Alma Mater Studiorum · Università di Bologna

Dipartimento di Fisica e Astronomia "Augusto Righi"
Corso di Laurea in Fisica

Decoherence in quantum systems and the Lindbladian

Relator:
Prof. Elisa Ercolessi

Presented by:
Susanna Bellentani

Academic Year 2024/2025

Abstract

This thesis presents an introduction to the concept of decoherence in quantum systems and how it can be described by the Lindbladian master equation. The paper includes a preparatory section about the foundations of quantum mechanics and the concept of qubit, other than a discussion about how to describe the state, evolution, and measurement of a quantum system by using the density matrix formalism. Then, superoperators are defined as a tool to study the evolution of a system coupled to its environment, as well as a mathematical instrument that is able to describe both the concept of unitary (coherent) evolution and the measurement process. By using the properties of superoperators, the Lindbladian is introduced as an alternative to the Schrödinger equation, as it is capable of describing the temporal evolution of an open quantum system interacting with its environment. Lastly, as an application, this paper provides an analysis of phase damping, as it is an explicative example of the effects of decoherence.

Abstract

La seguente tesi presenta una trattazione introduttiva al concetto di decoerenza nei sistemi quantistici e di come questa possa essere descritta dall'equazione di Lindblad. Lo scritto è complessivo di una parte propedeutica riguardante i fondamenti di meccanica quantistica e il concetto di qubit, oltre a una trattazione su come descrivere stato, evoluzione e misura di un sistema quantistico tramite il formalismo delle matrici densità. Proseguendo, si definiscono i superoperatori come strumento per permettere di studiare l'evoluzione di un sistema immerso nell'ambiente, nonché come strumento matematico in grado di racchiudere sia il concetto di evoluzione unitaria (coerente) che il processo di misura. Tramite il concetto di superoperatore, si introduce l'equazione di Lindblad come alternativa all'equazione di Schrödinger, in quanto in grado di descrivere l'evoluzione temporale di un sistema quantistico aperto, che interagisce con l'ambiente. Infine si porta, come applicazione, l'analisi del fenomeno dello smorzamento di fase, in quanto esemplificativo degli effetti della decoerenza.

Contents

Introduction

In the context of physics, an endless number of theories have been developed through the years, as endless is the number of ways in which we can approach the same problem, without ever getting to the end of it. For example, something that will always be central is the study of the evolution of the state of a system; at university, we have gone through this process multiple times, starting from classical mechanics, proceeding to analytical mechanics, and ending at quantum mechanics.

On a smaller scale, this process is the same that physicists went through through the years, by trying to expand the boundaries of our knowledge. The branch of quantum mechanics emerged as, in the early 20th century, some problems were found that Newtonian mechanics and Maxwell's electrodynamics were unable to answer to; for example, the photoelectric effect or the black body radiation, where classical predictions did not match the experimental results.

Something unique about quantum mechanics is the non-deterministic interpretation that it gives of reality; while classical physics affirms that, given the initial conditions, any process is deterministic, quantum mechanics is intrinsically probabilistic: I cannot observe or interact with a quantum system without inevitably making its state collapse in one of many possible states, each occurring with a certain probability, causing it to lose its coherence. This process is called *decoherence*, and we will see it to be central in order to understand how interactions between our system and the environment (which can be, for example, me trying to measure a property of the system under observation, or a particle being hit by a large number of photons) inevitably cause a loss of information.

We will see how it is possible to mathematically describe this loss of information, but there is no way, from our human macroscopical perspective, to stop decoherence from happening. From my point of view, the decohering process evidences the limits that we hop into by being really big and massive in comparison to the processes we study through quantum mechanics, and the fact that we can not pretend to describe reality in its totality since we too are part of it.

Before we can further discuss these concepts, we will describe all that is necessary to define what a quantum system is and the different ways in which its state can change.

Chapter 1

Foundations of quantum information

In this chapter, we will briefly go through what quantum mechanics is based on and introduce the qubit as the quantum version of a classical bit in the context of quantum computation and information. If needed, a deeper look at the foundations necessary for our purposes is given by [1].

1.1 Axioms of quantum mechanics

In order to talk about quantum theory, we need to define what states, observables, measurement, and dynamics are in this mathematical model of the physical world.

1. **States**: the state of a physical system (at a certain time t) is a ray in an appropriate Hilbert space. A Hilbert space is a vector space over the complex numbers $\mathbb{C}$ where we use Dirac's ket notation to write his vectors: $|\psi\rangle \in \mathcal{H}$.

   Hilbert spaces have an inner product $\langle\psi|\phi\rangle \in \mathbb{C}, \forall\, |\phi\rangle, |\psi\rangle \in \mathcal{H}$ defined by the properties:

   (a) $\langle\lambda\psi|\phi\rangle = \overline{\lambda}\,\langle\psi|\phi\rangle$;
   (b) $\langle\psi|\lambda\phi\rangle = \lambda\,\langle\psi|\phi\rangle$;
   (c) $\langle\psi|\phi\rangle = \overline{\langle\phi|\psi\rangle}$;
   (d) $\langle\psi|\psi\rangle = ||\psi||^2 \geq 0$.

   We can now define a ray as an equivalence class of vectors that differ by multiplication by a nonzero complex scalar, and we choose a representative of this class to have unit norm: $\langle\psi|\psi\rangle = 1$. We note that $|\psi\rangle$ and $e^{i\alpha}|\psi\rangle$ describe the same physical state of the system and that given two vectors, we can form another as $a|\psi\rangle + b|\phi\rangle$, with $a, b \in \mathbb{C}$ (the *superposition principle*).

2. **Observables**: an observable is a physical quantity that we can decide to measure. In quantum mechanics, an observable is a self adjoint operator on $\mathcal{H}$: $\mathbf{A} = \mathbf{A}^\dagger$ with $\mathbf{A}^\dagger$ defined by $\langle\psi|\mathbf{A}\phi\rangle = \langle\mathbf{A}^\dagger\psi|\phi\rangle$.

   It is important to say that any self-adjoint operator has a spectral representation: $\mathbf{A}\,|\psi_i\rangle = \lambda_i\,|\psi_i\rangle$ with the important properties that all the eigenvalues $\lambda_i \in \mathbb{R}$, and the eigenvectors $|\psi_i\rangle$ form an orthonormal basis in $\mathcal{H}$.

   We can re-formulate these properties by using orthogonal projectors $\mathbf{P}_i$ (with $\mathbf{P}_i^\dagger = \mathbf{P}_i$) which project on the associated eigenvector $|\psi_i\rangle$ with corresponding eigenvalue λ_i. The advantage we obtain by using projectors is that we work with mathematical objects that are truly independent from total phases, as:

$$\mathbf{P}_{|\psi\rangle} = \frac{|\psi\rangle\,\langle\psi|}{\langle\psi|\psi\rangle},$$

   but because we imposed $\langle\psi|\psi\rangle = 1, \mathbf{P}_\psi = |\psi\rangle\,\langle\psi|$ and

$$\mathbf{P}_{|\psi\rangle} = e^{i\alpha}\,|\psi\rangle\,e^{-i\alpha}\,\langle\psi| = (e^{i\alpha}e^{-i\alpha})\,|\psi\rangle\,\langle\psi| = |\psi\rangle\,\langle\psi|\,.$$

   We re-express the spectral representation of a self-adjoint operator as $\mathbf{A} = \sum_i \lambda_i \mathbf{P}_i$ and point out that the property of the existence of a complete orthonormal basis can be found in the definition of orthogonal projectors: $\mathbf{P}_j\mathbf{P}_k = \delta_{jk}\mathbf{P}_j$; $\sum_i \mathbf{P}_i = \mathbb{I}$.

   Another useful thing to point out is that, given a normalized state $|\psi\rangle$, the expectation value of an observable $\mathbf{A}$:

$$\langle\mathbf{A}\rangle = \langle\psi|\mathbf{A}|\psi\rangle = tr[\mathbf{P}_\psi\mathbf{A}].$$

3. **Measurement**: Something unique about quantum mechanics is the probabilistic and destructive nature of measurement processes. The numerical outcome of an orthogonal measurement on a system described by the state $|\psi\rangle$ based on an observable $\mathbf{A}$ is an eigenvalue of $\mathbf{A}$, obtained with probability:

$$Prob(\lambda_i) = p_i = ||\mathbf{P}_i\,|\psi\rangle\,||^2 = \langle\psi|\mathbf{P}_i|\psi\rangle\,.$$

   We point out that the physical request $\langle\psi|\psi\rangle = 1$ grants us that $\sum_i p_i = 1$ which is essential to maintain the significance of the model.

   After the measurement has occurred, the initial state collapses, and our new quantum state becomes:

$$\frac{\mathbf{P}_i\,|\psi\rangle}{(\langle\psi|\mathbf{P}_i|\psi\rangle)^{\frac{1}{2}}}.$$

4. **Dynamics**: The time evolution of a quantum system is determined by the Hamiltonian, as described by the Schrödinger equation:

$$\frac{d}{dt}|\psi(t)\rangle = -i\mathbf{H}|\psi(t)\rangle$$

where we imposed $\hbar = 1$ to lighten up the formalism. This equation is linear and only contains first-order derivatives: this means that if $|\psi(0)\rangle$ is given, $|\psi(t)\rangle$ is completely determined.

