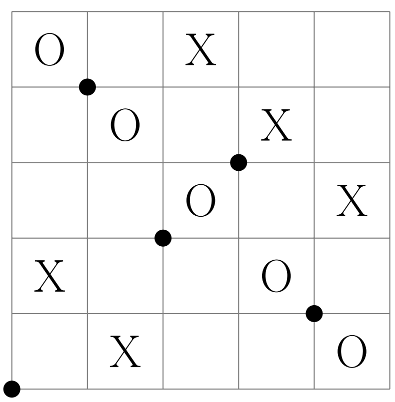
(b) Grid state c_2



(c) Grid state c_3



(d) Grid state c_4



(e) Grid state c_5

Figure 4.17: Grid states of the trefoil knot with Alexander grading 0.

(4.2) we obtain

$$\begin{aligned}
M(b) &= 0 \\
M(c_1) &= M(c_2) = M(c_3) = M(c_4) = M(c_5) = -1 \\
M(d) &= -6.
\end{aligned}$$

In order to compute the three flavours of grid homology we start with the simpler version of the grid homology $\widetilde{GH}(\mathbb{G})$. To avoid notational overload and since the grid diagram $\mathbb{G}$ is fixed we will denote $\widetilde{GH}_m(\mathbb{G}, a) = \widetilde{GH}_{m,a}$ and similarly for GH^- and $\widehat{GH}$. We have

$$\begin{aligned}
&\widetilde{GC}_{0,1} \quad \text{is the grid complex of dimension 1 generated by } b \\
&\widetilde{GC}_{-1,0} \quad \text{is the grid complex of dimension 5 generated by } c_i \\
&\widetilde{GC}_{-6,-5} \quad \text{is the grid complex of dimension 1 generated by } d
\end{aligned}$$

and we obtain the diagram

$$\begin{gathered}
\longrightarrow 0 \longrightarrow \widetilde{GC}_{0,1} \longrightarrow 0 \longrightarrow \\
b \\
\longrightarrow 0 \longrightarrow \widetilde{GC}_{-1,0} \longrightarrow 0 \longrightarrow \\
c_i \\
\dots \\
\longrightarrow 0 \longrightarrow \widetilde{GC}_{-6,-5} \longrightarrow 0 \longrightarrow \\
d
\end{gathered}$$

where the arrows are the differentials $\tilde{\partial}$ dropping the Maslov degree by 1. We need to compute the differential $\tilde{\partial}$ of the 6 generators, which means, find empty rectangles (i.e. not containing other points of the generator) not containing elements of $\mathbb{X}, \mathbb{O}$.

Observing Figure 4.16 and 4.17 we can see that each rectangle between the grid states and any other possible grid state contain an X or an O -marking, so we obtain

$$\begin{aligned}
\tilde{\partial}(b) &= 0 \\
\tilde{\partial}(c_i) &= 0 \quad \forall\, i = 1, \dots, 5 \\
\tilde{\partial}(d) &= 0.
\end{aligned}$$

To compute $\widetilde{GH}(\mathbb{G})$ we use Equation (4.18). Since the differential of each of the grid states is zero, we have that

$$\mathrm{Ker}(\tilde{\partial}) \cap \widetilde{GC}_{m,a} = \widetilde{GC}_{m,a} \quad \forall (m,a).$$

From the diagram of the grid complexes we deduce that $\mathrm{Im}(\tilde{\partial}) = 0$ for every differential, which implies

$$\mathrm{Im}(\tilde{\partial}) \cap \widetilde{GC}_{m,a} = 0 \quad \forall (m,a).$$

Thus we obtain the following grid homology groups:

$$\begin{aligned} \widetilde{GH}_{0,1} &= \mathbb{Z}_2\langle b \rangle \\ \widetilde{GH}_{-1,0} &= \bigoplus_{i=1}^{5} \mathbb{Z}_2\langle c_i \rangle \\ \widetilde{GH}_{-6,-5} &= \mathbb{Z}_2\langle d \rangle. \end{aligned}$$

Since $\widetilde{GH}(\mathbb{G})$ is not an invariant we do not calculate the grid homology groups in other bidegrees.
To obtain $GH^-(K)$ and $\widehat{GH}(K)$ we can repeat this procedure but it will be very expensive in terms of computational time, since we won't find that $\partial^-(\mathbf{x}) = 0$ for each grid state $\mathbf{x}$. To compute them we will use Properties 4.24 and 4.25.
Proposition 4.24 allows us to compute $\widehat{GH}(K)$ from $\widetilde{GH}(\mathbb{G})$. Using the Pascal triangle we obtain

$$\widetilde{GH}_{m,a} \cong \widehat{GH}_{m,a} \oplus 4\,\widehat{GH}_{m+1,a+1} \oplus 6\,\widehat{GH}_{m+2,a+2} \oplus 4\,\widehat{GH}_{m+3,a+3} \oplus \widehat{GH}_{m+4,a+4}$$

where $k\,\widehat{GH}_{m,a} = \bigoplus_{i=1}^{k} \widehat{GH}_{m,a}$.
Applying this equation to $\widetilde{GH}_{0,1}$ we find

$$\widetilde{GH}_{0,1} \cong \widehat{GH}_{0,1} \oplus 4\,\widehat{GH}_{1,2} \oplus 6\,\widehat{GH}_{2,3} \oplus 4\,\widehat{GH}_{3,4} \oplus \widehat{GH}_{4,5}$$

since we have no generators in Alexander degree greater than 1 all terms cancels out except for $\widehat{GH}_{0,1}$ which is so isomorphic to $\widetilde{GH}_{0,1}$, thus $\widehat{GH}_{0,1} \cong \mathbb{Z}_2$.
We repeat the same reasoning for the other $\widetilde{GH}_{m,a}$, erasing the zero terms.

$$\bigoplus_{i=1}^{5} \mathbb{Z}_2 \cong \widetilde{GH}_{-1,0} \cong \widehat{GH}_{-1,0} \oplus 4\,\widehat{GH}_{0,1}$$

so we obtain $\widehat{GH}_{-1,0} \cong \mathbb{Z}_2$. Moreover,

$$\mathbb{Z}_2 \cong \widetilde{GH}_{-6,-5} \cong \widehat{GH}_{-6,-5} \oplus 4\,\widehat{GH}_{-5,-4} \oplus 6\,\widehat{GH}_{-4,-3} \oplus 4\,\widehat{GH}_{-3,-2} \oplus \widehat{GH}_{-2,-1}$$

although we do not have information about the intermediate terms, they have to be zero, otherwise the right hand side cannot be isomorphic to $\mathbb{Z}_2$. Thus we obtain

$$\mathbb{Z}_2 \cong \widehat{GH}_{-6,-5} \oplus \widehat{GH}_{-2,-1}$$

which means that either $\widehat{GH}_{-6,-5} = 0$ or $\widehat{GH}_{-2,-1} = \mathbb{Z}_2$. Applying the equation to $\widetilde{GH}_{-7,-6} = 0$ we obtain that $\widehat{GH}_{-6,-5} = 0$, so we conclude that $\widehat{GH}_{-2,-1}$=0. Summarizing we have

$$\widehat{GH}_{m,a} = \begin{cases} \mathbb{Z}_2 & (m,a) = (0,1), (-1,0), (-2,-1) \\ 0 & \text{otherwise.} \end{cases}$$

We can now calculate $GH^-_{m,a}$ from $\widehat{GH}_{m,a}$ using Proposition 4.25.
Applied to $\widehat{GH}_{0,1}$ we obtain

$$\longrightarrow GH^-_{2,2} \longrightarrow GH^-_{0,1} \longrightarrow \widehat{GH}_{0,1} \longrightarrow GH^-_{1,2} \longrightarrow$$

which leads us to $GH^-_{0,1} \cong \widehat{GH}_{0,1} \cong \mathbb{Z}_2$ since we have no generators in Alexander degree 2.
Applied to $\widehat{GH}_{-1,0}$ we obtain

$$\begin{aligned} \longrightarrow GH^-_{1,1} \longrightarrow GH^-_{-1,0} \longrightarrow &\widehat{GH}_{-1,0} \longrightarrow GH^-_{0,1} \longrightarrow GH^-_{-2,0} \longrightarrow \\ 0 \longrightarrow GH^-_{-1,0} \longrightarrow &\mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow 0 \end{aligned}$$

so, using the properties of the short exact sequences we obtain $GH^-_{-1,0} = 0$.
Applied to $\widehat{GH}_{-2,-1}$ we obtain

$$\longrightarrow GH^-_{0,0} \longrightarrow GH^-_{-2,-1} \longrightarrow \widehat{GH}_{-2,-1} \longrightarrow GH^-_{-1,0} \longrightarrow$$

thus we have $GH^-_{-2,-1} \cong \widehat{GH}_{-2,-1} \cong \mathbb{Z}_2$.
We now apply the proposition to the bidegrees where $\widehat{GH}_{m,a} = 0$, obtaining the following

$$\begin{aligned} \longrightarrow \widehat{GH}_{0,0} \longrightarrow &GH^-_{-1,-1} \longrightarrow GH^-_{-3,-2} \longrightarrow \widehat{GH}_{-3,-2} \longrightarrow GH^-_{-2,-1} \longrightarrow GH^-_{-4,-2} \longrightarrow \widehat{GH}_{-4,-2} \longrightarrow \\ 0 \longrightarrow &GH^-_{-1,-1} \longrightarrow GH^-_{-3,-2} \longrightarrow 0 \longrightarrow \mathbb{Z}_2 \longrightarrow GH^-_{-4,-2} \longrightarrow 0. \end{aligned}$$

