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BRST-invariant FRG flow in Quantum Einstein Gravity

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Abstract

The thesis concerns a non-standard method of implementing the Functional Renormalization Group (FRG) framework in a quantum field theory of gravity which is capable of explicitly preserving the Becchi-Rouet-Stora-Tyutin (BRST) symmetry of the theory at all stages of the construction. In particular, we firstly review the functional Faddeev-Popov quantization and gauge-fixing of a classical field theory of gravity within the background field method and the role and properties of BRST symmetry. Secondly, we present the standard FRG framework implementation and we discuss how the standard regularization procedure, consisting in manually adding a quadratic regulator term in the metric fluctuation and Faddeev-Popov ghosts to the gauge-fixed action, breaks BRST symmetry and leads to a Wetterich-Morris equation incompatible with the constraint imposed by BRST symmetry, i.e. the Zinn-Justin equation. Finally, we present how a BRST-invariant FRG framework can be introduced by combining the regularization and gauge-fixing procedures in a single step, using a non-standard gauge-fixing choice, which allows to introduce quadratic mass terms in the gauge-fixed action without breaking BRST symmetry, to regularize the theory in an explicitly BRST-invariant manner. Then, we derive the Wetterich-Morris equation stemming from the construction, proving its compatibility with the Zinn-Justin equation and presenting its component form within the Einstein-Hilbert truncation.

Contents

Introduction

One of the most important challenges in modern theoretical physics is the one of developing a complete and consistent quantum theory of gravity, capable of describing the gravitational interaction at the quantum level and at all energy-length scales, from the low energy - long length infrared regime (IR) to the high energy - short length regime ultraviolet (UV). One of the most important approaches currently pursued in the research of a such theory is the one based on the framework of Quantum Field Theory (QFT), which already successfully describes all other fundamental interactions, electromagnetic, strong and weak, at the quantum level, both in special and general relativistic classic spacetimes, where the gravitational interaction is not quantized. In fact, as widely known, the quantization of Einstein's theory of gravity as a quantum field theory leads at the perturbative level to a non-renormalizable theory, unpredictive in the UV regime [1]. However, within the Renormalization Group framework [2] and according to the Asymptotic Safety conjecture by Steven Weinberg [3], it is possible to investigate the potential renormalizability at the non-perturbative level of a quantum field theory of gravity, searching for a complete theory predictive at all energy-length scales.
The thesis concerns a particular way of implementing these ideas in a quantum field theory of gravity, the Functional Renormalization Group (FRG), in which the scale dependence of the theory is described by an effective average action which interpolates between the classical theory, described by the classical action, and the quantum theory, described by the quantum effective action [4]. In particular, the thesis focuses on the problem of preserving in this framework the fundamental symmetry possessed by a theory of quantum gravity, the Becchi-Rouet-Stora-Tyutin (BRST) symmetry taking the place of gauge symmetry under diffeomorphisms at the quantum level after the gauge-fixing procedure. In fact, the standard methodology of applying this framework to quantum gravity [5] breaks BRST symmetry, which is lost in the scale-dependent theory for a general value of the scale. The purpose of this work is to develop an FRG framework for a theory of quantum gravity in which BRST symmetry is instead always manifestly preserved, generalizing the formalism introduced in [6], which allows to construct an explicitly BRST-invariant FRG for a quantum non-abelian gauge theory, to a theory of gravity.

As far as the structure is concerned, the thesis is articulated in three parts:

1. In the first part we synthetically review the elementary principles of the theories which constitute the basis of the second and third part, namely Quantum Field Theory and General Relativity and the framework of the Functional Renormalization Group.
As far as QFT is concerned, we adopt the functional formalism and we consider a scalar theory in flat spacetime to review the definition of the generating functionals, namely the path integral, its logarithm and the effective action, their properties and relations; we also

review the description of symmetries at the quantum level in the functional formalism, namely the distinction between non-anomalous and anomalous symmetry and the Ward identities.
As far the FRG framework is concerned, we review the fundamental idea and how to implement it in the scalar QFT previously presented, namely the regularization procedure of the generating functionals in terms of a scale-dependent quadratic regulator term added to the action; we also review the definition of the effective average action as a scale-dependent interpolating functional between the classic action and the quantum effective action, and the derivation of the exact functional differential equation which describes this interpolation, the Wetterich-Morris equation, also reviewing how it can be written in component form, in terms of a system of differential equations for scale-dependent couplings and how to tackle its solution in terms of truncation schemes.
As far as General Relativity is concerned, we adopt the metric formulation of the theory and we review the fundamental principles, namely the general relativity principle and the equivalence principle, and how those are translated in the language of differential geometry, recalling the definitions of the fundamental geometric objects and the Einstein equations; we also review the lagrangian formulation of the theory in terms of an action principle, i.e. the characterization of the theory as classical field theory of the gravitational interaction with the metric as dynamical field; finally, we review the characterization of the theory as gauge theory with diffeomorphisms of the metric, i.e. the functional variation induced by a general change of coordinates, as gauge transformations.

2. The second part of the thesis has a double role. Firstly, we review the definition of a Quantum Einstein Gravity (QEG) theory as a quantum field theory for the gravitational interaction compatible with the general relativity principle, and thus gauge-symmetric under diffeomorphisms, which is non-renormalizable at the perturbative level and whose renormalizability is intended to be studied with renormalization group techniques at the non-perturbative level, within the Weinberg's asymptotic safety conjecture. In particular, we highlight the difficulties which one has to overcome in the practical construction of the theory, namely the necessity of using a gauge-fixing procedure to remove gauge redundancy and obtain a well-defined path integral and the one of using the background field method in order to have a well-defined notion of coarse-graining in the FRG framework. We thus present the implementation of the background field method in terms of a background metric and a metric fluctuation and how infinitesimal diffeomorphisms are reinterpreted either as true or background gauge transformations; as gauge-fixing method we adopt the Faddeev-Popov quantization method, also in its variants in presence of a Nakanishi-Lautrup auxiliary field and a noise field. We then present the general form of the BRST transformations for the metric fields and the Faddeev-Popov ghosts and the BRST symmetry of the gauge-fixed action with its fundamental properties, at the classical and quantum levels, namely its nilpotency and its description at the quantum level in terms of a Ward-Takahashi equation; we also stress how the symmetry is present for any choice of gauge-fixing, and how the non-gauge-fixed action and the gauge-fixing sector introduced in the Faddeev-Popov quantization are separately BRST-invariant.
Secondly, we specialize the discussion to the theory with the standard choice of gauge-fixing, reviewing the various terms introduced in the gauge-fixed action and reviewing the specific appearance of BRST symmetry and its properties, in particular the Zinn-Justin equation

describing the constraint imposed by the symmetry on the quantum effective action. We then review the standard implementation of the FRG framework via the regularization of the gauge-fixed action with quadratic terms for the metric fluctuation and the ghosts, and we discuss how those produce an explicit breaking of BRST symmetry, which is then lost in the regularized scale-dependent theory, as represented by the emergence of a modified Zinn-Justin equation for the effective average action. Finally, we review the derivation of the standard Wetterich-Morris equation describing the FRG flow of the effective average action, stressing its incompatibility with BRST symmetry, due to the non-BRST-invariant nature of the regularization procedure; we also review the component form of the equation within the Einstein-Hilbert truncation.

3. In the third part of the thesis we present the central matter of the work and the main results obtained. In particular, we first review how the fundamental idea introduced in [6] is translated in a gravitational theory, namely combining the gauge-fixing and regularization procedures in a single step and using a non-standard gauge-fixing to introduce the regulators necessary for an FRG regularization directly as part of the gauge-fixing terms, so that BRST symmetry is always manifestly preserved, since we are formally performing just a particular gauge-fixing and the gauge-fixing sector of the action is always BRST-invariant, in particular no non-BRST-invariant regulator is manually added to the action.
We articulate the construction in two steps. Firstly, we present a non-standard gauge-fixing which allows to introduce in the action quadratic mass terms for the metric fluctuation and the Faddeev-Popov ghosts and also reproduce the standard gauge-fixing terms, describing the specific appearance of BRST symmetry and its properties, in particular the non-standard Zinn-Justin equation describing the constraint imposed by the symmetry on the quantum effective action.
Secondly, we implement the FRG regularization by simply promoting the mass parameters to suitably defined FRG regulators obtaining by construction a BRST-invariant regularized scaled-dependent theory, whose effective average action satisfies for each value of the scale the constraint imposed by the Zinn-Justin equation. Finally, we derive the non-standard Wetterich-Morris equation stemming from the construction, discussing its non-standard properties, proving its expected compatibility with the constraint imposed by the Zinn-Justin equation and deriving its component form within the Einstein-Hilbert truncation and employing a particular regularization scheme.

Notations

In the manuscript we adopt the following conventions and notations:

- We work in natural units in which the speed of light in the vacuum and the reduced Planck's constant are 1:

  $$c = 1 = \hbar$$

  The value of Newton's gravitational constant G is instead not set to 1 and its symbol left explicit.

- We work within the conventions regarding spacetime dimensionality and metric signature typically used in the applications of FRG methods, i.e. we work in D dimensions and euclidean signature: a generic curved spacetime is described as a riemannian manifold $(\mathcal{M}, g)$ with D-dimensional topological space and equipped with a metric tensor $g_{\mu\nu}$ with euclidean signature:

  $$\mathrm{sign} g_{\mu\nu} = (+, \dots, +)$$

  A flat spacetime is described as a D-dimensional euclidean spacetime $(\mathbb{R}^D, \delta)$ with metric:

  $$\delta_{\mu\nu} = \mathrm{diag}(1, \dots, 1)$$

- Indices appearing in the components of tensorial quantities are summed according to the Einstein notation of repeated indices.

- Partial, covariant and functional derivatives are respectively indicated as:

  $$\partial_\mu \qquad \nabla_\mu \qquad \frac{\delta}{\delta\phi(x)}$$

  When fields evaluated at multiple spacetime points are present in an expression, we indicate the spacetime variable of a partial and covariant derivative as:

  $$\partial_{(x)\mu} \qquad \nabla_{(x)\mu}$$

  In presence of Grassmann-graded fields, left and right functional derivatives are respectively indicated as:

  $$\frac{\delta}{\delta\psi(x)} \qquad \frac{\overleftarrow{\delta}}{\delta\psi(x)}$$

- A $\bullet$ symbol denotes an understood integration:

$$A \bullet B = \int d^D x A(x) B(x) \qquad\qquad A(x,\cdot) \bullet B = \int d^D y A(x,y) B(y)$$

The variant $\bullet_g$ denotes an understood integration with the invariant measure $d^D x \sqrt{g}$ on the spacetime manifold:

$$A \bullet B_g = \int d^D x \sqrt{g(x)} A(x) B(x) \qquad\qquad A(x,\cdot) \bullet_g B = \int d^D y \sqrt{g(y)} A(x,y) B(y)$$

- det and tr denote determinants and traces over operators with discrete indices, while Det and Tr denote functional determinants and traces of operators with continuous indices. In particular, Tr indicates:

$$\mathrm{Tr}[A(x,y)] = \int d^D x d^D y \delta(x-y) A(x,y)$$

The variant Tr_g contains an invariant measure $d^D x \sqrt{g}$ on the spacetime manifold:

$$\mathrm{Tr}[A(x,y)] = \int d^D x \sqrt{g(x)} d^D y \delta(x-y) A(x,y)$$

- In quantum field theory unnormalized, normalized and connected correlation functions are respectively indicated as:

$$\langle \cdot \rangle_u \qquad\qquad \langle \cdot \rangle \qquad\qquad \langle \cdot \rangle_c$$

If sources are not set to zero they are indicated as:

$$\langle \cdot \rangle_{u,J} \qquad\qquad \langle \cdot \rangle_J \qquad\qquad \langle \cdot \rangle_{c,J}$$

Part 1
Fundamentals

The first part of the thesis is devoted to a synthetic review of the elementary principles of Quantum Field Theory, General Relativity, and the Functional Renormalization Group. The part is divided in three sections. In the first section we consider a scalar theory in flat spacetime to review the elementary aspects of the functional formalism in QFT, namely generating functionals and the description of symmetries in terms of Ward-Takahashi identities. In the second section we review the implementation of the FRG framework, discussing in particular the regularization procedure and the Wetterich-Morris equation. In the third section we review the fundamentals of Einstein's General Relativity, discussing in particular its characterization as a gauge theory under diffeomorphisms.

1.1 Elements of Quantum Field Theory

In this section we list the elementary notions of QFT, in its functional formulation, to which we will refer in the following, considering for simplicity a scalar theory. In particular, we briefly review the various generating functionals and the description of symmetries at the quantum level in terms of Ward identities. We suggest for instance [7] for a detailed review.

1.1.1 Functional formalism in euclidean signature

Consider a classical field theory for a scalar field ϕ in a D-dimensional euclidean spacetime $(\mathbb{R}^D, \delta)$ described by a generic action $S[\phi]$, given by the sum of a quadratic term describing the free theory and one describing self-interactions:

$$S[\phi] = S_0[\phi] + S_{int}[\phi] = \int d^D x \; (\mathcal{L}_0 + \mathcal{L}_{int}) \tag{1.1.1}$$

In the functional formalism, quantizing the theory amounts to write and in principle compute the path integral, i.e. the functional integral of the exponential of the classical action. In euclidean signature the path integral is written as:

$$Z = \int \mathcal{D}\phi \, e^{-S[\phi]} \tag{1.1.2}$$

where the integration is extended to all possible configurations of the field and the path integral measure can be practically understood as:

$$\mathcal{D}\phi = \prod_x d\phi(x) \tag{1.1.3}$$

Starting from the path integral, one can define the quantum generating functionals from which all the information on the quantum theory can be extracted by taking functional derivatives. We now briefly recall their definition and main properties:

1. Path integral: Introduce a source field J for the scalar field, the path integral with sources is defined as:

$$Z = \int \mathcal{D}\phi \, e^{-S[\phi]-S_{source[\phi;J]}} \tag{1.1.4}$$

with the source term:

$$S_{source}[\phi; J] = -\int d^D x \, J \, \phi \tag{1.1.5}$$

It is the generator of unnormalized correlation functions:

$$\begin{aligned} \langle \phi(x_1) \dots \phi(x_n) \rangle_u &= \int \mathcal{D}\phi \, \phi(x_1) \dots \phi(x_n) e^{-S[\phi]} \\ &= \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} Z[J] \Big|_{J=0} \end{aligned} \tag{1.1.6}$$

In perturbation theory they can be expressed as sums of correlation functions with respect to the non-interacting path integral:

$$\begin{aligned} \langle \phi(x_1) \dots \phi(x_n) \rangle_u &= \left\langle \phi(x_1) \dots \phi(x_n) e^{-S_{int}[\phi]} \right\rangle_{0,u} = \\ &= \langle \phi(x_1) \dots \phi(x_n) \rangle_{0,u} \\ &\quad + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int d^D y_i \dots y_k \, \langle \phi(x_1) \dots \phi(x_n) \mathcal{L}_{int}(y_1) \dots \mathcal{L}_{int}(y_k) \rangle_{0,u} \end{aligned} \tag{1.1.7}$$

and similarly for the path integral itself:

$$Z = Z_0 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int d^D y_i \dots y_k \, \langle \mathcal{L}_{int}(y_1) \dots \mathcal{L}_{int}(y_k) \rangle_{0,u} \tag{1.1.8}$$

Non-interacting correlation functions can be then expressed according to the Wick theorem [8] as sums of products of free propagators, i.e. 2-point non-interacting correlation functions $\langle \phi(x)\phi(y) \rangle_0$:

$$\langle \phi(x_1) \dots \phi(x_n) \rangle_0 = \sum_{\substack{\text{Wick} \\ \text{contractions}}} \prod_{i,j} \langle \phi(x_i)\phi(x_j) \rangle_0 \tag{1.1.9}$$

In the Feynman diagrammatic representation, this is represented by identifying unnormalized correlation functions with the sum of all diagrams with n external points; in particular the path integral is equal to the sum of all vacuum diagrams, i.e. with zero external points. Vacuum diagrams correspond to factorizable integrals in the perturbative expansion of an unnormalized correlation, from which their sum, i.e. the path integral, can be thus factorized:

$$\langle \phi(x_1) \dots \phi(x_n) \rangle_u \equiv \langle \phi(x_1) \dots \phi(x_n) \rangle \, Z \tag{1.1.10}$$

leaving the sum of all diagrams with n external points and no vacuum diagrams, i.e. the normalized correlation function:

$$\begin{aligned} \langle \phi(x_1) \dots \phi(x_n) \rangle &= \frac{1}{Z} \int \mathcal{D}\phi \, \phi(x_1) \dots \phi(x_n) e^{-S[\phi]} \\ &= \frac{1}{Z} \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} Z[J] \Big|_{J=0} \end{aligned} \tag{1.1.11}$$

2. Path integral logarithm: The path integral logarithm is defined as:

$$W[J] = \log Z[J] \tag{1.1.12}$$

It is the generator of connected correlation functions:

$$\langle \phi(x_1) \dots \phi(x_n) \rangle_c = \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} W[J] \Big|_{J=0} \tag{1.1.13}$$

which in perturbation theory are given by the sum of all connected diagrams with n external points. Indeed, by expressing the functional according to the definition and applying the derivative rule of the logarithm, one finds that a generic normalized correlation function is given by the sum of all possible products of connected correlation functions:

$$\begin{aligned} \langle \phi(x_1) \dots \phi(x_n) \rangle = \langle \phi(x_1) \dots \phi(x_n) \rangle_c &+ \langle \phi(x_1) \rangle_c \langle \phi(x_2) \dots \phi(x_n) \rangle_c + \cdots \\ &+ \langle \phi(x_1)\phi(x_2) \rangle_c \langle \phi(x_3) \dots \phi(x_n) \rangle_c + \cdots \\ &\vdots \\ &+ \langle \phi(x_1) \rangle_c \dots \langle \phi(x_n) \rangle_c \end{aligned} \tag{1.1.14}$$

In particular, the relation:

$$\langle \phi(x)\phi(y) \rangle = \langle \phi(x)\phi(y) \rangle_c + \langle \phi(x) \rangle_c \langle \phi(y) \rangle_c \tag{1.1.15}$$

follows from setting the source to zero in the relation between second derivatives:

$$\frac{1}{Z[J]} \frac{\delta^2 Z[J]}{\delta J(x) \delta J(y)} = \frac{\delta^2 W[J]}{\delta J(x) \delta J(y)} + \frac{\delta W[J]}{\delta J(x)} \frac{\delta W[J]}{\delta J(y))} \tag{1.1.16}$$

The 2-point connected function $\langle \phi(x)\phi(y) \rangle_c = \langle \phi(x)\phi(y) \rangle - \langle \phi(x) \rangle_c \langle \phi(y) \rangle_c$ gives the exact propagator, expressed in perturbation theory as the free propagator plus all the connected quantum loop corrections. In general, connected correlations functions are those entering the LSZ reduction formula and contributing to the non-trivial scattering matrix elements in the S-matrix of the theory.

