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Non-Invertible Duality Defects and Lattice Symmetries in 2d CFTs

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To my grandfather,
Nonno Remo.

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Abstract

Generalized symmetries have become a powerful tool for understanding quantum field theory. In particular, two-dimensional conformal field theories provide an ideal setting for exploring these ideas, thanks to their solvability and rich defect algebras. This thesis precisely investigates non-invertible duality symmetries in compact boson models. After reviewing the $c = 1$ theory and the construction of duality symmetries via discrete gauging, we analyze their action on Dp-branes through their definition in terms of Ishibashi states. We then turn to the $c = 2$ compact boson, whose moduli space and $O(2, 2; \mathbb{Z})$ duality group exhibit a significantly richer structure. The central result of this work is the introduction of a systematic method for constructing non-invertible duality symmetries at rational points by exploiting the crystallographic symmetries of the underlying lattice to the compactification torus. This approach reveals a clear geometric origin for these defects. Finally, we analyze the action of duality symmetries on Dp-branes, showing that, unlike in the $c = 1$ case, they can induce genuinely non-trivial transformations on boundary conditions.

Contents

1 Introduction

Symmetries as a guiding principle in physics

Symmetries have always played a central role in our understanding of the physical world. From the earliest formulations of classical mechanics to the modern framework of quantum field theory, the notion of symmetry has been fundamental in understanding the laws of nature. In its most basic formulation, a symmetry expresses the invariance of a physical system under a specific class of transformations; it corresponds to a change of perspective that leaves the physical content of the theory unchanged. Mathematically, the set of all such transformations forms a group, encoding how successive symmetry operations combine and how they can be undone. This interplay between symmetries and group theory provides a powerful algebraic framework for classifying physical systems and constraining their possible behaviors. Indeed, much of modern theoretical physics can be understood as the study of how the symmetries of a system, expressed through their underlying group structure, restrict its dynamics and observable quantities. This relatively simple idea underlies some of the most fundamental developments in physics: it connects conservation laws to invariance principles, organizes the possible interactions between fundamental fields, and even constrains the geometry of spacetime itself.

The relation between continuous symmetries and conservation laws was first established by Emmy Noether in 1918 [1]. Noether's theorem shows that every continuous symmetry of the action of a physical system corresponds to a conserved current and an associated conserved charge. This insight explains why, for example, the conservation of energy, momentum, and angular momentum in classical mechanics is not an empirical coincidence, but rather a direct consequence of the invariance of physical laws under time translations, spatial translations, and spatial rotations, respectively [2]. This connection between symmetries and conservation laws is so fundamental and powerful that it has become one of the conceptual bases of modern theoretical physics.

During the 20th century, then, the concept of symmetry evolved from a useful mathematical tool to a guiding principle in the formulation of physical theories. One of the most peculiar examples is given by Einstein's special relativity [3, 4], which was built on the invariance of physical laws under Lorentz transformations, and its generalization to general relativity, which elevated this principle to full local covariance under arbitrary coordinate transformations [5, 6]. Similarly, in quantum mechanics, symmetries controlled the structure of the Hilbert space and the degeneracies in the energy spectrum [7]. With the subsequent development of quantum field theory, then, symmetry principles became even more central [8, 9]; in this framework, they acquired a structural relevance, dictating the form of the interactions allowed by the theory. Gauge symmetries, in particular, play a fundamental role. The requirement of invariance under local, spacetime-dependent, transformations, necessitates the introduction of gauge fields that mediate interactions and fixes the allowed couplings between matter and interaction mediators. The Standard Model of particle physics is the culmination of this symmetry-based construction: an $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge theory whose interaction structure is completely determined by the requirement of local gauge invariance [10]. In this sense, the dynamics of the fundamental interactions are not arbitrarily postulated but emerge as a direct consequence of symmetry principles.

Despite their central role, however, the traditional notion of symmetry as a group of transformations acting on local fields eventually proved to be too restrictive. As theoretical physics evolved, particularly in the study of strongly coupled gauge theories, topological phases of matter, and conformal field theories, new types of symmetries began to appear that could not be described by the usual group theory formalism.

Beyond groups: higher-form and non-invertible symmetries

As we said, while ordinary symmetries have long provided a foundational organizing principle in theoretical physics, the last decade has revealed that they represent only a small subset of a much richer class of transformations. An important step in this generalization came with the development of higher-form symmetries [11–13], which extend the notion of symmetry from transformations acting on point-like operators to transformations acting on extended objects. In a theory with a p-form global symmetry, the charged operators are p-dimensional: rather than local objects, the natural probes of the symmetry are extended operators supported on p-dimensional submanifolds.

A simple and important illustration is provided by Wilson lines in gauge theories. Given a gauge field A valued in the Lie algebra of some gauge group G, a Wilson operator is defined along a one-dimensional curve γ by

$$W_R(\gamma) = \mathrm{Tr}_R \, \mathcal{P} \exp\left(i \int_\gamma A \right) , \tag{1.0.1}$$

where R is a representation of G and $\mathcal{P}$ denotes normal ordering. Such operators are intrinsically extended, as their support is the one dimensional submanifold γ. In Maxwell theory, these Wilson lines are charged under an electric 1-form symmetry whose symmetry operators are topological codimension-two surfaces that can be linked with the Wilson line. Likewise, 't Hooft lines are charged under the magnetic 1-form symmetry, and the interplay between the two forms the familiar electric–magnetic duality structure of abelian gauge theories [14–16].

Pure Yang-Mills theory offers another example. When the gauge group is $SU(N)$, the theory possesses a $\mathbb{Z}_N$ 1-form center symmetry. This symmetry acts on Wilson lines in representations that transform non-trivially under the center of $SU(N)$, and the way these line operators transform under the center symmetry carries direct physical information about confinement.For instance, in the confining phase, the area law for Wilson loops is precisely a consequence of the unbroken $\mathbb{Z}_N$ 1-form symmetry, whereas in the deconfined or Higgs phases this symmetry is explicitly or spontaneously broken. Thus, the phase structure of Yang–Mills theory can be reinterpreted in terms of symmetry breaking of higher-form symmetries [11, 17–19].

These examples illustrate a fundamental change in perspective. Higher-form symmetries are still global symmetries, but their action is no longer on point-like operators. Instead, they determine how extended operators, such as lines, surfaces, and higher-dimensional defects, behave and transform. In practice, this means that they control the allowed correlation functions of such operators and often dictate the phase structure of gauge theories, including whether a theory confines or supports topological order. They therefore provide the natural language for situations in which the basic probes of the theory are extended geometric objects.

A second, possibly even more surprising, development was the discovery of non-invertible symmetries. These arise most naturally in two-dimensional conformal field theories and in topological phases of matter, where one encounters topological line or surface defects that can be freely deformed without modifying correlation functions, but whose composition laws fail to form a group structure. A paradigmatic example is the Kramers-Wannier (KW) duality [20–22] of the two-dimensional Ising model. The KW transformation exchanges the high-temperature (disordered) and low-temperature (ordered) phases of the model and acts non-trivially on the spin and disorder operators. In the CFT description, this duality is implemented by a topological line defect [23, 24]. The peculiarity is that this defect is non-invertible: fusing it with itself does not reproduce the identity defect, but instead yields a direct sum of defects corresponding to the identity and to the $\mathbb{Z}_2$ spin-flip symmetry. This is encoded in the fusion rule

$$D_{\mathrm{KW}} \times D_{\mathrm{KW}} \;=\; \mathbf{1} \oplus \sigma, \tag{1.0.2}$$

where σ denotes the $\mathbb{Z}_2$ symmetry defect [25–28].

More modern examples arise in four-dimensional gauge theories at strong coupling [29–32], where duality walls, which are codimension-one interfaces implementing electric-magnetic duality, act as non-invertible symmetries mixing Wilson and 't Hooft operators in a manner incompatible with a group-like composition law [33–35]. In all these cases, the transformations act consistently on the operator algebra and on the Hilbert space, constrain the dynamics of the theory, and define selection rules, despite lacking inverses in the usual sense.

The topological defects point of view also provides a unified framework for describing 't Hooft anomalies. In this language, anomalies are encoded in the categorical data appearing in the fusion and associativity relations of symmetry defects. Since these fusion rules are part of the RG-invariant topological data of the theory, they persist along the RG flow, provided the symmetry is not broken in the infrared. In particular, this perspective applies uniformly to ordinary, higher-form, and non-invertible symmetries, for which, in all cases, the anomaly is captured by the categorical structure of the symmetry defects fusion algebra.

These developments made it clear that the traditional definition of symmetry, as a set of transformations forming a group structure and acting on local fields, is too restrictive. The modern perspective, initiated in [11], describes symmetries as topological defect operators of codimension $(p+1)$, whose insertion in correlation functions can be freely deformed without affecting physical observables. Ordinary symmetries correspond to the case $p = 0$, higher-form symmetries arise for $p > 0$, and non-invertible symmetries appear when the corresponding topological defects do not admit inverses under fusion [36]. This unified framework, known as generalized global symmetries, is able to describe dualities, topological defects, extended operators, and higher-form transformations in the same way. In this way, generalized symmetries extend rather than replace the conventional notion of symmetry, providing a deeper and more flexible language for describing the global structure of quantum field theories [37, 12, 13].

Two-dimensional conformal field theories as a testing ground

To explore generalized symmetries concretely, two-dimensional conformal field theories (2d CFTs) provide an ideal setting. Indeed, these theories are central in theoretical physics; they are simple enough to be exactly solvable, but at the same time rich enough to exhibit highly non-trivial algebraic and geometric structures. In addition, in two dimensions, the infinite-dimensional extension of the conformal group, that is to say, the Virasoro algebra, provides these theories with an exceptional degree of symmetry, allowing for exact calculations of correlation functions, partition functions, and operator spectra [38–40].

Moreover, 2-dimensional CFTs naturally admit a description in terms of extended operators and topological defects [22, 41], making them a perfect laboratory for studying generalized and non-invertible symmetries. The operator content of a 2d CFT can be organized into representations of the chiral algebra, and its topological defects can be classified by the modular properties of the theory. In particular, rational conformal field theories (RCFTs), which possess a finite number of primary operators and well-defined modular S and T matrices, allow one to construct explicit examples of non-invertible topological lines satisfying categorical fusion rules, such as those of Tambara–Yamagami [42] or more general fusion categories. These same structures appear in condensed matter systems, topological quantum field theories, and string compactifications.

String theory and the role of non-linear sigma models

Beyond their intrinsic interest, two-dimensional conformal field theories play a central role in string theory. In this framework, the fundamental object is a one-dimensional string whose worldsheet sweeps out a two-dimensional surface as it propagates through spacetime. The dynamic of the string is governed by a two-dimensional quantum field theory defined on the worldsheet, known as a non-linear sigma model. The fields $X^i(\sigma,\tau)$ of this theory describe maps from the worldsheet Σ into a target manifold $\mathcal{M}$, which represents the spacetime in which the string moves.

The action of such a sigma model takes the general form

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma G_{ij}(X)\,\partial_\mu X^i \partial^\mu X^j + \frac{i}{4\pi\alpha'}\int_\Sigma B_{ij}(X)\,\epsilon^{\mu\nu}\partial_\mu X^i \partial_\nu X^j + \cdots\,, \tag{1.0.3}$$

where G_{ij} is the target-space metric and B_{ij} is the antisymmetric Kalb–Ramond two-form field [43, 44]. Consistency conditions of the worldsheet theory impose severe constraints on the background fields G_{ij} and B_{ij}; in particular, the requirement of conformal invariance translates into the Einstein equations in the target space [45]. In this way, spacetime geometry emerges as a condition for the quantum consistency of the two-dimensional worldsheet theory.

Non-linear sigma models thus serve as toy models for string theory, allowing the study of the interplay between geometry and quantum field theory. They encode the symmetries of the target space as global or local symmetries of the worldsheet, and their dualities, such as T-duality or mirror symmetry, are encoded by transformations acting on the worldsheet fields [46, 47]. An important example is T-duality: a sigma model describing strings propagating on a circle of radius R is equivalent, as a CFT, to one on a circle of radius $1/R$, with momentum modes exchanged with winding modes. This duality acts non-trivially on worldsheet operators, mapping vertex operators to one another and exchanging Neumann and Dirichlet boundary conditions, thereby relating different D-brane configurations. Because of this structure, 2d CFT techniques allow one to analyze D-branes, moduli spaces of string compactifications, orbifolds, and other phenomena in a mathematically precise way.

In summary, two-dimensional CFTs, in particular sigma models, are the basis of perturbative string theory: they encode the geometry and topology of the target space, determine which backgrounds are consistent, and capture the rich set of dualities and extended symmetries that characterize string dynamics.

Generalized symmetries in non-linear sigma models and string theory

The framework of non-linear sigma models provides, then, a natural setting for the appearance of generalized symmetries. Since the worldsheet fields $X^i(\sigma,\tau)$ describe maps from the two-dimensional surface Σ into a target manifold $\mathcal{M}$, extended operators arise very naturally. For instance, vertex operators create point-like excitations on the worldsheet, but their target-space counterparts can represent extended objects and involve momentum and winding modes. These extended objects transform non-trivially under generalized global symmetries acting on the worldsheet theory. For example, already in the simplest settings, such as strings propagating on a circle or torus, the momentum and winding sectors get exchanged under the action of T-duality, which is implemented by a codimension-one topological defect on the worldsheet. These topological defects act as higher-form or even non-invertible symmetries, depending on the transformation implemented [48].

This perspective highlights why non-linear sigma models and two-dimensional conformal field theories are so useful in the study of generalized symmetries. On the one hand, they provide a fully controlled and exactly solvable framework in which generalized symmetries can be realized explicitly, allowing one to identify their symmetry operators, fusion rules, and categorical structures. On the other hand, these theories underlie perturbative string theory, so the generalized symmetries that appear on the worldsheet have a direct interpretation in spacetime. In particular, the topological defects of the worldsheet CFT encode target-space dualities and non-trivial transformations of D-branes and string states. From this point of view, generalized symmetries of two-dimensional sigma models, at the same time, offer insight into the mathematical structure of fusion categories and defect algebras and allow the study of the rich symmetry structure of string theory itself.

Aim and structure of the thesis

The aim of this thesis is, then, to investigate the emergence, structure, and geometric interpretation of generalized global symmetries, in particular non-invertible ones, in two-dimensional non-linear sigma models. These theories, especially the compact boson models, provide a powerful testing ground for studying generalized symmetries. At the same time, these models arise naturally as worldsheet non-linear sigma models in perturbative string theory, meaning that the generalized symmetries emerging in two dimensions have a direct interpretation in terms of string dualities, D-brane transformations, and target-space geometric symmetries. The goal of this work is, therefore, to clarify the mathematical mechanisms through which non-invertible symmetries emerge in 2d CFTs and to understand their connection with the geometric symmetries of non-linear sigma model's target spaces. The structure of the thesis is as follows.

In chapter 2 we provide a broad introduction to generalized symmetries in quantum field theory and string theory. We begin by reviewing the traditional notion of symmetry, the associated Noether currents and conserved charges, and the modern reformulation of 0-form symmetries in terms of codimension-one topological defect operators. We then introduce higher-form symmetries, explaining how their action on extended operators naturally arises from conserved p-form currents and how their topological nature constrains the operator content of a theory. After this, we proceed with the introduction of the framework of fusion categories, which extends the mathematical structure of symmetry defects beyond the ordinary notion of groups and allows for the emergence of non-invertible symmetries. The role of quantum dimensions and fusion rules is discussed in detail, including explicit examples such as the gauging of finite groups and the emergence of $\mathrm{Rep}(G)$ fusion categories. We then conclude with a brief phenomenological discussion. We show how generalized symmetries have concrete physical implications and are not merely a reformulation of ordinary symmetries, but instead play an active role in constraining the dynamics, consistency conditions, and observable features of quantum field theories, with direct consequences for physical phenomena.

In chapter 3, then, we turn to the $c = 1$ compact boson. We present the basic structure of the theory, including its operator content, target space geometry, chiral algebra, and modular properties. The rationality condition is discussed in detail, identifying the points in moduli space where the theory becomes an RCFT with a finite number of primary fields. At these rational points, we analyze the emergence of non-invertible symmetries through gauging, self-duality, and half-space gauging constructions, with special emphasis on the emergence of the so-called Tambara–Yamagami fusion categories. The chapter also examines the interplay between duality symmetries and Dp-branes: we review the construction of Cardy and Ishibashi boundary states, describe how T-duality acts on Neumann and Dirichlet branes, and show how non-invertible defects implement transformations between different brane configurations. The $c = 1$ case thus serves as a testing ground to illustrate many of the categorical and geometric mechanisms that will be central in the study of the $c = 2$ theory.

In chapter 4, finally, we present the main novel results of the thesis. In this section we consider the $c = 2$ compact boson, realized as a non-linear sigma model with target-space a two dimensional torus. After reviewing its construction, the structure of the moduli space, and the action of the duality group $O(2, 2; \mathbb{Z})$, we analyze the rationality condition and identify the RCFT points where the chiral algebra gets enhanced, providing the natural setting in which non-invertible symmetries emerge. The main contribution of this work is to show that worldsheet duality symmetries can be directly related to the geometry of the target space. In particular, we introduce a systematic procedure to construct duality defects by studying the geometric symmetries of the two dimensional non-degenerate lattice underlying the target space torus, which yields a clear and unified method for generating non-invertible duality symmetries in the $c = 2$ theory. Finally, we investigate the action of these duality symmetries on Dp-branes; unlike the $c = 1$ case, where duality symmetries simply exchange D0- and D1-branes, here the action is significantly richer. Indeed, we find that the total number of Dp-branes need not be preserved, revealing a more intricate action on the boundary states of the theory.

2 Generalized Symmetries in Quantum Field Theory and String Theory

In this chapter, we explore the concept of generalized symmetries, which extends the traditional notion of symmetry to include both higher-form and non-invertible cases. We first begin by reviewing the ordinary notion of symmetry and how the concepts of conserved currents and charges naturally arise. We then introduce the modern definition of symmetries in terms of topological defects, showing how this perspective naturally leads to the inclusion of higher-form and non-invertible symmetries. Finally, we conclude the chapter with the discussion of a phenomenological application that highlights the physical relevance of generalized symmetries beyond the formal framework.

2.1 The concept of symmetry in physics

From classical mechanics to quantum field theory, symmetries have always played a central role in physics, working as a guiding principle in formulating physical laws. For example, in QFT, and more specifically in gauge theories, symmetries themselves are responsible for dictating the possible interactions between the fields and for the organization of the spectrum of the theory. In this section, we will revise some of the fundamental aspects of symmetries in QFT. For a comprehensive review, we refer to [8,49,10].

A first, naive, definition of symmetry can be given by the following statement: a symmetry is a transformation of the system's degrees of freedom such that the physical observables of the theory are left invariant. To make this statement more precise, let us start by considering a field theory given by an action $S[\phi]$, where ϕ denotes all the dynamical fields of the theory. Now, a transformation $g : \phi \mapsto g \cdot \phi$ is a symmetry of the theory if the action $S[\phi]$ is left invariant, that is to say $S[\phi] \mapsto S'[\phi] = S[g \cdot \phi] = S[\phi]$. In particular, the set of all such transformations naturally forms a group, usually denoted by G. Indeed, given two symmetries of the system, g_1 and g_2, the composition of the two will naturally lead to a new symmetry, defined as $g_3 \equiv g_2 \cdot g_1$. Moreover, the set of transformations contains also the identity element e which acts trivially on the fields, $e \cdot \phi = \phi$; finally, given a symmetry transformation g, there always exists an inverse element g^{-1} such that $g^{-1} \cdot g = e$. These properties naturally define a group structure over the set of such transformations.

To give an example of such a symmetry group, let us consider the theory of a free complex scalar field ϕ with mass m in 4-dimensions. The action takes the form

$$S[\phi] = \int d^4x \mathcal{L}(x) = \int d^4x \left(\partial_\mu \bar{\phi} \partial^\mu \phi - m^2 \bar{\phi} \phi \right) , \tag{2.1.1}$$

and it is easy to notice that it is invariant under the global field transformation given by

$$\begin{cases} \phi \mapsto e^{i\theta}\phi \\ \bar{\phi} \mapsto e^{-i\theta}\bar{\phi} \end{cases} \quad \text{for } \theta \in [0, 2\pi), \tag{2.1.2}$$

which implies that the theory presents a $U(1)$ global symmetry.

Now, the existence of a continuous global symmetry has an important physical consequence, that is the existence of a conserved current. The latter can be obtained following Noether's theorem. To derive it, let us consider a field theory with Lagrangian density $\mathcal{L}(x) = \mathcal{L}(\phi(x), \partial_\mu \phi(x))$, where $\phi(x)$ represents any dynamical field of the theory. Now, under the action of an infinitesimal symmetry transformation $\phi \mapsto \phi + \alpha\Delta\phi$, where α is the parameter of the transformation, the Lagrangian will be invariant up to a total derivative:

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha \partial_\mu \mathcal{J}^\mu(x), \tag{2.1.3}$$

for a given $\mathcal{J}^\mu(x)$. Similarly, we can also consider the explicit variation of the Lagrangian by varying the field:

$$\begin{aligned} \alpha\Delta\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi}(\alpha\Delta\phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu(\alpha\Delta\phi) \\ &= \alpha\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Delta\phi \right) + \alpha \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] \Delta\phi. \end{aligned} \tag{2.1.4}$$

Comparing the two expressions (2.1.3) and (2.1.4) we get:

$$\partial_\mu \left[\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Delta\phi \right) - \mathcal{J}^\mu \right] = - \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] \Delta\phi. \tag{2.1.5}$$

From the classical Euler-Lagrange equations, we easily see that the right-hand side of the above expression vanishes when considering an extremal field configuration, that is to say when the equations of motion are satisfied, which clearly implies the conservation of Noether's current defined as:

$$j^\mu = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Delta\phi \right) - \mathcal{J}^\mu \implies \partial_\mu j^\mu = 0. \tag{2.1.6}$$

The conservation law just derived can also be expressed in terms of a conserved charge, where Noether's charge is defined as:

$$Q = \int d^{d-1}x \, j^0, \tag{2.1.7}$$

that is, the integral of the Noether current over a spatial slice of the d-dimensional spacetime[1]. Indeed, if we assume that the fields vanish fast enough at infinity, we get:

$$\frac{dQ}{dx^0} = \int d^3x \partial_0 j^0 = -\int d^3x \partial_i j^i = 0, \;\; i = 1, \dots, 3, \tag{2.1.9}$$

implying that the charge Q is conserved in time.

We can now consider again our previous example, and work out the Noether's current and the conserved charge for this specific theory. In this case, the infinitesimal transformation induced by the $U(1)$ global symmetry over the fields ϕ and $\bar{\phi}$ is

$$\Delta\phi = i\alpha\phi, \quad \Delta\bar{\phi} = -i\alpha\bar{\phi}, \tag{2.1.10}$$

and from equation (2.1.6) we get:

$$\begin{aligned} j^\mu =& \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\frac{\delta\phi}{\delta\alpha} + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{\phi})}\frac{\delta\bar{\phi}}{\delta\alpha} \\ =& -i[(\partial^\mu\bar{\phi})\phi - \bar{\phi}\partial^\mu\phi]. \end{aligned} \tag{2.1.11}$$

This is the conserved current under the $U(1)$ global symmetry, and it is possible to check that $\partial_\mu j^\mu = 0$ when the field configuration satisfies the equations of motion. Moreover, we can also derive the conserved charge under this symmetry, that is:

$$Q = i\int d^3x \left(\frac{\partial\bar{\phi}}{\partial x^0}\phi - \bar{\phi}\frac{\partial\phi}{\partial x^0} \right). \tag{2.1.12}$$

We will not show it here, but it is possible to check that this conserved charge is related to the number operators as $Q = N_a - N_b$ and it implies the conservation of the total number of particles in the theory (with sign). For a more detailed discussion, we refer to [49].

So, why is it important to define Noether's current and charge? Because the charge operator is needed in order to define the action of the symmetry on the Hilbert space. Indeed, the symmetry is represented in the Hilbert space by a unitary operator $U(g) = e^{i\alpha Q}$, where α parametrizes the group element $g \in G$. In particular, the action of the symmetry on a local operator $\mathcal{O}(x)$ is given by the conjugation

$$U^{-1}(g)\mathcal{O}(x)U(g) = \mathcal{O}'(x), \tag{2.1.13}$$

[1]More in general, for a symmetry group G with elements $g = e^{i\alpha^a T_a}$, where T_a are the generators of the associated Lie algebra $\mathfrak{g}$, there will be a Noether's current for each generator T_a, j^μ_a, and a corresponding charge Q_a, such that they satisfy:

$$[Q_a, Q_b] = i f_{ab}{}^c Q_c, \tag{2.1.8}$$

where $f_{ab}{}^c$ are the structure constants of $\mathfrak{g}$

which, for an infinitesimal transformation, takes the form

$$\delta\mathcal{O}(x) = -i\epsilon[Q, \mathcal{O}(x)], \tag{2.1.14}$$

showing that the conserved charge precisely acts as the generator of the symmetry transformation on operators.

2.2 Symmetries as topological operators

In the previous section we reviewed the notion of symmetry in physics, how it is naturally associated to the concept of group, and how we can derive the conserved quantities given a symmetry of a theory. In this section we will instead introduce the modern formulation of symmetry, which will naturally lead us to the concept of generalized symmetry.

As we saw above, given a continuous global symmetry G of a field theory, Noether's theorem implies the existence of a conserved current j^μ and a conserved charge Q. However, in modern physics, it is more convenient to recast these results in terms of differential forms. Indeed, it is possible to notice that this current can be associated to a 1-form current defined as $\boldsymbol{j} = j_\mu dx^\mu$ for which the conservation of the current can be expressed as the fact that the 1-form is co-closed:

$$\partial_\mu j^\mu = 0 \implies d \star \boldsymbol{j} = 0, \tag{2.2.1}$$

where $\star$ is the Hodge star operator[2]. In this framework, to obtain the conserved charge we need to integrate $\star\boldsymbol{j}$ over a co-dimension 1 manifold (M_{d-1})

$$Q(M_{d-1}) = \int_{M_{d-1}} \star\boldsymbol{j}, \tag{2.2.5}$$

[2]Let (M, g) be a d-dimensional pseudo-Riemannian manifold with metric g. Then, the Hodge star operator is a map between the space of k-forms and the one of $(d-k)$-forms defined as:

$$\begin{aligned} \star : \Omega^k(M) &\to \Omega^{(d-k)}(M) \\ \xi &\mapsto \star\xi \text{ s.t. } \langle\eta, \xi\rangle = \int_M \eta \wedge \star\xi,\ \forall\eta \in \Omega^k(M) \end{aligned} \tag{2.2.2}$$

where $\langle -, - \rangle$ is the Hodge inner product naturally induced by g on $\Omega^k(M)$. In local coordinates, given a k-form

$$\xi = \frac{1}{p!}\xi_{\mu_1,\mu_2,\ldots,\mu_k} dx^{\mu_1} \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}, \tag{2.2.3}$$

the Hodge dual is defined as:

$$*\xi = \frac{\sqrt{|g|}}{k!(d-k)!}\xi^{\mu_1\ldots\mu_k}\epsilon_{\mu_1\ldots\mu_k\nu_1\ldots\nu_{d-k}}\, dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_{d-k}} \tag{2.2.4}$$

where $\epsilon_{\mu_1,\ldots,\mu_d}$ is the Levi-Civita symbol.

and the conservation of the charge is guaranteed by the Stokes-Cartan theorem.

What we want to do now is to derive the action of symmetries on local operators in this new setting. To do so, let us recall that in QFT we can insert conserved currents into correlation functions in order to obtain Ward identities; in particular, for a single operator $\mathcal{O}$ inserted at a point $\boldsymbol{p}$ we have:

$$\delta^{(d)}(\boldsymbol{x}-\boldsymbol{p})\langle\delta_a\mathcal{O}(\boldsymbol{p})\rangle = \langle d\star\boldsymbol{j}_a(\boldsymbol{x})\mathcal{O}(\boldsymbol{p})\rangle, \tag{2.2.6}$$

where δ_a is the infinitesimal transformation induced by the generator T_a of the symmetry. Now, let us consider a d-dimensional manifold N_d with boundary M_{d-1} surrounding the operator $\mathcal{O}$, and let us integrate the Ward identity over this manifold:

$$\begin{aligned}\int_{N_d} d^dx\delta^{(d)}(\boldsymbol{x}-\boldsymbol{p})\langle\delta_a\mathcal{O}(\boldsymbol{p})\rangle &= \int_{N_d}\langle d\star\boldsymbol{j}_a(\boldsymbol{x})\mathcal{O}(\boldsymbol{p})\rangle \\ &= \int_{M_{d-1}}\langle\star\boldsymbol{j}_a(\boldsymbol{x})\mathcal{O}(\boldsymbol{p})\rangle = \langle Q_a(M_{d-1})\mathcal{O}(\boldsymbol{p})\rangle. \end{aligned} \tag{2.2.7}$$

It is important to notice that this expression is non-zero only if the point $\boldsymbol{p}$ is inside the volume enclosed in M_{d-1}. This has a rigorous mathematical definition in terms of the linking number[3], which is a topological property, and in the case of a point $\boldsymbol{p}$ and a $(d-1)$-dimensional manifold is defined as:

$$\text{Link}(M_{d-1},\boldsymbol{p}) = \begin{cases}1, & \text{if } \boldsymbol{p}\in N_d\backslash M_{d-1},\\ 0, & \text{if } \boldsymbol{p}\notin N_d\backslash M_{d-1}.\end{cases} \tag{2.2.9}$$

In this way, equation (2.2.7) becomes:

$$\text{Link}(M_{d-1},\boldsymbol{p})\langle\delta_a\mathcal{O}(\boldsymbol{p})\rangle = \langle Q_a(M_{d-1})\mathcal{O}(\boldsymbol{p})\rangle, \tag{2.2.10}$$

and, thanks to the topological property of the linking number, the charge $Q(M_{d-1})$ is independent of M_{d-1} as long as $\boldsymbol{p}$ is contained in it.

At this point, it is natural to define a symmetry generator in terms of a topological

[3] Let M be a d-dimensional manifold. Let U_q and V_r be two oriented submanifolds of dimensions q and r such that $q+r=d-1$. Moreover, assume that the two manifolds do not intersect and that are homotopically trivial, that is to say both U_q and V_r are boundaries of manifolds of one higher dimension. Let $W_{r+1}\subset M$ be such that $V_r=\partial W_{r+1}$; then, in general, $W_{r+1}\cap U_q\neq\emptyset$, and they will intersect in a finite number of points p_i (with sign). Then, the linking number of U_q and V_r is defined as:

$$\text{Link}(U_q,V_r)\equiv\sum_i\text{sign}(p_i). \tag{2.2.8}$$

Note that this is a topological property, independent of the choice of W_{r+1} and invariant under homotopic deformations of the two submanifolds U_q and V_r, as long as they do not intersect each other under deformation. For a detailed discussion on linking numbers in QFT see [50].

operator of dimension $d-1$ (co-dimension 1). Indeed, given a continuous symmetry G, with associated conserved currents and charges $\boldsymbol{j}_a(x)$ and Q_a, we can define a topological operator that implements the action of the symmetry on the operators. To do so, let M_{d-1} be a $(d-1)$-dimensional submanifold in spacetime, and $g = e^{i\alpha^a T_a} \in G$ an element of the symmetry group; then, we can define the topological operator

$$D^{(g)}_{d-1}(M_{d-1}) = \exp\left[i\alpha^a Q_a(M_{d-1})\right] \tag{2.2.11}$$

which acts on local operators as:

$$D^{(g)}_{d-1}(M_{d-1})\mathcal{O}(x) = R(g)\mathcal{O}(x), \tag{2.2.12}$$

where $R(g)$ is the group element g in the operator $\mathcal{O}$ representation. For a visual representation of how topological defects act on local operators, see Fig. 1. Moreover, from equation (2.1.8), it is possible to check that the insertion of two such topological operators associated with two elements of the symmetry group, $g, h \in G$, is equivalent to the insertion of a single topological operator associated with the product of the two elements, $gh \in G$, leading to:

$$D^{(g)}_{d-1} \otimes D^{(h)}_{d-1} = D^{(gh)}_{d-1}. \tag{2.2.13}$$

The topological operator $D^{(g)}_{d-1}(M_{d-1})$ is called a symmetry defect operator (SDO).

The fundamental aspect of this reformulation of symmetries is that the operators we construct are topological. This is due to the conservation of Noether's current, which implies that any deformation of M_{d-1} on which the symmetry defect is supported, if the operator insertion is not crossed, has no physical effect. Moreover, because the action of SDOs on operators is controlled by the linking number, any SDO supported on a co-dimension 1 manifold will act on local operators $\mathcal{O}(x)$, that is to say operators supported on a point $x \in M_d$, where M_d is the manifold on which we define our QFT (the space-time). For this reason, co-dimension 1 operators $D^{(g)}_{d-1}(M_{d-1})$ are identified as generators of 0-form symmetries, that we call $G^{(0)}$. In particular, by reversing the process, and from the identification of 0-form symmetries with SDOs, it is also possible to construct topological operators associated with finite symmetries, for which it is not possible to define a conserved current.

2.3 Higher-form symmetries

In the previous section we saw how symmetry operators are naturally associated to topological operators of co-dimension 1 that act on local operators. In this section we will instead see how it is possible to generalize this construction in order to obtain higher-form symmetries.

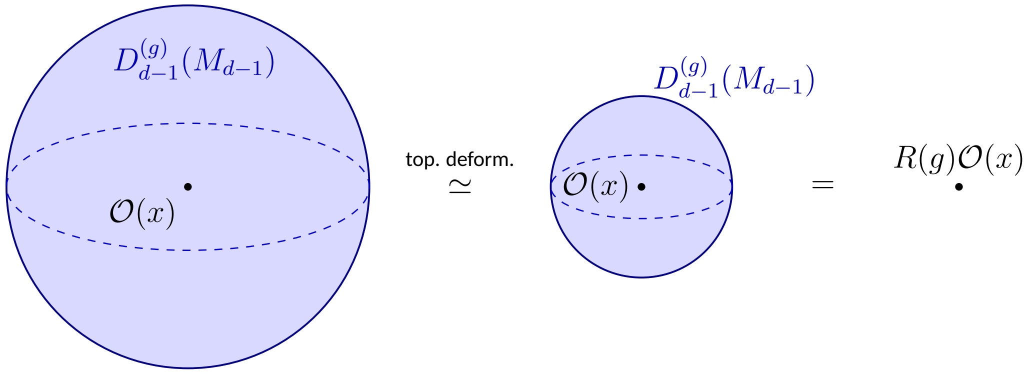


Figure 1: We present a visual representation of a co-dimension 1 topological operator $D^{(g)}_{d-1}(M_{d-1})$ surrounding a local operator $\mathcal{O}(x)$. In particular, continuous deformations of the symmetry defect that do not cross the local operator insertion have no physical effect, implying that the SDO is topological, and the action on $\mathcal{O}(x)$ is obtained by shrinking $D^{(g)}_{d-1}(M_{d-1})$ up to a point.

In equation (2.2.11) we defined symmetry defect operators supported on submanifolds of co-dimension 1. What we can do now is to relax the requirement on the dimension of the support and allow for defects supported on submanifolds of co-dimension $(p+1)$. What we get is a so-called $G^{(p)}$-form symmetry, which is generated by co-dimension $p+1$ topological defects $D^{(g)}_{d-p-1}(M_{d-p-1})$, $g \in G^{(p)}$, that satisfy the usual group law (2.2.13). As in the 0-form symmetry case, these topological operators can be constructed starting from a conserved current (if there exists one, that is to say for continuous symmetries), which corresponds to a $(p+1)$-form $\boldsymbol{j}_{p+1}$ satisfying the co-closure condition:

$$\partial_{\mu_1} j^{[\mu_1 \dots \mu_{p+1}]}(x) = 0 \implies d \star \boldsymbol{j}_{p+1}(x) = 0. \tag{2.3.1}$$

As we saw above, 0-form symmetries naturally act on local operators, which are the charged objects under the symmetry; what are instead the corresponding charged operators in the case of a p-form symmetry? As discussed earlier (see Footnote 3), the linking of an operator supported on a $d-(p+1)$-dimensional manifold can be non-trivial only with an operator supported on a p-dimensional one. This clearly implies that the only operators that can be charged under a p-form symmetry are p-dimensional operators $\mathcal{O}_p(W_p)$, where W_p is the world-volume submanifold, for which Link(M_{d-p-1}, W_p) can be different from zero. See Fig. 2 for a visual representation of how p-form symmetries act on p-dimensional operators.

However, we have a substantial difference between 0-form symmetries and p-form ones. Indeed, if we consider a non-abelian symmetry group, in order to have a well-defined action of the symmetry, we need a well-defined ordering of the symmetry elements acting on the operators; indeed, if a symmetry G is non-abelian, given $g, h \in G$ we have $[g, h] \neq 0$ in general. This implies that the ordering of the defect insertions

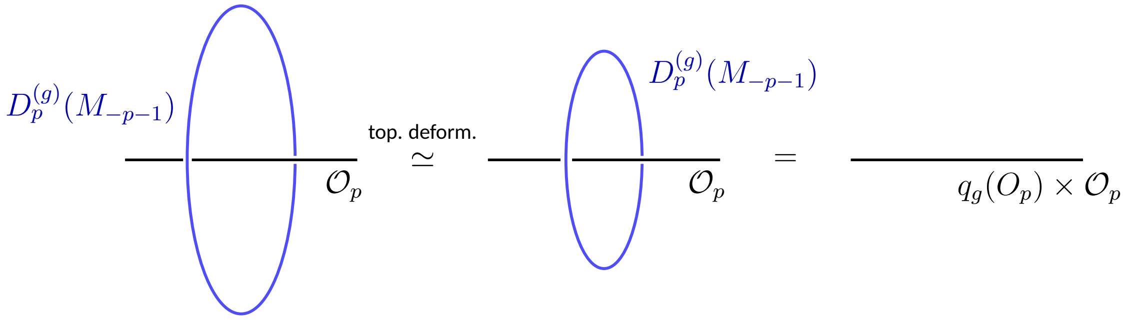


Figure 2: We show the action of a p-form symmetry operator $D_p^{(g)}(M_{-p-1})$, supported on a co-dimension $p+1$ submanifold, on a p-dimensional operator $\mathcal{O}_p$. The charge of the enclosed operator $q_g(O_p)$ is obtained by shrinking the surrounding topological operator.

is important for the resulting action on the operators. Now, in the 0-form symmetries case, since the topological defects are $(d-1)$-dimensional, we only have a single transverse direction to them, which implies that we have a well-defined ordering of the defect insertions. This is clearly understandable in the 3-dimensional case, in which 0-form symmetries are supported on 2-dimensional submanifolds; in this setup, given 2 SDOs surrounding a point x (we can consider 2 concentric spheres), it is not possible to continuously deform them in order to change their order (see Fig. 3 for a visual representation).