If we rewrite this equation to first order in the infinitesimal quantity dt, we obtain $|\psi(t+dt)\rangle = (\mathbb{I} - i\mathbf{H}dt)|\psi(t)\rangle$. The operator $\mathbf{U}(dt) \equiv \mathbb{I} - i\mathbf{H}dt$ is unitary ($\mathbf{U}^\dagger\mathbf{U} = 1$ because $\mathbf{H}$ is self adjoint). Since a product of unitary operators is finite, time evolution over a finite interval is also unitary. We can write $|\psi(t)\rangle = \mathbf{U}(t)|\psi(0)\rangle$ and if $\mathbf{H}$ is t-independent, $\mathbf{U} = e^{-i\mathbf{H}t}$. It is important to notice that unitary operators, unlike self-adjoint operators in the measurement process, preserve the scalar product, and consequently preserve the probabilistic interpretation:

$$|\psi_1{}'\rangle = \mathbf{U}|\psi_1\rangle \ ; \ |\psi_2{}'\rangle = \mathbf{U}|\psi_2\rangle$$

$$\langle\psi_1'|\psi_2'\rangle = \langle\psi_1|\mathbf{U}^\dagger\mathbf{U}|\psi_2\rangle = \langle\psi_1|\psi_2\rangle \ , \text{ as } \mathbf{U}^\dagger\mathbf{U} = 1$$

1.2 The qubit

We are now able to describe a generic quantum system, but in order to talk about quantum information, we need to restate the previous axioms in terms of a system made of *qubits*.

As the indivisible unit of classical information is called the *bit*, the corresponding unit of quantum information is called the "quantum bit", or *qubit* which describes the simplest possible quantum system. We will now briefly go through the Axioms of quantum mechanics in the case of a single qubit system.

1. **State**: The state of a qubit lives in the Hilbert space $\mathcal{H} = \mathbb{C}^2$, so that we can denote an orthonormal basis for a two dimensional vector space $\{|0\rangle\,;|1\rangle\}$ (computational basis); then, the most general normalized state of a qubit can be expressed as

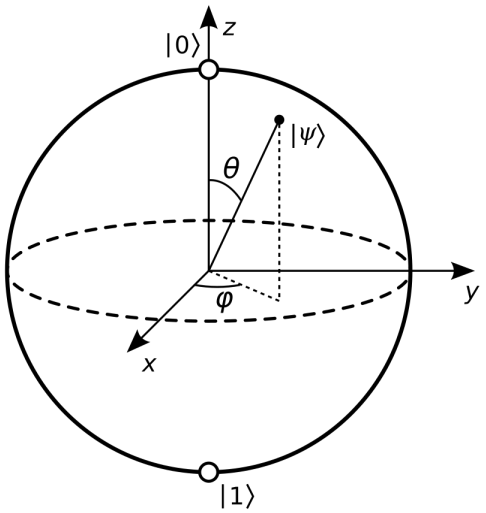
$$|\psi\rangle = a|0\rangle + b|1\rangle$$

with $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$. We can already see why the qubit is much stronger than his classical equivalent, as while in the classical model the unit can only assume the two values {0,1}, a single qubit can be found in an infinite number of different states.

A useful way to visualize the state of a qubit is by identifying it as a point on the Bloch sphere: let us get back to the equation $|\psi\rangle = a\,|0\rangle + b\,|1\rangle$ and re-define $a = |a|e^{i\phi_1}$; $b = |b|e^{i\phi_2}$, so that we can write: $|\psi\rangle = |a|e^{i\phi_1}\,|0\rangle + |b|e^{i\phi_2}\,|1\rangle$. If we factor out $e^{i\phi_1}$ and assume that $|a| = cos(\frac{\theta}{2})$ and $|b| = sin(\frac{\theta}{2})$, we obtain the state:

$$|\psi\rangle = cos\frac{\theta}{2}\,|0\rangle + sin\frac{\theta}{2}e^{i\phi}\,|1\rangle$$

with $\theta \in (0, \pi)$; $\phi \in (0, 2\pi)$ and $\phi = \phi_2 - \phi_1$ (we purposely ignored the factor $e^{i\phi_1}$, as it can be identified with a global phase, which is irrelevant). We can see that $|\psi\rangle$ can be interpreted as a point on S^2 (the Bloch sphere).



the Bloch Sphere

2. **Observable**: As we already said, any observable in quantum mechanics needs to be a self-adjoint operator; in the case of a single qubit, it needs to be a 2x2 matrix. A real basis to write all possible 2x2 self-adjoint matrices is given by the identity matrix and the Pauli matrices: $\{\mathbb{I}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$, so that any self-adjoint operator that can represent an observable of a single qubit system can be written as:

$$A = a_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + a_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with $a_0, a_1, a_2, a_3 \in \mathbb{R}$.

3. **Measurement:** We will further explore measurement on qubit systems in the next chapter, as it is more useful for our purposes to talk about it using the formalism of the *density matrices*.

4. **Evolution:** The evolution of the state of a single qubit system $|\psi\rangle$ can be identified as a rotation of the state around a chosen axis on the Bloch sphere. We know that any unitary operator can be written as $\mathbf{U}(t) = e^{-i\mathbf{A}}$, with $\mathbf{A}$ being a self-adjoint operator, so that, in the case of a single qubit, we can write:

$$\mathbf{U}(t) = e^{-i(a_0\mathbb{I}+\overrightarrow{a}\cdot\overrightarrow{\boldsymbol{\sigma}})}.$$

By assuming that $\overrightarrow{a} = \frac{\theta}{2}\hat{n}$ with $||\hat{n}|| = 1$, we re-write $\mathbf{U}$(t) as $\mathbf{U}(t) = e^{-ia_0}e^{-i\frac{\theta}{2}\hat{n}\cdot\overrightarrow{\boldsymbol{\sigma}}}$ where we can ignore e^{-ia_0} being a total phase. By expanding the exponential series, we find our final expression for any unitary operator acting on a single qubit:

$$\mathbf{U}(t) = e^{-i\frac{\theta}{2}\hat{n}\cdot\overrightarrow{\sigma}} = cos\frac{\theta}{2}\mathbb{I} - isin\frac{\theta}{2}\hat{n}\cdot\overrightarrow{\boldsymbol{\sigma}}$$

which, in fact, originates a rotation of the state of an angle θ around the $\hat{n}$ axis (on the Bloch sphere).

We are now ready to get into the next chapter and abandon single qubit systems to get into the studying of two (or more) qubit systems, which are much more complex. There is indeed something to consider that we did not have to before: the fact that qubits can *interact* with each other.

A two qubit system lives in a Hilbert space that is the tensor product of the Hilbert spaces of two single qubits $\mathcal{H}_T = \mathcal{H}_1 \otimes \mathcal{H}_2$ whose dimension scales as $d_T = d_1 \cdot d_2$. A generic state living in $\mathcal{H}_T$ can be written as: $|\psi\rangle = \sum_{ij} |\psi\rangle_\alpha |\phi\rangle_\beta$, where $|\psi\rangle_\alpha \in \mathcal{H}_1,\ |\phi\rangle_\beta \in \mathcal{H}_2$. If $|\psi\rangle$ can be written as the product between a vector in $\mathcal{H}_1$ and a vector in $\mathcal{H}_2$, we are dealing with a *separable* state (or *product* state), otherwise, if that's not possible, the state is *entangled*.
For example, the state

$$|\psi\rangle = \frac{|0\rangle_1 |0\rangle_2 + |0\rangle_1 |1\rangle_2}{\sqrt{2}}$$

is a product state, as it can be written as

$$|\psi\rangle = |0\rangle_1 \cdot \left(\frac{|0\rangle_2 + |1\rangle_2}{\sqrt{2}}\right) = |0\rangle_1 |+\rangle_2 ,$$

while the state

$$|\psi\rangle = \frac{|0\rangle_1 |0\rangle_2 + |1\rangle_1 |1\rangle_2}{\sqrt{2}}$$

is entangled, as it can't be written as a separable state and is characterized by a strong correlation between the two qubits. Entanglement only exists within the quantum mechanics formalism and allows us to develop systems that are impossible in the context of classical physics.

Chapter 2

The density matrix

In this chapter, we will see that even though the axioms presented in the previous section provide a perfectly reasonable formulation of the quantum theory, under many circumstances those appear to be violated. That happens because those axioms are intended to describe the quantum behavior of the whole universe, while our observations are always limited to a small part of a much larger quantum system. We need to consider the interactions between our small system and the environment in order to better understand the results given by this theory. We will now see that when we limit our study to just a part of a larger system, contrary to the axioms:

1. states are not rays, so that a state will not have to be characterized by having unit norm, but other options will be allowed;

2. measurements are not orthogonal projections, so that we needn't define an orthogonal set of operators in order to perform a measurement on our system;

3. evolution is not unitary, as it is impossible to completely avoid interference, so that reversibility will not be an option.

In order to study more complex systems, it is of great use to introduce the density matrix formalism.