Analysing the following sequence for $GH^-_{-1,-1}$

$$\longrightarrow GH^-_{1,0} \longrightarrow GH^-_{-1,-1} \longrightarrow \widehat{GH}_{-1,-1} \longrightarrow GH^-_{0,0} \longrightarrow$$

we find that $GH^-_{-1,-1} = 0$, so, in the previous sequence we find $GH^-_{-3,-2} = 0$ and $GH^-_{-4,-2} = \mathbb{Z}_2$. This suggests us to find the following pattern

$$GH^-_{-2k,-k} \cong \mathbb{Z}_2 \quad \forall k \geq 1. \tag{4.38}$$

We want to show this using induction: we already proved for $k = 1$, so we take $k \geq 2$ and suppose $GH^-_{-2(k-1),-(k-1)} \cong \mathbb{Z}_2$ and apply the proposition to $GH^-_{-2k,-k}$. We have

$$\longrightarrow \widehat{GH}_{-2k+1,-k} \longrightarrow GH^-_{-2(k-1),-(k-1)} \longrightarrow GH^-_{-2k,-k} \longrightarrow \widehat{GH}_{-2k,-k} \longrightarrow$$

where we know that $\widehat{GH}_{-2k,-k} = \widehat{GH}_{-2k+1,-k} = 0$ for $k \geq 2$ and for inductive hypothesis $GH^-_{-2(k-1),-(k-1)} \cong \mathbb{Z}_2$, so $GH^-_{-2k,-k} \cong \mathbb{Z}_2 \quad \forall k \geq 1$.
Thus we conclude that

$$GH^-_{m,a} = \begin{cases} \mathbb{Z}_2 & (m,a) = (0,1), (-2k,-k) \quad \forall k \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Due to Lemma 4.12, we can think GH^- as a $\mathbb{Z}_2[U]$ module, where U is one of the V_i, thus we can rewrite $GH^-_{m,a}$ as

$$GH^-(K) \cong (\mathbb{Z}_2[U]/U)_{(0,1)} \oplus \mathbb{Z}_2[U]_{(-2,-1)}$$

which agrees with Proposition 4.33 since $\tau = 1$.
Using Proposition 4.30 we can compute the simply blocked grid homology of the mirror image $m(K)$ of the trefoil

$$\widehat{GH}_{m,a}(m(K)) = \begin{cases} \mathbb{Z}_2 & (m,a) = (2,1), (1,0), (0,-1) \\ 0 & \text{otherwise.} \end{cases}$$

Since $\widehat{GH}$ is a knot invariant, this shows how the trefoil and its mirror image are not equivalent knots as questioned in Remark 4.

Although $GH^-, \widehat{GH}$ are powerful invariants, they are not classifying invariants, indeed there exists infinite family of knots with the same grid homology. An example is shown in Figure 4.18. We leave as reference for the reader [24, Section 9.5]

Nevertheless, from the equivalence between grid homology and Knot Floer Homology, it follows that the grid homology detects the unknot (see [24, Th. 1.3.1]); however, it is still and open problem to show this directly using grid diagrams.

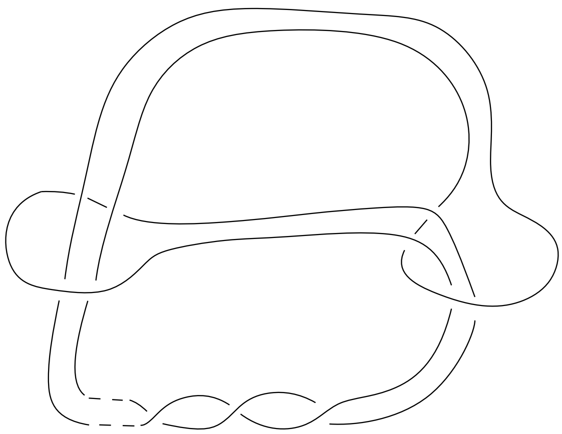


Figure 4.18: A family of distinct knots with the same grid homology.

4.4 Homologies in $L(p,q)$

As in the preceding section we consider the ring $\mathcal{R} = \mathbb{Z}_2[V_1, \dots, V_n]$. The differential are defined as in the S^3 case (Definitions 4.17, 4.19 and 4.22): in the next section we will show their behaviour and the behaviour of the multiplication by V_i with the three gradings. The main difference here is how we can connect generators with rectangle depending on their Spin^c degree.

4.4.1 Differentials

We start by recalling the definitions of the differentials.

Definition 4.4.1. Let's consider $\mathbf{x} \in S(\mathbb{G})$

$$\tilde{\partial} := \tilde{\partial}_{\mathbb{O},\mathbb{X}}(\mathbf{x}) = \sum_{\mathbf{y}\in S(\mathbb{G})} \sum_{\{r\in \mathrm{Rect}^\circ(\mathbf{x},\mathbf{y}) | r\cap\mathbb{O}=r\cap\mathbb{X}=\emptyset\}} \mathbf{y}. \tag{4.39}$$

$$\partial^- := \partial^-_{\mathbb{X}}(\mathbf{x}) = \sum_{\mathbf{y}\in S(\mathbb{G})} \sum_{\{r\in \mathrm{Rect}^\circ(\mathbf{x},\mathbf{y}) | r\cap\mathbb{X}=\emptyset\}} V_1^{O_1(r)} \dots V_n^{O_n(r)} \cdot \mathbf{y} \tag{4.40}$$

$$\hat{\partial} := \hat{\partial}_{\mathbb{X},O_i}(\mathbf{x}) = \sum_{\mathbf{y}\in S(\mathbb{G})} \sum_{\{r\in \mathrm{Rect}^\circ(\mathbf{x},\mathbf{y}) | r\cap\mathbb{X}=\emptyset, O_n(r)=0\}} V_1^{O_1(r)} \dots V_{n-1}^{O_{n-1}(r)} \cdot \mathbf{y} \tag{4.41}$$

where $O_i(r)$ is how many times the marking O_i appears in the rectangle r.

Definition 4.4.2. The complex $GC^-(\mathbb{G})$ is the free $\mathcal{R}$-module generated over $S(\mathbb{G})$. The complex $\widehat{GC}(\mathbb{G})$ is the free $\mathcal{R}/V_n$ module generated over $S(\mathbb{G})$. We extend the gradings to the whole module by setting the behaviour of the action for the variables in the ring:

$$\begin{aligned} A(V\mathbf{x}) &= A(\mathbf{x}) - 1 \\ M(V\mathbf{x}) &= M(\mathbf{x}) - 2 \\ S(V\mathbf{x}) &= S(\mathbf{x}) \end{aligned}$$

where V is any of the V_i.

If $\mathbf{x}$,$\mathbf{y}\in S(\mathbb{G})$, then $|\mathrm{Rect}(\mathbf{x},\mathbf{y})| \in \{0,2\}$ and in the next proposition we will prove that it can be non-zero only for generators in the same Spin^c degree, which differ by a single transposition. With the same hypothesis on the generators, $|\mathrm{Rect}^\circ(\mathbf{x},\mathbf{y})| \in \{0,1,2\}$.

Proposition 4.34. *Given a grid diagram $\mathbb{G}$ of parameters (n,p,q), the modules $GC^-(\mathbb{G})$ endowed with the endomorphs ∂^-, defined as in Equation (4.40) are chain complexes, that is $(\partial^-)^2 = 0$.*
Moreover ∂^- acts on the three gradings as follows:

1. $S(\partial^-(\boldsymbol{x})) = S(\boldsymbol{x})$
2. $M(\partial^-(\boldsymbol{x})) = M(\boldsymbol{x}) - 1$
3. $A(\partial^-(\boldsymbol{x})) = A(\boldsymbol{x})$.

An analogous proposition can be stated for $\widehat{GC}(\mathbb{G})$ endowed with the endomorphism $\hat{\partial}$. The proof is identical so we will show it only for $GC^-(\mathbb{G})$.

Proof. We start by proving that $(\partial^-)^2 = 0$, thus we have to study all the possible decompositions in rectangles of polygons, defined as in Definition 4.1.2, connecting two generators.
The proof follows step by step the proof of Lemma 4.6 with the construction of a differential $(\partial^-)^2 = 0$ counting domains between grid states. In lens spaces, due to the identifications, we have different cases to study.
If a domain ϕ does not cross one of the α curves, we can cut the torus open along it and think the domain as living in an $np \times np$ grid for S^3 and conclude with Lemma 4.6. Thus we only need to worry about polygons that intersect all α circles.
We define the quantity $Q = |\mathbf{x} \setminus (\mathbf{x} \cap \mathbf{z})| \in \{0,\ldots,4\}$

- $Q = 0$. Here $\mathbf{x}$ and $\mathbf{z}$ shares all 4 vertices and the only possible polygons are thin annulus (see Definition 4.1.2). Each one of them contains one X-marking, hence they do not contribute to the differential.
- $Q = 1$. This case can be dismissed since rectangles only connect generator which differ in exactly two points.

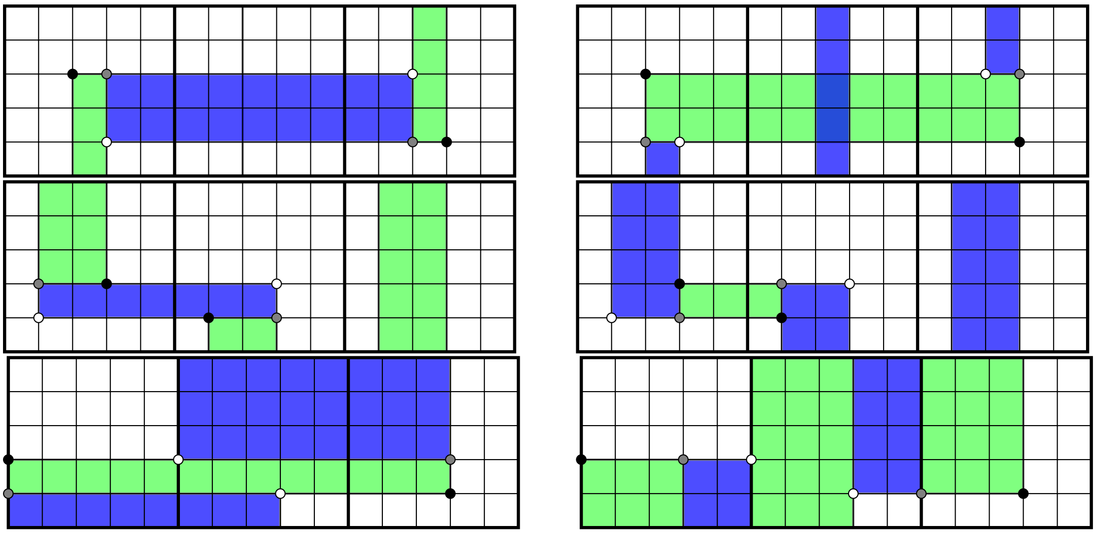


Figure 4.19: Relevant combinations for M=2. On each row the two possible decomposition are shown. **x**=white dots, **y**=grey dots, **z**=black dots.

