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A Look into AlphaGeometry

Tesi di Laurea in Machine Learning

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A mamma e papà.

Introduction

Since the birth of the concept of automated reasoning and AI research, proving mathematical theorems has always been one of the most sought-after objectives, and so since the 1950s different methods and approaches have been explored. As such studies proceeded, theorem proving has been shown to be a particularly non-trivial task for learning-based methods, due to the difficulties related to the translations of human proofs into objective machine-verifiable languages, implying the scarcity of the necessary training data.

Regarding theorem proving, an interesting sector is the Math Olympiads one, which has always been full of high-difficulty level problems that are served as a challenge for pre-university students from all over the world. As such, every time a new machine-learning approach appears, measuring its performance on such problems is always one of the most interesting analysis to perform.

Math Olympiads are annual competitions held in over 100 different countries, involving many pre-university students that take part in different national selections of increasing difficulty, ending with the selection of 6 students per participating country that will attend the International Math Olympiad [5] (IMO), the most difficult and important mathematical competition in the world. IMO problems are generally classified into four different topics: Algebra, Combinatorics, Geometry, and Number Theory. In the context of the IMO, *geometry* refers specifically to Euclidean geometry.

Analyzing Euclidean geometry Theorem Provers in particular, they fall in two branches: computer algebra methods and search/axiomatic methods. The first methods treat geometry statements as polynomial equations of their point coordinates, and the proving process consists of specialized transformations of large polynomials. Theoretically, these methods can determine with a 100% accuracy if a theorem is true or not for each problem, but require a lot of time to reach their conclusions and do not produce any proof, just the truth value of a theorem. Wu's method [17] is the most important one among the methods of this type, and since its introduction, this category has been largely considered solved. The second methods instead consist of proving the theorem by doing a step-by-step search using a set of geometry axioms, and usually return a proof accessible by human readers. Large Language Models such as GPT-5 can be considered in this category.

In this work, we will analyze AlphaGeometry[16], a theorem prover developed by Google DeepMind in 2023 that belongs to the second category of solver, since it takes as input a Euclidean geometry problem/theorem and attempts to produce a proof in a human-readable language.

AlphaGeometry is a significant milestone in the AI research, thanks to its approach to the problem of the scarcity of training data, which is particularly crucial for geometry problems. In fact, to create AlphaGeometry, no human proof was translated. An existing symbolic engine[3] was improved and used to generate more than 100 million synthetic theorems and their proofs. These proofs served as the training data for a neural language model that was pre-trained on all the generated synthetic data and then fine-tuned on the subset of these synthetic proofs that contained auxiliary constructions.

In fact, geometry problems' human solutions frequently present auxiliary constructions, something that a machine finds quite difficult to replicate, in particular for difficult problems like IMO ones, which usually require multiple auxiliary constructions. In the past, each attempt at implementing auxiliary construction into automated methods still relied on human-designed heuristic. So fine-tuning the model on those proof steps is a really useful process, since it helps the model to focus on suggesting auxiliary constructions, which is a task that is typically very challenging for machine-learning methods.

The neural language model works alongside the improved symbolic engine, suggesting an auxiliary construction each time the symbolic engine could not reach the conclusion on its own. This approach was discovered to be quite successful, since, considering a subset of 30 IMO geometry problems from 2000 to 2023, Wu's Method (the previous state-of-art) could only solve 10 of them within a 48h time limit, while AlphaGeometry could solve 25 of them within the competition's time limit of 1.5h per problem, returning a functioning proof for each of them. AlphaGeometry reached what is considered to be a new state-of-art, with a performance that is comparable to the average of a human Silver Medalist.

In its evaluation, AlphaGeometry was subjected to extensive testing, showing that even working with a fraction of the original data and budget, astonishing results were achieved. AlphaGeometry was in fact tested on a bigger set of problems and in different training conditions, comparing the results with the previous best methods and their variations, but these tests will not be discussed in this work. Still, it is advised to check them to better appreciate AlphaGeometry's importance in the literature.

Being aware of the potential of this approach, it is fundamental to thoroughly study AlphaGeometry, its functioning, creation, and training, since improving it and replicating this approach in other fields and topics could lead to enormous improvements in the entire machine learning world.

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Chapter 1

The Model

1.1 AlphaGeometry

AlphaGeometry[16] is a neuro-symbolic system developed by DeepMind that works as a theorem prover for Euclidean Geometry. It is composed of two different components that work alternately: a Neural Language Model and a Symbolic Deduction Engine.

AlphaGeometry is the first of its kind in automated theorem-proving, since it entirely bypasses the most important problem for learning-based methods: the scarcity of training data and the cost of translating human demonstrations into machine-verifiable format. AlphaGeometry, in fact, is trained on its own synthetically generated data, generated using a pre-existing symbolic engine, specifically improved for this task.

1.2 The Symbolic Engine

The first component we are going to analyze is the symbolic deduction engine mentioned above. The symbolic engine used in AlphaGeometry is a combination of two different components. The first is known as Deductive Database (DD), and was previously developed by Chou et al.[3] around 2000. It was perfected by DeepMind researchers by integrating Algebraic Reasoning (AR) into it, leading to the creation of the DDAR, AlphaGeometry's symbolic solver.

DDAR's purpose is to deduce new statements based on some geometric premises taken as input, combining both logical deduction to handle geometric rules and algebraic computation to handle angle, ratio, and distance chasing.

The output of the first component is fed into the second one and vice versa, each expanding the other's set of deduced statements until the joint deduction closure stops expanding.

1.2.1 Deductive Database (DD)

Deductive Database belongs to the category of geometric theorem solver based on synthetic deduction. It follows deduction rules in the form of Horn Clauses:

$$Q(x) \leftarrow P_1(x), \ldots, P_k(x) \tag{1.1}$$

where both Q and $P_1, \ldots, P_k$ are geometry predicates, and each x is a set of points. If each predicate P_i for $i = 1, \ldots, k$ is true for its own set of points, then $Q(x)$ is added to the known facts.

In its original version[3] the program used about 70 rules to describe basic geometric properties, but used only definite Horn clauses without function symbols. This limitation implied the total absence of algebraic computation.

Geometry problems have always been characterized by the presence of transitivity and symmetry properties. As a simple example, each property applied to a triangle can be applied to the same triangle just by exchanging its vertices order. This has always been an important problem from a computational point of view, but was solved by using simple mathematical structures such as sequences and equivalence classes to represent geometry rules in the database. A feature used for this purpose is the use of full-angles instead of regular angles.

Definition 1.1. A *full-angle* $\angle(u, v)$ is the angle from line u to line v

Full-angles were introduced in Wu's Method[18] because thanks to this definition two full-angles $\angle(u, v)$ and $\angle(l, m)$ are equal if there exists a rotation K such that $K(l) \parallel u$ and $K(m) \parallel v$, greatly simplifying the predicate of angle congruence. Moreover, using full-angles, supplementary angles are considered as the same angle.

The most crucial difference from the original implementation of DD is the use of a graph data structure instead of strings of canonical forms. This different approach allows to deal with problems as the symmetrical permutation of function arguments and the transitivity of geometric properties as collinearity and concyclicity. Implementing this approach made some of the original deduction rules useless, since the graph structure itself handled them, so the number of deduction rules in the final model decreased from the original 70 rules to 43 rules.

From now on, each angle mentioned in this work will be considered a full-angle and can be written analogously as $\angle(u, v)$ or $\angle(u - v)$.

1.2.2 Algebraic Reasoning (AR)

Algebraic proof steps are a fundamental part of geometry proofs but are not covered in geometric rules, so implementing them would cover a lot more proofs than DD only. To implement them, an approach similar to the one used in GeoLogic[10] was used, but enlarged a little to cover not only angle chasing, but also ratio and distance chasing too.

To start, a coefficient matrix $A \in \mathbb{R}^{m \times n}$ is created, with m as the number of input equations and n as the number of variables. It is necessary to bring every equation in the form '$a + b + c + d = 0$' taking every member of the equation on the same side equal to 0.

- For angles: $\angle ABC = \angle XYZ$ becomes $s(AB) - s(BC) = s(XY) - s(YZ)$, where $s(AB)$ is the angle between AB and the x-direction, modulo π;
- For ratios: $AB : CD = EF : GH$ is represented as $\log(AB) - \log(CD) = \log(EF) - \log(GH)$, where $\log(AB)$ is the log of the length of the segment AB;
- For distances: each variable is a (point, line) pair, representing a specific point on a specific line;

Having brought each equality into the desired form, we compile the matrix A with values corresponding to the coefficients of each variable for each equation, and then proceed to run the Gaussian elimination process, obtaining all new equalities that will be used to further proceed with the proof.

Example 1.2. If we have $a - b = b - c$, $d - c = a - d$ and $d - c = c - e$:

$$\begin{pmatrix} a & b & c & d & e \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ 0 & 1 & -2 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} a & b & c & d & e \\ 1 & 0 & 0 & -1.5 & 0.5 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -0.5 & -0.5 \end{pmatrix} \Rightarrow \begin{cases} a = 1.5d - 0.5e \\ b = d \\ c = 0.5d + 0.5e \end{cases}$$

In this specific example, Algebraic reasoning deduced $b = d$.

1.3 Dataset

Training AlphaGeometry's language model was probably the most difficult step to overcome, since a large-scale dataset for this purpose still does not exist. This led to the interesting approach used for the training of AlphaGeometry: the creation of an entirely synthetic dataset that did not make any use of any human-designed problem step, from statement to proof.

The first step in the creation of this dataset consisted of sampling a random set of theorem premises that serves as input to the symbolic deduction engine. In fact, given a random set of premises, the symbolic engine starts to generate its derivation by adding new facts to the known ones. This process returns a directed acyclic graph of all the reachable conclusions for each set of random premises. Taking into consideration a single one of these graphs, each one of its nodes is a reachable conclusion, which is connected by edges to its parent nodes. Having obtained this directed acyclic graph, it is possible for each node N to perform a traceback process that returns its dependency subgraph $G(N)$. This subgraph has the conclusion N as its root, and the minimal set of necessary premises P as its leaves. Thanks to this process, we obtain a perfect synthetic training example, which is presented as a triplet $(P, N, G(N)) =$ (premises, conclusion, proof).

An interesting part is the way auxiliary constructions are handled. Auxiliary constructions are a fundamental part in a lot of theorem proofs, especially in International Math Olympiads. Thanks to the graph structure of the symbolic engine solutions, we can trace all the proof terms (the geometric objects and logical statements) that are essential to the construction of the terms appearing in the conclusion statement N. The proof terms in the premises P that are not among these essential ones represent the dependency difference between the conclusion statement and the conclusion objects, so they are all new proof terms. Isolating the proof steps that use these objects provides us a way to select the specific proof steps that generate these terms.

In the end, a proof pruning process and a deduplication process are applied to ensure the quality of the proofs and the diversity of the dataset.

The entire data generation process was run in a highly parallelized environment and produced a dataset that counted 100 million theorem-proofs examples, and about 9 millions of them involve at least one auxiliary construction.

Traceback

As mentioned, a traceback algorithm is performed on each deduction step, to trace the minimal premises needed to deduce that specific step. This is essential to discard any unnecessary auxiliary construction.

This traceback process is used for both geometric rule deduction and algebraic deduction in different ways.