2.1 Bipartite quantum systems

Let us consider the Hilbert space of a bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B$ with $\mathcal{H}_A$ and $\mathcal{H}_B$ being the Hilbert spaces of the single systems. We can imagine, for example, for system A to be the one we are observing, and for system B to be our environment. Because our bipartite system is the tensor product between two Hilbert spaces, if $\{|i\rangle_A\}$ is an orthogonal basis for $\mathcal{H}_A$ and $\{|k\rangle_B\}$ is an orthogonal basis for $\mathcal{H}_B$, then $|i\rangle_A \otimes |k\rangle_B$ is an orthonormal

basis for the composite space. A general pure state of $\mathcal{H}_A \otimes \mathcal{H}_B$ can be written as:

$$|\psi\rangle_{AB} = \sum_{i,k} \lambda_{ik} |i\rangle_A \otimes |k\rangle_B \ , \text{ with } \sum_{i,k} |\lambda_{ik}|^2 = 1.$$

While doing observations, we often focus only on the subsystem A, as we care to measure the system we prepared, not the environment. Then, the expectation value of an observable acting only on subsystem A:

$$\begin{aligned}
\langle \mathbf{M}_A \rangle &= {}_{AB}\langle \psi | \mathbf{M}_A \otimes \mathbb{I}_\mathbf{B} | \psi \rangle_{AB} \\
&= \sum_{j,l} \lambda^*_{jl} ({}_A\langle j| \otimes {}_B\langle l|)(M_A \otimes \mathbb{I}_\mathbf{B}) \sum_{i,k} \lambda_{ik} (|i\rangle_A \otimes |k\rangle_B) \\
&= \sum_{i,j,k} \lambda^*_{jk} \lambda_{ik} {}_A\langle j | \mathbf{M}_A | i \rangle_A \ , \text{ as } {}_B\langle l | k \rangle_B \neq 0 \leftrightarrow l = k \\
&= tr(\mathbf{M}_A \boldsymbol{\rho}_A)
\end{aligned}$$

where

$$\boldsymbol{\rho}_A = tr_B(\boldsymbol{\rho}_{AB}) = tr_B(|\psi\rangle_{AB} \langle\psi|) = \sum_{i,j,k} \lambda^*_{jk} \lambda_{ik} |i\rangle_A \langle j| \, .$$

We call $\boldsymbol{\rho}_A$ the density matrix of system A, which is obtained by performing a partial trace over the density matrix of the bipartite system AB. From the definition we gave, three properties can be highlighted:

1. $\boldsymbol{\rho}_A$ is self-adjoint: $\boldsymbol{\rho}_A = \boldsymbol{\rho}_A^\dagger$ then, according to the spectral theorem, we can diagonalize the density matrix and obtain

$$\boldsymbol{\rho}_A = \sum_a p_a |\phi_a\rangle \langle \phi_a| \, ,$$

   with p_a being the eigenvalues of $\boldsymbol{\rho}_A$ and $|\phi_a\rangle$ the corresponding eigenvectors;

2. $\boldsymbol{\rho}_A$ is positive: $\forall |\psi\rangle_A \ ; \ {}_A\langle \psi | \boldsymbol{\rho}_A | \psi \rangle_A = \sum_k |\sum_i \lambda_{ikA} \langle \psi | i \rangle_A|^2 \geq 0$;

3. $tr(\boldsymbol{\rho}_A) = 1$: $tr(\boldsymbol{\rho}_A) = \sum_{i,k} |\lambda_{ik}|^2 = 1$, as $|\psi\rangle_{AB}$ is asked to be normalized.

Property n.2 is equivalent to asking that the eigenvalues $p_a \geq 0$ and property n.3 adds to it that we want $\sum_a p_a = 1$, then $0 \leq p_a \leq 1$.

The density matrix formalism allows us to describe both *pure* and *mixed* states: if the state $|\psi\rangle_A$ of subsystem A is a ray, we say that the state is *pure* and the corresponding density matrix $\boldsymbol{\rho}_A = |\psi\rangle_A \langle\psi|$ is the projector onto the one-dimensional space spanned by $|\psi\rangle_A$. In case of a density matrix of a pure state, then, we can add a property to the list:

4. $\boldsymbol{\rho}_A^2 = \boldsymbol{\rho}_A$

As, in the simplest case where the density matrix is diagonal, $\boldsymbol{\rho}_A^2 = (\sum_a p_a |\phi_a\rangle \langle\phi_a|)^2 = \sum_{a,a'} p_a p_{a'} |\phi_a\rangle \langle\phi_a|\phi_{a'}\rangle \langle\phi_{a'}| = p_a^2 |\phi_a\rangle \langle\phi_a|$. If we want this property to work out, p_a can only take on values 0 and 1. Then, because property n.3 still needs to be effective, we need all eigenvalues to be null except one, so that our density matrix becomes the projector onto the one-dimensional space spanned by the eigenvector corresponding to the only non-zero eigenvalue.

On the other hand, if our state is not a ray, it is called *mixed* and the sum $\boldsymbol{\rho}_A = \sum_a p_a |\phi_a\rangle \langle\phi_a|$ has two or more terms. Hence, in this case, $\boldsymbol{\rho}_A^2 \neq \boldsymbol{\rho}_A$. We may interpret the density matrix of a mixed state as describing an *ensemble* of pure quantum states, each occurring with his corresponding probability p_a.
This statistical aspect of quantum mechanics (*decoherence*) arises in system A when two systems interact with each other, or, in other words, when systems A and B are entangled and system B is inaccessible. Entanglement destroys the coherence of a superposition of states of A, so that some of the phases become inaccessible if we look at system A alone. In fact, every time we try to look at A alone, A collapses in one of a set of alternative pure states, each occurring with a specific probability.

2.1.1 Entanglement

We need to be careful to distinguish entanglement from simple correlation: if we consider a bipartite system of two qubits in the state $|0\rangle_A |0\rangle_B$, we can say that these two qubits are correlated, as they are both in the same state $|0\rangle$, but they are not entangled. We could easily obtain this state by asking the two preparers of systems A and B to prepare their qubits in the state $|0\rangle$, but entanglement cannot be created locally. The only way to entangle systems A and B is to allow the two systems to directly interact with each other. The only way to turn the state $|0\rangle_A |0\rangle_B$ into an entangled state like

$$\frac{|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B}{\sqrt{2}}$$

is by applying to it a collective unitary transformation $\mathbf{U}_{AB}$. Local unitary transformations like $\mathbf{U}_A \otimes \mathbf{U}_B$ and local measurement cannot entangle the state, we must bring the two qubits together and allow them to interact.

As we already said in the previous chapter, a bipartite pure state is entangled if it can't be written as the direct product of pure states in $\mathcal{H}_A$ and $\mathcal{H}_B$ (if it's not separable). If we are dealing with a separable state $|\psi\rangle_{AB} = |\xi\rangle_A \otimes |\chi\rangle_B$, the reduced density matrices $\boldsymbol{\rho}_A = |\xi\rangle_A \langle\xi|$ and $\boldsymbol{\rho}_B = |\chi\rangle_B \langle\chi|$ are pure. Any state that is not pure is separable, and in this case the reduced density matrices are mixed.

2.1.2 The density matrix for a single qubit

It is useful to see how pure and mixed states are represented on the Bloch sphere in the case of a single qubit system, as we can imagine it to be part of a two qubit system (a bipartite system) representing the environment and the system we prepared. By respecting properties 1-3, the density matrix for a single qubit can be written as:

$$\boldsymbol{\rho} = \frac{1}{2}\begin{pmatrix} 1+n_z & n_x - in_y \\ n_x + in_y & 1-n_z \end{pmatrix} = \frac{\mathbb{I} + \overrightarrow{n}\cdot\overrightarrow{\sigma}}{2}.$$

Due to property n.2 asking for positivity, the eigenvalues:

$$p_\pm = \frac{1 \pm |\overrightarrow{n}|}{2} \geq 0$$

which is the same as asking $|\overrightarrow{n}| \leq 1$. We can easily check that in the extreme case $|\overrightarrow{n}| = 1$, our eigenvalues are $p_+ = 1$; $p_- = 0$. From the previous section, we learned that when the eigenvalues of a density matrix can only take on these two values, property n.5 is in action, and the matrix represents a pure state. In all the other cases, when $|\overrightarrow{n}| < 1$ the density matrix describes a mixed state instead.
This leads us to the conclusion that, because no negative eigenvalues are allowed, there is a 1-1 correspondence between the possible states of a qubit and a *unit-3 ball* $0 \leq |\overrightarrow{n}| \leq 1$ which we can identify with the Bloch Sphere (even though it is in fact a ball, not a sphere). Mixed states are then points in the inner part of the ball, while pure states are, as said before, points on the surface of the ball.

A pure state is, in fact, a coherent superposition of states, as we are granted to have, with probability 1, a specific combination of rays with determined probabilities. For example,

$$\boldsymbol{\rho}_1 = |+\rangle\langle+|\,,\ \boldsymbol{\rho}_2 = |0\rangle\langle 0| \ \text{ with } \ |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

are both density matrices of pure states. If a system's state is pure, when a measurement is done on the sole system, we have certainty on the probabilities we will obtain from the different outcomes, which means there is no phase ambiguity on the state we find out. Differently, the density matrix of a mixed state is, for example:

$$\boldsymbol{\rho} = \frac{1}{3}\boldsymbol{\rho}_1 + \frac{2}{3}\boldsymbol{\rho}_2$$

this superposition is incoherent, as by not knowing which density matrix we are looking at each time we measure, there is no way to determine an expected probability from a measurement's outcome. In other words, there is no unique relative phase we can assign to every possible state as we are dealing with an ensemble of pure states.

