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PHYSICS

SOME INTEGRABILITY RESULTS FOR WRONSKIANS IN GRAVITY AND SUPERSYMMETRIC GAUGE THEORIES

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Abstract

In this thesis we study the Confluent Heun equation (CHE) with quantum integrability methods, with particular focus on the relevant wronskians solving it's monodromy problem. It is known that CHE is in correspondence with the quantization of Seiberg-Witten (SW) differential for $\mathcal{N} = 2$ Super-Yang-Mills (SYM) with number of flavours $N_f = 3$ in the Nekrasov-Shatashvili (NS) background. Besides, the same equation is crucial in black hole perturbation of Schwarzschild and Kerr geometry theory. In this context, the wronskians between the Floquet functions are computed in a new way, for both $N_f = 0$ and $N_f = 3$ theories and compared with results in the literature. By using them, we compute the wronskians between regular solutions, as well, which are their connection coefficients and play the role of the so called Q-functions in quantum integrability. To reach this goal, we built a new algorithmic way to compute the quantum momentum of the Floquet and regular solutions by means of some special polynomials that can be computed in a recursive way and enjoy interesting properties.

Contents

1 Introduction

Both Seiberg-Witten (SW) differential for $\mathcal{N} = 2$ Super-Yang- Mills (SYM) with number of flavours $N_f = 3$ in the Nekrasov-Shatashvili (NS) background and Teukolsky equation for Schwarzschild background (in its homogeneous version, with spin $s = 2$) are in correspondence with Confluent Heun equation (CHE).
Ordinary DIfferential Equations/Integrable Models (ODE/IM) prescribes an integrable model correspondent to the CHE as established in [7][1], this specific mo is associated to CHE with specific functionals and integral equations. In particular, the connection coefficients of the ode correspond to special integrability quantities and also to quantum cycles of the gauge theories[9][4].
Using ODE/IM methods applied to CHE[7] [5] we compute quantities useful for both gauge and gravity theories in correspondence.
The method is first tested for the simpler $N_f = 0$ that corresponds to the Mathieu equation and then applied to the more complicated (CHE).
For the gauge side, we compute the antisymmetrized acquired phase $\Phi(k)$ that is in correspondence with the dual gauge period, A_D, as given by the Nekrasov instanton function.
We also compute Q-functions $Q_i(\theta)$ as instantonic series and match them with ones obtained with the different approach of [8].
In addition, we compute exactly the wavefunctions $\psi_{-,0}(y)$, $\psi_{0,0}(y)$ and $\psi_{+,0}(y)$ that are the regular solutions at the singularities, both directly and in the Floquet basis.
These functions correspond to the solutions of the Teukolsky equation with certain boundary conditions needed to compute gravitational waveforms emitted by a particle moving in a Schwarzschild geometry [1] [9] .
In the process we found a systematical way to compute the solution of the Riccati equation of the Heun type equations in the istantonic expansion through a recursion relation. This method leads to interesting resummation formulae that are in turn used to compute the wronskian of the Floquet solutions $W[\psi_+, \psi_-]$ in an innovative way through a matching of regular solutions.
Moreover, this technology is also used to match and justify some results found in [1] [9] .
An interesting mathematical formula is also found for any smooth function $f(y)$

$$f(y) \cdot B_n \left(\frac{d}{dy} \ln f(y), \frac{d^2}{dy^2} \ln f(y), ..., \frac{d^n}{dy^n} \ln f(y) \right) = \frac{d^n}{dy^n} f(y) \tag{1.1}$$

The thesis is organized as follows

- In chapter 2 a presentation of Quantum Seiberg-Witten theory introduces the gauge theory involved and the key quantities.
- In chapter 3 is a brief introduction of the integrability tools needed for the specific case of the Heun type equations.
- In chapter 4 we show how our construction to build the wavefunctions, wronskian and Q-functions works for the simpler $N_f = 0$ theory
- In chapter 5 the construction is applied to the more complicated $N_f = 3$ case and is the core part of the thesis
- Chapter 6 introduces briefly the Black hole perturbation theory and his connection with CHE
- The Appendices contain some heavier calculations needed for computational view and deeper understanding

2 Quantum Seiberg-Witten theory

Supersymmetry is a conjectured symmetry of the nature that put in correspondence fermions and bosons.
It can be expressed it terms of the action of supersymmetric charges $Q_{i\alpha}$ that satisfy the graded algebra

$$\{Q_{i\alpha}, \bar{Q}^j_{\dot{\beta}}\} = 2\delta^j_i \sigma^\mu_{\alpha\dot{\beta}} P_\mu \tag{2.1}$$

$$[Q_{i\alpha}, P_\mu] = 0 \tag{2.2}$$

$$\left[\bar{Q}_{i\dot{\alpha}}, P_\mu\right] = 0 \tag{2.3}$$

$$[Q_{i\dot{\alpha}}, M^{\mu\nu}] = (\sigma^{\mu\nu})^\beta_\alpha Q_{i\beta} \tag{2.4}$$

$$\left[\bar{Q}^{\dot{\alpha}}_i, M^{\mu\nu}\right] = (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}} Q^{\dot{\beta}}_i \tag{2.5}$$

An $\mathcal{N}=1$ chiral superfield ϕ is denoted in terms of spin components $(0, \frac{1}{2})$ and is made of a scalar z, a fermion ψ and an auxiliary field f.
Under SUSY transformations, his components vary as

$$\delta z = \sqrt{2}\epsilon\psi \tag{2.6}$$

$$\delta\psi = \sqrt{2}i\partial_\mu z\sigma^\mu\bar{\epsilon} - \sqrt{2}f\epsilon \tag{2.7}$$

$$\delta f = \sqrt{2}i\partial_\mu\psi\sigma^\mu\bar{\epsilon} \tag{2.8}$$

A $\mathcal{N}=1$ vector superfield V has spin components $(\frac{1}{2}, 1)$ and can be expanded in terms of components fields as (in the Wes-Zumino gauge)

$$V = \theta\sigma^\mu\bar{\theta}v_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) \tag{2.9}$$

The $\mathcal{N}=2$ vector superfield is defined as the sum of a chiral and a vector superfield in $\mathcal{N}=1$.
The exact Lagrangian for the Yang-Mills theory $\mathcal{N}=2$ SUSY is

$$\mathcal{L} = Tr\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu\bar{\lambda} - i\psi\sigma^\mu D_\mu\bar{\psi} + (D_\mu z)^\dagger D^\mu z + \frac{\theta}{32\pi^2}g^2 F_{\mu\nu}\bar{F}^{\mu\nu} + \tag{2.10}$$

$$+\frac{1}{2}D^2 + f^\dagger f + i\sqrt{2}gz^\dagger\{\lambda,\psi\} - i\sqrt{2}g\{\bar{\lambda},\bar{\psi}\}z + gD[z,z^\dagger]\Big) \tag{2.11}$$

With

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \tag{2.12}$$

And

$$F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - \frac{i}{2}[v_\mu, v_\nu] \tag{2.13}$$

Instanton corrections break $U(1)$ symmetry group to $\mathbf{Z_8}$, if the equations of motion auxiliary fields are inserted in the Lagrangian $\mathcal{L}$ produce the scalar potential

$$V(z,z^\dagger) = \frac{g^2}{2}Tr([z,z^\dagger])^2 \tag{2.14}$$

Unbroken SUSY requires $V(z,z^\dagger)=0$, for $SU(2)$ we have the expectation value

$$< z > = \frac{1}{2}a\sigma_3 \tag{2.15}$$

However gauge transformations can take $a \to -a$ and therefore we have to be label gauge inequivalent vacua (the moduli space $\mathcal{M}$) through

$$u = < trz^2 > \tag{2.16}$$

We focus now on the low energy description via the Wilsonian effective action S_W that for $U(1)$ is

$$S_W = \frac{1}{16\pi}Im\int d^4x\left(\frac{1}{2}\int d^2\theta\mathcal{F}''(\phi)W^\alpha W_\alpha + \int d^2\theta d^2\bar{\theta}\phi^\dagger\mathcal{F}'(\phi)\right) \tag{2.17}$$

With the prepotential $\mathcal{F}$ and the kinetic term for the vector field

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V \tag{2.18}$$

One can also define the dual field and the dual prepotential as

$$\phi_D = \frac{\partial \mathcal{F}(\phi)}{\partial \phi} \qquad \phi_= - \frac{\partial \mathcal{F}_D(\phi_D)}{\partial \phi_D} \tag{2.19}$$

or

$$a_D = \frac{\partial \mathcal{F}}{\partial a} \qquad a = -\frac{\partial \mathcal{F}_D}{\partial a_D} \tag{2.20}$$

a and a_D are also called Seiberg-Witten cycles or Seiberg-Witten periods, in [14] the authors calculated them exactly as integrals of the SW differential

$$\lambda_{SW} = \sqrt{2u - 2\Lambda^2 \cos z} \tag{2.21}$$

$$a(u,\Lambda) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lambda_{SW} dz = \Lambda\sqrt{2\left(\frac{u}{\Lambda^2}+1\right)}{}_2F_1\left(-\frac{1}{2},\frac{1}{2},1,\frac{2}{1+\frac{u}{\Lambda^2}}\right) \tag{2.22}$$

$$a_D(u,\Lambda) = \frac{1}{2\pi}\int_{-\arccos(u/\Lambda^2)-i0^+}^{\arccos\left(u/\Lambda^2\right)-i0^+} \lambda_{SW} dz = i\Lambda\frac{1-\frac{u}{\Lambda^2}}{2}{}_2F_1\left(\frac{1}{2},\frac{1}{2},2,\frac{1-\frac{u}{\Lambda^2}}{2}\right) \tag{2.23}$$

One can then invert $a(u)$ and use the relation $a_D(a) = \frac{\partial \mathcal{F}}{\partial a}$ to obtain the prepotential $\mathcal{F}$ of the theory.
Seiberg-Witten theory can be deformed to the QSW theory with the spacetime deformation called Ω background, introducing the supergravity parameters ϵ_1 and ϵ_2 [13].
We are just interested in the Nekrasov-Shatashvili (NS) limit where $\epsilon_2 \to 0$ and ϵ_1 plays the role of the Planck constant.
In this framework, the quantum SW period $a(\epsilon_1, u, \Lambda)$ can still be computed and is found to correspond also to the Floquet exponent for the Mathieu equation, which is just the Seiberg-Witten curve

$$-\frac{\epsilon_1^2}{2}\frac{d}{dz^2}\psi(z) + \left(\Lambda^2 \cos z - u\right)\psi(z) = 0 \tag{2.24}$$

We will see that this is just an aspect of the general ODE/IM correspondence between this two theories.
We can also define a new quantity

$$A_D = \frac{\partial \mathcal{F}_{NS}}{\partial a} \tag{2.25}$$

Where $\mathcal{F}_{NS}$ is our deformed prepotential, this new quantity A_D differs from the dual deformed cycle period a_D, but will be useful later on.
The Seiberg-Witten curve for $\mathcal{N} = 2$ $SU(2)$ is studied also with N_f fundamental matter flavour hypermultiplets and a correspondence with Ordinary differential equation (ODE) is found for some of them.
For example $N_f = 0$ correspond to the Mathieu equation as we already said, $N_f = 2$ correspond to the doubly confluent Heun equation, $N_f = 3$ correspond to the confluent Heun equation (CHE) and finally $N_f = 4$ correspond to the Heun equation (HE). In this work we will focus mainly on CHE since it also enjoys a correspondence with the Teukolsky equation that governs black hole perturbations and in gauge variables reads

$$\begin{aligned}&-\hbar^2\frac{d^2}{dy^2}\psi(y) + \Big(e^{2y}\Lambda_3\left(4(m_1-m_2)^2\right) + 4e^y\sqrt{\Lambda_3}\left(-2\hbar^2 + 8m_1m_2 + \Lambda_3 m_3 - 8u\right) + \\ &+ \left(\Lambda_3^2 - 24\Lambda_3 m_3 + 64u\right) + 4e^{-y}\sqrt{\Lambda_3}\left(8m_3 - \Lambda_3\right) + 4\Lambda_3 e^{-2y}\Big)\frac{1}{16\left(\sqrt{\Lambda_3}e^y - 2\right)^2}\psi(y) = 0\end{aligned} \tag{2.26}$$

3 Integrability

3.1 Mathieu equation

The Mathieu equation (2.24) can be written in the "integrability form"[5]

$$\left(-\frac{d^2}{dy^2} + e^{2\theta}\cosh y + P^2\right)\psi(y) = 0 \tag{3.1}$$

That is in the Schrodinger-like equation form

$$\left(-\frac{d^2}{dy^2} + V(y)\right)\psi(y) = 0 \tag{3.2}$$

This equation has two irregular singular points at $y \to \pm\infty$ and two subdominant asymptotic solutions

$$U_0 \simeq \frac{1}{\sqrt{2}}\exp\left(-\frac{\theta}{2} - \frac{y}{4}\right)\exp\left(-2e^{\theta+\frac{y}{2}}\right) \qquad y \to +\infty \tag{3.3}$$

$$V_0 \simeq \frac{1}{\sqrt{2}}\exp\left(-\frac{\theta}{2} + \frac{y}{4}\right)\exp\left(-2e^{\theta-\frac{y}{2}}\right) \qquad y \to -\infty \tag{3.4}$$

Other solutions can be generated by applying on these the following discrete symmetries of (3.1)

$$\Lambda : \theta \to \theta + \frac{i\pi}{2} \qquad y \to y + i\pi \qquad , \qquad \Omega : \theta \to \theta + \frac{i\pi}{2} \qquad y \to y - i\pi \tag{3.5}$$

Concisely $U_1 = \Lambda U_0$ is invariant under Ω and $V_1 = \Omega V_0$ is invariant under Λ.
We may interpret this phenomenon as spontaneous symmetry breaking for ODE.
Now, following the ODE/IM prescription, one can define a Q-function as the connection coefficient between the two singularities, that is

$$Q(\theta) = W[U_0, V_0] \tag{3.6}$$

Acting with the symmetries, one gets

$$W[U_1, V_0]U_0 - W[U_0, V_0]U_1 = W[U_1, U_0]V_0 \tag{3.7}$$

If we use $W[U_1, V_0] = Q(\theta + i\pi)$, $W[U_0, V_0] = Q(\theta)$ and $W[U_1, U_0] = i$, we get

$$iV_0(y) = Q(\theta + i\pi)U_0(y) - Q(\theta)U_1(y) \tag{3.8}$$

In the same way, one can also find

$$iU_0 = Q\left(\theta + \frac{i\pi}{2}\right)V_0(y) - Q(\theta)V_1(y) \tag{3.9}$$

From that, one can also find the QQ-relation

$$1 + Q^2\left(\theta + \frac{i\pi}{2}\right) = Q(\theta)Q(\theta + i\pi) \tag{3.10}$$

Also Baxter's TQ relations and Liouville Y-system can be find [5], however, we will not use them, we will just need to use

$$Q(\theta) = -i\lim_{y\to-\infty}\frac{U_0}{V_1} = \sqrt{2}e^{\frac{\theta}{2}}\lim_{y\to-\infty} e^{-\frac{y}{4}-2e^{\theta-\frac{y}{2}}}U_0 \tag{3.11}$$

And

$$Q(\theta) = -i\lim_{y\to+\infty}\frac{V_0}{U_1} = \sqrt{2}e^{\frac{\theta}{2}}\lim_{y\to+\infty} e^{\frac{y}{4}-2e^{\theta+\frac{y}{2}}}V_0(y) \tag{3.12}$$

Obtained through the limits of (3.8) and(3.9).

3.2 Confluent Heun equation

Similarly, equation (2.26) has an irregular singularity at $y \to -\infty$ and 2 regular singularities at $y \to +\infty$ and at $y \to \ln \frac{2}{\sqrt{\Lambda_3}}$.

It is convenient to change independent variables and use the "integrability parameters", defined analogously to lower N_f theories as

$$y = t - \frac{1}{2}\ln\frac{\Lambda_3}{\hbar}, \quad \frac{\Lambda_3}{4\hbar} = e^{\theta}, \quad \frac{m_k}{\hbar} = q_k, \quad \frac{u}{\hbar^2} = P^2. \tag{3.13}$$

Through this map, we arrive at the "integrability form"

$$-\frac{d^2}{dt^2}\psi(t) + \left(\frac{1}{4}e^{2t}(q_1-q_2)^2 + e^t\left(-\frac{1}{2} + 2q_1q_2 + e^{\theta}q_3 - 2P^2\right)\right.$$
$$\left. + \left(e^{2\theta} - 6e^{\theta}q_3 + 4P^2\right) + e^{-t}\left(8e^{\theta}q_3 - 4e^{2\theta}\right) + 4e^{-2t}e^{2\theta}\right)\frac{1}{(e^t-2)^2}\psi(t) = 0 \tag{3.14}$$

In [8] the authors found that the CHE (3.14) enjoys the following discrete symmetries

$$\begin{array}{l} \Omega_-: \quad \theta \to \theta + i\pi, \quad t \to t, \quad q_1 \to +q_1, \quad q_2 \to +q_2, \quad q_3 \to -q_3, \\ \Omega_+: \quad \theta \to \theta, \quad t \to t, \quad q_1 \to -q_1, \quad q_2 \to -q_2, \quad q_3 \to +q_3, \\ E: \quad \theta \to \theta, \quad t \to t, \quad q_1 \to +q_2, \quad q_2 \to +q_1, \quad q_3 \to +q_3 \end{array} \tag{3.15}$$

Using these on the solutions with the following regular behavior at the singular points

$$\psi_{-,0}(y) \approx e^{-(q_3+\frac{1}{2})\theta + (q_3+\frac{1}{2})y}\exp\left(-e^{\theta-y}\right) \qquad y \to -\infty \tag{3.16}$$

$$\psi_{0,0}(y) \approx \frac{1}{\sqrt{2}}\left(e^y-2\right)^{\frac{1}{2}(1+q_1+q_2)} \qquad y \to \ln(2) \tag{3.17}$$

$$\psi_{+,0}(y) \approx \exp\left(-\frac{q_1-q_2}{2}y\right) \qquad y \to +\infty \tag{3.18}$$

From these solutions, using the symmetries, we can also obtain new independent solutions (typically irregular).

$$\psi_{0,1} = \Omega_+\psi_{0,0} \tag{3.19}$$
$$\psi_{-,1} = \Omega_-\psi_{-,0} \tag{3.20}$$
$$\tilde{\psi}_{+,1} = E\psi_{+,0} \tag{3.21}$$

It was found in [8] that one can define, following the ODE/IM prescription, some integrability Q functions [1] as CHE connection coefficients, meaning as wrosnkians between the regular solutions

$$Q_1(\theta) = W[\psi_{0,0}, \psi_{-,0}] \tag{3.22}$$
$$Q_2(\theta) = W[\psi_{+,0}, \psi_{0,0}] \tag{3.23}$$
$$Q_3(\theta) = W[\psi_{+,0}, \psi_{-,0}] \tag{3.24}$$

Upon acting on the Q-functions with the symmetries (3.15) and using properties of the wronskians we can write the following linear relations

$$\psi_{0,0} = -\frac{ie^{-i\pi q_3}}{2}Q_1(\theta+i\pi, -q_1, -q_2, -q_3)\psi_{-,0} + \frac{ie^{-i\pi q_3}}{2}Q_1(\theta, q_1, q_2, q_3)\psi_{-,1} \tag{3.25}$$

$$\psi_{+,0} = -\frac{1}{q_1+q_2}Q_2(\theta, -q_2, -q_1, q_3)\psi_{0,0} + \frac{1}{q_1+q_2}Q_2(\theta, q_1, q_2, q_3)\psi_{0,1} \tag{3.26}$$

$$\psi_{-,0} = -\frac{1}{q_1-q_2}Q_3(\theta, q_2, q_1, q_3)\psi_{+,0} + \frac{1}{q_1-q_2}Q_3(\theta, q_1, q_2, q_3)\tilde{\psi}_{+,1} \tag{3.27}$$

[1] We don't specify which integrable model for the moment.

