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Mathematical Programming Approaches to Net Interest Income Optimization in Bank Balance Sheets

Master thesis in Operations Research

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To those who have been
part of this journey.

Introduction

In the modern banking system, the management of risk and return is strictly governed by regulatory standards, these regulations influence how banks plan investments, funding strategies, and capital allocation decisions. All these activities fall within the discipline of Asset and Liability Management (ALM), which provides the economic and strategic framework through which banks coordinate their balance-sheet decisions to ensure profitability, solvency, and short term and long term financial stability. ALM aims to balance risk and return through the integrated management of interest-rate exposure, liquidity, and capital constraints. The introduction of the Basel III framework by the Basel Committee significantly reshaped the regulatory landscape after the 2008 crisis, among the most influential quantitative measures are risk-weighted assets (RWA), the Liquidity Coverage Ratio (LCR), and the Net Stable Funding Ratio (NSFR). The thesis develops a rigorous mathematical programming framework to model these interactions over a one year horizon. The theoretical basis from which we start is a general constrained optimization problem:

$$\min_{x \in \mathbb{R}^n} f(x) \qquad \text{s.t.} \qquad g_i(x) \le 0, i \in I, \quad h_j(x) = 0, j \in E,$$

which provides the foundation for the subsequent developments. We present the necessary and, under suitable regularity assumptions, sufficient conditions for a feasible point to be optimal. In particular, we introduce the Karush Kuhn Tucker conditions, which state that a feasible point $x^\star$ is optimal if there exist Lagrange multipliers $\lambda_i \ge 0$ and $\mu_j \in \mathbb{R}$ such that:

$$\text{(Stationarity)} \qquad \nabla f(x^\star) + \sum_{i \in I} \lambda_i \nabla g_i(x^\star) + \sum_{j \in E} \mu_j \nabla h_j(x^\star) = 0,$$

$$\text{(Primal feasibility)} \qquad g_i(x^\star) \le 0, \quad \forall\, i \in \mathcal{I}, \qquad h_j(x^\star) = 0, \quad \forall\, j \in \mathcal{J},$$

$$\text{(Dual feasibility)} \qquad \lambda_i \ge 0, \quad \forall\, i \in \mathcal{I},$$

$$\text{(Complementary slackness)} \qquad \lambda_i g_i(x^\star) = 0 \quad \forall\, i \in \mathcal{I}.$$

These conditions play a central role in the analysis of the model developed in the thesis, they motivate the construction of our numerical algorithm for the resolution of constrained optimization problems. In particular, when appropriate simplifications are introduced, the nonlinear balance sheet optimization problem can be reformulated as a linear program, whose polyhedral structure allows the use of classical methods such as the “Simplex Algorithm”. After we presents the structure of a bank’s balance sheet, where each financial aggregate is characterized by vector

$$\theta_k = (a_k, t_k, \tau_k, m_k, f_k, r_k, s_k),$$

encoding: asset/liability sign, interest rate type, maturity, payment frequency, reference parameter, and repayment structure. For each aggregate k, and for each month $t = 1, \dots, 12$, the decision variable

$$x_{k,t} \in \mathbb{R},$$

represents a variation in exposure and its actual monthly exposure is determined by the function $V_k(x_k; \theta_k, t)$. The objective function, Net Interest Income, is then written as

$$\mathrm{NII}(x) = \sum_{k=1}^{N} a_k \left(\mathrm{NII}_k^{\mathrm{ACTUAL}} + \sum_{t=1}^{12} e_{k,t} x_{k,t} \right),$$

where it can be decomposed into a constant component, coming from the initial portfolio, and a variable part driven by the decision variables and the parameters $e_{k,t}$, which denote the interest rate associated with aggregate k at time t. The model incorporates several structural and regulatory constraints. The monthly balance sheet identity and capital requirements introduce the inequality, respectively,

$$\sum_{k=1}^{N} a_k \, V_k(x_k; \theta_k, t) = \mathrm{E}, \qquad \forall\, t = 1, \dots, 12,$$
$$\sum_{k=1}^{N} \mathrm{rw}_k \, V_k(x_k; \theta_k, t) \le Cap, \qquad \forall\, t = 1, \dots, 12,$$

where E denotes the equity level, ensuring the fundamental balance sheet relation between assets and liabilities, *Cap* represents the available regulatory capital as required by Basel III, and rw_k denotes the risk–weight factor associated with aggregate k. Liquidity regulation is captured through the nonlinear expressions of the Liquidity Coverage Ratio,

$$\mathrm{LCR}_t(x) = \frac{\mathrm{HQLA}_t(x)}{\mathrm{NCF}_t(x)} \ge 1.1,$$

and the Net Stable Funding Ratio,

$$\mathrm{NSFR}_t(x) = \frac{\mathrm{ASF}_t(x)}{\mathrm{RSF}_t(x)} \ge 1.1,$$

and in conclusion we have some operational limits that impose the box constraints

$$\underline{x}_{k,t} \leq x_{k,t} \leq \overline{x}_{k,t}.$$

By collecting all the above components, a complete nonlinear optimization model is formed:

$$\begin{aligned}
\max_{x_{k,t}} \quad & \sum_{k=1}^{N} a_k \left(\mathrm{NII}_k^{\mathrm{ACTUAL}} + \sum_{t=1}^{12} x_{k,t} e_{k,t} \right) \\
\text{s.t.} \quad & \sum_{k=1}^{N} a_k \, V_k(x_{k,1}, \ldots, x_{k,12}; \theta_k, t) = E, \qquad \forall\, t = 1, \ldots, 12, \\
& \sum_{k=1}^{N} \mathrm{rw}_k \, V_k(x_{k,1}, \ldots, x_{k,12}; \theta_k, t) \leq Cap, \qquad \forall\, t = 1, \ldots, 12, \\
& LCR_t(x) \geq 1.1, \qquad NSFR_t(x) \geq 1.1, \qquad \forall\, t = 1, \ldots, 12, \\
& \underline{x}_{k,t} \leq x_{k,t} \leq \overline{x}_{k,t}, \qquad \forall\, k = 1, \ldots, N, \quad \forall\, t = 1, \ldots, 12.
\end{aligned}$$

The final section of the thesis presents the numerical implementation of the constructed model. After translating the theoretical model into a computational framework and reconstructing the bank's balance sheet aggregates, the optimization problems are coded and solved in *Python*, under different sets of regulatory constraints. By analyzing different model configurations, the study illustrates how each regulatory component shapes the space of feasible solutions and clarifies the tradeoffs imposed by regulatory constraints. It is therefore hoped that the optimization model will be viewed not only as a theoretical construct, but also as a practical decision support tool for strategic balance sheet management.

Contents

Chapter 1

Preliminaries concepts of Operational Research and Mathematical Programming

In this chapter we introduce the fundamental concepts of Operational Research and Mathematical Programming, starting from the general formulation of an optimization problem and the definitions of feasible set, local solutions, and global solutions. We distinguish between linear and nonlinear problems, highlighting the role of the gradient and the Hessian in characterizing optimality conditions for unconstrained optimization. When moving to constrained problems, we present geometric notions such as active sets, tangent cones, and linearized feasible directions, which naturally lead to the Karush–Kuhn–Tucker conditions and the corresponding second-order criteria. The concepts outlined here provide the theoretical and methodological foundations for the algorithms and applications developed in the following chapters. We mainly follow the classical treatments presented in [5], whose structure and notation are aligned with ours, for the theoretical foundations of nonlinear programming we refer to [2], while for the linear and integer programming aspects we rely on the comprehensive exposition of [6].

1.1 General Optimization Problem

Consider the general optimization problem

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq 0, \quad i \in \mathcal{I}, \\ & h_j(x) = 0, \quad j \in \mathcal{E}, \end{aligned} \tag{1.1}$$

where

$$f:\mathbb{R}^n\to\mathbb{R},\quad g_i:\mathbb{R}^n\to\mathbb{R},\quad h_j:\mathbb{R}^n\to\mathbb{R},$$

and $\mathcal{E}$ and $\mathcal{I}$ are two finite sets of indices. From now on, we fix their cardinalities as

$$|\mathcal{I}|=m,\qquad |\mathcal{E}|=p,$$

where $p,m\in\mathbb{N}$ are fixed but arbitrary. Here the vector $x=(x_1,\dots,x_n)$ is the *optimization variable* of the problem, the function f is the *cost*, or *objective*, function and the functions g_i and h_i determine inequality constraints and equality constraints, respectively. For simplicity of notation, since the sets $\mathcal{I}$ and $\mathcal{E}$ are finite, from now on we assume that they are enumerated as

$$\mathcal{I}=\{i_1,i_2,\dots,i_m\},\qquad \mathcal{E}=\{j_1,j_2,\dots,j_p\}.$$

Thus, for each $k=1,\dots,m$ the index i_k denotes an element of $\mathcal{I}$, and the index j_k denotes an element of $\mathcal{E}$.

Definition 1.1 (Feasible set)**.** *The "Feasible set" (or search space) of problem* (1.1) *is defined as*

$$\mathcal{D}:=\Big\{x\in\mathbb{R}^n\Big|g(x)\le 0,\; h(x)=0\Big\},$$

where

$$g(x):=\begin{bmatrix} g_{i_1}(x)\\ \vdots\\ g_{i_m}(x)\end{bmatrix}\in\mathbb{R}^m,\quad h(x):=\begin{bmatrix} h_{j_1}(x)\\ \vdots\\ h_{j_p}(x)\end{bmatrix}\in\mathbb{R}^p.$$

Definition 1.2 (Solution, Global solution, Local solution)**.** *A vector $x^*\in\mathcal{D}$ is called a "Feasible solution" to problem* (1.1) *if it satisfies all the constraints, i.e., $x^*\in\mathcal{D}$. It is said to be a "Global (or optimal) solution" if it achieves the minimum of the objective function over the feasible set:*

$$f(x^*)\le f(x)\quad\forall\, x\in\mathcal{D}.$$

In this case, the optimal value is denoted by $f^*:=f(x^*)$. If the inequality is strict for all $x\in\mathcal{D}$ with $x\neq x^*$, then x^* is called a "strong global solution":

$$f(x^*)<f(x)\quad\forall\, x\in\mathcal{D},\; x\neq x^*.$$

A vector $x^*\in\mathcal{D}$ is a "local solution" to problem (1.1) *if there exists a neighborhood $\mathcal{N}$ of x^* such that*

$$f(x^*)\le f(x)\quad\forall\, x\in\mathcal{N}\cap\mathcal{D}.$$

If the inequality is strict for all $x \in \mathcal{N} \cap \mathcal{D}$ with $x \neq x^*$, then x^* is called a "strong local solution":

$$f(x^*) < f(x) \quad \forall\, x \in \mathcal{N} \cap \mathcal{D},\, x \neq x^*.$$

Moreover, if x^* is the only local solution in $\mathcal{N} \cap \mathcal{D}$, it is called an "isolated local solution".

Different classes of problems are distinguished according to the properties of the objective function and constraints (linear, nonlinear, convex), the number of variables (large or small), the smoothness of the functions (differentiable or non-differentiable), and so on.

Definition 1.3 (LP and NLP). *If all functions f, g_i, h_j can be written in affine form, i.e.,*

$$f(x) = c^\top x + c_0, \quad g_i(x) = a_i^\top x + b_i, \quad h_j(x) = d_j^\top x + e_j,$$

where $c, a_i, d_j \in \mathbb{R}^n$ and $c_0, b_i, e_j \in \mathbb{R}$, then problem (1.1) *is called a "linear programming problem" (LP). If at least one among f, g_i, h_j is nonlinear, then the problem is called a "nonlinear programming problem" (NLP).*

1.2 Unconstrained Optimization

One of the main methods for classifying optimization problems is based on the presence or absence of constraints on the variables. Unconstrained problems, for which we have $\mathcal{D} = \mathbb{R}^n$, do not impose explicit restrictions on the variables. Even when natural constraints exist, they are often negligible in practice, allowing the use of unconstrained methods. Additionally, unconstrained formulations frequently arise from constrained problems via penalization, incorporating the constraints into the objective function to discourage violations without restricting the search space explicitly. Constrained problems, by contrast, require all constraints to be satisfied.

Henceforth, we assume that objective function f is sufficiently smooth. Efficient identification of local minima can then rely on first and second order information: if f is twice continuously differentiable, a point x^* can be classified as a local minimizer (or strict local minimizer) by examining the gradient $\nabla f(x^*)$ and the Hessian $\nabla^2 f(x^*)$.

Definition 1.4 (Descent Direction). *A vector $d \in \mathbb{R}^n$ is called a "descent direction" for a function f at a point x if there exists $\delta > 0$ such that*

$$f(x + \alpha d) < f(x) \quad \forall\, \alpha \in (0, \delta).$$

The "directional derivative" of f at x along the direction d is defined as

$$f'(x, d) := \lim_{\lambda \to 0^+} \frac{f(x + \lambda d) - f(x)}{\lambda} = \nabla f(x)^\top d.$$

Hence, d is a descent direction at x if

$$\nabla f(x)^\top d < 0.$$

Lemma 1.5 (First-Order Necessary Conditions)**.** *If x^* is a local minimizer of f and f is continuously differentiable in an open neighborhood of x^*, then*

$$\nabla f(x^*) = 0.$$

Lemma 1.6 (Second-Order Necessary Conditions)**.** *If x^* is a local minimizer of f and $\nabla^2 f$ exists and is continuous in an open neighborhood of x^*, then*

$$\nabla f(x^*) = 0 \quad \textit{and} \quad \nabla^2 f(x^*) \succeq 0,$$

i.e., the Hessian is positive semi-definite.

Lemma 1.7 (Second-Order Sufficient Conditions)**.** *Suppose $\nabla^2 f$ is continuous in an open neighborhood of x^*. If*

$$\nabla f(x^*) = 0 \quad \textit{and} \quad \nabla^2 f(x^*) \succ 0,$$

i.e., the Hessian is positive definite, then x^* is a strict local minimizer of f.

The necessary and sufficient conditions are based on the Taylor series expansion and are valid only in a neighborhood of the current solution x^*. An important consequence of the necessary conditions is that they can be extended to a global statement: if a point x does not satisfy the necessary conditions for being a local minimum, then it cannot be a global minimum.

All unconstrained optimization algorithms require an initial guess x_0, which may be chosen based on prior knowledge of the problem or automatically by the algorithm. From this starting point, a sequence of iterates $\{x_k\}_{k\geq 0}$ is generated with the aim of converging to a minimizer of f. The procedure stops when no further progress can be made or when the iterates are sufficiently close to a solution. Each step from x_k to x_{k+1} is determined using information about f at the current point, possibly combined with data from previous iterations, to obtain a new point with a lower objective value. At each iteration k, let x_k denote the current point. The next iterate is obtained as

$$x_{k+1} = x_k + \alpha_k d_k,$$

where $d_k \in \mathbb{R}^n$, with $\|d_k\| = 1$, is the *search direction*, and $\alpha_k > 0$ is the *step size*.

Two fundamental principles underlie the construction of the new iterate:

- **Line search strategy**, which define the path along which the algorithm moves from x_k, determines the search direction and then the step size,

- **Trust region strategy**, which determine how far along this direction the next point x_{k+1} is chosen, determines the step size and then the search direction.

Most algorithms for unconstrained optimization can be understood as variations or combinations of these two basic ideas.

1.3 Constrained Optimization

Let us now consider the problem in the general form proposed in (1.1), with the assumption that the functions f, g_i and h_j are sufficient smooth, real-valued functions on a subset of $\mathbb{R}^n$.

Definition 1.8 (Active set)**.** *Let $x \in \mathcal{D}$ be a feasible solution. The "active set" at x is defined as*

$$\mathcal{A}(x) := \mathcal{E} \cup \{i \in \mathcal{I} \mid g_i(x) = 0\}.$$

In other words, $\mathcal{A}(x)$ consists of:

- *all equality constraints, which are always active,*

- *together with those inequality constraints that are satisfied as equalities at x.*

Inequality constraints with $g_i(x) < 0$ are not in $A(x)$ and are called "inactive".

It is useful to recall some geometric notions that describe how the feasible set looks locally around a feasible solution. In particular, we need to formalize the idea of the directions along which it is possible (or at least approximately possible) to move while remaining feasible.

Definition 1.9 (Tangent cone)**.** *Let $\mathcal{X} \subseteq \mathbb{R}^n$ a closed convex set and let $x \in \mathcal{X}$. The "tangent cone" to $\mathcal{X}$ at x, denoted $T_{\mathcal{X}}(x)$, is defined as the set of all vectors $d \in \mathbb{R}^n$ such that there is:*

- *a sequence $\{x_k\}_{k\in\mathbb{N}} \subseteq \mathcal{X}$ with $x_k \to x$ for $k \to +\infty$,*

- *a sequence $\{t_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}_{>0}$ with $t_k \to 0^+$ for $k \to +\infty$,*

for which the limit

$$\lim_{k\to+\infty} \frac{x_k - x}{t_k} = d$$

holds.

In other words, the tangent cone is a geometric object that generalizes the notion of the tangent space to a manifold to the case of certain spaces with singularities, defined as all limits of vectors connecting a point x to points x_k of the set, normalized by scalars $t_k \to 0$, thus capturing all directions in which it is possible to move away from x while remaining within the feasible set.