- $Q = 2$. Here we don't have an S^3 counterpart so we need to distinguish two subcase

  1. The rectangle starting from **x** does not cross all α curves, so up to vertical/horizontal translation it can be placed in such a way that it does not intersect the boundary of the planar grid diagram.
  2. The rectangle starting from **x** intersects all the α curves at least once.

  Either way the second rectangle joining the intermediate generator to **z** must end and start on the same α curve of the first rectangle as shown in Figure 4.19.

- $Q = 3$. In this case the rectangles shares only one vertex and we can again distinguish two possibilities as in the previous case (see Figure 4.20).

- $Q = 4$. The corners of the two rectangles are all distinct, here there are two ways of counting them, which correspond to taking the two rectangles in either order. We remark that one rectangle might wrap around the other according to the identifications.

Since for each Q there are two possible decompositions in rectangles of the polygon, the differential will be zero since we are working with $\mathbb{Z}_1$ coefficiets.

We are left to show how the differential behaves with the gradings.
We begin by examining the behaviour with the Spin^c degree. From Equation

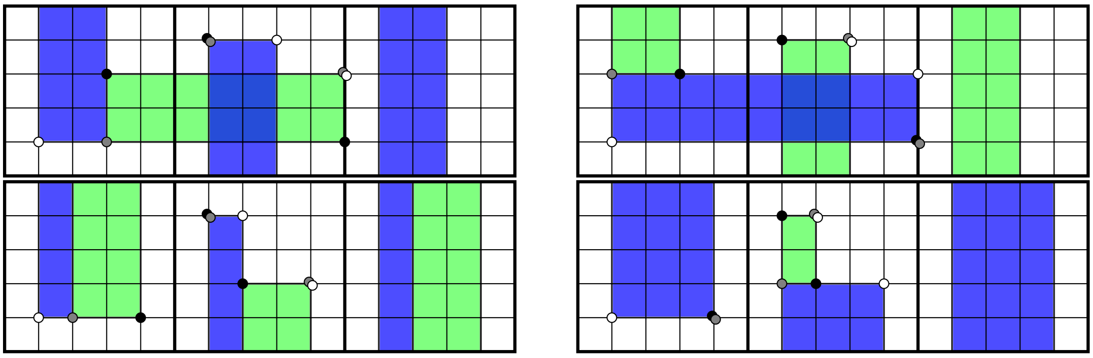


Figure 4.20: Relevant combinations for M=3. On each row the two possible decomposition are shown. $\mathbf{x}$=white dots, $\mathbf{y}$=grey dots, $\mathbf{z}$=black dots.

(4.16) the only relevant part of a generator $\mathbf{x}$ for the computation of S($\mathbf{x}$) is given by its p-coordinates. If $\mathbf{y}$ appears in the differential of $\mathbf{x}$, call x_i, x_j, y_i, y_j the p-coordinates where $\mathbf{x}$ and $\mathbf{y}$ differ. If $x_i = k$ and $x_j = l$, with $k, l \in \mathbb{Z}_p$, then, since they are connected by a rectangle, $y_i = k + t$ and $y_j = l + t \mod p$ for some $t \in \{0, \ldots, p-1\}$ so $S(\mathbf{x}) = S(\mathbf{y})$.
Let's consider now the Maslov and the Alexander grading. If $\mathbf{x}$,$\mathbf{y}$$\in S(\mathbb{G})$ are connected by and empty rectangle r directed from $\mathbf{x}$ to $\mathbf{y}$, then their lifts $\tilde{\mathbf{x}}, \tilde{\mathbf{y}}$ will differ in 2p positions. This implies the following equations:

$$I(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}) = I(\tilde{\mathbf{y}}, \tilde{\mathbf{y}}) + p$$
$$I(\tilde{\mathbf{x}}, \tilde{\mathbb{O}}) = I(\tilde{\mathbf{y}}, \tilde{\mathbb{O}}) + p \sum_{i=1}^{n} O_i(r)$$
$$I(\tilde{\mathbb{O}}, \tilde{\mathbf{x}}) = I(\tilde{\mathbb{O}}, \tilde{\mathbf{y}}) + p \sum_{i=1}^{n} O_i(r).$$

Thus we obtain that

$$M(\mathbf{y}) = M(\mathbf{x}) + \frac{1}{p}[-p + 2p \sum_{i=1}^{n} O_i(r)].$$

Iterating Definition 4.4.2 we obtain

$$M(V_1^{O_1(r)} \cdots V_n^{o_n(r)} \mathbf{x}) = M(\mathbf{x}) - 2 \sum_{i=1}^{n} O_i(r).$$

Combining all these equation we obtain

$$\begin{aligned}
M(\partial^-(\mathbf{x})) &= \sum_{\mathbf{y}\in S(\mathbb{G})} \sum_{\{r\in \mathrm{Rect}^\circ(\mathbf{x},\mathbf{y}) | r\cap \mathbb{X}=\emptyset\}} \left(M(\mathbf{y}) - 2\sum_{i=1}^{n} O_i(r) \right) \\
&= \sum_{\mathbf{y}\in S(\mathbb{G})} \sum_{\{r\in \mathrm{Rect}^\circ(\mathbf{x},\mathbf{y}) | r\cap \mathbb{X}=\emptyset\}} \left(M(\mathbf{x}) - 1 + 2\sum_{i=1}^{n} O_i(r) - 2\sum_{i=1}^{n} O_i(r) \right) \\
&= M(\mathbf{x}) - 1.
\end{aligned}$$

With analogous calculations we obtain the result for the Alexander degree. □

An analogous proof can be given for the differential defined in Equation (4.39) as we did in the case of S^3.

Proposition 4.35. *Let $\mathbb{G}$ be a grid of parameters (n,p,q) for a knot K. Then the action of multiplication by V_i on the complex $GC^-(\mathbb{G})$ is quasi-isomorphic to multiplication by V_j.*

Proof. The idea is the same of Lemma 4.12, which consist in defining an operator counting the rectangles between two grid states, described in Proposition 4.34. □

We can now define the homologies for the three complexes $(\widetilde{GC}(\mathbb{G}), \tilde{\partial})$,$(GC^-(\mathbb{G}), \partial^-)$, $(\widehat{GC}(\mathbb{G}), \hat{\partial})$.

Definition 4.4.3. We define the grid homology

$$\widetilde{GH}(\mathbb{G}) = (\widetilde{GC}(\mathbb{G}), \tilde{\partial})$$

regarded as a $(\mathbb{Q}, \mathbb{Q}, \mathbb{Z}_p)$ graded module over $\mathbb{Z}_2$.

Theorem 4.36. *The homologies*

$$\begin{aligned}
GH^-(\mathbb{G}) &= H(GC^-(\mathbb{G}), \partial^-) \\
\widehat{GH}(\mathbb{G}) &= H(\widehat{GC}(\mathbb{G}), \hat{\partial})
\end{aligned}$$

regarded as $(\mathbb{Q}, \mathbb{Q}, \mathbb{Z}_p)$ graded modules over $\mathcal{R}$ are invariants of the knot $(L(p,q), K)$. Moreover, $(GC^-(\mathbb{G}), \partial^-)$ is quasi-isomorphic (Definition A.3.5) to a finitely generated $\mathbb{Z}_2[U]$-module, where U acts as any of the V_i, and $(\widehat{GC}(\mathbb{G}), \hat{\partial})$ is quasi-isomorphic to a finitely generated $\mathbb{Z}_2$-module.

Proof. The complete proof can be found in [2], the idea is to consider an Heegaard diagram for the knot in $L(p,q)$ and show that the complexes defined above using grid diagrams are isomorphic to the complexes defined for the Knot Floer Homology, and that the gradings corresponds. □

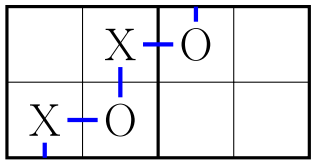


Figure 4.21: Knot K in $L(2,1)$ with grid number 2.

Sometimes we will refer to the grid homology of a knot K$\subset L(p,q)$ as $GH^*(\mathbb{G})$ referring to its grid presentation or as $GH^*(L(p,q),K)$ to specify the ambient space and the knot, where * is either $\tilde{}, ^-, \hat{}$.
Since the differential preserves the decomposition of the complex in the Spin^c structure, we can write:

$$\begin{aligned}
\widetilde{GH}(L(p,q),K) &= \bigoplus_{s\in\mathbb{Z}_p} \widetilde{GH}(L(p,q),K,s)\\
GH^-(L(p,q),K) &= \bigoplus_{s\in\mathbb{Z}_p} GH^-(L(p,q),K,s)\\
\widehat{GH}(L(p,q),K) &= \bigoplus_{s\in\mathbb{Z}_p} \widehat{GH}(L(p,q),K,s).
\end{aligned}$$

As in the previous section we have some properties for the grid homologies.

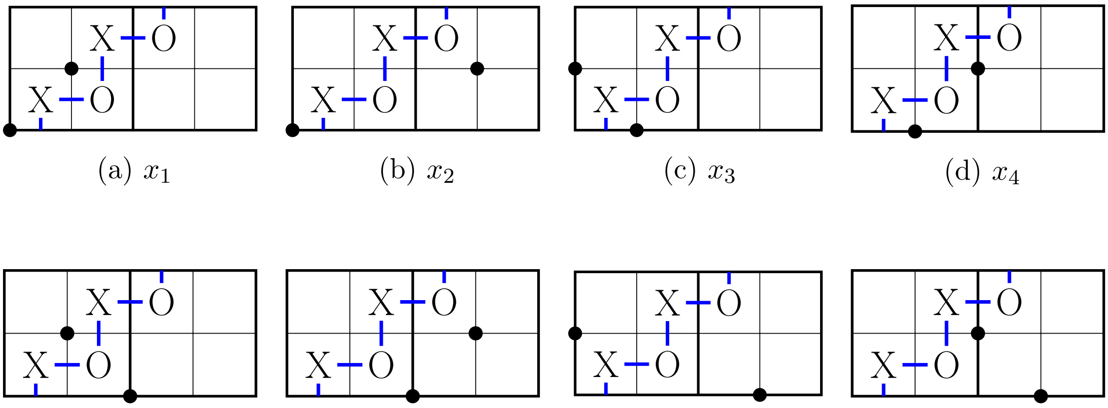
Proposition 4.37. *Let's consider a grid presentation $\mathbb{G}$ for a knot K$\subset L(p,q)$ with grid number n. Then we have the following result*

$$\widetilde{GH}(\mathbb{G}) \cong \widehat{GH}(\mathbb{G}) \otimes W^{\otimes(n-1)}$$

where W is a two-dimensional bigraded $\mathbb{Z}_2$-vector space with one generator in bigrading $(0,0)$ and another in bigrading $(-1,-1)$.