Doing the limits to the singularities, we can also find the following forms for the Q functions

$$Q_1(\theta) = -2ie^{i\pi q_3} \lim_{y\to-\infty} \frac{\psi_{0,0}}{\psi_{-,1}} = -2ie^{i\pi q_3} \lim_{y\to-\infty} e^{-i\pi(q_3-\frac{1}{2})} e^{(q_3-\frac{1}{2})(y-\theta)} e^{-e^{\theta-y}} \psi_{0,0} \tag{3.28}$$

$$Q_2(\theta) = (q_1+q_2) \lim_{y\to ln(2)} \frac{\psi_{+,0}}{\psi_{0,1}} = (q_1+q_2) \lim_{y\to ln(2)} \sqrt{2}\,(e^y-2)^{-\frac{1}{2}(1-q_1-q_2)}\,\psi_{+,0} \tag{3.29}$$

$$Q_3(\theta) = (q_1-q_2) \lim_{y\to+\infty} \frac{\psi_{-,0}}{\tilde{\psi}_{+,1}} = (q_1-q_2) \lim_{y\to+\infty} e^{-\frac{q_1-q_2}{2}y}\psi_{-,0} \tag{3.30}$$

In [8] it was also shown that these Q-functions satisfy some functional relations, also called QQ-relations, which we report here

$$\begin{aligned} &Q_1(\theta+i\pi,q_1,q_2,-q_3)\,Q_1(\theta,-q_1,-q_2,q_3) - Q_1(\theta,q_1,q_2,q_3)\,Q_1(\theta+i\pi,-q_1,-q_2,-q_3) \\ &\quad = 2i\,(q_1+q_2)\,e^{i\pi q_3} \end{aligned} \tag{3.31}$$

$$Q_2(\theta,-q_2,-q_1,q_3)\,Q_2(\theta,q_2,q_1,q_3) - Q_2(\theta,q_1,q_2,q_3)\,Q_2(\theta,-q_1,-q_2,q_3,P) = q_1^2 - q_2^2 \tag{3.32}$$

$$Q_3(\theta,q_2,q_1,q_3)\,Q_3(\theta+i\pi,q_1,q_2,-q_3) - Q_3(\theta,q_1,q_2,q_3)\,Q_3(\theta+i\pi,q_2,q_1,-q_3) = -2i\,(q_1-q_2)\,e^{i\pi q_3} \tag{3.33}$$

This QQ-system allows us to work only on Q_1 and Q_2, since Q_3 can be obtained by

$$Q_3(\theta,q_1,q_2,q_3) = \frac{1}{q_1+q_2}\left[Q_1(\theta,-q_1,-q_2,q_3)\,Q_2(\theta,q_1,q_2,q_3) - Q_1(\theta,q_1,q_2,q_3)\,Q_2(\theta,-q_2,-q_1,q_3)\right] \tag{3.34}$$

Q-functions are objects that reveal a lot about the structure of the equation and consequently about the properties of the correspondent theories.
For this reason, we will work on them starting from the easier case $N_f = 0$

4 Q-function for $N_f = 0$

In this chapter, we will follow mainly [7] with the eyes on the generalization for the more complicated $N_f = 3$. From (3.2) the Riccati equation is found through the eikonal ansatz $\Pi(y) = \frac{d}{dy}\ln(\psi(y))$:

$$\Pi(y)^2 + \frac{d}{dy}\Pi(y) = V(y) \tag{4.1}$$

The behaviors at the singularities are

$$\Pi(y) = \pm e^{-\frac{y}{2}+\theta} + \frac{1}{4} \pm \frac{4P^2-1}{32e^\theta}e^{\frac{y}{2}} - \frac{4P^2-1}{64e^{2\theta}}e^y + O(e^{\frac{3}{2}y}) \tag{4.2}$$

For $y \to -\infty$ and

$$\Pi(y) = \pm e^{\frac{y}{2}+\theta} - \frac{1}{4} \pm \frac{4P^2-1}{32e^\theta}e^{-\frac{y}{2}} + \frac{4P^2-1}{64e^{2\theta}}e^{-y} + O(e^{-\frac{3}{2}y}) \tag{4.3}$$

for $y \to +\infty$

We look now at the Floquet solutions, that are functions that diverge at both singular points and follow the property

$$\psi_\pm(y+2\pi i) = e^{\pm 2\pi ik}\psi_\pm(y) \tag{4.4}$$

Where k is the Floquet exponent that will be proportional to the gauge period

$$\pm 2\pi ik = \int_y^{y+2\pi i} dy'\Pi_\pm(y') \tag{4.5}$$

The quantum momentum of the Floquet solutions $\Pi_\pm(y)$ to be divegrent need to behave as

$$\Pi_\pm(y) = +e^{\frac{y}{2}+\theta} - \frac{1}{4} + \frac{4P^2-1}{32e^\theta} e^{-\frac{y}{2}} + \frac{4P^2-1}{64e^{2\theta}} e^{-y} + O(e^{-\frac{3}{2}y}) \tag{4.6}$$

$$\Pi_\pm(y) = -e^{-\frac{y}{2}+\theta} + \frac{1}{4} - \frac{4P^2-1}{32e^\theta} e^{\frac{y}{2}} - \frac{4P^2-1}{64e^{2\theta}} e^{y} + O(e^{\frac{3}{2}y}) \tag{4.7}$$

The fact that the two Floquet solutions behave in the same way at the singularities will be crucial.
The wave functions are then constructed in this way

$$\psi_\pm(y) = e^{c^\pm} e^{2e^\theta e^{-\frac{y}{2}}} e^{\frac{y}{4}} \exp\left(\int_{-\infty}^{y} dy' \left(\Pi_\pm(y') + e^\theta e^{-\frac{y'}{2}} - \frac{1}{4}\right)\right) \tag{4.8}$$

Here $c^\pm$ is a constant of the integration and we have added a regulator in the exponent.
This clearly behaves as

$$\psi_\pm(y) \approx e^{c^\pm} e^{\frac{y}{4}} e^{2e^\theta e^{-\frac{y}{2}}} \qquad y \to -\infty \tag{4.9}$$

If we want to catch the behavior at $y = +\infty$ we have to manipulate the integral as follows

$$\begin{aligned}\int_{-\infty}^{y} dy' \left(\Pi_\pm(y') + e^\theta e^{-\frac{y'}{2}} - \frac{1}{4}\right) = \int_{-\infty}^{0} dy' \left(\Pi_\pm(y') + e^\theta e^{-\frac{y'}{2}} - \frac{1}{4}\right) + \\ + \int_{0}^{y} dy' \left(\Pi_\pm(y') - e^\theta e^{\frac{y'}{2}} + \frac{1}{4}\right) - \frac{y}{2}\end{aligned} \tag{4.10}$$

It is not easy to read, since it is highly symmetric, but the procedure is

- We divide the integration region in two (the middle point is conveniently set to zero, but could be any finite number)
- we leave the regulator for $-\infty$ for the left region
- we add and subtract the regulator for $+\infty$ for the right region

At the end, the behavior of the wavefunction at $y = +\infty$ will look like this

$$\psi_\pm(y) \approx e^{c^\pm} e^{-\frac{y}{4}} e^{2e^\theta e^{\frac{y}{2}}} e^{\varphi(\pm k)} \tag{4.11}$$

Where we have defined the acquired phase

$$\varphi(\pm k) = \int_{-\infty}^{0} dy' \left(\Pi_\pm(y') + e^\theta e^{-\frac{y'}{2}} - \frac{1}{4}\right) + \int_{0}^{+\infty} dy' \left(\Pi_\pm(y') - e^\theta e^{\frac{y'}{2}} + \frac{1}{4}\right) \equiv \varphi_<(\pm k) + \varphi_>(\pm k) \tag{4.12}$$

In [7] is showed how one can one get the the acquired phase of a Floquet solution from the on of the other solution with the change $k \to -k$. This can also be done using the parity symmetry $y \to -y$ or his generalization for higher N_f but we will stick with the first one since can be easily applied to all flavours. For $N_f = 3$, however, isn't that easy, so we prefer to use the sign change of the Floquet exponent.
The acquired phase plays an important role for the theory.
Since the Floquet solutions form a basis, we can write

$$U_0(y) = \frac{1}{s_+}\left(e^{-c^+} e^{-\varphi(k)} \psi_+(y) - e^{-c^-} e^{-\varphi(-k)} \psi_-(y)\right) \tag{4.13}$$

$$V_0(y) = \frac{1}{s_-}\left(e^{-c^+} \psi_+(y) - e^{-c^-} \psi_-(y)\right) \tag{4.14}$$

With s_+ and s_- coefficients to find. From the definition (3.6) and this expansion, we can write

$$Q(\theta) = W[U_0, V_0] = \frac{e^{-c^+ - c^- - \varphi^+ - \varphi^-}}{s_+ s_-}\left(e^{\varphi(k)} - e^{\varphi(-k)}\right) W[\psi_+, \psi_-] \tag{4.15}$$

Using then the alternative forms (3.11) and (3.12) we can fix s_+ and s_-.

$$s_+ = \frac{e^{-c^+-c^--\varphi^+-\varphi^-}}{\sqrt{2}e^{\frac{\theta}{2}}} W[\psi_+, \psi_-] \tag{4.16}$$

$$s_- = -\frac{e^{-c^+-c^-}}{\sqrt{2}e^{\frac{\theta}{2}}} W[\psi_+, \psi_-] \tag{4.17}$$

That eventually gives us

$$Q(\theta) = -\frac{2e^{\theta}e^{c^++c^-}}{W[\psi_+, \psi_-]}\left(e^{\varphi(k)} - e^{\varphi(-k)}\right) \tag{4.18}$$

4.1 Large $\theta \to -\infty$

It is interesting to look at the limit $\theta \to -\infty$ and, for finite y, we can safely do it, since the term $e^{2\theta}\cosh y$ is always smaller than P^2.
So we suppose the expansion

$$\Pi_+(y) = \sum_{n=0}^{+\infty} e^{2n\theta}\Pi_+^{(n)}(y) \tag{4.19}$$

Plugging this ansatz into the Riccati equation, we have the set of equations

$$\sum_{m=0}^{n} \Pi_+^{(m)}(y)\Pi_+^{(n-m)}(y) + \frac{d}{dy}\Pi_+^{(n)}(y) = V^{(n)}(y) \tag{4.20}$$

Where $V^{(0)}(y) = P^2$, $V^{(1)}(y) = 2\cosh y$ and $V^{(n)}(y) = 0$ for $n \geq 2$. $\Pi_+^{(n)}(y)$ is a sum from $-n$ to n of powers of e^{ny} with the same parity of n

$$\Pi_+^{(0)}(y) = P \tag{4.21}$$

$$\Pi_+^{(1)}(y) = \frac{e^{-y}}{2P-1} + \frac{e^{y}}{2P+1} \tag{4.22}$$

$$\Pi_+^{(2)}(y) = \frac{e^{-2y}}{(2P-1)^2(2-2P)} - \frac{1}{P(4P^2-1)} - \frac{e^{2y}}{(2P+1)^2(2+2P)} \tag{4.23}$$

We recall that we have

$$2\pi i k = \int_y^{y+2\pi i} dy'\Pi_+(y') \tag{4.24}$$

We can also obtain k by means of integration in the complex plane z on a circumference of unitary radius.

$$2\pi i k = i\int_0^{2\pi} dy'\Pi_+(y = iy') = \oint \frac{dz}{z}\Pi_+(z = e^y) = \operatorname*{Res}_{z=0}\left(\frac{1}{z}\Pi_+(z)\right) \tag{4.25}$$

Now, since k receives contributions only from constant terms, then only even n terms contribute, and we have

$$k(\theta, P) = \sum_{n=0}^{+\infty} e^{4n\theta}\operatorname*{Res}_{z=0}\left(\frac{1}{z}\Pi_+^{(2n)}(z)\right) \tag{4.26}$$

For the first orders, we have

$$k(\theta, P) = P - \frac{e^{4\theta}}{P(4P^2-1)} + \frac{e^{8\theta}(-60P^4+35P^2-2)}{4(4P^2-1)^3(P^2-1)P^3} + O(e^{12\theta}) \tag{4.27}$$

This relation can also be inverted

$$P^2 = \sum_{n=0}^{+\infty} p_0^{(n)}(k)e^{4n\theta} = k^2 + \frac{2}{4k^2-1}e^{4\theta} + O(e^{8\theta}) \tag{4.28}$$

From now on, we will use this expression for P^2

4.2 The kink trick

However, for $y \to +\infty$, the term $\frac{1}{2}e^{2\theta}e^{y}$ is ambiguous, so we can do the shift $y \to y - 2\theta$, inspired by the "kink trick" of TBA.
The Riccati equation now becomes

$$\Pi_{>}(y)^2 + \frac{d}{dy}\Pi_{>}(y) = e^{y} + P^2 + e^{4\theta}e^{-y} \tag{4.29}$$

Where $\Pi_{>}(y) = \Pi(y - 2\theta)$. Now we can safely do the limit $y \to +\infty$ that is present, for example, in $\varphi_{>}(k)$.
Again we can solve this equation with the ansatz $\Pi_{>}(y) = \sum_{n=0}^{+\infty} e^{4n\theta}\Pi_{>}^{(n)}(y)$, that plugged into (4.29) leads to

$$\sum_{m=0}^{n} \Pi_{>}^{(m)}(y)\Pi_{>}^{(n-m)}(y) + \frac{d}{dy}\Pi_{>}^{(n)}(y) = V_{>}^{(n)}(y) \tag{4.30}$$

Where $V_{>}^{(0)}(y) = e^{y} + k^2$, $V_{>}^{(1)}(y) = e^{-y} + p_2^{(1)}(k)$ and $V_{>}^{(n)}(y) = p_0^{(n)}(k)$ for $n \geq 2$. The solution for $\Pi_{>}^{(0)}(y)$ is

$$\Pi_{>}^{(0)}(y) = \frac{d}{dy}\ln\left(a_{+}J_{2k}\left(2ie^{\frac{y}{2}}\right) + a_{-}\ln J_{-2k}\left(2ie^{\frac{y}{2}}\right)\right) \tag{4.31}$$

We can state that $\psi_{\pm}^{(0)}(y) = J_{\pm 2k}\left(2ie^{\frac{y}{2}}\right)$ this is because we have

$$\psi_{\pm>}^{(0)}(y + 2\pi i) = J_{\pm 2k}\left(2ie^{\frac{y}{2}}e^{\pi i}\right) = e^{\pm 2k\pi i}J_{\pm 2k}\left(2ie^{\frac{y}{2}}\right) = e^{\pm 2k\pi i}\psi_{\pm>}^{(0)}(y) \tag{4.32}$$

Where we used the following property of the Bessel function that works for $m \in \mathbf{Z}$

$$J_{\nu}\left(ze^{m\pi i}\right) = e^{m\pi i\nu}J_{\nu}(z) \tag{4.33}$$

So we have

$$\Pi_{+>}^{(0)}(y) = \frac{d}{dy}\ln J_{2k}\left(2ie^{\frac{y}{2}}\right) \tag{4.34}$$

$$\Pi_{->}^{(0)}(y) = \frac{d}{dy}\ln J_{-2k}\left(2ie^{\frac{y}{2}}\right) \tag{4.35}$$

For $y \to -\infty$ we have the same issue, so we do the shift $y \to y + 2\theta$. The same procedure leads to

$$\Pi_{+<}^{(0)}(y) = \frac{d}{dy}\ln J_{-2k}\left(2ie^{-\frac{y}{2}}\right) \tag{4.36}$$

$$\Pi_{-<}^{(0)}(y) = \frac{d}{dy}\ln J_{2k}\left(2ie^{-\frac{y}{2}}\right) \tag{4.37}$$

With $\sum_{n=0}^{+\infty} e^{4n\theta}\Pi_{<}^{(n)}(y) = \Pi_{<}(y) = \Pi(y + 2\theta)$, in (A) we found $\Pi_{<}^{(n)}(y)$ and $\Pi_{>}^{(n)}(y)$ to be

$$\Pi_{>}^{(n)}(y) = \frac{d}{dy}\left(P_0^{(n)}(y) + \sum_{i=1}^{n} P_i^{(n)}(y)\frac{d^{i-1}}{dy^{i-1}}\Pi_{>}^{(0)}(y)\right) \tag{4.38}$$

$$\Pi_{<}^{(n)}(y) = \frac{d}{dy}\left(P_0^{(n)}(-y) + \sum_{i=1}^{n} P_i^{(n)}(-y)\frac{d^{i-1}}{dy^{i-1}}\Pi_{<}^{(0)}(y)\right) \tag{4.39}$$

4.3 Computing $\varphi(k)$

We can now compute $\varphi(k)$

$$
\begin{aligned}
\varphi(k) &= \int_{-\infty}^{0} dy' \left(\Pi_+(y') + e^{\theta} e^{-\frac{y'}{2}} - \frac{1}{4}\right) + \int_{0}^{+\infty} dy' \left(\Pi_+(y') - e^{\theta} e^{\frac{y'}{2}} + \frac{1}{4}\right) = \\
&= \int_{-\infty}^{-2\theta} dy' \left(\Pi_{<+}(y') + e^{-\frac{y'}{2}} - \frac{1}{4}\right) + \int_{2\theta}^{+\infty} dy' \left(\Pi_{>+}(y') - e^{\frac{y'}{2}} + \frac{1}{4}\right) = \\
&= - \left[\ln J_{-2k}\left(2ie^{-\frac{y}{2}}\right) - 2e^{-\frac{y}{2}} - \frac{y}{4}\right]_{y=-\infty} + \left[\ln J_{2k}\left(2ie^{\frac{y}{2}}\right) - 2e^{\frac{y}{2}} + \frac{y}{4}\right]_{y=+\infty} + \\
&\quad + \ln J_{-2k}\left(2ie^{\theta}\right) - 2e^{\theta} + \frac{\theta}{2} - \ln J_{2k}\left(2ie^{\theta}\right) + 2e^{\theta} - \frac{\theta}{2} + S_>(\theta,k) + S_<(\theta,k)
\end{aligned}
\tag{4.40}
$$

We have defined

$$
S_<(\theta,k) = \sum_{n=1}^{+\infty} e^{4n\theta} \int_{-\infty}^{-2\theta} dy' \Pi^{(n)}_{+<}(y') \tag{4.41}
$$

$$
S_>(\theta,k) = \sum_{n=1}^{+\infty} e^{4n\theta} \int_{2\theta}^{+\infty} dy' \Pi^{(n)}_{+>}(y') \tag{4.42}
$$

Now if we use the following expansion

$$
J_{\pm 2k}\left(2ie^{\frac{y}{2}}\right) \approx \frac{e^{\pm i\pi k} e^{-\frac{y}{4}} e^{2e^{\frac{y}{2}}}}{2\sqrt{\pi}} \qquad y \to +\infty \tag{4.43}
$$

We arrive to

$$
\varphi(k) = \left(i\pi k - \ln 2\sqrt{\pi}\right) - \left(-i\pi k - \ln 2\sqrt{\pi}\right) + \ln J_{-2k}\left(2ie^{\theta}\right) - \ln J_{2k}\left(2ie^{\theta}\right) + S_>(\theta,k) + S_<(\theta,k) \tag{4.44}
$$

This finally becomes

$$
\varphi(k) = 2i\pi k + \ln J_{-2k}\left(2ie^{\theta}\right) - \ln J_{2k}\left(2ie^{\theta}\right) + S_>(\theta,k) + S_<(\theta,k) \tag{4.45}
$$

We also have that

$$
\Pi^{(n)}_<(y,k) = -\Pi^{(n)}_>(-y,-k) \tag{4.46}
$$

This tells us that

$$
\begin{aligned}
S_>(\theta,k) &= \sum_{n=1}^{+\infty} e^{4n\theta} \int_{2\theta}^{+\infty} dy' \Pi^{(n)}_{+>}(y') = \\
&= -\sum_{n=1}^{+\infty} e^{4n\theta} \int_{2\theta}^{+\infty} dy' \Pi^{(n)}_{-<}(-y') = \\
&= -\sum_{n=1}^{+\infty} e^{4n\theta} \int_{-\infty}^{-2\theta} dy' \Pi^{(n)}_{-<}(y') = -S_<(\theta,-k)
\end{aligned}
\tag{4.47}
$$

So we have

$$
S_>(\theta,k) = -S_<(\theta,-k) \tag{4.48}
$$

And thus

$$
\varphi(k) = 2i\pi k + \ln J_{-2k}\left(2ie^{\theta}\right) - \ln J_{2k}\left(2ie^{\theta}\right) + S_>(\theta,k) - S_>(\theta,-k) \tag{4.49}
$$

Both the Bessel functions and the functions $S(\theta,k)$ are series in powers of $e^{2\theta}$, expanding both functions, and using the explicit form of $\Pi^{(n)}_{+>}(y)$ that we give in (A) we get

$$
\varphi(k) = 2i\pi k + -4\theta k + \ln\frac{\Gamma(1+2k)}{\Gamma(1-2k)} + \frac{8k}{(1-4k^2)^2} e^{4\theta} + O(e^{8\theta}) \tag{4.50}
$$

This can be recognized to be A_D for Seiberg-Witten theory for $\mathcal{N} = 2$ $SU(2)$ $N_f = 0$

4.4 Wronskian from decaying solution matching

We constructed $\psi^{(0)}_{\pm<}(y)$ from the equation

$$\frac{d^2}{dy^2}\psi^{(0)}_{<\pm}(y) = V^{(0)}_{<}\psi^{(0)}_{<\pm}(y) \tag{4.51}$$

With the 2 solutions that both diverge at $y \to -\infty$

$$\psi^{(0)}_{\pm<}(y) = J_{\mp 2k}(2ie^{-\frac{y}{2}}) \tag{4.52}$$

From that we can construct another solution

$$\psi^{(0)}_{0<}(y) = b_+\psi^{(0)}_{+<}(y) - b_-\psi^{(0)}_{-<}(y) \tag{4.53}$$