3. Effective action: The effective action is defined as Legendre transform of the path integral logarithm with respect to the sources:

$$\begin{aligned}\Gamma[\Phi] &= \sup_J \left\{ \int d^D x\, J\,\Phi - W[J] \right\} \\ &= \int d^D x\, J\,\Phi - W[J]\end{aligned} \tag{1.1.17}$$

In the second it is understood that the source is expressed as $J = J(\Phi)$ by inverting the relation given by imposing the extremality condition of the argument of the Legendre transform:

$$\Phi(x) = \langle \phi(x) \rangle_J = \frac{\delta W[J]}{\delta J(x)} \tag{1.1.18}$$

which is given by the average field in presence of sources. Similarly, we can express the path integral logarithm via the inverse Legendre transform:

$$\begin{aligned}W[J] &= \sup_\Phi \left\{ \int d^D x\, J\,\Phi - \Gamma[\Phi] \right\} \\ &= \int d^D x\, J\,\Phi - \Gamma[\Phi]\end{aligned} \tag{1.1.19}$$

where now in the second it is understood that the multiplet of fields is expressed as $\Phi = \Phi(J)$ by inverting the relations:

$$J(x) = \frac{\delta \Gamma[\Phi]}{\delta \Phi(x)} \tag{1.1.20}$$

The effective action can be defined also as solution of the integro-differential equation:

$$e^{-\Gamma[\Phi]} = \int \mathcal{D}\phi \, \exp\left\{ -S[\phi] + \int d^D x \frac{\delta \Gamma[\Phi]}{\delta \Phi} (\phi - \Phi) \right\} \tag{1.1.21}$$

which follows from equating the definition of the path integral logarithm (1.1.12) and its expression given by the inverse Legendre transform (1.1.19), computed in $J = J(\Phi)$ given by (1.1.20). This equation can be used to compute perturbatively the effective action, which can be expressed as a sum in quantum loop corrections in increasing powers of $\hbar$ (considering for a moment units in which it is not set to 1) to the classical action:

$$\Gamma[\Phi] = S[\Phi] + \sum_{n=1}^{\infty} \hbar^n \Delta_n[\Phi] \tag{1.1.22}$$

The second derivative of the effective action is the inverse operator of the second derivative of the path integral logarithm:

$$\frac{\delta^2 W[J]}{\delta J(x) \delta J(y)} = \left(\frac{\delta^2 \Gamma[\Phi]}{\delta \Phi(x) \delta \Phi(y)} \right)^{-1} \tag{1.1.23}$$

in the sense:

$$\begin{aligned}
\int d^D z \, \frac{\delta^2 W[J]}{\delta J(x)\delta J(y)} \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi(x)\delta \Phi(y)} &= \delta(x-y) \\
\int d^D z \, \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi(x)\delta \Phi(y)} \frac{\delta^2 W[J]}{\delta J(x)\delta J(y)} &= \delta(x-y)
\end{aligned} \tag{1.1.24}$$

In particular, setting the fields to zero, it gives the operator:

$$\left. \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi(x)\delta \Phi(y)} \right|_{\Phi=0} = \Gamma^{(2)}(x,y) \tag{1.1.25}$$

whose Green function is the exact propagator:

$$\int d^D z \, \Gamma^{(2)}(x,z) \, \langle \phi(z)\phi(y) \rangle = \delta(x-y) \tag{1.1.26}$$

similarly to the second derivative of the classical action which, setting the fields to zero, gives the kinetic operator of the free theory:

$$\left. \frac{\delta^2 S[\phi]}{\delta \phi(x)\delta \phi(y)} \right|_{\phi=0} = S^{(2)}(x,y) \tag{1.1.27}$$

whose Green function is the free propagator:

$$\int d^D z \, S^{(2)}(x,z) \, \langle \phi(z)\phi(y) \rangle_0 = \delta(x-y) \tag{1.1.28}$$

Finally, the the effective action is also the generator of 1-particle irreducible (1PI) vertices, i.e. the vertex operators which appear after factorizing from a connected correlation function all possible exact propagators. This follows essentially from the identities which can be obtained by taking derivatives of the inverse rule (**??**); for instance taking one derivative with respect to the field, one obtains the identity:

$$\begin{aligned}
\frac{\delta^3 W[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)} = \int & d^D y_1 d^D y_2 d^D y_3 \\
& \frac{\delta^3 \Gamma[\Phi]}{\delta \Phi(y_1)\delta \Phi(y_2)\delta \Phi(y_3)} \frac{\delta^2 W[J]}{\delta J(x)\delta J(y)} \frac{\delta^2 W[J]}{\delta J(x)\delta J(y)} \frac{\delta^2 W[J]}{\delta J(x)\delta J(y)}
\end{aligned} \tag{1.1.29}$$

which, setting sources and fields to zero, implies that the 3-point connected correlation function can be expressed as:

$$\begin{aligned}
\langle \phi(x_1)\phi(x_2)\phi(x_3) \rangle_c = \int & d^D y_1 d^D y_2 d^D y_3 \\
& \Gamma^{(2)}(y_1,y_2,y_3) \, \langle \phi(y_1)\phi(x_1) \rangle_c \, \langle \phi(y_2)\phi(x_2) \rangle_c \, \langle \phi(y_3)\phi(x_3) \rangle_c
\end{aligned} \tag{1.1.30}$$

i.e. as 3-point 1PI vertex attached to three exact propagators; in perturbation theory the vertex is a sum of 1PI diagrams, i.e. amputated diagrams without external lines which cannot be disconnected by cutting one of their internal lines. In general, the n-th functional derivative of the effective action gives n-point 1PI vertex:

$$\Gamma^{(n)}(x_1, \dots, x_n) = \frac{\delta}{\delta \Phi(x_1)} \cdots \frac{\delta}{\delta \Phi(x_n)} \Gamma[\Phi]\Big|_{\Phi=0} \tag{1.1.31}$$

which in perturbation theory is expressed as the local classical n-point vertex:

$$S^{(n)}(x_1, \dots, x_n) = \frac{\delta}{\delta \phi(x_1)} \cdots \frac{\delta}{\delta \phi(x_n)} S[\phi]\Big|_{\phi=0} \tag{1.1.32}$$

plus all the quantum loop corrections, generally non-local. A generic connected correlation function can be written as a finite sum of tree diagrams constructed with exact propagators and 1PI vertices.

1.1.2 Symmetries and Ward identities

Consider the scalar QFT presented in the previous subsection and a generic 1-parameter Lie transformation of the field, i.e. a transformation dependent on a continuous parameter which is smoothly connected to the identity and can be thus expressed in terms of an infinitesimal variation:

$$\phi(x) \quad \to \quad \phi^{(\alpha)}(x) = F_\alpha(\phi, \partial\phi, x) = \phi(x) + \delta_\alpha \phi(x) + O(\alpha^2) \tag{1.1.33}$$

where the parameter is now regarded as infinitesimal and the variation is of order $O(\alpha)$, i.e. it can be expressed as $\delta_\alpha \phi = \alpha \mathcal{G}(\phi)$, with $\mathcal{G}$ the generator of the transformation. Assume that at the classical level the theory is symmetric under such transformation:

$$S\left[\phi^{(\alpha)}\right] = S[\phi] \tag{1.1.34}$$

or in terms of the infinitesimal variation:

$$\delta_\alpha S[\phi] = 0 \tag{1.1.35}$$

At the quantum level, the behavior of the theory under the transformation can be evaluated by means of the path integral with sources (1.1.4). In particular, we can formally perform a change of variables in the path integral, which clearly does not affect its value, from the fields to the transformed fields; depending on the behavior of the path integral measure we have two possibilities, the transformation is non-anomalous, i.e. the measure is invariant:

$$\mathcal{D}\phi \quad \to \quad \mathcal{D}\phi^{(\alpha)} = \text{Det}\left[\frac{\delta \phi^{(\alpha)}}{\delta \phi}\right] \mathcal{D}\phi = \mathcal{D}\phi \tag{1.1.36}$$

vice versa the transformation is anomalous, i.e. the measure is not invariant:

$$\mathcal{D}\phi \quad \to \quad \mathcal{D}\phi^{(\alpha)} = \text{Det}\left[\frac{\delta \phi^{(\alpha)}}{\delta \phi}\right] \mathcal{D}\phi \neq \mathcal{D}\phi \tag{1.1.37}$$

In the first case we have that the symmetry is preserved also at the quantum level, in particular at the level of correlation functions. Indeed, changing variable from the field to the transformed field and then using the invariance of the action and the measure, the path integral can be also written as:

$$\begin{aligned} Z[J] &= \int \mathcal{D}\phi \, e^{-S[\phi]+\int d^D x \, J \, \phi} = \\ &= \int \mathcal{D}\phi^{(\alpha)} \, e^{-S[\phi^{(\alpha)}]+\int d^D x \, J \, \phi^{(\alpha)}} = \\ &= \int \mathcal{D}\phi \, e^{-S[\phi;]+\int d^D x \, J \, \phi^{(\alpha)}} \end{aligned} \tag{1.1.38}$$

where in the last the transformed field is now regarded as function of the non-transformed one. Comparing the first and last expression and taking arbitrary functionals derivatives with respect to the source, it follows that generic correlation functions of transformed and non-transformed fields are equal:

$$\left\langle \phi^{(\alpha)}(x_1) \dots \phi^{(\alpha)}(x_N) \right\rangle = \langle \phi(x_1) \dots \phi(x_N) \rangle \tag{1.1.39}$$

Vice versa, if the transformation is anomalous we have that the symmetry is broken at the quantum level:

$$\left\langle \phi^{(\alpha)}(x_1) \dots \phi^{(\alpha)}(x_N) \right\rangle \neq \langle \phi(x_1) \dots \phi(x_N) \rangle \tag{1.1.40}$$

This behavior holds also in presence of an arbitrary symmetry transformation, i.e. non-necessarily a Lie transformation smoothly connected to the identity. Making now use of the possibility of considering an infinitesimal transformation, we have, if the symmetry is non-anomalous:

$$\begin{aligned} Z[J] &= \int \mathcal{D}\phi \, e^{\int d^D x \, J \delta_\alpha \phi} \, e^{-S[\phi]+\int d^D x \, J \, \phi} = \\ &= Z[J] \left\langle 1 + \int d^D x \, J \, \delta_\alpha \phi + O(\alpha^2) \right\rangle_J = \\ &= Z[J] + Z[J] \left\langle \int d^D x \, J \, \delta_\alpha \phi \right\rangle_J + O(\alpha^2) \end{aligned}$$

from which it follows the Ward-Takahashi equation describing the infinitesimal symmetry at the quantum level:

$$\int d^D x \, J(x) \, \langle \delta_\theta \phi(x) \rangle_J = 0 \tag{1.1.41}$$

By taking arbitrary derivatives with respect to the sources and then setting them to zero, the equation can be used to generate a series of identities between correlation functions representing the constraints in which the symmetry is encoded at the quantum level; in particular, by taking N functional derivatives $\delta/\delta J(x_1) \dots \delta/\delta J(x_N)$ and setting the sources to zero, one obtains the Ward-Takahashi identity:

$$\begin{aligned} 0 &= \sum_{n=1}^{N} \langle \phi(x_1) \dots \delta_\alpha \phi(x_n) \dots \phi(x_N) \rangle \\ &= \langle \delta_\alpha \left(\phi(x_1) \dots \phi(x_n) \dots \phi(x_N) \right) \rangle \\ &= \delta_\alpha \left\langle \phi(x_1) \dots \phi(x_n) \dots \phi^{i_N}(x_N) \right\rangle \end{aligned} \tag{1.1.42}$$

The third expression follows from the assumed invariance of the path integral measure and can be seen as the infinitesimal form of (1.1.39).
If the symmetry is anomalous one must take into account the variation of the measure given by the jacobian of the transformation; the functional derivative inside the determinant can be rewritten, in explicit notation, as:

$$\frac{\delta\phi^{(\alpha)}(x)}{\delta\phi(y)} = \delta(x-y) + \frac{\delta}{\delta\phi(y)}\left(\delta_\alpha\phi(x)\right) \tag{1.1.43}$$

with the first term coming from the functional derivative $\delta\phi(x)/\delta\phi(y)$. Using now the functional extension of the matrix identity $\det\left(\mathbb{1}+\epsilon\mathrm{M}\right) = 1 + \epsilon\mathrm{trM} + O(\epsilon^2)$, with ϵ an expansion parameter (it is the $O(\epsilon)$-order approximation of the exact matrix identity $\det e^{\epsilon\mathrm{M}} = e^{\epsilon\mathrm{trM}}$), the jacobian is rewritten, in explicit notation, as:

$$\mathrm{Det}\left[\frac{\delta\phi^{(\alpha)}(x)}{\delta\phi(y)}\right] = 1 + \mathrm{Tr}\left[\frac{\delta}{\delta\phi(y)}\left(\delta_\alpha\phi(x)\right)\right] + O(\alpha^2) \tag{1.1.44}$$

Taking into account the additional term appearing at order $O(\alpha)$, we have that the non-anomalous Ward-Takahashi equation is substituted by:

$$\int d^Dx\, J(x)\left\langle\delta_\alpha\phi(x)\right\rangle_J = -\left\langle \mathrm{Tr}\left[\frac{\delta}{\delta\phi(y)}\left(\delta_\alpha\phi(x)\right)\right]\right\rangle_J \tag{1.1.45}$$

The value of the trace on the right hand side is known as anomaly and represents the fact that the anomalous infinitesimal symmetry is broken at the quantum level.

1.2 Elements of Functional Renormalization Group

In this section we briefly review the implementation of the FRG machinery in a QFT, considering for simplicity a scalar theory, in order to anticipate the fundamental ideas at the base of the method which in the following will be applied to the gravitational interaction. In particular, we introduce the concepts of FRG regularization and Wetterich-Morris equation. We suggest for instance [5] for a detailed review.

1.2.1 Regularization

The FRG is a particular methodology of implementing the fundamental idea at the base of the wilsonian Renormalization Group framework [2]. This corresponds to tackling the problem of solving the QFT not by trying to compute the path integral integrating all the quantum fluctuations at once, but by integrating portions of fluctuations, which typically results in an scale-dependent average theory in between the classical theory, i.e. no fluctuation integrated, and the quantum theory, i.e. all fluctuations integrated, and then studying the scale-dependence of such average theory.

In the wilsonian approach the idea is essentially implemented by imposing a UV cut-off k_{UV} in the momentum of integrated field modes in momentum space in the path integral and then performing a coarse-graining of modes, i.e. formally integrating the modes in a momentum shell below the cut-off, which results in lowering the cut-off by a certain scale factor $x > 1$, $k'_{UV} = k_{UV}/x$. The result can be interpreted as the path integral of a wilsonian effective action which describes the original theory at an energy scale lowered by the scale factor x; formally iterating the procedure and going in the continuum limit in which the scale is changed by an infinitesimal amount, one obtains the scale dependent theory which reproduces the complete one (still UV-regulated) in the limit in which all modes are integrated.
In the FRG approach the coarse-graining idea is implemented not by formally integrating in steps subsequent shells of modes, but by formally integrating one "extendable" shell of modes between a UV cut-off k_{UV} and a floating IR cut off k, obtaining a scale dependent theory which reproduces the complete one (still UV-regulated) when k is lowered from k_{UV} to zero and all modes are integrated.

In order to make the construction precise, consider now the scalar QFT described in section 1.1. The first step of the FRG method is the regularization procedure of the path integral and the generating functionals introduced in subsection 1.1.1. Consider the expansion of the field in momentum space:

$$\phi(x) = \int \frac{d^D p}{(2\pi)^D} e^{ipx} \tilde{\phi}(p) \tag{1.2.1}$$

i.e. in the basis of Fourier plane waves $u_p(x) = e^{ipx}$, eigenfunctions of the laplacian operator $\Box = \partial^2$:

$$-\Box\, e^{ipx} = p^2\, e^{ipx} \tag{1.2.2}$$

The path integral (1.1.4) is rewritten in momentum space by changing variables from the fields to the Fourier weights (which produces an irrelevant multiplicative constant in the measure):

$$Z \propto \int \mathcal{D}\tilde{\phi}\, e^{-S[\tilde{\phi}]} \tag{1.2.3}$$

In this form the contributions to the path integral are ordered by the value of momentum p^2 of the Fourier weights, i.e. the eigenvalue of the negative laplacian of the corresponding modes. The UV regularization is achieved with a sharp UV cut-off k_{UV} on the momenta of modes considered in the integration:

$$\mathcal{D}\tilde{\phi} = \prod_{0\leq|p|\leq|k_{UV}|} d\tilde{\phi}(p) \tag{1.2.4}$$

which in the following we leave implicit. The IR regularization is achieved with a smooth and floating IR cut-off k implemented by deforming the action in the path integral with a suitable quadratic regulator term:

$$\begin{aligned}\Delta S_{(k)} &= \frac{1}{2}\int d^D x \phi(x) \mathcal{R}_{(k)}\left(-\Box\right)\phi(x) \\ &= \frac{1}{2}\int \frac{d^D p}{(2\pi)^D} \tilde{\phi}(-p)\mathcal{R}_{(k)}(p^2)\tilde{\phi}(p)\end{aligned} \tag{1.2.5}$$

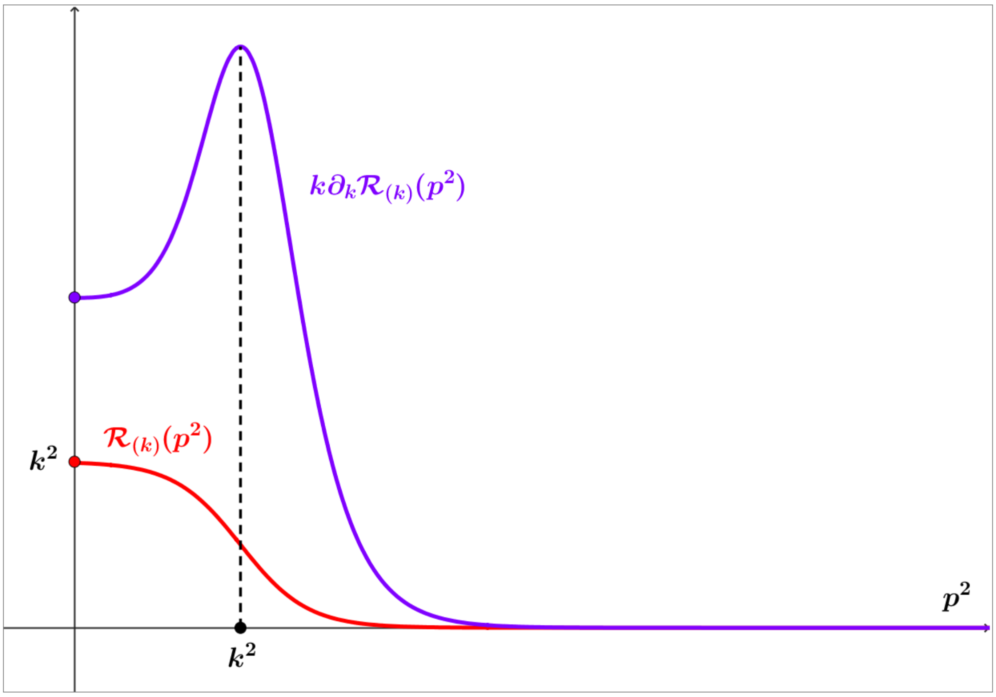

Figure 1: *Qualitative functional form of a regulator $\mathcal{R}_{(k)}(p^2)$ (red line) and its logarithmic derivative $k\partial_k\mathcal{R}_{(k)}(p^2)$ (purple line).*

which has the effect of blocking the integration of modes which have momentum squared less than the floating scale k^2; this is achieved by choosing the functional form of the regulator in momentum space to satisfy the properties:

$$\begin{cases} \mathbf{1.} \quad \mathcal{R}_{(k)}(p^2) \quad \to \quad > 0 \quad , \quad \frac{p^2}{k^2} \to 0 \\ \mathbf{2.} \quad \mathcal{R}_{(k)}(p^2) \quad \to \quad 0 \quad , \quad \frac{p^2}{k^2} \to \infty \\ \mathbf{3.} \quad \mathcal{R}_{(k)}(p^2) \quad \to \quad \infty \quad , \quad k^2 \to k^2_{UV} \to \infty \\ \mathbf{4.} \quad \mathcal{R}_{(k)}(p^2) \quad \to \quad 0 \quad , \quad k^2 \to 0 \end{cases} \tag{1.2.6}$$