2.2 Evolution of the density operator

Now that we have introduced bipartite systems, we need to discuss how they evolve in time.
Let us consider a bipartite pure state living in the space $\mathcal{H}_A \otimes \mathcal{H}_B$ whose Hamiltonian has the form:

$$\mathbf{H}_{AB} = \mathbf{H}_A \otimes \mathbb{I}_B + \mathbb{I}_A \otimes \mathbf{H}_B$$

If the Hamiltonian takes this form, the two systems are uncoupled and therefore evolve independently. Then, the time evolution operator for the bipartite system can be decomposed into separate unitary operators:

$$\mathbf{U}_{AB} = \mathbf{U}_A(t) \otimes \mathbf{U}_B(t).$$

According to Schrodinger's equation, an initial pure state:

$$|\psi(0)\rangle_{AB} = \sum_{i,k} \lambda_{ik} |i(0)\rangle_A \otimes |k(0)\rangle_B$$

evolves to

$$|\psi(t)\rangle_{AB} = \sum_{i,k} \lambda_{ik} \mathbf{U}_A(t) |i(0)\rangle_A \otimes \mathbf{U}_B(t) |k(0)\rangle_B .$$

By writing $|i(t)\rangle = \mathbf{U}_A(t) |i(0)\rangle_A$ and $|k(t)\rangle = \mathbf{U}_B(t) |k(0)\rangle_B$ and by taking the partial trace over system B, we obtain the evolved density matrix for system A:

$$\boldsymbol{\rho}_A(t) = tr_B(\boldsymbol{\rho}_{AB}(t)) = tr_B(|\psi(t)\rangle_{AB} \langle\psi(t)|) = \sum_{i,j,k} \lambda_{jk}\lambda_{ik}^* |i(t)\rangle_A \langle j(t)| = \mathbf{U}_A(t)\boldsymbol{\rho}_A\mathbf{U}_A(t)^\dagger.$$

In particular, in the basis in which $\boldsymbol{\rho}_A(0)$ is diagonal, we have:

$$\boldsymbol{\rho}_A(t) = \sum_a p_a \mathbf{U}_A(t) |\phi_a(0)\rangle_A \langle\phi_a(0)| \mathbf{U}_A(t)^\dagger$$

which is consistent with the ensemble interpretation. We see that if a state occurs with probability p_a at $t = 0$, then the probability is preserved by the unitary operator. Each pure state of the ensemble can always be found with the same probability in every stage of his time evolution.

2.3 Measurement

Our objective in this section is to examine the properties of generalized measurements on a system A that can be realized by performing an orthogonal measurement on the bipartite system AB.

2.3.1 Orthogonal measurements for density matrices

In the first chapter, we already talked about orthogonal measurement for rays in a Hilbert space; we will now restate those concepts by transposing them in the density matrix formalism. Given a set of operators $\{\mathbf{E}_a\}$ with the properties:

$$\mathbf{E}_a = \mathbf{E}_a^\dagger, \;\; \mathbf{E}_a\mathbf{E}_b = \delta_{ab}\mathbf{E}_a, \;\; \sum_a \mathbf{E}_a = \mathbb{I},$$

we can carry out a measurement procedure that will take the pure state described by the density matrix $\boldsymbol{\rho} = |\psi\rangle\langle\psi|$ to

$$\frac{\mathbf{E}_a |\psi\rangle\langle\psi| \mathbf{E}_a}{\langle\psi| \mathbf{E}_a |\psi\rangle}$$

with probability $Prob(a) = \langle\psi| \mathbf{E}_a |\psi\rangle$. The measurement results can be described by a density matrix obtained by summing over the various possible outcomes weighted by the probability of that outcome, in which case the measurements affects the initial pure state according to

$$|\psi\rangle\langle\psi| \rightarrow \sum_a \mathbf{E}_a |\psi\rangle\langle\psi| \mathbf{E}_a$$

that is an ensemble of pure states describing the measurement outcomes.
The problem now is that the set of operators $\{\mathbf{E}_a\}$, measuring the observable $\mathbf{M}_A$, is formed by the orthogonal projectors onto the eigenstates of the chosen observable. Hence, unless our initial pure state $\boldsymbol{\rho} = |\psi\rangle\langle\psi|$ happened to be an eigenstaste of the observable being measured (in which case the density matrix we obtain is pure), it must be a mixed state after the measurement. Therefore, the result of the measurement is

$$\boldsymbol{\rho} \rightarrow \sum_a \mathbf{E}_a \boldsymbol{\rho} \mathbf{E}_a.$$

2.3.2 Generalized measurement (POVM)

We would now like to bring the measurement concept beyond the orthogonal measurement we just defined. In order to do so, we will consider system A as part of a larger quantum system. There are many ways in which we can realize this request, we could, for example, suppose that our system A is extended to a tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$ or that the larger space has the structure of a direct sum $\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_A^\perp$. We will start by considering this second case, which is more intuitive.

Let us suppose that we are dealing with a space with structure $\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_A^\perp$ and that our observer "lives" in system A, so that he only has access to observables $\mathbf{M}_A$ with support in $\mathcal{H}_A$, such that

$$\mathbf{M}_A |\psi^\perp\rangle = 0 = \langle\psi^\perp| \mathbf{M}_A, \; \forall |\psi^\perp\rangle \in \mathcal{H}_A^\perp.$$

Let's now consider the bases $\{|i\rangle\}$ for $\mathcal{H}_A$ and $\{|k\rangle\}$ for $\mathcal{H}_A^\perp$. To obtain a generalized measurement in $\mathcal{H}_A$, we need to perform an orthogonal measurement on $\mathcal{H}$. Thus, we will consider once again the set of orthogonal projectors $\{\mathbf{E}_a\}$, where each projector can be written as $\mathbf{E}_a = |u_a\rangle\langle u_a|$, with $|u_a\rangle$ being a normalized vector in $\mathcal{H}$ (we will consider the special case of operators with only one non-vanishing eigenvalue). Each vector $|u_a\rangle$ has a unique orthogonal decomposition:

$$|u_a\rangle = |\psi_a\rangle + |\psi_a^\perp\rangle\,,$$

where $|\psi_a\rangle$ and $|\psi_a^\perp\rangle$ are unnormalized vectors in $\mathcal{H}_A$ and $\mathcal{H}_A^\perp$ respectively.
After the orthogonal measurement, the new density matrix will be $|u_a\rangle\langle u_a|$ with probability $\langle u_a|\boldsymbol{\rho}_A|u_a\rangle$, but because our observer will only know about the component of the obtained state in his space $\mathcal{H}_A$ (as he knows nothing about $\mathcal{H}_A^\perp$), from his point of view the outcome of the measurement is $|\psi_a\rangle\langle\psi_a|$ with probability $\langle\psi_a|\boldsymbol{\rho}_A|\psi_a\rangle$.
By writing $|\psi_a\rangle = \sqrt{\lambda_a}\,|\phi_a\rangle$, where $|\phi_a\rangle$ is a normalized state, we can define the operator

$$\mathbf{F}_a = \mathbf{E}_A\mathbf{E}_a\mathbf{E}_A = |\psi_a\rangle\langle\psi_a| = \lambda_a\,|\phi_a\rangle\langle\phi_a|\,,$$

with $\mathbf{E}_A$ being the orthogonal projector taking $\mathcal{H}$ to $\mathcal{H}_A$. We may then say that outcome a occurs with probability $tr(\mathbf{F}_a\boldsymbol{\rho})$ and that each $\mathbf{F}_a$ is hermitian and nonnegative, but it only is a projection on state $|\phi_a\rangle\langle\phi_a|$ if $\lambda_a = 1$. Furthermore, the $\mathbf{F}_a$'s sum to the identity on $\mathcal{H}_\mathcal{A}$:

$$\sum_a \mathbf{F}_a = \mathbf{E}_A(\sum_a \mathbf{E}_a)\mathbf{E}_A = \mathbf{E}_A = \mathbb{I}_A.$$

A partition of unity by nonnegative operators is called a *positive operator valued measurement* (POVM). In the generalized measurement theory (not in the case of one-dimensional operators), each outcome has a probability:

$$Prob(a) = tr(\boldsymbol{\rho}\mathbf{F}_a).$$

The positivity of $\mathbf{F}_a$ grants us the consistency of the physical interpretation by ensuring positive probabilities, while $\sum_a \mathbf{F}_a = \mathbb{I}_A$ ensures that probabilities sum to unity.
It is also useful to see the effect that a POVM by one dimensional operators has on the density matrix of our original quantum state:

$$\begin{aligned}
\boldsymbol{\rho} \to \boldsymbol{\rho}' &= \sum_a |\phi_a\rangle\langle\phi_a|\,(\lambda_a\langle\phi_a|\boldsymbol{\rho}|\phi_a\rangle) \\
&= \sum_a (\sqrt{\lambda_a}\,|\phi_a\rangle\langle\phi_a|)\boldsymbol{\rho}(\sqrt{\lambda_a}\,|\phi_a\rangle\langle\phi_a|) \\
&= \sum_a \sqrt{\mathbf{F}_a}\boldsymbol{\rho}\sqrt{\mathbf{F}_a},
\end{aligned}$$

which generalizes the result of the orthogonal measurement ($\sum_a \mathbf{E}_a\boldsymbol{\rho}\mathbf{E}_a$) in the case where the $\mathbf{F}_a$'s are not projectors.

2.3.3 Orthogonal measurement on a tensor product

In this section, we will see how we can perform a POVM on system A by performing an orthogonal measurement on the bipartite system AB. Let's consider two isolated systems A and B described by the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$ and suppose that we perform an orthogonal measurement on the tensor product with the set of mutually orthogonal projectors $\sum_a \mathbf{E}_a = 1$. If the two systems are not entangled, their state is described by a density matrix such as

$$\boldsymbol{\rho}_{AB} = \boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B;$$

the outcome a of the measurement occurs with probability

$$Prob(a) = tr_{AB}[\mathbf{E}_a(\boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B)]$$

and the measurement process transforms the density matrix as

$$\boldsymbol{\rho}'_{AB}(a) = \frac{\mathbf{E}_a(\boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B)\mathbf{E}_a}{tr_{AB}[\mathbf{E}_a(\boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B)]}.$$

This is the density matrix we would observe if we were observing both systems A and B, but we see that if we only have access to system A, the new density matrix for this system is given by the partial trace of the above:

$$\boldsymbol{\rho}'_A(a) = \frac{tr_B[\mathbf{E}_a(\boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B)\mathbf{E}_a]}{tr_{AB}[\mathbf{E}_a(\boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B)]}$$

and each outcome a, if watched by system A only, occurs with probability

$$Prob(a) = tr_A\{tr_{AB}[\mathbf{E}_a(\boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B)]\} = tr_A(\mathbf{F}_a\boldsymbol{\rho}_A).$$

We see that the same event a can be described by an orthogonal measurement on ρ_{AB} or by a POVM on ρ_A, since in both cases it occurs with the same probability.
By introducing orthonormal bases $\{|i\rangle_A\}$ for $\mathcal{H}_A$ and $\{|k\rangle_B\}$ for $\mathcal{H}_B$, we can rewrite the equation $tr_{AB}[\mathbf{E}_a(\boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B)] = \mathbf{F}_a\boldsymbol{\rho}_A$:

$$\sum_{ijkl}(E_a)_{jl,ik}(\rho_A)_{ij}(\rho_B)_{kl} = \sum_{ij}(F_a)_{ij}(\rho_A)_{ij},$$

from which we can state

$$(F_a)_{ij} = \sum_{kl}(E_a)_{jl,ik}(\rho_B)_{kl}.$$

From this equation we obtained, we notice that each operator $\mathbf{F}_a$ has the following properties:

1. **Hermiticity**:

$$(F_a)^*_{ij} = \sum_{kl} (E_a)^*_{il,jk} (\rho_B)^*_{kl} = \sum_{kl} (E_a)_{jk,il} (\rho_B)_{lk} = (F_a)_{ji}$$

   as both $\mathbf{E}_a$ and $\boldsymbol{\rho}_B$ are hermitian.