We can choose the two constants so that this wavefunction decays at $y \to -\infty$.
For example, we choose

$$b_\pm = \frac{\pi}{2}\frac{e^{\pm i\pi k}}{\sin 2\pi k} \tag{4.54}$$

In that way we have

$$\psi^{(0)}_{0<}(y) = \frac{\pi}{2}\frac{e^{i\pi k}J_{-2k}(2ie^{-\frac{y}{2}}) - e^{-i\pi k}J_{2k}(2ie^{-\frac{y}{2}})}{\sin(2\pi k)} = K_{2k}(2e^{-\frac{y}{2}}) \tag{4.55}$$

And from that we can construct the momentum

$$\Pi^{(0)}_{0<}(y) = \frac{d}{dy}\ln\psi^{(0)}_{0<}(y) = \frac{d}{dy}\ln K_{2k}(2e^{-\frac{y}{2}}) \tag{4.56}$$

that is solution of the Riccati equation at the first order.
Now from this last function we can construct the total wavefunction in this way

$$\begin{aligned} V_0(y) &= e^{c_0}e^{-2e^{\theta-\frac{y}{2}}}e^{\frac{y}{4}}\exp\left(\int_{-\infty}^{y}dy'\left(\Pi_0(y') - e^{-\frac{y'}{2}+\theta} - \frac{1}{4}\right)\right) \\ &= e^{c_0}e^{-2e^{\theta-\frac{y}{2}}}e^{\frac{y}{4}}\exp\left(\int_{-\infty}^{y-2\theta}dy'\left(\Pi_{0<}(y') - e^{-\frac{y'}{2}} - \frac{1}{4}\right)\right) \end{aligned} \tag{4.57}$$

With the definition $\Pi_{0<}(y) = \sum_{n=0}^{+\infty}e^{4n\theta}\Pi^{(n)}_{0<}(y)$. $\Pi^{(n)}_{0<}(y)$ are created with the usual formula

$$\Pi^{(n)}_{0<}(y) = \frac{d}{dy}\left(P^{(n)}_0(y) + \sum_{i=1}^{n}P^{(n)}_i(y)\frac{d^{i-1}}{dy^{i-1}}\Pi^{(0)}_{0<}(y)\right) \tag{4.58}$$

Polynomials are the same, since the two solutions satisfy the same set of equations.
We can take the $y \to -\infty$ limit and match it with the limit for $V_0(y)$, this will set $e^{c_0} = \frac{e^{-\frac{\theta}{2}}}{\sqrt{2}}$.
We can then manipulate the wavefunction as follows

$$\begin{aligned} V_0(y) =& e^{c_0}e^{-2e^{\theta-\frac{y}{2}}}e^{\frac{y}{4}}\exp\left(\int_{-\infty}^{y-2\theta}dy'\left(\Pi_{0<}(y') - e^{-\frac{y'}{2}} - \frac{1}{4}\right)\right) = \\ =& e^{c_0}e^{-2e^{\theta-\frac{y}{2}}}e^{\frac{y}{4}}\exp\left(\int_{-\infty}^{y-2\theta}dy'\left(\sum_{n=0}^{+\infty}e^{4n\theta}\Pi^{(n)}_{0<}(y) - e^{-\frac{y'}{2}} - \frac{1}{4}\right)\right) = \\ =& \frac{2e^{c_0}e^{\frac{\theta}{2}}}{\sqrt{\pi}}K_{2k}\left(2e^{-\frac{y}{2}+\theta}\right)\exp\left(\sum_{n=1}^{+\infty}e^{4n\theta}\left(P^{(n)}_0(y-2\theta) + \sum_{i=1}^{n}P^{(n)}_i(y-2\theta)\left[\frac{d^{i-1}}{dy'^{i-1}}\Pi^{(0)}_{0<}(y')\right]_{y'=y-2\theta}\right)\right) = \\ =& \frac{2e^{c_0}e^{\frac{\theta}{2}}}{\sqrt{\pi}}K_{2k}\left(2e^{-\frac{y}{2}+\theta}\right)\exp\left(\sum_{n=1}^{+\infty}e^{4n\theta}P^{(n)}_0(y-2\theta)\right)\exp\left(\sum_{j=1}^{+\infty}\left[\frac{d^{j-1}}{dy'^{j-1}}\Pi^{(0)}_{0<}(y')\right]_{y'=y-2\theta}\sum_{n=j}^{+\infty}e^{4n\theta}P^{(n)}_j(y-2\theta)\right) \end{aligned} \tag{4.59}$$

Now we can use the following (A.23) and

$$\sum_{n=j}^{+\infty} \frac{t^n}{n!} B_{n,j}(x_1, ..., x_{n-k+1}) = \frac{1}{j!} \left(\sum_{n=1}^{+\infty} x_n \frac{t^n}{n!} \right)^j \tag{4.60}$$

To write

$$\sum_{j=1}^{+\infty} \left[\frac{d^{j-1}}{dy'^{j-1}} \Pi^{(0)}_{0<}(y') \right]_{y'=y-2\theta} \sum_{n=j}^{+\infty} e^{4n\theta} P^{(n)}_j (y-2\theta) = \sum_{j=1}^{+\infty} \left[\frac{d^{j-1}}{dy'^{j-1}} \Pi^{(0)}_{0<}(y') \right]_{y'=y-2\theta} \frac{1}{j!} \left(\sum_{n=1}^{+\infty} e^{4n\theta} P^{(n)}_1 (y-2\theta) \right)^j \tag{4.61}$$

Now, we can substitute the explicit form of the momentum to get

$$V_0(y) = \frac{2e^{c_0} e^{\frac{\theta}{2}}}{\sqrt{\pi}} \exp\left(\sum_{n=1}^{+\infty} e^{4n\theta} P^{(n)}_0 (y-2\theta) \right) \exp\left(\sum_{j=0}^{+\infty} \left[\frac{d^j}{dy'^j} \ln K_{2k}(2e^{-\frac{y'}{2}}) \right]_{y'=y-2\theta} \frac{1}{j!} \left(\sum_{n=1}^{+\infty} e^{4n\theta} P^{(n)}_1 (y-2\theta) \right)^j \right) \tag{4.62}$$

This reconstructs a taylor series at the exponent

$$f(x) = \sum_{j=0}^{+\infty} \frac{f^{(j)}(x_0)}{j!} (x-x_0)^j \tag{4.63}$$

With the definitions

$$x_0 = y - 2\theta \tag{4.64}$$

$$x - x_0 = \sum_{n=1}^{+\infty} e^{4n\theta} P^{(n)}_1 (y-2\theta) \tag{4.65}$$

$$f(x) = \ln K_{2k}\left(2e^{-\frac{x}{2}}\right) \tag{4.66}$$

That means that we have

$$\begin{aligned} V_0(y) =& \frac{2e^{c_0} e^{\frac{\theta}{2}}}{\sqrt{\pi}} \exp\left(\sum_{n=1}^{+\infty} e^{4n\theta} P^{(n)}_0 (y-2\theta) \right) \exp\left(\ln K_{2k} \left(2e^{-\frac{y}{2}+\theta} \exp\left(-\frac{1}{2} \sum_{n=1}^{+\infty} e^{4n\theta} P^{(n)}_1 (y-2\theta) \right) \right) \right) = \\ =& \frac{2e^{c_0} e^{\frac{\theta}{2}}}{\sqrt{\pi}} \exp\left(\sum_{n=1}^{+\infty} e^{4n\theta} P^{(n)}_0 (y-2\theta) \right) K_{2k} \left(2e^{-\frac{y}{2}+\theta} \exp\left(-\frac{1}{2} \sum_{n=1}^{+\infty} e^{4n\theta} P^{(n)}_1 (y-2\theta) \right) \right) \end{aligned} \tag{4.67}$$

We can also define

$$L(y-2\theta) = y - 2\theta + \sum_{n=1}^{+\infty} e^{4n\theta} P^{(n)}_1 (y-2\theta) \tag{4.68}$$

In this way, we have

$$V_0(y) = \frac{2e^{c_0} e^{\frac{\theta}{2}}}{\sqrt{\pi}} \exp\left(\sum_{n=1}^{+\infty} e^{4n\theta} P^{(n)}_0 (y-2\theta) \right) K_{2k}\left(2e^{-\frac{1}{2}L(y-2\theta)} \right) \tag{4.69}$$

On the other hand, we recall the relation

$$V_0 = \frac{1}{s_-} \left(e^{-c^+} \psi_+(y) - e^{-c^-} \psi_-(y) \right) \tag{4.70}$$

With the definition

$$\begin{aligned} \psi_+(y) =& e^{c^+} e^{2e^{\theta} e^{-\frac{y}{2}}} e^{\frac{y}{4}} \exp\left(\int_{-\infty}^{y-2\theta} dy' \left(\Pi_{<+}(y') + e^{-\frac{y'}{2}} - \frac{1}{4} \right) \right) = \\ =& e^{c^+} 2\sqrt{\pi} e^{\frac{\theta}{2}} e^{i\pi k} \exp\left(\ln J_{-2k}(2ie^{-\frac{y}{2}+\theta}) + \sum_{n=1}^{+\infty} e^{4n\theta} \left(P^{(n)}_0 (y-2\theta) + \sum_{i=1}^{n} P^{(n)}_i (y-2\theta) \left[\frac{d^{i-1}}{dy'^{i-1}} \Pi^{(0)}_{+<}(y') \right]_{y'=y-2\theta} \right) \right) \end{aligned} \tag{4.71}$$

Now, if we follow the same steps as before, we get

$$\psi_+(y) = e^{c^+} e^{\frac{\theta}{2}} 2\sqrt{\pi} e^{i\pi k} \exp\left(\sum_{n=1}^{+\infty} e^{4n\theta} P_0^{(n)}(y-2\theta)\right) J_{-2k}\left(2ie^{-\frac{1}{2}L(y-2\theta)}\right) \tag{4.72}$$

And for the other Floquet we have

$$\psi_-(y) = e^{c^-} e^{\frac{\theta}{2}} 2\sqrt{\pi} e^{-i\pi k} \exp\left(\sum_{n=1}^{+\infty} e^{4n\theta} P_0^{(n)}(y-2\theta)\right) J_{2k}\left(2ie^{-\frac{1}{2}L(y-2\theta)}\right) \tag{4.73}$$

That leads to

$$\begin{aligned} V_0(y) &= -\frac{\sqrt{2}e^{\frac{\theta}{2}} e^{c^+ + c^-}}{W[\psi_+, \psi_-]} \left(e^{-c^+}\psi_+(y) - e^{-c^-}\psi_-(y)\right) = \\ &= -\frac{e^{\theta} 2\sqrt{2\pi} e^{c^+ + c^-}}{W[\psi_+, \psi_-]} \exp\left(\sum_{n=1}^{+\infty} e^{4n\theta} P_0^{(n)}(y-2\theta)\right)\left(e^{i\pi k} J_{-2k}\left(2ie^{-\frac{1}{2}L(y-2\theta)}\right) - e^{-i\pi k} J_{2k}\left(2ie^{-\frac{1}{2}L(y-2\theta)}\right)\right) = \\ &= -\frac{e^{\theta} 2\sqrt{2\pi} e^{c^+ + c^-}}{W[\psi_+, \psi_-]} \frac{2\sin(2\pi k)}{\pi} \exp\left(\sum_{n=1}^{+\infty} e^{4n\theta} P_0^{(n)}(y-2\theta)\right) K_{2k}\left(2e^{-\frac{1}{2}L(y-2\theta)}\right) \end{aligned} \tag{4.74}$$

If we compare it with (4.69), we get

$$W[\psi_+, \psi_-] = -e^{c_+ + c_-} 4e^{\theta} \sin(2\pi k) \tag{4.75}$$

4.5 Computing $Q(\theta)$

We have all the ingredients to compute $Q(\theta)$ (4.18), in addition, using the symmetry (4.48) we can write

$$e^{\varphi(k)} - e^{\varphi(-k)} = 2\sinh\varphi(k) \tag{4.76}$$

This ultimately leads to

$$Q(\theta) = \frac{\sinh\varphi(k)}{\sin(2\pi k)} \tag{4.77}$$

This coincides with the result obtain in [7]

5 Q-function for $N_f = 3$

5.1 Floquet basis and the acquired phases

We now do the same steps for the $N_f = 3$ theory.
We take the Riccati solution of the CHE (3.14) with the following divergent behaviors

$$\Pi(y) \approx -e^{\theta}e^{-y} + \frac{1}{2} - q_3 \qquad for \qquad y \to -\infty \tag{5.1}$$

$$\Pi(y) \approx \frac{1-(q_1+q_2)}{e^y - 2} \qquad for \qquad y \to \ln 2 \tag{5.2}$$

$$\Pi(y) \approx \frac{q_1 - q_2}{2} \qquad for \qquad y \to +\infty \tag{5.3}$$

In this way we can construct the Floquet solutions, that satisfy the quasi-periodic condition $\psi_\pm(y+2\pi i) = e^{\pm 2\pi i k}\psi_\pm(y)$.
These are constructed as follows

$$\psi_\pm(y) = e^{c^\pm} e^{e^{\theta - y} + \left(\frac{1}{2} - q_3\right)y} \cdot \exp\left(\int_{-\infty}^{y} dy' \left(\Pi_\pm(y') + e^{\theta}e^{-y'} - \left(\frac{1}{2} - q_3\right)\right)\right) \tag{5.4}$$

with $c^\pm$ constants.
The Floquet exponent k is given by

$$\pm 2\pi i k = \int_y^{y+2\pi i} dy' \Pi_\pm(y') \tag{5.5}$$

The wavefunction at the irregular singularity behaves as we want

$$\psi_\pm(y) \approx e^{c^\pm} e^{e^{\theta-y}+\left(\frac{1}{2}-q_3\right)y} \qquad y \to -\infty \tag{5.6}$$

In order to catch the behavior at the other singularities, we need to manipulate the integrand as follows

$$\begin{aligned}\int_{-\infty}^{y} dy' \left(\Pi_\pm(y') + e^{\theta-y'} - \left(\frac{1}{2} - q_3\right)\right) &= \int_{-\infty}^{a} dy' \left(\Pi_\pm(y') + e^{\theta-y'} - \left(\frac{1}{2} - q_3\right)\right) + \\ + \int_a^y dy' \left(\Pi_\pm(y') - \frac{1-(q_1+q_2)}{e^{y'}-2}\right) &+ \int_a^y dy' \left(\frac{1-(q_1+q_2)}{e^{y'}-2} + e^{\theta-y'} - \left(\frac{1}{2} - q_3\right)\right)\end{aligned} \tag{5.7}$$

where $-\infty < a < \ln 2$, moreover, the last integral can be computed to be

$$\int_a^y dy' \left(\frac{1-(q_1+q_2)}{e^{y'}-2} + e^{\theta-y'} - \left(\frac{1}{2} - q_3\right)\right) = -e^{\theta-y} - \left(\frac{1}{2} - q_3\right) y + \frac{1-(q_1+q_2)}{2} \left(\ln|e^y - 2| - y\right) + \tag{5.8}$$

$$+ e^{\theta-a} + \left(\frac{1}{2} - q_3\right) a - \frac{1-(q_1+q_2)}{2} \left(\ln|e^a - 2| - a\right) \tag{5.9}$$

The behavior at $y = \ln 2$ is then

$$\psi_\pm(y) \approx e^{c^\pm} (e^y - 2)^{\frac{1-(q_1+q_2)}{2}} e^{\varphi(\pm k)} \qquad y \to \ln 2 \tag{5.10}$$

Where $\varphi(k)$ is the acquired phase from $-\infty$ to $\ln 2$

$$\begin{aligned}\varphi(\pm k) = &\int_{-\infty}^{a} dy' \left(\Pi_\pm(y') + e^{\theta-y'} - (\frac{1}{2} - q_3)\right) + \int_a^{\ln 2} dy' \left(\Pi_\pm(y') + \frac{-1+q_1+q_2}{e^{y'}-2}\right) + \\ &\frac{-1+q_1+q_2}{2} \ln 2 + \left(\frac{1-q_1-q_2}{2} + \frac{1}{2} - q_3\right) a + e^{\theta-a} + \frac{-1+q_1+q_2}{2} \ln|e^a - 2|\end{aligned} \tag{5.11}$$

As shown in [7], when sending $k \to -k$ one gets $\Pi_+(y) \to \Pi_-(y)$.
We can also define the acquired phase from $\ln 2$ to $+\infty$, given by

$$\begin{aligned}\varphi_2(k) = &+ \int_{\ln 2}^{y_*} dy' \left(\Pi_+(y') + \frac{-1+q_1+q_2}{e^{y'}-2}\right) + \int_{y*}^{+\infty} dy' \left(\Pi_+(y') - \frac{q_1-q_2}{2}\right) + \\ &- \frac{q_1-q_2}{2} y_* + \frac{-1+q_1+q_2}{2} \left(y_* - \ln|e^{y_*} - 2| - \ln 2\right)\end{aligned} \tag{5.12}$$

where $\ln 2 < y_* < +\infty$. This gives us the behavior

$$\psi_\pm(y) \approx e^{c^\pm} e^{\frac{q_1-q_2}{2} y} e^{\varphi(\pm k)+\varphi_2(\pm k)} \qquad y \to +\infty \tag{5.13}$$

Since the Floquet solutions $\psi_\pm(y)$ form a basis, we can expand the regular solutions in terms of the Floquet basis

$$\psi_{-,0}(y) = \frac{1}{s_-} \left(e^{-c^+} \psi_+(y) - e^{-c^-} \psi_-(y)\right) \tag{5.14}$$

$$\psi_{0,0}(y) = \frac{1}{s_0} \left(e^{-c^+} e^{-\varphi(k)} \psi_+(y) - e^{-c^-} e^{-\varphi(-k)} \psi_-(y)\right) \tag{5.15}$$

$$\psi_{+,0}(y) = \frac{1}{s_+} \left(e^{-c^+} e^{-\varphi(k)-\varphi_2(k)} \psi_+(y) - e^{-c^-} e^{-\varphi(-k)-\varphi_2(-k)} \psi_-(y)\right) \tag{5.16}$$

We can fix the normalization constants as $c^{\pm} = -\frac{\varphi(\pm k)}{2}$ so we have

$$\psi_{-,0}(y) = \frac{1}{s_-}\left(e^{\frac{\varphi(k)}{2}}\psi_+(y) - e^{\frac{\varphi(-k)}{2}}\psi_-(y)\right) \tag{5.17}$$

$$\psi_{0,0}(y) = \frac{1}{s_0}\left(e^{-\frac{\varphi(k)}{2}}\psi_+(y) - e^{-\frac{\varphi(-k)}{2}}\psi_-(y)\right) \tag{5.18}$$

$$\psi_{+,0}(y) = \frac{1}{s_+}\left(e^{-\frac{\varphi(k)}{2}-\varphi_2(k)}\psi_+(y) - e^{-\frac{\varphi(-k)}{2}-\varphi_2(-k)}\psi_-(y)\right) \tag{5.19}$$

In order to find s_i we use the definitions $Q_i(\theta)$ given in (3.22) (3.23) (3.24).
For example, we can explicitly write $Q_1(\theta)$ using (5.17) and (5.18)

$$Q_1(\theta) = \frac{e^{-\frac{\varphi(k)}{2}}e^{-\frac{\varphi(-k)}{2}}}{s_0 s_-}\left(e^{\varphi(k)} - e^{\varphi(-k)}\right)W[\psi_+, \psi_-] \tag{5.20}$$

Using the form (3.28) and the definition (5.18) we can also write

$$Q_1(\theta) = -\frac{2e^{-(q_3-\frac{1}{2})\theta}}{s_0}e^{-\varphi(k)}e^{-\varphi(-k)}\left(e^{\varphi(k)} - e^{\varphi(-k)}\right) \tag{5.21}$$

If we compare it with (5.20), we get

$$\frac{1}{s_-} = \frac{-2e^{-(q_3-\frac{1}{2})\theta}e^{-\frac{\varphi(k)}{2}}e^{-\frac{\varphi(-k)}{2}}}{W[\psi_+, \psi_-]} \tag{5.22}$$

