Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE
Corso di Laurea Magistrale in Matematica

Torelli Theorem for K3 Surfaces

Complex Algebraic Geometry

Relatore:
Prof. Fatighenti Enrico

Presentata da:
Nasello Francesco

Anno Accademico 2024-2025

Introduction

The study of K3 surfaces plays a key role in complex algebraic geometry, lying at the crossroads of algebraic, differential, and Hodge-theoretic methods. Originally emerging in the Enriques–Kodaira classification of complex surfaces, K3 surfaces soon were recognized as the two-dimensional prototypes of a much richer class of varieties, namely compact hyperkähler manifolds.

The name "K3" was introduced by André Weil, reportedly inspired by an analogy with the mountain K2 and in honor of Kummer, Kähler, and Kodaira. From the algebro–geometric viewpoint, K3 surfaces are compact complex surfaces with trivial canonical bundle and vanishing first Betti number. The Kähler property follows from a theorem of Siu (see [Siu83]), which shows that any compact complex surface carrying a nowhere–degenerate holomorphic 2-form is necessarily Kähler. In complex dimension two, compact simply connected holomorphic symplectic manifolds coincide with K3 surfaces, making them the simplest examples of irreducible holomorphic symplectic manifolds.

A decisive step in understanding K3 surfaces was the development of Hodge theory. It gradually became clear that the second cohomology group of a K3 surface carries far more geometric information than one might a priori expect. In particular, its Hodge structure provides a cohomological encoding of the subtle features of the underlying complex surface.

These ideas found a precise formulation in the global Torelli theorem for K3 surfaces. Originally proved by Piatetski–Shapiro and Shafarevich, the statement was subsequently refined by Burns–Rapoport, Kulikov, and Todorov, leading to its modern form. At its core, this theorem establishes a correspondence between complex structures on a fixed differentiable manifold and suitable Hodge-theoretic data. Such a link is further strengthened by including the information provided by the Kähler cone.

Although the theorem bears the name of Torelli — in analogy with the classical case for algebraic curves — its proof for K3 surfaces relies on these subsequent advancements.

A modern exposition is available in Huybrechts' monograph [Huy16].

Beauville subsequently generalized the theory of K3 surfaces to higher dimensions by introducing hyperkähler manifolds [Bea83]. These varieties provide a natural extension of the two-dimensional case. The global Torelli theorem was later extended to this broader setting in Verbitsky's work [Ver13a], building on deep results on monodromy groups that were subsequently clarified by Markman. Expository accounts of hyperkähler geometry and its relation to the Torelli problem are available in the lecture notes of O'Grady [OGr13].

The aim of this thesis is to analyse K3 surfaces starting from their projective realizations and moving towards a global Hodge-theoretic perspective. A fundamental feature of these surfaces is that the group $H^2(X, \mathbb{Z})$ carries a natural lattice structure given by the cup–product pairing. The latter is determined by the underlying differentiable manifold and is therefore independent of the complex structure. By contrast, the Hodge decomposition varies holomorphically, and the line $H^{2,0}(X)$ provides a cohomological encoding of the geometry of the surface.

In this setting, the Néron–Severi lattice $\mathrm{NS}(X)$ significantly comes into play. Divisors on a K3 surface correspond to integral $(1,1)$–classes and generate $\mathrm{NS}(X)$, which captures the algebraic part of $H^2(X, \mathbb{Z})$. In this way, projective models of K3 surfaces naturally give rise to polarized surfaces whose geometric features are encoded in the corresponding lattice. In particular, along Noether–Lefschetz loci, the appearance of additional integral $(1,1)$–classes causes the Picard rank to jump. This phenomenon imposes lattice–theoretic constraints on the cohomology, directly influencing the structure of the associated moduli spaces.

To understand the dynamics of these geometric features, one is led to consider families of K3 surfaces and the associated variation of Hodge structure. The period map assigns to a marked surface the line $H^{2,0}(X)$ inside the fixed lattice $H^2(X, \mathbb{Z})$, translating the description of complex structures into the study of the geometry of a period domain.

The global Torelli theorem serves as the focal point of this thesis. It states that the Hodge structure on $H^2(X, \mathbb{Z})$ uniquely determines the isomorphism class of a marked K3 surface, provided that the positivity conditions of the Kähler cone are satisfied. In this light, the period map acts as the bridge between the geometric constructions of K3 surfaces and their moduli, providing a guiding principle for the developments presented in this work.

The exposition is organized as follows. We begin in Chapter 1 by establishing the geometric foundations of K3 surfaces. After recalling their fundamental properties, we examine several concrete constructions, including complete intersections in Fano varieties, double covers, elliptic fibrations, and Kummer surfaces. For each of these examples, we provide an explicit count of the parameters, showing how their geometric degrees of freedom relate to the Picard rank of the surface.

The second part of the thesis introduces the Hodge-theoretic framework. We describe the K3 lattice and its weight-two Hodge structure, leading to the construction of the period domain and the moduli space of marked surfaces. The analysis of the period map then provides the basis for the local Torelli theorem and the study of Noether-Lefschetz loci.

The final part is devoted to the global Torelli theorem. After discussing the Kähler cone and the global structure of the moduli space, we adopt a twistor-based approach to the proof. This perspective highlights the existence of Calabi-Yau metrics and provides a differential-geometric framework that complements classical arguments, ultimately leading to the proof of the global Torelli theorem.

Contents

Chapter 1

K3 surfaces

In this chapter we study K3 surfaces and present several explicit constructions and examples. The aim is to illustrate their basic geometric properties and to provide concrete models of K3 surfaces arising from different geometric constructions.

In complex dimension two, the notion of a K3 surface coincides with that of a hyperkähler manifold. Indeed, a K3 surface is a compact simply connected complex surface admitting a nowhere vanishing holomorphic 2-form. By [Siu83, Thm. 1], every such surface is Kähler, and therefore hyperkähler. Conversely, any hyperkähler surface necessarily has trivial canonical bundle and vanishing first Betti number, and hence is a K3 surface. For this reason, K3 surfaces provide the simplest and most accessible examples of hyperkähler manifolds.

Throughout this chapter, all varieties are defined over $\mathbb{C}$. By GAGA, we will freely identify smooth projective complex varieties with projective complex manifolds, and regular morphisms with holomorphic maps.

The presentation follows mainly [OGr12].

Definition 1.1 (Linearly normal projective variety)**.** Let $X \subset \mathbb{P}^N$ be an irreducible projective variety. We say that X is *linearly normal* if the natural restriction map

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \longrightarrow H^0(X, \mathcal{O}_X(1))$$

is surjective. Equivalently, the embedding $X \hookrightarrow \mathbb{P}^N$ is induced by the complete linear system $|\mathcal{O}_X(1)|$, i.e.

$$\mathbb{P}^N = \mathbb{P}\big(H^0(X, \mathcal{O}_X(1))^\vee\big).$$

Example 1.2. Consider the twisted cubic curve

$$\nu_3 : \mathbb{P}^1 \longrightarrow \mathbb{P}^3, \qquad [s : t] \longmapsto [s^3 : s^2t : st^2 : t^3].$$

Denote its image by $C = \nu_3(\mathbb{P}^1) \subset \mathbb{P}^3$. Then $\nu_3^*\mathcal{O}_C(1) \cong \mathcal{O}_{\mathbb{P}^1}(3)$, thus

$$H^0(C, \mathcal{O}_C(1)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3)).$$

X is linearly normal in $\mathbb{P}^3$.

If $p = [0:0:0:1] \notin C$, the projection

$$\pi_p : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2, \qquad [x_0 : x_1 : x_2 : x_3] \longmapsto [x_0 : x_1 : x_2],$$

induces the morphism

$$\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^2, \qquad [s:t] \longmapsto [s^3 : s^2t : st^2],$$

whose image

$$\widetilde{C} := \varphi(\mathbb{P}^1) = V(x_0x_2^2 - x_1^3) \subset \mathbb{P}^2$$

is a rational cubic curve that is not linearly normal, since the system inducing the embedding of $\widetilde{C}$ in $\mathbb{P}^2$ is generated by the three sections:

$$V = \mathrm{Span}\{s^3,\ s^2t,\ st^2\} \subset H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3)).$$

Hence the corresponding map

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \longrightarrow H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(1))$$

has image V, which is a proper subspace of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$. Therefore, the embedding of $\widetilde{C}$ in $\mathbb{P}^2$ is induced by an *incomplete* linear system.

Definition 1.3 (K3 surface)**.** A *K3 surface* is a compact (connected) surface with vanishing first Betti number and trivial canonical bundle.

Remark 1.4. It follows from [Siu83, Thm. 1] that any compact complex surface admitting a nowhere vanishing holomorphic 2-form is Kähler. In particular, every K3 surface is Kähler.

Remark 1.5 (Hodge diamond of a K3 surface). Let X be a complex K3 surface. By definition $K_X \cong \mathcal{O}_X$ and $b_1(X) = 0$. Hence $h^{1,0}(X) = 0$ and

$$h^{2,0}(X) = h^0(X, K_X) = 1.$$

It follows that

$$\chi(\mathcal{O}_X) = 1 - h^{1,0}(X) + h^{2,0}(X) = 2.$$

By Noether's formula,

$$\chi(\mathcal{O}_X) = \frac{K_X^2 + e(X)}{12}.$$

Since $K_X^2 = 0$, one obtains $e(X) = 24$.

Using Poincaré duality and $b_1(X) = 0$, we have

$$e(X) = b_0 - b_1 + b_2 - b_3 + b_4 = 2 + b_2,$$

so $b_2(X) = 22$. By Hodge decomposition,

$$b_2(X) = h^{2,0}(X) + h^{1,1}(X) + h^{0,2}(X),$$

and therefore $h^{1,1}(X) = 20$.

The Hodge diamond of X is thus

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1. \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Definition 1.6 (Canonical linear system and canonical curve)**.** Let C be a smooth projective curve of genus $g \geq 2$. The *canonical linear system* of C is the complete linear system

$$|K_C| = \mathbb{P}\big(H^0(C, K_C)\big),$$

where K_C is the canonical line bundle of C. The corresponding morphism

$$\varphi_{|K_C|} : C \longrightarrow \mathbb{P}^{g-1}$$

is called the *canonical map* of C. If this morphism is an embedding, its image

$$\varphi_{|K_C|}(C) \subset \mathbb{P}^{g-1}$$

is called the *canonical curve* associated with C.

Lemma 1.7. *Let $X \subset \mathbb{P}^g$ be a linearly normal K3 surface, and denote by $L := \mathcal{O}_X(1)$ the restriction of the hyperplane bundle. Then every smooth hyperplane section of X is a canonical curve of genus g.*

Proof. Since $X \subset \mathbb{P}^g$ is linearly normal, the embedding $X \hookrightarrow \mathbb{P}^g$ is induced by the complete linear system $|L|$. In particular, the associated morphism is

$$\varphi_{|L|} \colon X \longrightarrow \mathbb{P}^{h^0(X,L)-1},$$

and it identifies X with its image in $\mathbb{P}^g$. Hence

$$h^0(X,L) - 1 = g \quad \Longrightarrow \quad h^0(X,L) = g+1.$$

Because X is a K3 surface, we have $K_X \cong \mathcal{O}_X$ and $H^1(X,\mathcal{O}_X) = 0$, so $\chi(\mathcal{O}_X) = 2$. For an ample line bundle L on a K3 surface, Riemann–Roch gives

$$\chi(L) = \chi(\mathcal{O}_X) + \tfrac{1}{2}c_1(L) \cdot \big(c_1(L) - c_1(K_X)\big) = 2 + \tfrac{1}{2}L^2.$$

Using Serre duality and $K_X \cong \mathcal{O}_X$, we have

$$h^2(X,L) = h^0(X, K_X \otimes L^\vee) = h^0(X, L^\vee) = 0,$$

since L is ample, and moreover $h^1(X,L) = 0$ by Kodaira vanishing. Hence

$$h^0(X,L) = \chi(L) = 2 + \tfrac{1}{2}c_1(L)^2.$$

Combining this with $h^0(X,L) = g+1$, we obtain

$$g + 1 = 2 + \tfrac{1}{2}c_1(L)^2, \qquad \text{i.e.} \qquad L^2 = 2g-2.$$

Let $H \subset \mathbb{P}^g$ be a hyperplane and set $C := X \cap H$. For a general choice of H, Bertini's theorem implies that $C = X \cap H$ is a smooth irreducible curve on X (see [Har77, Cor. III.10.9]). As a divisor on X, we have $C \in |L|$, and the restriction of L to C is

$$L|_C \cong \mathcal{O}_C(1).$$

By the adjunction formula, for a smooth curve $C \subset X$ one has

$$K_C \cong (K_X \otimes \mathcal{O}_X(C))\big|_C.$$

Since X is a K3 surface, $K_X \cong \mathcal{O}_X$, and $C \in |L|$, so

$$K_C \cong \mathcal{O}_X(C)\big|_C \cong L|_C \cong \mathcal{O}_C(1).$$

Taking degrees, we get

$$\begin{aligned} \deg K_C &= \int_C c_1(K_C) = \int_C c_1(L|_C), \\ &= \int_X c_1(L) \smile [C] = \int_X c_1(L)^2 = 2g-2. \end{aligned}$$

The adjunction formula for curves also gives

$$\deg K_C = 2g(C) - 2,$$

where $g(C)$ is the genus of C. Comparing, we obtain

$$2g(C) - 2 = 2g - 2 \quad \Longrightarrow \quad g(C) = g.$$

Thus $K_C \cong \mathcal{O}_C(1)$ and $g(C) = g$, so

$$h^0(C, K_C) = g.$$

The hyperplane embedding $C \hookrightarrow H \cong \mathbb{P}^{g-1}$ is induced by the sections of $\mathcal{O}_C(1)$, and this linear system has dimension $g-1$, which equals $\dim |K_C|$. Hence the hyperplane system coincides with the complete canonical linear system $|K_C|$, and the embedding

$$C \hookrightarrow \mathbb{P}^{g-1}$$

is the canonical embedding.

Therefore every smooth hyperplane section C of X is a canonical curve of genus g. □

Definition 1.8 (Néron–Severi group and Picard number)**.** Let X be a smooth projective variety. The *Néron–Severi group* of X is the image of the first Chern class map

$$c_1 : \mathrm{Pic}(X) \longrightarrow H^2(X, \mathbb{Z}),$$

and is denoted by

$$\mathrm{NS}(X) := \mathrm{Im}(c_1).$$

Its rank

$$\rho(X) := \mathrm{rank}\, \mathrm{NS}(X)$$

is called the *Picard number* of X. Two line bundles (or divisors) are said to be *numerically equivalent* if they have the same first Chern class in $H^2(X, \mathbb{Z})$.

Remark 1.9. From the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0,$$

one obtains the long exact sequence in cohomology

$$\cdots \to H^1(X, \mathcal{O}_X) \to \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to \ldots.$$

The kernel of c_1 is denoted by $\mathrm{Pic}^0(X)$ and satisfies

$$\mathrm{Pic}^0(X) \cong H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}),$$

so that $\mathrm{Pic}^0(X)$ has the structure of a complex torus. It follows that

$$\mathrm{NS}(X) \cong \mathrm{Pic}(X)/\mathrm{Pic}^0(X).$$

The group $\mathrm{NS}(X)$ is finitely generated but may contain torsion.

Definition 1.10 (Polarization)**.** A *polarization* on a smooth projective variety X is an ample line bundle L on X, or equivalently its first Chern class $c_1(L) \in \mathrm{NS}(X)$. The pair (X, L) is called a *polarized variety.* When X is a K3 surface, the integer

$$g = 1 + \tfrac{1}{2}L^2$$

is called the *genus* of the polarization L.

1.1 K3 surfaces as complete intersections in Fano varieties

Definition 1.11 (Fano variety)**.** A *Fano variety* is a smooth projective variety V such that its anticanonical divisor $-K_V$ is ample.

The *index* of a Fano variety V is the largest positive integer r such that there exists an ample divisor class $H \in \mathrm{Pic}(V)$ satisfying

$$-K_V \sim rH.$$

Remark 1.12. For a smooth projective variety V and any ample line bundle L on V, Kodaira vanishing gives

$$H^i(V,\, K_V \otimes L) = 0 \qquad \text{for all } i > 0.$$

If V is a smooth Fano variety, then $-K_V$ is ample. Taking $L = -K_V$, we obtain

$$H^1(V, \mathcal{O}_V) = 0.$$

By the exponential exact sequence, the first Chern class map

$$c_1\colon \mathrm{Pic}(V) \longrightarrow H^2(V, \mathbb{Z})$$

is injective. Moreover, smooth Fano varieties over $\mathbb{C}$ are simply connected (see [Kol96, Thm. 4.13]); hence $H_1(V, \mathbb{Z}) = 0$. By the universal coefficient theorem, this implies that $H^2(V, \mathbb{Z})$ is torsion-free. Consequently, the Picard group $\mathrm{Pic}(V)$ is torsion-free and is canonically isomorphic to the Néron–Severi group $\mathrm{NS}(V)$.

Therefore the condition

$$-K_V \sim rH$$

means that the anticanonical divisor is divisible by r in $\mathrm{Pic}(V)$ (or equivalently in $\mathrm{NS}(V)$) in a well-defined sense.

Remark 1.13. If the Picard number of V is $\rho(V) = 1$, then $\mathrm{Pic}(V) \cong \mathbb{Z}$ and is generated by an ample divisor H whose class is primitive. In this case the expression

$$-K_V \sim rH$$

determines H uniquely up to linear equivalence; the divisor class H is called the *fundamental divisor* of V and will be referred to as the *ambient polarization*.

Example 1.14 (Computation of index). We compute the index of some smooth Fano varieties with Picard number strictly greater than one.

1. Let $V = \mathbb{P}^n \times \mathbb{P}^m$, and denote by

   $$H_1 = p_1^*\mathcal{O}_{\mathbb{P}^n}(1), \qquad H_2 = p_2^*\mathcal{O}_{\mathbb{P}^m}(1)$$

   the pull-backs of the hyperplane classes. Then

   $$\mathrm{Pic}(V) \cong \mathbb{Z}H_1 \oplus \mathbb{Z}H_2, \qquad -K_V = (n+1)H_1 + (m+1)H_2.$$

   Hence the index of V is

   $$r(V) = \gcd(n+1, m+1).$$

2. Let

   $$V = F(k_1, k_2; n) = \{\, 0 \subset U_{k_1} \subset U_{k_2} \subset \mathbb{C}^n \,\}, \qquad 1 \le k_1 < k_2 < n,$$

   be the variety parametrizing chains of linear subspaces of dimensions k_1 and k_2.

   There are two natural morphisms

   $$\pi_1 \colon V \longrightarrow \mathrm{Gr}(k_1, n), \qquad \pi_2 \colon V \longrightarrow \mathrm{Gr}(k_2, n),$$

obtained by forgetting U_{k_2} and U_{k_1}, respectively. The fibres of π_1 are isomorphic to $\mathrm{Gr}(k_2 - k_1, n - k_1)$, while the fibres of π_2 are isomorphic to $\mathrm{Gr}(k_1, k_2)$. Hence both morphisms realise V as a Grassmann bundle.

For a Grassmann bundle $\mathrm{Gr}_Y(r, E) \to Y$, the Picard group is given by

$$\mathrm{Pic}(\mathrm{Gr}_Y(r, E)) \cong \pi^*\mathrm{Pic}(Y) \oplus \mathbb{Z}\,[\mathcal{O}(1)],$$

where $\mathcal{O}(1)$ denotes the line bundle inducing the Plücker embedding on the fibres. Since $\mathrm{Pic}(\mathrm{Gr}(k, n)) \cong \mathbb{Z}$, it follows that

$$\mathrm{Pic}(V) \cong \mathbb{Z}H_1 \oplus \mathbb{Z}H_2,$$

where

$$H_1 := \pi_1^*\mathcal{O}_{\mathrm{Gr}(k_1,n)}(1), \qquad H_2 := \pi_2^*\mathcal{O}_{\mathrm{Gr}(k_2,n)}(1).$$

The canonical divisor of V can be computed by viewing V as a Grassmann bundle over $\mathrm{Gr}(k_2, n)$, namely

$$V \cong \mathrm{Gr}_{\mathrm{Gr}(k_2,n)}(k_1, \mathcal{S}_2),$$

where $\mathcal{S}_2$ denotes the universal (tautological) rank-k_2 subbundle on $\mathrm{Gr}(k_2, n)$. Applying the standard formula

$$K_{\mathrm{Gr}_Y(r,E)} \cong \pi^* K_Y \otimes \pi^*(\det E)^{\,\mathrm{rk}(E)-r} \otimes \mathcal{O}_\pi(-\mathrm{rk}(E)),$$

one obtains, after a straightforward simplification,

$$-K_V \sim k_2\, H_1 + (n - k_1)\, H_2.$$

Therefore the index of V is

$$r(V) = \gcd(k_2,\ n - k_1).$$

3. Let $V = \mathrm{Bl}_{\mathbb{P}^k}\mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ along a linear subspace $\mathbb{P}^k$, with $0 \le k \le n-2$. Denote by H the pull-back of the hyperplane class on $\mathbb{P}^n$ and by E the exceptional divisor. Then

$$\mathrm{Pic}(V) \cong \mathbb{Z}H \oplus \mathbb{Z}E, \qquad -K_V = (n+1)H - (n-k-1)E.$$

We claim that V is Fano, namely that $-K_V$ is ample. Since $\rho(V) = 2$, Kleiman's ampleness criterion (see [Har77, Thm. III.4.7]) reduces the problem to checking positivity of $-K_V$ on the following two curve classes.

Let ℓ be the strict transform of a line in $\mathbb{P}^n$ not meeting $\mathbb{P}^k$, and let f be a line contained in a fiber of the exceptional divisor $E \to \mathbb{P}^k$. Then

$$H \cdot \ell = 1, \quad E \cdot \ell = 0, \qquad H \cdot f = 0, \quad E \cdot f = -1,$$

hence

$$(-K_V) \cdot \ell = n + 1 > 0, \qquad (-K_V) \cdot f = n - k - 1 > 0,$$

where the last inequality follows from $k \leq n-2$. Therefore $-K_V$ is ample and V is Fano. In particular, its index is

$$r(V) = \gcd(n+1, n-k-1).$$

Remark 1.15. Let V be a smooth n-dimensional Fano variety with Picard number $\rho(V) = 1$ and index r; that is,

$$-K_V \sim rH,$$

where H is an ample divisor whose class is primitive in $\mathrm{Pic}(V)$.

Let $a_1, \dots, a_{n-2}$ be strictly positive integers such that

$$a_1 + \cdots + a_{n-2} = r.$$

Suppose that for each $i = 1, \dots, n-2$ we have $D_i \in |a_i H|$, and that these divisors intersect transversely, so that

$$X := D_1 \cap \cdots \cap D_{n-2}$$

is a smooth surface.

By the adjunction formula,

$$K_X \cong \big(K_V \otimes \mathcal{O}_V(D_1 + \cdots + D_{n-2})\big)\big|_X.$$

Since $D_i \sim a_i H$ and $-K_V \sim rH$ with $\sum a_i = r$, we have

$$K_V + D_1 + \cdots + D_{n-2} \sim 0,$$

hence $K_X \cong \mathcal{O}_X$.

As V is Fano, its anticanonical bundle $-K_V$ is ample. By the Kodaira vanishing theorem, one has

$$H^i(V, \mathcal{O}_V) = 0 \quad \text{for all } i > 0.$$

In particular, $H^1(V, \mathcal{O}_V) = 0$, hence $b_1(V) = 2h^{1,0}(V) = 0$. By the Lefschetz hyperplane theorem for smooth ample complete intersections, the natural map $H^i(V, \mathbb{Z}) \to H^i(X, \mathbb{Z})$ is an isomorphism for $i < 2$, and thus

$$b_0(X) = b_0(V) = 1, \qquad b_1(X) = b_1(V) = 0.$$

Therefore X is a K3 surface.

Example 1.16 (K3 surfaces as complete intersections in Fano varieties). The following table lists some smooth K3 surfaces that can be obtained as complete intersections in Fano varieties of index r, according to their genus

$$g = \tfrac{1}{2}L^2 + 1,$$

where $L := H|_X$ denotes the polarization induced on X by the ambient ample divisor.

V	$\dim(V)$	K_V	X	**genus** g
$\mathbb{P}^3$	3	$\mathcal{O}_{\mathbb{P}^3}(-4)$	Quartic surface ($d = 4$)	3
$\mathbb{P}^4$	4	$\mathcal{O}_{\mathbb{P}^4}(-5)$	$(2,3)$ complete intersection	4
$\mathbb{P}^5$	5	$\mathcal{O}_{\mathbb{P}^5}(-6)$	$(2,2,2)$ complete intersection	5
$\mathrm{Gr}(2,5)$	6	$\mathcal{O}_{\mathrm{Gr}(2,5)}(-5)$	$(1,1,1,2)$ complete intersection	6
$\mathrm{Gr}(2,6)$	8	$\mathcal{O}_{\mathrm{Gr}(2,6)}(-6)$	Linear section of codim 6	8

Remark 1.17 (Computation of the genus). We explain how to compute the genus

$$g = 1 + \tfrac{1}{2}L^2 = 1 + \tfrac{1}{2}\int_X c_1(L)^2$$

for the K3 surfaces appearing in the table.

For the quartic surface $X \subset \mathbb{P}^3$, we have $V = \mathbb{P}^3$, $H = \mathcal{O}_{\mathbb{P}^3}(1)$ and $[X] = 4H$. Then

$$\begin{aligned} L^2 &= \int_X c_1(L)^2 = \int_{\mathbb{P}^3} c_1(H)^2 \smile [X] \\ &= \int_{\mathbb{P}^3} H^2 \cdot 4H = 4 \int_{\mathbb{P}^3} H^3 = 4, \end{aligned}$$

so

$$g = 1 + \tfrac{1}{2}L^2 = 1 + \tfrac{1}{2} \cdot 4 = 3.$$

The two complete intersections $(2,3) \subset \mathbb{P}^4$ and $(2,2,2) \subset \mathbb{P}^5$ are computed in the same way. If H denotes the hyperplane class, then the fundamental classes are

$$[X] = (2H)(3H) = 6H^2, \qquad [X] = (2H)^3 = 8H^3.$$

It follows that

$$L^2 = 6,\ g = 4 \qquad \text{and} \qquad L^2 = 8,\ g = 5,$$

respectively.

For the Grassmannian cases we use the known intersection numbers $\int_{\mathrm{Gr}(2,5)} H^6 = 5$ and $\int_{\mathrm{Gr}(2,6)} H^8 = 14$ (see [EH16, Ch. 14]).

For $X \subset \mathrm{Gr}(2,5)$ of type $(1,1,1,2)$, we have

$$[X] = H \cdot H \cdot H \cdot (2H) = 2H^4,$$

and

$$\begin{aligned} L^2 &= \int_X c_1(L)^2 = \int_{\mathrm{Gr}(2,5)} c_1(H)^2 \smile [X] \\ &= \int_{\mathrm{Gr}(2,5)} H^2 \cdot 2H^4 = 2\int_{\mathrm{Gr}(2,5)} H^6 = 10, \end{aligned}$$

so

$$g = 1 + \tfrac{1}{2}L^2 = 1 + \tfrac{1}{2} \cdot 10 = 6.$$

For $X \subset \mathrm{Gr}(2,6)$ given by a linear section of codimension 6, we have

$$[X] = H^6,$$

and

$$\begin{aligned} L^2 &= \int_X c_1(L)^2 = \int_{\mathrm{Gr}(2,6)} c_1(H)^2 \smile [X] \\ &= \int_{\mathrm{Gr}(2,6)} H^2 \cdot H^6 = \int_{\mathrm{Gr}(2,6)} H^8 = 14, \end{aligned}$$

so

$$g = 1 + \tfrac{1}{2}L^2 = 1 + \tfrac{1}{2} \cdot 14 = 8.$$

1.2 K3 surfaces as double covers

Let Y be a complex manifold and let L be a holomorphic line bundle on Y. By abuse of notation, we write L also for the total space of the line bundle. Let

$$p\colon L \longrightarrow Y$$

be the natural projection.

Definition 1.18 (Squaring map)**.** The *squaring map* is the holomorphic map

$$m\colon L \longrightarrow L^{\otimes 2}$$

defined fibrewise by

$$m(\ell) = \ell \otimes \ell.$$

Remark 1.19. If $U \subset Y$ is an open subset over which L is trivial, and we identify $L|_U \cong U \times \mathbb{C}$, then under this trivialization the squaring map is

$$m(y, w) = (y, w^2).$$

Definition 1.20 (Double cover and branch divisor)**.** Assume that there exists a nonzero section

$$\sigma \in H^0(Y, L^{\otimes 2}).$$

Let

$$X := m^{-1}(\sigma(Y)) \subset L, \qquad \pi := p|_X \colon X \longrightarrow Y.$$

The map π is called the *double cover associated with* σ, and the divisor

$$B := \operatorname{div}(\sigma) \subset Y$$

is called its *branch divisor.*

Remark 1.21. Let $U \subset Y$ be an open subset over which L is trivial and write $\sigma|_U = f$. Then

$$X \cap p^{-1}(U) = \{(y, w) \in U \times \mathbb{C} \mid w^2 = f(y)\}.$$

Proposition 1.22. *The map $\pi\colon X \to Y$ is a finite holomorphic map of degree 2, étale*[1] *over $Y \setminus B$ and ramified along B.*

Proof. Let $y \in Y$ and choose a neighbourhood U of y over which L is trivial. Writing $\sigma|_U = f$, the space X is locally defined by

$$F(y, w) = w^2 - f(y) = 0.$$

If $y \notin B$, then $f(y) \neq 0$, hence the equation $w^2 = f(y)$ has two distinct solutions and $w \neq 0$ at every point of $\pi^{-1}(y)$. Therefore

$$\frac{\partial F}{\partial w}(y, w) = 2w \neq 0,$$

[1] A holomorphic map between smooth complex varieties is étale if its differential is everywhere invertible; equivalently it is a local biholomorphism.

and by the holomorphic implicit function theorem the projection π is a local biholomorphism. Thus π is étale over $Y \setminus B$.

If $y \in B$, then $f(y) = 0$ and the unique point in the fibre is $(y, 0)$. At this point $\partial F/\partial w = 0$, so π is ramified along $\pi^{-1}(B)$. □

Proposition 1.23. *The space X is smooth if and only if the branch divisor B is smooth.*

Proof. Let $(y, w) \in X$ and let $U \subset Y$ be an open subset over which L is trivial. Write $\sigma|_U = f$, so that $X \cap p^{-1}(U) \subset U \times \mathbb{C}$ is locally defined by

$$F(y, w) = w^2 - f(y) = 0.$$

A point $(y, w) \in X$ is singular if and only if

$$\frac{\partial F}{\partial w}(y, w) = 0 \qquad \text{and} \qquad df_y = 0.$$

The first condition implies $w = 0$, while the second means that y is a singular point of $B = \{f = 0\}$. □

Definition 1.24 (Ramification divisor)**.** Assume that B is smooth. Let $R \subset X$ be the reduced divisor supported on $\pi^{-1}(B)$, that is, a divisor whose irreducible components all appear with multiplicity one. It is called the *ramification divisor.*

Proposition 1.25. *Assume that the branch divisor B is smooth and let $R \subset X$ be the ramification divisor. Then there is a natural isomorphism of line bundles*

$$\mathcal{O}_X(R) \cong \pi^* L.$$

Proof. Consider the total space of the line bundle L and the projection $p \colon L \to Y$. There exists a tautological holomorphic section

$$\tau \in H^0(L, p^* L),$$

defined by $\tau(\ell) = \ell$ for every $\ell \in L$.

Restricting τ to the subspace $X \subset L$, we obtain a holomorphic section

$$\tau|_X \in H^0(X, \pi^* L).$$

We claim that the zero divisor of $\tau|_X$ coincides with the ramification divisor R.

Indeed, let $U \subset Y$ be an open subset over which L is trivial and write $\sigma|_U = f$. On $p^{-1}(U) \cong U \times \mathbb{C}$, the section τ is represented by the coordinate function w, and X is locally given by the equation

$$w^2 = f(y).$$

Hence $\tau|_X$ vanishes precisely along $\{w = 0\}$, which is equal to $\pi^{-1}(B) \cap p^{-1}(U)$. Since B is smooth, this vanishing occurs with multiplicity one, and therefore $\mathrm{div}(\tau|_X) = R$.

It follows that $\tau|_X$ defines an isomorphism

$$\mathcal{O}_X(R) \cong \pi^* L,$$

as claimed. □

Remark 1.26. Since $\pi\colon X \to Y$ is finite with ramification divisor R, the Hurwitz formula yields

$$K_X \cong \pi^* K_Y \otimes \mathcal{O}_X(R).$$

Using the isomorphism $\mathcal{O}_X(R) \cong \pi^* L$, one obtains

$$K_X \cong \pi^*(K_Y \otimes L).$$

Lemma 1.27. *Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_Y$-modules over $\mathbb{C}$, endowed with an involution $\varphi\colon \mathcal{F} \to \mathcal{F}$ such that $\varphi^2 = \mathrm{id}$. Then $\mathcal{F}$ decomposes as a direct sum*

$$\mathcal{F} = \mathcal{F}^+ \oplus \mathcal{F}^-,$$

where

$$\mathcal{F}^+ := \ker(\varphi - \mathrm{id}), \qquad \mathcal{F}^- := \ker(\varphi + \mathrm{id}).$$

The subsheaves $\mathcal{F}^+$ and $\mathcal{F}^-$ are called the $(+1)$-eigensheaf and the (-1)-eigensheaf of φ, respectively.

Reference. See [AM69, Ch. 2]. □

Proposition 1.28 (Decomposition of the structure sheaf)**.** *There is a natural decomposition*

$$\pi_* \mathcal{O}_X \cong \mathcal{O}_Y \oplus L^{-1},$$

corresponding to the $(+1)$ and (-1) eigensheaves for the action of the covering involution $\iota\colon X \to X$ defined fibrewise by $\ell \mapsto -\ell$.