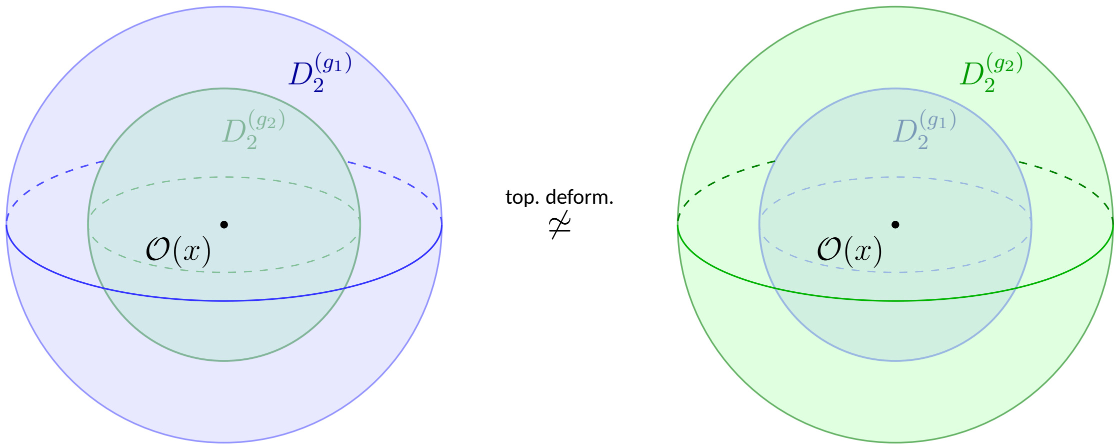


Figure 3: We show that, given two 0-form symmetry generators surrounding a local operator, their order cannot be exchanged by continuous deformations. In particular, in three dimensions two concentric spherical symmetry surfaces cannot be topologically deformed into each other, as the inner and outer spheres cannot be inverted continuously. This establishes that there is a well-defined notion of ordering for 0-form symmetry operators, and therefore 0-form symmetries can be non-abelian.

However, given a p-form symmetry, the topological operators implementing the symmetry are now $d-(p+1)$-dimensional, implying that the transverse space is a $p+1$ dimensional plane. In particular, this higher dimensionality allows for continuous transformations of the topological defects that reverse their order, and for this reason, there is no well-defined notion of ordering:

$$D^{(g_1)}_{d-(p+1)} \cdot D^{(g_2)}_{d-(p+1)} \overset{\text{top. deform.}}{\simeq} D^{(g_2)}_{d-(p+1)} \cdot D^{(g_1)}_{d-(p+1)}. \tag{2.3.2}$$

This is easily understandable in the 3-dimensional case again, by considering a 1-form symmetry supported on a 1-dimensional submanifold. In this case, given 2 SDOs surrounding a line γ (we can consider 2 concentric circles), we are always able to change their order via continuous transformations (see Fig. 4). The possibility of reversing the insertion order of topological operators clearly implies that p-form symmetries $G^{(p)}$, for $p \geq 1$, must be abelian.

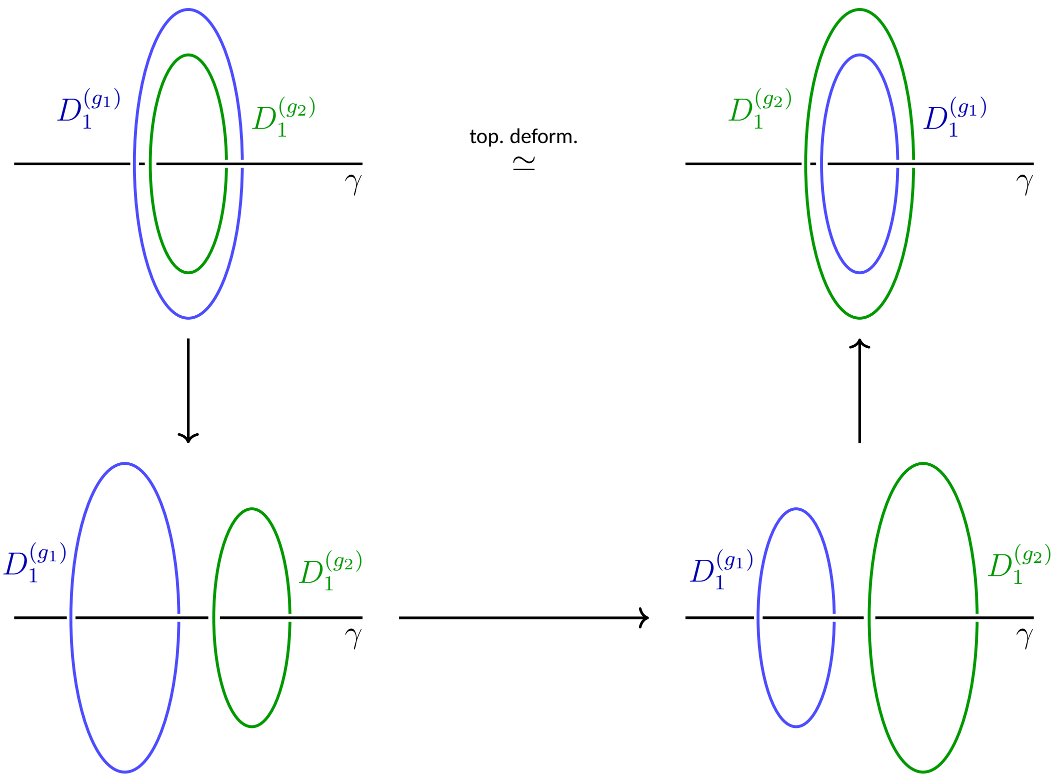


Figure 4: Given two p-form symmetry operators acting on a p-dimensional charged operator, their order can be continuously exchanged. Indeed, since p-form symmetry operators are supported on submanifolds of co-dimension $p+1$, the transverse space has dimension greater than one, allowing us to smoothly deform one defect around the other without intersecting them. As a result, the action of the two generators commutes.

2.3.1 An example: the 4-dimensional Maxwell theory

We can now illustrate an example of a p-form symmetry. To do so, let us consider the 4d Maxwell theory, that is a $U(1)$ gauge theory, with gauge field $\boldsymbol{A}$ and coupling g. The action takes the form

$$S = \frac{1}{2g^2}\int \boldsymbol{F}\wedge\star\boldsymbol{F} = -\frac{1}{4g^2}\int F_{\mu\nu}F^{\mu\nu}, \tag{2.3.3}$$

where $\boldsymbol{F} = d\boldsymbol{A}$ is the 2-form field strength. It is easy to check that the equations of motion are

$$d\star\boldsymbol{F} = 0. \tag{2.3.4}$$

In particular, this equation can be identified with the conservation of a 2-form current $\boldsymbol{j}_2^e = \frac{1}{g^2}F$, where the prefactor $\frac{1}{g^2}$ comes from normalization conditions. The conservation of this current corresponds to a $U(1)$ 1-form symmetry called electric (or central) 1-form symmetry (that we call $U_e^{(1)}(1)$). To obtain the corresponding symmetry defect operator, we can follow what we did around equation (2.2.11). We take the exponential of the Hodge dual current integrated over a 2-cycle, which is the support of the topological defect, since we are integrating a 2-current, and we get:

$$D_e^{(g)}(M_2) = \exp\left(i\alpha\oint_{M_2}\star\boldsymbol{j}_2^e\right), \tag{2.3.5}$$

where $g\in U_e^{(1)}$ such that $g = e^{i\alpha},\ \alpha\in[0,2\pi)$. In particular, $\oint_{M_2}\star\boldsymbol{j}_2^e$ measures the charge enclosed in the 2-dimensional manifold M_2. Moreover, since $\boldsymbol{F}$ is a $U(1)$ gauge field strength, the symmetry defect operators will satisfy the $U(1)$ group multiplication law and, for this reason, we say that this corresponds to a $U(1)$ 1-form symmetry.

We can now look at the charged operators. From the above discussion, since we are considering a 1-form symmetry, the charged object will be line operators. In particular, the charged line operators are the Wilson lines $W_{q_e}(M_1)$ defined as

$$W_{q_e}(M_1) = \exp\left(iq_e\int_{M_1}\boldsymbol{A}\right), \tag{2.3.6}$$

which couple with the symmetry defect operators as:

$$D_2^{(g)}(M_2)W_{q_e}(M_1) = e^{i\alpha q_e \text{Link}(M_2,M_1)}W_{q_e}(M_1), \tag{2.3.7}$$

where Link(M_2, M_1) is the linking number between the support of the defect M_2 and the support of the line operator M_1.

Let us now look more closely to the physical meaning of equation (2.3.7). A Wilson line can be thought of as the world-line of a heavy probe particle carrying electric charge, and, as we know from Maxwell's equations, an electrically charged particle acts as a source of electromagnetic field. To see this, let us consider a Wilson line supported on a 1-dimensional submanifold M_1; the effect of this charged line operator on Maxwell's equations can be obtained by inserting the Wilson line in the path integral as:

$$\int [d\boldsymbol{A}]\, e^{iq_e \int_{M_1} \boldsymbol{A}} e^{\frac{1}{2g^2}\int_{M_4} \boldsymbol{F}\wedge\star\boldsymbol{F}} = \int [d\boldsymbol{A}]\, e^{\int_{M_4}\left(iq_e\delta^3(M_1)\wedge\boldsymbol{A}+\frac{1}{2g^2}\boldsymbol{F}\wedge\star\boldsymbol{F}\right)}, \qquad (2.3.8)$$

Where $\delta^3(M_1)$ is a 3-form delta function such that

$$\int_{\Sigma_3^T} \delta^3(M_1) = 1 \qquad (2.3.9)$$

where Σ_3^T is a 3d submanifold transverse to M_1 and intersecting it exactly once. In this way, it is easy to see that the Wilson line acts as an electric source and Maxwell's equations are modified as:

$$d\star\boldsymbol{F} = q_e g^2 \delta^3(M_1). \qquad (2.3.10)$$

At this point, integrating this equation over a 3-manifold Σ_3 with boundary $\partial\Sigma_3$, we get:

$$\int_{\Sigma_3} d\star\boldsymbol{F} = \int_{\partial\Sigma_3} \star\boldsymbol{F} = q_e g^2 \text{Link}(\partial\Sigma_3, M_1) \qquad (2.3.11)$$

which can be identified with the electric flux generated by the Wilson line insertion at the boundary $\partial\Sigma_3$. This clearly implies that the charge probed by the symmetry defect operator insertion around a Wilson line precisely measures the electrical charge carried by the Wilson line itself.

Moreover, $d\star\boldsymbol{F} = 0$ is only one of the two differential form equations that control the dynamics of the electromagnetic field. The second one comes instead from the Bianchi identity and takes the form:

$$d\boldsymbol{F} = 0. \qquad (2.3.12)$$

It is easy to see that this is a conservation equation for a 2-form current too, namely $\boldsymbol{j}_2^m = \frac{1}{2\pi}\star\boldsymbol{F}$, and is associated to the $U(1)$ 1-form magnetic symmetry, the dual symmetry of the electric 1-form symmetry. Again, we can define the 2-dimensional symmetry defect operators, $D_m^{(g)}(M_2)$, as before, but this time the charged operator under this symmetry are the so-called 't Hooft lines, defined as

$$T_{q_m}(M_1) = e^{im\int e_{M_1}\boldsymbol{A^D}}, \qquad (2.3.13)$$

where $\boldsymbol{A^D}$ is defined as the 1-form such that $\star\boldsymbol{F} = d\boldsymbol{A^D}$, that is to say the dual field of the 1-form field $\boldsymbol{A}$. We will not go into details in this case, since the resulting symmetry is equivalent to the previous one, simply associated with magnetic charges.

For a more detailed discussion on higher-form symmetries and on the 4d Maxwell's theory we refer to [13, 12].

2.4 Non-invertible symmetries

In the previous section, we showed how the notion of symmetry can be extended to a broader class of transformations acting on higher-dimensional operators. This generalization was made possible by reformulating the action of symmetry groups in terms of topological operators that act on the physical, charged operators of the theory. Within this framework, the emergence of higher-form symmetries appeared as a natural, geometrically motivated extension of ordinary global symmetries. In the present section, instead, we turn to a second class of generalized symmetries, the so-called non-invertible symmetries.

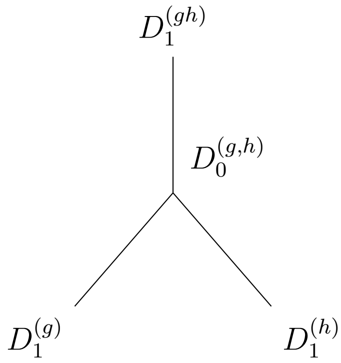
2.4.1 An introduction to fusion categories

In order to make this generalization, as we did with the introduction of symmetry defect operators to allow for the definition of higher-form symmetries, we need to introduce a new framework in which non-invertible symmetries are possible, the one of the fusion categories. To do so, let us start by recasting the well-known case of a 0-form symmetry $G^{(0)}$ in 2d within this new framework (we refer to [12] for a more detailed discussion); the category is defined in terms of a set of objects, morphisms and a fusion product:

- The simple objects of the category are the topological defect lines (TDLs) $D_1^{(g)}$, $g \in G^{(0)}$, representing distinct isomorphism classes. Moreover, topological defect lines have a natural direct sum structure (we denote it by $\oplus$) that is defined as:
$$\langle\left(D_1^{(g)} \oplus D_1^{(h)}\right)\cdots\rangle = \langle D_1^{(g)}\cdots\rangle + \langle D_1^{(h)}\cdots\rangle, \tag{2.4.1}$$
and we can define the semisimple objects of the category (that we denote with $\mathcal{L}$) as a finite direct sum of simple objects.

- The morphisms are topological local operators D_0, called topological junctions, that act as homomorphism between two TDLs and form a vector space over $\mathbb{C}$, Hom $(\mathcal{L}_i, \mathcal{L}_j)$. In particular, given two simple objects, we have:
$$\text{Hom}\left(D_1^{(g)}, D_1^{(h)}\right) = \begin{cases} \text{Id if } \ h = g, \\ \emptyset \text{ if } \ h \neq g, \end{cases} \tag{2.4.2}$$
where Id is the identity morphism.

- The fusion of the 0-form symmetry TDLs is:
$$D_1^{(g)} \otimes D_1^{(h)} = D_1^{(gh)}, \tag{2.4.3}$$
where $g, h \in G^{(0)}$. The fusion of two TDLs can also be written in terms of a local operator $D_0^{(g,h)} \in \text{Hom}(D_1^{(g)} \otimes D_1^{(h)}, D_1^{(gh)})$ as:



In particular, the fusion category we just defined is characterized by the fact that TDLs fuse according to a group multiplication; this is the fusion category of a G-graded vector space Vec$_G$.

What we just showed is simply a reformulation of a standard 0-form symmetry in terms of a fusion category characterized by a group-like fusion rule; so, why is this new framework useful in order to introduce non-invertible symmetries? This is due to the fact that, in general, fusion categories are characterized by more involved fusion rules. Indeed, let $\mathcal{C}$ be a fusion category and let the set of simple objects be denoted by $\{\mathcal{L}_i\}_{i\in\mathcal{I}}$, where $\mathcal{I} = \{i, j, k, \dots\}$; then, given two TDLs, the fusion rule takes the general form:

$$\mathcal{L}_i \otimes \mathcal{L}_j = \bigoplus_{k\in\mathcal{I}} N_{ij}^k \mathcal{L}_k, \tag{2.4.4}$$

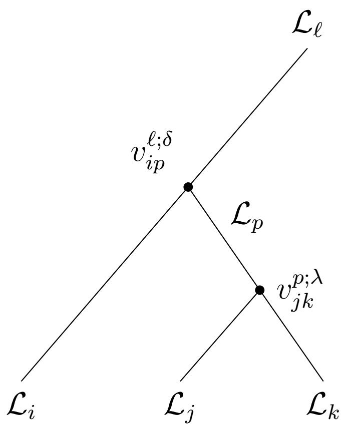
where $N_{ij}^k \in \mathbb{Z}_{\geq 0}$ are non-negative integers and are called fusion coefficients. These are defined as the dimension of the trivalent junction vector space

$$N_{ij}^k = \text{dim}_{\mathbb{C}}\text{Hom}(\mathcal{L}_i \otimes \mathcal{L}_j, \mathcal{L}_k). \tag{2.4.5}$$

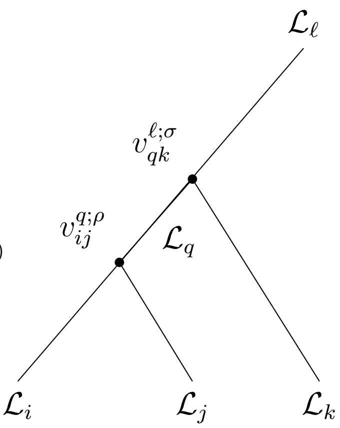
Indeed, as we said above, Hom$(\mathcal{L}_i \otimes \mathcal{L}_j, \mathcal{L}_k)$ is a vector space, and we can denote the basis vectors of this vector space as $v_{ij}^{k;\delta}$, where $\delta = 1, \dots, N_{ij}^k$, and these represent the local operators responsible for the trivalent junction. Moreover, in a generic fusion category, the tensor product $\otimes$ is associative up to an isomorphism, called the associator. Indeed, given X, Y, Z objects in $\mathcal{C}$ (not necessarily simple), the associator α_{XYZ} is an isomorphism:

$$\alpha_{XYZ} \in \text{Hom}((X \otimes Y) \otimes Z,\ X \otimes (Y \otimes Z)), \tag{2.4.6}$$

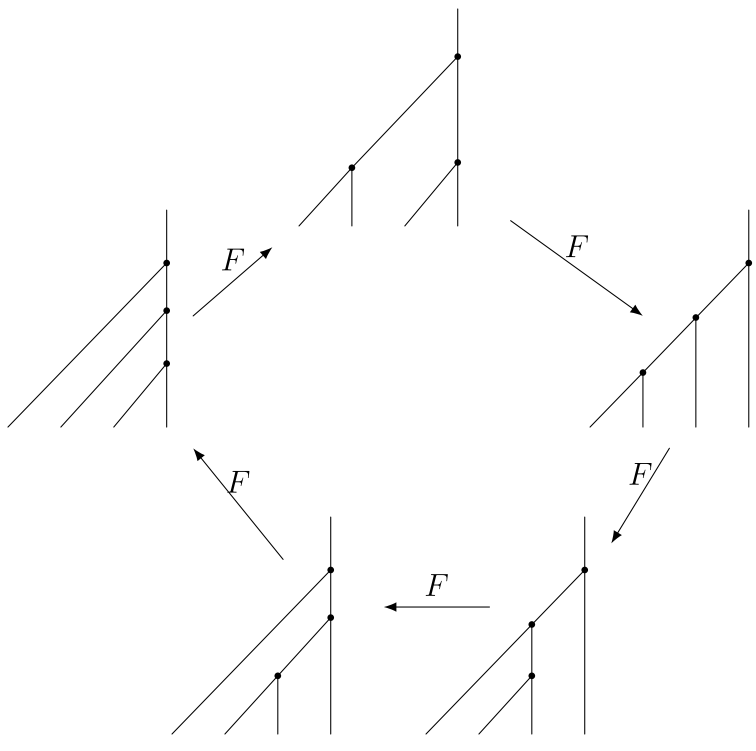
and, given three simple objects, we denote it by α_{ijk}, with $i, j, k \in \mathcal{I}$. In particular, given a fixed basis of the trivalent junctions, the matrix element of the associators are called F-symbols, and are defined in such a way that:



$$= \sum_{q\in\mathcal{I}} \sum_{\rho=1}^{N_{ij}^{q}} \sum_{\sigma=1}^{N_{qk}^{\ell}} \left[F_{ijk}^{\ell}\right]_{(p,\lambda,\delta)\,(q,\rho,\sigma)}$$



The F-symbols can be regarded as the basis transformation matrix elements of Hom$(\mathcal{L}_i \otimes \mathcal{L}_j \otimes \mathcal{L}_k, \mathcal{L}_\ell)$ which relate two inequivalent ways of decomposing the four-point junction into two trivalent junctions, and such a basis transformation is called an F-move. The F-symbols must also satisfy an algebraic constraint, called the pentagon identity, coming from the consistency condition under consecutive changes of basis in the five-point junction vector space. This algebraic constraint guarantees that different ways of performing F-moves give rise to the same correlation function. The pentagon identity can be represented as



and the algebraic consistency condition can be written as:

$$\sum_{\rho=1}^{N_{qm}^{p}} \left[F_{ijm}^{p}\right]_{(n,\delta,\lambda)(q,\eta,\rho)} \left[F_{qkl}^{p}\right]_{(m,\rho,\sigma)(r,\pi,\xi)}$$
$$= \sum_{s\in\mathcal{I}} \sum_{\nu=1}^{N_{jk}^{s}} \sum_{\tau=1}^{N_{sl}^{n}} \sum_{\omega=1}^{N_{is}^{r}} \left[F_{jkl}^{n}\right]_{(m,\lambda,\sigma)(s,\nu,\tau)} \left[F_{isl}^{p}\right]_{(n,\delta,\tau)(r,\omega,\xi)} \left[F_{ijk}^{r}\right]_{(s,\omega,\nu)(q,\eta,\pi)} . \tag{2.4.7}$$

In particular, the data $(\{\mathcal{L}\}_{i\in\mathcal{I}}, N_{ij}^{k}, \left[F_{ijm}^{p}\right]_{(n,\delta,\lambda)(q,\eta,\rho)})$, namely the set of simple objects, the fusion coefficients and the F-symbols are what define a fusion category $\mathcal{C}$ (For a more detailed discussion on fusion categories we refer to [23, 51–53]).

2.4.2 Non-invertibility from the fusion rules

In the previous section we introduced the framework of fusion categories, which allows for the description of non-invertible symmetries. In this section, we explain what we mean by a non-invertible symmetry and how to determine whether a symmetry is non-invertible, given the associated fusion category.

In order to discuss invertibility, we first need to introduce the notion of quantum dimension, which is defined in terms of the Perron–Frobenius eigenvalue of the fusion matrices of the category. Indeed, for every simple object $\mathcal{L}_i$, we can define the fusion matrix

$$(N_i)_{jk} = N_{ij}^{k}, \qquad \text{such that} \qquad \mathcal{L}_i \otimes \mathcal{L}_j = \bigoplus_k N_{ij}^{k}\, \mathcal{L}_k, \tag{2.4.8}$$

where the entries $N_{ij}^{k} \in \mathbb{Z}_{\geq 0}$ are the fusion coefficients. Each N_i is therefore a non-negative integer matrix that encodes the action of fusing $\mathcal{L}_i$ with any other object of the category. By the Perron–Frobenius theorem, any non-negative, irreducible matrix N_i admits a unique largest real positive eigenvalue d_i, called the Perron–Frobenius eigenvalue, whose associated eigenvector $\vec{d} = (d_j)_{j\in\mathcal{I}}$ has strictly positive entries:

$$N_i\, \vec{d} = d_i\, \vec{d}, \qquad d_i > 0. \tag{2.4.9}$$

In the context of fusion categories, this eigenvalue d_i is called the quantum dimension of the simple object $\mathcal{L}_i$. It provides a measure of the "size" of an object, or equivalently, of the number of fusion channels it generates when fused with other lines. In particular, the quantum dimension satisfies a multiplicative property:

$$d(\mathcal{L}_i \otimes \mathcal{L}_j) = d_i\, d_j, \tag{2.4.10}$$

and $d_{\mathbf{1}} = 1$ for the trivial object $\mathbf{1}$. We refer to [54] for a comprehensive exposition on the quantum dimension of fusion categories.

With the notion of quantum dimension, we are finally able to define invertibility for a TDL. A simple object $\mathcal{L}_i$ is said to be invertible if there exists another simple object $\mathcal{L}_{i^{-1}}$ such that

$$\mathcal{L}_i \otimes \mathcal{L}_{i^{-1}} = \mathbf{1}, \qquad \mathcal{L}_{i^{-1}} \otimes \mathcal{L}_i = \mathbf{1}, \tag{2.4.11}$$

where $\mathbf{1}$ is the identity SDO that acts trivially on the charged operators. In terms of fusion coefficients, this means

$$N_{i\,i^{-1}}^{\mathbf{1}} = 1, \qquad N_{i\,i^{-1}}^{k\neq\mathbf{1}} = 0, \tag{2.4.12}$$

so that the fusion of $\mathcal{L}_i$ with the inverse defect line yields exactly one object, the identity $\mathbf{1}$. However, this condition has an important consequence; this is the fact that, since there exists an inverse defect line that gives the identity operator when fused with $\mathcal{L}_i$, then the fusion of $\mathcal{L}_i$ with any other simple object must yield exactly one simple object, in such a way that it can be uniquely undone. In particular, this implies that the corresponding fusion matrix N_i is thus a permutation matrix, since each of its columns contains exactly one nonzero entry, which, moreover, must be equal to one. For a permutation matrix, all eigenvalues have unit modulus, and the Perron–Frobenius eigenvalue is

$$d_i = 1. \tag{2.4.13}$$

Hence, invertibility implies that the quantum dimension equals one.

Conversely, suppose that $d_i > 1$. Then N_i cannot be a permutation matrix: there must exist at least one object $\mathcal{L}_j$ such that

$$\mathcal{L}_i \otimes \mathcal{L}_j = \bigoplus_k N_{ij}^k \, \mathcal{L}_k, \tag{2.4.14}$$

with at least two nonzero fusion coefficients N_{ij}^k (or at least one fusion coefficient $N_{ij}^k > 1$). Fusion with $\mathcal{L}_i$ therefore produces a direct sum of multiple simple objects rather than a single one. Now assume, for contradiction, that $\mathcal{L}_{i^{-1}}$ exists and satisfies $\mathcal{L}_i \otimes \mathcal{L}_{i^{-1}} = \mathbf{1}$. Applying the multiplicativity of the quantum dimension gives

$$d(\mathcal{L}_i \otimes \mathcal{L}_{i^{-1}}) = d_i \, d_{i^{-1}} = d_{\mathbf{1}} = 1, \tag{2.4.15}$$

which immediately implies $d_i = d_{i^{-1}} = 1$ (since $d_i \geq 1$ for any fusion category; we refer to [55] for the proof). This contradicts the assumption $d_i > 1$, and therefore no inverse can exist.

To summarize, a simple object with $d_i > 1$ cannot be invertible, and, for this reason, if a fusion category associated to a symmetry of the theory contains such an object, we say that the symmetry is non-invertible. Indeed, intuitively, the quantum dimension d_i measures how the number of fusion channels grows when repeatedly fusing with $\mathcal{L}_i$. If $d_i = 1$, fusion acts as a one-to-one relabeling of simple lines, corresponding to an ordinary (invertible) symmetry. If instead $d_i > 1$, fusion with $\mathcal{L}_i$ increases the dimension of the space of topological lines, and the operation cannot be inverted: $\mathcal{L}_i$ then represents a non-invertible symmetry generator.

2.4.3 An example of non-invertible symmetry: Rep(G)

In this section we want to show an example of non-invertible symmetry. In particular, one of the simplest ways in order to construct new symmetries, that, in general, turn out to be non-invertible, is by gauging an invertible symmetry. In this example, we will focus on a 2d theory $\mathcal{T}$ with a 0-form global symmetry G.

To gauge a 0-form symmetry G means to promote the global symmetry transformations of G to local gauge redundancies by coupling the theory to a G gauge field and summing over all possible configurations of that field. Concretely, one introduces a background G-bundle for the global symmetry, couples it to the fields of the theory $\mathcal{T}$, and then integrates over all gauge-inequivalent G-bundles with appropriate topological weights. The resulting path integral defines a new theory, denoted $\mathcal{T}/G$.

Before gauging, the theory $\mathcal{T}$ admits a set of invertible topological defect lines $\{D^{(g)}\}_{g\in G}$ implementing the symmetry transformations of the group G and generating the fusion category Vec_G, as was shown in (2.4.1). Since all symmetry lines fuse according to the group multiplication law, they are invertible, each having quantum dimension $d_g = 1$.

After gauging the symmetry, the resulting theory $\mathcal{T}/G$ no longer possesses the invertible group symmetry Vec_G, but instead exhibits a new symmetry. The topological defect lines of this new symmetry are the resulting Wilson lines in the irreducible representations $\mathbf{R}$ of G, given by

$$D_1^{(\mathbf{R})}(M_1) = \mathrm{Tr}_{(\mathbf{R})} e^{\int_{M_1} b}, \tag{2.4.16}$$

where b is the G gauge field introduced in order to gauge the theory. In particular, the fusion rules of the Wilson lines take the form

$$D_1^{(\mathbf{R}_1)} \otimes D_1^{(\mathbf{R}_2)} = \bigoplus_{\mathbf{R}_3} N_{\mathbf{R}_3}^{\mathbf{R}_1\mathbf{R}_2} D_1^{(\mathbf{R}_3)}, \tag{2.4.17}$$

where $N_{\mathbf{R}_3}^{\mathbf{R}_1\mathbf{R}_2}$ are the Clebsch-Gordan coefficients of the tensor product of the representations:

$$\mathbf{R}_1 \otimes \mathbf{R}_2 = \bigoplus_{\mathbf{R}_3} N_{\mathbf{R}_3}^{\mathbf{R}_1\mathbf{R}_2} \mathbf{R}_3, \tag{2.4.18}$$

and, in this case, the identity $\mathbf{1}$ is the trivial representation. In particular, these topological defect lines define a new fusion category $\mathrm{Rep}(G)$, the category of finite-dimensional representations of G (see [12,54] for a more detailed explanation on the emergence and definition of the Rep(G) fusion category).

The crucial difference between the pre-gauged and post-gauged theories lies in the invertibility of the symmetry lines. In Vec_G, every object $D^{(g)}$ has a unique inverse $D^{(g^{-1})}$, following from the group-like fusion rule. In $\mathrm{Rep}(G)$, by contrast, only the one-dimensional representations remain invertible. Instead, any representation R_i of higher dimension $\dim R_i > 1$ gives rise to a non-invertible line, since its fusion decomposes into a direct sum of several simple objects rather than a single one. Indeed, the corresponding fusion matrix N_i has Perron-Frobenius eigenvalue

$$d_i = \dim R_i > 1, \tag{2.4.19}$$

and hence violates the invertibility condition $d_i = 1$. In particular, from the reletion of the quantum dimension of the TDLs and the well-known fact from group theory that

$$G \text{ abelian} \iff \text{dim}(R) = 1\ \forall R \text{ irreducible representation} \tag{2.4.20}$$

(we refer to [56] for a proof of this statement), we get that the gauging of a non-abelian 0-form symmetry results in a non-invertible fusion category Rep(G).

To give an explicit example of non-invertible symmetry arising from the gauging of a non-abelian 0-form symmetry, we can consider the gauging of the simplest non-abelian finite group, the symmetric group S_3. This is generated by 3 generators as

$$S_3 = \langle \text{id}, a, b | a^3 = b^2 = \text{id},\ bab = a^2 \rangle. \tag{2.4.21}$$

The irreducible representations of this group are the trivial one, $\mathbf{1}_+$, the sign representation (depending on the sign of the permutation), $\mathbf{1}_-$, and the 2d representation $\mathbf{2}$ (which is the representation of the group acting as the symmetry group of the equilateral triangle). The tensor product rules of these representations are:

$$\begin{aligned} \mathbf{1}_+ \otimes \mathbf{1}_\pm &= \mathbf{1}_\pm \\ \mathbf{1}_- \otimes \mathbf{1}_- &= \mathbf{1}_+ \\ \mathbf{2} \otimes \mathbf{1}_\pm &= \mathbf{2} \\ \mathbf{2} \otimes \mathbf{2} &= \mathbf{1}_+ \oplus \mathbf{1}_- \oplus \mathbf{2}. \end{aligned} \tag{2.4.22}$$

In particular, since the fusion rules of the TDLs follow from the above coefficients, we get

$$D_1^{\mathbf{2}} \otimes D_1^{\mathbf{2}} = D_1^{\mathbf{1}_+} \oplus D_1^{\mathbf{1}_-} \oplus D_1^{\mathbf{2}} \tag{2.4.23}$$

which is the fusion rule of a non-invertible topological defect line.

2.4.4 Non-invertibility from non-genuine operators

As we saw in the previous section, the non-invertibility property of a topological defect can be defined in terms of its quantum dimension. In particular, let $\mathcal{L}$ be a topological defect, then if its quantum dimension $d_{\mathcal{L}} > 1$ the defect is not invertible. However, this is just one of the possible definitions. Indeed, the non-invertibility property of a topological defect can also be defined in terms of its action on the operators of the theory.

To do so, we first need to introduce the notion of non-genuine operator. A non-genuine p-dimensional operator is an operator that, in order to be well-defined, needs to be attached to the end of a $p+1$-dimensional operator. To better understand this definition, we illustrate an example. Consider a d-dimensional $U(1)$-gauge theory, and let A be the gauge field (as in the example in Section 2.3.1). Let $\phi(x)$ be a matter field with charge $q \in \mathbb{Z}$ under the $U(1)$-gauge symmetry. Now, the insertion of the field $\phi(x)$ cannot define a local operator of the theory, since it is not gauge invariant. Indeed, under the gauge transformation

$$A(x) \to A(x) - d\theta(x) \tag{2.4.24}$$

$\phi(x)$ would transform as:

$$\phi(x) \to e^{iq\theta(x)}\phi(x). \tag{2.4.25}$$

In order to make the insertion gauge invariant, we need to attach this local operator to a charge q Wilson line defined on the 1-dimensional submanifold M_1 such that $\partial M_1 = x$, that is to say the Wilson line must end on the local operator $\phi(x)$. Indeed, in this way, the Wilson line transforms as

$$W_q(M_1) = \exp\left(iq\int_{M_1} A\right) \to \exp\left(-iq\int_{M_1} d\theta\right) W_q(M_1) = e^{-iq\theta(x)}W_q(M_1) \tag{2.4.26}$$

and the combination of the two $\phi(x)W_q(M_1)$ is gauge invariant. This implies that $\phi(x)$ is a non-genuine operator and it must be defined as the ending point of a Wilson line (see [57] for a more detailed discussion). This is illustrated in Fig. 5.

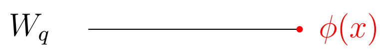


Figure 5: The matter field operator $\phi(x)$ of charge q is not gauge invariant, and for this reason it lives at the end of a Wilson line of charge q, defining a non-genuine operator.

We are now able to define a non-invertible defect in terms of its action on the operators of the theory. In particular, a symmetry defect $\mathcal{D}(M_p)$ is said to be non-invertible if, given some genuine operator $\mathcal{O}(M_q)$, the action of $\mathcal{D}(M_p)$ on $\mathcal{O}(M_q)$ yields a non-genuine operator, meaning that the resulting operator must be attached to some higher-dimensional operator $\mathcal{O}'(N_{q+1})$, whose boundary satisfies $M_q = \partial N_{q+1}$. This is due to the fact that the action of a non-invertible defect on genuine operators is not an automorphism, and, for this reason, it acts non-invertibly.

To understand this better, let us consider the 2d Ising model, which is characterized by the existence of a non-invertible defect line D, called the Kramers-Wannier duality defect. This comes from the fact that the Ising model is isomorphic to its $\mathbb{Z}_2$ orbifold, and this gives rise to the non-invertible defect D with fusion rule:

$$D \times D = \text{Id} + \eta, \tag{2.4.27}$$

where η is the topological defect line implementing the $\mathbb{Z}_2$ invertible symmetry, satisfying $\eta \times \eta = \text{Id}$ (in our discussion below we will only need the fusion rule of the non-invertible TDL D; for a complete discussion on the 2d Ising model we refer to [37]). Now, let us consider the insertion of this TDL in the 2d theory, together with a genuine operator $\mathcal{O}(x)$, as depicted in Fig. 6.(a). Since the defect D is topological, we can de-

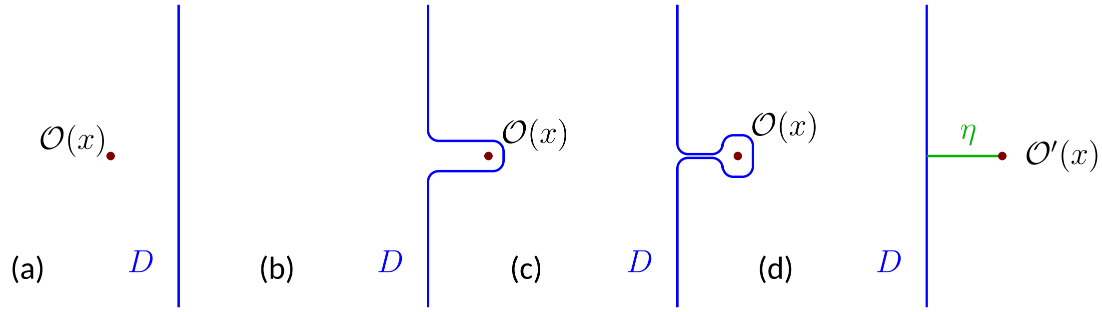


Figure 6: The action of the non-invertible Kramers-Wannier topological defect D in the 2d Ising model on a genuine operator $\mathcal{O}(x)$.

form it (as long as it doesn't cross the local operator $\mathcal{O}(x)$) in order that it surrounds the local operator insertion. At this point, using the fusion rule for D on the two horizontal segments, we get that the genuine local operator $\mathcal{O}(x)$ is mapped to a non-genuine operator $\mathcal{O}'(x)$, which, in order to be well-defined, needs to be attached at the end of the η-line (for a more complete discussion, we refer to [12]).

2.5 Generalized symmetries and phenomenology

Before moving to the main section of this thesis, namely the emergence of non-invertible symmetries in the 2d compact boson, we want to make the connection between the landscape of generalized symmetries and the possible phenomenological implications that these can have. Indeed, throughout this chapter, we have presented the mathematical framework in which the construction of generalized symmetries is possible; to make this generalization clearer, we have also presented some examples of higher form and non-invertible symmetries. However, the examples we presented in the previous sections were created ad hoc in order to show some peculiar properties of generalized symmetries, without the ambition of deriving any realistic phenomenological implication. The goal of this final section is instead to present a phenomenologically relevant example of generalized symmetry application, and how these broader types of symmetries can be helpful in order to construct a realistic physical model.

The application of generalized symmetries we want to discuss here has been first presented in [58] and is an axion-less generalized symmetry-driven attempt to solve the strong CP problem.