2. **Positivity**: if we choose to work with the basis that diagonalizes $\boldsymbol{\rho}_B = \sum_k p_k \left|k\right\rangle_B \left\langle k\right|$,

$$_A\left\langle \psi|\mathbf{F}_a|\psi\right\rangle_A = \sum_k p_k (_A\left\langle \psi\right| \otimes_B \left\langle k\right|) \mathbf{E}_a (\left|\psi\right\rangle_A \otimes \left|k\right\rangle_B) \geq 0$$

   as $\mathbf{E}_a$ is positive.

3. **Completeness**:

$$\sum_a \mathbf{F}_a = \sum_k p_k {}_B\left\langle k\right| \sum_a \mathbf{E}_a \left|k\right\rangle_B = \mathbb{I}_A,$$

   as $\sum_a \mathbf{E}_a = \mathbb{I}_{AB}$ and $tr(\boldsymbol{\rho}_B) = 1$.

We do not request for the $\mathbf{F}_a$'s to be mutually orthogonal as the allowed number of operators defining a POVM on system A is only limited by the dimension of $\mathcal{H}_A \otimes \mathcal{H}_B$, which can be much greater than the dimension of A, as explained in [2].

Chapter 3

Superoperators

In the previous chapter, we tried to understand the behavior of one part of a bipartite quantum system: we saw that a pure state of a bipartite system may behave like a mixed state when only subsystem A undergoes an observation process, and that an orthogonal measurement on a bipartite system may be a non-orthogonal POVM on A alone.

We now want to understand how to describe the evolution of system A when the bipartite system undergoes a unitary evolution (we previously just saw the special case where $\mathbf{U}_{AB} = \mathbf{U}_A \otimes \mathbf{U}_B$, so that systems A and B evolve independently).

3.1 The operator-sum representation

Suppose we are dealing with a bipartite system whose density matrix is a tensor product state of the form

$$\boldsymbol{\rho}_A \otimes |0\rangle_B \langle 0| ,$$

so that $\boldsymbol{\rho}_A$ is the density matrix of system A and system B is assumed to be in the pure state $|0\rangle_B$. When the bipartite system AB undergoes a unitary evolution $\mathbf{U}_{AB}$, the density matrix of the bipartite system evolves as

$$\mathbf{U}_{AB}(\boldsymbol{\rho}_A \otimes |0\rangle_B \langle 0|)\mathbf{U}_{AB}^\dagger.$$

Now, in order to find the density matrix of system A, we perform the partial trace over system B:

$$\begin{aligned} \boldsymbol{\rho}'_A &= tr_B[\mathbf{U}_{AB}(\boldsymbol{\rho}_A \otimes |0\rangle_B \langle 0|)\mathbf{U}_{AB}^\dagger] \\ &= \sum_k {}_B\langle k|\mathbf{U}_{AB}|0\rangle_B \, \boldsymbol{\rho}_A \, \langle 0|\mathbf{U}_{AB}|k\rangle_B \end{aligned}$$

where $\{k\}_B$ is an orthonormal basis for system B. We see that $\langle k|\mathbf{U}_{AB}|0\rangle_B$ is an operator acting on $\boldsymbol{\rho}_A$ whose matrix elements are ${}_A\langle i|\,({}_B\langle k|\mathbf{U}_{AB}|l\rangle_B)\,|j\rangle_A$. If we denote this matrix

$$\mathbf{M}_k =_B \langle k|\mathbf{U}_{AB}|0\rangle_B$$

then we may express $\boldsymbol{\rho}'_A$ as

$$\$(\boldsymbol{\rho}_A) \equiv \boldsymbol{\rho}'_A = \sum_k \mathbf{M}_k \boldsymbol{\rho}_A \mathbf{M}_k^\dagger.$$

This equation defines a linear map \$ that takes linear operators to linear operators. We can see that if $\mathbf{U}_{AB}$ is unitary, then

$$\sum_k \mathbf{M}_k^\dagger \mathbf{M}_k = \sum_k {}_B\langle 0|\mathbf{U}_{AB}|k\rangle_B\,\langle k|\mathbf{U}_{AB}|0\rangle_B =_B \langle 0|\mathbf{U}_{AB}^\dagger \mathbf{U}_{AB}|0\rangle_B = \mathbb{I}_A.$$

If this property is satisfied, the linear map \$ is called a *superoperator* and $\$(\boldsymbol{\rho}_A) = \sum_k \mathbf{M}_k \boldsymbol{\rho}_A \mathbf{M}_k^\dagger$ is called *operator-sum* representation (or *Kraus* representation) of the superoperator \$. As this linear map brings linear operators to linear operators, $\boldsymbol{\rho}'_A$ is a density matrix if $\boldsymbol{\rho}_A$ is, so that:

1. $\boldsymbol{\rho}'_A$ is hermitian: $\boldsymbol{\rho}_A^{'\dagger} = \sum_k \mathbf{M}_k \boldsymbol{\rho}_A^\dagger \mathbf{M}_k^\dagger = \boldsymbol{\rho}_A$;
2. $\boldsymbol{\rho}'_A$ has unit trace: $tr\boldsymbol{\rho}'_A = \sum_k tr(\boldsymbol{\rho}_A \mathbf{M}_k^\dagger \mathbf{M}_k) = tr\boldsymbol{\rho}_A = 1$;
3. $\boldsymbol{\rho}'_A$ is positive: ${}_A\langle\psi|\boldsymbol{\rho}'_A|\psi\rangle_A = \sum_k (\langle\psi|\,\mathbf{M}_k)\boldsymbol{\rho}_A(M_k^\dagger\,|\psi\rangle) \geq 0$.

It is clear that the Kraus representation of a superoperator is not unique, as we obtained it by performing a partial trace over system B in a particular basis, but we can perform the same operation in any basis we please. For example, if we use the basis $\{{}_B\langle l| = \sum_k U_{lkB}\,\langle k|\}$, we obtain the representation

$$\$(\boldsymbol{\rho}_A) = \sum_l \mathbf{N}_l \boldsymbol{\rho}_A \mathbf{N}_l^\dagger$$

where $\mathbf{N}_l = U_{lk}\mathbf{M}_k$ with U_{lk} being unitary. The Kraus representation theorem provides a proof that two operator-sum representations of the same superoperator are always related this way. A demonstration of the Kraus representation theorem can be found in [2].

The superoperator formalism is essential to understand decoherence, that is the evolution of pure states into mixed states. Unitary evolution of system A is just a special case of a superoperator that only contains one term in the Kraus representation. In every other case, when the representation has more than one term, there are pure states of $\mathcal{H}_A$ that become entangled with $\mathcal{H}_B$ and evolve under unitary evolution of the bipartite

system $\mathbf{U}_{AB}$. If A and B are entangled, there is no way to write the unitary operator $\mathbf{U}_{AB}$ as a tensor product (as we already said, entanglement cannot be created locally). We will now see that this theory is able to unify under the same formalism both unitary evolution and measurement, which seem to be really far concepts. Still, these two concepts must be distinguishable, as unitary evolution on system A alone does not bring decoherence, while measurement does. The answer is that a superoperator \$ is only invertible if it is unitary (or if the Kraus representation it's just made up of one term, causing the density matrix to evolve coherently). In all the other cases, decoherence occurs, and the superoperator is not invertible, so that there is an arrow of time, even at a microscopical level: decoherence causes an irremediable loss of quantum information.