Proof. Let $U \subset Y$ be an open subset over which L is trivial, and write $\sigma|_U = f$. Then

$$\mathcal{O}_X(\pi^{-1}(U)) \;\cong\; \mathcal{O}_Y(U)[w]/(w^2 - f).$$

The covering involution ι acts by $w \mapsto -w$. Applying Lemma 1.27 to the induced action of ι on $\pi_*\mathcal{O}_X$, we obtain a decomposition into invariant and anti-invariant parts. The invariant part is $\mathcal{O}_Y(U)$, while the anti-invariant part is generated by w. Since w transforms as a local generator of L^{-1}, this identifies

$$(\pi_*\mathcal{O}_X)|_U \cong \mathcal{O}_U \oplus L^{-1}|_U.$$

Gluing over Y yields the claimed decomposition. □

Proposition 1.29. *For every $p \geq 0$ one has*

$$H^p(X, \mathcal{O}_X) \cong H^p(Y, \mathcal{O}_Y) \oplus H^p(Y, L^{-1}).$$

Proof. Let $\{U_i\}_{i\in I}$ be an open Stein covering of Y. Since π is finite, $\{\pi^{-1}(U_i)\}_{i\in I}$ is an open Stein covering of X. The Čech complexes computing $H^p(X, \mathcal{O}_X)$ and $H^p(Y, \pi_*\mathcal{O}_X)$ are canonically identified, hence

$$H^p(X, \mathcal{O}_X) \cong H^p(Y, \pi_*\mathcal{O}_X).$$

Using the decomposition of $\pi_*\mathcal{O}_X$, the result follows. □

Remark 1.30 (Construction of K3 surfaces as double covers). Let Y be a smooth projective surface such that

$$H^1(Y, \mathcal{O}_Y) = 0,$$

and assume that there exists a smooth divisor

$$B \in |-2K_Y|.$$

Let $L := -K_Y$, so that $L^{\otimes 2} \cong \mathcal{O}_Y(B)$, and let

$$\pi \colon X \to Y$$

be the associated double cover branched along B.

Since B is smooth, the surface X is smooth. By the canonical bundle formula for double covers one has

$$K_X \cong \pi^*(K_Y \otimes L).$$

As $L = -K_Y$, this gives

$$K_X \cong \mathcal{O}_X.$$

Moreover, using the decomposition

$$\pi_* \mathcal{O}_X \cong \mathcal{O}_Y \oplus L^{-1}$$

and Serre duality, we obtain

$$H^1(X, \mathcal{O}_X) \cong H^1(Y, \mathcal{O}_Y) \oplus H^1(Y, K_Y) = 0.$$

Therefore X is a K3 surface.

We recall the following definitions concerning positivity properties of line bundles on surfaces.

Definition 1.31 (Nef, big and primitive line bundles)**.** Let X be a smooth projective surface and let L be a line bundle on X.

1. L is said to be *nef* if
$$L \cdot C \geq 0$$
for every irreducible curve $C \subset X$.

2. L is said to be *big* if
$$L^2 > 0.$$

3. L is said to be *primitive* if $c_1(L)$ is primitive in $\mathrm{NS}(X)$, i.e. the following implication holds:
$$c_1(L) = m\beta \in \mathrm{NS}(X),\ m \in \mathbb{Z} \implies m = \pm 1.$$

Example 1.32. 1. Let $Y = \mathbb{P}^2$. The canonical bundle of $\mathbb{P}^2$ is

$$K_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-3),$$

hence

$$-2K_Y \cong \mathcal{O}_{\mathbb{P}^2}(6).$$

Choosing a smooth divisor

$$B \in |\mathcal{O}_{\mathbb{P}^2}(6)|,$$

i.e. a smooth plane sextic curve, we obtain a smooth double cover

$$\pi\colon X \to \mathbb{P}^2$$

branched along B. By the previous remark, the surface X is a K3 surface. Such a surface is usually called a *double plane*. Let $H := \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$. Since $H^2 = 2$, the Riemann–Roch formula on a K3 surface gives

$$\chi(mH) = \frac{1}{2}(mH)^2 + 2 = m^2 + 2.$$

As H is nef and big, one has $h^1(mH) = h^2(mH) = 0$ for all $m \geq 1$, hence

$$h^0(mH) = m^2 + 2.$$

In particular, $h^0(H) = 3$, $h^0(2H) = 6$, and $h^0(3H) = 11$. The graded ring

$$R(X, H) := \bigoplus_{m \geq 0} H^0(X, mH)$$

is therefore generated by three elements of degree 1 and one additional generator of degree 3. Consequently, X is realized as a hypersurface in the weighted projective space $\mathbb{P}(1, 1, 1, 3)$.

Moreover, since the first relation in $R(X, H)$ occurs in degree 6, the defining equation of $X \subset \mathbb{P}(1, 1, 1, 3)$ has weighted degree 6, and X is given by an equation of the form

$$w^2 = f_6(x_0, x_1, x_2),$$

where f_6 is a homogeneous polynomial of degree 6.

2. Let $Y = \mathbb{P}^1 \times \mathbb{P}^1$. Its canonical bundle is

$$K_Y \cong \mathcal{O}_Y(-2, -2),$$

so that

$$-2K_Y \cong \mathcal{O}_Y(4, 4).$$

Choosing a smooth divisor

$$B \in |\mathcal{O}_Y(4, 4)|,$$

the associated double cover

$$\pi\colon X \to Y$$

is smooth and, by the previous construction, is a K3 surface. This type of K3 surface is often referred to as a *double quadric*.

1.3 Elliptic K3 surfaces

Definition 1.33 (Elliptic fibration)**.** Let X be a K3 surface. An *elliptic fibration* on X is a regular non-constant morphism

$$f\colon X \longrightarrow T$$

to a smooth connected curve T such that the fibre over a non-critical value is a smooth curve of genus 1. A K3 surface is said to be *elliptic* if it admits an elliptic fibration.

Remark 1.34. Let X be a K3 surface and let

$$f\colon X \longrightarrow T$$

be an elliptic fibration. Then necessarily $T \cong \mathbb{P}^1$. Indeed, the pull-back of holomorphic 1-forms induces an injection

$$f^*\colon H^0(T, \Omega^1_T) \hookrightarrow H^0(X, \Omega^1_X).$$

Since $H^0(X, \Omega^1_X) = 0$ for a K3 surface, it follows that $H^0(T, \Omega^1_T) = 0$. Hence the genus of T is zero, and therefore $T \cong \mathbb{P}^1$.

On the other hand, let $f\colon X \to \mathbb{P}^1$ be a regular dominant morphism with connected fibres. Let F be a smooth fibre of f. Then F is a smooth curve and one has $F^2 = 0$, since distinct fibres are disjoint and linearly equivalent. By the adjunction formula,

$$K_F \cong (K_X \otimes \mathcal{O}_X(F))|_F.$$

Taking degrees yields

$$2g(F) - 2 = \deg K_F = (K_X + F) \cdot F.$$

Since X is a K3 surface, $K_X \cong \mathcal{O}_X$, and $F^2 = 0$, one has $2g(F) - 2 = 0$, hence $g(F) = 1$.

Remark 1.35. Let X be a K3 surface and let $L \in \mathrm{Pic}(X)$ be a nef and effective divisor such that $L^2 = 0$. By the Riemann–Roch formula on X,

$$\chi(L) = \tfrac{1}{2}L^2 + 2 = 2,$$

hence $h^0(X, L) = 2$ and the complete linear system $|L|$ is a pencil. Since L is nef, $|L|$ has no fixed components and therefore defines a morphism

$$\varphi_{|L|}\colon X \longrightarrow \mathbb{P}^1.$$

By the previous remark, the general fibre of $\varphi_{|L|}$ has genus 1, so $\varphi_{|L|}$ is an elliptic fibration on X.

Conversely, if $f\colon X \to \mathbb{P}^1$ is an elliptic fibration and F denotes the class of a fibre, then distinct fibres are disjoint and linearly equivalent, hence

$$F^2 = 0.$$

Example 1.36. Let $X \subset \mathbb{P}^1 \times \mathbb{P}^2$ be a smooth divisor of bidegree $(2, 3)$. We denote by H_1 and H_2 the pull-backs of the hyperplane classes from $\mathbb{P}^1$ and $\mathbb{P}^2$, respectively. Then

$$X \in |2H_1 + 3H_2|.$$

By the adjunction formula, the canonical bundle of X is

$$K_X \cong (K_{\mathbb{P}^1\times\mathbb{P}^2} + 2H_1 + 3H_2)|_X.$$

Since $K_{\mathbb{P}^1\times\mathbb{P}^2} \cong -2H_1 - 3H_2$, it follows that $K_X \cong \mathcal{O}_X$, and hence X is a K3 surface.

Consider the projection

$$p_1\colon X \longrightarrow \mathbb{P}^1.$$

For a point $t \in \mathbb{P}^1$, the fibre $X_t = p_1^{-1}(t)$ is a plane curve of degree 3, hence a smooth curve of genus 1 for t general. Therefore p_1 defines an elliptic fibration on X.

Example 1.37 (A pencil $|H - \ell|$ on a quartic K3). Let $X \subset \mathbb{P}^3$ be a smooth quartic surface, and let $H := \mathcal{O}_X(1)$ be the hyperplane class. Assume that X contains a line $\ell \subset X$. This assumption forces the Picard rank of X to be at least 2, since the classes of H and of the line ℓ are linearly independent in $\mathrm{Pic}(X)$. Consider the linear system $|H - \ell|$.

The divisor class $H - \ell$ is effective and its self-intersection is given by

$$(H - \ell)^2 = H^2 - 2H \cdot \ell + \ell^2.$$

Here $H^2 = 4$, since X has degree 4, and $H \cdot \ell = 1$, because ℓ is a line in $\mathbb{P}^3$. Moreover, since $\ell \cong \mathbb{P}^1$ and $K_X \cong \mathcal{O}_X$, the adjunction formula gives

$$2g(\ell) - 2 = (K_X + \ell) \cdot \ell,$$

hence $\ell^2 = -2$. Therefore

$$(H - \ell)^2 = 4 - 2 \cdot 1 + (-2) = 0.$$

The linear system $|H-\ell|$ consists of hyperplane sections of X containing ℓ. Since the hyperplanes of $\mathbb{P}^3$ containing a fixed line form a $\mathbb{P}^1$, the system $|H-\ell|$ is a pencil. Moreover, ℓ is a fixed component of the hyperplane sections containing it, hence $|H-\ell|$ has no base points. Therefore $|H-\ell|$ defines a morphism

$$\varphi = \varphi_{|H-\ell|}\colon X \longrightarrow \mathbb{P}^1.$$

Let $\{s_0, s_1\}$ be a basis of $H^0(X, \mathcal{O}_X(H-\ell))$. For $[a:b] \in \mathbb{P}^1$, let $H_{[a:b]} \subset \mathbb{P}^3$ be the hyperplane containing ℓ corresponding to the section $as_0 + bs_1$. Then the hyperplane section of X decomposes as

$$H_{[a:b]} \cap X = \ell + C_{[a:b]},$$

where $C_{[a:b]}$ is the residual curve. In particular, $C_{[a:b]} \in |H-\ell|$ and

$$\varphi^{-1}([a:b]) = C_{[a:b]}.$$

For a general point $[a:b] \in \mathbb{P}^1$, the curve $C_{[a:b]}$ is smooth. By adjunction,

$$2g(C_{[a:b]}) - 2 = (K_X + C_{[a:b]}) \cdot C_{[a:b]} = C_{[a:b]}^2 = 0,$$

hence $g(C_{[a:b]}) = 1$. Therefore φ is an elliptic fibration on X.

Definition 1.38 (Jacobian elliptic fibration)**.** Let $f\colon X \to \mathbb{P}^1$ be an elliptic fibration on a smooth projective surface. The fibration f is said to be *Jacobian* if it admits a section.

Remark 1.39. If an elliptic fibration admits a section, then the section intersects each fibre in exactly one point. Equivalently, denoting by F the numerical class of a fibre, a section corresponds to a divisor $S \subset X$ such that

$$S \cdot F = 1.$$

In this case, each smooth fibre carries a distinguished point and hence a natural structure of an elliptic curve as a group.

In general, an elliptic fibration need not admit a section. When no section exists, the general fibre is still a smooth curve of genus 1, but there is no canonical choice of a base point on the fibres, and the fibration is not Jacobian.

Remark 1.40. In the example of a smooth quartic surface $X \subset \mathbb{P}^3$ containing a line ℓ, the pencil $|H-\ell|$ defines an elliptic fibration on X. For a general such surface, this elliptic fibration does not admit a section, and hence it is not Jacobian. This phenomenon will reappear below and shows that elliptic K3 surfaces may carry elliptic fibrations with or without section.

1.3.1 An elliptic K3 from a double cover of $\mathbb{F}_4$

Definition 1.41 (Hirzebruch surface)**.** For an integer $n \geq 0$, the *Hirzebruch surface* $\mathbb{F}_n$ is the rational ruled surface

$$\mathbb{F}_n := \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)\big).$$

Denote by

$$p\colon \mathbb{F}_n \longrightarrow \mathbb{P}^1$$

the natural projection, by F the numerical class of a fibre of p, and by

$$\Sigma := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}) \subset \mathbb{F}_n$$

the distinguished section.

Remark 1.42. Let

$$E := \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n).$$

A point of $\mathbb{P}(E)$ can be described as a pair (t, ℓ), where $t \in \mathbb{P}^1$ and $\ell \subset E_t$ is a one-dimensional vector subspace. The natural projection

$$p\colon \mathbb{P}(E) \to \mathbb{P}^1$$

sends such a pair to the point t.

Since E splits as a direct sum, there is a canonical inclusion

$$\mathcal{O}_{\mathbb{P}^1} \hookrightarrow E$$

given fibrewise by inclusion into the first summand. For each $t \in \mathbb{P}^1$, this determines a distinguished one-dimensional subspace

$$\mathcal{O}_{\mathbb{P}^1,t} \subset E_t,$$

and hence a distinguished point of the fibre $p^{-1}(t) \cong \mathbb{P}^1$. In this way one obtains a morphism

$$s\colon \mathbb{P}^1 \longrightarrow \mathbb{P}(E), \qquad t \longmapsto (\mathcal{O}_{\mathbb{P}^1,t} \subset E_t),$$

satisfying $p \circ s = \mathrm{id}_{\mathbb{P}^1}$.

We denote by

$$\Sigma := s(\mathbb{P}^1) \subset \mathbb{P}(E)$$

the image of this morphism. Equivalently, Σ can be described as the projectivization

$$\mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1}\big) \subset \mathbb{P}(E).$$

Indeed, for a fixed $t \in \mathbb{P}^1$, the fibre $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1})_t$ consists of the unique one-dimensional subspace of $\mathcal{O}_{\mathbb{P}^1,t}$, which is canonically included in E_t. Thus the two descriptions define the same subset Σ of $\mathbb{P}(E)$.

Remark 1.43. The intersection numbers of the basic divisor classes on $\mathbb{F}_n$ are

$$F^2 = 0, \qquad \Sigma \cdot F = 1, \qquad \Sigma^2 = -n.$$

Indeed, $F^2 = 0$ since distinct fibres are disjoint, and $\Sigma \cdot F = 1$ because Σ meets each fibre transversely in one point. Moreover, the normal bundle of Σ in $\mathbb{F}_n$ is

$$\mathcal{N}_{\Sigma/\mathbb{F}_n} \;\cong\; \mathcal{O}_{\mathbb{P}^1}(-n).$$

Therefore

$$\Sigma^2 = \deg \mathcal{N}_{\Sigma/\mathbb{F}_n} = -n.$$

Lemma 1.44. *Let $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$, with projection $p\colon \mathbb{F}_n \to \mathbb{P}^1$. Then*

$$K_{\mathbb{F}_n} \;\sim\; -2\Sigma - (n+2)F.$$

Proof. Let $E := \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$, so that $\mathbb{F}_n = \mathbb{P}(E)$. For a projective bundle one has the canonical bundle formula

$$K_{\mathbb{P}(E)} \;\cong\; \mathcal{O}_{\mathbb{P}(E)}(-2) \otimes p^*(K_{\mathbb{P}^1} \otimes \det E).$$

Here $\det E \cong \mathcal{O}_{\mathbb{P}^1}(-n)$ and $K_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$, hence

$$K_{\mathbb{F}_n} \;\cong\; \mathcal{O}_{\mathbb{F}_n}(-2) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(-n-2).$$

Denote $\xi := c_1(\mathcal{O}_{\mathbb{F}_n}(1))$. Then $c_1(\mathcal{O}_{\mathbb{F}_n}(-2)) = -2\xi$ and $c_1(p^*\mathcal{O}_{\mathbb{P}^1}(1)) = F$, so numerically

$$K_{\mathbb{F}_n} \;\equiv\; -2\xi - (n+2)F.$$

On the other hand, the section Σ corresponding to the subbundle $\mathcal{O}_{\mathbb{P}^1} \hookrightarrow E$ satisfies

$$\mathcal{O}_{\mathbb{F}_n}(\Sigma) \;\cong\; \mathcal{O}_{\mathbb{F}_n}(1),$$

hence $\Sigma \equiv \xi$. Substituting gives

$$K_{\mathbb{F}_n} \;\equiv\; -2\Sigma - (n+2)F,$$

as claimed. □

Example 1.45 (Double cover of $\mathbb{F}_4$ giving an elliptic K3 with section). Let

$$B \in |-2K_{\mathbb{F}_4}| = |4\Sigma + 12F|$$

be a smooth divisor and let $\pi\colon X \to \mathbb{F}_4$ be the double cover ramified along B. Then X is a K3 surface (as in the general double cover construction), and the composition

$$f := p \circ \pi\colon X \longrightarrow \mathbb{P}^1$$

is an elliptic fibration.

Moreover, one can choose B in such a way that f admits a section. Namely, one can arrange

$$B = \Sigma + B_0, \qquad B_0 \in |3\Sigma + 12F|$$

with B_0 smooth and disjoint from Σ. Indeed, note that

$$\Sigma \cdot (3\Sigma + 12F) = 3\Sigma^2 + 12(\Sigma \cdot F) = 3(-4) + 12 \cdot 1 = 0,$$

so it is numerically possible for B_0 to avoid Σ. With this choice, the curve Σ is a reduced component of the branch divisor. Consequently, its preimage $\pi^{-1}(\Sigma)$ is a smooth reduced curve, and the restriction

$$\pi|_{\pi^{-1}(\Sigma)}\colon \pi^{-1}(\Sigma) \to \Sigma$$

is an isomorphism. Identifying $\Sigma \cong \mathbb{P}^1$ via $p|_\Sigma$, the curve $\pi^{-1}(\Sigma)$ therefore defines a section of the elliptic fibration $f = p \circ \pi$.

Remark 1.46 (Geometric production of B_0). One way to obtain such a smooth $B_0 \in |3\Sigma + 12F|$ disjoint from Σ is the following. Let $H \subset \mathbb{P}^5$ be a hyperplane and let $C \subset H$ be a rational normal quartic curve. Choose a point $p \in \mathbb{P}^5 \setminus H$ and let $\mathcal{C} \subset \mathbb{P}^5$ be the cone over C with vertex p. Let $\rho\colon \widetilde{\mathcal{C}} \to \mathcal{C}$ be the blow-up of the vertex. Then

$$\widetilde{\mathcal{C}} \;\cong\; \mathbb{F}_4, \qquad \rho^*\mathcal{O}_{\mathcal{C}}(1) \;\cong\; \mathcal{O}_{\mathbb{F}_4}(\Sigma + 4F).$$

If $V \subset \mathbb{P}^5$ is a cubic hypersurface intersecting $\mathcal{C}$ transversely (in particular V does not contain p), then

$$B_0 := \rho^*(V|_{\mathcal{C}}) \in |3(\Sigma + 4F)| = |3\Sigma + 12F|$$

is smooth and can be chosen disjoint from Σ.

Remark 1.47. The elliptic K3 surfaces in Example 1.37 and Example 1.45 are genuinely different: in the second construction we can ensure the existence of a section, while for a general quartic containing a line the fibration $|H - L|$ has no section.

1.4 Kummer surfaces

Definition 1.48 (Abelian surface)**.** An *abelian surface* is a smooth projective surface T such that

$$K_T \cong \mathcal{O}_T \qquad \text{and} \qquad h^1(T, \mathcal{O}_T) = 2.$$

Remark 1.49. The category of abelian varieties over $\mathbb{C}$ is equivalent to the category of projective complex tori. In particular, abelian surfaces can be identified with projective complex tori of dimension 2, that is, quotients

$$T \cong V/\Lambda,$$

where $V \cong \mathbb{C}^2$ and $\Lambda \subset V$ is a lattice of rank 4 such that V/Λ is projective. See [GH78, Chapter 2, §6].

From this description one can compute the Hodge numbers. Since

$$H^k(T, \mathbb{C}) \cong \wedge^k H^1(T, \mathbb{C})$$

and $h^{1,0}(T) = h^{0,1}(T) = 2$, it follows that

$$h^{2,0}(T) = h^{0,2}(T) = 1, \qquad h^{1,1}(T) = 4.$$

Therefore the Hodge diamond of an abelian surface is

$$\begin{matrix} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{matrix}$$

Let T be an abelian surface over $\mathbb{C}$.

Definition 1.50 (Singularity of type A_1)**.** Let X be a normal surface over $\mathbb{C}$ and let $p \in X$ be a singular point. We say that p is a *singularity of type* A_1 if the completed local ring $\widehat{\mathcal{O}}_{X,p}$ is isomorphic, as a $\mathbb{C}$-algebra, to

$$\mathbb{C}[\![x, y, z]\!]/(xy - z^2).$$

Proposition 1.51. *Let T be an abelian surface and let*

$$\iota\colon T \to T, \qquad x \mapsto -x$$

be the involution given by inversion. Then the quotient

$$K(T) := T/\langle\iota\rangle$$

is a normal projective surface with exactly 16 *singular points, all of which are singularities of type* A_1.

Proof. The involution ι is induced by the group inversion on the abelian surface T.

A point $x \in T$ is fixed by ι if and only if $x = -x$, equivalently $2x = 0$. Hence the fixed locus is the finite subgroup

$$T[2] := \{\, x \in T \mid 2x = 0 \,\}.$$

Since T is an abelian surface, one has

$$T[2] \cong (\mathbb{Z}/2\mathbb{Z})^4,$$

so $T[2]$ consists of exactly 16 points. The action of ι is free on $T \setminus T[2]$, hence the quotient $K(T)$ is smooth away from the images of the points of $T[2]$. Therefore $K(T)$ has exactly 16 singular points.

Let $x_0 \in T[2]$ and let $p \in K(T)$ be its image. Since T is smooth, we may choose étale local coordinates (u, v) on T centered at x_0 such that, in these coordinates, the involution ι acts as

$$(u, v) \longmapsto (-u, -v).$$

Since the group $\{\pm 1\}$ is finite, the completed local ring of the quotient is the invariant subring of the completed local ring upstairs. Hence

$$\widehat{\mathcal{O}}_{K(T),p} \;\cong\; \widehat{\mathcal{O}}_{T,x_0}^{\iota} \;\cong\; \mathbb{C}[\![u, v]\!]^{\{\pm 1\}},$$

where the involution acts by $u \mapsto -u$, $v \mapsto -v$.

Set

$$x := u^2, \qquad y := v^2, \qquad z := uv.$$

These elements are $\{\pm 1\}$-invariant and satisfy the relation $xy - z^2 = 0$. As before, every invariant monomial is a polynomial in u^2, v^2, and uv. Therefore

$$\widehat{\mathcal{O}}_{K(T),p} \;\cong\; \mathbb{C}[\![x, y, z]\!]/(xy - z^2),$$

and the point p is a singularity of type A_1. □

Definition 1.52 (Kummer surface)**.** The *Kummer surface associated with T* is the smooth surface

$$\widetilde{K}(T)$$

obtained by blowing up the 16 singular points of $K(T)$.

Remark 1.53 (Blow-up of an A_1-singularity)*.* Let

$$X := \{\, xy - z^2 = 0 \,\} \subset \mathbb{C}^3$$

and let $0 \in X$ be its unique singular point. Consider the blow-up

$$\pi\colon \mathrm{Bl}_0\mathbb{C}^3 \longrightarrow \mathbb{C}^3$$

of the origin, with exceptional divisor

$$E_{\mathbb{C}^3} := \pi^{-1}(0) \cong \mathbb{P}^2.$$

Denote by $\widetilde{X} \subset \mathrm{Bl}_0\mathbb{C}^3$ the strict transform of X.

The blow-up $\mathrm{Bl}_0\mathbb{C}^3$ is covered by three affine charts corresponding to $X \neq 0$, $Y \neq 0$, and $Z \neq 0$ in homogeneous coordinates $[X : Y : Z]$ on $\mathbb{P}^2$.

On the chart $X \neq 0$, set

$$u := \frac{Y}{X}, \qquad v := \frac{Z}{X}.$$

The defining equations of the blow-up give

$$y = xu, \qquad z = xv.$$

Substituting into the equation of X, one obtains

$$xy - z^2 = x(xu) - (xv)^2 = x^2(u - v^2).$$

Thus the total transform of X is defined by $x^2(u - v^2) = 0$, and the strict transform $\widetilde{X}$ is given by the equation

$$u - v^2 = 0,$$

which defines a smooth surface in this chart.

On the chart $Y \neq 0$, writing

$$u' := \frac{X}{Y}, \qquad v' := \frac{Z}{Y},$$

one has

$$x = yu', \qquad z = yv',$$

and the equation of X becomes

$$y^2(u' - v'^2) = 0.$$

Hence the strict transform is given by $u' - v'^2 = 0$, which is smooth.

On the chart $Z \neq 0$, writing

$$u'' := \frac{X}{Z}, \qquad v'' := \frac{Y}{Z},$$

one has

$$x = zu'', \qquad y = zv'',$$

and the equation of X becomes

$$z^2(u''v'' - 1) = 0.$$

Thus the strict transform is given by $u''v'' - 1 = 0$, which is smooth.

These local computations show that $\widetilde{X}$ is smooth.

The exceptional divisor $E_{\mathbb{C}^3} \cong \mathbb{P}^2$ has homogeneous coordinates $[X : Y : Z]$ corresponding to the directions of (x, y, z). The exceptional curve

$$E := \widetilde{X} \cap E_{\mathbb{C}^3}$$

is given by the homogeneous degree–2 part of the equation of X, namely

$$E = \{\, XY - Z^2 = 0 \,\} \subset \mathbb{P}^2.$$

This is a smooth conic, hence $E \cong \mathbb{P}^1$.

To compute the self-intersection of E, we compute its normal bundle. On the chart $X \neq 0$, the curve E is given by $x = 0$ and is parametrized by the coordinate v, so x defines a local generator of $\mathcal{N}_{E/\widetilde{X}}$. On the chart $Y \neq 0$, the curve E is given by $y = 0$ and is parametrized by v', so y defines a local generator.

On the overlap one has $y = xu$, and restricting to E gives $y = xv^2$. Hence the transition function of $\mathcal{N}_{E/\widetilde{X}}$ is multiplication by v^2, which shows that

$$\mathcal{N}_{E/\widetilde{X}} \cong \mathcal{O}_{\mathbb{P}^1}(-2).$$

Therefore $E^2 = -2$. Thus, resolving an A_1-singularity replaces the singular point by a smooth rational curve with self-intersection -2.

Remark 1.54. Let $\pi\colon \widetilde{T} \to T$ be the blow-up of T at the 16 points of $T[2]$. The involution ι lifts to an involution

$$\widetilde{\iota}\colon \widetilde{T} \to \widetilde{T},$$

which acts trivially on each exceptional divisor. The quotient

$$\widetilde{T}/\langle\widetilde{\iota}\rangle$$

is naturally isomorphic to $\widetilde{K}(T)$. Thus the Kummer surface can be obtained either by first taking the quotient and then blowing up, or by first blowing up and then taking the quotient.

Remark 1.55. If T is projective, then $\widetilde{T}$ is projective, hence $\widetilde{K}(T)$ is projective and in particular Kähler.

Theorem 1.56. *Let T be a two-dimensional complex torus. Then the associated Kummer surface $\widetilde{K}(T)$ is a K3 surface.*

Proof. The surface $\widetilde{K}(T)$ is connected and Kähler.

Let $\pi\colon \widetilde{T} \to T$ be the blow-up of T at the 16 points of $T[2]$. Since the blow-up of a complex surface at finitely many points does not change the first integral cohomology group, one has

$$H^1(\widetilde{T}, \mathbb{Z}) \cong H^1(T, \mathbb{Z}).$$

The involution $\iota\colon T \to T$, $x \mapsto -x$, lifts to an involution $\widetilde{\iota}\colon \widetilde{T} \to \widetilde{T}$. The induced action of $\widetilde{\iota}$ on $H^1(\widetilde{T}, \mathbb{Z})$ coincides with the action of ι on $H^1(T, \mathbb{Z})$. Since ι acts as multiplication by -1 on $H^1(T, \mathbb{Z})$, it follows that

$$H^1(\widetilde{T}, \mathbb{Z})^{\widetilde{\iota}} = 0.$$

Let $\rho\colon \widetilde{T} \to \widetilde{K}(T)$ be the quotient map. For a finite group action one has a natural injection

$$H^1(\widetilde{K}(T), \mathbb{Z}) \hookrightarrow H^1(\widetilde{T}, \mathbb{Z})^{\widetilde{\iota}}.$$

Therefore

$$H^1(\widetilde{K}(T), \mathbb{Z}) = 0,$$

and hence $b_1(\widetilde{K}(T)) = 0$.

We now show that the canonical bundle of $\widetilde{K}(T)$ is trivial. Since T is a complex torus, its canonical bundle is trivial and there exists a nowhere vanishing holomorphic 2-form

$$\tau \in H^0(T, \Omega_T^2).$$

Pulling back via the blow-up map, we obtain a holomorphic 2-form

$$\widetilde{\tau} := \pi^*\tau \in H^0(\widetilde{T}, \Omega^2_{\widetilde{T}}),$$

which is not identically zero.

The involution ι acts as -1 on holomorphic 1-forms on T, and therefore acts trivially on holomorphic 2-forms. Consequently, $\widetilde{\tau}$ is invariant under the lifted involution $\widetilde{\iota}$. It follows that there exists a holomorphic 2-form

$$\sigma \in H^0(\widetilde{K}(T), \Omega^2_{\widetilde{K}(T)})$$

such that

$$\rho^*\sigma = \widetilde{\tau}.$$

The form σ is non-zero outside the exceptional locus. Let $R_1, \ldots, R_{16} \subset \widetilde{K}(T)$ be the exceptional curves arising from the resolution of the singular points of $K(T)$. We can write the divisor of zeros of σ as

$$\operatorname{div}(\sigma) = \sum_{i=1}^{16} a_i R_i, \qquad a_i \geq 0.$$

Since σ is a section of the canonical bundle, one has

$$K_{\widetilde{K}(T)} \sim \sum_{i=1}^{16} a_i R_i.$$

For each i, the curve R_i is a smooth rational curve with self-intersection $R_i^2 = -2$. By the adjunction formula,

$$2g(R_i) - 2 = (K_{\widetilde{K}(T)} + R_i) \cdot R_i.$$

Since $g(R_i) = 0$, this gives

$$-2 = (K_{\widetilde{K}(T)} + R_i) \cdot R_i = a_i R_i^2 + R_i^2 = (a_i + 1)(-2),$$

hence $a_i = 0$ for all i.

Therefore $\operatorname{div}(\sigma) = 0$, and σ is nowhere vanishing. It follows that

$$K_{\widetilde{K}(T)} \cong \mathcal{O}_{\widetilde{K}(T)}.$$

Since $\widetilde{K}(T)$ is a compact Kähler surface with vanishing first Betti number and trivial canonical bundle, it is a K3 surface. □

1.5 Dimension of the parameter spaces

We conclude this chapter by estimating, for each of the constructions above, the number of *effective parameters*, namely the dimension of the corresponding family of K3 surfaces after taking into account the natural equivalences built into the construction (such as projective automorphisms of the ambient variety).

The number of effective parameters is determined by the dimension of the relevant moduli space and by the Picard number of a very general member of the family. Indeed, for polarized K3 surfaces the Picard number measures how many independent divisor classes are forced by the construction, and therefore controls the dimension of the corresponding family inside the moduli space.

In the following we compute the effective number of parameters case by case, and we collect the results in a summary table.

Remark 1.57. Recall that

$$h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \binom{n+d}{d}, \qquad \dim \mathrm{PGL}_{n+1} = (n+1)^2 - 1.$$

In the examples below, when the construction produces a family of projective K3 surfaces whose very general member has Picard number $\rho(X) = 1$, the effective dimension is obtained by subtracting the dimension of the relevant automorphism group of the ambient space from the dimension of the parameter space of defining equations.

Example 1.58 (Quartic surfaces in $\mathbb{P}^3$). A quartic surface is defined by a homogeneous polynomial of degree 4 in 4 variables, up to scaling. Hence the parameter space is

$$\mathbb{P}\big(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4))\big), \qquad \dim |\mathcal{O}_{\mathbb{P}^3}(4)| = \binom{7}{4} - 1 = 35 - 1 = 34.$$

Quotienting by projectivities, we subtract $\dim \mathrm{PGL}_4 = 15$, and obtain

$$34 - 15 = 19$$

effective parameters.

Example 1.59 (Complete intersections $(2,3) \subset \mathbb{P}^4$). A surface $X \subset \mathbb{P}^4$ of type $(2,3)$ is the complete intersection of a quadric Q and a cubic C, that is,

$$X = V(Q) \cap V(C),$$

with

$$Q \in H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)), \qquad C \in H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)).$$

The space of quadrics in $\mathbb{P}^4$ has dimension

$$h^0(\mathbb{P}^4, \mathcal{O}(2)) = \binom{6}{4} = 15,$$

so $|\mathcal{O}_{\mathbb{P}^4}(2)|$ has dimension 14.

Fix a smooth quadric threefold $Y := V(Q)$. Geometrically, once Q is fixed, the surface X is a divisor of type $\mathcal{O}_Y(3)$ on Y, and two cubics define the same divisor on Y if and only if they differ by $Q \cdot L$ for some linear form L.

This parameter count can be expressed intrinsically in terms of X. Since X is a smooth complete intersection of type $(2,3)$, one has

$$\mathcal{N}_{X/\mathbb{P}^4} \cong \mathcal{O}_X(2) \oplus \mathcal{O}_X(3),$$

hence the space of defining equations modulo the natural relations is

$$H^0\big(X, \mathcal{O}_X(2) \oplus \mathcal{O}_X(3)\big).$$

To compute its dimension, we use the Koszul resolution of $\mathcal{O}_X$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-5) \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-2) \oplus \mathcal{O}_{\mathbb{P}^4}(-3) \longrightarrow \mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Twisting by $\mathcal{O}_{\mathbb{P}^4}(2)$ and taking global sections, using $H^1(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(-k)) = 0$ for $k > 0$, we obtain

$$h^0(X, \mathcal{O}_X(2)) = 15 - 1 = 14.$$

Similarly, twisting by $\mathcal{O}_{\mathbb{P}^4}(3)$ yields

$$h^0(X, \mathcal{O}_X(3)) = 35 - (1 + 5) = 29.$$

Therefore

$$h^0\big(X, \mathcal{O}_X(2) \oplus \mathcal{O}_X(3)\big) = 14 + 29 = 43,$$

which coincides with the dimension of the parameter space before quotienting by PGL_5. Finally,

$$43 - 24 = 19$$

effective parameters.