The strong CP problem derives from the fact that quantum chromodynamics (QCD) Lagrangian admits a CP-violating term

$$\mathcal{L}_\theta = \frac{\theta g_s^2}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{a\,\mu\nu}, \tag{2.5.1}$$

where $G^a_{\mu\nu}$ is the gluon field strength, $\tilde{G}^{a\,\mu\nu}$ its dual, g_s is the strong coupling constant and $\theta \in [0, 2\pi)$ is a parameter. This term violates the combination of parity (P) and charge-conjugation (C) symmetries, hence CP, leading to observable effects such as a nonzero neutron electric dipole moment. In addition to this term, CP violation can also be originated by the complex phases in the quark mass matrices, and it is proportional to arg$[\det(M_u M_d)]$, where M_u and M_d are the up and down-type quark mass matrices. In particular, the observable CP-violating parameter in strong interactions is the sum of these two effects, resulting in:

$$\bar{\theta} = \theta + \text{arg}[\det(M_u M_d)]. \tag{2.5.2}$$

The experimental measure of $\bar{\theta}$, however, puts an upper bound to its value at $|\bar{\theta}| \lesssim 10^{-10}$. Since CP symmetry is violated in the weak interactions (as was already experimentally proven by Cronin and Fitch in 1964, see [59] for the original paper), such a fine-tuned value in the strong sector appears unnatural. For this reason, explaining why QCD preserves CP constitutes the so-called strong CP problem.

One of the possible solutions, and the most popular one, is via the Peccei-Quinn mechanism, which was introduced by R. D. Peccei and H. R. Quinn in 1977 (see [60] for the original work). This mechanism relies on the introduction of a global chiral symmetry $U(1)_{PQ}$ which is spontaneously broken. This spontaneous symmetry breaking (SSB) gives rise to a pseudo-Goldstone boson known as the axion a which dynamically ensures the vanishing of $\bar{\theta}$ at the minimum of its potential. The problem in the Peccei-Quinn mechanism, however, is that it gives rise to a new particle, the axion, whose existence has long been ruled out by experiments. For this reason, the exploration of axion-less mechanisms that are able to impose the $\bar{\theta} = 0$ condition has been an active research field in the last decades. In the following, we will present such a mechanism driven by generalized symmetries arguments.

As we said above, there are two possible sources of CP violation in QCD, namely θ and arg$[\det(M_u M_d)]$. Regarding the first term, there are strong theoretical reasons to impose the vanishing of θ. Indeed, in most UV-complete frameworks (such as string theory, higher-dimensional gauge theories, supersymmetric grand unification theories), CP is an exact symmetry of the action before compactification or spontaneous breaking, implying the vanishing of θ at the tree level. In addition to this, if the quark mass matrices are real, arg$[\det(M_u M_d)] = 0$, there will be no radiative corrections that can generate θ, and so, in this case, $\theta = 0$ remains valid also at higher loop orders. This implies that, in order to address the strong CP problem, we need a theoretical motivation to impose the vanishing of arg$[\det(M_u M_d)]$, or, at least, to guarantee arg$[\det(M_u M_d)] \lesssim 10^{-10}$.

In the cited article [58], the authors propose a GUT-motivated model for the axionless solution of the strong CP problem, following from the $SU(5)$ GUT-inspired SM (see [61] for a consistent formulation of this model). In particular, the central point of this paper is the gauging of a finite symmetry, which gives rise to a non-invertible symmetry, imposing new constraints on the mass matrices resulting from the theory. More precisely, this method relies on the gauging of the $\mathbb{Z}_2$ automorphism of the $\mathbb{Z}_5$ symmetry of the theory, giving rise to the new symmetry $\tilde{Z}_5$. This new symmetry is described by a fusion category, whose objects are the invariant conjugacy classes of $D_5 = \mathbb{Z}_5 \rtimes \mathbb{Z}_2$ under the $\mathbb{Z}_2$ automorphism; these are $\{[g^0], [g^1], [g^2]\}$, with fusion rules:

$$\begin{aligned} &[g^0] \otimes [g^0] = [g^0], && [g^0] \otimes [g^1] = [g^1] \otimes [g^0] = [g^1], \\ &[g^0] \otimes [g^2] = [g^2] \otimes [g^0] = [g^2], && [g^1] \otimes [g^1] = [g^0] + [g^2], \\ &[g^1] \otimes [g^2] = [g^2] \otimes [g^1] = [g^1] + [g^2], && [g^2] \otimes [g^2] = [g^0] + [g^1], \end{aligned} \tag{2.5.3}$$

and we can clearly see the non-invertibility property. More importantly, this generalized (non-invertible) symmetry constrains the possible quark mass matrices that can be obtained in the theory, that turn out to be of the form:

$$\begin{pmatrix} \checkmark & 0 & 0 \\ 0 & \checkmark & 0 \\ 0 & 0 & \checkmark \end{pmatrix}, \quad \begin{pmatrix} 0 & \checkmark & 0 \\ \checkmark & 0 & \checkmark \\ 0 & \checkmark & \checkmark \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & \checkmark & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.5.4}$$

where the ✓ denotes the allowed entries, as was shown in [62]. Following the charge assignment proposed in [58], the quark mass matrices turn out to be of the form:

$$M^0_{u,d} = \begin{pmatrix} 0 & a_{u,d} & 0 \\ a'_{u,d} & 0 & c_{u,d} \\ 0 & c'_{u,d} & d_{u,d} \end{pmatrix}, \tag{2.5.5}$$

where $a_{u,d}, a'_{u,d}, c_{u,d}, c'_{u,d}, d_{u,d} \in \mathbb{R}$. These constraints on the quark mass matrices clearly imply arg$[\det(M_u M_d)] = 0$, which, together with the assumption that the underlying fundamental theory is CP invariant, that is to say $\theta = 0$, gives $\bar{\theta} = 0$, without introducing any new field and, consequently, giving rise to an axion-free CP invariant theory. Moreover, in this framework, the authors show that it is also possible to introduce a new complex scalar field, η, charged under the non-invertible symmetry, whose vacuum expectation value $\langle \eta \rangle$ allows for an imaginary term $be^{\phi_{u,d}}$ in the quark mass matrices. Surprisingly, the two phases are such that arg$[\det(M_u M_d)] = 0$, and so the CP symmetry is not violated while giving a natural explanation of the correct sign of the baryon-number asymmetry in the present universe.

Conceptually, this model shows how generalized (non-invertible) symmetries, initially formulated in the context of topological defect operators, can play a direct phenomenological role. By constraining quark mass matrices through non-invertible fusion rules, they realize an axion-less solution to the strong CP problem that is UV-motivated and consistent with experimental data.

3 The $c = 1$ Compact Boson: Rationality and Symmetries

In the previous chapter, we presented the notion of generalized symmetry. We started by reviewing the classical concept of symmetry in QFT; then, by reformulating symmetries in terms of topological operators and fusion categories, we have been able to define higher-form and non-invertible symmetries. Finally, we presented a phenomenological application of generalized symmetries.

With this new framework, we are now able to move to the study of the $c = 1$ compact boson and how non-invertible symmetries can arise in this type of theory. We will start with a general introduction to the $c = 1$ compact boson, presenting its action, the operator content, the partition function and its main characteristics (for a complete review of the c=1 compact boson, we refer to [38,63,43,47,64]); we will then move to the concept of rational CFT and we will present the rationality condition for the $c = 1$ compact boson. Finally, we will show how to construct non-invertible symmetries at these rational points, their categorical formulation, and how these act on the Dp-branes of the theory.

This section will serve as a testing ground for some of the techniques that will be used in the next section, in which we will apply the same analysis to the construction of non-invertible symmetries for the $c = 2$ compact boson.

3.1 The $c = 1$ compact boson

The $c = 1$ compact boson is one of the simplest two-dimensional conformal field theories (CFT). This is a 2-dimensional QFT that describes a single real scalar field $\phi(\sigma, \tau)$ compactified on a circle of radius R, S^1_R. It can be interpreted as a non-linear sigma model with a one-dimensional target space S^1_R. In general, a non-linear sigma model is a quantum field theory whose fundamental field $X : \Sigma \to \mathcal{M}$ represents a map from the two-dimensional worldsheet Σ to a target manifold $\mathcal{M}$, and whose dynamics is governed by the pullback of the target-space metric G_{ij}. The compact boson thus corresponds to the special case in which the target manifold is a circle, $\mathcal{M} = S^1_R$, with metric $ds^2 = R^2 d\phi^2$, where R is the radius of the circle. The worldsheet action can be written in terms of the compact scalar field $\phi \sim \phi + 2\pi$ as:

$$S = \frac{R^2}{4\pi} \int d\phi \wedge \star d\phi = \frac{R^2}{4\pi} \int d\sigma d\tau \; \partial_\mu \phi \partial^\mu \phi = \frac{1}{4\pi} \int d\sigma d\tau \; \partial_\mu X \partial^\mu X, \qquad (3.1.1)$$

where we introduced the new compact scalar field X, defined as $X = R\phi$, which satisfies $X \sim X + 2\pi R$; we do this for simplicity in the control of the scale factor R in the following. The equations of motion are those of a free massless field,

$$\partial_\tau^2 X - \partial_\sigma^2 X = 0, \tag{3.1.2}$$

whose general solution is the sum of a left and a right moving component,

$$X(\sigma, \tau) = \frac{1}{\sqrt{2}} \left(X_L(\tau + \sigma) + X_R(\tau - \sigma) \right). \tag{3.1.3}$$

The $\frac{1}{\sqrt{2}}$ is a normalization factor; this is the standard normalization used in the CFT and string theory literature and it is needed in order to guarantee that the 2-point function $\langle X_L(z) X_L(w) \rangle$ has the standard operator product expansion with leading term $\langle X_L(z) X_L(w) \rangle \sim -ln(z - w)$ (here z and w are complex coordinates on the 2d plane on which the CFT is defined; these can be obtained by Wick rotating the time coordinate τ).

Because the target space is compact, the field configurations can wind around the circle as one moves along the spatial direction of the worldsheet. This leads to topologically non-trivial field configurations in the theory. If we consider the worldsheet to be a cylinder with coordinates (σ, τ), such that $\sigma \sim \sigma + 2\pi$, the boundary condition takes the form

$$X(\sigma + 2\pi, \tau) = X(\sigma, \tau) + 2\pi R w, \qquad w \in \mathbb{Z}, \tag{3.1.4}$$

where w is the winding number. The field can also carry momentum along the compact direction, and the requirement for the vertex operator e^{ipX} to be single-valued under $X \to X + 2\pi R$ quantizes the momentum as

$$p = \frac{n}{R}, \qquad n \in \mathbb{Z}. \tag{3.1.5}$$

Each sector of the theory is therefore labeled by a pair of integers (n, w), which correspond respectively to the momentum and winding numbers.

The general solution to the above equations of motion (3.1.2) is

$$X(\sigma, \tau) = X_0 + p\,\tau + wR\,\sigma + i \sum_{m \neq 0} \frac{1}{m} \left(\alpha_m e^{-im(\tau+\sigma)} + \tilde{\alpha}_m e^{-im(\tau-\sigma)} \right), \tag{3.1.6}$$

where the first term is an integration constant, the second and third ones describe the zero-mode sector (with $p = \frac{n}{R}$) while the oscillatory part describes the excitations of the field. By introducing the new coordinates $\xi^\pm = \tau \pm \sigma$, the left and right moving components can be written as

$$\begin{aligned} X_L(\xi^+) &= X_{0L} + p_L\,\xi^+ + i\sqrt{2} \sum_{m \neq 0} \frac{1}{m} \alpha_m e^{-im\xi^+}, \\ X_R(\xi^-) &= X_{0R} + p_R\,\xi^- + i\sqrt{2} \sum_{m \neq 0} \frac{1}{m} \tilde{\alpha}_m e^{-im\xi^-}, \end{aligned} \tag{3.1.7}$$

with

$$p_L = \frac{1}{\sqrt{2}}\left(\frac{n}{R} + wR\right), \qquad p_R = \frac{1}{\sqrt{2}}\left(\frac{n}{R} - wR\right). \tag{3.1.8}$$

It is also possible to introduce a second scalar field $\tilde{X}$, called the dual field, defined locally as

$$\partial_\mu \tilde{X} = \epsilon_{\mu\nu}\, \partial^\nu X, \tag{3.1.9}$$

or equivalently $d\tilde{X} = \star dX$. This relation exchanges the roles of space and time derivatives. Both X and $\tilde{X}$ satisfy the same equation of motion and are compact fields, with $\tilde{X} \sim \tilde{X} + \frac{2\pi}{R}$.

In particular, the compact boson presents two independent global shift symmetries:

$$X \to X + R\epsilon, \tag{3.1.10}$$

$$\tilde{X} \to \tilde{X} + \frac{1}{R}\tilde{\epsilon}. \tag{3.1.11}$$

The first, acting on X, generates the Noether current

$$j^\mu_X = \frac{1}{2\pi}\,\partial^\mu X, \qquad Q_X = \int_0^{2\pi} d\sigma\, j^\tau_X = \frac{n}{R}, \tag{3.1.12}$$

while the second, acting on $\tilde{X}$, arises from the Bianchi identity $\epsilon^{\mu\nu}\partial_\mu\partial_\nu X = 0$ and has a conserved current

$$j^\mu_{\tilde{X}} = \frac{1}{2\pi}\,\epsilon^{\mu\nu}\partial_\nu X, \qquad Q_{\tilde{X}} = \int_0^{2\pi} d\sigma\, j^\tau_{\tilde{X}} = wR. \tag{3.1.13}$$

These shift symmetries give rise to two $U(1)$ symmetries whose conserved charges are $\frac{n}{R}$ and Rw, labeled by the two integers (n, w), and the global symmetry of the theory is denoted by $U(1)_n \times U(1)_w$. In terms of chiral components,

$$X = \frac{1}{\sqrt{2}}(X_L + X_R), \qquad \tilde{X} = \frac{1}{\sqrt{2}}(X_L - X_R), \tag{3.1.14}$$

so that the left and right moving fields can be regarded as linear combinations of the original and dual field.

The two global $U(1)$ symmetries just described can also be rewritten in terms of two conserved chiral currents,

$$J_L(\xi^+) = \partial_+ X_L(\xi^+), \qquad J_R(\xi^-) = \partial_- X_R(\xi^-), \tag{3.1.15}$$

(where $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$) which generate the two independent shift symmetries of the left and right moving sectors. In particular, these correspond to chiral operators of conformal dimension $(h_L, h_R) = (1, 0)$ and $(h_L, h_R) = (0, 1)$ respectively, which define the global $U(1)_L \times U(1)_R$ chiral algebra of the compact boson (this is simply a redefinition of the global symmetry $U(1)_n \times U(1)_w$ introduced above in terms of left and right momenta).

The spectrum of primary operators can be written in terms of p_L and p_R, and the vertex operators take the form

$$V_{n,w} = e^{i\frac{n}{R}X} e^{iwR\tilde{X}} = e^{ip_L X_L} e^{ip_R X_R}. \tag{3.1.16}$$

The conformal dimension Δ and spin s of each operator can be written in terms of $h_L = \frac{1}{2}p_L^2$ and $h_R = \frac{1}{2}p_R^2$ as:

$$\Delta = h_L + h_R = \frac{1}{2}\left(\frac{n^2}{R^2} + w^2R^2\right), \quad s = h_L - h_R = nw, \tag{3.1.17}$$

and the Dirac quantization condition, which in this case takes the form $nw \in \mathbb{Z}$, is trivially satisfied. Moreover, the primary operators of the theory are in one-to-one correspondence with the sites of the even, self-dual integer momentum lattice $\Gamma^{1,1}$, in which the left and right momentum take value

$$\Gamma^{1,1} \ni \begin{pmatrix} p_L \\ p_R \end{pmatrix} = L\begin{pmatrix} n \\ w \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{R} & R \\ \frac{1}{R} & -R \end{pmatrix}\begin{pmatrix} n \\ w \end{pmatrix}, \tag{3.1.18}$$

where L is the lattice generating matrix. An important thing to notice is that it is possible to use the $\Gamma^{1,1}$ lattice in order to compute the conformal dimension Δ and the spin s of a primary operator. Indeed, these can be computed as

$$\Delta = h_L + h_R = \frac{1}{2}\begin{pmatrix} p_L & p_R \end{pmatrix}\begin{pmatrix} p_L \\ p_R \end{pmatrix} = \frac{1}{2}\begin{pmatrix} n & w \end{pmatrix} L^T L \begin{pmatrix} n \\ w \end{pmatrix}, \tag{3.1.19}$$

$$s = h_L - h_R = \frac{1}{2}\begin{pmatrix} p_L & p_R \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} p_L \\ p_R \end{pmatrix} = \frac{1}{2}\begin{pmatrix} n & w \end{pmatrix} L^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} L \begin{pmatrix} n \\ w \end{pmatrix}, \tag{3.1.20}$$

and so $\frac{1}{2}L^T L$ and $\frac{1}{2}L^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} L$ can be regarded as the Δ and s pairing over the (n, w) space.

The partition function of the compact boson encodes the full spectrum of momentum and winding excitations. By taking the worldsheet to be the Euclidean torus with modular parameter $\tau = \tau_1 + i\tau_2$, the partition function is defined as

$$Z(R; \tau, \bar{\tau}) = \mathrm{Tr}_{\mathcal{H}}\left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}\right), \qquad q = e^{2\pi i\tau}, \tag{3.1.21}$$

where the trace runs over all states of the Hilbert space labeled by integer momentum and winding numbers (n, w); L_0 and $\bar{L}_0$ are the 0-mode of the left and right Virasoro algebras, that is to say the holomorphic and anti-holomorphic ones. The total partition function can then be written as

$$Z(R;\tau_1,\tau_2) = \frac{1}{|\eta(\tau)|^2} \sum_{(n,w)\in\mathbb{Z}^2} e^{-\pi\tau_2\left(\frac{n^2}{R^2}+w^2R^2\right)} e^{2\pi i\tau_1\, nw}, \tag{3.1.22}$$

where $\eta(\tau)$ is the Dedekind eta-function defined as

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \tag{3.1.23}$$

and it accounts for the contribution of the oscillator modes. Now, each sector contributes according to the left and right moving momentum (3.1.8) in such a way that the sum over (n, w) can also be rewritten in terms of the momentum lattice $\Gamma^{1,1}$. In particular, in this way the partition function takes the much more familiar and easier form

$$Z(\tau,\bar{\tau};R) = \frac{1}{|\eta(\tau)|^2} \sum_{(p_L,p_R)\in\Gamma^{1,1}} q^{\frac{p_L^2}{2}}\,\bar{q}^{\frac{p_R^2}{2}}. \tag{3.1.24}$$

A fundamental property of the partition function (3.1.24) is its invariance under modular transformations of the torus, which reflect redundancies in the parametrization of the worldsheet geometry. The modular group $\mathrm{SL}(2,\mathbb{Z})$ acts on the complex structure parameter as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : \tau \longrightarrow \frac{a\tau+b}{c\tau+d} \tag{3.1.25}$$

and the physical consistency of the theory requires the partition function to satisfy

$$Z\left(\tfrac{a\tau+b}{c\tau+d}, \tfrac{a\bar{\tau}+b}{c\bar{\tau}+d}; R\right) = Z(\tau,\bar{\tau};R), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}). \tag{3.1.26}$$

In particular, the group $SL(2,\mathbb{Z})$ is generated by T and S, which are defined as

$$T:\ \tau \to \tau+1, \qquad S:\ \tau \to -\frac{1}{\tau}, \tag{3.1.27}$$

The invariance under the T transformation follows from the quantization of spin, $s = h_L - h_R = nw \in \mathbb{Z}$, which ensures $q^{h_L}\bar{q}^{h_R}$ acquires only an integer phase under $\tau \to \tau + 1$. The S transformation, $\tau \to -\frac{1}{\tau}$, can be regarded as the exchange of the temporal and spatial cycles of the torus. In the $c = 1$ compact boson, this operation interchanges momentum and winding modes, acting on the lattice labels as $(n, w) \mapsto (w, -n)$. Since the partition function (3.1.24) includes a symmetric sum over all integer pairs (n, w) (that is, over the self-dual integer momentum lattice), it is manifestly invariant under this exchange. This modular invariance guarantees the consistency of the compact boson CFT.

Another important feature of the compact boson is its invariance under T-duality, a discrete symmetry that exchanges momentum and winding modes. In terms of the compact scalar X and its dual field $\tilde{X}$, the theory depends on the compactification radius R through the left and right moving momentum (3.1.8) spanned by $n, w \in \mathbb{Z}$. However, the conformal dimension and spin of a state depend only on p_L^2 and p_R^2, and the same is true for the partition function, implying that the theory remains invariant under the transformation

$$R \longrightarrow \frac{1}{R}, \qquad n \longleftrightarrow w, \tag{3.1.28}$$

which exchanges the roles of momentum and winding. This is the T-duality symmetry of the compact boson.

From a geometric point of view, T-duality maps the theory on a circle of radius R to a dual theory on a circle of radius $1/R$. In terms of the fields, it acts as

$$X \ \longleftrightarrow \ \tilde{X}, \tag{3.1.29}$$

interchanging the two $U(1)$ symmetries of the model. This can be easily seen from the mode expansion of the fields X_L and X_R, resulting in the overall map

$$X_L \to X_L \quad \text{and} \quad X_R \to -X_R. \tag{3.1.30}$$

At the self-dual point $R = 1$ (where both X and $\tilde{X}$ are maps to the unit circle), the left and right moving momenta become $p_L = \frac{1}{\sqrt{2}}(n + w)$ and $p_R = \frac{1}{\sqrt{2}}(n - w)$, and the global symmetry $U(1)_L \times U(1)_R$ is enhanced to $SU(2)_L \times SU(2)_R$. Indeed, the spectrum contains additional dimension-one operators that define new conserved currents. In particular, from the conformal weights $h_L = \frac{1}{2}p_L^2$ and $h_R = \frac{1}{2}p_R^2$, one finds that for $R = 1$ the vertex operators with momenta $(p_L, p_R) = (\pm\sqrt{2}, 0)$ and $(0, \pm\sqrt{2})$ have $h_L = 1$ and $h_R = 1$, respectively. These can therefore be identified with new left and right moving currents,

$$J_L^{\pm}(\xi^+) =: e^{\pm i\sqrt{2}X_L(\xi^+)} :, \qquad J_R^{\pm}(\xi^-) =: e^{\pm i\sqrt{2}X_R(\xi^-)} :, \tag{3.1.31}$$

which, together with $J_L(\xi^+)$ and $J_R(\xi^-)$, enlarge the left and right chiral algebras to $SU(2)_L$ and $SU(2)_R$ respectively. This enhancement reflects the fact that, at $R = 1$, T-duality acts as an automorphism of the chiral current algebra, rotating the $U(1)$ currents into each other.

As we saw above, the $c = 1$ compact boson is characterized by the radius R of the circle S^1_R on which the field X is compactified. In particular, the value of the radius spans a continuous set of theories, defined by $R \in \mathbb{R}^+$; however, we also have to account for T-duality, which states that two theories defined by the radius R and $1/R$ are equivalent up to T-duality. This implies that the fundamental domain in which R

takes value is given by the quotient of $\mathbb{R}^+$ by the T-duality identification, resulting in $\mathbb{R}_{\geq 1}$, and it is called the circle branch of the moduli space. However, in addition to the circle compactification on S^1_R, the $c = 1$ compact boson admits a second continuous family of modular-invariant theories, obtained by quotienting the circle by the discrete reflection symmetry $X \to -X$. This construction defines the so-called orbifold branch of the moduli space, corresponding to the target space $S^1_R/\mathbb{Z}_2$; the circle and orbifold branches are connected at the point $R = 2$, and they are the only two branches in the $c = 1$ moduli space. Geometrically, this orbifold can be viewed as a line segment of length πR, with two fixed points under the $\mathbb{Z}_2$ action located at $X = 0$ and $X = \pi R$ (see Fig. 7). The orbifold theory is defined by projecting the Hilbert space of the circle

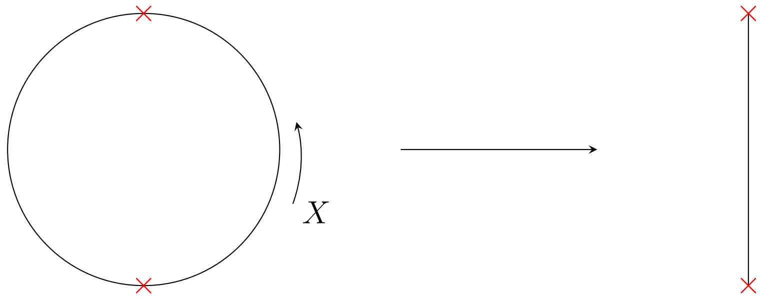


Figure 7: The action of $\mathbb{Z}_2$ orbifold on S^1_R. On the left we show the circle of radius R, on the right we show the resulting line segment $S^1_R/\mathbb{Z}_2$ with the two fixed points.

model onto $\mathbb{Z}_2$-invariant states and by including new twisted sectors associated with field configurations that are periodic up to the reflection symmetry. More explicitly, the untwisted sector consists of the $\mathbb{Z}_2$-even subset of the original Hilbert space, where the field satisfies $X(\sigma + 2\pi, \tau) = X(\sigma, \tau)$, while the twisted sector is generated by configurations obeying the boundary condition

$$X(\sigma + 2\pi, \tau) = -X(\sigma, \tau), \tag{3.1.32}$$

which implies a mode expansion in terms of half-integer oscillators and no momentum or winding zero modes. The twisted sector therefore contains two degenerate vacua $|0\rangle_{1,2}$, localized at the orbifold fixed points. The full Hilbert space of the orbifold theory is the sum of the projected untwisted sector and the twisted sector,

$$\mathcal{H}_{\text{orb}} = \mathcal{H}_{\text{untw}} \oplus \mathcal{H}_{\text{tw}}, \tag{3.1.33}$$

and the torus partition function can be obtained by summing over all sectors weighted by their corresponding $\mathbb{Z}_2$ boundary conditions along the two cycles of the torus. The result can be expressed as

$$Z_{\text{orb}}(R) = \frac{1}{2}\Big(Z_{00} + Z_{01} + Z_{10} + Z_{11}\Big), \tag{3.1.34}$$

where Z_{00} is the untwisted contribution of the original circle theory, Z_{01} and Z_{10} correspond to the twisted insertions along the spatial and temporal cycles, and Z_{11} represents the double twisted sector. We will not go into the details of the $\mathbb{Z}_2$ orbifold Hilbert space and partition function here. We refer to [63, 39] for a more detailed discussion and for the explicit form of the partition function.

The fact that the $\mathbb{Z}_2$ orbifold represents the only additional branch of the $c = 1$ moduli space is not accidental but has a clear geometric origin. Indeed, the circle S^1_R can be viewed as the quotient of the real line by a one-dimensional lattice of period $2\pi R$, that is

$$S^1_R = \mathbb{R}/(2\pi R\mathbb{Z}). \tag{3.1.35}$$

The geometry of this lattice completely determines the possible discrete symmetries that can act on the target space. In one dimension, the only non-trivial automorphism of the lattice $\Lambda = 2\pi R\mathbb{Z}$ is the reflection $x \mapsto -x$ (where x represents the coordinate on the real axis), which generates the group $\mathbb{Z}_2$. Therefore, within this geometric realization of the compact boson, the only admissible orbifold is the $\mathbb{Z}_2$ orbifold. This explains why the $\mathbb{Z}_2$ orbifold is the unique non-trivial extension of the circle branch and why the $c = 1$ moduli space consists precisely of these two connected components: the circle branch and the $\mathbb{Z}_2$ orbifold branch[4] (see Fig. 8).

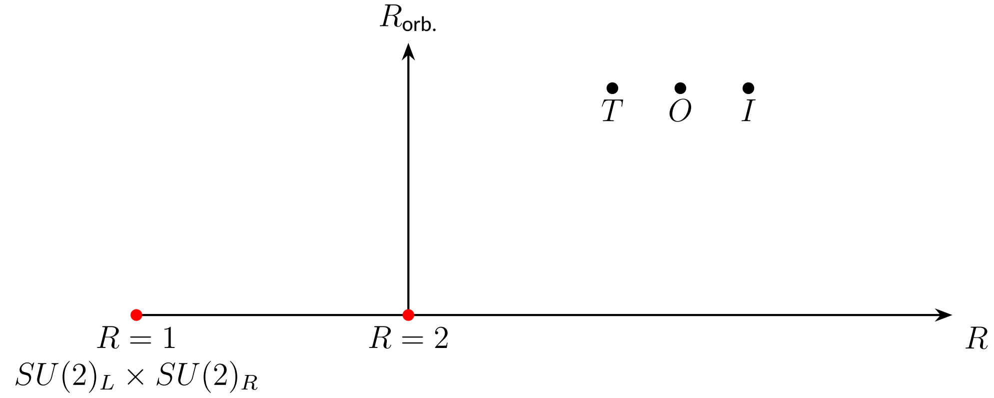


Figure 8: The $c = 1$ compact boson moduli space. It presents 2 connected components, namely the circle and the $\mathbb{Z}_2$ orbifold branches, that join at the $R = 2$ point. In addition, the moduli space features the enhanced symmetry point at $R = 1$, corresponding to $SU(2) \times SU(2)$ chiral symmetry, as well as the three exceptional isolated components T, O and I.

[4] In addition to the 2 connected components, the $c = 1$ moduli space presents 3 isolated exceptional component, denoted by T, O and I. These come from the gauging of the three special symmetry subgroups of $SU(2)$. For a detailed discussion we refer to [39, 26].

3.2 The rationality condition

In the previous section, we discussed the $c = 1$ compact boson, which is one of the simplest 2-dimensional CFTs and which will serve as a testing ground for the next chapter, in which we will discuss the emergence of generalized symmetries in the $c = 2$ compact boson. In this section, we want instead to introduce the concept of rational conformal field theory (RCFT), which will be central in the construction of non-invertible symmetries, as we will see later.

We can start by giving the definition of RCFT: a two-dimensional conformal field theory is said to be rational if its chiral algebra $\mathcal{A}$ admits only a finite number of inequivalent irreducible representations. In such a theory, the Hilbert space of states decomposes into a finite direct sum of tensor products of left and right moving representations of the chiral algebras,

$$\mathcal{H} = \bigoplus_{i,\bar{j}=1}^{N} M_{i\bar{j}}\, \mathcal{V}_i^L \otimes \mathcal{V}_{\bar{j}}^R, \tag{3.2.1}$$

where $\mathcal{V}_i^L$ and $\mathcal{V}_{\bar{j}}^R$ denote the irreducible representations of $\mathcal{A}^L$ and $\mathcal{A}^R$, respectively the left and right chiral algebras, and $M_{i\bar{j}} \in \mathbb{Z}_{\geq 0}$ are non-negative integers specifying the multiplicity of each sector. The number N of distinct primary operators is therefore finite, and the operator product expansion closes among this finite set of primaries. The characters of these representations,

$$\chi_i(\tau) = \mathrm{Tr}_{\mathcal{V}_i}\left(q^{L_0 - \frac{c}{24}}\right), \qquad q = e^{2\pi i \tau}, \tag{3.2.2}$$

form a finite-dimensional representation of the modular group $\mathrm{SL}(2, \mathbb{Z})$. Indeed, under a generic modular transformation of the form (3.1.25), the characters transform into each other; in particular, by looking at the transformation rules under the two generators of $SL(2, \mathbb{Z})$ S and T, we can construct two matrices S_{ij} and T_{ij} such that

$$\chi_i(-1/\tau) = \sum_j S_{ij} \chi_j(\tau), \qquad \chi_i(\tau + 1) = \sum_j T_{ij} \chi_j(\tau). \tag{3.2.3}$$

The full torus partition function can then be written as

$$Z(\tau, \bar{\tau}) = \sum_{i,\bar{j}} M_{i\bar{j}}\, \chi_i(\tau)\, \bar{\chi}_{\bar{j}}(\bar{\tau}), \tag{3.2.4}$$

and the modular invariance condition imposes strong algebraic constraints on the allowed combinations of left and right representations; these are given by

$$[M, T] = 0, \quad [M, S] = 0 \tag{3.2.5}$$

and, together with the requirement of the uniqueness of the vacuum $M_{00} = 1$, these conditions constrain the form of the matrix M.

Moreover, in rational conformal field theories, the primary fields of the chiral algebra $\mathcal{A}$ obey discrete fusion rules that describe how the finite set of representations is closed under operator product. These rules take the form

$$\mathcal{V}_i \times \mathcal{V}_j = \sum_k N_{ij}{}^k, \mathcal{V}_k, \qquad N_{ij}{}^k \in \mathbb{Z}_{\geq 0}, \tag{3.2.6}$$

where the non-negative integers $N_{ij}{}^k$ are known as fusion coefficients. They define an associative and commutative algebra, called the fusion algebra, or Verlinde algebra, whose structure is entirely determined by the modular S-matrix, defined in equation (3.2.3), through the Verlinde formula

$$N_{ij}{}^k = \sum_m \frac{S_{im} S_{jm} S^*_{km}}{S_{0m}}. \tag{3.2.7}$$

This algebra encodes how inequivalent representations of the enhanced chiral algebra combine under fusion.

An important subclass of rational conformal field theories is given by the so-called diagonal RCFTs, in which the left and right chiral sectors are paired one-to-one. In this case, the multiplicity matrix takes the form $M_{i\bar{j}} = \delta_{i\bar{j}}$, and the partition function reduces to

$$Z_{\text{diag}}(\tau, \bar{\tau}) = \sum_i |\chi_i(\tau)|^2. \tag{3.2.8}$$

Diagonal RCFTs correspond to theories whose spectrum is built from the tensor product of each chiral representation with its right-moving counterpart, ensuring that all primary operators have integer spin.

RCFTs play a central role in conformal field theory and string theory because they are exactly solvable models. For a complete exposition on rational conformal field theories, we refer to [65, 66].

With the general definition of rational CFT, we are now able to study the specific case of the $c = 1$ compact boson. As we saw in the previous section, for generic values of R, the left and right momenta are given by equation (3.1.8) and they are in one-to-one correspondence with the sites of the infinite lattice (3.1.18). For generic R, the left and right chiral algebras are generated by the primary chiral operators J_L and J_R defined in (3.1.15), which are the only two chiral primary operators of the theory, in general. However, we saw that, at the self-dual radius, the chiral algebra is enhanced to $SU(2)_L \times SU(2)_R$ thanks to the existence of a larger set of chiral operators, which includes $J_L^\pm$ and $J_R^\pm$; this enhanced chiral symmetry is due to the fact that, at $R = 1$, the primary operators obtained with $(n, w) = (1, \pm 1)$ have conformal weights $(h_L, h_R) = (1, 0)$ and $(h_L, h_R) = (0, 1)$. We can now ask ourselves if $R = 1$ is the only point in the moduli space with enhanced chiral algebra, and the answer to this question is no. Indeed, there exists an infinite set of radii $R = \sqrt{\frac{N}{M}}$, for $N, M \in \mathbb{Z}$ coprime, at which the theory presents an enhanced chiral algebra. This is due to the fact that, if R^2 is rational, we can construct a larger set of chiral operators, besides J_L and J_R. To construct them, we have to impose the chirality (anti-chirality) condition, that is $p_R = 0$ ($p_L = 0$). This takes the form

$$p_R = \frac{1}{\sqrt{2}}\left(\frac{n}{R} - wR\right) = 0 \implies n = wR^2 = w\frac{N}{M}, \tag{3.2.9}$$

from which we clearly see the importance of the rationality condition for R^2, since both n and w need to be integers. The smallest non-trivial couple of (n, w) satisfying this condition is given by $w = M$, which imposes $n = N$. The resulting chiral operators are

$$J_L^+ = e^{i\sqrt{2K}X_L} \quad \text{and} \quad J_L^- = e^{-i\sqrt{2K}X_L}, \tag{3.2.10}$$

where $K = NM$, and which have conformal dimension $(h_L, h_R) = (K, 0)$. In the same way, by imposing the anti-chirality condition $p_L = 0$, we obtain the constraint $n = -w\frac{N}{M}$, which leads to two anti-chiral operators $J_R^\pm$ with conformal dimension $(h_L, h_R) = (0, K)$. In particular, including this new set of chiral operators enhances the chiral algebra to $U(1)_{2K}$, generated by $(J_L, J_L^\pm)$ (the same thing is valid for the anti-chiral one, the right-moving one); this is an affine $U(1)$ Kac-Moody algebra at level $2K$, and it is an extension of the chiral algebra $U(1)$ (we will not go into the details of this type of algebras; for a complete exposition of the subject, see [38, 63]).

We can now show explicitly that, when the compactification radius satisfies the rationality condition above, the number of primary operators of the theory becomes finite, corresponding to a rational CFT. As we have seen, at these points the chiral algebra is enhanced to $U(1)_{2K}$, with $K = NM$, due to the existence of additional chiral fields $J_L^\pm = e^{\pm i\sqrt{2K}X_L}$ (and their right-moving counterparts). The action of these chiral operators on the Hilbert space results in a shifting of the momentum and winding numbers as $(n, w) \to (n + N, w + M)$. This implies that states related by this transformation are identified within the same representation of the extended chiral algebra, defining a finite set of equivalence classes under the identification $p_L \sim p_L + \sqrt{2K}$. In particular, it is possible to define an invariant combination of n and w under this identification,

$$m \equiv nM + wN \pmod{2K}, \tag{3.2.11}$$

which labels distinct representations of the extended algebra. The allowed values of m then take value in $\mathbb{Z}/(2K\mathbb{Z})$, resulting in precisely $2K$ inequivalent chiral sectors. The same analysis can be applied to the right moving sector, yielding another finite set of $2K$ anti-chiral representations. In this way, the full Hilbert space decomposes into a finite sum of tensor products of left and right modules,

$$\mathcal{H} = \bigoplus_{m,\bar{m}=0}^{2K-1} M_{m\bar{m}}\, \mathcal{V}_m^L \otimes \mathcal{V}_{\bar{m}}^R, \tag{3.2.12}$$

implying that the RCFTs in the $c = 1$ compact boson model correspond to those theories that satisfy $R^2 \in \mathbb{Q}$. Moreover, the fusion ring of the $c = 1$ compact boson RCFT is given by $\mathbb{Z}_{2K}$ in general. This means that theories with the same chiral and fusion algebras, but different modular invariants, correspond to different ways in which we can decompose K into the product of two coprime integer numbers, leading to different compactification radii. Finally, the theory is diagonal, $M_{m\bar{m}} = \delta_{m\bar{m}}$, if the radius satisfies $R^2 \in \mathbb{N}$ (or, equivalently, $R^2 = 1/N,\ N \in \mathbb{N}$, because of T-duality).