Indeed, in the path integral thanks to property **1.** Fourier weights relative to modes with eigenvalue $p^2 \lesssim k^2$ are suppressed, since their contributions receives an exponential damping $e^{-\Delta S_{(k)}} \sim 0$; and thanks to property **2.** and those relative to modes with eigenvalue $p^2 \gtrsim k^2$ are left untouched, since the exponential damping disappears in this regime $e^{-\Delta S_{(k)}} \sim 1$:

$$\begin{aligned} \tilde{Z}_{(k)}[\bar{g}] &= \int \mathcal{D}\phi \, e^{-S[\phi]-\Delta S_{(k)}[\phi]} = \\ &= \int \mathcal{D}\phi \, e^{-S[\phi]-\frac{1}{2}\int d^Dx\, \phi \mathcal{R}_{(k)}(-\Box)\phi} = \\ &\propto \int \mathcal{D}\tilde{\phi} \, e^{-S[\phi]-\frac{1}{2}\int \frac{d^Dp}{(2\pi)^D}\tilde{\phi}(-p)\mathcal{R}_{(k)}(p^2)\tilde{\phi}(p)} = \end{aligned}$$

$$\sim \int \prod_{|k|\lesssim|p|\lesssim|k_{UV}|} d\tilde{\phi}(p)\, e^{-S[\phi]} \tag{1.2.7}$$

The role of the additional properties required will be discussed below. In order to guarantee those properties one can consider a functional dependence for the regulators of the type:

$$\mathcal{R}_{(k)}(p^2) = k^2 \mathcal{R}_0 \left(\frac{p^2}{k^2}\right) \tag{1.2.8}$$

where $\mathcal{R}_0(x)$ is a dimensionless and positive shape function which interpolates between $\mathcal{R}_0(0) = 1$ and $\mathcal{R}_0(\infty) = 0$:

$$\mathcal{R}_0(x) \begin{cases} \to 1 & , \quad x \to 0 \\ \to 0 & , \quad x \to \infty \end{cases} \tag{1.2.9}$$

The typical behavior of a regulator and its derivative in k is depicted in figure 1. In particular, given this functional dependence, the quadratic regulator term can be seen as a momentum-dependent mass term, which goes from k^2 to zero increasing the momentum. Adding the regulator term to the action we define the regulated action:

$$\tilde{S}_{(k)}[\phi] = S[\phi] + \Delta S_{(k)}[\phi] \tag{1.2.10}$$

The quantum theory is described by the generating functionals introduced in subsection 1.1.1 constructed with the regulated action and with the implicit UV cut-off k_{UV}:

$$\tilde{Z}_{(k)}[J] = \int \mathcal{D}\phi\, e^{-\tilde{S}_{(k)}[\phi] - S_{source}[\phi;J]} \tag{1.2.11}$$

$$\tilde{W}_{(k)}[J] = \log \tilde{Z}_{(k)}[J] \tag{1.2.12}$$

$$\begin{aligned} \tilde{\Gamma}_{(k)}[\Phi] &= \sup_J \left\{ \int d^D x\, J\, \Phi - \tilde{W}_{(k)}[J] \right\} \\ &= \int d^D x\, J\, \Phi - \tilde{W}_{(k)}[J] \end{aligned} \tag{1.2.13}$$

$$\begin{aligned} \tilde{W}_{(k)}[J] &= \sup_\Phi \left\{ \int d^D x\, J\, \Phi - \tilde{\Gamma}_{(k)}[\Phi] \right\} \\ &= \int d^D x\, J\, \Phi - \tilde{\Gamma}_{(k)}[\Phi] \end{aligned} \tag{1.2.14}$$

Now all averages are k-dependent, as well as the relation between field and source in the Legendre transform, $J = J_{(k)}(\Phi)$, $\Phi = \Phi_{(k)}(J)$:

$$\Phi(x) = \langle \phi(x) \rangle_J = \frac{\delta \tilde{W}_{(k)}[J]}{\delta J(x)} \tag{1.2.15}$$

$$J(x) = \frac{\delta \tilde{\Gamma}_{(k)}[\Phi]}{\delta \Phi(x)} \tag{1.2.16}$$

The integro-differential equation for the regulated effective action is:

$$e^{-\tilde{\Gamma}_{(k)}[\Phi]} = \int \mathcal{D}\phi \, \exp\left\{ -\tilde{S}_{(k)}[\phi] + \int d^D x \frac{\delta \tilde{\Gamma}_{(k)}[\Phi]}{\delta \Phi} (\phi - \Phi) \right\} \tag{1.2.17}$$

In the FRG framework the object which describes the scale-dependent average theory, similarly to the wilsonian effective action in the wilsonian approach, is the effective average action, whose formal definition, i.e. based on the above generating functionals, is:

$$\Gamma_{(k)}[\Phi] = \tilde{\Gamma}_{(k)}[\Phi] - \Delta S_{(k)}[\Phi] \tag{1.2.18}$$

Thanks to the second couple of properties in (1.2.6), this object interpolates by construction between the classical and the quantum theory, since we have the limits:

$$\Gamma_{(k \to 0)}[\Phi] = \Gamma[\Phi] \tag{1.2.19}$$

$$\Gamma_{(k \to k_{UV} \to \infty)}[\Phi] = S[\Phi] + \cdots \tag{1.2.20}$$

In particular, for $k \to 0$, the effective average action tends to the unregulated quantum effective action since $\Delta S_{(k \to 0)} \to 0$ (property **4.**):

$$\Gamma_{(k \to 0)}[\Phi] = \tilde{\Gamma}_{(k \to 0)}[\Phi] - \Delta S_{(k \to 0)}[\Phi] = \Gamma[\Phi] - 0$$

For $k \to k_{UV} \to \infty$, the effective average action tends approximately to the unregulated classic action; this can be seen from the integro-differential equation (1.2.17) rewritten for the effective average action:

$$\begin{aligned} e^{-\Gamma_{(k)}[\Phi]} = \int \mathcal{D}\phi \, \exp\Bigg\{ & -S[\phi] - \Delta S_{(k)}[\phi] + \Delta S_{(k)}[\Phi] \\ & + \int d^D x \frac{\delta \Delta S_{(k)}[\Phi]}{\delta \Phi} (\phi - \Phi) + \int d^D x \frac{\delta \Gamma_{(k)}[\Phi]}{\delta \Phi} (\phi - \Phi) \Bigg\} \end{aligned} \tag{1.2.21}$$

where the difference of regulator terms plus the piece with the derivative gives precisely the regulator term itself computed in the difference between the integrated and average field $\phi - \Phi$:

$$-\Delta S_{(k)}[\phi] + \Delta S_{(k)}[\Phi] + \int d^D x \frac{\delta \Delta S_{(k)}[\Phi]}{\delta \Phi} (\phi - \Phi) = -\Delta S_{(k)}[\phi - \Phi] \tag{1.2.22}$$

Therefore, since $\mathcal{R}_{(k \to k_{UV} \to \infty)} \to \infty$ (property **3.**), one can recognize the functional equivalent of the gaussian limit representation of a Dirac delta $\delta(x-y) = \lim_{R\to\infty} \sqrt{R/2\pi} \exp[-R/2(x-y)^2]$, and we have approximately:

$$e^{-\Delta S_{(k)}[\phi - \Phi]} \overset{k \to k_{UV} \to \infty}{\longrightarrow} \sim \delta[\phi - \Phi] \tag{1.2.23}$$

Therefore in the integro-differential equation, in the limit $k \to k_{UV} \to \infty$, we obtain as dominant term the integrand computed in $\phi = \Phi$, i.e. $\exp(-S[\Phi])$, from which it follows (1.2.20). Determining the precise relation between the effective average action in the limit $k \to k_{UV} \to \infty$ and the classical action, i.e. determining the corrective terms in (1.2.20), is known in FRG theory as reconstruction problem and will not be discussed in the thesis, see for instance [5] for a general treatment.

1.2.2 Wetterich-Morris equation

The second step of the FRG method is deriving from the formal definition of effective average action (1.2.18) a differential equation describing its flow as the FRG scale k varies, so that the problem of solving the theory, i.e. computing the path integral and the generating functionals, can be then translated into finding a solution of the equation. The equation follows from computing the derivative in FRG time:

$$t = \log k \tag{1.2.24}$$

of the regulated effective action and then rewriting the result in terms of the effective average action according to the formal definition. From the definitions of the regulated effective action (1.2.13) and the path integral logarithm (1.2.14) we have the equalities:

$$\begin{aligned}\partial_t \tilde{\Gamma}_{(k)}[\Phi] &= -\partial_t \tilde{W}_{(k)}[J = -\frac{\partial_t \tilde{Z}_{(k)}[J]}{\tilde{Z}_{(k)}[J]} \\ &= \left\langle \partial_t \Delta S_{(k)}[\phi] \right\rangle_J\end{aligned} \tag{1.2.25}$$

where it is understood that the time derivatives of source-dependent objects are computed in $J = J_{(k)}(\Phi)$. In particular, the first equality comes from the properties of the Legendre transform recalling that $J = J_{(k)}(\Phi, K)$ according to (1.2.15):

$$\partial_t \tilde{\Gamma}_{(k)} = \int d^D x \, \cancel{\partial_t J_{(k)} \Phi} - \left(\partial_t \tilde{W}_{(k)}|_{J_{(k)}} + \int d^D x \, \cancel{\partial_t J_{(k)} \frac{\delta \tilde{W}_{(k)}}{\delta J}} \right)$$

In particular:

$$\partial_t \tilde{\Gamma}_{(k)}[\Phi] = \left\langle \partial_t \Delta S_{(k)}[\phi] \right\rangle_J \tag{1.2.26}$$

Substituting the explicit expression of the regulator term (1.2.5) and introducing an additional integration to separate it from the field on which it acts, we have:

$$\begin{aligned}\partial_t \tilde{\Gamma}_{(k)}[\Phi] &= \frac{1}{2} \int d^D x \left\langle \phi(x) \partial_t \mathcal{R}_{(k)} \left(-\Box\right) \phi(x) \right\rangle_J = \\ &= \frac{1}{2} \int d^D x d^D y \delta(x-y) \partial_t \mathcal{R}_{(k)} \left(-\Box_{(y)}\right) \langle \phi(x) \phi(y) \rangle_J \\ &= \mathrm{Tr} \left[\frac{1}{2} \partial_t \mathcal{R}_{(k)} \left(-\Box\right) \langle \phi \otimes \phi \rangle_J \right]\end{aligned} \tag{1.2.27}$$

In the last expression we introduced the compact notation for the functional trace and it is understood that the regulator acts in the second of the two terms in the direct product.

One can now write the 2-point correlation function (derivatives of the regulated path integral) in terms of connected correlation functions (derivatives of the regulated path integral logarithm) according to (1.1.16):

$$\frac{1}{\tilde{Z}_{(k)}[J]}\frac{\delta^2 Z_{(k)}[J]}{\delta J(x)\delta J(y)} = \frac{\delta^2 \tilde{W}_{(k)}[J]}{\delta J(x)\delta J(y)} + \frac{\delta \tilde{W}_{(k)}[J]}{\delta J(x)}\frac{\delta \tilde{W}_{(k)}[J]}{\delta J(y)} \tag{1.2.28}$$

The product term simply gives a product of average fields $\langle\phi(x)\rangle_J \langle\phi(y)\rangle_J = \Phi(x)\Phi(y)$ which traced give back the derivative of the regulator term (1.2.5) computed in the average field $\partial_t \Delta S[\Phi]$:

$$\mathrm{Tr}\left[\frac{1}{2}\partial_t \mathcal{R}_{(k)}\left(-\Box\right)\Phi\otimes\Phi\right] = \partial_t \Delta S[\Phi] \tag{1.2.29}$$

the first term can be written as inverse of the second derivative of the regulated effective action according to (1.1.23):

$$\frac{\delta^2 \tilde{W}_{(k)}[J]}{\delta J(x)\delta J(y)} = \left(\frac{\delta^2 \tilde{\Gamma}_{(k)}[\Phi]}{\delta\Phi(x)\delta\Phi(y)}\right)^{-1} \tag{1.2.30}$$

The result is the flow equation for the regulated effective action:

$$\partial_t \tilde{\Gamma}_{(k)}[\Phi] = \mathrm{Tr}\left[\frac{1}{2}\partial_t \mathcal{R}_{(k)}\left(-\Box\right)\left(\frac{\delta^2 \tilde{\Gamma}_{(k)}[\Phi]}{\delta\Phi\delta\Phi}\right)^{-1}\right] + \partial_t \Delta S_{(k)}[\Phi] \tag{1.2.31}$$

Finally, using the definition to write the regulated effective action as $\tilde{\Gamma}_{(k)} = \Gamma_{(k)} + \Delta S_{(k)}$, and the relation:

$$\frac{\delta^2 \tilde{\Gamma}_{(k)}[\Phi]}{\delta\Phi(x)\delta\Phi(y)} = \frac{\delta^2 \Gamma_{(k)}[\Phi]}{\delta\Phi(x)\delta\Phi(y)} + \mathcal{R}_{(k)}\left(-\Box_{(x)}\right)\delta(x-y) \tag{1.2.32}$$

one obtains the Wetterich-Morris equation for the effective average action [4][9]:

$$\partial_t \Gamma_{(k)}[\Phi] = \mathrm{Tr}\left[\frac{1}{2}\partial_t \mathcal{R}_{(k)}\left(-\Box\right)\left(\frac{\delta^2 \Gamma_{(k)}[\Phi]}{\delta\Phi\delta\Phi} + \mathcal{R}_{(k)}\left(-\Box\right)\right)^{-1}\right] \tag{1.2.33}$$

We make the following remarks:

1. The equation is derived from the formal definition of effective average action (1.2.18) based on the generating functionals, which define the quantum theory. At this point it possible to reverse the perspective and consider the Wetterich-Morris equation as the fundamental object defining the quantum theory and the effective average action as its solution: according to the limits (1.2.19), (1.2.20), given the classical theory described by the classical action S, the solution of the equation describes a flow which leads to the quantum theory described by the effective action Γ, i.e. $\sim S \xrightarrow{\infty\to k} \Gamma_{(k)} \xrightarrow{k\to 0} \Gamma$. In particular, the classical action does not enter in the derivation of the equation and plays only the role of formal initial condition of the flow.

2. The equation formally inherits from the formal definition of effective average action (1.2.18) an implicit dependence on the UV cut-off k_{UV}, however it can be safely removed by letting $k_{UV} \to \infty$: formally expressing the traced operator in momentum space and the trace as a sum over momenta, the derivative of the regulator has, thanks to properties (1.2.6), the qualitative behavior depicted in figure 1; therefore, the dominant contributions to the sum come only from a narrow band of momenta centered around k and those from the UV region are suppressed.

3. In the derivation no approximation is introduced and the Wetterich-Morris equation is thus an exact functional differential equation with second order derivatives. In particular, it has a 1-loop structure, since the right hand side is given by tracing the exact regulated propagator $(\delta^2\Gamma/\delta\Phi\delta\Phi)^{-1} = \delta^2 W/\delta J\delta J = \langle\phi\phi\rangle_J$, with an insertion of the regulator. The equation contains only second derivatives due to the quadratic nature of the regulator (1.2.5), which makes appear inside $\left\langle\partial_t \Delta S_{(k)}[\phi]\right\rangle_J$ in (2.4.37) only 2-point correlation functions.

4. The flow described by the equation can be visualized as a trajectory inside the theory space, or space of actions, i.e. the abstract infinite-dimensional space spanned by all possible operators compatible with the symmetries of the theory which in principle can appear inside the action. By construction, the flow starts and ends in the two points representing respectively the classical action (plus counter terms) and the quantum effective action; the specific shape of the trajectory depends on the regulator used (figure 2).

Picking a basis of linear independent action functionals $\{I_a[\Phi]\}$ in the theory space:

$$\mathcal{T} = \text{span}\{I_a[\Phi]\} \tag{1.2.34}$$

a generic action functional can be expanded as:

$$A[\Phi] = \sum_{a=1}^{\infty} u^a I_a[\Phi] \tag{1.2.35}$$

where the so called generalized coupling constants $\{u^a\}$ can be seen as coordinates of the action functional in theory space. The effective average action can be itself expressed, for any fixed value of k, in the basis of functionals $\{I_a[\Phi]\}$:

$$\Gamma_{(k)}[\Phi] = \sum_{a=1}^{\infty} u^a(k) I_a[\Phi] \tag{1.2.36}$$

The k-dependence is contained in the so called running coupling constants $\{u^a(k)\}$. Substituting the formal expression in the Wetterich-Morris equation, one obtains:

$$\begin{aligned}\sum_{a=1}^{\infty} \partial_t u^a(k) I_a[\Phi] &= \text{Tr}\left[\frac{1}{2}\partial_t \mathcal{R}_{(k)}(-\Box)\left(\sum_{a=1}^{\infty} u^a(k)\frac{\delta^2 I[\Phi]}{\delta\Phi\delta\Phi} + \mathcal{R}_{(k)}(-\Box)\right)^{-1}\right] = \\ &= \sum_{a=1}^{\infty} b^a\left(\left\{u^b(k)\right\}, k\right) I_a[\Phi]\end{aligned} \tag{1.2.37}$$

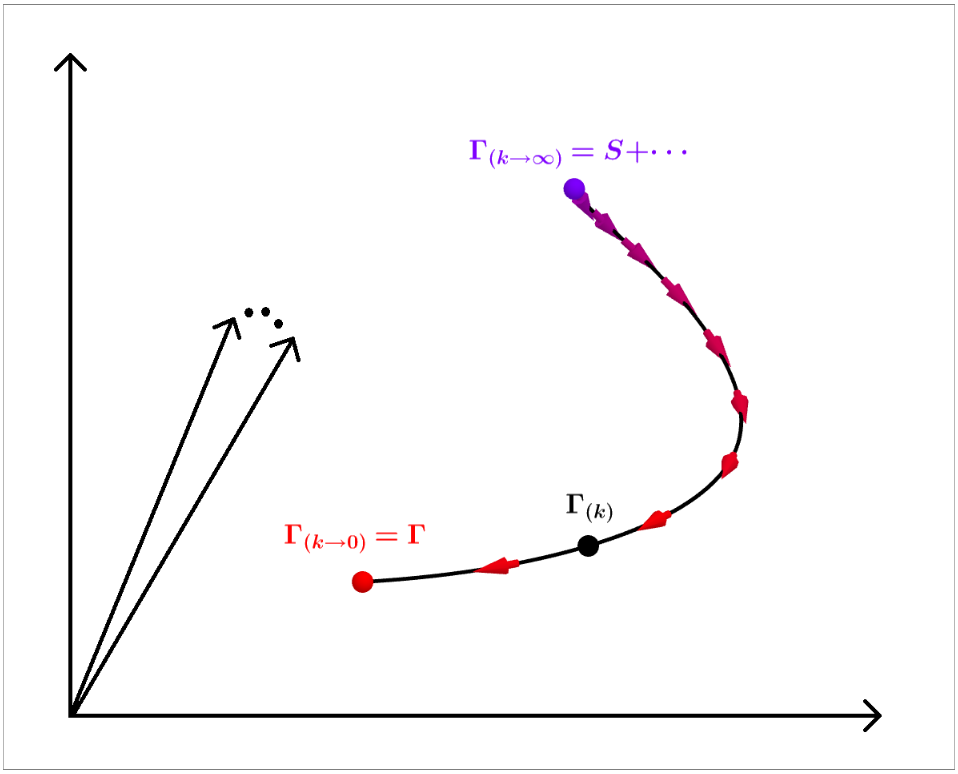

Figure 2: *Pictorial representation of the FRG flow in theory space described by the Wetterich-Morris equation for the effective average action* $\Gamma_{(k)}$*, which unfolds from the classic action* S *(plus counter terms) to the quantum effective action* Γ*.*

since also the right hand side must be expressible in terms of the basis functionals; the functions $b^a\left(\left\{u^b(k)\right\}, k\right)$ are called beta functions and have in general both an explicit and an implicit k-dependence, respectively due to the presence of the regulator and the dependence on the set of all running coupling constants. Projecting in the basis functionals, one obtains the Wetterich-Morris equation in component form:

$$\partial_t u^a(k) = b^a\left(\left\{u^b(k)\right\}, k\right) \tag{1.2.38}$$

which is a coupled system of infinitely many ordinary differential equations for the running coupling constants.
In order to tackle the problem of solving the system, approximation methods are required. The common strategy is the method of truncation, i.e. projecting the Wetterich-Morris equation in a subspace of the full theory space $\mathcal{T}_{trunc} \subset \mathcal{T}$, by proposing a restrictive ansatz for the solution:

$$\Gamma_{(k)}[\Phi] = \sum_{i=1}^{N} u^i(k) I_i[\Phi] \tag{1.2.39}$$

Inserting in the Wetterich-Morris equation, the right hand side generates in general also terms outside the subspace, which have no counter part in the left hand side:

$$\sum_{i=1}^{N} \partial_t u^i(k) I_i[\Phi] = \mathrm{Tr}\left[\frac{1}{2}\partial_t \mathcal{R}_{(k)}(-\Box)\left(\sum_{i=1}^{N} u^i(k)\frac{\delta^2 I[\Phi]}{\delta\Phi\delta\Phi} + \mathcal{R}_{(k)}(-\Box)\right)^{-1}\right] =$$

$$= \sum_{i=1}^{N} b^i \left(\left\{ u^j(k) \right\}, k \right) I_i[\Phi] + \sum_{a\,,\, I_a \notin \mathcal{T}_{trunc}}^{\infty} b^a \left(\left\{ u^j(k) \right\}, k \right) I_a[\Phi] \qquad (1.2.40)$$

therefore, in order to render the approximation consistent, one must assume that it is legitimate to neglect the scale-dependence in the directions leaving from the subspace:

$$b^a \equiv 0 \qquad \forall\, a \;:\; I_a \notin \mathcal{T}_{trunc} \qquad (1.2.41)$$

and the system of differential equations for the running couplings becomes a closed system of N equations:

$$\partial_t u^i(k) = b^i \left(\left\{ u^j(k) \right\}, k \right) \qquad (1.2.42)$$

which can be then solved with standard methods. The solution is an approximate non-perturbative solution and therefore it can still give information beyond the realm of perturbation theory.

