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Gravity and Yang-Mills in $d = 2 + \epsilon$

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Abstract

This thesis explores the ultraviolet behavior of gravity coupled to Yang-Mills fields within the Asymptotic Safety scenario. We employ a perturbative approach based on the dimensional expansion in $d = 2 + \epsilon$, which allows for a controlled analytical continuation and circumvents ambiguities associated with heat kernel methods for quantizing the metric. We calculate the one-loop beta functions for the Einstein-Yang-Mills system and analyze the renormalization group flow. To interpret the physical implications, we evaluate the flow on-shell using two different schemes to handle the equations of motion. In both schemes, we identify a non-Gaussian fixed point, suggesting the theory could be asymptotically safe. However, we discover a discrepancy between the schemes in the limit $d \to 4$, where in one case the non-Gaussian fixed point merges with the Gaussian fixed point. This result highlights a potential scheme dependence in the on-shell analysis. A parallel investigation in $d = 4 - \epsilon$ dimensions confirms the perturbative non-renormalizability of the theory, as no interacting fixed point is found. Our findings support the utility of the $d = 2 + \epsilon$ expansion as a tool to investigate quantum gravity while also underscoring the challenges in extrapolating results to four dimensions.

Contents

Introduction

Finding a consistent and fundamental quantum theory of gravity remains one of the most challenging open problems in theoretical high-energy physics. One motivation for pursuing such a theory arises from seemingly simple questions, such as: How do elementary particles interact through the gravitational field? Further motivation comes from the breakdown of both the Standard Model (SM) of particle physics and General Relativity (GR). Both the theories are affected by singularities. In GR, curvature singularities appear in both black hole and cosmological spacetimes. Their resolution is expected to require quantum effects, thus motivating the search for a quantum theory of gravity. From dimensional analysis, such a theory is expected to become relevant at the Planck scale,

$$M_{\text{Planck}} = \sqrt{\hbar c/G_N} \approx 10^{19}\text{GeV}.$$

In quantum field theory it is well known that coupling constants become functions of the energy scales entering the renormalization process. In turn this implies a modification of the classical scaling properties of a theory. Such energy dependence of a coupling a is encoded in its beta function

$$\beta_a = \mu\frac{\partial a}{\partial\mu},$$

where μ is the renormalization scale. If a coupling grows indefinitely as the energy scale increases, i.e. as distances become shorter, we have a breakdown of perturbation theory. If this happens at finite energy we say that the theory has a Landau pole. One may hope that the theory still makes sense in a non-perturbative regime. However, if a non-perturbative ultraviolet (UV) completion exists, one must consider the appearance of new physics. A hint about the missing physics comes from the scale of the Landau poles: experimental results from the LHC indicate that the SM remains internally consistent up to the scale where quantum gravity is expected to become relevant, while the Landau poles occur far above the Planck scale. This motivates the idea that an ultraviolet completion could emerge from the inclusion of quantum gravity in the SM. A conservative approach is building a quantum field theory of the metric which avoids the introduction of new fields for gravity and relies on the QFT framework that has shown to be successful for all other fundamental interactions. Following this approach, the perturbative quantization of the classical description for gravity results in a non-renormalizable theory: logarithmic divergencies of the theory require the addition of new terms to the Lagrangian in order to absorb the divergencies. Each new term comes with a coupling which has a low-energy value as a free parameter of the theory that needs to be fixed by observations. Einstein-Hilbert gravity requires an infinite number of free parameters making the theory non-predictive at high energies [8, 20, 38]. One could react differently to this outcome. A possible reaction is the acceptance of failure of perturbative approach and the pursue of a non-perturbative quantization, for example, this approach is the one of Loop Quantum Gravity. Another possibility is accepting that quantum gravity constitutes an effective field theory valid at low energies, whose UV completion requires the introduction of new degrees of freedom and symmetries as in String Theory. In a less radical approach one could retains the fields known from GR and ask whether there is a symmetry principle that one can impose to reduce to a finite number the free parameters. One proposal along this line is the Asymptotic Safety scenario [40], which is a quantum realization of scale symmetry [14]. As we said before, in quantum field theory, couplings are scale-dependent, so it is not guaranteed that a theory that is consistent at one scale remains consistent as we change to a smaller distance scale/larger momentum scale. A restoration of scale symmetry can be achieved in theories where the effect of quantum fluctuations balances out at finite values of couplings, corresponding to non-Gaussian fixed point of the renormalization group (RG). At these values, this quantum scale symmetry allows one to construct models which hold up to arbitrarily short distance scales. The name asymptotic safety is related to the fact that this quantum scale symmetry is almost as good as asymptotic freedom, where quantum fluctuations vanish asymptotically. Most importantly, this symmetry allows to recover predictivity of effective field theories. In fact, one can think of quantum scale symmetry as just another symmetry one imposes on the dynamics of the theory, restricting the possible interaction structures and thereby reducing the number of undetermined couplings, i.e., the free parameters of the model. In the case of Einstein's gravity, the asymptotic safety conjecture is based on the premise that the theory, if seen as a quantum field theory of the metric tensor, is ultraviolet complete thanks to the presence of a suitable fixed point of the renormalization group. This was confirmed by Reuter [33] using background and Wilsonian RG methods, on which most of the recent literature of the topic is now based. However, the application of these methods often comes at the price of having to deal with effective action that is scheme- and gauge-dependent. A way to overcome this problem would be to address scheme- and gauge-dependence in a setting in which the UV fixed point is still perturbative. As originally suggested by Weinberg [40], this setting is provided by gravity in $d = 2 + \epsilon$ dimensions, that exhibits an UV fixed point for Newton's constant motivating the asymptotic safety since its inception. Recently the approach in $d = 2 + \epsilon$ was reconsidered [25, 26, 27] in light of gauge and parametric dependence induced by the background splitting. Considering the obvious limitation of the continuation to $d = 4$, which requires the limit $\epsilon \to 2$, certainly outside the validity of perturbation theory, this approach should be regarded as a complementary approach to the functional one. In this work we want to extend this approach considering Yang-Mills theory together with gravity. We structure the thesis as follows:

In Chapter 1 we review the idea of Asymptotic Safety, starting from the definition of the theory space and explaining how the existence of a non-Gaussian fixed point is related to a UV completion. Finally, following the original Weinberg's idea, a motivation the study of gravity in $d = 2 + \epsilon$ is given.

In Chapter 2 we review the methods used to compute the RG flow of the theory. In particular we will review how heat kernel methods and the background field method can be used to compute the effective action at one-loop. We will also explain how to analytically continue the theory to $d = 2 + \epsilon$ in dimensional regularization together with modified minimal subtraction scheme.

In Chapter 3 we test the methods introduced in the previous chapter for the simple case of Yang-Mills in flat spacetime. An extension for the curved spacetime is given. This analysis leads to the necessity to include dynamical degrees of freedom for gravity.

In Chapter 4 we finally consider the case of the Einstein-Yang-Mills theory, we compute the beta functions of the theory in $d = 2 + \epsilon$ keeping as on-shell essential coupling once the Yang-Mills coupling and the Newton's constant, and again, the cosmological constant and the Newton's constant. We analyze how the presence of the Yang-Mills interaction affect the Non-Gaussian fixed point and the Asymptotic Safety scenario.

Chapter 1

Asymptotic Safety

In this chapter we want to review the key idea of asymptotic safety [6, 13, 14, 30, 31, 33, 34, 35],. In particular, we emphasize that its definition is independent from the functional RG approach and the Wilsonian way of thinking of path integrals. We will also justify the setting of $d = 2 + \epsilon$ as a complementary approach to assess the asymptotic safety scenario, which gives further motivation for the work of this thesis.

1.1 Theory space

To present the idea of asymptotic safety, it is necessary to introduce some useful definitions and tools. First, we need to define *theory space*. Given a general set of fields $\phi(x)$, the theory space consists of all action functionals

$$A : \phi \longmapsto A[\phi] \tag{1.1}$$

depending on this set. The functionals are subject to certain symmetry requirements, for example, $\mathbb{Z}_2$-symmetry for a single scalar, or diffeomorphism invariance if ϕ denotes a spacetime metric. The theory space $\{A[\,\cdot\,]\}$ is fixed once the field content and the symmetries are fixed. Now we can assume that it is possible to find a set of "basis functionals" $\{P_\alpha[\,\cdot\,]\}$, typically local operators constructed with the fields ϕ and their derivative. Every point of the theory space has an expansion [34] of the from

$$A[\phi] = \sum_\alpha = \bar{u}_\alpha P_\alpha[\phi]\,. \tag{1.2}$$

The coefficients $\bar{u}_\alpha$ are called *generalized couplings* and are the local coordinates of the theory space. More precisely, one usually consider the subset of *essential couplings*, i.e., those coordinates which cannot be absorbed by a field reparametrization. The couplings that can be eliminated by field redefinitions are called *redundant couplings*. At this point, we need to assume that the RG defines a vector field $\vec{\beta}$ on the theory space [30]. The RG

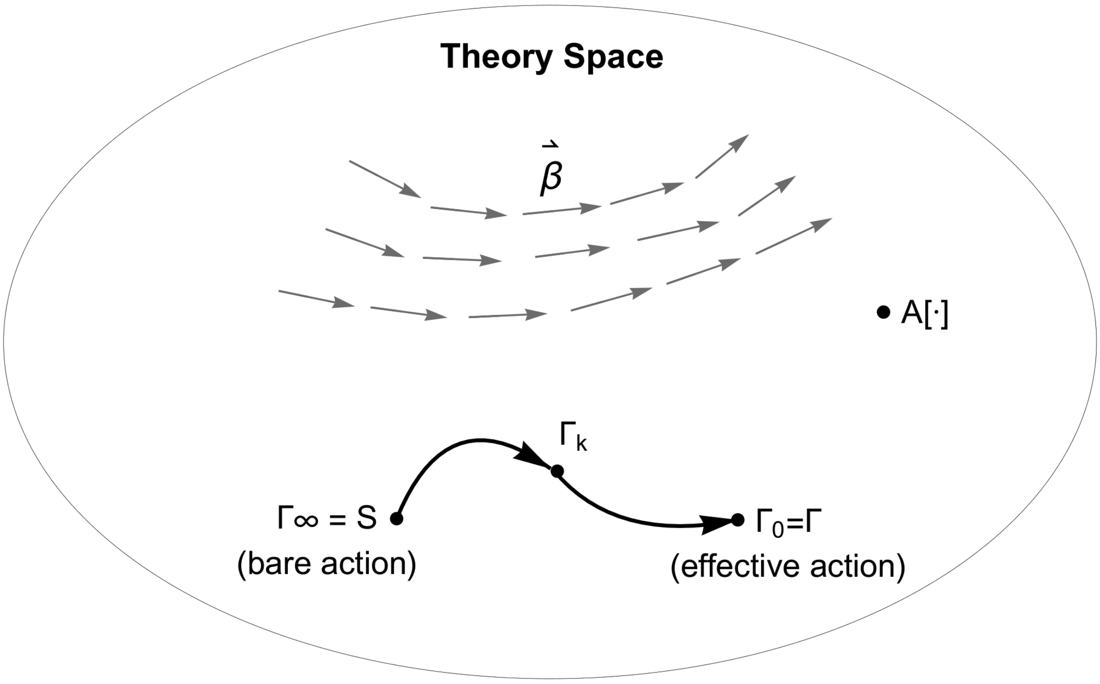

Figure 1.1: Schematic sketch of the theory space. The points of the theory space are the action functionals $A[\,\cdot\,]$. The RG defines a vector field $\vec{\beta}$ on the theory space. The corresponding RG flow consists of the RG trajectories $k \mapsto \Gamma_k$. They start at S and end at the standard effective action Γ.

flow, given by the integral curves of the vector field $\vec{\beta}$, describe the dependence of the action functionals on an energy scale k, or alternatively, one can consider a "RG time" $t = \log k$. We identify theories with RG trajectories $k \mapsto \Gamma_k$. They start, for $k \to \infty$, at the bare action S and terminate at the ordinary effective action Γ at $k = 0$. Since only the essential couplings are coordinates on the theory space, Γ_∞ and S may differ by a simple, explicitly known functional. The natural orientation of the trajectories is from higher to lower scales k, the direction of increasing "coarse graining". Expanding Γ_k as in (1.2),

$$\Gamma_k[\phi] = \sum_\alpha \bar{u}_\alpha(k)\; P_\alpha[\phi]\,, \tag{1.3}$$

the trajectory is defined by the running couplings $\bar{u}_\alpha(k)$. In standard jargon one would refer to $\bar{u}_\alpha(k = \infty)$ as the "bare" parameters and to $\bar{u}_\alpha(k = 0)$ as the "renormalized" parameters. Fig. 1.2 gives a schematic summary of the structures of the theory space. It is useful to re-express the couplings in terms of their dimensionless counterparts $u_\alpha \equiv k^{-d_\alpha}\bar{u}_\alpha$, where d_α is the canonical mass dimension of $\bar{u}_\alpha$. It can generally be expected that when k goes to infinity some couplings $\bar{u}_\alpha(k)$ also go to infinity. What we want to avoid is that the dimensionless couplings u_α diverge. This can happen even at some finite scale $k_{\max}$, as in QED and ϕ^4 theory, signaling a breakdown of the theory. In this case the theory holds for a finite energy range and it is said to be an Effective Field Theory.

1.2 The idea of Asymptotic Safety

The basic idea of asymptotic safety can be understood as follows. Naively, the boundary of the theory space sketched in Fig. 1.2 separates points with all essential (dimensionless) coordinates $\{u_\alpha\}$ well defined, from points with undefined, divergent couplings. In this context, the task of renormalization theory consists in constructing theories corresponding to "infinitely long" RG trajectories. These trajectories should lie entirely in the theory space and should not leave the theory space in the UV limit $k \to \infty$ nor in the infrared (IR) limit $k \to 0$. Every such trajectory defines one possible quantum theory. We can consider the case in which the RG flow admits a fixed point (FP), which is defined as a point u^*_α in the theory space such that the beta functions of the dimensionless couplings vanish, i.e.

$$\beta_{u_\alpha} = k\frac{\partial u_\alpha(k)}{\partial k} = 0 \quad \text{at} \quad u_\alpha = u^*_\alpha. \tag{1.4}$$

The RG trajectories have small "velocity" near a fixed point because β_α are small there; and directly at the fixed point, the running stops completely and scale invariance is recovered. As a result, the theory corresponding to the trajectory running into such a fixed point, does not escape the theory space for $k \to \infty$ and has a well behaved action functional. Such a theory does not suffer from divergent couplings and is said to be *asymptotically safe* from unphysical divergences and represents a UV complete theory. Weinberg proposed, in the context of gravity [40], to use a Non-Gaussian fixed point (NGFP) to take the limit $k \to \infty$. A NGFP is a fixed point where not all couplings u^*_α vanish. Alternatively, in a Gaussian fixed point (GFP) we have $u^*_\alpha = 0,\ \forall\alpha = 1, 2\ldots$. One important aspect is that dimensionful couplings keep running according to a power law involving their canonical mass dimensions d_α:

$$\bar{u}_\alpha(k) = u^*_\alpha k^{d_\alpha}. \tag{1.5}$$

Furthermore, non-essential dimensionless couplings are not required to reach the fixed point.

1.3 UV critical hypersurface

One can try to evaluate how many asymptotically safe trajectories there are in theory space. To address this task one important concept is the one of *UV critical hypersurface* associated to a FP. Given a NGFP, its UV critical hypersurface $\mathcal{S}_{UV}$ consists of all points of theory space which are pulled into the NGFP by the inverse RG flow, i.e. for increasing k. Assuming that this surface is a smooth manifold, its dimension is equal to the dimension dim $\mathcal{S}_{UV}$ of its tangent space at the FP. The latter can be computed in the following way. In the vicinity of the fixed point the flow can be linearized:

$$k\frac{\partial u_\alpha(k)}{\partial k} = \sum_\beta M_{\alpha\beta}(u_\beta(k) - u^*_\beta), \tag{1.6}$$

where

$$M_{\alpha\beta} = \left.\frac{\partial \beta_\alpha}{\partial u_\beta}\right|_{u=u*}. \tag{1.7}$$

The general solution to this equation reads

$$u_\alpha(k) = u^*_\alpha + \sum_i c_i v^i_\alpha \left(\frac{k_0}{k}\right)^{\lambda_i} \tag{1.8}$$

where v^i are the right-eigenvectors of the matrix M with eigenvalues $-\lambda_i$, i.e.,

$$\sum_\beta M_{\alpha\beta}\ v^i_\beta = -\lambda_i\, v^i_\alpha\,. \tag{1.9}$$

Since the matrix M is not symmetric in general the eigenvalues are not guaranteed to be real. However, we can assume that the eigenvectors from a complete basis. Furthermore k_0 is a reference scale, and c_i are constants of integration. If $u_\alpha(k)$ describes a trajectory corresponding to an asymptotically safe theory, it must lie in $\mathcal{S}_{UV}$ and approach u^*_α as $k \to \infty$. As a result, we must set $c_i = 0$ for all i corresponding to eigenvalues with positive real part, the ones with Re $\lambda_i < 0$. On the other hand, the dimensionality of the critical hypersurface is given by the number of eigenvalues with negative real part, i.e., Re $\lambda_i > 0$. The corresponding eigenvectors span the tangent space to $\mathcal{S}_{UV}$ at the NGFP. The number of free parameters of the theory is equal to the dimension of $\mathcal{S}_{UV}$. Thus, the theory is more predictive when the $\mathcal{S}_{UV}$ has lower dimension. The ideal situation would be a theory with a one dimensional critical hypersurface. In this case there would be a single renormalizable trajectory and once we have determined the initial position at some scale k, the theory is completely determined. At the opposite extreme, if $\mathcal{S}_{UV}$ was infinite dimensional, the theory would not be predictive. The intermediate case is a theory with finite dimensional critical surface. Such a theory space would have the same good properties of perturbatively renormalizable and asymptotically free theory, because it would be well behaved in the UV and it would have only a finite number of undetermined parameters. To conclude this section we show how asymptotic safety represents the generalization of renormalizability and asymptotic freedom to the case

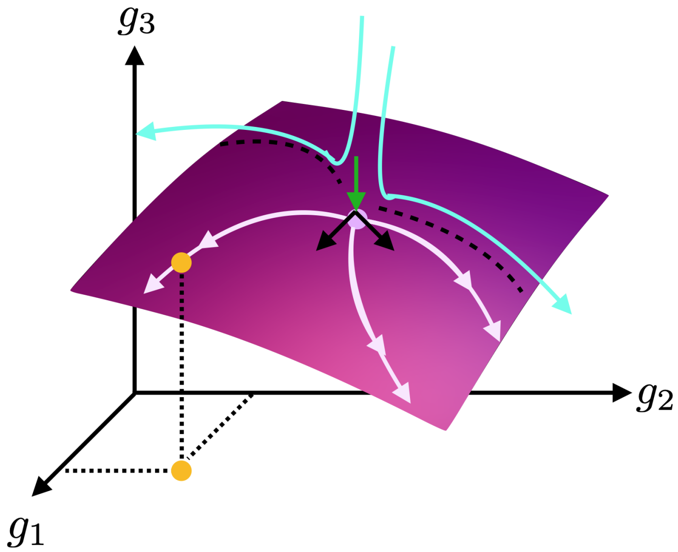

Figure 1.2: Graphics taken from [13]. Illustration of a fixed point (light purple dot) with its UV critical hypersurface (purple). RG trajectories starting off the critical hypersurface (teal) are pulled towards the fixed point along the irrelevant direction (roughly aligned with g_3), before the IR repulsive directions g_1 and g_2 kick in and drive the flow away from the fixed point. The linearized flow is indicated by the black (relevant directions) and green (irrelevant direction) arrows.

when the FP does not correspond simply to a free theory [30]. To do so we can consider the example of a GFP, corresponding to a free theory. The beta functions have the form

$$k\frac{\partial u_\alpha}{\partial k} = -d_\alpha u_\alpha + k^{-d_\alpha}\beta_\alpha\,. \tag{1.10}$$

The functions $\beta_\alpha = k(\partial \bar{u}_\alpha/\partial k)$ represent the loop corrections, which vanish at the GFP. In this case the eigenvalues of the matrix M are just given by the canonical mass dimension

$$-\lambda_i = -d_i\,. \tag{1.11}$$

The relevant couplings are the ones that are power counting renormalizable, and the critical surface consists of the power counting renormalizable actions. Fig. 1.3 gives an illustration of fixed point together with its critical hypersurface.

1.4 Why $d = 2 + \epsilon$

Gravity is the domain of fundamental physics where the problem of finding a UV completion is most acute, and so it is here that most work on asymptotic safety concentrated, following Weinberg original suggestion [40]. The first important observation is that the Newton's constant G_N is not dimensionful for every spacetime dimension d. In $d = 2$ the G_N is dimensionless and it has asymptotically free beta function,

$$\beta_{G_N} \propto - G_N^2, \tag{1.12}$$

meaning that in principle one could obtain consistent predictions from perturbation theory that are valid up to arbitrarily high energies. $d = 2$ is referred as the *critical dimension* of the Einstein-Hilbert action. This result is not particularly useful for the physically interesting $d = 4$ case, unless one realizes that in $d = 2 + \epsilon$ one can re-instate the canonical mass dimension of the Newton's constant and find that its RG running, in units of of an RG scale k, it is

$$-\epsilon G_N + \beta_{G_N} \propto - G_N^2. \tag{1.13}$$

This means that we have a scale invariant value $G_N^* \sim \mathcal{O}(\epsilon)$, arising as a fixed point solution of $\beta_{G_N} = 0$, and so, an asymptotically safe theory. Weinberg conjectured that gravity could be asymptotically safe in $d \geqslant 2$ and, most importantly, in $d = 4$. The existence of the non trivial UV fixed point G_N^* guarantees that, at least for small ϵ there is a UV completion. This has given a reason to push forward the investigation of the asymptotic safety conjecture which has received increasing attention over the past few decades [6]. Most of the literature has eventually settled on the use of a non-perturbative method known as functional RG [34, 35]. However, the non perturbative approach suffers from a severe renormalization scheme dependence which mixes both with gauge and parametrization dependence making unclear which are the physical predictions of the theory in terms of observables. In [26] was suggested that asymptotic safety conjecture pursuit should couple the functional RG with a less scheme-dependent approach such as the perturbative framework, following Weinberg's original idea. They addressed the problem of the analytical continuation from $d = 2 + \epsilon$ to $d = 4$ proposing an original procedure that will be explained in Chapter 2. In this work, we will follow this approach.

Chapter 2

Heat Kernel, Effective Action, and analytical continuation

The heat kernel finds many applications in physics and mathematics [12, 23, 39, 2, 3, 4],. An application in quantum field theory is the calculation of effective actions incorporating quantum corrections to the classical results. The definition of effective action and the computation of the one-loop approximation can be found in Appendix A. The heat kernel was first introduced in QFT by J. Schwinger who proposed that the Green's functions could be related to the dynamical properties of a fictitious particle with spacetime coordinates depending upon a proper time parameter. The relation was obtained originally for a Dirac field in flat spacetime and in this case the heat kernel naturally arises. B. DeWitt extended this procedure to curved spacetime and found recurrence relations between heat kernel coefficients. In this chapter we will review the heat kernel techniques with the Seeley-DeWitt expansion in curved spacetime, and their role in the computation of one-loop perturbative contributions to the effective action. We will also review the background field method as a technique for the computation of the effective action. Finally, we will also introduce the procedure to analytically continue metric theories from $d = 2$.