3.3 Non-invertible symmetries at RCFT points

In the previous section, we introduced the notion of rational conformal field theory and discussed its realization in the $c = 1$ compact boson. In particular, we showed that when the compactification radius satisfies $R^2 \in \mathbb{Q}$, the theory exhibits an enhanced chiral algebra of the form $U(1)_{2K}$, with $K = NM$ and $R = \sqrt{N/M}$, which leads to a finite number of primary operators. This precisely corresponds to the condition that a CFT must satisfy in order to be rational. In this section, instead, we will show how non-invertible symmetries emerge at these rational points of the moduli space. We will begin by explaining why rationality is a necessary condition for the existence of such generalized symmetries, and then describe how topological defects arise and what type of fusion category they generate.

3.3.1 Gauging and self-duality

As we saw in the previous chapter, one of the simplest ways to construct non-invertible symmetries is by gauging an (invertible) symmetry. This is the technique that we are going to apply in the following, together with the concept of self-duality. So, let us start by presenting the procedure of gauging a global symmetry in the case of the $c = 1$ compact boson. We will consider the gauging of a finite subgroup $\mathbb{Z}_N \times \mathbb{Z}_M$ of $U(1)_n \times U(1)_w$[5], that we will denote by $\sigma_{N,M}$. The action of this subgroup on the field X and the dual one $\tilde{X}$ is:

$$\sigma_{N,M} \ni (a,b) : \left(X, \tilde{X}\right) \mapsto \left(X + \frac{2\pi R}{N} a, \tilde{X} + \frac{2\pi R}{M} b\right). \tag{3.3.1}$$

What we want to do now is to gauge this subgroup of the global symmetry, and, in order to do so, let us review the procedure of gauging a global symmetry of a CFT (this is the same procedure that is used to construct the $\mathbb{Z}_2$ orbifold that we introduced above around equation (3.1.32); we refer to [63] for a detailed discussion). The starting point is the standard partition function of a CFT defined in (3.1.21)

$$Z(\tau, \bar{\tau}) = \mathrm{Tr}_{\mathcal{H}}\left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}\right), \qquad q = e^{2\pi i \tau}. \tag{3.3.2}$$

[5] We consider only diagonal subgroups of the global symmetry, that is to say with $\mathbb{Z}_N \subset U(1)_n$ and $\mathbb{Z}_M \subset U(1)_w$. Moreover, we require that gcd$(N, M) = 1$ in order for the symmetry to be anomaly free.

Now, let us consider a global symmetry H of the theory; then, the gauged partition function will take the form:

$$Z_{gauged}(\tau,\bar{\tau}) = \sum_{g,h\in H\ s.t.\ [g,h]=0} \mathrm{Tr}_{\mathcal{H}_h}\left(g q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}\right), \qquad q = e^{2\pi i\tau}, \tag{3.3.3}$$

where we sum over the elements of the symmetry group H. In particular, $\mathcal{H}_h$ refers to the invariant subspace of the Hilbert space under the action of h and, for non-trivial g, $\mathrm{Tr}_{\mathcal{H}_h}\left(g q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}\right)$ are called g-twisted sectors. So, the gauging procedure consists in identifying the subspace of the Hilbert space invariant under the action of $h \in H$, which, since the primary operators are in one-to-one correspondence with the sites of the lattice $\Gamma^{1,1}$, is equivalent to determining the invariant sublattice; the following step is then to introduce the so-called g-twisted states, with $g \in H$ such that $[g,h] = 0$, in order to guarantee modular invariance of the gauged theory.

We can now apply this construction to the case of the subgroup (3.3.1). Let us start by considering the gauging of the subgroup $\mathbb{Z}_N \subset U(1)_n$; to do so, we need to identify the invariant subspace of the Hilbert space under the action of the symmetry generator, which is given by

$$V_{n,w} = e^{i\frac{n}{R}X}e^{iwR\tilde{X}} \mapsto e^{i\frac{n}{R}\left(X+\frac{2\pi R}{N}\right)}e^{iwR\tilde{X}}. \tag{3.3.4}$$

It is clear that the invariant operators will be those with a momentum number of the form $n = n'N$, with $n' \in \mathbb{Z}$. Indeed, for this subset of operators, the action of $\mathbb{Z}_N$ will be the introduction of a phase $e^{2\pi i}$, which leaves the operator unchanged. Following the procedure described above, we have now to construct the twisted states; the states in this sector are characterized by an additional phase $e^{\frac{2\pi i}{N}}$ coming from the periodic shift $\tilde{X} \mapsto \tilde{X} + \frac{2\pi}{R}$, implying that the winding number must be fractional and of the form $w = \frac{w'}{N}$, with $w' \in \mathbb{Z}$.

Let us now analyze more carefully the structure of the twisted sector introduced by the gauging procedure. When we gauge a finite subgroup $\mathbb{Z}_N \subset U(1)_n$, the original Hilbert space of the theory $\mathcal{H}$ is decomposed into sectors labeled by the boundary conditions of the field X. The untwisted sector corresponds to the set of configurations that are invariant under the action of the gauged symmetry. However, modular invariance of the full partition function requires the inclusion of additional sectors, the twisted sectors, whose field configurations are periodic only up to the action of a group element $g \in \mathbb{Z}_N$. In the present case, this corresponds to imposing the boundary condition

$$X(\sigma + 2\pi, \tau) = X(\sigma,\tau) + \frac{2\pi R}{N}k, \qquad k = 0,1,\ldots,N-1, \tag{3.3.5}$$

where k labels the element of the group $\mathbb{Z}_N$. The case $k = 0$ corresponds to the untwisted sector, while the $N-1$ remaining values of k define the distinct twisted sectors of the theory.

Each twisted sector can be interpreted as describing strings that close only up to a discrete shift by $\frac{2\pi R}{N}k$ in the target space. Equivalently, these configurations carry fractional winding numbers, of the form

$$w = \frac{k}{N}, \qquad k = 1, \dots, N-1, \tag{3.3.6}$$

as we said above. As a consequence, the twisted Hilbert spaces $\mathcal{H}_k$ are generated by states whose left- and right-moving momenta are given by

$$(p_L, p_R) = \frac{1}{\sqrt{2}}\left(\frac{nN}{R} + \left(w + \frac{k}{N}\right)R,\, \frac{nN}{R} - \left(w + \frac{k}{N}\right)R\right), \qquad n, w \in \mathbb{Z}. \tag{3.3.7}$$

These fractional shifts modify the structure of the momentum lattice $\Gamma^{1,1}$, producing N distinct sectors (one untwisted and $N-1$ twisted) that together form a modular-invariant completion of the gauged theory.

Overall, the action of the gauging can be written in terms of left and right moving momentum as:

$$(p_L, p_R) = \frac{1}{\sqrt{2}}\left(\frac{n'N}{R} + \frac{w'R}{N}, n'NR - \frac{w'R}{N}\right), \quad n', w' \in \mathbb{Z}. \tag{3.3.8}$$

From this expression, we see that these states can be identified with the sites of the momentum lattice of a compact boson with target space a circle of radius $\frac{R}{N}$. In the same way, it is also possible to apply the same analysis to the subgroup $\mathbb{Z}_M \subset U(1)_w$, leading to a theory compactified on a circle of radius RM. This clearly implies that, after gauging the diagonal subgroup $\mathbb{Z}_N \times \mathbb{Z}_M \subset U(1)_n \times U(1)_w$, the untwisted and twisted sectors rearrange in such a way that they correspond to a theory of a compact boson compactified on a circle of radius $\frac{M}{N}R$ (we refer to [26, 67] for a more detailed discussion).

Now, we just saw that the gauging of a finite subgroup $\mathbb{Z}_N \times \mathbb{Z}_M \subset U(1)_n \times U(1)_w$ has the overall effect of rescaling the radius as $\frac{M}{N}R$; this result, together with the notion of rational CFT, is central for the construction of non-invertible symmetries. Indeed, if we consider the RCFT with radius $R = \sqrt{\frac{N}{M}}$, by gauging the finite subgroup $\mathbb{Z}_N \times \mathbb{Z}_M \subset U(1)_n \times U(1)_w$, we end up with a $c = 1$ compact boson compactified on a circle of radius $R' = \frac{M}{N}\sqrt{\frac{N}{M}} = \sqrt{\frac{M}{N}}$; the important thing to notice here is that the resulting radius is simply the inverse of the original one, that is $R' = \sqrt{\frac{M}{N}} = R^{-1}$. This is central because, as we saw in the previous section, the $c = 1$ compact boson enjoys another discrete symmetry, T-duality. This symmetry states that the theory at R and the one at $\frac{1}{R}$ are equivalent, up to a redefinition of the momentum and winding numbers (indeed, this identification implies the exchange of the field X with its dual $\tilde{X}$). This implies that, if we combine the gauging $\sigma_{N,M}$ of the $\mathbb{Z}_N \times \mathbb{Z}_M$ subgroup of the chiral symmetry with the action of T-duality, that we denote with T, we get that the overall action of $T \circ \sigma_{N,M}$ leaves the theory invariant; the combination of gauging and T-duality becomes then a symmetry of the theory and is called a duality symmetry.

3.3.2 Half-space gauging and Tambara-Yamagami categories

In the previous section we just saw that, given a $c = 1$ compact boson RCFT, by combining the gauging of a finite subgroup of the global chiral symmetry with T-duality, it is possible to construct a duality symmetry of the theory. We want now to show how it is possible to define a non-invertible symmetry starting from this property of the theory.

In order to obtain the non-invertible topological defect, we implement the half-space gauging method. To understand this construction, consider a CFT $\mathcal{T}$ on a 2d spacetime with a global symmetry G. Let us now divide the spacetime in 2 halves and gauge the symmetry on only one side of the co-dimension one submanifold (which, since we are considering a 2d spacetime, is a line). If the gauged theory $\mathcal{T}/\mathcal{G}$ is isomorphic to the original one, $\mathcal{T}/G \cong \mathcal{T}$, then we can impose Dirichlet boundary conditions for the gauge field associated to G, and the interface will define a topological defect $\mathcal{D}$ in the theory (see Fig. 9 for a visual representation; for a complete exposition of half-space gauging in arbitrary dimensions, we refer to [37]).

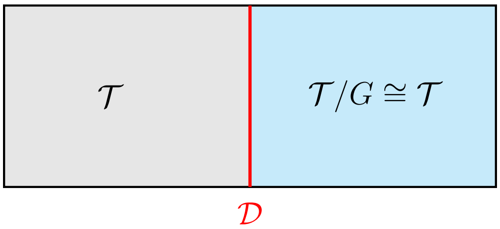


Figure 9: Visual representation of half-space gauging. The spacetime is divided in two halves by a co-dimension 1 submanifold. After gauging the global symmetry on one side, the submanifold becomes the support of a topological defect operator that we denote by $\mathcal{D}$.

Now, from the half-space gauging prescription, we get a new topological defect line $\mathcal{D}$. However, we need to figure out the fusion rules that govern the resulting fusion category. It turns out that, for any 2d theory self-dual under the gauging of a non-anomalous abelian group A, the half-space gauging procedure gives rise to a Tambara-Yamagami fusion category TY(A, χ, ϵ) (we will denote this type of fusion category with TY). A TY category can be considered as a minimal extension of the fusion category of the abelian group A, Vec$_A$, by the inclusion of an additional non-invertible object $\mathcal{D}$ that encodes the duality symmetry operation. The simple objects of the category are thus the TDLs g corresponding to the simple objects of the fusion category Vec$_A$ (which are in one-to-one correspondence with the elements of the abelian group A), together with a non-invertible line $\mathcal{D}$. The fusion rules are given by

$$g \times h = gh, \qquad g \times \mathcal{D} = \mathcal{D} \times g = \mathcal{D}, \qquad \mathcal{D} \times \mathcal{D} = \bigoplus_{g \in A} g. \tag{3.3.9}$$

From the fusion rules of the duality defect $\mathcal{D}$, it is easy to compute its quantum dimension $d_{\mathcal{D}} = \sqrt{|A|}$, which is given in terms of the order of the group, implying the non-invertibility of the defect. The only non-trivial F-symbols of the theory are

$$\left[F_{\mathcal{D}}^{g\mathcal{D}h}\right]_{\mathcal{D},\mathcal{D}} = \left[F_{h}^{\mathcal{D}g\mathcal{D}}\right]_{\mathcal{D},\mathcal{D}} = \chi(g,h), \qquad \left[F_{\mathcal{D}}^{\mathcal{D}\mathcal{D}\mathcal{D}}\right]_{g,h} = \frac{\epsilon}{\sqrt{|A|}}\chi(g,h)^{-1}, \tag{3.3.10}$$

which are defined in terms of:

- $\epsilon = \pm 1$, the Frobenius-Schur indicator for $\mathcal{D}$, which is a class $\epsilon \in H^3(\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$. The Frobenius–Schur indicator is a $\mathbb{Z}_2$-valued invariant that characterizes whether $\mathcal{D}$ behaves as a real or pseudo-real object under fusion. The Frobenius-Schur indicator will not play a central role in our discussion and for this reason we will not go into details. For a detailed introduction to this object, we refer to [68];
- a symmetric, non-degenerate bilinear character $\chi : A \times A \to U(1)$, that satisfies:

$$\chi(g,h) = \chi(h,g), \;\; \chi(gh,k) = \chi(g,k)\chi(h,k), \;\; \chi(g,hk) = \chi(g,h)\chi(g,k). \tag{3.3.11}$$

To summarize, from the set of simple objects, the form of the fusion rules and the F-symbols, we clearly see that a Tambara-Yamagami category is fully determined in terms of the gauged abelian group A, the bicharacter χ and the Frobenius-Schur indicator ϵ, justifying the usual notation TY(A, χ, ϵ). For a detailed discussion of Tambara-Yamagami categories, we refer to the original paper [42]; for a more recent introduction to the topic, we refer instead to [55, 69, 70].

3.3.3 The non-invertible line $\mathcal{D}$ in the $c = 1$ compact boson

Now that we know how to obtain the non-invertible topological defect line implementing a duality symmetry, we can apply this construction to the rational $c = 1$ compact boson. Let us consider an RCFT point in the circle branch at radius $R = \sqrt{\frac{N}{M}}$. As we saw above, we can gauge the finite diagonal subgroup $\mathbb{Z}_N \times \mathbb{Z}_M \subset U(1)_n \times U(1)_w$, and the resulting theory will be isomorphic to the original one up to T-duality. We denote by η the TDL generating the $\mathbb{Z}_N \subset U(1)_n$ subgroup, such that $\eta^N = \text{Id}$ (and the simple objects of Vec$_{\mathbb{Z}_N}$ are defined as η^k for $k \in \{0, 1, \ldots, N-1\}$), and by $\tilde{\eta}$ the line generating the fusion category associated with $\mathbb{Z}_M \subset U(1)_M$, such that $\tilde{\eta}^M = \text{Id}$. If we apply the half-space gauging procedure to this theory, we obtain a new topological defect line, that we denote by $\mathcal{D}$, implementing the duality symmetry. The fusion rules of the resulting Tambara-Yamagami category (following equation (3.3.9)) take the form

$$\mathcal{D} \otimes \eta = \eta \otimes \mathcal{D} = \mathcal{D} \otimes \tilde{\eta} = \tilde{\eta} \otimes \mathcal{D} = \mathcal{D}, \quad \mathcal{D} \otimes \mathcal{D} = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \eta^n \tilde{\eta}^m. \tag{3.3.12}$$

At this point, we are able to study the action of the non-invertible topological defect on the primary operators of the theory. From equation (3.3.8), we know that the overall action of gauging a subgroup of the chiral symmetry is rescaling the compactification radius as $R \mapsto \frac{M}{N} R$, which implies a redefinition of the momentum and winding numbers as $(n, w) \mapsto \left(\frac{M}{N} n, \frac{N}{M} w\right)$. In addition to this, the duality symmetry requires also the implementation of T-duality, which maps the compactification radius R to its inverse R^{-1} and which accounts instead for an exchange of the momentum and winding numbers, as we showed in (3.1.28). The action of $\mathcal{D}$ on a primary operator $V_{n,w}$ is then the combination of these two operations, with an overall map of the charges given by

$$(n, w) \mapsto \left(\frac{N}{M} w, \frac{M}{N} n \right). \tag{3.3.13}$$

However, we need to distinguish two cases:

- $n = \tilde{n} N$ and $w = \tilde{w} M$, with $\tilde{n}, \tilde{w} \in \mathbb{N}$: these are the primary operators characterized by momentum and winding numbers which are integer multiples of N and M, respectively. This subset of primary operators, after the gauging, will be mapped to a new set of genuine operators with integer momentum and winding numbers
$$V_{n,w} \mapsto V_{n',w'} = V_{\tilde{w} N, \tilde{n} M}. \tag{3.3.14}$$
- $n = \tilde{n} N + k$ and $w = \tilde{w} M + k'$, for $k \in \{0, \ldots, N-1\}$, $k' \in \{0, \ldots, M-1\}$ and $(k, k') \neq (0, 0)$: these operators are characterized by the fact that (at least one between) the momentum and winding numbers are not integer multiples of N and M, respectively. This implies that, when we drag the topological defect $\mathcal{D}$ trough the operator insertion, we end up with an operator characterized by fractional values of momentum and/or winding number; that is to say, we end up with a non-genuine operator. This happens because this set of operators is not gauge invariant. If we consider, for example, the operator $V_{k,0} = e^{ikX/R}$ for $k \in \{1, \ldots, N-1\}$, under the gauging of $\mathbb{Z}_N \subset U(1)_n$ we get the non-genuine operator $V_{k/N,0}$; this is non-genuine because, under the $2\pi R$ translation of the X field, the transformed operator is no more single valued and acquires a phase $e^{2i\pi k/N}$. Indeed, the primary operators with these values of (n, w) are elements of the twisted sectors. This means that, as we saw in Section 2.4.4,

after passing trough the defect line $\mathcal{D}$, these operators must be placed at the end point of a line operator, which is itself attached to $\mathcal{D}$ (see Fig. 10 for a visual representation). In particular, in order to guarantee gauge invariance, the line operator must be a combination of the gauged symmetry generators η and $\tilde{\eta}$; the precise combination is $\eta^u\tilde{\eta}^v$, where $u = n$ mod N and $v = w$ mod M, and, since $\eta^N = \text{Id}$ and $\tilde{\eta}^M = \text{Id}$, this is equivalent to $\eta^n\tilde{\eta}^w$ (we refer to [48] for the proof). From the form of the line operators that must be introduce in order to guarantee gauge invariance, we also understand why the operators in the untwisted sector end up to be genuine operators after the action of $\mathcal{D}$; indeed, if we apply the same analysis in that case, the required line operators would be just some power of the identity operator, which can be omitted (since both n and w are integer multiples of N and M, respectively).

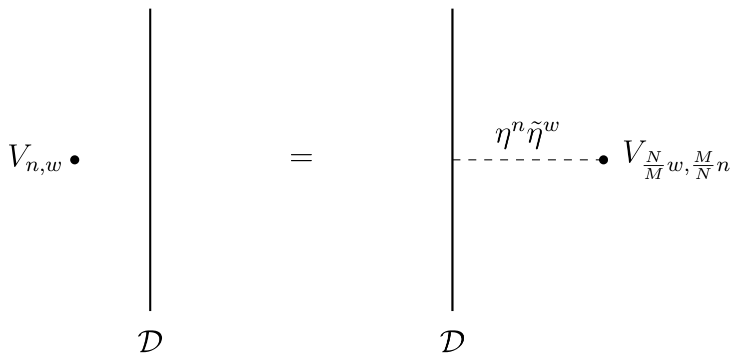


Figure 10: Action of the non-invertible line $\mathcal{D}$ on the vertex operator $V_{m,w}$. If the operator $V_{n,w}$ is not in the untwisted sector, then $\eta^n \neq \text{Id}$ and/or $\tilde{\eta}^w \neq \text{Id}$ and the introduction of the line operator $\eta^n\tilde{\eta}^w$ is necessary for gauge invariance.

Finally, the action of the duality symmetry on the primary operators can also be interpreted in terms of the momentum lattice $\Gamma^{1,1}$. Indeed, as we saw in equation (3.1.18), the primary operators in the $c = 1$ compact boson theory are in one-to-one correspondence with the sites of the integer, self-dual momentum lattice. The action on the momentum and winding numbers can be written in matrix form as:

$$\begin{pmatrix} n \\ w \end{pmatrix} \mapsto \begin{pmatrix} n' \\ w' \end{pmatrix} = \begin{pmatrix} 0 & \frac{N}{M} \\ \frac{M}{N} & 0 \end{pmatrix} \begin{pmatrix} n \\ w \end{pmatrix} \equiv D \begin{pmatrix} n \\ w \end{pmatrix}. \tag{3.3.15}$$

Now, from the definition of the momentum lattice $\Gamma^{1,1}$ in equation (3.1.18), the duality symmetry can then be seen acting on the lattice generating matrix L, instead of acting on the quantum numbers. This results in the overall matrix transformation (for the

$c = 1$ compact boson with radius $R = \sqrt{\frac{N}{M}}$ for which $T \circ \sigma_{N,M}$ is a duality symmetry)

$$L = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{M}{N}} & \sqrt{\frac{N}{M}} \\ \sqrt{\frac{M}{N}} & -\sqrt{\frac{N}{M}} \end{pmatrix} \mapsto L' = LD = L \begin{pmatrix} 0 & \frac{N}{M} \\ \frac{M}{N} & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{M}{N}} & \sqrt{\frac{N}{M}} \\ -\sqrt{\frac{M}{N}} & \sqrt{\frac{N}{M}} \end{pmatrix}, \tag{3.3.16}$$

which can be clearly rewritten as

$$L' = RL = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} L. \tag{3.3.17}$$

This way of rewriting the action of the duality symmetry on the lattice generator in terms of a new matrix R will be important in the next section and, mostly, in the $c = 2$ compact boson, as we will see in the next chapter. For the moment, there are two important things to notice. First of all, the matrix D belongs to $O(1, 1, \mathbb{Q})$, which guarantees that the spin s of the primary operators, defined as $s = h_L - h_R = L^T \eta L$ with $\eta = \text{diag}(1, -1)$, is preserved under the action of the duality symmetry; this is equivalent to the fact that R is in $O(1, 1, \mathbb{Q})$ too (notice that the two matrices are both in $O(1, 1, \mathbb{Q})$, but in two different representations; indeed, D preserves the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, while R leaves invariant the diagonal form diag$(1, -1)$). Secondly, the fact that $R^T R = \text{Id}$ guarantees instead that the conformal dimension of the primary operators, defined as $\Delta = h_L + h_R = L^T L$, is preserved too.

3.3.4 An example of duality symmetry in the $c = 1$ compact boson

At this point, we want to give an example of duality symmetry for the $c = 1$ compact boson. As we saw, in order to be able to construct such a non-invertible symmetry, we need the theory to be a rational CFT, which is equivalent to the requirement that $R^2 \in \mathbb{Q}$. For this reason, let's consider the theory of a boson compactified on a circle of radius $R = \sqrt{3}$; the action then takes the form:

$$S = \frac{3}{4\pi} \int d\sigma d\tau \, \partial_\mu \phi \partial^\mu \phi. \tag{3.3.18}$$

The left and right-moving momentum, instead, are given by

$$p_L = \frac{1}{\sqrt{2}} \left(\frac{n}{\sqrt{3}} + w\sqrt{3} \right), \qquad p_R = \frac{1}{\sqrt{2}} \left(\frac{n}{\sqrt{3}} - w\sqrt{3} \right), \tag{3.3.19}$$

which can be rewritten in terms of the self-dual integer momentum lattice $\Gamma^{1,1}$ generated by the matrix

$$L = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{3}} & \sqrt{3} \\ \frac{1}{\sqrt{3}} & -\sqrt{3} \end{pmatrix}. \tag{3.3.20}$$

From the rationality property, we know that the chiral algebra gets enhanced to $U(1)_{6,L} \times U(1)_{6,R}$ thanks to the existence of the additional chiral operators

$$J_L^{\pm} = e^{\pm i\sqrt{2K}X_L} \quad \text{and} \quad J_R^{\pm} = e^{\pm i\sqrt{2K}X_R}, \tag{3.3.21}$$

which gives rise to a finite number of primary operators associated with the irreducible representations of the chiral symmetry. The irreducible representations are then labeled by $m \in \mathbb{Z}/6\mathbb{Z} = \{0, \dots, 5\}$, corresponding to the left-moving momentum

$$p_L = \frac{m}{\sqrt{6}} \tag{3.3.22}$$

and similarly for the right-moving one.

Now that we have constructed an example of RCFT, we are able to apply the procedure described above in order to construct the non-invertible symmetry. From the general discussion, it is easy to notice that the gauging of the diagonal subgroup $\mathbb{Z}_3 \subset U(1)_n$ maps the theory to $R' = \frac{1}{\sqrt{3}}$, which is T-dual to the original one. In particular, the gauging procedure gives rise to one untwisted sector, which corresponds to those states with winding number $w = 3w'$ for $w' \in \mathbb{Z}$, and 2 twisted sectors, characterized by fractional values of the winding number coming from those states with $w = 3w' + k$ for $w' \in \mathbb{Z}$ and $k = \{1, 2\}$. From the fact that the resulting theory can be considered as the compactification on a circle of radius $R' = \frac{1}{\sqrt{3}}$, denoting the gauging of $\mathbb{Z}_3$ by $\sigma_{3,1}$, the combination of this with T-duality $T \circ \sigma_{3,1}$ defines a duality symmetry of the theory denoted by $\mathcal{D}$. This is described by the fusion category with set of simple objects $\{\eta, \mathcal{D}\}$ and fusion rules

$$\eta^3 = \text{Id}, \quad \mathcal{D} \otimes \eta = \eta \otimes \mathcal{D} = \mathcal{D}, \quad \mathcal{D} \otimes \mathcal{D} = \sum_{n=0}^{2} \eta^n, \tag{3.3.23}$$

where η is the generator of the $\mathbb{Z}_3$ symmetry. The duality symmetry can be represented by the $O(1,1,\mathbb{Q})$ matrix

$$D = \begin{pmatrix} 0 & 3 \\ \frac{1}{3} & 0 \end{pmatrix}, \tag{3.3.24}$$

which acts on $\Gamma^{1,1}$ as

$$L \mapsto L' = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{3}} & \sqrt{3} \\ -\frac{1}{\sqrt{3}} & \sqrt{3} \end{pmatrix}. \tag{3.3.25}$$

In Fig. 11 we give a visual representation of the two lattices L and L'. It is interesting to notice that from the plot of the two lattices we can infer the non-invertibility of the duality symmetry. Indeed, by looking at Fig. 11, we can distinguish two cases. The first one corresponds to those states that, after the action of $\mathcal{D}$, are mapped to genuine operators. These are those sites of the lattice L' that overlap with the sites of the original

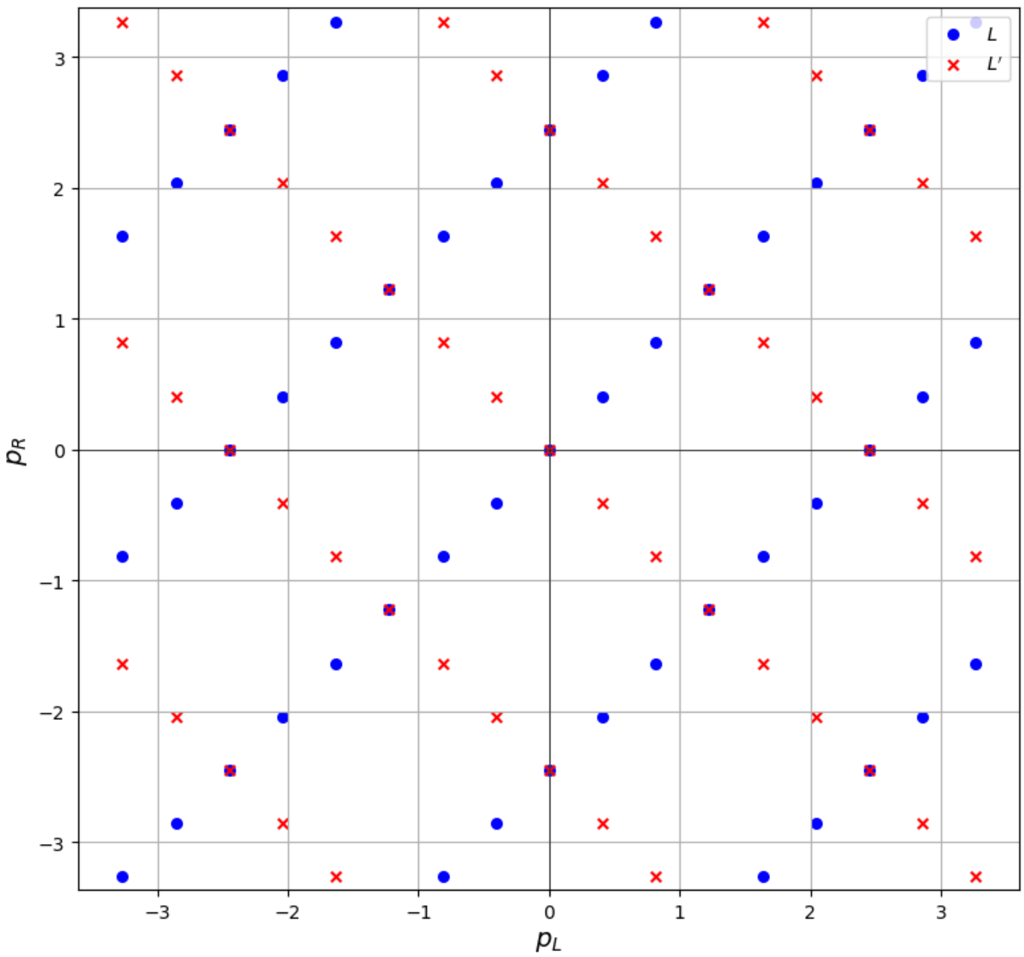


Figure 11: Plot of the self-dual integer momentum lattice $\Gamma^{1,1}$ before and after the action of the duality symmetry $\mathcal{D}$. With blue dots we represented the original lattice spanned by L, with the red crosses the transformed one $L' = LD$.

lattice L, implying that the transformed operator can be identified with an operator of the original theory. As a result, the new operator is gauge invariant and belongs to the untwisted sector. The second case, instead, are those lattice points of L' that do not overlap with any site of lattice L. These correspond instead to the non-genuine operators arising from the non-invertibility of the duality symmetry $\mathcal{D}$. Indeed, these sites come from the twisted sectors of the theory, which must be included in the spectrum of the gauged theory in order to guarantee modular invariance, but that are not gauge invariant and for this reason must be placed at the end of a line operator, implying that they are non-genuine.

3.4 Dp-branes and non-invertible symmetries

Before moving to the main section of the thesis, namely the emergence of duality symmetries in the $c = 2$ compact boson, we first want to study how such symmetries act on the boundary states of the theory. In rational conformal field theories, boundary states are special conformally invariant states that encode the possible consistent boundary conditions preserving the bulk symmetry. These states are in one-to-one correspondence with Dp-branes, which in string theory are extended objects on which open strings can end. In the worldsheet formulation, Dp-branes are realized through specific boundary conditions on the embedding fields of the string: Dirichlet boundary conditions fix the string endpoints along directions transverse to the brane, while Neumann boundary conditions allow free motion along directions parallel to it.

Boundary states in RCFTs therefore provide the conformal field theory realization of Dp-branes. In this section, we will introduce how these states are defined in the RCFT framework, beginning with the construction of Ishibashi and Cardy states. This will allow us to understand how duality defects act on boundary conditions and how they transform Dp-brane configurations within the theory.

3.4.1 Dp-branes in RCFTs

In rational CFTs, the boundary states, which are the field theory realization of the Dp-branes, can be constructed in terms of the so-called Ishibashi states (we refer to [71] for the original work); we will denote these states by $|B\rangle\rangle$. These are states that must satisfy specific boundary conditions imposed by the preservation of the CFT's symmetry. First of all, in order to preserve conformal invariance, we need to guarantee that no energy-momentum flows through the boundary, which is equivalent to the requirement that $T(\xi^+) = \bar{T}(\xi^-)$, where T and $\bar{T}$ are the holomorphic and anti-holomorphic components of the stress energy tensor, while ξ^+ and ξ^- take values on the boundary. Now, by considering the mode expansion of the energy-momentum tensor in terms of the Virasoro generators $L_n,\ \bar{L}_n$, we get the condition

$$\left(L_n - \bar{L}_{-n}\right)|B\rangle\rangle = 0. \tag{3.4.1}$$

However, in general, a consistent boundary condition must preserve not only the Virasoro algebra, but also a larger chiral algebra $\mathcal{A}$ containing additional left and right-moving currents $W^L(\xi^+)$ and $W^R(\xi^-)$ with modes W^L_n and W^R_n. The preservation of this extended symmetry requires identifying the left and right chiral sectors along the boundary through a linear map Ω acting as an automorphism of the chiral algebra:

$$\left(W^L_n - (-1)^{h_W}\,\Omega(W^R_{-n})\right)|B\rangle\rangle = 0. \tag{3.4.2}$$

This is the so-called gluing condition, which generalizes (3.4.1). Here h_W denotes the conformal weight of the chiral field $W^L(\xi^+)$, and the sign factor $(-1)^{h_W}$ arises from the reflection of the worldsheet coordinate $\xi^- \mapsto \xi^+$ at the boundary (see [72] for a exhaustive review on Dp-branes in rational CFTs). The automorphism Ω is what specifies the type of boundary condition. In particular, we can consider the mode expansion of this condition; to do so, let's consider W to be the spin 1 current generating the chiral algebra, that is to say J_L (and, in the same way, J_R for W_R). By considering the explicit expression of the algebra generators in terms of left and right moving fields given in (3.1.15), equation (3.4.2) can be written as:

$$\left(\alpha_n^\mu + R^\mu_\nu \tilde{\alpha}^\nu_{-n}\right) |B\rangle\rangle = 0, \tag{3.4.3}$$

where $\mu, \nu = 1, \ldots, d$ are the spacetime indices and $R \in O(d)$ is an automorphism of the chiral algebra (here we are considering the general expression of the Ishibashi states condition valid for an arbitrary target space; in the $c = 1$ compact boson case that we are considering, in which the target space is a circle, we will drop the spacetime indices). In particular, focusing on the zero mode gluing condition, we get

$$\left(p_L^\mu + R^\mu_\nu p_R^\nu\right) |B\rangle\rangle = 0, \tag{3.4.4}$$

which is a relation between the left and the right-moving momentum of the state. The matrix R, then, encodes the geometric reflection properties of the boundary, which are given by the eigenvalues of the matrix. In particular:

- Eigenvalues $+1$ correspond to Neumann directions, for which the derivative normal to the boundary vanishes, $\partial_\sigma X^\mu = 0$. These are directions along which the endpoints of the open string are free to move, that is to say, parallel to the Dp-brane.

- Eigenvalues -1 correspond to Dirichlet directions, for which $\partial_\tau X^\mu = 0$, i.e. the coordinate X^μ is fixed at the boundary. These are the directions transverse to the Dp-brane, where the endpoints of the string are fixed.

Hence, R specifies the orientation and dimensionality of the branes. In particular, given a gluing condition Ω represented by a matrix R with p eigenvalues $+1$ and $d-p$ eigenvalues -1, this will correspond to a Dp-brane in target space.

Now, in a rational conformal field theory, the chiral algebra $\mathcal{A}$ admits only a finite number of inequivalent irreducible representations $\{\mathcal{V}_i\}$. For each representation i, one can construct an Ishibashi state $|i\rangle\rangle$ that satisfies (3.4.2) within the sector $\mathcal{V}_i^L \otimes \mathcal{V}^R_{\Omega(i)}$. These Ishibashi states form a complete basis of solutions to the gluing equations. However, since we are considering a rational CFT, the chiral algebra is enhanced; in order for the Dp-branes to preserve the full chiral algebra, it is not enough to consider the Ishibashi states satisfying the gluing conditions (3.4.3). Instead, the physical boundary states, called Cardy states, will correspond to linear combinations of the Ishibashi ones of the form:

$$|B_a\rangle = \sum_j \beta_{aj}\, |j\rangle\rangle, \tag{3.4.5}$$

where β_{ai} are complex coefficients that must satisfy some precise consistency conditions, called Cardy consistency conditions (such as modular invariance on the worldsheet; we refer to [72] for a complete exposition). In particular, the combinations of the Ishibashi states with these specific coefficients give rise to the physical boundary states of the theory. Now, if we restrict ourselves to the case of the diagonal modular invariant theories, it was shown by Cardy in [73] that the coefficients can be rewritten in terms of the entries of the S-matrix defined in (3.2.3) as:

$$|i\rangle = \sum_j \frac{S_{ij}}{\sqrt{S_{0j}}} |j\rangle\rangle. \tag{3.4.6}$$

It is important to notice that, in the case of diagonal RCFTs, the number of Ishibashi states is precisely the same as the number of the Cardy ones, that is to say, the number of physical boundary states corresponding to the Dp-branes.

3.4.2 Dp-branes in the $c = 1$ compact boson

Now that we have introduced the notion of Dp-brane for RCFTs, we are able to analyze them in the $c = 1$ compact boson case. The first thing to notice is that, since the target space is a circle S^1, we can drop the space-time indexes from equations (3.4.3) and (3.4.4), and the latter will take the form:

$$(p_L + Rp_R)|B\rangle\rangle = 0, \quad R = \pm 1. \tag{3.4.7}$$

In particular, in the 1-dimensional spacetime case, we clearly see that we will only have two types of boundary conditions, due to the fact that $R \in O(d) \implies R = \pm 1$, and they correspond to either Neumann or Dirichlet boundary conditions, respectively. We can now look at the specific solutions of the gluing conditions, and we will get the two sets of Ishibashi states given by:

- $R = -1$: for the gluing condition represented by the matrix R with eigenvalue -1 we get the condition on the momentum

$$p_L = p_R \implies \frac{1}{\sqrt{2}}\left(\frac{n}{R} + wR\right) = \frac{1}{\sqrt{2}}\left(\frac{n}{R} - wR\right) \implies wR = 0. \tag{3.4.8}$$

  As we said, these Ishibashi states will be related to the Cardy states representing D0-branes. In particular, we see that the gluing condition gives $p_L = p_R = \frac{n}{R}$ and, since we are considering an RCFT with $R = \sqrt{\frac{N}{M}}$, these correspond to $2K = 2NM$ different Ishibashi states, one for each distinct representation of the chiral algebra labeled by $n \in \{0, \ldots, 2K-1\}$. By solving the gluing condition for every level in the mode expansion (3.4.3), we get that the explicit expression for these Ishibashi states is:

$$|n\rangle\rangle = \exp\left[+\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}\tilde{\alpha}_{-n}\right]\times\sum_{m}\left|\frac{r}{R}+mR,\frac{r}{R}+mR\right\rangle, \qquad (3.4.9)$$

for $r \in \{0,\dots,2K-1\}$.