Example 15. We compute the flavours of the grid homology of the knot in Figure 4.21 where O_1, O_2 are the O-markings in the first and in the second row respectively.
Since $K \subset L(2,1)$ and $n = 2$ we have $n!p^n = 8$ generators as shown in Figure 4.22, defined as

$$\begin{aligned}
\mathbf{x}_1 &= [(\mathrm{Id}),(0,0)] & \mathbf{x}_5 &= [(\mathrm{Id}),(1,0)]\\
\mathbf{x}_2 &= [(\mathrm{Id}),(0,1)] & \mathbf{x}_6 &= [(\mathrm{Id}),(1,1)]\\
\mathbf{x}_3 &= [(1,2),(0,0)] & \mathbf{x}_7 &= [(1,2),(1,0)]\\
\mathbf{x}_4 &= [(1,2),(0,1)] & \mathbf{x}_8 &= [(1,2),(1,1)]
\end{aligned}$$



(a) x_1 (b) x_2 (c) x_3 (d) x_4

(e) x_5 (f) x_6 (g) x_7 (h) x_8

Figure 4.22: Grid states of K of Figure 4.21.

where Id is the identity permutation.
Firstly we compute the Spin^c grading of the generators using Equation (4.16)

$$S(\mathbf{x}) = 1 - 1 + \sum_{i=1}^{8}(x_i - x_i^{\mathbb{O}})$$

where $\mathbf{x}_{\mathbb{O}} = [(1,2),(0,0)] = \mathbf{x}_4$, so

$$\begin{aligned} S(\mathbf{x}_1) &= 1 \qquad & S(\mathbf{x}_5) &= 0 \\ S(\mathbf{x}_2) &= 0 \qquad & S(\mathbf{x}_6) &= 1 \\ S(\mathbf{x}_3) &= 1 \qquad & S(\mathbf{x}_7) &= 0 \\ S(\mathbf{x}_4) &= 0 \qquad & S(\mathbf{x}_8) &= 1. \end{aligned}$$

The correction term is $d(2,1,0) = -\frac{1}{4}$.
To compute the Maslov and the Alexander gradings we need to consider the lift of the generators to S^3 and use Equations (4.15) and (4.14), thus we obtain:

$$\begin{aligned} M(\mathbf{x}_1) &= \frac{1}{4} & A(\mathbf{x}_1) &= \frac{1}{4} & M(\mathbf{x}_5) &= -\frac{1}{4} & A(\mathbf{x}_5) &= -\frac{1}{4} \\ M(\mathbf{x}_2) &= -\frac{1}{4} & A(\mathbf{x}_2) &= -\frac{1}{4} & M(\mathbf{x}_6) &= \frac{1}{4} & A(\mathbf{x}_6) &= -\frac{3}{4} \\ M(\mathbf{x}_3 &= -\frac{3}{4} & A(\mathbf{x}_3) &= -\frac{3}{4} & M(\mathbf{x}_7) &= \frac{3}{4} & A(\mathbf{x}_7) &= -\frac{1}{4} \\ M(\mathbf{x}_4) &= -\frac{5}{4} & A(\mathbf{x}_4) &= -\frac{5}{4} & M(\mathbf{x}_8) &= -\frac{3}{4} & A(\mathbf{x}_8) &= -\frac{3}{4}. \end{aligned}$$

We now distinguish the two cases when $\mathrm{Spin}^c = 0$ and $\mathrm{Spin}^c = 1$ since we count rectangle only between generators in the same Spin^c degree which differ of a transposition.
Starting with $\mathrm{Spin}^c = 0$ we can compute $\tilde{\partial}, \partial^-, \hat{\partial}$ and obtain:

$$\begin{aligned}
&\tilde{\partial}(\mathbf{x}_2) = 0 && \partial^-(\mathbf{x}_2) = 0 && \hat{\partial}(\mathbf{x}_2) = 0\\
&\tilde{\partial}(\mathbf{x}_4) = 0 && \partial^-(\mathbf{x}_4) = V_1\mathbf{x}_5 + V_2\mathbf{x}_2 && \hat{\partial}(\mathbf{x}_4) = V_1\mathbf{x}_5\\
&\tilde{\partial}(\mathbf{x}_5) = 0 && \partial^-(\mathbf{x}_5) = 0 && \hat{\partial}(\mathbf{x}_5) = 0\\
&\tilde{\partial}(\mathbf{x}_7) = \mathbf{x}_2 + \mathbf{x}_5 && \partial^-(\mathbf{x}_7) = \mathbf{x}_2 + \mathbf{x}_5 && \hat{\partial}(\mathbf{x}_7) = \mathbf{x}_2 + \mathbf{x}_5.
\end{aligned}$$

Thus, denoting $\widetilde{GH}_0, GH_0^-, \widehat{GH}_0$ the three flavours in the $\mathrm{Spin}^c = 0$ grading we obtain

$$\begin{aligned}
\widetilde{GH}_0(K) &= \frac{\langle \mathbf{x}_2, \mathbf{x}_4, \mathbf{x}_5\rangle}{\langle \mathbf{x}_2 + \mathbf{x}_5\rangle} \cong \mathbb{Z}_2\langle \mathbf{x}_2\rangle \oplus \mathbb{Z}_2\langle \mathbf{x}_4\rangle\\
GH_0^-(K) &= \frac{\langle \mathbf{x}_2, \mathbf{x}_5, V_1\mathbf{x}_2, V_2\mathbf{x}_2, V_1\mathbf{x}_5, V_2\mathbf{x}_5\rangle}{\langle \mathbf{x}_2 + \mathbf{x}_5, V_1\mathbf{x}_5 + V_2\mathbf{x}_2\rangle} \cong \mathbb{Z}_2[U]\langle \mathbf{x}_2\rangle\\
\widehat{GH}_0(K) &= \frac{\langle \mathbf{x}_2, \mathbf{x}_5, V_1\mathbf{x}_2, V_1\mathbf{x}_5\rangle}{\langle \mathbf{x}_2 + \mathbf{x}_5, V_1\mathbf{x}_5\rangle} \cong \mathbb{Z}_2\langle \mathbf{x}_2\rangle
\end{aligned}$$

where to simplify the generators we used that $\mathbf{x}_2 \sim \mathbf{x}_5$ in the quotient and where we identified the multiplication by V_1 with U.
For $\mathrm{Spin}^c = 1$ we have the differentials:

$$\begin{aligned}
&\tilde{\partial}(\mathbf{x}_1) = 0 && \partial^-(\mathbf{x}_1) = 0 && \hat{\partial}(\mathbf{x}_1) = 0\\
&\tilde{\partial}(\mathbf{x}_3) = 0 && \partial^-(\mathbf{x}_3) = (V_1 + V_2)\mathbf{x}_1 && \hat{\partial}(\mathbf{x}_3) = V_1\mathbf{x}_1\\
&\tilde{\partial}(\mathbf{x}_6) = \mathbf{x}_3 + \mathbf{x}_8 && \partial^-(\mathbf{x}_6) = \mathbf{x}_3 + \mathbf{x}_8 && \hat{\partial}(\mathbf{x}_6) = \mathbf{x}_3 + \mathbf{x}_8\\
&\tilde{\partial}(\mathbf{x}_8) = 0 && \partial^-(\mathbf{x}_8) = (V_1 + V_2)\mathbf{x}_1 && \hat{\partial}(\mathbf{x}_8) = V_1\mathbf{x}_1.
\end{aligned}$$

Thus, denoting $\widetilde{GH}_1, GH_1^-, \widehat{GH}_1$ the three flavours in the $\mathrm{Spin}^c = 1$ grading we obtain

$$\begin{aligned}
\widetilde{GH}_1(K) &= \frac{\langle \mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_8\rangle}{\langle \mathbf{x}_3 + \mathbf{x}_8\rangle} \cong \mathbb{Z}_2\langle \mathbf{x}_1\rangle \oplus \mathbb{Z}_2\langle \mathbf{x}_3\rangle\\
GH_1^-(K) &= \frac{\langle \mathbf{x}_1, V_1\mathbf{x}_1, V_2\mathbf{x}_2, \mathbf{x}_3 + \mathbf{x}_8\rangle}{\langle (V_1 + V_2)\mathbf{x}_1, \mathbf{x}_3 + \mathbf{x}_8\rangle} \cong \mathbb{Z}_2[U]\langle \mathbf{x}_1\rangle\\
\widehat{GH}_1(K) &= \frac{\langle \mathbf{x}_1, V_1\mathbf{x}_1, \mathbf{x}_3 + \mathbf{x}_8\rangle}{\langle V_1\mathbf{x}_1, \mathbf{x}_3 + \mathbf{x}_8\rangle} \cong \mathbb{Z}_2\langle \mathbf{x}_1\rangle.
\end{aligned}$$

Remark 25. We developed our theory for knots and considering $\mathbb{Z}_2$ coefficients in our calculation. The theory can be extended to links and to $\mathbb{Z}$ coefficients for both S^3 and $L(p,q)$ as done in [9, 24]

4.5 Grid Homology and Alexander polynomial

In this chapter we want to investigate the relation between the Alexander polynomial of a knot and its grid homology groups $\widetilde{GH}(K), GH^-(K), \widehat{GH}(K)$. This relation is widely known for knots in S^3 but is still unknown for knots in $L(p,q)$. We start by recalling the definition of graded Euler characteristic of a bigraded vector space (see Definition A.3.9).

Proposition 4.38. *Let $\mathbb{G}$ be a grid diagram for a knot $K \subset S^3$ with grid number n. The graded Euler characteristic of the bigraded vector space $\widetilde{GH}(\mathbb{G})$ is given by*

$$\chi(\widetilde{GH}(\mathbb{G})) = (1-t^{-1})^{n-1} \cdot \nabla_K(t)$$

where $\nabla_K(t)$ is the Alexander-Conway polynomial defined as in Theorem 3.13.