2.1 Heat equation and Seeley-DeWitt expansion

In the following we will be interested in elliptic differential operators that are supposed to be defined over a riemannian d-dimensional manifold $\mathcal{M}^d$, and are assumed to have the general form

$$\mathcal{O}_x = -D_x^2 + E(x), \qquad D_\mu = \nabla_\mu + A_\mu, \tag{2.1}$$

where A_μ is a matrix-valued vector gauge connection, E is endomorphism acting on the multi component fields. These may be scalars, spinors, vectors etc. and can be regarded as sections of some vector bundle over $\mathcal{M}^d$. In the end, $\nabla_\mu = \partial_\mu + \Gamma_\mu$ is the covariant derivative including the appropriate spin connection on $\mathcal{M}^d$ according to the type of field on which it acts. To extend the results to the usual spacetime it is necessary to assume an analytic continuation of any minkowskian metric to one of euclidean signature. The Green function G for the operator $\mathcal{O}$ on $\mathcal{M}^d$ is formally defined by requiring it to satisfy

$$\mathcal{O}_x G(x,x') = \delta^d(x,x'), \tag{2.2}$$

in which δ is the biscalar δ-function generalizing the usual flat space Dirac delta, considering a scalar function $\phi(x)$

$$\int d^dx' \sqrt{g'}\delta^d(x,x')\phi(x') = \phi(x). \tag{2.3}$$

The heat kernel function is defined as the solution of the following differential equation

$$\frac{\partial\,\mathcal{G}(s;x,x')}{\partial s} + \mathcal{O}_x\mathcal{G}(s;x,x') = 0 \tag{2.4}$$

with initial condition

$$\mathcal{G}(0;x,x') = \delta^{(d)}(x,x'). \tag{2.5}$$

If we solve the diffusion equation (2.4) implicitly

$$\mathcal{G}(s;x,x') = \langle x'|\, e^{-s\mathcal{O}}\, |x\rangle\,, \tag{2.6}$$

we can see that the heat kernel is related to the Green function G by

$$G(x,x') = \int_0^\infty ds\; \mathcal{G}(s;x,x'). \tag{2.7}$$

The heat kernel function has an asymptotic expansion for $s \to 0^+$ which captures the ultraviolet properties of the Green function. Following DeWitt [12], it has the form

$$\mathcal{G}(s;x,x') = \frac{\Delta(x,x')^{1/2}}{(4\pi s)^{d/2}} e^{-\frac{\sigma(x,x')}{2s}} \sum_{k\geqslant 0} a_k(x,x') s^k. \tag{2.8}$$

In Eq.(2.8) several bitensors are introduced, the most fundamental is $\sigma(x,x')$ called *geodetic interval* or *Synge-DeWitt's world function*. It is defined as half of the square of the geodesic distance between x and x'. The bitensor $\Delta(x,x')$ is known as *van Vleck determinant* and is related to the world function and the determinant metric by

$$\Delta(x,x') = -\frac{1}{g(x)^{1/2}\, g(x')^{1/2}} \det\left(-\frac{\partial^2}{\partial x^\alpha \partial x'^\beta}\sigma(x,x')\right). \tag{2.9}$$

Together, σ and Δ ensure that the leading term of the Seeley-DeWitt parametrization covariantly generalizes the solution of the heat equation in flat space. They are constructed only from the metric and satisfy the so called *crucial relations*

$$\sigma^\mu \sigma_\mu = 2\sigma, \tag{2.10}$$

$$\Delta^{1/2}\sigma_\mu{}^\mu + 2\sigma^\mu \nabla_\mu \Delta^{1/2} = d\Delta^{1/2}, \tag{2.11}$$

for which we suppressed the bitensor coordinates and we used the notation in which subscripts of σ indicate covariant derivatives, i.e. $\sigma_{\mu_1 \dots \mu_n} := \nabla_{\mu_n} \dots \nabla_{\mu_1} \sigma$. Finally, the bitensors $a_k(x, x')$ are the coefficients of the asymptotic expansion, also known as *Seeley coefficients*, they depend on the detailed form of the operator $\mathcal{O}$ and contain its geometrical information, which includes curvatures, connections and interactions. They are determined by the equations

$$\begin{gathered} k a_k + \sigma^\mu D_\mu a_k + \Delta^{-1/2} \mathcal{O}(\Delta^{1/2} a_{k-1}) = 0, \\ \sigma^\mu D_\mu a_0 = 0, \end{gathered} \tag{2.12}$$

obtained from (2.4),(2.5),(2.8) in conjunction with (2.10),(2.11).
The ultraviolet properties are local in renormalizable theories and for the case of the heat kernel locality correspond to $x \sim x'$ and it is captured by the coincidence limit in which $x' \to x$. Given any bitensor $B(x, x')$, its coincidence limit is defined as

$$[B] := \lim_{x' \longrightarrow x} B(x, x'). \tag{2.13}$$

One important note is that covariant derivatives do not generally commute with the coincidence limit, so

$$\nabla [B] \neq [\nabla B]. \tag{2.14}$$

The coincidence limits of the bitensors $\sigma(x, x')$ and $\Delta(x, x')$ and their derivatives can be obtained by repeated differentiation of the *crucial relations*, and the same can be done with the Seeley coefficients differentiating (2.12). The calculation for the case of a scalar field can be found in can be found in Appendix B. Here we report the first coincidence limits for the Seeley coefficients [22, 39] which are used later

$$\begin{aligned} [a_0] &= \mathbb{1} \\ [a_1] &= \mathbb{1}\frac{R}{6} - E, \\ [a_2] &= \frac{1}{12}\Omega_{\mu\nu}\Omega^{\mu\nu} + \frac{1}{2}\left(\mathbb{1}\frac{1}{6}R - E\right)^2 - \frac{1}{6}D^2 E \\ &\quad + \mathbb{1}\frac{1}{30}\left[\frac{1}{6}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - R_{\mu\nu}R^{\mu\nu}) + \nabla^2 R\right], \end{aligned} \tag{2.15}$$

where $\mathbb{1}$ is the identity acting on the fields and $\Omega_{\mu\nu} := [D_\mu, D_\nu]$ is the curvature associated to the covariant derivative D_μ.

2.2 Background field method and one-loop contribution

A useful technique for computing the effective action is the background field method [1, 37, 32]. The generating functional for a generic field ϕ with action $S[\phi]$ and source J is

$$Z[J] = e^iW[J] = \int D\phi \, e^{iS[\phi]+i\int d^dx\, J\phi}\,, \tag{2.16}$$

where $W[J]$ is the generating functional of the connected diagrams. The effective action is defined by

$$\Gamma[\varphi] = \min_J \left\{ W[J] - \int d^dx\, J\phi \right\} \quad \text{where} \quad \varphi = \frac{\delta W[J]}{\delta J}\,. \tag{2.17}$$

In the background field method one splits the field into a background and a fluctuation, i.e.

$$\phi(x) = \varphi(x) + \tilde{\phi}(x)\,. \tag{2.18}$$

The background has to be considered as an inert spectator in the quantization process, while the quantum fluctuation is the field has to be path-integrated over. The background functionals are defined as

$$\begin{aligned} Z_B[\tilde{J};\varphi] &= e^{iW_B[\tilde{J},\varphi]} = \int D\tilde{\phi}\; e^{iS[\varphi+\tilde{\phi}]+i\int d^dx\tilde{J}\tilde{\phi}}\,, \\ \Gamma_B[\tilde{\varphi};\varphi] &= \min_{\tilde{J}} \left\{ W_B[\tilde{J};\varphi \int d^dx\tilde{J}\tilde{\phi}\;] \right\}. \end{aligned} \tag{2.19}$$

It is possible to obtain an useful relation performing the following change of variable in Eq.(2.19),

$$\tilde{\phi} \to \phi = \varphi + \tilde{\phi}\,, \tag{2.20}$$

and considering the measure as translational invariant. One finds

$$\begin{aligned} Z_B[\tilde{J};\varphi] &= Z[\tilde{J}]e^{-i\int d^dx\, \tilde{J}\varphi}\,, \\ W_B[\tilde{J};\varphi] &= W[\tilde{J} - \int d^dx\; \tilde{J}\varphi]\,, \\ \Gamma_B[\tilde{\varphi};\varphi] &= \Gamma[\tilde{\varphi}+\varphi]\,. \end{aligned} \tag{2.21}$$

Considering $\tilde{\varphi} = 0$ in the last equation of (2.21), we get the following identity

$$\Gamma[\varphi] = \Gamma_B[0;\varphi]\,. \tag{2.22}$$

This last equation says that to compute the effective action $\Gamma[\varphi]$ we can compute the vacuum effective action the presence of the background φ. We remind the reader that we are not working in the Lorentzian signature, so in the following we will consider the analog definitions for the Euclidean signature. In Appendix A we show how the one-loop perturbative contributions to the vacuum effective action for given background fields are given in terms of the determinants of operators such as $\mathcal{O}$. These can be naturally defined in general in terms of the functional trace of the heat kernel by

$$\log \text{Det } \mathcal{O} = \text{Tr } \log \mathcal{O} = -\int_0^\infty \frac{ds}{s} \text{Tr } e^{-s\mathcal{O}}, \tag{2.23}$$

where $e^{-s\mathcal{O}}$ admits the asymptotic expansion (2.8). We get

$$\text{Tr } \log \mathcal{O} = -\int_0^\infty \frac{ds}{s} \frac{1}{(4\pi s)^{d/2}} \sum_{k\geqslant 0} \text{tr } [a_k] s^k = -\sum_{k\geqslant 0} \int_0^\infty \frac{ds}{(4\pi)^{d/2}} s^{k-1-\frac{d}{2}} \text{tr } [a_k], \tag{2.24}$$

where the remaining trace is on the remaining indices of the heat kernel coefficients. Eq. (2.24) allows to write quantum corrections in terms of the heat kernel coefficients.

2.3 Three steps for analytical continuation

The strategy adopted for dealing with the regularization of the theories discussed in this work is dimensional regularization (DR) with modified minimal subtraction ($\overline{\text{MS}}$) of the divergences close to the critical dimension of the theory. For example, in gravity $d_{\text{crit}} = 2$, so we subtract the poles $1/d-2$ and a finite part after analytic continuation of the results in the dimensionality. However, some difficulties arise when one tries to apply $\overline{\text{MS}}$ to a quantum theory of the metric. The most prominent one is that several tensor contraction as $g_\mu{}^\mu = d$ appear when taking the trace of the heat kernel coefficients. These might change the finite part of the subtractions when multiplying a pole, or in the worst case, entirely remove a divergence. This could make ambiguous the status of some divergences. To deal with these problems we will follow three steps for the analytical continuation as described in [26]. The idea behind this procedure is that, at the end of the day, we want to consider analytical continuation of our results above $d = 2$.

1. The first step is the analytical continuation of the covariant Feynman diagrams, or quantum corrections, in the dimension, i.e. $d = 2 \to d = 2 - \zeta$. At this moment $\zeta \neq -\epsilon$ introduced previously. Practically this means to analitically continue the d appearing in (2.24). For $\zeta > 0$ the diagrams that are relevant for perturbation theory converge. This is the regime in which we compute radiative corrections. The divergences thus appear as poles $1/\zeta$ and must be subtracted with counterterm operators. Finally, their coefficients assemble into beta functions of renormalized couplings.

2. The second step of the procedure is related to the dimensionality appearing in $\mathrm{tr}\,[a_k]$. Any time a tensorial contraction returns the dimension of spacetime, we denote that dimension as D using a different notation with respect to the previous one to emphasize the different treatment. The simplest example is of course $g_\mu{}^\mu = D$. It is very imporant at this point *not* to substitute $D = 2$ *nor* $D = 2 - \zeta$ or $D = 2 + \epsilon$ when computing divergences. This has the advantage that D appears parametrically in computations, much like N appears in the renormalization of $SU(N)$ gauge theory. Similarly to gauge theories, by setting $D = 2$ or $D = 4$ the metric fluctuactions have the expected degrees of freedom in a given dimension.

3. The third step is finally the continuation of the results to $d > 2$. This is done by continuing $D = 2 + \epsilon$ with ϵ which corresponds the the forbidden region $\zeta < 0$. This explains why this operation is separated from the process of dimensionally regularizing the theory. This is done *after* having regulated and renormalized the model and obtained a beta function. This step introduces the dimensionless coupling through the replacement $G \to G\mu^{-\epsilon}$, where μ is the RG scale, effectively measuring the coupling constants in units of μ.

A summary of the general strategy is the following: to eliminate poles in ζ coming from diagrams entirely through $\overline{MS}$ subtraction (first step), so the express the beta functions as D-dependent objects (second step), that can be continued to $d > 2$ (third step). Using these steps one could investigate the two dimensional limit by taking $D = 2 + \epsilon$ and $\epsilon \to 0$, but can also estimate the four dimensional limit by taking $D = 2 + \epsilon$ and $\epsilon \to 2$. Clearly the limit $\epsilon \to 2$ can be dangerous, for this reason, we recall that the results based on perturbation theory must be considered together with results coming from non-perturbative methods. The advantage of this procedure is that it breaks down the problematic dimensional continuation of gravitational theory in manageable steps, in a way that it is under control and can be discussed at separate moments.

Chapter 3

Yang-Mills

The following chapter will involve the analysis of Yang-Mills theory. In the first part we'll restrict our analysis to flat-space case in order to the review the renormalization scheme proposed in Chapter 2 with a well-known result. In the second part the focus will be the curved spacetime. It will emerge the need to include dynamical degrees of freedom related to the metric.

3.1 Yang-Mills theory in flat spacetime

The objective of this section is to obtain one-loop divergences of a non-abelian gauge field theory in four dimensions. The action in the Euclidean formulation is

$$S_{YM} = \frac{1}{4g^2} \int d^d x \, F^a_{\mu\nu} F^{\mu\nu a} \tag{3.1}$$

where g is the coupling constant, ${F_{\mu\nu}}^a$ are the components of the field strength tensor $F_{\mu\nu}$, i.e.

$$F_{\mu\nu} = -i{F_{\mu\nu}}^a t^a, \tag{3.2}$$

t^a are the *generators* of the $SU(N)$ group. The field strength tensor is defined by the commutator of the covariant derivative

$$D_\mu = \partial_\mu + A_\mu = \partial_\mu - iA^a_\mu t^a, \tag{3.3}$$

so it takes the form

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{3.4}$$

Expanding in components and considering the relation

$$[t^a, t^b] = if^{abc}t^c, \tag{3.5}$$

where f^{abc} are called *structure constants* of the Lie group, we obtain

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu. \tag{3.6}$$

To conclude this set of definitions we give also the infinitesimal transformation of the gauge field introducing the *adjoint representation* for the generators of the Lie algebra, i.e. $(t^c_{\text{adj}})^{ab} = i f^{acb}$,

$$\delta A^a_\mu = D_\mu \alpha^a = \partial_\mu \alpha^a + f^{abc} A^b_\mu \alpha^c. \tag{3.7}$$

Note that using this normalization the gauge coupling is removed from the covariant derivative ad moved as a coefficient of the whole action.

3.1.1 Background field method

The effective action will be computed through the background field method [32, 36]. We split the gauge field into a background field and a fluctuating quantum field, namely

$$A^a_\mu = \bar{A}^a_\mu + a^a_\mu. \tag{3.8}$$

The field strength decomposes as

$$\begin{aligned}
F^a_{\mu\nu} &= \partial_\mu \bar{A}^a_\nu - \partial_\nu \bar{A}^a_\nu + f^{abc} \bar{A}^b_\mu \bar{A}^c_\nu \\
&\quad + \partial_\mu a^a_\nu + f^{abc} \bar{A}^b_\mu a^c_\nu - \partial_\nu a^a_\mu + f^{abc} a^b_\mu \bar{A}^c_\nu + f^{abc} a^b_\mu a^c_\nu \\
&= \partial_\mu \bar{A}^a_\nu - \partial_\nu \bar{A}^a_\nu + f^{abc} \bar{A}^b_\mu \bar{A}^c_\nu \\
&\quad + \partial_\mu a^a_\nu + f^{abc} \bar{A}^b_\mu a^c_\nu - \partial_\nu a^a_\mu - f^{acb} \bar{A}^c_\nu a^b_\mu + f^{abc} a^b_\mu a^c_\nu \\
&= \bar{F}_{\mu\nu} + \bar{D}_\mu a^a_\nu - \bar{D}_\nu a^a_\mu + f^{abc} a^b_\mu a^c_\nu,
\end{aligned} \tag{3.9}$$

where, in the last line, $\bar{F}_{\mu\nu} = \partial_\mu \bar{A}^a_\nu - \partial_\nu \bar{A}^a_\nu + f^{abc} \bar{A}^b_\mu \bar{A}^c_\nu$ is the field strength of the background field, and $\bar{D}_\mu a^a_\nu = \partial_\mu a^a_\nu + f^{abc} \bar{A}^b_\mu a^c_\nu$ is the covariant derivative with respect to the background field in the adjoint representation. One important observation is the transformation (3.7) can be split in different ways between background and fluctuation. One way is to keep the background fixed and attribute all the variation to the quantum field:

$$\begin{aligned}
\delta^{(Q)}_\alpha \bar{A}^a_\mu &= 0, \\
\delta^{(Q)}_\alpha a^a_\mu &= D_\mu \alpha^a.
\end{aligned} \tag{3.10}$$

These are called""quantum gauge transformations". Another way to split the transformation is

$$\begin{aligned}
\delta^{(B)}_\alpha \bar{A}^a_\mu &= \bar{D}_\mu \alpha^a = \partial_\mu \alpha^a + f^{abc} \bar{A}^b_\mu \alpha^c, \\
\delta^{(B)}_\alpha a^a_\mu &= f^{abc} a^b_\mu \alpha^c,
\end{aligned} \tag{3.11}$$

so that the background transforms as a connection and the quantum field as a matter field in the adjoint representation. These are called "background gauge transformations". The gauge fixing is meant to break the quantum gauge transformations but is possible to choose it in such a way to preserve the background gauge invariance. This is extremely advantageous since it will constraint the effective action. Using Faddeev-Popov method [32] or BRST quantization method [37] we can obtain the following gauge-fixing term covariant with respect to the background gauge field

$$\mathcal{L}_{GF} = \frac{1}{2g^2\xi}(\bar{D}^\mu a^a_\mu)^2, \tag{3.12}$$

and the ghost term

$$\mathcal{L}_{GH} = \bar{c}^a \bar{D}^\mu D_\mu c^a. \tag{3.13}$$

Then the gauge fixed Lagrangian is

$$\mathcal{L} = \frac{1}{4g^2}(\bar{F}_{\mu\nu} + \bar{D}_\mu a^a_\nu - \bar{D}_\nu a^a_\mu + f^{abc} a^b_\mu a^c_\nu)^2 + \frac{1}{2g^2\xi}(\bar{D}^\mu a^a_\mu)^2 + \bar{c}^a \bar{D}^\mu D_\mu c^a. \tag{3.14}$$

To compute the effective action $\Gamma[\bar{A}]$ to one-loop order we drop linear terms in the fluctuating field a^a_μ and then integrate over the terms quadratic in a^a_μ and the ghost fields (see Appendix A). To integrate the quadratic terms is necessary to work out the terms in (3.14) quadratic in each of the various fields. It is convenient to choose the Feynman gauge $\xi = 1$. The terms quadratic in a^a_μ are

$$\mathcal{L}^{(2)}_a = \frac{1}{2g^2}\left[\frac{1}{2}(\bar{D}_\mu a^a_\nu - \bar{D}_\nu a^a_\mu)^2 + \bar{F}^{\mu\nu a} f^{abc} a^b_\mu a^c_\nu + (\bar{D}^\mu a^a_\mu)^2\right]. \tag{3.15}$$

After integrating by parts and using anti-symmetry of the structure constants we can rewrite

$$\mathcal{L}^{(2)}_a = \frac{1}{2g^2}\left\{a^a_\mu[-(\bar{D}^2)^{ab} g^{\mu\nu} + (\bar{D}^\nu \bar{D}^\mu)^{ab} - (\bar{D}^\mu \bar{D}^\nu)^{ab}]a^b_\nu - a^a_\mu f^{abc} \bar{F}^{b\mu\nu} a^c_\nu\right\}. \tag{3.16}$$

We can recognize the commutator of covariant derivatives and use the relation

$$[\bar{D}^\nu, \bar{D}^\mu]^{ab} = -i\bar{F}^{\nu\mu\, c}(t^c_{adj})^{ab} = \bar{F}^{\nu\mu\, c} f^{acb} \tag{3.17}$$

where we used the adjoint representation for the generators. Substituting in (3.16) we get

$$\mathcal{L}^{(2)}_a = \frac{1}{2g^2}\left\{a^a_\mu[-(\bar{D}^2)^{ac} g^{\mu\nu} - 2f^{abc}\bar{F}^{b\mu\nu}]a^c_\nu\right\}. \tag{3.18}$$

We can rewrite this equation as

$$\mathcal{L}^{(2)}_a = \frac{1}{2g^2} a^a_\mu \Delta^{\mu\nu\, ac} a^c_\nu, \tag{3.19}$$

where

$$\Delta^{\mu\nu\,ac} = -g^{\mu\nu}(\bar{D}^2)^{ac} + E^{\mu\nu\,ac}; \qquad E^{\mu\nu\,ac} = -2f^{abc}F^{b\mu\nu} \tag{3.20}$$

The effective action will be a functional of two fields: $\Gamma[a;\bar{A}]$. We are interested in the special case $a^a_\mu = 0$, in this case $D_\mu = \bar{D}_\mu$ and the ghost term is simply

$$\mathcal{L}_{GH} = \bar{c}^a \Delta^{ab}_{GH} c^b, \tag{3.21}$$

where

$$\Delta_{GH} = \bar{D}^2. \tag{3.22}$$

3.1.2 Effective action and beta function

The effective action at one-loop is given by

$$\begin{aligned} e^{-\Gamma[\bar{A}]} &= \int Da\, D\bar{c}\, Dc\, e^{-(S_{YM}[\bar{A}]+S^{(2)}_a+S_{GH})} \\ &= e^{-S_{YM}[\bar{A}]} \int Da\, e^{-\int d^4x\, \mathcal{L}^{(2)}_a} \int D\bar{c}\, Dc\, e^{-\int d^4x \mathcal{L}_{GH}} \\ &= e^{-S_{YM}[\bar{A}]} (\mathrm{Det}\,\Delta^{\mu\nu\,ab})^{-\frac{1}{2}} (\mathrm{Det}\,(-\Delta^{ab}_{GH})). \end{aligned} \tag{3.23}$$

Taking the logarithm we get

$$\Gamma[\bar{A}] = S_{YM}[\bar{A}] + \frac{1}{2}\mathrm{Tr}\,\ln\Delta - \mathrm{Tr}\,\ln(-\Delta_{GH}). \tag{3.24}$$

Following [5] and [39] we can use the integral representation to relate the Tr ln Δ to the heat kernel coefficients. We have

$$\frac{1}{2}\mathrm{Tr}\,\ln\,\Delta - \mathrm{Tr}\,\ln\,(-\Delta_{GH}) = -\frac{1}{2}\int_0^\infty \frac{ds}{s}\mathrm{Tr}\; e^{-s\Delta} + \int_0^\infty \frac{ds}{s}\mathrm{Tr}\; e^{-s(-\Delta_{GH})} \tag{3.25}$$

and recognizing $e^{-s\Delta}$ as the heat kernel function we can expand (see Appendix B)

$$\frac{1}{2}\mathrm{Tr}\,\ln\,\Delta - \mathrm{Tr}\,\ln\,(-\Delta_{GH}) = -\frac{1}{2}\int_0^\infty \frac{ds}{s}\frac{1}{(4\pi s)^{\frac{d}{2}}}\sum_{k\geqslant 0}(\mathrm{Tr}\; a_k(\Delta) - 2\,\mathrm{Tr}\; a_k(-\Delta_{GH}))\, s^k \tag{3.26}$$