Example 1.60 (Complete intersections $(2,2,2) \subset \mathbb{P}^5$). A complete intersection $X \subset \mathbb{P}^5$ of type $(2,2,2)$ is cut out by three independent quadrics, namely by a 3-dimensional linear subspace of $H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2))$, which has dimension

$$h^0(\mathbb{P}^5, \mathcal{O}(2)) = \binom{7}{5} = 21.$$

Thus the parameter space is the Grassmannian $\mathrm{Gr}(3,21)$, of dimension $3(21-3)=54$.

This parameter count can again be expressed intrinsically in terms of X. Since X is a smooth complete intersection of three quadrics, its normal bundle splits as

$$\mathcal{N}_{X/\mathbb{P}^5} \cong \mathcal{O}_X(2)^{\oplus 3}.$$

Hence the space of defining equations modulo the natural relations is

$$H^0\big(X, \mathcal{O}_X(2)^{\oplus 3}\big).$$

To compute its dimension, consider the Koszul resolution of $\mathcal{O}_X$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^5}(-6) \longrightarrow \mathcal{O}_{\mathbb{P}^5}(-4)^{\oplus 3} \longrightarrow \mathcal{O}_{\mathbb{P}^5}(-2)^{\oplus 3} \longrightarrow \mathcal{O}_{\mathbb{P}^5} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Twisting by $\mathcal{O}_{\mathbb{P}^5}(2)$ and taking global sections, using $H^1(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(-k))=0$ for $k>0$, we obtain

$$0 \to H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}) \to H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2))^{\oplus 3} \to H^0\big(X, \mathcal{O}_X(2)^{\oplus 3}\big) \to 0,$$

hence

$$h^0\big(X, \mathcal{O}_X(2)^{\oplus 3}\big) = 3\cdot 21 - 3 = 63 - 9 = 54.$$

Therefore the dimension of the parameter space is 54, in agreement with the Grassmannian description. Finally, quotienting by PGL_6 (whose dimension is 35) gives

$$54 - 35 = 19$$

effective parameters.

Example 1.61 (Double planes). A double cover $\pi\colon X \to \mathbb{P}^2$ is uniquely determined by its branch divisor $B \in |\mathcal{O}_{\mathbb{P}^2}(6)|$. Therefore, in order to determine the number of parameters of the K3 surface X, it is sufficient to count the parameters of the branch curve B.

The surface X is smooth if and only if the branch sextic B is smooth. The locus of smooth sextic curves is a Zariski open and dense subset of $|\mathcal{O}_{\mathbb{P}^2}(6)|$.

Hence

$$\dim|\mathcal{O}_{\mathbb{P}^2}(6)| = \binom{8}{2} - 1 = 28 - 1 = 27.$$

Quotienting by PGL_3 (dimension 8) yields

$$27 - 8 = 19$$

effective parameters.

Example 1.62 (Divisors of type $(2,3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$). Let $X \subset \mathbb{P}^1 \times \mathbb{P}^2$ be a smooth divisor of type $(2,3)$, that is,

$$X \in \left|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(2,3)\right|.$$

Since

$$K_{\mathbb{P}^1\times\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(-2,-3),$$

adjunction gives $K_X \cong \mathcal{O}_X$, so X is a K3 surface.

By the Künneth formula,

$$h^0\big(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}(2,3)\big) = h^0(\mathbb{P}^1, \mathcal{O}(2)) \cdot h^0(\mathbb{P}^2, \mathcal{O}(3)) = 3 \cdot 10 = 30,$$

so the linear system has dimension 29. Quotienting by

$$\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^2) = \mathrm{PGL}_2 \times \mathrm{PGL}_3,$$

whose dimension is $3 + 8 = 11$, we obtain

$$29 - 11 = 18$$

effective parameters.

Moreover, the Picard group of $\mathbb{P}^1 \times \mathbb{P}^2$ is generated by the two pullbacks of the hyperplane classes, and their restrictions to X remain independent. Hence a very general member of this family satisfies

$$\rho(X) = 2.$$

Example 1.63 (Kummer surfaces). A Kummer surface $\widetilde{K}(T)$ is determined by the isomorphism class of the underlying two-dimensional complex torus T. The moduli space of complex 2-tori has dimension 4, hence Kummer surfaces form a 4-dimensional family.

Remark 1.64. All the remaining constructions in this chapter impose the existence of at least one additional algebraic class in the Néron–Severi group besides the polarization. Equivalently, they correspond to families of polarized K3 surfaces with Picard number

$$\rho(X) \geq 2.$$

By the Noether–Lefschetz theorem, the locus of polarized K3 surfaces with Picard number at least 2 has codimension at least one inside the 19-dimensional moduli space. Consequently, these families depend on at most

$$19 - 1 = 18$$

effective parameters.

This applies for instance to:

- quartic K3 surfaces containing a line and endowed with the elliptic pencil $|H - \ell|$ (Example 1.37);
- double covers of $\mathbb{F}_4$ admitting a section (Example 1.45);
- elliptic K3 surfaces with a prescribed section.

We collect in the following table all the examples of K3 surfaces discussed in this chapter, indicating the Picard number of a very general member and the number of effective parameters.

K3 surface	$\rho(X)$	**Effective parameters**
Quartic surface in $\mathbb{P}^3$	1	19
$(2,3)$ complete intersection in $\mathbb{P}^4$	1	19
$(2,2,2)$ complete intersection in $\mathbb{P}^5$	1	19
$(1,1,1,2)$ complete intersection in $\mathrm{Gr}(2,5)$	1	19
Linear section of codim 6 of $\mathrm{Gr}(2,6)$	1	19
Double plane (smooth sextic branch)	1	19
Double quadric (smooth $(4,4)$ branch)	≥ 2	18
Quartic surface containing a line	≥ 2	18
Elliptic K3 from the pencil $\lvert H - \ell \rvert$	≥ 2	18
Double cover of $\mathbb{F}_4$ admitting a section	≥ 2	18
Kummer surface $\widetilde{K}(T)$	≥ 17	4

Chapter 2

Periods and local Torelli Theorems for K3 surfaces

This chapter develops the period theory of K3 surfaces and the deformation–theoretic interpretation of their Hodge structures, culminating in the infinitesimal and local Torelli theorems.

For a K3 surface X, the Hodge structure on $H^2(X, \mathbb{Z})$ encodes a large part of the geometry of X. The period map associates to a marked K3 surface (X, ϕ) the complex line $H^{2,0}(X)$ inside the K3 lattice, viewed as a point in the period domain Ω_Λ. A central question is to what extent the complex structure of a K3 surface is determined by this Hodge data.

The infinitesimal Torelli theorem describes the differential of the period map in terms of deformation theory and the Gauss–Manin connection developed in Appendix A and Appendix B. For K3 surfaces, this leads to the local Torelli theorem, stating that the period map is locally a biholomorphism.

We also introduce Noether–Lefschetz loci in the period domain and use deformation theory to study families of K3 surfaces and their Néron–Severi lattices. These tools will be applied later in the study of isomorphisms of K3 surfaces and the Torelli theorem.

The exposition builds on [OGr12] and [Voi02].

2.1 Lattices and period domains

Definition 2.1 (Lattice)**.** A *lattice* is a free abelian group L of finite rank endowed with a non-degenerate symmetric bilinear form

$$q_L : L \times L \longrightarrow \mathbb{Z}.$$

We will often denote this bilinear form by $(\cdot, \cdot)$.

The lattice is said to be:

- *even* if $q_L(x, x) \in 2\mathbb{Z}$ for all $x \in L$;
- *unimodular* if the induced homomorphism

$$L \to L^\vee := \mathrm{Hom}(L, \mathbb{Z})$$

is an isomorphism;
- of *signature* (p, q) if the bilinear form has p positive and q negative eigenvalues after extension to $L \otimes \mathbb{R}$.

Definition 2.2 (Isometry of lattices)**.** Let (L, q_L) and $(L', q_{L'})$ be lattices. An *isometry of lattices* is a group isomorphism

$$\varphi : L \longrightarrow L'$$

such that

$$q_{L'}(\varphi(x), \varphi(y)) = q_L(x, y) \qquad \text{for all } x, y \in L.$$

When $L = L'$, an isometry is called a *lattice automorphism.* The group of all automorphism of L is denoted by $O(L)$.

Definition 2.3 (Period domain)**.** Let (L, q_L) be a lattice of signature (p, q) with $p \geq 2$. The *period domain associated with* L is the subset

$$\Omega_L := \{[\omega] \in \mathbb{P}(L_{\mathbb{C}}) \mid q_L(\omega) = 0,\ q_L(\omega, \overline{\omega}) > 0\}.$$

Remark 2.4. Let L be a lattice of signature (p, q) with $p \geq 2$. Denote by

$$\mathrm{Gr}^{\mathrm{or}}_{+}(2, L_{\mathbb{R}})$$

the set of pairs (V, τ) where $V \subset L_{\mathbb{R}}$ is a 2-dimensional subspace on which the quadratic form q_L is positive definite and τ is an orientation of V.

There is a natural diffeomorphism

$$\Omega_L \xrightarrow{\sim} \mathrm{Gr}^{\mathrm{or}}_{+}(2, L_{\mathbb{R}}), \qquad [\alpha] \mapsto (V_\alpha, \tau_\alpha). \tag{2.1.1}$$

Let $\alpha \in L_{\mathbb{C}}$ represent a point of Ω_L and write $\alpha = x+iy$ with $x, y \in L_{\mathbb{R}}$. The relations

$$(\alpha, \alpha) = 0, \qquad (\alpha, \overline{\alpha}) > 0$$

imply

$$(x, x) = (y, y) > 0, \qquad (x, y) = 0.$$

Hence $V_\alpha := \langle x, y\rangle$ is a positive definite plane. We equip it with the orientation τ_α for which (x, y) is a positively oriented basis. This depends only on the class $[\alpha]$.

Conversely, let $(V, \tau) \in \mathrm{Gr}^{\mathrm{or}}_{+}(2, L_{\mathbb{R}})$. Choose a τ-oriented orthonormal basis (e_1, e_2) of V and set $\alpha = e_1 + ie_2$. Then

$$(\alpha, \alpha) = 0, \qquad (\alpha, \overline{\alpha}) > 0,$$

so $[\alpha] \in \Omega_L$. The resulting class depends only on the orientation τ of V. The two constructions are inverse to each other and define the claimed diffeomorphism.

Proposition 2.5. *Let L be a lattice of signature $(3, q)$. Then the period domain Ω_L has the homotopy type of S^2. In particular it is simply connected.*

Proof. Write

$$L_{\mathbb{R}} = P \oplus N$$

where P is a maximal positive subspace and N is negative definite. Then $\dim P = 3$.

Let $\pi : L_{\mathbb{R}} \to P$ be the projection. If $V \in \mathrm{Gr}_{+}(2, L_{\mathbb{R}})$, then $V \cap N = \{0\}$ and therefore $\pi(V) \in \mathrm{Gr}(2, P)$. This defines a continuous map

$$f : \mathrm{Gr}_{+}(2, L_{\mathbb{R}}) \longrightarrow \mathrm{Gr}(2, P).$$

The inclusion $i : \mathrm{Gr}(2, P) \hookrightarrow \mathrm{Gr}_{+}(2, L_{\mathbb{R}})$ satisfies $f \circ i = \mathrm{Id}$, and $i \circ f$ is homotopic to the identity. Hence $\mathrm{Gr}_{+}(2, L_{\mathbb{R}})$ has the same homotopy type as $\mathrm{Gr}(2, P)$.

Since $\dim P = 3$, we have

$$\mathrm{Gr}(2, P) \cong \mathbb{P}^2_{\mathbb{R}}.$$

Therefore $\mathrm{Gr}_{+}(2, L_{\mathbb{R}})$ has the homotopy type of $\mathbb{P}^2_{\mathbb{R}}$.

The space $\mathrm{Gr}^{\mathrm{or}}_{+}(2, L_{\mathbb{R}})$ is the double cover corresponding to the choice of an orientation of the positive 2-plane. Hence it has the homotopy type of the universal cover of $\mathbb{P}^2_{\mathbb{R}}$, namely S^2.

By the previous remark

$$\Omega_L \cong \mathrm{Gr}_+^{\mathrm{or}}(2, L_{\mathbb{R}}),$$

and the statement follows. □

Remark 2.6. If L is a lattice of signature $(2, q)$, then Ω_L has two connected components.

Indeed, let

$$L_{\mathbb{R}} = P \oplus N$$

where P is a maximal positive subspace and N is negative definite. Then $\dim P = 2$. Arguing as in the proof above, one sees that $\mathrm{Gr}_+(2, L_{\mathbb{R}})$ is homotopy equivalent to $\mathrm{Gr}(2, P)$, which is a point. Hence $\mathrm{Gr}_+(2, L_{\mathbb{R}})$ is connected.

On the other hand, by the previous remark one has

$$\Omega_L \cong \mathrm{Gr}_+^{\mathrm{or}}(2, L_{\mathbb{R}}),$$

and $\mathrm{Gr}_+^{\mathrm{or}}(2, L_{\mathbb{R}}) \to \mathrm{Gr}_+(2, L_{\mathbb{R}})$ is a double cover. Since the base is connected, it follows that $\mathrm{Gr}_+^{\mathrm{or}}(2, L_{\mathbb{R}})$ has exactly two connected components. Therefore Ω_L has two connected components.

Definition 2.7 (Orientation preserving subgroup)**.** Let (L, q_L) be a lattice of signature (p, q) with $p \geq 1$, and let Ω_L be the associated period domain. The subgroup

$$O^+(L) \subset O(L)$$

consists of those isometries that preserve each connected component of Ω_L.

Remark 2.8 (Functoriality under lattice isomorphisms). Let (L, q_L) and $(L', q_{L'})$ be lattices and let

$$\psi : L \xrightarrow{\sim} L'$$

be an isometry. Then conjugation by ψ induces a canonical group isomorphism

$$\psi_* : O(L) \xrightarrow{\sim} O(L'), \qquad g \longmapsto \psi g \psi^{-1}.$$

Moreover, ψ_* restricts to an isomorphism

$$O^+(L) \xrightarrow{\sim} O^+(L').$$

2.1.1 The K3 lattice

Let X be a smooth compact complex surface. Since X has real dimension 4, the cup product induces a bilinear pairing

$$\smile : H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \longrightarrow H^4(X,\mathbb{Z}).$$

Identifying $H^4(X,\mathbb{Z})$ with $\mathbb{Z}$ via the fundamental class $[X] \in H_4(X,\mathbb{Z})$, this pairing defines a symmetric bilinear form

$$(\,\cdot\,,\,\cdot\,) \colon H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \longrightarrow \mathbb{Z}, \qquad (\alpha,\beta) = \int_X \alpha \smile \beta,$$

called the *intersection form* of X.

As the intersection form takes integral values, Poincaré duality implies that the induced pairing on the torsion–free part of $H^2(X,\mathbb{Z})$ is non-degenerate, and hence defines a non-degenerate lattice structure on $H^2(X,\mathbb{Z})/\mathrm{tors}$.

Definition 2.9 (Spin manifold)**.** Let M be a smooth oriented real manifold. The manifold M is said to be *spin* if the structure group of its tangent bundle TM admits a lift from $\mathrm{SO}(n)$ to its universal double cover $\mathrm{Spin}(n)$[1].

Remark 2.10. The existence of a spin structure on the tangent bundle of M is a purely topological property. The obstruction to the existence of such a structure is given by the second Stiefel–Whitney class

$$w_2(TM) \in H^2(M,\mathbb{Z}/2\mathbb{Z}).$$

A classical result in differential topology states that a smooth oriented manifold admits a spin structure precisely when this class vanishes (see [MS74, Chapter 4]).

Remark 2.11. Let X be a complex surface. The real tangent bundle $TX_{\mathbb{R}}$ is naturally induced by the holomorphic tangent bundle T_X. In this situation, the second Stiefel–Whitney class of $TX_{\mathbb{R}}$ is related to the first Chern class of T_X by the congruence

$$w_2(TX_{\mathbb{R}}) \equiv c_1(T_X) \pmod 2.$$

Since $c_1(T_X) = -c_1(K_X)$, it follows that the vanishing of $w_2(TX_{\mathbb{R}})$ is equivalent to the vanishing of $c_1(K_X)$ modulo 2.

[1]The group $\mathrm{Spin}(n)$ is the simply connected double cover of $\mathrm{SO}(n)$, constructed inside the Clifford algebra, and fits into the exact sequence $1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to 1$.

Remark 2.12. If X is a K3 surface, then the canonical bundle is trivial, $K_X \cong \mathcal{O}_X$, and therefore $c_1(K_X) = 0$. As a consequence, the second Stiefel–Whitney class of T_X vanishes and every K3 surface admits a spin structure.

Proposition 2.13 (Spin and parity of the intersection form)**.** *Let M be a smooth compact oriented real 4–manifold. Then M is spin if and only if the intersection form on $H^2(M, \mathbb{Z})/\mathrm{tors}$ is even.*

Reference. See [Hir78, Ch. I, §4]. □

Theorem 2.14 (The cohomology lattice of a K3 surface)**.** *Let X be a complex K3 surface. Then the lattice*

$$\big(H^2(X, \mathbb{Z})/\mathrm{tors}, \smile\big)$$

is even, unimodular, of rank 22 and signature $(3, 19)$.

Proof. By Poincaré duality, the cup product pairing on $H^2(X, \mathbb{Z})/\mathrm{tors}$ is unimodular.

From the computation of the Hodge diamond in Chapter 1, we know that

$$b_2(X) = 22, \qquad h^{2,0}(X) = h^{0,2}(X) = 1, \qquad h^{1,1}(X) = 20.$$

The Hodge decomposition induces an orthogonal decomposition of $H^2(X, \mathbb{R})$ with respect to the intersection form

$$H^2(X, \mathbb{R}) = \Big(\big(H^{2,0}(X) \oplus H^{0,2}(X)\big) \cap H^2(X, \mathbb{R})\Big) \;\oplus\; H^{1,1}(X, \mathbb{R}).$$

The intersection form is positive definite on the real subspace

$$\big(H^{2,0}(X) \oplus H^{0,2}(X)\big) \cap H^2(X, \mathbb{R}),$$

which has real dimension 2. Indeed, if $0 \neq \sigma \in H^{2,0}(X)$ and $\sigma = x + iy$ with $x, y \in H^2(X, \mathbb{R})$, then

$$(\sigma, \sigma) = (x, x) - (y, y) + 2i(x, y) = 0$$

and

$$(\sigma, \overline{\sigma}) = (x, x) + (y, y) > 0.$$

Hence $(x, x) = (y, y) > 0$ and $(x, y) = 0$.

Moreover, if $\omega \in H^{1,1}(X, \mathbb{R})$ is a Kähler class, then

$$(\omega, \omega) = \int_X \omega^2 > 0,$$

so the restriction of the intersection form to $H^{1,1}(X,\mathbb{R})$ has a positive direction.

By the Hodge index theorem, the restriction of the intersection form to $H^{1,1}(X,\mathbb{R})$ has signature $(1,19)$. Therefore, combining the two positive directions coming from $H^{2,0}(X)\oplus H^{0,2}(X)$ with the Kähler direction in $H^{1,1}(X,\mathbb{R})$, one obtains that the intersection form on $H^2(X,\mathbb{R})$ has signature $(3,19)$.

Finally, as noted above, a K3 surface is spin. By Proposition 2.13, this implies that the intersection form on $H^2(X,\mathbb{Z})/\mathrm{tors}$ is even. □

Definition 2.15 (The K3 lattice)**.** The *K3 lattice* is the abstract lattice

$$\Lambda := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2},$$

where U denotes the hyperbolic plane and $E_8(-1)$ denotes the lattice obtained from the positive definite E_8 lattice by multiplying the bilinear form by -1.

Remark 2.16. With respect to the decomposition in Definition 2.15, the bilinear form on Λ is represented by the block diagonal matrix

$$\begin{pmatrix} U & 0 & 0 & 0 & 0 \\ 0 & U & 0 & 0 & 0 \\ 0 & 0 & U & 0 & 0 \\ 0 & 0 & 0 & -E_8 & 0 \\ 0 & 0 & 0 & 0 & -E_8 \end{pmatrix},$$

where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad E_8 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

is the Cartan matrix of type E_8.

Remark 2.17. The K3 lattice Λ is an even unimodular lattice of rank 22 and signature $(3,19)$. Indeed, the hyperbolic plane U is even and unimodular of signature $(1,1)$, while the lattice $E_8(-1)$ is even, unimodular and negative definite of rank 8. Since these properties are preserved under orthogonal direct sums, the claim follows.

Theorem 2.18 (Uniqueness of the K3 lattice)**.** *For every complex K3 surface X there exists an isometry of lattices*

$$\big(H^2(X,\mathbb{Z})/\mathrm{tors}, \smile\big) \;\cong\; \Lambda.$$

Proof. By Theorem 2.14, the lattice $H^2(X,\mathbb{Z})/\mathrm{tors}$ is even, unimodular, of rank 22 and signature $(3,19)$. The classification of even unimodular indefinite lattices (see [MH73, Chapter II, Theorem 5.3]) states that an even unimodular lattice of signature (p,q) is unique up to isometry whenever $p-q\equiv 0 \pmod 8$. Since Λ satisfies these conditions, the claim follows. □

Remark 2.19. For the $K3$ lattice Λ the period domain Ω_Λ is simply connected by Proposition 2.5.

Let (X,h) be a polarized $K3$ surface and let $\varphi : H^2(X,\mathbb{Z})\to\Lambda$ be a marking. Since the polarization class h is of type $(1,1)$, the period σ_X satisfies $(\sigma_X,h)=0$. Therefore the period lies in $\mathbb{P}(h^\perp\otimes\mathbb{C})$.

The relevant lattice is thus $L := h^\perp\subset\Lambda$, which has signature $(2,19)$. By Remark 2.6, the corresponding period domain Ω_L has two connected components.

Definition 2.20 (Marking)**.** Let X be a K3 surface. A *marking* of X is an isometry of lattices

$$\phi\colon H^2(X,\mathbb{Z})/\mathrm{tors}\longrightarrow\Lambda.$$

A pair (X,ϕ) is called a *marked K3 surface.*

Definition 2.21 (Period point)**.** Let (X,ϕ) be a marked K3 surface. The *period point* (or simply the *period*) of (X,ϕ) is the complex line

$$[\phi(H^{2,0}(X))] \;\in\; \mathbb{P}(\Lambda_\mathbb{C}).$$

Remark 2.22. By Definition 2.3, the period point of a marked K3 surface lies in the period domain Ω_Λ. The domain Ω_Λ is called the *period domain of K3 surfaces.*

Definition 2.23 (Hodge structure)**.** Let $H_\mathbb{Z}$ be a free abelian group of finite rank. A *(pure) Hodge structure of weight k* on $H_\mathbb{Z}$ is a decomposition

$$H_\mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$$

such that

$$\overline{H^{p,q}} = H^{q,p} \qquad \text{for all } p,q.$$

Definition 2.24 (Hodge structure of K3 type)**.** Let Λ be a lattice. A *Hodge structure of K3 type* on Λ is a Hodge structure of weight 2

$$\Lambda_{\mathbb{C}} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$$

satisfying the following conditions:

- $\dim_{\mathbb{C}} H^{2,0} = \dim_{\mathbb{C}} H^{0,2} = 1$;
- $H^{0,2} = \overline{H^{2,0}}$;
- the decomposition is orthogonal with respect to the bilinear form on $\Lambda_{\mathbb{C}}$;
- the bilinear form is positive definite on $(H^{2,0} \oplus H^{0,2}) \cap \Lambda_{\mathbb{R}}$.

Remark 2.25. Given a complex K3 surface X together with a marking

$$\phi\colon H^2(X,\mathbb{Z})/\mathrm{tors} \longrightarrow \Lambda,$$

the Hodge decomposition of $H^2(X,\mathbb{C})$ induces a Hodge structure of K3 type on Λ via ϕ.

2.2 The moduli space of marked K3 surfaces

We use the deformation–theoretic framework introduced in Appendix A. In particular, we will use Kuranishi theory and the unobstructedness criterion discussed there to construct a locally universal deformation of a marked K3 surface.

Let $\mathcal{M}$ denote the moduli space of marked K3 surfaces. By definition, $\mathcal{M}$ parametrizes isomorphism classes of pairs (X,ϕ), where X is a complex K3 surface and

$$\phi\colon H^2(X,\mathbb{Z})/\mathrm{tors} \longrightarrow \Lambda$$

is an isometry of lattices. Here two marked K3 surfaces (X,ϕ) and (X',ϕ') are considered isomorphic if there exists an isomorphism $f\colon X \to X'$ such that $\phi' \circ f^* = \phi$.

Lemma 2.26. *Let X be a complex surface and let*

$$0 \neq \sigma \in H^0(X,\Omega_X^2)$$

be a nowhere vanishing holomorphic 2–form. Then contraction with σ defines an isomorphism of holomorphic vector bundles

$$L_\sigma\colon T_X \xrightarrow{\sim} \Omega_X^1, \qquad v \longmapsto v \lrcorner \sigma,$$

where

$$\big(v_p \lrcorner \sigma_p\big)(w) = \sigma_p(v_p, w), \qquad v_p, w \in T_{X,p}.$$

Conversely, any skew-symmetric[2] *bundle isomorphism*

$$\varphi\colon T_X \xrightarrow{\sim} \Omega^1_X$$

is given by contraction with a nowhere vanishing holomorphic 2*–form on* X*.*

Reference. See [Huy05, Ch. 11]. □

Remark 2.27. Lemma 2.26 applies to K3 surfaces since they admit a nowhere vanishing holomorphic 2–form $\sigma \in H^0(X, \Omega^2_X)$. Equivalently, for a K3 surface one has $K_X \cong \mathcal{O}_X$, and the natural wedge pairing

$$\Omega^1_X \otimes \Omega^1_X \longrightarrow K_X$$

is a perfect pairing. After choosing a trivialization $K_X \cong \mathcal{O}_X$ given by σ, this identifies Ω^1_X with its dual, hence $T_X \cong \Omega^1_X$.

Definition 2.28 (Deformation of a marked K3 surface)**.** Let (X, ϕ) be a marked K3 surface. *A deformation of* (X, ϕ) *over a complex space* S consists of

1. a proper holomorphic submersion

   $$\pi : \mathcal{X} \to S$$

   whose fibers are K3 surfaces,

2. an isometry of local systems

   $$\Phi : (R^2\pi_*\mathbb{Z})/\mathrm{tors} \xrightarrow{\ \sim\ } \underline{\Lambda}_S,$$

3. a point $s_0 \in S$ together with an isomorphism

   $$X_{s_0} \cong X$$

   such that the induced marking

   $$H^2(X, \mathbb{Z})/\mathrm{tors} \to \Lambda$$

   coincides with ϕ.

[2] An isomorphism $\varphi : T_X \to \Omega^1_X$ corresponds to a bilinear form $B(v, w) = \varphi(v)(w)$ on T_X. It is called skew-symmetric if $B(v, w) = -B(w, v)$ for all $v, w \in T_{X,p}$, i.e. it defines a section of Ω^2_X.

Definition 2.29 (Locally universal deformation)**.** A deformation

$$(\pi : \mathcal{X} \to U, \Phi)$$

of a marked K3 surface (X, ϕ) is called *locally universal* if for every deformation $(\pi' : \mathcal{X}' \to S, \Phi')$ of (X, ϕ) with distinguished point $s_0 \in S$, there exist an open neighborhood $V \subseteq S$ of s_0 and a holomorphic map

$$f : V \to U, \qquad f(s_0) = 0,$$

such that there is an isomorphism of families over V

$$(\mathcal{X}', \Phi')|_V \cong f^*(\mathcal{X}, \Phi).$$

Proposition 2.30 (Local moduli of marked K3 surfaces)**.** *Let (X, ϕ) be a marked K3 surface. Then there exists a complex manifold U, smooth of dimension* 20*, together with a proper holomorphic submersion*

$$\pi\colon \mathcal{X} \longrightarrow U$$

and an isometry of local systems

$$\Phi\colon (R^2\pi_*\mathbb{Z})/\mathrm{tors} \;\longrightarrow\; \underline{\Lambda}_U$$

such that:

1. *the fiber $X_0 := \pi^{-1}(0)$ over a distinguished point $0 \in U$ is isomorphic to X;*
2. *the induced marking on X_0 agrees with ϕ under this identification;*
3. *(π, Φ) is locally universal among deformations of the marked surface (X, ϕ).*

Proof. According to Theorem A.19, the space of first–order deformations of the complex structure of X is naturally identified with $H^1(X, T_X)$, and obstructions lie in $H^2(X, T_X)$. For a K3 surface one has $K_X \cong \mathcal{O}_X$ and hence, by Serre duality,

$$H^2(X, T_X) \cong H^0(X, \Omega^1_X)^\vee = 0,$$

so deformations of X are unobstructed. By Theorem A.25, there exists a (germ of a) smooth complex manifold U whose tangent space at 0 identifies with $H^1(X, T_X)$, and a holomorphic family $\pi\colon \mathcal{X} \to U$ which is locally universal.

It remains to incorporate the marking. Since $\pi\colon \mathcal{X} \to U$ is a proper holomorphic submersion with connected fibers, Theorem A.6 implies that π is a C^∞ locally trivial fibration. After possibly shrinking U, this yields differentiable identifications of the fibers X_t with X_0, unique up to isotopy. In particular, the cohomology groups $H^2(X_t, \mathbb{Z})$ are naturally identified with $H^2(X_0, \mathbb{Z})$. Passing to the torsion–free quotients, this induces natural identifications

$$H^2(X_t, \mathbb{Z})/\text{tors} \;\cong\; H^2(X_0, \mathbb{Z})/\text{tors}.$$

Moreover, the local system $R^2\pi_*\mathbb{Z}$, i.e. the locally constant sheaf on U whose fiber at a point $t \in U$ is $H^2(X_t, \mathbb{Z})$, is therefore constant on U. Fixing a biholomorphic identification $f\colon X_0 \xrightarrow{\sim} X$, and composing the induced pull-back in cohomology with the marking $\phi\colon H^2(X, \mathbb{Z})/\text{tors} \to \Lambda$, we obtain an isometry of local systems

$$\Phi\colon R^2\pi_*\mathbb{Z}/\text{tors} \longrightarrow \underline{\Lambda}_U,$$

whose restriction to the central fiber agrees with ϕ.

By Lemma 2.26, one has

$$T_X \cong \Omega^1_X,$$

hence

$$H^1(X, T_X) \cong H^1(X, \Omega^1_X).$$

By Remark 1.5, one has

$$h^1(X, \Omega^1_X) = h^{1,1}(X) = 20.$$

Therefore $h^1(X, T_X) = 20$, and hence U is a smooth complex manifold of dimension 20. □

Remark 2.31. A deformation $(\pi\colon \mathcal{X} \to U, \Phi)$ as in Proposition 2.30 is called a *Kuranishi family* of the marked K3 surface (X, ϕ).

The base U of such a family is a smooth complex manifold of dimension 20, and the set

$$U(\pi, \Phi) := \{(X_t, \phi_t) \mid t \in U\}$$

of marked fibers defines a subset of the moduli space $\mathcal{M}$. Via the identification $(X_t, \phi_t) \leftrightarrow t$, it inherits the structure of a complex manifold.

If (π', Φ') is another Kuranishi family, the corresponding sets $U(\pi, \Phi)$ and $U(\pi', \Phi')$ overlap precisely when the corresponding marked fibers are isomorphic. By the local universality property, the induced transition maps between the corresponding open subsets of U and U' are holomorphic. Therefore these sets form a holomorphic atlas on $\mathcal{M}$.

In this way $\mathcal{M}$ acquires the structure of a complex manifold of dimension 20.

2.3 Period map and infinitesimal Torelli theorem

We use the language of relative de Rham cohomology, Hodge bundles, and the Gauss–Manin connection introduced in Appendix B.

Let $\mathcal{M}$ be the moduli space of marked K3 surfaces. We define the *period map*

$$\mathcal{P}\colon \mathcal{M} \longrightarrow \Omega_\Lambda$$

as the map assigning to each marked K3 surface the complex line of its holomorphic 2–forms, viewed as a point in the period domain:

$$\mathcal{P}(X,\phi) := [\,\phi(H^{2,0}(X))\,].$$

By construction, the image of $\mathcal{P}$ is contained in the period domain Ω_Λ. Indeed, if $0 \neq \omega \in H^{2,0}(X)$, then $(\omega,\omega) = 0$ and $(\omega,\overline{\omega}) > 0$ with respect to the cup product pairing.

Fix a point $(X,\phi) \in \mathcal{M}$ and let $U \subset \mathcal{M}$ be a neighborhood of (X,ϕ) as in Proposition 2.30. Thus there exists a smooth family of K3 surfaces

$$\pi\colon \mathcal{X} \longrightarrow U$$

together with an isometry of local systems

$$\Phi\colon R^2\pi_*\mathbb{Z}/\mathrm{tors} \xrightarrow{\sim} \underline{\Lambda}_U$$

whose restriction to the fiber over (X,ϕ) coincides with the marking ϕ. Let

$$\mathcal{H}^2 := R^2\pi_*\Omega^\bullet_{\mathcal{X}/U}$$

be the holomorphic vector bundle introduced in Appendix B, endowed with the Hodge filtration $F^\bullet\mathcal{H}^2$. By Theorem B.7 and Theorem B.8, for every $t \in U$ one has a canonical identification

$$(\mathcal{H}^2)_t \cong H^2(X_t,\mathbb{C}).$$

The family π determines a *local period map*

$$\mathcal{P}_U\colon U \longrightarrow \mathbb{P}(\mathcal{H}^2), \qquad t \longmapsto [\,(F^2\mathcal{H}^2)_t\,],$$

which, after tensoring Φ with $\mathbb{C}$, identifies $\mathcal{H}^2$ with $\underline{\Lambda}_U \otimes_{\mathbb{Z}} \mathcal{O}_U = \underline{\Lambda_{\mathbb{C}}}_U$, and hence coincides with the restriction of the global period map $\mathcal{P}$ to U.