This procedure can also be done for Q_2 and Q_3 to find all s_i. One finds

$$Q_1(\theta) = \frac{1}{W[\psi_+, \psi_-]}4\sqrt{2}(q_1+q_2)e^{-(q_3-\frac{1}{2})\theta}\sinh(\Phi) \tag{5.23}$$

$$Q_2(\theta) = -\frac{e^{\frac{\varphi(k)+\varphi(-k)+\varphi_2(k)+\varphi_2(-k)}{2}}}{W[\psi_+, \psi_-]}2\sqrt{2}(q_1+q_2)(q_1-q_2)\sinh(\Phi_2) \tag{5.24}$$

$$Q_3(\theta) = -\frac{e^{\frac{\varphi_2(k)+\varphi_2(-k)}{2}}}{W[\psi_+, \psi_-]}4e^{-(q_3-\frac{1}{2})\theta}(q_1-q_2)\sinh(\Phi_3) \tag{5.25}$$

with

$$\Phi = \frac{\varphi(k)-\varphi(-k)}{2} \tag{5.26}$$

$$\Phi_2 = \frac{\varphi_2(k)-\varphi_2(-k)}{2} \tag{5.27}$$

$$\Phi_3 = \Phi + \Phi_2 \tag{5.28}$$

We want to point out that we could have found the formula (5.25) for Q_3 by inserting the formulae for Q_1 and Q_2 inside the (3.34).
Lastly, we can write

$$\psi_{-,0}(y) = \frac{2e^{-(q_3-\frac{1}{2})\theta}}{W[\psi_+, \psi_-]}e^{-\frac{\varphi(k)}{2}}\left(\psi_-(y) - e^{\Phi}\psi_+(y)\right) \tag{5.29}$$

$$\psi_{0,0}(y) = \frac{\sqrt{2}(q_1+q_2)}{W[\psi_+, \psi_-]}e^{\frac{\varphi(k)}{2}}\left(\psi_-(y) - e^{-\Phi}\psi_+(y)\right) \tag{5.30}$$

$$\psi_{+,0}(y) = \frac{(q_2-q_1)e^{\frac{\varphi(k)}{2}}e^{+\varphi_2(k)}}{W[\psi_+, \psi_-]}\left(\psi_-(y) - e^{-\Phi-2\Phi_2}\psi_+(y)\right) \tag{5.31}$$

5.2 Large $\theta \to -\infty$

It is interesting to look at the limit $\theta \to -\infty$ and for finite y we can safely do it.
We suppose the expansion

$$\Pi_{\pm}(y) = \sum_{n=0}^{+\infty} e^{n\theta} \Pi_{\pm}^{(n)}(y) \tag{5.32}$$

Plugging this ansatz into the Riccati equation generates the set of equations

$$\sum_{m=0}^{n} \Pi_{\pm}^{(m)}(y)\Pi_{\pm}^{(n-m)}(y) + \frac{d}{dy}\Pi_{\pm}^{(n)}(y) = V_3^{(n)}(y) \tag{5.33}$$

This can be solved order by order after the substitution

$$P^2 = k^2 + \sum_{n=0}^{+\infty} e^{n\theta} p_3^{(n)}(k, q_1, q_2, q_3) \tag{5.34}$$

The coefficients $p_3^{(n)}(k, q_1, q_2, q_3)$ can be found similarly to the $N_f = 0$ case by inverting (5.5), expanding (5.33) we have

$$2\Pi_{+}^{(0)}(y)\Pi_{+}^{(1)}(y) + \frac{d}{dy}\Pi_{+}^{(1)}(y) = V_3^{(1)} = \frac{(q_3 - 2p_3^{(1)}(k))e^y - 6q_3 + 8q_3e^{-y} + 4p_3^{(1)}}{(e^y-2)^2} = \frac{q_3 - 2p_3^{(1)} - 4e^{-y}q_3}{(e^y-2)} \tag{5.35}$$

$$2\Pi_{+}^{(0)}(y)\Pi_{+}^{(2)}(y) + [\Pi_{+}^{(1)}(y)]^2 + \frac{d}{dy}\Pi_{+}^{(2)}(y) = V_3^{(2)} = \frac{-2p_3^{(2)}(k)e^y + 1 + 4p_3^{(2)}(k) - 4e^{-y} + 4e^{-2y}}{(e^y-2)^2} \tag{5.36}$$

$$\vdots$$

$$2\Pi_{+}^{(0)}(y)\Pi_{+}^{(n)}(y) + \frac{d}{dy}\Pi_{+}^{(n)}(y) + \sum_{m=1}^{n-1}\Pi_{+}^{(m)}(y)\Pi_{+}^{(n-m)}(y) = V_3^{(n)} = \frac{-2p_3^{(n)}e^y + 4p_3^{(n)}}{(e^y-2)^2}, \quad n \geq 3 \tag{5.37}$$

At zeroth order, we have the solution

$$\Pi_{\pm}^{(0)}(y) = \frac{d}{dy}\ln\left[(e^y-2)^{\frac{1-q_1-q_2}{2}} e^{\pm ky} {}_2F_1\left(\frac{1}{2} \pm k - q_1, \frac{1}{2} \pm k - q_2; 1 \pm 2k; \frac{e^y}{2}\right)\right] \tag{5.38}$$

This indeed satisfies the Floquet condition, we notice that doing $k \to -k$ leads to $\Pi_{\mp}^{(0)}(y)$ as we know.
For higher order we can use the following non trivial ansatz like we did in previous section

$$\Pi_{+}^{(n)}(y) = \frac{d}{dy}\left(P_0^{(n,r)}(y) + \sum_{i=1}^{n} P_i^{(n,r)}(y)\frac{d^{i-1}}{dy^{i-1}}\Pi_{+}^{(0)}(y)\right) \tag{5.39}$$

$P_j^{n,r}(y)$ turns out to be polynomials like in $N_f = 0$ (A).
When we cannot do $\theta \to -\infty$ when $y \to -\infty$, this is important when we compute $\varphi(k)$ for example.
Inspired by the "kink method" for solving TBA equations, we shift $y \to y + \theta$ and we call it the "kink shift", in this way we can do the $y \to -\infty$ limit, we have

$$\Pi_{<}(y) = \Pi(y+\theta) = \sum_{n=0}^{+\infty} e^{n\theta}\Pi_{<}^{(n)}(y) \tag{5.40}$$

and

$$V_{<}(y) = V_3(y+\theta) = \sum_{n=0}^{+\infty} e^{n\theta}V_{<}^{(n)}(y) \tag{5.41}$$

where we have

$$V_<^{(0)} = k^2 + 2q_3 e^{-y} + e^{-2y} \tag{5.42}$$

$$V_<^{(1)} = \frac{1}{8}\left[(-1+4k^2+4q_1q_2)e^y + 4(q_3+2p_3^{(1)}(k))\right] \tag{5.43}$$

$$V_<^{(2)} = \frac{1}{16}\left[(-2+4k^2+q_1^2+6q_1q_2+q_2^2)e^{2y} + 4(q_3+2p_3^{(1)}(k))e^y + 16p_3^{(2)}(k)\right] \tag{5.44}$$

$$V_<^{(3)} = \frac{1}{32}\left[(-3+4k^2+2q_1^2+8q_1q_2+2q_2^2)e^{3y} + 4(q_3+2p_3^{(1)}(k))e^{2y} + 16e^y p_3^{(2)}(k) + 32p_3^{(3)}(k)\right] \tag{5.45}$$

$$V_<^{(4)} = \frac{1}{64}\left[(-4+4k^2+3q_1^2+10q_1q_2+3q_2^2)e^{4y} + 4(q_3+2p_3^{(1)}(k))e^{3y} + 16e^{2y}p_3^{(2)}(k) + 32e^y p_3^{(3)}(k) + 64p_3^{(4)}(k)\right] \tag{5.46}$$

The n-th order Riccati becomes

$$\sum_{m=0}^{n} \Pi_<^{(m)}(y)\Pi_<^{(n-m)}(y) + \frac{d}{dy}\Pi_<^{(n)}(y) = V_<^{(n)}(y) \tag{5.47}$$

Again at zeroth order we find the two solutions

$$\Pi_{\pm<}^{(0)}(y) = \frac{d}{dy}\ln\left[e^{\pm ky}e^{-e^{-y}}\,{}_1F_1(\frac{1}{2}\mp k+q_3, 1\mp 2k, 2e^{-y})\right] \tag{5.48}$$

that again are the Floquet solutions.
For higher order, we can use the following ansatz, as we did above

$$\Pi_<^{(n)}(y) = \frac{d}{dy}\left(P_0^{(n,l)}(y) + \sum_{i=1}^{n} P_i^{(n,l)}(y)\frac{d^{i-1}}{dy^{i-1}}\Pi_<^{(0)}(y)\right) \tag{5.49}$$

Polynomials $P_j^{(n,l)}$ and $P_j^{(n,r)}$ are computed in detail in (A)

5.3 Wronskian from decaying solution matching

We start from the 2 wavefunctions that both diverge at $y \to -\infty$

$$\psi_{\pm<}^{(0)}(y) = e^{\pm ky}e^{-e^{-y}}\,{}_1F_1\left(\frac{1}{2}\mp k+q_3, 1\mp 2k, 2e^{-y}\right) \tag{5.50}$$

From that we can construct another wavefunction with the two constants so that this wavefunction decays at $y \to -\infty$

$$\psi_{-,0<}^{(0)}(y) = a_-(k)\psi_{+<}^{(0)}(y) - a_-(-k)\psi_{-<}^{(0)}(y) \tag{5.51}$$

We choose $a_-(\pm k)$ such that we have

$$\psi_{\pm<}^{(0)}(y) \approx \frac{1}{a_-(\pm k)}e^{e^{-y}}e^{\left(\frac{1}{2}-q_3\right)y} \qquad y \to -\infty \tag{5.52}$$

That explicitly is

$$a_-(k) = \frac{\Gamma\left(\frac{1}{2}-k+q_3\right)}{\Gamma\left(1-2k\right)}2^{\frac{1}{2}-k-q_3} \tag{5.53}$$

In this way, we get

$$\psi_{-,0<}^{(0)}(y) \approx e^{-e^{-y}}e^{\left(\frac{1}{2}+q_3\right)y}2^{-2q_3}\Gamma\left(\frac{1}{2}-k+q_3\right)\Gamma\left(\frac{1}{2}+k+q_3\right)\frac{\sin(2k\pi)}{\pi} \qquad y \to -\infty \tag{5.54}$$

From that we can construct the momentum

$$\Pi_{-,0<}^{(0)}(y) = \frac{d}{dy}\ln\psi_{-,0<}^{(0)}(y) \tag{5.55}$$

that is solution of the Riccati equation at the first order.
Now from this last function we can construct the total wavefunction in this way

$$\begin{aligned}\psi_{-,0}(y) &= e^{c_0} e^{-e^{\theta-y}} e^{\left(\frac{1}{2}+q_3\right)y} \exp\left(\int_{-\infty}^{y} dy' \left(\Pi_{-,0}(y') - e^{-y'+\theta} - \left(\frac{1}{2}+q_3\right)\right)\right) \\ &= e^{c_0} e^{-e^{\theta-y}} e^{\left(\frac{1}{2}+q_3\right)y} \exp\left(\int_{-\infty}^{y-\theta} dy' \left(\Pi_{-,0<}(y') - e^{-y'} - \left(\frac{1}{2}+q_3\right)\right)\right)\end{aligned} \tag{5.56}$$

With the definition $\Pi_{-,0<}(y) = \sum_{n=0}^{+\infty} e^{n\theta} \Pi_{-,0<}^{(n)}(y)$.
$\Pi_{-,0<}^{(n)}(y)$ are created with the usual formula

$$\Pi_{-,0<}^{(n)}(y) = \frac{d}{dy}\left(P_0^{(n,l)}(y) + \sum_{i=1}^{n} P_i^{(n,l)}(y) \frac{d^{i-1}}{dy^{i-1}} \Pi_{-,0<}^{(0)}(y)\right) \tag{5.57}$$

We can take the $y \to -\infty$ limit and match it with (3.16), this set $e^{c_0} = e^{-\left(\frac{1}{2}+q_3\right)\theta}$.
Using the explicit form (5.57) and (5.54) we can rearrange $\psi_{-,0}(y)$ in this way

$$\psi_{-,0}(y) = \frac{\pi 2^{2q_3} \psi_{0<}^{(0)}(y-\theta) e^{-\tilde{S}_-}}{\Gamma\left(\frac{1}{2}-k+q_3\right)\Gamma\left(\frac{1}{2}+k+q_3\right)\sin(2k\pi)} \exp\left(\sum_{n=1}^{+\infty} e^{n\theta} P_0^{(n,l)}(y-\theta)\right) \exp\left(\sum_{j=1}^{+\infty} \ln\psi_{-,0<}^{(0)}(y-\theta)^{(j)} \sum_{n=j}^{+\infty} e^{n\theta} P_j^{(n,l)}(y-\theta)\right) \tag{5.58}$$

Where

$$\ln\psi_{-,0<}^{(0)}(y-\theta)^{(j)} = \left[\frac{d^j}{dy'^j} \ln\psi_{-,0<}^{(0)}(y')\right]_{y'=y-\theta} \tag{5.59}$$

And

$$\tilde{S}_- = \left[\sum_{n=1}^{+\infty} e^{n\theta}\left(P_0^{(n,l)}(y') + \sum_{i=1}^{n} P_i^{(n,l)}(y') \frac{d^i}{dy'^i} \ln\psi_{-,0<}^{(0)}(y')\right)\right]_{y'=-\infty} \tag{5.60}$$

Now we can use (A.23) and the known mathematical relation

$$\sum_{n=j}^{+\infty} \frac{t^n}{n!} B_{n,j}(x_1, ..., x_{n-k+1}) = \frac{1}{j!}\left(\sum_{n=1}^{+\infty} x_n \frac{t^n}{n!}\right)^j \tag{5.61}$$

to write

$$\sum_{j=0}^{+\infty} \ln\psi_{-,0<}^{(0)}(y-\theta)^{(j)} \sum_{n=j}^{+\infty} e^{n\theta} P_j^{(n,l)}(y-\theta) = \sum_{j=0}^{+\infty} \ln\psi_{-,0<}^{(0)}(y-\theta)^{(j)} \frac{1}{j!}\left(\sum_{n=1}^{+\infty} e^{n\theta} P_1^{(n,l)}(y-\theta)\right)^j \tag{5.62}$$

This reconstructs a Taylor series at the exponent

$$f(x) = \sum_{j=0}^{+\infty} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j \tag{5.63}$$

With the definitions

$$x_0 = y - \theta \tag{5.64}$$

$$x - x_0 = \sum_{n=1}^{+\infty} e^{n\theta} P_1^{(n)}(y-\theta) \tag{5.65}$$

$$f(x) = \ln\psi_{-,0<}^{(0)}(x) \tag{5.66}$$

That means that we have

$$\psi_{-,0}(y) = \frac{\pi 2^{2q_3} e^{-\tilde{S}_-}}{\Gamma\left(\frac{1}{2}-k+q_3\right)\Gamma\left(\frac{1}{2}+k+q_3\right)\sin(2k\pi)} \exp\left(\sum_{n=1}^{+\infty} e^{n\theta} P_0^{(n,l)}(y-\theta)\right) \psi^{(0)}_{-,0<}\left(L(y-\theta)\right) \tag{5.67}$$

With the definition

$$L(x) = x + \sum_{n=1}^{+\infty} e^{n\theta} P_1^{(n,l)}(x) \tag{5.68}$$

On the other hand, we recall that we can write $\psi_{-,0}(y)$ in the Floquet basis, we can follow the same procedure for the Floquet solutions

$$\begin{aligned}\psi_{\pm}(y) &= e^{c^{\pm}} e^{e^{\theta-y}+(\frac{1}{2}-q_3)y} \exp\left(\int_{-\infty}^{y-\theta} dy' \left(\Pi_{\pm<}(y') + e^{-y'} + q_3 - \frac{1}{2}\right)\right) = \\ &= e^{c^{\pm}} e^{-(q_3-\frac{1}{2})\theta} e^{-S_-} a_-(\pm k) \exp\left(\sum_{n=1}^{+\infty} e^{n\theta} P_0^{(n,l)}(y-\theta)\right) \psi^{(0)}_{\pm<}\left(L(y-\theta)\right)\end{aligned} \tag{5.69}$$

Where we defined

$$S_- = \left[\sum_{n=1}^{+\infty} e^{n\theta}\left(P_0^{(n,l)}(y') + \sum_{i=1}^{n} P_i^{(n,l)}(y') \frac{d^i}{dy'^i} \ln \psi^{(0)}_{\pm<}(y')\right)\right]_{y'=-\infty} \tag{5.70}$$

We notice how the combination (5.51) is in this way reconstructed.
This leads to

$$\psi_{-,0}(y) = \frac{-2e^{c_+}e^{c_-}e^{-2(q_3-\frac{1}{2})\theta}e^{-S_-}}{W[\psi_+,\psi_-]} \exp\left(\sum_{n=1}^{+\infty} e^{n\theta} P_0^{(n)}(y-\theta)\right) \psi^{(0)}_{-,0<}\left(L(y-\theta)\right) \tag{5.71}$$

It can be shown that (see (C))

$$\tilde{S}_- = -S_- \tag{5.72}$$

If we use this and compare the latter expression for $\psi_{-,0}(y)$ with (5.67) we get the wronskian

$$\boxed{W[\psi_+,\psi_-] = -e^{-\frac{\varphi(k)+\varphi(-k)}{2}} 2^{2\left(\frac{1}{2}-q_3\right)} e^{-2(q_3-\frac{1}{2})\theta} e^{-2S_-} \Gamma\left(\frac{1}{2}-k+q_3\right)\Gamma\left(\frac{1}{2}+k+q_3\right)\frac{\sin(2\pi k)}{\pi}} \tag{5.73}$$

5.4 Computing $\varphi(k)$ and Φ

To compute the acquired phase $\varphi(k)$, we just need the left kink, since it's valid up to $\ln 2$.
Starting from the definition (5.11) we can send $a \to \ln 2$ and get

$$\begin{aligned}\varphi(k) &= \int_{-\infty}^{\ln 2-\theta} dy' \left(\Pi^{(0)}_{+<}(y') + e^{-y'} - (\frac{1}{2}-q_3)\right) + \left(\frac{1}{2}-q_3\right)\ln 2 + e^{\theta-\ln 2} + \sum_{n=1}^{+\infty} e^{n\theta}\int_{-\infty}^{\ln 2-\theta} dy' \left(\Pi^{(n)}_{+<}(y')\right) = \\ &= \left(k\ln 2 - k\theta - e^{-\ln 2}e^{\theta} + \ln\left({}_1F_1(\frac{1}{2}-k+q_3, 1-2k, 2e^{-\ln 2}e^{\theta})\right) - e^{-\ln 2}e^{\theta} + (\frac{1}{2}-q_3)\theta - (\frac{1}{2}-q_3)\ln 2\right) - \\ &+ \left[ky - e^{-y} + \ln\left({}_1F_1(\frac{1}{2}-k+q_3, 1-2k, 2e^{-y})\right) - e^{-y} - (\frac{1}{2}-q_3)y\right]_{-\infty} + \left(\frac{1}{2}-q_3\right)\ln 2 + e^{\theta-\ln 2} + \\ &+ \sum_{n=1}^{+\infty} e^{n\theta}\int_{-\infty}^{\ln 2-\theta} dy' \left(\Pi^{(n)}_{+<}(y')\right) = \end{aligned} \tag{5.74}$$

$$= \varphi^{(0)}(k) + \sum_{n=1}^{+\infty} e^{n\theta}\int_{-\infty}^{\ln 2-\theta} dy' \left(\Pi^{(n)}_{+<}(y')\right) \tag{5.75}$$

Where $\varphi^{(0)}(k)$ is the part of $\varphi(k)$ that doesn't contain $\Pi^{(n)}_{+<}(y')$ with $n > 0$.
Using the expansion of the Generalized Hypergeometric function we get

$$\varphi^{(0)}(k) = \left(\frac{1}{2} - q_3\right)\theta - k\theta - \ln\frac{\Gamma(1-2k)}{\Gamma\left(\frac{1}{2} - k + q_3\right)} - \frac{2q_3}{2k-1}e^{\theta - \ln 2} + O(e^{2\theta}) \tag{5.76}$$

If we also include the first instanton contribution we find

$$\varphi(\theta,k) = \left(\frac{1}{2} - q_3 - k\right)\theta + \left(\frac{1}{2} - q_3\right)\ln 2 + \ln\frac{\Gamma(1+2k)\Gamma(q_1+q_2)\Gamma(\frac{1}{2}-k+q_3)}{\Gamma(1-2k)\Gamma\left(\frac{1}{2}+k+q_1\right)\Gamma\left(\frac{1}{2}+k+q_2\right)} + \frac{-1+q_1+q_2}{2}\ln 2+$$
$$+\frac{8kq_1q_2q_3}{(4k^2-1)^2}e^{\theta} + \frac{q_1q_3+q_2q_3-q_1q_2}{1-4k^2}e^{\theta} + O(e^{2\theta}) \tag{5.77}$$

If we define

$$\mathcal{F}^{(1)}_{inst} = \frac{q_1q_2q_3}{1-4k^2}e^{\theta} \tag{5.78}$$

We can also write the last line as

$$\frac{\partial\mathcal{F}^{(1)}_{inst}}{\partial k} + \frac{\partial\mathcal{F}^{(1)}_{inst}}{\partial q_1} + \frac{\partial\mathcal{F}^{(1)}_{inst}}{\partial q_2} - \frac{\partial\mathcal{F}^{(1)}_{inst}}{\partial q_3} \tag{5.79}$$