1.3 Elements of General Relativity

In this section we list the elementary notions of General Relativity, in its basic metric formulation, to which we will refer in the following, assuming prior knowledge of the language of differential geometry. In particular, we briefly review the foundations, the lagrangian formulation and role of diffeomorphisms as gauge transformations. We suggest for instance [10] for a detailed review.

1.3.1 Principles

General Relativity is the theory developed by Einstein to overcome the limitations of Special Relativity, namely the extension of the relativity principle to inertial and global observers - reference frames only, and the absence of a theory of gravity due to the incompatibility with newtonian gravity. As a result the theory has a double role: it is the theory of relativity which generalizes the relativity principle to all observes - reference frames, in particular those which are arbitrary accelerated and local; and it is the classical field theory describing gravitational interaction compatibly with the generalized relativity principle. The first aspect of the theory is founded on the general relativity and equivalence principles:

> **General relativity principle**: *The laws of physics have the same form in all reference frames.*
>
> **Equivalence principle (strong)**: *For each event in spacetime there always exists a local reference frame in which all gravitational effects vanish.*

The general relativity principle is mathematically translated in the description of physics in the language of differential geometry:

1. Events in spacetime are identified with points in a 4-dimensional[1] differentiable manifold, i.e. a 4-dimensional topological set $\mathcal{M}$ of points P which can be covered with charts of arbitrary coordinates $x^\mu : \mathcal{M} \to \mathbb{R}^4$ interchangeable according to general coordinate transformations, differentiable and invertible:

$$\begin{cases} x^\mu \quad \to \quad x'^\mu = x'^\mu(x) \\ \qquad \det \dfrac{\partial x'^\mu}{\partial x^\nu} \neq 0 \end{cases} \tag{1.3.1}$$

The local structure of the manifold in a point P, i.e. the tangent space $T_P\mathcal{M}$, is isomorphic to $\mathbb{R}^4$, $T_P\mathcal{M} \cong \mathbb{R}^4$.

2. A reference frame is identified with a set of vector fields e_μ which provide a basis for vectors in the tangent spaces $T_P\mathcal{M}$ of the manifold; the dual 1-forms fields $\tilde{e}^\mu$, i.e. $\tilde{e}^\mu(e_\nu) = \delta^\mu_\nu$, provide a basis for 1-forms in the cotangent spaces $T^*_P\mathcal{M}$. The choice of a coordinate system x^μ induces a reference frame given by the coordinate basis vectors $e_\mu = \partial_\mu$ and the coordinate basis 1-forms $\tilde{e}^\mu = \tilde{dx}^\mu$. Reference frames and observers are not in 1-1 correspondence. In particular, physical observers, i.e. finitely extended measurement apparatus, are identified with local reference frames.

3. Laws of physics are written in terms of tensors and tensorial operations:

$$A = B \tag{1.3.2}$$

and can be written in any reference frame:

$$A^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n}(x) = B^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n}(x) \tag{1.3.3}$$

by using the corresponding basis to expand tensors in components:

$$A = A^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n} e_{\mu_1} \otimes \cdots \otimes e_{\mu_m} \otimes \tilde{e}^{\nu_1} \otimes \cdots \otimes \tilde{e}^{\nu_n} \tag{1.3.4}$$

The form of the law is covariant under an arbitrary change of coordinates:

$$\begin{aligned} x^\mu \quad &\to \quad x'^\mu = x'^\mu(x) \\ e_\mu \quad &\to \quad e'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} e_\nu \\ \tilde{e}^\mu \quad &\to \quad \tilde{e}'^{\,\mu} = \frac{\partial x'^\mu}{\partial x^\nu} \tilde{e}^\nu \end{aligned} \tag{1.3.5}$$

according to the transformation rule of tensor components:

$$A^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n}(x) \quad \to \quad A'^{\,\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n}(x') = \frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\mu_m}}{\partial x^{\alpha_m}} \frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\beta_n}}{\partial x'^{\nu_n}} A^{\alpha_1\ldots\alpha_m}{}_{\beta_1\ldots\beta_n}(x) \tag{1.3.6}$$

[1] In this subsection, in order to review the physical foundations of General relativity, we consider $D = 4$ and the lorentzian signature $(-,+,+,+)$ for the metric.

4. The spacetime manifold must also have a riemannian metric structure $(\mathcal{M}, g)$, i.e. it must be equipped with a metric tensor field $g_{\mu\nu}$:

$$g = g_{\mu\nu}\tilde{e}^{\mu} \otimes \tilde{e}^{\nu} \tag{1.3.7}$$

i.e. a rank-$(0, 2)$ tensor defining a non-degenerate scalar product between vectors, i.e. a measure of angles:

$$g(v, w) = v \cdot w = g_{\mu\nu}v^{\mu}w^{\nu} \tag{1.3.8}$$

a measure of lengths, via the line element (in a coordinate basis):

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} \tag{1.3.9}$$

and a measure of volumes, via the volume differential form (in a coordinate basis):

$$\omega = \sqrt{|g|}\, \tilde{dx}^1 \wedge \tilde{dx}^2 \wedge \tilde{dx}^3 \wedge \tilde{dx}^4 \tag{1.3.10}$$

with the metric determinant $g = \det g_{\mu\nu}$. We recall the result that there is always a reference frame for which the matrix of components of the metric tensor at a point P is in canonical form, i.e. diagonalized with ± 1 on the diagonal, depending on its signature:

$$g_{\mu\nu}|_P = \pm\delta_{\mu\nu} \tag{1.3.11}$$

5. The spacetime manifold is in general curved, i.e. characterized by an intrinsic curvature: defined a notion of parallel transport for vectors, one has that vectors parallely transported in a closed path return at the starting point rotated with respect to their original orientation. In mathematical terms, this is represented by the non-commutativity of the covariant derivative ∇_{μ} defining the notion of parallel transport, i.e. a vector w is said to be parallely transported along a curve with tangent vector v if $\nabla_v w^{\nu} = v^{\mu}\nabla_{\mu}w^{\nu} = 0$. The non-commutativity of a covariant derivative, and thus the intrinsic curvature of a manifold, is described by the Riemann tensor $R^{\alpha}{}_{\mu\beta\nu}$, which is defined by the identity:

$$[\nabla_{\mu}, \nabla_{\nu}]v^{\alpha} = R^{\alpha}{}_{\beta\mu\nu}v^{\beta} - T^{\lambda}{}_{\mu\nu}\nabla_{\lambda}v^{\alpha} \tag{1.3.12}$$

where $T^{\lambda}{}_{\mu\nu}$ is the torsion tensor:

$$T^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} \tag{1.3.13}$$

with $\Gamma^{\lambda}_{\mu\nu}$ the affine connection associated the covariant derivative. The components of the Riemann tensor are given by:

$$R^{\alpha}{}_{\mu\beta\nu} = \partial_{\beta}\Gamma^{\alpha}_{\mu\nu} - \partial_{\nu}\Gamma^{\alpha}_{\mu\beta} + \Gamma^{\lambda}_{\beta\mu}\Gamma^{\alpha}_{\lambda\nu} - \Gamma^{\lambda}_{\nu\mu}\Gamma^{\alpha}_{\lambda\beta} \tag{1.3.14}$$

They are characterized by the symmetry properties:

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} \tag{1.3.15}$$
$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta} \tag{1.3.16}$$

and by the algebraic and derivative Bianchi identities:

$$R_{\alpha\beta\mu\nu} + R_{\alpha\mu\nu\beta} + R_{\alpha\nu\beta\mu} = 0 \tag{1.3.17}$$
$$\nabla_\lambda R_{\alpha\beta\mu\nu} + \nabla_\alpha R_{\beta\lambda\mu\nu} + \nabla_\beta R_{\lambda\alpha\mu\nu} = 0 \tag{1.3.18}$$

As a result of those constraints the Riemann tensor has, in D dimensions, $D^2(D^2-1)/12$ independent components. The contractions of the Riemann tensor are the Ricci tensor:

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} \tag{1.3.19}$$

which is symmetric, and the Ricci scalar curvature:

$$R = g^{\mu\nu} R_{\mu\nu} \tag{1.3.20}$$

We recall that the affine connection transforms non-tensorially under a change of coordinates as:

$$\Gamma^{\lambda}_{\mu\nu}(x) \quad \rightarrow \quad \Gamma'^{\,\lambda}_{\mu\nu}(x') = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \Gamma^{\rho}_{\alpha\beta}(x) + \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\nu}} \tag{1.3.21}$$

A difference of affine connections does instead transform tensorially, and in particular an antisymmetrized affine connection, which in fact gives the torsion tensor. In the thesis we will consider only the torsion-less metric connection, defined by the compatibility condition of the covariant derivative with the metric:

$$\nabla_\lambda g_{\mu\nu} = 0 \tag{1.3.22}$$

and given by the expression:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{g^{\lambda\rho}}{2} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} \right) \tag{1.3.23}$$

We recall also the the action of the covariant derivative on a generic rank-(m,n) tensor:

$$\begin{aligned} \nabla_\lambda A^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} = {} & \partial_\lambda A^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} \\ & + \Gamma^{\mu_1}_{\rho\lambda} A^{\rho \dots \mu_m}{}_{\nu_1 \dots \nu_n} + \dots + \Gamma^{\mu_m}_{\rho\lambda} A^{\mu_1 \dots \rho}{}_{\nu_1 \dots \nu_n} \\ & - \Gamma^{\rho}_{\nu_1\lambda} A^{\mu_1 \dots \mu_m}{}_{\rho \dots \nu_n} - \dots - \Gamma^{\rho}_{\nu_n\lambda} A^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \rho} \end{aligned} \tag{1.3.24}$$

and the generalization of (1.3.12) for the commutator of covariant derivatives on rank-(m,n) tensor:

$$\begin{aligned} [\nabla_\alpha, \nabla_\beta] A^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} = & - T^{\lambda}{}_{\mu\nu} \nabla_\lambda A^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} \\ & + R^{\mu_1}{}_{\rho\alpha\beta} A^{\rho \dots \mu_m}{}_{\nu_1 \dots \nu_n} + \dots + R^{\mu_m}{}_{\rho\alpha\beta} A^{\mu_1 \dots \rho}{}_{\nu_1 \dots \nu_n} \\ & - R^{\rho}{}_{\nu_1\alpha\beta} A^{\mu_1 \dots \mu_m}{}_{\rho \dots \nu_n} - \dots - R^{\rho}{}_{\nu_n\alpha\beta} A^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \rho} \end{aligned} \tag{1.3.25}$$

Finally, we recall the result that given a point P, it is always possible to find a so called normal or gaussian reference frame in which the metric is in canonical form and the affine connection, i.e. the first derivatives of the metric, vanish:

$$\Gamma^{\lambda}_{\mu\nu}\big|_P = 0 \tag{1.3.26}$$

and the action of the covariant derivative tends to the ordinary derivative.

The equivalence principle represents the requirement that in each event in spacetime it is possible to find a local reference frame in which general laws of physics can be expressed in the language of Special Relativity, namely in terms of tensors covariant under the Lorentz group $SO(1,3)$. It is mathematically translated in the requirement that the signature of the metric for the spacetime manifold is lorentzian $(-,+,+,+)$; indeed, in this way we have that for each point P one can construct a normal frame in which the metric is in canonical form and coincides with Minkowski metric:

$$g_{\mu\nu}|_P = \eta_{\mu\nu} \tag{1.3.27}$$

and the affine connection vanish, so that one can replace covariant derivatives with ordinary derivatives, which behave tensorially in Special relativity. Moreover, one has also a family of normal frames in the point P with vanishing affine connection, those connected by changes of reference frame preserve the canonical form of the metric, i.e. Lorentz transformations. Those local reference frames correspond to free falling observers which do not experience gravitational effects and describe physics according to Special Relativity, and are thus considered inertial in General Relativity.

The equivalence principle also suggests the connection of gravity with the curvature of spacetime, since the effects of both can be locally made to vanish by considering an appropriate reference frame. This connection is the foundation of General Relativity as theory for the gravitational interaction compatible with the general relativity principle. The connection is quantitatively realized in the equations which connect the curvature of the spacetime manifold to the sources of gravity, the Einstein field equations:

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu} \tag{1.3.28}$$

where $G_{\mu\nu}$ is the Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, R\, g_{\mu\nu} \tag{1.3.29}$$

and $T_{\mu\nu}$ the stress-energy tensor of the matter distribution acting as source of gravity. The equations form a system of ten second order and non-linear partial differential equations in the components of the metric tensor field as unknown functions; they are not all independent as a result of the Bianchi identity satisfied by the Einstein tensor:

$$\nabla_\mu G^{\mu\nu} = 0 \tag{1.3.30}$$

which imposes the continuity equation which a stress-energy tensor must satisfy:

$$\nabla_\mu T^{\mu\nu} = 0 \tag{1.3.31}$$

1.3.2 Lagrangian formulation in euclidean signature

Consider Einstein's theory of gravity in a D-dimensional riemannian spacetime manifold $(\mathcal{M}, g)$ equipped with a metric tensor with euclidean signature $(+,\dots,+)$, whose evolution is governed by the Einstein field equations. With respect to the previous section we also introduce the cosmological constant Λ, in presence of which the Einstein field equations are written as:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu} \tag{1.3.32}$$

In these conventions, the Einstein's theory of gravity is rewritten in lagrangian formulation in terms of the action:

$$S_{GR}[g] = S_{EH}[g] + S_M[g] \tag{1.3.33}$$

where the first term is the Einstein-Hilbert action:

$$S_{EH}[g] = \frac{1}{16\pi G} \int d^D x \sqrt{g}\; (-R + 2\Lambda) \tag{1.3.34}$$

the second the action term describing gravitational sources:

$$S_M[g] = \int d^D x \sqrt{g}\, \mathcal{L}_M \tag{1.3.35}$$

with $\mathcal{L}_M$ a generic matter lagrangian. In natural units, $c = 1 = \hbar$, and D dimensions we have the mass dimensions $[R] = 2$, $[\Lambda] = 2$, $[G] = 2 - D$, $[\mathcal{L}_M] = D$. The coordinates x^μ used to express the integrals are arbitrary; $d^D x \sqrt{g}$ is the invariant volume element in D dimensions and euclidean signature, with the metric determinant $g = \det g_{\mu\nu} > 0$. According to lagrangian mechanics, the classical evolution of the metric field is given by the stationary configuration of the action, i.e. the one for which the first order variation of the action under an arbitrary variation $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ is vanishing:

$$\delta S_{GR}[g] = 0 \tag{1.3.36}$$

or equivalently, the one satisfying the Euler-Lagrange equations of motion:

$$\frac{1}{\sqrt{g}} \frac{\delta S_{GR}[g]}{\delta g^{\mu\nu}} = 0 \tag{1.3.37}$$

Imposing such action principle we recover Einstein field equations, since the first order variation of the action gives:

$$\delta S_{GR}[g] = -\frac{1}{16\pi G} \int d^D x \sqrt{g}\; (G_{\mu\nu} + \Lambda g_{\mu\nu} - 8\pi G\, T_{\mu\nu})\, \delta g^{\mu\nu} \tag{1.3.38}$$

and:

$$\frac{1}{\sqrt{g}} \frac{\delta S_{GR}[g]}{\delta g^{\mu\nu}} = -\frac{1}{16\pi G}\; (G_{\mu\nu} + \Lambda g_{\mu\nu} - 8\pi G\, T_{\mu\nu}) = 0 \tag{1.3.39}$$

The matter section comes from the first order variation of the matter action:

$$\delta S_M[g] = \frac{1}{2} \int d^D x \sqrt{g}\, T_{\mu\nu} \tag{1.3.40}$$

where the energy-momentum tensor is defined as:

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S_M[g]}{\delta g_{\mu\nu}} \tag{1.3.41}$$

The geometric sector follows from the first order variation of the Einstein-Hilbert action:

$$\delta S_{EH}[g] = -\frac{1}{16\pi G}\int d^D x\sqrt{g}\,\left(G_{\mu\nu} + \Lambda g_{\mu\nu}\right)\delta g^{\mu\nu} \tag{1.3.42}$$

which can be computed according to the variation rules:

$$\delta\sqrt{g} = -\frac{1}{2}\sqrt{g}g_{\mu\nu}\delta g^{\mu\nu} \tag{1.3.43}$$

$$\delta\left(\sqrt{g}g^{\mu\nu}\right) = \sqrt{g}\left(\delta g^{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\delta g^{\alpha\beta}\right) \tag{1.3.44}$$

$$\delta\left(\sqrt{g}R\right) = \delta\left(\sqrt{g}g^{\mu\nu}\right)R_{\mu\nu} + \sqrt{g}g^{\mu\nu}\delta R_{\mu\nu} = \sqrt{g}G_{\mu\nu}\delta g^{\mu\nu} + \sqrt{g}g^{\mu\nu}\delta R_{\mu\nu} \tag{1.3.45}$$

In particular, in the third one the term $\sqrt{g}g^{\mu\nu}\delta R_{\mu\nu}$ can be neglected since it gives a total divergence:

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\lambda X^\lambda \tag{1.3.46}$$

with:

$$X^\lambda = g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} - g^{\lambda\alpha}\delta\Gamma^\beta_{\alpha\beta} \tag{1.3.47}$$

as a consequence of the variation rule of the Riemann tensor (note that the variation of an affine connection $\delta\Gamma^\lambda_{\mu\nu}$ is a tensor):

$$\delta R^\alpha{}_{\beta\mu\nu} = \nabla_\beta\delta\Gamma^\alpha_{\mu\nu} - \nabla_\nu\delta\Gamma^\alpha_{\mu\beta} \tag{1.3.48}$$

which implies:

$$\delta R_{\mu\nu} = \nabla_\alpha\delta\Gamma^\alpha_{\mu\nu} - \nabla_\nu\delta\Gamma^\alpha_{\mu\alpha} \tag{1.3.49}$$

Indeed, according to Gauss theorem, the integral of the total divergence gives a boundary term:

$$\int d^D x\sqrt{g}\,\nabla_\lambda X^\lambda = \int_{\partial\mathcal{M}} d^{D-1}\sqrt{g^{(D-1)}}\,n_\lambda X^\lambda \tag{1.3.50}$$

with n_λ the normal to the boundary. Such term can be neglected, assuming that the variation principle is formulated with the condition that the metric variation vanish on the boundary $\delta g_{\mu\nu}|_{\partial\mathcal{M}} = 0$, or for instance assuming that the spacetime is closed, i.e. compact and without boundary. Considering more general variations and spacetimes, the term is non-vanishing a constitutes the Brown-York-Gibbons-Hawking term [11][12] which must be subtracted from the gravitational action in order to obtain from the action principle the correct equations of motion, i.e. Einstein field equations.
We stress that in the following we will always consider the gravitational action in absence of the Brown-York-Gibbons-Hawking term, assuming that boundary terms, originating for instance from integration by parts, can be neglected.