Proof. Since the Euler characteristic of a chain complex agrees with that of its homology, we have $\chi(\widetilde{GH}(\mathbb{G})) = \chi(\widetilde{GC}(\mathbb{G}))$.
Defining $\sigma_{\mathbf{x}\mapsto\mathbf{y}}$ the permutation connecting the grid state $\mathbf{x}$ to the grid state $\mathbf{y}$ and using Lemma 4.3 we can write

$$\begin{aligned}(-1)^{M(\mathbf{x})} &= \mathrm{sgn}(\sigma_{\mathbf{x}\mapsto\mathbf{x}^{NWO}}) = \mathrm{sgn}(\sigma_{\mathbf{x}\mapsto\mathbf{I}})\mathrm{sgn}(\sigma_{\mathbf{I}\mapsto\mathbf{I}'})\mathrm{sgn}(\sigma_{\mathbf{I}'\mapsto\mathbf{x}^{SWO}})\mathrm{sgn}(\sigma_{\mathbf{x}^{SWO}\mapsto\mathbf{x}^{NWO}})\\ &= \mathrm{sgn}(\sigma_{\mathbf{x}\mapsto\mathbf{I}})(-1)^{n(n-1)/2}\epsilon(\mathbb{G})(-1)^{n-1}\end{aligned}$$

where we have the permutations $\mathbf{I} = (1,2,...,n)$, $\mathbf{I}' = (n, n-1, ..., 1)$ and going from the permutation $\mathbf{x}^{SWO}$ to $\mathbf{x}^{NWO}$ means to move each element i to $i+1$.
Thus we can write

$$\begin{aligned}\chi(\widetilde{GH}(\mathbb{G})) &= \sum_{\mathbf{x}\in S(\mathbb{G})} (-1)^{M(\mathbf{x})} t^{A(\mathbf{x})}\\ &= \sum_{\mathbf{x}\in S(\mathbb{G})} \mathrm{sgn}(\sigma_{\mathbf{x}\mapsto\mathbf{x}^{NWO}}) t^{-\sum_{x\in\mathbf{x}} w_D(x)} t^a(\mathbb{G}) t^{-(n-1)/2}\\ &= \epsilon(\mathbb{G})(-1)^{n-1}\det(M(\mathbb{G}))t^a(\mathbb{G})t^{(1-n)/2}\end{aligned}$$

using Definition 4.2.5 for the t-power and the Leibniz formula for the determinant of $M(\mathbb{G})$. The result now follows from Theorem 3.13 since $(1-t^{-1})^{n-1}(t^{-\frac{1}{2}} - t^{\frac{1}{2}})^{1-n} = (-1)^{n-1}t^{(1-n)/2}$. □

Theorem 4.39. *Let $K \subset S^3$ be a knot. The graded Euler characteristic of the simply blocked grid homology $\widehat{GH}(K)$ is equal to the Alexander-Conway polynomial $\nabla_K(t)$.*

Proof. Considering the bigraded vector space W from Proposition 4.24, its graded Euler characteristic is $\chi(W) = 1 - t^{-1}$, so by the previous proposition follows the identity. □

We can consider the graded Euler characteristic of GH^-

$$\chi(GH^-(K)) = \sum_{m,a} (-1)^m \dim_{\mathbb{Z}_2} GH^-_m(K, a) \cdot t^a.$$

By Lemma 4.32 for each $m, a \in \mathbb{Z}$ the chain complex $GC^-_m(\mathbb{G}, a)$ is finite dimensional and is equal to 0 if m or s are sufficiently large, thus we can define the graded Euler characteristic at the chain level

$$\chi(GC^-(\mathbb{G})) = \sum_{m,a} (-1)^m \dim_{\mathbb{Z}_2} GC^-_m(\mathbb{G}, a) \cdot t^a \in \mathbb{Z}[t, t^{-1}]$$

and general properties of Euler characteristic ensures that $\chi(GH^-(K)) = \chi(GC^-(\mathbb{G}))$.

Proposition 4.40. *Let $K \subset S^3$ be a knot. The graded Euler characteristic of $GH^-(K)$ is given by*

$$\chi(GH^-(K)) = \frac{\nabla_K(t)}{1 - t^{-1}}.$$

Proof. The short exact sequence, defined in Proposition 4.25

$$0 \longrightarrow GC^-(\mathbb{G})[\![2,1]\!] \xrightarrow{V_i} GC^-(\mathbb{G}) \longrightarrow \widehat{GC}(\mathbb{G}) \longrightarrow 0$$

implies the relation

$$\begin{aligned}\chi(GC^-(\mathbb{G})) &= \chi(GC^-(\mathbb{G})[\![2,1]\!]) + \chi(\widehat{GC}(\mathbb{G})) \\ &= \chi(GC^-(\mathbb{G}))t^{-1} + \chi(\widehat{GC}(\mathbb{G})).\end{aligned}$$

Since the graded Euler characteristic is unchanged under homology and recalling that $\frac{1}{1-t^{-1}} = \sum_{i \geq 0} t^{-i} \in \mathbb{Z}[t, t^{-1}]$ we conclude. □

Remark 26. With this connection between the Alexander-Conway polynomial and the grid homology, we can find analogies between their properties. Proposition 4.29 can be seen as a manifestation in grid homology of the symmetry of the Alexander-Conway polynomial; on the contrary, Proposition 4.30 contrasts the behaviour of the mirroring in the Alexander polynomial showing that, as expected, the grid homology is a more powerful invariant.

Example 16. Previously we computed the three flavours of the grid homology for the unknot and the trefoil knot. We want to compute their Alexander-Conway polynomial using the graded Euler characteristic.
Recall that, using the grid diagram $\mathbb{G}$ of dimension 3 in Figure 4.7 of the unknot $\mathcal{O}$,

$$\begin{aligned}\widetilde{GH}(\mathbb{G}) &= \mathbb{Z}_{2(0,0)} \oplus 2\mathbb{Z}_{2(-1,-1)} \oplus \mathbb{Z}_{2(-2,-2)}\\ GH^{-}(\mathcal{O}) &= \mathbb{Z}_2[U]_{(0,0)}\\ \widehat{GH}(\mathcal{O}) &= \mathbb{Z}_{2(0,0)}\end{aligned}$$

where $\mathbb{Z}_{2(i,j)}$ means that $\mathbb{Z}_2$ has a generator in bigrading (i,j).
Thus, applying the previous propositions we obtain

$$\begin{aligned}\chi(\widetilde{GH}(\mathbb{G})) &= 1 - 2t^{-1} + t^{-2} = (1-t^{-1})^2 \cdot 1\\ \chi(GH^{-}(\mathcal{O})) &= 1 + t^{-1} + t^{-2} + t^{-3} + \cdots = \frac{1}{1-t^{-1}}\\ \chi(\widehat{GH}(\mathcal{O})) &= 1 \equiv \nabla_{\mathcal{O}}(t).\end{aligned}$$

Using the grid diagram $\mathbb{G}$ in Figure 2.7 for the trefoil knot K we obtain

$$\begin{aligned}GH^{-}(K) &= \mathbb{Z}_{2(0,1)} \oplus \mathbb{Z}_2[U]_{(-2,-1)}\\ \widehat{GH}(K) &= \mathbb{Z}_{2(0,1)} \oplus \mathbb{Z}_{2(-1,0)} \oplus \mathbb{Z}_{2(-2,-1)}\end{aligned}$$

thus

$$\begin{aligned}\chi(\widehat{GH}(K)) &= t - 1 + t^{-1} \equiv \nabla_{\mathcal{O}}(t)\\ \chi(GH^{-}(K)) &= t + t^{-1} + t^{-2} + \cdots = \frac{t-1-t^{-1}}{1-t^{-1}}.\end{aligned}$$

As enunciated at the beginning of this chapter, this relation between the Alexander polynomial and Grid Homology in lens spaces (other than S^3) is not yet known. A good hint may come from [17], where the authors describe an algorithm to compute Knot Floer Homology from the knot group of $(1,1)$-knots in S^3. These are knots that can be placed in a one-bridge position with respect to a genus-one Heegaard splitting of the 3-sphere; thus, they can be defined for lens spaces as well.
Considering that the twisted Alexander polynomial can also be computed from a presentation of the knot group, there may be a way to compute the Alexander polynomial directly from Grid Homology. Furthermore, we observe a symmetry between the fact that different Euler characteristics can be defined using Spin^c structures, and that different Alexander polynomials can be defined using the torsion of $L(p,q)$.

Appendix A

Appendix

We collect here some definitions and results purely algebraic that are used in the previous chapters and may help the reader.

A.1 Groups and rings

We start by giving some basic definition about groups.

Definition A.1.1. A commutative group G (written additively) is said to be **finitely generated** if there exist elements $g_1, g_2, \ldots, g_n \in G$ such that every $g \in G$ can be written as an integer linear combination of $g_1, g_2, \ldots, g_n$:

$$g = a_1 g_1 + a_2 g_2 + \cdots + a_n g_n$$

with $a_1, a_2, \ldots, a_n \in \mathbb{Z}$. The set $\{g_1, g_2, \ldots, g_n\}$ is called a set of generators for G.

Theorem A.1. *Let G be a finitely generated abelian group.*

1. *Either $G \cong \mathbb{Z}^k$ with $k \geq 0$, or*

$$G \cong \mathbb{Z}^k \oplus \bigoplus_{i=1}^{r} \mathbb{Z}_{d_i}$$

*where $k \geq 0$, the integers d_i satisfy $d_i \geq 2$ and, if $i < j$, then d_i divides d_j. The integer k is called the **rank** of G and denoted as $\mathrm{rk}(G)$.*

2. *The numbers $k, d_1, d_2, \ldots, d_r$ are uniquely determined by G.*

Definition A.1.2. [12, Ch. 4] Given a group G we define a **presentation** $\langle \mathbf{x} \mid \mathbf{r} \rangle$ to be a method to specify a group, composed of a set of *generators* $\mathbf{x}$ and a set of *relations* $\mathbf{r}$.

More formally we say that G has a presentation $\langle \mathbf{x} \mid \mathbf{r} \rangle$ if it is isomorphic to the quotient of the free group generated by $\mathbf{x}$, by the normal subgroup generated by the relations $\mathbf{r}$, that are words in the letter $\mathbf{x}$.

Given a group, it is natural to see whether two presentations specify the same group. To do so we introduce the Tietze transformations.