Let $(X, \phi) \in \mathcal{M}$ and set $[\omega] = \mathcal{P}(X, \phi)$, with $0 \neq \omega \in \Lambda_{\mathbb{C}}$. Since Ω_Λ is an open subset of the smooth quadric

$$Q = \{[\alpha] \in \mathbb{P}(\Lambda_{\mathbb{C}}) \mid (\alpha, \alpha) = 0\},$$

the tangent space at $[\omega]$ admits a canonical description

$$T_{[\omega]}\Omega_\Lambda \;\cong\; \operatorname{Hom}\big(\mathbb{C}\omega,\; \omega^\perp/\mathbb{C}\omega\big), \qquad \omega^\perp := \{v \in \Lambda_{\mathbb{C}} \mid (v, \omega) = 0\}.$$

Since the Hodge decomposition is orthogonal with respect to the intersection pairing, one has

$$\omega^\perp = H^{2,0}(X) \oplus H^{1,1}(X),$$

which yields

$$\omega^\perp/\mathbb{C}\omega \cong H^{1,1}(X).$$

Hence one obtains a natural identification

$$T_{\mathcal{P}(X,\phi)}\Omega_\Lambda \;\cong\; \operatorname{Hom}\big(H^{2,0}(X),\, H^{1,1}(X)\big).$$

For the family $\pi\colon \mathcal{X} \to U$, each point $t \in U$ carries a Kodaira–Spencer map

$$\kappa_t\colon T_tU \longrightarrow H^1(X_t, T_{X_t}).$$

At the base point $t_0 = (X, \phi)$, local universality implies that κ_{t_0} is an isomorphism; under the standard identification

$$T_{(X,\phi)}\mathcal{M} \cong H^1(X, T_X),$$

the Kodaira–Spencer map at t_0 coincides with the identity.

We now describe the differential of the period map at the point $t_0 = (X, \phi)$ in the direction $v \in T_{t_0}U$.

Let $0 \neq \sigma \in H^{2,0}(X) = (F^2\mathcal{H}^2)_{t_0}$ and choose a local holomorphic section $\tilde{\sigma}$ of the line bundle $F^2\mathcal{H}^2$ defined near t_0 with $\tilde{\sigma}(t_0) = \sigma$. By definition of the local period map

$$\mathcal{P}_U(t) = [\,(F^2\mathcal{H}^2)_t\,],$$

the differential $d\mathcal{P}_{t_0}(v)$ measures the first–order variation of the line spanned by $\tilde{\sigma}(t)$ inside $(\mathcal{H}^2)_t$.

Let

$$\nabla\colon \mathcal{H}^2 \longrightarrow \mathcal{H}^2 \otimes \Omega^1_U$$

be the Gauss–Manin connection. Evaluating $\nabla\tilde{\sigma}$ at t_0 and pairing with v gives an element $\nabla_v(\tilde{\sigma})(t_0) \in (\mathcal{H}^2)_{t_0} = H^2(X,\mathbb{C})$. Since we are varying a *line*, the corresponding tangent vector in the projective space depends only on the class of $\nabla_v(\tilde{\sigma})(t_0)$ modulo the line $\mathbb{C}\sigma = (F^2\mathcal{H}^2)_{t_0}$. Thus $d\mathcal{P}_{t_0}(v)$ is represented by the class

$$[\nabla_v(\tilde{\sigma})(t_0)] \;\in\; (\mathcal{H}^2)_{t_0}/(F^2\mathcal{H}^2)_{t_0}.$$

By Theorem B.19 one has

$$\nabla(F^2\mathcal{H}^2) \subset F^1\mathcal{H}^2 \otimes \Omega^1_U,$$

hence, after evaluating at t_0 and contracting with v,

$$\nabla_v(\tilde{\sigma})(t_0) \in (F^1\mathcal{H}^2)_{t_0} = F^1H^2(X,\mathbb{C}).$$

Therefore the class $[\nabla_v(\tilde{\sigma})(t_0)]$ actually lies in the smaller quotient

$$(F^1\mathcal{H}^2/F^2\mathcal{H}^2)_{t_0} \;=\; F^1H^2(X,\mathbb{C})/F^2H^2(X,\mathbb{C}) \;\cong\; H^{1,1}(X),$$

where the last identification uses the Hodge decomposition of $H^2(X,\mathbb{C})$.

The classical computation of Griffiths identifies this class explicitly in terms of the Kodaira–Spencer map (see [Gri69, Sec. 2]): for $v \in T_{t_0}U$ one has

$$[\nabla_v(\tilde{\sigma})(t_0)] = \big[\; \kappa_{t_0}(v) \lrcorner\, \sigma \;\big] \in H^1(X,\Omega^1_X) \cong H^{1,1}(X).$$

Under the identification

$$T_{\mathcal{P}(X,\phi)}\Omega_\Lambda \;\cong\; \operatorname{Hom}\big(H^{2,0}(X),\, H^{1,1}(X)\big),$$

this means that $d\mathcal{P}_{t_0}(v)$ is the homomorphism sending σ to the class $[\,\kappa_{t_0}(v) \lrcorner\, \sigma\,]$.

Theorem 2.32 (Infinitesimal Torelli theorem for K3 surfaces)**.** *Let (X,ϕ) be a marked K3 surface. Then the differential of the period map*

$$d\mathcal{P}_{(X,\phi)}\colon T_{(X,\phi)}\mathcal{M} \longrightarrow T_{\mathcal{P}(X,\phi)}\Omega_\Lambda$$

is injective.

Proof. Using the identification

$$T_{(X,\phi)}\mathcal{M} \cong H^1(X, T_X),$$

let $\xi \in H^1(X, T_X)$ and assume that $d\mathcal{P}_{(X,\phi)}(\xi) = 0$. Choose a generator $0 \neq \sigma \in H^{2,0}(X)$. Griffiths' formula for the differential of the period map gives

$$\left[\, \xi \lrcorner \sigma \,\right] = 0 \in H^1(X, \Omega^1_X).$$

Since contraction with σ induces an isomorphism

$$H^1(X, T_X) \xrightarrow{\sim} H^1(X, \Omega^1_X)$$

by Lemma 2.26, it follows that $\xi = 0$. □

Corollary 2.33 (Local Torelli for K3 surfaces)**.** *The period map* $\mathcal{P} \colon \mathcal{M} \to \Omega_\Lambda$ *is a local biholomorphism.*

Proof. The moduli space $\mathcal{M}$ is smooth of dimension 20, and Ω_Λ is an open subset of a smooth quadric hypersurface in $\mathbb{P}^{21}$, hence $\dim \Omega_\Lambda = 20$. By Theorem 2.32, the differential of $\mathcal{P}$ is injective at every point, and therefore an isomorphism. The holomorphic inverse function theorem implies that $\mathcal{P}$ is a local biholomorphism. □

Historically, in the terminology of Andreotti–Weil, the local Torelli theorem asserts that the period map is locally an immersion.

In this chapter we follow an approach close to those of [Voi02] and [OGr12]: we first prove the infinitesimal Torelli theorem (Theorem 2.32), and then deduce the local Torelli theorem from the equality of the dimensions of the deformation space and the period domain.

2.4 Noether–Lefschetz loci and deformation theory of K3 surfaces

Let X be a complex K3 surface. We denote by

$$H^{1,1}_{\mathbb{Z}}(X) := H^{1,1}(X) \cap H^2(X, \mathbb{Z})$$

the group of integral $(1,1)$–classes, and by

$$H^{1,1}_{\mathbb{Z}}(X)/\mathrm{tors}$$

its quotient modulo the torsion subgroup.

Definition 2.34 (Néron–Severi lattice)**.** The *Néron–Severi lattice* of X is defined as

$$S_X := \mathrm{NS}(X)/\mathrm{tors},$$

endowed with the bilinear form induced by the intersection pairing on $H^2(X,\mathbb{Z})$.

Remark 2.35. By the Lefschetz $(1,1)$–Theorem, the image of the Chern class map coincides with the group of integral $(1,1)$–classes. Hence

$$S_X \cong H^{1,1}_{\mathbb{Z}}(X)/\mathrm{tors},$$

Definition 2.36 (Transcendental lattice)**.** The *transcendental lattice* of X is the smallest sublattice

$$T_X \subset H^2(X,\mathbb{Z})/\mathrm{tors}$$

such that

$$H^{2,0}(X) \subset T_X \otimes_{\mathbb{Z}} \mathbb{C}$$

and the quotient $(H^2(X,\mathbb{Z})/\mathrm{tors})/T_X$ is torsion–free.

Remark 2.37. By definition, the transcendental lattice T_X is the smallest sublattice of $H^2(X,\mathbb{Z})/\mathrm{tors}$ whose complexification contains $H^{2,0}(X)$, and such that the quotient $\big(H^2(X,\mathbb{Z})/\mathrm{tors}\big)/T_X$ is torsion–free.

Let $L \subset T_X$ be an integral Hodge substructure, i.e. a sublattice such that $L \otimes \mathbb{C}$ is stable under the Hodge decomposition. Since $H^{2,0}(X)$ is a one–dimensional Hodge summand, either $L = 0$ or $H^{2,0}(X) \subset L \otimes \mathbb{C}$. In the latter case, the minimality of T_X forces $L = T_X$. Hence T_X has no non–trivial integral Hodge substructures (equivalently, none non–trivial over $\mathbb{Q}$).

Next, since $S_X \subset H^{1,1}(X)$, it is orthogonal to $H^{2,0}(X)$ with respect to the intersection pairing. Indeed, for $\alpha \in H^{1,1}(X)$ and $\sigma \in H^{2,0}(X)$ one has

$$\alpha \smile \sigma \in H^{3,1}(X) = 0,$$

so that $(\alpha, \sigma) = 0$. In particular,

$$S_X \subset \big(H^{2,0}(X)\big)^{\perp} \quad \Rightarrow \quad T_X \subset S_X^{\perp}.$$

If the restriction of the intersection form to S_X is non–degenerate (for instance, if X is projective), then taking orthogonal complements yields

$$T_X = S_X^{\perp}.$$

Finally, by Lefschetz $(1,1)$, the Néron–Severi lattice is determined by the period: if σ generates $H^{2,0}(X)$, then

$$S_X = H^2(X,\mathbb{Z}) \cap \sigma^{\perp}. \tag{2.4.1}$$

Definition 2.38 (Noether–Lefschetz locus)**.** Let $S \subset \Lambda$ be a sublattice. The *Noether–Lefschetz locus* associated with S is the subset

$$\Omega_S := \big\{[\alpha] \in \Omega_\Lambda \;\big|\; (\alpha, s) = 0 \text{ for all } s \in S\big\}.$$

Remark 2.39. Let (X,ϕ) be a marked K3 surface, and let $S \subset \Lambda$ be a sublattice. By the characterization (2.4.1) of the Néron–Severi lattice in terms of the period, one has

$$\mathcal{P}(X,\phi) \in \Omega_S \quad \Longleftrightarrow \quad \phi^{-1}(S) \subset S_X.$$

Indeed, the condition $\mathcal{P}(X,\phi) \in \Omega_S$ means that, if $\sigma \in H^{2,0}(X)$ is a generator and $\omega = \phi(\sigma)$, then $(\omega, s) = 0$ for all $s \in S$. Equivalently, one has

$$(\sigma, \phi^{-1}(s)) = 0 \qquad \text{for all } s \in S,$$

that is, $\phi^{-1}(S) \subset \sigma^{\perp} \cap H^2(X,\mathbb{Z})$. By the Lefschetz $(1,1)$–Theorem, this intersection coincides with the Néron–Severi lattice S_X, and the equivalence follows.

In particular, for a non–zero vector $v \in \Lambda$, the locus

$$\Omega_v := \Omega_{\mathbb{Z}v}$$

parametrizes marked K3 surfaces whose Néron–Severi lattice contains the class $\phi^{-1}(v)$.

Remark 2.40 (Picard rank and Noether–Lefschetz loci). Let (X,ϕ) be a marked K3 surface and let $v \in \Lambda$ be a non–zero vector. If $(X,\phi) \in \Omega_v$, then by Remark 2.39 one has

$$\phi^{-1}(v) \in S_X.$$

Thus the Néron–Severi lattice contains the class $\phi^{-1}(v)$.

In particular, if S_X already contains a non–trivial class (for instance a polarization), the appearance of $\phi^{-1}(v)$ may increase the rank of S_X. Geometrically, this corresponds to the fact that along Noether–Lefschetz loci additional integral $(1,1)$–classes appear, and therefore the Picard rank can jump.

2.4.1 Projective models of polarized K3 surfaces

We now state some of Saint–Donat's results on the projective models of polarized K3 surfaces.

Theorem 2.41 (Saint–Donat)**.** *Let X be a K3 surface and let L be a nef line bundle on X such that $L^2 \geq 2$. Then $|L|$ is base–point free, unless there exists an irreducible curve $E \subset X$ such that $E^2 = -2$ and $L \cdot E = 0$.*

Reference. See [Sai74, Thm. 2.7]. □

Theorem 2.42 (Saint–Donat, projective models)**.** *Let X be a K3 surface and let L be a primitive nef line bundle on X with $L^2 \geq 2$.*

1. *If $L^2 = 2$, then $|L|$ is base–point free and defines a morphism*

$$\varphi_{|L|} : X \longrightarrow \mathbb{P}^2$$

of degree 2. In particular, X is realized as a double cover of $\mathbb{P}^2$ branched along a plane sextic curve.

2. *If $L^2 = 4$, then $|L|$ is very ample unless there exists an irreducible curve E with $E^2 = 0$ and $L \cdot E = 2$. In the very ample case, $|L|$ embeds X as a smooth quartic surface in $\mathbb{P}^3$.*

3. *If $L^2 \geq 6$, then L is very ample unless one of the following exceptional cases occurs:*

   (a) there exists an irreducible curve E with $E^2 = 0$ and $L \cdot E = 2$;

   (b) there exists an irreducible curve E with $E^2 = -2$ and $L \cdot E = 0$.

Proof. The classification of projective models of polarized K3 surfaces is contained in [Sai74, Thm. 5.2, Thm. 6.1]. We only recall the arguments needed for our applications, namely the cases $L^2 = 2$ and $L^2 = 4$.

Assume $L^2 = 2$. By Theorem 2.41, the linear system $|L|$ is base–point free. By Riemann–Roch on a K3 surface,

$$\chi(L) = 2 + \frac{1}{2}L^2 = 3.$$

Since L is nef and $L^2 > 0$, the line bundle L^{-1} cannot be effective, hence

$$h^0(L^{-1}) = 0.$$

By Serre duality, $h^2(L) = h^0(L^{-1}) = 0$. Moreover, since L is big and nef, $h^1(L) = 0$ by the Kawamata–Viehweg vanishing theorem (see [Laz04, Thm. 4.3.1]). Therefore $h^0(L) = \chi(L) = 3$. Thus $|L|$ defines a morphism

$$\varphi_{|L|} : X \longrightarrow \mathbb{P}^2.$$

Moreover, if $H \in H^2(\mathbb{P}^2, \mathbb{Z})$ denotes the hyperplane class, then $\varphi_{|L|}^* \mathcal{O}_{\mathbb{P}^2}(1) \cong L$, hence

$$(\varphi_{|L|}^* H)^2 = L^2 = 2.$$

Since $(\varphi_{|L|}^* H)^2 = \deg(\varphi_{|L|}) \cdot H^2$ and $H^2 = 1$ on $\mathbb{P}^2$, it follows that $\deg(\varphi_{|L|}) = 2$. In particular, $\varphi_{|L|}$ realizes X as a double cover of $\mathbb{P}^2$.

Let $B \subset \mathbb{P}^2$ be the branch divisor of $\varphi_{|L|}$. Since $\varphi_{|L|} : X \to \mathbb{P}^2$ is a double cover, the branch divisor satisfies $B \sim 2D$ for some divisor D on $\mathbb{P}^2$, and the canonical bundle formula gives

$$K_X = \varphi_{|L|}^*(K_{\mathbb{P}^2} + D).$$

Since X is a K3 surface, $K_X \cong \mathcal{O}_X$, hence

$$K_{\mathbb{P}^2} + D \sim 0.$$

Because $K_{\mathbb{P}^2} \sim -3H$, we obtain $D \sim 3H$, and therefore

$$B \sim 2D \sim 6H.$$

Thus the branch divisor is a sextic curve in $\mathbb{P}^2$.

Assume now that $L^2 = 4$ and that L is primitive and ample. By Theorem 2.41, the linear system $|L|$ is base–point free.

Since L is ample, there are no irreducible curves E with $L \cdot E = 0$. If no curve E with $E^2 = 0$ and $L \cdot E = 2$ exists, then none of the exceptional cases in [Sai74, Thm. 5.2] occurs, and therefore L is very ample.

By Riemann–Roch on a K3 surface,

$$\chi(L) = 2 + \frac{1}{2} L^2 = 4.$$

As before, $h^0(L^{-1}) = 0$ and $h^2(L) = 0$. Again, $h^1(L) = 0$ because L is big and nef, hence $h^0(L) = 4$. Thus $|L|$ defines an embedding

$$X \hookrightarrow \mathbb{P}^3,$$

whose image is a smooth quartic surface. □

Example 2.43 (Quartic K3 surfaces and a Noether–Lefschetz locus). Let $X_0 \subset \mathbb{P}^3$ be a smooth quartic surface, and let

$$\phi_0 : H^2(X_0, \mathbb{Z}) \xrightarrow{\sim} \Lambda$$

be a marking. Set

$$h_0 := c_1\big(\mathcal{O}_{X_0}(1)\big), \qquad v := \phi_0(h_0).$$

Since $h_0 \in H^{1,1}_{\mathbb{Z}}(X_0)$, the characterization $S_{X_0} = H^2(X_0, \mathbb{Z}) \cap \sigma^{\perp}$ implies that the period point satisfies

$$\mathcal{P}(X_0, \phi_0) \in \Omega_v.$$

Let

$$\pi : \mathcal{X} \to U$$

be a representative of the Kuranishi family of X_0, endowed with a marking extending ϕ_0. After possibly shrinking U, we may assume that the local period map

$$\mathcal{P}_\pi : U \to \Omega_\Lambda$$

is an embedding.

Let

$$t \in \mathcal{P}_\pi^{-1}(\Omega_v), \qquad h_t := \phi_t^{-1}(v) \in H^2(X_t, \mathbb{Z}).$$

By definition of Ω_v, the class h_t is of Hodge type $(1,1)$, hence there exists a line bundle L_t on X_t such that

$$c_1(L_t) = h_t.$$

For $t = 0$, the line bundle $L_0 \cong \mathcal{O}_{X_0}(1)$ is very ample on the smooth quartic X_0, hence it satisfies the hypotheses of Theorem 2.42. Since the existence of exceptional curves E intersecting a line bundle with low degree is a closed condition in flat families (see [Har77, p. III.12]), for t sufficiently close to 0 the line bundle L_t continues to satisfy the hypotheses of Theorem 2.42 (meaning no such exceptional curves appear on X_t). Therefore, L_t is very ample and the complete linear system $|L_t|$ defines an embedding

$$\varphi_{|L_t|} : X_t \hookrightarrow \mathbb{P}^3,$$

realizing X_t as a smooth quartic surface.

Conversely, any smooth quartic surface sufficiently close to X_0 arises in this way from a point $t \in \mathcal{P}_\pi^{-1}(\Omega_v)$. It follows that the locus of periods of quartic K3 surfaces coincides locally with the Noether–Lefschetz locus Ω_v.

Since Ω_v has codimension 1 in Ω_Λ, this shows that the set of periods of quartic K3 surfaces is a 19–dimensional analytic subvariety of the period domain.

2.4.2 Density of Noether–Lefschetz loci

As observed above, for a fixed primitive class $v \in \Lambda$, the Noether–Lefschetz locus

$$\Omega_v = \{[\alpha] \in \Omega_\Lambda \mid (\alpha, v) = 0\}$$

parametrizes marked K3 surfaces whose Néron–Severi lattice contains the class corresponding to v. Geometrically, this means that an additional integral $(1,1)$–class appears, so that the Picard rank may increase.

The purpose of this subsection is to show that these loci are dense in the period domain. More precisely, we prove that for every fixed integer k, the union of the hypersurfaces Ω_v corresponding to primitive classes v with $(v, v) = 2k$ is dense in Ω_Λ.

This density statement will be a key ingredient in the proof that all K3 surfaces belong to a single deformation class (Theorem 2.46).

For $k \in \mathbb{Z}$, let

$$\mathcal{P}_{2k} \subset \Lambda$$

be the set of non–zero primitive vectors v, i.e. vectors which are not non–trivial integral multiples of another vector of Λ, such that $(v, v) = 2k$.

Lemma 2.44. *Let $w \in \mathcal{P}_0$ and $k \in \mathbb{Z}$. There exists a sequence $\{w_n\}$ with $w_n \in \mathcal{P}_{2k}$ such that*

$$\lim_{n\to\infty} [w_n] = [w]$$

in $\mathbb{P}(\Lambda_\mathbb{R})$.

Proof. Since Λ is unimodular and w is primitive, there exists an element $w_1 \in \Lambda$ such that $(w, w_1) = 1$. Hence the sublattice generated by w and w_1 is isometric to the hyperbolic plane U.

By the classification of even unimodular lattices and the description of the K3 lattice, one has an orthogonal decomposition

$$\Lambda \cong U \oplus \Lambda', \qquad \Lambda' \cong U^{\oplus 2} \oplus E_8(-1)^{\oplus 2},$$

with w contained in the first summand. Let $\{u, u'\}$ be a basis of one of the copies of U appearing in Λ'. For $n \in \mathbb{N}$, set

$$w_n := nw + u + ku'.$$

Then $(w_n, w_n) = 2k$ and $(w_n, u') = 1$, hence w_n is primitive. Moreover,

$$\lim_{n\to\infty} [w_n] = [w]$$

in $\mathbb{P}(\Lambda_{\mathbb{R}})$. □

Proposition 2.45. *For every $k \in \mathbb{Z}$, the union*

$$\bigcup_{v\in\mathcal{P}_{2k}} \Omega_v$$

is dense in Ω_Λ.

Proof. Let $[\alpha] \in \Omega_\Lambda$ and write $\alpha = x + iy$ with $x, y \in \Lambda_{\mathbb{R}}$. The real plane

$$P := \langle x, y\rangle \subset \Lambda_{\mathbb{R}}$$

is positive definite, hence its orthogonal complement $P^\perp$ has signature $(1, 19)$. In particular, $P^\perp$ contains non–zero isotropic real vectors.

In particular, $P^\perp$ contains non–zero isotropic real vectors. Let $v \in P^\perp$ be one such real isotropic vector.

Since the intersection form on Λ is indefinite, the set of rational isotropic lines is dense in the set of real isotropic lines. Hence, we can find a sequence of non–zero isotropic rational vectors $w^{(m)} \in \Lambda_{\mathbb{Q}}$ such that $[w^{(m)}] \longrightarrow [v]$ in $\mathbb{P}(\Lambda_{\mathbb{R}})$.

By Lemma 2.44, for each m we can approximate $w^{(m)}$ by a primitive vector in $\mathcal{P}_{2k}$. By a diagonal selection argument, we obtain a sequence of primitive vectors $w_n \in \mathcal{P}_{2k}$ such that

$$[w_n] \longrightarrow [v]$$

in $\mathbb{P}(\Lambda_{\mathbb{R}})$.

Since $v \in P^\perp$, the real plane P is orthogonal to v. Therefore, for n sufficiently large, there exist positive definite oriented real 2–planes $P_n \subset w_n^\perp$ arbitrarily close to P. Let $[\alpha_n] \in \Omega_{w_n}$ be the period points corresponding to P_n. Then

$$[\alpha_n] \to [\alpha]$$

in Ω_Λ, showing that every neighborhood of $[\alpha]$ intersects Ω_{w_n} for n sufficiently large.

This proves that $\bigcup_{v\in\mathcal{P}_{2k}} \Omega_v$ is dense in Ω_Λ. □

2.4.3 Deformation equivalence of K3 surfaces

Theorem 2.46 (Kodaira). *All K3 surfaces belong to a single deformation class.*

Proof. Let X_0 be a K3 surface and let $\pi : \mathcal{X} \to U$ be its Kuranishi family. After possibly shrinking U, we may assume that the local period map identifies U with an open subset of the period domain Ω_Λ, by the local Torelli theorem (Corollary 2.33).

Fix $k = 1$. By Proposition 2.45, the union

$$\bigcup_{v \in \mathcal{P}_2} \Omega_v$$

is dense in Ω_Λ. Hence the period point of X_0 can be approximated by periods lying in some Ω_v with $(v, v) = 2$. Using that the period map is a local biholomorphism, we may therefore choose $t \in U$ such that the period of X_t lies in Ω_v for some $v \in \mathcal{P}_2$. Equivalently, the class

$$h_t := \phi_t^{-1}(v) \in H^2(X_t, \mathbb{Z})$$

is of Hodge type $(1, 1)$, primitive, and satisfies $(h_t, h_t) = 2$.

Moreover, we may assume that $\rho(X_t) = 1$. Indeed, for fixed v the locus Ω_v parametrizes K3 surfaces whose Néron–Severi lattice contains $\mathbb{Z}h_t$; the condition $\rho(X_t) > 1$ corresponds to the intersection with further Noether–Lefschetz loci, hence to a countable union of proper analytic subsets of Ω_v. Choosing t outside this union yields $\mathrm{NS}(X_t) = \mathbb{Z}h_t$.

By the Lefschetz $(1, 1)$–Theorem, there exists a line bundle L_t on X_t such that $c_1(L_t) = h_t$. Since $\chi(L_t) = 2 + \frac{1}{2}(h_t, h_t) = 3$, Riemann–Roch gives

$$h^0(L_t) - h^1(L_t) + h^0(L_t^{-1}) = 3.$$

By Serre duality, $h^2(L_t) = h^0(L_t^{-1})$. As $(h_t, h_t) > 0$, the line bundle L_t^{-1} cannot be effective, hence $h^0(L_t^{-1}) = 0$. Moreover, since L_t is big and nef, $h^1(L_t) = 0$ by the Kawamata–Viehweg vanishing theorem. Therefore, $h^0(L_t) = 3$.

Because $c_1(L_t)$ generates $\mathrm{NS}(X_t)$, the line bundle L_t is nef and primitive. Since $(L_t^2) = 2$, Theorem 2.42 (1) implies that $|L_t|$ is base–point free and defines a morphism

$$\varphi_{|L_t|} : X_t \longrightarrow \mathbb{P}^2.$$

Since $(L_t^2) = 2$ and $h^0(L_t) = 3$, the image is $\mathbb{P}^2$ and the map has degree 2. In particular, $\varphi_{|L_t|}$ is a double cover of $\mathbb{P}^2$, and its branch locus is a plane sextic. For t general (still within Ω_v), the branch sextic is smooth, so X_t is a double plane ramified over a smooth sextic.

Since all such double covers form a connected family, they are deformation equivalent. Thus X_0 is deformation equivalent to a double plane, and hence all K3 surfaces belong to a single deformation class. □

Corollary 2.47. *Let X be a K3 surface. Then X is simply connected and $H^2(X,\mathbb{Z})$ is torsion–free.*

Proof. By Theorem 2.46, X is diffeomorphic to a smooth quartic surface in $\mathbb{P}^3$, which is simply connected. Since $H^1(X,\mathbb{Z}) = 0$, the universal coefficient theorem implies that $H^2(X,\mathbb{Z})$ has no torsion. □

2.5 Integral Hodge isometries

Definition 2.48 (Integral Hodge isometry)**.** Let X and Y be K3 surfaces. An *integral Hodge isometry* is an isomorphism of lattices

$$\varphi : H^2(Y,\mathbb{Z}) \xrightarrow{\sim} H^2(X,\mathbb{Z})$$

such that:

1. φ is an isometry with respect to the intersection forms;
2. φ is a morphism of Hodge structures, i.e. $\varphi(H^{p,q}(Y)) = H^{p,q}(X)$ for all p, q.

We denote by

$$\mathrm{HIsom}\big(H^2(Y,\mathbb{Z}),\ H^2(X,\mathbb{Z})\big)$$

the set of integral Hodge isometries.

Remark 2.49. Since a weight–two Hodge structure of K3 type is determined by the complex line $H^{2,0}$ together with complex conjugation, condition *(2)* in Definition 2.48 is equivalent to requiring that

$$\varphi\big(H^{2,0}(Y)\big) = H^{2,0}(X).$$

2.5.1 Parallel transport

Let $\pi : \mathcal{X} \to B$ be a smooth proper holomorphic submersion with K3 fibers. Since π is a proper holomorphic submersion, Theorem A.6 implies that π is locally trivial in the C^∞ category. In particular, the groups $H^2(X_b,\mathbb{Z})$ form a local system $R^2\pi_*\mathbb{Z}$ on B.

Fix a base point $b_0 \in B$. For any $b \in B$ and any path $\gamma : [0,1] \to B$ with $\gamma(0) = b_0$ and $\gamma(1) = b$, parallel transport (equivalently, a choice of trivialization of $R^2\pi_*\mathbb{Z}$ along γ) gives an isomorphism

$$\gamma_b : H^2(X_{b_0}, \mathbb{Z}) \xrightarrow{\sim} H^2(X_b, \mathbb{Z}),$$

called the *parallel transport operator* along γ. Since the intersection form is a topological invariant, γ_b is an isometry of lattices.

If B is simply connected (or after restricting to a simply connected neighborhood), parallel transport is independent of the path, and the local system $R^2\pi_*\mathbb{Z}$ becomes canonically trivial. Given $b_1, b_2 \in B$, the induced isometry

$$\gamma_{b_2} \circ \gamma_{b_1}^{-1} : H^2(X_{b_1}, \mathbb{Z}) \xrightarrow{\sim} H^2(X_{b_2}, \mathbb{Z})$$

is therefore canonically defined.

In general, parallel transport does not preserve the Hodge decomposition: indeed the Hodge filtration varies holomorphically and, by Theorem B.19, one only has $\nabla(F^p) \subset F^{p-1} \otimes \Omega^1_B$, so the subbundle F^2 is not flat in families with non-constant period map. Thus $\gamma_{b_2} \circ \gamma_{b_1}^{-1}$ need not be a Hodge isometry.

However, if $\gamma_{b_2} \circ \gamma_{b_1}^{-1}$ is an integral Hodge isometry, then X_{b_1} and X_{b_2} have the same period. After shrinking B so that the local period map becomes an isomorphism onto an open subset of Ω_Λ (Local Torelli, Corollary 2.33), it follows that $X_{b_1} \cong X_{b_2}$.

2.5.2 From isomorphisms to Hodge isometries

Let X and Y be K3 surfaces. Any isomorphism

$$f : X \xrightarrow{\sim} Y$$

induces, by functoriality of singular cohomology, an isomorphism

$$f^* : H^2(Y, \mathbb{Z}) \xrightarrow{\sim} H^2(X, \mathbb{Z})$$

which preserves the cup–product pairing and the Hodge decomposition. In particular, f^* is an integral Hodge isometry.

Thus one obtains a natural map

$$\operatorname{Iso}(X, Y) \longrightarrow \operatorname{HIsom}\big(H^2(Y, \mathbb{Z}),\ H^2(X, \mathbb{Z})\big), \tag{2.5.1}$$

from the set of isomorphisms between X and Y to the set of integral Hodge isometries between their second cohomology lattices.

Proposition 2.50. *The map* (2.5.1) *is injective.*

Proof. If $\mathrm{Iso}(X,Y)=\varnothing$, there is nothing to prove. Otherwise, fix an isomorphism $\psi_0 : X \to Y$. Then any other isomorphism $X \to Y$ may be written uniquely as $\psi_0 \circ \varphi$, with $\varphi \in \mathrm{Aut}(X)$. Hence it suffices to prove injectivity in the case $X = Y$, that is, to show that an automorphism $\varphi \in \mathrm{Aut}(X)$ acting trivially on $H^2(X,\mathbb{Z})$ must be the identity.

Assume therefore that

$$\varphi^* = \mathrm{id}_{H^2(X,\mathbb{Z})}.$$

Let

$$\pi : \mathcal{X} \to U$$

be a representative of the Kuranishi family of X, with reference point $0 \in U$ such that $X_0 \cong X$. By functoriality of the deformation space, the automorphism φ induces the following data: there exist an open neighborhood $0 \in V \subset U$, a holomorphic map $m : V \to U$, and an isomorphism of families

$$\Phi : \mathcal{X}|_V \xrightarrow{\sim} \mathcal{X}$$

fitting into a commutative diagram

$$\begin{array}{ccc} \mathcal{X}|_V & \xrightarrow{\Phi} & \mathcal{X} \\ {\scriptstyle \pi_V}\downarrow & & \downarrow{\scriptstyle \pi} \\ V & \xrightarrow{m} & U, \end{array}$$

such that $\Phi|_{X_0} = \varphi$.

Since φ^* acts trivially on $H^2(X,\mathbb{Z})$, the induced action on periods is trivial. Equivalently, the local period map satisfies

$$\mathcal{P}_\pi \circ m = \mathcal{P}_\pi|_V.$$

By the local Torelli theorem, the period map is locally injective; hence m coincides with the inclusion of V into U. Shrinking U if necessary, we may therefore assume $V = U$ and $m = \mathrm{id}_U$. This implies that φ extends to a family of automorphisms

$$\varphi_t : X_t \to X_t \qquad (t \in U).$$

By Proposition 2.45, the Noether–Lefschetz loci corresponding to primitive classes of square 2 are dense in U. Hence there exists a hypersurface $D \subset U$ such that for every

$t \in D$ the surface X_t admits a primitive class

$$h_t \in H^{1,1}(X_t, \mathbb{Z}), \qquad (h_t, h_t) = 2.$$

For very general $t \in D$, one has $\rho(X_t) = 1$, so that

$$\mathrm{NS}(X_t) = \mathbb{Z}h_t$$

and $\pm h_t$ is ample. By the results recalled in the proof of Theorem 2.46, the surface X_t is then realized as a double cover of $\mathbb{P}^2$ branched along a smooth sextic curve B_t.

Since φ_t^* acts trivially on $H^2(X_t, \mathbb{Z})$, it must fix h_t. Therefore φ_t preserves the polarization and descends to an automorphism of the plane sextic B_t. For t very general, the sextic B_t has no non–trivial automorphisms. It follows that $\varphi_t = \mathrm{id}_{X_t}$ for all t in a non–empty open subset of D.

By holomorphicity, this implies $\varphi_t = \mathrm{id}_{X_t}$ for all $t \in U$, and in particular $\varphi = \mathrm{id}_X$. This concludes the proof. □

The proposition shows that the geometry of a K3 surface is faithfully reflected in the Hodge structure of its second cohomology. The next chapter addresses the converse problem: to what extent an integral Hodge isometry between the second cohomology groups of two K3 surfaces is induced by an isomorphism of the surfaces themselves.