Definition 1.10 (Linearized Feasible Direction Set). *Let $x \in \mathbb{R}^n$ a feasible solution, the "linearized feasible direction set" of x, denoted $F(x)$, is*

$$F(x) = \left\{ d \in \mathbb{R}^n \middle| \nabla h_j(x)^\top d = 0, \, \forall\, j \in \mathcal{E}, \quad \nabla g_i(x)^\top d \leq 0, \, \forall\, i \in A(x) \cap \mathcal{I} \right\}.$$

Intuitively, the set $F(x)$ represents all directions $d \in \mathbb{R}^n$ along which it is possible to move infinitesimally from the point x while remaining, to first order, within the linearized approximation of the active constraints. These are the directions that satisfy all linearized equality constraints and do not violate the linearized active inequality constraints. The relation between $T_{\mathcal{X}}(x)$ and $F(x)$ is central in constrained optimization: under appropriate regularity assumptions (constraint qualifications) the two sets coincide. One of the most common qualifications ensuring this property is the following.

Lemma 1.11. *If x^* is a local solution of general optimization problem* (1.1)*, then we have:*

$$\nabla f(x^*)^\top d \geq 0, \qquad \forall\, d \in T_{\mathcal{D}}(x^*).$$

Proof. Assume, by contradiction, that there exists a tangent direction $d \in T_{\mathcal{X}}(x^*)$ such that

$$\nabla f(x^*)^\top d < 0,$$

from the definition of tangent cone given above, there are sequences $\{z_k\}_{k \in \mathbb{N}} \subset \mathcal{D}$ and $\{t_k\}_{k \in \mathbb{N}} \in \mathbb{R}_{>0}$ with, $t_k \to 0^+$, satisfying

$$z_k \to x^*, \qquad \frac{z_k - x^*}{t_k} \to d, \qquad \text{for } k \to +\infty.$$

Consider the Taylor expansion of f around x^*:

$$f(z_k) = f(x^*) + \nabla f(x^*)^\top (z_k - x^*) + o(\|z_k - x^*\|),$$

since $z_k - x^* = t_k d + o(t_k)$, this becomes

$$f(z_k) = f(x^*) + t_k \nabla f(x^*)^\top d + o(t_k).$$

Because $\nabla f(x^*)^\top d < 0$, the first order term dominates the remainder for k sufficiently large. Hence, there exists $K \in \mathbb{N}$ such that for all $k \geq K$,

$$f(z_k) < f(x^*) + \tfrac{1}{2} t_k \nabla f(x^*)^\top d < f(x^*).$$

Thus, within any neighborhood of x^*, we can find points z_k that lie in $\mathcal{D}$ and satisfy $f(z_k) < f(x^*)$. This contradicts the assumption that x^* is a local solution. Therefore,

$$\nabla f(x^*)^\top d \geq 0, \qquad \forall\, d \in T_{\mathcal{X}}(x^*),$$

which completes the proof. □

Definition 1.12 (Linear Independence Constraint Qualification). *Given a feasible solution x and its corresponding active set $\mathcal{A}(x)$, the "Linear Independence Constraint Qualification (LICQ)" is said to hold at x if the set of active constraint gradients*

$$\{\nabla g_i(x) : i \in \mathcal{A}(x) \cap \mathcal{I}\} \cup \{\nabla h_j : j \in \mathcal{E}\}$$

is linearly independent. That is, no active constraint gradient can be expressed as a linear combination of the others.

Remark 1.13. Let x^* a feasible solution of general problem (1.1). Then the following hold:

1. $T_{\mathcal{D}}(x^*) \subseteq F(x^*)$,
2. if the Linear Independence Constraint Qualification (LICQ) holds at x^*, then

$$T_{\mathcal{D}}(x^*) = F(x^*).$$

We now give necessary conditions, defined in the following theorem, called *first and second order conditions* because they relate to the properties of the gradients and of the Hessian of the objective and constraint functions respectively. These conditions are the foundation for many of the algorithms described later in the thesis.

Definition 1.14 (Lagrangian). *The "Lagrangian function" associated with problem* (1.1) *is*

$$\mathcal{L}(x, \lambda, \mu) := f(x) + \sum_{k=1}^{m} \lambda_k g_{i_k}(x) + \sum_{k=1}^{p} \mu_k h_{j_k}(x), \tag{1.2}$$

where λ_k and μ_k are called "Lagrange multipliers" associated with the corresponding constraints. The gradient of the Lagrangian with respect to x is given by

$$\nabla_x \mathcal{L}(x, \lambda, \mu) = \nabla f(x) + \sum_{k=1}^{m} \lambda_k \nabla g_{i_k}(x) + \sum_{k=1}^{p} \mu_k \nabla h_{j_k}(x).$$

Theorem 1.15 (First and Second Order Necessary Conditions). *Given the general problem* (1.1)*, assume that the functions f, g_i, and h_j for each i and j are C^1, and LICQ holds at a local solution x^*. Then, there exist Lagrange multipliers $\lambda \in \mathbb{R}^m$ for the inequality constraints and $\mu \in \mathbb{R}^p$ for the equality constraints such that the following conditions, named Karush-Kuhn-Tucker conditions [9], are satisfied:*

1. ***Stationarity:***
$$\nabla_x \mathcal{L}(x^*, \lambda, \mu) = 0. \tag{1.3}$$

2. ***Primal feasibility:***
$$g_i(x^*) \leq 0, \quad i \in \mathcal{I}, \qquad h_j(x^*) = 0, \quad j \in \mathcal{E}. \tag{1.4}$$

3. ***Dual feasibility:***
$$\lambda_k \geq 0, \quad k = 1, \dots, m. \tag{1.5}$$

4. ***Complementary slackness:***
$$\lambda_k g_{i_k}(x^*) = 0, \quad k = 1, \dots, m. \tag{1.6}$$

If in addition the functions f, g_i, and h_j for each i and j are C^2, there holds
$$w^\top \nabla^2_{xx} \mathcal{L}(x^*, \lambda, \mu) w \geq 0, \tag{1.7}$$
for all $w \in \mathbb{R}^n$ such that
$$\nabla g_i(x^*)^\top w = 0, \, \forall \, i \in \mathcal{A}(x^*) \cap \mathcal{I}, \qquad \nabla h_j(x^*)^\top w = 0, \, \forall \, j \in \mathcal{E}.$$

Remark 1.16. The complementary slackness implies that for each inequality constraint, either the constraint is active ($g_{i_k}(x^*) = 0$) or the corresponding multiplier is zero ($\lambda_k = 0$), or both. For inactive constraints, $\lambda_k = 0$, and the stationarity condition can equivalently be written summing only over the active constraints:
$$\nabla f(x^*) + \sum_{k: i_k \in \mathcal{A}(x^*)} \lambda_k \nabla g_{i_k}(x^*) + \sum_{k=1}^{p} \mu_k \nabla h_{j_k}(x^*) = 0.$$

We will call a point x^* satisfying the KKT conditions also "KKT point or critical point" for the NLP (1.1).

Lemma 1.17 (Farkas' Lemma)**.** *Consider the cone*
$$K = K_{B,C} := \{By + Cw : y \in \mathbb{R}^q_{\geq 0}, \, w \in \mathbb{R}^p\},$$
where $B \in \mathbb{R}^{n \times q}$, $C \in \mathbb{R}^{n \times p}$. Then, for any vector $g \in \mathbb{R}^n$, the following holds:
$$g \in K \quad \Longleftrightarrow \quad \nexists \, d \in \mathbb{R}^n \quad s.\ t. \quad g^\top d < 0, \quad B^\top d \geq 0, \quad C^\top d = 0.$$

In general, all vector equalities and inequalities such as $B^\top d \geq 0$ or $C^\top d = 0$ must be interpreted component by component, that is, each condition holds for every component of the corresponding vector.

Remark 1.18. The second alternative has a geometric interpretation: the vector d defines a hyperplane in $\mathbb{R}^n$ that separates g from the cone K. In other words, either g lies inside the cone, or there exists a separating hyperplane witnessing that g is outside the cone but never both.

The Farkas' lemma is a classical and well known result in mathematical programming. Several equivalent formulations can be found, for instance, in [6], Chapter 7. The proof of the version used here follows closely the exposition in [5], Lemma 12.4, where the argument is presented in a particularly clear manner.

Proof of Theorem (1.15). We first prove the first-order part of the statement, that is, the existence of Lagrange multipliers satisfying the Karush–Kuhn–Tucker conditions. Let $x^* \in \mathcal{D}$ be a local solution of problem (1.1) and assume that LICQ holds at x^*. By Lemma 1.11 we have

$$\nabla f(x^*)^\top d \geq 0, \qquad \forall\, d \in T_{\mathcal{D}}(x^*), \tag{1.8}$$

moreover, by the remark on the relation between the tangent cone and the linearized feasible direction set, LICQ implies

$$T_{\mathcal{D}}(x^*) = F(x^*). \tag{1.9}$$

Combining (1.8) and (1.9) we obtain

$$\nabla f(x^*)^\top d \geq 0, \qquad \forall\, d \in F(x^*). \tag{1.10}$$

For convenience, let us enumerate the set of inequality constraints in effect on x^*:

$$\mathcal{A}(x^*) \cap \mathcal{I} = \{\, i \in \mathcal{I} \mid g_i(x^*) = 0 \,\} = \{\hat{i}_1, \dots, \hat{i}_q\},$$

for some $q \in \mathbb{N}$ and recall that the inequality and equality constraints are indexed by

$$\mathcal{I} = \{i_1, i_2, \dots, i_m\}, \qquad \mathcal{E} = \{j_1, j_2, \dots, j_p\}.$$

We now define the matrices

$$B := \begin{bmatrix} -\nabla g_{\hat{i}_1}(x^*) & \cdots & -\nabla g_{\hat{i}_q}(x^*) \end{bmatrix} \in \mathbb{R}^{n\times q}, \qquad C := \begin{bmatrix} \nabla h_{j_1}(x^*) & \cdots & \nabla h_{j_p}(x^*) \end{bmatrix} \in \mathbb{R}^{n\times p}.$$

With this choice, the cone K appearing in Lemma 1.17 is

$$K = \Big\{ -\sum_{\ell=1}^{q} y_\ell \nabla g_{\hat{i}_\ell}(x^*) + \sum_{k=1}^{p} w_k \nabla h_{j_k}(x^*) : y \in \mathbb{R}^q_{\geq 0},\ w \in \mathbb{R}^p \Big\},$$

and if we apply this Lemma with

$$g := \nabla f(x^*), \qquad B, C \text{ as above},$$

we obtain the equivalence

$$g \in K \quad \Longleftrightarrow \quad \nexists\, d \in \mathbb{R}^n \quad \text{s. t.} \quad g^\top d < 0, \quad B^\top d \geq 0, \quad C^\top d = 0.$$

Let us analyze the negation of this statement. Assume, for contradiction, that there exists a vector $d \in \mathbb{R}^n$ such that

$$\nabla f(x^*)^\top d < 0, \qquad B^\top d \geq 0, \qquad C^\top d = 0. \tag{1.11}$$

Using the definitions of B and C, the conditions $B^\top d \geq 0$ and $C^\top d = 0$ are equivalent to

$$(-\nabla g_{\hat{i}_\ell}(x^*))^\top d \geq 0, \quad \ell = 1, \ldots, q, \qquad \nabla h_{j_k}(x^*)^\top d = 0, \quad k = 1, \ldots, p,$$

that is to say,

$$\nabla g_{\hat{i}_\ell}(x^*)^\top d \leq 0, \quad \forall\, \hat{i}_\ell \in \mathcal{A}(x^*) \cap \mathcal{I}, \qquad \nabla h_j(x^*)^\top d = 0, \quad \forall\, j \in \mathcal{E}.$$

By the definition of the linearized feasible direction set $F(x^*)$, these relations mean that $d \in F(x^*)$. Hence, under (1.11), we obtain simultaneously

$$d \in F(x^*), \qquad \nabla f(x^*)^\top d < 0,$$

which contradicts (1.10). Therefore, such a vector d cannot exist by the formulation of Lemma 1.17, it follows that

$$\nabla f(x^*) \in K,$$

i.e., there exist multipliers $y \in \mathbb{R}^q_{\geq 0}$ and $w \in \mathbb{R}^p$ such that

$$\nabla f(x^*) = -\sum_{\ell=1}^{q} y_\ell \, \nabla g_{\hat{i}_\ell}(x^*) + \sum_{k=1}^{p} w_k \, \nabla h_{j_k}(x^*). \tag{1.12}$$

We now define Lagrange multipliers for all inequality and equality constraints. For the inequality multipliers, we set

$$\hat{\lambda}_\ell := y_\ell, \qquad \forall\, \ell = 1, \ldots, q,$$

and for the equality multipliers we define

$$\mu_{j_k} := -w_k, \quad \forall\, k = 1, \ldots, p.$$

With this notation, (1.12) becomes

$$\nabla f(x^*) = -\sum_{\ell=1}^{q} \hat{\lambda}_\ell \nabla g_{\hat{i}_\ell}(x^*) - \sum_{k=1}^{p} \mu_k \nabla h_{j_k}(x^*),$$

we now construct the full vector of multipliers $\lambda \in \mathbb{R}^m$ associated with the inequality constraints by setting for each $k = 1, \dots, p$:

$$\lambda_k := \begin{cases} \hat{\lambda}_\ell, & \text{if } i_k = \hat{i}_\ell \text{ for some } \ell \in \{1, \dots, q\}, \\ 0, & \text{otherwise,} \end{cases}$$

since $\{\hat{i}_1, \dots, \hat{i}_q\} \subseteq \{i_1, \dots, i_m\}$, the above identity can be rewritten as

$$\nabla f(x^*) + \sum_{k=1}^{m} \lambda_k \nabla g_{i_k}(x^*) + \sum_{k=1}^{p} \mu_k \nabla h_{j_k}(x^*) = 0.$$

That is precisely the stationarity condition

$$\nabla_x \mathcal{L}(x^*, \lambda, \mu) = 0.$$

We now check the remaining KKT conditions:

- *Primal feasibility.* By assumption, $x^* \in \mathcal{D}$ is feasible for problem (1.1), hence

$$g_i(x^*) \leq 0, \quad \forall\, i \in \mathcal{I}, \qquad h_j(x^*) = 0, \quad \forall\, j \in \mathcal{E}.$$

- *Dual feasibility.* By construction.

- *Complementary slackness.* For $i_k \in \mathcal{A}(x^*)$ we have $g_{i_k}(x^*) = 0$ by definition of active inequality constraints, hence

$$\lambda_k g_i(x^*) = \lambda_k \cdot 0 = 0.$$

  For $i_k \in \mathcal{I} \setminus (\mathcal{A}(x^*) \cap \mathcal{I})$ we have $\lambda_k = 0$, therefore,

$$\lambda_k g_{i_k}(x^*) = 0, \qquad \forall\, k = 1, \dots, m.$$

This completes the proof of the first-order necessary conditions. We now assume in addition that f, g_i and h_j are of class C^2 in a neighborhood of the local solution x^*, and we prove (1.7). Let $\lambda \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^p$ be the Lagrange multipliers associated with x^* given by the first-order part of the theorem, and fix any direction $w \in \mathbb{R}^n$ such that

$$\nabla g_i(x^*)^\top w = 0, \quad \forall\, i \in \mathcal{A}(x^*) \cap \mathcal{I}, \qquad \nabla h_j(x^*)^\top w = 0, \quad \forall\, j \in \mathcal{E}. \tag{1.13}$$

Consider the mapping

$$F(x) := \begin{pmatrix} \big(g_i(x)\big)_{i \in \mathcal{A}(x^*) \cap \mathcal{I}} \\ \big(h_j(x)\big)_{j \in \mathcal{E}} \end{pmatrix}.$$

By definition of the active set we have $F(x^*) = 0$. Moreover, LICQ at x^* means that the gradients

$$\{\nabla g_i(x^*) : i \in \mathcal{A}(x^*) \cap \mathcal{I}\} \cup \{\nabla h_j(x^*) : j \in \mathcal{E}\}$$

are linearly independent, that is, the $\nabla F(x^*)$ has full row rank. Since F is C^2, by the "Implicit Function Theorem" the level set

$$\mathcal{M} := \{x \in \mathbb{R}^n \mid F(x) = 0\}$$

is a C^2 manifold in a neighborhood of x^*, and its tangent space at x^* is

$$T_{\mathcal{M}}(x^*) = \{d \in \mathbb{R}^n \mid \nabla g_i(x^*)^\top d = 0, \forall\, i \in \mathcal{A}(x^*) \cap \mathcal{I}, \quad \nabla h_j(x^*)^\top d = 0, \forall\, j \in \mathcal{E}\}.$$

By (1.13) we have $w \in T_{\mathcal{M}}(x^*)$. A standard result as a consequence of the Implicit Function Theorem ensures that for every $w \in T_{\mathcal{M}}(x^*)$ there exists a C^2 curve

$$\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$$

such that:

$$\gamma(0) = x^*, \qquad \gamma'(0) = w.$$

By construction we have, for all t sufficiently small,

$$g_i(\gamma(t)) = 0, \quad \forall\, i \in \mathcal{A}(x^*) \cap \mathcal{I}, \qquad h_j(\gamma(t)) = 0, \quad \forall\, j \in \mathcal{E}.$$

For the inactive inequality constraints $i \in \mathcal{I} \setminus \mathcal{A}(x^*) \cap \mathcal{I}$ we have $g_i(x^*) < 0$, by continuity of g_i there exists $\delta > 0$ such that $g_i(\gamma(t)) < 0$ for all $|t| \leq \delta$. Hence, for $|t|$ small enough, $\gamma(t)$ satisfies all constraints of (1.1), and therefore $\gamma(t)$ is a curve of feasible solution passing through x^* with initial direction w. Now we define a new scalar function as a composition of the objective function and this curve,

$$\varphi(t) := f(\gamma(t)), \qquad t \in (-\varepsilon, \varepsilon).$$

Since x^* is a local solution of (1.1) and $\gamma(t)$ is feasible for $|t|$ small, the point $t = 0$ is a local minimizer of φ, i.e.,

$$\varphi'(0) = 0, \qquad \varphi''(0) \geq 0.$$

By the chain rule we have

$$\varphi'(t) = \nabla f(\gamma(t))^\top \gamma'(t) \qquad \Longrightarrow \qquad \varphi'(0) = \nabla f(x^*)^\top w.$$