Definition A.1.3. Given a presentation of a group G we define the **Tietze transformations** as follows

1. If a relation can be derived from the existing relations then it can be added to the presentation
2. If a relation can be derived from the other relations then it can be added to the presentation
3. It is possible to add a generator that is expressed as a word in the original generators
4. If a generator can be expressed as a word in the other generators then it can be removed.

The Tietze theorem ensures that two finite presentations, i.e., presentation where both $\mathbf{x}$ ans $\mathbf{r}$ are finite set, of the same group can be obtained one from the other by a finite sequence of Tietze transformations (see [12, Th. 3.2]).
Fox in [12, Ch. 7] give a more general construction of the Alexander matrix (see Section 3.1.3) using a "free calculus", called Fox calculus. We briefly recall the definition of group ring and of Fox calculus.

Definition A.1.4. Given a group G and a ring of integers R, we define the **group ring RG** as the ring where the elements are maps $f : G \longrightarrow R$ such that $f(g) = 0$ except for at most a finite number of $g \in G$. The two operations of RG are defined as follows:

$$
\begin{aligned}
(r_1 + r_2)g &= r_1 g + r_2 r \\
(r_1 r_2)g &= \sum_{h \in G} (r_1 h)(r_2 h^{-1} g) \\
(nr)g &= n(rg)
\end{aligned}
$$

for any $r_1, r_2 \in R$, $g \in G$ and n an integer.

Choosing a mapping $\varphi : G \longrightarrow RG$ which assign to each $g \in G$ the function $g^* \in RG$ defined as $g^*(h) = \delta_{hg}$, i.e., that is 1 if $h = g$ and zero otherwise, allow us to see the group ring RG as an additive group. Indeed, given a non zero element

$r \in RG$, if $g_1, \ldots, g_k$, with $k \geq 1$, are the elements of G such that $r(g_i) \neq 0$, we can write $r = n_1 g_1^* + \cdots + n_k g_k^*$, with $n_i = r(g_i)$ for $i = 1, \ldots, k$.
Identifying g and g^* we can write elements of RG as finite integral combinations of elements of G.

Definition A.1.5. We call **abelianizer** $a : G \longrightarrow G/[G, G]$ the group homomorphism, where $[G, G] = \{aba^{-1}b^{-1} | a, b \in G\}$ is the commutator subgroup.
Considered the group homomorphism $t : G \longrightarrow R$ defined by $t(g) = 1$, for all $g \in G$, we call **trivializer** the unique extension to the ring homomorphism $t : RG \longrightarrow R$ defined as $t(\sum n_i g) = \sum n_i$.

We now give a brief explanation of the Fox calculus in the group ring RG.

Definition A.1.6. We define a **derivative** in RG as a map $D : RG \longrightarrow RG$ which satisfies:

$$D(r_1 + r_2) = D(r_1) + D(r_2) \tag{A.1}$$
$$D(r_1 r_2) = D(r_1)t(r_2) + r_1 D(r_2) \tag{A.2}$$
$$D(g_1 g_2) = D(g_1) + g_1 D(g_2). \tag{A.3}$$

Where t is the trivializer, $r_1, r_2 \in RG$ and $g_1, g_2 \in G$.

Proposition A.2. *There are some basic consequences of this axioms:*

- $D(\sum_i n_i g_i) = \sum_i n_i D(g_i)$.
- $D(n) = 0$ *for any integer* n.
- $D(g^{-1}) = -g^{-1} D(g)$ *for any* $g \in G$.

We are interested now in consider a particular situation where to perform our derivative. Starting by considering a free group $F = \langle x_1, \ldots, x_n, \ldots \rangle$ we can consider the group ring RF as the finite sum of finite products of powers of x_i. To each generator we can associate a derivative such that

$$D_j(x_i) := \frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

where $D_j(x_i)$ is the derivative of the element x_i with respect to the element x_j. (see [12, Chap. 7, Prop. 29.]).
Moreover, $\forall f \in RF$

$$D(f) = \sum_j \partial f / \partial x_j.$$

Consider a group presentation for $G = \langle \mathbf{x} \mid \mathbf{r} \rangle = \langle x_1, \dots, x_n, \cdots \mid r_1, \dots, r_m, \dots \rangle$ and call F the free group generated by $\mathbf{x}$. The group presentation of G can be seen as the group F factored by the canonical homomorphism $\gamma : F \longrightarrow F/R$, where R is the normal subgroup generated by $\mathbf{r}$.
The function γ possess an unique extension to homomorphism between the respective group ring. Denoting as $H = G/[G, G]$ the abelianization of G, for each i we can consider the composition:

$$RF \xrightarrow{\frac{\partial}{\partial x_i}} RF \xrightarrow{\gamma} RG \xrightarrow{a} RH. \tag{A.4}$$

Definition A.1.7. We define the **Alexander matrix** of $G = \langle \mathbf{x} \mid \mathbf{r} \rangle$, with $\mathbf{x}, \mathbf{r}$ finite, as the matrix

$$A = (a_{i,j})_{i,j} \text{ where } a_{i,j} = a\gamma\left(\frac{\partial r_i}{\partial x_j}\right). \tag{A.5}$$

We state some properties of the g.c.d domains. All the proof can be found in [12, Ch. 8].

Definition A.1.8. A ring R will be called a **g.c.d. domain** if it is an integral domain and every finite set of elements has a greatest common divisor (g.c.d).

Lemma A.3. *The group ring of an infinite cyclic group is a g.c.d. domain and has only trivial units, i.e. powers of a generator and their negatives.*

Lemma A.4. *In a g.c.d domain with identity, the g.c.d of any finite set of elements is the generator of the smallest principal ideal that contains them.*

Definition A.1.9. A commutative ring D is said to be Noetherian or to satisfy the ascending chain condition on submodules (or A.C.C. on submodules) if there is no infinite increasing chain of submodules in D, i.e., whenever

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

is an increasing chain of submodules of D, then there is a positive integer m such that $I_k = I_m$ for all $k \geq m$.

Theorem A.5. *The following are equivalent:*

1. *D is a Noetherian ring.*
2. *Every nonempty set of submodules of D contains a maximal element under inclusion.*
3. *Every submodule of D is finitely generated.*

We end this part by recalling some results and definition on chain complexes of abelian groups. Since each abelian group is a Module over $\mathbb{Z}$, that is a PID, the following notions are a particular case of those described in the next section.

Definition A.1.10. A sequence of homomorphisms of abelian groups

$$\xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\cdots} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

satisfying $\partial_n \partial_{n+1} = 0$ for each $n \geq 0$ is called a **chain complex**.

The condition $\partial_n \partial_{n+1} = 0$ is equivalent to the inclusion $\operatorname{Im} \partial_{n+1} \subset \operatorname{Ker} \partial_n$. The ***n*-th homology group** of the chain complex is defined to be the quotient group:

$$H_n = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}.$$

- Elements of $\operatorname{Ker} \partial_n$ are called **cycles**.
- Elements of $\operatorname{Im} \partial_{n+1}$ are called **boundaries**.
- Elements of H_n are cosets of $\operatorname{Im} \partial_{n+1}$, called **homology classes**.

Two cycles representing the same homology class are said to be **homologous**, meaning that their difference is a boundary.
We say that a chain complex is **finitely generated** if $n < \infty$ and if $\operatorname{rk}(C_i)$ is finite (see Theorem A.1).

Definition A.1.11. Let $C = (C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_1} C_0)$ be a finitely generated chain complex , then the **Euler characteristic of** X is

$$\chi(X) = \sum_{i=1}^{n} (-1)^i \operatorname{rk}(X_i).$$

Lemma A.6. *If X is a chain complex of finitely generated abelian groups, then its Euler characteristic agrees with the one of its homology groups.*

Proof. Consider X to be the chain complex $X_n \xrightarrow{\partial_n} X_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_1} X_0$.
Define $Z_i = \operatorname{Ker} \partial_i$ and $B_{i-1} = \operatorname{Im} \partial_i$ and consider the following short exact sequences

$$0 \longrightarrow Z_i \longrightarrow X_i \longrightarrow B_{i-1} \longrightarrow 0$$
$$0 \longrightarrow B_i \longrightarrow Z_i \longrightarrow H(X_i) \longrightarrow 0.$$

Due to the properties of the rank we obtain

$$\operatorname{rk}(X_i) = \operatorname{rk}(B_{i-1}) - \operatorname{rk}(Z_i) \qquad \operatorname{rk}(H(X_i)) = \operatorname{rk}(Z_i) - \operatorname{rk}(B_i).$$

Observing that $B_{-1} = 0 = B_n$ we find

$$\chi(X) - \chi(H(X)) = \sum_{i=0}^{n} (-1)^i \operatorname{rk}(X_i) - \operatorname{rk}(H(X_i)) = (-1)^n B_n - B_{-1} = 0.$$

$\square$

A.2 Modules

We now state the basic definitions a some useful properties of the module. We leave as reference [13, 18, 24].

Definition A.2.1. Let D be a ring (not necessarily commutative nor with 1). A left D-module or a left module over D is a set M together with:

1. a binary operation $+$ on M under which M is an abelian group, and
2. an action of D on M (that is, a map $D \times M \to M$) denoted by dm, for all $d \in D$ and for all $m \in M$ which satisfies:
   (a) $(d+s)m = dm + sm$, for all $d, s \in D$, $m \in M$,
   (b) $(ds)m = d(sm)$, for all $d, s \in D$, $m \in M$, and
   (c) $d(m+n) = dm + dn$, for all $d \in D$, $m, n \in M$.

If the ring D has a 1 we impose the additional axiom:

(d) $1m = m$, for all $m \in M$.

Remark 27. If D is a field, then M is in particular a vector space.
If we consider $D = \mathbb{Z}$, then the $\mathbb{Z}$-modules are the abelian groups.

Definition A.2.2. A D-module F is said to be **free** on the subset $A \subseteq F$ if for every nonzero element $x \in F$, there exist unique nonzero elements $d_1, d_2, \ldots, d_n \in D$ and unique distinct elements $a_1, a_2, \ldots, a_n \in A$ such that

$$x = d_1a_1 + d_2a_2 + \cdots + d_na_n$$

for some $n \in \mathbb{Z}^+$. In this situation, we say that A is a **basis** or a set of **free generators** for F. If D is a commutative ring, the cardinality of A is called the **rank** of F.