Chapter 3

Global Torelli theorem for K3 surfaces

In this chapter we study the global Torelli theorem for K3 surfaces, which describes the precise relationship between isomorphisms of K3 surfaces and Hodge isometries of their second cohomology lattices.

The treatment is based on [OGr12] and [Huy16].

Theorem 3.1 (Weak Global Torelli). *Let X and X' be K3 surfaces. Then X is isomorphic to X' if and only if there exists an integral Hodge isometry*

$$\varphi\colon H^2(X',\mathbb{Z}) \xrightarrow{\sim} H^2(X,\mathbb{Z}).$$

Remark 3.2. If $f\colon X \to X'$ is an isomorphism of K3 surfaces, then the induced map

$$f^*\colon H^2(X',\mathbb{Z}) \longrightarrow H^2(X,\mathbb{Z})$$

is an integral Hodge isometry. The weak global Torelli theorem asserts that the existence of a Hodge isometry implies that the two surfaces are isomorphic.

However, the theorem does not guarantee that a given Hodge isometry φ is induced by an isomorphism. In general, there may exist Hodge isometries which are not of the form f^* for any isomorphism f.

Example 3.3 (The involution $-\mathrm{id}$). The map

$$-\mathrm{id}\colon H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathbb{Z})$$

is an integral Hodge isometry. Nevertheless, it cannot be induced by an automorphism of X. Indeed, any automorphism of a K3 surface sends Kähler classes to Kähler classes, while $-\mathrm{id}$ sends a Kähler class to its negative.

Definition 3.4 (Weyl group)**.** Let X be a K3 surface. For a class

$$\delta \in \Delta_X \subset H^{1,1}_{\mathbb{Z}}(X)$$

with $(\delta, \delta) = -2$, the *reflection in* δ is the isometry

$$r_\delta(\alpha) := \alpha + (\alpha, \delta)\,\delta, \qquad \alpha \in H^2(X, \mathbb{Z}).$$

The *Weyl group* of X is the subgroup

$$W_X \subset O\big(H^2(X, \mathbb{Z})\big)$$

generated by the reflections r_δ.

Example 3.5 (Reflections in (−2)–classes). Suppose that X is a K3 surface and that

$$\delta \in H^{1,1}_{\mathbb{Z}}(X)$$

satisfies $(\delta, \delta) = -2$. Let

$$r_\delta \in W_X$$

be the corresponding reflection. It is an integral Hodge isometry which acts as the identity on $\delta^\perp$ and sends δ to $-\delta$.

We claim that r_δ is not induced by an automorphism of X. By the Lefschetz $(1,1)$–Theorem, there exists a line bundle L on X such that $c_1(L) = \delta$. Since $(\delta, \delta) = -2$, the Riemann–Roch formula on a K3 surface gives

$$\chi(L) = 2 + \tfrac{1}{2}(L, L) = 1.$$

Using Serre duality and the triviality of K_X, we obtain

$$1 = \chi(L) = h^0(X, L) - h^1(X, L) + h^0(X, L^{-1}),$$

hence

$$h^0(X, L) + h^0(X, L^{-1}) \geq 1.$$

Thus either L or L^{-1} has a non-zero section.

Suppose that an automorphism f satisfies $f^* = r_\delta$. Then $f^*L \cong L^{-1}$, and since f is an isomorphism,

$$h^0(X, L) = h^0(X, f^*L) = h^0(X, L^{-1}).$$

It follows that both L and L^{-1} have non-zero sections. Hence both are effective, which forces $L \cong \mathcal{O}_X$. Thus $c_1(L) = 0$, contradicting $(\delta, \delta) = -2$.

Theorem 3.6 (Strong Global Torelli)**.** *Let X and X' be K3 surfaces and let*

$$\varphi\colon H^2(X',\mathbb{Z}) \xrightarrow{\sim} H^2(X,\mathbb{Z})$$

be an integral Hodge isometry. Then there exists an (unique) isomorphism

$$f\colon X \xrightarrow{\sim} X'$$

such that $f^* = \varphi$ if and only if

$$\varphi(\mathcal{K}_{X'}) = \mathcal{K}_X,$$

where $\mathcal{K}_X$ and $\mathcal{K}_{X'}$ denote the Kähler cones.

The Kähler cone admits a purely Hodge–theoretic description inside $H^{1,1}_{\mathbb{R}}(X)$ (see Proposition 3.18), so the condition in Theorem 3.6 is intrinsic to the Hodge structure.

As observed above, not every Hodge isometry is induced by an isomorphism. The strong global Torelli theorem clarifies the situation: although a given Hodge isometry

$$\varphi\colon H^2(X',\mathbb{Z}) \xrightarrow{\sim} H^2(X,\mathbb{Z})$$

need not preserve the Kähler cones, the Weyl group generated by reflections in (-2)–classes acts transitively on the chambers of the positive cone (see [Huy16, Ch. 8, Prop. 2.1]). Thus, after composing φ with a suitable element of the Weyl group, one obtains a Hodge isometry sending $\mathcal{K}_{X'}$ onto $\mathcal{K}_X$. The strong Torelli theorem then implies that this composition is induced by an isomorphism. In particular, the strong global Torelli theorem implies the weak version.

3.1 The Kähler cone

Definition 3.7 (Cone)**.** Let V be a finite–dimensional real vector space. A *cone* in V is a subset $C \subset V$ which is stable under multiplication by any strictly positive real number. In particular, a cone need not contain the origin.

Let X be a K3 surface. We consider the real vector space $H^{1,1}_{\mathbb{R}}(X)$ endowed with the intersection form. We define the cone

$$C_X := \left\{ z \in H^{1,1}_{\mathbb{R}}(X) \;\middle|\; (z,z) > 0 \right\}. \tag{3.1.1}$$

Remark 3.8 (Connected components of C_X). Since the intersection form on $H^{1,1}_{\mathbb{R}}(X)$ has signature $(1, h^{1,1}(X) - 1)$, there exists a vector $e \in H^{1,1}_{\mathbb{R}}(X)$ with $(e, e) > 0$. For every $z \in C_X$, the scalar product (z, e) is non–zero. Moreover, the sign of (z, e) cannot change along a continuous path inside C_X. Hence the subsets

$$\{z \in C_X \mid (z, e) > 0\} \quad \text{and} \quad \{z \in C_X \mid (z, e) < 0\}$$

are exactly the two connected components of C_X. These correspond to the two possible orientations of the one–dimensional subspaces of $H^{1,1}_{\mathbb{R}}(X)$ on which the intersection form is positive.

Definition 3.9 (Kähler cone)**.** The *Kähler cone* of X is the subset

$$\mathcal{K}_X \subset H^{1,1}_{\mathbb{R}}(X)$$

consisting of cohomology classes represented by Kähler forms.

By definition, one has $\mathcal{K}_X \subset C_X$.

Definition 3.10 (Positive cone)**.** The connected component of C_X containing the Kähler cone $\mathcal{K}_X$ is called the *positive cone* of X and is denoted by C_X^+.

It follows that

$$C_X = C_X^+ \sqcup (-C_X^+).$$

Definition 3.11 ((−2)–classes)**.** Set

$$\Delta_X := \big\{\delta \in H^{1,1}_{\mathbb{Z}}(X) \bigm| (\delta, \delta) = -2\big\}, \qquad \Delta_X^+ := \big\{\delta \in \Delta_X \bigm| \delta = c_1(\mathcal{O}_X(R)),\ R \in \mathrm{Eff}(X)\big\},$$

where $\mathrm{Eff}(X)$ denotes the set of effective divisors on X.

Remark 3.12. Let $\delta \in \Delta_X$. By the same Riemann–Roch argument as in Example 3.5, either δ or $-\delta$ is effective. In particular,

$$\Delta_X = \Delta_X^+ \sqcup (-\Delta_X^+).$$

Remark 3.13. Let $\delta \in \Delta_X$. The hyperplane

$$\delta^\perp \subset H^{1,1}_{\mathbb{R}}(X),$$

which will be referred to as a *wall*, satisfies $\delta^\perp \cap C_X \neq \varnothing$.

Indeed, since $(\delta, \delta) < 0$ and the intersection form on $H^{1,1}_{\mathbb{R}}(X)$ has signature $(1, h^{1,1}(X) - 1)$, the restriction of the form to $\delta^\perp$ has signature $(1, h^{1,1}(X) - 2)$. In particular, $\delta^\perp$ contains non–zero vectors of positive square, and hence $\delta^\perp \cap C_X \neq \varnothing$.

Lemma 3.14 (Local finiteness of the walls)**.** *The family of subsets*

$$\{\delta^{\perp} \cap C_X\}_{\delta \in \Delta_X}$$

is locally finite in C_X.

Reference. See [Huy16, Ch. 8]. □

Since each $\delta^{\perp}$ is closed and the family is locally finite, the complement

$$C_X \setminus \bigcup_{\delta \in \Delta_X} \delta^{\perp} \tag{3.1.2}$$

is an open subset of C_X.

Definition 3.15 (Chambers of the positive cone)**.** A connected component of the open set (3.1.2) is called an *open chamber* of C_X.

Remark 3.16. Let $\omega \in H^{1,1}_{\mathbb{R}}(X)$ be a Kähler class. Then

$$(\omega, \delta) > 0 \qquad \text{for every } \delta \in \Delta_X^+,$$

since $\delta = c_1(\mathcal{O}_X(R))$ for some effective curve $R \in \mathrm{Eff}(X)$. In particular, $\mathcal{K}_X$ does not meet any wall $\delta^{\perp}$ with $\delta \in \Delta_X$. Since $\mathcal{K}_X$ is connected, it is contained in a unique open chamber of C_X.

Lemma 3.17 (Kähler criterion for K3 surfaces)**.** *Let X be a K3 surface and let $\beta \in H^{1,1}_{\mathbb{R}}(X)$. Assume that $(\beta, \beta) > 0$, that $(\beta, \omega) > 0$ for some Kähler class ω, and that $(\beta, \delta) > 0$ for every $\delta \in \Delta_X^+$. Then β is a Kähler class.*

Reference. See [Huy16, Ch. 8]. □

Proposition 3.18. *Let X be a K3 surface. Then the Kähler cone $\mathcal{K}_X$ is an open chamber of the positive cone C_X^+.*

Proof. Let A_X be the open chamber of C_X containing $\mathcal{K}_X$. We claim that $A_X = \mathcal{K}_X$.

Fix $\alpha \in A_X$ and choose a Kähler class $\omega \in \mathcal{K}_X$. Let $R \subset X$ be an irreducible curve with $R^2 < 0$ and set

$$\delta := c_1(\mathcal{O}_X(R)) \in H^{1,1}_{\mathbb{Z}}(X).$$

Since $K_X \cong \mathcal{O}_X$, adjunction gives

$$2g(R) - 2 = (K_X + R) \cdot R = R^2.$$

As $g(R) \geq 0$ and $R^2 < 0$, it follows that $g(R) = 0$ and $R^2 = -2$, hence $\delta \in \Delta_X^+$. Because ω is Kähler and R is effective one has $(\omega, \delta) > 0$. Moreover, the segment $t \mapsto (1-t)\omega + t\alpha$ is contained in A_X, hence it does not meet the wall $\delta^\perp$; in particular the real-valued function

$$t \longmapsto \big((1-t)\omega + t\alpha, \delta\big)$$

never vanishes for $t \in [0,1]$. Since its value at $t = 0$ is positive, it remains positive for all t, and in particular

$$(\alpha, \delta) > 0, \qquad \text{i.e.} \qquad \alpha \cdot R > 0$$

for every irreducible curve $R \subset X$ with $R^2 < 0$.

We also have $(\alpha, \omega) > 0$. Indeed, if α and ω are linearly dependent then $\alpha \in C_X^+$ implies $\alpha = \lambda\omega$ for some $\lambda > 0$ and the claim is clear. Otherwise, the real plane $\langle \alpha, \omega \rangle$ has signature $(1,1)$ by the Hodge index theorem, hence the locus of vectors of positive square in this plane has exactly two connected components. Since $\alpha, \omega \in C_X^+$, they belong to the same component, and it follows that $(\alpha, \omega) > 0$.

Applying Lemma 3.17 to $\beta = \alpha$, we conclude that $\alpha \in \mathcal{K}_X$.

Therefore $A_X \subset \mathcal{K}_X$, while the reverse inclusion holds by definition of A_X. Hence $A_X = \mathcal{K}_X$, and $\mathcal{K}_X$ is an open chamber of C_X. Since $\mathcal{K}_X \subset C_X^+$, it is in fact an open chamber of the positive cone C_X^+. □

3.2 Global properties of the moduli space of marked K3 surfaces

Let $\mathcal{M}$ denote the moduli space of marked K3 surfaces introduced in Section 2.2. By definition, $\mathcal{M}$ parametrizes isomorphism classes of pairs (X, ϕ), where X is a complex K3 surface and

$$\phi : H^2(X, \mathbb{Z}) \xrightarrow{\sim} \Lambda$$

is a marking.

As explained in Proposition 2.30 and Remark 2.31, the space $\mathcal{M}$ carries a natural structure of complex manifold of dimension 20, obtained by gluing the base spaces of local Kuranishi families of marked K3 surfaces.

We now consider the period map

$$\mathcal{P} : \mathcal{M} \longrightarrow \Omega_\Lambda, \qquad (X, \phi) \longmapsto [\phi(H^{2,0}(X))].$$

By the local Torelli theorem (Corollary 2.33), this map is a local biholomorphism. A natural strategy would be to prove the global Torelli theorem by showing that $\mathcal{P}$ is a topological covering. Indeed, since the period domain Ω_Λ is simply connected, a covering map $\mathcal{P} : \mathcal{M} \to \Omega_\Lambda$ would restrict to a homeomorphism on each connected component of $\mathcal{M}$. In particular, if $\mathcal{M}$ were connected, $\mathcal{P}$ would be a global biholomorphism.

However, this approach cannot be applied directly, since the moduli space $\mathcal{M}$ turns out not to be Hausdorff. Moreover, $\mathcal{M}$ is not connected.

3.2.1 Non-Hausdorffness

Our goal is to prove that the moduli space $\mathcal{M}$ of marked K3 surfaces is not Hausdorff.

The strategy is to analyze the behavior of the period map in a suitable degeneration. The Picard–Lefschetz construction will allow us to produce distinct marked K3 surfaces with the same period that cannot be separated by disjoint neighborhoods in $\mathcal{M}$.

Definition 3.19 (Ordinary double point)**.** Let Y be a complex analytic space of dimension n, and let $p \in Y$. We say that p is an *ordinary double point* of Y if the tangent cone of Y at p is a cone over a smooth quadric hypersurface. Equivalently, the germ (Y, p) is analytically isomorphic to the germ at the origin of

$$\{x_1^2 + \cdots + x_{n+1}^2 = 0\} \subset \mathbb{C}^{n+1}.$$

Remark 3.20. Let $\varphi : X \to Y$ be the blow–up of an ordinary double point $p \in Y$. Then the exceptional divisor $\varphi^{-1}(p)$ is isomorphic to a smooth projective quadric of dimension $n-1$, and X is smooth along $\varphi^{-1}(p)$.

Example 3.21 (Resolution of a surface node). Let

$$Y = \{xy - z^2 = 0\} \subset \mathbb{C}^3,$$

which has an ordinary double point at the origin. This is an A_1–singularity.

Let $\pi : \widetilde{Y} \to Y$ be the blow–up of the origin. Then $\widetilde{Y}$ is smooth and the exceptional divisor

$$E = \pi^{-1}(0)$$

is a smooth conic in $\mathbb{P}^2$, hence $E \cong \mathbb{P}^1$. Moreover, as computed in Remark 1.53,

$$\mathcal{N}_{E/\widetilde{Y}} \cong \mathcal{O}_{\mathbb{P}^1}(-2),$$

so that $E^2 = -2$.

Thus, resolving an ordinary double point on a surface replaces the singularity by a smooth rational curve of self–intersection -2.

A Picard–Lefschetz degeneration

By a local analysis of ordinary double points due to Atiyah (see [Ati58]), after a quadratic base change the degeneration admits a simultaneous resolution, and in complex dimension 2 the smooth fiber is deformation equivalent to the blow–up of the singular fiber at its double points.

In our situation, we consider the following one–parameter degeneration of compact complex surfaces. Let $B \subset \mathbb{C}$ be a small disc centered at the origin, and let

$$f : \mathcal{Y} \to B$$

be a holomorphic family of compact complex surfaces such that:

1. $\mathcal{Y}$ is smooth;
2. all fibers $Y_t := f^{-1}(t)$ are smooth for $t \neq 0$;
3. the central fiber Y_0 has exactly one singular point, which is an ordinary double point.

Let p be the singular point of Y_0. Under these assumptions, there exist local coordinates (x, y, z) on $\mathcal{Y}$ centered at p in which the map f is given by

$$f(x, y, z) = x^2 + y^2 + z^2.$$

Set $B_0 := B \setminus \{0\}$ and $\mathcal{Y}^0 := f^{-1}(B_0)$, and denote by $f_0 := f|_{\mathcal{Y}^0}$. Since all fibers over B_0 are smooth, the groups $H^2(Y_t, \mathbb{Z})$ form a local system $R^2 f_{0*}\mathbb{Z}$ on B_0.

Fix a base point $a \in B_0$. The monodromy action of a generator of $\pi_1(B_0, a)$ on $H^2(Y_a, \mathbb{Z})$ is described by the Picard–Lefschetz formula: there exists a class

$$\delta \in H^2(Y_a, \mathbb{Z}), \qquad (\delta, \delta) = -2,$$

called the *vanishing class*, such that the monodromy transformation is

$$\gamma(\alpha) = \alpha + (\alpha, \delta)\, \delta, \qquad \alpha \in H^2(Y_a, \mathbb{Z}).$$

The Picard–Lefschetz formula shows that the monodromy transformation γ is a reflection in the class δ, hence $\gamma^2 = \mathrm{id}$. In particular, the monodromy has order two. It follows that after a quadratic base change the pulled–back local system has trivial monodromy. We now perform a base change in order to eliminate the monodromy. Let $\widetilde{B} := D(0, r^{1/2})$ and consider the double cover

$$m : \widetilde{B} \to B, \qquad m(s) = s^2.$$

Let

$$\widetilde{\mathcal{Y}} := \mathcal{Y} \times_B \widetilde{B}$$

be the family obtained from $f : \mathcal{Y} \to B$ by base change.

There is a unique point $\widetilde{p} \in \widetilde{\mathcal{Y}}$ mapping to p, and it is the only singular point of $\widetilde{\mathcal{Y}}$. In the above local coordinates, the germ of $\widetilde{\mathcal{Y}}$ at $\widetilde{p}$ is isomorphic to the germ at the origin of

$$\{x^2 + y^2 + z^2 - s^2 = 0\} \subset \mathbb{C}^4.$$

Consider the blow–up

$$\widehat{\mathcal{Y}} := \mathrm{Bl}_{\widetilde{p}}\widetilde{\mathcal{Y}} \xrightarrow{\pi} \widetilde{\mathcal{Y}}.$$

The blow–up resolves the singularity at $\widetilde{p}$, and $\widehat{\mathcal{Y}}$ is smooth.

The exceptional divisor

$$E := \pi^{-1}(\widetilde{p})$$

is a smooth quadric surface in $\mathbb{P}^3$, hence

$$E \cong \mathbb{P}^1 \times \mathbb{P}^1, \qquad \mathcal{N}_{E/\widehat{\mathcal{Y}}} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1).$$

By Nakano's contraction theorem [Nak55], such a divisor can be contracted along either ruling. Thus there exists a smooth threefold X and a birational morphism

$$\widehat{\mathcal{Y}} \longrightarrow X$$

contracting E onto a smooth rational curve

$$C \cong \mathbb{P}^1, \qquad \mathcal{N}_{C/X} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

The base change morphism $\widetilde{\mathcal{Y}} \longrightarrow \mathcal{Y}$ and the birational map $\widetilde{\mathcal{Y}} \dashrightarrow X$ induce a rational map

$$X \dashrightarrow \mathcal{Y}.$$

[1] This rational map extends to a morphism

$$X \longrightarrow \mathcal{Y}.$$

[1]Locally analytically, this birational transformation is the classical Atiyah flop associated with a threefold ordinary double point; see [Ati58].

The constructions fit into the commutative diagram over $m : \widetilde{B} \to B$:

$$\begin{array}{ccccc} \widehat{\mathcal{Y}} & \longrightarrow & \widetilde{\mathcal{Y}} & \longrightarrow & \mathcal{Y} \\ \downarrow & \searrow & \downarrow & & \downarrow \\ \widetilde{B} & = & \widetilde{B} & \xrightarrow{m} & B \end{array}$$

Let $g : X \to \widetilde{B}$ be the induced morphism. Then g is a family of smooth surfaces, the morphism $X_0 \to Y_0$ is the blow–up of the singular point of Y_0, and for $s \neq 0$ one has

$$X_s \cong Y_{s^2}.$$

Therefore the blow–up of Y_0 at its singular point is a deformation of Y_t for $t \neq 0$.

This is compatible with the Picard–Lefschetz formula. Since $g : X \to \widetilde{B}$ is a family of smooth surfaces over a disc, there is no monodromy. On the other hand, the induced homomorphism

$$m_* : \pi_1(\widetilde{B}, a^{1/2}) \to \pi_1(B, a)$$

has image $2\pi_1(B, a)$, and the Picard–Lefschetz formula implies that twice a generator of $\pi_1(B, a)$ acts trivially on $H^2(Y_a, \mathbb{Z})$.

Finally, let $C \subset X_0$ be the exceptional divisor of the blow–up $X_0 \to Y_0$. Then C is a smooth rational curve with

$$C \cdot C = -2.$$

Via parallel transport in the local system, one has natural identifications

$$H^2(X_0, \mathbb{Z}) \cong H^2(X_{a^{1/2}}, \mathbb{Z}) \cong H^2(Y_a, \mathbb{Z}),$$

under which the Poincaré dual of C corresponds, up to sign, to the vanishing cycle.

In summary, we start with a degeneration with an ordinary double point. After a quadratic base change and resolution, one obtains a smooth family whose central fiber is the blow–up of the singular fiber at the node. In particular, the exceptional curve is a smooth rational curve with self–intersection -2, whose class corresponds (up to sign) to the vanishing cycle.

Lemma 3.22 (Grauert contraction criterion)**.** *Let X be a compact complex surface and let $R \subset X$ be a smooth rational curve with $R^2 < 0$. Then there exists a proper holomorphic map*

$$\varphi : X \to Y$$

which is an isomorphism over $X \setminus R$ and contracts R to a point $p \in Y$. Moreover, $\varphi_* \mathcal{O}_X = \mathcal{O}_Y$.

Reference. See [Bar+04, Chapter III, §2]. □

Proposition 3.23. *Let X be a K3 surface containing a smooth rational curve $R \subset X$ such that $R^2 = -2$. Then:*

1. *there exists a contraction*
$$\varphi : X \to Y$$
sending R to a point $p \in Y$;

2. *Y is smooth outside p and has an ordinary double point at p;*

3. $\mathrm{Def}(Y)$ *is unobstructed and has the same dimension as* $\mathrm{Def}(X)$*;*

4. *the versal deformation families of X and Y are related by the Picard–Lefschetz construction described above.*

Proof. Since R is a smooth rational curve with negative self–intersection, Grauert's contraction criterion (Lemma 3.22) produces a proper morphism

$$\varphi : X \to Y$$

which contracts R to a point p and is an isomorphism away from R.

Because $R^2 = -2$, the singularity of Y at p is a rational double point of type A_1, that is, an ordinary double point. Thus Y is smooth outside p.

Deformations of X are unobstructed, since $H^2(X, T_X) = 0$. Moreover, the Picard–Lefschetz construction described above shows that X and Y appear as fibers of the same deformation family. In particular, the deformation space of Y is unobstructed and has the same dimension as that of X. Moreover, there exists a natural map

$$\mathrm{Def}(X) \to \mathrm{Def}(Y)$$

relating the corresponding versal families.

More precisely, there exist a family

$$f : \mathcal{Y} \to B$$

with central fiber $Y_0 \cong Y$, and a family

$$\widetilde{\pi} : \mathcal{X} \to \widetilde{B}$$

representing $\mathrm{Def}(X)$ and obtained from $\mathcal{Y}$ by quadratic base change and resolution of the resulting singularity, as shown in Subsection 3.2.1. The central fiber $\mathcal{X}_0$ is the blow–up of Y_0 at p. ☐

Example 3.24 (Degeneration of quartic K3 surfaces). Let

$$Y_0 = \{F_0 = 0\}, \qquad Y_\infty = \{F_\infty = 0\} \subset \mathbb{P}^3$$

be quartic surfaces such that Y_0 has a single ordinary double point p and Y_∞ is smooth and intersects Y_0 transversely. Let $D \subset \mathbb{C}$ be a small disc centered at 0 and consider the family

$$\mathcal{Y} := \{(x,t) \in \mathbb{P}^3 \times D \mid F_0(x) + tF_\infty(x) = 0\}.$$

Then Y_t is smooth for $t \neq 0$, while Y_0 has a single ordinary double point. The blow–up of Y_0 at p, denoted X_0, appears as the central fiber of the family obtained from $\mathcal{Y}$ by quadratic base change and resolution. Hence X_0 is a deformation of the smooth quartic surface Y_∞.

Projection from the point p defines a morphism

$$\psi : X_0 \to \mathbb{P}^2.$$

Assume that Y_0 contains no line through p. Then ψ is finite. Indeed, the fibers of the projection correspond to the strict transforms of the lines $\ell \subset \mathbb{P}^3$ through p. If ψ were not finite, some curve in X_0 would be contracted, hence would lie in the strict transform of a line through p, which would imply that Y_0 contains such a line, contradicting the assumption.

For a general line ℓ through p one has

$$\ell \cdot Y_0 = 4.$$

Since p is an ordinary double point, the intersection multiplicity at p is 2, and therefore ℓ meets Y_0 in exactly two further points. This shows that ψ has degree 2.

The ramification occurs when the residual intersection has multiplicity 2, i.e. when ℓ is tangent to Y_0 away from p. The image of the ramification divisor is a plane curve $B \subset \mathbb{P}^2$.

Since X_0 is a K3 surface, one has $K_{X_0} \cong \mathcal{O}_{X_0}$. For a double cover $\psi : X_0 \to \mathbb{P}^2$ branched along B,

$$K_{X_0} \cong \psi^*\left(K_{\mathbb{P}^2} + \tfrac{1}{2}B\right).$$

As $K_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$, it follows that $B \in |\mathcal{O}_{\mathbb{P}^2}(6)|$. Hence ψ is ramified along a plane sextic.

Thus X_0 is a double cover of $\mathbb{P}^2$ branched along a sextic curve, recovering independently that X_0 is deformation equivalent to a smooth quartic K3 surface by Kodaira's theorem (Theorem 2.46).

Corollary 3.25. *The moduli space $\mathcal{M}$ of marked K3 surfaces is not Hausdorff.*

Proof. Let X be a K3 surface containing a smooth rational curve $R \subset X$, and set

$$\gamma := c_1(\mathcal{O}_X(R)).$$

By adjunction one has $R^2 = -2$, and deformations of X are unobstructed.

Consider the Picard–Lefschetz family

$$\widetilde{\pi} : \mathcal{X} \to \widetilde{B},$$

whose central fiber satisfies $\mathcal{X}_0 \cong X$. Since $\widetilde{B}$ is simply connected and the monodromy has been eliminated by the quadratic base change, the local system $R^2\widetilde{\pi}_*\mathbb{Z}$ is trivial. Fixing a marking on the central fiber therefore induces a marking

$$F : R^2\widetilde{\pi}_*\mathbb{Z} \xrightarrow{\sim} \underline{\Lambda}_{\widetilde{B}}.$$

Set

$$\eta := F_0(\gamma) \in \Lambda, \qquad G := r_\eta \circ F.$$

For every $b \in \widetilde{B}$ we obtain marked K3 surfaces (X_b, f_b) and (X_b, g_b).

At $b = 0$ the two markings differ by the reflection in the (-2)–class γ, hence

$$[(X_0, f_0)] \neq [(X_0, g_0)].$$

Let $\iota : \widetilde{B} \to \widetilde{B}$ be the covering involution. By the Picard–Lefschetz construction described in 3.2.1, for every $b \neq 0$ there exists an isomorphism

$$\psi_b : X_b \xrightarrow{\sim} X_{\iota(b)}$$

such that

$$f_b \circ H^2(\psi_b) = g_{\iota(b)}.$$

Hence

$$(X_b, f_b) \cong (X_{\iota(b)}, g_{\iota(b)}) \qquad \text{for all } b \neq 0.$$

Let $U, V \subset \mathcal{M}$ be neighborhoods of $[(X_0, f_0)]$ and $[(X_0, g_0)]$. Since $[(X_0, f_0)]$ lies in the chart determined by $(\widetilde{\pi}, F)$, after shrinking $\widetilde{B}$ we may assume that for b sufficiently small one has $[(X_b, f_b)] \in U$. Choose such $b \neq 0$ sufficiently close to 0. Since ι is continuous and $\iota(0) = 0$, also $\iota(b)$ is arbitrarily close to 0. Hence, for b small enough, $[(X_{\iota(b)}, g_{\iota(b)})] \in V$. Since the two marked surfaces above coincide in $\mathcal{M}$, it follows that $U \cap V \neq \varnothing$.

Therefore the two points $[(X_0, f_0)]$ and $[(X_0, g_0)]$ are inseparable, and $\mathcal{M}$ is not Hausdorff. □

3.2.2 Hausdorffization

In this subsection we construct a natural Hausdorff quotient of $\mathcal{M}$ that preserves the complex structure and through which the period map factors.

This is achieved by identifying inseparable points and showing that the resulting quotient is a Hausdorff complex manifold.

Definition 3.26 (Hausdorff relation)**.** Let Z be a topological space. Two points $x, y \in Z$ are said to be *inseparable* if every pair of open neighborhoods $x \in U \subset Z$, $y \in V \subset Z$ satisfies $U \cap V \neq \varnothing$. We write

$$x \sim y$$

and call $\sim$ the *Hausdorff relation* on Z.

Let $\Delta_Z \subset Z \times Z$ denote the diagonal. By definition one has

$$\overline{\Delta_Z} = \{(x, y) \in Z \times Z \mid x \sim y\}.$$

Clearly the relation $\sim$ is reflexive and symmetric.

Proposition 3.27. *Let Z be a topological space and suppose that:*

1. *the Hausdorff relation $\sim$ is an equivalence relation, so that the quotient topological space*

$$\overline{Z} := Z/\sim$$

is defined;

2. *the quotient map*

$$\pi : Z \to \overline{Z}$$

is open.

Then $\overline{Z}$ is Hausdorff.

Proof. It suffices to show that the diagonal

$$\Delta_{\overline{Z}} \subset \overline{Z} \times \overline{Z}$$

is closed.

Consider the map

$$\varphi : Z \times Z \longrightarrow \overline{Z} \times \overline{Z}, \qquad (z_1, z_2) \longmapsto (\pi(z_1), \pi(z_2)).$$

This map is continuous and is the set–theoretic quotient map for the equivalence relation on $Z \times Z$ defined by

$$(z_1, z_2) \sim (z_1', z_2') \quad \Longleftrightarrow \quad z_1 \sim z_1' \text{ and } z_2 \sim z_2'.$$

We claim that φ is the quotient map in the category of topological spaces. Let $U \subset \overline{Z} \times \overline{Z}$ be such that $\varphi^{-1}(U)$ is open. Since π is open, it follows that U is open. Thus φ is a quotient map.

Therefore, it suffices to prove that $\varphi^{-1}(\Delta_{\overline{Z}})$ is closed in $Z \times Z$. But by definition of the Hausdorff relation,

$$\varphi^{-1}(\Delta_{\overline{Z}}) = \{(x, y) \in Z \times Z \mid x \sim y\} = \overline{\Delta_Z},$$

which is closed in $Z \times Z$. This proves that $\overline{Z}$ is Hausdorff. □

Remark 3.28 (Universal property of the Hausdorff quotient). Suppose that the hypotheses of Proposition 3.27 are satisfied, and let

$$\pi : Z \to \overline{Z}$$

be the quotient map.

Then π satisfies the following universal property: if W is a Hausdorff topological space and $f : Z \to W$ is a continuous map, there exists a unique continuous map $\overline{f} : \overline{Z} \to W$ such that the following diagram commutes:

$$\begin{array}{ccc} Z & \xrightarrow{\;f\;} & W \\ {\scriptstyle \pi}\big\downarrow & \nearrow_{\overline{f}} & \\ \overline{Z} & & \end{array}$$

that is, $f = \overline{f} \circ \pi$.

The next step is to check that the Hausdorff relation $\sim$ on $\mathcal{M}$ satisfies the assumptions of Proposition 3.27.

Definition 3.29 (Analytic k–cycle)**.** Let X be a complex analytic space. An *analytic k–cycle* on X is a finite formal sum

$$Z = \sum_i n_i V_i,$$

where $V_i \subset X$ are irreducible analytic subvarieties of complex dimension k and $n_i \in \mathbb{Z}$.

The union $\bigcup_i V_i$ is called the *support* of Z. If all coefficients $n_i \geq 0$, the cycle is said to be *effective*.

Remark 3.30. If X has dimension n, analytic $(n-1)$–cycles are precisely divisors on X.

Proposition 3.31. *Let $\pi : \mathcal{X} \to B$ and $\rho : \mathcal{Y} \to B$ be families of K3 surfaces over a smooth complex base B. Assume that the local systems $R^2\pi_*\mathbb{Z}, R^2\rho_*\mathbb{Z}$ are trivial and that there exists an isometry of local systems*

$$\Psi : R^2\rho_*\mathbb{Z} \xrightarrow{\sim} R^2\pi_*\mathbb{Z}.$$

Suppose moreover that there exists a sequence $\{b_i\} \subset B$ converging to $0 \in B$ and isomorphisms $\varphi_i : X_{b_i} \xrightarrow{\sim} Y_{b_i}$ such that, for every i, $H^2(\varphi_i) = \Psi(b_i)$.