Traceback for geometric-rule deduction

To study how traceback is performed for geometric rule deduction, it is necessary to analyze how the equality, concyclicity and collinearity transitivity graph are managed in the graph structure.

- Equality: if '$a = b$', '$b = c$','$c = d$' and '$a = d$' are deduced, this results in nodes a, b, c, d connected to the same "equality node" e, so we maintain a graph within e that has edges $[(a, b), (b, c), (c, d), (a, d)]$. This way it is possible to perform a breadth-first search to find the shortest path of transitivity of equality between any pair of variables.

- Collinearity and Concyclicity: we use hypergraphs $G(V, E)$ with respectively 3-edges and 4-edges. This way traceback is equivalent to finding the Minimum Spanning Tree (MST) for the target set of nodes S whose weight is the cardinality of the union of its hyperedges e':

$$\text{MST}(S) = \min_{T \subset E} \left| \bigcup_{e' \subset T} w(e') \right| \text{ s.t. } S \subset T$$

Traceback on other deduction rules works by returning the immediate parents of each rule.

Traceback for algebraic deduction

After Gaussian elimination derives a new equality, performing traceback is equivalent to solving a mixed-integer linear program (MILP): we select which of the original equations to combine so that the target equality is reproduced, while minimizing the number of premises used.

Considering the coefficient matrix $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^n$ the target equation, the minimal set of premises for b can be obtained by considering two non-negative integer vectors $x, y \in \mathbb{Z}^n$ and solving the problem:

$$x, y = \arg\min_{x,y} \sum_i (x_i + y_i) \quad \text{s.t. } A^\top (x - y) = b.$$

This means that we will consider the i-th rows where $(x_i - y_i) \neq 0$.

Proof pruning

The traceback process we performed grants that the set of immediate ancestors to any node is minimal, but it does not guarantee that the fully traced back dependency subgraph $G(N)$ and the necessary premises P are minimal. To obtain minimality, another step is performed: proof pruning.

To analyze this process in detail, we first define minimality.

Definition 1.3 (Minimality)**.** Minimality is the property that $G(N)$ and P cannot be further pruned without losing the reachability of the conclusion.

To obtain minimality, a trial and error approach is taken: each subset of auxiliary points contained in a proof will be discarded alternately, and the symbolic engine DDAR will be tested on that smaller subsect of premises to verify reachability of the conclusion. After this process, the proof that uses the smallest set of points across all possible trials is returned as the final one.

This process grants that any unused auxiliary construction will be entirely discarded from the proof.

1.4 The Neural Language Model

The Neural Language Model is the second component that takes part in the solution of a problem, alongside the symbolic solver.

It is a transformer language model trained on 100 million proofs. These proofs, that appear in the dataset as the triplet $(P, N, G(N))$, are serialized into a text string of the form '`<premises><conclusion><proof>`', and by training on such sequence of symbols, the model effectively learns to generate the proof based on the theorem premises and conclusion. Having pretrained the model on these 100 million synthetic generated proofs, the model is fine-tuned on the subset of 9 million proofs that require auxiliary constructions, to help the model focus on this specific task: the suggestion of an adequate auxiliary construction that can help the symbolic engine to reach a conclusion.

Regarding hyperparameters, a detailed description can be found in the original paper. The only thing worth mentioning is that no hyperparameter tuning was performed, since the Meliad library[13] was used for transformer training with its base settings and most of the values were provided by default in the Meliad codebase. At the end of the training, the model has 151 million parameters, excluding embedding layers at its input and output heads. The customized tokenizer is trained with 'word' mode using SentencePiece[8], and has a vocabulary size of 757.

1.5 Proof search

We observed in detail how AlphaGeometry was created; now we will see how it works: we will analyze the proof searching process.

Proof search is a process that can be summarized as a loop in which the symbolic engine and the language model run in turns until the theorem is proven or the number of loops exceeds a maximum value.

When AlphaGeometry is tested on a new problem, the loop starts. First, the symbolic engine tries to solve it. Then, if the conclusion statement is not proven, the language model is called into action. The problem statement string and past constructions (if any) are fed into the language model which generates one extra sentence at each turn, describing a new auxiliary construction. Then the symbolic engine starts again its deduction process using the new input from the language model, and this process continues until the symbolic engine proves the conclusion statement.

There is an important fact that needs to be specified. The symbolic solver is created to "understand" geometry, working on logical deduction and algebraic reasoning. The language model, on the other hand, does not perform any geometric reasoning but more of a pattern-matching task based on the 100 million proofs it was trained on. In fact, being trained on text alone, it is not that good at suggesting the best possible geometric construction. This problem brings to light the necessity to analyze more outputs of the language model and choose the best one among them. So a beam search is implemented to explore the top k constructions generated.

Chapter 2

Problem Syntax & AlphaGeometry Solutions

In this chapter, we will study the problem solutions obtained by AlphaGeometry. First, we will observe how problems are translated into AlphaGeometry's syntax, and then we will compare different problems' solutions.

2.1 Problem Syntax

List of possible problem hypotheses

Here is a list of possible geometric constructions and their explanations. These are used to translate human-written problem statements into AlphaGeometry's language.

Construction	**Description**
X = angle_bisector(A, B, C)	Construct a point X on the angle bisector of $\angle ABC$
X = angle_mirror(A, B, C)	Construct a point X such that BC is the bisector of $\angle ABX$
X = circle(A, B, C)	Construct point X as the circumcenter of A, B, C
A, B, C, D = eq_quadrilateral()	Construct quadrilateral $ABCD$ with $AD = BC$
A, B, C, D = eq_trapezoid()	Construct trapezoid $ABCD$ with $AD = BC$
X = eqtriangle(B, C)	Construct X such that ABC is an equilateral triangle
X = equangle2(A, B, C)	Construct X such that $\angle BAX = \angle XCB$
A, B, C, D = eqdia_equadrilateral()	Construct quadrilateral $ABCD$ with $AC = BD$
X = eqdistance(A, B, C)	Construct X such that $XA = BC$
X = foot(A, B, C)	Construct X as the foot of A on BC

Continues on next page

Construction	Description
X = free	Construct a free point X
X = incenter(A, B, C)	Construct X as the incenter of ABC
X, Y, Z, I = incenter2(A, B, C)	Construct I as the incenter of ABC with touchpoints X, Y, Z
X = excenter(A, B, C)	Construct X as the excenter of ABC
X , Y, Z, I = excenter2(A, B, C)	Construct X as the excenter of ABC with touchpoints X, Y, Z
X = centroid(A,B,C)	Construct X as the centroid of ABC
X, Y, Z, I = midpointcircle(A, B, C)	Construct X, Y, Z as the midpoints of triangle ABC, and I as the circumcenter of XYZ
A, B, C = isos()	Construct A, B, C such that $AB = AC$
X = tangent(O, A)	Construct X such that OA is perpendicular to AX
X = midpoint(A, B)	Construct X as the midpoint of AB
X = mirror(A, B)	Construct X such that B is the midpoint of AX
X = rotate90(A, B)	Construct X such that AXB is a right isosceles triangle
X = on_aline(A, B, C, D, E)	Construct X such that $\angle XAB = \angle CDE$
X = on_bline(X, A, B)	Construct X on the perpendicular bisector of AB
X = on_circle(O, A)	Construct X such that $OA = OX$
X = on_line(A, B)	Construct X on line AB
X = on_pline(A, B, C)	Construct X such that XA is parallel to BC
X = on_tline(A, B, C)	Construct X such that XA is perpendicular to BC
X = orthocentrer(A, B, C)	Construct X as the orthocentre of ABC
X = parallelogram(A, B, C)	Construct X such that $ABCX$ is a parallelogram
A, B, C, D, E = pentagon()	Construct pentagon $ABCDE$
A, B, C, D = quadrilateral()	Construct quadrilateral $ABCD$
A, B, C, D = trapezoid()	Construct right trapezoid $ABCD$
A, B, C = r_triangle()	Construct right triangle ABC
A, B, C, D = rectangle()	Construct rectangle $ABCD$
X = reflect(A, B, C)	Construct X as the reflection of A about BC
A, B, C = risos()	Construct right isosceles triangle ABC
X = angle(A, B, α)	Construct X such that $\angle ABX = \alpha$
A, B = segment()	Construct two distinct points A, B
X = shift(B, C, D)	Construct point X such that $XB = CD$ and $XC = BD$
X, Y = square(A, B)	Construct X, Y such that $XYAB$ is a square
A, B, C, D = init_square()	Construct square $ABCD$
A, B, C, D = trapezoid()	Construct trapezoid $ABCD$

Continues on next page

Construction	Description
A, B, C = triangle()	Construct triangle ABC
A, B, C = triangle12()	Construct triangle ABC with $AB : AC = 1 : 2$
X, Y, Z, I = 2L1C(A, B, C, O)	Construct circle center I that touches line AC and line BC and circle (O, A) at X, Y, Z
X, Y, Z = 3PEQ(A, B, C)	Construct X, Y, Z on three sides of triangle ABC such that Y is the midpoint of XZ
X, Y = trisect(A, B, C)	Construct X, Y on AC such that BX and BY trisect $\angle ABC$
X, Y = trisegment(A, B)	Construct X, Y on segment AB such that $AX = XY = YB$
X = on_dia(A, B)	Construct point X such that AX is perpendicular to BX
A, B, C = iectriangle()	Construct equilateral triangle ABC
X, Y, Z, T = cc_tangent(O, A, W, B)	Construct common tangents of circles (O, A) and (W, B) with touchpoints X, Y for one tangent and Z, T for the other.
X = eqangle3(A, B, D, E, F)	Construct point X such that $\angle AXB = \angle EDF$
X, Y = tangent(A, O, B)	Construct points X, Y as the tangent touch points from A to circle (O, B)

Table 2.1: List of possible geometric constructions.[16]

There are other possible constructions, but these above are the most used.

During the definition of a construction, the variable name can be repeated, e.g.:

```
a b c = triangle;                  a b c = triangle a b c;
x = incenter a b c;                x = incenter x a b c;
x y z i = incenter2 a b c;         x y z i = incenter2 x y z i a b c;
```

The elements in the left column equal the ones in the right column.

List of possible problem thesis

Problems are translated using the constructions listed above, and the thesis predicate is written after a question mark; then the translated problem is inserted into a text document.

Remark 2.1. It is essential to respect spacing and punctuation exactly; the syntax is sensitive to token order.

The problem thesis can be one of these predicates:

Predicates	Description
coll a b c	A, B, C are collinear
cong o a o b	$OA = OB$
cyclic a b c d	A, B, C, D are on the same circle
eqangle a b c d m n p q	the angle between AB and CD is equal to the angle between MN and PQ
eqratio m a a b n c c d	$MA : AB = NC : CD$
midp e a b	E is the midpoint of AB
para a b c d	$AB \parallel CD$
perp a b c d	$AB \perp CD$
simtri a b c p q r	$\triangle ABC$ is similiar to $\triangle PQR$
contri a b c p q r	$\triangle ABC$ is congruent to $\triangle PQR$

Table 2.2: List of possible predicates.[3]

A simple problem example

This is the human-written version of the well-known Euler's Line Theorem.