We show that $\varphi'(0) = 0$ is automatically implied by the KKT conditions. Indeed, from stationarity,

$$\nabla f(x^*)^\top w = -\sum_{k=1}^{m} \lambda_k \nabla g_{i_k}(x^*)^\top w - \sum_{k=1}^{p} \mu_k \nabla h_{j_k}(x^*)^\top w.$$

For $i \in \mathcal{A}(x^*) \cap \mathcal{I}$ we have $\nabla g_i(x^*)^\top w = 0$ by assumption (1.13). For $i \in \mathcal{I} \setminus \mathcal{A}(x^*) \cap \mathcal{I}$ we have $g_i(x^*) < 0$, hence $\lambda_i = 0$ by complementary slackness. For all $j \in \mathcal{E}$ we have $\nabla h_j(x^*)^\top w = 0$ again by (1.13). Therefore both sums vanish and

$$\nabla f(x^*)^\top w = 0,$$

so that indeed $\varphi'(0) = 0$, consistently with the fact that $t = 0$ is a local minimum of φ. Differentiating φ a second time and using again the chain rule, we obtain

$$\varphi''(t) = \gamma'(t)^\top \nabla^2 f(\gamma(t))\, \gamma'(t) + \nabla f(\gamma(t))^\top \gamma''(t),$$

evaluating at $t = 0$ gives

$$\varphi''(0) = w^\top \nabla^2 f(x^*)\, w + \nabla f(x^*)^\top \gamma''(0). \tag{1.14}$$

We now express $\nabla f(x^*)^\top \gamma''(0)$ in terms of the constraints and the multipliers, from stationarity we have

$$\nabla f(x^*)^\top \gamma''(0) = -\sum_{i \in \mathcal{I}} \lambda_i \nabla g_i(x^*)^\top \gamma''(0) - \sum_{j \in \mathcal{E}} \mu_j \nabla h_j(x^*)^\top \gamma''(0). \tag{1.15}$$

Differentiating twice g_i with respect to t gives, for each $i \in \mathcal{A}(x^*) \cap \mathcal{I}$,

$$\frac{d}{dt} g_i(\gamma(t)) = \nabla g_i(\gamma(t))^\top \gamma'(t), \qquad \frac{d^2}{dt^2} g_i(\gamma(t)) = \gamma'(t)^\top \nabla^2 g_i(\gamma(t))\, \gamma'(t) + \nabla g_i(\gamma(t))^\top \gamma''(t),$$

evaluating at $t = 0$ and using $\gamma'(0) = w$ we obtain

$$0 = \frac{d^2}{dt^2} g_i(\gamma(t))\bigg|_{t=0} = w^\top \nabla^2 g_i(x^*)\, w + \nabla g_i(x^*)^\top \gamma''(0),$$

from which

$$\nabla g_i(x^*)^\top \gamma''(0) = -w^\top \nabla^2 g_i(x^*)\, w, \qquad i \in \mathcal{A}(x^*) \cap \mathcal{I}. \tag{1.16}$$

For all k such that $i_k \in \mathcal{I} \setminus \mathcal{A}(x^*) \cap \mathcal{I}$ we have $\lambda_k = 0$, so these indices do not contribute to (1.15). Similarly, for h_j fixed $j \in \mathcal{E}$,

$$0 = \frac{d^2}{dt^2} h_j(\gamma(t))\bigg|_{t=0} = w^\top \nabla^2 h_j(x^*)\, w + \nabla h_j(x^*)^\top \gamma''(0),$$

from which

$$\nabla h_j(x^*)^\top \gamma''(0) = -w^\top \nabla^2 h_j(x^*)\, w, \qquad j \in \mathcal{E}. \tag{1.17}$$

Substituting (1.16) and (1.17) in (1.15) we have

$$\nabla f(x^*)^\top \gamma''(0) = \sum_{i \in \mathcal{A}(x^*) \cap \mathcal{I}} \lambda_i\, w^\top \nabla^2 g_i(x^*)\, w + \sum_{j \in \mathcal{E}} \mu_j\, w^\top \nabla^2 h_j(x^*)\, w.$$

Inserting this expression into (1.14), it follows

$$\varphi''(0) = w^\top \nabla^2 f(x^*)\, w + \sum_{i \in \mathcal{A}(x^*) \cap \mathcal{I}} \lambda_i\, w^\top \nabla^2 g_i(x^*)\, w + \sum_{j \in \mathcal{E}} \mu_j\, w^\top \nabla^2 h_j(x^*)\, w.$$

Noting that the terms with $i \notin \mathcal{A}(x^*) \cap \mathcal{I}$ are zero because $\lambda_i = 0$, we can rewrite the whole thing as

$$\varphi''(0) = w^\top \Big(\nabla^2 f(x^*) + \sum_{i \in \mathcal{I}} \lambda_i \nabla^2 g_i(x^*) + \sum_{j \in \mathcal{E}} \mu_j \nabla^2 h_j(x^*) \Big) w = w^\top \nabla^2_{xx} \mathcal{L}(x^*, \lambda, \mu)\, w.$$

Since x^* is a local minimum and $\gamma(t)$ is a feasible curve passing through x^*, we have that $t = 0$ is a local minimum of $\varphi(t) = f(\gamma(t))$, and therefore

$$\varphi''(0) \geq 0,$$

By the previous formula for $\varphi''(0)$ we deduce

$$w^\top \nabla^2_{xx} \mathcal{L}(x^*, \lambda, \mu)\, w \geq 0$$

for every $w \in \mathbb{R}^n$ that satisfies (1.13), that is, for every direction that is tangent to the surfaces of the active constraints in x^*. □

We introduce a sufficient condition, formulated in terms of the objective function and the constraints which guarantees that x^* is a local solution to the problem. However, the second-order sufficient condition closely resembles the necessary condition just introduced but differs in that the qualification of the constraint is not required and the Hessian inequality is replaced by a strict inequality.

Definition 1.19 (Critical Cone). *Let x^* be a feasible solution and let (λ^*, μ^*) be a pair of Lagrange multipliers satisfying the KKT conditions. The "critical cone" at (x^*, λ^*, μ^*) is defined as*

$$w \in C(x^*, \lambda^*) \iff \begin{cases} -\nabla h_j(x^*)^\top w \leq 0, & \forall\, j \in \mathcal{E}, \\ -\nabla g_{i_k}(x^*)^\top w \leq 0, & \forall\, i_k \in \mathcal{A}(x) \text{ with } \lambda^*_{i_k} > 0, \\ -\nabla g_{i_k}(x^*)^\top w \geq 0, & \forall\, i_k \in \mathcal{A}(x) \text{ with } \lambda^*_{i_k} = 0. \end{cases} \tag{1.18}$$

Lemma 1.20. *Let $\{x_k\}_{k \in \mathbb{N}}$ be a sequence of feasible solution with $x_k \to x^*$ and let*

$$d = \lim_{k \to \infty} \frac{x_k - x^*}{\|x_k - x^*\|}.$$

Then $d \in C(x^*, \lambda^*)$, that is, the limit direction belongs to the critical cone.

Theorem 1.21 (Second-Order Sufficiency Conditions)**.** *Given the general problem* (1.1)*, assume that the functions* f*,* g_i*, and* h_j *for each* i *and* j *are* C^2*, and let* $x^* \in \mathbb{R}^n$ *be a feasible solution. Let us assume that* $\lambda \in \mathbb{R}^m$ *and* $\mu \in \mathbb{R}^p$ *exist such that* x^* *satisfies the KKT conditions for the problem, and additionally,*

$$w^\top \nabla^2_{xx}\mathcal{L}(x^*, \lambda, \mu)\, w > 0, \qquad \forall\, w \in C(x, \lambda),\ w \neq 0, \tag{1.19}$$

then x^* *is a strict local minimum of problem* (1.1)*.*

Proof. We define the compact subset of C as

$$\bar{C} = \{w \in C(x^*, \lambda^*) \mid \|w\| = 1\},$$

then by hypothesis (1.19) we can define

$$\sigma := \min_{w \in \bar{C}} w^\top \nabla^2_{xx}\mathcal{L}(x^*, \lambda^*, \mu^*)\, w > 0.$$

Moreover, since $C(x^*, \lambda^*)$ is a cone, it follows that $w/\|w\| \in \bar{C}$ if and only if $w \in C(x^*, \lambda^*)$, $w \neq 0$. Therefore, for any such w we have

$$w^\top \nabla^2_{xx}\mathcal{L}(x^*, \lambda^*, \mu^*)\, w \geq \sigma\, \|w\|^2.$$

Suppose that there exists a feasible sequence $\{x_k\}_{k\in\mathbb{N}}$ converging to x^* such that

$$f(x_k) < f(x^*) + \frac{\sigma}{4}\|x_k - x^*\|^2, \qquad \forall\, k > K, \text{ with } K \in \mathbb{N} \text{ sufficiently large,}$$

which is equivalent to assuming that x^* is not a strict local solution, now we consider the limiting direction

$$d = \lim_{k\to\infty} \frac{x_k - x^*}{\|x_k - x^*\|}.$$

Using the Taylor expansion of the Lagrangian combined with the KKT conditions, we have

$$f(x_k) \geq f(x^*) + \frac{1}{2}(x_k - x^*)^T \nabla^2_{xx} L(x^*, \lambda^*)(x_k - x^*) + o(\|x_k - x^*\|^2),$$

since $d \in C(x^*, \lambda^*)$ by previews lemma and by the definition of σ, it follows

$$f(x_k) \geq f(x^*) + \frac{\sigma}{2}\|x_k - x^*\|^2 + o(\|x_k - x^*\|^2),$$

which again contradicts the initial assumption. Every feasible sequence $x_k \to x^*$ satisfies

$$f(x_k) \geq f(x^*) + \frac{\sigma}{4}\|x_k - x^*\|^2$$

for k sufficiently large, so x^* is a strict local solution. □

The complete proof of this theorem, as well as that of the previous lemma, can be found in [5], Theorem 12.6. For brevity, we omit further details here.

After presenting conditions the second order conditions (1.7) and (1.19), it is possible to express them in a slightly weaker but also more convenient form to verify. This alternative form is based on a two sided projection of the Lagrangian Hessian onto subspaces associated with critical cone, it is a subset of the Linearized Feasible Direction Set, and consists of those directions that tend to remain aligned with the active inequality constraints, particularly for indices $i_k \in \mathcal{I}$ where the corresponding Lagrange multiplier $\lambda^*_{i_k}$ is positive. It also incorporates the equality constraints. In this way, the critical cone represents directions in $\mathcal{F}(x^*)$ along which first-order information alone does not indicate whether the objective function will increase or decrease. We introduce the notation $A(x^*)$ to denote the matrix whose rows correspond to the gradients of the active constraints:

$$A(x^*)^\top := \Big[\cdots \ \nabla h_j(x^*) \ \cdots \ \nabla g_i(x^*) \ \cdots \Big]_{j\in\mathcal{E},\, i\in\mathcal{A}(x^*)\cap\mathcal{I}}.$$

The simplest case occurs when the Lagrange multiplier vector λ^* that satisfies the KKT conditions is unique (for example, when the LICQ condition holds) and strict complementarity is satisfied. In this situation, the definition (1.18) of the critical cone reduces to

$$C(x^*, \lambda^*) = \{w \in \mathbb{R}^n \mid A(x^*)^\top w = 0\},$$

where $A(x^*)$ is defined as above, in other words, the *critical cone* coincides with the null space of the active constraint gradients and in this context we can define a full-column-rank matrix $Z \in \mathbb{R}^{n\times(n-m)}$ whose columns form a basis of $C(x^*, \lambda^*)$, i.e.,

$$C(x^*, \lambda^*) = \{Zu \mid u \in \mathbb{R}^{\dim(C(x^*,\lambda^*))}\}.$$

Using this notation, the positive semidefinite second-order condition (1.7) can be rewritten as

$$u^\top Z^\top \nabla^2_{xx}\mathcal{L}(x^*, \lambda^*)Zu \geq 0, \quad \forall\, u,$$

or more concisely,

$$Z^\top \nabla^2_{xx}\mathcal{L}(x^*, \lambda^*)Z \succeq 0.$$

Similarly, the positive definite condition (1.19) becomes

$$Z^\top \nabla^2_{xx}\mathcal{L}(x^*, \lambda^*)Z \succ 0.$$

From a numerical point of view, the matrix Z can be computed, which allows the positive definiteness or semi-definiteness conditions to be checked by forming the matrices and computing their eigenvalues. One way to obtain Z is by applying a QR factorization to the transpose of the active constraint gradients, as we show next:

$$A(x^*)^\top = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix} = Q_1 R$$

where R is a square upper triangular matrix and Q is an $n \times n$ orthogonal matrix. If R is non-singular, we can set $Z = Q_2$. If R is singular (indicating that the active constraint gradients are linearly dependent), a slight modification of the QR procedure with column pivoting can be used to correctly identify Z.

1.4 Linear Programming

Linear programming problems are characterized by a linear objective function and a set of linear constraints, which may include both equalities and inequalities. The feasible set defined by these constraints is a polytope, namely a convex and connected set bounded by flat faces. The objective function can be represented by a family of hyperplanes whose level sets translate uniformly across the variable space. In general, the optimal solution corresponds to a vertex of the feasible polytope. However, uniqueness is not guaranteed: the optimal value may be attained over an entire edge or face, and in higher dimensions the set of optimal solutions can extend beyond a single point. For both theoretical analysis and algorithmic implementation, it is customary to express a linear program in the following "standard form":

$$\min c^\top x \qquad \text{subject to} \quad Ax = b,\, x \geq 0, \tag{1.20}$$

where $c \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ are vector, and $A \in \mathbb{R}^{m \times n}$ is a real matrix, with n and m denote the number of variables and constraints, respectively. Through elementary transformations, any linear program can be reformulated into this canonical representation. In many situations, linear programs are initially formulated with inequality constraints and without explicit sign restrictions on the decision variables. For example,

$$\min_{x \in \mathbb{R}^n} c^\top x \qquad \text{subject to} \quad Ax \leq b.$$

Let us transform this problem into the canonical form often required by linear optimization algorithms, which we will see later, through two operations. First, each inequality constraint can be rewritten as an equality by introducing a non-negative "slack variable",

$$Ax \leq b \quad \Longleftrightarrow \quad Ax + z = b, \qquad z \in \mathbb{R}^m,\, z \geq 0.$$

Secondly, since the variables x are unrestricted in sign, each component can be written as the difference of two non-negative variables:

$$x = x^+ - x^-, \qquad x^+ \geq 0,\ x^- \geq 0.$$

Substituting this decomposition into the constraints and the objective function we get

$$A(x^+ - x^-) + z = \begin{bmatrix} A & -A & I_m \end{bmatrix} \begin{pmatrix} x^+ \\ x^- \\ z \end{pmatrix} = b, \qquad c^\top(x^+ - x^-) = \begin{pmatrix} c \\ -c \\ 0 \end{pmatrix}^\top \begin{pmatrix} x^+ \\ x^- \\ z \end{pmatrix},$$

with all variables non-negative. This procedure shows that any linear program with inequality constraints and variables free in sign can be systematically rewritten in the canonical form

$$\min \tilde{c}^\top \tilde{x} \qquad \text{subject to} \quad \tilde{A}\tilde{x} = b,\ \tilde{x} \geq 0,$$

where $\tilde{x}$ collects all non-negative variables (x^+, x^-, z). In this way, every inequality can be converted into an equality with non-negative additional variables, ensuring that the problem fits the standard framework for linear programming methods.

1.4.1 Optimality Conditions

The optimality of a feasible solution for the linear program

$$\min c^\top x \qquad \text{subject to} \quad Ax \leq b,\ x \geq 0, \tag{1.21}$$

can be characterized entirely by the first-order Karush–Kuhn–Tucker (KKT) conditions. Since the problem is convex and the objective function is linear, second-order conditions are not informative in this setting: the Hessian of the Lagrangian vanishes identically.

To formulate the KKT system, we introduce Lagrange multipliers $\lambda \in \mathbb{R}^m$ for the inequality constraints $Ax \leq b$ and multipliers $s \in \mathbb{R}^n$ for the non negativity bounds $x \geq 0$. The Lagrangian associated with (1.21) is

$$L(x, \lambda, s) = c^\top x - \lambda^\top (Ax - b) - s^\top x. \tag{1.22}$$

The KKT conditions state that a feasible vector x is optimal if and only if there exist multipliers $\lambda \in \mathbb{R}^m$ and $s \in \mathbb{R}^n$ such that

$$A^\top \lambda + s = c, \tag{1.23}$$

$$Ax \leq b, \tag{1.24}$$

$$x \geq 0, \tag{1.25}$$

$$s \geq 0, \tag{1.26}$$

$$x_i s_i = 0, \quad i = 1, \dots, n. \tag{1.27}$$

Condition 1.27, known as *complementarity*, enforces that for each component either the primal variable x_i or the corresponding dual variable s_i vanishes, it can be expressed as the inner product of the vectors x and s. Combining the stationarity condition (1.23) with feasibility and complementarity, one obtains the identity

$$c^\top x = (A^\top \lambda + s)^\top x = (Ax)^\top \lambda = b^\top \lambda.$$

The equality $c^\top x = b^\top \lambda$ reflects the fact that primal and dual objective values coincide at optimality. Moreover, for any feasible $\bar{x}$ we have

$$c^\top \bar{x} = (A^\top \lambda + s)^\top \bar{x} = (A\bar{x})^\top \lambda + (s)^\top \bar{x} \geq b^\top \lambda = c^\top x,$$

where the inequality follows from $A\bar{x} \leq b$, $\lambda \geq 0$, $\bar{x} \geq 0$ and $s \geq 0$. This shows that no feasible solution can yield a strictly lower objective value than x. Finally, note that the optimality of $\bar{x}$ requires

$$\bar{x}^\top s = 0,$$

so that any positive component of s enforces the vanishing of the corresponding component of every optimal solution $\bar{x}$. This characterization highlights the close interplay between primal feasibility, dual feasibility, and complementarity in the theory of linear programming.