1.3.3 Diffeomorphisms

Einstein's theory of gravity, and more generally a theory compatible with the general relativity principle, enjoys a gauge symmetry, i.e. the action is symmetric under a transformation dependent on a set of local functions, related to general changes of coordinates.

Specifically, the symmetry transformation is given by diffeomorphisms, i.e. changes of coordinates regarded in the active sense as differentiable and invertible automorphisms from the manifold in itself. In the thesis we will consider diffeomorphisms only from the passive view point, as transformation induced by changes of coordinates.

Consider a change of coordinates:

$$x^\mu \quad \to \quad x'^\mu = (\phi^{-1})^\mu(x) \tag{1.3.51}$$

where, for each value taken by the index, $(\phi^{-1})^\mu$ is an invertible function with inverse ϕ^μ. We refer as diffeomorphism to the functional transformation induced by the change of coordinates on generic tensors:

$$A^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n}(x) \quad \to \quad A^{(\phi)\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n}(x) = A'^{\,\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n}(x' = x) \tag{1.3.52}$$

Explicitly the transformation rule for a rank-(m,n) tensor is:

$$\begin{aligned} A^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n}(x) \quad \to \quad A^{(\phi)\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n}(x) =& \frac{\partial(\phi^{-1})^{\mu_1}}{\partial x^{\alpha_1}}(\phi(x))\ldots\frac{\partial(\phi^{-1})^{\mu_m}}{\partial x^{\alpha_m}}(\phi(x))\cdot \\ &\cdot\frac{\partial\phi^{\beta_1}}{\partial x^{\nu_1}}(x)\ldots\frac{\partial\phi^{\beta_n}}{\partial x^{\nu_n}}(x)A^{\alpha_1\ldots\alpha_m}{}_{\beta_1\ldots\beta_n}(\phi(x)) \end{aligned} \tag{1.3.53}$$

We write compactly:

$$A^{(\phi)} = D_\phi(A) \tag{1.3.54}$$

The set of diffeomorphisms for the various possible functions ϕ^μ forms a group of transformations, Diff$(\mathcal{M})$, with composition rule:

$$D_\psi(D_\phi(\cdot)) = D_{\phi\circ\psi}(\cdot) \tag{1.3.55}$$

where $\circ$ denotes the composition of functions $(\phi\circ\psi)^\mu(x) = \phi^\mu(\psi(x))$. The identity e corresponds to the trivial diffeomorphism associated to the trivial change of coordinates with functions $(\phi^{-1})^\mu(x) = x^\mu$; the inverse of a diffeomorphism associated to a change of coordinates is given by the one associated to the inverse change of coordinates with functions. The composition rule follows trivially for scalars:

$$A^{(\phi)(\psi)}(x) = A(\phi(\psi(x))) =)A(\phi\circ\psi(x)) \tag{1.3.56}$$

and for higher tank tensors from the derivative rule of the composition of functions, for instance:

$$\begin{aligned} A^{(\phi)(\psi)}{}_\mu(x) &= \frac{\partial\psi^\lambda}{\partial x^\mu}(x)\frac{\partial\phi^\nu}{\partial x^\lambda}(\psi(x))A_\nu(\phi(\psi(x))) \\ &= \frac{\partial}{\partial x^\mu}(\phi\circ\psi)^\nu(x)A_\nu(\phi\circ\psi(x)) \end{aligned} \tag{1.3.57}$$

$$\begin{aligned} A^{(\phi)(\psi)\mu}(x) &= \frac{\partial(\psi^{-1})^\mu}{\partial x^\lambda}(\psi(x))\frac{\partial(\phi^{-1})^\lambda}{\partial x^\nu}(\phi(\psi(x)))A_\nu(\phi(\psi(x))) \\ &= \frac{\partial}{\partial x^\mu}((\phi\circ\psi)^{-1})^\nu(\phi\circ\psi(x))A_\nu(\phi\circ\psi(x)) \end{aligned} \tag{1.3.58}$$

where the notation indicates $A^{(\phi)(\psi)} = \left(A^{(\phi)}\right)^{(\psi)}$. The diffeomorphism group contains a part non-connected to the identity, constituted by "large" diffeomorphisms associated to "large" changes of coordinates, and a part connected to the identity, $\mathrm{Diff}_e(\mathcal{M})$, constituted by diffeomorphisms associated to changes of coordinates which can be smoothly connected to the identity, i.e. coordinate shifts along arbitrary smooth vector fields:

$$x^\mu \quad \rightarrow \quad x'^\mu = (\phi_{(\xi)}^{-1})^\mu(x) = x^\mu - \xi^\mu(x) \tag{1.3.59}$$

which indeed reduce continuously to the identity by letting the components of the vector field go to zero[2]. So, the set diffeomorphisms for the various possible vector fields ξ^μ:

$$\begin{aligned} A^{\mu_1\dots\mu_m}{}_{\nu_1\dots\nu_n}(x) \quad \rightarrow \quad A^{(\xi)\mu_1\dots\mu_m}{}_{\nu_1\dots\nu_n}(x) =& \frac{\partial(\phi_{(\xi)}^{-1})^{\mu_1}}{\partial x^{\alpha_1}}(\phi_{(\xi)}(x))\dots\frac{\partial(\phi_{(\xi)}^{-1})^{\mu_m}}{\partial x^{\alpha_m}}(\phi_{(\xi)}(x))\cdot \\ &\cdot\frac{\partial\phi_{(\xi)}^{\beta_1}}{\partial x^{\nu_1}}(x)\dots\frac{\partial\phi_{(\xi)}^{\beta_n}}{\partial x^{\nu_n}}(x)A^{\alpha_1\dots\alpha_m}{}_{\beta_1\dots\beta_n}(\phi_{(\xi)}(x)) \end{aligned} \tag{1.3.60}$$

form a local Lie group of transformations, $\mathrm{Diff}_e(\mathcal{M})$, with local parameters given by the components of the vector field. We write compactly:

$$A^{(\xi)} = D^e_\xi(A) \tag{1.3.61}$$

The composition rule is the one inherited from the diffeomorphism group:

$$D^e_\eta(D^e_\xi(\cdot)) = D^e_{\xi\circ\eta}(\cdot) \tag{1.3.62}$$

where the notation is a shorthand for $\xi\circ\eta \equiv \phi_{(\xi)}\circ\phi_{(\eta)}$. Due to the connectedness with identity, one can consider the infinitesimal form of a diffeomorphism connected to the identity, i.e. the order $O(\xi)$ of the transformation rule (1.3.60)[2]:

$$A^{\mu_1\dots\mu_m}{}_{\nu_1\dots\nu_n}(x) \quad \rightarrow \quad A^{(\xi)\mu_1\dots\mu_m}{}_{\nu_1\dots\nu_n}(x) = A^{\mu_1\dots\mu_m}{}_{\nu_1\dots\nu_n}(x) + \delta_\xi A^{\mu_1\dots\mu_m}{}_{\nu_1\dots\nu_n}(x) \tag{1.3.63}$$

It can be seen that the variation is given by the Lie derivative of the tensor along the vector field:

$$\delta_\xi A^{\mu_1\dots\mu_m}{}_{\nu_1\dots\nu_n}(x) = \pounds_\xi A^{\mu_1\dots\mu_m}{}_{\nu_1\dots\nu_n}(x) \tag{1.3.64}$$

We recall that the Lie derivative of a generic rank-(m,n) tensor is:

$$\begin{aligned} \pounds_\xi A^{\mu_1\dots\mu_m}{}_{\nu_1\dots\nu_n} =& \xi^\lambda\partial_\lambda A^{\mu_1\dots\mu_m}{}_{\nu_1\dots\nu_n} \\ &- \partial_\rho\xi^{\mu_1}A^{\rho\dots\mu_m}{}_{\nu_1\dots\nu_n} - \dots - \partial_\rho\xi^{\mu_m}A^{\mu_1\dots\rho}{}_{\nu_1\dots\nu_n} \\ &+ \partial_{\mu_1}\xi^\rho A^{\mu_1\dots\mu_m}{}_{\rho\dots\nu_n} + \dots + \partial_{\mu_n}\xi^\rho A^{\mu_1\dots\mu_m}{}_{\nu_1\dots\rho} \end{aligned} \tag{1.3.65}$$

and in spite of the presence of partial derivatives, it is a tensor. Indeed, introducing any covariant derivative, the Lie derivative (which is an independent notion and does

[2]More formally, one considers as change of coordinates $x^\mu \rightarrow x'^\mu = x^\mu - \epsilon\xi^\mu(x)$, where ϵ is a continuous parameter. The identity is recovered in the limit $\epsilon \rightarrow 0$. Tensors transform under an infinitesimal diffeomorphism by their Lie derivative at order $O(\epsilon)$.

not require the presence of an affine connection on the manifold) can be rewritten in a manifest covariant form by trading ordinary derivatives for covariant ones; in particular, if the covariant derivative is associated to a torsion-free connection, the trading amounts to simply substituting ordinary derivative with covariant ones:

$$\begin{aligned}\pounds_\xi A^{\mu_1\dots\mu_m}{}_{\nu_1\dots\nu_n} = \xi^\lambda \nabla_\lambda A^{\mu_1\dots\mu_m}{}_{\nu_1\dots\nu_n} \\ - \nabla_\rho \xi^{\mu_1} A^{\rho\dots\mu_m}{}_{\nu_1\dots\nu_n} - \dots - \nabla_\rho \xi^{\mu_m} A^{\mu_1\dots\rho}{}_{\nu_1\dots\nu_n} \\ + \nabla_{\mu_1} \xi^\rho A^{\mu_1\dots\mu_m}{}_{\rho\dots\nu_n} + \dots + \nabla_{\mu_n} \xi^\rho A^{\mu_1\dots\mu_m}{}_{\nu_1\dots\rho}\end{aligned} \tag{1.3.66}$$

Consider now Einstein's theory of gravity described by the action (1.3.33). The theory enjoys a gauge symmetry under diffeomorphisms of the metric as gauge transformations with local functions ϕ^μ, i.e. under the transformation:

$$g_{\mu\nu}(x) \quad \to \quad g^{(\phi)}_{\mu\nu}(x) = \frac{\partial \phi^\alpha}{\partial x^\mu}(x) \frac{\partial \phi^\beta}{\partial x^\nu}(x) g_{\alpha\beta}(\phi(x)) \tag{1.3.67}$$

the action is invariant:

$$S_{GR}[g] = S_{GR}[g^{(\phi)}] \tag{1.3.68}$$

The invariance follows simply from the possibility of formally changing variable in the action from x^μ to x'^μ which are precisely the new coordinates entering the change of coordinates inducing the diffeomorphism, and then using the covariance of the measure and the scalars in the integrand (the matter lagrangian is constructed with the metric and matter fields, and globally must be a scalar):

$$\begin{aligned}S_{GR}[g^{(\phi)}] &= \int d^D x \sqrt{g^{(\phi)}(x)} \left(\frac{-R^{(\phi)}(x) + 2\Lambda}{16\pi G} + \mathcal{L}^{(\phi)}_M(x) \right) = \\ &= \int d^D x' \sqrt{g^{(\phi)}(x')} \left(\frac{-R^{(\phi)}(x') + 2\Lambda}{16\pi G} + \mathcal{L}^{(\phi)}_M(x') \right) = \\ &= \int d^D x \sqrt{g(x)} \left(\frac{-R(x) + 2\Lambda}{16\pi G} + \mathcal{L}_M(x) \right) = S_{GR}[g]\end{aligned}$$

In fact, this gauge symmetry represents that the theory is compatible with the general covariance principle. Clearly, the symmetry holds also for diffeomorphisms connected to the identity and in particular at the infinitesimal level, under infinitesimal diffeomorphisms of the metric as infinitesimal gauge transformations:

$$g_{\mu\nu}(x) \quad \to \quad g^{(\xi)}_{\mu\nu}(x) = g_{\mu\nu}(x) + \delta_\xi g_{\mu\nu}(x) \tag{1.3.69}$$

where the variation is:

$$\begin{aligned}\delta_\xi g_{\mu\nu} = \pounds_\xi g_{\mu\nu} &= \xi^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu \xi^\lambda g_{\lambda\nu} + \partial_\nu \xi^\lambda g_{\lambda\mu} \\ &= \nabla_\mu \xi^\lambda g_{\lambda\nu} + \nabla_\nu \xi^\lambda g_{\lambda\mu}\end{aligned} \tag{1.3.70}$$

The variation of the action is vanishing:

$$\delta_\xi S_{GR}[g] = 0 \tag{1.3.71}$$

which follows from (1.3.68), but can be also verified by computing the variation (1.3.38) with $\delta g_{\mu\nu} = \delta_\xi g_{\mu\nu}$, integrating by parts and using the Bianchi identity - continuity equation satisfied by the Einstein tensor and the stress energy tensor:

$$\begin{aligned}
\delta_\xi S_{GR}[g] &= -\int d^D x \sqrt{g} \left(\frac{G_{\mu\nu} + \Lambda g_{\mu\nu}}{16\pi G} - \frac{1}{2} T_{\mu\nu} \right) \delta_\xi g^{\mu\nu} = \\
&= -\int d^D x \sqrt{g} \left(\frac{G_{\mu\nu} + \Lambda g_{\mu\nu}}{16\pi G} - \frac{1}{2} T_{\mu\nu} \right) (\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu) = \\
&= \int d^D x \sqrt{g} \left(\frac{\xi^\nu \nabla^\mu G_{\mu\nu} + \xi^\mu \nabla^\nu G_{\mu\nu}}{16\pi G} - \frac{\xi^\nu \nabla^\mu T_{\mu\nu} + \xi^\mu \nabla^\nu T_{\mu\nu}}{2} \right) = 0
\end{aligned}$$

Part 2
FRG flows in Quantum Einstein Gravity

The second part of the thesis is devoted to the definition and construction of a Quantum Einstein Gravity theory and its symmetry properties, and to the presentation of the standard implementation method of the Functional Renormalization Group.
The part is divided in four sections. In the first section we review the definition of a QEG theory, highlighting how the formal construction of the theory requires a gauge-fixing procedure and how the application of the FRG framework requires the background field method in order to have a properly-defined coarse-graining. In the second section we present the implementation of the background field method and the quantization procedure with gauge-fixing via the Faddeev-Popov method; we also present BRST symmetry and its general properties, at the classical and quantum levels. In the third section we present the standard gauge-fixing choice for the theory. In the fourth section we implement the standard FRG regularization procedure and we present the associated Wetterich-Morris equation, discussing in particular its incompatibility with BRST symmetry and presenting its component form within the Einstein-Hilbert truncation.

2.1 Premises

2.1.1 Quantum Einstein Gravity

A theory of Quantum Einstein Gravity (QEG), or metric quantum gravity, is in general a quantum field theory for the gravitational interaction containing a quantum field whose expectation value, for certain quantum states, is expected to have the properties of a classical riemannian metric field, i.e. a symmetric rank-$(0,2)$ tensor field defining a non-degenerate and positive-definite (working in euclidean signature) bilinear form, capable of describing a gravitational field in a spacetime manifold, according to Einstein's General Relativity [5]. The theory is in general non-renormalizable at the perturbative level and its renormalizability is intended to be studied at the non-perturbative level with renormalization group techniques within the asymptotic safety paradigm. The quantum field whose average should produce a classical metric field could be both an elementary field, i.e. a quantum metric field $g_{\mu\nu}$, or a composite object built of elementary fields (for instance quantum tetrads, employing the tetrad formalism). In the thesis we consider the first case. Within this choice, in the functional formalism, a QEG theory thus corresponds to a quantum field theory for a Grassmann-even and symmetric field $g_{\mu\nu}$, described by a path integral of the type:

$$Z_{QEG} = \int \mathcal{D}g\, e^{-S[g]} \tag{2.1.1}$$

with the properties:

1. Gauge symmetry: *S is a classical action for the gravitational interaction required to be compatible with the principle of general relativity and thus enjoying a gauge symmetry under diffeomorphisms associated to general coordinate transformations*:

$$S\left[g^{(\phi)}\right] = S[g] \tag{2.1.2}$$

This is required in order to reproduce in the classical regime a classical field theory of gravity. In particular this could be also a more general theory than Einstein's theory of gravity; in fact, assuming the FRG perspective, the fundamental concepts defining the quantum theory are the flow equation, where the classical action enters just as initial condition, and the theory space, fixed by the symmetries of the theory.
We also notice that as a result of this requirement, the theory is a gauge theory and thus affected in the quantization process by the standard problem of resolving the ambiguities due to gauge redundancy. In particular, in the path integral formalism, this appears as the fact that, according to the naive definition (2.1.1) of the path integral, for each integrated configuration we are also redundantly integrating over the infinitely many associated gauge equivalent configurations which contribute with equal weight.

$$e^{-S[g^{(\phi)}]} = e^{-S[g]} \quad , \quad \forall \phi \tag{2.1.3}$$

resulting in a divergent, but non-physically-meaningful, contribution to the path integral:

$$Z_{QEG} = \int \mathcal{D}g\, e^{-S[g]} = \infty \tag{2.1.4}$$

In order to properly quantize the theory and extract from the naive definition a well-defined path integral is thus necessary to apply a gauge-fixing quantization method, restricting the path integration over the configurations satisfying a gauge-fixing condition breaking diffeomorphism symmetry. In the thesis we will adopt the Faddeev-Popov quantization method in its standard formulation (in particular we will not discuss the Gribov problem). In the rest of this introductory section all concepts are introduced referring for simplicity to the naive path integral.

2. Field variable: *the field variable integrated in the path integral is a generic symmetric tensor, in particular non-necessarily a non-degenerate and positive-definite metric.*

In the thesis we assume this requirement. It is justified by the fact that, according to the definition, there should be "classically appearing" states, specified by some boundary conditions on the path integral, for which the average:

$$\mathsf{g}_{(\psi)\mu\nu}(x) = \langle g_{\mu\nu}(x) \rangle_\psi = \frac{1}{Z_{QEG}} \int_\psi \mathcal{D}g\, g_{\mu\nu}(x)\, e^{-S[g]} \tag{2.1.5}$$

(correspondent in the operatorial formalism to some expectation value $\mathtt{g}_{(\psi)\mu\nu}(x) = \langle\psi|\hat{g}_{\mu\nu}(x)|\psi\rangle$) is a classical metric. However, in general for other states the expectation value is not expected to have an interpretation as classical metric, being possibly not smooth or degenerate or for instance identically vanishing.
In the literature both possibilities are actually considered: the integration is extended over all symmetric tensor fields or over the subset of all metrics, non-degenerate and positive-definite. The main difference in the corresponding theories is the measure: in the first case the domain of integration is a linear space and a standard translationally-invariant measure can be considered:

$$\mathcal{D}g = \prod_x \prod_{\mu,\nu\geq\mu} dg_{\mu\nu}(x) \tag{2.1.6}$$

in the second it is a non-trivial curved space and one has to construct the measure by finding an atlas of coordinate charts parametrizing the metrics [13]. According to our choice, in the thesis we will consider a standard translationally-invariant measure[3].