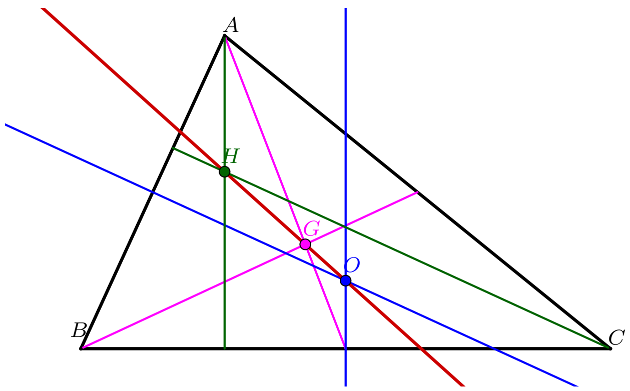

Figure 2.1: *Euler's Line (in red).*

Theorem 2.2 (Euler's Line). Let ABC be a non-equilateral triangle. Consider G its centroid, H its orthocenter, and O its circumcenter. Then H, G, O are collinear.

The following is a translation into AlphaGeometry's syntax:

```
a b c = triangle a b c; h = orthocenter a b c; g = centroid g a b c;
o = circle a b c ? coll h g o
```

The variable will automatically be renamed in alphabetical order by AlphaGeometry, based on their original order of appearance in the translated statement. *From now on, in every translated statement, its variables will be already named in alphabetic order.*

2.2 Problem Solutions

Having analyzed the syntax, we can proceed to analyze a few solutions produced by AlphaGeometry. In particular, since AlphaGeometry was tested on International Math Olympiad's problems, we will observe how it solves problems belonging to different levels of Math Olympiads, specifically on some Italian Distrectual-level and National-level problems of the recent years. We will also test AlphaGeometry on the 2024 IMO's geometry problem, since the model was tested only on previous years problems.

Remark 2.3. In AlphaGeometry's proofs, sometimes (SSS), (SAS) or (ASA) could appear. These acronyms refer to the Triangle Congruence Theorems.

2.2.1 Distrectual-Level Problem

Original statement

Problem 2.4 (Gara di Febbraio 2025 – problem 16). Let ABC be an acute triangle. Let H be the foot of the altitude from B to AC, and let K be the foot of the altitude from C to AB. Let E be the reflection of K with respect to line BC, and let P be the intersection between EH and BC.

(a) Show that the quadrilateral $CEBH$ can be inscribed in a circle.

(b) Show that the triangle PBE is similar to triangle ABC.

(c) Show that the line AP is perpendicular to the line BC.

Translated statement

```
a b c = triangle a b c; d = foot d b a c; e = foot e c a b;
f = reflect f e b c; g = on_line g f d, on_line g b c ? perp a g b c
```

In this type of competition, problems are almost beginner level, so the first two questions were just intermediate steps to prove the third one. That is the reason we tested AlphaGeometry directly on the third question.

Official Solution[7]

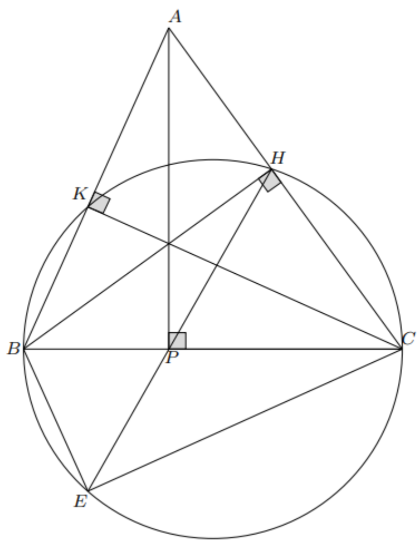

Figure 2.2: *Gara di Febbraio 2025, problem 16. Original figure from the solution paper.*[7]

Proof. Since ABC is acute-angled, the points K, H lie on the segments AB, AC respectively.

(a) By definition of H, K, the angles $\angle BKC, \angle BHC$ are right angles. Since E is obtained reflecting K with respect to BC, we have $BE = BK$ and $EC = KC$. Therefore, by the third congruence criterion, the triangles $\triangle BCE$ and $\triangle BCK$ are congruent and, in particular, $\angle BEC = \angle BKC = 90°$. Hence, the quadrilateral $CEBH$ has supplementary opposite angles and is therefore cyclic. Since also $\angle BKC = 90°$, the pentagon $CEBKH$ is cyclic as well.

(b) Since the triangles $\triangle BCE$ and $\triangle BCK$ are congruent, we have $\angle KBC = \angle CBE$. Moreover, by the cyclicity of quadrilateral $BECH$, we have $\angle BEH = \angle BCH$, since they subtend the same chord BH. The triangles $\triangle PBE$ and $\triangle ABC$ are then similar by the second similarity criterion.

(c) For point (b), we have $\angle BPE = \angle BAC$ and therefore $\angle BPH = 180° - \angle BAC$. The quadrilateral $BPHA$ has supplementary opposite angles and is therefore cyclic.

We observe that $\angle BPA = \angle BHA$, since they both subtend the same chord AB, but $\angle BHA = 90°$ because BH is an altitude of ABC. This concludes the proof. □

AlphaGeometry's solution

Theorem premises

Points: $A\ B\ C\ D\ E\ F\ G$

[00]	D, C, A are collinear	[04]	$BE = BF$
[01]	$BD \perp AC$	[05]	$CE = CF$
[02]	B, A, E are collinear	[06]	B, C, G are collinear
[03]	$CE \perp AB$	[07]	F, D, G are collinear

Auxiliary construction

No auxiliary construction was needed for the solution of this problem.

Proof steps

1. $BE = BF$ [04] & $CE = CF$ [05] (SSS)⇒ $\angle EBC = \angle CBF$ [08]
2. $BE = BF$ [04] & $CE = CF$ [05] (SSS)⇒ $\angle BEC = \angle CFB$ [09]
3. B, A, E are collinear [02] & B, C, G are collinear [06] & $\angle EBC = \angle CBF$ [08] ⇒ $\angle EBG = \angle GBF$ [10]
4. $BE = BF$ [04] & $\angle EBG = \angle GBF$ [10] (SAS)⇒ $\angle BEG = \angle GFB$ [11]
5. D, C, A are collinear [00] & $AC \perp BD$ [01] ⇒ $DC \perp BD$ [12]
6. $CE \perp AB$ [03] & $DC \perp BD$ [12] ⇒ $\angle(CE - BD) = \angle(BA - DC)$ [13]
7. $CE \perp AB$ [03] & $DC \perp BD$ [12] ⇒ $\angle ECD = \angle ABD$ [14]
8. D, C, A are collinear [00] & $\angle(CE - BD) = \angle(BA - DC)$ [13] & $\angle BEC = \angle CFB$ [09] & B, A, E are collinear [02] ⇒ $\angle BFC = \angle BDC$ [15]
9. $\angle BFC = \angle BDC$ [15] ⇒ B, D, C, F are concyclic [16]
10. B, D, C, F are concyclic [16] ⇒ $\angle BCD = \angle BFD$ [17]
11. B, C, G are collinear [06] & B, A, E are collinear [02] & $\angle BEG = \angle GFB$ [11] & F, D, G are collinear [07] & $\angle BCD = \angle BFD$ [17] & D, C, A are collinear [00] ⇒ $\angle ACG = \angle AEG$ [18]

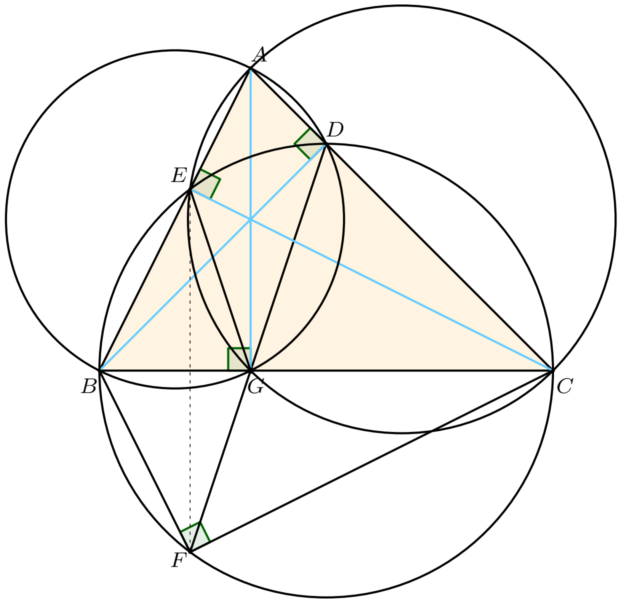

Figure 2.3: *Gara di Febbraio 2025, problem 16. AlphaGeometry's solution.*

12. $\angle ACG = \angle AEG$ [18] $\Rightarrow$ G, C, A, E are concyclic [19]

13. G, C, A, E are concyclic [19] $\Rightarrow$ $\angle GCE = \angle GAE$ [20]

14. B, C, G are collinear [06] & F, D, G are collinear [07] & D, C, A are collinear [00] & $\angle BCD = \angle BFD$ [17] & $\angle EBC = \angle CBF$ [08] & B, A, E are collinear [02] $\Rightarrow$ $\angle GBA = \angle GDA$ [21]

15. $\angle GBA = \angle GDA$ [21] $\Rightarrow$ B, D, A, G are concyclic [22]

16. B, D, A, G are concyclic [22] $\Rightarrow$ $\angle BDA = \angle BGA$ [23]

17. $\angle GCE = \angle GAE$ [20] & B, C, G are collinear [06] & B, A, E are collinear [02] & $\angle ECD = \angle ABD$ [14] & D, C, A are collinear [00] & $\angle BDA = \angle BGA$ [23] $\Rightarrow$ $BC \perp AG$ □

Comments and Differences

Here are the main differences between the human solution and AlphaGeometry's one.

Problem statement: As mentioned before, we tested AlphaGeometry only on the third question of the original problem.

Proof length: AlphaGeometry's proof is even shorter than the original one, and, while question (a) was proven as an intermediate step (step [16]), question (b) was not proven as part of the proof. This does not indicate an error in AlphaGeometry's approach, which is valid although different.

Auxiliary constructions: No auxiliary construction was needed, so this means that the problem was solved by the symbolic engine on the first attempt, without any intervention of the neural language model.

2.2.2 National-level Problem

Original statement

Problem 2.5 (Cesenatico 2021 – problem 2). Let ABC be a triangle, and let I be the center of its incircle. Let D be the reflection of I with respect to side AB, and let E be the reflection of I with respect to side AC. Prove that the circumcircles of triangles BID and CIE are tangent to each other.

Translated statement

```
a b c = triangle a b c; d = incenter d a b c;
e = reflect e d a b; f = reflect f d a c;
g = circle g b d e; h = circle h c d f  ? coll g d h
```

Official Solution[9]

Proof. If the circles circumscribed about triangles $\triangle BID$ and $\triangle CIE$ are tangent, then the point of tangency is necessarily I. Denoting by O_1 and O_2 the circumcenters of $\triangle BID$ and $\triangle CIE$, the thesis holds if and only if the points O_1, I, and O_2 are collinear.

Let α, β, γ denote the measures of the angles in A, B, and C, respectively. Triangle $\triangle BDI$ is isosceles, so O_1 lies on the perpendicular bisector of DI, that is, on line AB. The segments O_1I and O_1B are equal as radii, hence triangle $\triangle O_1IB$ is isosceles. Moreover, line BI is the bisector of angle at B. It follows that

$$\angle O_1IB = \angle IBO_1 = \angle IBC = \frac{\beta}{2},$$

and therefore lines BC and O_1I are parallel, since they form equal alternate interior angles with transversal IB. In an analogous manner, one shows that line O_2I is parallel to BC.