1.4.2 Bases and Basic feasible solutions

We write the linear problem in standard form

$$\min c^\top x \qquad \text{subject to} \quad Ax = b,\, x \geq 0,$$

and we assume that the number of variables is larger than the number of constraints so $n > m$. The matrix A has full rank so it contains m linearly independent columns. In practice the data can be prepared in advance to remove redundant constraints or to eliminate some variables and the use of slack surplus or artificial variables can also make the matrix A satisfy this condition.

Definition 1.22 (Basic Feasible Solution and Basis Matrix). *A vector x is called a "basic feasible solution" if it satisfies the feasibility conditions and there is a subset $\mathcal{B}$ of the indices $\{1, \ldots, n\}$ such that:*

1. *B has exactly m elements,*
2. *$x_i = 0$ for every $i \notin \mathcal{B}$,*

3. the matrix $B = [A_i]_{i\in\mathcal{B}}$ is not singular.

Under this property, $\mathcal{B}$ and B are called the "basis" and the "basis matrix", respectively.

Theorem 1.23. *Let $x \in \mathcal{D} := \{x \geq 0 : Ax = b\}$, $\mathcal{D}$ is the feasible polytope, then x is a vertex of P if and only x is a basic feasible solution associated with a basis $\mathcal{B}$ of matrix A.*

Proof. ($\Leftarrow$) Suppose, by contradiction, that x is a basic feasible solution but not a vertex of $\mathcal{D}$. Then there exist distinct $y, z \in \mathcal{D}$ and a scalar $0 < \lambda < 1$ such that

$$x = \lambda y + (1-\lambda)z.$$

Since $y, z \in \mathcal{D}$, it follows that $Ay = Az = b$ and $y, z \geq 0$. Let $\mathcal{B} := \{j : x_j > 0\}$ denote the set of indices corresponding to the basis of basic variables solution x. By definition of a BFS, the columns $\{A_j : j \in \mathcal{B}\}$ are linearly independent and $|\mathcal{B}| = m$. For all $j \notin \mathcal{B}$ we have $x_j = 0$, hence,

$$0 = x_j = \lambda y_j + (1-\lambda)z_j.$$

Since $\lambda, 1-\lambda > 0$ and $y_j, z_j \geq 0$, it follows that $y_j = z_j = 0$ for all $j \notin \mathcal{B}$. Therefore, y and z have nonzero components only in the positions indexed by $\mathcal{B}$. Because $y, z \in \mathcal{D}$, one can write

$$\sum_{j\in\mathcal{B}} y_j A_j = b, \qquad \sum_{j\in\mathcal{B}} z_j A_j = b.$$

Subtracting these two equalities yields

$$\sum_{j\in\mathcal{B}} (y_j - z_j) A_j = 0.$$

Let $d := y - z$. Then $Ad = 0$ and $d \neq 0$, since $y \neq z$. This shows that the columns $\{A_j : j \in \mathcal{B}\}$ are linearly dependent, which contradicts the definition of a basis associated with a basic feasible solution. Hence our initial assumption was false, and x must be a vertex of $\mathcal{D}$.

($\Rightarrow$) Assume that $x \in \mathcal{D}$ is a vertex. Let $\mathcal{B} := \{j : x_j > 0\}$, and denote by $\{A_j : j \in \mathcal{B}\}$ the corresponding columns of A. Let $|\mathcal{B}| = p$. Suppose, by contradiction, that these columns are linearly dependent. Then there exists a nonzero vector $d \in \mathbb{R}^n$ such that

$$Ad = 0,\ d_j = 0, \quad \forall\, j \notin \mathcal{B}.$$

For sufficiently small $|\theta|$, the points $x + \theta d$ and $x - \theta d$ both satisfy $A(x \pm \theta d) = b$ and remain non-negative, so that $x \pm \theta d \in \mathcal{D}$. Since

$$x = \tfrac{1}{2}\big[(x + \theta d) + (x - \theta d)\big],$$

the point x lies on the line segment connecting two distinct feasible points of D. This contradicts the assumption that x is a vertex. Hence the columns $\{A_j : j \in \mathcal{B}\}$ must be linearly independent, and therefore $p \leq m$. If $p < m$, we can add $m - p$ additional indices to $\mathcal{B}$ in order to form a full basis matrix B of A. Consequently, x is the basic feasible solution associated with B. $\square$

Chapter 2

Balance Sheet Model and Regulatory Constraints

In the modern banking system, risk and return management are strictly governed by prudential rules that not only determine the amount of capital and liquidity to be held, but also define the operational methods by which financial institutions plan their investment, raising, and capital allocation strategies.

All of these activities fall within a broader discipline called Asset and Liability Management (ALM), which represents the economic and strategic framework through which banks coordinate budgetary decisions to simultaneously ensure profitability and financial stability. ALM aims to balance risk and return through integrated management of interest rate exposure, liquidity, and capital constraints.

The introduction of the Basel III principles by the Basel Committee on Banking Supervision has profoundly changed the framework within which banks must operate. Among the main regulatory constraints, we will encounter risk-weighted assets (RWA), the Liquidity Coverage Ratio (LCR), and the Net Stable Funding Ratio (NSFR). These quantitative measures significantly influence decisions regarding balance sheet structure and banking book composition, simultaneously influencing profitability and liquidity management.

We will first study the structure of the bank balance sheet, the calculation of Net Interest Income (NII), and the main types of fixed- and floating-rate financial instruments, along with their technical characteristics, maturities, and repayment terms. The combination of constraints with return objectives creates a complex problem in which asset and liability portfolio planning must simultaneously meet capital and liquidity requirements while maximizing the economic outcome. This formally translates into a constrained optimization problem. For a more detailed analysis of the theoretical foundations of bank risk management and practical approaches to optimizing asset and liability strategies, see [3] and [7], respectively.

2.1 The Balance Sheet Model

A bank's balance sheet is an accounting document that captures the institution's financial position at a given point in time, such as the end of a month, quarter, or year. In simple terms, it shows where the bank's resources come from and how these resources are used. It is a basic tool for understanding the bank's solidity and liquidity, and its exposure to risk. It follows the basic accounting rule that total assets always equal total liabilities plus equity, or

$$A - L = E,$$

this balance reflects the accounting principle that every use of resources has a corresponding source of funding. We now provide an example for illustrative purposes only, where we can see how the three aforementioned categories are clearly defined and distinct, and how this equation is verified. The main balance sheet items, such as loans, investments, deposits, and debt, include various types of contracts. These contracts can vary in terms of currency, duration, interest rate, creditworthiness, and accounting method. To understand how the balance sheet will change over time, it is important to analyze these individual contracts. In financial modeling, the way we aggregate balance sheet items may differ from the standard accounting classification. For example, we can break down "Loans and advances to customers" by repayment type such as amortizing versus at maturity or categorize "Debt issued" by fixed versus floating rates to see their sensitivity to interest rate changes. This helps assess the effect of each strategy on the bank's risk and return. To build a mathematical model for balance sheet optimization, a stable and clear accounting structure is required. The model's logic, including its constraints and relationships, must not depend on the input data. However, account names and organization influence the definition of contracts and decision variables, so it is important to use a consistent input structure.

> *Different versions of the balance sheet, aggregated or disaggregated at will, can be output for targeted analysis at the level of interest.*

This way, the balance sheet is no longer just a static snapshot, it becomes a dynamic tool that evolves based on the bank's strategies for profitability, risk management, and regulatory compliance.

Balance sheet

USD m	31.12.24
Assets	
Cash and balances at central banks	223,329
Amounts due from banks	18,903
Receivables from securities financing transactions measured at amortized cost	118,301
Cash collateral receivables on derivative instruments	43,959
Loans and advances to customers	579,967
Other financial assets measured at amortized cost	58,835
Total financial assets measured at amortized cost	1,043,293
Financial assets at fair value held for trading	159,065
of which: assets pledged as collateral that may be sold or repledged by counterparties	*38,532*
Derivative financial instruments	185,551
Brokerage receivables	25,858
Financial assets at fair value not held for trading	95,472
Total financial assets measured at fair value through profit or loss	465,947
Financial assets measured at fair value through other comprehensive income	2,195
Investments in associates	2,306
Property, equipment and software	15,498
Goodwill and intangible assets	6,887
Deferred tax assets	11,134
Other non-financial assets	17,766
Total assets	1,565,028
Liabilities	
Amounts due to banks	23,347
Payables from securities financing transactions measured at amortized cost	14,833
Cash collateral payables on derivative instruments	35,490
Customer deposits	745,777
Debt issued measured at amortized cost	214,219
Other financial liabilities measured at amortized cost	21,033
Total financial liabilities measured at amortized cost	1,054,698
Financial liabilities at fair value held for trading	35,247
Derivative financial instruments	180,636
Brokerage payables designated at fair value	49,023
Debt issued designated at fair value	107,909
Other financial liabilities designated at fair value	28,699
Total financial liabilities measured at fair value through profit or loss	401,514
Provisions and contingent liabilities	8,409
Other non-financial liabilities	14,834
Total liabilities	1,479,454
Equity	
Share capital	346
Share premium	12,012
Treasury shares	(6,402)
Retained earnings	78,035
Other comprehensive income recognized directly in equity, net of tax	1,088
Equity attributable to shareholders	85,079
Equity attributable to non-controlling interests	494
Total equity	85,574
Total liabilities and equity	1,565,028

Figure 2.1: Balance sheet UBS [13]

2.2 Classification of Financial Instruments

After analyzing the structure of the balance sheet, it is necessary to introduce a formal framework for classifying and aggregating the various balance sheet items consistently. Such a classification is crucial in a mathematical and operational research context, as it provides the information base upon which optimization and data analysis models can be constructed, and it was chosen based on the problem and the input data. Let us denote by $\mathcal{BS}$ the set of all financial instruments considered in the balance sheet: where each element in $\mathcal{BS}$ represents a single financial instrument, and N^* is the total number of instruments associated to the following way of aggregating.

The first segmentation is based on the *accounting category* of the instrument:

- **Assets**: An economic resource owned or controlled by the bank that has value and from which a future economic benefit is expected (e.g. interest, dividends, or capital gains).

- **Liabilities**: A debt or obligation of the bank arising from past events, the fulfillment of which will result in an outflow of economic resources in the future (e.g. payment of debt or repayment of funds).

Equity instruments can be considered a residual category, because in our case they represent a single value fixed at the beginning of the optimization, not subject to any changes during the simulation time horizon.

In addition to this primary classification, financial instruments are also classified based on the *type of interest rate*, which can be fixed, floating, or mixed/hybrid. A fixed rate instrument maintains the same value up to its maturity, while a floating rate instrument adjusts periodically based on a market parameter. A mixed rate instrument combines both, for example, with initial fixed-rate periods followed by floating-rate periods.

The *reference rate*, or *pivot rate* is the market parameter used to determine the interest rate in **floating**, **fixed**, or **mixed** contracts. It is based on official indices that reflect current monetary conditions: in the euro area, the Eurirs is commonly used for variable rates, while the Eurirs represents the reference rate for fixed rates. These indices are standardized and recognized in financial markets.

Another way to classify financial instruments is by their repayment term structure, which describes both the overall duration of the instrument and the terms by which interest and principal are repaid over time. The *maturity* indicates the period between the issuance of the instrument and the final repayment of the principal. Likewise, the *payment frequency* defines the time intervals in which interest is paid, determining the timing of periodic cash flows. The *repayment capital type* specifies the method of principal repayment: it can be a lump sum at maturity, typically referred to as a bullet repayment, or through periodic payments over the life of the instrument, according to a linear or an exogenous amortizing plan.

Financial instruments are also classified according to the *record type*,which identifies their accounting nature:

- **Bonds**: bonds issued by public entities or corporations, with predetermined credit rights.
- **Current Accounts and Sight Deposits**: checking accounts and demand deposits, liquid instruments that generate short-term fixed or floating interest.
- **Loans and Receivables at Maturity / Time Deposits**: loans and term deposits, with defined duration and repayment schedules, generating programmed interest flows.
- **Data Entry Operation – 700**: accounting records of specific operations, often internal to the bank, used for monitoring or internal management purposes.
- **Stocks, Funds, Options**: equity instruments, fund shares, or derivatives, which exhibit risk and return characteristics different from fixed-income instruments.

To each instrument we associate a "vector of characteristics":

$$\theta_k = \big(a_k, t_k, \tau_k, m_k, f_k, r_k, s_k, p_k\big), \tag{2.1}$$

where each component captures a fundamental property of the instrument:

- a_k: accounting category,
- t_k: interest rate type (fixed, floating, mixed),
- τ_k: pivot rate or reference index (Euribor, CMS),
- m_k: maturity,
- f_k: interest payment frequency,
- r_k: repayment type (bullet, amortizing, etc.),
- s_k: record type (bonds, loans, stocks, etc.).

This representation allows a structured and flexible way to aggregate instruments according to shared characteristics.

2.2.1 Aggregation of Instruments

This multidimensional classification structure allows instruments with similar economic, temporal, and accounting characteristics to be aggregated, facilitating both mathematical modeling of the portfolio and the assessment of interest rate, liquidity, and credit risks. Let $\mathcal{I}_g \subseteq \mathcal{I}$ denote the subset of instruments sharing a specific feature g described above. For any quantitative characteristic x_k of an instrument k for example, outstanding amount, interest contribution, or any relevant risk or liquidity metric, the aggregated value for the group at a given time or evaluation date can be expressed as:

$$X_g = \sum_{k \in \mathcal{I}_g} x_k,$$

where X_g represents the total or combined contribution of the characteristic x_k for all instruments in the group. This framework can be applied to a wide range of numerical attributes, enabling consistent metric calculations while maintaining the desired aggregation.

2.3 Net Interest Income (NII)

In the context of Asset-Liability Management (ALM), the profit generated by the difference between the return on assets and the cost of liabilities is a key measure of financial performance. The main indicator of this profitability is the "Net Interest Income (NII)", defined as:

$$NII = \text{Interest Income} - \text{Interest Liabilities}. \tag{2.2}$$

The interest income represents the profits from interest-bearing assets while interest expenses reflect the cost of funding and other sources of financing. The NII is the main component of a bank's operating income and forms the basis of commercial banks' operating results. It reflects the bank's ability to convert its balance sheet structure and interest rate profile into profit by capturing the so-called spread between assets and liabilities.

From a financial perspective, the net interest income (NII) summarizes the efficiency with which a bank uses its resources to generate value, being influenced by multiple factors: the volume and composition of loans and deposits, the interest rate structure, the timing of cash flows (repricing), and interest rate risk management policies. The NII also serves as a stability indicator: a reduction in the net interest income, caused for example by high financing costs or low asset returns, can reduce the bank's ability to generate capital and withstand market shocks.

In balance sheet optimization, the NII becomes the objective function to be maximized. The goal is to find a combination of assets and liabilities that, given regulatory and operational constraints, maximizes the net interest income over time. This also requires considering risk variables: interest rate changes, maturity mismatches, duration, options embedded in contracts, and regulatory limits (e.g. on liquidity and capital requirements). These factors do not change the NII formula, but they influence its stability and long-term sustainability.

From a mathematical point of view, considering a set of positions with exposures x_k and a_k as a weighting indicator to determine whether the i-th instrument is an asset or a liability, the objective function can be represented as:

$$\max_x NII(x) = \sum_k a_k e_k x_k, \tag{2.3}$$

where e_k are the interest rates of the corresponding instruments.

A closely related indicator to NII is "Net Interest Margin (NIM)". While NII measures the absolute value of the interest income generated by the difference between financial income and expenses, NIM represents the relative measure, comparing this income to the bank's total interest-bearing assets. It is defined as:

$$NIM = \frac{\text{NII}}{\text{Earning Assets}}. \tag{2.4}$$

NIM therefore represents the efficiency with which a bank uses its assets to generate interest income. Higher NIM values indicate a greater ability to transform assets into profitable margins, while lower values may reflect an increase in the cost of funding, a reduction in lending rates, or increased competition in the credit market.

2.4 Regulatory Constraints: Basel III Framework

Following the 2008 global financial crisis, the Basel Committee on Banking Supervision (BCBS) introduced a set of international standards, known as Basel III [1], with the aim of strengthening the resilience of the banking system and reducing systemic risk. The framework evolved from previous agreements (Basel I and II) and broadened the focus from just credit risk to an integrated approach that includes market, liquidity, and leverage risks. The core principle of Basel III is the definition of adequate capital and liquidity requirements based on the structure of banks' balance sheets. The regulatory framework is based on three main pillars:

- **Pillar 1**: called "Minimum Capital Requirements", it establishes the minimum amount of capital a bank should hold to cover credit, market, and operational risks.

It provides guidelines for calculating the risk exposure of a bank's balance sheet assets (the "risk-weighted assets", RWA and their components) and establishes the minimum capital requirements;

- **Pillar 2**: called "Supervisory Review and Evaluation Process" involves both banks and regulators taking a view on whether a firm should hold additional capital against risks not covered in Pillar 1. Part of the Pillar 2 process is the 'Internal Capital Adequacy Assessment Process' (ICAAP), which is a bank's self-assessment of risks not captured by Pillar 1;
- **Pillar 3**: called "Market Discipline", aims to encourage market discipline by requiring banks to disclose specific, prescribed details of their risks, capital, and risk management.