Then there exist curves $C_j \subset X_0$ and $D_j \subset Y_0$, for j in a finite index set J, and an analytic 2–cycle

$$\Gamma_\infty = \Delta + \sum_{j \in J} n_j \left(C_j \times D_j\right) \subset X_0 \times Y_0,$$

such that:

1. *Δ is the graph of an isomorphism $X_0 \xrightarrow{\sim} Y_0$;*
2. *the Poincaré dual of Γ_∞ in $H^4(X_0 \times Y_0, \mathbb{Z})$ coincides with the Poincaré dual of the graph of φ_i.*

Sketch of proof. Consider the graphs $\Gamma_i \subset X_{b_i} \times Y_{b_i}$ of the isomorphisms φ_i. Using the Hausdorff topology on closed subsets of $\mathcal{X} \times_B \mathcal{Y}$ and compactness over a small disc $K \subset B$, one extracts a subsequence converging to a closed subset $Z \subset X_0 \times Y_0$. By Bishop's theorem on limits of analytic cycles [Bis64], Z is an analytic set, hence defines an analytic 2–cycle

$$\Gamma_\infty = \sum n_j Z_j.$$

Using the Künneth decomposition and Poincaré duality, one interprets the cohomology class of Γ_∞ as a correspondence inducing the same maps on cohomology as the graphs Γ_i. Since the φ_i are isomorphisms preserving the Hodge decomposition, there must exist a component Δ dominating both factors, and it appears with multiplicity 1.

All other components are contained in products $C_j \times D_j$ of curves. Thus

$$\Gamma_\infty = \Delta + \sum_{j \in J} n_j (C_j \times D_j).$$

Hence Δ is the graph of a bimeromorphic map $X_0 \dashrightarrow Y_0$, which turns out to be an isomorphism. For the details we refer to [OGr12, Proposition 1.4.7]. □

Remark 3.32. Let $[(X,f)], [(Y,g)] \in \mathcal{M}$ such that

$$\mathcal{P}(X,f) = \mathcal{P}(Y,g).$$

Let

$$\pi : \mathcal{X} \to U, \qquad \rho : \mathcal{Y} \to V$$

be Kuranishi families of X and Y respectively.

Shrinking U and V if necessary, we may assume that the local systems $R^2\pi_*\mathbb{Z}$ and $R^2\rho_*\mathbb{Z}$ are trivial. Choose trivializations

$$F : R^2\pi_*\mathbb{Z} \xrightarrow{\sim} \underline{\Lambda}_U, \qquad G : R^2\rho_*\mathbb{Z} \xrightarrow{\sim} \underline{\Lambda}_V$$

extending the markings f and g.

By the local Torelli theorem (Corollary 2.33), the period maps

$$\mathcal{P}_\pi : U \to \Omega_\Lambda, \qquad \mathcal{P}_\rho : V \to \Omega_\Lambda$$

are biholomorphisms onto open neighborhoods of

$$\mathcal{P}(X,f) = \mathcal{P}(Y,g).$$

After shrinking again, we may assume

$$\mathcal{P}_\pi(U) = \mathcal{P}_\rho(V).$$

Hence

$$\mathcal{P}_\rho^{-1} \circ \mathcal{P}_\pi : U \longrightarrow V$$

is an isomorphism.

Replacing V with U via this identification, we obtain representatives

$$\pi : \mathcal{X} \to B, \qquad \rho : \mathcal{Y} \to B$$

over the same base B, with $\mathcal{P}_\pi = \mathcal{P}_\rho$. Let $0 \in B$ correspond to the central fibers $X_0 \cong X$ and $Y_0 \cong Y$.

Corollary 3.33. *Let $[(X, f)], [(Y, g)] \in \mathcal{M}$ be inseparable. Then:*

1. $\mathcal{P}(X, f) = \mathcal{P}(Y, g)$;
2. $X \cong Y$.

If in addition $[(X, f)] \neq [(Y, g)]$, then $H^{1,1}_{\mathbb{Z}}(X) \neq 0$ (i.e. $H^{1,1}_{\mathbb{Z}}(Y) \neq 0$), or equivalently, there exists a nonzero class $v \in \Lambda$ such that $\mathcal{P}(X, f) = \mathcal{P}(Y, g) \in \Omega_v$.

Proof. Since the period map

$$\mathcal{P} : \mathcal{M} \to \Omega_\Lambda$$

is continuous and Ω_Λ is Hausdorff, inseparable points must have the same period. This proves (1).

To prove (2), choose representatives as in Remark 3.32, which applies by (1): let

$$i_\pi : B \hookrightarrow \mathcal{M}, \qquad i_\rho : B \hookrightarrow \mathcal{M}$$

be the open embeddings defined by the trivializations F and G.

Since points of $\mathcal{M}$ admit countable bases of neighborhoods and $[(X, f)]$, $[(Y, g)]$ are inseparable, there exists a sequence

$$[(Z_i, h_i)] \in i_\pi(B) \cap i_\rho(B)$$

converging to both $[(X, f)]$ and $[(Y, g)]$. Because $\mathcal{P}_\pi = \mathcal{P}_\rho$, we have

$$i_\pi^{-1}([(Z_i, h_i)]) = i_\rho^{-1}([(Z_i, h_i)]).$$

Set

$$b_i := i_\pi^{-1}([(Z_i, h_i)]).$$

Then $b_i \to 0$ in B.

The hypotheses of Proposition 3.31 are therefore satisfied by π and ρ, with

$$\Psi = F^{-1} \circ G.$$

Moreover, for each i there exists an isomorphism

$$\varphi_i : X_{b_i} \xrightarrow{\sim} Y_{b_i}.$$

Proposition 3.31 then yields an isomorphism

$$X_0 \cong Y_0,$$

hence $X \cong Y$.

Finally, suppose that $[(X, f)] \neq [(Y, g)]$. We claim that $H^{1,1}_{\mathbb{Z}}(X) \neq 0$.

Assume by contradiction that $H^{1,1}_{\mathbb{Z}}(X) = 0$. By Proposition 3.31, inseparability of $[(X, f)]$ and $[(Y, g)]$ yields an analytic 2–cycle

$$\Gamma_\infty = \Delta + \sum_{j \in J} n_j \, (C_j \times D_j) \subset X_0 \times Y_0,$$

where Δ is the graph of an isomorphism $X_0 \cong Y_0$ and $C_j \subset X_0$ are curves.

Since $H^{1,1}_{\mathbb{Z}}(X) = 0$, the surface X contains no nontrivial curves. Hence no such C_j can exist, and therefore

$$\Gamma_\infty = \Delta.$$

It follows that the two markings coincide, so $[(X, f)] = [(Y, g)]$, contradicting the assumption that they are distinct. Therefore $H^{1,1}_{\mathbb{Z}}(X) \neq 0$. □

Proposition 3.34. *Let $[(X, f)], [(Y, g)] \in \mathcal{M}$ be inseparable, and let $U, V \subset \mathcal{M}$ be neighborhoods of these points. Then the closure of $U \cap V$ contains neighborhoods of $[(X, f)]$ and $[(Y, g)]$.*

Proof. Work in the setup of Remark 3.32, and assume B is a ball in $\mathbb{C}^{20}$.

Let $i_\pi : B \hookrightarrow \mathcal{M}$ be the inclusion defined by F. Set

$$T := i_\pi^{-1}(V) \subset B.$$

It suffices to prove that $T = B$.

By Corollary 3.33, the boundary ∂T is contained in the locus

$$N := \{\, b \in B \mid h^{1,1}_{\mathbb{Z}}(X_b) > 0 \,\}.$$

This set has real codimension 2 in B. Assume by contradiction that $T \neq B$. Since T is open, there exists $b_1 \in T$ with $h^{1,1}_{\mathbb{Z}}(X_{b_1}) = 0$.

Choose a complex disc $D \subset B$ containing b_1 and a point $b_2 \in B \setminus T$. The set

$$N(D) := \{\, b \in D \mid h^{1,1}_{\mathbb{Z}}(X_b) > 0 \,\}$$

is countable, since it is a countable union of proper analytic subsets of the disc D. Moreover,

$$(\partial T) \cap D \subset N(D).$$

Since $D \backslash N(D)$ is connected, $(\partial T) \cap D$ cannot separate b_1 from b_2, which is a contradiction. Hence $T = B$. □

Corollary 3.35. *The inseparability relation on $\mathcal{M}$ is an equivalence relation, and the quotient map*

$$\mathcal{M} \longrightarrow \overline{\mathcal{M}}$$

is open. In particular, $\overline{\mathcal{M}}$ is a Hausdorff complex manifold.

Proof. By Corollary 3.33, inseparability is reflexive and symmetric, and transitivity follows from Proposition 3.34. Hence inseparability defines an equivalence relation on $\mathcal{M}$.

Proposition 3.34 also implies that the quotient map is open. The conclusion then follows from Proposition 3.27. □

By Corollary 3.35, the period map

$$\mathcal{P} : \mathcal{M} \to \Omega_\Lambda$$

descends to a local isomorphism

$$\overline{\mathcal{P}} : \overline{\mathcal{M}} \longrightarrow \Omega_\Lambda$$

between Hausdorff complex manifolds.

3.3 A topological criterion for covering maps

In order to prove that the period map

$$\overline{\mathcal{P}} : \overline{\mathcal{M}} \longrightarrow \Omega_\Lambda$$

is a topological covering, we use a general criterion due to Verbitsky (see [Ver13b]). The statement is given in Proposition 3.38. We first prove a general uniqueness lemma.

Lemma 3.36. *Let $f : M \to N$ be a local homeomorphism of topological spaces and assume that M is Hausdorff. Let X be a connected topological space and let $x_0 \in X$. Suppose that $\sigma, \tau : X \to M$ are continuous maps such that*

$$\sigma(x_0) = \tau(x_0) \qquad \textit{and} \qquad f \circ \sigma = f \circ \tau.$$

Then $\sigma = \tau$.

Proof. Consider the subset

$$A := \{\, x \in X \mid \sigma(x) = \tau(x) \,\}.$$

Since M is Hausdorff, A is closed in X.

We claim that A is open. Let $x \in A$ and set $m := \sigma(x) = \tau(x)$. Since f is a local homeomorphism, there exists an open neighborhood $U \subset M$ of m such that

$$f|_U : U \longrightarrow f(U)$$

is a homeomorphism.

By continuity of σ and τ, after shrinking X around x we may assume that both maps take values in U. On this neighborhood one has

$$f \circ \sigma = f \circ \tau,$$

and since $f|_U$ is injective, it follows that $\sigma = \tau$ there. Thus A is open.

Since X is connected and A contains x_0, we conclude that $A = X$. □

Definition 3.37 (Closed ball)**.** Let M be a topological manifold. A *closed ball* in M is a closed subset $D \subset M$ contained in a coordinate chart (U, φ), where

$$\varphi : U \xrightarrow{\sim} \mathbb{R}^n$$

is a homeomorphism, such that $\varphi(D)$ is a closed Euclidean ball

$$\overline{B_R(a)} \subset \mathbb{R}^n$$

of strictly positive radius. Its interior will be denoted by B and called an *open ball.* We also write $D = \overline{B}$.

Proposition 3.38 (Verbitsky's criterion)**.** *Let $f : M \to N$ be a local homeomorphism of topological manifolds and assume that M is Hausdorff. Then f is a topological covering if and only if the following condition holds: for every closed ball $B \subset N$ and every connected component C of $f^{-1}(B)$, one has*

$$f(C) = B.$$

Proof. The necessity is clear.

Conversely, since N is covered by coordinate charts, we may assume that $N = \mathbb{R}^n$. It suffices to prove that if M is connected, then

$$f : M \longrightarrow \mathbb{R}^n$$

is a homeomorphism. The general case follows by restricting f to the connected components of M.

Fix $m \in M$ and set $a := f(m)$. Let

$$I \subset [0, +\infty)$$

be the set of radii R such that there exists a continuous section

$$s_R : \overline{B_R(a)} \longrightarrow M$$

with

$$s_R(a) = m \qquad \text{and} \qquad f \circ s_R = \mathrm{id}_{\overline{B_R(a)}}.$$

Clearly $0 \in I$, and I is an interval. A compactness argument, together with Lemma 3.36, shows that I is open. Thus it suffices to prove that $\sup I = +\infty$.

Suppose by contradiction that $R_0 := \sup I < +\infty$. There exists a section

$$t_0 : \overline{B_{R_0}(a)} \longrightarrow M$$

through m. Set

$$C_0 := \mathrm{Im}(t_0) \cap f^{-1}(\overline{B_{R_0}(a)}).$$

One shows that $f|_{C_0}$ is injective and that C_0 is open in $f^{-1}(\overline{B_{R_0}(a)})$. The injectivity follows from the fact that t_0 is a section and that M is Hausdorff. The openness is proved using the local homeomorphism property together with Lemma 3.36.

By construction C_0 is closed and non–empty, hence it is a connected component of $f^{-1}(\overline{B_{R_0}(a)})$. By hypothesis one has

$$f(C_0) = \overline{B_{R_0}(a)}.$$

Thus $f|_{C_0}$ is bijective, and the previous argument shows that its inverse is continuous.

This produces a section over $\overline{B_{R_0}(a)}$, contradicting the definition of R_0. Hence $\sup I = +\infty$, and f is a homeomorphism. □

3.4 Holonomy of Calabi–Yau metrics on K3 surfaces

We now relate the holonomy results of Appendix C to the case of K3 surfaces.

Let X be a K3 surface and let ω be a Kähler class. Since X is simply connected and $c_1(X) = 0$, Theorem C.21 provides a unique Calabi–Yau metric h such that $[\omega_h] = [\omega]$. Let g denote the underlying Riemannian metric. By Theorem C.24, for every $p \in X$ one has

$$\mathrm{Hol}_p(X, g) \subset \mathrm{SU}(T_pX, h).$$

As X has complex dimension 2, one has

$$\mathrm{SU}(T_pX, h) \cong \mathrm{SU}(2),$$

where SU(2) denotes the group of complex 2×2 unitary matrices of determinant 1.

3.4.1 A quaternionic description

In order to make the inclusion

$$\mathrm{Hol}_p(X, g) \subset \mathrm{SU}(2) \subset \mathrm{SO}(4)$$

more explicit, we describe SO(4) using the quaternionic structure of $\mathbb{R}^4$. This will allow us to identify the subgroup corresponding to SU(2) and to determine its centralizer. These facts will be used in the next subsection to construct the twistor family associated with the Calabi–Yau metric on X.

Proposition 3.39. *There exists a double covering*

$$\mathrm{SU}(2) \times \mathrm{SU}(2) \longrightarrow \mathrm{SO}(4)$$

with kernel

$$\{(I_2, I_2), (-I_2, -I_2)\}.$$

Proof. Let $\mathbb{H}$ be the algebra of quaternions, with basis $\{1, i, j, k\}$ over $\mathbb{R}$ satisfying

$$i^2 = j^2 = k^2 = -1, \qquad ij = k, \quad ji = -k, \qquad jk = i, \quad kj = -i.$$

For

$$x = x_1 + x_2 i + x_3 j + x_4 k$$

define the conjugate

$$\overline{x} = x_1 - x_2 i - x_3 j - x_4 k.$$

One checks that

$$x\overline{x} = \overline{x}x.$$

The quaternionic Hermitian product is defined by

$$\langle w, z \rangle := \overline{w} z.$$

Its real part coincides with the standard Euclidean scalar product on $\mathbb{R}^4 \cong \mathbb{H}$.

Let

$$\mathrm{Sp}(1) := \{ z \in \mathbb{H} \mid z\overline{z} = 1 \}$$

be the group of unit quaternions.

Right multiplication endows $\mathbb{H}$ with the structure of a complex vector space of dimension 2. Choosing the complex basis $\{1, j\}$, one verifies that

$$\langle w_1 + w_2 j,\ z_1 + z_2 j \rangle = w_1 \overline{z_1} + w_2 \overline{z_2} + (w_1 \overline{z_2} - w_2 \overline{z_1}) j.$$

Thus the quaternionic Hermitian product decomposes into the standard Hermitian product and the standard volume form. It follows that left multiplication by a unit quaternion defines an element of U(2). Since Sp(1) is connected and the determinant is 1 at the identity, it is identically 1, hence the action lies in SU(2). This yields an isomorphism

$$\mathrm{Sp}(1) \cong \mathrm{SU}(2). \tag{3.4.1}$$

On the other hand, right multiplication by a unit quaternion defines an $\mathbb{R}$-linear automorphism of $\mathbb{H}$ which preserves the real part of the quaternionic product. Indeed,

$$\mathrm{Re}\langle w\mu, z\mu \rangle = \mathrm{Re}\langle w, z \rangle.$$

Hence right multiplication determines an element of SO(4).

Combining left and right multiplication, one obtains a homomorphism

$$\mathrm{Sp}(1) \times \mathrm{Sp}(1) \longrightarrow \mathrm{SO}(4), \qquad (\mu, \lambda) \longmapsto L_\mu \circ R_{\lambda^{-1}}.$$

Since the kernel is finite, its Lie algebra is zero, hence the differential at the identity is injective. As

$$\dim\big(\mathrm{Sp}(1) \times \mathrm{Sp}(1)\big) = \dim \mathrm{SO}(4) = 6,$$

it follows that the differential is an isomorphism. Therefore the image is an open subgroup of SO(4). Since $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ is compact, the image is compact and hence closed. Being both open and closed in the connected group SO(4), the image must be all of SO(4). □

Remark 3.40. The double covering

$$\mathrm{SU}(2) \times \mathrm{SU}(2) \longrightarrow \mathrm{SO}(4)$$

admits a complex analogue. Indeed, there exists a double covering

$$\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{SO}_4(\mathbb{C}).$$

Geometrically this can be understood using the classical identification of a smooth quadric surface

$$Q \subset \mathbb{P}^3$$

with

$$Q \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

The projections

$$\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$$

exhibit Q as a ruled surface in two different ways. Accordingly, the variety of lines contained in Q is

$$F_1(Q) \cong \mathbb{P}^1 \sqcup \mathbb{P}^1,$$

where each component parametrizes one of the two rulings of lines.

The group $\mathrm{SL}_2(\mathbb{C})$ acts on $\mathbb{P}^1$ by projective transformations, hence the two factors of $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ act on the two factors of $\mathbb{P}^1 \times \mathbb{P}^1$ and therefore on the two families of lines. This action induces the homomorphism

$$\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{SO}_4(\mathbb{C}),$$

whose kernel is $\{(I_2, I_2), (-I_2, -I_2)\}$, giving the double covering.

Proposition 3.41. *Let*

$$\mathrm{SU}(2) \subset \mathrm{SO}(4)$$

be the subgroup corresponding to left multiplication by unit quaternions via the isomorphism (3.4.1)*, that is*

$$\mathrm{SU}(2) = \{L_\mu \mid \mu \in \mathrm{Sp}(1)\}.$$

Then the centralizer of $\mathrm{SU}(2)$ *in* $\mathrm{SO}(4)$ *is*

$$\{A \in \mathrm{SO}(4) \mid AU = UA \text{ for all } U \in \mathrm{SU}(2)\} = \{R_\lambda \mid \lambda \in \mathrm{Sp}(1)\}.$$

In particular, an element $I \in \mathrm{SO}(4)$ *satisfying*

$$I^2 = -\mathrm{Id}$$

commutes with $\mathrm{SU}(2)$ *if and only if* $I = R_\lambda$ *for some purely imaginary unit quaternion* $\lambda \in \mathbb{H}$.

Proof. By the construction of the homomorphism

$$\mathrm{Sp}(1) \times \mathrm{Sp}(1) \longrightarrow \mathrm{SO}(4), \qquad (\mu, \lambda) \longmapsto L_\mu \circ R_{\lambda^{-1}},$$

given in the previous proposition, right multiplication R_λ commutes with every left multiplication L_μ. Hence the right-hand side is contained in the left-hand side.

Conversely, suppose that an element of $\mathrm{SO}(4)$ of the form

$$L_{\mu_0} \circ R_{\lambda_0}$$

commutes with all L_μ, $\mu \in \mathrm{Sp}(1)$. Since R_{λ_0} already commutes with all left multiplications, it follows that L_{μ_0} commutes with every L_μ. Thus L_{μ_0} lies in the center of $\mathrm{SU}(2)$. As the center of $\mathrm{SU}(2)$ is $\{\pm\mathrm{Id}\}$, we obtain $L_{\mu_0} = \pm\mathrm{Id}$. Using that $(-1,-1)$ lies in the kernel of the covering constructed above, we conclude that the commuting elements are precisely the R_λ.

For the last statement, observe that $R_\lambda^2 = R_{\lambda^2}$. Hence $R_\lambda^2 = -\mathrm{Id}$ if and only if $\lambda^2 = -1$, which is equivalent to λ being purely imaginary and of norm 1. □

3.4.2 The twistor family and its periods

We now return to the K3 surface X endowed with a Calabi–Yau metric h. Keeping the notation of the previous subsection, fix a point $p \in X$. By Proposition 3.41,

$$\mathrm{Hol}_p(X, g) \subset \mathrm{SU}(2) \subset \mathrm{SO}(4),$$

and the centralizer of $\mathrm{SU}(2)$ inside $\mathrm{SO}(4)$ is the subgroup

$$\{R_\lambda \mid \lambda \in \mathrm{Sp}(1)\}.$$

Proposition 3.42. *The orthogonal endomorphisms of* T_pX *of square* $-\mathrm{Id}$ *which commute with* $\mathrm{Hol}_p(X, g)$ *are precisely the maps* R_λ *with* $\lambda \in \mathrm{Sp}(1)$ *satisfying* $\lambda^2 = -1$. *In particular, they are parametrized by*

$$S^2 = \{\lambda \in \mathbb{H} \mid \lambda\overline{\lambda} = 1,\ \Re(\lambda) = 0\}.$$

Proof. By the description of the centralizer, every orthogonal endomorphism commuting with $\mathrm{Hol}_p(X,g)$ is of the form R_λ with $\lambda \in \mathrm{Sp}(1)$. The condition that $R_\lambda^2 = -\mathrm{Id}$ is equivalent to $R_{\lambda^2} = -\mathrm{Id}$, hence to $\lambda^2 = -1$. This holds if and only if λ is purely imaginary and of unit norm. □

Proposition 3.43. *The sphere*

$$S^2 = \{\lambda \in \mathbb{H} \mid \lambda\overline{\lambda} = 1,\ \Re(\lambda) = 0\}$$

admits a canonical orientation and is canonically identified with $\mathbb{P}^1$.

Proof. If $I_p \in S^2$ and $J_p \in S^2$ is orthogonal to I_p, then $\{J_p, J_p \circ I_p\}$ is declared to be a positively oriented basis of $T_{I_p}S^2$. This determines an orientation on S^2. Equipped with this orientation, S^2 admits a canonical identification $S^2 \cong \mathbb{P}^1$, where $\mathbb{P}^1$ is viewed as the Riemann sphere. □

Proposition 3.44. *Let* $I_p \in S^2$. *There exists a unique parallel orthogonal endomorphism*

$$I : TX \longrightarrow TX$$

such that $I^2 = -\mathrm{Id}$ *and* $I(p) = I_p$. *Moreover,* I *defines an integrable complex structure on the smooth manifold underlying* X.

Reference. See [Lee97, Ch. 6]. □

Proposition 3.45. *The complex structures obtained above fit together*[2] *to give a complex structure on the smooth manifold* $X \times \mathbb{P}^1$ *such that the projection*

$$\pi : \mathcal{X} \longrightarrow \mathbb{P}^1$$

is holomorphic and the fiber over $I \in \mathbb{P}^1$ *is the complex surface* (X, I).

Reference. See [Bes87, Ch. 14.G, Thm. 14.40]. □

[2]The twistor space $\mathcal{X}$ is identified with $X \times S^2$ as a smooth manifold. Proposition 3.45 asserts the integrability of the almost complex structure $\mathcal{J}$ on $\mathcal{X}$, which is defined by the family $\{I_\lambda\}_{\lambda \in S^2}$ on the tangent bundle of the fibers and by the standard complex structure of $\mathbb{P}^1$ on the tangent bundle of the base. Consequently, the transition maps between local charts are holomorphic, and $\mathcal{X}$ inherits the structure of a 3-dimensional complex manifold such that the projection π onto $\mathbb{P}^1$ is a holomorphic submersion.

Definition 3.46 (Twistor family)**.** The holomorphic family

$$\pi : \mathcal{X} \longrightarrow \mathbb{P}^1$$

constructed in Proposition 3.45 is called the *twistor family* associated with the Calabi–Yau metric h.

Proposition 3.47. *For every $I \in \mathbb{P}^1$, the complex surface (X, I) is a K3 surface.*

Proof. Let $J \in \mathbb{P}^1$ be orthogonal to I and set $K = J \circ I$. Define

$$\omega_I(v, w) = g(Iv, w), \qquad \sigma_I = \omega_J + \sqrt{-1}\,\omega_K.$$

Since I, J, K are parallel, the forms $\omega_I, \omega_J, \omega_K$ are parallel. Hence σ_I is parallel and therefore closed. A direct computation shows that σ_I is of type $(2, 0)$ with respect to I. Being parallel, it is nowhere vanishing. Thus (X, I) admits a nowhere vanishing holomorphic 2-form, and hence is a K3 surface. □

Proposition 3.48. *Let*

$$W(X, h) = \{a\,\omega_I + b\,\omega_J + c\,\omega_K \mid (a, b, c) \in \mathbb{R}^3\}.$$

Then the natural map

$$W(X, h) \longrightarrow H^2_{\mathrm{dR}}(X)$$

is injective.

Proof. Since ω_I is a Kähler form,

$$\int_X \omega_I \wedge \omega_I > 0.$$

Moreover,

$$0 = \int_X \omega_I \wedge \sigma_I = \int_X \omega_I \wedge \omega_J + \sqrt{-1} \int_X \omega_I \wedge \omega_K.$$

Thus

$$\int_X \omega_I \wedge \omega_J = \int_X \omega_I \wedge \omega_K = 0.$$

By cyclic permutation of I, J, K the same argument shows that all mixed pairings vanish. Hence the classes of $\omega_I, \omega_J, \omega_K$ are non-zero and mutually orthogonal, and the map is injective. □

Remark 3.49. Since $\mathcal{X}$ is the smooth manifold $X \times \mathbb{P}^1$, each fiber $\pi^{-1}(I)$ is diffeomorphic to X. Hence there is a natural identification

$$H^2(\pi^{-1}(I), \mathbb{Z}) \cong H^2(X, \mathbb{Z}).$$

Under this identification, the periods of the fibers of π are precisely the isotropic lines in

$$\mathbb{P}(W(X, h)_{\mathbb{C}})$$

with respect to the intersection form.

Definition 3.50 (Twistor line)**.** Let $W \subset H^2(X, \mathbb{R})$ be a 3–dimensional subspace on which the intersection form is positive definite. The associated *twistor line* is denoted by T_W and is the conic

$$T_W = \{[\alpha] \in \mathbb{P}(W_{\mathbb{C}}) \mid q_\Lambda(\alpha) = 0\}.$$

Proposition 3.51. *Let*

$$W := W(X, h) \subset H^2(X, \mathbb{R})$$

be the associated 3*–dimensional positive subspace. Then the restriction of the period map*

$$\mathcal{P} : \overline{\mathcal{M}} \longrightarrow \Omega_\Lambda$$

to the twistor family

$$\pi : \mathcal{X} \longrightarrow \mathbb{P}^1$$

associated with h identifies $\mathbb{P}^1$ biholomorphically with the twistor line T_W.

Proof. By construction of the twistor family, for each $I \in \mathbb{P}^1$ the period of the fiber (X, I) is generated by the holomorphic 2–form

$$\sigma_I = \omega_J + \sqrt{-1}\,\omega_K,$$

hence lies in $\mathbb{P}(W_{\mathbb{C}})$ and satisfies $(\sigma_I, \sigma_I) = 0$. Thus the image of $\mathbb{P}^1$ under $\mathcal{P}$ is contained in T_W.

Conversely, every isotropic line in $\mathbb{P}(W_{\mathbb{C}})$ is generated by a complex linear combination $a\,\omega_I + b\,\omega_J + c\,\omega_K$ with $(a, b, c) \in S^2$, which corresponds to a complex structure in the twistor family. Hence the image coincides with T_W.

Since the period map is a local biholomorphism and both $\mathbb{P}^1$ and T_W are compact Riemann surfaces, the map is a biholomorphism. □

Proposition 3.52. *Let*

$$W \subset \Lambda_{\mathbb{R}}$$

be a 3–dimensional positive definite subspace and assume that

$$W^{\perp} \cap \Lambda = \{0\}.$$

Suppose that there exists $[(X,f)] \in \overline{\mathcal{M}}$ *such that* $\overline{\mathcal{P}}([(X,f)]) \in T_W$. *Then there exists one and only one lift of* T_W *to* $\overline{\mathcal{M}}$ *through* $[(X,f)]$.

Proof. Uniqueness follows from Lemma 3.36, since $\overline{\mathcal{M}}$ is Hausdorff, $\overline{\mathcal{P}}$ is a local homeomorphism, and T_W is connected.

We prove the existence.

Set

$$0 \neq v \in \Lambda.$$

Since $W^{\perp} \cap \Lambda = \{0\}$, the set

$$T_W \cap \Omega_v = \{\, [\alpha] \in T_W \mid (\alpha, v) = 0 \,\}$$

is finite (in fact it consists of at most two points). As Λ is countable, it follows that there exists a dense subset

$$T_W^{\mathrm{gen}} \subset T_W$$

such that for every $[\alpha] \in T_W^{\mathrm{gen}}$ one has

$$(\Re(\alpha) \oplus \Im(\alpha))^{\perp} \cap \Lambda = \{0\}.$$

Let

$$\rho : \mathcal{X} \to U$$

be a representative of $\mathrm{Def}(X)$, and choose a trivialization

$$F : R^2\rho_*\mathbb{Z} \xrightarrow{\sim} \underline{\Lambda}_U$$

extending the marking f. Since $\mathcal{P}_\rho$ is a local biholomorphism and

$$\mathcal{P}_\rho(0) = \overline{\mathcal{P}}([(X,f)]) \in T_W,$$

after shrinking U we may assume that

$$\mathcal{P}_\rho(U) \cap T_W$$

is an open neighborhood of $\mathcal{P}_\rho(0)$ in T_W. As T_W^{gen} is dense in T_W, there exists $t \in U$ such that

$$\mathcal{P}_\rho(t) \in T_W^{\text{gen}}.$$

Denote by (X_t, f_t) the corresponding marked surface.

Let $\beta \in W_{\mathbb{C}}$ be a generator of

$$W \cap \mathcal{P}_\rho(t)^\perp.$$

Then

$$\omega_t := f_t^{-1}(\beta) \in H^{1,1}_{\mathbb{R}}(X_t).$$

Since $\mathcal{P}_\rho(t) \in T_W^{\text{gen}}$, we have

$$H^{1,1}_{\mathbb{Z}}(X_t) = \{0\}.$$

By the description of the Kähler cone of a K3 surface, either ω_t or $-\omega_t$ is a Kähler class; replacing β by $-\beta$ if necessary, we may assume that ω_t is Kähler.

Let h be the Calabi–Yau metric in the class ω_t (Theorem C.21), and let

$$\pi : \mathcal{Y} \to \mathbb{P}^1$$

be the associated twistor family. By Proposition 3.51, the period map sends the base $\mathbb{P}^1$ biholomorphically onto T_W.

Hence π determines a holomorphic map

$$\mathbb{P}^1 \longrightarrow \overline{\mathcal{M}}$$

lifting T_W.

By construction, one fiber of this family has period $\overline{\mathcal{P}}([(X, f)])$. Applying Proposition 3.31, the corresponding marked K3 surface is inseparable from (X, f), hence represents the same point in $\overline{\mathcal{M}}$.

Therefore the image of $\mathbb{P}^1$ contains $[(X, f)]$, and we obtain the desired lift. □

3.5 Geometry of twistor conics

Twistor families will be used to verify that the hypotheses of Proposition 3.38 are satisfied by the period map

$$\overline{\mathcal{P}} : \overline{\mathcal{M}} \longrightarrow \Omega_\Lambda.$$

We therefore begin with a discussion of the geometry of twistor conics in the period domain.

Let L be an even lattice of signature $(3, b-3)$, where $b = \text{rk}(L)$.

Definition 3.53 (Twistor conic)**.** Let $W \subset L_{\mathbb{R}}$ be a 3–dimensional subspace on which the quadratic form is positive definite. The associated *twistor conic* is

$$T_W := \{ [\sigma] \in \Omega_L \mid \Re(\sigma), \Im(\sigma) \subset W \}.$$

Equivalently, under the identification (2.1.1) with the Grassmannian,

$$T_W = \{ V \in \mathrm{Gr}^{\mathrm{or}}_{+}(2, L_{\mathbb{R}}) \mid V \subset W \}.$$

Remark 3.54. The set T_W is called a *twistor conic* because it is a smooth conic in the projective plane $\mathbb{P}(W_{\mathbb{C}})$.

Indeed, if $[\sigma] \in T_W$, then $\Re(\sigma), \Im(\sigma) \subset W$, hence $\sigma \in W_{\mathbb{C}}$. Thus

$$T_W \subset \mathbb{P}(W_{\mathbb{C}}) \cong \mathbb{P}^2.$$

Inside this plane it is defined by the quadratic equation

$$(\sigma, \sigma) = 0.$$

Since the quadratic form is nondegenerate on $W_{\mathbb{C}}$, the corresponding projective quadric is smooth. Hence T_W is a smooth conic and therefore isomorphic to $\mathbb{P}^1$.

Definition 3.55 (Generic twistor conic)**.** A twistor conic T_W is said to be *generic* if

$$W^{\perp} \cap L = \{0\}.$$

Remark 3.56. If T_W is generic, then there exists a dense open subset of T_W consisting of points $[\sigma]$ such that

$$(\Re(\sigma) \oplus \Im(\sigma))^{\perp} \cap L = \{0\}.$$

In particular, for such points the corresponding oriented positive 2–planes contain no nonzero integral classes.

Definition 3.57 (Equivalence relative to a disc)**.** Let $D \subset \Omega_L$ be a closed disc, that is, a closed subset contained in a coordinate chart and homeomorphic to a closed Euclidean ball of strictly positive radius. Let B be its interior, so that $D = \overline{B}$.

Let $x, y \in D$. We say that *x is equivalent to y relative to D* if there exist generic twistor conics

$$T_{W_1}, \ldots, T_{W_n} \subset \Omega_L$$

and points $x_1, \ldots, x_n \in \Omega_L$ such that:

1. $x_1 = x$ and $x_n = y$;
2. for each $i = 1, \ldots, n-1$ one has $x_i, x_{i+1} \in T_{W_i}$, and moreover $x_n \in T_{W_n}$;
3. for each $i = 1, \ldots, n-1$, the points x_i and x_{i+1} belong to the same connected component of $T_{W_i} \cap B$.

We denote this relation by

$$x \sim_D y.$$

Remark 3.58. The relation $\sim_D$ is an equivalence relation on D.