But then O_1I and O_2I are parallel lines passing through I, hence they coincide; that is, the points O_1, I, O_2 are collinear, from which the thesis follows. □

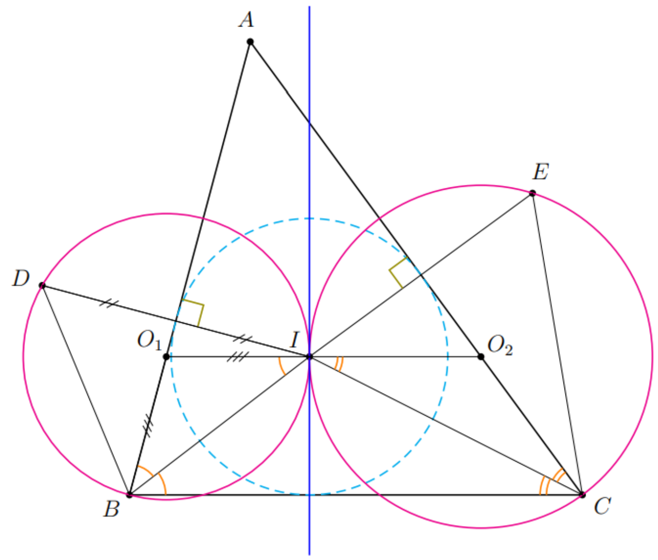

Figure 2.4: *Cesenatico 2021, problem 2. Original figure in the solution paper.*[9]

AlphaGeometry's solution

Theorem premises

Points: A B C D E F G H

[00]	$\angle DCA = \angle BCD$	[05]	$CD = CF$
[01]	$\angle BAD = \angle DAC$	[06]	$GB = GD$
[02]	$AD = AE$	[07]	$GD = GE$
[03]	$BD = BE$	[08]	$HD = HF$
[04]	$AD = AF$	[09]	$HC = HD$

Auxiliary construction

```
i = on_circle i c a, on_circle i d a
```

Point I such that $DI = DA$ [10] and $CI = CA$ [11]

Proof steps

1. $GB = GD$ [06] $\Rightarrow \angle DBG = \angle GDB$ [12]

2. $DI = DA$ [10] & $CI = CA$ [11] (SSS)$\Rightarrow \angle ICD = \angle DCA$ [13]

3. $DI = DA$ [10] & $CI = CA$ [11] (SSS)$\Rightarrow \angle(AC - DI) = \angle(AD - CI)$ [14]

4. $\angle ICD = \angle DCA$ [13] & $\angle DCA = \angle BCD$ [00] $\Rightarrow \angle BCD = \angle ICD$ [15]

5. $\angle BCD = \angle ICD$ [15] $\Rightarrow CB \parallel IC$ [16]

6. $BC \parallel CI$ [16] $\Rightarrow I, C, B$ are collinear [17]

7. $AD = AF$ [04] & $HD = HF$ [08] $\Rightarrow DF \perp AH$ [18]

8. $AD = AF$ [04] & $CD = CF$ [05] $\Rightarrow DF \perp AC$ [19]

9. $AD = AE$ [02] & $GD = GE$ [07] $\Rightarrow DE \perp AG$ [20]

10. $AD = AE$ [02] & $BD = BE$ [03] $\Rightarrow DE \perp AB$ [21]

11. $DF \perp AH$ [18] & $DF \perp AC$ [19] & $DE \perp AG$ [20] & $DE \perp AB$ [21] & $\angle DAC = \angle BAD$ [01] $\Rightarrow \angle DAH = \angle GAD$ [22]

12. $DF \perp AH$ [18] & $DF \perp AC$ [19] & $\angle(AC - DI) = \angle(AD - CI)$ [14] $\Rightarrow \angle(AH - ID) = \angle(AD - IC)$ [23]

13. $\angle DAH = \angle GAD$ [22] & $\angle(AH - ID) = \angle(AD - IC)$ [23] $\Rightarrow \angle ADI = \angle(AG - IC)$ [24]

14. I, C, B are collinear [17] & $\angle ADI = \angle(AG - IC)$ [24] & $DE \perp AG$ [20] & $DE \perp AB$ [21] & $CI \parallel BC$ [16] $\Rightarrow \angle DAB = \angle DIB$ [25]

15. $\angle DAB = \angle DIB$ [25] $\Rightarrow A, D, I, B$ are concyclic [26]

16. A, D, I, B are concyclic [26] $\Rightarrow \angle AID = \angle ABD$ [27]

17. A, D, I, B are concyclic [26] $\Rightarrow \angle ADB = \angle AIB$ [28]

18. $DI = DA$ [10] $\Rightarrow \angle DIA = \angle IAD$ [29]

19. $HC = HD$ [09] (SSS)$\Rightarrow \angle HCD = \angle CDH$ [30]

20. $\angle HCD = \angle CDH$ [30] & $DF \perp AH$ [18] & $DF \perp AC$ [19] & $\angle DCA = \angle BCD$ [00] $\Rightarrow \angle BCD = \angle HDC$ [31]

21. $\angle BCD = \angle HDC$ [31] $\Rightarrow CB \parallel HD$ [32]

22. $\angle DBG = \angle GDB$ [12] & $\angle AID = \angle ABD$ [27] & $DE \perp AG$ [20] & $DE \perp AB$ [21] & $\angle DIA = \angle IAD$ [29] & $\angle ADB = \angle AIB$ [28] & I, C, B are collinear [17] & $BC \parallel DH$ [32] $\Rightarrow \angle HDB = \angle GDB$ [33]

23. $\angle HDB = \angle GDB$ [33] $\Rightarrow HD \parallel GD$ [34]

24. $DH \parallel DG$ [34] $\Rightarrow H, G, D$ are collinear □

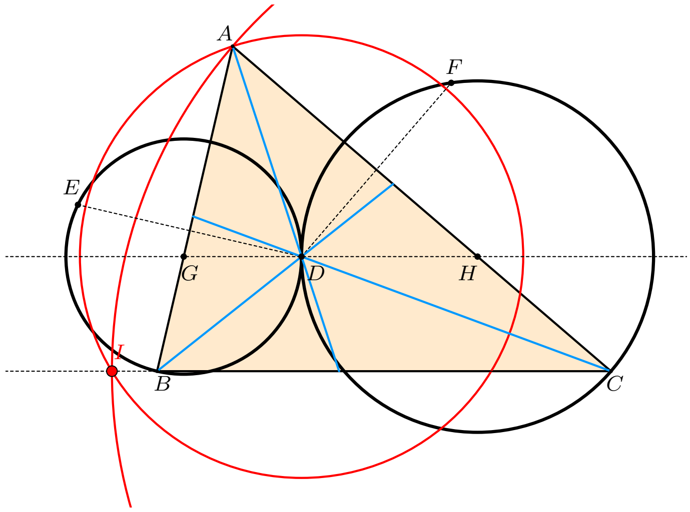

Figure 2.5: *Cesenatico 2021, problem 2. AlphaGeometry's solution.*

Comments and Differences

Here are the main differences between the human solution and AlphaGeometry's one.

Problem statement: The only noticeable difference between the original statement and the translated one lies in the statement of the proof's conclusion.

Since tangency between two circles is not a predicate in AlphaGeometry's syntax, the condition was translated into the equivalent geometric condition: the collinearity of the two circles' centers and their point of tangency, which is the common point D (the incenter).

Proof length: AlphaGeometry's proof length is quite similar to the original proof's length, and both of them are quite easy to read.

Auxiliary constructions: In the original paper the only auxiliary construction is the common tangent of the two circles. In AlphaGeometry's solution, the model creates the point I, since the symbolic engine does not arrive at the solution by itself.

The auxiliary point I introduced by the model is an unconventional but valid construction. I ran the test more times, gradually increasing the beam search size and depth, but the added auxiliary point continued to be the same.

2.2.3 IMO-level Problem

Original statement

Problem 2.6 (IMO 2024 – problem 4). Let ABC be a triangle with $AB < AC < BC$. Let the incenter and incircle of triangle ABC be I and ω, respectively. Let X be the point on line BC different from C such that the line through X parallel to AC is tangent to ω. Similarly, let Y be the point on line BC different from B such that the line through Y parallel to AB is tangent to ω. Let AI intersect the circumcircle of triangle ABC again at $P \neq A$. Let K and L be the midpoints of AC and AB, respectively. Prove that $\angle KIL + \angle YPX = 180°$.

Translated statement

```
a b c = triangle a b c; d = incenter d a b c;
e = foot e d a c; f = foot f d a b;
g = on_line g d e, on_circle g d e;
h = on_line h d f, on_circle h d f;
i = on_line i b c, on_pline i g a c;
j = on_line j b c, on_pline j h a b;
k = midpoint k a c; l = midpoint l a b;
m = on_line m a d, on_circum m a b c
? eqangle m j m i d l d k
```

Official Solution[6]

Proof. Let A_1 be the reflection of A in I, then A_1 lies on the angle bisector AP. Lines A_1X and A_1Y are the reflections of AC and AB in I, respectively, and so they are the tangents to ω from X and Y. As is well-known, $PB = PC = PI$, and since $\angle BAP = \angle PAC > 30°$, $PB = PC$ is greater than the circumradius. Hence $PI > \frac{1}{2}AP > AI$; we conclude that A_1 lies in the interior of segment AP.

We have $\angle APB = \angle ACB$ in the circumcircle and $\angle ACB = \angle A_1XC$ because $A_1X \parallel AC$. Hence, $\angle APB = \angle A_1XC$, and so quadrilateral BPA_1X is cyclic. Similarly, it follows that CYA_1P is cyclic.