Basel III therefore includes various indicators that allow for the rigorous quantification of liquidity and credit risk and the direct linking of asset and liability allocation decisions to regulatory constraints. We will now define some of these indicators that were subsequently introduced into the optimization process.

2.4.1 Liquidity Coverage Ratio (LCR)

The "Liquidity Coverage Ratio" (LCR) is designed to ensure that a bank has an adequate stock of unencumbered high-quality liquid assets (HQLA) to meet its liquidity needs in a short-term, 30-day liquidity stress scenario. HQLA consists of cash or assets that can be converted into cash with little or no loss in value in private markets. The underlying idea is that the stock of unencumbered HQLA should allow the bank to survive until day 30 of the stress scenario, by which time it is assumed that management and supervisors can take appropriate corrective action or that the bank can be resolved in an orderly manner. Formally, the Basel Framework defines the LCR as follows:

$$LCR = \frac{\text{Stock of HQLA}}{\text{Total net cash outflows over the next 30 calendar days}}. \tag{2.5}$$

A minimum standard of 100% is required, implying that the stock of HQLA must at least equal the total net cash outflows under a severe stress scenario.

The numerator of the LCR, the *stock of High-Quality Liquid Assets (HQLA)*, represents the volume of assets that can be easily and immediately converted into cash in private markets, even during periods of stress, and that are ideally eligible for central bank operations. Assets qualifying as HQLA must exhibit low credit and market risk, be easily and consistently valued, and be traded in deep and active markets with low volatility. These assets are divided into three distinct levels, each subject to specific composition and haircut requirements:

- **Level 1 Assets (0% haircut):**
  - Cash, central bank reserves, and marketable securities backed by sovereigns or central banks.
  - Must be of the highest liquidity and credit quality (rated AA^- or higher).
  - No limit on their composition within the HQLA stock.
  - Examples include: coins and banknotes, reserves held at the central bank, and sovereign bonds issued in the domestic currency.
- **Level 2A Assets (15% haircut):**
  - Comprise high-quality corporate and covered bonds not included in Level 1.
  - Minimum external rating of AA^- or equivalent.
  - The total composition of Level 2 assets (2A + 2B) in HQLA cannot exceed this 40% threshold.
- **Level 2B Assets (25–50% haircut):**
  - Include lower-rated corporate bonds (A^+ to BBB^-), certain equities, and residential mortgage-backed securities (RMBS) that meet strict criteria.
  - A maximum of 15% of the total HQLA stock can consist of Level 2B assets.
  - Haircuts vary depending on the asset class:
    * 25% for corporate bonds (rated A^+ to BBB^-),
    * 50% for eligible equities listed on major stock indices,
    * 25% for high-quality RMBS with a minimum rating of AA and a maximum loan-to-value ratio below 80%.

Furthermore, to qualify as HQLA, all assets must be unencumbered, legally and operationally accessible to the liquidity management function, and under the control of the treasury. Supervisors may impose additional requirements or exclude assets that do not meet local market liquidity standards. After haircuts and composition limits are applied, the adjusted stock of HQLA represents the numerator of the LCR:

$$\text{Stock of HQLA} = \text{Level 1} + 0.85 \times \text{Level 2A} + (0.5 \text{ or } 0.75) \times \text{Level 2B}, \tag{2.6}$$

where each multiplier reflects the regulatory haircut assigned to the respective level.

The denominator of the LCR, the *total net cash outflows*, represents the amount of liquidity that a bank is expected to need under a severe 30-day stress scenario. It is calculated as the total expected cash outflows minus the total expected cash inflows during the period, with the inflows being capped at 75% of the outflows to ensure a conservative liquidity buffer. Formally, [LCR40], this can be expressed as:

$$\text{Total Net Cash Outflows} = \text{Expected Outflows} - \min\big(\text{Expected Inflows},\ 0.75 \times \text{Expected Outflows}\big). \tag{2.7}$$

Cash outflows reflect the estimated loss of funding that a bank would face due to the withdrawal of deposits, the non-renewal of maturing liabilities, and the potential use of committed credit and liquidity facilities. They are determined under stress assumptions that take into account both contractual obligations and behavioral factors observed during periods of market tension. Retail deposits generally receive lower run-off rates—typically between 3% and 10%—depending on their stability and the presence of deposit insurance schemes. In contrast, wholesale funding provided by financial institutions or corporations, particularly when not related to operational purposes, is assigned much higher run-off rates, often up to 100%, reflecting its lower stability in stressed markets. Additional outflows may also arise from potential collateral calls, derivatives margin requirements, or drawdowns on committed lines of credit to customers. Cash inflows, conversely, represent the liquidity that a bank can reasonably expect to receive within the same 30-day horizon from performing assets, contractual repayments, and other cash-generating activities. However, the Basel Framework requires that such inflows be treated conservatively. Only inflows from fully performing exposures are included, and they must be adjusted for the likelihood of delayed or partial repayment under stress conditions. For instance, inflows from loans and advances may be recognized at 50% of the contractual amount, while inflows from maturing securities or deposits at other financial institutions are considered only if they are certain and contractually enforceable. Inflows from derivatives are recognized net of collateral and margin obligations. By limiting the recognition of inflows to 75% of total outflows, the LCR ensures that a bank's liquidity position does not rely excessively on potentially uncertain incoming cash flows.

This conservative treatment reinforces the prudential goal of the ratio: to ensure that the institution holds a sufficient stock of High-Quality Liquid Assets (HQLA) to cover the net liquidity gap that would emerge under an acute 30-day stress scenario.

The resulting metric therefore captures a realistic and prudent estimate of the bank's short-term funding needs, ensuring resilience even under severe market disruptions.

2.4.2 Net Stable Funding Ratio (NSFR)

The "Net Stable Funding Ratio" (NSFR) is a key standard for long-term structural liquidity introduced by Basel III. While the LCR focuses on short-term liquidity over a 30-day time horizon, the NSFR ensures that banks maintain a sustainable funding structure over a longer time horizon, typically one year. Its objective is to reduce maturity transformation risk, i.e., the tendency of banks to finance long-term illiquid assets with short-term unstable liabilities, which has been a major cause of liquidity crises. Formally, the ratio is expressed as:

$$\text{NSFR} = \frac{\text{Available Stable Funding}}{\text{Required Stable Funding}}. \tag{2.8}$$

A value equal to or greater than 100% indicates that the bank's available stable funding sources are sufficient to finance the portion of assets that requires stable funding under normal and stressed market conditions.

The numerator, "Available Stable Funding", measures the stability of a bank's funding profile. Each liability category is assigned a so-called *ASF factor*, which ranges approximately from 0% for the least stable sources to 100% for the most stable ones. In general, the higher the reliability and the longer the maturity of a funding source, the higher its ASF factor. For instance, regulatory capital and long-term debt instruments are considered the most stable sources of funding and therefore receive the highest factors. Retail deposits — especially those that are insured or belong to customers with long-term relationships with the bank — are also viewed as relatively stable and receive intermediate factors. Conversely, short-term wholesale funding obtained from financial institutions or interbank markets is highly sensitive to market confidence and thus receives low or even zero ASF factors. This structure encourages banks to diversify their liabilities and to rely more heavily on stable, long-term funding sources rather than volatile short-term borrowing. In essence, the ASF reflects the expected behavior of investors and depositors under stressed market conditions, taking into account contractual maturities, funding concentration, and withdrawal patterns.

The denominator, "Required Stable Funding", represents the amount of stable funding a bank must hold to support its assets and off-balance-sheet exposures. Each asset or exposure is assigned an *RSF factor*, which ranges from very low values (for assets that are highly liquid and can be rapidly sold or used as collateral) to high values (for illiquid or long-term assets that cannot be easily converted into cash). In practical terms, cash and central bank reserves are considered to require little or no stable funding, since they can be used immediately to meet obligations. On the other hand, long-term loans, real estate holdings, and intangible assets require a higher proportion of stable funding because they cannot be liquidated quickly without incurring significant losses. Similarly, assets encumbered or pledged as collateral also increase the RSF requirement, as they are not readily available for liquidity management. The RSF thus captures the liquidity characteristics and residual maturity of a bank's asset portfolio. Increases in the average maturity or illiquidity of assets lead to higher RSF factors, while holding more marketable or short-term assets reduces the overall funding requirement.

2.4.3 RWA-based Capital Constraint

Within the Basel regulatory framework, "Risk-Weighted Assets" (RWA) represent the measure through which the risk profile of a bank's exposures is translated into capital requirements. Each asset held by a bank is weighted by a coefficient that reflects its credit, market, or operational risk, determining how much regulatory capital must be maintained to absorb potential losses. The total RWA thus acts as a constraint on a bank's balance sheet: the higher the overall riskiness of its assets, the greater the amount of capital that must be held. This mechanism ensures that the institution maintains solvency even under adverse conditions, thereby protecting depositors and the financial system as a whole. The total amount of Risk-Weighted Assets can be expressed as:

$$\text{RWA}_{\text{total}} = \text{RWA}_{\text{credit}} + \text{RWA}_{\text{market}} + \text{RWA}_{\text{operational}}. \tag{2.9}$$

Each component captures a specific dimension of financial risk and is computed either through a standardized regulatory methodology or an internal model approach, depending on the institution's complexity and supervisory approval. The main components can be described as follows:

- **Credit Risk RWA:** This represents the capital requirement associated with potential losses arising from counterparty default or credit deterioration. Under the Basel framework, each exposure is assigned to a risk weight that increases with the probability of default (PD) and the severity of the expected loss, measured by the loss given default (LGD). Highly rated sovereign or institutional exposures receive lower risk weights, while unsecured or low-rated exposures receive higher ones. This component ensures that capital requirements are sensitive to credit quality and asset collateralization.

- **Market Risk RWA:** This captures exposure to losses caused by adverse movements in market variables such as interest rates, exchange rates, equity prices, or commodity prices. They are determined through standardized sensitivity-based approaches or internal models approved by regulators, typically based on statistical risk measures such as Value-at-Risk (VaR) or Expected Shortfall (ES). These measures ensure that the bank holds adequate capital to withstand potential market volatility and extreme price shocks.

- **RWA for Operational Risk:** Quantify the amount of capital required to cover losses resulting from operational failures, including process deficiencies, human errors, system failures, or external events. The calculation is based on a business indicator that reflects the scale and complexity of the bank's operations, adjusted for its historical loss experience. This ensures proportionality between operational complexity and capital absorption.

In the context of balance sheet optimization the concept of risk-weighted assets can be interpreted as a capital-absorbing operator that maps the portfolio composition to an equivalent measure of risk-weighted exposure. Each instrument x_k on the balance sheet is associated with a regulatory risk weight rw_k, and within the optimization model, these coefficients act as scaling factors that translate nominal exposures into their risk-weighted equivalents, thus linking the decision variables to the capital requirement. In this sense, the RWA component introduces a structural constraint on the feasible set of portfolio allocations: the aggregate risk-weighted exposure resulting from the selected positions must remain compatible with the available regulatory capital.

2.4.4 Quantitative Aspects and Implications for Asset–Liability Management

From a quantitative standpoint, liquidity and capital ratios such as the LCR, NSFR, and RWA introduce explicit regulatory constraints within bank balance sheet optimization models. While the LCR and NSFR impose nonlinear relationships between assets and liabilities by linking portfolio composition to short-term and long-term liquidity requirements, the RWA framework directly affects the capital constraint by determining the minimum amount of own funds required to support risk exposures.

In the context of Asset–Liability Management, these regulatory measures act as structural stability constraints. As a result, modern ALM optimization must integrate these prudential ratios as binding conditions, ensuring that strategic decisions on funding, investment, and capital structure remain consistent with regulatory solvency and liquidity standards.

Chapter 3

Mathematical Programming Models

Optimizing a bank's balance sheet is a central problem in Asset-Liability Management (ALM), the objective of which is to determine the optimal allocation of assets and liabilities. The growing complexity of prudential regulation and the interplay between earnings generation and regulatory ratios (see previous chapter) have rendered traditional static approaches insufficient. Consequently, optimization-based frameworks have become a key decision-making tool in strategic balance sheet management, where the problem can be formulated as a constrained optimization problem, where the decision variables represent monthly changes in portfolio exposures. The objective function maximizes the net interest income (NII) subject to balance sheet consistency, regulatory capital adequacy, and liquidity constraints. This naturally leads to a linear optimization problem in its basic form and a nonlinear extension when liquidity constraints are incorporated.

In this chapter, we present both formulations. The first, purely linear, describes the optimization of exposures given the portfolio structure and regulatory capital constraints. The second introduces simplified liquidity constraints, derived from regulatory ratios, leading to a nonlinear problem that integrates solvency and liquidity dimensions into a unified optimization framework. The models are built over a one-year time horizon (12 monthly steps) and are based on simplifying assumptions necessary to bridge the different levels of aggregation between the income, capital, and liquidity simulation modules.

3.1 Notations

To formalize the optimization framework developed in this chapter, we introduce the notation and mathematical objects that will be used throughout the chapter. The notation follows standard conventions for optimization and mathematical modeling [2].

Each symbol is uniquely defined to ensure consistency between the algebraic formulation of the model and its economic interpretation.

Indices.

k index for financial instruments in the balance sheet, $k = 1, \dots, N$;

t index for discrete time periods (months), $t = 1, \dots, T$, with $T = 12$ over the considered horizon.

Decision variables.

$x_{k,t}$ variation in the exposure of instrument k at time t; positive values indicate an increase in exposure, negative values a reduction.

Parameters.

a_k asset/liability indicator, taking value $+1$ for assets and -1 for liabilities;

$\mathrm{NII}_k^{\mathrm{ACTUAL}}$ net interest income associated with the initial exposure of instrument k;

$e_{k,t}$ expected yield (interest rate or spread) of instrument k at time t;

s_k residual or surviving exposure of instrument k beyond the considered time horizon;

rw_k regulatory risk weight assigned to instrument k;

Cap total capital coverage available across all exposures, defined as

$$Cap = \sum_{k=1}^{N} \mathrm{CET1}_k \mathrm{OUT}_k,$$

where:

- $\mathrm{CET1}_k$ regulatory capital component allocated to instrument k,
- OUT_k cut-off exposure associated with instrument k;

E constant exogenous equity component ensuring balance sheet consistency;

$\underline{x}_{k,t}, \overline{x}_{k,t}$ lower and upper bounds on the variation of the exposure $x_{k,t}$;

θ_k set of characteristics of instrument k.

We can now formalize the optimization problem at the heart of this framework, expressing it in the general form presented in (1.1). This representation provides a compact yet rigorous way to describe the model, in which the objective function formalizes the maximization of net interest income (NII), and the set of constraints captures the regulatory and balance sheet structure through the balance sheet equation, RWA limits, and liquidity ratios.

Within the balance sheet, all financial instruments have been disaggregated into a granular representation that reflects their relevant contractual and accounting characteristics. These attributes, collected in the characteristic vector θ_k defined in (2.1), include the accounting category, the interest rate type t_k, the benchmark parameter τ_k, the maturity m_k, the interest payment frequency f_k, the repayment structure r_k ,and the record type s_k , in numerical terms, we will see later, only some of these really come into play, the remaining ones, as already mentioned, serve to make each aggregate unique. This disaggregation ensures that each balance sheet position is represented consistently and that the optimization model correctly reflects its financial behavior over time.

Furthermore, the optimization is not performed on the entire portfolio, but only on the subset of instruments belonging to the *loans and receivables record type*. All remaining balance sheet categories are treated as constants within the optimization framework, contributing fixed components to the objective function and constraints, while remaining unchanged over time.

Based on this formulation, two model variants are developed. The first is a fully linear model that maximizes the NII subject only to budget consistency, RWA limits, and bound on decision variables. The second expands the model by introducing LCR and NSFR constraints, whose inherently nonlinear nature requires further analytical treatment and specific modeling assumptions.