The quantity $\mathcal{F}^{(1)}_{inst}$ coincides with the first instantonic correction of the prepotential written in [8].
The acquired phase $\varphi(k)$ agrees with the one found in [7] with the confluence limit of Heun theory. The anti-symmetrised phase is then given by

$$\Phi = \frac{\varphi(k)-\varphi(-k)}{2} = -k\theta + \ln\frac{\Gamma(1+2k)}{\Gamma(1-2k)} + \frac{1}{2}\ln\frac{\Gamma\left(\frac{1}{2}-k+q_3\right)\Gamma\left(\frac{1}{2}-k+q_1\right)\Gamma\left(\frac{1}{2}-k+q_2\right)}{\Gamma\left(\frac{1}{2}+k+q_3\right)\Gamma\left(\frac{1}{2}+k+q_1\right)\Gamma\left(\frac{1}{2}+k+q_2\right)} + O(e^{\theta}) \tag{5.80}$$

We used the non trivial formula (C.11), Φ coincides with the perturbative contribution of the dual gauge period A_D of $N_f = 3$ [8] [7].
One can also check the first instanton contribution:

$$\Phi = -k\theta + \ln\frac{\Gamma(1+2k)}{\Gamma(1-2k)} + \frac{1}{2}\ln\frac{\Gamma\left(\frac{1}{2}-k+q_3\right)\Gamma\left(\frac{1}{2}-k+q_1\right)\Gamma\left(\frac{1}{2}-k+q_2\right)}{\Gamma\left(\frac{1}{2}+k+q_3\right)\Gamma\left(\frac{1}{2}+k+q_1\right)\Gamma\left(\frac{1}{2}+k+q_2\right)} + \frac{8kq_1q_2q_3}{(4k^2-1)^2}e^{\theta} + O(e^{2\theta})$$

and again it coincides with A_D.
However, this procedure forces us to resum all the contributions to get the n-th instanton contribution.
In order to avoid this one can compute $\varphi(k)$ keeping a generic, this, however, forces us to compute the integral present in (5.11)

$$\int_a^{\ln 2} dy'\left(\Pi_{\pm}(y') + \frac{-1+q_1+q_2}{e^{y'}-2}\right) \tag{5.81}$$

where we have to use the non-kinked function. In this way we get

$$\Phi^{(0)} = -k\theta + \ln\frac{\Gamma(1+2k)}{\Gamma(1-2k)} + \frac{1}{2}\ln\frac{\Gamma(\frac{1}{2}-k+q_1)\Gamma(\frac{1}{2}-k+q_2)\Gamma(\frac{1}{2}-k+q_3)}{\Gamma(\frac{1}{2}+k+q_1)\Gamma(\frac{1}{2}+k+q_2)\Gamma(\frac{1}{2}+k+q_3)} \tag{5.82}$$

and

$$\Phi = \sum_{n=0}^{+\infty}\Phi^{(n)} = \frac{1}{2}\ln\frac{{}_1F_1(\frac{1}{2}-k+q_3, 1-2k, 2e^{-a}e^{\theta})}{{}_1F_1(\frac{1}{2}+k+q_3, 1+2k, 2e^{-a}e^{\theta})} - \frac{1}{2}\ln\left(\frac{{}_2F_1\left(\frac{1}{2}+k-q_1, \frac{1}{2}+k-q_2; 1+2k; \frac{e^a}{2}\right)}{{}_2F_1\left(\frac{1}{2}-k-q_1, \frac{1}{2}-k-q_2; 1-2k; \frac{e^a}{2}\right)}\right) +$$
$$+\sum_{n=1}^{+\infty}e^{n\theta}\int_{-\infty}^{a-\theta}dy'\left[\frac{\Pi^{(n)}_{<}(y',k)-\Pi^{(n)}_{<}(y',-k)}{2}\right] + \sum_{n=1}^{+\infty}e^{n\theta}\int_a^{\ln 2}dy'\left[\frac{\Pi^{(n)}_{+}(y',k)-\Pi^{(n)}_{+}(y',-k)}{2}\right] \tag{5.83}$$

We notice how we recover the perturbative result with just the contribution from $\Phi^{(0)}$ without resumming infinite terms at all orders.

Nevertheless, we need infinite terms resummations to cancel some terms that depend on a.
In fact, if we look at Φ one can think that the term $-\frac{1}{2}\ln\left(\frac{{}_2F_1\left(\frac{1}{2}+k-q_1,\frac{1}{2}+k-q_2;1+2k;\frac{e^a}{2}\right)}{{}_2F_1\left(\frac{1}{2}-k-q_1,\frac{1}{2}-k-q_2;1-2k;\frac{e^a}{2}\right)}\right)$ should contribute to 1-loop, however, using (C.11) one can show that it is completely canceled by some terms coming from $\sum_{n=1}^{+\infty} e^{n\theta}\int_{-\infty}^{a-\theta} dy' \left[\frac{\Pi_<^{(n)}(y',k)-\Pi_<^{(n)}(y'$
.
With the same idea, we can also show that $\frac{1}{2}\ln\frac{{}_1F_1(\frac{1}{2}-k+q_3,1-2k,2e^{-a}e^{\theta})}{{}_1F_1(\frac{1}{2}+k+q_3,1+2k,2e^{-a}e^{\theta})}$ is canceled by terms that come from $\sum_{n=1}^{+\infty} e^{n\theta}\int_a^{\ln 2} dy' \left[\frac{\Pi_+^{(n)}(y',k)-\Pi_+^{(n)}(y',-k)}{2}\right]$.
This leads us to a result independent of a, that matches with the dual gauge period A_D of $N_f = 3$ at least up to 2-instantons.
We can also compute $\varphi(k)$ in another way, using the techniques of the previous section:

$$\varphi(k) = \left(\frac{1}{2}-q_3\right)\theta + \ln a_-(k) - \ln a_0(k) - S_- + S_0 + \ln\left(\frac{\psi_{+<}^{(0)}(L(a-\theta))}{\psi_+^{(0)}(R(a))}\right) + \sum_{n=1}^{+\infty} e^{n\theta}\left(P_0^{(n,l)}(a-\theta) - P_0^{(n,r)}(a)\right) \tag{5.84}$$

Where we used (5.68) and defined

$$S_0 = \left[\sum_{n=1}^{+\infty} e^{n\theta}\left(P_0^{(n,r)}(y) + \sum_{i=1}^{n} P_i^{(n,r)}(y)\frac{d^i}{dy^i}\ln\psi_\pm^{(0)}(y)\right)\right]_{y=\ln 2} \tag{5.85}$$

And

$$R(y) = y + \sum_{n=1}^{+\infty} e^{n\theta} P_1^{(n,r)}(y) \tag{5.86}$$

We also used used

$$\psi_\pm^{(0)}(y) \approx \frac{1}{a_0(\pm k)}(e^y-2)^{\frac{1-q_1-q_2}{2}} \qquad y \to \ln 2 \tag{5.87}$$

with

$$a_0(k) = \frac{\Gamma\left(\frac{1}{2}+k+q_1\right)\Gamma\left(\frac{1}{2}+k+q_2\right)}{\Gamma(1+2k)\,\Gamma(q_1+q_2)} 2^{-k} \tag{5.88}$$

This can be expanded to

$$\begin{aligned}\varphi(k) = &\left(\frac{1}{2}-q_3\right)\theta + \ln a_-(k) - \ln a_0(k) - S_- + S_0 + \sum_{n=1}^{+\infty} e^{n\theta}\left(P_0^{(n,l)}(a-\theta) - P_0^{(n,r)}(a)\right) + \\ &\mp k\theta \pm k\sum_{n=1}^{+\infty} e^{n\theta}P_1^{(n,l)}(a-\theta) - e^{\theta-a}e^{-\sum_{n=1}^{+\infty} e^{n\theta}P_1^{(n,l)}(a-\theta)} + \sum_{n=1}^{+\infty} L_n(\pm k)\frac{e^{n(\theta-a)}}{n!}e^{-n\sum_{n=1}^{+\infty} e^{n\theta}P_1^{(n,l)}(a-\theta)} + \\ &\mp k\sum_{n=1}^{+\infty} e^{n\theta}P_1^{(n,r)}(a) - \frac{1-q_1-q_2}{2}\ln\left(e^{a+\sum_{n=1}^{+\infty} e^{n\theta}P_1^{(n,r)}(a)} - 2\right) - \sum_{n=1}^{+\infty} R_n(\pm k)\frac{e^{na}}{n!}e^{n\sum_{n=1}^{+\infty} e^{n\theta}P_1^{(n,r)}(a)}\end{aligned} \tag{5.89}$$

The symmetric sum is

$$\frac{\varphi(k)+\varphi(-k)}{2} = \left(\frac{1}{2}-q_3\right)\theta + \frac{1}{2}\ln(a_-(k)a_-(-k)) - \frac{1}{2}\ln(a_0(k)a_0(-k)) - S_- + S_0 + A_+(\theta) \tag{5.90}$$

While the antisymmetric sum Φ is

$$\Phi = \frac{1}{2}\ln(a_-(k)a_0(-k)) - \frac{1}{2}(\ln a_0(k)a_-(-k)) - k\theta + A_-(\theta) \tag{5.91}$$

We try to compute the quantities $A_\pm(\theta)$ in (C), but we already know that they can't depend on a

5.5 Computing $\varphi_2(k)$ and Φ_2

We now compute the acquired phase $\varphi_2(k)$ (5.12). We need to consider the behavior

$$\psi_{\pm}^{(0)}(y) \approx \frac{1}{a_+(\pm k)} e^{\frac{q_1-q_2}{2}y} \qquad y \to +\infty \tag{5.92}$$

with

$$a_+(k) = \frac{\Gamma\left(\frac{1}{2}+k+q_1\right)\Gamma\left(\frac{1}{2}+k-q_2\right)}{\Gamma\left(1+2k\right)\Gamma(q_1-q_2)} \left(-\frac{1}{2}\right)^{\frac{1}{2}+k-q_1} \tag{5.93}$$

Since the integral in (5.12) cancels out in y^* we can write

$$\begin{aligned}\varphi_2(k) &= \ln a_0(k) - \ln a_+(k) - S_0 + S_+ \\ &= \ln\left(\frac{\Gamma(q_1-q_2)\Gamma\left(\frac{1}{2}+k+q_2\right)}{\Gamma(q_1+q_2)\Gamma\left(\frac{1}{2}+k-q_2\right)}\right) + \ln\Big((-1)^{-\frac{1}{2}+k-q_1} 2^{\frac{1}{2}-q_1}\Big) - S_0 + S_+\end{aligned} \tag{5.94}$$

with (5.85) and

$$S_+ = \left[\sum_{n=1}^{+\infty} e^{n\theta}\left(P_0^{(n,r)}(y) + \sum_{i=1}^{n} P_i^{(n,r)}(y)\frac{d^i}{dy^i}\ln\psi_{\pm}^{(0)}(y)\right)\right]_{y=+\infty} \tag{5.95}$$

This leads interestingly at

$$\Phi_2 = -k\ln(-1) + \ln\sqrt{\frac{\cos(\pi(k-q_2))}{\cos(\pi(k+q_2))}} \tag{5.96}$$

That means

$$\sinh(\Phi_2) = \frac{\sin(2\pi k)\sqrt{\Gamma(\frac{1}{2}-k-q_2)\Gamma\left(\frac{1}{2}-k+q_2\right)\Gamma(\frac{1}{2}+k-q_2)\Gamma\left(\frac{1}{2}+k+q_2\right)}}{2\pi i} e^{i\pi q_2} \tag{5.97}$$

5.6 Computing $Q_i(\theta)$

We now have all the ingredients to compute $Q_i(\theta)$, we recall that we have

$$Q_1(\theta) = \frac{1}{W[\psi_+,\psi_-]} 4\sqrt{2}(q_1+q_2)e^{-(q_3-\frac{1}{2})\theta}\sinh(\Phi) \tag{5.98}$$

If we substitute the formula for the Wronksian (5.73), we have

$$Q_1(\theta) = -\frac{\pi e^{\frac{\varphi(k)+\varphi(-k)}{2}} e^{(q_3-\frac{1}{2})\theta}}{\Gamma\left(\frac{1}{2}-k+q_3\right)\Gamma\left(\frac{1}{2}+k+q_3\right)} 2^{2\left(q_3-\frac{1}{2}\right)} 4\sqrt{2}(q_1+q_2)e^{2S_-}\frac{\sinh(\Phi)}{\sin(2\pi k)} \tag{5.99}$$

Now we can also use the symmetrized phase (5.90) to obtain

$$Q_1(\theta) = -\frac{\pi e^{A_+(\theta)} 2^{\left(q_3-\frac{1}{2}\right)} 4\sqrt{2}\Gamma(1+q_1+q_2)}{\left[\prod_{i=1}^{3}\Gamma\left(\frac{1}{2}+k+q_i\right)\Gamma\left(\frac{1}{2}-k+q_i\right)\right]^{\frac{1}{2}}} e^{S_-+S_0}\frac{\sinh(\Phi)}{\sin(2\pi k)} \tag{5.100}$$

Using $A_+(\theta)$ computed in (C) and (B.25) (B.24) we get a match with the hypothesized result found in [8]

$$Q_1(\theta) = 2\pi\exp\left(\sum_{j=1}^{3}\left(\frac{\partial\mathcal{F}_{inst}}{\partial q_j} + \frac{\partial\mathcal{F}_{pert}}{\partial q_j}\right)\right)\frac{\sinh\Phi}{\sin 2\pi k} \tag{5.101}$$

We can also compute $Q_2(\theta)$, but it is easier to follow another path and we will do it in the next section; in any case the result in both ways is

$$Q_2(\theta) = \frac{\Gamma(1+q_1+q_2)\Gamma(1+q_1-q_2)}{\Gamma\left(\frac{1}{2}-k+q_1\right)\Gamma\left(\frac{1}{2}+k+q_1\right)} 2^{q_2} e^{S_0+S_+} \tag{5.102}$$

This is consistent with the known result [8]

$$Q_2(\theta) = \exp\left[2\left(\frac{\partial \mathcal{F}_{pert}}{\partial q_1} + \frac{\partial \mathcal{F}_{inst}}{\partial q_1}\right)\right] = \exp\left[2\frac{\partial \mathcal{F}}{\partial q_1}\right] \tag{5.103}$$

with

$$\exp\left(2\frac{\partial \mathcal{F}_{pert}}{\partial q_1}\right) = \frac{\Gamma(1+q_1+q_2)\Gamma(1+q_1-q_2)}{\Gamma\left(\frac{1}{2}-k+q_1\right)\Gamma\left(\frac{1}{2}+k+q_1\right)}2^{q_2} \tag{5.104}$$

To get $Q_3(\theta)$ one could just use the QQ-system (3.34)

5.7 Alternative path to compute $Q_2(\theta)$

We want now to construct the Floquet basis starting from $+\infty$ instead of $-\infty$, in this way, when we calculate Q_2, we don't have to do any kink. We consider the Floquet solutions

$$\tilde{\psi}_\pm(y) = e^{\tilde{c}_\pm} e^{\frac{q_1-q_2}{2}y} \exp\left(-\int_y^{+\infty} dy'\left(\Pi_\pm(y') - \frac{q_1-q_2}{2}\right)\right) \tag{5.105}$$

with $\tilde{c}_\pm$ constant. The wavefunction at $+\infty$ behaves as

$$\tilde{\psi}_\pm(y) \approx e^{\tilde{c}_\pm} e^{\frac{q_1-q_2}{2}y} \qquad y \to +\infty \tag{5.106}$$

The Floquet solutions $\psi_\pm(y)$ and $\tilde{\psi}_\pm(y)$ are the same apart from a normalization

$$\tilde{\psi}_\pm(y) = C\psi_\pm(y) \tag{5.107}$$

In order to catch the behavior at the other singularities, we need to manipulate the integrand as we did in the previous section

$$\int_y^{+\infty} dy'\left(\Pi_\pm(y') - \frac{q_1-q_2}{2}\right) = \int_{y^*}^{+\infty} dy'\left(\Pi_\pm(y') - \frac{q_1-q_2}{2}\right) + \int_y^{y^*} dy'\left(\Pi_\pm(y') - \frac{1-(q_1+q_2)}{e^y-2}\right) + \tag{5.108}$$

$$+\frac{q_1-q_2}{2}(y-y^*) + \frac{1-(q_1+q_2)}{2}\left(\ln|e^{y^*}-2| - y^*\right) - \frac{1-(q_1+q_2)}{2}\left(\ln|e^y-2| - y\right) \tag{5.109}$$

The behavior at $y = \ln 2$ is then

$$\tilde{\psi}_\pm(y) \approx e^{\tilde{c}_\pm}(e^y-2)^{\frac{1-(q_1+q_2)}{2}} e^{-\varphi_2(\pm k)} \qquad y \to \ln 2 \tag{5.110}$$

We can do the same procedure for $-\infty$ finding

$$\tilde{\psi}_\pm(y) \approx e^{\tilde{c}_\pm} e^{e^{\theta-y}+(\frac{1}{2}-q_3)y} e^{-\varphi_2(\pm k)-\varphi(\pm k)} \qquad y \to -\infty \tag{5.111}$$

Since the Floquet solutions $\tilde{\psi}_\pm(y)$ form a basis, we can expand the regular solutions in terms of the Floquet basis:

$$\psi_{-,0}(y) = \frac{1}{\tilde{s}_-}\left(e^{-\tilde{c}_+}e^{\varphi_2(k)}e^{\varphi(k)}\tilde{\psi}_+(y) - e^{-\tilde{c}_-}e^{\varphi_2(-k)}e^{\varphi(-k)}\tilde{\psi}_-(y)\right) \tag{5.112}$$

$$\psi_{0,0}(y) = \frac{1}{\tilde{s}_0}\left(e^{-\tilde{c}_+}e^{\varphi_2(k)}\tilde{\psi}_+(y) - e^{-\tilde{c}_-}e^{\varphi_2(-k)}\tilde{\psi}_-(y)\right) \tag{5.113}$$

$$\psi_{+,0}(y) = \frac{1}{\tilde{s}_+}\left(e^{-\tilde{c}_+}\tilde{\psi}_+(y) - e^{-\tilde{c}_-}\tilde{\psi}_-(y)\right) \tag{5.114}$$

Again we do the change of basis that we did in previous sections

$$Q_2(\theta) = \frac{e^{-\tilde{c}_+}e^{-\tilde{c}_-}e^{\frac{\varphi_2(k)+\varphi_2(-k)}{2}}}{\tilde{s}_0\tilde{s}_+}\left(e^{\frac{\varphi_2(k)-\varphi_2(-k)}{2}} - e^{-\frac{\varphi_2(k)-\varphi_2(-k)}{2}}\right)W[\tilde{\psi}_+, \tilde{\psi}_-] \tag{5.115}$$

Using the form (3.29) and the definition (5.19) we have

$$Q_2(\theta) = (q_1+q_2)\lim_{y\to ln(2)} \sqrt{2}\,(e^y-2)^{-\frac{1}{2}(1-q_1-q_2)}\,\frac{1}{\tilde{s}_+}\left(e^{-\tilde{c}_+}\tilde{\psi}_+(y) - e^{-\tilde{c}_-}\tilde{\psi}_-(y)\right) =$$
$$= -\sqrt{2}\frac{q_1+q_2}{\tilde{s}_+}e^{-\frac{\varphi_2(k)+\varphi_2(-k)}{2}}\left(e^{\frac{\varphi_2(k)-\varphi_2(-k)}{2}} - e^{-\frac{\varphi_2(k)-\varphi_2(-k)}{2}}\right) \tag{5.116}$$

If we compare it with (5.115), we get

$$\frac{1}{\tilde{s}_0} = \frac{-\sqrt{2}(q_1+q_2)e^{\tilde{c}_+}e^{\tilde{c}_-}e^{-\varphi_2(k)}e^{-\varphi_2(-k)}}{W[\tilde{\psi}_+,\tilde{\psi}_-]} \tag{5.117}$$

We do the same thing for $Q_3(\theta)$

$$Q_3(\theta) = \frac{e^{-\tilde{c}_+}e^{-\tilde{c}_-}}{\tilde{s}_+\tilde{s}_-}\left(e^{\varphi(k)+\varphi_2(k)} - e^{\varphi(-k)+\varphi_2(-k)}\right)W[\tilde{\psi}_+,\tilde{\psi}_-] \tag{5.118}$$

$$Q_3(\theta) = (q_1-q_2)\lim_{y\to+\infty} e^{-\frac{q_1-q_2}{2}y}\frac{1}{\tilde{s}_-}\left(e^{-\tilde{c}_+ + \varphi(k)+\varphi_2(k)}\tilde{\psi}_+(y) - e^{-\tilde{c}_- + \varphi(-k)+\varphi_2(-k)}\tilde{\psi}_-(y)\right) =$$
$$= \frac{q_1-q_2}{\tilde{s}_-}\left(e^{\varphi(k)+\varphi_2(k)} - e^{\varphi(-k)+\varphi_2(-k)}\right) \tag{5.119}$$

That leads to

$$\frac{1}{\tilde{s}_+} = \frac{(q_1-q_2)e^{\tilde{c}_+}e^{\tilde{c}_-}}{W[\tilde{\psi}_+,\tilde{\psi}_-]} \tag{5.120}$$