3.1.1 POVM as a superoperator

In the previous chapter, we saw that a POVM on system A can be realized in many ways, now we will see that a unitary transformation on system AB (which is not local and therefore entangles A and B) followed by an orthogonal measurement on system B, can be described as a POVM in A.
Let us consider once again the bipartite system AB, where $\boldsymbol{\rho}_A = |\psi\rangle_A \langle\psi|$, so that

$$|\psi\rangle_A |0\rangle_B \to \sum_k \mathbf{M}_k |\psi\rangle_A |k\rangle_B$$

and if we make an orthogonal measurement on the $\{|k\rangle_B\}$ basis, we obtain the outcome k with probability:

$$Prob(k) =_A \langle\psi|\mathbf{M}_k^\dagger \mathbf{M}_k|\psi\rangle_A .$$

By expressing $\boldsymbol{\rho}_A$ as an ensemble of pure states, we have probability

$$Prob(k) = tr(\mathbf{F}_k \boldsymbol{\rho}_A), \text{ with } \mathbf{F}_k = \mathbf{M}_k^\dagger \mathbf{M}_k$$

to obtain outcome k. From the properties of the operator-sum representation, it is clear that $\mathbf{F}_k$ is positive and $\sum_k \mathbf{F}_k = \mathbb{I}$, so this is indeed a POVM on system A. The density matrix evolves as:

$$\boldsymbol{\rho} \to \sum_k \sqrt{\mathbf{F}_k} \boldsymbol{\rho} \sqrt{\mathbf{F}_k}$$

that is just a special case of a superoperator. Since each $\sqrt{\mathbf{F}_k}$ is hermitian, the condition $\sum_k \mathbf{F}_k = \mathbb{I}$ is just the operator-sum normalization condition.
This implementation of the POVM is not the most efficient, as the dimension of the required space $\mathcal{H}_A \otimes \mathcal{H}_B$ to perform such measurement is the product of the dimension of $\mathcal{H}_A$ and the number of possible outcomes k. Still, we have found out that a POVM is the most general measurement we can perform in system A by first entangling systems

A and B and then performing an orthogonal measurement on system B. We are basically discovering information about system A without touching it, just by measuring the environment B, thanks to the fact that we allowed A and B to interact and entangle.

3.2 Quantum channels

We will now talk about two examples of superoperators acting on a single qubit system A. We are going to refer to these superoperators as *quantum channels*, as we may imagine that \$ describes the fate of quantum information that is transmitted with some loss of fidelity from a sender to a receiver. We can also imagine a quantum channel to describe a time evolution (not a spatial one) and that \$ is the evolution of a system interacting with his environment.

In this section, in order to better understand what a quantum channel does, we are going to work with two-qubit systems (a qubit to describe our system, a qubit to describe the environment).

3.2.1 Depolarizing channel

This quantum channel describes the evolution of the state of a qubit which remains intact with probability $1-p$ and encounters an "error" with probability p. This error can be any one of three types:

1. Bit flip error: $|\psi\rangle \to \boldsymbol{\sigma}_1 |\psi\rangle$, or $|0\rangle \to |1\rangle\,;\ |1\rangle \to |0\rangle$
2. Phase flip error: $|\psi\rangle \to \boldsymbol{\sigma}_3 |\psi\rangle$, or $|0\rangle \to |0\rangle\,;\ |1\rangle \to -|1\rangle$
3. Bit and phase flip error:$|\psi\rangle \to \boldsymbol{\sigma}_2 |\psi\rangle$, or $|0\rangle \to +i\,|1\rangle\,;\ |1\rangle \to -i\,|0\rangle$

where we are using the computational basis $\{|0\rangle, |1\rangle\}$.
As we have seen, a superoperator can be defined as a unitary operator on a bipartite system AE (system A + Environment) or as an operator-sum. The unitary operator $\mathbf{U}_{AE}$ describing the depolarizing channel lives in a space $\mathcal{H}_A \otimes \mathcal{H}_E$ where $\mathcal{H}_E$ has dimension 4. This unitary operator can be defined as follows:

$$|\psi\rangle_A |0\rangle_E \to \sqrt{1-p}\,|\psi\rangle_A |0\rangle_E + \sqrt{\frac{p}{3}}(\boldsymbol{\sigma}_1 |\psi\rangle_A |1\rangle_E + \boldsymbol{\sigma}_2 |\psi\rangle_A |2\rangle_E + \boldsymbol{\sigma}_1 |\psi\rangle_A |3\rangle_E).$$

The environment, as we can see, evolves in one of 4 mutually orthogonal states, so that if we perform a measurement on E alone in the basis $\{|k\rangle_E,\ k=0,1,2,3\}$ we can know what error occurred and eventually reverse it.

In order to obtain the Kraus representation of this channel we need to perform the partial trace over the environment in the $|k\rangle_E$ basis, so that we obtain:

$$\mathbf{M}_0 = \sqrt{1-p}\,\mathbb{I}, \mathbf{M}_1 = \sqrt{\frac{p}{3}}\boldsymbol{\sigma}_1,\ \mathbf{M}_2 = \sqrt{\frac{p}{3}}\boldsymbol{\sigma}_2,\ \mathbf{M}_3 = \sqrt{\frac{p}{3}}\boldsymbol{\sigma}_3.$$

Then, a general density matrix describing the system A qubit evolves as the ensemble of pure states:

$$\boldsymbol{\rho} \to \$(\boldsymbol{\rho}) = (1-p)\boldsymbol{\rho} + \frac{p}{3}(\boldsymbol{\sigma}_1\boldsymbol{\rho}\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2\boldsymbol{\rho}\boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_3\boldsymbol{\rho}\boldsymbol{\sigma}_3)$$

where we are summing over the four (in principle distinguishable) ways in which the environment could evolve. It is also easy to check the normalization condition of the operator-sum by using the property of the Pauli matrices $\boldsymbol{\sigma_i^2} = \mathbb{I}$, as

$$\sum_k \mathbf{M}_k^\dagger \mathbf{M}_k = \left[(1-p) + 3\frac{p}{3}\right]\mathbb{I} = \mathbb{I}.$$

3.2.2 Phase-damping channel

As another example, we will get into the description of the phase-damping channel, which is particularly useful as it reveals decoherence in realistic physical situations. A unitary representation of this channel:

$$|0\rangle_A |0\rangle_E \to \sqrt{1-p}\,|0\rangle_A |0\rangle_E + \sqrt{p}\,|0\rangle_A |1\rangle_E\,,$$
$$|1\rangle_A |0\rangle_E \to \sqrt{1-p}\,|1\rangle_A |0\rangle_E + \sqrt{p}\,|1\rangle_A |2\rangle_E\,.$$

In this case, unlike the depolarizing channel, qubit A does not make any transitions; it is the environment that occasionally scatters off of the qubit being kicked into state $|1\rangle_E$ if system A is in the state $|0\rangle_A$, and into the state $|2\rangle_E$ if A is in the state $|1\rangle_A$. Furthermore, also unlike the depolarizing channel, we can see this channel picks out a preferred basis for qubit A, as in the computational basis $\{|0\rangle_A, |1\rangle_A\}$ the bit flip error never occurs.

The operator-sum representation can be found by performing the partial trace over the environment in the $\{|0\rangle_E, |1\rangle_E, |2\rangle_E\}$ basis for $\mathcal{H}_E$, so that we obtain the Kraus operators:

$$\mathbf{M}_0 = \sqrt{1-p}\,\mathbb{I}, \mathbf{M}_1 = \sqrt{p}\boldsymbol{P}_0,\ \mathbf{M}_2 = \sqrt{p}\mathbf{P}_1,$$

with $\mathbf{P}_0$ and $\mathbf{P}_1$ being the orthogonal projectors respectively on $|0\rangle$ and $|1\rangle$. We can easily check the normalization condition as $\mathbf{M}_0^2 + \mathbf{M}_1^2 + \mathbf{M}_2^2 = \mathbb{I}$. The phase-damping channel evolves the initial density matrix $\boldsymbol{\rho}$ to

$$\begin{aligned}\$(\boldsymbol{\rho}) &= \mathbf{M}_0\boldsymbol{\rho}\mathbf{M}_0 + \mathbf{M}_1\boldsymbol{\rho}\mathbf{M}_1 + \mathbf{M}_2\boldsymbol{\rho}\mathbf{M}_2 \\ &= (1-p)\boldsymbol{\rho} + p\begin{pmatrix}\rho_{00} & 0\\ 0 & \rho_{11}\end{pmatrix} = \begin{pmatrix}\rho_{00} & (1-p)\rho_{01}\\ (1-p)\rho_{10} & \rho_{11}\end{pmatrix},\end{aligned}$$

so that the on-diagonal terms remain invariant while the off-diagonal terms decay every time a scattering event occurs. But what is the consequence of this process?

Let us suppose that Γ is the probability per unit time for a scattering event to happen, so that $p = \Gamma\Delta t \ll 1$ when a time Δt has passed; then, the evolution of the density matrix $\boldsymbol{\rho}$ over a longer time $t = n\Delta t$ is obtained by applying our superoperator n times ($\$^n$). By doing so, our on-diagonal terms are preserved, but the non-diagonal ones are suppressed by

$$(1-p)^n = (1-\Gamma\Delta t)^{t/\Delta t} \to e^{-\Gamma t}, \text{ as } \Delta t \to 0.$$

This means that if we prepare an initial pure state $a\,|0\rangle + b\,|1\rangle$, after a time $t \gg \Gamma^{-1}$ has passed, the state decays to the incoherent superposition $\boldsymbol{\rho'} = |a|^2\,|0\rangle\langle 0| + |b|^2\,|1\rangle\langle 1|$. We see that decoherence occurs in the preferred basis $\{|0\rangle, |1\rangle\}$, as the in principle non-diagonal density matrix evolves by getting projected onto the computational basis, causing an inevitable loss of information. In fact, regardless of the initial density operator, after iterating $\$ a sufficient number of times, we always end up in a preferred basis (depending on the locality of the interaction between our system and the environment).

Physical interpretation

We can imagine for system A to be a heavy particle (for example, an interstellar dust grain) that interacts with a background gas of light particles (for example, the 3^oK microwave photons) that represents our environment. The interaction between the dust grain and the photons occurs as, while moving, the photons hit the heavy particle and scatter from it.

Let us suppose that system A is initially prepared in a superposition of position eigenstates

$$|\psi\rangle = \frac{|x\rangle + |-x\rangle}{\sqrt{2}},$$

so that we might be able to monitor the heavy particle's position, but it is impossible to keep track of the evolution of the quantum states of all the photons that scatter form it. For our purposes then, the state of the particle can be described by the density matrix $\boldsymbol{\rho}$ which is obtained by tracing over the environment.