3.2 Linear Case

As a general assumption of our optimization setting, we consider a finite time horizon of one year, divided into monthly time buckets. Each bucket represents a discrete evaluation point at which the relevant financial quantities and constraints are assessed. The optimization portfolio, whose structure has been defined above, is assumed to consist of N instruments, each characterized by a unique set of attributes as described in the characteristic vector θ_k. The optimization model can therefore be written as follows:

$$
\begin{aligned}
\max_{x_{k,t}} \quad & \sum_{k=1}^{N} a_k \left(\mathrm{NII}_k^{\mathrm{ACTUAL}} + \sum_{t=1}^{T} x_{k,t} e_{k,t} \right) \\
\text{s.t.} \quad & \sum_{k=1}^{N} a_k \, V_k(x_{k,1}, \ldots, x_{k,T}; \theta_k, t) = E, \qquad \forall\, t = 1, \ldots, T, \\
& \sum_{k=1}^{N} \mathrm{rw}_k \, V_k(x_{k,1}, \ldots, x_{k,T}; \theta_k, t) \leq Cap, \qquad \forall\, t = 1, \ldots, T, \\
& \underline{x}_{k,t} \leq x_{k,t} \leq \overline{x}_{k,t}, \qquad \forall\, k = 1, \ldots, N, \ \forall\, t = 1, \ldots, T.
\end{aligned}
$$

The first equality constraint requires the basic accounting rule, balancing assets and liabilities for each month, ensuring that the bank's total exposure remains consistent with its level of equity. In this expression, the function $V_k(x_{k,1}, \ldots, x_{k,T}; \theta_k, t)$ represents the effective exposure of instrument k at time t, taking into account its maturity, repayment type, interest payment frequency, and any residual outstanding. This ensures that changes in $x_{k,t}$ are incorporated consistently with the contractual characteristics of each instrument. The second constraint imposes a limit on risk-weighted assets, requiring that the sum of risk-weighted exposures do not exceed available regulatory capital. Here, each instrument is multiplied by its risk weight rw_k, which reflects the regulatory risk associated with that position. Finally, the upper and lower bounds of $x_{k,t}$ define the allowable operating range for exposure changes. The lower bound $\underline{x}_{k,t}$ represents the maximum allowable reduction (shrink), while the upper bound $\overline{x}_{k,t}$ represents the maximum allowable increase (growth) of exposure in each month. As mentioned, the function $V_k(\cdot)$ represents the effective exposure of instrument k at time t, and is defined piecewise as follows:

$$
V_k(x_{k,1}, \ldots, x_{k,T}; \theta_k, t) =
\begin{cases}
x_{k,t-1}, & \text{if } m = 1 \ , \\
\displaystyle\sum_{r=t-m}^{T} x_{k,r}, & \text{if } m \in \{3, 6\} \text{ and bullet repayment}, \\
\displaystyle\sum_{r=t-m}^{T} x_{k,r} - \sum_{r=1}^{m} w_r x_{k,t-r}, & \text{if } m \in \{3, 6\} \text{ and linear amortization}, \\
s_k + \displaystyle\sum_{r=1}^{T} x_{k,r}, & \text{if } m > 12 \text{ and bullet repayment}, \\
s_k + \displaystyle\sum_{r=1}^{T} x_{k,r} - \sum_{r=1}^{T} w_r x_{k,r}, & \text{if } m > 12 \text{ and linear amortization}.
\end{cases}
$$

The weights w_r depend on the repayment frequency parameter.

3.3 Non Linear Case

The nonlinear extension of the previous model maintains the same objective function and the same bounds on the decision variables. In other words, we still aim to maximize the NII while respecting the above mentioned constraints, but in this case, additional objects are introduced to capture regulatory liquidity requirements, leading to two nonlinear constraints: the Liquidity Coverage Ratio and the Net Stable Funding Ratio. The first one ensures that high-quality liquid assets (HQLA) are sufficient to cover net cash outflows over a short-term (30-day) stress horizon. The second one ensures that the available stable financing (ASF) is sufficient relative to the required stable financing (RSF) over a time horizon of one year, for more details see Chapter (2).

$$\text{LCR} = \frac{\text{HQLA}(x)}{\text{Net Cash Outflows}(x)} \geq 1.1, \qquad \text{NSFR} = \frac{\text{ASF}(x)}{\text{RSF}(x)} \geq 1.1.$$

To incorporate liquidity requirements into the optimization model, several simplifications were made due to the limitations of the available data sets. We started with a given portfolio at the cut-off date, from which we defined a constant-volume baseline strategy as the starting point for optimization, which by regulatory design is a feasible solution to the problem. For the aggregate, the contributions to the numerators and denominators of LCR and NSFR are then calculated based on the baseline and then scaled proportionally based on the instrument's portfolio share.

3.3.1 Liquidity Coverage Ratio

As a first simplification,we use a formula that thinks in terms of separate contributions (included and not included in optimization) to calculate the values of the indicator, for each month, the LCR is computed as:

$$\text{LCR}_t(x) = \frac{\left| \sum_{k=1}^{N} HQLA_{k,t}(x_{k,t}) + \text{Else\_N}_t \right|}{\left| \sum_{k=1}^{N} NCF_{k,t}(x_{k,t}) + \text{Else\_D}_t \right|} \geq 1.1, \qquad \forall\, t = 1, \dots, T.$$

where:

- $x_{k,t}$: exposure variation of instrument k in month t (decision variable).
- $HQLA_{k,t}(x_{k,t})$: contribution of instrument k to high-quality liquid assets in month t.
- $NCF_{k,t}(x_{k,t})$: contribution of instrument k to net cash outflows in month t.

- Else_N_t: contribution to HQLA from instruments not included in the optimization.
- Else_D_t: contribution to NCF from instruments not included in the optimization.

In the absence of complete data, we approximate the contributions $HQLA_{k,t}$ and $NCF_{k,t}$ for each instrument starting from the constant-volume baseline strategy. Let $Buffer\_base_{k,t}$ and $Den\_base_{k,t}$ be the contributions to the numerator and denominator of instrument k with respect to the baseline strategy, and val_base_k be the corresponding portfolio values. Therefore:

$$HQLA_{k,t}(x_{k,t}) \approx \frac{Buffer\_base_{k,t}}{val\_base_k} x_{k,t}, \qquad NCF_{k,t}(x_{k,t}) \approx \frac{Den\_base_{k,t}}{val\_base_k} x_{k,t}.$$

While this is a reasonable first-order approximation, it does not fully capture all the interactions between portfolio values and liquidity contributions due to incompleteness and inconsistencies in the available data. Nonetheless, this approach allows us to calculate the monthly contributions for the optimized instruments and add them to the constant contributions of the non-optimized aggregates.

3.3.2 Net Stable Funding Ratio

In a similar way to what was done, the NSFR is calculated in the same way.

$$\text{NSFR}_t(x) = \frac{\left|\sum_{k=1}^{N} ASF_{k,t}(x_{k,t}) + \text{Else\_ASF}_t\right|}{\left|\sum_{k=1}^{N} RSF_{k,t}(x_{k,t}) + \text{Else\_RSF}_t\right|} \geq 1.1, \qquad \forall\, t = 1, \ldots, T.$$

where:

- $x_{k,t}$: exposure variation of instrument k in month t (decision variable),
- $ASF_k(x_k)$: contribution of instrument k to available stable funding at time t,
- $RSF_k(x_k)$: contribution of instrument k to required stable funding at time t,
- Else_ASF_t: fixed ASF contribution from non-optimized instruments at time t,
- Else_RSF_t: fixed RSF contribution from non-optimized instruments at time t.

Let $ASF\_base_{k,t}$ and $RSF\_base_{k,t}$ denote the baseline contributions of instrument k to ASF and RSF, and val_base_k the corresponding portfolio values. Then we approximate:

$$ASF_{k,t}(x_{k,t}) \approx \frac{ASF\_base_{k,t}}{val\_base_k} x_{k,t}, \qquad RSF_{k,t}(x_{k,t}) \approx \frac{RSF\_base_{k,t}}{val\_base_k} x_{k,t}.$$

This allows computing the NSFR contribution of the optimized instruments proportionally to the portfolio values, while constant contributions from non-optimized instruments are added to maintain consistency. We now incorporate these constraints into the previous linear formulation, leading to a more extensive and complete representation of the optimization problem:

$$
\begin{aligned}
\max_{x_{k,t}} \quad & \sum_{k=1}^{N} a_k \left(\mathrm{NII}_k^{\mathrm{ACTUAL}} + \sum_{t=1}^{T} x_{k,t} e_{k,t} \right) \\
\text{s.t.} \quad & \sum_{k=1}^{N} a_k \, V_k(x_{k,1}, \dots, x_{k,T}; \theta_k, t) = E, \qquad \forall\, t = 1, \dots, T, \\
& \sum_{k=1}^{N} \mathrm{rw}_k \, V_k(x_{k,1}, \dots, x_{k,T}; \theta_k, t) \leq Cap, \qquad \forall\, t = 1, \dots, T, \\
& \mathrm{LCR}_t(x) = \frac{\left| \sum_{k=1}^{N} \frac{\text{Buffer\_base}_{k,t}}{\text{val\_base}_{k,t}} x_{k,t} + \text{Else\_N}_t \right|}{\left| \sum_{k=1}^{N} \frac{\text{Den\_base}_{k,t}}{\text{val\_base}_{k,t}} x_{k,t} + \text{Else\_D}_t \right|} \geq 1.1, \qquad \forall\, t = 1, \dots, T, \\
& \mathrm{NSFR}_t(x) = \frac{\left| \sum_{k=1}^{N} \frac{\text{ASF\_base}_{k,t}}{\text{val\_base}_{k,t}} x_{k,t} + \text{Else\_ASF}_t \right|}{\left| \sum_{k=1}^{N} \frac{\text{RSF\_base}_{k,t}}{\text{val\_base}_{k,t}} x_{k,t} + \text{Else\_RSF}_t \right|} \geq 1.1, \qquad \forall\, t = 1, \dots, T, \\
& \underline{x}_{k,t} \leq x_{k,t} \leq \overline{x}_{k,t}, \qquad \forall\, k = 1, \dots, N, \quad \forall\, t = 1, \dots, T.
\end{aligned}
$$

3.3.3 Assumption of constant sign and rewriting of liquidity constraints

Lemma 3.1 (Linearization with known signs)**.** *Let the problem be given as in* (1.1)*. Let* $\kappa > 0$ *and consider the inequality constraint*

$$g(x) = \frac{|A(x)|}{|B(x)|} - \kappa \leq 0,$$

where $A(x)$ *and* $B(x)$ *are linear functions of* x*. If there exist* $s_A, s_B \in \{+1, -1\}$ *such that* $s_A A(x) \geq 0$ *and* $s_B B(x) > 0$ *for all feasible solution* x *of the problem, then*

$$\frac{|A(x)|}{|B(x)|} \leq \kappa \quad \Longleftrightarrow \quad s_A A(x) \leq \kappa s_B B(x),$$

which is an affine (linear plus constant) constraint. Then we have four possible cases:

$$\text{I: } A(x) \geq 0, B(x) > 0: \quad A(x) \leq \kappa B(x).$$

$$\text{II: } A(x) \geq 0, B(x) < 0: \quad A(x) \leq -\kappa B(x) \quad \Longleftrightarrow \quad A(x) + \kappa B(x) \leq 0.$$

$$\text{III: } A(x) \leq 0, B(x) > 0: \quad -A(x) \leq \kappa B(x) \quad \Longleftrightarrow \quad A(x) + \kappa B(x) \geq 0.$$

$$\text{IV: } A(x) \leq 0, B(x) < 0: \quad -A(x) \leq -\kappa B(x) \quad \Longleftrightarrow \quad A(x) \geq \kappa B(x).$$

In what follows, we fix the signs to obtain a single linear form that is valid on the whole feasible set.

In the balance sheet configurations typically observed in banking practice, the components entering the regulatory liquidity ratios exhibit stable and well-defined signs at the aggregate level. This property is not merely an empirical regularity, but rather derives from the structural constraints of the bank's balance sheet and from the regulatory definitions underlying the Liquidity Coverage Ratio (LCR) and the Net Stable Funding Ratio (NSFR). More specifically:

- **LCR.** The numerator (HQLA) has a positive sign: it consists of eligible and unencumbered assets (cash, central bank reserves, eligible securities), and any haircuts, while reducing its recognized value, do not reduce it to zero. Short Positions on assets (that can generate negative Net Liquidity Position) must not be represented in this section. The denominator (NCF, total net cash outflows) has a negative sign, it measures 30-day net outflows, and in Commercial Banks (where maturity transformation is carried out) and under regulatory caps, inflows do not exceed gross outflows to the point of inverting the sign of the aggregate.

- **NSFR.** The denominator, Required Stable Funding (RSF), is associated with the asset side of the balance sheet and is structurally positive, as it represents the weighted sum of exposures requiring funding stability. The numerator, Available Stable Funding (ASF), is associated with the liability side of the balance sheet and thus carries a structurally negative sign, as it represents the stable sources of funding on the liability side of the balance sheet.

Therefore, it is reasonable to assume that, inside the feasible set given by the bounds $\underline{x}_{k,t} \leq x_{k,t} \leq \overline{x}_{k,t}$, the aggregate signs of the terms in LCR and NSFR are constant and known. This structure enables the reformulation of the ratio-based and absolute-value expressions to be reformulated into linear form, preserving the same feasible set while improving its tractability for optimization and simulation purposes.

We rewrite that constraints, and for notational simplicity let us set for $t = 1, \dots, T$:

$$N_t^{LCR}(x) = \sum_{k=1}^{N} \frac{Buffer\_base_{k,t}}{val\_base_{k,t}} x_{k,t} + \text{Else\_N}_t,$$

$$D_t^{LCR}(x) = \sum_{k=1}^{N} \frac{Den\_base_{k,t}}{val\_base_{k,t}} x_{k,t} + \text{Else\_D}_t,$$

$$N_t^{NSFR}(x) = \sum_{k=1}^{N} \frac{ASF\_base_{k,t}}{val\_base_{k,t}} x_{k,t} + \text{Else\_ASF}_t,$$

$$D_t^{NSFR}(x) = \sum_{k=1}^{N} \frac{RSF\_base_{k,t}}{val\_base_{k,t}} x_{k,t} + \text{Else\_RSF}_t,$$

so the original constraints in compact form as

$$\text{LCR}_t(x) = \frac{|N_t^{LCR}(x)|}{|D_t^{LCR}(x)|} \geq 1.1, \qquad \text{NSFR}_t(x) = \frac{|N_t^{NSFR}(x)|}{|D_t^{NSFR}(x)|} \geq 1.1.$$

In the LCR constraint we assume, consistently with the described convention,

$$N_t^{LCR}(x) \geq 0 \quad \text{and} \quad D_t^{LCR}(x) < 0, \qquad \forall\, x \in \mathcal{D}, \quad \forall\, t = 1, \dots T,$$

then

$$|N_t^{LCR}(x)| = N_t^{LCR}(x), \qquad |D_t^{LCR}(x)| = -D_t^{LCR}(x).$$

For a fixed threshold $\kappa = 1.1$, we are in case (II) of Lemma (3.1), numerator non-negative and denominator negative, therefore

$$\frac{|N_t^{LCR}(x)|}{|D_t^{LCR}(x)|} = \frac{N_t^{LCR}(x)}{-D_t^{LCR}(x)} \geq 1.1 \quad \iff \quad N_t^{LCR}(x) + 1.1\, D_t^{LCR}(x) \geq 0.$$

Expanding both terms and doing simple algebraic steps, for each $t = 1, \dots, T$ we obtain a new formulation for the constraints related to the LCR:

$$\sum_{k=1}^{N} \left(\frac{Buffer\_base_{k,t}}{val\_base_{k,t}} + 1.1\, \frac{Den\_base_{k,t}}{val\_base_{k,t}} \right) x_{k,t} + \text{Else\_N}_t + 1.1\, \text{Else\_D}_t \geq 0.$$

Analogously, we assume

$$N_t^{NSFR}(x) \leq 0 \quad \text{and} \quad D_t^{NSFR}(x) > 0, \qquad \forall\, x \in \mathcal{D}, \quad \forall\, t = 1, \dots T.$$

Hence

$$|N_t^{NSFR}(x)| = -N_t^{NSFR}(x), \qquad |D_t^{NSFR}(x)| = D_t^{NSFR}(x).$$

With $\kappa = 1.1$, we are now in case (III) of Lemma (3.1), and thus

$$\frac{|N_t^{NSFR}(x)|}{|D_t^{NSFR}(x)|} = \frac{-N_t^{NSFR}(x)}{D_t^{NSFR}(x)} \geq 1.1 \quad \iff \quad N_t^{NSFR}(x) + 1.1\, D_t^{NSFR}(x) \leq 0.$$

Here too, by substituting and doing some algebraic steps for each $t = 1, \dots, T$ we obtain a new formulation for the constraints related to the NSFR:

$$\sum_{k=1}^{N} \left(\frac{ASF\_base_{k,t}}{val\_base_{k,t}} + 1.1 \, \frac{RSF\_base_{k,t}}{val\_base_{k,t}} \right) x_{k,t} + \text{Else\_ASF}_t + 1.1 \, \text{Else\_RSF}_t \leq 0.$$

We have therefore transformed the liquidity constraints, initially nonlinear due to the presence of absolute values and ratios, into affine linear constraints valid over the feasible set $\mathcal{D}$. Therefore, the entire optimization problem is now linearized, with both the objective function and constraints all linear. In this form, it belongs to the class of linear programming (LP) problems and can therefore be solved efficiently and guaranteed using the simplex algorithm.

3.4 The Simplex Method

For a more complete treatment of the Simplex algorithm and its theoretical foundations, the reader is referred to the classic manuals by [10] and [6], from which the following exposition draws some inspiration.

We can express a generic linear optimization problem in the form:

$$\min c^\top x \qquad \text{subject to} \quad Ax \leq b,\ x \geq 0, \tag{3.1}$$

where $c \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$.

Theorem 3.2. *Let $\mathcal{D} = \{x \geq 0 : Ax = b\}$ the feasible polytope of the linear program* (3.1)*, assume that $\mathcal{D}$ is non-empty and bounded (in the direction of optimization, i.e., from below for minimization problems). If the linear program* (3.1) *admits at least one optimal solution then there exists at least one* vertex $x^* \in \mathcal{D}$ *such that*

$$c^\top x^* = \min_{x \in \mathcal{D}} c^\top x.$$

In other words, if an optimal solution exists, then at least one of the vertices of the feasible polytope $\mathcal{D}$ is optimal.