Let's work on the right hand side of the equation

$$-\frac{1}{2}\int_0^\infty \frac{ds}{s}\frac{1}{(4\pi s)^{d/2}}\sum_{k\geqslant 0}(\mathrm{tr}\;[a_k(\Delta)] - 2\;\mathrm{tr}\;[a_k(-\Delta_{GH})])s^k, \tag{3.27}$$

note the coincidence limit on the heat kernel coefficients. To regularize this integral it is necessary to introduce a mass parameter m using the exponential e^{-sm^2}, i.e.

$$-\frac{1}{2}\sum_{k\geqslant 0}\int_0^\infty \frac{ds}{(4\pi)^{d/2}} s^{k-1-d/2} e^{-sm^2}(\mathrm{tr}\;[a_k(\Delta)] - 2\;\mathrm{tr}\;[a_k(-\Delta_{GH})]). \tag{3.28}$$

We can rewrite equation (3.28) as

$$-\frac{1}{2}\sum_k(\text{tr }[a_k(\Delta)] - 2\text{ tr }[a_k(-\Delta_{GH})])\frac{1}{(4\pi)^{d/2}}\int_0^\infty ds\, s^{k-1-d/2}e^{-sm^2} \tag{3.29}$$

and recalling the definition of the *Gamma function*,

$$\Gamma(z) = \int_0^\infty dt\; t^{z-1}e^{-t}, \tag{3.30}$$

we have

$$\int_0^\infty ds\, s^{k-1-d/2}e^{-sm^2} = m^{d-2k}\,\Gamma\left(k - \frac{d}{2}\right). \tag{3.31}$$

Substituting in (3.29), we get

$$-\frac{1}{2}\sum_k(\text{tr }[a_k(\Delta)] - 2\text{ tr }[a_k(-\Delta_{GH})])\frac{1}{(4\pi)^{d/2}}\; m^{d-2k}\Gamma\left(k - \frac{d}{2}\right) \tag{3.32}$$

Since we are interested in the case $d = 4$, it is clear from (2.24) that quartic divergences appear for $k = 0$, the quadratic ones for $k = 1$, and the logarithmic ones for $k = 2$. The quartic divergences are field-independent so will neglect them. There are no quadratic divergences because tr $E = 0$ and both a_1 vanish. The logarithmic divergences are given by the last one of (2.15). For the rest of this section we will drop the bar notation for the background field. For the operator Δ we get

$$\begin{aligned}
\text{tr }[a_2(\Delta)] &= \int d^dx\,\left(\frac{D}{12}F_{\rho\sigma}F^{\rho\sigma} + \frac{1}{2}E^{\rho\sigma\,ab}E_{\rho\sigma}{}^{ab}\right)\\
&= \int d^dx\,\left(\frac{D}{12}F^a_{\rho\sigma}F^{b\rho\sigma}f^{cad}f^{dbe}\delta^{ce} + \frac{4}{2}f^{acb}F^{c\rho\sigma}f^{adb}F^d_{\rho\sigma}\right)\\
&= \int d^dx\left(-\frac{D}{12}F^{a\rho\sigma}F^b_{\rho\sigma}C_2\delta^{ab} + 2F^{c\rho\sigma}F^d_{\rho\sigma}C_2\delta^{cd}\right)\\
&= \int d^dx\left(\frac{24-D}{12}C_2F^{a\rho\sigma}F^a_{\rho\sigma}\right),
\end{aligned} \tag{3.33}$$

while, for the ghost operator $-\Delta_{GH}$, we get

$$\text{tr }[a_2(-\Delta_{GH})] = \int d^dx\,\frac{1}{12}F^{\rho\sigma}F_{\rho\sigma} = \int d^dx\,\left(-\frac{C_2}{12}F^{a\rho\sigma}F^a_{\rho\sigma}\right). \tag{3.34}$$

In both these equations we introduced the Casimir invariant C_2 through the relation

$$f^{acd}f^{bcd} = C_2\delta^{ab}. \tag{3.35}$$

For the adjoint representation of the group $SU(N)$ we have $C_2 = 2$. Substituting (3.33) and (3.34) in (3.32), we have

$$-\frac{1}{2}\frac{1}{(4\pi)^{d/2}}m^{d-4}\,\Gamma\left(2-\frac{d}{2}\right)\int d^dx\,\left(\frac{26-D}{12}\right)C_2\,F_{\mu\nu}{}^{a}F^{\mu\nu a} \tag{3.36}$$

Now we can perform the analytic continuation $d \to d = 4-\epsilon$, resulting in

$$-\frac{1}{2}\frac{1}{(4\pi)^2}(4\pi)^{\frac{\epsilon}{2}}m^{-\epsilon}\,\Gamma\left(\frac{\epsilon}{2}\right)\int d^dx\,\left(\frac{26-D}{12}\right)C_2\,F_{\mu\nu}{}^{a}F^{\mu\nu a}. \tag{3.37}$$

Considering the expansions

$$\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon), \tag{3.38}$$

where γ is the Euler-Mascheroni constant,

$$m^{-\epsilon} = 1 - \epsilon \log\, m + \mathcal{O}(\epsilon^2), \tag{3.39}$$

and

$$(4\pi)^{\frac{\epsilon}{2}} = 1 + \frac{\epsilon}{2}\log\, 4\pi + \mathcal{O}(\epsilon^2), \tag{3.40}$$

we get

$$-\frac{1}{2}\lim_{m\longrightarrow\infty}\frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma + 2\log\,\frac{1}{m} + \log\, 4\pi\right)\int d^dx\left(\frac{26-D}{12}\right)C_2 F^a_{\mu\nu}F^{a\mu\nu} \tag{3.41}$$

Using $\overline{\text{MS}}$ scheme we are left with

$$-\frac{1}{(4\pi)^2}\left(\frac{26-D}{12}\right)C_2\log\,\frac{\mu}{m}\int d^dx\; F^a_{\mu\nu}F^{a\mu\nu} \tag{3.42}$$

The one-loop effective action has the form

$$\Gamma[A] = S_{YM}[A] - \frac{1}{(4\pi)^2}\left(\frac{26-D}{12}\right)C_2\log\,\frac{\mu}{m}\int d^dx\; F^a_{\mu\nu}F^{a\mu\nu}. \tag{3.43}$$

We can define the renormalized coupling g_R as

$$\frac{1}{4g_R^2} = \frac{1}{4g^2} - \frac{1}{(4\pi)^2}\left(\frac{26-D}{12}\right)C_2\log\,\frac{\mu}{m}, \tag{3.44}$$

from which

$$g_R(m) = \frac{g}{\sqrt{1-\frac{g^2}{(4\pi)^2}\frac{26-D}{3}C_2\log\,\frac{\mu}{m}}}. \tag{3.45}$$

We can finally compute the beta function for the coupling constant

$$m\frac{dg_R}{dm}\bigg|_{m=\mu} = \frac{mg}{2}\left(1-\frac{g^2}{(4\pi)^2}\frac{26-D}{3}C_2\log\,\frac{\mu}{m}\right)^{-\frac{3}{2}}\left(\frac{g^2}{(4\pi)^2}\frac{26-D}{3}C_2\frac{m}{\mu}\frac{-1}{m^2}\right)\bigg|_{m=\mu} \tag{3.46}$$

$$\beta(g_R) = -\frac{1}{2}\frac{1}{(4\pi)^2}\frac{26-D}{3}C_2 g_R^3. \tag{3.47}$$

We recovered the well known Yang-Mills beta function. At this point we could follow the third step for the analytical continuation described in Chapter 2 and move form $d = 4$. In Chapter 4 we will perform the explicit computation for Einstein-Yang-Mills theory in $d = 2 + \epsilon$.

3.2 Yang-Mills in curved space

In the following section we will compute UV divergences for Yang-Mills theory in curved space. The relevance of such calculations, without considering also the fluctuations of the metric and hence quantum gravity, has been disputed. Nevertheless, the considerations of field theories with classical background fields on curved space occurs in semiclassical contexts and the techniques involved are necessary preliminary to considering quantum gravity. The action in presence of a general background metric $\bar{g}_{\mu\nu}$ takes the form

$$S_{YM} = \int d^4x\,\sqrt{\bar{g}}\frac{1}{4\eta^2}F^a_{\mu\nu}F^{a\mu\nu} \tag{3.48}$$

where

$$F^a_{\mu\nu} = \nabla_\mu A^a_\nu - \nabla_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu \tag{3.49}$$

and ∇_μ is the covariant derivative related to the metric $\bar{g}_{\mu\nu}$. We changed the notation from g to η for the coupling constant to avoid confusion with the metric. The last equality is due to the fact that we are studying the torsionless case in which the Christoffel symbols are symmetric $\bar{\Gamma}^\rho_{\mu\nu} = \bar{\Gamma}^\rho_{\nu\mu}$.

3.2.1 Background field method

As in the previous section we can use background field method, define $A^a_\mu = \bar{A}^a_\mu + a^a_\mu$ and keep only quadratic terms in a since we are interested in one-loop calculations. We can write

$$F^a_{\mu\nu} = \bar{F}^a_{\mu\nu} + \bar{D}_\mu a^a_\nu - \bar{D}_\nu a^a_\mu + f^{abc}a^b_\mu a^c_\nu \tag{3.50}$$

where now D_μ is a covariant derivative which contains both the connections due to the metric $\bar{g}_{\mu\nu}$ and the gauge field $\bar{A}^a_\mu$, namely

$$\bar{D}_\mu a^a_\nu = \bar{\nabla}_\mu a^a_\nu + f^{abc}\bar{A}^b_\mu a^c_\nu = \partial_\mu a^a_\nu - \bar{\Gamma}^\rho_{\mu\nu}a^a_\rho + f^{abc}\bar{A}^b_\mu a^c_\nu. \tag{3.51}$$

We can choose to consider a gauge fixing term related to the condition

$$\bar{D}_\mu a^{a\mu} = 0. \tag{3.52}$$

Following the previous calculation as in the flat space case we arrive to the following quadratic contribution to the action

$$S_2[\bar{A},\bar{g};a] = \frac{1}{\eta^2}\int d^4x\,\sqrt{\bar{g}}\frac{1}{2}\left[a^a_\mu(-g^{\mu\nu}\bar{D}^2)^{ab}a^b_\nu + a^a_\mu[\bar{D}^\nu,\bar{D}^\mu]^{ab}a^b_\nu + \bar{F}^a_{\mu\nu}f^{abc}a^{b\mu}a^{c\nu}\right]. \tag{3.53}$$

Differently with respect to the previous case, the commutator is not just the Yang Mills field strength, considering the matrix valued form we have

$$[\bar{D}^\nu,\bar{D}^\mu] = [\bar{\nabla}^\nu,\bar{\nabla}^\mu] + [\bar{\nabla}^\nu,\bar{A}^\mu] + [\bar{A}^\nu,\bar{\nabla}^\mu] + [\bar{A}^\nu,\bar{A}^\mu] = [\bar{\nabla}^\nu,\bar{\nabla}^\mu] + \bar{F}^{\nu\mu}. \tag{3.54}$$

Recalling that the commutation relation of covariant derivatives of a controvariant vector field V^ρ is

$$[\nabla_\mu,\nabla_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma, \tag{3.55}$$

we can rewrite the commutator term in (3.53) explicitly in the components

$$\begin{aligned} a^a_\mu[\bar{\nabla}^\nu,\bar{\nabla}^\mu]^{ab}a^b_\nu + a^a_\mu(\bar{F}^{\nu\mu})^{ab}a^b_\nu &= a^a_\mu\bar{R}_\nu{}^{\rho\nu\mu}\delta^{ab}a^b_\rho + \bar{F}^{\nu\mu c}f^{acb}a^a_\mu a^b_\nu \\ &= a^a_\mu\bar{R}^{\mu\nu}\delta^{ab}a^b_\nu + \bar{F}^{\mu\nu a}f^{abc}a^b_\mu a^c_\nu. \end{aligned} \tag{3.56}$$

Substituting in (3.53) we get

$$\frac{1}{\eta^2}\int d^4x\,\sqrt{\bar{g}}\frac{1}{2}\left[a^a_\mu(-g^{\mu\nu}\bar{D}^2)^{ab}a^b_\nu + a^a_\mu\bar{R}^{\mu\nu}\delta^{ab}a^b_\nu + 2\bar{F}^a_{\mu\nu}f^{abc}a^{b\mu}a^{c\nu}\right]. \tag{3.57}$$

At the end, we are left with an elliptic operator of Laplace type

$$\Delta = -g^{\mu\nu}(\bar{D}^2)^{ab} + E^{\mu\nu ab} \tag{3.58}$$

with a covariant derivative containing the connection

$$(\omega_\mu)_\nu{}^{\rho ac} = f^{abc}\bar{A}^b_\mu\delta_\nu{}^\rho - \bar{\Gamma}^\rho_{\mu\nu}\delta^{ac} \tag{3.59}$$

and an endomorphism

$$E^{\mu\nu bc} = \bar{R}^{\mu\nu}\delta^{bc} + 2\bar{F}^{a\mu\nu}f^{abc}. \tag{3.60}$$

The field strength corresponding to the connection (3.59) is

$$(\Omega_{\mu\nu})_\rho{}^{\sigma ac} = \bar{R}_\rho{}^\sigma{}_{\mu\nu}\,\delta^{ac} + \bar{F}^b_{\mu\nu}\,f^{abc}\delta_\rho{}^\sigma. \tag{3.61}$$

The ghost term corresponding to the gauge condition (3.52) is just

$$\mathcal{L}_{\rm GH} = \bar{c}^a\Delta^{ab}_{\rm GH}c^b \tag{3.62}$$

with

$$\Delta^{ab}_{GH} = (\mathcal{D}^2)^{ab} \tag{3.63}$$

where

$$\mathcal{D}^{ac} = \partial_\mu\delta^{ac} + f^{abc}\bar{A}^b_\mu. \tag{3.64}$$

3.2.2 Effective action and divergences

Following the exact same procedure as in the previous section we get

$$\Gamma[\bar{A};\bar{g}] = S_{YM}[\bar{A};\bar{g}] + \frac{1}{2}\mathrm{Tr}\,\log\,\Delta - \mathrm{Tr}\,\log\,(-\Delta_{GH}). \tag{3.65}$$

Following the previous section, in $d = 4$ the we have quartic, quadratic and logarithmic divergences. The quartic one are proportional to

$$\mathrm{tr}\,[a_0(\Delta)] - 2\,\mathrm{tr}\,[a_0(-\Delta_{GH})]. \tag{3.66}$$

The coefficients are

$$\mathrm{tr}\,[a_0(\Delta)] = \int d^dx\,\sqrt{\bar{g}}D(N^2-1)\,, \tag{3.67}$$

and

$$\mathrm{tr}\,[a_0(-\Delta)_{GH}] = \int d^dx\,\sqrt{\bar{g}}(N^2-1)\,. \tag{3.68}$$

Again, as in the flat case, the quartic divergences are field-independent and can be neglected. The quadratic divergences are instead proportional to

$$\mathrm{tr}\,[a_1(\Delta)] - 2\,\mathrm{tr}\,[a_1(-\Delta_{GH})]\,, \tag{3.69}$$

and the corresponding coefficients are

$$\mathrm{tr}\,[a_1(\Delta)] = \int d^dx\,\sqrt{\bar{g}}\frac{\bar{R}}{6}(N^2-1)\,(D-6)\ , \tag{3.70}$$

and

$$\mathrm{tr}\,[a_1(-\Delta_{GH}) = \int d^dx\,\sqrt{\bar{g}}\frac{\bar{R}}{6}(N^2-1)\ . \tag{3.71}$$

Putting these coefficients into (3.69) we get the following quadratic divergence.

$$\int d^d x \, \frac{\bar{R}}{6}(N^2-1)(D-8) \, . \tag{3.72}$$

Finally, the logarithmic divergences are proportional to

$$\mathrm{tr}\,\left[a_2(\Delta)\right] - 2\ \mathrm{tr}\,\left[a_2(-\Delta_{GH})\right], \tag{3.73}$$

and the coefficients are

$$\begin{aligned} \mathrm{tr}\,\left[a_2(\Delta)\right] = \int d^d x \, \sqrt{\bar{g}} \Bigg[& \left(\frac{24-D}{12}\right) C_2 \bar{F}^{\mu\nu a} \bar{F}_{\mu\nu}{}^{a} + \left(\frac{D-15}{180}\right)(N^2-1)\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma} \\ & \left(\frac{90-D}{180}\right)(N^2-1)\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + \left(\frac{D-12}{72}\right)(N^2-1)\bar{R}^2 \Bigg], \end{aligned} \tag{3.74}$$

and

$$\mathrm{tr}\,\left[a_2(-\Delta_{GH}]\right) = \int d^d x \, \sqrt{\bar{g}} \left[\frac{N^2-1}{180}\left(\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma} - \bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + \frac{5}{2}\bar{R}^2\right) - \frac{1}{12} C_2 \bar{F}^{a\mu\nu}\bar{F}^{a}_{\mu\nu} \right]. \tag{3.75}$$

Putting (3.74) and (3.75) in (3.73), we get

$$\begin{aligned} \int d^d x \, \sqrt{\bar{g}} \Bigg[& \left(\frac{26-D}{12}\right) C_2 \bar{F}^{\mu\nu a} \bar{F}_{\mu\nu}{}^{a} + \left(\frac{D-17}{180}\right)(N^2-1)\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma} \\ & \left(\frac{92-D}{180}\right)(N^2-1)\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + \left(\frac{D-14}{72}\right)(N^2-1)\bar{R}^2 \Bigg]. \end{aligned} \tag{3.76}$$

Looking at these divergences, it is clear the need to introduce the dynamical terms for gravity in the action. We will consider the Einstein-Yang-Mills theory in the next chapter.

Chapter 4

Gravity + Yang Mills

To introduce the dynamical degrees of freedom of gravity it is necessary to consider a gravitational term in the action, in particular we choose the Euclidean Einstein-Hilbert action:

$$S_{EH}[g] = \int d^dx \, \sqrt{g}(g_0 - g_1 R), \tag{4.1}$$

in which we introduced the curvature scalar R and the couplings g_0, g_1 related to the cosmological and Newton's constant. We choose g_0 to be exactly the cosmological constant, while the strength of the gravitational interaction is weighted by G, which we choose to be $g_1 = G^{-1}$. Action (4.1) is manifestly invariant under diffeomorphisms. Infinitesimal diffeomorphisms act as changes of coordinates and are generated by a vector field, $x^\mu \to x^\mu + \xi^\mu(x)$, resulting in a transformation of the metric

$$\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \, . \tag{4.2}$$

Since the composition of two transformations is still a transformation, the diffeomorphisms from a group, whose algebra is obviously closed

$$[\delta_{\xi_1}, \delta_{\xi_2}] = \delta_{[\delta_{\xi_1}, \delta_{\xi_2}]}, \tag{4.3}$$

where one the right hand side the commutator denotes the standard Lie brackets of two vector fields. Considering also the Yang Mills part we have an action of the form

$$S[g, A] = \int d^dx \, \sqrt{g} \left(g_0 - g_1 R + \frac{1}{4\eta^2} F^a_{\mu\nu} F^{\mu\nu a} \right) \tag{4.4}$$

The analogous calculation for Lorentzian metrics differs by an overall sign. The equations of motion for the action (4.4) are

$$\frac{1}{2} g_{\mu\nu}(g_0 - g_1 R) + g_1 R_{\mu\nu} = T_{\mu\nu}, \tag{4.5}$$

where

$$T_{\mu\nu} = \frac{1}{2g^2}\left(F_\mu{}^{\alpha a}F_{\nu\alpha}{}^a - \frac{1}{4}F_{\alpha\beta}{}^a F^{\alpha\beta a}g_{\mu\nu}\right) , \tag{4.6}$$

and

$$D_\nu F^{\mu\nu a} = 0 . \tag{4.7}$$

4.1 The gravitational path integral

The path integral is formally given by

$$Z = \int DgDA\, e^{iS[g,A]}. \tag{4.8}$$

This integral is divergent due to the invariance under diffeomorphisms and gauge transformations for the action. To make sense of this integral one could follow the Faddeev-Popov procedure or the BRST quantization method. Both the procedures are analogous to the constructions for Yang-Mills theories, with a crucial difference: in Yang-Mills theory there is no difficulty in taking a "zero connection" and therefore the use of background field method is optional. In the case of gravity it is not clear how to make sense of the action for a "zero metric", or more generally degenerate metrics. It is debatable whether such configurations should be taken into account or not. As a consequence, the use of the background field method is almost unavoidable. Let's therefore split

$$\begin{aligned} g_{\mu\nu} &= \bar{g}_{\mu\nu} + h_{\mu\nu} + \frac{\lambda}{2}h_{\mu\rho}\bar{g}^{\rho\sigma}h_{\sigma\nu} \\ A^a_\mu &= \bar{A}^a_\mu + a^a_\mu \end{aligned} \tag{4.9}$$

where $\bar{g}$ and $\bar{A}$ are the classical backgrounds, h and a are the quantum fields or fluctuations and λ is an arbitrary parameter to test the parametric dependence of results. Notice that when $\lambda = 1$, (4.9) matches the exponential parametrization to the quadratic order. The split allows us to preserve manifest covariance under the background version of (4.2). The gauge fixing term must fix (4.2) seen as a transformation of $h_{\mu\nu}$ at fixed arbitrary $\bar{g}_{\mu\nu}$, which is non-linear because of the right hand side of (4.9). To construct the correct transformation of $h_{\mu\nu}$ order-by-order in $h_{\mu\nu}$ itself, we need to invert the relation

$$g_{\nu\rho}\nabla_\mu\xi^\rho + g_{\mu\rho}\nabla_\nu\xi^\rho = \delta_\xi h_{\mu\nu} + \lambda h_{(\mu\rho}\bar{g}^{\rho\theta}\delta_\xi h_{\theta\nu)} + \mathcal{O}(h^3). \tag{4.10}$$

Using (4.9) on metrics and connections on the left hand side, we find

$$\delta_\xi h_{\mu\nu} = \bar{g}_{\rho\nu}\bar{\nabla}_\mu\xi^\rho + \bar{g}_{\rho\mu}\bar{\nabla}_\nu\xi^\rho + \mathcal{O}(h), \tag{4.11}$$

where indices can be raised and lowered by the background metric. In order to preserve the background symmetry we will consider the following gauge fixing terms

$$\begin{aligned} S_{GF,\,h} &= \frac{g_1}{2\alpha}\int d^dx\,\sqrt{\bar{g}}\bar{g}^{\mu\nu}f_\mu f_\nu, \\ f_\mu &= \bar{\nabla}^\rho h_{\rho\mu} - \frac{1}{2}\bar{\nabla}_\mu h, \end{aligned} \tag{4.12}$$

and

$$S_{GF,\,a} = \frac{1}{2\gamma\eta^2}\int d^dx\,\sqrt{\bar{g}}(\bar{D}_\mu a^{\mu a})^2, \tag{4.13}$$

where α and γ are gauge fixing parameters, $h = \bar{g}^{\mu\nu}h_{\mu\nu}$ and D_μ is a covariant derivative containing both the connections due to the spacetime metric and the $SU(N)$ group as in the previous chapter. In the following we will consider $\gamma = 1$ and $\alpha = 1$ for convenience. The gauge fixing terms (4.13) and (4.12) come with two ghost terms. Introducing the ghost fields $\bar{c}^\mu, c^\mu$ for the diffeomorphisms, and $\bar{b}^a, b^a$ for the gauge transformations, we have

$$\begin{aligned} S_{GH,h} &= \int d^dx\,\sqrt{\bar{g}}\,\bar{c}^\mu\;\delta_\xi f_\mu|_{\xi=c} \\ &= \int d^dx\,\sqrt{\bar{g}}\,\bar{c}_\mu\,(\bar{\nabla}^2\bar{g}^{\mu\nu} + \bar{R}^{\mu\nu})\,c_\nu. \end{aligned} \tag{4.14}$$

and

$$S_{GH,a} = \int d^dx\,\sqrt{\bar{g}}\,\bar{b}^a(\bar{D}^2)^{ab}b^b. \tag{4.15}$$