- $R = 1$: for the gluing condition represented by the matrix R with eigenvalue $+1$ we get instead the condition

$$p_L = -p_R \implies \frac{1}{\sqrt{2}}\left(\frac{n}{R}+wR\right) = -\frac{1}{\sqrt{2}}\left(\frac{n}{R}-wR\right) \implies \frac{n}{R} = 0, \qquad (3.4.10)$$

and these Ishibashi states will be related instead to the Cardy states representing D1-branes. In the same way as above, we see that the gluing condition gives $p_L = -p_R = wR$ and this time it will correspond to only two different Ishibashi states, given by $w \in \{0, K\}$. By solving the gluing condition for every level in the mode expansion (3.4.3), we get:

$$|w\rangle\rangle = \exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}\tilde{\alpha}_{-n}\right]\times\sum_{m}\left|\frac{r}{R}+mR,-\frac{r}{R}-mR\right\rangle, \qquad (3.4.11)$$

for $r \in \{0, K\}$.

Now, the solutions we just obtained are the Ishibashi states of the theory, which do not correspond to physical states, but are related to the physical boundary states of the theory, that is to say the Dp-branes, via the Cardy's formula (3.4.5). However, as we pointed out in the previous section, in the diagonal RCFT case, the Cardy's formula takes the easier form given in (3.4.6), and, most importantly, the number of Ishibashi states will correspond to the number of Cardy states. So, if we restrict our analysis to the diagonal RCFT cases of the $c = 1$ compact boson, which are characterized by the condition $R = \sqrt{N},\ N \in \mathbb{Z}$, we will get a total number of Dp-branes equal to $2(N+1)$, which are precisely $2N$ D0-branes, located at equidistant points on the circle, and 2 D1-branes, wrapping the whole circle and characterized by different values of the Wilson line. In Fig. 12 we give a visual representation of the Ishibashi states (and, consequently, of the Dp-branes) in the $c = 1$ compact boson case at radius $R = \sqrt{3}$.

3.4.3 Duality symmetries and Dp-branes

In this final section, we will focus on the diagonal RCFT case of the $c = 1$ compact boson. Now that we have seen how to construct the Ishibashi states, and, consequently, the Dp-branes, we want to study how the duality symmetries we defined above act on these

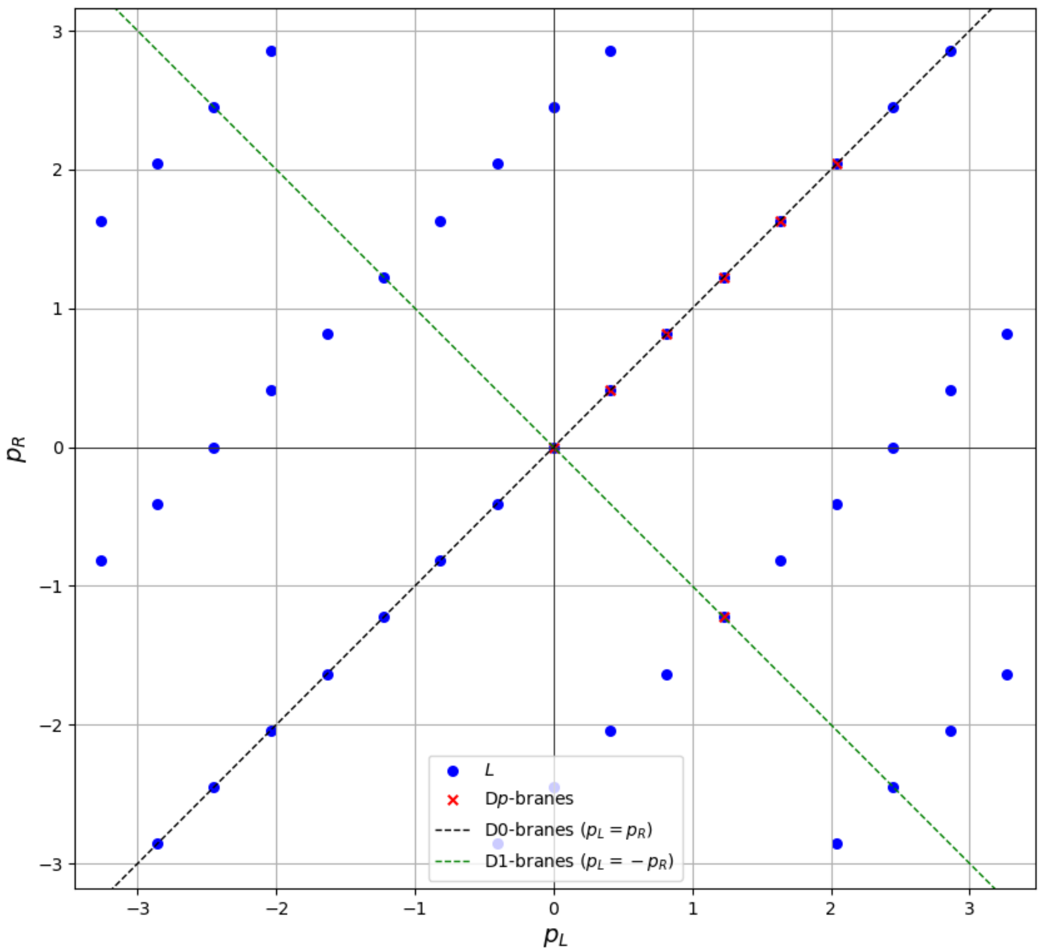


Figure 12: Plot of the self-dual integer momentum lattice $\Gamma^{1,1}$ of the $c = 1$ compact boson at radius $R = \sqrt{3}$. With blue dots we represented the lattice sites spanned by L defined in (3.3.20). With the red crosses, instead, we highlighted the points corresponding to the Ishibashi states, which are related to the Dp-branes. In particular, the points lying on the $p_L = p_R$, the black line in the plot, correspond to the D0-branes, while those on the $p_L = -p_R$ line, represented in green, are associated to the D1-branes. It is important to notice that the point $(p_L, p_R) = (0, 0)$ is associated to both a D0 and a D1-branes, and the distinction between the two different states is given by the oscillatory modes prefactor shown in (3.4.9) and (3.4.11).

states. As we saw, the action of a duality symmetry on the primary operators of the theory can be represented by a matrix $D \in O(1, 1, \mathbb{Q})$ acting on the lattice generating matrix L as $L \mapsto L' = LD$. Now, from the generic expression of D, we have been able to write the explicit action of this matrix on L in (3.3.16). Interestingly, as we observed in (3.3.17), the transformed lattice generating matrix L' can equivalently be written as

$$L' = LD = RL, \quad R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.4.12}$$

and this expression holds for every duality symmetry of the $c = 1$ compact boson. From this rewriting of the duality symmetry in terms of a left multiplying matrix R, it is then easier to understand the action of duality symmetries on the Ishibashi states. Indeed, we can clearly notice that the duality symmetry acts on the left and right-moving momentum of the theory in an asymmetric way, leaving the left momentum invariant while changing the sign of the right-moving one

$$\mathcal{D}: \quad p_L \mapsto p_L, \quad p_R \mapsto -p_R. \tag{3.4.13}$$

This asymmetric action on the left and right-moving momentum then clearly affects the Ishibashi states condition, in particular by mapping the Ishibashi states corresponding to D0-branes to those corresponding to D1-branes, and those associated with D1-branes to D0 ones. Indeed, given a state $|B\rangle\rangle$ solving equation (3.4.7) with $R = \pm 1$, then the action of the duality symmetry on the state transforms it in such a way that the new state will now satisfy the same Ishibashi state condition but with $R = \mp 1$:

$$\mathcal{D}: \quad (p_L \pm p_R)\,|B\rangle\rangle = 0 \quad \mapsto \quad (p_L \mp p_R)\,|B\rangle\rangle = 0. \tag{3.4.14}$$

From this overall action on the Ishibashi states, we clearly see that we get an exchange between D0 and D1-branes under the action of the duality symmetry, with the total number of branes preserved. Going back to the example of the $c = 1$ compact boson with target space a circle of radius $R = \sqrt{3}$, we immediately observe that the resulting theory will contain 2 D0-branes and 6 D1-branes.

The overall exchange between the Ishibashi states solving the Ishibashi condition for $R = +1$ with those solving the gluing condition given by $R = -1$ is clearly a consequence of the possibility to rewrite any duality symmetry of the form discussed in Section 3.3.3 in terms of the same matrix $R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ left-multiplying the lattice generating matrix L. This is peculiar of the $c = 1$ compact boson and the simple action of T-duality on the parameters of the theory. In the next section, instead, in which we will apply this analysis to the $c = 2$ compact boson, we will see how more intricate relations between Ishibashi states can emerge.

4 The $c = 2$ Compact Boson, RCFT Points and Non-Invertible Symmetries

In the previous chapter, we analyzed the $c = 1$ compact boson, focusing in particular on the rational conformal field theory (RCFT) points of its moduli space and on how non-invertible symmetries can arise at these special loci. We saw that, at rational points, the theory can become self-dual under the gauging of a finite subgroup of the chiral algebra, leading to the emergence of the so-called duality symmetries. These symmetries are in general non-invertible and are described by a Tambara-Yamagami fusion category. In addition, we examined how Dp-branes transform under the action of such duality defects.

In this chapter, we extend this analysis to the case of the $c = 2$ compact boson. The presence of an enlarged duality group in this theory allows for a richer and more intricate realization of non-invertible symmetries. We will begin by introducing the general framework of the $c = 2$ compact boson theory and describing the structure of its moduli space. We will then discuss the rationality condition and show how duality symmetries emerge at these enhanced symmetry points. Finally, we will study the behavior of Dp-branes under the action of these non-invertible symmetry transformations.

4.1 The $c = 2$ compact boson

The $c = 2$ compact boson represents the natural generalization of the $c = 1$ compact boson to a two-dimensional target space. It is one of the simplest examples of a non-linear sigma model with multiple compact directions and provides an ideal setting for studying the interplay between geometry, duality, and generalized symmetries in two-dimensional conformal field theories. This theory describes two real scalar fields $\phi^1(\sigma,\tau)$ and $\phi^2(\sigma,\tau)$ compactified on a two-dimensional torus T^2, and it can be interpreted as the worldsheet theory of a string propagating on a flat toroidal target space. Let us now introduce the non-linear sigma model more precisely; for a comprehensive review we refer to [74,75]. Consider a two-dimensional non-degenerate lattice $\Lambda \subset \mathbb{R}^2$ generated by $\lambda_1,\ \lambda_2 \in \mathbb{R}^2$. This can be defined in terms of a matrix Λ given by:

$$\Lambda = R\begin{pmatrix}1 & \tau_1\\ 0 & \tau_2\end{pmatrix},\ R \in \mathbb{R}_{>0} \quad \text{and} \quad \tau_2 \neq 0 \tag{4.1.1}$$

where we require τ_2 to be different from zero in order to have a non-degenerate lattice. The simple form of the first column, namely $\begin{pmatrix}1\\0\end{pmatrix}$, is due to the fact that we always have the freedom to rotate the lattice in $\mathbb{R}^2$ in such a way that one generator lies on the x-axis; this implies that $\Lambda \in O(2)\backslash GL(2)$, where the $O(2)$ group identifies lattices generating the same torus up to rotations and reflections. A two-dimensional torus can then be defined in terms of the non-degenerate lattice Λ as $T^2 = \mathbb{R}^2/\Lambda$; its metric is given by $G_{\mu\nu} = \langle \lambda_\mu, \lambda_\nu \rangle = \Lambda^T \Lambda$, which is a constant symmetric two-tensor. The sigma model, instead, is defined on a two-dimensional worldsheet Σ with coordinates (τ, σ), as in the $c = 1$ compact boson, by two compact scalar fields $\phi^\mu(\tau, \sigma),\ \mu, \nu \in \{1, 2\},\ \ \phi^\mu \sim \phi^\mu + 2\pi$. These represent the embedding of the worldsheet in the target space and define a map $\phi : \Sigma \to T^2$. The action that describes their dynamics takes the form:

$$S = \frac{1}{4\pi} \int d\sigma d\tau \delta^{\mu\nu} G_{ij} \partial_\mu \phi^i \partial_\nu \phi^j + \frac{i}{4\pi} \int d\sigma d\tau \epsilon^{\mu\nu} B_{ij} \partial_\mu \phi^i \partial_\nu \phi^j, \tag{4.1.2}$$

where $B_{\mu\nu}$ is an antisymmetric tensor known as the B-field, which can be interpreted as the analog of the Kalb-Ramond field from String Theory. From the action written above, the parameters of the theory are $(\Lambda, B) \in O(2)\backslash GL(2) \times \text{Skew}(2)$ and their explicit expression is given by:

$$G = R^2 \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & \tau_1^2 + \tau_2^2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}. \tag{4.1.3}$$

For toroidal sigma models, it is customary and useful to recast the metric G_{ij} and the antisymmetric tensor B_{ij} in terms of two complex parameters τ and ρ defined as

$$\tau = \frac{G_{12}}{G_{11}} + i\frac{\sqrt{G}}{G_{11}} = \tau_1 + i\tau_2, \quad \rho = \rho_1 + i\rho_2 = b + i\sqrt{G}, \quad G = det(G). \tag{4.1.4}$$

These are the so-called complex structure and complexified Kähler modulus, respectively; the former encodes the information regarding the shape of the torus, while the latter combines the antisymmetric B field with the area of the torus T^2. These 2 complex parameters parametrize the 4-dimensional toroidal branch of the $c = 2$ moduli space. It is also possible to notice that, given τ and ρ such that $\tau_1 = \rho_1 = 0$, the theory factorizes in the tensor product of two $c = 1$ compact bosons, with radii R and $R\tau_2$ respectively.

Analogously to the $c = 1$ compact boson, the $c = 2$ compact boson admits an equivalent description of the theory in terms of dual fields $\tilde{\phi}^i \sim \tilde{\phi}^i + 2\pi,\ i = 1, 2$ (see [76,77] for the general definition of dual fields in non-linear sigma models). Again, both the dual fields and the original ones admit a decomposition in terms of left and right-moving fields, which this time takes the more intricate form:

$$\begin{aligned}\phi^1 &= \frac{1}{\sqrt{2\tau_2\rho_2}}\left[\tau_2(X^1+\overline{X}^1)-\tau_1(X^2+\overline{X}^2)\right]\\ \phi^2 &= \frac{1}{\sqrt{2\tau_2\rho_2}}\left(X^2+\overline{X}^2\right)\\ \tilde{\phi}^1 &= \frac{1}{\sqrt{2\tau_2\rho_2}}\left[\rho_2(X^1-\overline{X}^1)-\rho_1(X^2+\overline{X}^2)\right]\\ \tilde{\phi}^2 &= \frac{1}{\sqrt{2\tau_2\rho_2}}\Big[(\rho_1\tau_2+\rho_2\tau_1)X^1+(\rho_1\tau_2-\rho_2\tau_1)\overline{X}^1\\ &\qquad +(\rho_2\tau_2-\rho_1\tau_1)X^2-(\rho_1\tau_1+\rho_2\tau_2)\overline{X}^2\Big].\end{aligned} \tag{4.1.5}$$

This apparently involved combination is due to the fact that the 2-torus has non-trivial complex structure and Kähler modulus and, in order to obtain a well-defined decomposition of the fields in terms of the chiral components, we need to keep track of the geometry. For future purposes, it is convenient to rewrite this in matrix form as

$$\begin{pmatrix}\phi^1\\ \phi^2\\ \tilde{\phi}^1\\ \tilde{\phi}^2\end{pmatrix} = L^T\begin{pmatrix}X^1\\ X^2\\ \bar{X}^1\\ \bar{X}^2\end{pmatrix}, \tag{4.1.6}$$

where we defined the matrix L as the transpose of the transformation matrix between the fields ϕ^i, $\tilde{\phi}^i$ and the left and right-moving fields X^i, $\bar{X}^i$, which can be written in terms of the complex structure and the complexified Kähler modulus as

$$L = \frac{1}{\sqrt{2\tau_2\rho_2}}\begin{pmatrix}\tau_2 & 0 & \rho_2 & \rho_1\tau_2+\rho_2\tau_1\\ -\tau_1 & 1 & -\rho_1 & -\rho_1\tau_1+\rho_2\tau_2\\ \tau_2 & 0 & -\rho_2 & \rho_1\tau_2-\rho_2\tau_1\\ -\tau_1 & 1 & -\rho_1 & -\rho_1\tau_1-\rho_2\tau_2\end{pmatrix} \tag{4.1.7}$$

As in the $c = 1$ compact boson, at generic values of τ and ρ, this theory is invariant under the four global shifts of the fields $\phi^i \mapsto \phi^i + \alpha^i$ and $\tilde{\phi}^i \mapsto \tilde{\phi}^i + \tilde{\alpha}^i$. This invariance gives rise to four global symmetries, which result in

$$U(1)^2_{\boldsymbol{n}} \times U(1)^2_{\boldsymbol{w}} = U(1)_{n_1} \times U(1)_{n_2} \times U(1)_{w_1} \times U(1)_{w_2}, \tag{4.1.8}$$

and the charges under these symmetries are the momentum and the winding numbers $\boldsymbol{n} = (n_1, n_2)^T$ and $\boldsymbol{w} = (w_1, w_2)^T$, respectively. The primary operators are then characterized by these four integer charges and take the form:

$$V_{\boldsymbol{n},\boldsymbol{w}} = e^{i\boldsymbol{n}^T\boldsymbol{\phi} + i\boldsymbol{w}^T\tilde{\boldsymbol{\phi}}}, \tag{4.1.9}$$

where $\boldsymbol{\phi} = (\phi^1, \phi^2)^T$ and $\tilde{\boldsymbol{\phi}} = (\tilde{\phi}^1, \tilde{\phi}^2)^T$. However, as we saw above, the fields and their duals can be written in terms of the chiral fields X^i and $\bar{X}^i$, with the explicit relation (4.1.6). In this way, we can rewrite the primary operators in terms of the chiral fields by defining the left and right moving momenta $\boldsymbol{p}_L = (p_{L,1}, p_{L,2})$ and $\boldsymbol{p}_R = (p_{R,1}, p_{R,2})$ as:

$$\begin{pmatrix} \boldsymbol{n} & \boldsymbol{w} \end{pmatrix} \begin{pmatrix} \boldsymbol{\phi} \\ \tilde{\boldsymbol{\phi}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{n} & \boldsymbol{w} \end{pmatrix} L^T \begin{pmatrix} \boldsymbol{X} \\ \bar{\boldsymbol{X}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{p}_L & \boldsymbol{p}_R \end{pmatrix} \begin{pmatrix} \boldsymbol{X} \\ \bar{\boldsymbol{X}} \end{pmatrix}, \tag{4.1.10}$$

where $\begin{pmatrix} \boldsymbol{X} \\ \bar{\boldsymbol{X}} \end{pmatrix} = (X^1, X^2, \bar{X}^1, \bar{X}^2)^T$ is the vector of chiral fields. In particular, we see that $\boldsymbol{p}_{L,R}$ are given in terms of the matrix L, and they take values in the even, self-dual, integer momentum lattice $\Gamma^{2,2}$ defined as:

$$\Gamma^{(2,2)} \ni \begin{pmatrix} \boldsymbol{p}_L \\ \boldsymbol{p}_R \end{pmatrix} = L \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{w} \end{pmatrix}. \tag{4.1.11}$$

In this way, the primary operators, as in the $c = 1$ compact boson, are in one-to-one correspondence with the sites of this lattice.

As we saw in the previous section, the $c = 1$ compact boson is characterized by its invariance under T-duality, which acts by exchanging momentum and winding numbers, together with the exchange of the defining field X with its dual $\tilde{X}$. In particular, T-duality implies the identification of a theory at radius R with the one at radius $1/R$.

In the $c = 2$ compact boson case, we have a similar property; however, since the target space of this class of sigma models is 2-dimensional, namely a torus, this type of theory will be characterized by a larger set of duality transformations. As discussed above, the torus is specified by two complex parameters, τ and ρ, representing the complex structure and the complexified Kähler modulus, respectively. Each of them takes values in the complex upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$, such that the continuous moduli space is $\mathbb{H}_\tau \times \mathbb{H}_\rho$. If we consider the discrete group $SL(2,\mathbb{Z})_\tau \times SL(2,\mathbb{Z})_\rho$ acting on these parameters as

$$(\tau, \rho) \longmapsto \left(\frac{a\tau + b}{c\tau + d}, \frac{a'\rho + b'}{c'\rho + d'} \right), \tag{4.1.12}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})_\tau, \quad \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL(2,\mathbb{Z})_\rho,$$

this leaves the conformal field theory invariant, as it preserves the even, self-dual momentum lattice $\Gamma^{(2,2)}$. In addition to this, the theory also admits two additional discrete symmetries: mirror symmetry and spatial inversion. Mirror symmetry corresponds to the exchange of the two complex parameters, $\tau \leftrightarrow \rho$, which geometrically relates a compactification on a torus to its mirror dual, interchanging complex and Kähler structures; this will be denoted by $\mathbb{Z}_2^M$ and it acts as $(\tau, \rho) \mapsto (\rho, \tau)$. Spatial inversion, instead, acts by reversing the torus orientation and corresponds to the $\mathbb{Z}_2^I$ operation $(\tau, \rho) \mapsto (-\bar{\tau}, -\bar{\rho})$.

Taken together, these transformations generate the full discrete duality group of the $c = 2$ compact boson, given by

$$O(2,2;\mathbb{Z}) \simeq P(SL(2,\mathbb{Z})_\tau \times SL(2,\mathbb{Z})_\rho) \rtimes (\mathbb{Z}_2^M \times \mathbb{Z}_2^I), \tag{4.1.13}$$

which, overall, is equivalent to $O(2,2,\mathbb{Z})$. This group acts discretely on the continuous parameter space $(\tau, \rho) \in \mathbb{H}_\tau \times \mathbb{H}_\rho$, preserving the conformal spectrum. Consequently, the moduli space of inequivalent $c = 2$ compact boson toroidal CFTs is obtained by quotienting the continuous moduli space by the full duality group, leading to

$$\mathcal{T}^2 = (\mathbb{H}_\tau / PSL(2,\mathbb{Z})_\tau \times \mathbb{H}_\rho / PSL(2,\mathbb{Z})_\rho)/(\mathbb{Z}_2^M \times \mathbb{Z}_2^I). \tag{4.1.14}$$

The resulting moduli space identifies all backgrounds related by the action of the discrete duality group and represents the higher-dimensional generalization of the $R \leftrightarrow 1/R$ T-duality of the $c = 1$ compact boson, where both the shape and the size of the torus now participate in the duality transformations.

Now, given a primary operator of the form $V_{\boldsymbol{n},\boldsymbol{w}} = e^{i\boldsymbol{n}^T\boldsymbol{\phi} + i\boldsymbol{w}^T\tilde{\boldsymbol{\phi}}} = e^{i\boldsymbol{p}_L^T\boldsymbol{X} + i\boldsymbol{p}_R^T\bar{\boldsymbol{X}}}$, we can define $h_L = \frac{1}{2}(p_{L,1}^2 + p_{L,2}^2)$ and $h_R = \frac{1}{2}(p_{R,1}^2 + p_{R,2}^2)$, and its conformal dimension and spin are given by

$$\Delta = h_L + h_R, \quad s = h_L - h_R. \tag{4.1.15}$$

These two numbers that characterize a primary operator, can also be obtained from the momentum lattice $\Gamma^{2,2}$. Indeed, let us consider a primary operator characterized by momentum and winding numbers $(\boldsymbol{n}, \boldsymbol{w})$, which correspond to left and right-moving momentum $(\boldsymbol{p}_L, \boldsymbol{p}_R)$. The spin can then be written as

$$s = h_L - h_R = (\boldsymbol{p}_L \ \boldsymbol{p}_R)\, \eta \begin{pmatrix} \boldsymbol{p}_L \\ \boldsymbol{p}_R \end{pmatrix} = (\boldsymbol{n} \ \boldsymbol{w})\, L^T \eta L \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{w} \end{pmatrix} = \sum_i n_i w_i, \tag{4.1.16}$$

where

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad L^T \eta L = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{4.1.17}$$

and the latter represents the inner product of the momentum lattice. For the conformal dimension, instead, we can write it in terms of the quadratic form

$$\Delta = h_L + h_R = (\boldsymbol{p}_L \ \boldsymbol{p}_R) \begin{pmatrix} \boldsymbol{p}_L \\ \boldsymbol{p}_R \end{pmatrix} = (\boldsymbol{n} \ \boldsymbol{w})\, L^T L \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{w} \end{pmatrix} \equiv (\boldsymbol{n} \ \boldsymbol{w})\, \mathcal{E} \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{w} \end{pmatrix} \tag{4.1.18}$$

where we defined the matrix $\mathcal{E}$ in terms of the lattice generating matrix L. This matrix can be written in terms of the torus metric G and the antisymmetric tensor B as

$$\mathcal{E} = \begin{pmatrix} G^{-1} & G^{-1}B \\ -BG^{-1} & G - BG^{-1}B \end{pmatrix} \tag{4.1.19}$$

and it is called the generalized metric. This will be important in the following to define duality symmetries in the $c = 2$ compact boson theory. Its definition comes from the double field theory formalism, which is a useful approach to describe theories invariant under a duality group (we refer to [78] for a detailed discussion of the double field theory formalism). In particular, as we saw in (4.1.13), the duality group of the $c = 2$ compact boson is given by the discrete group $O(2, 2, \mathbb{Z})$, which has a natural action on the generalized metric given by

$$O(2, 2, \mathbb{Z}) \ni O : \mathcal{E} \mapsto \mathcal{E}' = O^T \mathcal{E} O. \tag{4.1.20}$$

This action mixes the metric and the B-field while preserving the conformal spectrum, as it corresponds to an automorphism of the even, self-dual, integer momentum lattice $\Gamma^{(2,2)}$. The elements of $O(2, 2; \mathbb{Z})$ exchange and mix momentum and winding excitations in a way that leaves the worldsheet CFT invariant, equivalently to the action of the duality group on (τ, ρ). In particular, the 4-dimensional representations of the elements of the duality group is given by:

$$\begin{gathered}
\begin{pmatrix} a & -c & 0 & 0 \\ -b & d & 0 & 0 \\ 0 & 0 & d & b \\ 0 & 0 & c & a \end{pmatrix} \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})_\tau \\
\begin{pmatrix} d' & 0 & 0 & b' \\ 0 & d' & -b' & 0 \\ 0 & -c' & a' & 0 \\ c' & 0 & 0 & a' \end{pmatrix} \quad \text{for} \quad \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL(2, \mathbb{Z})_\rho \\
M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{Z}_2^M, \quad I = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{Z}_2^I
\end{gathered} \tag{4.1.21}$$

Finally, we can present the partition function of the theory. The lattice $\Gamma^{2,2}$ defined above encodes all physical states of the theory. As we obtained for the $c = 1$ compact boson, the corresponding partition function takes the standard form

$$Z(\sigma, \bar{\sigma}; G, B) = \frac{1}{|\eta(\sigma)|^4} \sum_{(p_L, p_R) \in \Gamma^{2,2}} q^{\frac{1}{2}p_L^2} \, \bar{q}^{\frac{1}{2}p_R^2}, \tag{4.1.22}$$

where $q = e^{2\pi i\sigma}$, σ is the complex structure of the worldsheet torus on which we define the partition function, and $\eta(\sigma)$ is the Dedekind function defined in (3.1.23). This partition function depends only on the background moduli (G, B) through the lattice $\Gamma^{(2,2)}$ and is manifestly invariant under the modular group of the worldsheet torus as well as the target-space duality group $O(2, 2; \mathbb{Z})$, reflecting the exact equivalence of all tori related by these transformations.

4.2 The $c = 2$ moduli space

As we saw in Section 3.1, the conformal manifold of the $c = 1$ compact boson consists of two continuous branches, namely the circle branch and the orbifold branch, which meet at the point $R = 2$ (we recall that, in our conventions, the self-dual radius corresponds to $R = 1$). The latter is obtained by orbifolding the $\mathbb{Z}_2$ symmetry acting as $X \mapsto -X$ on the embedding field. This reflection is a symmetry of the theory for any point on the circle branch. This originates from the fact that the target space S^1 is defined in terms of a one-dimensional lattice as the quotient $\mathbb{R}/(2\pi R\mathbb{Z})$, and the only nontrivial automorphism of a one-dimensional lattice is the $\mathbb{Z}_2$ reflection symmetry, which acts by $x \mapsto -x$, where x is the coordinate on $\mathbb{R}$, and whose orbifolding generates the only additional continuous branch of the moduli space, namely the orbifold one. In addition to these connected components, the moduli space also contains three isolated exceptional orbifold points, obtained by gauging the remaining finite subgroups of $SU(2)$ at the self-dual radius: the tetrahedral, octahedral, and icosahedral groups. The gauging of the cyclic subgroups $\mathbf{Z}_k$ and of the dihedral subgroups $\mathbf{D}_k$ of $SU(2)$ instead produces points at $R = k$ lying, respectively, on the circle branch and on the orbifold branch.

In this section, we want to present an equivalent analysis for the $c = 2$ compact boson moduli space; for the original work, we refer to [74]. In this article, the authors show that the moduli space $\mathcal{C}^2$ of unitary conformal field theories with central charge $c = 2$ is composed of 29 non-exceptional, non-isolated components, meeting at different multicritical points and lines, leading to a much more involved moduli space than the $c = 1$ compact boson one.

In the previous section, we focused on the $c = 2$ compact boson theories defined on the toroidal branch; these correspond to the set of non-linear sigma models with target space a two-dimensional torus T^2 endowed with a flat metric G and a constant antisymmetric tensor B. As explained above, this branch is parametrized by two complex moduli: the complex structure $\tau = \tau_1 + i\tau_2$ and the complexified Kähler modulus $\rho = \rho_1 + i\rho_2$, each taking values in the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$. Before identifying physically equivalent points, the continuous parameter space is thus $\mathbb{H}_\tau \times \mathbb{H}_\rho$, which, however, is quotiented by the duality group $O(2, 2; \mathbb{Z}) \simeq P(SL(2, \mathbb{Z})_\tau \times SL(2, \mathbb{Z})_\rho) \rtimes (\mathbb{Z}_2^M \times \mathbb{Z}_2^I)$. The theories obtained by varying (τ, ρ) within this quotient space constitute the continuous toroidal branch $\mathcal{T}^2 \subset \mathcal{C}^2$ defined in (4.1.14).

However, as in the $c = 1$ case, where an additional component arises from the orbifold procedure of the discrete lattice symmetry $\mathbb{Z}_2$, the $c = 2$ moduli space also contains further branches obtained by orbifolding the possible symmetries of the underlying lattice defining the 2-torus. Indeed, as we saw in the previous section, a two-torus T^2 can be defined as the quotient $T^2 = \mathbb{R}^2/\Lambda$, where Λ is a non-degenerate two-dimensional lattice. Now, if we consider a discrete symmetry of Λ, that is to say, an automorphism of the lattice, the orbifolding procedure will lead to a new conformal field theory while keeping the central charge $c = 2$ invariant. However, the resulting theory will no longer be associated with a point on the toroidal branch but will take value on a different branch of the moduli space. This implies that, as in the $c = 1$ case, the orbifolding procedure of the discrete symmetries of the underlying lattice generates all the possible branches in the moduli space.

In particular, the possible symmetries of a two-dimensional lattice are classified and correspond to the 17 inequivalent crystallographic plane groups, also known as the wallpaper groups. Each of them is characterized by a finite point group $P \subset O(2)$ and a translational subgroup $\Delta \subset \mathbb{R}^2$, which together describe all possible combinations of rotational, reflectional, and translational symmetries compatible with lattice periodicity. Consequently, only a finite number of orbifold constructions are possible, leading to a finite number of additional branches in the $c = 2$ moduli space $\mathcal{C}^2$ (just as in the one-dimensional case, where the single reflection symmetry gives rise to one extra orbifold branch).

A crucial difference, however, is that while the $\mathbb{Z}_2$ inversion of the one-dimensional lattice is a symmetry for every value of the compactification radius R, each crystallographic plane group is a symmetry of a given lattice $\Lambda(\tau, \rho)$ only for special values of the moduli τ and ρ. For example, a subset of the possible lattice symmetries is given by the rotations $R(\theta)$ acting as $(x_1, x_2) \mapsto R(\theta)(x_1, x_2)$ on the $\mathbb{R}^2$ coordinates, with $R(\theta) \in SO(2)$. In order for $R(\theta)$ to be a symmetry of a given lattice Λ, it must satisfy the condition $R_\theta \Lambda = \Lambda$.

Now, the possible rotation angles are constrained by the two-dimensional crystallographic restriction theorem, allowing only for $\theta = \pi,\ 2\pi/3,\ \pi/2,$ and $\pi/3$ (we refer to [79] for the original work on the classification of the crystallographic plane groups). Consequently, the admissible rotational symmetry groups of a two-dimensional lattice are $\mathbb{Z}_2,\ \mathbb{Z}_3,\ \mathbb{Z}_4,$ and $\mathbb{Z}_6$. Among these, the $\mathbb{Z}_2$ group generated by the inversion $(x_1, x_2) \mapsto (-x_1, -x_2)$ always represents a symmetry of a two-dimensional lattice (equivalently to the one-dimensional case), as it leaves any 2d lattice invariant, and thus the corresponding $\mathbb{Z}_2$ orbifold exists everywhere on the toroidal branch. The $\mathbb{Z}_3$ symmetry, corresponding to rotations by $2\pi/3$, instead, is compatible only with the equilateral (hexagonal) lattice defined by $\tau = \exp(2\pi i/3)$. This implies that the toroidal compactifications characterized by $(\tau, \rho) = (\exp(2\pi i/3), \rho)$ will enjoy an additional symmetry that, after applying the orbifold procedure, leads to a new, well-defined conformal field theory. Similarly, the $\mathbb{Z}_4$ symmetry, which acts as a $\pi/2$ rotation, exists only for the square lattice, realized at $\tau = i$, while the $\mathbb{Z}_6$ symmetry acts on the same hexagonal lattice at $\tau = \exp(\pi i/3)$.

Each of these cases corresponds to fixed loci of enhanced symmetry in the toroidal branch, where an additional orbifold construction is possible. In particular, the orbifolding procedure of such crystallographic symmetries yields new families of conformal field theories, which form separate connected components of the full moduli space. Each component is a continuous branch parameterized by the remaining modulus (for example, by ρ when τ is fixed to one of the symmetric values) and is connected to other branches at multicritical points where multiple lattice symmetries coexist. For example, the intersection between the $\mathbb{Z}_2$ and $\mathbb{Z}_4$ orbifold branches occurs at $(\tau, \rho) = (i, \rho)$, corresponding to the square lattice, where both the π and $\pi/2$ rotations act as symmetries.

It is important to notice that the rotations are not the only possible crystallographic symmetries; indeed, as we mentioned above, there are 17 wallpaper groups. These, in addition to the aforementioned rotations, account for spatial reflections, which act on the coordinates as $(x_1, x_2) \mapsto (-x_1, x_2)$ or $(x_1, x_2) \mapsto (x_1, -x_2)$ and represent a symmetry of the lattices with $\tau_1 = \{0, -1/2\}$, dihedral groups and translations. Moreover, one can consider more intricate constructions, such as combinations of rotational and translational symmetries, corresponding to nontrivial elements of the wallpaper groups. These produce additional continuous families, and the network of all such branches, together with their intersections, accounts for the 29 non-exceptional components identified in [74].

The intersection loci, where several components meet, correspond to theories with extended chiral algebras and enhanced non-abelian symmetry, generalizing the self-dual and exceptional point of the $c = 1$ moduli space. In summary, the moduli space of $c = 2$ compact bosons can be described as the union of the toroidal branch, given by the quotient $(\mathbb{H}_\tau \times \mathbb{H}_\rho)/O(2, 2; \mathbb{Z})$, together with its orbifold descendants obtained by gauging the crystallographic plane groups acting at special values of (τ, ρ). These various components are connected through multicritical lines and points of enhanced symmetry, generating a highly non-trivial and interconnected conformal manifold for the $c = 2$ compact boson, denoted by $\mathcal{C}^2$.

4.3 The rationality condition

In the previous chapter, we analyzed the emergence of duality symmetries in the $c = 1$ compact boson. To construct these topological defects, we focused on special loci of the moduli space, namely the non-linear sigma models satisfying the rationality condition $R^2 \in \mathbb{Q}$. These points correspond to rational conformal field theories (RCFTs), characterized by an enhanced chiral algebra and, consequently, a finite number of primary operators. The rationality condition is essential for the definition of duality symmetries; indeed, although a finite subgroup of the chiral algebra can, in principle, be gauged at any point in the moduli space, only at RCFT points does this gauging produce a new theory that is dual to the original one, and, therefore, we can construct the corresponding duality symmetry (as discussed in Section 3.3.1). In this chapter, we extend the construction of duality symmetries to the $c = 2$ compact boson. Once again, the rationality condition plays a crucial role in this type of symmetry, and we will present its explicit form in this section.

Let us now focus on the $c = 2$ compact boson; as we saw in the previous section, due to the higher dimensional target space, this type of non-linear sigma model presents a richer moduli space. In particular, the toroidal compactifications are specified by the value of the complex structure τ and the complexified Kähler modulus ρ. These two parameters specify a point on the toroidal branch in the moduli space $\mathbb{H}_\tau \times \mathbb{H}_\rho / O(2, 2; \mathbb{Z})$, and the conformal spectrum is determined by the self-dual integer momentum lattice $\Gamma^{(2,2)}(\tau, \rho)$ defined as

$$\Gamma^{(2,2)} \ni \begin{pmatrix} \boldsymbol{p}_L \\ \boldsymbol{p}_R \end{pmatrix} = L \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{w} \end{pmatrix} , \;\; \boldsymbol{n}, \boldsymbol{w} \in \mathbb{Z}^2, \tag{4.3.1}$$

where $L = L(\tau, \rho)$ is the lattice generating matrix introduced in (4.1.6). For generic (τ, ρ) the theory possesses a chiral algebra of the form $U(1)^2_L \times U(1)^2_R$, generated by the two left and right currents $J^i_L = \partial_+ X^i$ and $J^i_R = \partial_- \bar{X}^i$ (we did not discuss the chiral currents explicitly for the $c = 2$ compact boson, but they can be defined analogously to (3.1.15) of the $c = 1$ case). However, at special points in the moduli space, the lattice $\Gamma^{(2,2)}$ acquires additional automorphisms that generate new chiral fields and hence enlarge the chiral algebra, leading to rational theories.