Proposition 3.59. *Let $D \subset \Omega_L$ be a closed disc. Then D consists of a single equivalence class with respect to $\sim_D$.*

Proof. Since D is connected, it suffices to prove that $\sim_D$–equivalence classes are open in D.

We use the real interpretation of the period domain. There is a natural identification

$$\Omega_L \cong \mathrm{Gr}^{\mathrm{or}}_+(2, L_{\mathbb{R}}),$$

given by the isomorphism (2.1.1). Consequently, a twistor conic associated with a positive 3–dimensional subspace $W \subset L_{\mathbb{R}}$ is given by

$$T_W = \{\, (V, \tau) \in \mathrm{Gr}^{\mathrm{or}}_+(2, L_{\mathbb{R}}) \mid V \subset W \,\}.$$

Let

$$\pi : \mathrm{Gr}^{\mathrm{or}}_+(2, L_{\mathbb{R}}) \longrightarrow \mathrm{Gr}_+(2, L_{\mathbb{R}})$$

be the natural double covering forgetting the orientation. Fix a point $V \in D$. Since π is a covering map, there exists an open neighbourhood $U \subset D$ of V such that $\pi|_U$ is a homeomorphism onto its image. Replacing D with U, it suffices to prove openness of equivalence inside $\mathrm{Gr}_+(2, L_{\mathbb{R}})$. Hence from now on we work in

$$\mathrm{Gr}_+(2, L_{\mathbb{R}}),$$

the open subset of $\mathrm{Gr}(2, L_{\mathbb{R}})$ parametrizing positive definite 2–planes.

Choose once and for all a positive definite scalar product on $L_{\mathbb{R}}$ and denote the corresponding norm by $\| \cdot \|$. Let $V \in D$ and fix a basis $\{u, w\}$ of V. For $r > 0$ define

$$N_r = \big\{ \langle u', w' \rangle \;\big|\; \|u' - u\| < r,\ \|w' - w\| < r \big\}.$$

For r sufficiently small, N_r is an open neighbourhood of V contained in $\mathrm{Gr}_+(2, L_{\mathbb{R}})$, and moreover $N_r \subset D$.

We claim that, for r small enough, every element of N_r is equivalent to V relative to D.

Let $\varepsilon > 0$ be small. One can choose vectors $c, d \in L_{\mathbb{R}}$ with

$$\|c - w\| < \varepsilon, \qquad \|d - u\| < \varepsilon,$$

such that:

1. the 3–spaces $\langle u, w, c\rangle$ and $\langle u, w, d\rangle$ are positive definite;
2. $\langle u, w, c\rangle^{\perp} \cap L = \{0\}$, $\langle u, w, d\rangle^{\perp} \cap L = \{0\}$.

The second condition ensures that the corresponding twistor conics are generic.

Let $\delta > 0$ be small and let

$$V' = \langle u', w'\rangle \in N_{\delta}.$$

For δ sufficiently small, the 3–spaces

$$\langle u', w', c\rangle, \qquad \langle u', w', d\rangle$$

are positive and satisfy the same genericity condition.

Define the positive 3–spaces

$$W_1 = \langle u, w, c\rangle, \quad W_2 = \langle u, w', c\rangle, \quad W_3 = \langle u, w', d\rangle, \quad W_4 = \langle u', w', d\rangle.$$

Each T_{W_i} is a generic twistor conic.

We now exhibit a chain of points connecting V and V':

$$V,\ \langle u, c\rangle,\ \langle u, w'\rangle,\ \langle d, w'\rangle,\ V'.$$

Consecutive elements lie in the same twistor conic T_{W_i}.

It remains to check that, for ε and δ sufficiently small, each pair of consecutive points lies in the same connected component of $T_{W_i} \cap N_r$.

We treat the first step. For $0 \le t \le 1$ consider

$$\gamma_1(t) = \langle u,\ tw + (1 - t)c\rangle.$$

One has

$$\|tw + (1 - t)c - w\| = \|(1 - t)(c - w)\| \le 2\varepsilon.$$

Hence, if $2\varepsilon < r$, the path $\gamma_1(t)$ is contained in N_r, and therefore V and $\langle u, c\rangle$ belong to the same connected component of $T_{W_1} \cap N_r$.

For the second step, consider

$$\gamma_2(t) = \langle u,\, tc + (1-t)w'\rangle.$$

Using

$$\|tc + (1-t)w' - w\| \leq 2\delta + \varepsilon,$$

we conclude that, if $2\delta + \varepsilon < r$, the planes $\langle u, c\rangle$ and $\langle u, w'\rangle$ lie in the same connected component of $T_{W_2} \cap N_r$.

The remaining two steps are handled similarly, using analogous estimates.

Choosing ε and δ small compared with r, we conclude that every element of N_r is equivalent to V relative to D. Hence the equivalence class of V is open in D.

Since D is connected, there is only one equivalence class in D. □

Proposition 3.60. *Let $D \subset \Omega_L$ be a closed smooth disc and let $(V, \tau) \in \partial D$.*

Then there exists a generic twistor conic T_W such that

$$(V, \tau) \in \partial\big(T_W \cap D\big).$$

Proof. Let V be the oriented positive plane corresponding to (V, τ).

The tangent space to $\mathrm{Gr}^{\mathrm{or}}_+(2, L_\mathbb{R})$ at V is canonically identified with

$$\mathrm{Hom}(V, V^\perp),$$

where orthogonality is taken with respect to the quadratic form q_L. Indeed, $\mathrm{Gr}^{\mathrm{or}}_+(2, L_\mathbb{R})$ is an open subset of the oriented Grassmannian of 2–planes, since positivity of the quadratic form is an open condition, and therefore the tangent space coincides with that of the usual Grassmannian.

Let $u \in V^\perp$ be a vector satisfying

$$q_L(u) > 0.$$

Denote by $W_u \subset L_\mathbb{R}$ the subspace spanned by V and u. Then W_u is a positive 3–dimensional subspace and hence determines a twistor conic T_{W_u}.

The tangent space at V to T_{W_u} is naturally identified with

$$T_V(T_{W_u}) = \mathrm{Hom}(V, \mathbb{R}u) \subset \mathrm{Hom}(V, V^\perp).$$

We now show that one can choose u such that:

1. $W_u^\perp \cap L = \{0\}$, so that T_{W_u} is generic;

2. $\mathrm{Hom}(V, \mathbb{R}u)$ is not contained in the tangent space $T_V(\partial D)$.

Since D is a smooth closed disc, its boundary ∂D is a smooth hypersurface in Ω_L. Hence $T_V(\partial D)$ is a codimension–1 subspace of

$$T_V\Omega_L \cong \mathrm{Hom}(V, V^\perp).$$

Therefore it suffices to show that the subset

$$\mathcal{S} = \{\, \varphi \in \mathrm{Hom}(V, V^\perp) \mid \dim(\mathrm{Im}\,\varphi) = 1,\ q_L|_{\mathrm{Im}\,\varphi} > 0\}$$

spans the whole vector space $\mathrm{Hom}(V, V^\perp)$.

To see this, observe that $\mathrm{Hom}(V, V^\perp)$ is naturally isomorphic to $V^\vee \otimes V^\perp$. Elements of $\mathcal{S}$ are precisely the rank–1 tensors $v^\vee \otimes u$ where $u \in V^\perp$ satisfies $q_L(u) > 0$. Since $V^\perp$ contains a nonempty open cone of positive vectors and rank–1 tensors span $V^\vee \otimes V^\perp$, it follows that $\mathcal{S}$ spans all of $\mathrm{Hom}(V, V^\perp)$.

Consequently, one can choose u so that $\mathrm{Hom}(V, \mathbb{R}u)$ is not contained in $T_V(\partial D)$. For such a choice of u, the corresponding twistor conic T_{W_u} intersects D transversely at (V, τ), and hence

$$(V, \tau) \in \partial\big(T_{W_u} \cap D\big).$$

This completes the proof. □

3.6 Proof of the global Torelli theorem

We are now in a position to complete the proof of the global Torelli theorem. The key step is to show that the period map

$$\overline{\mathcal{P}} : \overline{\mathcal{M}} \longrightarrow \Omega_\Lambda$$

is a topological covering. This will be achieved by applying Verbitsky's criterion, using the geometry of generic twistor conics developed in the previous section.

Theorem 3.61. *The period map*

$$\overline{\mathcal{P}} : \overline{\mathcal{M}} \longrightarrow \Omega_\Lambda$$

is a topological covering.

Proof. By Corollary 3.35, $\overline{\mathcal{M}}$ is a Hausdorff complex manifold and $\overline{\mathcal{P}}$ is a local biholomorphism; in particular, $\overline{\mathcal{P}}$ is a local homeomorphism.

We apply Verbitsky's criterion (Proposition 3.38). Let $D \subset \Omega_\Lambda$ be a closed ball and let C be a connected component of $\overline{\mathcal{P}}^{-1}(D)$. We must show that $\overline{\mathcal{P}}(C) = D$.

Set $S := \overline{\mathcal{P}}(C) \subset D$. Since $\overline{\mathcal{P}}$ is a local homeomorphism, S is nonempty and open.

We claim that S is saturated with respect to the equivalence relation $\sim_D$ introduced in the previous section. Assuming the claim, Proposition 3.59 implies that D consists of a single $\sim_D$–equivalence class, hence $S = D$. Therefore $\overline{\mathcal{P}}(C) = D$, and Verbitsky's criterion yields that $\overline{\mathcal{P}}$ is a topological covering.

It remains to prove the saturation claim. Let $x \in S$ and let $y \in D$ with $x \sim_D y$. Choose $m \in C$ such that $\overline{\mathcal{P}}(m) = x$. By definition of $\sim_D$ there exist generic twistor conics $T_{W_1}, \dots, T_{W_n} \subset \Omega_\Lambda$ and points $x_1, \dots, x_n \in D$ such that $x_1 = x$, $x_n = y$, and for each i the points x_i and x_{i+1} lie in the same connected component of $T_{W_i} \cap \operatorname{int}(D)$.

We prove by induction that $x_i \in S$ for all i. Assume that $x_i \in S$ and choose $m_i \in C$ with $\overline{\mathcal{P}}(m_i) = x_i$.

Since T_{W_i} is generic and contains x_i, Proposition 3.52 provides a unique lift of T_{W_i} to $\overline{\mathcal{M}}$ through m_i, that is, a holomorphic map

$$\iota_i : \mathbb{P}^1 \to \overline{\mathcal{M}}$$

whose image $Z_i := \iota_i(\mathbb{P}^1)$ contains m_i and such that $\overline{\mathcal{P}}|_{Z_i} : Z_i \to T_{W_i}$ is a biholomorphism.

Let U_i be the connected component of $T_{W_i} \cap \operatorname{int}(D)$ containing x_i. Then the connected component of $Z_i \cap \overline{\mathcal{P}}^{-1}(D)$ containing m_i maps homeomorphically onto U_i. In particular, it is a connected subset of $\overline{\mathcal{P}}^{-1}(D)$ containing m_i, hence it is contained in the connected component C. Therefore $x_{i+1} \in S$.

By induction $y = x_n \in S$. Hence S is $\sim_D$–saturated, completing the proof. □

Remark 3.62. Since the period domain Ω_Λ is simply connected, any topological covering over it is a disjoint union of homeomorphisms. It follows that the restriction of $\overline{\mathcal{P}}$ to any connected component of $\overline{\mathcal{M}}$ is a biholomorphism onto Ω_Λ.

Now let Λ be the K3 lattice. If (X, ϕ) is a marked K3 surface, the marking

$$\phi : H^2(X, \mathbb{Z}) \xrightarrow{\sim} \Lambda$$

induces canonical identifications

$$O\big(H^2(X,\mathbb{Z})\big) \cong O(\Lambda), \qquad O^+\big(H^2(X,\mathbb{Z})\big) \cong O^+(\Lambda).$$

Definition 3.63 (Action on marked K3 surfaces)**.** The group $O(\Lambda)$ acts on the moduli space $\mathcal{M}$ of marked K3 surfaces by changing the marking:

$$\gamma \cdot (X, \phi) := (X, \gamma \circ \phi), \qquad \gamma \in O(\Lambda).$$

Since the Hausdorff relation is preserved by the action of $O(\Lambda)$, the action descends to the Hausdorff quotient

$$O(\Lambda) \curvearrowright \overline{\mathcal{M}}.$$

Remark 3.64 (Equivariance of the period map). The period map is $O(\Lambda)$–equivariant:

$$\mathcal{P}(\gamma \cdot (X, \phi)) = \gamma \mathcal{P}(X, \phi).$$

Proof of Theorem 3.1. The necessity is immediate: if $f \colon X \xrightarrow{\sim} X'$ is an isomorphism, the induced map $f^* \colon H^2(X', \mathbb{Z}) \xrightarrow{\sim} H^2(X, \mathbb{Z})$ is an integral Hodge isometry.

Conversely, assume there exists an integral Hodge isometry $\varphi \colon H^2(X', \mathbb{Z}) \xrightarrow{\sim} H^2(X, \mathbb{Z})$. Choose markings $\phi : H^2(X, \mathbb{Z}) \xrightarrow{\sim} \Lambda$ and $\phi' : H^2(X', \mathbb{Z}) \xrightarrow{\sim} \Lambda$, and set:

$$\gamma := \phi \circ \varphi \circ (\phi')^{-1} \in O(\Lambda).$$

By construction, γ is a Hodge isometry that identifies the period points:

$$\overline{\mathcal{P}}(m) = \overline{\mathcal{P}}(\gamma \cdot m'),$$

where $m = [(X, \phi)]$ and $m' = [(X', \phi')]$. Let $\overline{\mathcal{M}}^0$ be the connected component of $\overline{\mathcal{M}}$ containing m.

By Remark 3.62, $\overline{\mathcal{P}}$ restricts to a biholomorphism between $\overline{\mathcal{M}}^0$ and Ω_Λ. Since the isometry $-\mathrm{id} \in O(\Lambda)$ acts as the identity on the period domain but exchanges the connected components of $\overline{\mathcal{M}}$[3], we may assume (possibly replacing γ with $-\gamma$) that $\gamma \cdot m'$ belongs to $\overline{\mathcal{M}}^0$.

The injectivity of $\overline{\mathcal{P}}$ on the connected component $\overline{\mathcal{M}}^0$ implies that:

$$m = \gamma \cdot m' \in \overline{\mathcal{M}}.$$

By the construction of the Hausdorff quotient $\overline{\mathcal{M}} = \mathcal{M}/\sim$, the identity $m = \gamma \cdot m'$ in $\overline{\mathcal{M}}$ implies that the marked K3 surfaces m and $\gamma \cdot m'$ are in the same equivalence class. This means that they either coincide as points in the moduli space $\mathcal{M}$ or are distinct but inseparable. In the first case, the surfaces X and X' are isomorphic by definition; in the second case, their isomorphism is guaranteed by Corollary 3.33. In both situations, we conclude that $X \cong X'$, which completes the proof of the weak global Torelli theorem. □

[3] For a rigorous justification of this fact, see [Huy16, Ch. 7, Cor. 5.1, Prop. 5.4, and Prop. 5.5].

Proof of Theorem 3.6. The necessity is clear: if $f\colon X \xrightarrow{\sim} X'$ is an isomorphism, its pullback f^* preserves the Hodge structure and the intersection form. Furthermore, the pullback of a Kähler metric on X' is a Kähler metric on X, hence $f^*(\mathcal{K}_{X'}) = \mathcal{K}_X$. Thus $\varphi = f^*$ satisfies the required conditions.

Conversely, assume that $\varphi\colon H^2(X',\mathbb{Z}) \xrightarrow{\sim} H^2(X,\mathbb{Z})$ is an integral Hodge isometry such that $\varphi(\mathcal{K}_{X'}) = \mathcal{K}_X$. Choose markings $\phi : H^2(X,\mathbb{Z}) \xrightarrow{\sim} \Lambda$ and $\phi' : H^2(X',\mathbb{Z}) \xrightarrow{\sim} \Lambda$, and set

$$m = [(X,\phi)], \qquad m' = [(X',\phi')], \qquad \gamma := \phi \circ \varphi \circ (\phi')^{-1} \in O(\Lambda).$$

As in the proof of Theorem 3.1, after possibly replacing γ with $-\gamma$, we may assume that m and $\gamma \cdot m'$ lie in the same connected component of $\overline{\mathcal{M}}$ and have the same period. Hence they define the same point of $\overline{\mathcal{M}}$. Therefore, either they coincide in $\mathcal{M}$, or they are inseparable.

If they coincide in $\mathcal{M}$, then there exists an isomorphism $f : X \xrightarrow{\sim} X'$ such that $\varphi = f^*$, and there is nothing to prove. Thus we may assume that m and $\gamma \cdot m'$ are distinct and inseparable. By Proposition 3.31, applied to a pair of families realizing this inseparability, there exists an effective analytic 2–cycle

$$\Gamma = \Delta + \sum_{j\in J} n_j (C_j \times C'_j) \subset X \times X'$$

whose fundamental class induces the map

$$\varphi^{-1} : H^2(X,\mathbb{Z}) \longrightarrow H^2(X',\mathbb{Z}).$$

Here, Δ is the graph of an isomorphism $f\colon X \xrightarrow{\sim} X'$, the subsets $C_j \subset X$ and $C'_j \subset X'$ are curves, and the coefficients n_j are non-negative integers.

We must show that $n_j = 0$ for all j, yielding $\Gamma = \Delta$ and consequently $\varphi = f^*$.

Let $\alpha \in \mathcal{K}_X$ be a Kähler class on X. The action of the correspondence Γ on α gives

$$\beta := \varphi^{-1}(\alpha) = f_*\alpha + \sum_{j\in J} n_j (\alpha \cdot C_j) C'_j \in H^{1,1}_{\mathbb{R}}(X').$$

Let $\alpha' := f_*\alpha$. Since f is an isomorphism, α' is a Kähler class on X'. Furthermore, define $D := \sum_{j\in J} n_j(\alpha \cdot C_j)C'_j$. Since α is Kähler, we have $(\alpha \cdot C_j) > 0$. Since $n_j \geq 0$, D is either zero or an effective divisor on X'. We can write

$$\beta = \alpha' + [D].$$

By hypothesis $\varphi(\mathcal{K}_{X'}) = \mathcal{K}_X$, hence β is a Kähler class on X'. Since φ^{-1} is an isometry, we have $(\beta, \beta) = (\alpha, \alpha)$. As f is an isomorphism, $(\alpha, \alpha) = (\alpha', \alpha')$. It follows that

$$(\beta, \beta) = (\alpha', \alpha').$$

Expanding the self-intersection of β, we get

$$(\beta, \beta) = (\alpha' + D, \alpha' + D) = (\alpha', \alpha') + 2(\alpha', D) + (D, D).$$

Equating the two expressions for (β, β) yields

$$2(\alpha', D) + (D, D) = 0.$$

We know that α' is a Kähler class and D is an effective divisor, which guarantees $(\alpha', D) \geq 0$. Now, consider the intersection of β with D:

$$(\beta, D) = (\alpha' + D, D) = (\alpha', D) + (D, D).$$

Substituting $(D, D) = -2(\alpha', D)$ from the previous relation, we obtain

$$(\beta, D) = (\alpha', D) - 2(\alpha', D) = -(\alpha', D) \leq 0.$$

However, β is a Kähler class and D is an effective divisor. If $D \neq 0$, we must strictly have $(\beta, D) > 0$. This contradiction forces $D = 0$.

Since $D = 0$ and each term $n_j(\alpha \cdot C_j)$ is non-negative with $(\alpha \cdot C_j) > 0$, we conclude that $n_j = 0$ for all $j \in J$. Consequently, $\Gamma = \Delta$, which means the map φ^{-1} is exactly the pushforward f_*. Taking inverses, $\varphi = f^*$.

Finally, the uniqueness of the isomorphism f follows from a standard property of K3 surfaces (see [Huy16, Ch. 15]): the natural representation $\mathrm{Aut}(X) \to O\big(H^2(X, \mathbb{Z})\big)$ is injective. If $g\colon X \xrightarrow{\sim} X'$ is another isomorphism with $g^* = \varphi$, then $(f^{-1} \circ g)^* = \mathrm{id}_{H^2(X,\mathbb{Z})}$, which implies $f^{-1} \circ g = \mathrm{id}_X$, and hence $f = g$. □

The strong global Torelli theorem shows that the complex structure of a K3 surface is entirelyencodedintheHodgestructureofitssecondcohomology, oncethepositionofthe Kähler cone inside the positive cone is taken into account. In particular, the geometry of K3 surfaces can be recovered from purely lattice–theoretic and Hodge–theoretic data, and the period domain emerges as a global parameter space describing their moduli up to the natural action of the Weyl group.

Appendix A

Local systems and deformation theory

This appendix collects the basic tools from topology and deformation theory that underlie the construction of the local moduli space of marked K3 surfaces.

We first discuss higher direct images and Ehresmann's theorem, which explain the local constancy of cohomology in smooth proper families. We then develop the deformation theory of complex manifolds, including the deformation functor, Kodaira–Spencer theory, obstruction theory, and the existence of locally universal deformations.

These results provide the structural framework for the study of families of K3 surfaces and their moduli. Standard references for the material in this appendix include [Har10; Kod05; Ser06; Voi02; Bar+04].

A.1 Ehresmann's theorem and local systems

Definition A.1 (Higher direct image)**.** Let $f : Y \to X$ be a continuous map of topological spaces and let $\mathcal{F}$ be a sheaf of abelian groups on Y. For $k \geq 0$, the *k–th higher direct image of* $\mathcal{F}$ is the sheaf $R^k f_* \mathcal{F}$ on X associated to the presheaf

$$U \longmapsto H^k(f^{-1}(U), \mathcal{F}|_{f^{-1}(U)}), \qquad U \subseteq X \text{ open}.$$

Remark A.2. The sheaf $R^k f_* \mathcal{F}$ can equivalently be described as the k-th right derived functor of the direct image functor

$$f_* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X).$$

More precisely, if $\mathcal{F} \to I^\bullet$ is an injective resolution, then

$$R^k f_* \mathcal{F} \cong H^k\big(f_* I^\bullet\big).$$

For the equivalence between this derived-functor construction and the description via cohomology on the inverse images of open sets, see [Voi02].

Definition A.3 (Local system)**.** Let X be a topological space. A *local system* of abelian groups on X is a locally constant sheaf $\mathcal{L}$ of abelian groups on X, i.e. for every $x \in X$ there exists an open neighborhood U of x and an abelian group A such that

$$\mathcal{L}|_U \cong \underline{A}_U.$$

Definition A.4 (Local system of lattices)**.** Let X be a topological space. A *local system of lattices* on X is a locally constant sheaf $\mathcal{L}$ of free $\mathbb{Z}$–modules of finite rank, together with a morphism of sheaves

$$q : \mathcal{L} \otimes \mathcal{L} \to \underline{\mathbb{Z}}_X$$

such that for every $x \in X$ the induced pairing

$$q_x : \mathcal{L}_x \times \mathcal{L}_x \to \mathbb{Z}$$

is nondegenerate.

Definition A.5 (Isometry of local systems)**.** Let $(\mathcal{L}_1, q_1)$ and $(\mathcal{L}_2, q_2)$ be local systems of lattices on X. An *isometry of local systems* is an isomorphism of sheaves

$$\Phi : \mathcal{L}_1 \xrightarrow{\sim} \mathcal{L}_2$$

such that for every open set $U \subseteq X$ and all $a, b \in \mathcal{L}_1(U)$,

$$q_2(\Phi(a), \Phi(b)) = q_1(a, b).$$

Equivalently, for every $x \in X$ the induced map on the stalk

$$\Phi_x : (\mathcal{L}_1)_x \to (\mathcal{L}_2)_x$$

is an isometry of lattices.

Theorem A.6 (Ehresmann)**.** *Let*

$$\pi : \mathcal{X} \to B$$

be a proper holomorphic submersion between complex manifolds. Then π is a C^∞ locally trivial fibration.

Reference. See [Voi02, Chapter 9, Proposition 9.3]. □

Corollary A.7. *Let*

$$\pi : \mathcal{X} \to B$$

be a proper holomorphic submersion with connected fibers. Then for every $k \geq 0$, the sheaf $R^k\pi_*\mathbb{Z}$ is a local system on B whose fiber at $b \in B$ is $H^k(X_b, \mathbb{Z})$.

Proof. By Ehresmann's theorem, π is a differentiably locally trivial fibration. Hence the fibers are identified up to isotopy over sufficiently small open subsets of B, and their singular cohomology groups vary locally constantly. This shows that $R^k\pi_*\mathbb{Z}$ is a local system with fiber $H^k(X_b, \mathbb{Z})$. □

A.2 Deformations of complex manifolds

Definition A.8 (Deformation)**.** Let X be a complex manifold. A *deformation of* X over a complex space S consists of

1. a proper holomorphic submersion

$$\pi : \mathcal{X} \to S,$$

2. a point $0 \in S$,
3. an isomorphism of complex manifolds

$$\pi^{-1}(0) \cong X.$$

Definition A.9 (Isomorphism of deformations)**.** Let

$$\pi : \mathcal{X} \to S, \qquad \pi' : \mathcal{X}' \to S$$

be deformations of X over the same base S, with fixed identifications

$$i : \pi^{-1}(0) \xrightarrow{\sim} X, \qquad i' : (\pi')^{-1}(0) \xrightarrow{\sim} X.$$

An *isomorphism of deformations* is a biholomorphism

$$F : \mathcal{X} \to \mathcal{X}'$$

such that $\pi' \circ F = \pi$ and $i' \circ F|_{\pi^{-1}(0)} = i$.

Definition A.10 (Locally universal deformation)**.** A deformation

$$\pi : \mathcal{X} \to S$$

of X is called *locally universal* if for any deformation $\pi' : \mathcal{X}' \to T$ of X with distinguished point $t_0 \in T$, there exist an open neighborhood $V \subseteq T$ of t_0 and a holomorphic map

$$f : V \to S, \qquad f(t_0) = 0,$$

such that

$$\mathcal{X}'|_V \cong f^*\mathcal{X}.$$

Definition A.11 (Deformation equivalence)**.** Let X and Y be complex manifolds. We say that X and Y are *deformation equivalent* if there exist a connected complex space S, a proper holomorphic submersion

$$\pi : \mathcal{X} \to S,$$

and points $s_1, s_2 \in S$ such that

$$\mathcal{X}_{s_1} \cong X, \qquad \mathcal{X}_{s_2} \cong Y.$$

Definition A.12 (Versal deformation)**.** Let X be a complex manifold. A deformation

$$\pi : \mathcal{X} \to S$$

of X with distinguished point $0 \in S$ is called *versal* if for any deformation

$$\pi' : \mathcal{X}' \to T$$

of X with distinguished point $t_0 \in T$, there exist an open neighborhood $V \subseteq T$ of t_0 and a holomorphic map

$$f : V \to S, \qquad f(t_0) = 0,$$

such that

$$\mathcal{X}'|_V \cong f^*\mathcal{X}.$$

Remark A.13. A locally universal deformation is in particular versal.

Remark A.14. Let

$$\pi : \mathcal{X} \to S$$

be a deformation of a complex manifold X. Since π is a proper holomorphic submersion, by Theorem A.6 it is a C^∞ locally trivial fibration. In particular, all the fibers $\mathcal{X}_s$ are diffeomorphic as smooth manifolds.

A.2.1 The deformation functor and Kodaira–Spencer theory

Let $\mathrm{Art}_{\mathbb{C}}$ denote the category of local Artinian $\mathbb{C}$–algebras with residue field $\mathbb{C}$, whose morphisms are local $\mathbb{C}$–algebra homomorphisms.

Definition A.15 (Infinitesimal deformation)**.** Let X be a compact complex manifold. An *infinitesimal deformation of* X is a deformation over $\mathrm{Spec}A$, where $A \in \mathrm{Art}_{\mathbb{C}}$.

Definition A.16 (Deformation functor)**.** Let X be a compact complex manifold. The *deformation functor of* X is the functor

$$\mathrm{Def}_X : \mathrm{Art}_{\mathbb{C}} \to \mathrm{Sets}$$

defined as follows.

For $A \in \mathrm{Art}_{\mathbb{C}}$, $\mathrm{Def}_X(A)$ is the set of isomorphism classes of infinitesimal deformations

$$\pi : \mathcal{X} \to \mathrm{Spec}A$$

together with a fixed identification

$$\mathcal{X} \times_{\mathrm{Spec}A} \mathrm{Spec}\mathbb{C} \cong X.$$

If $\varphi : A \to B$ is a morphism in $\mathrm{Art}_{\mathbb{C}}$ and

$$\pi : \mathcal{X} \to \mathrm{Spec}A$$

represents an element of $\mathrm{Def}_X(A)$, the induced element of $\mathrm{Def}_X(B)$ is represented by the base–changed deformation

$$\pi_B : \mathcal{X}_B \to \mathrm{Spec}B, \qquad \mathcal{X}_B := \mathcal{X} \times_{\mathrm{Spec}A} \mathrm{Spec}B,$$

where $\mathrm{Spec}B \to \mathrm{Spec}A$ is the morphism induced by φ.

Definition A.17 (Infinitesimal deformation of order k)**.** Let X be a compact complex manifold and let

$$A_k := \mathbb{C}[\varepsilon]/(\varepsilon^{k+1}).$$

An *infinitesimal deformation of* X *of order* k is a deformation of X over $\mathrm{Spec}A_k$, i.e. an element of

$$\mathrm{Def}_X(A_k).$$

Remark A.18. The natural quotient morphisms

$$A_{k+1} \to A_k$$

induce restriction maps

$$\mathrm{Def}_X(A_{k+1}) \to \mathrm{Def}_X(A_k).$$

A deformation of order $k+1$ mapping to a given element of $\mathrm{Def}_X(A_k)$ is called a lifting (or extension) of that deformation.

Theorem A.19 (Kodaira–Spencer)**.** *Let X be a compact complex manifold. There is a natural isomorphism*

$$\mathrm{Def}_X\big(\mathbb{C}[\varepsilon]/(\varepsilon^2)\big) \cong H^1(X, T_X).$$

Reference. See [Har10, Chapter 1, Theorem 3.4.1]. □

Definition A.20 (Small extension)**.** A *small extension* in $\mathrm{Art}_{\mathbb{C}}$ is an exact sequence

$$0 \to I \to A' \to A \to 0$$

such that $\mathfrak{m}_{A'} I = 0$, where $\mathfrak{m}_{A'}$ denotes the maximal ideal of the local Artinian ring A'.

Definition A.21 (Lifting of a deformation)**.** Let

$$0 \to I \to A' \to A \to 0$$

be a small extension in $\mathrm{Art}_{\mathbb{C}}$ and let $\xi \in \mathrm{Def}_X(A)$. A *lifting (or extension) of ξ to A'* is an element $\xi' \in \mathrm{Def}_X(A')$ whose image under the natural restriction map

$$\mathrm{Def}_X(A') \to \mathrm{Def}_X(A)$$

is ξ.

Definition A.22 (Obstruction class)**.** Let

$$0 \to I \to A' \to A \to 0$$

be a small extension in $\mathrm{Art}_{\mathbb{C}}$ and let $\xi \in \mathrm{Def}_X(A)$. An *obstruction class for lifting ξ to A'* is an element

$$\mathrm{ob}(\xi, A') \in H^2(X, T_X) \otimes_{\mathbb{C}} I$$

such that $\mathrm{ob}(\xi, A') = 0$ if and only if ξ admits a lifting to an element of $\mathrm{Def}_X(A')$.

Theorem A.23 (Obstruction theory)**.** *Let X be a compact complex manifold. For every small extension*

$$0 \to I \to A' \to A \to 0$$

and every deformation $\xi \in \mathrm{Def}_X(A)$, there exists an obstruction class

$$\mathrm{ob}(\xi, A') \in H^2(X, T_X) \otimes_{\mathbb{C}} I$$

whose vanishing is necessary and sufficient for the existence of a lifting of ξ to $\mathrm{Def}_X(A')$.

Reference. See [Ser06, Theorem 2.4.6]. □

Corollary A.24. *If*

$$H^2(X, T_X) = 0,$$

then the deformation functor Def_X is unobstructed.

Theorem A.25 (Kuranishi)**.** *Let X be a compact complex manifold. There exists a germ of a complex analytic space $(U, 0)$ and a deformation*

$$\pi : \mathcal{X} \to U$$

of X which is locally universal and satisfies

$$T_0 U \cong H^1(X, T_X).$$

If $H^2(X, T_X) = 0$, then U is smooth.

Reference. See [Huy16, Thm. 4.1.2]. □

Remark A.26. The cohomology groups of the tangent bundle control deformation theory:

- $H^0(X, T_X)$ corresponds to infinitesimal automorphisms;
- $H^1(X, T_X)$ parametrizes first–order deformations;
- $H^2(X, T_X)$ contains obstruction classes.

Appendix B

Variation of Hodge structure and the Gauss–Manin connection

This appendix develops the Hodge–de Rham theory for smooth proper holomorphic families. We explain how the Hodge decomposition of the cohomology of a compact Kähler manifold behaves in such families and how it is encoded in the Hodge filtration.

We introduce the relative de Rham complex and the associated cohomology bundles, describe the Hodge filtration, and construct the Gauss–Manin connection. We then establish Griffiths transversality and show how these structures give rise to variations of Hodge structure.

The material presented here forms the analytic and cohomological background for the theory of the period map and for the infinitesimal Torelli theorem. The main references are [Gri69; Lit14; Voi02].

B.1 Hodge–de Rham theory in families

Theorem B.1 (Hodge decomposition)**.** *Let X be a compact Kähler manifold. Then, for every $k \geq 0$, the complexified de Rham cohomology of X admits a decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

where $H^{p,q}(X)$ denotes the space of cohomology classes represented by harmonic forms of type (p, q). Moreover, one has $\overline{H^{p,q}(X)} = H^{q,p}(X)$.

Definition B.2 (Hodge filtration)**.** The Hodge decomposition induces a decreasing filtration on cohomology, called the *Hodge filtration*, defined by

$$F^p H^k(X,\mathbb{C}) := \bigoplus_{r\geq p} H^{r,k-r}(X).$$

Remark B.3. For compact Kähler manifolds, the Hodge filtration and the Hodge decomposition determine each other.