As we have already seen in the previous section, phase damping can be caused by a scattering event such as this, and the scattering rate Γ determines the amount of time $t \gg \Gamma^{-1}$ after which the off-diagonal terms of the density matrix $\boldsymbol{\rho}$ are completely negligible (as they decay like $e^{-\Gamma t}$). We can say that the scattering rate Γ determines the *decoherence time scale*.

Another important time scale to consider in order to understand the dynamics of the dust grain is the *damping time scale*, which depends on the time that it takes for a significant amount of the particle's momentum to be transferred to the photons; this is

quite a long time as, we are dealing with a heavy particle. In this model, the decoherence time scale is determined by the amount of time that it takes for a single photon to be scattered by the dust grain, so that we can firmly say that

$$\Gamma_{dec} \gg \Gamma_{damp}.$$

As we have already noted, decoherence occurs on a preferred basis that in our interpretation we assumed to be the position-eigenstate basis. This physically means that dust grains located in distinguishable positions scatter the photons of the environment into mutually orthogonal states.

What we can learn from this example is that a coherent superposition of states of a heavy object decoheres very rapidly compared to how slowly the damping process happens, and that because of the spatial locality of the interactions between system A and the environment, decoherence picks up a preferred local basis. We can imagine that the same would happen if we studied decoherence for a heavy object such as the Schrödinger's cat; by taking a cat state $\frac{1}{\sqrt{2}}|dead\rangle + |alive\rangle$ decoherence would occur in a preferred basis, since "deadness" and "aliveness" can only be verified locally.

Chapter 4

Master equation

In the previous chapter, we saw how superoperators generally describe the evolution of a density matrix, including how pure states can evolve into mixed states (that is, decoherence) if we only focus on a small part of a bigger system. In a similar way, unitary transformations allow us to generally describe how coherent quantum evolution happens. However, in the case of coherent evolution, we find it convenient to study the evolution of a system A by associating its dynamics to a differential equation.

In the context of quantum mechanics, the dynamics of a system is characterized by the Hamiltonian, which describes the evolution over an infinitesimal time interval; then, we can associate the Hamiltonian to a differential equation, the Schrödinger equation, that if integrated finally gives the evolution of our density matrix $\boldsymbol{\rho}_A$ over a finite time interval.

In this chapter, we will try to describe, to at least a good approximation, the not necessarily coherent (unitary) evolution of $\boldsymbol{\rho}_A(t)$ by using a differential equation.
Such a description of the dynamics of a system is only possible if the evolution is local in time (or *Markovian*), which means if the evolution from $\boldsymbol{\rho}_A(t)$ to $\boldsymbol{\rho}_A(t+dt)$ is governed by a first order differential equation (so that $\boldsymbol{\rho}_A(t+dt)$ is completely determined by $\boldsymbol{\rho}_A(t)$). We know from the previous chapter that we can affirm for the bipartite system A+E to evolve coherently (so that the evolution of the bipartite system can be described by the Schrödinger equation), but this is not sufficient to grant locality in time for the evolution of system A alone: if the evolution of a state living in $\mathcal{H}_A \otimes \mathcal{H}_E$ evolves by unitary evolution, this does not mean that a state living in $\mathcal{H}_A$ evolves coherently too.

Indeed, the density operator $\boldsymbol{\rho}_A(t+dt)$ depends not only on $\boldsymbol{\rho}_A(t)$, but also on $\boldsymbol{\rho}_A$ at earlier times, as the environment keeps memory of this information for a while and can transfer it back to system A. This can happen because as information can flow out from system A to the environment, it can also flow back in, resulting in fluctuations of the system that are local in time, in which case the linearity of the process is compromised.

Then, except for the case of coherent evolution, the fluctuations are inevitable and it is impossible to describe the dynamics of the quantum system with an exact differential equation, but, in many contexts, it can still be a very good approximation. We will call this approximated equation *master equation*.

But how do we state in which cases this approximation is valid? Let us suppose that it takes a time Δt_{env} for the environment to "forget" the information lost by system A, so that after this time has passed, we are sure that this information cannot feed back into system A and influence its evolution. Then, we may incorporate "coarse graining" in time into our description, so that we choose our "differential unit" to be $\Delta t_{coarse} \gg \Delta t_{env}$. By doing this, we neglect the memory of the environment as we are filtering all the high frequency components of motion with $\omega \gg (\Delta t_{coarse})^{-1}$.

This "Markovian approximation" allows us to describe the evolution of a fluctuating system (which is not unitary) by a master equation, and will be useful if the time scale of the dynamics of the system that we want to describe is long compared to Δt_{coarse}; for example, the *damping* scale satisfies

$$\Delta t_{damp} \gg \Delta t_{coarse} \gg \Delta t_{res},$$

and can consequently be described as an approximated Markovian process.

4.1 The Lindbladian

In light of these statements, we will now try to obtain the master equation by assuming the validity of the Markovian approximation. We know that unitary time evolution of a density matrix is governed by the Schrödinger equation

$$\dot{\boldsymbol{\rho}} = -i[\mathbf{H}, \boldsymbol{\rho}],$$

which, if $\mathbf{H}$ is time independent, has solution:

$$\boldsymbol{\rho}(t) = e^{-i\mathbf{H}t}\boldsymbol{\rho}(0)e^{i\mathbf{H}t}.$$

We want to obtain a generalized version of the Schrödinger equation that is able to describe the non unitary evolution of an approximately Markovian system. In order to do so, we consider the linear operator $\mathcal{L}$, which we will call the *Lindbladian*, that generates a finite superoperator in the same sense that a Hamiltonian generates unitary operators (that describe coherent time evolution). We will then have

$$\dot{\boldsymbol{\rho}} = \mathcal{L}(\boldsymbol{\rho}),$$

that if $\mathcal{L}$ is time independent has formal solution:

$$\boldsymbol{\rho}(t) = e^{\mathcal{L}t}[\boldsymbol{\rho}(0)].$$

In order to compute the Lindbladian, we start by considering the operator-sum representation of a general superoperator:

$$\boldsymbol{\rho}(t) = \$(\boldsymbol{\rho}(0)) = \sum_k \mathbf{M}_k(t)\boldsymbol{\rho}(0)\mathbf{M}_k^\dagger(t)$$

for which we know that $\$_{t=0} = \mathbb{I}$. We can expand this equation to first order and obtain

$$\boldsymbol{\rho}(t+dt) = \boldsymbol{\rho}(t) + dt\dot{\boldsymbol{\rho}}(t) = \sum_k \mathbf{M}_k(t)\boldsymbol{\rho}(t)\mathbf{M}_k^\dagger(t).$$

Then, by considering an infinitesimal interval dt, the Kraus operators will be $\mathbf{M}_0 = \mathbb{I} + O(dt)$, as $\$_{t=0} = \mathbb{I}$, while the other operators $\mathbf{M}_k$, $k > 0$ will be of order $\sqrt{dt}$ and describe the "quantum jumps" that the system might undergo (with probability dt). We may then write

$$\mathbf{M}_k = \sqrt{dt}\mathbf{L}_k,\ k = 1, 2, 3, ...$$
$$\mathbf{M}_0 = \mathbb{I} + (-i\mathbf{H} + \mathbf{K})dt,$$

where $\mathbf{H}$ and $\mathbf{K}$ are both hermitian and $\mathbf{H}$, $\mathbf{K}$ and $\mathbf{L}_k$ are zeroth order in dt. We can determine what $\mathbf{K}$ is by applying the Kraus normalization condition (and neglecting terms of order dt^2):

$$\mathbb{I} = \sum_k \mathbf{M}_k^\dagger\mathbf{M}_k = \mathbb{I} + dt(2\mathbf{K} + \sum_{k>0}\mathbf{L}_k^\dagger\mathbf{L}_k),$$

which is satisfied if

$$2\mathbf{K} + \sum_{k>0}\mathbf{L}_k^\dagger\mathbf{L}_k = 0, \text{ or } \mathbf{K} = -\frac{1}{2}\sum_k \mathbf{L}_k^\dagger\mathbf{L}_k,$$

so that $\mathbf{M}_0 = \mathbb{I} + (-i\mathbf{H} - \frac{1}{2}\sum_k \mathbf{L}_k^\dagger\mathbf{L}_k)dt$. By substituting $\mathbf{M}_0$ and $\mathbf{M}_k$ as we just defined them into the Kraus representation of $\boldsymbol{\rho} + dt\dot{\boldsymbol{\rho}} = \sum_k \mathbf{M}_k(t)\boldsymbol{\rho}\mathbf{M}_k^\dagger(t)$ and by equating the terms of order dt of this expression, we obtain the Lindblad equation:

$$\dot{\boldsymbol{\rho}} \equiv \mathcal{L}[\boldsymbol{\rho}] = -i[\mathbf{H}, \boldsymbol{\rho}] + \sum_{k>0}\left(\mathbf{L}_k\boldsymbol{\rho}\mathbf{L}_k^\dagger - \frac{1}{2}\mathbf{L}_k^\dagger\mathbf{L}_k\boldsymbol{\rho} - \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\rho}\mathbf{L}_{\boldsymbol{k}}^\dagger\mathbf{L}_{\boldsymbol{k}}\right).$$

The first term of the Lindbladian is the usual term proper of the Schrödinger equation, while the other terms describe the possible transitions that may occur because of the interaction between our system and the environment. The operators $\mathbf{L}_k$ are called *Lindblad operators* or *quantum jump operators*. Each operator $\mathbf{M}_k$ produces a term $\mathbf{L}_k\boldsymbol{\rho}\mathbf{L}_k^\dagger$ describing one of the possible quantum jumps, while the rest of the terms involving the

Lindblad operators are needed to normalize properly in case that no quantum jumps occur. From the Kraus representation, it follows that $\boldsymbol{\rho}(t+dt)$ is a density matrix if $\boldsymbol{\rho}(t)$ is, as $\dot{\boldsymbol{\rho}}$ is hermitian and $tr\dot{\boldsymbol{\rho}} = 0$. Also as a consequence of the operator-sum representation, the Lindbladian preserves positivity.