Now we are ready to transform $\angle KIL + \angle YPX$ to the sum of angles in triangle A_1CB. By a homothety of factor 2 at A we have $\angle KIL = \angle CA_1B$. In circles BPA_1X

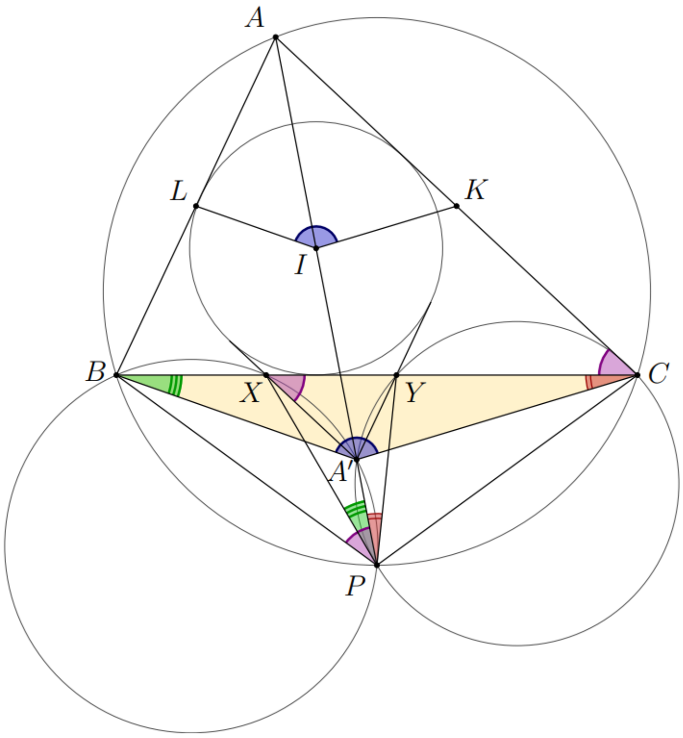

Figure 2.6: *IMO 2024, problem 4. Original figure in the solution paper.*[6]

and CYA_1P we have $\angle APX = \angle A_1BC$ and $\angle YPA = \angle BCA_1$, therefore

$$\angle KIL + \angle YPX = \angle CA_1B + (\angle YPA + \angle APX) = \angle CA_1B + \angle BCA_1 + \angle A_1BC = 180°.$$

□

AlphaGeometry's Solution

Theorem premises

Points: $A\ B\ C\ D\ E\ F\ G\ H\ I\ J\ K\ L\ M$

[00] $\angle BAD = \angle DAC$
[01] $\angle ACD = \angle DCB$
[02] A, C, E are collinear
[03] $DE \perp AC$
[04] A, F, B are collinear
[05] $DF \perp AB$
[06] $DG = DE$
[07] G, D, E are collinear
[08] $DH = DF$
[09] H, D, F are collinear
[10] I, B, C are collinear
[11] $IG \parallel AC$
[12] B, C, J are collinear
[13] $JH \parallel AB$
[14] A, K, C are collinear
[15] $KA = KC$
[16] L, A, B are collinear
[17] $LA = LB$
[18] A, M, D are collinear
[19] A, M, C, B are concyclic

Auxiliary construction

```
n = on_line n b c, on_tline n d b c
```

Point N such that $DN \perp BC$ [20] and B, C, N are collinear [21]

Proof steps

1. A, C, E are collinear [02] & A, B, F are collinear [04] & $DF \perp AB$ [05] & $DE \perp AC$ [03] $\Rightarrow \angle AED = \angle AFD$ [22]
2. $\angle AED = \angle AFD$ [22] $\Rightarrow A, D, E, F$ are concyclic [23]
3. A, D, E, F are concyclic [23] $\Rightarrow \angle ADE = \angle AFE$ [24]
4. $\angle ADE = \angle AFE$ [24] & A, F, B are collinear [04] $\Rightarrow \angle ADE = \angle(AB - EF)$ [25]
5. A, C, E are collinear [02] & B, N, C are collinear [21] & $DN \perp BC$ [20] & $DE \perp AC$ [03] $\Rightarrow \angle CED = \angle CND$ [26]
6. $\angle CED = \angle CND$ [26] $\Rightarrow N, C, D, E$ are concyclic [27]
7. N, C, D, E are concyclic [27] $\Rightarrow \angle NDC = \angle NEC$ [28]
8. N, C, D, E are concyclic [27] $\Rightarrow \angle NCD = \angle NED$ [29]
9. $\angle NDC = \angle NEC$ [28] & A, C, E are collinear [02] $\Rightarrow \angle NDC = \angle(EN - AC)$ [30]
10. A, B, F are collinear [04] & A, C, E are collinear [02] & $DF \perp AB$ [05] & $DE \perp AC$ [03] $\Rightarrow \angle DFA = \angle AED$ [31]
11. A, B, F are collinear [04] & A, C, E are collinear [02] & $\angle DAB = \angle CAD$ [00] $\Rightarrow$ $\angle DAF = \angle EAD$ [32]
12. $\angle DFA = \angle AED$ [31] & $\angle DAF = \angle EAD$ [32] (Similar Triangles)$\Rightarrow DF = DE$ [33]
13. $\angle DFA = \angle AED$ [31] & $\angle DAF = \angle EAD$ [32] (Similar Triangles)$\Rightarrow AF = AE$ [34]
14. B, N, C are collinear [21] & A, C, E are collinear [02] & $DN \perp BC$ [20] & $DE \perp AC$ [03] $\Rightarrow \angle DNC = \angle CED$ [35]
15. B, C, N are collinear [21] & A, C, E are collinear [02] & $\angle DCB = \angle ACD$ [01] $\Rightarrow$ $\angle DCN = \angle ECD$ [36]

16. $\angle DNC = \angle CED$ [35] & $\angle DCN = \angle ECD$ [36] (Similar Triangles)$\Rightarrow$ $DN = DE$ [37]

17. $\angle DNC = \angle CED$ [35] & $\angle DCN = \angle ECD$ [36] (Similar Triangles)$\Rightarrow$ $CN = CE$ [38]

18. $DH = DF$ [08] & $DF = DE$ [33] & $DN = DE$ [37] & $DG = DE$ [06] $\Rightarrow$ G, N, H, E are concyclic [39]

19. $DH = DF$ [08] & $DF = DE$ [33] & $DN = DE$ [37] & $DG = DE$ [06] $\Rightarrow$ D is the circumcenter of ΔGNH [40]

20. G, H, E, N are concyclic [39] $\Rightarrow$ $\angle GEH = \angle GNH$ [41]

21. G, H, E, N are concyclic [39] $\Rightarrow$ $\angle GHE = \angle GNE$ [42]

22. G, H, E, N are concyclic [39] $\Rightarrow$ $\angle GHN = \angle GEN$ [43]

23. $\angle GEH = \angle GNH$ [41] & G, D, E are collinear [07] $\Rightarrow$ $\angle DEH = \angle GNH$ [44]

24. G, D, E are collinear [07] & $DG = DE$ [06] $\Rightarrow$ D is midpoint of GE [45]

25. H, D, F are collinear [09] & $DH = DF$ [08] $\Rightarrow$ D is midpoint of HF [46]

26. D is midpoint of GE [45] & D is midpoint of HF [46] $\Rightarrow$ $GH \parallel EF$ [47]

27. D is midpoint of GE [45] & D is midpoint of HF [46] $\Rightarrow$ $GF \parallel EH$ [48]

28. $\angle GHE = \angle GNE$ [42] & $GH \parallel EF$ [47] $\Rightarrow$ $\angle FEH = \angle GNE$ [49]

29. A, M, C, B are concyclic [19] $\Rightarrow$ $\angle AMC = \angle ABC$ [50]

30. A, M, C, B are concyclic [19] $\Rightarrow$ $\angle CAB = \angle CMB$ [51]

31. A, M, C, B are concyclic [19] $\Rightarrow$ $\angle AMB = \angle ACB$ [52]

32. B, N, C are collinear [21] & B, A, F are collinear [04] & A, M, D are collinear [18] & $\angle AMC = \angle ABC$ [50] $\Rightarrow$ $\angle NBF = \angle CMD$ [53]

33. $DN = DE$ [37] & $DG = DE$ [06] & $DF = DE$ [33] $\Rightarrow$ D is the circumcenter of ΔGNF [54]

34. $DN = DE$ [37] & $DG = DE$ [06] & $DF = DE$ [33] $\Rightarrow$ G, N, E, F are concyclic [55]

35. L, A, B are collinear [16] & A, F, B are collinear [04] & $DF \perp AB$ [05] $\Rightarrow DF \perp FL$ [56]

36. D is the circumcenter of ΔGNF [54] & $DF \perp FL$ [56] $\Rightarrow \angle LFG = \angle FNG$ [57]

37. $DN = DE$ [37] & $CN = CE$ [38] $\Rightarrow NE \perp DC$ [58]

38. $DN = DE$ [37] & $DG = DE$ [06] $\Rightarrow D$ is the circumcenter of ΔGNE [59]

39. D is the circumcenter of ΔGNE [59] & G, D, E are collinear [07] $\Rightarrow GN \perp EN$ [60]

40. $DF = DE$ [33] & $AF = AE$ [34] $\Rightarrow FE \perp DA$ [61]

41. $DH = DF$ [08] & $DF = DE$ [33] & $DG = DE$ [06] $\Rightarrow D$ is the circumcenter of ΔGHE [62]

42. D is the circumcenter of ΔGHE [62] & G, D, E are collinear [07] $\Rightarrow GH \perp EH$ [63]

43. B, A, F are collinear [04] & A, M, D are collinear [18] & $\angle LFG = \angle FNG$ [57] & L, A, B are collinear [16] & $FG \parallel EH$ [48] & $NE \perp DC$ [58] & $GN \perp EN$ [60] & $FE \perp DA$ [61] & $GH \perp EH$ [63] & $GH \parallel EF$ [47] $\Rightarrow \angle NFB = \angle CDM$ [64]

44. $\angle NBF = \angle CMD$ [53] & $\angle NFB = \angle CDM$ [64] (Similar Triangles)$\Rightarrow FN : FB = DC : DM$ [65]

45. $\angle NBF = \angle CMD$ [53] & $\angle NFB = \angle CDM$ [64] (Similar Triangles)$\Rightarrow NB : NF = CM : CD$ [66]

46. $\angle NBF = \angle CMD$ [53] & $\angle NFB = \angle CDM$ [64] (Similar Triangles)$\Rightarrow BN : MC = BF : MD$ [67]

47. G, N, E, F are concyclic [55] $\Rightarrow \angle GEN = \angle GFN$ [68]

48. B, C, J are collinear [12] & $\angle GEN = \angle GFN$ [68] & G, D, E are collinear [07] & $FG \parallel EH$ [48] & $\angle NCD = \angle NED$ [29] & B, C, N are collinear [21] $\Rightarrow \angle DCJ = \angle GFN$ [69]

49. G, N, H, E are concyclic [39] & G, N, E, F are concyclic [55] $\Rightarrow G, H, E, F$ are concyclic [70]

50. G, N, E, F are concyclic [55] & G, H, E, F are concyclic [70] $\Rightarrow G, N, H, F$ are concyclic [71]

51. G, H, N, F are concyclic [71] $\Rightarrow \angle GNH = \angle GFH$ [72]

52. G, H, N, F are concyclic [71] $\Rightarrow \angle GHF = \angle GNF$ [73]

53. B, N, C are collinear [21] & B, C, J are collinear [12] & H, D, F are collinear [09] & $DF \perp AB$ [05] & $DN \perp BC$ [20] & $AB \parallel HJ$ [13] $\Rightarrow \angle DNJ = \angle DHJ$ [74]

54. $\angle DNJ = \angle DHJ$ [74] $\Rightarrow N, H, D, J$ are concyclic [75]

55. N, H, D, J are concyclic [75] $\Rightarrow \angle NHD = \angle NJD$ [76]

56. N, H, D, J are concyclic [75] $\Rightarrow \angle HND = \angle HJD$ [77]

57. B, C, J are collinear [12] & $\angle GNH = \angle GFH$ [72] & H, D, F are collinear [09] & $FG \parallel EH$ [48] & $\angle NHD = \angle NJD$ [76] & B, C, N are collinear [21] $\Rightarrow \angle DJC = \angle FGN$ [78]

58. $\angle DCJ = \angle GFN$ [69] & $\angle DJC = \angle FGN$ [78] (Similar Triangles)$\Rightarrow CD : DJ = NF : GN$ [79]

59. $MD : CD = BF : NF$ [65] & $CD : DJ = NF : GN$ [79] $\Rightarrow MD : DJ = BF : GN$ [80]

60. B, A, F are collinear [04] & B, N, C are collinear [21] & $DF \perp AB$ [05] & $DN \perp BC$ [20] $\Rightarrow \angle BFD = \angle BND$ [81]

61. $\angle BFD = \angle BND$ [81] $\Rightarrow B, N, D, F$ are concyclic [82]