3. Background independence: *The theory is expected to be formulated in a background-independent way, i.e. it should explain rather than presuppose the existence and the properties of spacetime.*

In particular, more concretely, assuming that the topological and differential structure of the spacetime manifold as a set of events is given and fixed, no preferred riemannian metric structure, i.e. no preferred metric tensor, should play a distinguished role in the theory.

2.1.2 Coarse-graining in Quantum Einstein Gravity

The implementation of the FRG framework presented for the scalar theory in section 1.2 cannot be straightforwardly applied to a QEG theory; essentially, this is due to the requirements, outlined in the previous subsection, regarding gauge symmetry and background independence.
As seen for the scalar theory, in a non-gauge theory in flat spacetime the coarse-graining of contributions from the various field configurations to the path integral is usually performed by first going in momentum space, i.e. expanding fields in eigenfunctions $u_p(x) = e^{ipx}$ of the negative laplacian $-\Box = \partial^2$ and rewriting the integration measure in terms of the corresponding Fourier weights $\prod_p d\phi(p)$, then declaring the weights to be IR or UV on whether they correspond to low or high momentum eigenfunctions and finally by blocking the integration of those with eigenvalue p^2 smaller than a floating scale k^2. In a gauge theory, as in this case, a properly-defined coarse-graining should be a gauge-invariant notion, i.e. the distinction between "coarse" and "fine" should be gauge-invariant; this is possible considering in the place of the laplacian in flat space a gauge-covariant operator. In this case the flat space laplacian should thus be substituted with a covariant laplacian

[3] In subsection 2.2.2 we will comment again on the path integral measure in relation to the implementation of the Faddeev-Popov method. In particular, the standard translationally-invariant measure (2.1.6) turns out to be non-diffeomorphism-invariant and technically should be substituted with a diffeomorphism-invariant one; practically, however, this will not be an issue as far as the matter of the thesis is concerned.

constructed with a general metric. However, due to the requirement of background independence, we cannot introduce any "reference" metric and following this route we are forced to use the covariant laplacian constructed with the dynamical metric field itself:

$$\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu \tag{2.1.7}$$

which is not a viable option. Firstly, the operator would be not defined integrating over symmetric tensors in the path integral, since those could be degenerate; moreover, we could consider a straightforward, correctly gauge-invariant, generalization of the regulator term (1.2.5) for the scalar theory:

$$\Delta S_{(k)}[g] \propto \frac{1}{2}\int d^Dx\sqrt{g}\, g^{\mu\nu}\mathcal{R}_{(k)}\left(-\Box\right) g_{\mu\nu} = 0 \tag{2.1.8}$$

with $\mathcal{R}_{(k)}(p^2)$ a regulating function of the type depicted in figure 1, but the term is actually identically zero due to the compatibility of the covariant derivative with the metric.

A natural solution to the problem of implementing a well-defined coarse graining in QEG is furnished by the background field method. The method consists in splitting the dynamical metric field $g_{\mu\nu}$ in the "sum" of: **1.** a background metric field $\bar{g}_{\mu\nu}$, regarded as a non-dynamical quantity which remains a spectator in the quantization; it is required to be a generic, i.e possibly non-flat, metric; **2.** a metric fluctuation field $h_{\mu\nu}$, considered as new dynamical field variable of the theory to be quantized, i.e. integrated in the path integral (or promoted to an operator in the operatorial formalism):

$$Z_{QEG}[\bar{g}] = \int \mathcal{D}h\, e^{-S[h;\bar{g}]} \tag{2.1.9}$$

The "sum" can be realized with different parametrizations, depending on whether we are integrating over symmetric tensors or metric tensors. In the first case, one can use a linear split:

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x) \tag{2.1.10}$$

where the metric fluctuation is an arbitrary symmetric tensor, in particular not required to be a metric. In the second case, one has to find a parametrization in the curved space of metrics in terms of the metric fluctuation as unconstrained variable; for instance, we mention the exponential split, typically used in the literature [14]:

$$g_{\mu\nu}(x) = \bar{g}_{\mu\lambda}(x)\left(e^h\right)^\lambda{}_\nu(x) \tag{2.1.11}$$

As we will see more explicitly in section 2.2, when adopting the background field method the QEG theory, describing a spacetime with a quantum metric $(\mathcal{M}, g)$, becomes formally equal to an "ordinary" QFT for a symmetric matter field $h_{\mu\nu}$ in a classical curved background with a given metric $\bar{g}_{\mu\nu}$ $(\mathcal{M}, \bar{g})$ (as stated above, at the topological level the set of events $\mathcal{M}$ can be assumed to be the same). The requirement of background independence is thus translated in the requirement that the background metric has to be considered arbitrary, so that the QEG theory can be thought as equivalent to the infinite set of QFTs for a symmetric matter field in an arbitrary classical curved spacetime.

Adopting the background field method, we have now a natural notion of gauge-invariant coarse-graining, one for each of the theories labeled by a given background metric. It is the one provided by the background laplacian operator:

$$\bar{\Box} = \bar{g}^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu \tag{2.1.12}$$

and its spectrum of eigenfunctions and eigenvalues (a curved laplacian is a negative-definite self-adjoint operator):

$$-\bar{\Box}u_p(x) = p^2 u_p(x) \tag{2.1.13}$$

which are implicitly parametrically dependent on the background metric; the eigenvalues p^2 of the negative background laplacian can be interpreted as generalized momenta. Formally expanding the fields in the eingenfunctions of the background laplacian, i.e. in generalized momentum space (we may consider the spectrum to be discrete, for simplicity, assuming that the spacetime is a compact space):

$$h_{\mu\nu}(x) = \sum_p \tilde{h}_{\mu\nu}(p)u_p(x) \tag{2.1.14}$$

we can thus order the modes and Fourier weights as in flat spacetime according to the value of the generalized momentum:

$$\begin{array}{ccccccccccccc} & 0 & \leq & p_1^2 & \leq & \cdots & \leq & p_n^2 & \leq & \cdots & \leq & \infty & \\ & & & \updownarrow & & & & \updownarrow & & & & & \\ \boxed{\text{IR}} & & & u_{p_1}(x) & & \cdots & & u_{p_n}(x) & & \cdots & & & \boxed{\text{UV}} \\ & & & \updownarrow & & & & \updownarrow & & & & & \\ & & & \tilde{h}(p_1) & & \cdots & & \tilde{h}(p_n) & & \cdots & & & \end{array} \tag{2.1.15}$$

"IR" or "coarse" modes are those with a low generalized momentum, "UV" or "fine" modes those with a high generalized momentum. According to this ordering, we can now implement the FRG machinery in the QEG theory, similarly to the scalar theory in subsection 1.2.1, by regulating the path integral with a quadratic regulator term:

$$\Delta S_{(k)}[h;\bar{g}] \propto \frac{1}{2}\int d^Dx\sqrt{\bar{g}}\, h_{\alpha\beta}\mathcal{R}^{\alpha\beta\mu\nu}\mathcal{R}_{(k)}\left(-\bar{\Box}\right)h_{\mu\nu} \tag{2.1.16}$$

where $\mathcal{R}^{\alpha\beta\mu\nu}$ a suitable structure tensor constructed with the background metric and $\mathcal{R}_{(k)}(p^2)$ is again a regulating function of the type depicted in figure 1, so that the integration of weights which are IR with respect to a floating scale k^2 is suppressed:

$$\begin{aligned} Z_{QEG(k)}[\bar{g}] &= \int \mathcal{D}h\, e^{-S[h;\bar{g}]-\Delta S_{(k)}[h;\bar{g}]} = \\ &= \int \mathcal{D}h\, e^{-S[h;\bar{g}]-\frac{1}{2}\int d^Dx\sqrt{\bar{g}}\,h_{\alpha\beta}\bar{\mathcal{R}}^{\alpha\beta\mu\nu}\mathcal{R}_{(k)}\left(-\bar{\Box}\right)h_{\mu\nu}} = \\ &\propto \int \mathcal{D}\tilde{h}\, e^{-S[h;\bar{g}]-\frac{1}{2}\sum_p \tilde{h}_{\alpha\beta}(-p)\bar{\mathcal{R}}^{\alpha\beta\mu\nu}\mathcal{R}_{(k)}\left(p^2\right)\tilde{h}_{\mu\nu}(p)} = \\ &\sim \int \prod_{|k|\lesssim|p|\lesssim|k_{UV}|} d\tilde{h}(p)\, e^{-S[h;\bar{g}]} \end{aligned} \tag{2.1.17}$$

In subsection 2.3.1, after having properly quantized the theory via the Faddeev-Popov method, we will give the precise details fo the standard regularization procedure considering also the additional fields introduced in the quantization, i.e. the Faddeev-Popov ghosts.

2.2 Background field method and Faddeev-Popov quantization method

In light of the preliminary concepts introduced in the previous section, we now present how to implement the two methods which are required in order to pass from a classical field theory of gravity to a QEG theory usable as input for the FRG machinery: the Faddeev-Popov quantization method, to obtain a well-defined gauge-fixed path integral, and the background field method, to guarantee a formally well-defined coarse-graining. In particular, in order to obtain a QEG theory already prepared for the implementation of the FRG machinery, we will first present how a classical field theory of gravity is reinterpreted after splitting the metric according to the background field method, remaining at the classical level, and then how to quantize the theory via the Faddeev-Popov quantization method, directly in combination with the background field method.
Although, as explained, we could consider a generic action compatible with the general relativity principle, we consider from now on Einstein's theory of gravity as classical field theory of gravity, in order to make ideas more concrete (in particular in the presentation of the background field method); specifically, we consider pure gravity in absence of matter sources, so the pure Einstein-Hilbert theory described at the classical level by the Einstein-Hilbert action.

2.2.1 Background field method

Consider as spacetime a riemannian metric manifold $(\mathcal{M}, g)$ with topological space $\mathcal{M}$, with dimension $\dim\mathcal{M} = D$, and metric tensor field g, and introduce a generic coordinate system $x^\mu : \mathcal{M} \to \mathbb{R}^D$ with associated basis vectors e_μ and basis 1-forms $\tilde{e}^\mu$. Let the classical dynamics of the metric tensor field be governed by the pure Einstein-Hilbert theory, described by the Einstein-Hilbert action (1.3.34), which we rewrite as:

$$S_{EH}[g] = \frac{2}{\kappa^2} \int d^D x \sqrt{g}\, (-R + 2\Lambda) \tag{2.2.1}$$

introducing, for later convenience, the constant:

$$\kappa = \sqrt{32\pi G} \tag{2.2.2}$$

which in natural units, $c = 1 = \hbar$, and D dimensions has mass dimension $[\kappa] = (2-D)/2$, since Newton's constant has mass dimension $2 - D$.

Background field method

According to the analysis in 2.1.2, we implement the background field method via the linear split (2.1.10):

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x) \tag{2.2.3}$$

where $\bar{g}_{\mu\nu}$ is a generic, i.e non-flat, background metric tensor field, regarded as a non-dynamical quantity, and $h_{\mu\nu}$ is a symmetric metric fluctuation tensor field, considered as new dynamical field variable of the theory. Substituting the decomposition in the Einstein-Hilbert action, we introduce the following notation to emphasize the role of the metric fluctuation as elementary field variable and the one of the background metric as a parametric field variable:

$$S_{EH}[g = \bar{g} + h] \equiv S_{EH}[h; \bar{g}] \tag{2.2.4}$$

The content of the theory, i.e. the kinetic and interaction terms for the metric fluctuation, can be explicitly obtained by expanding the action in a functional Taylor series in powers of the metric fluctuation around the background metric:

$$S_{EH}[h; \bar{g}] = \sum_{n=0}^{\infty} S_{EH,n}[h; \bar{g}] \tag{2.2.5}$$

where the Taylor term of n-th order can be formally expressed as:

$$S_{EH,n}[h; \bar{g}] = \frac{1}{n!} \int d^D x_1 ... d^D x_n \frac{\delta^n S_{EH}[g]}{\delta g_{\mu_1 \nu_1}(x_1) ... \delta g_{\mu_n \nu_n}(x_n)} \bigg|_{g=\bar{g}} h_{\mu_1 \nu_1}(x_1) ... h_{\mu_n \nu_n}(x_n) \tag{2.2.6}$$

In particular, the Taylor coefficient of n-th order is given by the n-th functional derivative of the action computed in the background metric.

Before examining the first terms in the background expansion, we discuss how the gauge symmetry of the theory is reinterpreted. We focus in particular on infinitesimal diffeomorphisms (1.3.59), i.e. infinitesimal gauge transformations, since those are the ones to consider in the Faddeev-Popov method. Due to the linearity of the Lie derivative, the variation (1.3.70) is linear in the metric and performing a linear split of the type (2.2.3), we can distribute the derivative on the background metric and the metric fluctuation:

$$\delta_\xi(\bar{g}_{\mu\nu} + h_{\mu\nu}) = \pounds_\xi(\bar{g}_{\mu\nu} + h_{\mu\nu}) = \pounds_\xi \bar{g}_{\mu\nu} + \pounds_\xi h_{\mu\nu} \tag{2.2.7}$$

The individual variations of the background metric and the metric fluctuation can be defined in multiple ways, as long as their sum is equal to the complete variation of the metric; in particular, there are two meaningful ways of defining those variations, depending on how the two Lie derivatives in (2.2.7) are split:

1. <u>True gauge transformation</u>: the total variation, i.e. the sum of the two Lie derivatives, is entirely ascribed to the metric perturbation, the infinitesimal diffeomorphism is rewritten as:

$$\begin{cases} h_{\mu\nu}(x) & \to & h^{(\xi)}_{\mu\nu}(x) = h_{\mu\nu}(x) + \delta_\xi h_{\mu\nu}(x) \\ \bar{g}_{\mu\nu}(x) & \to & \bar{g}^{(\xi)}_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \delta_\xi \bar{g}_{\mu\nu}(x) \end{cases} \tag{2.2.8}$$

With variations:

$$\begin{cases} \delta_\xi h_{\mu\nu} = \pounds_\xi(\bar{g}_{\mu\nu} + h_{\mu\nu}) \\ \qquad\quad = \left(\xi^\lambda \partial_\lambda \bar{g}_{\mu\nu} + \partial_\mu \xi^\lambda \bar{g}_{\lambda\nu} + \partial_\nu \xi^\lambda \bar{g}_{\lambda\mu}\right) + \left(\xi^\lambda \partial_\lambda h_{\mu\nu} + \partial_\mu \xi^\lambda h_{\lambda\nu} + \partial_\nu \xi^\lambda h_{\lambda\mu}\right) \\ \qquad\quad = \left(\bar{\nabla}_\mu \xi^\lambda \bar{g}_{\lambda\nu} + \bar{\nabla}_\nu \xi^\lambda \bar{g}_{\lambda\mu}\right) + \left(\xi^\lambda \bar{\nabla}_\lambda h_{\mu\nu} + \bar{\nabla}_\mu \xi^\lambda h_{\lambda\nu} + \bar{\nabla}_\nu \xi^\lambda h_{\lambda\mu}\right) \\ \delta_\xi \bar{g}_{\mu\nu} = \qquad 0 \end{cases} \tag{2.2.9}$$

In the first explicit expression for the variation of the metric fluctuation, we wrote the two Lie derivatives in terms of ordinary derivatives, while in the second we used the general property (1.3.66) to substitute the ordinary derivatives with covariant derivatives compatible with the background metric, so that the Lie derivative of the background metric contains in this form only two pieces.
This transformation is typically called "true" gauge transformation since it leaves invariant the background metric, which is regarded as a non-dynamical object in the background-expanded theory; indeed, it will be the one to be gauge fixed with the Faddeev-Popov method in the next subsection. We notice that, in order to achieve this result, the transformation is "anomalous", in the sense that we cannot think anymore the transformation as induced by a change of coordinates of the type (1.3.59); in fact, as seen in subsection 1.3.3, the corresponding functional variation of any tensor of any rank should be given by its Lie derivative, while in this case we are defining the variation of two tensors, the background metric and the metric fluctuation, to be different from their Lie derivative (respectively equal to zero and to the Lie derivative of another tensor, the full metric).
We indicate with $\widehat{\mathrm{Diff}}_e(\mathcal{M})$ the Lie group associated to diffeomorphisms connected to the identity acting on the metric fields in the form of true gauge transformations, with the composition rule:

$$\left\{h_{\mu\nu}^{(\xi)(\eta)}, \bar{g}_{\mu\nu}\right\} = \left\{h_{\mu\nu}^{(\xi\,\circ\,\eta)}, \bar{g}_{\mu\nu}\right\} \tag{2.2.10}$$

where we recall the shorthand notation introduced in subsection 1.3.3 $\xi \circ \eta \equiv \phi_{(\xi)} \circ \phi_{(\eta)}$.

2. Background gauge transformation: both the background metric and the metric fluctuation are varied according to the corresponding Lie derivative, the infinitesimal diffeomorphism is rewritten as:

$$\begin{cases} h_{\mu\nu}(x) \quad \rightarrow \quad h_{\mu\nu}^{\overline{(\xi)}}(x) = h_{\mu\nu}(x) + \bar{\delta}_\xi h_{\mu\nu}(x) \\ \bar{g}_{\mu\nu}(x) \quad \rightarrow \quad \bar{g}_{\mu\nu}^{\overline{(\xi)}}(x) = \bar{g}_{\mu\nu}(x) + \bar{\delta}_\xi \bar{g}_{\mu\nu}(x) \end{cases} \tag{2.2.11}$$

With variations:

$$\begin{cases} \bar{\delta}_\xi h_{\mu\nu} = \pounds_\xi h_{\mu\nu} = \xi^\lambda \partial_\lambda h_{\mu\nu} + \partial_\mu \xi^\lambda h_{\lambda\nu} + \partial_\nu \xi^\lambda h_{\lambda\mu} \\ \qquad\qquad\qquad\;\; = \xi^\lambda \bar{\nabla}_\lambda h_{\mu\nu} + \bar{\nabla}_\mu \xi^\lambda h_{\lambda\nu} + \bar{\nabla}_\nu \xi^\lambda h_{\lambda\mu} \\ \bar{\delta}_\xi \bar{g}_{\mu\nu} = \pounds_\xi \bar{g}_{\mu\nu} = \xi^\lambda \partial_\lambda \bar{g}_{\mu\nu} + \partial_\mu \xi^\lambda \bar{g}_{\lambda\nu} + \partial_\nu \xi^\lambda \bar{g}_{\lambda\mu} \\ \qquad\qquad\qquad\;\; = \bar{\nabla}_\mu \xi^\lambda \bar{g}_{\lambda\nu} + \bar{\nabla}_\nu \xi^\lambda \bar{g}_{\lambda\mu} \end{cases} \tag{2.2.12}$$

Again, in the first explicit expression of the variations the Lie derivatives are written in terms of ordinary derivatives and in the second in terms of covariant derivatives compatible with the background metric.
Contrary to the true gauge transformation, this transformation is typically called "background" gauge transformation since also the background metric transforms. Moreover, in this case the transformation can be regarded as "non-anomalous" and interpreted as the result induced by a change of coordinates of the type (1.3.59), since now the functional variations of the background metric and metric fluctuation are correctly

given by the corresponding Lie derivatives.
We indicate with $\overline{\text{Diff}}_e(\mathcal{M})$ the Lie group associated to diffeomorphisms connected to the identity acting on metric fields in the form of background gauge transformations, with composition rule:

$$\left\{h_{\mu\nu}^{\overline{(\xi)}\,\overline{(\eta)}}, \bar{g}_{\mu\nu}^{\overline{(\xi)}\,\overline{(\eta)}}\right\} = \left\{h_{\mu\nu}^{\overline{(\xi\circ\eta)}}, \bar{g}_{\mu\nu}^{\overline{(\xi\circ\eta)}}\right\} \tag{2.2.13}$$

According to the comment above, the action of this group is equal to the one of $\text{Diff}_e(\mathcal{M})$.