The corresponding operators are, respectively,

$$\Delta^{\mu\nu}_{GH,h} = \bar{\nabla}^2\bar{g}^{\mu\nu} + \bar{R}^{\mu\nu} \tag{4.16}$$

and

$$\Delta^{ab}_{GH,a} = (\bar{D}^2)^{ab}. \tag{4.17}$$

4.1.1 Quadratic expansion

The second order perturbation of the action (4.4) together with the corresponding gauge fixing terms (4.12) and (4.13), contains three contributions due to the quadratic contribution in h, in a, and a product of linear contributions in h and a, so we can write

$$S^{(2)}[h,a;\bar{g},\bar{A}] = S^{(2)}_{a,a} + S^{(2)}_{h,h} + S^{(2)}_{a,h}. \tag{4.18}$$

The term $S^{(2)}_{a,a}$ is analogue to what we obtained in the previous chapter, i.e.

$$S^{(2)}_{a,a} = \int d^dx\,\sqrt{\bar{g}}\frac{1}{2\eta^2}a^a_\mu\,\Delta^{\mu\nu ab}_{aa}\,a^b_\nu, \tag{4.19}$$

where

$$\Delta_{aa}^{\mu\nu ab} = -\bar{g}^{\mu\nu}(\bar{D}^2)^{ab} + E^{\mu\nu ab}, \tag{4.20}$$

and

$$E^{\mu\nu ab} = \bar{R}^{\mu\nu}\delta^{ab} + 2\bar{F}^{c\mu\nu}f^{cab}. \tag{4.21}$$

The $S^{(2)}_{h,h}$ is more complex due to the structure of the operator. It reads as follows.

$$S^{(2)}_{h,h} = \int d^dx\, \sqrt{\bar{g}}\frac{g_1}{2}\, h_{\mu\nu}\, \mathcal{O}_{hh}^{\mu\nu\rho\sigma}\, h_{\rho\sigma} \tag{4.22}$$

where

$$\mathcal{O}_{hh}^{\mu\nu\rho\sigma} = -K^{\mu\nu\rho\sigma}\,\bar{\nabla}^2 + U^{\mu\nu\rho\sigma}\,. \tag{4.23}$$

We have

$$K^{\mu\nu\rho\sigma} = \frac{1}{4}\left(\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma} + \bar{g}^{\mu\sigma}\bar{g}^{\nu\rho} - \bar{g}^{\mu\nu}\bar{g}^{\rho\sigma}\right) \tag{4.24}$$

and

$$\begin{aligned}
U^{\mu\nu\rho\sigma} = {} & \frac{\bar{F}^{\mu\sigma a}\bar{F}^{\nu\rho a}}{2g_1\ \eta^2} + \frac{\bar{F}^{\mu\rho a}\bar{F}^{\nu\sigma a}}{2g_1\eta^2} - \frac{\bar{F}^{\rho\lambda a}\bar{F}^{\sigma}{}_{\lambda}{}^{a}\bar{g}^{\mu\nu}}{2\ g_1\eta^2} + - \frac{(\lambda-2)\bar{F}^{\nu\lambda a}\ \bar{F}^{\sigma}{}_{\lambda}{}^{a}\bar{g}^{\mu\rho}}{4g_1\eta^2} \\
& - \frac{(\lambda-2)\bar{F}^{\nu\lambda a}\bar{F}^{\rho}{}_{\lambda}{}^{a}\bar{g}^{\mu\sigma}}{4g_1\eta^2} - \frac{(\lambda-2)\bar{F}^{\mu\lambda}\bar{F}^{\sigma}{}_{\lambda}{}^{a}\bar{g}^{\nu\rho}}{4g_1\eta^2} + \frac{g_0\ (\lambda-1)\bar{g}^{\mu\sigma}\bar{g}^{\nu\rho}}{2g_1} \\
& + \frac{(\lambda-1)\bar{F}_{\kappa\lambda}{}^{a}\bar{F}^{\kappa\lambda a}\bar{g}^{\mu\sigma}\bar{g}^{\nu\rho}}{8\ g_1\eta^2} - \frac{(\lambda-2)\bar{F}^{\mu\lambda a}\ \bar{F}^{\rho}{}_{\lambda}{}^{a}\bar{g}^{\nu\sigma}}{4g_1\eta^2} + \frac{g_0(\lambda-1)\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}}{2g_1} \\
& + \frac{(\lambda-1)\bar{F}_{\kappa\lambda}{}^{a}\bar{F}^{\kappa\lambda a}\bar{g}^{\mu\rho}\bar{g}^{\nu\ \sigma}}{8g_1\eta^2} - \frac{\bar{F}^{\mu\lambda a}\bar{F}^{\ \nu}{}_{\lambda}{}^{a}\bar{g}^{\rho\sigma}}{2g_1\eta^2} + \frac{g_0\bar{g}^{\mu\nu}\bar{g}^{\rho\sigma}}{2g_1} \\
& + \frac{\bar{F}_{\kappa\lambda}{}^{a}\bar{F}^{\kappa\lambda a}\bar{g}^{\mu\nu}\bar{g}^{\rho\sigma}}{8g_1\eta^2} + \bar{g}^{\rho\sigma}\bar{R}^{\mu\nu} + \frac{\lambda-1}{2}\bar{g}^{\nu\sigma}\bar{R}^{\mu\rho} + \frac{\lambda-1}{2}\bar{g}^{\nu\rho}\bar{R}^{\mu\sigma} + \frac{\lambda-1}{2}\bar{g}^{\mu\sigma}\bar{R}^{\nu\rho} \\
& + \frac{\lambda-1}{2}\bar{g}^{\mu\rho}\bar{R}^{\nu\sigma} + \bar{g}^{\mu\ \nu}\bar{R}^{\rho\sigma} + \frac{1-\lambda}{2}\bar{g}^{\mu\sigma}\bar{g}^{\nu\rho}\bar{R} + \frac{1-\lambda}{2}\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}\bar{R} - \frac{1}{2}\bar{g}^{\mu\nu}\bar{g}^{\rho\sigma}\bar{R} \\
& - \bar{R}^{\mu\rho\nu\sigma} - \bar{R}^{\mu\sigma\nu\rho}
\end{aligned} \tag{4.25}$$

At this point gravity exhibits a new aspect that was not present in the previous cases. The form of the operator $\mathcal{O}_{hh}^{\mu\nu\rho\sigma}$ is not immediately useful for the computation. This happens because the general formula to compute the one-loop effective action is given in terms of a functional determinant of a differential operator, In this context, a differential operator is an object mapping from the space of symmetric tensors to itself. What we computed above is instead an bilinear form, mapping two copies of symmetric tensors to real numbers. In the standard language of differential geometry the $\mathcal{O}_{hh}^{\mu\nu\rho\sigma}$ is a covariant symmetric tensor, while a differential operator is a tensor with one covariant and one controvariant index. The importance of this observation lies in the fact the while the determinant of a differential operator is basis-independent whereas the determinant of a covariant tensor is not. The reason why this can be confusing is that the position of indices in the sense of four-dimensional tensors may be opposite to the one in the functional sense. The solution is given defining an ultralocal metric in the space of symmetric tensors, namely the *DeWitt super-metric*

$$\gamma_{\mu\nu,\rho\sigma} = \bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} + \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho} - \frac{2}{D-2}\bar{g}_{\mu\nu}\bar{g}_{\rho\sigma}, \tag{4.26}$$

that is manifestly symmetric under the exchange of the first two indices. It satisfies the equation

$$\gamma_{\mu\nu,\rho\sigma}\gamma^{\rho\sigma,\alpha\beta} = \mathbf{1}^{\alpha\beta}_{\mu\nu} \equiv \frac{1}{2}(\delta^\alpha_\mu\delta^\beta_\nu + \delta^\alpha_\nu\delta^\beta_\mu). \tag{4.27}$$

With this choice we have

$$K^{\mu\nu\rho\sigma} = \gamma^{\mu\nu,\rho\sigma}, \qquad K^{-1}_{\mu\nu\rho\sigma} = \gamma_{\mu\nu,\rho\sigma}, \tag{4.28}$$

and the definition of the correct kinetic operator Δ_{hh} comes from

$$(\Delta_{hh})_{\mu\nu}{}^{\rho\sigma} \equiv K^{-1}_{\mu\nu\alpha\beta}\mathcal{O}^{\alpha\beta\rho\sigma}_{hh} = -\bar{\nabla}^2\mathbf{1}^{\rho\sigma}_{\mu\nu} + W_{\mu\nu}{}^{\rho\sigma}, \tag{4.29}$$

where

$$\begin{aligned}
W_{\mu\nu}{}^{\gamma\kappa} = {} & \frac{g_0(\lambda-1)\delta^\gamma{}_\nu\delta^{\ \kappa}{}_\mu}{g_1} + \frac{g_0(\lambda-1)\delta^{\ \gamma}{}_\mu\delta^\kappa{}_\nu}{g_1} + \frac{(\lambda-1)\delta^\gamma{}_\nu\delta^{\ \kappa}{}_\mu\bar{F}_{\alpha\beta}{}^a\bar{F}^{\alpha\beta a}}{4\eta^2 g_1} \\
& + \frac{(\lambda-1)\delta^\gamma{}_\mu\ \delta^\kappa{}_\nu\bar{F}_{\alpha\beta}{}^a\bar{F}^{\alpha\beta a}}{4\eta^2 g_1} + \frac{\bar{F}^\gamma{}_\nu{}^a\ \bar{F}^\kappa{}_\mu{}^a}{\eta^2 g_1} + \frac{\bar{F}^\gamma{}_\mu{}^a\bar{F}^\kappa{}_\nu{}^a}{\eta^2 g_1} \\
& - \frac{(\lambda-2)\ \delta^\kappa{}_\nu\bar{F}^{\gamma\alpha a}\bar{F}_{\mu\alpha}{}^a}{2\eta^2 g_1} - \frac{(\lambda-2)\delta^\gamma{}_\nu\bar{F}^{\kappa\alpha a}\bar{F}_{\mu\alpha}{}^a}{2\eta^2 g_1} \\
& - \frac{(\lambda-2)\delta^\kappa{}_\mu\bar{F}^{\gamma\ \alpha a}\bar{F}_{\nu\alpha}{}^a}{2\eta^2 g_1} - \frac{(\lambda-2)\delta^\gamma{}_\mu\bar{F}^{\kappa\alpha a}\bar{F}_{\nu\alpha}{}^a}{2\eta^2 g_1} \\
& - \frac{\bar{F}_\mu{}^{\alpha a}\bar{F}_{\nu\ \alpha}{}^a\bar{g}^{\gamma\kappa}}{\eta^2 g_1} + \frac{2(\lambda-2)\bar{F}^{\gamma\alpha a}\bar{F}^\kappa{}_\alpha{}^a\bar{g}_{\mu\nu}}{(d-2)\eta^2 g_1} + \frac{2g_0\lambda\bar{g}^{\gamma\kappa}\bar{g}_{\mu\nu}}{g_1(2-d)} \\
& - \frac{(\lambda-2)\ \bar{F}_{\alpha\beta}{}^a\bar{F}^{\alpha\beta a}\bar{g}^{\gamma\kappa}\bar{g}_{\mu\nu}}{2(d-2)\eta^2 g_1} - \frac{4(\lambda-1)\bar{g}_{\mu\nu}\bar{R}^{\gamma\kappa}}{d-2} \\
& + (\lambda-1)\delta^\kappa{}_\nu\bar{R}^\gamma{}_\mu + (\lambda-1)\delta^\kappa{}_\mu\bar{R}^\gamma{}_\nu + (\lambda-1)\delta^{\ \gamma}{}_\nu\bar{R}^\kappa{}_\mu \\
& + (\lambda-1)\delta^\gamma{}_\mu\bar{R}^\kappa{}_\nu + 2g^{\gamma\kappa}\bar{R}_{\mu\nu} + (1-\lambda)\delta^\gamma{}_\nu\delta^\kappa{}_\mu\bar{R} \\
& + (1-\lambda)\delta^\gamma{}_\mu\delta^\kappa{}_\nu\bar{R} + \frac{2(\lambda-1)\bar{g}^{\gamma\kappa}\bar{g}_{\mu\nu}\bar{R}}{d-2} \\
& - 2(\bar{R}^\gamma{}_\mu{}^\kappa{}_\nu + \bar{R}^\gamma{}_\nu{}^\kappa{}_\mu)\,.
\end{aligned} \tag{4.30}$$

The mixed term $S^{(2)}_{a,h}$ can be written as

$$S^{(2)}_{a,h} = \int d^d x\, \sqrt{\bar{g}} \frac{\sqrt{g_1}}{2\eta} h_{\mu\nu}\, \mathcal{O}^{\mu\nu\rho\, b}_{ha}\, a^b_\rho + \int d^d x\, \sqrt{\bar{g}} \frac{\sqrt{g_1}}{2\eta} a^b_\rho\, \mathcal{O}^{\rho\, b\, \mu\nu}_{ah}\, h_{\mu\nu}. \tag{4.31}$$

The operators $\mathcal{O}^{\mu\nu\rho\, b}_{ha}$ and $\mathcal{O}^{\rho\, b\, \mu\nu}_{ah}$ can be separated into an endomorphism plus a piece proportional to a covariant derivative, namely

$$\begin{aligned} \mathcal{O}^{\mu\nu\rho\, b}_{ha} &= P^{\mu\nu\rho\, b} + V^{\mu\nu\rho\, b}_{(ha)\ \ \lambda} \bar{\nabla}^\lambda, \\ \mathcal{O}^{\rho\, b\, \mu\nu}_{ah} &= P^{\rho\, b\, \mu\nu} + V^{\rho\, b\, \mu\nu}_{(ah)\ \ \lambda} \bar{\nabla}^\lambda, \end{aligned} \tag{4.32}$$

where the endomorphism part $P^{\mu\nu\,\rho\, b}$ is

$$\begin{aligned} P^{\mu\nu\,\rho\, b} = P^{\rho\, b\, \mu\nu} = \frac{1}{2\sqrt{g_1}\eta} \Bigg(&\bar{A}^{\nu a} \bar{F}^{\mu\rho c} f^{bac} + \bar{A}^{\mu a} \bar{F}^{\nu\rho c} f^{bac} + \bar{A}^{\alpha a} \bar{F}^{\rho\ \ c}_{\ \alpha} f^{bac} \bar{g}^{\mu\nu} \\ &- \bar{A}^{\alpha a} \bar{F}^{\nu\ \ c}_{\ \alpha} f^{bac} \bar{g}^{\mu\rho} - \bar{A}^{\alpha a} \bar{F}^{\mu\ \ c}_{\ \alpha} f^{bac} \bar{g}^{\nu\rho} - \frac{1}{2} \bar{g}^{\nu\rho} \bar{\nabla}_\alpha \bar{F}^{\mu\alpha b} \\ &+ \frac{1}{2} \bar{g}^{\nu\alpha} \bar{\nabla}_\alpha \bar{F}^{\mu\rho b} - \frac{1}{2} \bar{g}^{\mu\rho} \bar{\nabla}_\alpha \bar{F}^{\nu\alpha b} + \frac{1}{2} \bar{g}^{\mu\alpha} \bar{\nabla}_\alpha \bar{F}^{\nu\rho b} + \frac{1}{2} \bar{g}^{\mu\nu} \bar{\nabla}_\alpha \bar{F}^{\rho\alpha b} \Bigg), \end{aligned} \tag{4.33}$$

and the vector parts are

$$V^{\mu\nu\rho\, b}_{(ha)\ \ \lambda} = \frac{1}{2\sqrt{g_1}\eta} (\bar{F}^{\nu\ \ b}_{\ \lambda} \bar{g}^{\mu\rho} + \bar{F}^{\mu\ \ b}_{\ \lambda} \bar{g}^{\nu\rho} - \bar{F}^{\rho\ \ b}_{\ \lambda} \bar{g}^{\mu\nu} - \bar{F}^{\nu\rho b} \delta^\mu_\lambda - \bar{F}^{\mu\rho b} \delta^\nu_\lambda) \tag{4.34}$$

and

$$V^{\rho\, b\, \mu\nu}_{(ah)\ \ \lambda} = \frac{1}{2\sqrt{g_1}\eta} (\bar{F}^{\rho\ \ b}_{\ \lambda} \bar{g}^{\mu\nu} - \bar{F}^{\rho\mu b} \delta^\nu_\lambda - \bar{F}^{\rho\nu b} \delta^\mu_\lambda - \bar{F}^{\mu\ \ b}_{\ \lambda} \bar{g}^{\rho\nu} - \bar{F}^{\nu\ \ b}_{\ \lambda} \bar{g}^{\rho\mu})\,. \tag{4.35}$$

The vector terms satisfy

$$V^{\mu\nu\,\rho\, b}_{(ha)} = -V^{\rho\, b\, \mu\nu}_{(ah)}. \tag{4.36}$$

In the same spirit of what we did earlier, we lower the indexes of the operator $\mathcal{O}^{\mu\nu\,\rho\, b}_{(ha)}$ defining the operator $\Delta_{(ha)}$, namely

$$\Delta_{(ha)\mu\nu}{}^{\rho\sigma} = \gamma_{\mu\nu,\alpha\beta} \mathcal{O}^{\alpha\beta\rho\sigma}_{(ha)}. \tag{4.37}$$

The endomorphism takes the form

$$\begin{aligned} P_{\mu\nu}{}^{\rho\, b} = \frac{1}{\sqrt{g_1}\eta} \Bigg(&\bar{A}_\nu{}^a \bar{F}_\mu{}^{\rho c} f^{bac} - \bar{A}^{\alpha a} \delta^\rho_\nu \bar{F}_{\mu\alpha}{}^c f^{bac} - \bar{A}^{\alpha a} \delta^\rho_\mu \bar{F}_{\nu\alpha}{}^c f^{bac} \\ &+ \bar{A}_\mu{}^a \bar{F}_\nu{}^{\rho c} f^{bac} + \frac{2}{d-2} \bar{A}^{\alpha a} \bar{F}^{\rho\ \ c}_{\ \alpha} f^{bac} \bar{g}_{\mu\nu} - \frac{1}{2} \delta^\rho_\nu \bar{\nabla}_\alpha \bar{F}_\mu{}^{\alpha b} \\ &- \frac{1}{2} \delta^\rho_\mu \bar{\nabla}_\alpha \bar{F}_\nu{}^{\alpha b} + \frac{1}{d-2} \bar{g}_{\mu\nu} \bar{\nabla}_\alpha \bar{F}^{\rho\alpha b} + \frac{1}{2} \bar{\nabla}_\mu \bar{F}_\nu{}^{\rho b} + \frac{1}{2} \bar{\nabla}_\nu \bar{F}_\mu{}^{\rho b} \Bigg) \end{aligned} \tag{4.38}$$

and the vector

$$V_{(ha)\,\mu\nu}{}^{\rho\,b}{}_{\lambda} = \frac{1}{\sqrt{g_1}\eta}\left(\delta^{\rho}_{\nu}\bar{F}_{\mu\lambda}{}^{b} + \delta^{\rho}_{\mu}\bar{F}_{\nu\lambda}{}^{b} - \frac{2\bar{F}^{\rho}{}_{\lambda}{}^{b}\bar{g}_{\mu\nu}}{d-2} - \bar{F}_{\nu}{}^{\rho b}\bar{g}_{\mu\lambda} - \bar{F}_{\mu}{}^{\rho b}\bar{g}_{\nu\lambda}\right). \tag{4.39}$$

We remind that this operation doesn't solely lower indexes but transform hessians into differential operators. We verified that it is possible to make $\bar{A}$ disappear by using the gauge covariant derivative and making the equations explicitly invariant under background transformations. However, we chose to use the covariant derivative $\bar{\nabla}$ on both a^{ρ}_{μ} and $h_{\mu\nu}$. The operator $\mathcal{O}^{\rho\,b\,\mu\nu}_{ah}$ is already in the correct form since it acts on tensors $h_{\mu\nu}$ with lower indexes, so we can write simply

$$\Delta^{\rho\,b\,\mu\nu}_{(ah)} = \mathcal{O}^{\rho\,b\,\mu\nu}_{ah}. \tag{4.40}$$

Now we can define the fields Ψ and $\Psi^{\dagger}$ as

$$\Psi^{A} = \begin{bmatrix} \frac{1}{\eta}a_{\rho}{}^{a} \\ \sqrt{g_1}h_{\mu\nu} \end{bmatrix} \qquad \Psi^{\dagger A} = \begin{bmatrix} \frac{1}{\eta}a_{\rho}{}^{a} & \sqrt{g_1}h^{\mu\nu} \end{bmatrix}, \qquad A = 1,2. \tag{4.41}$$

together with the operator

$$\mathcal{O}^{AB}_{\Psi} = \begin{bmatrix} \Delta^{\kappa\lambda\,ab}_{aa} & \Delta^{\kappa\,a\,\rho\sigma}_{(ah)} \\ \Delta_{(ha)\mu\nu}{}^{\lambda\,b} & (\Delta_{hh})_{\mu\nu}{}^{\rho\sigma} \end{bmatrix} \qquad A,B = 1,2. \tag{4.42}$$

Using these definitions and neglecting for the moment the β-dependent term, the quadratic expansion of the action takes the particular simple form

$$S^{(2)} = \int d^{d}x\,\sqrt{\bar{g}}\,\frac{1}{2}\Psi^{\dagger A}\gamma^{AB}\,\mathcal{O}^{BC}_{\Psi}\,\Psi^{C}, \tag{4.43}$$

where

$$\gamma^{AB} = \begin{bmatrix} 1 & 0 \\ 0 & \gamma^{\alpha\beta,\mu\nu} \end{bmatrix}. \tag{4.44}$$

4.2 Effective action

The one loop correction effective action is given by

$$\Gamma[\bar{g},\bar{A}] = S[\bar{g},\bar{A}] + \frac{1}{2}\mathrm{Tr}\,\log\,\mathcal{O}_{\Psi} - \mathrm{Tr}\,\log\,(-\Delta_{GH,h}) - \mathrm{Tr}\,\log\,(-\Delta_{GH,a}) \tag{4.45}$$

and can be expressed in terms of the heat kernel coefficients.