Definition B.4 (Relative de Rham complex)**.** Let

$$\pi : \mathcal{X} \to B$$

be a smooth holomorphic submersion. The *relative de Rham complex* of the family is the complex of sheaves

$$\Omega^\bullet_{\mathcal{X}/B} : \quad \mathcal{O}_{\mathcal{X}} \longrightarrow \Omega^1_{\mathcal{X}/B} \longrightarrow \Omega^2_{\mathcal{X}/B} \longrightarrow \cdots,$$

where

$$\Omega^p_{\mathcal{X}/B} := \Omega^p_{\mathcal{X}} \big/ \pi^*\Omega^1_B \wedge \Omega^{p-1}_{\mathcal{X}}.$$

Remark B.5. Let $\pi : \mathcal{X} \to B$ be the morphism and, for $b \in B$, denote by $X_b := \pi^{-1}(b)$ the fibre over b. By construction, for every $b \in B$ one has canonical identifications

$$\Omega^p_{\mathcal{X}/B}\big|_{X_b} \cong \Omega^p_{X_b}, \qquad \Omega^\bullet_{\mathcal{X}/B}\big|_{X_b} \cong \Omega^\bullet_{X_b}.$$

Definition B.6 (Hypercohomology)**.** Let X be a complex manifold and let $K^\bullet$ be a complex of sheaves on X. Choose a resolution of $K^\bullet$ by a complex of injective (or Γ-acyclic) sheaves

$$K^\bullet \longrightarrow I^\bullet.$$

The *hypercohomology of* $K^\bullet$ is defined by

$$\mathbb{H}^k(X, K^\bullet) := H^k\big(\Gamma(X, I^\bullet)\big).$$

Theorem B.7 (Relative de Rham base change)**.** *Let $\pi : \mathcal{X} \to B$ be a smooth proper holomorphic submersion. For every $k \geq 0$ and $b \in B$, there is a canonical isomorphism*

$$\big(R^k\pi_*\Omega^\bullet_{\mathcal{X}/B}\big)_b \;\cong\; \mathbb{H}^k(X_b, \Omega^\bullet_{X_b}).$$

Reference. Apply [Voi02, Thm. 10.10] to the relative holomorphic de Rham complex $\Omega^\bullet_{\mathcal{X}/B}$. □

Theorem B.8 (de Rham theorem for compact Kähler manifolds)**.** *Let X be a compact Kähler manifold. Then for every $k \geq 0$ there is a canonical isomorphism*

$$\mathbb{H}^k(X, \Omega_X^\bullet) \cong H^k(X, \mathbb{C}).$$

Reference. See [Voi02, Cor. 8.14]. □

Remark B.9. Combining Theorem B.7 with the de Rham theorem for compact Kähler manifolds, one obtains, for every $k \geq 0$ and $b \in B$, a canonical identification

$$\left(R^k \pi_* \Omega^\bullet_{\mathcal{X}/B}\right)_b \cong H^k(X_b, \mathbb{C}).$$

In order to relate the relative de Rham complex to the Hodge filtration on the cohomology of the fibers, we introduce the natural filtration on a complex given by truncation.

Definition B.10 (Stupid filtration)**.** Let $K^\bullet$ be a cochain complex. The *stupid filtration* (also called the *naive filtration*) on $K^\bullet$ is the decreasing filtration defined by

$$F^p K^\bullet := K^{\geq p},$$

where $K^{\geq p}$ denotes the subcomplex

$$K^{\geq p}: \quad 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow K^p \longrightarrow K^{p+1} \longrightarrow K^{p+2} \longrightarrow \cdots,$$

with K^p placed in degree p.

Definition B.11 (Spectral sequence)**.** A *spectral sequence* in an abelian category $\mathcal{A}$ consists of the following data:

1. for each integer $r \geq r_0$, a bigraded family of objects

   $$\{E_r^{p,q}\}_{p,q \in \mathbb{Z}},$$

   where, for fixed r, the bigraded object $E_r^{\bullet,\bullet}$ is called the *r-th page* of the spectral sequence;

2. for each $r \geq r_0$, differentials

   $$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,\, q-r+1}$$

   satisfying $d_r \circ d_r = 0$;

3. isomorphisms
$$E_{r+1}^{p,q} \cong \frac{\ker(d_r^{p,q}\colon E_r^{p,q} \to E_r^{p+r,q-r+1})}{\operatorname{im}\left(d_r^{p-r,q+r-1}\colon E_r^{p-r,q+r-1} \to E_r^{p,q}\right)}.$$

Definition B.12 (Convergence of a spectral sequence)**.** Let $\{E_r^{p,q}, d_r\}_{r\geq r_0}$ be a spectral sequence. We say that it *converges* to a graded object

$$H^\bullet = \{H^n\}_{n\in\mathbb{Z}},$$

and we write

$$E_r^{p,q} \Rightarrow H^{p+q},$$

if for each integer n there exists an exhaustive and separated decreasing filtration

$$\cdots \subset F^{p+1}H^n \subset F^pH^n \subset \cdots \subset H^n$$

such that

$$E_\infty^{p,q} \cong \operatorname{gr}_F^p H^{p+q} := F^pH^{p+q}/F^{p+1}H^{p+q},$$

where $E_\infty^{p,q}$ denotes the limit term of $E_r^{p,q}$ for $r \gg 0$.

Remark B.13. The stupid filtration

$$\Omega_{\mathcal{X}/B}^{\geq p}: \quad 0 \to \cdots \to 0 \to \Omega_{\mathcal{X}/B}^{p} \to \Omega_{\mathcal{X}/B}^{p+1} \to \cdots$$

on the relative de Rham complex $\Omega_{\mathcal{X}/B}^\bullet$ induces a spectral sequence, that is, the spectral sequence canonically associated with the filtered complex $(\Omega_{\mathcal{X}/B}^\bullet, \Omega_{\mathcal{X}/B}^{\geq\bullet})$,

$$E_1^{p,q} = R^q\pi_*\Omega_{\mathcal{X}/B}^p \quad\Rightarrow\quad R^{p+q}\pi_*\Omega_{\mathcal{X}/B}^\bullet. \tag{B.1.1}$$

This spectral sequence is called the *Hodge–de Rham spectral sequence.*

The relation between the relative de Rham complex and the Hodge filtration is governed by the Hodge–de Rham spectral sequence introduced above. The key result is the degeneration of this spectral sequence at the first page.

Theorem B.14 (Degeneration at E_1)**.** *Let $\pi : \mathcal{X} \to B$ be a smooth proper holomorphic map whose fibers are compact Kähler manifolds. Then the Hodge–de Rham spectral sequence*

$$E_1^{p,q} = R^q\pi_*\Omega_{\mathcal{X}/B}^p \Rightarrow R^{p+q}\pi_*\Omega_{\mathcal{X}/B}^\bullet$$

degenerates at the E_1–page.

Reference. See [Voi02, Thm. 10.21]. □

Corollary B.15 (Consequences of the E_1–degeneration)**.** *Under the assumptions of Theorem B.14, the following statements hold.*

1. *For every $k \geq 0$, the induced filtration on $R^k\pi_*\Omega^\bullet_{\mathcal{X}/B}$ satisfies*

$$\operatorname{gr}^p_F R^k\pi_*\Omega^\bullet_{\mathcal{X}/B} \cong R^{k-p}\pi_*\Omega^p_{\mathcal{X}/B}.$$

2. *The Hodge filtration on $R^k\pi_*\Omega^\bullet_{\mathcal{X}/B}$ is induced by the stupid filtration on the relative de Rham complex and is given by*

$$F^p R^k\pi_*\Omega^\bullet_{\mathcal{X}/B} = \operatorname{Im}\Big(R^k\pi_*\Omega^{\geq p}_{\mathcal{X}/B} \to R^k\pi_*\Omega^\bullet_{\mathcal{X}/B}\Big).$$

3. *Fiberwise, this filtration coincides with the Hodge filtration on $H^k(X_b, \mathbb{C})$.*

Reference. See [Voi02, Prop. 10.19, Cor. 10.20 and Thm. 10.21]. □

B.2 Gauss–Manin connection and Griffiths transversality

In this subsection we construct the Gauss–Manin connection on the relative de Rham cohomology of a smooth proper holomorphic family. This connection describes how the cohomology of the fibers varies holomorphically with the parameter.

We then study its interaction with the Hodge filtration, leading to Griffiths transversality.

Let $\pi : \mathcal{X} \to B$ be a smooth proper holomorphic submersion of complex manifolds, with compact Kähler fibers. Under these assumptions, Theorem B.7, the de Rham theorem for compact Kähler manifolds, and Theorem B.14 apply. In particular, the relative de Rham complex $\Omega^\bullet_{\mathcal{X}/B}$ computes the cohomology of the fibers and induces the Hodge filtration on cohomology.

We set

$$\mathcal{H}^k := R^k\pi_*\Omega^\bullet_{\mathcal{X}/B}.$$

For every $b \in B$ one has canonical isomorphisms

$$(\mathcal{H}^k)_b \cong \mathbb{H}^k(X_b, \Omega^\bullet_{X_b}) \cong H^k(X_b, \mathbb{C}).$$

By Corollary B.15, the Hodge filtration is induced by holomorphic subbundles

$$F^p\mathcal{H}^k \subset \mathcal{H}^k,$$

called the *Hodge bundles* of the family, with fibers

$$(F^p\mathcal{H}^k)_b = F^pH^k(X_b, \mathbb{C}).$$

We denote by $\Omega^\bullet_{\mathcal{X}}$ the absolute holomorphic de Rham complex and by $\Omega^\bullet_{\mathcal{X}/B}$ the relative de Rham complex.

Definition B.16 (Holomorphic connection)**.** Let B be a complex manifold and let $\mathcal{E}$ be a holomorphic vector bundle on B. A *holomorphic connection* on $\mathcal{E}$ is a $\mathbb{C}$–linear morphism of sheaves of abelian groups

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_B} \Omega^1_B$$

such that for every open set $U \subset B$, every holomorphic function $f \in \mathcal{O}_B(U)$ and every section $s \in \mathcal{E}(U)$, the Leibniz rule

$$\nabla(f \cdot s) = f \cdot \nabla(s) + s \otimes df$$

holds in $\mathcal{E}(U) \otimes_{\mathcal{O}_B(U)} \Omega^1_B(U)$.

Definition B.17 (Flat connection)**.** Let $\mathcal{E}$ be a holomorphic vector bundle on a complex manifold B, equipped with a holomorphic connection

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_B} \Omega^1_B.$$

The curvature of ∇ is the $\mathcal{O}_B$–linear morphism

$$\nabla^2 : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_B} \Omega^2_B.$$

The connection ∇ is called *flat* if

$$\nabla^2 = 0.$$

B.2.1 Construction of the Gauss–Manin connection

The Gauss–Manin connection is defined via hypercohomology as the connecting morphism associated with a natural short exact sequence of complexes obtained from the absolute de Rham complex and the cotangent sequence of π. Since π is a submersion, the cotangent sequence

$$0 \longrightarrow \pi^*\Omega^1_B \longrightarrow \Omega^1_{\mathcal{X}} \longrightarrow \Omega^1_{\mathcal{X}/B} \longrightarrow 0$$

is exact. For $p \geq 0$ this induces a decreasing filtration on the absolute de Rham complex $\Omega^\bullet_{\mathcal{X}}$ by subcomplexes, defined by

$$F^p\Omega^\bullet_{\mathcal{X}} := \operatorname{Im}\bigl(\pi^*\Omega^p_B \wedge \Omega^{\bullet-p}_{\mathcal{X}} \longrightarrow \Omega^\bullet_{\mathcal{X}}\bigr).$$

By construction, the filtration is multiplicative and compatible with the exterior differential.

The quotient of the absolute de Rham complex by the first step of the filtration identifies canonically with the relative de Rham complex:

$$\Omega^\bullet_{\mathcal{X}}/F^1\Omega^\bullet_{\mathcal{X}} \;\cong\; \Omega^\bullet_{\mathcal{X}/B}.$$

Moreover, the first graded piece admits a natural description:

$$F^1\Omega^\bullet_{\mathcal{X}}/F^2\Omega^\bullet_{\mathcal{X}} \;\cong\; \pi^*\Omega^1_B \otimes_{\mathcal{O}_{\mathcal{X}}} \Omega^{\bullet-1}_{\mathcal{X}/B}.$$

Since $F^1\Omega^\bullet_{\mathcal{X}}$ is a subcomplex of the absolute de Rham complex, there is a short exact sequence of complexes

$$0 \longrightarrow F^1\Omega^\bullet_{\mathcal{X}} \longrightarrow \Omega^\bullet_{\mathcal{X}} \longrightarrow \Omega^\bullet_{\mathcal{X}}/F^1\Omega^\bullet_{\mathcal{X}} \longrightarrow 0.$$

The filtration being compatible with the differential, we may quotient this sequence by the subcomplex $F^2\Omega^\bullet_{\mathcal{X}}$, obtaining

$$0 \longrightarrow F^1\Omega^\bullet_{\mathcal{X}}/F^2\Omega^\bullet_{\mathcal{X}} \longrightarrow \Omega^\bullet_{\mathcal{X}}/F^2\Omega^\bullet_{\mathcal{X}} \longrightarrow \Omega^\bullet_{\mathcal{X}}/F^1\Omega^\bullet_{\mathcal{X}} \longrightarrow 0.$$

Using the canonical identifications described above for the quotients $\Omega^\bullet_{\mathcal{X}}/F^1\Omega^\bullet_{\mathcal{X}}$ and $F^1\Omega^\bullet_{\mathcal{X}}/F^2\Omega^\bullet_{\mathcal{X}}$, we obtain the following short exact sequence of complexes:

$$0 \longrightarrow \pi^*\Omega^1_B \otimes \Omega^{\bullet-1}_{\mathcal{X}/B} \longrightarrow \Omega^\bullet_{\mathcal{X}}/F^2\Omega^\bullet_{\mathcal{X}} \longrightarrow \Omega^\bullet_{\mathcal{X}/B} \longrightarrow 0. \tag{B.2.1}$$

Let $U \subset B$ be a sufficiently small open subset. Applying hypercohomology to the restriction of (B.2.1) over $\pi^{-1}(U)$ yields a long exact sequence

$$0 \longrightarrow \mathbb{H}^0\Big(\pi^{-1}(U),\, \pi^*\Omega^1_B\otimes\Omega^{\bullet-1}_{\mathcal{X}/B}\Big) \longrightarrow \mathbb{H}^0(\pi^{-1}(U),\, \Omega^\bullet_{\mathcal{X}}/F^2\Omega^\bullet_{\mathcal{X}}) \longrightarrow \mathbb{H}^0\Big(\pi^{-1}(U),\, \Omega^\bullet_{\mathcal{X}/B}\Big)$$

$$\hookrightarrow \mathbb{H}^1\Big(\pi^{-1}(U),\, \pi^*\Omega^1_B\otimes\Omega^{\bullet-1}_{\mathcal{X}/B}\Big) \longrightarrow \mathbb{H}^1(\pi^{-1}(U),\, \Omega^\bullet_{\mathcal{X}}/F^2\Omega^\bullet_{\mathcal{X}}) \longrightarrow \mathbb{H}^1\Big(\pi^{-1}(U),\, \Omega^\bullet_{\mathcal{X}/B}\Big)$$

$$\hookrightarrow \mathbb{H}^2\Big(\pi^{-1}(U),\, \pi^*\Omega^1_B\otimes\Omega^{\bullet-1}_{\mathcal{X}/B}\Big) \longrightarrow \mathbb{H}^2(\pi^{-1}(U),\, \Omega^\bullet_{\mathcal{X}}/F^2\Omega^\bullet_{\mathcal{X}}) \longrightarrow \mathbb{H}^2\Big(\pi^{-1}(U),\, \Omega^\bullet_{\mathcal{X}/B}\Big) \longrightarrow \cdots$$

The connecting morphism

$$\delta_U : \mathbb{H}^k\big(\pi^{-1}(U),\, \Omega^\bullet_{\mathcal{X}/B}\big) \longrightarrow \mathbb{H}^{k+1}\Big(\pi^{-1}(U),\, \pi^*\Omega^1_B\otimes\Omega^{\bullet-1}_{\mathcal{X}/B}\Big)$$

is, by definition, the Gauss–Manin connection in hypercohomology over U.

Since $\Omega^1_B|_U$ is a free $\mathcal{O}_U$–module, we can choose a basis and write

$$\Omega^1_B|_U \cong \mathcal{O}_U^{\oplus r}, \qquad \pi^*\Omega^1_B|_{\pi^{-1}(U)} \cong \mathcal{O}_{\pi^{-1}(U)}^{\oplus r}.$$

It follows that, for any complex $K^\bullet$ of $\mathcal{O}_{\mathcal{X}}$–modules,

$$\mathbb{H}^m\big(\pi^{-1}(U),\, \pi^*\Omega^1_B \otimes K^\bullet\big) \cong \Omega^1_B(U) \otimes_{\mathcal{O}_B(U)} \mathbb{H}^m\big(\pi^{-1}(U),\, K^\bullet\big)\,.$$

Applying this with $K^\bullet = \Omega^{\bullet-1}_{\mathcal{X}/B}$, and using the canonical identification $\Omega^{\bullet-1}_{\mathcal{X}/B} = \Omega^\bullet_{\mathcal{X}/B}[-1]$, which induces an isomorphism

$$\mathbb{H}^{k+1}\Big(\pi^{-1}(U),\, \Omega^{\bullet-1}_{\mathcal{X}/B}\Big) \cong \mathbb{H}^k\big(\pi^{-1}(U),\, \Omega^\bullet_{\mathcal{X}/B}\big)\,,$$

the connecting morphism

$$\delta_U : \mathbb{H}^k\big(\pi^{-1}(U),\, \Omega^\bullet_{\mathcal{X}/B}\big) \longrightarrow \mathbb{H}^{k+1}\Big(\pi^{-1}(U),\, \pi^*\Omega^1_B\otimes\Omega^{\bullet-1}_{\mathcal{X}/B}\Big)$$

may be viewed as a $\mathbb{C}$–linear map

$$\nabla_U : \mathbb{H}^k\big(\pi^{-1}(U),\, \Omega^\bullet_{\mathcal{X}/B}\big) \longrightarrow \Omega^1_B(U) \otimes \mathbb{H}^k\big(\pi^{-1}(U),\, \Omega^\bullet_{\mathcal{X}/B}\big)\,.$$

This map satisfies the Leibniz rule and therefore defines a holomorphic connection on the bundle $\mathcal{H}^k$.

Varying U, these local morphisms glue to a holomorphic connection

$$\nabla : \mathcal{H}^k \longrightarrow \Omega^1_B \otimes \mathcal{H}^k.$$

This connection is called the *Gauss–Manin connection.*

Proposition B.18. *The Gauss–Manin connection*

$$\nabla : \mathcal{H}^k \to \mathcal{H}^k \otimes \Omega^1_B$$

is flat.

Reference. See [Voi02, §9.2]. □

We now describe how the Gauss–Manin connection interacts with the Hodge filtration, a relationship governed by the following fundamental property, known as *Griffiths transversality.*

Theorem B.19 (Griffiths transversality)**.** *Let $\pi : \mathcal{X} \to B$ be a smooth proper holomorphic submersion with compact Kähler fibers, and let*

$$\nabla : \mathcal{H}^k \longrightarrow \mathcal{H}^k \otimes \Omega^1_B$$

be the Gauss–Manin connection constructed above. Then ∇ satisfies the Griffiths transversality condition

$$\nabla\big(F^p\mathcal{H}^k\big) \ \subset \ F^{p-1}\mathcal{H}^k \otimes \Omega^1_B \qquad \textit{for all } p.$$

Proof. Recall from Corollary B.15 that the Hodge filtration on

$$\mathcal{H}^k \cong R^k\pi_*\Omega^\bullet_{\mathcal{X}/B}$$

is induced by the stupid filtration on the relative de Rham complex and is given by

$$F^p\mathcal{H}^k = \mathrm{Im}\Big(R^k\pi_*\Omega^{\geq p}_{\mathcal{X}/B} \longrightarrow R^k\pi_*\Omega^\bullet_{\mathcal{X}/B}\Big).$$

By construction, the short exact sequence of complexes (B.2.1) is compatible with the stupid filtration on $\Omega^\bullet_{\mathcal{X}/B}$. More precisely, for each p the subcomplex $\Omega^{\geq p}_{\mathcal{X}/B} \subset \Omega^\bullet_{\mathcal{X}/B}$ admits a natural lift to a subcomplex of $\Omega^\bullet_{\mathcal{X}}/F^2\Omega^\bullet_{\mathcal{X}}$, whose preimage under the projection

$$\Omega^\bullet_{\mathcal{X}}/F^2\Omega^\bullet_{\mathcal{X}} \longrightarrow \Omega^\bullet_{\mathcal{X}/B}$$

is given by $\Omega^{\geq p}_{\mathcal{X}}/F^2\Omega^\bullet_{\mathcal{X}}$.

Applying hypercohomology to this filtered version of the short exact sequence (B.2.1) over $\pi^{-1}(U)$, we obtain that the connecting morphism

$$\delta_U : \mathbb{H}^k\big(\pi^{-1}(U),\, \Omega^\bullet_{\mathcal{X}/B}\big) \longrightarrow \mathbb{H}^{k+1}\Big(\pi^{-1}(U),\, \pi^*\Omega^1_B \otimes \Omega^{\bullet-1}_{\mathcal{X}/B}\Big)$$

maps

$$\mathbb{H}^k\Big(\pi^{-1}(U),\, \Omega^{\geq p}_{\mathcal{X}/B}\Big) \xrightarrow{\delta_U|} \mathbb{H}^{k+1}\Big(\pi^{-1}(U),\, \pi^*\Omega^1_B \otimes \Omega^{\geq p-1}_{\mathcal{X}/B}\Big)\,.$$

Using the canonical identification of hypercohomology described in the construction of the Gauss–Manin connection, this implies

$$\nabla_U\Big(\mathbb{H}^k\Big(\pi^{-1}(U),\, \Omega^{\geq p}_{\mathcal{X}/B}\Big)\Big) \subset \Omega^1_B(U) \otimes \mathbb{H}^k\Big(\pi^{-1}(U),\, \Omega^{\geq p-1}_{\mathcal{X}/B}\Big)\,.$$

Passing to the image in cohomology and using the description of the Hodge filtration, we conclude that

$$\nabla\big(F^p\mathcal{H}^k\big) \;\subset\; F^{p-1}\mathcal{H}^k \otimes \Omega^1_B,$$

as claimed. □

Appendix C

Holonomy and Calabi–Yau Geometry

The aim of this appendix is to relate the Riemannian holonomy of a Kähler manifold to the existence of Calabi–Yau metrics. The key tool is the holonomy principle, which allows one to translate geometric structures into reductions of the holonomy group.

We begin with the construction of parallel transport, the Levi–Civita connection and the holonomy group. We then specialize to Kähler manifolds and show that the holonomy is contained in the unitary group. Finally, we prove that reduction of the holonomy to the special unitary group is equivalent to the existence of a Calabi–Yau metric.

These results will be applied to the case of K3 surfaces.

References include [KN63; Lee97; Kob87; OGr12; Yau78].

C.1 Levi–Civita connection and holonomy group

Let (M, g) be a Riemannian manifold and let

$$\gamma : [a, b] \to M$$

be a smooth curve.

A *vector field along* γ is a smooth map

$$V : [a, b] \to TM$$

such that

$$V(t) \in T_{\gamma(t)}M \quad \text{for all } t \in [a, b].$$

Definition C.1 (Parallel vector field)**.** Let ∇ be a connection on TM and let $\gamma : [a, b] \to M$ be a smooth curve. A vector field $V(t)$ along γ is said to be *parallel with respect to*

∇ if

$$\nabla_{\dot{\gamma}(t)} V(t)^1 = 0 \qquad \text{for all } t \in [a,b].$$

Proposition C.2. *Let $\gamma : [a,b] \to M$ be a smooth curve and let $v \in T_{\gamma(a)}M$. There exists a unique vector field $V(t)$ along γ which is parallel and satisfies*

$$V(a) = v.$$

Reference. See [Lee97, Prop. 4.5]. □

Definition C.3 (Parallel transport)**.** Let $\gamma : [a,b] \to M$ be a smooth curve. For $v \in T_{\gamma(a)}M$, let $V(t)$ be the unique parallel vector field along γ such that $V(a) = v$. The vector

$$V(b) \in T_{\gamma(b)}M$$

is called the *parallel transport of v along γ.*

The associated linear map

$$\varphi_\gamma : T_{\gamma(a)}M \longrightarrow T_{\gamma(b)}M, \qquad v \longmapsto V(b),$$

is called the *parallel transport map.*

Definition C.4 (Torsion of a connection)**.** Let ∇ be a connection on TM. The *torsion tensor* of ∇ is the $(1,2)$-tensor[2]

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

for all vector fields X, Y on M.

The connection ∇ is said to be *torsion free* if

$$T(X,Y) = 0 \qquad \text{for all } X, Y.$$

Theorem C.5 (Fundamental theorem of Riemannian geometry)**.** *Let (M,g) be a Riemannian manifold. There exists a unique connection ∇ on TM such that:*

[1]Here $\nabla_{\dot{\gamma}(t)} V(t)$ denotes the covariant derivative of the vector field V along the curve γ. In local coordinates, if $V(t) = V^i(t)\frac{\partial}{\partial x^i}$ and $\dot{\gamma}(t) = \dot{\gamma}^j(t)\frac{\partial}{\partial x^j}$, then

$$\nabla_{\dot{\gamma}(t)} V(t) = \left(\frac{dV^k}{dt} + \Gamma^k_{ij}\dot{\gamma}^i V^j\right)\frac{\partial}{\partial x^k},$$

where Γ^k_{ij} are the Christoffel symbols of the connection.

[2]That is, a bilinear map $T : TM \times TM \to TM$.

1. *∇ is torsion free;*
2. *$\nabla g = 0$.*

Reference. See [Lee97, Thm. 3.6]. □

Definition C.6. The connection given by the previous theorem is called the *Levi–Civita connection* of g.

Definition C.7 (Holonomy group)**.** Let (M, g) be a Riemannian manifold with Levi–Civita connection ∇, and let $p \in M$.

For every piecewise smooth loop

$$\gamma : [a, b] \to M, \qquad \gamma(a) = \gamma(b) = p,$$

parallel transport with respect to ∇ along γ defines a linear map

$$\varphi_\gamma : T_pM \longrightarrow T_pM.$$

The *holonomy group at p* is the subgroup

$$\mathrm{Hol}_p(M, g) := \{\varphi_\gamma \mid \gamma \text{ loop based at } p\} \subset \mathrm{GL}(T_pM).$$

Proposition C.8. *Let (M, g) be a Riemannian manifold and let $p \in M$. For the Levi–Civita connection, parallel transport along any loop based at p preserves the Riemannian metric. In particular,*

$$\mathrm{Hol}_p(M, g) \subset \mathrm{O}(T_pM).$$

Reference. See [Lee97, Prop. 4.7]. □

Remark C.9. If M is connected and $n = \dim M$, then $\mathrm{Hol}_p(M, g)$ is a subgroup of $\mathrm{O}(T_pM) \cong \mathrm{O}(n)$. For different choices of p, the holonomy groups are conjugate.

Remark C.10. The Levi–Civita connection ∇ on TM induces in a natural way a connection, still denoted by ∇, on every tensor bundle

$$TM^{\otimes r} \otimes (TM^\vee)^{\otimes s}.$$

This extension is determined by the Leibniz rule and by duality. More precisely, for tensor fields S and T one sets

$$\nabla(S \otimes T) = (\nabla S) \otimes T + S \otimes (\nabla T),$$

and the induced connection on the dual bundle $TM^\vee$ is defined by

$$X\big(\alpha(Y)\big) = (\nabla_X \alpha)(Y) + \alpha(\nabla_X Y),$$

for vector fields X, Y and 1–forms α.

Definition C.11. Let $U \subset M$ be an open set and let

$$T \in \Gamma\big(U, TM^{\otimes r} \otimes (TM^\vee)^{\otimes s}\big).$$

The tensor field T is said to be *parallel* on U if

$$\nabla T = 0 \quad \text{on } U.$$

If $U = M$, we denote by

$$\Gamma_{\text{par}}\big(M, TM^{\otimes r} \otimes (TM^\vee)^{\otimes s}\big)$$

the space of global parallel tensor fields of type (r, s).

Theorem C.12 (Holonomy principle)**.** *Let (M, g) be a connected Riemannian manifold and let $p \in M$. For any integers $r, s \geq 0$, consider the space of parallel tensor fields*

$$\Gamma_{\text{par}}\big(M, TM^{\otimes r} \otimes (TM^\vee)^{\otimes s}\big).$$

Evaluation at p defines a linear map

$$\Gamma_{\text{par}}\big(M, TM^{\otimes r} \otimes (TM^\vee)^{\otimes s}\big) \longrightarrow T_pM^{\otimes r} \otimes (T_pM^\vee)^{\otimes s}.$$

This map is injective, and its image coincides with the subspace of tensors invariant under the natural action of $\mathrm{Hol}_p(M, g)$.

Reference. See [KN63, Ch. II, §4]. □

C.2 Holonomy of Kähler manifolds

Let (X, h) be a Kähler manifold of complex dimension m. Let g be the associated Riemannian metric and

$$I : TX \longrightarrow TX$$

the complex structure of X. Thus $I^2 = -\mathrm{Id}$ and

$$g(Iv, Iw) = g(v, w) \qquad \text{for all } v, w \in TX.$$

The Kähler form ω is defined by

$$\omega(v, w) = g(Iv, w).$$

For a Kähler manifold, giving a Hermitian metric h is equivalent to giving the associated Riemannian metric g together with the complex structure I. Accordingly, we will denote a Kähler manifold interchangeably by (X, h) or (X, g, I).

Proposition C.13. *Let (X, h) be a Kähler manifold and let g be the associated Riemannian metric. Then the complex structure I is parallel with respect to the Levi–Civita connection:*

$$\nabla I = 0.$$

Reference. See [Lee97, Ch. 6]. □

Remark C.14. Since $\nabla I = 0$, the Levi–Civita connection preserves the decomposition of the complexified tangent bundle

$$TX \otimes \mathbb{C} = T^{1,0}X \oplus T^{0,1}X.$$

In particular, ∇ induces a connection on the holomorphic tangent bundle $T^{1,0}X$ and on its dual bundle $(T^{1,0}X)^\vee$. By functoriality, it therefore induces connections on all tensor powers of these bundles, and in particular on the canonical bundle

$$K_X = \det\big((T^{1,0}X)^\vee\big).$$

We recall the following definitions.

Definition C.15. Let V be a complex vector space of dimension m endowed with a Hermitian inner product

$$h : V \times V \longrightarrow \mathbb{C}.$$

The *unitary group* of (V, h) is

$$\mathrm{U}(V, h) := \{A \in \mathrm{GL}_{\mathbb{C}}(V) \mid h(Av, Aw) = h(v, w) \text{ for all } v, w \in V\}.$$

The *special unitary group* of (V, h) is the subgroup

$$\mathrm{SU}(V, h) := \{A \in \mathrm{U}(V, h) \mid \det(A) = 1\}.$$

For a Kähler manifold (X, g, I) and $p \in X$ we will also write

$$\mathrm{U}(T_pX, g, I) \quad \text{and} \quad \mathrm{SU}(T_pX, g, I)$$

to denote the groups $\mathrm{U}(T_pX, h)$ and $\mathrm{SU}(T_pX, h)$, respectively, where h is the induced Hermitian metric.

Theorem C.16. *Let (X, g) be a connected Riemannian manifold and let*

$$I : TX \longrightarrow TX$$

be an orthogonal endomorphism satisfying $I^2 = -\mathrm{Id}$.

If (X, g, I) is Kähler, then for every $p \in X$ one has

$$\mathrm{Hol}_p(X, g) \subset \mathrm{U}(T_pX, g, I).$$

Conversely, if for some $p \in X$ one has $\mathrm{Hol}_p(X, g) \subset \mathrm{U}(T_pX, g, I)$, then I is parallel with respect to the Levi–Civita connection. In particular, (X, g, I) is Kähler.

Reference. See [OGr12, §1.4]. □

C.3 Calabi–Yau metrics

Let X be a compact Kähler manifold. We denote by

$$c_1^{\mathbb{R}}(X) \in H^2_{\mathrm{dR}}(X, \mathbb{R})$$

the first Chern class in de Rham cohomology.

Remark C.17. The condition $c_1^{\mathbb{R}}(X) = 0$ is equivalent to requiring that the integral first Chern class

$$c_1(X) \in H^2(X, \mathbb{Z})$$

is torsion.

Theorem C.18. *Let $\mathcal{E} \to X$ be a holomorphic vector bundle over a complex manifold X, endowed with a Hermitian metric h. There exists a unique connection*

$$\nabla : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E} \otimes \Omega^1_X)$$

such that:

1. *∇ is compatible with h, i.e.*

$$d\,h(s,t) = h(\nabla s, t) + h(s, \nabla t) \qquad \textit{for all } s,t \in \Gamma(\mathcal{E});$$

2. *the $(0,1)$–component of ∇ agrees with the holomorphic structure, i.e.*

$$\nabla^{0,1} = \bar{\partial}_E.$$

Reference. See [Kob87, Ch. I, §5]. □

Definition C.19 (Chern connection)**.** The connection in Theorem C.18 is called the *Chern connection* of $(\mathcal{E}, h)$.

Definition C.20 (Calabi–Yau metric)**.** Let X be a compact Kähler manifold. A *Calabi–Yau metric* on X is a Hermitian metric h such that the Chern connection of (K_X, h) is flat, that is, its curvature vanishes[3].

Theorem C.21 (Yau)**.** *Let X be a compact Kähler manifold such that*

$$c_1^{\mathbb{R}}(X) = 0.$$

For every Kähler class $[\omega] \in H^2_{\mathrm{dR}}(X, \mathbb{R})$ there exists a unique Calabi–Yau metric h on X whose Kähler form ω_h satisfies

$$[\omega_h] = [\omega].$$

Reference. See [Yau78]. □

Remark C.22. The theorem is a result of existence. In general it is not possible to write down explicitly a Calabi–Yau metric.

Remark C.23. In general, even if X admits a Calabi–Yau metric, the Levi–Civita connection on the tangent bundle is not flat.

Theorem C.24. *Let (X, h) be a Kähler manifold of complex dimension m, and let g be the associated Riemannian metric. Fix $p \in X$.*

Assume that

$$\mathrm{Hol}_p(X, g) \subset \mathrm{SU}(T_pX, h).$$

Then the canonical bundle K_X admits a non-zero parallel section. In particular, the Chern connection on K_X is flat, hence h is a Calabi–Yau metric.

[3]Equivalently, the associated Kähler metric is Ricci-flat.

Conversely, assume that X is simply connected and that h is a Calabi–Yau metric. Then

$$\mathrm{Hol}_p(X, g) \subset \mathrm{SU}(T_pX, h).$$

Reference. See [OGr12, §1.4]. □

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