Definition A.2.3. A D-module M is said to be **finitely generated** if there exists a finite set of elements $\{m_1, m_2, \ldots, m_n\} \subseteq M$ such that every element $m \in M$ can be written as a D-linear combination of $m_1, m_2, \ldots, m_n$:

$$m = d_1m_1 + d_2m_2 + \cdots + d_nm_n$$

where $d_1, d_2, \ldots, d_n \in D$. In this case, the set $\{m_1, m_2, \ldots, m_n\}$ is called a **set of generators** for M.

This observation inspire us in the following classification theorem.

Theorem A.7. *Let D be a PID and let M be a finitely generated D-module.*

1. *Then M is isomorphic to the direct sum of finitely many cyclic modules. More precisely,*

$$M \cong D^r \oplus D/(a_1) \oplus D/(a_2) \oplus \cdots \oplus D/(a_m)$$

for some integer $r \geq 0$ and nonzero elements $a_1, a_2, \ldots, a_m$ of D which are not units in D and which satisfy the divisibility relations $a_1 \mid a_2 \mid \cdots \mid a_m$.

2. *In the decomposition in (1),*

$$\operatorname{Tor}(M) \cong D/(a_1) \oplus D/(a_2) \oplus \cdots \oplus D/(a_m)$$

and M is a torsion module if and only if $r = 0$.

3. *M is torsion-free, i.e.* $\operatorname{Tor}(M) \cong 0$ *if and only if M is free.*

Definition A.2.4. A **chain complex** is an D-module C, equipped with an D-module homomorphism $\partial : C \to C$ with the property that $\partial \circ \partial = 0$. The map ∂ is called the **boundary map** or the **differential** for C. A **cycle** is an element $z \in C$ with $\partial z = 0$ and a **boundary** is an element b of the form $b = \partial a$ for some $a \in C$; i.e. the cycles are the elements in the kernel $\operatorname{Ker} \partial$, and the boundaries are the elements in the image $\operatorname{Im} \partial$.

Definition A.2.5. An D-submodule $C' \subset C$ of a chain complex (C, ∂) is a **subcomplex** if $\partial(C') \subset C'$. In this case, the pair $(C', \partial|_{C'})$ is a chain complex.

Similarly, if $(C', \partial|_{C'})$ is a subcomplex, the quotient module C/C' inherits a boundary operator $\partial_{C/C'}$ induced from ∂. The pair $(C/C', \partial_{C/C'})$ is the **quotient complex** of C by C'.

The condition $\partial^2 = 0$ says $\operatorname{Im} \partial \subset \operatorname{Ker} \partial$, so we can make the following definition:

Definition A.2.6. The **homology** $H(C, \partial)$ of the chain complex (C, ∂) is the quotient D-module:

$$H(C, \partial) = \operatorname{Ker} \partial / \operatorname{Im} \partial.$$

Definition A.2.7. Let (C, ∂) and (C', ∂') be two chain complexes over D. A **chain map** is an D-module map $f : (C, \partial) \to (C', \partial')$ is a homomorphism of D-modules, satisfying the property that $\partial' \circ f = f \circ \partial$.
A chain map $f : C \to C'$ maps cycles to cycles and boundaries to boundaries, and so it induces an D-module map $H(f) : H(C, \partial) \to H(C', \partial')$ on the homology modules.

Definition A.2.8. A sequence $\{C_i\}_{i\in\mathbb{Z}}$ of D-modules equipped with D-module maps $f_i\colon C_i \to C_{i+1}$ is called an **exact sequence** of D-modules if $\operatorname{Im} f_i = \operatorname{Ker} f_{i+1}$.

A special case is when C is a **short exact sequence**, that is, $C_i = 0$ unless $i \in \{1, 2, 3\}$. In this case, the maps $f_1\colon C_1 \to C_2$ and $f_2\colon C_2 \to C_3$ satisfy that:

- f_1 is injective,
- f_2 is surjective, and
- $\operatorname{Im} f_1 = \operatorname{Ker} f_2$.

An **exact triangle** is a 3-periodic exact sequence; i.e., a sequence in which there are three D-modules C_1, C_2, and C_3, and maps $f_1\colon C_1 \to C_2$, $f_2\colon C_2 \to C_3$, and $f_3\colon C_3 \to C_1$, with $\operatorname{Ker}(f_i) = \operatorname{Im}(f_{i-1})$, where $C_i = C_{i+3}$ and $f_i = f_{i+3}$.

An exact sequence of chain complexes (C_i, ∂_i) over D is an exact sequence of D-modules, where the maps $f_i\colon C_i \to C_{i+1}$ are also chain maps, i.e. $\partial_{i+1} \circ f_i = f_i \circ \partial_i$.

A short exact sequence of chain complexes induces a long exact sequence in homology, according to the following standard result

Lemma A.8. *To each short exact sequence of chain complexes of D-modules*

$$0 \longrightarrow (C, \partial) \xrightarrow{\ f\ } (C', \partial') \xrightarrow{\ g\ } (C'', \partial'') \longrightarrow 0$$

there is an associated D-module homomorphism $\delta\colon H(C'', \partial'') \to H(C, \partial)$, called the ***connecting homomorphism****, such that*

$$\begin{tikzcd} H(C,\partial) \arrow[rr, "H(f)"] & & H(C',\partial') \arrow[dl, "H(g)"] \\ & H(C'',\partial'') \arrow[ul, "\delta"] & \end{tikzcd}$$

is an exact triangle.

Let D be a commutative ring with unity and $F = \langle f_1, \dots, f_k \rangle$ a free module over D with (free) basis $f_1, \dots, f_k$, i.e. every $x \in F$ can be written as $\sum_{i=1}^{k} a_i f_i$ for an unique choice of $a_i \in D$.

Let M be a finitely presented module, then it exists a finite presentation, i.e. an exact sequence $F \xrightarrow{\alpha} E \longrightarrow M \longrightarrow 0$ with E, F two finitely generated free modules over D. Essentially the generators of M are the images of the generators of E in M, while the relations are the images in E of the generators of F. Fixing bases for E and F, the map $\alpha : F \longrightarrow E$ can be represented by a matrix $A \in M_{m,n}(D)$ with m and n the cardinality of the basis of E and F, respectively. The matrix A is called **presentation matrix** for M.

This presentation matrix is clearly not unique and we can define an equivalence relation between presentation matrix as follows:

Definition A.2.9. Given A, A' presentation matrixes for a finitely presented module M, we say that A' is **equivalent** to A if it can be obtained from A by a finite sequence of the following moves (and their inverses):

1. Permuting rows or columns
2. Adjoining a column of zeros
3. Adding to a row (columns) a linear combination of the other rows (columns)
4. Adjoining a new row and column, such that the entry in the intersection of the new row and column is 1, and the remaining entries in the rows and column are all 0.

Proposition A.9. *Equivalent matrices present the same module.*

Proof. Consider two matrices A, A' representing the module M, corresponding, with respect to some basis, to the maps α, α' in the following presentation

$$\begin{tikzcd} F \arrow[r, "\alpha"] \arrow[d, shift left, "\gamma"] & E \arrow[r, two heads, "\phi"] \arrow[d, shift left, "\beta"] & M \arrow[r] \arrow[d, "Id"] & 0 \\ F' \arrow[r, "\alpha'"] \arrow[u, shift left, "\gamma'"] & E' \arrow[r, two heads, "\phi'"] \arrow[u, shift left, "\beta'"] & M \arrow[r] & 0 \end{tikzcd}$$

The basis of E and the surjectivity of ϕ' can be used to construct a linear map $\beta : E \longrightarrow E'$ such that $\phi'\beta = \phi$. Similarly, the freeness of F and the exactness of the sequence at E, E' produces a map $\gamma : F \longrightarrow F'$ such that $\beta\alpha = \alpha'\gamma$.
If β, γ are represented by matrices B, C with respect to the given bases, then $BA = A'C$. Analogously, a symmetrical argument, produces $\gamma' : F' \longrightarrow F$ and $\beta' : E' \longrightarrow E$, with matrices C', B' such that $B'A' = AC'$.
Denoting with $\sim$ the equivalence by the moves of Definition A.2.9, we obtain:

$$\begin{aligned} A &\sim \begin{pmatrix} A & B' \\ 0 & I \end{pmatrix} \quad \text{by move (4) and (3)} \\ &\sim \begin{pmatrix} A & B' & B'A' \\ 0 & I & A' \end{pmatrix} \quad \text{by move (2) and (3)} \\ &\sim \begin{pmatrix} A & B' & 0 \\ 0 & I & A' \end{pmatrix} \quad \text{by move (3) as } AC' = B'A' \\ &\sim \begin{pmatrix} A & B' & 0 & B'B \\ 0 & I & A' & B \end{pmatrix} \quad \text{by move (2) and (3)} \end{aligned}$$

Now, $\forall e \in E$, by construction of β' we have $\phi\beta'\beta(e) = \phi'\beta(e) = \phi(e)$ so, by the exactness at E, we get $(\beta'\beta - 1_E) \in \operatorname{Ker}\phi = \operatorname{Im}\alpha$. Because E is free, there is a map $\delta : E \longrightarrow F$ such that $\alpha\delta = \beta'\beta - 1_E$, thus, if D is a matrix representing δ, we have $AD = B'B - I$. Hence we obtain

$$A \sim \begin{pmatrix} A & B' & 0 & B'B \\ 0 & I & A' & B \end{pmatrix} \sim \begin{pmatrix} A & B' & 0 & I \\ 0 & I & A' & B \end{pmatrix} \sim \begin{pmatrix} A' & B & 0 & I \\ 0 & I & A & B' \end{pmatrix} \sim A'$$

where the first equivalence is obtained using (3) multiplying by D, the second is obtained by (1) and the last is obtained repeating the whole argument with the roles of A, A' interchanged. $\square$

Definition A.2.10. Let M be a finitely presented D-module and A a presentation matrix for M with m rows. The **r-th elementary ideal** $E_r(M)$ of M is the ideal in D generated by the determinant of the $(m-r+1) \times (m-r+1)$ minors of A.