The condition for rationality in the $c = 2$ compact boson can be stated in several equivalent ways. Similarly to the $c = 1$ case, this is given by the requirement that both the metric G and the antisymmetric tensor B are characterized by rational entries, that is to say, $G \in GL(2, \mathbb{Q})$ and $B \in \text{Skew}(2, \mathbb{Q})$. However, as we saw in the previous section, the moduli space of the $c = 2$ compact boson can be better expressed in terms of the two complex moduli τ and ρ; for this reason, we can rewrite the rationality condition in terms of these parameters. As was shown in [80], the rationality condition then corresponds to the requirement that both the complex structure τ and the complexified Kähler modulus ρ take values in the same imaginary quadratic number field, namely

$$\tau, \rho \in \mathbb{Q}(\sqrt{D}), \qquad D < 0, \tag{4.3.2}$$

where

$$x \in \mathbb{Q}(\sqrt{D}) \iff x = x_1 + x_2\sqrt{D}, \;\; x_1, x_2 \in \mathbb{Q}. \tag{4.3.3}$$

In particular, τ can be written in terms of a quadratic equation with integer coefficients

$$a\tau^2 + b\tau + c = 0, \quad a, b, c \in \mathbb{Z}, \quad \text{gcd}(a, b, c) = 1, \tag{4.3.4}$$

such that

$$\tau = \frac{-b}{2a} + \frac{i}{2a}\sqrt{-D}, \quad D = b^2 - 4ac; \tag{4.3.5}$$

a similar equation then holds for ρ too, which can be characterized by different integer coefficients a', b', c' under the condition that $b'^2 - 4a'c' = D$, in such a way that the two complex parameters belong to the same number field. This number-theoretic condition admits a natural geometric interpretation in terms of the complex multiplication (CM) property of the torus. To see this, notice that the complex structure τ determines a two-dimensional lattice $\Lambda_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$, which defines the complex torus $E_\tau = \mathbb{C}/\Lambda_\tau$. This is a one-dimensional complex torus, that is to say, a flat elliptic curve. Any holomorphic endomorphism of E_τ is then induced by multiplication by some complex number $\alpha \in \mathbb{C}$ on $\mathbb{C}$, that is to say, $z \mapsto \alpha z$, where z is the complex coordinate over the complex plane $\mathbb{C}$; in order for this linear map to be an endomorphism of the complex torus, it must preserve the underlying lattice Λ_τ:

$$\alpha\Lambda_\tau \subseteq \Lambda_\tau. \tag{4.3.6}$$

For a generic value of τ this is only possible when $\alpha \in \mathbb{Z}$, implying that the endomorphism ring corresponds to

$$\mathrm{End}(E_\tau) \cong \mathbb{Z}. \tag{4.3.7}$$

However, if τ satisfies a quadratic equation with integer coefficients of the form (4.3.4), then the lattice Λ_τ is preserved under multiplication by the complex number

$$\alpha = a\tau, \tag{4.3.8}$$

which is proportional to the complex structure itself. More generally, the lattice Λ_τ is preserved under the multiplication by all the elements of the order

$$\mathcal{O}_D = \mathbb{Z} + \mathbb{Z}\,\frac{b + \sqrt{D}}{2} = \mathbb{Z} + \mathbb{Z}a\tau. \tag{4.3.9}$$

In this case the ring of endomorphisms of the torus is strictly larger than $\mathbb{Z}$, namely

$$\mathrm{End}(E_\tau) \cong \mathcal{O}_D \supsetneq \mathbb{Z}. \tag{4.3.10}$$

A complex torus with this property is said to have complex multiplication. From the lattice perspective, this means that multiplication by any $\alpha \in \mathcal{O}_D$ acts on the basis $(1, \tau)$ by an integral 2×2 matrix

$$\alpha \begin{pmatrix} 1 \\ \tau \end{pmatrix} = \begin{pmatrix} 1 \\ \tau \end{pmatrix} A_\alpha, \qquad A_\alpha \in GL(2, \mathbb{Z}), \tag{4.3.11}$$

and thus produces a genuine lattice automorphism. This enhancement of the endomorphism ring of the torus is precisely what distinguishes CM points from generic ones.

The rationality condition (4.3.2) now simply asserts that both complex parameters of the T^2 sigma model lie in the same imaginary quadratic field:

$$\tau, \rho \in \mathbb{Q}(\sqrt{D}), \qquad D < 0. \tag{4.3.12}$$

Geometrically, this means that not only the complex structure torus E_τ, but also the Kähler torus E_ρ, which can be defined in terms of the lattice $\Lambda_\rho = \mathbb{Z} \oplus \rho\mathbb{Z}$, admits complex multiplication by the same order $\mathcal{O}_D$. The non-linear sigma models characterized by these values of τ and ρ are precisely the RCFT points in the moduli space.

To make the connection between complex multiplication and rationality more precise, let us now describe how the momentum lattice reorganizes at CM points. Recall that the spectrum of primary operators of the theory is in one-to-one correspondence with the even, self-dual integer momentum lattice $\Gamma^{(2,2)} \subset \mathbb{R}^{2,2}$, generated by the vectors $(\boldsymbol{p}_L, \boldsymbol{p}_R)$ with

$$\begin{pmatrix} \boldsymbol{p}_L \\ \boldsymbol{p}_R \end{pmatrix} = L(\tau, \rho) \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{w} \end{pmatrix}, \qquad \boldsymbol{n}, \boldsymbol{w} \in \mathbb{Z}^2. \tag{4.3.13}$$

This is a rank-four lattice equipped with the bilinear form of signature $(2, 2)$. From this, it is possible to define the left and right-moving set of charges as the projections

$$\Gamma_L = \{\boldsymbol{p}_L \mid \begin{pmatrix} \boldsymbol{p}_L \\ \boldsymbol{p}_R \end{pmatrix} \in \Gamma^{(2,2)}\}, \qquad \Gamma_R = \{\boldsymbol{p}_R \mid \begin{pmatrix} \boldsymbol{p}_L \\ \boldsymbol{p}_R \end{pmatrix} \in \Gamma^{(2,2)}\}. \tag{4.3.14}$$

At generic points in moduli space, these lattices contain infinitely many inequivalent orbits under the chiral algebra $U(1)^2_L \times U(1)^2_R$.

In addition to these two sets of states, it is possible to introduce the sublattice of purely left-moving states,

$$\Gamma_0 = \{\boldsymbol{p}_L \mid \begin{pmatrix} \boldsymbol{p}_L \\ \boldsymbol{0} \end{pmatrix} \in \Gamma^{(2,2)}\}, \tag{4.3.15}$$

that is, the set of momentum–winding pairs for which the right-moving component vanishes (equivalently, it is possible to define the sublattice of purely right-moving states, that we denote by $\tilde{\Gamma}_0$, which contains the states in $\Gamma^{2,2}$ that satisfy $\boldsymbol{p}_L = 0$). At generic values of (τ, ρ), the only solution to $\boldsymbol{p}_R = 0$ is given by momentum and winding numbers $\boldsymbol{n} = \boldsymbol{w} = 0$, and therefore $\Gamma_0 = \{0\}$, that is to say, a rank zero sublattice of the full momentum lattice $\Gamma^{2,2}$. Consequently, the only chiral fields of the theory are the currents $J^i_L = \partial_+ X^i$, generating an abelian chiral algebra $U(1)^2_L$.

However, at complex multiplication points, the set of states satisfying the defining property of Γ_0 is enhanced. Indeed, the condition $\boldsymbol{p}_R(\boldsymbol{n}, \boldsymbol{w}) = 0$ now gives rise to two diophantine equations characterized by an infinite number of solutions. In particular, the lattice Γ_0 becomes a non-trivial rank-two sublattice of $\Gamma^{2,2}$. The emergence of a non-trivial sublattice Γ_0 of purely left-moving charges has an immediate consequence for the chiral algebra of the theory. Each non-zero vector $\boldsymbol{p}_L \in \Gamma_0$ corresponds to a chiral primary operator of the form

$$V_{\boldsymbol{p}_L}(z) = \exp(i\,\boldsymbol{p}_L \cdot \boldsymbol{X}_L(z))\,, \tag{4.3.16}$$

with conformal weights $(h_L, h_R) = \left(\frac{1}{2}\boldsymbol{p}_L^2, 0\right)$. At generic points in the moduli space the only solutions to $p_R = 0$ are trivial, and thus the only chiral primaries are the conserved currents $\partial_+ X_L^i$, generating the abelian chiral algebra $U(1)_L^2$. At CM points, instead, the presence of a rank-two lattice of non-trivial solutions to $p_R = 0$ implies the existence of additional chiral fields of non-zero conformal weight. These operators extend the chiral algebra beyond $U(1)_L^2$, producing a genuinely larger Kac–Moody algebra generated by the elements of Γ_0 (in the same way as in the $c = 1$ compact boson, in which the solutions to the chirality condition generated the enhanced chiral algebra $U(1)_{2K} \times U(1)_{2K}$; see the discussion around (3.2.10)).

The crucial point is that the sublattice Γ_0 becomes a finite index sublattice in Γ_L, and the number of distinct primary representations of the extended chiral algebra is given by the index $[\Gamma_L : \Gamma_0]$. More precisely, the quotient

$$\Gamma_L/\Gamma_0 \tag{4.3.17}$$

is a finite abelian group, whose elements label the distinct representations of the enhanced chiral algebra. This finiteness property is exactly the defining feature of a rational conformal field theory: the left-moving spectrum decomposes into finitely many irreducible representations of the enhanced chiral algebra, and the operator product expansion closes among this finite set of primaries. A completely analogous construction applies to the right-moving sector, where the sublattice $\tilde{\Gamma}_0$ controls the enhancement of the right-moving chiral algebra. An explicit example of the structure of Γ_L and Γ_0 at the CM point $(\tau, \rho) = (i, 2i)$ is shown in Fig. 13. We also represent the identification of lattice points under the action of the extended chiral algebra, which we denote with the black arrows, where ζ indicates an element of the enhanced chiral algebra that accounts for the specific identification.

From the perspective of the full momentum lattice $\Gamma^{(2,2)}$, the rationality condition is therefore equivalent to the statement that Γ_L (and similarly Γ_R) contains a rank-two sublattice of purely chiral states of finite index. Consequently, the full Hilbert space of the theory decomposes into a finite direct sum of tensor products of left and right chiral representations,

$$\mathcal{H} = \bigoplus_{\mu,\bar{\mu}\,\in\,\Gamma_L/\Gamma_0} M_{\mu\bar{\mu}}\,\mathcal{V}_\mu^L \otimes \mathcal{V}_{\bar{\mu}}^R, \tag{4.3.18}$$

implying that the toroidal sigma model has become an RCFT.

Finally, let us briefly discuss the special subclass of diagonal RCFTs within the $c = 2$

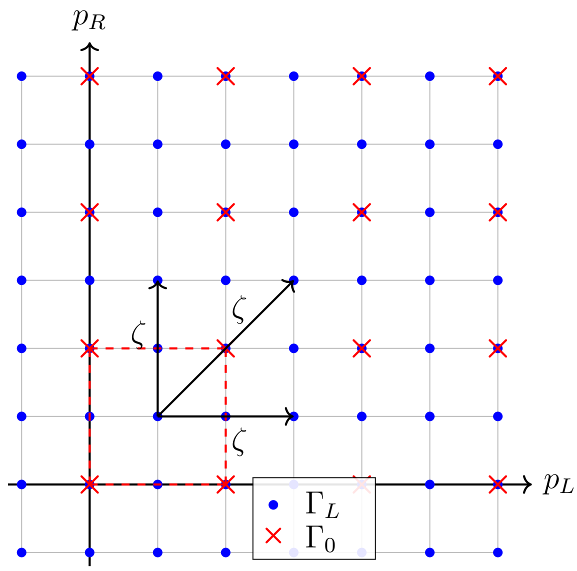


Figure 13: Example of the momentum lattice projections Γ_L (blue points) and its purely left-moving sublattice Γ_0 (red crosses) for the rational CFT point $(\tau, \rho) = (i, 2i)$. In this case Γ_0 is a rank-two sublattice of finite index inside Γ_L; as we said, the index of $\Gamma_0 \subset \Gamma_L$ is associated the number of primary operators, which, in the current example, are is $[\Gamma_L : \Gamma_0] = 4$. The dashed red square highlights one choice of fundamental domain for the quotient Γ_L/Γ_0. The black arrows represent the action of generators of the enhanced chiral algebra on Γ_L, showing how different lattice points are identified into a finite number of orbits. This finiteness is precisely what ensures that only a finite number of left-moving primaries appear at this rational point of the moduli space.

compact boson. By definition, a rational conformal field theory is diagonal when the left and right-moving chiral algebras coincide and every left-moving representation is paired with the corresponding right-moving counterpart. The multiplicity matrix then reduces to

$$M_{\mu\bar{\mu}} = \delta_{\mu\bar{\mu}}, \tag{4.3.19}$$

and the full torus partition function factorizes as

$$Z_{\text{diag}}(\tau, \rho) = \sum_{\mu \in \Gamma_L/\Gamma_0} \left|\chi_\mu(\sigma)\right|^2, \tag{4.3.20}$$

where χ_μ are the characters of the irreducible representations of the extended chiral algebra. In particular, in the $c = 2$ compact boson, the diagonal RCFT points in the moduli space correspond to the toroidal compactifications characterized by

$$(\tau, \rho) = (\tau, fa\tau), \quad \tau = \frac{-b + \sqrt{D}}{2a}, \tag{4.3.21}$$

where $f \in \mathbb{N}$ is a positive integer. These are characterized by the value of the complexified Kähler structure, which is proportional to the complex structure; in particular, the proportionality constant is an integer.

Diagonal RCFTs essentially implies the identification of left and right momentum sublattices, namely Γ_L and Γ_R. In particular, this implies that the full momentum lattice $\Gamma^{2,2}$ can then be written as

$$\Gamma^{2,2} = (\Gamma_L, \Gamma'_L), \tag{4.3.22}$$

with the relation

$$\Gamma'_L - \Gamma_L = \Gamma_0. \tag{4.3.23}$$

That is to say, the left and right-moving sublattices can differ at most by elements of the chiral sublattice Γ_0, which implies that the left and right-moving representations must be identified one-to-one.

4.4 The emergence of non-invertible symmetries

In the previous section, we introduced the notion of rational conformal field theory for the $c = 2$ compact boson, showing that the theory becomes rational precisely when the complex structure τ and the complexified Kähler modulus ρ both lie in the same imaginary quadratic number field $\mathbb{Q}(\sqrt{D})$ with $D < 0$. As discussed, this algebraic condition on the complex parameters admits a natural geometric interpretation in terms of the momentum lattice and the structure of its left and right-moving projections.

Having presented the rationality condition for the $c = 2$ model, we are now able to construct the duality defects that arise in this class of non-linear sigma models. We will begin by reviewing the role of gauging and the notion of self-duality, and then proceed to the explicit construction of the non-invertible symmetries that appear at these points. We will then continue with a concrete example of such a duality symmetry and will provide a geometric interpretation of its action on the torus. In particular, we will be able to connect the geometric symmetries of the target space torus with the emergence of duality symmetries on the worldsheet, leading to a lattice-based construction of such duality defects.

4.4.1 Gauging and self-duality in the $c = 2$ compact boson

As we saw in Section 3.3, the construction of duality symmetries is based on the gauging of a finite subgroup of the chiral algebra and the notion of self-duality. In this section, we will present these two concepts in the $c = 2$ case.

Let us start by recalling that, at a generic point (τ, ρ) on the toroidal branch, the $c = 2$ compact boson enjoys the chiral global symmetry

$$U(1)^2_{\boldsymbol{n}} \times U(1)^2_{\boldsymbol{w}} = U(1)_{n_1} \times U(1)_{n_2} \times U(1)_{w_1} \times U(1)_{w_2}, \tag{4.4.1}$$

whose conserved charges are the momentum and winding numbers $\boldsymbol{n} = (n_1, n_2)^T$ and $\boldsymbol{w} = (w_1, w_2)^T$, respectively. These charges label the primary operators, which take the form

$$V_{\boldsymbol{n},\boldsymbol{w}} = \exp\big(i\,\boldsymbol{n}^T\boldsymbol{\phi} + i\,\boldsymbol{w}^T\tilde{\boldsymbol{\phi}}\big) = \exp\big(i\,\boldsymbol{p}_L^T\boldsymbol{X} + i\,\boldsymbol{p}_R^T\bar{\boldsymbol{X}}\big). \tag{4.4.2}$$

The left and right-moving momenta $(\boldsymbol{p}_L, \boldsymbol{p}_R)$ are related to $(\boldsymbol{n}, \boldsymbol{w})$ by

$$\begin{pmatrix}\boldsymbol{p}_L\\ \boldsymbol{p}_R\end{pmatrix} = L(\tau,\rho)\begin{pmatrix}\boldsymbol{n}\\ \boldsymbol{w}\end{pmatrix}, \qquad \boldsymbol{n},\boldsymbol{w} \in \mathbb{Z}^2, \tag{4.4.3}$$

where $L(\tau, \rho)$ is defined in (4.1.7), and they take values in the even self-dual momentum lattice $\Gamma^{(2,2)}$.

At this point, we can consider the gauging of a finite subgroup of the chiral symmetry. As in the $c = 1$ compact boson case, we will consider the gauging of non-anomalous, diagonal subgroups of the chiral algebra of the form:

$$\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{M_1} \times \mathbb{Z}_{M_2} \subset U(1)^2_{\boldsymbol{n}} \times U(1)^2_{\boldsymbol{w}}, \tag{4.4.4}$$

where $\mathbb{Z}_{N_i} \subset U(1)_{n_i},\;\; \mathbb{Z}_{M_i} \subset U(1)_{w_i}$. Similarly to what we obtained in Section 3.3.1, in which the gauging of the chiral subgroup $\mathbb{Z}_N \times \mathbb{Z}_M \subset U(1)_n \times U(1)_w$ led to the mapping of the original target space radius R to $R' = \frac{M}{N}R$, also in this case the action of the gauging will act non trivially on the parameters of the theory. In particular, the overall action of (4.4.4) on τ and ρ is given by the maps

$$\tau \mapsto \tau' = \frac{N_1 M_2}{N_2 M_1}\tau, \quad \rho \mapsto \rho' = \frac{M_1 M_2}{N_1 N_2}\rho. \tag{4.4.5}$$

These can be derived analogously to the $c = 1$ case by analyzing the action of gauging on the fields $\phi^i, \tilde{\phi}^i$; we refer to [67] for the explicit derivation. Equivalently, we can describe the action of the gauging (4.4.4) on the momentum and winding numbers. Indeed, as in the $c = 1$ case, the effect of gauging a finite subgroup of the chiral algebra can be encoded in a linear action on the charge vector $\begin{pmatrix}\boldsymbol{n}\\ \boldsymbol{w}\end{pmatrix}$. Following the general formalism of [70], any discrete gauging of a subgroup of $U(1)^2_{\boldsymbol{n}} \times U(1)^2_{\boldsymbol{w}}$ is represented by a rational matrix

$$\sigma \;\in\; O(2,2;\mathbb{Q}), \tag{4.4.6}$$

acting on the charges as

$$\begin{pmatrix}\boldsymbol{n}\\ \boldsymbol{w}\end{pmatrix} \;\longmapsto\; \begin{pmatrix}\boldsymbol{n}'\\ \boldsymbol{w}'\end{pmatrix} = \sigma\begin{pmatrix}\boldsymbol{n}\\ \boldsymbol{w}\end{pmatrix}. \tag{4.4.7}$$

The appearance of a rational matrix, rather than an integral one, is expected after the $c = 1$ analysis; indeed, the gauging procedure produces fractional momentum and winding numbers corresponding to the twisted sectors of the orbifold theory, which are required for modular invariance. The matrix σ corresponding to the diagonal gauging (4.4.4), then, is given by

$$\sigma = \begin{pmatrix} \frac{N_1}{M_1} & 0 & 0 & 0 \\ 0 & \frac{N_2}{M_2} & 0 & 0 \\ 0 & 0 & \frac{M_1}{N_1} & 0 \\ 0 & 0 & 0 & \frac{M_2}{N_2} \end{pmatrix}. \tag{4.4.8}$$

As we saw in the $c = 1$ compact boson, in order to construct a duality symmetry, the gauging of the chiral algebra subgroup must be combined with T-duality. Indeed, the overall action on the radius $R \mapsto \frac{M}{N}R$, for specific values of M, N and R, can be undone by performing a T-duality transformation that maps the compactification radius to its original value. In the same way, the effect of the gauging just described in the $c = 2$ case is, in general, to map the original theory, defined by (τ, ρ), to a different toroidal conformal field theory, that is to say, to a different point in the toroidal branch (τ', ρ'). Now, for a generic choice of the chiral algebra subgroup and values of the moduli parameters, the orbifold theory obtained after applying σ is not related to the original one by any duality transformation. However, the gauging of specific finite subgroups of the chiral algebra, performed at special points on the toroidal branch, leads to a set of target space parameters that can be mapped back to the original theory by means of a duality transformation.

A duality symmetry is then obtained precisely in this situation, that is to say, by combining the gauging with an element of the duality group,

$$T \in O(2,2;\mathbb{Z}), \tag{4.4.9}$$

and by requiring that the composite transformation

$$D \;=\; \sigma \circ T \tag{4.4.10}$$

maps the theory back to its original moduli parameters. However, unlike the $c = 1$ case, in which the moduli space contains only T-duality that acts simply as $R \leftrightarrow 1/R$, the $c = 2$ compact boson duality group $O(2,2;\mathbb{Z})$ is significantly richer in structure. This implies that finding a duality transformation T that restores the original parameters after gauging is a much subtler problem. As a consequence, explicitly constructing duality symmetries in the $c = 2$ case requires solving more involved algebraic constraints and is only possible at special loci in moduli space.

The requirement that $D \in O(2,2,\mathbb{Q})$ defines a duality symmetry of the original theory can be expressed compactly in terms of the generalized metric $\mathcal{E}$ introduced in (4.1.19). Indeed, recall that $\mathcal{E}$ encodes both the target-space metric and B-field, and transforms as

$$\mathcal{E} \;\longmapsto\; O^T \, \mathcal{E} \, O \qquad \text{for} \quad O \in O(2,2;\mathbb{Z}). \tag{4.4.11}$$

This implies that the theory is left invariant by D if and only if the generalized metric is preserved under conjugation:

$$D^T \mathcal{E} D = \mathcal{E}. \tag{4.4.12}$$

This is the defining condition for a matrix $D \in O(2, 2, \mathbb{Q})$ to represent a duality symmetry of the theory. Indeed, similarly to what we had in the $c = 1$ case, the requirement for D to take value in $O(2, 2, \mathbb{Q})$ guarantees the invariance of the spin of primary operators; instead, the invariance of the generalized metric under conjugation by this matrix guarantees that the conformal dimension of the primaries is preserved too.

From the worldsheet perspective, the operator D has a natural interpretation as the matrix representation of a topological duality defect $\mathcal{D}$. Such a defect implements the action of D on physical states, mapping local operators to genuine and non-genuine ones. In particular, the fact that D belongs to $O(2, 2; \mathbb{Q})$ but is not, in general, an element of $O(2, 2; \mathbb{Z})$ reflects the non-invertibility of $\mathcal{D}$ and, thus, that it does not admit a proper inverse. This should be regarded as the direct generalization of the $c = 1$ compact boson analysis discussed in Section 3.3.3. There, duality symmetries were encoded by matrices $D \in O(1, 1; \mathbb{Q})$ acting on the charge lattice $\Gamma^{1,1}$, or equivalently by left multiplication of the lattice generator L by a matrix R; see equations (3.3.16) and (3.3.17). The same matrices D were identified as the matrix representations of the non-invertible duality defect $\mathcal{D}$. In the present $c = 2$ setting, the matrix $D \in O(2, 2; \mathbb{Q})$ plays exactly the same role, now acting on the higher-rank lattice $\Gamma^{2,2}$ and implementing the duality defect $\mathcal{D}$ that, similarly to the $c = 1$ compact boson, is described by a Tambara-Yamagami category, as defined in Section 3.3.2.

Moreover, using the decomposition of the generalized metric in terms of the lattice generating matrix, $\mathcal{E} = L^T L$, we can equivalently regard the duality symmetry as acting directly on L. Under the action of D, the lattice generating matrix L transforms as

$$D : \quad L \longmapsto L' = LD. \tag{4.4.13}$$

In other words, the duality symmetry maps the original lattice generated by L to a new lattice generated by L', and the condition

$$(L')^T L' = L^T L = \mathcal{E} \tag{4.4.14}$$

ensures that both lattices encode the same generalized metric and, therefore, describe the same target-space torus.

Finally, in close analogy to what we found in (3.3.17) for the $c = 1$ case, the action of the duality symmetry on the lattice generator can be rewritten in terms of a left–multiplying matrix $R \in O(2,2)$[6], so that

$$D : \quad L \longmapsto L' = LD = RL. \tag{4.4.16}$$

Thus, as in the $c = 1$ compact boson, the duality symmetry can be viewed either as a right action on the charge lattice, via D, or as a left action on the lattice generator, via R. However, we have an important difference with the $c = 1$ compact boson. Indeed, in the one-dimensional case, we were able to explicitly compute the left–multiplying matrix R appearing in (3.3.17), and this matrix was universal, in the sense that it applied to any duality defect arising from the composition of gauging and T-duality. By contrast, in the $c = 2$ case, the structure of $O(2,2;\mathbb{Z})$ is sufficiently rich that the corresponding matrix R is no longer universal; in this case, it depends on the specific gauging and duality transformation required to construct the duality symmetry. In particular, the identification of R requires solving the condition $LD = RL$ for each individual duality defect, and the solution may vary from case to case. This emphasizes once more that the construction of duality symmetries in the $c = 2$ compact boson is much more intricate than in the $c = 1$ case, reflecting both the enlarged duality group and the richer structure of the underlying momentum lattice. In the next section, we will present a few example of duality symmetries and we will be able to compute the left multiplying matrix R explicitly. In particular, we will examine how these matrices are related to geometric symmetries of the target-space torus, and how this connection can be exploited to construct new duality symmetries in the $c = 2$ compact boson.

4.4.2 Examples of duality symmetries in the $c = 2$ compact boson

In the previous section, we presented the general construction of duality symmetries in the $c = 2$ compact boson, highlighting their emergence as combination of discrete gauging and elements of the duality group $O(2,2;\mathbb{Z})$. We now turn to explicit examples of such symmetries. In what follows, we will focus on two concrete cases, in order to present different peculiarities of such symmetries. For each example, we will compute the corresponding duality matrix D and determine the associated left–multiplying matrix R, making manifest the action of the duality symmetry on the momentum lattice.

[6] Both matrices D and R belong to $O(2,2)$ in general. However, they are two different representations of the same group: D preserves the off–diagonal metric

$$\eta_{\text{off}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{4.4.15}$$

while R preserves the diagonal form $\eta_{\text{diag}} = \text{diag}(1,1,-1,-1)$. These two metrics are related by a change of basis, so that D and R describe the same $O(2,2)$ transformation in different frames.

The orthogonal torus case: $(\tau, \rho) = (i, 3i)$

In this first example, we focus on the toroidal compactification characterized by moduli parameters $(\tau, \rho) = (i, 3i)$. Using the definitions of τ and ρ in terms of the metric and the antisymmetric tensor B given in (4.1.4), these take the form:

$$G = 9 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.4.17}$$

This corresponds to an orthogonal torus, obtained as the product of two circles, $T^2 = S^1_R \times S^1_{R'}$, of radii $R = 1$ and $R' = \sqrt{3}$, respectively. Since the metric is diagonal and the B-field vanishes, the compactification factorizes into two decoupled $c = 1$ compact bosons. This simple structure makes it a convenient starting point for studying duality symmetries before moving to more general non-factorized compactifications. From the explicit form of the lattice generating matrix L given in (4.1.7), we get

$$L = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & -3 \end{pmatrix}, \tag{4.4.18}$$

while the generalized metric is given by

$$\mathcal{E} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}. \tag{4.4.19}$$

Note that the structure of the lattice generating matrix L simplifies significantly in this orthogonal case. In particular, if we restrict ourselves to the sublattice generated by the first and third columns of L, spanned by the charges n_1 and w_1, we recover precisely the charge lattice of a single compact boson at radius $R = \sqrt{3}$, as discussed in the example of Section 3.3.4. This property corresponds to the fact that the theory factorizes into two decoupled $c = 1$ compact bosons. This identifies the first circle $S^1_{R=1}$ with a trivial factor, while the second circle $S^1_{R'=\sqrt{3}}$ realizes a non-trivial radius where non-invertible duality symmetries can arise. This observation implies that a possible duality symmetry of the full $(2, 2)$-dimensional lattice can be built by appropriately embedding the duality symmetry of the $c = 1$ compact boson at radius $R = \sqrt{3}$ into the higher-rank lattice. In other words, the factorization of the torus allows us to exploit the structure of duality defects previously identified in the $c = 1$ case and extend them to the full $c = 2$ setting.

As we saw in Section 3.3.4, the duality symmetry in the $c = 1$ case was constructed by taking the combination of gauging the $\mathbb{Z}_3 \subset U(1)_n$ subgroup of the chiral symmetry and T-duality. Let us now translate this construction into the $c = 2$ compact boson formalism. As we said, the $\Gamma^{2,2}$ sublattice generated by $(n_1, w_1)^T$ is equivalent to the momentum lattice $\Gamma^{1,1}$ of the $R = \sqrt{3}$ compact boson; this implies that the required gauging in this case is given by $\mathbb{Z}_3 \subset U(1)_{n_1}$, which we denote by $\sigma_{3,1}$. The T-duality transformation, instead, is implemented by mirror symmetry, which acts by exchanging the Kähler and complex structure moduli:

$$(\tau, \rho) \;\longleftrightarrow\; (\rho, \tau); \tag{4.4.20}$$

in matrix form, this is given by M presented in (4.1.21). Indeed, when restricted to the decoupled factors of the torus, this is equivalent to the familiar $R \leftrightarrow 1/R$ transformation of the $c = 1$ compact boson. At this point, combining the gauging $\sigma_{3,1}$ with mirror symmetry allows us to construct a duality symmetry of the full $c = 2$ compact boson at $(\tau, \rho) = (i, 3i)$. Explicitly, this duality symmetry is given by

$$D_1 \;=\; \sigma_{3,1} \circ M, \tag{4.4.21}$$

acting on the charge lattice via right-multiplication and preserving the generalized metric according to the constraint (4.4.12). Indeed, from (4.4.8) and (4.1.21), we can easily obtain the matrix form of the duality defect, which is given by

$$D_1 = \begin{pmatrix} 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.4.22}$$

and it is possible to explicitly check that $\mathcal{E} \longmapsto D_1^T \mathcal{E} D_1 = \mathcal{E}$.

As we pointed out above, from the decomposition of the generalized metric in terms of the lattice generating matrix L, we can compute the action of the duality symmetry on the momentum lattice $\Gamma^{2,2}$. This is given by:

$$L \longmapsto L_1 = LD_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \\ -1 & 0 & 3 & 0 \\ 0 & 1 & 0 & -3 \end{pmatrix}. \tag{4.4.23}$$

As before, let us focus on the sublattice generated by the first and third columns of L and spanned by $(n_1, w_1)^T$, that is to say, the $c = 1$ factor characterized by $R = \sqrt{3}$. In this setting, the transformation introduced by D_1 has the overall action of flipping the sign of the third row, which is precisely the same mapping observed in the $c = 1$ compact boson example at radius $R = \sqrt{3}$. In fact, if we project onto this sublattice and ignore the trivial $S^1_{R=1}$ factor, the lattice transformation is given by the same matrix D_1 as in (3.3.24). In particular, if we restrict ourselves to the 2-dimensional sublattice, the transformation $L \mapsto L_1$ can be equivalently represented by Fig. 11, where the blue and red lattices correspond to L and L_1, respectively. From the explicit expression (4.1.11), we clearly see that the overall action of the duality symmetry is to flip the sign of the right-moving momentum $p_{R,1}$, associated with the $S^1_{R=\sqrt{3}}$ factor. More precisely, as in the $c = 1$ case, we can rephrase the transformation of the lattice generating matrix in terms of a left-multiplying matrix $R_1 \in O(2,2)$, satisfying $L_1 = R_1 L$. For this orthogonal torus example, we easily find

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{4.4.24}$$

As we highlighted above, due to the higher-dimensionality of the target space, this explicit form of the left-multiplying matrix R_1 is not universal but rather depends on the specific duality symmetry we consider. However, as we will see in the next section, the form of these matrices is not completely arbitrary but is dictated by the geometric symmetries of the target-space torus.

While the above duality symmetry arises naturally from the factorization of the target space into two circles, it is important to emphasize that the orthogonal torus admits additional duality symmetries that cannot be simply reduced to the composition of duality symmetries of the individual S^1 factors. Indeed, even though $T^2 = S^1_R \times S^1_{R'}$ is geometrically a product of two circles, the full duality group $O(2,2;\mathbb{Z})$ is strictly larger than the product of the $c = 1$ duality groups associated with each circle. In particular, the circle with $R = 1$ does not admit a non-invertible duality symmetry at all, and even for the circle with $R' = \sqrt{3}$, we found a single duality symmetry, while in the $c = 2$ compact boson case we can have multiple ones at a single point in the moduli space.

More intricate duality symmetries can arise from the interplay of both momentum and winding modes in the two directions, and generically involve nontrivial mixing between the two $c = 1$ sectors. These symmetries are connected to the structure of the $O(2,2;\mathbb{Z})$ duality group and cannot be decomposed into separate duality transformations on each circle. In this orthogonal torus example, one can explicitly compute an additional duality symmetry of this type. However, rather than presenting its full construction, we focus on its implications for the structure of the lattice generating matrix. For the complete derivation, we refer the reader to [70], where the full duality group of the orthogonal torus is extracted from the graded fusion category associated with the model. The duality symmetry we want to consider is given by the composition

$$D_2 = \tilde{\sigma}_{3,1}\sigma_{3,1}S_\rho S_\tau, \tag{4.4.25}$$

where $\sigma_{3,1}$ is the $\mathbb{Z}_3 \subset U(1)_{n_1}$ gauging considered above, $\tilde{\sigma}_{3,1}$ is, equivalently, the $\mathbb{Z}_3 \subset U(1)_{n_2}$ gauging, while S_τ and S_ρ are the S-duality transformations presented in (3.1.27), which act as $\tau \mapsto -\frac{1}{\tau}$ and $\rho \mapsto -\frac{1}{\rho}$ respectively. In matrix form, this duality symmetry is given by

$$D_2 = \begin{pmatrix} 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \\ \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \end{pmatrix}, \tag{4.4.26}$$

which acts on the lattice generating matrix L as

$$L \longmapsto L_2 = LD_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \\ -1 & 0 & 3 & 0 \\ 0 & -1 & 0 & 3 \end{pmatrix}. \tag{4.4.27}$$

From the explicit form of D_2 and its action on L, we clearly see that this duality symmetry acts non-trivially on both $c = 1$ factors of the decomposition, mixing left and right-moving momenta in both orthogonal directions. In particular, when we write the associated left-multiplying matrix R_2, defined by $L_2 = R_2 L$, we find

$$R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4.4.28}$$

Unlike the matrix R_1 obtained above, which flipped only the third row corresponding to the right-moving momentum $p_{R,1}$ of the $R = \sqrt{3}$ circle, the duality symmetry described by R_2 flips the signs of both the third and fourth rows. Physically, this corresponds to a simultaneous sign reversal of both right-moving momenta, $p_{R,1}$ and $p_{R,2}$, associated with the two orthogonal directions. This duality symmetry, D_2, involves both chiralities and both directions at once and cannot be decomposed into independent dualities acting on each circle. This demonstrates in a concrete way the more subtle interplay of $O(2, 2; \mathbb{Z})$ transformations in two-dimensions. Such dualities reflect deeper geometric symmetries of the target torus, going beyond the simple composition of the $c = 1$ dualities.

The hexagonal torus case: $(\tau, \rho) = (e^{2\pi i/3}, -1/2 + i3\sqrt{3}/2)$

Let us now consider a more intricate example of a toroidal compactification, characterized by moduli parameters corresponding to the so-called hexagonal torus:

$$(\tau, \rho) = \left(e^{\frac{2\pi i}{3}}, -\frac{1}{2} + i\frac{3\sqrt{3}}{2} \right), \tag{4.4.29}$$

where $\rho = 3\tau$ up to a duality transformation. We can explicitly compute the lattice generating matrix L, which takes the form

$$L = \begin{pmatrix} \frac{\sqrt{6}}{6} & 0 & \frac{\sqrt{6}}{2} & -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{2}}{6} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{6} & \frac{2\sqrt{2}}{3} \\ \frac{\sqrt{6}}{6} & 0 & -\frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{2}}{6} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{6} & -\frac{5\sqrt{2}}{6} \end{pmatrix} . \tag{4.4.30}$$

In contrast to the orthogonal case discussed previously, this toroidal CFT does not factorize into the product of two decoupled $c = 1$ compact bosons. Instead, the geometry exhibits a non-trivial coupling between the two directions, which makes the analysis of duality symmetries significantly more subtle. This structure is typical for generic points in the moduli space of the $c = 2$ compact boson, where such non-factorized tori exhibit more intricate duality symmetries. Because of this coupling, the lattice does not decompose into simpler sublattices associated with independent $c = 1$ sectors, and duality symmetries cannot be straightforwardly inherited from the $c = 1$ case. Non-invertible duality defects in such setups often rely on a more intricate interplay between momentum and winding modes in both directions, and their construction generally requires solving (4.4.12) explicitly.

In what follows, we will present a specific example of a duality symmetry for the hexagonal torus, and we will explicitly compute its action on the momentum lattice $\Gamma^{2,2}$. The explicit derivation of such a non-invertible symmetry can be found in [70]. The duality defect is represented by the matrix

$$D = \begin{pmatrix} 1 & \frac{2}{3} & \frac{4}{3} & -2 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix} , \tag{4.4.31}$$

which belongs to $O(2, 2; \mathbb{Q})$ and satisfies the duality condition (4.4.12), consequently defining a non-invertible symmetry of the theory. Unlike the orthogonal torus example, however, the nontrivial structure of the lattice generating matrix L and the duality symmetry itself make it difficult to explicitly identify the decomposition of the latter in terms of gauging and duality transformations. Nevertheless, a general result proved in [81] ensures that any matrix $D \in O(2, 2; \mathbb{Q})$ satisfying (4.4.12) can always be expressed as a finite combination of gaugings of finite subgroups of the chiral symmetry and duality transformations of $O(2, 2; \mathbb{Z})$. This guarantees that D indeed corresponds to a legitimate duality defect, even when its explicit decomposition, such as the one in (4.4.31), is not immediately derivable.