4.2.1 Heat kernel coefficients

To compute the heat kernel coefficients of the operator $\mathcal{O}_\Psi$ it is convenient to write in a form that is compatible with heat kernel coefficients formulae (2.15), namely

$$\mathcal{O}_\Psi = -D^2_\Psi + E_\Psi + (V_\Psi)_\beta \bar{\nabla}^\beta, \tag{4.46}$$

where

$$-D^2_\Psi = \begin{bmatrix} -\bar{g}^{\kappa\lambda}(\bar{D}^2)^{ab} & 0 \\ 0 & -\bar{\nabla}^2 \mathbf{1}^{\rho\sigma}_{\mu\nu} \end{bmatrix}, \tag{4.47}$$

$$E_\Psi = \begin{bmatrix} E^{\kappa\lambda\, ab} & P^{\kappa\, a\, \rho\sigma} \\ P_{\mu\nu}{}^{\lambda\, b} & W_{\mu\nu}{}^{\rho\sigma} \end{bmatrix}, \tag{4.48}$$

and

$$V_{\Psi\beta} = \begin{bmatrix} 0 & V^{\kappa\, a\, \rho\sigma}_{(ah)\ \ \ \beta} \\ V_{(ha)\mu\nu}{}^{\lambda\, b}{}_\beta & 0 \end{bmatrix}. \tag{4.49}$$

At this point, it is important to notice the presence of a term which was not considered in the computation of the heat kernel coefficients (2.15), namely the vector part ∇_Ψ. To take care of this term we will follow [29]. Considering an operator of the form (4.46), i.e.

$$\Delta = -D^2 + V_\mu \nabla^\mu + E, \tag{4.50}$$

where D is a covariant derivative and E, V_μ are two matrices, the heat kernel coefficients are

$$\begin{aligned}
[a_0(\Delta)] &= \mathbf{1}, \\
[a_1(\Delta)] &= Z + \frac{1}{6} R\mathbf{1}, \\
[a_2(\Delta)] &= \frac{1}{6}\nabla^2 \left(Z + \frac{1}{5} R \right) + \left(\frac{1}{180} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - \frac{1}{180} R_{\mu\nu} R^{\mu\nu} + \frac{1}{72} R^2 \right) \mathbf{1} \\
&\quad + \frac{1}{2} Z^2 + \frac{1}{6} RZ + \frac{1}{12} Y_{\mu\nu} Y^{\mu\nu},
\end{aligned} \tag{4.51}$$

with

$$\begin{aligned}
Z &= -E - \nabla_\mu S^\mu - S_\mu S^\mu, \\
Y_{\mu\nu} &= [D_\mu, D_\nu] + G_{\mu\nu}, \\
G_{\mu\nu} &= \nabla_\mu S_\nu - \nabla_\nu S_\mu + S_\mu S_\nu - S_\nu S_\mu,
\end{aligned} \tag{4.52}$$

and

$$S_\mu = -\frac{1}{2} V_\mu. \tag{4.53}$$

Now we use formulae (4.51) for the operator $\mathcal{O}_\Psi$. The first coefficient is

$$\mathrm{tr}\,[a_0(\mathcal{O}_\Psi)] = \mathrm{tr}\,\begin{bmatrix}\delta^{ab}\bar{g}^{\kappa\lambda} & 0\\ 0 & \mathbf{1}^{\rho\sigma}_{\mu\nu}\end{bmatrix} = \int d^dx\,\sqrt{\bar{g}}\left[(N^2-1)D + \frac{D(D+1)}{2}\right]. \qquad (4.54)$$

Before computing the second coefficient, we show as intermediate result the matrix Z for the operator $\mathcal{O}_\Psi$. It reads

$$Z = \begin{bmatrix} -E^{\kappa\lambda\,ab} - \frac{1}{4}(V^2_{aa})^{\kappa\lambda\,ab} & -P^{\kappa\,a\,\rho\sigma} + \frac{1}{2}\bar{\nabla}_\beta V^{\kappa\,a\,\rho\sigma\,\beta}_{(ah)} \\ -P_{\mu\nu}{}^{\lambda\,b} + \frac{1}{2}\bar{\nabla}_\beta V_{(ha)\mu\nu}{}^{\lambda\,b\,\beta} & -W_{\mu\nu}{}^{\rho\sigma} - \frac{1}{4}(V^2_{hh})_{\mu\nu}{}^{\rho\sigma}\end{bmatrix}, \qquad (4.55)$$

where

$$(V^2_{aa})^{\kappa\lambda\,ab} = \frac{1}{g_1\eta^2}\left(-\frac{(D^2-4D+2)\bar{F}^\kappa{}_\alpha{}^a\,\bar{F}^{\lambda\alpha b}}{(D-2)} - \bar{F}^\kappa{}_\alpha{}^b\bar{F}^{\lambda\alpha a} - \bar{F}_{\alpha\beta}{}^b\bar{F}^{\alpha\beta a}\bar{g}^{\lambda\kappa}\right) \qquad (4.56)$$

and

$$\begin{aligned}(V^2_{hh})_{\mu\nu}{}^{\rho\sigma} = \frac{1}{g_1\eta^2}\Bigg(&\bar{F}_\nu{}^{\alpha a}\bar{F}^\rho{}_\alpha{}^a\bar{g}_\mu{}^\sigma - \bar{F}_\mu{}^{\alpha a}\bar{F}^\sigma{}_\alpha{}^a\bar{g}_\nu{}^\rho - \bar{F}_\mu{}^{\alpha a}\bar{F}^\rho{}_\alpha{}^a\bar{g}_\nu{}^\sigma\\ &+ 2\bar{F}_\mu{}^{\alpha a}\bar{F}_{\nu\alpha}^{\;\;a}\bar{g}^{\rho\sigma} - \frac{\bar{F}_{\alpha\beta}{}^a\bar{F}^{\alpha\beta a}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}}{(D-2)} - 2\bar{F}_\mu{}^{\sigma a}\bar{F}_\nu{}^{\rho a}\\ &- 2\bar{F}_\mu{}^{\rho a}\bar{F}_\nu{}^{\sigma a} + \frac{4\bar{F}^{\rho\alpha a}\bar{F}^\sigma{}_\alpha{}^a\bar{g}_{\mu\nu}}{(D-2)} - \bar{F}_\nu{}^{\alpha a}\bar{F}^\sigma{}_\alpha{}^a\bar{g}_\mu{}^\rho\Bigg).\end{aligned} \qquad (4.57)$$

Now we can trace Z and put it in the second of (4.51). We get

$$\begin{aligned}\mathrm{tr}\,[a_1(\mathcal{O}_\Psi)] = \int d^dx\,\sqrt{\bar{g}}\Bigg[&\frac{D\,g_0(D^2(\lambda-1)+D(1-\lambda)+2-4\lambda)}{g_1(D-2)}\\ &+\frac{(D^3(\lambda-1)+D^2(13-5\lambda)-20D+16(\lambda-1))\bar{F}_{\alpha\beta}{}^a\bar{F}^{\alpha\beta a}}{4g_1\eta^2(D-2)}\\ &+\left(D^2\left(\frac{13}{12}-\lambda\right)+D\left(\frac{N^2}{6}-\frac{13}{12}+\lambda\right)+1-N^2-4\lambda\right)\bar{R}\Bigg].\end{aligned} \qquad (4.58)$$

Again, before computing the coefficient $a_2(\mathcal{O}_\Psi)$, we give as intermediate result Z^2 and $Y_{\mu\nu}Y^{\mu\nu}$. Since we are interested in tracing these matrices we can just focus on their diagonals elements. Neglecting explicit indexes, we can write the diagonal part of Z^2 as

$$Z^2_{\mathrm{diag}} = \begin{bmatrix}(E+\frac{1}{4}V^2_{aa})^2 + (P-\frac{1}{2}\bar{\nabla}_\beta V^\beta_{ah})(P-\frac{1}{2}\bar{\nabla}_\beta V^\beta_{ha}) & 0\\ 0 & (P-\frac{1}{2}\bar{\nabla}_\beta V^\beta_{ha})(P-\frac{1}{2}\bar{\nabla}_\beta V^\beta_{ah}) + (W+\frac{1}{4}V^2_{hh})^2\end{bmatrix}, \qquad (4.59)$$

and the diagonal part of $Y_{\mu\nu}Y^{\mu\nu}$ is

$$(Y_{\mu\nu}Y^{\mu\nu})_{\text{diag}} = \begin{bmatrix} (Y_{\mu\nu}Y^{\mu\nu})^{11} & 0 \\ 0 & (Y_{\mu\nu}Y^{\mu\nu})^{22} \end{bmatrix} , \tag{4.60}$$

where

$$\begin{aligned}(Y_{\mu\nu}Y^{\mu\nu})^{11} = &\left([\bar{D}_\mu, \bar{D}_\nu] + \frac{1}{4}(V_{(ah)\mu}V_{(ha)\nu} - V_{(ah)\nu}V_{(ha)\mu})\right)^2 \\ &+ \frac{1}{4}(\bar{\nabla}_\mu V_{(ah)\nu} - \bar{\nabla}_\nu V_{(ah)\mu})(\bar{\nabla}^\mu V_{(ha)}{}^\nu - \bar{\nabla}^\nu V_{(ha)}{}^\mu)\end{aligned} \tag{4.61}$$

and

$$\begin{aligned}(Y_{\mu\nu}Y^{\mu\nu})^{22} = &\left([\bar{\nabla}_\mu, \bar{\nabla}_\nu] + \frac{1}{4}(V_{(ha)\mu}V_{(aa)\nu} - V_{(ha)\nu}V_{(ah)\mu})\right)^2 \\ &+ \frac{1}{4}(\bar{\nabla}_\mu V_{(ha)\nu} - \bar{\nabla}_\nu V_{(ha)\mu})(\bar{\nabla}^\mu V_{(ah)}{}^\nu - \bar{\nabla}^\nu V_{(ah)}{}^\mu) .\end{aligned} \tag{4.62}$$

Tracing these matrices and putting the results into (4.51) we find

$$
\begin{aligned}
\mathrm{tr}\,[a_2(\mathcal{O}_\Psi)] = \int d^dx\,\sqrt{\bar{g}}\Bigg[& c_1(D,\lambda)\,\frac{g_0^2}{g_1^2} + c_2(D,\lambda)\,\frac{g_0\bar{F}_{\mu\nu}{}^a\bar{F}^{\mu\nu a}}{g_1^2\eta^2} + c_3(D,\lambda)\frac{\bar{F}_{\mu\nu}{}^a\bar{F}^{\nu}_{\ \alpha}{}^b\bar{F}^{\alpha}_{\ \beta}{}^a\bar{F}^{\beta\mu b}}{g_1^2\eta^4} \\
& + c_4(D,\lambda)\frac{\bar{F}_{\mu\nu}{}^a\bar{F}^{\nu}_{\ \alpha}{}^b\bar{F}^{\alpha}_{\ \beta}{}^b\bar{F}^{\beta\mu a}}{g_1^2\eta^4} + c_5(D,\lambda)\frac{\bar{F}_{\mu\nu}{}^a\bar{F}^{\mu\nu b}\bar{F}_{\alpha\beta}{}^a\bar{F}^{\alpha\beta b}}{g_1^2\eta^4} \\
& + c_6(D,\lambda)\frac{\bar{F}_{\mu\nu}{}^a\bar{F}^{\mu\nu a}\bar{F}_{\alpha\beta}{}^b\bar{F}^{\alpha\beta b}}{g_1^2\eta^4} + c_7(D,\lambda)\frac{\bar{F}_{\mu\nu}{}^a\bar{F}^{\nu}_{\ \alpha}{}^b\bar{F}^{\alpha\mu c}f^{abc}}{g_1\eta^2} \\
& + c_8(D,\lambda)C_2\bar{F}_{\mu\nu}{}^a\bar{F}^{\mu\nu a} + c_9(D,\lambda)\frac{\bar{A}^{\mu a}\bar{A}^{\nu b}\bar{F}_{\mu}{}^{\alpha c}\bar{F}_{\nu\alpha}{}^d f^{ade}f^{bce}}{g_1\eta^2} \\
& + c_{10}(D,\lambda)\frac{\bar{A}^{\mu a}\bar{A}^{\nu b}\bar{F}_{\mu}{}^{\alpha c}\bar{F}_{\nu\alpha}{}^d f^{ace}f^{bde}}{g_1\eta^2} + c_{11}(D,\lambda)\frac{\bar{A}^a_\alpha\bar{A}^{\alpha b}\bar{F}_{\mu\nu}{}^c\bar{F}^{\mu\nu d}f^{ace}f^{bde}}{g_1\eta^2} \\
& + c_{12}(D,\lambda)\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + c_{13}(D,\lambda)\frac{\bar{F}_{\alpha}{}^{\mu a}\bar{F}^{\alpha\nu a}\bar{R}_{\mu\nu}}{g_1\eta^2} + c_{14}(D,\lambda)\frac{g_0\bar{R}}{g_1} \\
& + c_{15}(D,\lambda)\frac{\bar{F}_{\mu\nu}{}^a\bar{F}^{\mu\nu a}\bar{R}}{g_1\eta^2} + c_{16}(D,\lambda)\bar{R}^2 + c_{17}(D,\lambda)\frac{\bar{F}^{\mu\nu a}\bar{F}^{\rho\sigma a}\bar{R}_{\mu\nu\rho\sigma}}{g_1\eta^2} \\
& + c_{18}(D,\lambda)\frac{\bar{F}^{\mu\nu a}\bar{F}^{\rho\sigma a}\bar{R}_{\mu\rho\nu\sigma}}{g_1\eta^2} + c_{19}(D,\lambda)\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma} \\
& + c_{20}(D,\lambda)\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\rho\nu\sigma} + c_{21}(D,\lambda)\frac{\bar{A}^{\alpha a}\bar{F}^{\mu\nu v}f^{abc}\bar{\nabla}_\alpha\bar{F}_{\mu\nu}{}^c}{g_1\eta^2} \\
& + c_{22}(D,\lambda)\frac{\bar{\nabla}_\alpha\bar{F}_{\mu\nu}{}^a\bar{\nabla}^\alpha\bar{F}^{\mu\nu a}}{g_1\eta^2} + c_{23}(D,\lambda)\frac{\bar{A}^{\alpha a}\bar{F}^{\mu\nu v}f^{abc}\bar{\nabla}_\nu\bar{F}_{\alpha\mu}{}^c}{g_1\eta^2} \\
& + c_{24}(D,\lambda)\frac{\bar{\nabla}_\alpha\bar{F}^{\alpha\mu a}\bar{\nabla}_\beta\bar{F}_{\mu}{}^{\beta a}}{g_1\eta^2} + c_{25}(D,\lambda)\frac{\bar{A}^a_\alpha\bar{F}^{\alpha\mu b}f^{abc}\bar{\nabla}_\beta\bar{F}_{\mu}{}^{\beta c}}{g_1\eta^2} \\
& + c_{26}(D,\lambda)\frac{\bar{\nabla}_\alpha\bar{F}_{\mu\nu}{}^a\bar{\nabla}^\nu\bar{F}^{\mu\alpha a}}{g_1\eta^2} + c_{27}(D,\lambda)\frac{\bar{\nabla}_\alpha\bar{F}_{\mu\nu}{}^a\bar{\nabla}^\alpha\bar{F}^{\mu\nu a}}{g_1\eta^2}\Bigg],
\end{aligned}
\tag{4.63}
$$

where we neglected total derivatives. The explicit coefficients c_j can be found in Appendix B. The heat kernel coefficients for the operator $-\Delta_{GH,a}$ were already computed in the previous chapter and are given by (3.68),(3.71) and (3.75). Finally, the heat kernel coefficients of the operator $-\Delta_{GH,h}$ are

$$\mathrm{tr}\,[a_0(-\mathcal{O}_{GH,h})] = \int d^dx\,\sqrt{\bar{g}}D\,, \tag{4.64}$$

$$\mathrm{tr}\,[a_1(-\mathcal{O}_{GH,h})] = \int d^dx\,\sqrt{\bar{g}}\frac{\bar{R}}{6}(D+6)\,, \tag{4.65}$$

and

$$\text{tr}\left[a_2(-\mathcal{O}_{GH,h})\right] = \int d^dx\,\sqrt{\bar{g}}\left[\left(\frac{D+15}{180}\right)\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma}+\left(\frac{90-D}{180}\right)\bar{R}_{\mu\nu}\bar{R}^{\mu\nu}+\left(\frac{D+12}{72}\right)\bar{R}^2\right]. \tag{4.66}$$

4.3 Beta function in $d = 2 + \epsilon$

We can now write explicitly the quantum corrections in (4.45). Following the same computation present in Chapter 3 we get

$$-\sum_k \frac{1}{2}H_k\frac{m^{d-2k}}{(4\pi)^{d/2}}\Gamma\left(k-\frac{d}{2}\right), \tag{4.67}$$

where we defined

$$H_k \equiv \text{tr}\left[a_k(\mathcal{O}_\Psi)\right] - 2\ \text{tr}\left[a_k(-\Delta_{GH,h})\right] - 2\ \text{tr}\left[a_k(-\Delta_{GH,a})\right]. \tag{4.68}$$

We can now set $d = 2 - \epsilon$ and remember from (2.24) that in $d = 2$ the logarithmic divergences arise for $k = 1$. Recalling the expansions (3.38),(3.39),(3.40), and using $\overline{\text{MS}}$ scheme we get

$$\Gamma[\bar{A},\bar{g}] = S[\bar{A},\bar{g}] - \frac{1}{4\pi}\log\frac{\mu}{m}\,H_1\,, \tag{4.69}$$

with

$$\begin{aligned} H_1 = \int d^dx\Bigg\{ & \frac{D(D^2(\lambda-1)+D(1-\lambda)+2-4\lambda)}{D-2}\frac{g_0}{g_1} \\ & \frac{D^3(\lambda-1)+D^2(13-5\lambda)-20D+16(\lambda-1)}{4(D-2)}\frac{\bar{F}_{\mu\nu}{}^a\bar{F}^{\mu\nu a}}{g_1\eta^2} \\ & \frac{D^2(13-12\lambda)-8(2N^2-1-6\lambda)+D(2N^2-17+12\lambda)}{12}\bar{R}\Bigg\}\,. \end{aligned} \tag{4.70}$$

Physical results are obtained by going on-shell. This means that the metric must be expanded around a stationary point of the action, in other words the background metric $\bar{g}_{\mu\nu}$ and the background vector field $\bar{A}_\mu$ must satisfy the equations of motion (4.5) and (4.7) respectively. At this point, we choose to use the equation of motion (4.5) in two different ways, the first time we express the volume operator in terms of the field strength F and the Ricci scalar R, effectively trading g_0 for η and g_1. The second time we solve the equation of motion for F, ending up with counterterms that only depend on the geometrical content of the theory.. Performing the first substitution in (4.70) we get

$$\begin{aligned} H_1^{(1)} &= \int d^d x \left\{ A_1 \, \frac{\bar{F}_{\mu\nu}{}^a \bar{F}^{\mu\nu a}}{g_1 \eta^2} + B_1 \, \bar{R} \right\} \\ &:= \int d^d x \left\{ \frac{4 + 9D - 4D^2}{2(D-2)} \frac{\bar{F}_{\mu\nu}{}^a \bar{F}^{\mu\nu a}}{g_1 \eta^2} + \frac{D^2 + D(2N^2 - 5) + 16(1 - N^2)}{12} \bar{R} \right\} . \end{aligned} \tag{4.71}$$

Notice how the dependence on the parametrization parameter vanished, accordingly to the fact the physical results are independent from the parametrization. Recalling that g_1 is the inverse of the Newton constant G_N, we can define the following renormalized couplings

$$\frac{1}{(G_N)_R} = \frac{1}{G_N} + \frac{1}{4\pi} B_1 \log \frac{\mu}{m} \, , \tag{4.72}$$

and

$$\frac{1}{4\eta_R^2} = \frac{1}{4\eta^2} - \frac{1}{4\pi} \frac{G_N A_1}{\eta^2} \log \frac{\mu}{m} . \tag{4.73}$$

We can rewrite

$$(G_N)_R = \frac{G_N}{1 + \frac{G_N}{4\pi} B_1 \, \log \frac{\mu}{m}} \, , \tag{4.74}$$

and

$$\eta_R = \frac{\eta}{\sqrt{1 - \frac{G_n}{\pi} A_1 \, \log \frac{\mu}{m}}} \, . \tag{4.75}$$

The beta functions follow straightforwardly

$$\beta((G_N)_R) = \frac{B_1}{4\pi} (G_N)_R^2 \, , \tag{4.76}$$

and

$$\beta(\eta_R) = -\frac{A_1}{2\pi} (G_N)_R \, \eta_R \, . \tag{4.77}$$

We can re-instate the canonical mass [25, 26], as explained in Section 2.3, and find the beta functions of the dimensionless couplings constants $(\bar{G}_N)_R$ and $\bar{\eta}_R$, we get

$$\beta((\bar{G}_N)_R) = \frac{B_1}{4\pi} (\bar{G}_N)_R^2 + \epsilon \bar{G}_N \, , \tag{4.78}$$

and

$$\beta(\bar{\eta}_R) = -\frac{A_1}{2\pi} (\bar{G}_N)_R \, \bar{\eta}_R - \left(1 - \frac{\epsilon}{2}\right) \bar{\eta}_R \, . \tag{4.79}$$

It is important to realize that having regulated the theory close two dimensions we can now identify d and D. In this case we can write $\epsilon = D - 2$, and the beta functions become

$$\beta((\bar{G}_N)_R) = \frac{B_1}{4\pi} (\bar{G}_N)_R^2 + (D-2) \bar{G}_N \, , \tag{4.80}$$

and

$$\beta(\bar{\eta}_R) = -\frac{A_1}{2\pi}(\bar{G}_N)_R\,\bar{\eta}_R - \left(\frac{4-D}{2}\right)\bar{\eta}_R\,. \tag{4.81}$$

To find the fixed point we need to set $\beta((\bar{G}_N)_R) = 0 = \beta(\bar{\eta}_R)$, we find the coordinates G^* and η^* for two fixed points, a Gaussian one and a UV one. The Gaussian point is

$$\mathbf{FP}_{\text{Gauss}} = (0,0)\,, \tag{4.82}$$

the UV one is

$$\mathbf{FP}_{\text{UV}} = (G^*,0)\,, \tag{4.83}$$

with

$$G^* = -\frac{48\pi(D-2)}{D^2 + D(2N^2-5) + 16(1-N^2)}\,. \tag{4.84}$$

G^* has a complicated dependence on D and N, in Fig. 4.1 we plot its graph. We can observe that fixing $D=4$, G^* exists and has a positive value for $N > \sqrt{3}/2 \approx 1.224$. The Gaussian fixed point is instead indeterminate, all possible values of η lead to vanishing beta functions taken $G_N = 0$. This is a symptom of the fact that Yang-Mills is asymptotically free in $D=4$.We notice that exactly in $D=2$ the beta function of η is ill-defined because A_1 has a pole.
To find the critical exponents we need to linearize the flow around $\mathbf{FP}_{\text{UV}}$, namely

$$\begin{bmatrix} \partial_{G_N}\beta(G_N) & \partial_\eta\beta(G_N) \\ \partial_{G_N}\beta(\eta) & \partial_\eta\beta(\eta) \end{bmatrix}\Bigg|_{(G^*,0)} = \begin{bmatrix} \frac{B_1}{2\pi}G^* + (D-2) & 0 \\ 0 & -\frac{A_1}{2\pi}G^* - \frac{4-D}{2} \end{bmatrix}\,, \tag{4.85}$$

where we dropped the bar notation and the subscript R to refer to dimensionless renormalized couplings. The matrix is in diagonal form and the critical exponents are given by its eigenvalues

$$\begin{aligned} \lambda_1 &= 2 - D\,, \\ \lambda_2 &= -\frac{D^3 + (2N^2+87)\,D^2 + -12D(2N^2+15) + 32\,(2N^2-5)}{2\,(D^2 + D(2N^2-5) + 16(1-N^2))}\,. \end{aligned} \tag{4.86}$$

For $D > 2$ the first eigenvalue λ_1 is always negative and relevant. The second eigenvalue related to η has a complicated dependence on N and D. In $D=2$ it is negative for $N > \sqrt{41/6} \approx 2.614$. In $D=4$ it is negative for $N > \sqrt{3}/2 \approx 1.221$. Finally, in the case of Quantum Chromodynamics (QCD) $N=3$ and the critical exponent λ_2 is negative and relevant in the whole range $2 \leqslant D \leqslant 4$. Alternatively, we can substitute the equation of