Thanks to Proposition A.9 and to the properties of the determinant this definition is well posed, that is, the elementary ideal does not depend on the presentation matrix chosen. Moreover, by Laplace Theorem the elementary ideals of A form an ascending chain

$$0 = E_0(A) \subset E_1(A) \subset \cdots E_m(A) \subset E_{m+1}(A) = \cdots = D. \tag{A.6}$$

A.3 Bigraded modules

Consider the field $\mathbb{Z}_2$ with unit and consider the polynomial ring $R = \mathbb{Z}_2[V_1, \dots, V_n]$ in n formal variables $V_1, \dots, V_n$, including the case $n = 0$, so that $R = \mathbb{Z}_2$. In the present thesis, the action of the ring $R = \mathbb{Z}_2[V_1, \dots, V_n]$ changes the grading on the chain complex. In fact, quite often our complexes come naturally with two different gradings.

Definition A.3.1. A **bigraded *R*-module** M is an R-module, together with a splitting $M = \bigoplus_{m,a\in\mathbb{Z}} M_{m,a}$ as a K-modules, so that for each $i = 1, \dots, n$, the formal variable V_i maps $M_{m,a}$ into $M_{m-2,a-1}$.
A **bigraded R-module homomorphism** is a homomorphism $f : M \to M'$ between two bigraded R-modules that sends $M_{m,a}$ to $M'_{m,a}$ for all $m, a \in \mathbb{Z}$. More generally, an R-module homomorphism $f : M \to M'$ is said to be homogeneous of degree (q, t) if it sends $M_{m,a}$ to $M'_{m+q,a+t}$ for all $m, a \in \mathbb{Z}$.

The case where $n = 1$ will be of particular relevance to us. In this case, we write the algebra R simply as $K[U]$. When $n = 0$ and $K = \mathbb{Z}_2$, the bigraded modules are **bigraded vector spaces**.

Definition A.3.2. A **bigraded chain complex** over $\mathbb{Z}_2[V_1, \dots, V_n]$ is a chain complex (C, ∂) over $\mathbb{Z}_2$, equipped with endomorphisms $V_i : C \to C$ for $i = 1, \dots, n$, called the **algebra actions**, and a splitting $C = \bigoplus_{(m,a)\in\mathbb{Z}\oplus\mathbb{Z}} C_{m,a}$, satisfying the following compatibility conditions:

- for $i = 1, \ldots, n$, $\partial \circ V_i = V_i \circ \partial$;
- for all $i, j \in \{1, \ldots, n\}$, $V_i \circ V j = V_j \circ V_i$;
- ∂ maps $C_{m,a}$ to $C_{m-1,a}$;
- V_i maps $C_{m,a}$ to $C_{m-2,a-1}$.

In particular, if $n = 0$, the bigraded chain complex can be thought as a **bigraded vector space**

Definition A.3.3. Let X be a bigraded vector space, and fix integers a, b. The corresponding **shift** of X, denoted $X[\![a, b]\!]$ is the bigraded vector space that is isomorphic to X as a vector space and given the bigrading $X[\![a, b]\!]_{m,a} = X_{d+a,s+b}$

Let W be the two-dimensional bigraded vector space with one generator in bigrading $(0, 0)$ and another in bigrading $(-1, -1)$, and let X be any other bigraded vector space, then the tensor product $X \otimes W$ is identified with two copies of X, one of which is equipped with a shift in degree

$$X \otimes W \cong X \oplus X[\![1, 1]\!].$$

This can be iterated following the Pascal's triangle where i^{th} the power of W correspond to the $(i + 1)^{th}$ line of the triangle.

Definition A.3.4. If C and C' are bigraded complexes over R, then a **bigraded chain map** is a chain map f that maps $C_{m,a}$ into $C'_{m,a} \subset C'$.
More generally, if $f : C \to C'$ is a chain map, and there is a pair of integers (q, t) so that f maps $C_{m,a}$ into $C'_{m+q,a+t}$, then we say that f is a **homogeneous map with bidegree** (q, t).
An isomorphism of bigraded chain complexes is a bigraded chain map $f : (C, \partial) \to (C', \partial')$ for which there is another bigraded chain map $g : (C', \partial') \to (C, \partial)$ with $f \circ g = \mathrm{Id}_{C'}$ and $g \circ f = \mathrm{Id}_C$. If there is an isomorphism from (C, ∂) to (C', ∂'), we say that they are isomorphic bigraded chain complexes, and write $(C, \partial) \cong (C', \partial')$.

When C and C' are bigraded complexes over R and $f : C \to C'$ is a bigraded chain map, the induced map $H(f)$ respects the induced bigrading on the homology modules.
Given two bigraded chain maps $f : (C, \partial) \to (C', \partial')$ and $g : (C', \partial') \to (C'', \partial'')$, their composite $g \circ f$ is another bigraded chain map, whose induced map on homology satisfies:

$$H(g \circ f) = H(g) \circ H(f).$$

The identity map Id_C is a chain map that induces the identity map on $H(C, \partial)$. Consequently, an isomorphism between two chain complexes obviously induces an isomorphism between the corresponding homology modules. More generally:

Definition A.3.5. Let (C, ∂) and (C', ∂') be two chain complexes. A chain map $f : C \to C'$ is a **quasi-isomorphism** if it induces an isomorphism on homology. Two chain complexes (C, ∂) and (C', ∂') are said to be **quasi-isomorphic** if there is a chain complex (C'', ∂'') and two quasi-isomorphisms $f : C'' \to C$ and $g : C'' \to C'$. In cases where (C, ∂) and (C', ∂') are bigraded complexes over R, we require our quasi-isomorphisms to be bigraded maps over R.

Special kinds of quasi-isomorphisms are supplied by the following:

Definition A.3.6. Suppose that $f, g : (C, \partial) \to (C', \partial')$ be two bigraded chain maps between two bigraded chain complexes over R. The maps f and g are said to be **chain homotopic** if there is an R-module homomorphism $h : C \to C'$ that is homogeneous of degree $(1, 0)$, and that satisfies the formula

$$f - g = \partial' \circ h + h \circ \partial. \tag{A.7}$$

In this case, h is called a **chain homotopy** from g to f. More generally, if $f, g : (C, \partial) \to (C', \partial')$ are two chain maps that are homogeneous of degree (q, t), they are called chain homotopic if there is a map $h : C \to C'$ that is an R-module homomorphism homogeneous of degree $(q + 1, t)$ and satisfies Equation (A.7).

Lemma A.10. *Let (C, ∂) and (C', ∂') be chain complexes. Then chain homotopic maps from C to C' induce the same map in homology. Consequently, a chain homotopy equivalence is a quasi-isomorphism.*

Definition A.3.7. A chain map $f : C \to C'$ is a **chain homotopy equivalence** if there is a chain map $\varphi : C' \to C$, called a chain homotopy inverse to f, with the property that $f \circ \varphi$ and $\varphi \circ f$ are both chain homotopic to the respective identity maps. If there is a chain homotopy equivalence from C to C', then C and C' are said to be chain homotopy equivalent complexes.

A short exact sequence of chain complexes induces a long exact sequence on homology, according to the following lemma

Lemma A.11. *Let (C, ∂), (C', ∂'), and (C'', ∂'') be three bigraded chain complexes over $R = \mathbb{Z}_2[V_1, \dots, V_n]$. Suppose that $f : C \to C'$ is a chain map which is homogeneous of degree (m, t), and $g : C' \to C''$ is a bigraded chain map, both of which fit into a short exact sequence*

$$0 \longrightarrow C \xrightarrow{\quad f \quad} C' \xrightarrow{\quad g \quad} C'' \longrightarrow 0. \tag{A.8}$$

Then, there is a homomorphism of R-modules $\delta : H(C'') \to H(C)$ that is homogeneous of degree $(-q-1, -t)$, which fits into a long exact sequence

$$\longrightarrow H_{m-q,a-t}(C) \xrightarrow{H(f)} H_{m,a}(C') \xrightarrow{H(g)} H_{m,a}(C'') \xrightarrow{\quad \delta \quad} H_{m-q-1,a-t}(C) \longrightarrow \tag{A.9}$$

The proof of the above standard result can be found in [24, Lemma 4.5.3].

Remark 28. For example, if (C, ∂) is a bigraded chain complex over $R = \mathbb{Z}_2[V_1, \ldots, V_n]$ with $n \geq 1$, then multiplication by V_i for $i \in \{1, \ldots, n\}$ is a chain map $V_i : (C, \partial) \to (C, \partial)$. In this case, the quotient complex is denoted $\frac{C}{V_i}$, or more suggestively $C/\{V_i = 0\}$. This construction can be iterated; i.e. we can take the quotient of the chain complex by the map $V_j : \frac{C}{V_i} \to \frac{C}{V_i}$; the corresponding quotient will be denoted $\frac{C}{V_i = V_j = 0}$.

Proposition A.12. *Let C and C' be two bigraded chain complexes of $R = \mathbb{Z}_2[V_1, \ldots, V_n]$-modules. A chain map $f : C \to C'$, homogeneous of degree (q, t), naturally induces a chain map $\bar{f} : \frac{C}{V_i} \to \frac{C'}{V_i}$ that is also homogeneous of degree (q, t). Moreover, if g is another chain map that is homogeneous of degree (q, t), a chain homotopy h from f to g induces a chain homotopy $\bar{h}$ from $\bar{f}$ to $\bar{g}$.*

Lemma A.13. *Let C be a free chain complex over $\mathbb{Z}_2[V_1, \ldots, V_n]$, and suppose that multiplication by V_1 is chain homotopic to V_2. Then, there is a quasi-isomorphism of chain complexes over $\mathbb{F}[V_1, \ldots, V_n]$:*

$$C \otimes W \longrightarrow C/(V_1 - V_2).$$

Proposition A.14. *If C is a finitely generated chain complex over the ring $R = \mathbb{Z}_2[V_1, \ldots, V_n]$, then its homology $H(C)$ is also a finitely generated R-module.*

Consider now a bigraded module Y of bigradings (m, a).

Definition A.3.8. We define the **Poincaré polynomial** P_Y to be the Laurent polynomial in q and t

$$P_Y(q, t) = \sum_{m, a \in \mathbb{Z}} \dim Y_{m,a} \cdot q^m t^a.$$

Definition A.3.9. Let $X = \bigoplus_{m,a} X_{m,a}$ be a bigraded vector space. We define the **graded Euler characteristic of X** to be the Laurent polynomial in t given by

$$\chi(X) = \sum_{m,a} (-1)^m \dim X_{m,a} \cdot t^a. \tag{A.10}$$

Remark 29. Lemma A.6 remains valid for bigraded chain complexes.

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