62. B, N, C are collinear [21] & I, B, C are collinear [10] & $DN \perp BC$ [20] $\Rightarrow DN \perp NI$ [83]

63. D is the circumcenter of ΔGNF [54] & $DN \perp NI$ [83] $\Rightarrow \angle ING = \angle NFG$ [84]

64. B, N, D, F are concyclic [82] $\Rightarrow \angle BDN = \angle BFN$ [85]

65. B, N, D, F are concyclic [82] $\Rightarrow \angle BNF = \angle BDF$ [86]

66. B, N, C are collinear [21] & $\angle ING = \angle NFG$ [84] & I, B, C are collinear [10] & $FG \parallel EH$ [48] & $\angle LFG = \angle FNG$ [57] & L, A, B are collinear [16] & A, F, B are collinear [04] & $\angle BDN = \angle BFN$ [85] $\Rightarrow \angle BDN = \angle FNB$ [87]

67. B, N, D, F are concyclic [82] & $\angle BDN = \angle FNB$ [87] $\Rightarrow BN = FB$ [88]

68. $MD : DJ = BF : GN$ [80] & $BN = FB$ [88] $\Rightarrow MD : DJ = BN : GN$ [89]

69. $DH = DF$ [08] & $DF = DE$ [33] & $DN = DE$ [37] $\Rightarrow$ $DN = DH$ [90]

70. $DH = DF$ [08] & $DF = DE$ [33] & $DN = DE$ [37] $\Rightarrow$ D is the circumcenter of ΔNHF [91]

71. $DH = DF$ [08] & $DF = DE$ [33] & $DN = DE$ [37] $\Rightarrow$ D is the circumcenter of ΔNHE [92]

72. $DN = DH$ [90] $\Rightarrow$ $\angle DNH = \angle NHD$ [93]

73. D is the circumcenter of ΔNHF [91] & $DF \perp FL$ [56] $\Rightarrow$ $\angle LFN = \angle FHN$ [94]

74. L, A, B are collinear [16] & $\angle HND = \angle HJD$ [77] & $HJ \parallel AB$ [13] & $\angle DNH = \angle NHD$ [93] & H, D, F are collinear [09] & $\angle LFN = \angle FHN$ [94] & A, F, B are collinear [04] $\Rightarrow$ $\angle(NF - LA) = \angle(DJ - LA)$ [95]

75. $\angle(NF - LA) = \angle(DJ - LA)$ [95] $\Rightarrow$ $NF \parallel DJ$ [96]

76. B, N, C are collinear [21] & A, M, D are collinear [18] & $\angle ING = \angle NFG$ [84] & I, B, C are collinear [10] & $FG \parallel EH$ [48] & $FN \parallel DJ$ [96] & $FE \perp DA$ [61] & $GH \perp EH$ [63] & $GH \parallel EF$ [47] $\Rightarrow$ $\angle BNG = \angle JDM$ [97]

77. $MD : DJ = BN : GN$ [89] & $\angle BNG = \angle JDM$ [97] (Similar Triangles)$\Rightarrow$ $\angle NBG = \angle JMD$ [98]

78. $\angle NBG = \angle JMD$ [98] & B, C, N are collinear [21] & A, M, D are collinear [18] $\Rightarrow$ $\angle CBG = \angle(JM - AD)$ [99]

79. G, D, E are collinear [07] & B, N, C are collinear [21] & I, B, C are collinear [10] & $DN \perp BC$ [20] & $DE \perp AC$ [03] & $AC \parallel GI$ [11] $\Rightarrow$ $\angle DGI = \angle DNI$ [100]

80. $\angle DGI = \angle DNI$ [100] $\Rightarrow$ I, G, N, D are concyclic [101]

81. I, G, N, D are concyclic [101] $\Rightarrow$ $\angle ING = \angle IDG$ [102]

82. I, B, C are collinear [10] & $\angle ING = \angle IDG$ [102] & B, N, C are collinear [21] & G, D, E are collinear [07] & $\angle GEH = \angle GNH$ [41] $\Rightarrow$ $\angle NHE = \angle BID$ [103]

83. G, N, H, E are concyclic [39] & G, N, H, F are concyclic [71] $\Rightarrow$ H, E, N, F are concyclic [104]

84. H, E, N, F are concyclic [104] $\Rightarrow$ $\angle HEN = \angle HFN$ [105]

85. I, B, C are collinear [10] & $\angle BNF = \angle BDF$ [86] & B, C, N are collinear [21] & $\angle HEN = \angle HFN$ [105] & H, D, F are collinear [09] $\Rightarrow \angle NEH = \angle IBD$ [106]

86. $\angle NHE = \angle BID$ [103] & $\angle NEH = \angle IBD$ [106] (Similar Triangles)$\Rightarrow HN : NE = ID : BD$ [107]

87. $DN = DE$ [37] & $DG = DE$ [06] & $CE = CN$ [38] $\Rightarrow DG : DN = CE : CN$ [108]

88. G, D, E are collinear [07] & $DE \perp AC$ [03] & $AC \parallel GI$ [11] $\Rightarrow IG \perp GD$ [109]

89. I, B, C are collinear [10] & $DN \perp BC$ [20] $\Rightarrow ND \perp IB$ [110]

90. $IG \perp GD$ [109] & $ND \perp IB$ [110] $\Rightarrow \angle GIB = \angle GDN$ [111]

91. G, D, E are collinear [07] & A, C, E are collinear [02] & B, C, N are collinear [21] & $\angle GIB = \angle GDN$ [111] & I, B, C are collinear [10] & $GI \parallel AC$ [11] $\Rightarrow \angle GDN = \angle ECN$ [112]

92. $DG : DN = CE : CN$ [108] & $\angle GDN = \angle ECN$ [112] (Similar Triangles)$\Rightarrow GD : EC = GN : EN$ [113]

93. $DG : DN = CE : CN$ [108] & $\angle GDN = \angle ECN$ [112] (Similar Triangles)$\Rightarrow GD : GN = EC : EN$ [114]

94. $DG : DN = CE : CN$ [108] & $\angle GDN = \angle ECN$ [112] (Similar Triangles)$\Rightarrow \angle GND = \angle ENC$ [115]

95. $\angle GHN = \angle GEN$ [43] & G, D, E are collinear [07] & $GH \parallel EF$ [47] & $\angle NCD = \angle NED$ [29] & B, C, N are collinear [21] $\Rightarrow \angle DCB = \angle GHN$ [116]

96. $\angle GHF = \angle GNF$ [73] & H, D, F are collinear [09] & $GH \parallel EF$ [47] & $\angle BNF = \angle BDF$ [86] & B, C, N are collinear [21] $\Rightarrow \angle DBC = \angle HGN$ [117]

97. $\angle DCB = \angle GHN$ [116] & $\angle DBC = \angle HGN$ [117] (Similar Triangles)$\Rightarrow GN : BD = GH : BC$ [118]

98. A, M, D are collinear [18] & $\angle DAC = \angle BAD$ [00] $\Rightarrow \angle MAC = \angle BAM$ [119]

99. A, M, C, B are concyclic [19] & $\angle MAC = \angle BAM$ [119] $\Rightarrow MB = MC$ [120]

100. $DH = DF$ [08] & $DF = DE$ [33] & $DG = DE$ [06] & $MB = MC$ [120] $\Rightarrow DG : DH = MB : MC$ [121]

101. H, D, F are collinear [09] & L, A, B are collinear [16] & $DF \perp AB$ [05] $\Rightarrow HD \perp LA$ [122]

102. $IG \perp GD$ [109] & $HD \perp LA$ [122] $\Rightarrow \angle(IG - LA) = \angle GDH$ [123]

103. G, D, E are collinear [07] & H, D, F are collinear [09] & $\angle CAB = \angle CMB$ [51] & $\angle(IG - LA) = \angle GDH$ [123] & L, A, B are collinear [16] & $GI \parallel AC$ [11] $\Rightarrow$ $\angle GDH = \angle CMB$ [124]

104. $DG : DH = MB : MC$ [121] & $\angle GDH = \angle CMB$ [124] (Similar Triangles)$\Rightarrow$ $GD : BM = GH : BC$ [125]

105. L, A, B are collinear [16] & $LA = LB$ [17] $\Rightarrow$ L is midpoint of AB [126]

106. A, K, C are collinear [14] & $KA = KC$ [15] $\Rightarrow$ K is midpoint of AC [127]

107. L is midpoint of AB [126] & K is midpoint of AC [127] $\Rightarrow LK \parallel BC$ [128]

108. H, D, F are collinear [09] & L, A, B are collinear [16] & $DF \perp AB$ [05] & $DN \perp BC$ [20] & $BC \parallel KL$ [128] $\Rightarrow \angle(ND - LK) = \angle(HD - LA)$ [129]

109. L, A, B are collinear [16] & A, M, D are collinear [18] & $\angle AMC = \angle ABC$ [50] & $BC \parallel KL$ [128] $\Rightarrow \angle(LK - MC) = \angle LAM$ [130]

110. $\angle(ND - LK) = \angle(HD - LA)$ [129] & $\angle(LK - MC) = \angle LAM$ [130] $\Rightarrow \angle(ND - MC) = \angle(HD - AM)$ [131]

111. H, D, F are collinear [09] & A, M, D are collinear [18] & $\angle(ND - MC) = \angle(HD - AM)$ [131] $\Rightarrow \angle NDH = \angle CMD$ [132]

112. H, D, F are collinear [09] & A, M, D are collinear [18] & $\angle GNH = \angle GFH$ [72] & $FG \parallel EH$ [48] & $NE \perp DC$ [58] & $GN \perp EN$ [60] & $FE \perp DA$ [61] & $GH \perp EH$ [63] & $GH \parallel EF$ [47] $\Rightarrow \angle NHD = \angle CDM$ [133]

113. $\angle NDH = \angle CMD$ [132] & $\angle NHD = \angle CDM$ [133] (Similar Triangles)$\Rightarrow$ $DN : DH = MC : MD$ [134]

114. $GN : BD = GH : BC$ [118] & $GD : BM = GH : BC$ [125] & $DF = DE$ [33] & $DG = DE$ [06] & $MC = BM$ [120] & $DN : DH = MC : MD$ [134] & $DN = DH$ [90] $\Rightarrow GD : MD = GN : BD$ [135]

115. $GD : EC = GN : NE$ [113] & $GD : MD = GN : BD$ [135] $\Rightarrow CE : NE = MD : BD$ [136]

116. $HN : NE = ID : BD$ [107] & $NE : CE = BD : MD$ [136] $\Rightarrow$ $HN : CE = ID : MD$ [137]

117. $HN : CE = ID : MD$ [137] & $CN = CE$ [38] $\Rightarrow$ $HN : CN = ID : MD$ [138]

118. D is the circumcenter of ΔNHE [92] & $DN \perp NI$ [83] $\Rightarrow$ $\angle INH = \angle NEH$ [139]

119. D is the circumcenter of ΔGNE [59] & $DN \perp NI$ [83] $\Rightarrow$ $\angle ING = \angle NEG$ [140]

120. G, D, E are collinear [07] & $\angle ING = \angle IDG$ [102] & B, N, C are collinear [21] & I, B, C are collinear [10] & $\angle ING = \angle NEG$ [140] $\Rightarrow$ $\angle(NE - GD) = \angle IDG$ [141]

121. $\angle(NE - GD) = \angle IDG$ [141] $\Rightarrow$ $NE \parallel ID$ [142]