Proof. Since problem (3.1) is a linear programming problem, the objective function $\varphi(x) := c^\top x = (c_1, c_2, \dots, c_n)^\top x$ is linear, and the feasible set $\mathcal{D}$, by hypothesis, is non-empty and bounded. Let $x^{(1)}, x^{(2)}, \dots, x^{(p)}$ be the vertices of the polytope $\mathcal{D}$. If exists at least one optimal solution $x^{(0)} \in \mathcal{D}$ such that

$$\varphi(x^{(0)}) = \min_{x \in \mathcal{D}} c^\top x.$$

Due to the linear and therefore convex nature of $\mathcal{D}$ any point $x^{(0)} \in \mathcal{D}$ can be expressed as a convex combination of the vertices of $\mathcal{D}$, that is,

$$x^{(0)} = \sum_{i=1}^{p} \lambda_i x^{(i)}, \qquad \text{with} \quad \lambda_i \geq 0, \quad \sum_{i=1}^{p} \lambda_i = 1.$$

Let $x^{(k)}$ s.t.

$$c^\top x^{(k)} = \min_{1 \leq i \leq p} c^\top x^{(i)}.$$

We can now evaluate the value of objective function in $x^{(0)}$:

$$c^\top x^{(0)} = c^\top \left(\sum_{i=1}^{p} \lambda_i x^{(i)} \right) = \sum_{i=1}^{p} \lambda_i c^\top x^{(i)} \geq \sum_{i=1}^{p} \lambda_i c^\top x^{(k)} = c^\top x^{(k)} \sum_{i=1}^{p} \lambda_i = c^\top x^{(k)},$$

where the inequality is true since $c^\top x^{(k)}$ is the vertex with the smallest objective function and all $\lambda_i \geq 0$. However, $x^{(0)}$ is by definition an optimal solution, hence its cost cannot be strictly greater than that of any feasible solution, and in particular

$$c^\top x^{(0)} \leq c^\top x^{(k)}.$$

Combining the two inequalities, we obtain

$$c^\top x^{(0)} = c^\top x^{(k)}.$$

Therefore, the vertex $x^{(k)}$ attains the same optimal cost as $x^{(0)}$, proving that at least one vertex of the feasible polytope $\mathcal{D}$ is optimal. □

This theorem provides the theoretical foundation on which the Simplex Method is based. The resulting structural property is crucial from an algorithmic perspective: although the feasible set $\mathcal{D}$ can contain infinitely many feasible solutions, its vertices are finite, corresponding to the basic feasible solutions. The Simplex Method exploits this fact, limiting the search for the optimum to this finite set of vertices. Starting from an initial basic feasible solution, the algorithm moves along the edges of the polytope to an adjacent vertex that improves (or maintains) the value of the objective function. The process continues until no adjacent vertex produces an improvement, which implies that the current vertex is an optimal point.

3.4.1 Optimality conditions and reduced costs

To formalize this idea, consider the standard form of a generic linear problem, as introduced in Chapter (1),

$$\min c^\top x \qquad \text{s.t.} \quad Ax = b, x \geq 0,$$

where $A \in \mathbb{R}^{m\times n}$ has full row rank m, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. Now we consider a basic feasible solution x, and for each we have the corresponding basis $\mathcal{B} \subseteq \{1,\dots,n\}$, the goal is to determine which index should leave the basis in order to move to an adjacent basic feasible solution that yields an improvement in the objective function. We define the *non-basic* index set $\mathcal{N}$ as the complement of $\mathcal{B}$, that is,

$$\mathcal{N} = \{1,\dots,n\} \setminus \mathcal{B}.$$

Analogously to the basic matrix B, which consists of the columns of A indexed by $\mathcal{B}$, we define the *non-basic matrix* N as the sub-matrix of A containing the remaining columns, i.e.,

$$A = [\,B \mid N\,].$$

Accordingly, we partition the vectors x and c into their basic and non-basic components as follows:

$$x = \begin{bmatrix} x_{\mathcal{B}} \\ x_{\mathcal{N}} \end{bmatrix}, \qquad c = \begin{bmatrix} c_{\mathcal{B}} \\ c_{\mathcal{N}} \end{bmatrix},$$

then the linear system can thus be rewritten as

$$Bx_{\mathcal{B}} + Nx_{\mathcal{N}} = b,$$

but from definition of basic feasible solution it reduces to

$$x_{\mathcal{B}} = B^{-1}b, \qquad x_{\mathcal{N}} = 0.$$

Recalling the KKT conditions for this linear problem, a feasible solution x is an optimal feasible solution if there exist vectors $\lambda \in \mathbb{R}^m$ and $s \in \mathbb{R}^n$ such that

$$A^\top \lambda + s = c, \tag{3.2}$$

$$Ax = b, \tag{3.3}$$

$$x \geq 0, \tag{3.4}$$

$$s \geq 0, \tag{3.5}$$

$$x_i s_i = 0, \quad i = 1,\dots,n. \tag{3.6}$$

Since x is a basic feasible solution, we have

$$x_{\mathcal{B}} = B^{-1}b \geq 0, \qquad x_{\mathcal{N}} = 0.$$

We impose the complementarity condition on the basic variables by setting

$$s_{\mathcal{B}} = 0.$$

Considering now the stationarity condition (3.2) separately for the basic and non-basic index sets $\mathcal{B}$ and $\mathcal{N}$, we obtain

$$B^\top \lambda = c_\mathcal{B}, \qquad N^\top \lambda + s_\mathcal{N} = c_\mathcal{N}.$$

Since B is square and non-singular, the first equation uniquely determines λ as

$$\lambda = B^{-\top} c_\mathcal{B},$$

and substituting this into the second equation gives

$$s_\mathcal{N} = c_\mathcal{N} - N^\top \lambda = c_\mathcal{N} - (B^{-1}N)^\top c_\mathcal{B}.$$

The computation of the vector $s_\mathcal{N}$ is often referred to as *pricing*, and its components are called the *reduced costs* of the non-basic variables $x_\mathcal{N}$. If all components of $s_\mathcal{N}$ are non-negative, then together with $s_\mathcal{B} = 0$, $x_\mathcal{B} = B^{-1}b \geq 0$, and $x_\mathcal{N} = 0$, all the KKT conditions (3.2)–(3.6) are satisfied. Hence, the current basic feasible solution is optimal.

3.4.2 Pivot operations, unboundedness and degeneracy

However, if one or more components of $s_\mathcal{N}$ are negative, the complementarity condition (3.6) cannot hold at optimality. Since the corresponding non-basic variables are currently fixed at zero ($x_\mathcal{N} = 0$), in this case, at least one non-basic variable can be increased from zero to obtain a feasible direction that decreases the objective function value. Let $q \in \mathcal{N}$ be an index such that $s_q < 0$, this index is called the "entering index". The choice of this index is not unique; several choices are possible. A common selection strategy, originally proposed in [10], consists of choosing the most negative reduced cost, namely:

$$q = \arg\min_{j \in \mathcal{N}} s_j.$$

This rule, known as the "Dantzig rule", corresponds to selecting the non-basic variable that provides the greatest instantaneous decrease in the objective function per unit increase, this is reflected in choosing the steepest descent with respect to the current reduced costs. We then allow the corresponding non-basic variable x_q to increase from zero while preserving feasibility with respect to the equality constraints $Ax = b$. Since the remaining non-basic variables stay at zero, we have for a new iterate x^*,

$$Ax^* = Bx^*_\mathcal{B} + A_q x^*_q = b = Bx_\mathcal{B} = Ax.$$

Multiplying both sides by B^{-1} yields

$$x^*_\mathcal{B} = x_\mathcal{B} - B^{-1}A_q\, x^*_q.$$

Geometrically, this relation represents a move along an edge of the feasible polytope that decreases the objective value $c^\top x$. The simplex algorithm continues to move along this edge until a new vertex of the feasible region is reached, at this vertex, a new constraint $x_p \geq 0$ must have become active, in other words, one of the basic components x_p, with $p \in \mathcal{B}$, has decreased to zero. We then remove the index p, called "leaving index", from the basis $\mathcal{B}$ and replace it with the entering index q. The corresponding new basis is obtained by exchanging the indices p and q, that is,

$$\mathcal{B}^* = \mathcal{B} \setminus \{p\} \cup \{q\}, \qquad \mathcal{N}^* = \mathcal{N} \setminus \{q\} \cup \{p\}.$$

Finally, we can analyze how the objective function changes during this step. Using the stationarity condition (3.2), we can write

$$c^\top x = (A^\top \lambda + s)^\top x = \lambda^\top (Ax) + s^\top x = \lambda^\top b + s^\top x.$$

Since $Ax^* = Ax = b$, the variation of the objective function depends only on s and the change in x:

$$c^\top x^* - c^\top x = \lambda^\top (Ax^* - Ax) + s^\top (x^* - x) = s^\top (x^* - x).$$

Because $s_\mathcal{B} = 0$ and all non-basic variables remain zero except for the entering variable x_q, we obtain

$$s^\top (x^* - x) = s_q (x_q^* - 0) = s_q \, x_q^*.$$

Hence,

$$c^\top x^* - c^\top x = s_q \, x_q^*.$$

Because s_q is negative, this term is negative, implying that the objective function value strictly decreases, whenever $x_q^* > 0$. This argument, based solely on the KKT conditions (3.2)–(3.6), shows that whenever there exists a non-basic variable with negative s_q, moving along the feasible edge defined by the direction $-B^{-1} A_q$ decreases the objective value while maintaining primal feasibility.

If all components of $s_\mathcal{N}$ are non-negative, then all KKT conditions are satisfied, and the current basic feasible solution is optimal. It may happen that, for a chosen entering index $q \in \mathcal{N}$, all components of the direction vector $d = B^{-1} A_q$ are nonpositive. In this case, the constraint

$$x_\mathcal{B}^* = x_\mathcal{B} - B^{-1} A_q \, x_q^* \geq 0$$

is satisfied for all $x_q^* > 0$. Consequently, we can increase x_q indefinitely without violating feasibility. When this occurs, the objective function $c^\top x$ decreases without bound along the feasible direction $-B^{-1} A_q$, and the linear program is said to be *unbounded*. If the problem is bounded, the algorithm proceeds by moving from one vertex of the feasible region to another. The path traced by successive basic feasible solutions can be viewed as a sequence of edges on the boundary of the polytope, each step producing a reduction in the objective value until an optimal vertex x^* is reached.

Definition 3.3 (Degenerate Basis and Degenerate LP). *A basis $\mathcal{B}$ is said to be "degenerate" if at least one component of the associated basic feasible solution $x_\mathcal{B} = B^{-1}b$ is zero. A linear program is called "degenerate" if it possesses at least one degenerate basis.*

When the current basis $\mathcal{B}$ is nondegenerate, meaning that all basic components of $x_\mathcal{B} = B^{-1}b$ are strictly positive, therefore, each pivot produces a genuine change in the basic feasible solution and a strict decrease in the objective function $c^\top x$. Degeneracy can cause a pivot to change the basis without altering the current feasible point, which may lead to cycling. In practice, anti-cycling rules such as Bland's rule [8] or lexicographic pivoting are employed to ensure progress and avoid infinite loops.

Theorem 3.4 (Termination of the simplex method). *If the linear program is bounded and nondegenerate, the simplex method terminates after a finite number of iterations at a basic optimal solution.*

Proof. This is a classic result in linear programming theory, the proof of which can be found, for example, in [6], Chapter 11. □

3.4.3 Initialization: the two-phase method

The simplex method requires an initial basic feasible solution and finding one is not always easy, its construction may be as difficult as solving the original problem itself. To address this, practical implementations employ the "two-phase method".

In "Phase I", an auxiliary linear program is introduced by adding artificial variables $z \in \mathbb{R}^m$ to guarantee feasibility of the constraints:

$$\min_{x,z} e^\top z \quad \text{s.t.} \quad Ax + Ez = b, \quad (x, z) \geq 0,$$

where $e = (1, \ldots, 1)^\top$ and E is a diagonal matrix with

$$E_{jj} = 1 \text{ if } b_j \geq 0, \qquad E_{jj} = -1 \text{ if } b_j < 0.$$

The Phase-I problem is designed so that an initial basis and an initial basic feasible point are trivial to find, and so that its optimal solution provides a basic feasible starting point for the second phase. Indeed, the point

$$x = 0, \quad \text{and} \quad z_j = |b_j|, \text{ for } j = 1, \ldots, m,$$

is a basic feasible solution for this auxiliary problem, with $B = E$ as the initial basis matrix. It satisfies $Ax + Ez = b$ by construction and ensures $(x, z) \geq 0$. The artificial variables z quantify the violations of the original equality constraints $Ax = b$, for any feasible pair (x, z) we can rewrite

$$Ax = b - Ez.$$

Hence, z represents the amount by which the original constraints are not yet satisfied. Minimizing the objective $e^\top z = \sum_{j=1}^{m} z_j$ therefore forces all violations to vanish, driving the point x toward feasibility with respect to the original system. At optimality, if the auxiliary problem yields an objective value

$$e^\top z^* = 0,$$

then necessarily $z^* = 0$, because all components $z_j \geq 0$. Substituting $z^* = 0$ into the constraints $Ax + Ez = b$ gives

$$Ax^* = b, \qquad x^* \geq 0,$$

which means that x^* is feasible for the original problem. Conversely, if the original linear program admits a feasible vector x_f, then the pair $(x_f, 0)$ is feasible for the auxiliary problem and achieves $e^\top z = 0$. Therefore, the auxiliary problem has an optimal objective value of zero if and only if the original problem is feasible.

If Phase I terminates with an optimal solution $(\tilde{x}, \tilde{z})$ such that $e^\top \tilde{z} = 0$, then $\tilde{z} = 0$ and $\tilde{x}$ satisfies $A\tilde{x} = b$ and $\tilde{x} \geq 0$. This point, together with the final basis obtained in Phase I, provides the starting basic feasible solution for "Phase II". In this second phase, the original linear program

$$\min_x c^\top x \quad \text{s.t.} \quad Ax = b, \quad x \geq 0,$$

is solved using the standard simplex procedure initialized at $(x, z) = (\tilde{x}, 0)$. In practice, it is common that some artificial variables remain in the optimal basis at the end of Phase I, even though their corresponding values are zero. These variables are kept in the initial Phase-II basis to preserve non-singularity of the basis matrix. During the first few pivots of Phase II, they are gradually replaced by genuine variables of the original problem. Once an artificial variable leaves the basis, it can be safely removed from the system.

If, on the other hand, the Phase-I problem terminates with a strictly positive optimal value, that is,

$$e^\top z^* > 0,$$

then at least one component of z^* is positive, implying that the equality $Ax = b$ cannot be satisfied with any $x \geq 0$. Hence, the original problem is called "infeasible".

3.4.4 Complexity and practical performance

In optimization point of view, the simplex method is not a polynomial-time algorithm in terms of computational complexity. Even if it is very efficient in practice, there are some linear programming problems for which the simplex method requires an exponential number of iterations with respect to the problem size. Consider the standard form problem:

$$\min c^\top x \quad \text{s.t.} \quad Ax = b, x \geq 0,$$

where $A \in \mathbb{R}^{m\times n}$ has rank m. The number of possible bases is

$$\binom{n}{m},$$

because each basis corresponds to a choice of m linearly independent columns of A. In the worst case, the simplex method could visit an exponential number of bases in m, and the execution time would be

$$O\left(\binom{n}{m}\right) = O(2^n),$$

which means exponential in the problem size. A classical artificial example that shows this behavior was given by [11], in the so-called "Klee-Minty cube", a specially constructed linear programming problem of dimension n, the simplex method using Dantzig's rule (that is, choosing the most negative reduced cost) visits all 2^n bases before reaching the optimal solution. This result proves that, in general, the simplex method does not belong to the polynomial-time complexity class.

However, despite this theoretical worst case, empirical results show that the simplex method is extremely efficient in practice. In most real-world and randomly generated instances, the number of pivot operations required to reach optimality grows moderately with problem size, often following a nearly linear or low-order polynomial trend [12].

Chapter 4

Numerical Results

The purpose of this chapter is to evaluate, from a numerical perspective, the optimization framework developed in the previous chapters. Starting from the general theory of linear constrained optimization discussed in Chapter (1), and the balance sheet representation and regulatory framework introduced in Chapter (2), the mathematical programming models of Chapter (3) are applied here to a realistic banking context. A first step to verify the feasibility of the problem is to construct a reference portfolio based on the condition that the aggregates have a constant balance sheet volume. The two previously proposed formulations are solved, and using the results obtained, an analysis is performed that highlights both the effects of the optimization choices on the profitability profile and the implications for liquidity.

4.1 Base Scenario and Data Construction

The guiding question of the project can be stated as follows:

> *Given a certain Balance Sheet today, what is the best new business strategy that the bank should set up, over a one year horizon, in order to maximize Net Interest Income while satisfying all regulatory constraints?*

We start from a snapshot of the bank's balance sheet at the reference date t_0, already aggregated according to the assumptions and classification introduced in the previous chapter. In particular, for optimization purposes, we consider only classified operations with the record type *loans and receivables*, see Section (3.1), while all other balance sheet items are treated as exogenous and kept fixed over time. The analysis horizon is discrete and equal to one year, divided into $T = 12$ monthly buckets indexed by $t = 1, \ldots, 12$, consistent with the notation used previously. On this horizon, we first construct and simulate a deterministic *new business* strategy, which aims to keep the volume profile constant over time for each aggregate being optimized. More precisely, let

$$\mathcal{K}^{opt} \subseteq \mathcal{BS}, \qquad |\mathcal{K}^{opt}| = N,$$

denote the set of portfolio aggregates that can be optimized. Since the elements of $\mathcal{K}^{opt}$ do not require structured notation, we introduce explicit indexing and consider a bijection

$$\iota : \{1, \ldots, N\} \longrightarrow \mathcal{K}^{opt},$$

so that the k-th aggregate is identified by

$$K_k := \iota(k), \qquad k = 1, \ldots, N.$$

In the following, we will therefore use the index k to refer to the generic aggregate $K_k \in \mathcal{K}^{opt}$, then we denote with $\Delta V_{k,t}^{\mathrm{mat}}$ the amount of capital that matures (or is amortized) in month t in the aggregate k, and with $\Delta V_{k,t}^{\mathrm{NB}}$ the new business originated in the same month on the same instrument. The basic strategy, with constant volumes, imposes, by construction, the following identity:

$$\Delta V_{k,t}^{\mathrm{NB}} = \Delta V_{k,t}^{\mathrm{mat}} \qquad \forall\, k = 1, \ldots, N, \quad \forall\, t = 1, \ldots, T, \tag{4.1}$$

so that the overall stock of exposure associated with each aggregate k remains unchanged from the level observed in t_0. In other words, each month the bank renews exactly, for each individual aggregate, the volume that has matured, maintaining the balance sheet activity profile constant over time on the subset $\mathcal{K}^{\mathrm{opt}}$.