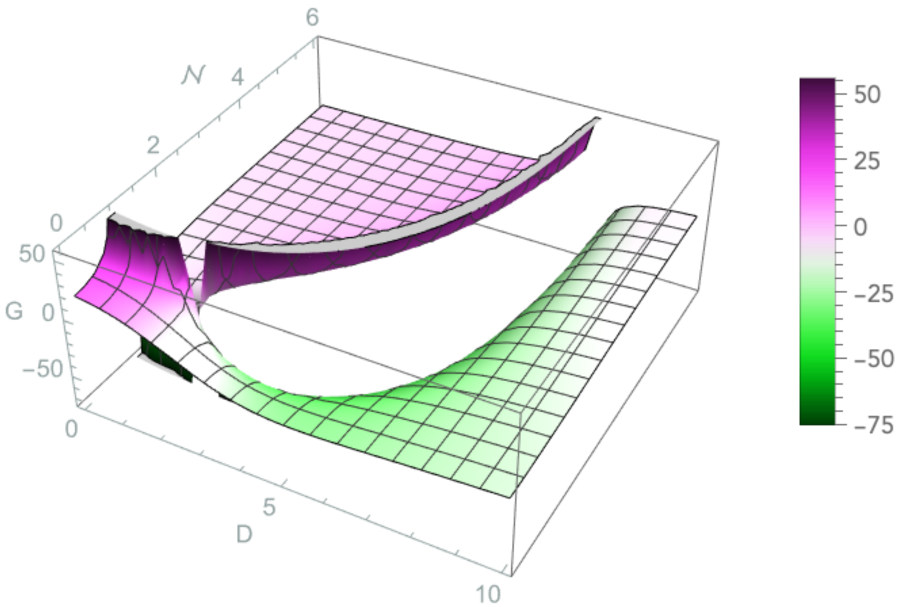

Figure 4.1: UV fixed point G^*. G^* depends on D and N. In this picture we consider the case of the first on-shell substitution. We observe regions in which G^* exists "safely" and is positive (pink) and regions in which G^* is not defined or is negative (green).

motion (4.5) in (4.70), writing F in terms of g_0/g_1 and R in (4.70), we get

$$\begin{aligned} H_1^{(2)} &= \int d^d x \left\{ A_2 \frac{g_0}{g_1} + B_2 \bar{R} \right\} \\ &:= \int d^d x \left\{ \frac{2D(4+9D-4D^2)}{(D-4)(D-2)} \frac{g_0}{g_1} \right. \\ &\qquad \left. + \frac{(D^3 + D^2(2N^2+87) - 12D(2N^2+15) + 32(2N^2-5))}{12(D-4)} \bar{R} \right\} . \end{aligned} \tag{4.87}$$

As above, we can define the following renormalized coupling

$$g_{0_R} = g_0 - \frac{A_2}{4\pi} G_N g_0 \log \frac{\mu}{m} , \tag{4.88}$$

and

$$\frac{1}{(G_N)_R} = \frac{1}{G_N} + \frac{B_2}{4\pi} \log \frac{\mu}{m} . \tag{4.89}$$

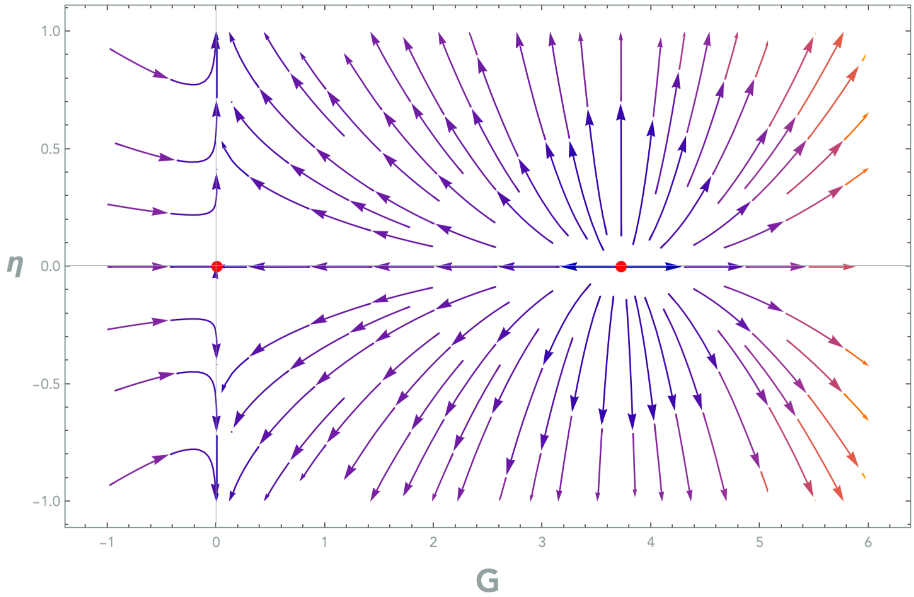

Figure 4.2: One-loop phase diagram of Gravity + Yang-Mills. G is the Newton coupling and η is the Yang-Mills coupling, they are coordinates in the theory space. Red dots indicate the non trivial fixed point $\mathbf{FP}_{\rm UV}$ and the Gaussian fixed point $\mathbf{FP}_{\rm Gauss}$. We set $D \to 3.656$ and $N \to 3$.

The corresponding beta functions are respectively

$$\beta(g_{0_R}) = \frac{A_2}{4\pi} g_{0_R} (G_N)_R \,, \tag{4.90}$$

and

$$\beta((G_N)_R) = \frac{B_2}{4\pi} (G_N)_R^2 \,. \tag{4.91}$$

Considering the dimensionless couplings in $d = 2 + \epsilon$ we get

$$\beta(\bar{g}_0) = \frac{A_2}{4\pi} \bar{g}_{0_R} (\bar{G}_N)_R - (2+\epsilon)\bar{g}_0 \,, \tag{4.92}$$

and

$$\beta((\bar{G}_N)_R) = \frac{B_2}{4\pi} (\bar{G}_N)_R^2 + \epsilon \bar{G}_N \,. \tag{4.93}$$

As in the previous calculation we consider in this case the possibility that $\epsilon = D - 2$ has a finite value, we can thus write

$$\beta(\bar{g}_0) = \frac{A_2}{4\pi} \bar{g}_{0_R} (\bar{G}_N)_R - D\bar{g}_0 \,, \tag{4.94}$$

and

$$\beta((\bar{G}_N)_R) = \frac{B_2}{4\pi}(\bar{G}_N)_R^2 + (D-2)\bar{G}_N \,. \tag{4.95}$$

Setting $\beta(\bar{g}_{0_R}) = 0 = \beta((\bar{G}_N)_R)$, we find the fixed points

$$\mathbf{FP}_{\text{Gauss}} = (0,0)\,, \tag{4.96}$$

and

$$\mathbf{FP}_{\text{UV}} = (G^*, 0)\,, \tag{4.97}$$

where now G^* is

$$G^* = -\frac{48\pi(D^2 - 6D + 8)}{D^3 + D^2(87 + 2N^2) - D(180 + 24N^2) - 160 + 64N^2}\,. \tag{4.98}$$

In Fig. 4.3 we plot its graph. Again we have a complicated dependence on D and N. This time $G^* = 0$ for both $D = 2$ and $D = 4$. A promising observation is that G^* exists and is positive for $N = 3$ in the range $2 < D < 4$. However, the discontinuity in $D = 2$ seems to suggest that our perturbative approach is not enough to trust the analytical continuation in this scheme. For $D > 4$ we find only negative values of G^* differently to the previous case. In Fig. 4.4 we plot the flow of the beta functions near $D = 4$. We notice that the two fixed points are very close, this is a hint of the fact that for $D = 4$ they will merge. We can now linearize the flow around $\mathbf{FP}_{\text{UV}}$, we get

$$\begin{bmatrix} \partial_{G_N}\beta(G_N) & \partial_{g_0}\beta(G_N) \\ \partial_{G_N}\beta(g_0) & \partial_{g_0}\beta(g_0) \end{bmatrix}\Bigg|_{(G^*,0)} = \begin{bmatrix} \frac{B_2}{2\pi}G^* + (D-2) & 0 \\ 0 & \frac{A_2}{2\pi}G^* - D \end{bmatrix}\,. \tag{4.99}$$

The critical exponents are

$$\lambda_1 = 2 - D \tag{4.100}$$

and

$$\lambda_2 = -\frac{D(D^3 + D^2(2N^2 - 105) + D(252 - 24N^2) + 32 + 64N^2)}{D^3 + D^2(87 + 2N^2) - 12D(15 + 2N^2) + 32(2N^2 - 5)}\,. \tag{4.101}$$

The first eigenvalue is the same as before. For $N = 3$ the second eigenvalue in negative around $D = 2$ and positive around $D = 4$. The coupling g_0 is relevant when $\epsilon \to 0$ and becomes irrelevant as $\epsilon \to 2$.

4.4 Beta function in $d = 4 - \epsilon$

To compute the beta function in $d = 4 - \epsilon$ we have to consider $k = 4$ in (4.67) and perform the expansion in ϵ, remembering the $\overline{MS}$ scheme we get

$$\Gamma[\bar{A}, \bar{g}] = S[\bar{A}, \bar{g}] - \frac{1}{(4\pi)^2}H_2 \log\frac{\mu}{m}\,, \tag{4.102}$$

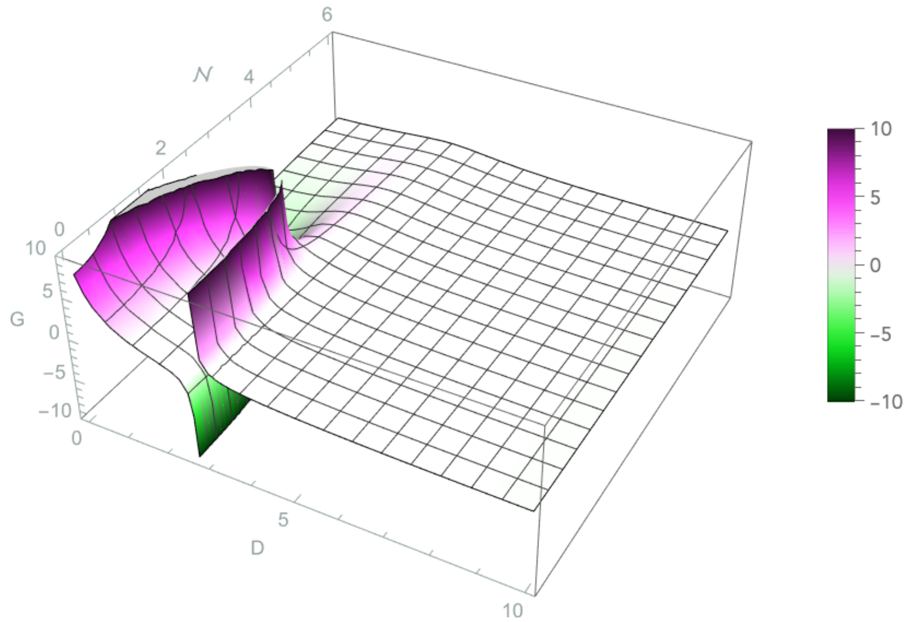

Figure 4.3: UV fixed point G^*. G^* depends on D and N. In this picture we consider the case of the second on-shell substitution. We observe regions in which G^* exists "safely" and is positive (pink) and regions in which G^* is not defined or is negative (green).

where H_2 can be explicitly calculated substituting (4.63), (3.75) and (4.66) into (4.68). Einstein-Yang-Mills theory is clearly non-renormalizable in $d = 4$, however, we can consider an effective field theory approach in which terms quadratic in curvatures $R, R_{\mu\nu}, R_{\mu\nu\rho\sigma}$, terms in powers of the tensor F greater than 2 and terms proportional to $(D_\mu F^{\alpha\beta\, a})^2$, are suppressed by a energy parameter. In $D = 4$ this should not be possible since the couplings of these quadratic curvatures are massless, however, we work in a general D and perform the truncation before considering the limit $D \to 4$. In this way, we could retain terms that can be reabsorbed in renormalized couplings. Differently with respect to the previous section, to go on-shell, it convenient to only consider the case in which one we solve the equation of motion for F. This is because proceeding with the first substitution, which keeps terms proportional to F and R, we obtain again terms that

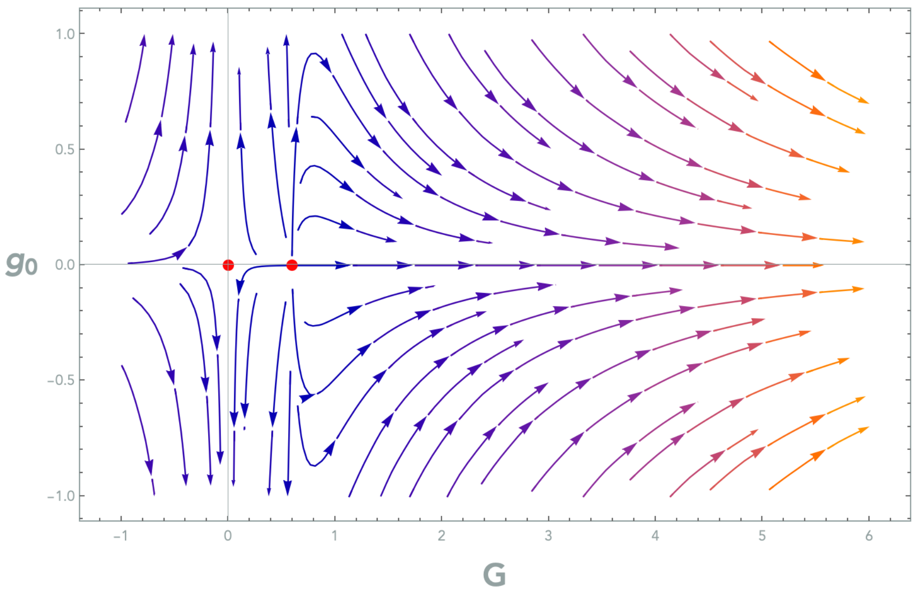

Figure 4.4: One-loop phase diagram of Gravity + Yang-Mills. G is the Newton coupling and η is the Yang-Mills coupling, they are coordinates in the theory space. Red dots indicate the non trivial fixed point $\mathbf{FP}_{\mathrm{UV}}$ and the Gaussian fixed point $\mathbf{FP}_{\mathrm{Gauss}}$. We set $D \to 3.17$ and $N \to 3$.

cannot be absorbed. Computing on-shell H_2 we get

$$\begin{aligned} H_2 = \int d^d x \Bigg\{ & \frac{D(416-180D+14D^2-2D^3+D^4)}{3(D-4)^2(D-2)^2}\frac{g_0^2}{g_1^2} \\ & + \frac{C_2 D(96-28D+D^2)}{3(D-4)^2} g_0 \eta^2 \\ & + \Bigg[\frac{-656+408D-107D^2+34D^3-6D^4}{3(D-4)^2(D-2)}\frac{g_0}{g_1} \\ & - \frac{C_2(96-28D+D^2)}{3(D-4)^2} g_1 \eta^2\Bigg]\bar{R}\Bigg\} . \end{aligned} \tag{4.103}$$

We can define the following renormalized couplings

$$\begin{aligned} g_{0_R} = & g_0 - \frac{1}{(4\pi)^2}\Bigg[\frac{D(416-180D+14D^2-2D^3+D^4)}{3(D-4)^2(D-2)^2} g_0^2 (G_N)^2 \\ & + \frac{C_2 D(96-28D+D^2)}{3(D-4)^2} g_0 \eta^2\Bigg] \log\frac{\mu}{m} , \end{aligned} \tag{4.104}$$

and

$$\frac{1}{(G_N)_R} = \frac{1}{G_N} + \frac{1}{(4\pi)^2}\Bigg[- \frac{C_2(96 - 28D + D^2)}{3(D-4)^2}\frac{\eta^2}{G_N} + \frac{-656 + 408D - 107D^2 + 34D^3 - 6D^4}{3(D-4)^2(D-2)} g_0 G_N \Bigg] \log\frac{\mu}{m} . \tag{4.105}$$

The corresponding beta functions are as follows

$$\begin{aligned}\beta(g_{0_R}) =& \frac{1}{(4\pi)^2}\frac{D(416 - 180D + 14D^2 - 2D^3 + D^4)}{3(D-4)^2(D-2)^2} g_{0_R}^2 (G_N)_R^2 \\ &+ \frac{1}{(4\pi)^2}\frac{C_2 D(96 - 28D + D^2)}{3(D-4)^2} g_{0_R}\eta_R^2 ,\end{aligned} \tag{4.106}$$

and

$$\begin{aligned}\beta((G_N)_R) =& -\frac{1}{(4\pi)^2}\frac{C_2(96 - 28D + D^2)}{3(D-4)^2}\eta_R^2 (G_N)_R \\ &+ \frac{1}{(4\pi)^2}\frac{-656 + 408D - 107D^2 + 34D^3 - 6D^4}{3(D-4)^2(D-2)} g_{0_R}(G_N)_R^3 .\end{aligned} \tag{4.107}$$

At this point we want to analytically continue our results to $D = 4 - \epsilon$, as before we can obtain the following beta functions for the massless couplings

$$\begin{aligned}\beta(\bar{g}_{0_R}) =& \frac{1}{(4\pi)^2}\frac{D(416 - 180D + 14D^2 - 2D^3 + D^4)}{3(D-4)^2(D-2)^2} \bar{g}_{0_R}^2 (\bar{G}_N)_R^2 \\ &+ \frac{1}{(4\pi)^2}\frac{C_2 D(96 - 28D + D^2)}{3(D-4)^2} \bar{g}_{0_R}\bar{\eta}_R^2 \\ &- (4-\epsilon)\bar{g}_{0_R} ,\end{aligned} \tag{4.108}$$

and

$$\begin{aligned}\beta((\bar{G}_N)_R) =& -\frac{1}{(4\pi)^2}\frac{C_2(96 - 28D + D^2)}{3(D-4)^2}\bar{\eta}_R^2 (\bar{G}_N)_R \\ &+ \frac{1}{(4\pi)^2}\frac{-656 + 408D - 107D^2 + 34D^3 - 6D^4}{3(D-4)^2(D-2)} \bar{g}_{0_R}(\bar{G}_N)_R^3 \\ &+ (2-\epsilon)(\bar{G}_N)_R .\end{aligned} \tag{4.109}$$

Considering the possibility of continuing to finite values of $\epsilon = 4 - D$, we can write

$$\begin{aligned}\beta(\bar{g}_{0_R}) =& \frac{1}{(4\pi)^2}\frac{D(416 - 180D + 14D^2 - 2D^3 + D^4)}{3(D-4)^2(D-2)^2} \bar{g}_{0_R}^2 (\bar{G}_N)_R^2 \\ &+ \frac{1}{(4\pi)^2}\frac{C_2 D(96 - 28D + D^2)}{3(D-4)^2} \bar{g}_{0_R}\bar{\eta}_R^2 \\ &- D\,\bar{g}_{0_R} ,\end{aligned} \tag{4.110}$$

and

$$
\begin{aligned}
\beta((\bar{G}_N)_R) = &- \frac{1}{(4\pi)^2} \frac{C_2(96 - 28D + D^2)}{3(D-4)^2} \bar{\eta}_R^2 (\bar{G}_N)_R \\
&+ \frac{1}{(4\pi)^2} \frac{-656 + 408D - 107D^2 + 34D^3 - 6D^4}{3(D-4)^2(D-2)} \bar{g}_{0_R} (\bar{G}_N)_R^3 \\
&+ (D-2)(\bar{G}_N)_R \,.
\end{aligned}
\tag{4.111}
$$

Unfortunately, we cannot find a Non-Gaussian fixed point for this set of beta functions. However, this was expected since Einstein-Yang-Mills in non-renormalizable in $D = 4$.

Conclusions

In this thesis, we investigated the Asymptotic Safety scenario for a theory of gravity coupled with Yang-Mills fields. Our analysis was conducted within the perturbative framework of a dimensional expansion around two dimensions, in $d = 2 + \epsilon$. This approach, recently reconsidered in [25, 26, 27], allows for a controlled analytical continuation of the theory's RG flow, overcoming some of the ambiguities that arise when applying heat kernel methods directly to the quantization of the metric field.

Our investigation began with preliminary analyses to test the methodology. We first reviewed the well-understood case of Yang-Mills theory in flat spacetime before extending the calculation to a curved spacetime background. The emergence of divergences proportional to geometric invariants, such as the Ricci scalar, in the curved space analysis confirmed the necessity of including dynamical degrees of freedom for gravity itself.

The core of our work focused on the full Einstein-Yang-Mills system. We computed the one-loop effective action and derived the beta functions for the theory's essential couplings. To analyze the physical properties of the RG flow, we evaluated the results on-shell by applying the equations of motion. This was performed using two distinct substitution schemes. In the first scheme, the cosmological constant was expressed in terms of the Yang-Mills field strength and the Ricci scalar. In the second, the field strength was written in terms of the cosmological constant and the Ricci scalar. In both cases, we identified a non-Gaussian fixed point (NGFP) that could serve as a UV completion for the theory. However, we observed a notable discrepancy between the two on-shell schemes when examining the limit $D \to 4$. While the first scheme maintained a distinct NGFP, the second scheme showed that the NGFP merges with the Gaussian fixed point in this limit, suggesting a return to a trivial UV behavior. This highlights a potential scheme dependence in the on-shell analysis within this framework.

Finally, we attempted to apply the same perturbative logic to the theory in $D = 4-\epsilon$ dimensions. As expected for a perturbatively non-renormalizable theory, we were unable to locate a non-Gaussian fixed point, confirming that, at this level of approximation, the theory does not exhibit asymptotic safety in four dimensions.

For future work, an interesting path would be to extend the analysis in $D = 4 - \epsilon$. As suggested by recent studies, quadratic divergences, typically disregarded in a minimal subtraction scheme, may play a crucial role and influence the beta functions

[24]. Adopting a suitable non-minimal renormalization scheme to properly account for these effects could potentially alter the UV properties of the theory and offer new insights into the asymptotic safety of gravity and matter.

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Thanks to ToBo, the people I love.

Appendices

Appendix A

Effective action and loop expansion

A.1 Effective action

The standard functionals in euclidean QFT are defined by

$$\begin{aligned} Z[J] = e^{\frac{1}{\hbar}W[J]} &= \int D\phi e^{-\frac{1}{\hbar}S[\phi]+\frac{1}{\hbar}J_i\phi^i} \\ \Gamma[\varphi] = J_i\varphi^i - W[J] & \quad \text{where} \quad \varphi^i = \frac{\delta W[J]}{\delta J_i}. \end{aligned} \tag{A.1}$$

$Z[J]$ is the standard generating functional of diagrams, $W[J]$ is the generating functional of connected diagrams and $\Gamma[\varphi]$ is the effective action. Note that we used hyper-condensed notation $J_i\phi^i = \int d^4x J(x)\phi(x)$. We keep $\hbar$ which we use as loop counting parameter, in fact, at loop oreder L one gets a factor $\hbar^{L-1}$. One may now invert the Legendre transform defining $\Gamma[\varphi]$ by

$$W[J] = J_i\varphi^i - \Gamma[\varphi], \qquad J_i = \frac{\delta\Gamma[\varphi]}{\delta J_i}. \tag{A.2}$$

Using these relations on finds an equation for the effective action

$$e^{-\frac{1}{\hbar}\left(\Gamma[\varphi]-\frac{\delta\Gamma[\varphi]}{\delta\varphi^i}\right)} = \int D\phi \;\; e^{-\frac{1}{\hbar}\left(S[\phi]-\frac{\delta\Gamma[\varphi]}{\delta\varphi^i}\phi^i\right)} \tag{A.3}$$

and after performing the shift $\phi \to \varphi + \phi$ in the path integral one has

$$e^{-\frac{1}{\hbar}\Gamma[\varphi]} = \int D\phi \;\; e^{-\frac{1}{\hbar}S[\varphi+\phi]+\frac{1}{\hbar}\frac{\delta\Gamma[\varphi]}{\delta\varphi^i}\phi^i}. \tag{A.4}$$

We can use this equation to study the expansion in loop, namely, the $\hbar$ expansion. It is convenient to use a compact notation and expand the classical action in a Taylor series

$$S[\varphi+\phi] = \sum_{n=0}^{\infty} = \frac{1}{n!}S_n[\varphi]\phi^n \tag{A.5}$$

where $S_n[\varphi] := \frac{\delta S[\varphi]}{\delta\varphi}$. We can use the same notation for $\Gamma_1[\varphi] := \frac{\delta\Gamma[\varphi]}{\delta\varphi}$. We get

$$\exp\left(-\frac{1}{\hbar}(\Gamma[\varphi]-S[\varphi])\right) = \int D\phi\ \exp\left(-\frac{1}{\hbar}\frac{1}{2}S_2[\varphi]\phi^2 - \frac{1}{\hbar}\sum_{n=3}^{\infty}\frac{1}{n!}S_n[\varphi]\phi^n + \frac{1}{\hbar}(\Gamma_1[\varphi]-S_1[\varphi])\phi\right) \tag{A.6}$$

and rescaling $\phi \to \sqrt{\hbar}\phi$:

$$\exp\left(-\frac{1}{\hbar}(\Gamma[\varphi]-S[\varphi])\right) = \int D\phi\ \exp\left(-\frac{1}{2}S_2[\varphi]\phi^2 - \sum_{n=3}^{\infty}\frac{\hbar^{\frac{n}{2}-1}}{n!}S_n[\varphi]\phi^n + \frac{1}{\sqrt{\hbar}}(\Gamma_1[\varphi]-S_1[\varphi])\phi\right). \tag{A.7}$$

The equation depends only on $\bar{\Gamma}[\varphi] := \Gamma[\varphi] - S[\varphi]$. Expanding $\bar{\Gamma}[\varphi]$ in powers of $\hbar$

$$\bar{\Gamma}[\varphi] = \sum_{n=1}^{\infty}\hbar^n\Gamma^{(n)}[\varphi] \tag{A.8}$$

where the sum start from $n = 1$ so that the bar over $\Gamma^{(n)}$ is not needed anymore, we get the following master equation

$$\exp\left(-\sum_{n=1}^{\infty}\hbar^{n-1}\Gamma^{(n)}[\varphi]\right) = \int D\phi\ \exp\left(-\frac{1}{2}S_2[\varphi]\phi^2 - \sum_{n=3}^{\infty}\frac{\hbar^{\frac{n}{2}-1}}{n!}S_n[\varphi]\phi^n + \sum_{n=1}^{\infty}\hbar^{n-\frac{1}{2}}\Gamma_1^{(n)}[\varphi]\phi\right). \tag{A.9}$$

This equation is analyzed by matching the powers of $\hbar$ in the perturbative expansion.