122. B, N, C are collinear [21] & A, M, D are collinear [18] & $\angle INH = \angle NEH$ [139] & I, B, C are collinear [10] & $FE \perp DA$ [61] & $GH \perp EH$ [63] & $GH \parallel EF$ [47] & $EN \parallel DI$ [142] $\Rightarrow$ $\angle HNC = \angle MDI$ [143]

123. $HN : CN = ID : MD$ [138] & $\angle HNC = \angle MDI$ [143] (Similar Triangles)$\Rightarrow$ $\angle NHC = \angle MID$ [144]

124. $HN : CN = ID : MD$ [138] & $\angle HNC = \angle MDI$ [143] (Similar Triangles)$\Rightarrow$ $\angle HCN = \angle DMI$ [145]

125. $\angle NHC = \angle MID$ [144] & $EN \parallel DI$ [142] $\Rightarrow$ $\angle NHC = \angle(IM - EN)$ [146]

126. $\angle BAD = \angle DAC$ [00] & $\angle ACD = \angle DCB$ [01] & $DN \perp BC$ [20] & $\angle ADE = \angle(AB - EF)$ [25] & $\angle NDC = \angle(EN - AC)$ [30] & $\angle DEH = \angle GNH$ [44] & $\angle FEH = \angle GNE$ [49] & $\angle CBG = \angle(JM - AD)$ [99] & $\angle NHC = \angle(IM - EN)$ [146] (Angle chase)$\Rightarrow$ $\angle JMI = \angle(CH - BG)$ [147]

127. L is midpoint of AB [126] & D is midpoint of GE [45] $\Rightarrow$ $LA : AB = DG : GE$ [148]

128. $LA : AB = DG : GE$ [148] & $LA = LB$ [17] & $DG = DE$ [06] $\Rightarrow$ $LB : AB = DE : GE$ [149]

129. $GD : GN = EC : EN$ [114] & $DF = DE$ [33] & $DG = DE$ [06] & $CN = CE$ [38] $\Rightarrow$ $DF : GN = CN : NE$ [150]

130. $ND \perp IB$ [110] & $HD \perp LA$ [122] $\Rightarrow$ $\angle(ND - LA) = \angle(IB - HD)$ [151]

131. H, D, F are collinear [09] & $DF \perp AB$ [05] & $AB \parallel HJ$ [13] $\Rightarrow$ $DH \perp HJ$ [152]

132. D is the circumcenter of ΔNHF [91] & $DH \perp HJ$ [152] $\Rightarrow \angle JHN = \angle HFN$ [153]

133. B, N, C are collinear [21] & $\angle(ND - LA) = \angle(IB - HD)$ [151] & L, A, B are collinear [16] & I, B, C are collinear [10] & H, D, F are collinear [09] & $\angle JHN = \angle HFN$ [153] & $HJ \parallel AB$ [13] $\Rightarrow \angle FNH = \angle BND$ [154]

134. H, D, F are collinear [09] & $\angle BDN = \angle BFN$ [85] & A, F, B are collinear [04] & $\angle LFN = \angle FHN$ [94] & L, A, B are collinear [16] $\Rightarrow \angle FHN = \angle BDN$ [155]

135. $\angle FNH = \angle BND$ [154] & $\angle FHN = \angle BDN$ [155] (Similar Triangles)$\Rightarrow NF : NH = NB : ND$ [156]

136. $DF = DE$ [33] & $DN = DE$ [37] $\Rightarrow DN = DF$ [157]

137. $NF : NH = NB : ND$ [156] & $FD = ND$ [157] $\Rightarrow NF : HN = BN : DF$ [158]

138. B, N, C are collinear [21] & $\angle GND = \angle ENC$ [115] $\Rightarrow \angle DNC = \angle GNE$ [159]

139. B, C, N are collinear [21] & G, D, E are collinear [07] & $\angle NCD = \angle NED$ [29] $\Rightarrow \angle DCN = \angle GEN$ [160]

140. $\angle DNC = \angle GNE$ [159] & $\angle DCN = \angle GEN$ [160] (Similar Triangles)$\Rightarrow DN : DC = GN : GE$ [161]

141. $DN : DC = GN : GE$ [161] & $FD = ND$ [157] $\Rightarrow DF : CD = GN : GE$ [162]

142. I, B, C are collinear [10] & A, C, E are collinear [02] & $\angle DCB = \angle ACD$ [01] $\Rightarrow \angle DCI = \angle ECD$ [163]

143. I, B, C are collinear [10] & $\angle ING = \angle IDG$ [102] & B, N, C are collinear [21] & G, D, E are collinear [07] & $NE \perp DC$ [58] & $GN \perp EN$ [60] $\Rightarrow \angle DIC = \angle EDC$ [164]

144. $\angle DCI = \angle ECD$ [163] & $\angle DIC = \angle EDC$ [164] (Similar Triangles)$\Rightarrow CD : CI = CE : CD$ [165]

145. $CD : CI = CE : CD$ [165] & $CN = CE$ [38] $\Rightarrow CD : IC = CN : CD$ [166]

146. $NB : NF = CM : CD$ [66] & $MC = BM$ [120] $\Rightarrow BN : NF = BM : CD$ [167]

147. $BN : MC = BF : MD$ [67] & $MC = BM$ [120] & $BN = FB$ [88] $\Rightarrow BN : BM = BN : MD$ [168]

148. D is the circumcenter of ΔNHE [92] & $DH \perp HJ$ [152] $\Rightarrow \angle JHE = \angle HNE$ [169]

149. $\angle JHE = \angle HNE$ [169] & $HJ \parallel AB$ [13] & $FE \perp DA$ [61] & $GH \perp EH$ [63] & $GH \parallel EF$ [47] $\Rightarrow \angle HNE = \angle BAD$ [170]

150. $DN = DF$ [157] $\Rightarrow \angle FND = \angle DFN$ [171]

151. $\angle BDN = \angle BFN$ [85] & A, F, B are collinear [04] & $\angle FND = \angle DFN$ [171] & $\angle HEN = \angle HFN$ [105] & H, D, F are collinear [09] $\Rightarrow \angle HEN = \angle ABD$ [172]

152. $\angle HNE = \angle BAD$ [170] & $\angle HEN = \angle ABD$ [172] (Similar Triangles)$\Rightarrow$ $HN : NE = AD : AB$ [173]

153. $LB : AB = DE : GE$ [149] & $DF : GN = CN : NE$ [150] & $BN = FB$ [88] & $NF : HN = BN : DF$ [158] & $DF : CD = GN : GE$ [162] & $CD : IC = CN : CD$ [166] & $BN : NF = BM : CD$ [167] & $BN : BM = BN : MD$ [168] & $HN : NE = AD : AB$ [173] (Ratio chase)$\Rightarrow$ $LA : AD = MD : IC$ [174]

154. $LA : AD = MD : IC$ [174] & $LA = LB$ [17] & $DN : DH = MC : MD$ [134] & $DN = DH$ [90] $\Rightarrow AL : AD = CM : CI$ [175]

155. L, A, B are collinear [16] & I, B, C are collinear [10] & $\angle AMC = \angle ABC$ [50] & A, M, D are collinear [18] $\Rightarrow \angle LAD = \angle ICM$ [176]

156. $AL : AD = CM : CI$ [175] & $\angle LAD = \angle ICM$ [176] (Similar Triangles)$\Rightarrow$ $\angle LDA = \angle CIM$ [177]

157. $\angle HCN = \angle DMI$ [145] & B, C, N are collinear [21] & A, M, D are collinear [18] & $\angle LDA = \angle CIM$ [177] & I, B, C are collinear [10] & $BC \parallel KL$ [128] $\Rightarrow \angle DLK = \angle(CH - LK)$ [178]

158. $\angle DLK = \angle(CH - LK)$ [178] $\Rightarrow LD \parallel CH$ [179]

159. L is midpoint of AB [126] & D is midpoint of HF [46] $\Rightarrow LA : AB = DH : HF$ [180]

160. $LA : AB = DH : HF$ [180] & $LA = LB$ [17] & $DH = DF$ [08] $\Rightarrow LB : AB = DF : HF$ [181]

161. K is midpoint of AC [127] & D is midpoint of GE [45] $\Rightarrow KA : AC = DG : GE$ [182]

162. $KA : AC = DG : GE$ [182] & $KA = KC$ [15] & $DG = DE$ [06] $\Rightarrow KC : AC = DE : GE$ [183]

163. D is the circumcenter of ΔGNH [40] & $DN \perp NI$ [83] $\Rightarrow \angle ING = \angle NHG$ [184]

164. D is the circumcenter of ΔGNH [40] & $DH \perp HJ$ [152] $\Rightarrow \angle JHN = \angle HGN$ [185]

165. H, D, F are collinear [09] & $\angle NHD = \angle NJD$ [76] & B, N, C are collinear [21] & B, C, J are collinear [12] & $\angle ING = \angle NHG$ [184] & I, B, C are collinear [10] & $GH \parallel EF$ [47] & $\angle JHN = \angle HGN$ [185] & $HJ \parallel AB$ [13] $\Rightarrow \angle NHJ = \angle JDH$ [186]

166. N, H, D, J are concyclic [75] & $\angle NHJ = \angle JDH$ [186] $\Rightarrow NJ = JH$ [187]

167. B, N, C are collinear [21] & B, C, J are collinear [12] & $DF \perp AB$ [05] & $DN \perp BC$ [20] & $\angle JHN = \angle HFN$ [153] & H, D, F are collinear [09] & $HJ \parallel AB$ [13] $\Rightarrow \angle FNH = \angle DNJ$ [188]

168. H, D, F are collinear [09] & B, N, C are collinear [21] & B, C, J are collinear [12] & $\angle NHD = \angle NJD$ [76] $\Rightarrow \angle FHN = \angle DJN$ [189]

169. $\angle FNH = \angle DNJ$ [188] & $\angle FHN = \angle DJN$ [189] (Similar Triangles)$\Rightarrow FN : FH = DN : DJ$ [190]

170. $\angle FNH = \angle DNJ$ [188] & $\angle FHN = \angle DJN$ [189] (Similar Triangles)$\Rightarrow NF : NH = ND : NJ$ [191]

171. $EH \parallel FG$ [48] $\Rightarrow \angle GHE = \angle HGF$ [192]

172. G, H, E, F are concyclic [70] & $\angle GHE = \angle HGF$ [192] $\Rightarrow GE = HF$ [193]

173. $FN : FH = DN : DJ$ [190] & $GE = HF$ [193] & $FD = ND$ [157] $\Rightarrow NF : GE = DF : DJ$ [194]

174. $NF : NH = ND : NJ$ [191] & $FD = ND$ [157] $\Rightarrow NF : HN = DF : NJ$ [195]

175. B, C, J are collinear [12] & $\angle DCB = \angle ACD$ [01] $\Rightarrow \angle DCJ = \angle ACD$ [196]

176. B, C, J are collinear [12] & $\angle GNH = \angle GFH$ [72] & H, D, F are collinear [09] & $FG \parallel EH$ [48] & $\angle NHD = \angle NJD$ [76] & B, N, C are collinear [21] & $FE \perp DA$ [61] & $GH \perp EH$ [63] & $GH \parallel EF$ [47] & $NE \perp DC$ [58] & $GN \perp EN$ [60] $\Rightarrow \angle DJC = \angle ADC$ [197]