As in the previous example, we can also consider the explicit action of the duality symmetry on the lattice generating matrix. This is encoded in the transformation

$$L' = LD = \begin{pmatrix} \frac{\sqrt{6}}{6} & 0 & \frac{\sqrt{6}}{2} & -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{2}}{6} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{6} & \frac{2\sqrt{2}}{3} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{2}}{6} & -\frac{\sqrt{2}}{6} & -\frac{5\sqrt{2}}{6} & \frac{2\sqrt{2}}{3} \end{pmatrix}. \tag{4.4.32}$$

Once again, we can reinterpret the action of the duality symmetry in terms of a left multiplying matrix $R \in O(2,2)$ such that $L' = R\,L$. For this specific duality symmetry, this matrix takes the form

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\left(\frac{\pi}{3}\right) & \sin\left(\frac{\pi}{3}\right) \\ 0 & 0 & \sin\left(\frac{\pi}{3}\right) & -\cos\left(\frac{\pi}{3}\right) \end{pmatrix}, \tag{4.4.33}$$

which admits the decomposition

$$R = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ 0 & 0 & \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{pmatrix}}_{\text{rotation by } \pi/3} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}_{\text{sign flip of } p_{R,2}}. \tag{4.4.34}$$

This expression highlights a crucial distinction from the orthogonal torus case, that is to say, the left–multiplying matrix is no longer a simple sign flip. Instead, R is the combination of a flip of the right-moving momentum component $p_{R,2}$, followed by a genuine geometric rotation by an angle $\pi/3$ in the right-moving momentum plane. This reflects the enhanced symmetry of the hexagonal torus and the fact that, in non-factorized compactifications, duality symmetries act non-trivially by mixing momentum and winding data across both directions.

As in the orthogonal case, the hexagonal torus admits additional duality symmetries besides the one presented here. These further non-invertible defects arise from other elaborate combinations of gauging operations and $O(2,2;\mathbb{Z})$ duality transformations and typically involve similar intricate mixing between momentum and winding modes in both directions. However, we will not discuss them in detail here. Instead, we refer the reader to [70], where a subset of the duality symmetries associated with this rational point in moduli space is derived from the underlying graded fusion category.

In the next section, we will examine more closely the structure and interpretation of the left–multiplying matrices R associated with these duality symmetries. In particular, we will explore how their form encodes geometric symmetries of the target-space torus and how this perspective can be used as a guiding principle for constructing new duality defects in more general compactifications.

4.4.3 The interplay between duality symmetries and lattice symmetries

As we already emphasized above, in contrast to the $c = 1$ compact boson, the $c = 2$ theories do not admit a universal form for the left–multiplying matrix R. Instead, the explicit form of R depends explicitly on the particular duality symmetry under consideration and on the geometry of the target-space torus. Now that we have worked out a few concrete examples, we are able to analyze these matrices more systematically and provide a clearer geometric interpretation of their action.

In Section 4.2, we reviewed the structure of the moduli space of the $c = 2$ compact boson, emphasizing that, in addition to the toroidal branch, it contains a finite set of additional components. These are the so–called orbifold branches, which arise from orbifolding the theory by discrete geometric symmetries of the two–dimensional lattice underlying the target–space torus. The classification of such symmetries is highly constrained; indeed, in two dimensions, there exist only seventeen inequivalent crystallographic symmetry groups. Each of these may admit one or more consistent orbifolding procedures, and they give rise to the twenty–eight distinct branches that compose the full moduli space of the $c = 2$ compact boson.

In particular, the discrete symmetries of a two–dimensional torus depend on the values of its moduli. While certain symmetries are present for every choice of (τ, ρ), such as the $\mathbb{Z}_2$ inversion symmetry, which flips the signs of both torus coordinates in the $\mathbb{R}^2$ plane as

$$\mathbb{Z}_2 : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \forall\, (\tau, \rho) \in \mathcal{T}^2, \tag{4.4.35}$$

that is to say, acting as a rotation by π, other discrete symmetries appear only at special, highly symmetric points of the moduli space. For example, the $\mathbb{Z}_4$ rotation is a symmetry of the underlying lattice only in the square torus case, which is characterized by the complex structure value $\tau = i$, and acts on $(x_1, x_2) \in \mathbb{R}^2$ as a $\pi/2$ rotation of the form

$$\mathbb{Z}_4 : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad \text{for } \tau = i. \tag{4.4.36}$$

Similarly, the $\mathbb{Z}_3$ symmetry appears only for the hexagonal torus, characterized by $\tau = e^{2\pi i/3}$; in the same way, this acts as a $\frac{2\pi}{3}$ rotation of the coordinates

$$\mathbb{Z}_3 : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \text{for } \tau = e^{2\pi i/3}. \tag{4.4.37}$$

The appearance of these discrete geometric symmetries is not only relevant for the classification of orbifold branches of the moduli space, but it also provides a geometric interpretation of the analysis of duality symmetries carried out in the previous section. There, we reinterpreted the duality symmetries in terms of left–multiplying matrices

$R \in O(2,2)$, which encode the action on the momentum lattice. Unlike the $c = 1$ case, where the left-multiplying matrix was universal and always reduced to a sign flip of the right–moving momentum, the $c = 2$ compact boson exhibits a richer structure. Indeed, depending on the specific duality symmetry we consider, the matrices R can represent simple sign reversals, as in the $c = 1$ case, but can also mix momenta in different directions implemented by geometric rotations, such as the $\pi/3$ rotation found in the hexagonal torus example.

These transformations, however, are not arbitrary. Indeed, from the previous examples, the structure of the left–multiplying matrices R appears to be strictly related to the same crystallographic symmetries of the underlying lattices of the target space tori. The key point emerging from the explicit examples presented above, and in most of the duality symmetries we analyzed while carrying out our research, is that the left–multiplying matrices R are not arbitrary elements of $O(2,2)$. In fact, they share a universal block structure of the form

$$R = \begin{pmatrix} \mathsf{Id}_2 & 0 \\ 0 & G \end{pmatrix}, \tag{4.4.38}$$

where Id_2 is the identity matrix acting on the left–moving momenta $(p_{L,1}, p_{L,2})$, while the lower block G acts on the right–moving momenta $(p_{R,1}, p_{R,2})$. In particular, the crucial point here is that the matrix G is not an arbitrary element of $GL(2,\mathbb{R})$. Rather, in most of the examples we studied[7], G is precisely a generator of one of the crystallographic symmetry groups of the underlying torus lattice, which, in addition, implies $G \in O(2)$. To understand the connection between duality symmetries and the geometric symmetries of the underlying lattice, let us re-examine the previous examples more carefully.

In the first example we considered above, we discussed the orthogonal torus $(\tau, \rho) = (i, 3i)$. For this RCFT, we presented two of the possible duality symmetries. For simplicity, let us start from the duality defect D_2 defined in (4.4.26). In this case, the left–multiplying matrix was found to be

$$R_2 = \begin{pmatrix} \mathsf{Id}_2 & 0 \\ 0 & -\mathsf{Id}_2 \end{pmatrix}, \tag{4.4.39}$$

[7]We would like to emphasize that, in this thesis, we have presented only a few examples of duality symmetries, selected for their clarity and specific properties. However, during our research, we analyzed a substantially larger set of duality symmetries arising at various rational points of the moduli space. In most of these cases, the corresponding left–multiplying matrices exhibited precisely the same block structure discussed here, with the lower block G being a crystallographic symmetry of the underlying torus lattice.

However, this pattern is not universal; there exist specific constructions in which the overall action of the duality symmetry maps the original theory to a different point on the toroidal branch, characterized by the same generalized metric. These more subtle examples will not be discussed in this thesis, but they will be analyzed in more detail in future works.

so that the lower block is given by $G = -\mathrm{Id}_2$, corresponding to a rotation of angle π in the right–moving momentum plane. As discussed above, this transformation generates the universal $\mathbb{Z}_2$ crystallographic symmetry of any two–dimensional lattice. In this example, the duality symmetry D_2 is therefore geometrically associated with the $\mathbb{Z}_2$ inversion symmetry of the target torus.

Let us now consider the second duality symmetry, D_1. In this case, the left–multiplying matrix was shown to be

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.4.40}$$

so that the lower block $G = \mathrm{diag}(-1, 1)$ acts as a reflection of one of the right–moving momentum components. This corresponds to a generator of the $\mathbb{Z}_4$ symmetry, which is present precisely for the subset of theories satisfying $\tau = i$. This $\mathbb{Z}_4$ manifests itself in the duality symmetry D_1.

We clearly see that, even within a single RCFT such as the orthogonal torus, different duality symmetries are associated with different crystallographic symmetries and, therefore, different blocks G. The universal $\mathbb{Z}_2$ inversion symmetry appears through D_2, while the enhanced $\mathbb{Z}_4$ symmetry specific to $\tau = i$ manifests itself in D_1. This already illustrates the general principle stated above; that is to say, the form of the left–multiplying matrices R is determined by the crystallographic symmetry groups of the underlying torus lattice, and different generators of these groups correspond to different non-invertible duality defects.

Let us now turn to the second example discussed above, namely the hexagonal torus with $(\tau, \rho) = \left(e^{2\pi i/3}, -\frac{1}{2} + i\frac{3\sqrt{3}}{2}\right)$. In this case, we presented a duality symmetry whose left–multiplying matrix takes the form

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\frac{\pi}{3}) & \sin(\frac{\pi}{3}) \\ 0 & 0 & \sin(\frac{\pi}{3}) & -\cos(\frac{\pi}{3}) \end{pmatrix}, \tag{4.4.41}$$

which in (4.4.34) we interpreted as the composition of a sign flip of $p_{R,2}$ followed by a genuine rotation of angle $\pi/3$ in the right–moving momentum plane. Then, in this case, the matrix G clearly takes the form

$$G = \underbrace{\begin{pmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{pmatrix}}_{\text{rotation by } \pi/3} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\text{sign flip of } p_{R,2}}. \tag{4.4.42}$$

and, again, this represents a discrete symmetry of the corresponding lattice underlying the target-space torus.

Indeed, as reviewed in Section 4.2, the hexagonal torus admits the non trivial rotational symmetries $\mathbb{Z}_3$ and $\mathbb{Z}_6$. Moreover, since the real part of τ is $\tau_1 = \text{Re}(\exp(\frac{2\pi i}{3})) = -\frac{1}{2}$, the hexagonal lattice also exhibits the reflection symmetries R_1 and R_2. As shown in [74], the rotation group $\mathbb{Z}_6$ along with the reflections R_1 and R_2, together generate the full dihedral group D_6; the matrix G above, then, corresponds precisely to a generator of a crystallographic symmetry group of the theory, in perfect agreement with the general block structure (4.4.38).

These three examples make clear that the matrices R encoding the left-action of duality symmetries on the momentum lattice are directly connected to the crystallographic symmetries of the target-space torus. This observation is particularly important because the crystallographic symmetries of two–dimensional lattices are completely classified, and only a finite number of them can occur. This implies that the lower block G appearing in the left–multiplying matrices R is not an arbitrary element of $O(2)$, but must belong to this finite list of admissible lattice symmetries. This reveals that the structure of the duality symmetries is ultimately controlled by the geometry of the target-space torus.

In the next section, we will take advantage of this connection to construct the duality defects. Instead of considering a duality matrix D and extracting the corresponding left–multiplying matrix R, we will reverse the logic: starting from the classified crystallographic symmetries of the torus lattice, we will show how to use them as a guiding principle to construct duality symmetries of the theory.

4.4.4 Towards a systematic construction of duality defects via lattice symmetries

In the previous section, we observed that duality symmetries and crystallographic symmetries of the torus lattice are deeply connected, and this relationship can be made explicit through the reformulation of the former in terms of left–multiplying matrices R acting on the lattice generating matrix L. However, the construction of duality symmetries is not straightforward. Indeed, given a $c = 2$ compact boson, determining a duality symmetry is a highly non-trivial procedure. Except for special situations, such as orthogonal lattices, where certain duality symmetries can be constructed directly by looking at the $c = 1$ sublattice and by embedding the corresponding duality symmetries in the full $c = 2$ theory, as we did around (4.4.21), the general problem requires solving the duality condition on the generalized metric presented in (4.4.12). However, this constraint equation is quadratic in the defect matrix D, and solving it explicitly is, in general, very difficult.

For this reason, a common strategy in the literature is to look for specific transformations that map a given point in the moduli space to another point describing a simpler toroidal compactification, where the duality symmetries are easier to identify. In general, the idea is to look for specific combinations of gaugings and duality transformations that map the original theory to an orthogonal point in the moduli space. Once the duality defects are constructed at that simpler point, one then maps them back to the original theory. However, even this procedure is highly non-trivial.

Following the observations of the previous section, we are now able to develop a more systematic and conceptually clear approach that exploits the connection with geometric lattice symmetries. The idea behind this method is that, instead of directly searching for solutions to (4.4.12), we will take advantage of the finite classification of crystallographic symmetries of two–dimensional lattices to construct the duality symmetries themselves.

Let us now present a systematic procedure to construct these duality symmetries. Consider a generic toroidal compactification on a two–torus T^2 defined in terms of a non-degenerate two dimensional lattice $\Lambda \subset \mathbb{R}^2$, where $T^2 = \mathbb{R}^2/\Lambda$, characterized by a choice of complex structure and complexified Kähler modulus (τ, ρ). Given the classification of two–dimensional crystallographic lattices presented in [74], we can immediately determine all possible discrete symmetries of Λ for that specific choice of modular parameters. In other words, the crystallographic symmetries associated with Λ are completely fixed once τ and ρ are given. In particular, as we saw above, these geometric symmetries admit a natural representation as $O(2)$ matrices acting on the coordinates of the torus as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto G \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad G \in O(2) \text{ a crystallographic symmetry of } \Lambda. \tag{4.4.43}$$

Now, as we reviewed in Section 4.1, once a toroidal compactification is specified by its moduli (τ, ρ), all the geometric and conformal data of the theory are fully determined. In particular, the choice of (τ, ρ) uniquely fixes the momentum lattice $\Gamma^{2,2}$ together with its lattice generating matrix, in such a way that the left and right-moving momentum can be defined as

$$\Gamma^{2,2} \ni \begin{pmatrix} \boldsymbol{p}_L \\ \boldsymbol{p}_R \end{pmatrix} = L(\tau, \rho) \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{w} \end{pmatrix}, \quad \boldsymbol{n}, \boldsymbol{w} \in \mathbb{Z}^2, \tag{4.4.44}$$

whose explicit form is completely dictated by the target-space torus metric and B-field. From $L(\tau, \rho)$, then, we can also construct the generalized metric defined in (4.1.19) as

$$\mathcal{E}(\tau, \rho) \; = \; L(\tau, \rho)^T \, L(\tau, \rho), \tag{4.4.45}$$

which encodes the kinetic terms and fully characterizes the CFT data of the $c = 2$ compact boson. Therefore, the entire structure of the model (that is to say, the momentum lattice, the spectrum of primary operators and the operator algebra) is fixed once the two complex parameters (τ, ρ) are chosen.

At this point, since the crystallographic symmetries of the target–space lattice are completely determined by the modular parameters (τ, ρ), we can now take any such symmetry generator

$$G \; \in \; O(2) \tag{4.4.46}$$

and use it as the starting point for constructing a duality symmetry of the full $c = 2$ compact boson. The key observation, motivated by the examples from the previous section, is that each crystallographic symmetry G can be extended to a matrix

$$R(G) \;=\; \begin{pmatrix} \mathrm{Id}_2 & 0 \\ 0 & G \end{pmatrix} \;\in\; O(2,2), \tag{4.4.47}$$

acting on the left– and right–moving momenta of $\Gamma^{2,2}$.

In particular, because $G \in O(2)$, it satisfies $G^T G = \mathrm{Id}_2$, and the full matrix $R(G)$ automatically obeys

$$R(G)^T R(G) = \mathrm{Id}_4, \tag{4.4.48}$$

and, more importantly,

$$R(G)^T \,\eta_{\text{diag}}\, R(G) = \eta_{\text{diag}}, \tag{4.4.49}$$

where $\eta_{\text{diag}} \;=\; \mathrm{diag}(1, 1, -1, -1)$ is the $O(2,2)$ metric in the diagonal form. Consequently, the transformation

$$L \;\longmapsto\; L' \;=\; R(G)\, L \tag{4.4.50}$$

preserves both the inner product on the momentum lattice $\Gamma^{2,2}$ and the generalized metric:

$$L^T \eta_{\text{diag}} L = \eta_{\text{off}} \;\longmapsto\; L'^T \eta_{\text{diag}} L' = L^T R(G)^T \eta_{\text{diag}} R(G) L = L^T \eta_{\text{diag}} L = \eta_{\text{off}}$$

$$\mathcal{E} \;\longmapsto\; \mathcal{E}' \;=\; L'^T L' \;=\; L^T R(G)^T R(G) L \;=\; L^T L \;=\; \mathcal{E}. \tag{4.4.51}$$

where η_{off} is the the off-diagonal representation of η defined in Footnote 6. In other words, every crystallographic symmetry G of the underlying torus automatically induces an $O(2,2)$ transformation $R(G)$ that preserves the full CFT data encoded in $\mathcal{E}$ and the momentum lattice $\Gamma^{2,2}$ inner product.

As we just saw, given a crystallographic symmetry $G \in O(2)$, this naturally induces an $O(2,2)$ transformation $R(G)$ that preserves both the inner product on $\Gamma^{2,2}$ and the generalized metric $\mathcal{E}$. We can now use this observation to construct the corresponding right–multiplying matrix D. Indeed, from the defining relation between the left and right–actions of duality symmetries presented in (4.4.16), we can invert that equation to obtain

$$D \;\equiv\; L^{-1} L' \;=\; L^{-1}\, R(G)\, L. \tag{4.4.52}$$

This expression immediately provides a natural candidate for the defect matrix acting on the lattice generating matrix L.

However, in general, the matrix D produced by this construction belongs to $O(2,2;\mathbb{R})$.

Instead, as emphasized in Section 4.4.1, a matrix defines a genuine duality symmetry of the $c = 2$ compact boson only if

$$D \;\in\; O(2,2,\mathbb{Q}), \tag{4.4.53}$$

in such a way that it can be interpreted as a finite combination of discrete gauging operations and elements of the duality group $O(2,2,\mathbb{Z})$. Thus, unless the conjugation in (4.4.52) yields a $O(2,2,\mathbb{Q})$ matrix, the transformation $R(G)$ does not lead to a duality defect of the CFT. In other words, the crystallographic symmetries of the target torus generate candidate $O(2,2)$ transformations; however, only in special cases, that is to say, when D happens to be rational, do they correspond to actual duality symmetries of the theory.

When the matrix obtained from (4.4.52) happens to lie in $O(2,2;\mathbb{Q})$, the situation, instead, is different. In this case, thanks to the general result proved in [81], we know that any rational $O(2,2)$ transformation satisfying the duality condition can always be decomposed into a finite sequence of discrete gaugings of finite subgroups of the chiral algebra together with elements of the duality group $O(2,2,\mathbb{Z})$. In other words, whenever the matrix $D = L^{-1}R(G)L$ is rational, this corresponds to a genuine duality symmetry of the $c = 2$ compact boson. Indeed, because $R(G)$ automatically satisfies the defining invariance conditions for a duality symmetry given in (4.4.51), the corresponding matrix D automatically satisfies the duality symmetry defining properties by construction. Thus, whenever $D \in O(2,2;\mathbb{Q})$, the induced transformation preserves the generalized metric and the momentum lattice inner product, and therefore defines, in general, a genuine non-invertible duality defect of the theory of the form introduced in Section 4.4.1.

As a final remark, let us emphasize that the matrix obtained from (4.4.52) may also lie in the integral group

$$D \;\in\; O(2,2;\mathbb{Z}). \tag{4.4.54}$$

When this occurs, the corresponding transformation does not represent a non-invertible duality defect, but simply an ordinary duality transformation that happens to be a genuine symmetry of the theory at the given point in moduli space. In this case, D implements an invertible duality identification and it does not involve any gauging.

Therefore, only when the matrix belongs to the rational group but is not integral, that is,

$$D \;\in\; O(2,2,\mathbb{Q}) \setminus O(2,2,\mathbb{Z}), \tag{4.4.55}$$

does it represent a truly non-invertible duality symmetry. In this situation, the transformation necessarily involves discrete gauging, together with ordinary dualities, defining a proper non-invertible symmetry of the $c = 2$ compact boson theory.

4.4.5 An example of duality defect generated by a lattice symmetry

As a last step in this section, we now want to apply the general construction presented above in order to explicitly generate a duality symmetry starting from the crystallographic symmetries of the target–space lattice. The idea is to begin with a toroidal compactification specified by its moduli (τ, ρ), determine the corresponding symmetry group of the lattice Λ, and then use the symmetry generator $G \in O(2)$ to construct the associated $O(2,2)$ matrix $R(G)$ of the form (4.4.38). Once $R(G)$ is known, the associated right–multiplying matrix D is obtained following equation (4.4.52), which, as discussed above, automatically satisfies the defining conditions for a duality symmetry of the $c = 2$ compact boson. The only remaining question is whether D belongs to $O(2, 2, \mathbb{R})$ or in $O(2, 2, \mathbb{Q})$, and, in the second case, the matrix represents precisely a duality symmetry of the theory.

Let us now consider again the hexagonal torus example presented in (4.4.29). In (4.4.31), we introduced a duality symmetry of the theory associated with a generator of the crystallographic symmetry group D_6 of the underlying lattice, for which the left–multiplying matrix R takes the form (4.4.33). This duality defect was originally obtained in [70][8] and its derivation required a rather intricate procedure. Indeed, in order to construct this duality symmetry, the authors first mapped the hexagonal point to a simpler orthogonal toroidal compactification, namely $(\tau, \rho) = (i\sqrt{3},\, i\,3\sqrt{3})$, where the analysis of duality symmetries is easier. After determining the non-invertible defect at that orthogonal point, they then mapped it back to the original hexagonal theory. This construction, however, is highly non-trivial and involves multiple steps.

Here, instead, we will exploit the crystallographic symmetries of the underlying lattice which, as already emphasized, are fully classified. This allows us to construct a duality symmetry of the theory directly, without the need to move to a different point in the moduli space. As we reviewed in Section 4.2, the hexagonal torus enjoys several non-trivial wallpaper symmetries, including the rotational symmetries $\mathbb{Z}_3$ and $\mathbb{Z}_6$.

In what follows, we will focus on the $\mathbb{Z}_3$ rotational symmetry. According to the general prescription described above, the first step is to identify the matrix representation of the crystallographic symmetry acting on the coordinates of the hexagonal lattice. For the $\mathbb{Z}_3$ symmetry, the generator is a rotation by an angle $\frac{2\pi}{3}$, whose matrix form is

$$G = \begin{pmatrix} \cos\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) \\ \sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{pmatrix} . \tag{4.4.56}$$

[8]The explicit matrix expression of the defect given in [70] differs from our representation. However, the two forms are related by elements of the duality group $O(2, 2, \mathbb{Z})$ and therefore describe the same physical duality symmetry.

Given this generator, we can now construct the associated left–multiplying matrix $R(G)$ using the block structure introduced in (4.4.47). Explicitly, we find

$$R(G) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) \\ 0 & 0 & \sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{pmatrix} . \tag{4.4.57}$$

This is precisely the left–multiplying matrix $R(G)$ that implements the action of the $\mathbb{Z}_3$ crystallographic symmetry on the momentum lattice.

Now that the left–multiplying matrix $R(G)$ has been constructed, we can act with it on the lattice generating matrix of the hexagonal torus, whose explicit form was given in (4.4.30). The transformed lattice is obtained by the action

$$L' \;=\; R(G)\, L, \tag{4.4.58}$$

and the straightforward computation gives us

$$L' = \begin{pmatrix} \frac{\sqrt{6}}{6} & 0 & \frac{\sqrt{6}}{2} & -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{2}}{6} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{6} & \frac{2\sqrt{2}}{3} \\ -\frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} \\ \frac{\sqrt{2}}{6} & -\frac{\sqrt{2}}{6} & -\frac{5\sqrt{2}}{6} & \frac{2\sqrt{2}}{3} \end{pmatrix} . \tag{4.4.59}$$

This matrix describes the transformed momentum lattice obtained by acting with the $\mathbb{Z}_3$ rotational symmetry on the right–moving momentum subspace while leaving the left–moving momentum invariant.

Now that we have L', the next step is to determine the corresponding right–multiplying matrix D that implements the same transformation on the charge lattice. As we said above in (4.4.52), this is obtained by inverting (4.4.16). The explicit computation gives us:

$$D = \begin{pmatrix} 0 & -\frac{1}{3} & \frac{7}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & -\frac{7}{3} & \frac{7}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix} \tag{4.4.60}$$

In particular, the matrix D we just obtained is an element of $O(2, 2, \mathbb{Q})$, and, most importantly, it contains non-integer rational entries. This immediately implies that D cannot correspond to an ordinary duality transformation. Instead, the presence of rational coefficients implies that the defect necessarily involves a gauging operation. In fact, by the general result proved in [81], every matrix

$$D \;\in\; O(2, 2, \mathbb{Q}) \setminus O(2, 2, \mathbb{Z}) \tag{4.4.61}$$

that satisfies the defining invariance condition of the generalized metric can always be decomposed into a finite sequence of discrete gaugings of subgroups of the chiral algebra, combined with ordinary duality transformations in $O(2, 2, \mathbb{Z})$. As a consequence, the matrix D constructed above represents a genuine non-invertible duality symmetry of the $c = 2$ compact boson at the hexagonal point.

This result is particularly impressive when compared with the standard approach for constructing duality symmetries used in the literature. As we recalled above, the explicit construction of non-invertible duality symmetries at the hexagonal point in [70] required a long and intricate procedure; the authors first mapped the theory to a special orthogonal point in the moduli space, determined the duality defect there, and then mapped it back to the hexagonal torus. Instead, the procedure developed in this section allows us to obtain a duality symmetry in only a few systematic steps and it completely relies on the classified crystallographic symmetries of the underlying target-space lattice.

4.5 Dp-branes in the $c = 2$ compact boson

In the previous section, we analyzed the structure of duality symmetries in the $c = 2$ compact boson, showing how they can be systematically constructed starting from the crystallographic symmetries of the torus and how the corresponding left–multiplying matrices encode the geometric transformations of the underlying lattice. In this last section of the chapter, instead, we turn to the study of the Dp-branes and how the duality symmetries act on these boundary states.

Let us start by briefly recalling the definition of Dp-branes in rational conformal field theories. As reviewed in Section 3.4.1, Dp-branes are described by boundary states that solve the gluing conditions relating the left and right-moving chiral algebras. These conditions are solved by the Ishibashi states, which represent a complete set of solutions but do not correspond to physical boundary states directly. Instead, they serve as the building blocks for the Cardy states, which satisfy the required consistency conditions and are in one-to-one correspondence with the physical Dp-branes of the theory. Moreover, in the diagonal RCFT case the number of Ishibashi states coincides with the number of Cardy states, allowing one to determine the total number of Dp-branes directly from the solutions to the gluing equations.

In the $c = 1$ compact boson, as we have seen, the situation is particularly simple; because this type of theories describe a one-dimensional target space, the gluing condition is represented by a single number $R = \pm 1$. This yields only two possible boundary conditions, corresponding to D0 and D1-branes.

In the $c = 2$ compact boson, however, the structure becomes considerably richer. The target space is now two-dimensional, and the gluing condition is represented not by a sign but by a real matrix $R \in O(2)$ acting on the left and right-moving oscillators, as shown in (3.4.3). To analyze the Ishibashi states explicitly, it is convenient to focus on the zero-mode gluing condition,

$$\left(p_{L,i} + R_i^{\,j}\, p_{R,j}\right) |B\rangle\rangle = 0, \tag{4.5.1}$$

where $i, j \in \{1, 2\}$ label the two target-space directions. This relation specifies a linear constraint on the momentum lattice $\Gamma^{2,2}$ of the theory and determines which lattice sites are compatible with a given boundary condition[9]. Solving this constraint is the first step in determining the Ishibashi states; each solution of the zero-mode condition corresponds to a sector in which an Ishibashi state can be constructed by imposing the full gluing condition on all oscillator modes.

As we saw around (3.4.4), the eigenvalues of the matrix R, which represents the gluing condition, determine the type of boundary state that solves it. In particular, each $+1$ eigenvalue corresponds to a Neumann direction, while each -1 eigenvalue corresponds to a Dirichlet direction; therefore, a matrix R with p eigenvalues $+1$ defines a Dp-brane. Since R is now a 2×2 orthogonal matrix, its spectrum can contain either two $+1$ eigenvalues, two -1 eigenvalues, or one eigenvalue of each sign. These three possibilities give rise to three distinct classes of branes in the $c = 2$ compact boson[10]. When both eigenvalues are $+1$, the only admissible matrix in $O(2)$ is the identity, $R = \text{Id} = \text{diag}(+1, +1)$, corresponding to Neumann boundary conditions in both directions and hence to D2-branes. This gluing condition leads to the relation between left and right-moving momentum

$$\boldsymbol{p}_L = \begin{pmatrix} p_{L,1} \\ p_{L,2} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \boldsymbol{p}_R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} p_{R,1} \\ p_{R,2} \end{pmatrix} = -\boldsymbol{p}_R. \tag{4.5.2}$$

Similarly, when both eigenvalues are -1, the only gluing matrix $R \in O(2)$ is given by $R = -\text{Id}$, yielding Dirichlet boundary conditions in both directions and, therefore, D0-branes; the condition on the left and right-momentum is given by:

$$\boldsymbol{p}_L = \begin{pmatrix} p_{L,1} \\ p_{L,2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{p}_R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{R,1} \\ p_{R,2} \end{pmatrix} = \boldsymbol{p}_R. \tag{4.5.3}$$

[9]As we already saw in Section 3.4.2, the left and right momentum do not uniquely characterize the boundary state, but we need the full combination of momentum and higher-oscillatory modes to be satisfied. However, this zero-mode condition already determines which momentum–winding combinations are compatible with the boundary condition; only those lattice points of $\Gamma^{2,2}$ that satisfy the relation above can appear in the corresponding Ishibashi state.

[10]Being more precise, the matrix R determines the gluing condition satisfied by the Ishibashi states, not the Dp-branes themselves. The Ishibashi states provide formal solutions to the boundary conditions, while the physical Dp-branes correspond to the Cardy states, obtained as specific linear combinations of Ishibashi states. However, since the eigenvalue structure of R fixes the number of Neumann and Dirichlet directions, it is customary to refer to R as defining a Dp-brane configuration. Moreover, in diagonal RCFTs the number of Ishibashi and Cardy states coincides, so this terminology is fully consistent in the settings considered here, since we will focus on this type of theories.

Finally, when the spectrum contains one $+1$ and one -1 eigenvalue, the gluing matrix takes the more general form

$$R = Q(\alpha) \equiv -M(\alpha)\,\mathrm{diag}(+1,-1), \qquad M(\alpha) \in SO(2), \tag{4.5.4}$$

which represents a reflection along one axis of the target space composed with a rotation by an angle α. In contrast to the previous two cases, this construction yields an infinite family of matrices $R \in O(2)$ with mixed eigenvalues: each choice of rotation angle corresponds to a different orientation of the D1-brane within the torus. The zero-mode equation takes the form:

$$\begin{aligned} \boldsymbol{p}_L = \begin{pmatrix} p_{L,1} \\ p_{L,2} \end{pmatrix} &= \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \boldsymbol{p}_R \\ &= \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix} \begin{pmatrix} p_{R,1} \\ p_{R,2} \end{pmatrix} \end{aligned} \tag{4.5.5}$$

However, despite the continuous parameter θ, only a finite subset of angles admits non-trivial solutions to the full gluing equations, when including also the higher oscillatory modes. As we will see explicitly in the example below, the zero-mode constraints restrict the allowed momenta and windings in such a way that only finitely many Ishibashi states appear at a given rational point in moduli space.

Summarizing the discussion above, the $c = 2$ compact boson admits three distinct types of Dp-branes: D0-branes corresponding to $R = -\mathrm{Id}$, D2-branes arising from $R = \mathrm{Id}$, and a continuous family of D1-branes associated with gluing matrices of the form $R = -M(\alpha)\,\mathrm{diag}(+1,-1)$, each value of θ defining a different orientation of the brane inside the torus. However, only finitely many of these orientations lead to consistent Ishibashi states, and this is due to the fact that D1-branes wrap the torus along different allowed angles determined by the structure of the momentum lattice.

At this point, we can turn to the action of the duality symmetries constructed in the previous section on these boundary states. As we emphasized earlier, each duality symmetry is associated with a crystallographic symmetry $G \in O(2)$ of the target-space lattice, which is implemented by a left-multiplying matrix that acts asymmetrically on the left and right-moving sectors. In particular, since the gluing condition for a Dp-brane identifies left and right-moving momenta through the matrix R, the corresponding transformation on branes is determined by how the duality symmetry modifies this relation. In particular, if the duality symmetry is associated with a crystallographic symmetry generated by G, and it is represented by a left-multiplying matrix $R(G)$, as defined in (4.4.47), acting on the momentum lattice, then its action maps a boundary condition encoded by R to a new one encoded by

$$R \;\longmapsto\; R' = RG, \tag{4.5.6}$$

which determines the resulting transformed Dp-brane. In this way, the effect of a duality symmetry on the space of Dp-branes is completely fixed by the corresponding crystallographic symmetry of the torus lattice.

As we clearly see, the behavior of Dp-branes in the $c = 2$ compact boson can be much more involved than the $c = 1$ case, where the action of duality symmetries on boundary states was particularly simple. Indeed, in that case, every duality transformation could be represented by the same left–multiplying matrix $R = \mathrm{diag}(1, -1)$, leading to a universal exchange between Neumann and Dirichlet conditions and hence to the overall action D0-branes $\longleftrightarrow$ D1-branes.

In the $c = 2$ theory, instead, the situation is much more complicated. Since the gluing matrix R may correspond to D0-, D1-, or D2-branes, and since D1-branes themselves appear in different classes, parametrized by the possible angles determined by the lattice geometry, a duality symmetry can act in more involved ways. Depending on the crystallographic generator G underlying the defect, it may rotate the Dp-branes or exchange different types of boundary conditions. Moreover, while in the $c = 1$ case the total number of Dp-branes was always preserved, since we only had an exchange of the two types, in this case we can have that the total number of $D0$, $D1$ and $D2$-branes can change under the action of a duality symmetry.

As an explicit example, let us consider the $\mathbb{Z}_2$ rotational symmetry discussed above in (4.4.35). Its action on the coordinates was implemented by the crystallographic generator $G = -\mathrm{Id}$, corresponding to a rotation by π on the target torus. When translated into the action on boundary states through the gluing condition, this transformation produces a simple pattern. First of all, D0-branes and D2-branes get exchanged: the condition $\boldsymbol{p}_L = \boldsymbol{p}_R$ characterizing D0-branes is mapped to the condition $\boldsymbol{p}_L = -\boldsymbol{p}_R$ of D2-branes, and conversely. Instead, the D1-branes remain D1-branes, but their orientation is rotated. Indeed, for a D1-brane specified by a gluing matrix of the form $R = Q(\alpha)$ as defined in (4.5.4), the action of the $\mathbb{Z}_2$ symmetry sends

$$\mathbb{Z}_2 : \; Q(\alpha) \;\longmapsto\; Q(\alpha + \pi), \tag{4.5.7}$$

reflecting the fact that a rotation by π flips the different Neumann and Dirichlet directions of the brane. Thus, while D1-branes preserve their nature, their angular classes get mixed in non-trivial ways. In this sense, the $\mathbb{Z}_2$ rotation may be viewed as the natural $c = 2$ analogue of the duality symmetry of the $c = 1$ compact boson. In the one–dimensional case, it exchanged D0- and D1-branes, leaving no room for more intricate transformations. Here, however, even this simple crystallographic symmetry already exhibits a more involved action. Indeed, in addition to exchanging D0- and D2-branes, it also permutes the classes of D1-branes among themselves by rotating their orientation $\alpha \mapsto \alpha + \pi$. This provides a first indication of the more elaborate interplay between duality symmetries and Dp-brane configurations that emerge in higher-dimensional compactifications.

4.5.1 An explicit example of duality symmetry acting on Dp-branes

To see how even more intricate behaviors can arise, we now turn to an explicit example. In particular, we will focus on the hexagonal torus introduced in (4.4.29), whose lattice enjoys the dihedral symmetry group D_6 as a crystallographic symmetry; in particular, this leads to the duality symmetry presented in (4.4.31), whose corresponding left-multiply matrix is given by (4.4.33). Below, we will analyze the action of this duality symmetry on the Dp-branes of the theory.

Before proceeding, let us note that the hexagonal torus corresponds to a diagonal RCFT. Indeed, since $\rho = 3\tau$ (up to a T–transformation of ρ, as defined in (3.1.27)), the theory represents a rational point in moduli space where the partition function is diagonal. As a consequence, the number of Ishibashi states equals the number of Cardy states, and therefore matches the number of physical Dp-branes. This allows us to classify the branes by directly counting the solutions to the gluing conditions.

We will start by constructing the Ishibashi states of the original theory at the hexagonal point, identifying the corresponding D0, D1 and D2-branes; for this analysis, we will follow the discussion in [80]. We will then apply the duality symmetry (4.4.31) to these states and analyze how the various branes are transformed under this action.

Dp-branes in the original hexagonal theory

To determine the Dp-branes of the compact boson at the hexagonal point

$$(\tau, \rho) = \left(e^{2\pi i/3},\, -\tfrac{1}{2} + i\,\tfrac{3\sqrt{3}}{2}\right), \tag{4.5.8}$$

we start from the momentum lattice generated by the matrix L introduced in (4.4.30). A generic state of the theory is characterized by four integers $(a, b, c, d) \in \mathbb{Z}^4$ specifying the momentum and winding numbers along the two cycles of the torus, which are respectively $\boldsymbol{n} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} c \\ d \end{pmatrix}$. The corresponding left and right-moving momenta are explicitly given by

$$\begin{pmatrix} p_{L,1} \\ p_{L,2} \\ p_{R,1} \\ p_{R,2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{6}}{6}a + \frac{\sqrt{6}}{2}c - \frac{\sqrt{6}}{3}d \\ \frac{\sqrt{2}}{6}a + \frac{\sqrt{2}}{3}b + \frac{\sqrt{2}}{6}c + \frac{2\sqrt{2}}{3}d \\ \frac{\sqrt{6}}{6}a - \frac{\sqrt{6}}{2}c + \frac{\sqrt{6}}{6}d \\ \frac{\sqrt{2}}{6}a + \frac{\sqrt{2}}{3}b + \frac{\sqrt{2}}{6}c - \frac{5\sqrt{2}}{6}d \end{pmatrix}, \tag{4.5.9}$$

which follows directly from the definition $\begin{pmatrix} \boldsymbol{p}_L \\ \boldsymbol{p}_R \end{pmatrix} = L \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{w} \end{pmatrix}$. The classification of the Dp-branes is then obtained by imposing the zero-mode gluing condition

$$p_{L,i} + R_i^{\,j} p_{R,j} = 0 \tag{4.5.10}$$

presented in (4.5.1), which is obtained from the general condition (3.4.4). For each choice of gluing matrix $R \in O(2)$, this equation imposes linear relations among the integers (a, b, c, d); every allowed solution identifies a set of momentum and winding numbers that allows for an Ishibashi state. Since the hexagonal point is a diagonal RCFT (because $\rho = 3\tau$ up to a T-transformation), the number of Ishibashi and Cardy states coincides, so each solution directly corresponds to a physical Dp-brane. We can now analyze the three types of gluing matrices, following the classification of the eigenvalues discussed above.