A.1.1 Approximation at 1-loop ($n = 1$)

Using $n = 1$ in Eq.(A.9) we keep the $\hbar$ independent terms and get

$$e^{-\Gamma^{(1)}[\varphi]} = \int D\phi\ e^{-\frac{1}{2}S_2[\varphi]\phi^2+O(\hbar^{1/2})} = (\mathrm{Det}\,S_2[\varphi])^{-\frac{1}{2}} = e^{-\frac{1}{2}\ln\mathrm{Det}\,S_2[\varphi]} = e^{-\frac{1}{2}\mathrm{Tr}\ln S_2[\varphi]} \tag{A.10}$$

so that

$$\Gamma^{(1)}[\varphi] = \frac{1}{2}\ln\mathrm{Det}\,S_2[\varphi] = \frac{1}{2}\mathrm{Tr}\ln S_2[\varphi]. \tag{A.11}$$

The effective action at 1-loop order is given by

$$\Gamma[\varphi] = S[\varphi] + \frac{\hbar}{2}\mathrm{Tr}\ln S_2[\varphi] + O(\hbar^2). \tag{A.12}$$

Appendix B

Heat Kernel (in curved space)

The heat kernel function is defined as the solution of the following differential equation

$$\frac{\partial\,\mathcal{G}(s;x,x')}{\partial s}+\mathcal{O}_x\mathcal{G}(s;x,x')=0 \tag{B.1}$$

with initial condition

$$\mathcal{G}(0;x,x')=\delta^{(d)}(x,x') \tag{B.2}$$

where $\delta^{(d)}(x,x')$ is the biscalar δ-function that generalizes the usual flat space Dirac delta, and x, x' are (coordinates of) points on an euclidean (i.e. riemannian) manifold of dimension d. To extend the results to the usual spacetime it is necessary to assume an analytic continuation of any Minkowski metric to one of euclidean signature. Restricting our attention to simple scalar fields, we can consider an operator $\mathcal{O}_x$ of Laplace-type

$$\mathcal{O}=-g^{\mu\nu}\nabla_\mu\partial_\nu+E \tag{B.3}$$

in which $E=E(x)$ is a local endomorphism acting multiplicatively on the scalar field's bundle. If we solve the diffusion equation (B.1) implicitly

$$\mathcal{G}(s;x,x')=\langle x'|\,e^{-s\mathcal{O}}\,|x\rangle\,, \tag{B.4}$$

we can see that heat kernel function is related to the Green function G, which is formally defined by

$$\mathcal{O}_xG(x,x')=\delta^{(d)}(x,x'). \tag{B.5}$$

Thus,

$$G(x,x')=\int_0^\infty ds\,\mathcal{G}(s;x,x'). \tag{B.6}$$

The heat kernel function has an asymptotic expansion for $s\to0^+$ which captures capture the ultraviolet properties of the Green function. Following DeWitt [12], it has the form

$$\mathcal{G}(x,x')=\frac{\Delta(x,x')^{1/2}}{(4\pi s)^{d/2}}e^{-\frac{\sigma(x,x')}{2s}}\sum_{k\geqslant0}a_k(x,x')s^k. \tag{B.7}$$

In Eq.(B.7) several bitensors are introduced, the most fundamental is $\sigma(x,x')$ called *geodetic interval* or *Synge-De Witt's world function.* It is defined as half of the square of the geodesic distance between x and x'. The bitensor $\Delta(x,x')$ is known as *van Vleck determinant* and is related to the world function and the determinant metric by

$$\Delta(x,x') = -\frac{1}{g(x)^{1/2}\,g(x')^{1/2}} \det\left(-\frac{\partial^2}{\partial x^\alpha \partial x'^\beta}\sigma(x,x')\right). \tag{B.8}$$

Together, σ and Δ ensure that the leading term of the Seely-De Witt parametrization covariantly generalizes the solution of the heat equation in flat space, where $\mathcal{O} \sim -\partial^2$. Finally, the bitensors $a_k(x,x')$ are the coefficients of the asymptotic expansion and contain the geometrical information of the operator $\mathcal{O}$. The ultraviolet properties are local in renormalizable theories and for the case of the heat kernel locality correspond to $x \sim x'$ and it is captured by the coincidence limit in which $x' \to x$. Given any bitensor $B(x,x')$, its coincidence limit is defined

$$[B] := \lim_{x' \longrightarrow x} B(x,x'). \tag{B.9}$$

One important note is that covariant derivatives do not generally commute with the coincidence limit, so

$$\nabla[B] \neq [\nabla B]. \tag{B.10}$$

The coincidence limits of the bitensors $\sigma(x,x')$ and $\Delta(x,x')$ and their derivatives can be obtained by repeated differentiation of the *crucial relations*, obtained in [12] by geometrical observations for geodesics, they read

$$\sigma_\mu \sigma^\mu = 2\sigma, \qquad \Delta^{1/2}\sigma_\mu{}^\mu + 2\sigma^\mu \nabla_\mu \Delta^{1/2} = d\Delta^{1/2}, \tag{B.11}$$

in which we suppressed the bitensor coordinates and we introduced the notation in which subscripts of σ indicate covariant derivative, i.e. $\sigma_{\mu_1\dots\mu_n} := \nabla_{\mu_n}\dots\nabla_{\mu_1}\sigma$. We can start with the first of Eq.(B.11) obtaining

$$\begin{aligned}
\sigma_\nu &= \sigma_{\mu\nu}\sigma^\mu && \text{(B.12)}\\
\sigma_{\nu\rho} &= \sigma_{\mu\nu\rho}\sigma^\mu + \sigma_{\mu\nu}\sigma^\mu{}_\rho && \text{(B.13)}\\
\sigma_{\nu\rho\sigma} &= \sigma_{\mu\nu\rho\sigma}\sigma^\mu + \sigma_{\mu\nu\rho}\sigma^\mu{}_\sigma + \sigma_{\mu\nu\sigma}\sigma^\mu{}_\rho + \sigma_{\mu\nu}\sigma^\mu{}_{\rho\sigma} && \text{(B.14)}\\
\sigma_{\nu\rho\sigma\tau} &= \sigma_{\mu\nu\rho\sigma\tau}\sigma^\mu + \sigma_{\mu\nu\rho\sigma}\sigma^\mu{}_\tau + \sigma_{\mu\nu\rho\tau}\sigma^\mu{}_\sigma + \sigma_{\mu\nu\rho}\sigma^\mu{}_{\sigma\tau} + \sigma_{\mu\nu\sigma\tau}\sigma^\mu{}_\rho && \text{(B.15)}\\
&\quad + \sigma_{\mu\nu\sigma}\sigma^\mu{}_{\rho\tau} + \sigma_{\mu\nu\tau}\sigma^\mu{}_{\rho\sigma} + \sigma_{\mu\nu}\sigma^\mu{}_{\rho\sigma\tau}\\
&\;\vdots
\end{aligned}$$

Now we can compute coincidence limits considering that we already know

$$[\sigma] := \lim_{x' \longrightarrow x} \sigma(x,x') = 0. \tag{B.16}$$

Using (B.16) and the first of (B.11) we get also

$$[\sigma_\mu] = 0. \tag{B.17}$$

We notice that using (B.17), the coincidence limit of (B.12) is a trivial identity, while for (B.13) we have

$$[\sigma_{\mu\nu}] = g_{\mu\nu}. \tag{B.18}$$

To compute coincidence limits for (B.14) and (B.15), we need to use some commutation laws for covariant differentiation together with (B.16),(B.17) and (B.18). We have

$$[\sigma_{\nu\rho\sigma}] = \lim_{x\longrightarrow x'}(\sigma_{\nu\rho\sigma} + \sigma_{\rho\nu\sigma} + \sigma_{\sigma\nu\rho}) \tag{B.19}$$

and we now subtract $3[\sigma_{\nu\rho\sigma}]$ and use the Ricci identity

$$\begin{aligned}
-2[\sigma_{\nu\rho\sigma}] &= [\sigma_{\rho\nu\sigma}] - [\sigma_{\nu\rho\sigma}] + [\sigma_{\sigma\nu\rho}] - [\sigma_{\nu\rho\sigma}] \\
&= ([\sigma_{\rho\nu\sigma}] - [\sigma_{\nu\rho\sigma}]) + ([\sigma_{\sigma\nu\rho}] - [\sigma_{\nu\sigma\rho}]) + ([\sigma_{\nu\sigma\rho}] - [\sigma_{\nu\rho\sigma}]) \\
&= [\nabla_\sigma[\nabla_\nu, \nabla_\rho]\sigma] + [\nabla_\rho[\nabla_\nu, \nabla_\sigma]\sigma] + [R_{\rho\sigma\nu}{}^{\mu}\sigma_\mu] = 0,
\end{aligned} \tag{B.20}$$

where in the last line the first two terms disappear because we assume a torsion-free theory, and the last term because of (B.17), so

$$[\sigma_{\nu\rho\sigma}] = 0. \tag{B.21}$$

A consequence of considering a torsion-free theory is reflected by the symmetry for the first two indexes closest to σ, namely $[\sigma_{\mu\nu\ldots}] = [\sigma_{\nu\mu\ldots}]$. For the four-derivative term, using (B.21), we have

$$[\sigma_{\nu\rho\sigma\tau}] = [\sigma_{\tau\nu\rho\sigma}] + [\sigma_{\sigma\nu\rho\tau}] + [\sigma_{\rho\nu\sigma\tau}] + [\sigma_{\nu\rho\sigma\tau}]. \tag{B.22}$$

Now we can subtract $4[\sigma_{\nu\rho\sigma\tau}]$ and use the relation

$$[\nabla_\mu, \nabla_\nu]T_{\rho\sigma} = R_{\mu\nu\rho}{}^{\lambda}T_{\lambda\sigma} + R_{\mu\nu\sigma}{}^{\lambda}T_{\rho\lambda}, \tag{B.23}$$

which, in the case of $T_{\mu\nu} = \sigma_{\mu\nu}$, gives simply

$$[\nabla_\mu, \nabla_\mu]\sigma_{\rho\sigma} = 0. \tag{B.24}$$

We get

$$\begin{aligned}
-3[\sigma_{\nu\rho\sigma\tau}] &= [\sigma_{\tau\nu\rho\sigma}] - [\sigma_{\nu\rho\sigma\tau}] + [\sigma_{\sigma\nu\rho\tau}] - [\sigma_{\nu\rho\sigma\tau}] + [\sigma_{\rho\nu\sigma\tau}] - [\sigma_{\nu\rho\sigma\tau}] \\
&= [\sigma_{\nu\tau\rho\sigma}] - [\sigma_{\nu\rho\sigma\tau}] + [\sigma_{\sigma\nu\rho\tau}] - [\sigma_{\nu\rho\sigma\tau}] \\
&= ([\sigma_{\nu\tau\rho\sigma}] - [\sigma_{\nu\rho\tau\sigma}]) + ([\sigma_{\nu\rho\tau\sigma}] - [\sigma_{\nu\rho\sigma\tau}]) + ([\sigma_{\nu\sigma\rho\tau}] - [\sigma_{\nu\rho\sigma\tau}]) \\
&= [\nabla_\sigma(R_{\rho\tau\nu}{}^{\mu}\sigma_\mu)] + [\nabla_\tau(R_{\rho\sigma\nu}{}^{\mu}\sigma_\mu)] + [[\nabla_\sigma, \nabla_\tau]\sigma_{\rho\nu}] \\
&= R_{\rho\tau\nu\sigma} + R_{\rho\sigma\nu\tau}.
\end{aligned} \tag{B.25}$$

Using symmetries of the Riemann tensor we get

$$[\sigma_{\nu\rho\sigma\tau}] = \frac{1}{3}(R_{\nu\sigma\tau\rho} + R_{\nu\tau\sigma\rho}). \tag{B.26}$$

Other relations can be obtained taking contractions in (B.15) and performing further differantiation, as

$$[\sigma_{\nu}{}^{\nu}{}_{\sigma}{}^{\sigma}{}_{\rho}] = \nabla_\rho R \tag{B.27}$$

and

$$[\sigma_{\mu}{}^{\mu}{}_{\nu}{}^{\nu}{}_{\sigma}{}^{\sigma}] = \frac{8}{5}\nabla^\mu\nabla_\mu R + \frac{4}{15}R_{\mu\nu}R^{\mu\nu} - \frac{4}{15}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \tag{B.28}$$

We can now analyze Δ in the same fashion, differentiating the second of (B.11), we get

$$d\nabla_\nu\Delta^{1/2} = \nabla_\nu\Delta^{1/2}\sigma_{\mu}{}^{\mu} + \Delta^{1/2}\sigma_{\mu}{}^{\mu}{}_{\nu} + 2\sigma^{\mu}_{\ \nu}\nabla_\mu\Delta^{1/2} + 2\sigma^\mu\nabla_\nu\nabla_\mu\Delta^{1/2} \tag{B.29}$$

$$\begin{aligned} d\nabla_\rho\nabla_\nu\Delta^{1/2} &= \nabla_\rho\nabla_\nu\Delta^{1/2}\sigma_{\mu}{}^{\mu} + \nabla_\nu\Delta^{1/2}\sigma_{\mu}{}^{\mu}{}_{\rho} + \nabla_\rho\Delta^{1/2}\sigma_{\mu}{}^{\mu}{}_{\nu} + \Delta^{1/2}\sigma_{\mu}{}^{\mu}{}_{\nu\rho} \\ &+ 2\sigma^{\mu}_{\ \nu\rho}\nabla_\mu\Delta^{1/2} + 2\sigma^{\mu}_{\ \nu}\nabla_\rho\nabla_\mu\Delta^{1/2} + 2\sigma^{\mu}_{\ \rho}\nabla_\nu\nabla_\mu\Delta^{1/2} + 2\sigma^\mu\nabla_\rho\nabla_\nu\nabla_\mu\Delta^{1/2}. \end{aligned} \tag{B.30}$$

Before computing coincidence limits we notice that by definition we have

$$[\Delta] = 1. \tag{B.31}$$

In Eq.(B.29), using coincidence limits of σ, we get trivially

$$[\nabla_\nu\Delta^{1/2}] = 0. \tag{B.32}$$

In Eq.(B.30), in particularly using (B.26) we get

$$[\nabla_\rho\nabla_\nu\Delta^{1/2}] = -\frac{1}{6}R_{\rho\nu}, \tag{B.33}$$

and similarly

$$[\nabla_\rho\nabla_\nu\Delta^{-1/2}] = \frac{1}{6}R_{\rho\nu}. \tag{B.34}$$

We can obtain an other useful relation taking the contraction and then differentiating Eq.(B.30), namely

$$\begin{aligned} d\nabla_\rho\nabla^\nu\nabla_\nu\Delta^{1/2} &= \nabla_\rho\nabla^\nu\nabla_\nu\Delta^{1/2}\sigma_{\mu}{}^{\mu} + \nabla^\nu\nabla_\nu\Delta^{1/2}\sigma_{\mu}{}^{\mu}{}_{\rho} + \nabla_\rho\nabla_\nu\Delta^{1/2}\sigma_{\mu}{}^{\mu\nu} + \nabla_\nu\Delta^{1/2}\sigma_{\mu}{}^{\mu\nu}{}_{\rho} \\ &+ \nabla_\rho\nabla^\nu\Delta^{1/2}\sigma_{\mu}{}^{\mu}{}_{\nu} + \nabla^\nu\Delta^{1/2}\sigma_{\mu}{}^{\mu}{}_{\nu\rho} + \nabla_\rho\Delta^{1/2}\sigma_{\mu}{}^{\mu}{}_{\nu}{}^{\nu} + \Delta^{1/2}\sigma_{\mu}{}^{\mu}{}_{\nu}{}^{\nu}{}_{\rho} \\ &+ 2\sigma^{\mu}{}_{\nu}{}^{\nu}{}_{\rho}\nabla_\mu\Delta^{1/2} + 2\sigma^{\mu}{}_{\nu}{}^{\nu}\nabla_\rho\nabla_\mu\Delta^{1/2} + 2\sigma^{\mu}{}_{\nu\rho}\nabla^\nu\nabla_\mu\Delta^{1/2} \\ &+ 2\sigma^{\mu}{}_{\nu}\nabla_\rho\nabla^\nu\nabla_\mu\Delta^{1/2} + 2\sigma^{\mu\nu}{}_{\rho}\nabla_\nu\nabla_\mu\Delta^{1/2} + 2\sigma^{\mu\nu}\nabla_\rho\nabla_\nu\nabla_\mu\Delta^{1/2} \\ &+ 2\sigma^{\mu}{}_{\rho}\nabla^\nu\nabla_\nu\nabla_\mu\Delta^{1/2} + 2\sigma^\mu\nabla_\rho\nabla^\nu\nabla_\nu\nabla_\mu\Delta^{1/2}, \end{aligned} \tag{B.35}$$

taking the coincidence limit we get

$$[\nabla_\rho \nabla^\nu \nabla_\nu \Delta^{1/2}] = -\frac{1}{6}([\sigma_{\mu\ \nu\ \rho}^{\ \mu\ \nu}] + [R_\rho^{\ \mu} \nabla_\mu \Delta^{1/2}]) = -\frac{1}{6}[\sigma_{\mu\ \nu\ \rho}^{\ \mu\ \nu}] = -\frac{1}{6}\nabla_\rho R. \tag{B.36}$$

Another differentiation gives

$$[\nabla^\nu \nabla_\nu \nabla^\mu \nabla_\mu \Delta^{1/2}] = -\frac{1}{5}\nabla^\mu \nabla_\mu R + \frac{1}{36}R^2 - \frac{1}{30}R_{\mu\nu}R^{\mu\nu} + \frac{1}{30}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \tag{B.37}$$

As we did for σ and Δ, we can compute coincidence limits for the coefficients $a_k(x, x')$ differentiating and inductively using

$$ka_k + \sigma^\mu \nabla_\mu a_k + \Delta^{-1/2}\mathcal{O}(\Delta^{1/2}a_{k-1}) = 0 \tag{B.38}$$

with the boundary condition $\sigma^\mu \nabla_\mu a_0$. The recursive equation (B.38) is obtained by (B.1), (B.7), together with (B.11). In the case of a simple scalar field the first coefficient is trivially $a_0(x, x') = 1$ because the Seely-De Witt expansion solves the diffusion equation in flat space. For an operator as (B.3) and for $k = 1$ we have

$$[a_1] = [\Delta^{-1/2}\nabla^\mu \partial_\mu \Delta^{1/2} - E] = -\frac{R}{6} - E. \tag{B.39}$$

Not without reason we can continue differentiating (B.38) in the case $k = 1$ to also get the coincidence limits for the first and the second derivative of the coefficient a_1. The resulting equation for the first derivative is

$$2[\nabla_\nu a_1] = [\Delta^{-1/2}\nabla_\nu \nabla^\mu \nabla_\mu \Delta^{1/2}] - \nabla_\nu E, \tag{B.40}$$

substituting the result (B.36), the equation takes the form

$$[\nabla_\nu a_1] = -\frac{1}{12}\nabla_\nu R - \nabla_\nu E. \tag{B.41}$$

The equation for the second derivative is

$$\begin{aligned} 3[\nabla^\nu \nabla_\nu a_1] = {} & [\nabla^\nu \nabla_\nu \Delta^{-1/2} \cdot \nabla^\mu \nabla_\mu \Delta^{1/2}] - [\nabla^\nu \nabla_\nu \Delta^{-1/2} \cdot E\Delta^{1/2}] \\ & + [\Delta^{-1/2}\nabla^\nu \nabla_\nu \nabla^\mu \nabla_\mu \Delta^{1/2}] + [\Delta^{-1/2} \cdot \nabla^\nu \nabla_\nu \nabla^\mu \nabla_\mu \Delta^{1/2}] \\ & - [\Delta^{-1/2} \cdot \nabla^\nu \nabla_\nu E \cdot \Delta^{1/2}] - [\Delta^{-1/2} \cdot E \cdot \nabla^\nu \nabla_\nu \Delta^{1/2}] \end{aligned} \tag{B.42}$$

using the previous results for Δ, we get

$$[\nabla^\nu \nabla_\nu a_1] = -\frac{1}{15}\nabla^2 R - \frac{1}{90}R_{\mu\nu}R^{\mu\nu} + \frac{1}{90}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{1}{3}\nabla^2 E. \tag{B.43}$$

We will conclude with the computation of the coincidence limit for the second coefficient, in the case $k = 2$, (B.38) becomes

$$2a_2 + \sigma^\mu \nabla_\mu a_2 + \Delta^{-1/2}(-g^{\mu\nu}\nabla_\mu \partial_\nu + E)(\Delta^{1/2} a_1), \tag{B.44}$$

and the coincidence limit is

$$\begin{aligned} 2[a_2] &= [\Delta^{-1/2} g^{\mu\nu} \nabla_\mu \partial_\nu (\Delta^{1/2} a_1) - \Delta^{-1/2} E a_1] \\ &= [g^{\mu\nu}(\nabla_\mu \partial_\nu \Delta^{1/2} \cdot a_1 + \partial_\nu \Delta^{1/2} \nabla_\mu a_1 + \nabla_\mu \Delta^{1/2} \partial_\nu a_1 + \Delta^{1/2} \nabla_\mu \partial_\nu a_1) + E a_1] \\ &= [\nabla^\mu \partial_\mu \Delta^{1/2} \cdot a_1 + \nabla^\mu \partial_\mu a_1 + E a_1], \end{aligned} \tag{B.45}$$

which using (B.39), (B.43) and (B.33), becomes

$$[a_2] = \frac{1}{72} R^2 - \frac{1}{30} \nabla^2 R - \frac{1}{180} R_{\mu\nu} R^{\mu\nu} + \frac{1}{180} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \frac{1}{2} E^2 - \frac{1}{6} \nabla^2 E + \frac{1}{6} RE. \tag{B.46}$$