Clearly, we have correctly:

$$\delta_\xi h_{\mu\nu} + \delta_\xi \bar{g}_{\mu\nu} = \bar{\delta}_\xi h_{\mu\nu} + \bar{\delta}_\xi \bar{g}_{\mu\nu} \quad \equiv \quad \delta_\xi g_{\mu\nu} = \pounds_\xi g_{\mu\nu} \tag{2.2.14}$$

and:

$$h_{\mu\nu}^{(\xi)} + \bar{g}_{\mu\nu}^{(\xi)} = h_{\mu\nu}^{\overline{(\xi)}} + \bar{g}_{\mu\nu}^{\overline{(\xi)}} \quad \equiv \quad g_{\mu\nu}^{(\xi)} = g_{\mu\nu} + \pounds_\xi g_{\mu\nu} \tag{2.2.15}$$

so that the two transformations represent the same original gauge symmetry:

$$\delta_\xi S_{EH}[h;\bar{g}] = \bar{\delta}_\xi S_{EH}[h;\bar{g}] \quad \equiv \quad \delta_\xi S_{EH}[g] = 0 \tag{2.2.16}$$

Background expansion

The Taylor term (2.2.6) in the background expansion is equal to the n-th order variation of the Einstein-Hilbert action under a variation $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ with $g_{\mu\nu} = \bar{g}_{\mu\nu}$, $\delta g_{\mu\nu} = h_{\mu\nu}$:

$$S_{EH,n}[h;\bar{g}] = \frac{1}{n!}\delta^n S_{EH}[g]\Big|_{g_{\mu\nu}=\bar{g}_{\mu\nu},\, \delta g_{\mu\nu}=h_{\mu\nu}} \tag{2.2.17}$$

which can be practically computed by expressing the variation of the integrand $\sqrt{g}(-R+2\Lambda)$ in terms of the variations of $\sqrt{g}$ and R, and then using their background expansions to compute the term of n-th order. In appendix A we give the necessary expansions to find the zeroth, first and second order term of the background expansion (2.2.5); in order to write concisely the explicit expressions, we introduce two definitions:

1. All tensors constructed with the metric with a "bar" on the symbol are defined as in subsection 1.3.1, but constructed with the background metric instead:

$$\bar{A}(\bar{g}) \equiv A(g = \bar{g}) \tag{2.2.18}$$

In particular, $\bar{g}^{\mu\nu}$ is the inverse background metric, $\bar{g}$ the background metric determinant, $\bar{\Gamma}^\lambda_{\mu\nu}$ the background metric connection:

$$\bar{\Gamma}^\lambda_{\mu\nu} = \frac{\bar{g}^{\lambda\rho}}{2}\left(\partial_\mu \bar{g}_{\nu\rho} + \partial_\nu \bar{g}_{\mu\rho} - \partial_\rho \bar{g}_{\mu\nu}\right) \tag{2.2.19}$$

and $\bar{R}^\alpha{}_{\mu\beta\nu}$, $\bar{R}_{\mu\nu}$, $\bar{R}$, $\bar{G}_{\mu\nu}$ the background curvature tensors, respectively, the background Riemann tensor, the background Ricci tensor, the background Ricci scalar and the background Einstein tensor:

$$\bar{R}^\alpha{}_{\mu\beta\nu} = \partial_\beta \bar{\Gamma}^\alpha_{\mu\nu} - \partial_\nu \bar{\Gamma}^\alpha_{\mu\beta} + \bar{\Gamma}^\lambda_{\beta\mu}\bar{\Gamma}^\alpha_{\lambda\nu} - \bar{\Gamma}^\lambda_{\nu\mu}\bar{\Gamma}^\alpha_{\lambda\beta} \tag{2.2.20}$$

$$\bar{R}_{\mu\nu} = \bar{R}^{\alpha}{}_{\mu\alpha\nu} \tag{2.2.21}$$

$$\bar{R} = \bar{g}^{\mu\nu}\bar{R}_{\mu\nu} \tag{2.2.22}$$

$$\bar{G}_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{1}{2}\,\bar{R}\,\bar{g}_{\mu\nu} \tag{2.2.23}$$

finally, $\bar{\nabla}_\mu$ is the covariant derivative associated to the background metric connection, compatible with the background metric:

$$\bar{\nabla}_\lambda \bar{g}_{\mu\nu} = 0 \tag{2.2.24}$$

and used to construct the associated background covariant laplacian operator:

$$\bar{\Box} = \bar{g}^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu \tag{2.2.25}$$

2. The indices of the metric fluctuation are defined to be raised and lowered with the background metric instead of the full metric; in particular, the contractions between the metric fluctuation and the inverse background metric and the trace of the metric fluctuation are defined to be, respectively:

$$h^{\mu}{}_{\nu} \equiv \bar{g}^{\mu\lambda}h_{\lambda\nu} \tag{2.2.26}$$

$$h^{\mu\nu} \equiv \bar{g}^{\mu\lambda}\bar{g}^{\nu\rho}h_{\lambda\rho} \tag{2.2.27}$$

$$h \equiv \bar{g}^{\mu\nu}h_{\mu\nu} \tag{2.2.28}$$

Given these notations, the zeroth order term is equal to the Einstein-Hilbert action computed in the background metric:

$$S_{EH,0}[h;\bar{g}] = \frac{2}{\kappa^2}\int d^Dx\sqrt{\bar{g}}\left(-\bar{R} + 2\Lambda\right) \tag{2.2.29}$$

The first order term is instead found to be:

$$S_{EH,1}[h;\bar{g}] = \frac{2}{\kappa^2}\int d^Dx\sqrt{\bar{g}}\,(\bar{G}_{\mu\nu} + \Lambda\bar{g}_{\mu\nu})h^{\mu\nu} \tag{2.2.30}$$

In particular, the first order Taylor coefficient is given by the left hand side of the Einstein equations computed in the background metric, according to (2.2.17) and the first order variation (1.3.42), which gives the equations of motion of the theory, i.e. Einstein field equations in the vacuum, in this case.
The second order term is given by the sum of three pieces:

$$S_{EH,2}[h;\bar{g}] = S_{EH,2-kin}[h;\bar{g}] + S_{EH,2-deD}[h;\bar{g}] + S_{EH,2-int}[h;\bar{g}] \tag{2.2.31}$$

which are found to be, respectively:

$$S_{EH,2-kin}[h;\bar{g}] = \frac{1}{\kappa^2}\int d^Dx\sqrt{\bar{g}}\left[-\frac{1}{2}\left(h^{\mu\nu}\bar{\Box}h_{\mu\nu} - \frac{1}{2}\,h\bar{\Box}h\right)\right] \tag{2.2.32}$$

$$S_{EH,2-deD}[h;\bar{g}] = \frac{1}{\kappa^2}\int d^Dx\sqrt{\bar{g}}\left[-\left(\bar{\nabla}^\mu h_{\mu\nu} - \frac{1}{2}\bar{\nabla}_\nu h\right)^2\right] \tag{2.2.33}$$

$$\begin{aligned} S_{EH,2-int}[h;\bar{g}] = \frac{1}{\kappa^2}\int d^Dx\sqrt{\bar{g}}\Big[&-\bar{R}_{\alpha\mu\beta\nu}h^{\alpha\beta}h^{\mu\nu} \\ &- \bar{R}_{\mu\nu}\left(h^{\mu\lambda}h_\lambda{}^\nu - hh^{\mu\nu}\right) \\ &+ \frac{1}{2}(\bar{R}-2\Lambda)\left(h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h^2\right)\Big] \end{aligned} \tag{2.2.34}$$

In the proper Taylor form (2.2.6), they can be compactly written as:

$$S_{EH,2-kin}[h;\bar{g}] = \frac{1}{\kappa^2}\int d^Dx\sqrt{\bar{g}}\,\frac{1}{2}\,h_{\alpha\beta}\left(-\bar{K}^{\alpha\beta,\mu\nu}\bar{\Box}\right)h_{\mu\nu} \tag{2.2.35}$$

$$S_{EH,2-deD}[h;\bar{g}] = \frac{1}{\kappa^2}\int d^Dx\sqrt{\bar{g}}\,\frac{1}{2}\,h_{\alpha\beta}\bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\rho\bar{\nabla}^\sigma h_{\mu\nu} \tag{2.2.36}$$

$$S_{EH,2-int}[h;\bar{g}] = \frac{1}{\kappa^2}\int d^Dx\sqrt{\bar{g}}\,\frac{1}{2}\,h_{\alpha\beta}\bar{O}_2^{\alpha\beta,\mu\nu}h_{\mu\nu} \tag{2.2.37}$$

Where the three structure tensors $\bar{K}^{\alpha\beta,\mu\nu}$, $\bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma}$, $\bar{O}_2^{\alpha\beta,\mu\nu}$ are defined as:

$$\bar{K}^{\alpha\beta,\mu\nu} = \frac{1}{2}\left(\bar{g}^{\alpha\mu}\bar{g}^{\beta\nu} + \bar{g}^{\alpha\nu}\bar{g}^{\beta\mu} - \bar{g}^{\alpha\beta}\bar{g}^{\mu\nu}\right) \tag{2.2.38}$$

$$\bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma} = 2\bar{K}^{\alpha\beta,\lambda}{}_\rho\bar{K}^{\mu\nu}{}_{\lambda\sigma} \tag{2.2.39}$$

$$\begin{aligned} \bar{O}_2^{\alpha\beta,\mu\nu} =& -\left(\bar{R}^{\alpha\mu\beta\nu} + \bar{R}^{\alpha\nu\beta\mu}\right) \\ &- \frac{1}{2}\left(\bar{g}^{\alpha\mu}\bar{R}^{\beta\nu} + \bar{g}^{\alpha\nu}\bar{R}^{\beta\mu} + \bar{g}^{\beta\mu}\bar{R}^{\alpha\nu} + \bar{g}^{\beta\nu}\bar{R}^{\alpha\mu} - 2\bar{g}^{\alpha\beta}\bar{R}^{\mu\nu} - 2\bar{g}^{\mu\nu}\bar{R}^{\alpha\beta}\right) \\ &+ (\bar{R}-2\Lambda)\bar{K}^{\alpha\beta,\mu\nu} \end{aligned} \tag{2.2.40}$$

They are all constructed with the background metric only, therefore they can be moved across any background covariant derivative and laplacian, since from the compatibility condition (2.2.24), their covariant derivative is also vanishing. They are also appropriately symmetrized so that $\bar{K}^{\alpha\beta,\mu\nu}\bar{\Box}$, $\bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\rho\bar{\nabla}^\sigma$ and $\bar{O}_2^{\alpha\beta,\mu\nu}$ are proper operators acting in the space of rank-$(0,2)$ symmetric tensors, namely symmetric under the exchanges $\alpha \leftrightarrow \beta$, $\mu \leftrightarrow \nu$, $\{\alpha\beta\} \leftrightarrow \{\mu\nu\}$. $\bar{K}^{\alpha\beta,\mu\nu}$ is obtained by rearranging the contractions in $S_{EH,2-kin}[h;\bar{g}]$ and its completely symmetrized. $\bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma}$ is obtained by recognizing that the integrand of $S_{EH,2-deD}[h;\bar{g}]$ is equal to $-\bar{K}^{\alpha\beta}{}_{\gamma\delta}\bar{\nabla}^\delta h_{\alpha\beta}\bar{g}^{\gamma\rho}\bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\sigma h_{\mu\nu}$ and integrating by parts the first covariant derivative, from which the change in sign; it is symmetric under $\alpha \leftrightarrow \beta$, $\mu \leftrightarrow \nu$ and $\{\alpha\beta\} \leftrightarrow \{\mu\nu\} + \rho \leftrightarrow \sigma$.
Recombining the three pieces, the second order term can be thus compactly expressed as:

$$S_{EH,2}[h;\bar{g}] = \frac{1}{\kappa^2}\int d^Dx\sqrt{\bar{g}}\,\frac{1}{2}\,h_{\alpha\beta}\left(-\bar{K}^{\alpha\beta,\mu\nu}\bar{\Box} + \bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\rho\bar{\nabla}^\sigma + \bar{O}_2^{\alpha\beta,\mu\nu}\right)h_{\mu\nu} \tag{2.2.41}$$

The infinitely many higher order terms have the same structure, i.e. action monomials containing an increasing number of metric perturbations multiplied by more and more complicated tensorial operators constructed with the background metric and various numbers of covariant derivatives, the square root of the background metric determinant $\sqrt{\bar{g}}$, which forms the overall integration measure $d^Dx\sqrt{\bar{g}}$, and the constant $1/\kappa^2$; to make explicit this structure, we introduce the following compact notation for the generic Taylor term of n-th order (2.2.6):

$$S_{EH,n}[h;\bar{g}] = \frac{1}{\kappa^2}\int d^Dx\sqrt{\bar{g}}\,\frac{1}{n!}\,\bar{O}_n h^n \tag{2.2.42}$$

Using this notation and substituting the explicit expressions found for the first three terms in the expansion (2.2.5), we can rewrite the full background-expanded Einstein-Hilbert action as:

$$\begin{aligned} S_{EH}[h;\bar{g}] =& \frac{1}{\kappa^2}\int d^Dx\sqrt{\bar{g}}\left[2\left(-\bar{R}+2\Lambda\right) + 2\left(\bar{G}_{\mu\nu}+\Lambda\bar{g}_{\mu\nu}\right)h_{\mu\nu}\right.\\ &+\frac{1}{2}\,h_{\alpha\beta}\left(-\bar{K}^{\alpha\beta,\mu\nu}\bar{\Box} + \bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\rho\bar{\nabla}^\sigma + \bar{O}_2^{\alpha\beta,\mu\nu}\right)h_{\mu\nu}\\ &\left.+\sum_{n=3}^{\infty}\frac{1}{n!}\,\bar{O}_n h^n\right] \end{aligned} \tag{2.2.43}$$

In conclusion, substituting the background split in the Einstein-Hilbert action and performing the background expansion, we are formally recasting the original theory as an effective theory for the metric fluctuation as a symmetric tensor field in a background spacetime equipped with the background metric $(\mathcal{M},\bar{g})$ (the invariant volume element is $d^Dx\sqrt{\bar{g}}$ and indices are raised and lowered by the background metric). The sum of the first two derivative pieces $S_{EH,2-kin}[h;\bar{g}]$ and $S_{EH,2-deD}[h;\bar{g}]$ of the second order term forms the kinetic term of the theory, the second piece in particular will be the one to be modified in the standard gauge-fixing procedure, presented in section 2.3, enforcing the de Donder gauge-fixing condition; the sum of the first order term $S_{EH,1}[h;\bar{g}]$, the third non-derivative piece of the second order term $S_{EH,2-int}[h;\bar{g}]$ and all the higher order terms $S_{EH,n\geq 3}[h;\bar{g}]$ constitutes a series of infinitely many n-point self-interactions of the metric fluctuation, entering also with various derivatives, with parametric dependence on the background metric (on top of the zeroth order term $S_{EH,0}[h;\bar{g}]$ which is just a constant as far as the metric fluctuation is considered).

We conclude the subsection with a remark. From a QFT point of view, one can notice that the metric fluctuation field is not canonically normalized due to the presence of the factor $1/\kappa^2$, which has a non-zero mass dimension, in front of the background-expanded action and in particular in front of the kinetic term. Therefore, to have a canonically normalized theory for the metric fluctuation we should operate the rescaling:

$$h_{\mu\nu} \to \kappa h_{\mu\nu} \tag{2.2.44}$$

and the background-expanded Einstein-Hilbert action becomes:

$$
\begin{aligned}
S_{EH}[h;\bar{g}] = \int d^D x \sqrt{\bar{g}} \Bigg[& \frac{2}{\kappa^2}\left(-\bar{R}+2\Lambda\right) + \frac{2}{\kappa}\left(\bar{G}_{\mu\nu}+\Lambda \bar{g}_{\mu\nu}\right) h^{\mu\nu} \\
& + \frac{1}{2} h_{\alpha\beta} \left(-\bar{K}^{\alpha\beta,\mu\nu}\bar{\Box} + \bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma} \bar{\nabla}^{\rho}\bar{\nabla}^{\sigma} + \bar{O}_2^{\alpha\beta,\mu\nu} \right) h_{\mu\nu} \\
& + \sum_{n=3}^{\infty} \frac{\kappa^{n-2}}{n!} \bar{O}_n h^n \Bigg]
\end{aligned}
\tag{2.2.45}
$$

The constant κ now appears as gravitational coupling constant in the various interaction terms. In particular, the interaction term of n-th order has coupling κ^{n-2} which in $D = 4$ dimensions has mass dimension $2 - n < 0$. According to the perturbative renormalization rules in terms of mass dimensionality of couplings and operators in the classical action, this implies the well known fact that Einstein's gravity is not perturbatively renormalizable, since the number of counter terms needed to renormalize divergences increases with the loop order and is thus infinite. The theory can still be used as an effective field theory to describe processes below the Planck scale, truncating the expansion at a certain order [15]. Adopting the FRG perspective, in the following we do not operate the rescaling and we do not introduce perturbative approximations, so that all terms in the expansion are formally considered.

2.2.2 Faddeev-Popov quantization method

Consider the pure Einstein-Hilbert theory without introducing, for the moment, the background field method. In order to extract a well-defined path integral from the naive, divergent definition:

$$
Z_{EH} = \int \mathcal{D}g \, e^{-S_{EH}[g]} = \infty \tag{2.2.46}
$$

we apply the Faddeev-Popov method to restrict the path integration over the configurations satisfying a gauge-fixing condition breaking diffeomorphism symmetry.
The method can be applied similarly to the standard case of a non-abelian gauge theory, namely the Yang-Mills theory [16]; however two observations are in order:

1. The method is typically applied to gauge theories with gauge group given by a local Lie group, i.e a parametric group of transformations connected to the identity and admitting an infinitesimal form. Therefore, in order to apply the method, we will consider the pure Einstein-Hilbert theory as a gauge theory with gauge group given the component of the diffeomorphisms group connected to the identity $\mathrm{Diff}_e(\mathcal{M})$; this will be sufficient to obtain as a final result a well defined path integral, since the gauge-fixing condition will break the gauge symmetry under infinitesimal diffeomorphisms and therefore also the one under general diffeomorphisms.