This finally gives us $Q_2(\theta)$

$$Q_2(\theta) = -\frac{e^{\tilde{c}_+}e^{\tilde{c}_-}e^{-\frac{\varphi_2(k)+\varphi_2(-k)}{2}}}{W[\tilde{\psi}_+,\tilde{\psi}_-]}2\sqrt{2}(q_1+q_2)(q_1-q_2)\sinh(\Phi_2) \tag{5.121}$$

$$\psi_{+,0}(y) = \frac{(q_1-q_2)e^{\tilde{c}_+\tilde{c}_-}}{W[\tilde{\psi}_+,\tilde{\psi}_-]}\left(e^{-\tilde{c}_+}\tilde{\psi}_+(y) - e^{-\tilde{c}_-}\tilde{\psi}_-(y)\right) \tag{5.122}$$

Fixing $\tilde{c}_\pm = \frac{\varphi_2(k)}{2}$ we get

$$Q_2(\theta) = -\frac{2\sqrt{2}(q_1+q_2)(q_1-q_2)}{W[\tilde{\psi}_+,\tilde{\psi}_-]}\sinh(\Phi_2) \tag{5.123}$$

We do the same procedure for the wronskian, but now we need to use

$$\psi_\pm^{(0)}(y) = (e^y-2)^{\frac{1-q_1-q_2}{2}}e^{\pm ky}\,{}_2F_1\left(\frac{1}{2}\pm k - q_1, \frac{1}{2}\pm k - q_2; 1\pm 2k; \frac{e^y}{2}\right) \tag{5.124}$$

From that we can construct another solution

$$\psi_{+,0}^{(0)}(y) = a_+(k)\psi_+^{(0)}(y) - a_+(-k)\psi_-^{(0)}(y) \tag{5.125}$$

With the definition (5.93), in this way we have

$$\psi_{+,0}^{(0)}(y) \approx \left(-\frac{1}{2}\right)^{q_2-q_1}\frac{\Gamma(q_2-q_1)}{\Gamma(q_1-q_2)}\left(\frac{\Gamma\left(\frac{1}{2}+k+q_1\right)\Gamma\left(\frac{1}{2}+k-q_2\right)}{\Gamma\left(\frac{1}{2}+k-q_1\right)\Gamma\left(\frac{1}{2}+k+q_2\right)} - \frac{\Gamma\left(\frac{1}{2}-k+q_1\right)\Gamma\left(\frac{1}{2}-k-q_2\right)}{\Gamma\left(\frac{1}{2}-k-q_1\right)\Gamma\left(\frac{1}{2}-k+q_2\right)}\right)e^{\frac{q_2-q_1}{2}y} \quad y\to+\infty \tag{5.126}$$

$$\psi_{+,0}^{(0)}(y) \approx \left(-\frac{1}{2}\right)^{q_2-q_1}\frac{\sin 2\pi k}{\pi\Gamma(q_1-q_2)^2(q_2-q_1)}\Gamma\left(\frac{1}{2}+k+q_1\right)\Gamma\left(\frac{1}{2}+k-q_2\right)\Gamma\left(\frac{1}{2}-k+q_1\right)\Gamma\left(\frac{1}{2}-k-q_2\right)e^{\frac{q_2-q_1}{2}y} \quad y\to+\infty \tag{5.127}$$

From that we can construct the momentum

$$\Pi^{(0)}_{+,0}(y) = \frac{d}{dy}\ln\psi^{(0)}_{+,0}(y) \tag{5.128}$$

and we know it is a solution of the Riccati equation at the first order.
Now from this last function we can construct the total wavefunction in this way

$$\psi_{+,0}(y) = e^{c_0}e^{-\frac{q_1-q_2}{2}y}\exp\left(-\int_y^{+\infty}dy'\left(\Pi_{+,0}(y')+\frac{q_1-q_2}{2}\right)\right) \tag{5.129}$$

with the definition $\Pi_{+,0}(y) = \sum_{n=0}^{+\infty} e^{4n\theta}\Pi^{(n)}_{+,0}(y)$ and with $\Pi^{(n)}_{+,0}(y)$ created with the usual formula

$$\Pi^{(n)}_{+,0}(y) = \frac{d}{dy}\left(P_0^{(n,r)}(y) + \sum_{i=1}^{n} P_i^{(n,r)}(y)\frac{d^{i-1}}{dy^{i-1}}\Pi^{(0)}_{+,0}(y)\right) \tag{5.130}$$

We can take the $y \to +\infty$ limit and match it with (3.18) and this set $e^{c_0} = 1$.
Again, we can get

$$\psi_{+,0}(y) = \frac{(-2)^{q_2-q_1}\Gamma(q_1-q_2)^2(q_2-q_1)e^{-\tilde{S}_+}}{\Gamma\left(\frac{1}{2}+k+q_1\right)\Gamma\left(\frac{1}{2}+k-q_2\right)\Gamma\left(\frac{1}{2}-k+q_1\right)\Gamma\left(\frac{1}{2}-k-q_2\right)}\frac{\pi}{\sin 2\pi k}\exp\left(\sum_{n=1}^{+\infty}e^{n\theta}P_0^{(n,r)}(y)\right)\psi^{(0)}_{+,0}\left(R(y)\right) \tag{5.131}$$

Where

$$\tilde{S}_+ = \left[\sum_{n=1}^{+\infty}e^{n\theta}\left(P_0^{(n,r)}(y') + \sum_{i=1}^{n}P_i^{(n,r)}(y')\frac{d^i}{dy'^i}\ln\psi^{(0)}_{+,0}(y')\right)\right]_{y'=+\infty} = -S_+ \tag{5.132}$$

And we used (5.86), on the other hand, we recall the relation

$$\psi_{+,0}(y) = \frac{(q_1-q_2)e^{\tilde{c}_+}e^{\tilde{c}_-}}{W[\psi_+,\psi_-]}\left(e^{-\tilde{c}_+}\tilde{\psi}_+(y) - e^{-\tilde{c}_-}\tilde{\psi}_-(y)\right) \tag{5.133}$$

And we can follow the same procedure for the Floquet solutions

$$\begin{aligned}\tilde{\psi}_\pm(y) &= e^{\tilde{c}_\pm}e^{\frac{q_1-q_2}{2}y}\exp\left(-\int_y^{+\infty}dy'\left(\Pi_\pm(y')-\frac{q_1-q_2}{2}\right)\right) = \\ &= e^{\tilde{c}_\pm}e^{-S_+}a_+(\pm k)\exp\left(\sum_{n=1}^{+\infty}e^{n\theta}P_0^{(n,r)}(y)\right)\psi^{(0)}_+\left(R(y)\right)\end{aligned} \tag{5.134}$$

that leads to

$$\psi_{+,0}(y) = \frac{(q_1-q_2)e^{\tilde{c}_+}e^{\tilde{c}_-}e^{-S_+}}{W[\tilde{\psi}_+,\tilde{\psi}_-]}\exp\left(\sum_{n=1}^{+\infty}e^{n\theta}P_0^{(n,r)}(y)\right)\psi^{(0)}_0\left(R(y)\right) \tag{5.135}$$

If we compare it with (5.131), we get a formula for the Wronskian of the Floquet solutions

$$W[\tilde{\psi}_+,\tilde{\psi}_-] = -\frac{e^{\tilde{c}_+}e^{\tilde{c}_-}e^{-2S_+}(-2)^{q_1-q_2}}{\Gamma(q_1-q_2)^2}\Gamma\left(\frac{1}{2}+k+q_1\right)\Gamma\left(\frac{1}{2}+k-q_2\right)\Gamma\left(\frac{1}{2}-k+q_1\right)\Gamma\left(\frac{1}{2}-k-q_2\right)\frac{\sin 2\pi k}{\pi} \tag{5.136}$$

Fixing $\tilde{c}_\pm = \frac{\varphi_2(k)}{2}$ we get

$$\boxed{W[\tilde{\psi}_+,\tilde{\psi}_-] = -\frac{e^{\frac{\varphi_2(k)+\varphi_2(-k)}{2}}e^{-2S_+}(-2)^{q_1-q_2}}{\Gamma(q_1-q_2)^2}\Gamma\left(\frac{1}{2}+k+q_1\right)\Gamma\left(\frac{1}{2}+k-q_2\right)\Gamma\left(\frac{1}{2}-k+q_1\right)\Gamma\left(\frac{1}{2}-k-q_2\right)\frac{\sin 2\pi k}{\pi}} \tag{5.137}$$

In this way, we have all the ingredients to compute $Q_2(\theta)$

$$Q_2(\theta) = -\frac{1}{W[\tilde{\psi}_+, \tilde{\psi}_-]} 2\sqrt{2}(q_1 + q_2)(q_1 - q_2)\sinh(\Phi_2) \tag{5.138}$$

Putting everything together, we obtain

$$Q_2(\theta) = \frac{\Gamma(1 + q_1 + q_2)\Gamma(1 + q_1 - q_2)}{\Gamma\left(\frac{1}{2} - k + q_1\right)\Gamma\left(\frac{1}{2} + k + q_1\right)} 2^{q_2} e^{S_0 + S_+} \tag{5.139}$$

which coincides with the formula (5.102) found before using different Floquet solutions. The advantage of this method is that we have considered only the interval $\ln 2 < y < +\infty$ which allowed to compute the Wronskian and $\varphi_2(k)$ without any kink transformation.

5.8 Alternative path starting from $\ln 2$

The last way to construct the Floquet basis is if we start from $y = \ln 2$. This is interesting since $\psi_{0,0}(y)$ is interesting in black hole physics. We consider the Floquet solutions

$$\hat{\psi}_\pm(y) = e^{\hat{c}_\pm}(e^y - 2)^{\frac{1-(q_1+q_2)}{2}} e^{\frac{1-(q_1+q_2)}{2}(\ln 2 - y)} \exp\left(-\int_y^{\ln 2} dy' \left(\Pi_\pm(y') - \frac{1-(q_1+q_2)}{e^y - 2}\right)\right) \tag{5.140}$$

with $\hat{c}_\pm$ constant. The wavefunction at $\ln 2$ behaves as

$$\hat{\psi}_\pm(y) \approx e^{\hat{c}_\pm}(e^y - 2)^{\frac{1-(q_1+q_2)}{2}} \qquad y \to \ln 2 \tag{5.141}$$

In order to catch the behavior at the other singularities, we need to manipulate the integrand as we did in the previous section, we get

$$\hat{\psi}_\pm(y) \approx e^{\hat{c}_\pm} e^{e^{\theta - y} + (\frac{1}{2} - q_3)y} e^{-\varphi(\pm k)} \qquad y \to -\infty \tag{5.142}$$

$$\hat{\psi}_\pm(y) \approx e^{\hat{c}_\pm} e^{\frac{q_1 - q_2}{2} y} e^{\varphi_2(\pm k)} \qquad y \to +\infty \tag{5.143}$$

Since the Floquet solutions $\hat{\psi}_\pm(y)$ form a basis, we can expand the regular solutions in terms of the Floquet basis:

$$\psi_{-,0}(y) = \frac{1}{\hat{s}_-}\left(e^{-\hat{c}_+} e^{\varphi(k)} \hat{\psi}_+(y) - e^{-\hat{c}_-} e^{\varphi(-k)} \hat{\psi}_-(y)\right) \tag{5.144}$$

$$\psi_{0,0}(y) = \frac{1}{\hat{s}_0}\left(e^{-\hat{c}_+} \hat{\psi}_+(y) - e^{-\hat{c}_-} \hat{\psi}_-(y)\right) \tag{5.145}$$

$$\psi_{+,0}(y) = \frac{1}{\hat{s}_+}\left(e^{-\hat{c}_+} e^{-\varphi_2(k)} \hat{\psi}_+(y) - e^{-\hat{c}_-} e^{-\varphi_2(-k)} \hat{\psi}_-(y)\right) \tag{5.146}$$

Again we can do the usual techniques to find

$$\frac{1}{\hat{s}_0} = \frac{-\sqrt{2}(q_1 + q_2) e^{\hat{c}_+} e^{\hat{c}_-}}{W[\hat{\psi}_+, \hat{\psi}_-]} \tag{5.147}$$

That leads to

$$\psi_{0,0}(y) = \frac{-\sqrt{2}(q_1 + q_2) e^{\hat{c}_+} e^{\hat{c}_-}}{W[\hat{\psi}_+, \hat{\psi}_-]}\left(e^{-\hat{c}_+} \hat{\psi}_+(y) - e^{-\hat{c}_-} \hat{\psi}_-(y)\right) \tag{5.148}$$

We do the same procedure for the wronskian, but now we need to use

$$\psi_\pm^{(0)}(y) = (e^y - 2)^{\frac{1-q_1-q_2}{2}} e^{\pm ky} {}_2F_1\left(\frac{1}{2} \pm k - q_1, \frac{1}{2} \pm k - q_2; 1 \pm 2k; \frac{e^y}{2}\right) \tag{5.149}$$

From that we can construct another solution

$$\psi_{0,0}^{(0)}(y) = a_0(k)\psi_+^{(0)}(y) - a_0(-k)\psi_-^{(0)}(y) \tag{5.150}$$

With the definition (5.88), in this way we have

$$\psi_{0,0}^{(0)}(y) \approx \left(-\frac{1}{2}\right)^{1+q_1+q_2} 2\frac{\sin 2\pi k}{\pi\Gamma(q_1+q_2)^2(q_1+q_2)} \prod_{\sigma=\pm1} \Gamma\left(\frac{1}{2}+\sigma k+q_1\right)\Gamma\left(\frac{1}{2}+\sigma k+q_2\right)(e^y-2)^{\frac{1+q_1+q_2}{2}} \quad y\to\ln 2 \tag{5.151}$$

From that we can construct the momentum

$$\Pi_{0,0}^{(0)}(y) = \frac{d}{dy}\ln\psi_{0,0}^{(0)}(y) \tag{5.152}$$

We know that it is a solution of the Riccati equation at the first order.
Now, from this last function we can construct the total wavefunction in this way

$$\psi_{0,0}(y) = e^{c_0}(e^y-2)^{\frac{1+(q_1+q_2)}{2}} e^{\frac{1+(q_1+q_2)}{2}(\ln 2-y)} \exp\left(-\int_y^{\ln 2} dy'\left(\Pi_{0,0}(y') - \frac{1+(q_1+q_2)}{e^y-2}\right)\right) \tag{5.153}$$

with the definition $\Pi_{0,0}(y) = \sum_{n=0}^{+\infty} e^{4n\theta}\Pi_{0,0}^{(n)}(y)$ and with $\Pi_{0,0}^{(n)}(y)$ created with the usual formula

$$\Pi_{0,0}^{(n)}(y) = \frac{d}{dy}\left(P_0^{(n,r)}(y) + \sum_{i=1}^{n} P_i^{(n,r)}(y)\frac{d^{i-1}}{dy^{i-1}}\Pi_{0,0}^{(0)}(y)\right) \tag{5.154}$$

We can take the $y\to\ln 2$ limit and match it with (3.17) and this set $e^{c_0} = \frac{1}{\sqrt{2}}$.
If we now insert the expansion $\Pi_{0,0}(y) = \sum_{n=0}^{+\infty} e^{4n\theta}\Pi_{0,0}^{(n)}(y)$ we arrive to

$$\psi_{0,0}(y) = \tilde{G}\psi_{0,0}^{(0)}(y)\exp\left(\sum_{n=1}^{+\infty} e^{n\theta}\left(P_0^{(n,r)}(y) + \sum_{i=1}^{n} P_i^{(n,r)}(y)\frac{d^{i-1}}{dy^{i-1}}\Pi_{0,0}^{(0)}(y)\right)\right) \tag{5.155}$$

With

$$\tilde{G} = (-2)^{1+q_1+q_2}\,\frac{\sqrt{2}\pi}{4\sin 2\pi k}\frac{\Gamma(q_1+q_2)^2(q_1+q_2)e^{S_0}}{\prod_{\sigma=\pm1}\Gamma\left(\frac{1}{2}+\sigma k+q_1\right)\Gamma\left(\frac{1}{2}+\sigma k+q_2\right)} \tag{5.156}$$

We used the definition (5.85)
We can rearrange the double sum at the exponent in this way

$$\psi_{0,0}(y) = \tilde{G}\psi_{0,0}^{(0)}(y)\exp\left(\sum_{n=1}^{+\infty} e^{n\theta}P_0^{(n,r)}(y) + \sum_{j=1}^{+\infty}\frac{d^j}{dy^j}\ln\psi_{0,0}^{(0)}(y)\sum_{n=j}^{+\infty} e^{n\theta}P_j^{(n,r)}(y)\right) \tag{5.157}$$

we use the following relations

$$\sum_{n=j}^{+\infty}\frac{t^n}{n!}B_{n,j}(x_1,...,x_{n-j+1}) = \frac{1}{j!}\left(\sum_{n=1}^{+\infty} x_n\frac{t^n}{n!}\right)^j \tag{5.158}$$

and

$$P_j^{(n,r)}(y) = \frac{1}{n!}B_{n,j}\left(1!P_1^{(1,r)}(y), 2!P_1^{(2,r)}(y), \ldots, (n-j+1)!P_1^{(n-j+1,r)}(y)\right), \qquad 1\le j\le n \tag{5.159}$$

$$\equiv \frac{1}{n!}B_{n,j}(P_1) \tag{5.160}$$

We created a shortcut in the writing for later use, in this way we can write

$$\psi_{0,0}(y) = \tilde{G}\psi_{0,0}^{(0)}(y)\exp\left(\sum_{n=1}^{+\infty} e^{n\theta}P_0^{(n,r)}(y)\right)\exp\left(\sum_{j=1}^{+\infty}\frac{d^j}{dy^j}\ln\psi_{0,0}^{(0)}(y)\frac{1}{j!}\left(\sum_{n=1}^{+\infty} e^{n\theta}P_1^{(n,r)}(y)\right)^j\right) \tag{5.161}$$

Now we can use the following mathematical formula

$$\exp\left(\sum_{n=1}^{+\infty} x_n \frac{t^n}{n!}\right) = \sum_{n=0}^{+\infty} \frac{t^n}{n!} B_n(x_1, ..., x_n) \tag{5.162}$$

to express the 2 exponentials as 2 series using the complete Bell polynomials

$$\psi_{0,0}(y) = \tilde{G}\psi_{0,0}^{(0)}(y) \left(\sum_{n=0}^{+\infty} \frac{e^{n\theta}}{n!} B_n(P_0)\right) \left(\sum_{j=0}^{+\infty} B_j \left(\ln \psi_{0,0}^{(0)}(y)\right) \frac{1}{j!} \left(\sum_{n=1}^{+\infty} e^{n\theta} P_1^{(n,r)}(y)\right)^j\right) \tag{5.163}$$

We again used the two shortcuts

$$B_n(P_0) \equiv B_n\left(1!P_0^{(1,r)}(y), 2!P_0^{(2,r)}(y), \ldots, (n-j+1)!P_0^{(n,r)}(y)\right) \tag{5.164}$$

And

$$B_n\left(\ln \psi_{0,0}^{(0)}(y)\right) \equiv B_n\left(\frac{d}{dy} \ln \psi_{0,0}^{(0)}(y), \frac{d^2}{dy^2} \ln \psi_{0,0}^{(0)}(y), \ldots, \frac{d^n}{dy^n} \ln \psi_{0,0}^{(0)}(y)\right) \tag{5.165}$$

Now we again use (5.158) (in the inverse direction) to write

$$\psi_{0,0}(y) = \tilde{G}\psi_{0,0}^{(0)}(y) \left(\sum_{n=0}^{+\infty} \frac{e^{n\theta}}{n!} B_n(P_0)\right) \left(\sum_{j=0}^{+\infty} B_j \left(\ln \psi_{0,0}^{(0)}(y)\right) \sum_{n=j}^{+\infty} \frac{t^n}{n!} B_{n,j}(P_1)\right) \tag{5.166}$$

Now we can invert the two sums in the second parentheses to write (one can check explicitly that the two sums can be rearranged like this)

$$\psi_{0,0}(y) = \tilde{G}\psi_{0,0}^{(0)}(y) \left(\sum_{n=0}^{+\infty} \frac{e^{n\theta}}{n!} B_n(P_0)\right) \left(\sum_{n=0}^{+\infty} \frac{e^{n\theta}}{n!} \sum_{j=1}^{n} B_j \left(\ln \psi_{0,0}^{(0)}(y)\right) B_{n,j}(P_1)\right) \tag{5.167}$$

Moreover, we can use the product of series formula

$$\left(\sum_{n=0}^{+\infty} a_n \frac{x^n}{n!}\right) \left(\sum_{n=0}^{+\infty} b_n \frac{x^n}{n!}\right) = \sum_{n=0}^{+\infty} \left(\sum_{j=0}^{n} \binom{n}{j} a_j b_{n-j}\right) \frac{x^n}{n!} \tag{5.168}$$

In this way, we write a unique series using the binomial coefficients

$$\psi_{0,0}(y) = \tilde{G}\psi_{0,0}^{(0)}(y) \left(\sum_{n=0}^{+\infty} D_n(y) \frac{e^{n\theta}}{n!}\right) \tag{5.169}$$