B.1 tr $[a_2(\mathcal{O}_\Psi)]$

In this section we write down the explicit coefficients c_j presented in subsection 4.2.1 Eq.(4.63).

$$c_1(D,\lambda) = \frac{D(4(1-4\lambda+3\lambda^2) - 2D()\lambda - 2) - 3D^2(\lambda-1)^2 + D^3(\lambda-1)^2}{(D-2)^2}, \tag{B.47}$$

$$\begin{aligned} c_2(D,\lambda) = &\frac{D^4(\lambda-1)^2 + D^3(-9+16\lambda-7\lambda^2) + D^2(19-28\lambda+10\lambda^2)}{2(D-2)^2} \\ &+ \frac{D(6-34\lambda+20\lambda^2) - 16(2-6\lambda+3\lambda^2)}{2(D-2)^2}, \end{aligned} \tag{B.48}$$

$$c_3(D,\lambda) = \frac{3D^2 - 2D - 16}{48(D-2)}, \tag{B.49}$$

$$\begin{aligned} c_4(D,\lambda) = &\frac{D^4 + 3D^2(15-24\lambda+8\lambda^2) - 6D^2(25-32\lambda+8\lambda^2)}{48(D-2)^2} \\ &+ \frac{-4D(95-168\lambda+72\lambda^2) + 32(31-48\lambda+18\lambda^2)}{48(D-2)^2}, \end{aligned} \tag{B.50}$$

$$c_5(D,\lambda) = \frac{3D^2 - 2D - 16}{48(D-2)}, \tag{B.51}$$

$$c_6(D,\lambda) = \frac{3D^4(\lambda-1)^2 + 3D^2(-45+78\lambda-33\lambda^2) + D^2(163-252\lambda+90\lambda^2)}{48(D-2)^2} + \frac{-4D(31-33\lambda+3\lambda^2) - 8(5-18\lambda+12\lambda^2)}{48(D-2^2)}\,, \tag{B.52}$$

$$c_7(D,\lambda) = -\frac{7(4-5D+D^2)}{12(D-2)}\,, \tag{B.53}$$

$$c_8(D,\lambda) = 2 - \frac{D}{12}\,, \tag{B.54}$$

$$c_9(D,\lambda) = 1\,, \tag{B.55}$$

$$c_{10}(D,\lambda) = \frac{2-4D+D^2}{D-2} \tag{B.56}$$

$$c_{11}(D,\lambda) = 1\,, \tag{B.57}$$

$$c_{12}(C,\lambda) = \frac{-D^3 + D^2(723-2N^2-1440\lambda+720\lambda)}{360(D-2)} + \frac{2D(-2971+92N^2+1440\lambda) - 360(-9+N^2-32\lambda+24\lambda^2)}{360(D-2)}\,, \tag{B.58}$$

$$c_{13}(D,\lambda) = \frac{-3D^2(13-20\lambda+8\lambda^2) + D(155-72\lambda) + 4(35-144\lambda+72\lambda^2)}{12(D-2)} \tag{B.59}$$

$$c_{14}(D,\lambda) = \frac{D^3(-13+25\lambda-12\lambda^2) + D^2(37-73\lambda+36\lambda^2)}{6(D-2)} + \frac{D(-22-28\lambda+24\lambda^2) + 48\lambda(4-3\lambda)}{6(D-2)}\,, \tag{B.60}$$

$$c_{15}(D,\lambda) = \frac{D^3(-13+25\lambda-12\lambda^2) + D^2(121-197\lambda+84\lambda^2)}{24(D-2)} + \frac{-12D(17-20\lambda+6\lambda^2) - 8(25-59\lambda+30\lambda^2)}{24(D-2)}\,, \tag{B.61}$$

$$c_{16}(D,\lambda) = \frac{D^3(13-12\lambda)^2 + D^2(-795+2N^2+1512\lambda-720\lambda^2)}{144(D-2)} + \frac{-2D(-613+14N^2+552\lambda-144\lambda^2) + 48(11+N^2-76\lambda+48\lambda^2)}{144(D-2)}\,, \tag{B.62}$$

$$c_{17}(D,\lambda) = 2\,, \tag{B.63}$$

$$c_{18}(D,\lambda) = \frac{(52-29D+D^2)}{12(D-2)}\,, \tag{B.64}$$

$$c_{19}(D,\lambda) = \frac{1}{360}(D^2 + D(-31+2N^2) - 30(-47+N^2))\,, \tag{B.65}$$

$$c_{20}(D,\lambda) = 4\,, \tag{B.66}$$

$$c_{21}(D,\lambda) = 1\,, \tag{B.67}$$

$$c_{22}(D,\lambda) = \frac{1}{24}\,, \tag{B.68}$$

$$c_{23}(D,\lambda) = -1\,, \tag{B.69}$$

$$c_{24}(D,\lambda) = -\frac{1}{12}\,, \tag{B.70}$$

$$c_{25}(D,\lambda) = -\frac{D^2-4D+2}{D-2}\,, \tag{B.71}$$

$$c_{26}(D,\lambda) = \frac{D^2-4D+2}{12(D-2)}\,, \tag{B.72}$$

$$c_{27}(D,\lambda) = -\frac{4D^2-11D+2}{24(D-2)}\,, \tag{B.73}$$

Appendix C

Metric perturbations

In this appendix we collect some formulae to compute metric perturbations. Since we are interested in the expansion of the gravitational action around a general background to second order in $h_{\mu\nu}$, I will omit $\mathcal{O}(h^3)$ in the following expressions. We will consider an expansion of the metric as in the main text

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} + \frac{\lambda}{2} h_{\mu\rho} \bar{g}^{\rho\sigma} h_{\sigma\nu}. \tag{C.1}$$

The inverse metric can be expanded as

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu} + \left(1 - \frac{\lambda}{2}\right) h^{\mu\rho} h_{\rho}^{\ \nu} \tag{C.2}$$

where the convention is that indices are raised and lowered with $\bar{g}_{\mu\nu}$. The variation of the volume element is

$$\sqrt{|g|} = \sqrt{|\bar{g}|} \left[1 + \frac{1}{2} h + \left(\frac{1}{8} h^2 + \frac{(\lambda - 1)}{4} h^{\mu\nu} h_{\mu\nu}\right)\right] \tag{C.3}$$

where $h = \bar{g}^{\mu\nu} h_{\mu\nu}$. The variations of the Christoffel symbols are

$$\Gamma^{\rho}_{\mu\nu} = \bar{\Gamma}^{\rho}_{\mu\nu} + \Gamma^{\rho(1)}_{\mu\nu} + \Gamma^{\rho(2)}_{\mu\nu}, \tag{C.4}$$

where

$$\Gamma^{\rho(1)}_{\mu\nu} = \frac{1}{2} (\bar{\nabla}_\mu h_\nu^{\ \rho} + \bar{\nabla}_\nu h_\mu^{\ \rho} - \bar{\nabla}^\rho h_{\mu\nu}), \tag{C.5}$$

$$\Gamma^{\rho(2)}_{\mu\nu} = \frac{1}{2} [h^{\rho\alpha} \bar{\nabla}_\alpha h_{\mu\nu} + (\lambda - 2) h^{\rho\alpha} \bar{\nabla}_{(\mu} h_{\nu)\alpha} + \lambda h_{(\mu}^{\ \ \alpha} \bar{\nabla}_{\nu)} h^{\rho\alpha} - \lambda h_{(\mu}^{\ \ \alpha} \bar{\nabla}^\rho h_{\nu)\alpha}]. \tag{C.6}$$

Parenthesis indicate symmetrization. From this expansion one can obtain the variation of the Riemann tensor.

$$R^{\mu}_{\ \nu\rho\sigma} = \bar{R}^{\mu}_{\ \nu\rho\sigma} + R^{\mu(1)}_{\nu\rho\sigma} + R^{\mu(2)}_{\nu\rho\sigma} \tag{C.7}$$

where

$$
\begin{aligned}
R^{\mu(1)}_{\nu\rho\sigma} = \bar{R}^{\mu}{}_{\nu\rho\alpha}h^{\alpha}{}_{\sigma} + \bar{R}_{\nu\alpha\rho\sigma}h^{\mu\alpha} + \frac{1}{2}\bar{\nabla}_{\nu}\bar{\nabla}^{\mu}h_{\rho\sigma} - \frac{1}{2}\bar{\nabla}^{\mu}\bar{\nabla}_{\nu}h_{\rho\sigma} \\
+ \frac{1}{2}\bar{\nabla}_{\nu}\bar{\nabla}_{\rho}h^{\mu}{}_{\sigma} - \frac{1}{2}\bar{\nabla}^{\mu}\bar{\nabla}_{\rho}h_{\nu\sigma} + \frac{1}{2}\bar{\nabla}^{\mu}\bar{\nabla}_{\sigma}h_{\nu\rho} - \frac{1}{2}\bar{\nabla}_{\nu}\bar{\nabla}_{\sigma}h^{\mu}{}_{\sigma},
\end{aligned}
\tag{C.8}
$$

and

$$
\begin{aligned}
R^{\mu(2)}_{\nu\rho\sigma} = {} & \frac{\lambda}{2}h_{\alpha}^{\ \beta}h_{\sigma}^{\ \alpha}\bar{R}^{\mu}{}_{\nu\rho\beta} + h^{\mu\alpha}h_{\sigma}^{\ \beta}\bar{R}_{\nu\alpha\rho\beta} + \frac{(\lambda-2)}{2}h_{\alpha}^{\ \beta}h^{\mu\alpha}\bar{R}_{\nu\beta\rho\sigma} \\
& + \frac{1}{2}h^{\mu\alpha}\bar{\nabla}_{\alpha}\bar{\nabla}_{\nu}h_{\rho\sigma} + \frac{1}{2}h^{\mu\alpha}\bar{\nabla}_{\alpha}\bar{\nabla}_{\rho}h_{\nu\sigma} - \frac{1}{2}h^{\mu\alpha}\bar{\nabla}_{\alpha}\bar{\nabla}_{\sigma}h_{\nu\rho} \\
& - \frac{1}{4}\bar{\nabla}_{\alpha}h_{\nu\sigma}\bar{\nabla}^{\alpha}h^{\mu}_{\rho} + \frac{1}{4}\bar{\nabla}_{\alpha}h_{\nu\rho}\bar{\nabla}^{\alpha}h^{\mu}_{\sigma} + \frac{1}{4}\bar{\nabla}^{\alpha}h_{\nu\sigma}\bar{\nabla}^{\mu}h_{\rho\alpha} \\
& - \frac{1}{4}\bar{\nabla}^{\alpha}h_{\nu\rho}\bar{\nabla}^{\mu}h_{\sigma\alpha} - \frac{\lambda}{4}h_{\sigma}^{\ \alpha}\bar{\nabla}^{\mu}\bar{\nabla}_{\nu}h_{\rho\alpha} - \frac{\lambda}{4}h_{\rho}^{\ \alpha}\bar{\nabla}^{\mu}\bar{\nabla}_{\nu}h_{\sigma\alpha} \\
& - \frac{\lambda}{4}h_{\sigma}^{\ \alpha}\bar{\nabla}^{\mu}\bar{\nabla}_{\rho}h_{\nu\alpha} - \frac{\lambda}{4}h_{\nu}^{\ \alpha}\bar{\nabla}^{\mu}\bar{\nabla}_{\rho}h_{\sigma\alpha} + \frac{\lambda}{4}h_{\rho}^{\ \alpha}\bar{\nabla}^{\mu}\bar{\nabla}_{\sigma}h_{\nu\alpha} \\
& + \frac{\lambda}{4}h_{\nu}^{\ \alpha}\bar{\nabla}^{\mu}\bar{\nabla}_{\sigma}h_{\rho\alpha} + \frac{1}{4}\bar{\nabla}^{\alpha}h^{\mu}_{\sigma}\bar{\nabla}_{\nu}h_{\rho\alpha} + \frac{1}{4}\bar{\nabla}^{\mu}h_{\sigma\alpha}\bar{\nabla}_{\nu}h_{\rho}^{\ \alpha} \\
& + \frac{1}{4}\bar{\nabla}^{\alpha}h^{\mu}_{\rho}\bar{\nabla}_{\nu}h_{\sigma\alpha} - \frac{1}{4}\bar{\nabla}^{\mu}h_{\rho}^{\ \alpha}\bar{\nabla}_{\nu}h_{\sigma\alpha} - \frac{1}{2}h^{\mu\alpha}\bar{\nabla}_{\nu}\bar{\nabla}_{\alpha}h_{\rho\sigma} \\
& + \frac{\lambda}{4}h_{\sigma}^{\ \alpha}\bar{\nabla}_{\nu}\bar{\nabla}^{\mu}h_{\rho\alpha} + \frac{\lambda}{4}h_{\rho}^{\ \alpha}\bar{\nabla}_{\nu}\bar{\nabla}^{\mu}h_{\sigma\alpha} + \frac{\lambda}{4}h_{\sigma}^{\ \alpha}\bar{\nabla}_{\nu}\bar{\nabla}_{\rho}h^{\mu}_{\alpha} \\
& + \frac{(\lambda-2)}{4}h^{\mu\alpha}\bar{\nabla}_{\nu}\bar{\nabla}_{\rho}h_{\sigma\alpha} + \frac{\lambda}{4}h_{\rho}^{\ \alpha}\bar{\nabla}_{\nu}\bar{\nabla}_{\sigma}h^{\mu}_{\alpha} + \left(\frac{2-\lambda}{4}\right)h^{\mu\alpha}\bar{\nabla}_{\nu}\bar{\nabla}_{\sigma}h_{\rho\alpha} \\
& + \frac{1}{4}\bar{\nabla}_{\alpha}h_{\nu\sigma}\bar{\nabla}_{\rho}h^{\mu\alpha} + \frac{(\lambda-1)}{4}\bar{\nabla}_{\nu}h_{\sigma\alpha}\bar{\nabla}_{\rho}h^{\mu\alpha} - \frac{1}{4}\bar{\nabla}^{\alpha}h^{\mu}_{\sigma}\bar{\nabla}_{\rho}h_{\nu\alpha} \\
& + \frac{(\lambda-1)}{4}\bar{\nabla}^{\mu}h_{\sigma\alpha}\bar{\nabla}_{\rho}h_{\nu}^{\ \alpha} - \frac{\lambda}{4}\bar{\nabla}^{\mu}h_{\nu}^{\ \alpha}\bar{\nabla}_{\rho}h_{\sigma\alpha} + \frac{\lambda}{4}\bar{\nabla}_{\nu}h^{\mu\alpha}\bar{\nabla}_{\rho}h_{\sigma\alpha} \\
& - \frac{1}{4}\bar{\nabla}_{\alpha}h_{\nu\rho}\bar{\nabla}_{\sigma}h^{\mu\alpha} + \frac{(\lambda-1)}{4}\bar{\nabla}_{\nu}h_{\rho\alpha}\bar{\nabla}_{\sigma}h^{\mu\alpha} + \frac{1}{4}\bar{\nabla}_{\rho}h_{\nu\alpha}\bar{\nabla}_{\sigma}h^{\mu\alpha} \\
& + \frac{1}{4}\bar{\nabla}^{\alpha}h^{\mu}_{\rho}\bar{\nabla}_{\sigma}h_{\nu\alpha} - \frac{1}{4}\bar{\nabla}_{\rho}h^{\mu\alpha}\bar{\nabla}_{\sigma}h_{\nu\alpha} + \frac{(\lambda-1)}{4}\bar{\nabla}^{\mu}h_{\rho\alpha}\bar{\nabla}_{\sigma}h_{\nu}^{\ \alpha} \\
& + \frac{\lambda}{4}\bar{\nabla}^{\mu}h_{\nu}^{\ \alpha}\bar{\nabla}_{\sigma}h_{\rho\alpha} - \frac{\lambda}{4}\bar{\nabla}_{\nu}h^{\mu\alpha}\bar{\nabla}_{\sigma}h_{\rho\alpha}.
\end{aligned}
\tag{C.9}
$$

The expansion of the Ricci tensor is

$$
R^{\mu}_{\nu} = \bar{R}^{\mu}_{\nu} + R^{\mu(1)}_{\nu} + R^{\mu(2)}_{\nu}. \tag{C.10}
$$

We find

$$
\begin{aligned}
R^{\mu(1)}_{\nu} = &-\frac{1}{2}h^{\mu\alpha}\bar{R}_{\nu\alpha} + \frac{1}{2}h_{\nu}{}^{\alpha}\bar{R}^{\mu}_{\alpha} - h^{\alpha\beta}\bar{R}_{\nu\alpha}{}^{\mu}{}_{\beta} \\
&- \frac{1}{2}\bar{\nabla}_{\alpha}\bar{\nabla}^{\alpha}h_{\nu}{}^{\mu} + \frac{1}{2}\bar{\nabla}_{\nu}\bar{\nabla}_{\alpha}h^{\mu\alpha} + \frac{1}{2}\bar{\nabla}^{\mu}\bar{\nabla}_{\alpha}h_{\nu}{}^{\alpha} - \frac{1}{2}\bar{\nabla}^{\mu}\bar{\nabla}_{\nu}h
\end{aligned}
\tag{C.11}
$$

and

$$
\begin{aligned}
R^{\mu(2)}_{\nu} = &\frac{1}{4}(2-\lambda)\,h_{\alpha}{}^{\beta}\,h^{\mu\alpha}\,\bar{R}_{\nu\beta} + \frac{1}{4}\lambda\,h_{\alpha}{}^{\beta}\,h_{\nu}{}^{\alpha}\,\bar{R}^{\mu}{}_{\beta} + \left(1-\frac{1}{2}\lambda\right)h_{\alpha}{}^{\kappa}\,h^{\alpha\beta}\,\bar{R}_{\nu\beta}{}^{\mu}{}_{\kappa} \\
&- \frac{1}{2}h^{\beta\kappa}\,h_{\nu}{}^{\alpha}\,\bar{R}^{\mu}{}_{\beta\alpha\kappa} + \frac{1}{2}h^{\mu\alpha}\,\bar{\nabla}_{\alpha}\bar{\nabla}_{\nu}h^{\beta}{}_{\beta} - \frac{1}{4}\bar{\nabla}_{\alpha}h^{\beta}{}_{\beta}\bar{\nabla}^{\alpha}h_{\nu}{}^{\mu} + \frac{1}{2}\bar{\nabla}^{\alpha}h_{\nu}{}^{\mu}\bar{\nabla}_{\beta}h_{\alpha}{}^{\beta} \\
&- \frac{1}{2}h^{\mu\alpha}\,\bar{\nabla}_{\beta}\bar{\nabla}_{\alpha}h_{\nu}{}^{\beta} + \frac{1}{2}h^{\alpha\beta}\,\bar{\nabla}_{\beta}\bar{\nabla}_{\alpha}h_{\nu}{}^{\mu} + \frac{1}{4}(2-\lambda)\,h^{\mu\alpha}\,\bar{\nabla}_{\beta}\bar{\nabla}^{\beta}h_{\nu\alpha} - \frac{1}{4}\lambda\,h_{\nu}{}^{\alpha}\,\bar{\nabla}_{\beta}\bar{\nabla}^{\beta}h^{\mu}{}_{\alpha} \\
&- \frac{1}{2}\bar{\nabla}_{\alpha}h^{\mu}{}_{\beta}\bar{\nabla}^{\beta}h_{\nu}{}^{\alpha} + \frac{1}{2}(1-\lambda)\,\bar{\nabla}_{\beta}h^{\mu}{}_{\alpha}\bar{\nabla}^{\beta}h_{\nu}{}^{\alpha} + \frac{1}{4}\lambda\,\bar{\nabla}^{\beta}h^{\mu\alpha}\bar{\nabla}_{\nu}h_{\alpha\beta}) \\
&+ \frac{1}{4}\bar{\nabla}_{\alpha}h^{\beta}{}_{\beta}\bar{\nabla}_{\nu}h^{\mu\alpha} + \frac{1}{4}(\lambda-2)\,\bar{\nabla}_{\beta}h_{\alpha}{}^{\beta}\bar{\nabla}_{\nu}h^{\mu\alpha} + \frac{1}{4}(\lambda-2)\,h^{\mu\alpha}\,\bar{\nabla}_{\nu}\bar{\nabla}_{\beta}h_{\alpha}{}^{\beta} \\
&+ \frac{1}{4}(\lambda-2)\,h^{\alpha\beta}\,\bar{\nabla}_{\nu}\bar{\nabla}_{\beta}h^{\mu}{}_{\alpha} + \frac{1}{4}\lambda\,\bar{\nabla}^{\beta}h_{\nu}{}^{\alpha}\bar{\nabla}^{\mu}h_{\alpha\beta}) + \frac{1}{4}(1-2\lambda)\,\bar{\nabla}_{\nu}h^{\alpha\beta}\bar{\nabla}^{\mu}h_{\alpha\beta} \\
&+ \frac{1}{4}\bar{\nabla}_{\alpha}h^{\beta}{}_{\beta}\bar{\nabla}^{\mu}h_{\nu}{}^{\alpha} + \frac{1}{4}(\lambda-2)\,\bar{\nabla}_{\beta}h_{\alpha}{}^{\beta}\bar{\nabla}^{\mu}h_{\nu}{}^{\alpha} + \frac{1}{4}\lambda\,h_{\nu}{}^{\alpha}\,\bar{\nabla}^{\mu}\bar{\nabla}_{\beta}h_{\alpha}{}^{\beta} \\
&+ \frac{1}{4}(\lambda-2)\,h^{\alpha\beta}\,\bar{\nabla}^{\mu}\bar{\nabla}_{\beta}h_{\nu\alpha} + \frac{1}{2}(1-\lambda)\,h^{\alpha\beta}\,\bar{\nabla}^{\mu}\bar{\nabla}_{\nu}h_{\alpha\beta}
\end{aligned}
\tag{C.12}
$$

Finally, the expansion of the Ricci scalar is

$$
R = \bar{R} + R^{(1)} + R^{(2)}, \tag{C.13}
$$

with

$$
R^{(1)} = \bar{\nabla}_{\nu}\bar{\nabla}_{\mu}h^{\mu\nu} - \bar{\nabla}^{2}h - h^{\mu\nu}\bar{R}_{\mu\nu} \tag{C.14}
$$

and

$$
\begin{aligned}
R^{(2)} = &(2-\lambda)h^{\rho}_{\mu}h^{\mu\nu}\bar{R}_{\nu\rho} + \frac{\lambda-2}{2}h^{\mu\nu}h^{\rho\sigma}\bar{R}_{\mu\rho\nu\sigma} + h^{\mu\nu}\bar{\nabla}_{\nu}\bar{\nabla}_{\mu}h - \frac{1}{4}\bar{\nabla}_{\mu}h\bar{\nabla}^{\mu}h \\
&+ \frac{\lambda-2}{2}\bar{\nabla}_{\mu}h^{\mu\nu}\bar{\nabla}_{\rho}h^{\rho}_{\nu} + \bar{\nabla}^{\mu}h\bar{\nabla}_{\nu}h^{\nu}_{\mu} + (\lambda-2)h^{\mu\nu}\bar{\nabla}_{\rho}\bar{\nabla}_{\nu}h^{\rho}_{\mu} \qquad . \\
&+ (1-\lambda)h^{\mu\nu\bar{\nabla}^{2}}h_{\mu\nu} + \frac{\lambda-1}{2}\bar{\nabla}_{\nu}h_{\mu\rho}\bar{\nabla}^{\rho}h^{\mu\nu} + \left(\frac{3}{4}-\lambda\right)\bar{\nabla}_{\rho}h_{\mu\nu}\bar{\nabla}^{\rho}h^{\mu\nu}
\end{aligned}
\tag{C.15}
$$

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