As we just saw, the D0-branes are related to the Ishibashi states satisfying the gluing condition with $R = -\mathrm{Id}$, as shown in (4.5.3). In this case, the zero-mode equation (4.5.10) reduces to $p_{L,i} = p_{R,i}$ for $i = 1, 2$, and substituting the explicit expressions for the left- and right-moving momenta given in (4.5.9) immediately yields the linear constraints

$$c = 0, \qquad d = 0, \tag{4.5.11}$$

while the integers $a, b \in \mathbb{Z}$ are free. At this point, it is useful to recall the results of Section 4.3, in which we analyzed the structure of rational points of the moduli space and the decomposition of the momentum lattice $\Gamma^{(2,2)}$ into its left and right-moving projections. In particular, we saw that at RCFT points the left-moving lattice Γ_L becomes a finite-index sublattice of $\mathbb{R}^2$, and its elements correspond precisely to the left-moving components of the momentum vectors in $\Gamma^{2,2}$. Since diagonal RCFTs obey the additional identification $\Gamma_L \simeq \Gamma_R$, the condition $p_L = p_R$ ensures that the admissible momentum–winding numbers give rise exactly to the two-dimensional sublattice Γ_L. In our present case, the relations (4.5.11) imply that the allowed Ishibashi states are labeled by vectors of the form $(a, b, 0, 0)$ and hence the D0-branes correspond exactly to the sites of this two-dimensional sublattice. In particular, this yield a total of 27 distinct D0-branes at the hexagonal point.

Let us now turn to the D2-branes, which correspond to the Ishibashi states solving the gluing condition with $R = \mathrm{Id}$. In this case the zero-mode equation (4.5.10) reduces to $p_{L,i} = -p_{R,i}$, as given in (4.5.2), and substituting the explicit expression for the momenta given in (4.5.9) yields the linear relations

$$c = -2b, \qquad d = 2a. \tag{4.5.12}$$

These constraints define the two-dimensional sublattice of $\Gamma^{(2,2)}$ denoted by Γ_0, that we defined in (4.3.15), consisting of the purely left-moving states of the theory. As discussed in Section 4.3, at diagonal RCFT points this sublattice has rank two and finite index in Γ_L, and each of its sites corresponds to a chiral primary. In the present case the relations (4.5.12) uniquely fix c and d in terms of a and b, and only a single independent Ishibashi state is compatible with the gluing condition, given by $a = b = 0$. Thus, the hexagonal compactification contains exactly one D2-brane.

Finally, let us turn to the D1-branes. In this case, the gluing matrix has one $+1$ and one -1 eigenvalue and can be written as $R = Q(\alpha) = -M(\alpha)\,\mathrm{diag}(+1,-1)$, with the corresponding zero-mode constraint given in (4.5.5). The parameter appearing in this equation, however, does not coincide with the physical orientation of the brane on the torus. We will not derive this fact here, but it can be shown that the geometric angle of the wrapped one-cycle is actually given by $\alpha/2$, rather than by α itself entering the gluing condition. Consequently, only those values of $\alpha/2$ that align with the crystallographic directions of the hexagonal lattice lead to non-trivial integer solutions of (4.5.5).

Solving the zero-mode equation (4.5.5) for the hexagonal compactification therefore reduces to determining the integer solutions of the corresponding linear constraints among (a, b, c, d). In particular, only six orientations are compatible with the momentum lattice $\Gamma^{(2,2)}$, and each yields a distinct class of D1-branes. These arise at the angles

$$\alpha \in \{0,\, \frac{\pi}{3},\, \frac{2\pi}{3},\, \pi,\, \frac{4\pi}{3},\, \frac{5\pi}{3}\} \tag{4.5.13}$$

for which the corresponding geometric orientations given by $\alpha/2$ correspond to the six crystallographic directions of the hexagonal torus. For each of these orientations, the zero-mode constraint fixes a two-dimensional sublattice of allowed momentum vectors, leading to the following relations:

- $\alpha = 0$:
$$a = -2b, \qquad d = 2c,$$
yielding 3 D1-branes at the physical angle $\alpha/2 = 0$.
- $\alpha = \frac{\pi}{3}$:
$$b = 0, \qquad c = 0,$$
yielding 9 D1-branes at the physical angle $\alpha/2 = \frac{\pi}{6}$.
- $\alpha = \frac{2\pi}{3}$:
$$a = b, \qquad d = -c,$$
yielding 3 D1-branes at the physical angle $\alpha/2 = \frac{\pi}{3}$.
- $\alpha = \pi$:
$$a = 0, \qquad d = 0,$$
yielding 9 D1-branes at the physical angle $\alpha/2 = \frac{\pi}{2}$.
- $\alpha = \frac{4\pi}{3}$:
$$b = -2a, \qquad c = 2d,$$
yielding 3 D1-branes at the physical angle $\alpha/2 = \frac{2\pi}{3}$.

- $\alpha = \frac{5\pi}{3}$:
$$b = -a, \qquad c = d,$$
yielding 9 D1-branes at the physical angle $\alpha/2 = \frac{5\pi}{6}$.

These six cases represent all the possible consistent D1-branes for the given torus defined by (4.4.29). Each admissible orientation corresponds to a distinct crystallographic direction of the torus, and the number of branes in each class matches the index of the corresponding solution sublattice inside $\Gamma^{(2,2)}$. In total, the theory contains $3+9+3+9+3+9 = 36$ D1-branes. We give a visual representation of the Dp-branes in Figure 14.

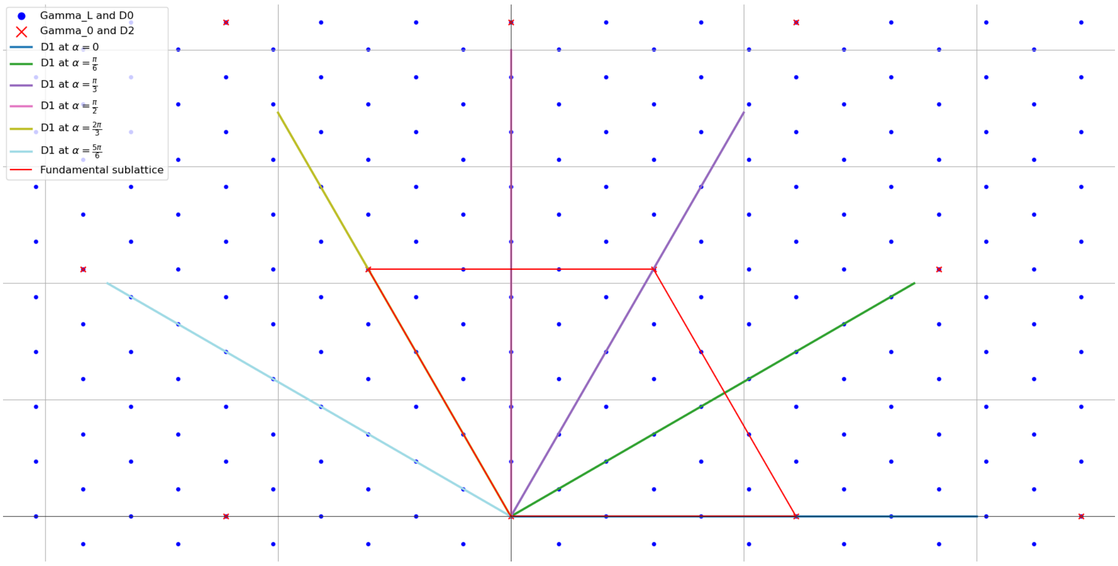


Figure 14: Representation of the Dp-branes for the hexagonal torus defined in (4.4.29). The blue dots denote the left-moving momentum sublattice Γ_L associated with the D0-branes, while the red crosses indicate the purely left-moving sublattice Γ_0 associated with the unique D2-brane. The colored line segments represent the six classes of D1-branes obtained from the integer solutions of the zero-mode gluing condition (4.5.5); each color corresponds to one of the admissible orientations, which align with the crystallographic directions of the hexagonal lattice. The angles formed by the different D1-branes are thus directly visible in the plot. Since the theory represents a point of enhanced chiral symmetry, lattice sites related by the action of the chiral algebra generators, as showed in Figure 13, are identified, and each line of solutions gives a finite number of D1-branes under this identification.

Summarizing the discussion above, the hexagonal compactification admits three types of Dp-branes: 27 D0-branes associated with the sublattice Γ_L, a single D2-brane corresponding to the chiral sublattice Γ_0 and 36 D1-branes organized into six different orientations. The total number of branes is therefore

$$N_{D0} + N_{D1} + N_{D2} = 27 + 36 + 1 = 64, \tag{4.5.14}$$

A summary of the complete set of Dp-branes is given by the table below.

Dp-branes	Number of branes
D0s	27
D2s	1
D1s($\alpha = 0$)	3
D1s($\alpha = \pi/6$)	9
D1s($\alpha = \pi/3$)	3
D1s($\alpha = \pi/2$)	9
D1s($\alpha = 2\pi/3$)	3
D1s($\alpha = 5\pi/6$)	9
Tot	64

Dp-branes under the action of the duality symmetry

Now that we have computed all Dp-branes of the original theory, we can proceed to analyze the action of the duality symmetry introduced in (4.4.31) on these boundary states. As discussed in Section 4.4.2, this duality transformation is implemented on the momentum lattice by the left–multiplying matrix defined in (4.4.33). This rewriting in terms of the matrix $R(G)$ allows us to determine the induced action on the gluing conditions characterizing each class of Dp-branes.

As we explained above, the action of the matrix $R(G)$ on the left and right-moving momentum is asymmetric and is given by

$$\boldsymbol{p}_L \longmapsto \boldsymbol{p}_L \quad \text{and} \quad \boldsymbol{p}_R \longmapsto G\boldsymbol{p}_R, \tag{4.5.15}$$

where G is a generator of a crystallographic symmetry of the corresponding torus. In this particular case, the corresponding matrix G, given in (4.4.42), admits a natural decomposition into two factors; these represent a sign flip of the second right-moving momentum component, $p_{R,2} \mapsto -p_{R,2}$, followed by a rotation of the right-moving momentum $\boldsymbol{p}_R$ by an angle $\pi/3$. This asymmetric action of the duality symmetry on the left and right-moving momentum implies that its overall effect on the boundary states is entirely determined by how the transformation modifies the gluing condition. As we explained in (4.5.6), this is equivalent to replacing the original gluing matrix R with a new one given by

$$R \;\longmapsto\; R' = RG, \tag{4.5.16}$$

which encodes the transformed boundary condition.

This formula allows us to determine explicitly how each type of Dp-brane is mapped under the duality symmetry. Since G is the composition of a sign flip with a rotation, the resulting transformation $R \mapsto RG$ modifies the Neumann and Dirichlet directions in a non-trivial way, and consequently induces a non-trivial permutation of the Dp-brane configurations.

We now examine this transformation explicitly, beginning by analyzing the D0-branes that we obtain after the action of the duality symmetry. After the duality symmetry transformation, the left- and right-moving momenta are given by

$$\boldsymbol{p}'_L = \boldsymbol{p}_L, \qquad \boldsymbol{p}'_R = G\,\boldsymbol{p}_R, \tag{4.5.17}$$

so the defining condition for a D0-brane in the transformed theory becomes

$$\boldsymbol{p}'_L = \boldsymbol{p}'_R \quad \Longleftrightarrow \quad \boldsymbol{p}_L = G\,\boldsymbol{p}_R. \tag{4.5.18}$$

Using the explicit form of the crystallographic generator G introduced in (4.4.42), we immediately recognize that (4.5.18) coincides precisely with the original D1-branes condition at physical angle $\pi/6$, that is to say, it satisfies the gluing condition (4.5.5) for $\alpha = \pi/3$. In other words, the resulting D0-branes in the transformed symmetry, after the action of the duality symmetry, satisfy the same relations that defined the D1$(\pi/6)$ class in the original theory.

It is also straightforward to verify that, in the same way, the D1-branes at the physical angle $\pi/6$ in the transformed theory correspond precisely to the original D0-branes. Indeed, recall that the D1-branes at physical angle $\pi/6$ are defined by the gluing matrix $R = Q(\alpha)$ given in (4.5.4) with $\alpha = \pi/3$. To determine the original corresponding branes, we must therefore examine how the relation

$$\boldsymbol{p}'_L = Q\Big(\frac{\pi}{3}\Big)\,\boldsymbol{p}'_R = M\Big(\frac{\pi}{3}\Big)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\boldsymbol{p}'_R \tag{4.5.19}$$

looks like in terms of the untransformed momenta. Using $\boldsymbol{p}'_L = \boldsymbol{p}_L$ and $\boldsymbol{p}'_R = G\boldsymbol{p}_R$, this becomes

$$\boldsymbol{p}_L = Q\Big(\frac{\pi}{3}\Big)\,G\,\boldsymbol{p}_R. \tag{4.5.20}$$

Now, using the explicit form of $Q(\alpha)$ and the decomposition of G given in (4.4.42), one finds

$$Q\Big(\frac{\pi}{3}\Big)\,G = M\Big(\frac{\pi}{3}\Big)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}M\Big(\frac{\pi}{3}\Big)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \mathrm{Id}. \tag{4.5.21}$$

Inserting this result into the gluing condition, we obtain

$$\boldsymbol{p}_L = \boldsymbol{p}_R, \tag{4.5.22}$$

which is precisely the defining equation of the original D0-branes in (4.5.3). Therefore, the D1-branes at the geometric angle $\pi/6$ appearing in the transformed theory are precisely the image of the original D0-branes under the duality symmetry action. In summary, the duality symmetry acts by exchanging the two classes

$$\text{D0} \longleftrightarrow \text{D1}\Big(\frac{\pi}{6}\Big), \tag{4.5.23}$$

implying that the transformed theory is characterized by 9 D0-branes and 27 D1-branes at the physical angle $\pi/6$.

Equivalently, let us now examine the transformation of the D2-branes. Recall that the original D2-branes equation is characterized by the gluing condition $\boldsymbol{p}_L = -\boldsymbol{p}_R$, that is to say by the matrix $R = \mathrm{Id}$ yielding (4.5.2). After the action of the duality symmetry, this condition becomes

$$\boldsymbol{p}'_L = -\,\boldsymbol{p}'_R \qquad \Longleftrightarrow \qquad \boldsymbol{p}_L = -\,G\,\boldsymbol{p}_R. \tag{4.5.24}$$

Using the explicit expression for G given in (4.4.42), one verifies that the matrix $-G$ coincides precisely with the D1-branes gluing matrix $Q(\alpha)$ for $\alpha = \frac{4\pi}{3}$, that is to say, $-\,G = Q\big(\frac{4\pi}{3}\big)$. Therefore, the condition (4.5.24) is exactly equivalent to the D1 gluing equation at the physical angle $\frac{2\pi}{3}$. In other words, the original D1-brane class at $\frac{2\pi}{3}$ is mapped to the D2-brane one.

In the same way, after applying the duality transformation, the D1-branes with $\alpha = \frac{4\pi}{3}$ are given by

$$\boldsymbol{p}'_L = Q\left(\frac{4\pi}{3}\right)\boldsymbol{p}'_R \qquad \Longleftrightarrow \qquad \boldsymbol{p}_L = Q\left(\frac{4\pi}{3}\right)G\,\boldsymbol{p}_R = -\boldsymbol{p}_R, \tag{4.5.25}$$

which is precisely the defining relation of the original D2-brane. Thus, we obtain the second exchange:

$$\text{D2} \longleftrightarrow \text{D1}\left(\frac{2\pi}{3}\right), \tag{4.5.26}$$

and the counting gives 3 D2-branes and 1 D1-brane at $2\pi/3$ in the transformed theory.

For the remaining D1-brane classes, namely those at physical angles 0, $\frac{2\pi}{3}$, $\frac{\pi}{2}$ and $\frac{5\pi}{6}$, it is not possible to perform an analysis equivalent to the one above. The reason is that, after the action of the duality symmetry, the corresponding gluing conditions do not correspond to any of the original Dp-brane equations. Indeed, the matrices $Q(\alpha)$ obtained in the transformed theory do not coincide with any of the admissible gluing matrices of the original model; these transformed D1-branes are not the image of any of the original D0, D1 or D2-brane classes.

Anyway, it is still possible to determine the number of resulting D1-branes by solving the transformed gluing conditions explicitly. This calculation is equivalent to the computation performed around (4.5.13), by solving the corresponding constraints among (a, b, c, d). This is done by using the resulting lattice generating matrix L' after the duality symmetry transformation given in (4.4.32). With this analysis, we find that the resulting theory contains exactly

$$1 \text{ D1-brane at the geometric angle } 0,\ 1 \text{ D1-brane at the geometric angle } \frac{2\pi}{3},$$
$$1 \text{ D1-brane at the geometric angle } \frac{\pi}{2},\ 3 \text{ D1-branes at the geometric angle } \frac{5\pi}{6}.$$

These classes represent all the remaining D1-branes in the transformed theory.

We can now summarize the action of the duality symmetry on the Dp-branes of the hexagonal torus. The table below shows, for each Dp-brane class, the total number of branes present in the resulting theory after the action of the duality symmetry and, where possible, their relation with the Dp-branes of the original theory.

Resulting Dp-branes	**Number of branes**	**Original Dp-branes**
D0s	9	*D1s($\alpha = \pi/6$)*
D2s	3	*D1s($\alpha = 2\pi/3$)*
D1s($\alpha = 0$)	1	
D1s($\alpha = \pi/6$)	27	D0s
D1s($\alpha = \pi/3$)	1	
D1s($\alpha = \pi$)	3	
D1s($\alpha = 2\pi/3$)	1	D2s
D1s($\alpha = 5\pi/6$)	3	
Total	48	

It is important to notice that, in addition to a mixing between different Dp-brane classes, the total number of Ishibashi states is not preserved by the duality symmetry. Indeed, the original theory contained 64 Ishibashi states, while the transformed one contains only 48 of them.

However, as we stated above, the Ishibashi states are not the physical boundary states of the theory; instead, these are given by the so-called Cardy states, which are obtained via the relation (3.4.5), and are in one-to-one correspondence with the Dp-branes. However, the number of Ishibashi states and Cardy states does not have to coincide in general; indeed, this is true only in the diagonal RCFT case. Thus, even though the number of Ishibashi states is not preserved under the action of the duality symmetry, it is still possible that the number of physical boundary states, that is to say, the number of Dp-branes, remains unchanged after the transformation due to the non-trivial mapping between Ishibashi and Cardy states. Indeed, the duality transformation could act in such a way that the diagonal RCFT relation between Ishibashi and Cardy states (3.4.6) no longer applies, thereby preserving the number of Dp-branes, even though the number of Ishibashi states is not preserved.

In this thesis, we have not performed a detailed analysis of the counting of Dp-branes after the duality transformation; our discussion is restricted to the behavior of the Ishibashi states. Nevertheless, our expectation is that the relation between Ishibashi and Cardy states reorganizes in such a way that the total number of Dp-branes is preserved. In particular, we intend to address this question in detail in future work.

5 Conclusions and Outlooks

In this thesis, we explored the structure of non-invertible duality symmetries in two-dimensional conformal field theories, with a particular focus on compact boson models.

In Chapter 2, we set the mathematical basis by reviewing the concept of generalized symmetry in quantum field theory and string theory, emphasizing the notion of higher-form and non-invertible symmetries, and by presenting their formulation in terms of topological defects and fusion categories.

In Chapter 3, we then examined the concept of duality symmetries in the $c = 1$ compact boson, reviewing how non-invertible duality defects arise from discrete gauging and how their action on Dp-branes can be understood through their action on Ishibashi states.

Finally, in Chapter 4, we presented the main conceptual results of this thesis. Indeed, we extended the previous analysis to the $c = 2$ compact boson, whose moduli space and duality group give rise to a much richer structure of rational points and duality symmetries. We observed that non-invertible duality symmetries of the $c = 2$ compact boson admit a clear geometric interpretation in terms of the crystallographic symmetries of the two dimensional lattice underlying the target-space torus. This led us to the introduction of a systematic approach for constructing duality symmetries. Instead of solving the non-linear constraint on the generalized metric, we showed how, starting from a lattice symmetry generator and extending it to a transformation on the integer self-dual momentum lattice, we are able to construct the matrix representation of the corresponding duality symmetry. Finally, following the discussion in the $c = 1$ compact boson case, we have been able to analyze the behavior of the Dp -branes in the $c = 2$ compact boson as well. In particular, we showed that, in this higher dimensional case, the action of the duality symmetries on boundary states can be much more involved and that, in some cases, the total number of Dp-branes is not preserved.

The results of this work open several future perspectives for further investigation. A first, natural direction is to refine and complete the understanding and classification of duality symmetries in toroidal theories. While our geometrically driven interpretation and construction provides a systematic way to generate a large class of non-invertible duality symmetries at rational points, a complete classification remains to be developed. In particular, as already noted above, not all duality symmetries arise from the crystallographic symmetries of the torus and, therefore, cannot be understood using the method introduced in this thesis. Understanding precisely which symmetries are excluded by this construction, and how to characterize them, is an open problem.

In this context, it would be especially interesting to clarify the relation between duality symmetries and the complex multiplication (CM) property of the tori appearing at RCFT points in moduli space. Since CM points are precisely those with enhanced endomorphism rings, and since our construction is based on the crystallographic symmetries of the torus lattice, one may expect a deep connection between duality symmetries and the geometric and algebraic features associated with complex multiplication.

A second natural direction is the extension of these ideas beyond toroidal compactifications to higher–dimensional Calabi-Yau target spaces, such as K3 surfaces or Calabi-Yau threefolds. Indeed, these models are more relevant from the point of view of string phenomenology, as they provide realistic compactifications with rich geometric and physical structures. However, unlike the torus case, the RCFT points in these moduli spaces do not appear to be dense, which makes the explicit analysis significantly more difficult [80]. In practice, much of the worldsheet control for these non-linear sigma models relies on special, exactly solvable points, in particular, Gepner models, which are directly related to the RCFT points of Calabi-Yau manifolds [82]. Determining whether non-invertible duality symmetries can be systematically constructed or classified in these settings would, therefore, represent a conceptual step forward in our understanding of string compactification.

Finally, it would be interesting to explore the potential implications for the Swampland program [83, 84]. Since duality defects encode non-invertible structures, monodromies, and degenerations in moduli space, they may provide new insights into aspects such as the distance conjecture [85] and the behavior of mixed Hodge structures near infinite-distance limits [86].

Taken together, the results of this thesis indicate that non-invertible duality symmetries occupy an important, but still only partially understood, position in two dimensional conformal field theories and string compactifications. Although the geometric construction developed here clarifies several aspects of duality symmetries in toroidal models, many questions remain open. A complete classification of these symmetries, a deeper understanding of their relation to complex multiplication and the extension of these ideas to higher-dimensional Calabi-Yau manifolds all represent important directions for future research. Addressing these problems may lead to a more systematic picture of symmetries in non-linear sigma models and could also provide us with new insights into broader issues, such as quantum-gravity constraints and swampland conjectures. For these reasons, non-invertible duality symmetries represent a promising direction for future developments in high-energy physics.

References

[1] E. Noether, *Invariante variationsprobleme*, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* **1918** (1918) 235–257.

[2] V. I. Arnold, *Mathematical Methods of Classical Mechanics*. Graduate Texts in Mathematics. Springer, 1989, 10.1007/978-1-4757-2063-1.

[3] A. Einstein, *Zur Elektrodynamik bewegter Körper*, *Annalen Phys.* **322** (1905) 891–921.

[4] W. Rindler, *Introduction to Special Relativity*. Clarendon., 1991.

[5] S. M. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*. Cambridge University Press, 7, 2019, 10.1017/9781108770385.

[6] R. M. Wald, *General Relativity*. Chicago Univ. Pr., Chicago, USA, 1984, 10.7208/chicago/9780226870373.001.0001.

[7] J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics*. Quantum physics, quantum information and quantum computation. Cambridge University Press, 3 ed., 10, 2020, 10.1017/9781108587280.

[8] M. E. Peskin and D. V. Schroeder, *An Introduction to quantum field theory*. Addison-Wesley, Reading, USA, 1995, 10.1201/9780429503559.

[9] S. Weinberg, *The Quantum theory of fields. Vol. 1: Foundations*. Cambridge University Press, 6, 2005, 10.1017/CBO9781139644167.

[10] M. D. Schwartz, *Quantum Field Theory and the Standard Model*. Cambridge University Press, 3, 2014.

[11] D. Gaiotto, A. Kapustin, N. Seiberg and B. Willett, *Generalized Global Symmetries*, *JHEP* **02** (2015) 172, [1412.5148].

[12] S. Schafer-Nameki, *ICTP lectures on (non-)invertible generalized symmetries*, *Phys. Rept.* **1063** (2024) 1–55, [2305.18296].

[13] T. D. Brennan and S. Hong, *Introduction to Generalized Global Symmetries in QFT and Particle Physics*, 2306.00912.

[14] G. 't Hooft, *On the phase transition towards permanent quark confinement*, *Nuclear Physics B* **138** (1978) 1–25.

[15] A. Kapustin and N. Seiberg, *Coupling a qft to a tqft and duality*, *Journal of High Energy Physics* **2014** (2014) 1.

[16] N. Kan, K. Kawabata and H. Wada, *Symmetry fractionalization and duality defects in Maxwell theory*, *JHEP* **10** (2024) 238, [2404.14481].

[17] D. M. Hofman and N. Iqbal, *Goldstone modes and photonization for higher form symmetries*, *SciPost Phys.* **6** (2019) 006.

[18] E. Lake, *Higher-form symmetries and spontaneous symmetry breaking*, 1802.07747.

[19] Y. Makeenko, *A Brief Introduction to Wilson Loops and Large N*, *Phys. Atom. Nucl.* **73** (2010) 878–894, [0906.4487].

[20] H. A. Kramers and G. H. Wannier, *Statistics of the two-dimensional ferromagnet. part i*, *Phys. Rev.* **60** (Aug, 1941) 252–262.

[21] J. Fröhlich, J. Fuchs, I. Runkel and C. Schweigert, *Kramers-wannier duality from conformal defects*, *Phys. Rev. Lett.* **93** (Aug, 2004) 070601.

[22] J. Frohlich, J. Fuchs, I. Runkel and C. Schweigert, *Duality and defects in rational conformal field theory*, *Nucl. Phys. B* **763** (2007) 354–430, [hep-th/0607247].

[23] L. Bhardwaj and Y. Tachikawa, *On finite symmetries and their gauging in two dimensions*, *JHEP* **03** (2018) 189, [1704.02330].

[24] *Topological defect lines and renormalization group flows in two dimensions*, *Journal of High Energy Physics* **2019** (2019) 26.

[25] R. Thorngren and Y. Wang, *Fusion category symmetry. Part I. Anomaly in-flow and gapped phases*, *JHEP* **04** (2024) 132, [1912.02817].

[26] R. Thorngren and Y. Wang, *Fusion category symmetry. part ii. categoriosities at c = 1 and beyond*, *Journal of High Energy Physics* **2024** (2024) 51.

[27] D. Gaiotto and J. Kulp, *Orbifold groupoids*, *JHEP* **02** (2021) 132, [2008.05960].

[28] T.-C. Huang, Y.-H. Lin and S. Seifnashri, *Construction of two-dimensional topological field theories with non-invertible symmetries*, *Journal of High Energy Physics* **2021** (2021) 28.

[29] M. Koide, Y. Nagoya and S. Yamaguchi, *Non-invertible topological defects in 4-dimensional $\mathbb{Z}_2$ pure lattice gauge theory*, *PTEP* **2022** (2022) 013B03, [2109.05992].

[30] Y. Choi, C. Cordova, P.-S. Hsin, H. T. Lam and S.-H. Shao, *Noninvertible duality defects in 3+1 dimensions*, *Phys. Rev. D* **105** (2022) 125016, [2111.01139].

[31] Y. Hayashi and Y. Tanizaki, *Non-invertible self-duality defects of Cardy-Rabinovici model and mixed gravitational anomaly*, *EPJ Web Conf.* **274** (2022) 02010.

[32] Y. Choi, C. Cordova, P.-S. Hsin, H. T. Lam and S.-H. Shao, *Non-invertible Condensation, Duality, and Triality Defects in 3+1 Dimensions*, *Commun. Math. Phys.* **402** (2023) 489–542, [2204.09025].

[33] J. Kaidi, K. Ohmori and Y. Zheng, *Kramers-wannier-like duality defects in $(3+1)d$ gauge theories*, *Phys. Rev. Lett.* **128** (Mar, 2022) 111601.

[34] A. A. Cox, E. Poppitz and S. S. Y. Wong, *Domain walls and deconfinement: a semiclassical picture of discrete anomaly inflow*, *JHEP* **12** (2019) 011, [1909.10979].

[35] D. Delmastro and J. Gomis, *Domain walls in 4d$\mathcal{N}$ = 1 SYM*, *JHEP* **03** (2021) 259, [2004.11395].

[36] L. Bhardwaj, L. E. Bottini, S. Schafer-Nameki and A. Tiwari, *Non-invertible higher-categorical symmetries*, *SciPost Phys.* **14** (2023) 007, [2204.06564].

[37] S.-H. Shao, *What's Done Cannot Be Undone: TASI Lectures on Non-Invertible Symmetries*, 2308.00747.

[38] P. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal field theory*. Graduate texts in contemporary physics. Springer, New York, NY, 1997, 10.1007/978-1-4612-2256-9.

[39] P. H. Ginsparg, *Curiosities at c = 1*, *Nucl. Phys. B* **295** (1988) 153–170.

[40] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, *Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory*, *Nucl. Phys. B* **241** (1984) 333–380.

[41] A. Kapustin and N. Saulina, *Surface operators in 3d Topological Field Theory and 2d Rational Conformal Field Theory*, 1012.0911.

[42] D. Tambara and S. Yamagami, *Tensor categories with fusion rules of self-duality for finite abelian groups*, *Journal of Algebra* **209** (1998) 692–707.

[43] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 12, 2007, 10.1017/CBO9780511816079.

[44] M. B. Green, J. H. Schwarz and E. Witten, *SUPERSTRING THEORY. VOL. 1: INTRODUCTION*. Cambridge Monographs on Mathematical Physics. 7, 1988.

[45] C. G. Callan, Jr., E. J. Martinec, M. J. Perry and D. Friedan, *Strings in Background Fields*, *Nucl. Phys. B* **262** (1985) 593–609.

[46] K. Hori and C. Vafa, *Mirror symmetry*, `hep-th/0002222`.

[47] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa et al., *Mirror symmetry*, vol. 1 of *Clay mathematics monographs*. AMS, Providence, USA, 2003.

[48] S. Bharadwaj, P. Niro and K. Roumpedakis, *Non-invertible defects on the worldsheet*, *Journal of High Energy Physics* **2025** (2025) 164.

[49] M. Srednicki, *Quantum field theory*. Cambridge University Press, 1, 2007, 10.1017/CBO9780511813917.

[50] G. T. Horowitz and M. Srednicki, *A Quantum Field Theoretic Description of Linking Numbers and Their Generalization*, *Commun. Math. Phys.* **130** (1990) 83–94.

[51] Y. Choi, B. C. Rayhaun, Y. Sanghavi and S.-H. Shao, *Remarks on boundaries, anomalies, and noninvertible symmetries*, *Phys. Rev. D* **108** (2023) 125005, [`2305.09713`].

[52] O. Diatlyk, C. Luo, Y. Wang and Q. Weller, *Gauging non-invertible symmetries: topological interfaces and generalized orbifold groupoid in 2d qft*, *Journal of High Energy Physics* **2024** (2024) 127.

[53] A. Perez-Lona, D. Robbins, E. Sharpe, T. Vandermeulen and X. Yu, *Notes on gauging noninvertible symmetries. Part I. Multiplicity-free cases*, *JHEP* **02** (2024) 154, [`2311.16230`].

[54] P. Etingof, D. Nikshych and V. Ostrik, *On fusion categories*, `math/0203060`.

[55] C.-M. Chang, Y.-H. Lin, S.-H. Shao, Y. Wang and X. Yin, *Topological Defect Lines and Renormalization Group Flows in Two Dimensions*, *JHEP* **01** (2019) 026, [`1802.04445`].

[56] J.-P. Serre, *Linear representations of finite groups*. Springer-Verlag, New York, 1977.

[57] L. Bhardwaj, L. E. Bottini, L. Fraser-Taliente, L. Gladden, D. S. W. Gould, A. Platschorre et al., *Lectures on generalized symmetries*, *Phys. Rept.* **1051** (2024) 1–87, [`2307.07547`].

[58] T. Kobayashi, H. Otsuka, M. Tanimoto and T. T. Yanagida, *GUT-motivated non-invertible symmetry as a solution to the strong CP problem and the neutrino CP-violating phase*, 2510.01680.

[59] J. H. Christenson, J. W. Cronin, V. L. Fitch and R. Turlay, *Evidence for the 2π decay of the k_2^0 meson*, *Phys. Rev. Lett.* **13** (Jul, 1964) 138–140.

[60] R. D. Peccei and H. R. Quinn, CP *conservation in the presence of pseudoparticles*, *Phys. Rev. Lett.* **38** (Jun, 1977) 1440–1443.

[61] M. Ibe, S. Shirai, M. Suzuki and T. T. Yanagida, *Novel gut with apparently complete su(5) multiplets*, *Phys. Rev. D* **100** (Sep, 2019) 055024.

[62] T. Kobayashi, H. Otsuka and T. T. Yanagida, *Non-invertible Symmetry as a Solution to the Strong CP Problem in a GUT-inspired Standard Model*, 2508.12287.

[63] P. H. Ginsparg, *APPLIED CONFORMAL FIELD THEORY*, in *Les Houches Summer School in Theoretical Physics: Fields, Strings, Critical Phenomena*, 9, 1988, hep-th/9108028.

[64] R. Dijkgraaf, E. Verlinde and H. Verlinde, *C=1 conformal field theories on riemann surfaces*, *Communications in Mathematical Physics* **115** (1988) 649–690.

[65] G. Moore and N. Seiberg, *Classical and quantum conformal field theory*, *Communications in Mathematical Physics* **123** (1989) 177–254.

[66] A. Sen, *An Introduction to rational conformal field theories*, *Comments Nucl. Part. Phys.* **20** (1991) 23–67.

[67] Y. Furuta, *On the classification of duality defects in c = 2 compact boson cfts with a discrete group orbifold*, *Journal of High Energy Physics* **2025** (2025) 219.

[68] P. Bantay, *The Frobenius-Schur indicator in conformal field theory*, *Phys. Lett. B* **394** (1997) 87–88, [hep-th/9610192].

[69] Y. Choi, D.-C. Lu and Z. Sun, *Self-duality under gauging a non-invertible symmetry*, *Journal of High Energy Physics* **2024** (2024) 142.

[70] J. A. Damia, G. Galati, O. Hulik and S. Mancani, *Exploring duality symmetries, multicriticality and rg flows at c = 2*, *Journal of High Energy Physics* **2024** (2024) 28.

[71] N. Ishibashi, *The Boundary and Crosscap States in Conformal Field Theories*, *Mod. Phys. Lett. A* **4** (1989) 251.

[72] A. Recknagel and V. Schomerus, *D-branes in Gepner models*, *Nucl. Phys. B* **531** (1998) 185–225, [hep-th/9712186].

[73] J. L. Cardy, *Boundary Conditions, Fusion Rules and the Verlinde Formula*, *Nucl. Phys. B* **324** (1989) 581–596.

[74] S. Dulat and K. Wendland, *Crystallographic orbifolds: Towards a classification of unitary conformal field theories with central charge c = 2*, *JHEP* **06** (2000) 012, [hep-th/0002227].

[75] A. Giveon, M. Porrati and E. Rabinovici, *Target space duality in string theory*, *Phys. Rept.* **244** (1994) 77–202, [hep-th/9401139].

[76] M. Rocek and E. P. Verlinde, *Duality, quotients, and currents*, *Nucl. Phys. B* **373** (1992) 630–646, [hep-th/9110053].

[77] E. Alvarez, L. Alvarez-Gaume and Y. Lozano, *An Introduction to T duality in string theory*, *Nucl. Phys. B Proc. Suppl.* **41** (1995) 1–20, [hep-th/9410237].

[78] O. Hohm, C. Hull and B. Zwiebach, *Generalized metric formulation of double field theory*, *JHEP* **08** (2010) 008, [1006.4823].

[79] G. Pólya, *Xii. Über die analogie der kristallsymmetrie in der ebene*, *Zeitschrift für Kristallographie - Crystalline Materials* **60** (1924) 278–282.

[80] S. Gukov and C. Vafa, *Rational conformal field theories and complex multiplication*, *Commun. Math. Phys.* **246** (2004) 181–210, [hep-th/0203213].

[81] S. Bharadwaj, P. Niro and K. Roumpedakis, *Non-invertible defects on the worldsheet*, *Journal of High Energy Physics* **2025** (2025) 164.

[82] C. Córdova and G. Rizi, *Non-invertible symmetry in calabi-yau conformal field theories*, *Journal of High Energy Physics* **2025** (2025) 45.

[83] C. Vafa, *The String landscape and the swampland*, hep-th/0509212.

[84] M. van Beest, J. Calderón-Infante, D. Mirfendereski and I. Valenzuela, *Lectures on the Swampland Program in String Compactifications*, *Phys. Rept.* **989** (2022) 1–50, [2102.01111].

[85] T. W. Grimm, E. Palti and I. Valenzuela, *Infinite distances in field space and massless towers of states*, *Journal of High Energy Physics* **2018** (2018) 143.

[86] T. W. Grimm, F. Ruehle and D. van de Heisteeg, *Classifying calabi–yau threefolds using infinite distance limits*, *Communications in Mathematical Physics* **382** (2021) 239–275.