177. $\angle DCJ = \angle ACD$ [196] & $\angle DJC = \angle ADC$ [197] (Similar Triangles)⇒ $CD : DJ = AC : AD$ [198]

178. B, C, J are collinear [12] & $\angle HND = \angle HJD$ [77] & $HJ \parallel AB$ [13] & $\angle DNH = \angle NHD$ [93] & H, D, F are collinear [09] & $\angle NHD = \angle NJD$ [76] & B, N, C are collinear [21] ⇒ $\angle DJB = \angle HJD$ [199]

179. D is the circumcenter of ΔNHF [91] & $DN \perp NI$ [83] ⇒ $\angle INH = \angle NFH$ [200]

180. B, C, J are collinear [12] & H, D, F are collinear [09] & $\angle NHD = \angle NJD$ [76] & B, N, C are collinear [21] & $\angle INH = \angle NFH$ [200] & I, B, C are collinear [10] & $\angle BNF = \angle BDF$ [86] ⇒ $\angle DBJ = \angle HDJ$ [201]

181. $\angle DJB = \angle HJD$ [199] & $\angle DBJ = \angle HDJ$ [201] (Similar Triangles)⇒ $JD : JB = JH : JD$ [202]

182. $JD : JB = JH : JD$ [202] & $NJ = JH$ [187] ⇒ $DJ : BJ = NJ : DJ$ [203]

183. $DH = DF$ [08] & $LB : AB = DE : GE$ [149] & $LB : AB = DF : HF$ [181] & $KC : AC = DE : GE$ [183] & $DN = DE$ [37] & $DF = DE$ [33] & $BN = FB$ [88] & $NJ = JH$ [187] & $NF : GE = DF : DJ$ [194] & $NF : HN = DF : NJ$ [195] & $NF : HN = BN : DF$ [158] & $CD : DJ = AC : AD$ [198] & $BN : NF = BM : CD$ [167] & $BN : BM = BN : MD$ [168] & $HN : NE = AD : AB$ [173] (Ratio chase)⇒ $AK : AD = MD : BJ$ [204]

184. $AK : AD = MD : BJ$ [204] & $KA = KC$ [15] & $MC = BM$ [120] & $DN : DH = MC : MD$ [134] & $DN = DH$ [90] ⇒ $BM : BJ = AK : AD$ [205]

185. B, C, J are collinear [12] & A, K, C are collinear [14] & $\angle AMB = \angle ACB$ [52] & A, M, D are collinear [18] ⇒ $\angle MBJ = \angle DAK$ [206]

186. $BM : BJ = AK : AD$ [205] & $\angle MBJ = \angle DAK$ [206] (Similar Triangles)⇒ $\angle MJB = \angle ADK$ [207]

187. $\angle GBC = \angle(AD - JM)$ [99] & $\angle MJB = \angle ADK$ [207] & B, C, J are collinear [12] & $BC \parallel KL$ [128] ⇒ $\angle DKL = \angle(BG - LK)$ [208]

188. $\angle DKL = \angle(BG - LK)$ [208] ⇒ $KD \parallel BG$ [209]

189. $\angle JMI = \angle(CH - BG)$ [147] & $DL \parallel CH$ [179] & $DK \parallel BG$ [209] ⇒ $\angle JMI = \angle LDK$ □

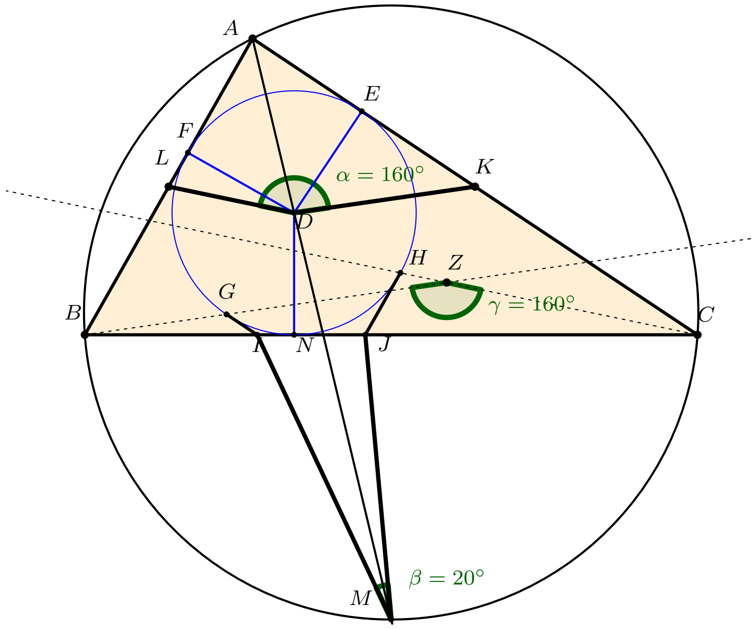

Figure 2.7: *IMO 2024, problem 4. AlphaGeometry's solution.*

Comments and Differences

We can clearly observe several differences between the human approach of the official solution and AlphaGeometry's approach.

Problem statement: It was necessary to add a few additional constructions such as points G and H to translate the problem into AlphaGeometry's syntax, since we had to specify a point for the construction of parallel lines GI and HJ. There is a small difference between the original thesis and the translated one, since AlphaGeometry works with full-angles.

Proof length: We can clearly see that the second proof is way longer than the first, having 189 steps. It should be noted that a big part of these steps consist of simple trivial passages that would not be specified in a human proof but are essential to the model, since the symbolic engine operates purely logically and proves every single minor relation from first principles.

For example, many steps are used just to prove that 4 points are cyclic. This happens because the machine does not have a geometrical visualization of the problem, whereas a human solver would use visual intuition to skip these steps.

Auxiliary constructions: In the official solution the introduction of point A' greatly simplifies the proof. In the model approach this point does not appear; the only auxiliary construction used is the point N, the tangent point between the incircle and BC, without trying to add other points.

This happens because of AlphaGeometry's inference process. Even if adding a point would have greatly simplified the proof, the DDAR could still deduce information just by adding a trivial point such as N. Since this process never ended, no point was added.

Chapter 3

Digressions

AlphaGeometry had a major impact on the automated reasoning world, being the first of its kind to obtain optimal results in the geometry theorem proving branch. As such, its been subject of research and studies from its first release.

Now we will briefly introduce three projects that are closely related to AlphaGeometry, discussing community developments and subsequent official advancements from DeepMind: AG4Masses[15], AlphaGeometry 2[2] and AlphaProof[1].

3.1 AG4Masses

Since January 2024, the date of AlphaGeometry's launch, its interest kept growing in the entire machine learning world. As such, in short time an active AlphaGeometry community formed on GitHub[4], where people started working together to solve new problems and errors obtained during the run of AlphaGeometry. In a short time, around April 2024, AG4Masses[15] was born, a GitHub repository where a slightly different version of AlphaGeometry appeared, one that solved some bugs and was studied as an easier-to-access version for everyone.

The differences from the original model are almost non-existent: more translated problem statements and solutions can be found, together with a Kaggle notebook[12] for running AG4Masses. Also, a lot of useful information is present, therefore researchers seeking to run AlphaGeometry for the first time are strongly encouraged to check that repository for setup guidance.

3.2 AlphaGeometry 2 & AlphaProof

In July 2024, Google DeepMind published an article[1] introducing AlphaProof and AlphaGeometry 2[2], two brand new theorem provers that achieved wonderful results when tested on IMO problems.

As expected, AlphaGeometry 2 solves geometry problems, obtaining even better results than its predecessor, while AlphaProof solves problems that belong to the other IMO's categories: Algebra, Number Theory, and Combinatorics.

Sadly, neither of these models has been released open source, but their results and some of their characteristics have been published.

AlphaProof

AlphaProof[1] is a reinforcement-learning based system designed for formal mathematical reasoning in the Lean[11] theorem prover, a programming language and interactive proof assistant that provides a rigorous framework for verifying proof correctness. AlphaProof combines a pre-trained language model with the AlphaZero[14] algorithm, enabling it to autonomously learn how to construct and verify mathematical proofs.

Working within formal frameworks like Lean ensures the verifiable correctness of the proofs, but their use has always been limited due to the scarcity of formal training data. To overcome this problem, a Gemini model was fine-tuned to translate natural-language problems into formal statements, creating a large and diverse set of training examples.

During problem solving, AlphaProof generates and tests candidate proofs by searching over possible proof steps in Lean. Verified proofs are then used to reinforce the model, gradually improving its performance on more complex tasks.

The system was trained on millions of problems across different mathematical domains, including those inspired by International Mathematical Olympiad (IMO) challenges, and then tested on IMO 2024 problems, solving three problems, including the most difficult one of that year.

AlphaGeometry 2

AlphaGeometry 2[2] is a significantly improved version of AlphaGeometry, released in mid-2024, more than a year after the release of its predecessor.

AlphaGeometry 2 solves many problems that were imbued into AlphaGeometry and enhances its performance significantly, in terms of efficiency, versatility, and obviously results.

Briefly speaking, almost any individual component of AlphaGeometry was improved, allowing the model to be tested on a bigger set of problems with ease, obtaining in shorter times proofs that are far more readable and original than the ones produced from its former model. A short summary of its improvements can be exposed as follows:

Domain language: Many predicates are added to the syntax used for the former model, improving the AlphaGeometry's language coverage from 66% to 88% on all 2000-2024 IMO geometry problems. In fact:

- More possible problem statements are covered, as the value of a specific angle or ratio;
- Linear equations of geometric quantities that appears in some geometry problems can now be expressed;
- The so-called locus problems, that talk about movements of objects such as points, lines and circles can be captured by the new syntax;
- Topological and non-degeneracy conditions can be verified with explicit predicates.

Automated formalization: The Gemini 2024 model is used to automate the problem formalization, so there is no need to manually translate input problems from natural language into domain specific language.

Stronger and faster symbolic engine: The previous symbolic engine received three major improvements:

- Capability to handle double points: the previous symbolic engine is unable to accept two points with different names and the same coordinates. Now this is not an issue.
- Faster algorithm: Many of the previous explicit rules worked on angles and distances. These rules are now totally discarded since all such deductions happen automatically in the AR engine, that works faster than the DD. A hard-coded search is implemented to improve efficiency for all the essential rules (this works particularly well for concyclicity and similarity).
- Faster implementation: Further speed improvements are obtained by implementing its core computation in C++ and exporting it into Python, reaching an astonishing 300 times faster symbolic engine

Better synthetic training data: A careful re-balancing of the data distribution allowed the covering of more complex diagrams, with problem and proofs that are significantly more complex, including more types of theorems. Switching from the previous proof pruning process to a greedy discarding algorithm improved the speed of the data generation algorithm.

Novel search algorithm: The original beam search performed in AlphaGeometry is replaced by a novel search algorithm, in which several differently configured beam searches are executed in parallel and are allowed to help each other through a knowledge-sharing mechanism.

Better language model: A new language model is obtained by training a transformer-based model that builds up on Gemini. The training consists of a single phase of unsupervised learning of all data, which consists of a 300 million synthetic theorems dataset.

Considering these improvements, it is not surprising to see that AlphaGeometry 2 solves 42 out of 50 of all 2000-2024 IMO geometry problems, a performance that surpasses an average gold medalist, showing solutions that, in the original article, were evaluated by expert mathematicians as demonstrating “superhuman creativity”.

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