With the definition

$$D_n(y) = \sum_{j=0}^{n} \binom{n}{j} B_{n-j}(P_0) \sum_{m=1}^{j} B_m \left(\ln \psi_{0,0}^{(0)}(y)\right) B_{j,m}(P_1) \tag{5.170}$$

This is hard to read, but it is easy to implement in Mathematica.
For the first terms we have,

$$D_0(y) = 1 \tag{5.171}$$

$$D_1(y) = B_1(P_0) + B_1 \left(\ln \psi_{0,0}^{(0)}(y)\right) B_{1,1}(P_1) \tag{5.172}$$

$$D_2(y) = \frac{B_2(P_0)}{2} + B_1(P_0) B_1 \left(\ln \psi_{0,0}^{(0)}(y)\right) B_{1,1}(P_1) + \frac{B_1 \left(\ln \psi_{0,0}^{(0)}(y)\right) B_{2,1}(P_1) + B_2 \left(\ln \psi_{0,0}^{(0)}(y)\right) B_{2,2}(P_1)}{2} \tag{5.173}$$

We can also say something interesting for $B_n\left(\ln\psi_{0,0}^{(0)}(y)\right)$

$$B_n\left(\ln\psi_{0,0}^{(0)}(y)\right)=\frac{\frac{d^n}{dy^n}\psi_{0,0}^{(0)}(y)}{\psi_{0,0}^{(0)}(y)} \tag{5.174}$$

This identity is explicitly checked up to $n=5$ for any smooth function $\psi_{0,0}^{(0)}(y)$, and could be a never written property for Bell polynomials.
Substituting this we get a prettier form (counting also the term $\psi_{0,0}^{(0)}(y)$ that cancels the denominator in (5.174))

$$\psi_{0,0}^{(0)}(y)D_0(y)=\psi_{0,0}^{(0)}(y) \tag{5.175}$$

$$\psi_{0,0}^{(0)}(y)D_1(y)=P_0^{(1,r)}(y)\psi_{0,0}^{(0)}(y)+P_1^{(1,r)}(y)\frac{d}{dy}\psi_{0,0}^{(0)}(y) \tag{5.176}$$

$$\psi_{0,0}^{(0)}(y)D_2(y)=\left(\frac{1}{2}P_0^{(1,r)}(y)^2+P_1^{(2,r)}(y)\right)\psi_{0,0}^{(0)}(y)+\left(P_0^{(1,r)}(y)P_1^{(1,r)}(y)+P_1^{(2,r)}(y)\right)\frac{d}{dy}\psi_{0,0}^{(0)}(y)+\frac{1}{2}P_1^{(1,r)}(y)^2\frac{d^2}{dy^2}\psi_{0,0}^{(0)}(y) \tag{5.177}$$

We already see that the $D_n(y)\frac{e^{n\theta}}{n!}$ term will be in the form

$$D_n(y)\frac{e^{n\theta}}{n!}=e^{n\theta}\sum_{j=1}^{n}\tilde{B}_j(y)\frac{d^j}{dy^j}\psi_{0,0}^{(0)}(y) \tag{5.178}$$

Where $\tilde{B}_j(y)$ are just polynomials that depend on $P_0^{(j,r)}(y)$ and $P_1^{(j,r)}(y)$, then we can use the CHE at the first order

$$\frac{d^2}{dy^2}\psi_{0,0}^{(0)}(y)=V^{(0)}(y)\psi_{0,0}^{(0)}(y) \tag{5.179}$$

to write all the derivatives in terms of just $\psi_{0,0}^{(0)}(y)$ and $\frac{d}{dy}\psi_{0,0}^{(0)}(y)$.
For example

$$\frac{d^3}{dy^3}\psi_{0,0}^{(0)}(y)=V^{(0)}(y)'\psi_{0,0}^{(0)}(y)+V^{(0)}(y)\frac{d}{dy}\psi_{0,0}^{(0)}(y) \tag{5.180}$$

$$\frac{d^4}{dy^4}\psi_{0,0}^{(0)}(y)=\left(V^{(0)}(y)^2+V^{(0)}(y)''\right)\psi_{0,0}^{(0)}(y)+2V^{(0)}(y)'\frac{d}{dy}\psi_{0,0}^{(0)}(y) \tag{5.181}$$

And so on. This leads us to

$$\psi_{0,0}(y)=\tilde{G}\left(\sum_{n=0}^{+\infty}\tilde{D}_n(y)e^{n\theta}\right)\psi_{0,0}^{(0)}(y)+\tilde{G}\left(\sum_{n=1}^{+\infty}\tilde{E}_n(y)e^{n\theta}\right)\frac{d}{dy}\psi_{0,0}^{(0)}(y) \tag{5.182}$$

With $\tilde{D}_n(y)$ and $\tilde{E}_n(y)$ some polynomials in e^y.
This method can also be used to compute other solutions (Floquet included) and will lead to a similar result.

6 Black hole perturbation

The Post-Newtonian (PN) and Multipole Post-Minkowskian (MPM) approaches apply when velocities are small and gravitational interactions are weak; black hole perturbation theory requires a small ratio between the masses of the two components of the binary system, but it makes no assumption on the strength of the gravitational interactions itself.
Finally, scattering amplitudes allow for finite velocities but require weak gravitational interactions and open trajectories ending on asymptotic states.
Black hole perturbation theory can be applied to the study of the gravitational waves produced by a light particle scattered in the Schwarzschild geometry.
The deformation produced by the particle motion can be viewed as a perturbation of the Schwarzschild geometry and at linear order it is described by a confluent Heun like equation (CHE).

6.1 Geodetic motion

We consider the motion of a particle of mass μ in the Schwarzschild metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{6.1}$$

With

$$f(r) = 1 - \frac{2M}{r} \tag{6.2}$$

M is the product of the Newton constant G and the black hole mass, i.e. a length rather than a mass scale. In the Hamiltonian formalism, the geodesic motion is governed by

$$\mathcal{H} = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu = -\frac{\mu^2}{2} \tag{6.3}$$

The affine parameter $d\tau = \mu^{-1}\sqrt{g_{\mu\nu}dx^\mu dx^\nu}$ is normalized such that the quadrivelocity u^μ and momentum p^μ coincide

$$u^\mu = p^\mu = \frac{dx^\mu}{d\tau} = \frac{\partial\mathcal{H}}{\partial p_\mu} \tag{6.4}$$

If we denote by $E = p_t$ and $J = p_\phi$ the conserved energy and angular momentum, the geodesic equations become

$$\frac{dt}{d\tau} = \frac{E}{f(r)} \quad , \quad \frac{d\phi}{d\tau} = \frac{J}{r^2} \quad , \quad \frac{d\theta}{d\tau} = 0 \quad , \quad \frac{dr}{d\tau} = \sqrt{E^2 - f(r)\left(\mu^2 + \frac{J^2}{r^2}\right)} \tag{6.5}$$

Without loss of generality, we can set $\theta = \pi/2$ as the orbital plane.
The stress energy tensor generated by a particle moving along the trajectory $x(\tau)$ reads

$$\mathcal{T}^{\mu\nu} = \int d\tau \frac{u^\mu v^\nu}{r^2\sin\theta}\delta^{(4)}(x - x(\tau)) \tag{6.6}$$

We are interested in the gravitational wave generated as a back reaction of the stress energy source on the geometry.
Post-Newtonian (PN) expansion: In this limit we consider scattering at small velocities and distances large compared to the Schwarzschild radius but still small with respect to the wavelength of the emitted wave.
We denote ω the frequency of the emitted wave.
To trace the PN order we introduce the bookkeeping parameter η and scale velocities with η and M with η^2 in such a way that the kinetic and potential energies scale in the same way, i.e.

$$p \to p\eta \quad , \quad M \to M\eta \quad , \quad \omega \to \omega\eta \quad , \quad J \to J\eta \quad , \quad \eta \to 0 \tag{6.7}$$

6.2 Teukolsky equation

We consider the perturbations of the metric generated by the motion of a light particle in the Schwarzschild geometry.
The metric perturbations can be conveniently decomposed according to their helicity s, with the physical components h_+, h related to the projection of the Weyl tensor with helicity $s = 2$, the so called ψ_4 mode.
At linear order, the Einstein equation for ψ can be separated into a radial and an angular part via the ansatz

$$\psi(X) = R^4\psi_4(X) = \int \frac{d\omega}{2\pi}\sum_{l,m} e^{-i\omega T}R_{lm}(R)Y^{lm}_{-2}(\Theta,\Phi) \tag{6.8}$$

with $X = (T, R, \Theta, \Phi)$ the coordinates of the observer and $Y^{lm}_{-2}(\Theta,\Phi)$ the spin weighted spherical harmonic functions. We use capital letters for the coordinates of the observer to distinguish them from the coordinates $x^\mu = (t, r, \theta, \phi)$ that will be used to describe the trajectory of the light particle.
The radial function $R_{lm}(r)$ satisfies the Teukolsky equation

$$r^4 f(r)^2\frac{d}{dr}\left(\frac{R'_{lm}}{r^2 f(r)}\right) + \left(\frac{\omega^2 r^2 + 4i\omega(r-M)}{f(r)} - 8i\omega r - (l+2)(l-1)\right)R_{lm}(r) = GT_{lm}(r) \tag{6.9}$$

with $T_{lm}(r)$ the harmonic decomposition of the stress energy source.

6.3 Confluent Heun equation correspondences

The confluent Heun differential equation (CHE)

$$g''(t) + g'(t)\left(\zeta + \frac{\gamma}{t} + \frac{\delta}{t-1}\right) + + \frac{\beta t - q}{(t-1)t} g(t) = 0 \tag{6.10}$$

is an ordinary differential equation of second order with two regular singularities at $t = 0, 1$ and an irregular one at infinity.
A basis of solutions is given by

$$HeunC(q, \beta, \gamma, \delta, \zeta, t) \tag{6.11}$$

$$t^{1-\gamma} HeunC((1-\gamma)(\zeta.\delta) + q, \beta + (1-\gamma)\zeta, 2-\gamma, \delta, \zeta, t) \tag{6.12}$$

With $HeunC$ the confluent Heun function.
One can also bring the irregular singularity to the origin, by taking $t = 1/z$, and introduce the parameters u, x, κ_i, such that

$$\beta = -x(\kappa_1 + \kappa_3) \quad , \quad \delta = \kappa_2 + \kappa_3 \quad , \quad \gamma = 1 - \kappa_2 + \kappa_3 \quad , \quad \zeta = -x \quad , \quad q = u - \kappa_3^2 - x(\kappa_1 + \kappa_3) \tag{6.13}$$

n this case the equation takes the form

$$G(z)'' + G'(z)\left(\frac{x}{z} + \frac{2\kappa_1 + 1}{z} + \frac{\kappa_2 + \kappa_3}{z-1}\right) + G(z)\frac{u - \kappa_1^2 + (\kappa_1 + \kappa_2)(\kappa_1 + \kappa_3)z}{(z-1)z^2} = 0 \tag{6.14}$$

with

$$G(z) = z^{-\kappa_1 - \kappa_3} g(\frac{1}{z}) \tag{6.15}$$

This is equivalent to the Teukolsky equation if we do the identifications

$$R_{lm}(z) = e^{-\frac{x}{2z}} (1-z)^{\frac{1+\kappa_2+\kappa_3}{2}} \left(\frac{x}{z}\right)^{\frac{3}{2}-\kappa_1} G(z) \quad , \quad u = \left(l + \frac{1}{2}\right)^2 + 10im\omega - 4\omega^2 M^2 \tag{6.16}$$

$$\kappa_1 = \frac{5}{2} + 2iM\omega = \frac{5}{2} + \frac{x}{2} \quad , \quad \kappa_2 = \frac{5}{2} - 2iM\omega = \frac{5}{2} - \frac{x}{2} \quad , \quad \kappa_3 = \frac{1}{2} - 2iM\omega = \frac{1}{2} - \frac{x}{2} \tag{6.17}$$

With $z = \frac{2M}{r}$ and $x = 4iM\omega$.
Ultimatley one can connect the Teukolsky (6.9) equation to the CHE (3.14) with the map

$$\begin{aligned} R_{\ell m}(r) &= \frac{r\sqrt{r-2M}}{\sqrt{2M}} \psi(y), \quad r = 4Me^{-y}, \\ e^{\theta} &= -4iM\omega, \quad P^2 = l(l+1) - 8M^2\omega^2 + \frac{1}{4}, \\ q_1 &= 2 - 2iM\omega, \quad q_2 = -2iM\omega, \quad q_3 = -2 - 2iM\omega. \end{aligned} \tag{6.18}$$

6.4 Correspondence

The Teukolsky-integrability correspondence maps the singularity at $y \to -\infty$ to $r \to +\infty$ that is the far region, the $y \to \ln 2$ singularity is mapped at $r \to 2M$ that is the horizon of the black hole and the singularity $y \to +\infty$ is mapped at $r \to 0$ the center of the black hole.
The connection coefficient $Q_2(\theta)$ that involves the two regular singularities is the easier to compute, since no kink trick is needed and doesn't contain A_D; this, however, in gravity language is computed inside of the black hole, which is physically less relevant.
The so called far away region is when $\frac{2M}{r} >> \omega M >> 0$, points away from the horizon, and corresponds in integrability at $y \to -\infty$ in the $\theta \to -\infty$ regime.
The so called Near horizon zone is when $\frac{2M}{r} \to 1$, which corresponds to $y \to \ln 2$. The near zone is when $\omega M >> \frac{2M}{r} >> 0$, in this region Post-Newtonian approximation applies that is $\theta \to -\infty$ while keeping y finite.
In [1], the authors find a way of writing the Floquet solutions in terms of $\psi_{\pm}^{(0)}(y)$ and his first derivative in 2 different regimes; in (5.8) we demonstrate that is indeed possible starting from the ansatz that we choose.
Demonstrating the exponential form (5.153) starting from the one used in [1] that is (5.182) is not that easy.
The advantage of (5.153) is that the convergence is more under control.

A $\Pi^{(n)}(y)$ terms

If we want to find terms up to nth instanton, we also have to find $\Pi_>^{(n)}(y)$ and $\Pi_<^{(n)}(y)$. We focus, for instance, on $\Pi_>^{(n)}(y)$, which is the solution of (4.30). We explicitly write the first two differential equations.

$$\Pi_>^{(0)}(y)^2 + \frac{d}{dy}\Pi_>^{(0)}(y) = V_>^{(0)}(y) \tag{A.1}$$

$$2\Pi_>^{(1)}(y)\Pi_>^{(0)}(y) + \frac{d}{dy}\Pi_>^{(1)}(y) = V_>^{(1)}(y) \tag{A.2}$$

We can now use the following ansatz:

$$\Pi_>^{(1)}(y) = \frac{d}{dy}\left(P_0^{(1)}(y) + P_1^{(1)}(y)\Pi_>^{(0)}(y)\right) \tag{A.3}$$

We can plug this in (A.2) and write the derivatives using

$$\Pi_>'^{(0)}(y) = V_>^{(0)} - \Pi_>^{(0)}(y)^2 \tag{A.4}$$

$$\Pi_>''^{(0)}(y) = V_>'^{(0)} - 2\Pi_>^{(0)}(y)V_>^{(0)} + 2\Pi_>^{(0)}(y)^3 \tag{A.5}$$

This gives

$$\begin{aligned}&P_0^{(1)}(y)'' + 2P_1^{(1)}(y)'V_>^{(0)} + P_1^{(1)}(y)V_>'^{(0)} + \left(2P_0^{(1)}(y)' + 2P_1^{(1)}(y)V_>^{(0)} + 2P_1^{(1)}(y)'' - 2P_1^{(1)}(y)V_>^{(0)}\right)\Pi_>^{(0)}(y) + \\ &+ \left(2P_1^{(1)}(y)' - 2P_1^{(1)}(y)'\right)\Pi_>^{(0)}(y)^2 + \left(2P_1^{(1)}(y) - 2P_1^{(1)}(y)\right)\Pi_>^{(0)}(y)^3 = V_>^{(1)}(y)\end{aligned} \tag{A.6}$$

Setting to zero the coefficient of $\Pi_>^{(0)}(y)$ we have the condition

$$2P_0^{(1)}(y)' + P_1^{(1)}(y)'' = 0 \tag{A.7}$$

This condition can be used in (**??**)to get a differential equation for $P_1^{(1)}(y)$

$$-\frac{1}{2}P_1^{(1)}(y)''' + 2P_1^{(1)}(y)'V_>^{(0)} + P_1^{(1)}(y)V_>'^{(0)} = V_>^{(1)} \tag{A.8}$$

We can continue with the third equation of (4.30), that is

$$2\Pi_>^{(2)}(y)\Pi_>^{(0)}(y) + \Pi_>^{(1)}(y)^2 + \frac{d}{dy}\Pi_>^{(2)}(y) = V_>^{(2)}(y) \tag{A.9}$$

And again, we make the ansatz

$$\Pi_>^{(2)}(y) = \frac{d}{dy}\left(P_0^{(2)}(y) + P_1^{(2)}(y)\Pi_>^{(0)}(y) + P_2^{(2)}(y)\Pi_>'^{(0)}(y)\right) \tag{A.10}$$

Substituting this into (A.9) and substituting the derivatives as before we get an equation with $\Pi_>^{(0)}(y)^4$ as the highest non zero power of $\Pi_>^{(0)}(y)$. The differential equations generated by imposing each coefficient of $\Pi_>^{(0)}(y)^n$ to zero are

$$\begin{aligned}&2P_2^{(2)}(y)'V_>^{(0)}(y)' + P_2^{(2)}(y)''V_>^{(0)}(y) - 2P_2^{(2)}(y)V_>^{(0)}(y)^2 + P_2^{(2)}(y)V_>^{(0)}(y)'' - V_>^{(2)}(y) + P_0^{(2)}(y)'' + \\ &+ P_1^{(1)}(y)^2V_>^{(0)}(y)^2 + 2P_1^{(1)}(y)V_>^{(0)}(y)P_0^{(1)}(y)' + P_0^{(1)}(y)'^2 + +2V_>^{(0)}(y)P_1^{(2)}(y)' + P_1^{(2)}(y)V_>^{(0)}(y)' = 0\end{aligned} \tag{A.11}$$

$$2P_1^{(1)}(y)'V_>^{(0)}(y)P_1^{(1)}(y)' + 2P_0^{(1)}(y)'P_1^{(1)}(y)' + 2P_0^{(2)}(y)' - 2P_2^{(2)}(y)V_>^{(0)}(y) + P_1^{(2)}(y)'' = 0 \tag{A.12}$$

$$- 2P_1^{(1)}(y)^2 V_>^{(0)}(y) + 4P_2^{(2)}(y)V_>^{(0)}(y) - 2P_1^{(1)}(y)P_0^{(1)}(y)' + P_1^{(1)}(y)'^2 - P_1^{(1)}(y)'' = 0 \tag{A.13}$$

$$- 2P_1^{(1)}(y)P_1^{(1)}(y)' + 2P_2^{(2)}(y)' = 0 \tag{A.14}$$

$$+ P_1^{(1)}(y)^2 - 2P_2^{(2)}(y) = 0 \tag{A.15}$$

If we use the conditions that we found for (A.2), we have that the last 3 equations are all equivalent to

$$P_2^{(2)}(y) = \frac{P_1^{(1)}(y)^2}{2} \tag{A.16}$$

We notice is a regular equation not a differential one. If we substitute this in the other ones, and use the previous conditions, we get another condition on the coefficient of $\Pi_>^{(0)}(y)$

$$P_0^{(2)}(y)' = -\frac{1}{2}P_1^{(2)}(y)'' - P_1^{(1)}(y)'P_0^{(1)}(y)' \tag{A.17}$$

Finally, using this equation, we are left with

$$+2P_1^{(2)}(y)'V_>^{(0)}(y) + P_1^{(2)}(y)V_>^{(0)}(y)' - \frac{1}{2}P_1^{(2)}(y)' = V_>^{(2)}(y) - F_2(y) \tag{A.18}$$

with $F_2(y)$ is just a function of the potential and $P_1^{(1)}(y)$

$$F_2(y) = P_1^{(1)}(y)'^2 V_>^{(0)}(y) + 2P_1^{(1)}(y)P_1^{(1)}(y)'V_>^{(0)}(y)' + \frac{3}{4}P_1^{(1)}(y)''^2 + \frac{1}{2}P_1^{(1)}(y)^2 V_>^{(0)}(y)'' + \frac{1}{2}P_1^{(1)}(y)'P_1^{(1)}(y)''' \tag{A.19}$$

We can go on with the next equations, the procedure is the same:

- Use an ansatz of the type

$$\Pi_>^{(n)}(y) = \frac{d}{dy}\left(P_0^{(n)}(y) + \sum_{i=1}^{n} P_i^{(n)}(y)\frac{d^{i-1}}{dy^{i-1}}\Pi_>^{(0)}(y)\right) \tag{A.20}$$

- Then we substitute the derivatives of $\Pi_>^{(0)}(y)$ with powers of $\Pi_>^{(0)}(y)$ using (A.1) and substitute it in their equation (4.30) together with all previous ansatze.
- From that we obtain a differential equation in powers of $\Pi_>^{(0)}(y)$, the coefficient of the highest power m gives a condition that can be replaced in the previous ones.
- We will be left with $m-1$ set of equations and the highest one will give another condition
- The procedure goes on until we are left with 2 differential equations, one for $P_1^{(n)}(y)$ and one for $P_0^{(n)}(y)$

At all orders we will have

$$P_0^{(n)}(y)' = -\frac{1}{2}P_1^{(n)}(y)'' + G_n(y) \tag{A.21}$$

$$-\frac{1}{2}P_1^{(n)}(y)''' + 2P_1^{(n)}(y)'V_>^{(0)}(y) + P_1^{(n)}(y)V_>^{(0)}(y)' = V_>^{(n)}(y) - F_n(y) \tag{A.22}$$

With G_n and F_n some functions of $P_1^{(i)}(y)$ with $i < n$ and of $V_>^{(0)}(y)$ For $P_i^{(n)}(y)$ for $1 < i < n$ we recognized this pattern

$$\boxed{P_j^{(n)}(y) = \frac{1}{n!}B_{n,j}\left(1!P_1^{(1)}(y), 2!P_1^{(2)}(y), \ldots, (n-j+1)!P_1^{(n-j+1)}(y)\right), \qquad 1 \le j \le n} \tag{A.23}$$

where $B_{n,j}$ are the ordinary partial Bell polynomials [2] that is checked up to $n = 6$. We stress that $V_>^{(n)}(y)$ is invariant under $k \to -k$ since it depends only on k^2. Up to this point we never used the explicit form of the potential, so these equations are true for all the flavours and for both kinks.