The relationship (4.1) can be directly matched with the decision variables introduced in Chapter 3. If we denote with $x_{k,t}$ the change in exposure on aggregate k in month t and with $V_k(x_{k,1}, \ldots, x_{k,T}; \theta_k, t)$ the effective exposure function, the constant volume strategy corresponds to fixing

$$x_{k,t}^{\mathrm{ref}} := \Delta V_{k,t}^{\mathrm{NB}} = \Delta V_{k,t}^{\mathrm{mat}}, \qquad \forall\, k = 1, \ldots N, \quad \forall\, t = 1, \ldots, T,$$

this strategy induces a deterministic trajectory of balance sheet variables. From a regulatory perspective, the initial portfolio observed at time t_0 is, by assumption, fully compliant with all supervisory requirements; by regulatory construction, it is a portfolio that is always sustainable for a bank. Specifically, the short-term liquidity measure (LCR), the stable funding measure (NSFR), and the RWA-based capital constraint are all satisfied in the actual balance sheet. Since the constant-volume strategy simply replaces maturing positions and keeps the exposure to each aggregate unchanged, it represents a feasible solution to the optimization problem by construction. Therefore, the reference point x^{ref} belongs to the feasible set of both optimization problems and satisfies all linear and nonlinear constraints. Finally, through balance sheet simulation induced by the constant-volume strategy combined with the outputs of a proprietary software of company "Prometeia", we create the basis for calibrating all the fixed parameters of the model, such as:

- the NII contributions associated with the initial stock, $\text{NII}_k^{\text{ACTUAL}}$ in the objective function, and the expected yield $e_{k,t}$ for each instrument and each month;
- the value of equity to be included in the balance sheet constraint;
- the risk weight coefficients rw_k, the corresponding cut-off exposures OUT_k and CET1_k the regulatory capital component, used to calculate the RWA constraint;
- the LCR and NSFR components relating to the aggregates, both regarding the non-optimized portfolio component Else_N_t, Else_D_t, Else_ASF_t, Else_RSF_t and regarding the constant-volume baseline strategy for calculating the contributions of the aggregates that are optimized, as $ASF\_base_{k,t}$, $RSF\_base_{k,t}$, $Den\_base_{k,t}$, $Buffer\_base_{k,t}$ and $val\_base_{k,t}$

In this way, the balance sheet snapshot at t_0 and the associated constant-volume strategy play a dual role: on one hand, they guarantee the existence of a feasible solution to the optimization problem, and on the other, they provide the data structure with which consistently parameterize all the components of the model presented in the previous chapters. The resulting linear program is implemented in `Python` and solved using the `scipy.optimize.linprog` routine with the HiGHS backend (`method=''highs''`). This algorithm provides a state-of-the-art implementation of linear programming methods, ensuring numerical robustness and efficient solution times even for problems with several thousand variables and constraints such as the one considered here.

4.2 Data Loading and Preprocessing

All portfolio information for the optimized aggregates $\mathcal{K}^{opt}$ is stored in a dedicated Excel file and imported into a tabular structure. Each row of this table corresponds to one aggregate k and contains all the relevant attributes:

- the vector of characteristic $\theta_k = \big(a_k, t_k, \tau_k, m_k, f_k, r_k, s_k, p_k\big)$,
- the cut-off exposure and the one year survival outstanding amounts,

- the calibrated NII contribution of the initial stock, $\text{NII}_k^{\text{ACTUAL}}$,
- the regulatory parameters: risk weights rw_k and capital coefficients used in the RWA constraint,
- the monthly baseline new-business volumes and the associated exposure coefficients that define the constant-volume strategy x^{ref},
- the lower and upper bounds $\underline{x}_{k,t}$ and $\overline{x}_{k,t}$ for each decision variable $x_{k,t}$.

The decision variables are arranged in a sub-matrix of that above:

$$x \in \mathbb{R}^{N \times T}, \qquad T = 12,$$

but for the interaction with the linear solver, this matrix is flattened into a vector $x \in \mathbb{R}^n$ with $n = N \times T$, and the bounds $\underline{x}_{k,t}$, $\overline{x}_{k,t}$ are collected into two vectors $\underline{x}, \overline{x} \in \mathbb{R}^n$ defining a simple box constraint. In the numerical case study presented in this chapter, the optimization set contains $N = 66$ aggregates. The analyzed dataset contains several columns, and we present a portion of them to illustrate its structure. For ease of readability, it is presented in three logical blocks. The first part of the dataset contains the identifying characteristics of each aggregate, Figure 4.1, i.e., the characteristics reported in the vector θ_k.

PDC Descr	A_L	Record type Descr	Rate Type Descr	Parameter	Repayment type	IntPayFreq	maturity years	maturity months
Asset	1.00	Loans and receivables	Fixed	EUR CMS 10Y	LinearAmortization	1	10	-
Asset	1.00	Loans and receivables	Fixed	EUR CMS 10Y	LinearAmortization	1	15	-
Asset	1.00	Loans and receivables	Fixed	EUR CMS 1Y	AtMaturity	1	1	-
Asset	1.00	Loans and receivables	Fixed	EUR CMS 1Y	LinearAmortization	3	-	3

Figure 4.1: Aggregate characteristics vector.

The second section of the dataset includes the parameters needed to calculate the NII and constraints. An excerpt of the structure is shown in Figure 4.2.

NII_BASELINE	OUTSANDING_CUT_OFF	Rate	rwa	cap	shrink_M1		shrink_M12	growth_M1		growth_M12	Exp_M1		Exp_M12
282'274'303.62	22'067'818'590.68	0.01	0.08	-	1'000'000.00		1'000'000.00	1'000'000'000'000.00		1'000'000'000'000.00	3.68%		0.17%
57'860'905.33	2'669'180'347.91	0.02	0.10	-	1'000'000.00		1'000'000.00	100'000'000'000.00		100'000'000'000.00	3.64%		0.17%
10'103'481.40	557'334'570.23	0.02	0.04	-	-		-	10'000'000'000.00		10'000'000'000.00	3.98%		0.15%
2'018'933.54	966'242'362.56	0.00	0.11	-	-		-	10'000'000'000.00		10'000'000'000.00	2.99%		0.14%

Figure 4.2: Excerpt of the external and management parameters.

The last part of the dataset reports the volumes of new business over a monthly horizon, generated by the constant-volume baseline strategy, Figure 4.3.

M1	M2	M3	M4	M5	M6	M7	M8	M9	M10	M11	M12
183'898'488.26	185'430'975.66	186'976'233.79	188'534'369.07	190'105'488.81	191'689'701.22	193'287'115.40	194'897'841.36	196'521'990.04	198'159'673.29	199'811'003.90	201'476'095.60
14'828'779.71	14'911'161.82	14'994'001.61	15'077'301.62	15'161'064.40	15'245'292.54	15'329'988.61	15'415'155.21	15'500'794.96	15'586'910.49	15'673'504.44	15'760'579.46
-	-	-	-	-	-	-	-	-	-	-	557'334'570.23
-	-	966'242'362.56	-	-	966'242'362.56	-	-	966'242'362.56	-	-	966'242'362.56

Figure 4.3: Monthly new business baseline volumes.

Starting from these data we are going to numerically implement the objective function that corresponds to the NII maximization introduced in Chapter 3:

$$\max_x \mathrm{NII}(x) = \sum_k a_k \left(\mathrm{NII}_k^{\mathrm{ACTUAL}} + \sum_{t=1}^{T} x_{k,t}\, e_{k,t} \right).$$

Since the linear solver operates in minimization form, the problem is recast as

$$\min_x \; c^\top x,$$

where the cost vector $c \in \mathbb{R}^n$ collects the coefficients

$$c_{k,t} = -\, a_k\, e_{k,t},$$

arranged consistently with the flattening of x. In this formulation, designed exclusively for the solver, we separate the optimization-dependent part from the constant part and use only the former. Later, during the evaluation phase, we will also add the constant component additively.

The balance sheet identities, which enforce the equality between assets and liabilities plus equity at each month t, are written in the form

$$\sum_k a_k\, V_k\big(x_{k,1}, \ldots, x_{k,T}; \theta_k, t\big) = \mathrm{E}, \qquad t = 1, \ldots, T.$$

For the linear solver, these T identities are expressed as

$$A^{eq} x = b^{eq},$$

where the matrix A^{eq} and the vector b^{eq} are obtained through simple algebraic operations done to isolate the variables $x_{k,t}$. The RWA–based capital constraint of Chapter 3 is treated in an analogous way. For each month t the inequality

$$\sum_k \mathrm{rw}_k\, V_k\big(x_{k,1}, \ldots, x_{k,T}; \theta_k, t\big) \leq Cap, \qquad t = 1, \ldots, T,$$

collecting all months, this family of constraints is written as

$$A^{ub} x \leq b^{ub}.$$

The liquidity constraints related to the LCR and NSFR are derived by combining the balance sheet structure with the regulatory requirements discussed, based on the assumptions presented in Section (3.3.3). For each optimized aggregate and for each month, the baseline are used to compute unit coefficients describing how one unit of exposure contributes to and we obtain:

$$\sum_{k=1}^{N} \left(\frac{Buffer\_base_{k,t}}{val\_base_{k,t}} + 1.1 \, \frac{Den\_base_{k,t}}{val\_base_{k,t}} \right) x_{k,t} + \text{Else\_N}_t + 1.1 \, \text{Else\_D}_t \geq 0,$$
$$\sum_{k=1}^{N} \left(\frac{ASF\_base_{k,t}}{val\_base_{k,t}} + 1.1 \, \frac{RSF\_base_{k,t}}{val\_base_{k,t}} \right) x_{k,t} + \text{Else\_ASF}_t + 1.1 \, \text{Else\_RSF}_t \leq 0.$$

Since the LCR and NSFR conditions are expressed in linear form, they can be rewritten in standard inequality form and incorporated into the global constraint system. By combining them with the RWA requirements, all linear constraints can be collected in the compact representation $A^{ub}x \leq b^{ub}$. The optimization problem submitted to the linear solver therefore reads:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & A^{eq}x = b^{eq}, \\ & A^{ub}x \leq b^{ub}, \\ & \underline{x} \leq x \leq \overline{x}. \end{aligned}$$

4.3 Numerical Evaluation

Once the data have been structured and the linear program has been generated as described in the previous section, we evaluate the behavior of the optimization framework under three different configurations. The analysis follows a progressive structure: we first solve the model under the constant-volume strategy used for calibration, then we consider the optimization with only RWA and balance sheet equation, and finally we impose the full set of constraints (adding LCR and NSFR). This staged approach allows us to disentangle the contribution of the constraints, highlighting how regulatory requirements shape the feasible region and interact with the objective of NII maximization.

The constant-volume strategy x^{ref} provides the benchmark against which all optimized solutions are compared. Since this strategy simply rolls over, period by period, the maturities of each aggregate, both the exposure profile and all associated regulatory quantities evolve in a deterministic and controlled manner. From the balance sheet simulation under x^{ref}, the following results are obtained:

- the Net Interest Income corresponds to an objective value of approximately

$$obj^{ref} \approx 9.655 \times 10^9,$$

consistent with the observed exposure and return structure. All quantities entering the objective function are expressed in euros, so that both $NII(x)$ and the reported optimal values can be interpreted directly as monetary amounts in euro;

- the RWA constraint is comfortably satisfied at each time bucket, with total RWA remaining close to capital;
- the balance sheet identity holds exactly, confirming internal consistency;
- the liquidity metrics are stable and well above the regulatory thresholds:

$$\mathrm{LCR}_t^{ref} \in [1.37,\ 1.58], \qquad \mathrm{NSFR}_t^{ref} \in [1.28,\ 1.43].$$

We next solve the optimization problem including all linear constraints (RWA, balance-sheet identities, and exposure bounds), but *excluding* the LCR and NSFR conditions. This configuration is designed to show the unconstrained profit-seeking behaviour of the model when liquidity regulation is ignored. The linear program converges rapidly:

$$\text{Status: Optimal}, \qquad \text{Iterations: } 34.$$

The objective function reaches

$$obj^{lin} \approx 1.7669 \times 10^{11},$$

which is nearly one order of magnitude larger than the baseline value. Such a substantial increase is the consequence of reallocating volumes toward the most profitable aggregates, unconstrained by funding stability or liquidity coverage concerns. However, the resulting liquidity indicators immediately reveal that the obtained solution is economically and regulatorily unimplementable:

$$\mathrm{LCR}_t^{lin} = \big[22.65,\ 22.25,\ 16.07, \ldots,\ 2.20,\ 1.46,\ -24.62\big],$$

$$\mathrm{NSFR}_t^{lin} = \big[0.74,\ 0.53,\ 0.43, \ldots, 0.06,\ 0.21,\ 0.14\big].$$

The LCR has out-of-scale and negative values in several months, and NSFR is constantly below the Basel III minimum threshold. This outcome shows that the model, in the absence of liquidity constraints, reallocates the portfolio into positions whose refinancing risk is extremely high, leveraging short-term and unstable funding. These results provide a crucial insight: merely imposing solvency constraints (RWA) and balance sheet equality is insufficient to guarantee a regulatory-feasible portfolio. Liquidity constraints are *structural* components of the feasible region and cannot be omitted without losing economic meaning.

Finally, we solve the complete optimization problem, including both the LCR and NSFR linearized conditions. This corresponds to the economically relevant configuration in which the bank aims to maximize NII while maintaining regulatory compliance across the entire one-year horizon. The solver reports:

$$\text{Status: Optimal}, \qquad \text{Iterations: } 165,$$

where the increase in iterations reflects the more restrictive feasible region. The optimal objective is

$$obj^{nl} \approx 1.7426 \times 10^{11},$$

which remains substantially higher than the constant-volume benchmark, although slightly lower than the unconstrained linear solution, but both LCR and NSFR sit exactly at the minimum allowed level:

$$\text{LCR}_t^{nl} = 1.1, \qquad \text{NSFR}_t^{nl} = 1.1, \qquad \forall t = 1, \dots, 12.$$

The model pushes the liquidity ratios to their regulatory floor in order to maximize NII within the admissible region, confirming the economic coherence of the optimization mechanism.

To complement the analysis, we also evaluate the Net Interest Margin (NIM), defined in Section (2.3), across the three configurations. Since NIM measures the interest income relative to the volume of earning assets, it provides a normalized indicator of profitability, allowing comparisons that are independent of the absolute size of the balance sheet. For the constant-volume benchmark portfolio, we obtain:

$$\text{NIM}^{ref} \approx 1.7002\%,$$

The results of our optimization show a reallocation of exposures toward the most profitable aggregates, producing a substantial improvement in relative profitability. In the first model, the one without liquidity constraints, the resulting margin rises to:

$$\text{NIM}^{lin} \approx 2.4491\%,$$

reflecting a more aggressive and less constrained asset composition. However, as previously noted, this configuration violates the liquidity requirements, and the resulting NIM must be interpreted as purely theoretical. In the full regulatory setting, including the LCR and NSFR linearized constraints, and we get as result:

$$\text{NIM}^{nl} \approx 2.4163\%.$$

The slight reduction compared to the unconstrained linear solution mirrors the tightening of the feasible region induced by liquidity regulation, nevertheless, the improvement relative to the constant-volume strategy remains substantial showing that the optimization framework is capable of enhancing efficiency even under strict supervisory requirements.

4.4 Comparison and Interpretation

In summary, the three scenarios offer a clear view of the role each constraint plays in defining the bank's optimal behavior. The constant volume solution represents the baseline configuration of the portfolio: it is prudent, fully compliant with all regulatory metrics, and devoid of optimization motives, serving primarily as a benchmark against which the effects of the subsequent models can be measured. The second scenario, free of liquidity constraints, highlights how a purely profit-oriented formulation leads the optimization algorithm to reallocate exposures toward the most remunerative aggregates, exploiting the linearity of the objective function and the absence of liquidity frictions. This configuration, disregards the funding stability of the bank and, therefore, does not constitute a sustainable policy within a regulatory environment shaped by Basel III. The introduction of the LCR and NSFR constraints fundamentally modifies the feasible region of the problem, their presence limits excessive concentration on high-yield assets when destabilize liquidity constraints. When both measures are enforced jointly, the resulting optimal allocation reflects the intrinsic trade-off at the core of Asset and Liability Management: the pursuit of profitability must be reconciled with the maintenance of a robust funding structure and the fulfillment of liquidity requirements over the entire planning horizon. The optimized portfolio obtained under the full regulatory framework is therefore more balanced, and ultimately more consistent with the operational logic of a banking institution. Finally, beyond the specific numerical results, the thesis underscores the value of optimization-based tools for decision-making in ALM. By formalizing balance-sheet management within a rigorous quantitative structure, the model enables practitioners to simulate alternative regulatory configurations, assess profitability liquidity trade-offs, and evaluate the impact of possible supervisory reforms. Although simplified in several respects, the framework developed here provides a foundation upon which more complex features, for these reasons, it is hoped that the proposed approach may serve not only as an academic contribution, but also as a practical component of a broader decision support architecture for modern balance-sheet management.

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