When evolving a system with the Lindbladian, we can imagine for us to be continuously projecting the environment onto the $|k\rangle_E$ basis, so that with probability $1 - O(dt)$ the environment remains in the initial state $|0\rangle_E$, while with a probability of order dt, the environment makes a quantum jump to one of the states $|k\rangle_E$, with $k > 0$.

4.1.1 Phase damping

We now want to use the Lindbladian master equation to describe phase damping; for this purpose, we will be using the creation and annihilation operators, whose functioning is investigated thoroughly in [3]. We will focus on the case in which system A is a single harmonic oscillator ($\mathbf{H}_A = \omega \mathbf{a}^\dagger \mathbf{a}$) and the environment E consists of many oscillators ($\mathbf{H}_E = \sum_i \omega_i \mathbf{b}_i^\dagger \mathbf{b}_i$), which are coupled to the system by a perturbation:

$$\mathbf{H}' = \left(\sum_i \lambda_i \mathbf{b}_i^\dagger \mathbf{b}_i \right) \mathbf{a}^\dagger \mathbf{a}.$$

We can imagine for our environment to be a free electromagnetic field with whom the single harmonic oscillator that is system A is interacting. We can identify the Lindblad operator

$$\mathbf{L} = \sqrt{\Gamma} \mathbf{a}^\dagger \mathbf{a}$$

so that the Lindbladian takes the form:

$$\dot{\boldsymbol{\rho}} = -i[\mathbf{H}_0, \boldsymbol{\rho}] + \Gamma \left(\mathbf{a}^\dagger \mathbf{a} \boldsymbol{\rho} \mathbf{a}^\dagger \mathbf{a} - \frac{1}{2} (\mathbf{a}^\dagger \mathbf{a})^2 \boldsymbol{\rho} - \frac{1}{2} \boldsymbol{\rho} (\mathbf{a}^\dagger \mathbf{a})^2 \right),$$

where $\mathbf{H}_0 = \mathbf{H}_A = \omega \mathbf{a}^\dagger \mathbf{a}$ is the Hamiltonian of the oscillator. We can then re-write the master equation by defining the interaction picture operator $\boldsymbol{\rho}_I$ through the relation

$$\boldsymbol{\rho} = e^{-i\mathbf{H}_0 t} \boldsymbol{\rho}_I(t) e^{i\mathbf{H}_0 t},$$

so that the Lindbladian in the interaction picture can be written as:

$$\dot{\boldsymbol{\rho}}_I = \Gamma \left(\mathbf{a}^\dagger \mathbf{a} \boldsymbol{\rho}_I \mathbf{a}^\dagger \mathbf{a} - \frac{1}{2} (\mathbf{a}^\dagger \mathbf{a})^2 \boldsymbol{\rho}_I - \frac{1}{2} \boldsymbol{\rho}_I (\mathbf{a}^\dagger \mathbf{a})^2 \right).$$

In this case, Γ can be interpreted as the rate at which the photons of the environment are scattered when the oscillator is singly occupied (when $\mathbf{a}^\dagger \mathbf{a} = n = 1$) . If the occupation number is n, then the scattering rate becomes Γn^2, which is accordant with the physical

interpretation, as the higher the occupation number, the greater the amount of photons from the environment that are scattered by the excited oscillator.

We can solve the Lindbladian for $\dot{\boldsymbol{\rho}}_I$ in the occupation number basis by expanding $\boldsymbol{\rho}_I = \sum_{n,m} |n\rangle \langle m|$; if we identify the occupation number operator $\mathbf{a}^\dagger \mathbf{a} |n\rangle = n |n\rangle$, the master equation becomes:

$$\dot{\rho}_{nm} = \Gamma\left(nm - \frac{1}{2}n^2 - \frac{1}{2}m^2\right)\rho_{nm} = -\frac{\Gamma}{2}(n-m)^2\rho_{nm}$$

which we can integrate to obtain

$$\rho_{nm}(t) = \rho_{nm}(0)\exp\left[-\frac{1}{2}\Gamma t(n-m)^2\right].$$

If we prepare a state that is a superposition of occupation number eigenstates with very different values of n, such as

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + |n\rangle\,,\; n \gg 1,$$

then, the off-diagonal terms of the density matrix decay like $\exp(-\frac{1}{2}\Gamma n^2 t)$, so we can say that the rate of decoherence is Γn^2. This is consistent with our analysis of phase damping, as Γ is the rate at which the photons scatter off the oscillator found in state $|n\rangle$. We also see that decoherence chooses a preferred basis depending on the environment: the occupation number basis is naturally favored, as it is the occupation number that appears in the coupling $\mathbf{H}'$ between system A and the environment. If needed, a more in-depth analysis of this example can be found in [2].

Conclusions and Outlooks

By making use of the theoretical concepts we introduced in the first two chapters, we have been able to study how decoherence can be described in many different ways. For instance, we can observe decoherence through the time evolution described by a superoperator, as pure states evolve into mixed states, or we can see how it affects the phase damping process, both through the Lindblad master equation and the operator-sum representation, as decoherence imposes a preferred local basis to our system's density matrix.

By studying phase damping, we have also seen how decoherence acts particularly fast when we are dealing with states having high energy difference, as in the case of the bipartite system made of a dust grain and a gas of light particle, or in the case of a highly populated harmonic oscillator interacting with an electromagnetic field. Still, I think that the most illustrative example to try and understand phase damping is the Schrödinger's cat.

As widely known, this mental experiment consists in placing a cat inside a box with an automatic device that breaks a vial of poison and kills the cat if an excited atomic state is observed to decay. The question that Schrödinger meant to ask is: what happens to the cat if the atom is in a superposition state? Is the cat alive or dead? And why do we never see superposition states in the real world? Let's suppose that our atom can either be found in the ground state $|0\rangle$ or in the excited state $|1\rangle$, so that the state of the bipartite system can be written as

$$|\psi\rangle = \frac{|alive\rangle_{cat}\,|1\rangle_{atom} + |dead\rangle_{cat}\,|0\rangle_{atom}}{\sqrt{2}};$$

we can see that the cat and the atom have become entangled, as their local interaction links their evolutions, and the density matrix of the system can be written as:

$$\boldsymbol{\rho} = \frac{1}{2}(|alive, 1\rangle\langle alive, 1| + |alive, 1\rangle\langle dead, 0| + |dead, 0\rangle\langle alive, 1| + |dead, 0\rangle\langle dead, 0|).$$

In a real world situation, it is impossible to perfectly isolate the cat and the atom in their box, and some kind of information will inevitably leak (for example, the heat from the cat's body could give indications to someone on the outside of the box). Such effects

may be modeled as phase damping, which exponentially damps out the two off-diagonal terms in our initial density matrix, leaving us with the state

$$\boldsymbol{\rho} = \frac{1}{2}(|alive, 1\rangle \langle alive, 1| + |dead, 0\rangle \langle dead, 0|),$$

which explains why we are only able to observe a cat that is either dead or alive.
As we said before, an important result about the decoherence of such a system is that coherence between states of high energy difference decays faster than between states with a lower energy difference, so that $\Gamma_{dec} \ggg \Gamma_{damp}$: decoherence is fast!

There still is another interesting thing to learn from this example; as we are observing a macroscopic system, decoherence occurs because it is impossible to keep track of all the microscopic degrees of freedom, and we have to build a *coarse grained* description of our system by tracing over the degrees of freedom of the environment with whom our system inevitably interacts (as we have done with the Lindbladian). The act of tracing by itself causes a loss of information, as we are ignoring part of the universe in order to focus on a smaller part. That is the same thing that happens when we make a measurement, as we choose to verify a very specific property of our system by ignoring many others.
It seems like a non-unitary evolution in an *open* system, including the collapse described by measurement processes, always arises from ignoring some degrees of freedom. We could then make the hypothesis that the evolution of a *closed* quantum system is always unitary, as loss of information cannot occur, and that the evolution of the quantum state of the whole universe (which we consider to be a closed system) is actually deterministic. If we accept this view, we also need to believe in the fact that all possible outcomes of a measurement have the same chance of "being real", so that it becomes hard to understand why, in our reality, only one outcome is actually realized, so that the cat is either dead or alive.

The paradoxical aspect of this idea can be solved if we are willing to include ourselves as part of the universe, so that our state should take part in the wave function that describes the quantum system: if we prepare the cat state and then look at the cat, the density operator actually becomes (after tracing away other degrees of freedom)

$$|1\rangle_{atom} |dead\rangle_{cat} |know\ it's\ dead\rangle_{me}, \text{ with probability } \frac{1}{2},$$

or

$$|0\rangle_{atom} |alive\rangle_{cat} |know\ it's\ alive\rangle_{me}, \text{ with probability } \frac{1}{2}.$$

So that $\boldsymbol{\rho}$ describes two alternatives where the choice "cat half dead/ half alive" is not an option. Then, by assuming that the wave function describes reality and that all evolution is unitary, we can give the "many worlds interpretation" of quantum physics: each time a measurement occurs, the wave function of the universe splits in two branches corresponding to the two possible outcomes; after many measurements, we are found with many branches, each with equal probability of describing reality.

Bibliography

[1] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information: 10th Anniversary Edition*. Cambridge University Press, 2011.

[2] John Preskill. "Lecture Notes for Physics 229: Quantum Information and Computation". In: (1998).

[3] Roberto Zucchini. "Quantum Mechanics: Lecture Notes".