A.1 $N_f = 0$

In the specific case of $N_f = 0$ we find that $P_1^{(n)}(y)$ are polynomials in e^{-y}

$$P_1^{(n)}(y) = \sum_{k=1}^{n} f_{n1k} e^{-ky} \tag{A.24}$$

This also implies

$$P_0^{(n)}(y) = \sum_{k=1}^{n} f_{n0k} e^{-ky} \tag{A.25}$$

explicitly one has

$$f_{111} = \frac{2}{1-4k^2} = -p_0^{(1)}(k) \tag{A.26}$$

The condition $f_{111} = -p_0^{(1)}(k)$ is redundant, this will be true for each f_{n11} and can be used to find $p_0^{(n)}(y)$ or viceversa to find f_{n11}. We also find interestingly

$$J_{\pm 2k}(2ie^{\frac{y}{2}}) = \frac{(i)^{\pm 2k}\, e^{\pm ky}}{\Gamma(1 \pm 2k)} \left(1 + \sum_{m=1}^{+\infty} \frac{e^{my}}{m!(1 \pm 2k)_m}\right) \tag{A.27}$$

where $(1 \pm 2k)_m$ are the Pochhammer symbols. This gives us

$$\ln J_{\pm 2k}(2ie^{\frac{y}{2}}) = \ln\left(\frac{(i)^{\pm 2k}\, e^{\pm ky}}{\Gamma(1 \pm 2k)}\right) + \ln\left(1 + \sum_{m=1}^{+\infty} \frac{e^{my}}{m!(1 \pm 2k)_m}\right) = \tag{A.28}$$

$$= \ln\left(\frac{(i)^{\pm 2k}\, e^{\pm ky}}{\Gamma(1 \pm 2k)}\right) + \sum_{m=1}^{+\infty} e^{my} A_m(\pm k) \tag{A.29}$$

where [2]

$$\boxed{A_m(k) = \sum_{1 \le p \le m} (-1)^{p-1}(p-1)! B_{m,p}(g_1, g_2, \ldots, g_{m-p+1})} \tag{A.30}$$

where $B_{m,p}$ are the partial Bell Polynomials and g_m are given by

$$g_m = \frac{1}{(1+2k)_m} = \frac{\Gamma(1+2k)}{\Gamma(1+2k+m)} \tag{A.31}$$

From this we noticed a relation with some coefficients of the $P_1^{(n)}(y)$ and $P_0^{(n)}(y)$ polynomials:

$$\boxed{f_{m1m} = \frac{1}{k}\frac{A_m(-k) - A_m(k)}{2}} \tag{A.32}$$

and

$$\boxed{f_{m0m} = \frac{A_m(-k) + A_m(k)}{2}} \tag{A.33}$$

We list some coefficients in terms of f_{111}

$$f_{212} = \frac{(5+4k^2) f_{111}^2}{16(k^2-1)} \tag{A.34}$$

$$f_{211} = -\frac{(7+20k^2) f_{111}^2}{8(k^2-1)(4k^2-1)} \tag{A.35}$$

$$f_{202} = \frac{1+8k^2}{16(k^2-1)} f_{111}^2 \tag{A.36}$$

$$f_{201} = \frac{7+20k^2}{16(1-5k^2+4k^4)} f_{111}^2 \tag{A.37}$$

$$f_{301} = \frac{29+232k^2+144k^4}{4(1-4k^2)^2(9-13k^2+4k^4)} f_{111}^3 \tag{A.38}$$

$$f_{302} = \frac{53+776k^2+656k^4}{48(9-49k^2+56k^4-16k^6)} f_{111}^3 \tag{A.39}$$

$$f_{303} = \frac{3+68k^2+64k^4}{24(9-13k^2+4k^4)} f_{111}^3 \tag{A.40}$$

$$f_{311} = \frac{29+232k^2+144k^4}{2(4k^2-1)^2(9-13k^2+4k^4)} f_{111}^3 \tag{A.41}$$

$$f_{312} = \frac{121+616k^2+208k^4}{24(9-49k^2+56k^4-16k^6)} f_{111}^3 \tag{A.42}$$

$$f_{313} = \frac{23+96k^2+16k^4}{24(9-13k^2+4k^4)} f_{111}^3 \tag{A.43}$$

A.2 $N_f = 3$

To get $\Pi_+^{(n)}(y)$ we have to solve iteratively (5.33). We can do that using the ansatz[2]

$$\Pi_+^{(n)}(y) = \frac{d}{dy}\left(P_0^{(n,r)}(y) + \sum_{i=1}^{n} P_i^{(n,r)}(y)\frac{d^{i-1}}{dy^{i-1}}\Pi_+^{(0)}(y)\right) \tag{A.44}$$

where $P_0^{(n,r)}(y) = \sum_{k=1}^{n} \tilde{f}_{n0k} e^{-ky}$ and $P_1^{(n,r)}(y) = \sum_{j=1}^{n} f_{n1j}\,(1-2e^{-y})^j$ for $j \geq 1$. $P_j^{(n,r)}(y)$ for $j > 1$ are found using the formula (A.23).
We eventually find

$$\Pi_+^{(1)}(y) = \frac{4q_3}{1-4k^2}\frac{d}{dy}\left[\frac{e^{-y}}{2} + \left(e^{-y} - \frac{1}{2}\right)\Pi_+^{(0)}(y)\right] \tag{A.45}$$

And we could go on with higher orders.
In order to get $\Pi_{<+}^{(n)}(y)$ we have to solve iteratively (5.47). We can do that using the ansatz:

$$\Pi_<^{(n)}(y) = \frac{d}{dy}\left(P_0^{(n,l)}(y) + \sum_{i=1}^{n} P_i^{(n,l)}(y)\frac{d^{i-1}}{dy^{i-1}}\Pi_<^{(0)}(y)\right) \tag{A.46}$$

where $P_j^{(n,l)}(e^y) = \sum_{k=1}^{n} f_{njk} e^{ky}$.
For example we find

$$\Pi_<^{(1)}(y) = \left(\frac{q_1 q_2}{2(4k^2-1)} + \frac{1}{8}\right)\frac{d}{dy}\left[-e^y + 2e^y \Pi_<^{(0)}(y)\right] \tag{A.47}$$

B Computing $S_\pm$ and S_0

We recall the definition

$$S_- = \left[\sum_{n=1}^{+\infty} e^{n\theta}\left(P_0^{(n,l)}(y') + \sum_{i=1}^{n} P_i^{(n,l)}(y')\frac{d^i}{dy'^i}\ln\psi_{\pm<}^{(0)}(y')\right)\right]_{y'=-\infty} \tag{B.1}$$

[2]We will focus on $\Pi_+(y)$, knowing that $\Pi_-(y,k) = \Pi_+(y,-k)$.

Since we have

$$\psi^{(0)}_{\pm<}(y) \approx \frac{1}{a_-(\pm k)} e^{e^{-y}} e^{\left(\frac{1}{2}-q_3\right)y} \qquad y \to -\infty \tag{B.2}$$

$$\frac{d}{dy} \ln \psi^{(0)}_{\pm<}(y) \approx -e^{-y} + \frac{1}{2} - q_3 \qquad y \to -\infty \tag{B.3}$$

We also know that

$$P_j^{(n,l)}(y) = \sum_{m=1}^{n} f_{njm} e^{my} \tag{B.4}$$

For $j = 0$ and for $j = 1$, while for $j > 1$, the polynomials start with e^{2y}.
In this way the contribution at $-\infty$ is just given by

$$S_- = \sum_{n=1}^{+\infty} e^{n\theta} \left(-f_{n11}\right) \tag{B.5}$$

For S_+ we recall that we have

$$S_+ = \left[\sum_{n=1}^{+\infty} e^{n\theta} \left(P_0^{(n,r)}(y') + \sum_{i=1}^{n} P_i'^{(n,r)}(y') \frac{d^i}{dy'^i} \ln \psi^{(0)}_{\pm}(y')\right)\right]_{y'=+\infty} \tag{B.6}$$

At $y \to +\infty$ we have

$$\psi^{(0)}_{\pm}(y) \approx \frac{1}{a_+(\pm k)} e^{\frac{q_1-q_2}{2} y} \qquad y \to +\infty \tag{B.7}$$

$$\frac{d}{dy} \ln \psi^{(0)}_{\pm}(y) \approx \frac{q_1 - q_2}{2} \qquad y \to +\infty \tag{B.8}$$

We also have

$$\lim_{y \to +\infty} P_0^{(n,r)}(y) = 0 \tag{B.9}$$

$$P_1^{(n,r)}(y) = \sum_{j=1}^{n} \tilde{f}_{n1j} \left(1 - 2e^{-y}\right)^j \tag{B.10}$$

In this way the contribution at $+\infty$ is just given by

$$S_+ = \frac{q_1 - q_2}{2} \sum_{n=1}^{+\infty} e^{n\theta} \left(\sum_{j=1}^{n} \tilde{f}_{n1j}\right) \tag{B.11}$$

Last we compute

$$S_0 = \left[\sum_{n=1}^{+\infty} e^{n\theta} \left(P_0^{(n,r)}(y') + \sum_{i=1}^{n} P_i^{(n,r)}(y') \frac{d^i}{dy'^i} \ln \psi^{(0)}_{\pm}(y')\right)\right]_{y'=\ln 2} \tag{B.12}$$

With the behavior

$$\psi^{(0)}_{\pm}(y) \approx \frac{1}{a_0(\pm k)} (e^y - 2)^{\frac{1-q_1-q_2}{2}} \qquad y \to \ln 2 \tag{B.13}$$

$$\frac{d}{dy} \ln \psi^{(0)}_{\pm}(y) \approx \frac{1 - q_1 - q_2}{2} \frac{d}{dy} \ln(e^y - 2) \qquad y \to \ln 2 \tag{B.14}$$

$$\frac{d^j}{dy^j} \ln \psi^{(0)}_{\pm}(y) \approx \frac{1 - q_1 - q_2}{2} \frac{d^j}{dy^j} \ln(e^y - 2) \approx \frac{1 - q_1 - q_2}{2} (-1)^{j-1}(j-1)! \left(\frac{e^y}{e^y - 2}\right)^j \qquad y \to \ln 2 \tag{B.15}$$

We know

$$\sum_{j=1}^{+\infty} \left(\frac{d^j}{dy^j} \ln \psi^{(0)}_0(y)\right) \sum_{n=j}^{+\infty} e^{n\theta} P_j^{(n,r)}(y) = \sum_{j=1}^{+\infty} \frac{1}{j!} \left(\sum_{n=1}^{+\infty} e^{n\theta} P_1^{(n,r)}(y)\right)^j \frac{d^j}{dy^j} \ln \psi^{(0)}_0(y) \tag{B.16}$$

And

$$P_1^{(n,r)}(y) = \sum_{j=1}^{n} \tilde{f}_{n1j}\left(1-2e^{-y}\right)^j = \sum_{j=1}^{n} \tilde{f}_{n1j}\left(\frac{e^y-2}{e^y}\right)^j \tag{B.17}$$

This leads to

$$\left[\sum_{j=1}^{+\infty}\left(\frac{d^j}{dy^j}\ln\psi_0^{(0)}(y)\right)\sum_{n=j}^{+\infty} e^{n\theta}P_j^{(n,r)}(y)\right]_{y=\ln 2} = \frac{1-q_1-q_2}{2}\left(e^{\theta}\tilde{f}_{111}+e^{2\theta}\left(-\frac{\tilde{f}_{111}^2}{2}+\tilde{f}_{211}\right)\right)+ \tag{B.18}$$

$$+\frac{1-q_1-q_2}{2}\left(e^{3\theta}\left(\frac{\tilde{f}_{111}^3}{3}-\tilde{f}_{211}\tilde{f}_{111}+\tilde{f}_{311}\right)+e^{4\theta}\left(-\frac{\tilde{f}_{111}^4}{4}+\tilde{f}_{111}^2\tilde{f}_{211}-\frac{\tilde{f}_{211}^2}{2}-2\tilde{f}_{311}\tilde{f}_{111}+\tilde{f}_{411}\right)\right)+O(e^{6\theta}) \tag{B.19}$$

We also have

$$\left[\sum_{n=1}^{+\infty} e^{n\theta}P_0^{(n,r)}(y)\right]_{y=\ln 2} = -\frac{1}{2}\left(e^{\theta}\tilde{f}_{111}+e^{2\theta}\left(-\frac{\tilde{f}_{111}^2}{2}+\tilde{f}_{211}\right)\right)+ \tag{B.20}$$

$$-\frac{1}{2}\left(e^{3\theta}\left(\frac{\tilde{f}_{111}^3}{3}-\tilde{f}_{211}\tilde{f}_{111}+\tilde{f}_{311}\right)+e^{4\theta}\left(-\frac{\tilde{f}_{111}^4}{4}+\tilde{f}_{111}^2\tilde{f}_{211}-\frac{\tilde{f}_{211}^2}{2}-2\tilde{f}_{311}\tilde{f}_{111}+\tilde{f}_{411}\right)\right)+O(e^{6\theta}) \tag{B.21}$$

That ultimately leads to

$$S_0 = -\frac{q_1+q_2}{2}\left(e^{\theta}\tilde{f}_{111}+e^{2\theta}\left(-\frac{\tilde{f}_{111}^2}{2}+\tilde{f}_{211}+\right)+e^{3\theta}\left(\frac{\tilde{f}_{111}^3}{3}-\tilde{f}_{211}\tilde{f}_{111}+\tilde{f}_{311}\right)\right)+ \tag{B.22}$$

$$-\frac{q_1+q_2}{2}\left(e^{4\theta}\left(-\frac{\tilde{f}_{111}^4}{4}+\tilde{f}_{111}^2\tilde{f}_{211}-\frac{\tilde{f}_{211}^2}{2}-2\tilde{f}_{311}\tilde{f}_{111}+\tilde{f}_{411}\right)\right)+O(e^{6\theta}) \tag{B.23}$$

These quantities are computed up to 3 istantons order and checked with the prepotential of $N_f = 3$ found in [8], we find

$$S_0 = \frac{\partial\mathcal{F}_{inst}}{\partial q_1}+\frac{\partial\mathcal{F}_{inst}}{\partial q_2} \tag{B.24}$$

$$S_- = \frac{\partial\mathcal{F}_{inst}}{\partial q_3} \tag{B.25}$$

$$S_+ = \frac{\partial\mathcal{F}_{inst}}{\partial q_1}-\frac{\partial\mathcal{F}_{inst}}{\partial q_2} \tag{B.26}$$

C $A_\pm(\theta)$

$$A_+(\theta) = +\sum_{n=1}^{+\infty} e^{n\theta}P_0^{(n,l)}(a-\theta) - e^{\theta-a}e^{-\sum_{n=1}^{+\infty} e^{n\theta}P_1^{(n,l)}(a-\theta)} + \sum_{n=1}^{+\infty}\frac{L_n(k)+L_n(-k)}{2}\frac{e^{n(\theta-a)}}{n!}e^{-n\sum_{n=1}^{+\infty} e^{n\theta}P_1^{(n,l)}(a-\theta)}+ \tag{C.1}$$

$$-\sum_{n=1}^{+\infty} e^{n\theta}P_0^{(n,r)}(a) - \frac{1-q_1-q_2}{2}\ln\left(e^{a+\sum_{n=1}^{+\infty} e^{n\theta}P_1^{(n,r)}(a)}-2\right) - \sum_{n=1}^{+\infty}\frac{R_n(k)+R_n(-k)}{2}\frac{e^{na}}{n!}e^{n\sum_{n=1}^{+\infty} e^{n\theta}P_1^{(n,r)}(a)} \tag{C.2}$$

$$A_-(\theta) = -k\sum_{n=1}^{+\infty} e^{n\theta}P_1^{(n,l)}(a-\theta) + \sum_{n=1}^{+\infty}\frac{L_n(k)-L_n(-k)}{2}\frac{e^{n(\theta-a)}}{n!}e^{-n\sum_{n=1}^{+\infty} e^{n\theta}P_1^{(n,l)}(a-\theta)}+ \tag{C.3}$$

$$-k\sum_{n=1}^{+\infty} e^{n\theta}P_1^{(n,r)}(a) - \sum_{n=1}^{+\infty}\frac{R_n(k)-R_n(-k)}{2}\frac{e^{na}}{n!}e^{n\sum_{n=1}^{+\infty} e^{n\theta}P_1^{(n,r)}(a)} \tag{C.4}$$

Where we have

$$\ln {}_1F_1\left(\frac{1}{2} \mp k + q_3, 1 \mp 2k, 2e^{-y}\right) = \sum_{n=1} L_n(\pm k)\frac{1}{n!}e^{-ny} \tag{C.5}$$

$$L_n(\pm k) = \sum_{j=1}^{n}(-1)^{j-1}(j-1)!B_{n,j}\left(\frac{\left(\frac{1}{2} \mp k + q_3\right)_1}{(1 \mp 2k)_1}2, \frac{\left(\frac{1}{2} \mp k + q_3\right)_2}{(1 \mp 2k)_2}2^2, ..., \frac{\left(\frac{1}{2} \mp k + q_3\right)_n}{(1 \mp 2k)_n}2^n\right) \tag{C.6}$$

And

$$\ln {}_2F_1\left(\frac{1}{2} \pm k - q_1, \frac{1}{2} \pm k - q_2; 1 \pm 2k; \frac{e^y}{2}\right) = \sum_{n=1}^{+\infty} R_n(\pm k)\frac{1}{n!}e^{ny} \tag{C.7}$$

$$R_n(\pm k) = \sum_{j=1}^{n}(-1)^{j-1}(j-1)!B_{n,j}\left(\frac{\left(\frac{1}{2} \pm k - q_1\right)_1\left(\frac{1}{2} \pm k - q_2\right)_n}{(1 \pm 2k)_1}\frac{1}{2}, ..., \frac{\left(\frac{1}{2} \pm k - q_1\right)_n\left(\frac{1}{2} \pm k - q_2\right)_n}{(1 \pm 2k)_n}\frac{1}{2^n}\right) \tag{C.8}$$

Computing order by order we find

$$A_+(\theta) = \frac{-1 + q_1 + q_2}{2}\ln(-2) + O(e^{2\theta}) \tag{C.9}$$

We check that is zero at e^θ and we think it could be true at all istantons, we also compute

$$A_-(\theta) = \frac{8kq_1q_2q_3}{(4k^2-1)^2}e^\theta + O(e^{2\theta}) \tag{C.10}$$

Some interesting cancellations are used

$$f_{n1n} = \frac{R_n(k) - R_n(-k)}{2k}\frac{1}{n!} \tag{C.11}$$

$$(-2)^n f_{n1n} = \frac{L_n(k) - L_n(-k)}{2k}\frac{1}{n!} \tag{C.12}$$

$$\sum_{n=1}^{+\infty} e^{n\theta} f_{n0n} = \sum_{n=1}^{+\infty}\frac{e^{n\theta}}{n!}\frac{R_n(k) + R_n(-k)}{2} + \frac{1 - q_1 - q_2}{2}\ln\left(e^\theta - 2\right) \tag{C.13}$$

$$\tilde{f}_{101} = \left(\frac{L_1(k) + L_1(-k)}{2} - 1\right) \tag{C.14}$$

$$\tilde{f}_{n0n} = \frac{L_n(k) + L_n(-k)}{2}\frac{1}{n!} \tag{C.15}$$

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