2. As stated in subsection 2.1.1, we consider the path integral to be performed over symmetric tensors. In principle, we can thus consider the standard translationally invariant measure (2.1.6) as in (2.2.46). However, this turns out to be effectively correct only from a practical point of view. The reason is that, strictly speaking, the standard path integral measure is not invariant under diffeomorphisms, since performing an infinitesimal diffeomorphism as change of variables in the path integral:

$$\mathcal{D}g \quad \to \quad \mathcal{D}g^{(\xi)} = \mathrm{Det}\left[\frac{\delta g^{(\xi)}}{\delta g}\right]\mathcal{D}g \neq \mathcal{D}g \tag{2.2.47}$$

one finds that the jacobian is not equal to 1. In particular, the functional derivative inside the determinant can be rewritten, in explicit notation, as:

$$\frac{\delta g^{(\xi)}_{\mu\nu}(x)}{\delta g_{\alpha\beta}(y)} = \delta^{\alpha\beta}_{\mu\nu}\delta(x-y) + \frac{\delta}{\delta g_{\alpha\beta}(y)}\left(\delta_\xi g_{\mu\nu}(x)\right) \tag{2.2.48}$$

where the first term, coming from the functional derivative $\delta g_{\mu\nu}(x)/\delta g_{\alpha\beta}(y)$ is the functional identity of the theory, with the symmetric Kronecker symbol:

$$\delta^{\alpha\beta}_{\mu\nu} = \frac{\delta^\alpha_\mu\delta^\beta_\nu + \delta^\alpha_\nu\delta^\beta_\mu}{2} \tag{2.2.49}$$

as identity in the $(D(D+1)/2)$-dimensional space of symmetric tensors and the Dirac delta $\delta(x-y)$ as identity in the infinite-dimensional space of functions of a spacetime argument. From the variation (1.3.70), written in terms of partial derivatives, the second term is given by:

$$\frac{\delta}{\delta g_{\alpha\beta}(y)}\left(\delta_\xi g_{\mu\nu}(x)\right) = \left(\xi(x)^\lambda\partial_{(x)\lambda}\delta^{\alpha\beta}_{\mu\nu} + \partial_{(x)\mu}\xi^\lambda(x)\delta^{\alpha\beta}_{\lambda\nu} + \partial_{(x)\nu}\xi^\lambda(x)\delta^{\alpha\beta}_{\lambda\mu}\right)\delta(x-y) \tag{2.2.50}$$

Using now the functional extension of the matrix identity $\det\left(\mathbb{1}+\epsilon\mathrm{M}\right) = 1+\epsilon\mathrm{tr}\mathrm{M}+O(\epsilon^2)$, the jacobian can be thus rewritten, in explicit notation, as:

$$\mathrm{Det}\left[\frac{\delta g^{(\xi)}_{\mu\nu}(x)}{\delta g_{\alpha\beta}(y)}\right] = 1 + \mathrm{Tr}\left[\frac{\delta}{\delta g_{\alpha\beta}(y)}\left(\delta_\xi g_{\mu\nu}(x)\right)\right] + O(\xi^2) \tag{2.2.51}$$

and the functional trace, given by:

$$\mathrm{Tr}\left[\frac{\delta}{\delta g_{\alpha\beta}(y)}\left(\delta_\xi g_{\mu\nu}(x)\right)\right] = \int d^Dx d^Dy\,\frac{\delta}{\delta g_{\alpha\beta}(y)}\left(\delta_\xi g_{\mu\nu}(x)\right)\delta^{\mu\nu}_{\alpha\beta}\delta(x-y) \tag{2.2.52}$$

turns out to be non-zero due to the term $\xi^\lambda\partial_\lambda g_{\mu\nu}$ in the variation, which produces the term:

$$\mathrm{Tr}\left[\frac{\delta}{\delta g_{\alpha\beta}(y)}\left(\delta_\xi g_{\mu\nu}(x)\right)\right] = \frac{D(D+1)}{2}\int d^Dx\xi(x)^\lambda\partial_{(x)\lambda}\delta(x-y)|_{x=y} \neq 0 \tag{2.2.53}$$

as first noticed in [17]. In conclusion, being the gauge-invariance of the measure one of the necessary requirements in order to apply the Faddeev-Popov method, the path integral (2.2.46) cannot be actually used as input of the procedure.
The problem of constructing a diffeomorphism-invariant measure $\mathcal{D}\mu[g]$, which goes into itself under a diffeomorphism as change of variables in the path integral:

$$\mathcal{D}\mu[g] \quad \to \quad \mathcal{D}\mu\left[g^{(\phi)}\right] = \mathcal{D}\mu[g] \tag{2.2.54}$$

and in particular, as far as the Faddeev-Popov method is concerned:

$$\mathcal{D}\mu[g] \quad \to \quad \mathcal{D}\mu\left[g^{(\xi)}\right] = \mathcal{D}\mu[g] \tag{2.2.55}$$

has been considered in the literature since the last century, in particular in the early works by Fradkin and Vilkovisky [18] and Fujikawa [19][20], which led to the two measures mostly used since then, and it is still discussed today [21] [22]. We also mention that in perturbative covariant computations in dimensional regularization $\delta(0)$ singularities are set to zero [21], therefore limiting to computations in this scenario, one is allowed to use the Faddeev-Popov method and its final result substituting the diffeomorphism-invariant measure with the original standard one, which, for the purpose of such computations, can be regarded as diffeomorphism-invariant, since setting the $\delta(0)$ to zero in the result (2.2.53), the functional trace (2.2.52) is vanishing and so the functional jacobian (2.2.51) is 1. As far as the thesis is concerned, in the following it will be sufficient to invoke the possibility of constructing and formally consider a diffeomorphism-invariant measure, without the necessity to specify an explicit expression.

Faddeev-Popov method

In light of those remarks, we redefine the naive path integral (2.2.46) substituting the standard measure with a suitable a suitable diffeomorphism-invariant measure:

$$Z_{EH} = \int \mathcal{D}\mu[g]\, e^{-S_{EH}[g]} \tag{2.2.56}$$

so that it can be now correctly used as input for the Faddeev-Popov method, considering the pure Einstein-Hilbert theory as gauge theory with $\mathrm{Diff}_e(\mathcal{M})$ as gauge group. Actually, in order to obtain directly a QEG theory prepared for the application of FRG methods, we also choose to apply immediately the linear background split (2.2.3):

$$Z_{EH}[\bar{g}] = \int \mathcal{D}\mu[h;\bar{g}]\, e^{-S_{EH}[h;\bar{g}]} \tag{2.2.57}$$

with the diffeomorphism-invariant measure rewritten as:

$$\mathcal{D}\mu[g = \bar{g} + h] \equiv \mathcal{D}\mu[h;\bar{g}] \tag{2.2.58}$$

which takes the place of the one coming from the standard translationally invariant measure $\mathcal{D}h$. Considering the metric fluctuation as the dynamical and elementary field to be quantized, the relevant gauge transformations are infinitesimal diffeomorphisms rewritten as true gauge transformations, while the symmetry under background gauge transformations is regarded as an additional symmetry of the theory, as explained at the end of subsection 2.2.1. So, we apply the Faddeev-Popov method on the path integral (2.2.57) considering the background-expanded pure Einstein-Hilbert theory as gauge theory with $\widehat{\mathrm{Diff}}_e(\mathcal{M})$ as gauge group, in order to break gauge symmetry under true gauge transformations:

1. Faddeev-Popov trick and Gribov problem: Consider a generic gauge-fixing condition of the type:

$$f(h;\bar{g}) = 0 \tag{2.2.59}$$

where $f(h;\bar{g})$ is a function of the metric fields breaking the symmetry under true gauge transformations. In order to restrict the path integration over configurations satisfying this condition and separate the non-physically-meaningful divergent contribution coming from gauge redundancy, we rewrite the path integral (2.2.57) introducing a trivial multiplication for 1 in the integrand, employing the identity:

$$\begin{aligned} 1 = \int \mathcal{D}f(h;\bar{g})\,\delta[f(h;\bar{g})] &= \int \mathcal{D}\omega[\xi]\,\delta[f\left(h^{(\xi)};\bar{g}\right)]\mathrm{Det}\left[\frac{\delta f\left(h^{(\xi)};\bar{g}\right)}{\delta\xi}\right] \\ &\overset{\star}{=} \Delta[h;\bar{g}]\int \mathcal{D}\omega[\xi]\,\delta\left[f\left(h^{(\xi)};\bar{g}\right)\right] \end{aligned} \tag{2.2.60}$$

which follows from the defining property of the functional Dirac delta distribution. In the second couple of equalities, the upper one is obtained by thinking the gauge transformed metric fields as functions of the vector field regarded as new integration variable, and $\mathcal{D}\omega[\xi]$ denotes the functional Haar invariant measure on the manifold given by the group $\widehat{\mathrm{Diff}}_e(\mathcal{M})$, where each gauge transformation is individuated by the associated vector field ξ^μ. This measure is characterized by the property of being invariant if one sends the generic integrated gauge transformation of $\widehat{\mathrm{Diff}}_e(\mathcal{M})$ into its composition with another fixed gauge transformation; so, if ξ^μ is the vector field associated to the integrated gauge transformation and η^μ the one associated to another fixed gauge transformation, we have:

$$\mathcal{D}\omega[\eta\circ\xi] = \mathcal{D}\omega[\xi\circ\eta] = \mathcal{D}\omega[\xi] \quad \forall\eta \tag{2.2.61}$$

In the lower equality $\Delta[h;\bar{g}]$ denotes the Faddeev-Popov determinant:

$$\Delta[h;\bar{g}] = \mathrm{Det}\left[\frac{\delta f\left(h^{(\xi)};\bar{g}\right)}{\delta\xi}\right]\Bigg|_{\xi=\tilde{\xi}\ :\ f\left(h^{(\tilde{\xi})};\bar{g}\right)=0} \tag{2.2.62}$$

obtained by enforcing on the functional jacobian the condition imposed by the functional Dirac delta and bringing it outside from the integral, since it no longer depends on the vector field. The symbol $\star$ on the equality denotes that this operation is possible only if there exists only one specific vector field $\tilde{\xi}$ for which the gauge-fixing condition is satisfied:

$$\exists!\ \tilde{\xi}\ :\ f\left(h^{(\tilde{\xi})};\bar{g}\right) = 0 \tag{2.2.63}$$

i.e. there exists, for each given metric configuration $\{h;\bar{g}\}$, only one metric configuration $\left\{\tilde{h}\equiv h^{(\tilde{\xi})};\tilde{\bar{g}}\equiv\bar{g}^{(\tilde{\xi})}\right\}$ between those equivalent to $\{h;\bar{g}\}$ which satisfies the gauge-fixing condition and it is thus selected to be integrated in the final path integral. Moreover, in order for the next steps to be valid, it is also necessary to assume that the Faddeev-Popov determinant is different from zero, which happens when the operator given by the functional derivative has zero as eigenvalue:

$$\Delta[h;\bar{g}] \neq 0 \tag{2.2.64}$$

Both conditions are in general not satisfied and this possibility corresponds to the Gribov problem [23]; in particular, the first possibility corresponds to the problem of Gribov copies, i.e. the fact that in general the gauge fixing condition (2.2.59) does not select a unique metric configuration inside each of the gauge orbits. Taking into account this problem in the Faddeev-Popov method is historically a very complicated task; therefore, we adopt the standard strategy of disregarding the problem, assuming that conditions (2.2.63) and (2.2.64) are satisfied, but then using a gauge-fixing function which does not necessarily respect them, taking advantage of the fact that the final result is nevertheless a correctly gauge-fixed path integral.

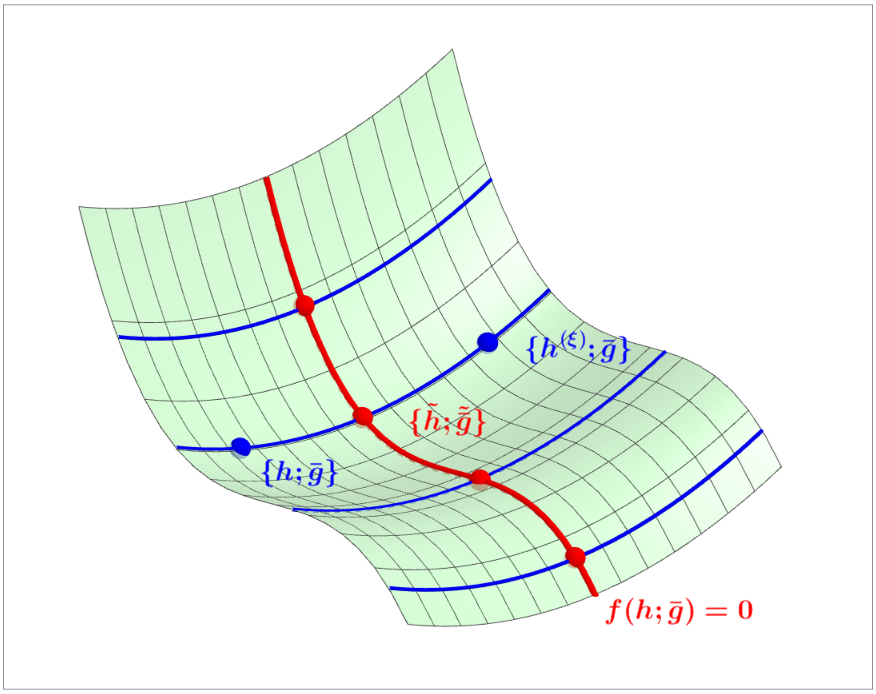

Figure 3: *Pictorial representation of the space of metric fields configurations. Configurations connected by gauge transformations lie on a gauge orbit (blue lines) and are physically equivalent. Assuming that the conditions preventing the Gribov problem are satisfied, the gauge-fixing condition* $f(h;\bar{g}) = 0$ *(red line) selects one representative configuration for each gauge orbit.*

Before proceeding with the method, we notice that the Faddeev-Popov determinant is characterized by the fundamental property of being gauge-invariant:

$$\Delta\big[h^{(\xi)};\bar{g}\big] = \Delta[h;\bar{g}] \quad \forall\xi \tag{2.2.65}$$

Indeed, from (2.2.60) we have that:

$$\frac{1}{\Delta[h;\bar{g}]} = \int \mathcal{D}\omega[\xi]\,\delta\big[f\left(h^{(\xi)};\bar{g}\right)\big] \tag{2.2.66}$$

and, recalling the composition rule of gauge transformations (2.2.10) and the property of the Haar measure (2.2.61):

$$\begin{aligned}\frac{1}{\Delta[h^{(\eta)};\bar{g}]} &= \int \mathcal{D}\omega[\xi]\,\delta\big[f\left(h^{(\eta)(\xi)};\bar{g}\right)\big] = \\ &= \int \mathcal{D}\omega[\xi]\,\delta\big[f\left(h^{(\eta\circ\xi)};\bar{g}\right)\big] = \\ &= \int \mathcal{D}\omega[\eta\circ\xi]\,\delta\big[f\left(h^{(\eta\circ\xi)};\bar{g}\right)\big] = \frac{1}{\Delta[h;\bar{g}]}\end{aligned}$$

We can now substitute the identity (2.2.60) inside the path integral (2.2.57):

$$\begin{aligned}Z_{EH}[\bar{g}] &= \int \mathcal{D}\mu[h;\bar{g}]\left(\Delta[h;\bar{g}]\int \mathcal{D}\omega[\xi]\,\delta\big[f\left(h^{(\xi)};\bar{g}\right)\big]\right)e^{-S_{EH}[h;\bar{g}]} = \\ &\stackrel{\mathbf{A}}{=} \int \mathcal{D}\omega[\xi]\left(\int \mathcal{D}\mu[h;\bar{g}]\Delta[h;\bar{g}]\,\delta\big[f\left(h^{(\xi)};\bar{g}\right)\big]\,e^{-S_{EH}[h;\bar{g}]}\right) = \\ &\stackrel{\mathbf{B}}{=} \int \mathcal{D}\omega[\xi]\left(\int \mathcal{D}\mu\big[h^{(\xi)};\bar{g}\big]\,\Delta\big[h^{(\xi)};\bar{g}\big]\,\delta\big[f\left(h^{(\xi)};\bar{g}\right)\big]\,e^{-S_{EH}[h^{(\xi)};\bar{g}]}\right) = \\ &\stackrel{\mathbf{C}}{=} \left(\int \mathcal{D}\omega[\xi]\right)\left(\int \mathcal{D}\mu[h;\bar{g}]\Delta[h;\bar{g}]\,\delta[f(h;\bar{g})]\,e^{-S_{EH}[h;\bar{g}]}\right)\end{aligned}$$

In step **A** we switched the two integrations, the one over metric configurations and the one over gauge transformations; in step **B** we used the gauge-invariance of the measure, the Faddeev-Popov determinant and the Einstein-Hilbert action to rewrite them in terms of the metric fields gauge-transformed according to the gauge transformation integrated in the external integral; in step **C** we renamed the integration variable in the integral over metric configurations recognizing that the dependence on the gauge transformation is actually fictitious, and therefore we separated the two integrals. The first, in particular, counts the number of gauge transformations which can be applied on a given metric configuration, or, in other words, the number of its gauge-equivalent configurations in its gauge orbit $\mathcal{O}_{\{h,\bar{g}\}}$:

$$\mathcal{O}_{\{h,\bar{g}\}} = \Big\{\{h'_{\mu\nu},\bar{g}_{\mu\nu}\} : \{h'_{\mu\nu},\bar{g}_{\mu\nu}\} = \{h^{(\xi)}_{\mu\nu},\bar{g}_{\mu\nu}\}\Big\} \tag{2.2.67}$$

so, the volume of a generic gauge orbit $\mathcal{O}$:

$$\mathrm{vol}\mathcal{O} = \int \mathcal{D}\omega[\xi] = \infty \tag{2.2.68}$$

which is responsible of the divergence of the naive path integral. Being this volume ultimately a constant, although infinite, it is irrelevant in the computation of correlation functions; therefore, it can be simply removed to define the Faddeev-Popov gauge-fixed path integral:

$$Z[\bar{g}] \equiv \frac{Z_{EH}[\bar{g}]}{\mathrm{vol}\mathcal{O}} \tag{2.2.69}$$

whose explicit form is:

$$Z[\bar{g}] = \int \mathcal{D}\mu[h;\bar{g}]\,\Delta[h;\bar{g}]\delta[f(h;\bar{g})]e^{-S_{EH}[h;\bar{g}]} \tag{2.2.70}$$

As desired, the path integral is now well defined, since the infinite contribution coming from gauge redundancy is removed and indeed the integration is effectively performed, thanks to the functional Dirac delta, only over the metric configurations satisfying the gauge-fixing condition (one per gauge orbit, assuming that the conditions preventing the Gribov problem are satisfied). We also notice that the Faddeev-Popov determinant can be effectively rewritten as:

$$\Delta[h;\bar{g}] = \mathrm{Det}\left[\frac{\delta f(h^{(\xi)};\bar{g})}{\delta\xi}\right]\Bigg|_{\xi=0} \tag{2.2.71}$$

since the functional Dirac delta enforces $f(h;\bar{g}) = 0$, therefore we have $f\left(h^{(\xi=0)};\bar{g}\right) = f(h;\bar{g}) = 0 \to \tilde{\xi} = 0$, again assuming that the conditions preventing the Gribov problem are satisfied and this is the only solution of the gauge-fixing equation.

2. Gauge-fixing term: The functional Dirac delta enforcing the gauge-fixing condition can be actually substituted by any functional of the gauge-fixing condition; indeed, shifting the gauge-fixing function by an arbitrary function χ independent of the metric fields, we obtain the same gauge-fixed path integral, with in particular the same Faddeev-Popov determinant (since it contains the derivative of the gauge-fixing function enters):

$$Z[\bar{g}] = \int \mathcal{D}\mu[h;\bar{g}]\,\Delta[h;\bar{g}]\delta[f(h;\bar{g}) - \chi]\,e^{-S_{EH}[h;\bar{g}]} \tag{2.2.72}$$

In this way we introduce a fictitious dependence on the function χ on which we can integrate, in order to smear the functional Dirac delta with a normalized functional distribution, which we write for convenience in the form of an exponential:

$$\begin{aligned}
Z[\bar{g}] = \frac{\int \mathcal{D}\chi\, e^{-S_{gf}[\chi]}}{\int \mathcal{D}\chi\, e^{-S_{gf}[\chi]}} Z[\bar{g}] \propto \int \mathcal{D}\chi\, Z[\bar{g}]\, e^{-S_{gf}[\chi]} &= \\
= \int \mathcal{D}\chi\mathcal{D}\mu[h;\bar{g}]\,\Delta[h;\bar{g}]\delta[f(h;\bar{g}) - \chi]e^{-S_{EH}[h;\bar{g}]-S_{gf}[\chi]}& \\
= \int \mathcal{D}\mu[h;\bar{g}]\,\Delta[h;\bar{g}]\,e^{-S_{EH}[h;\bar{g}]-S_{gf}[f(h;\bar{g})]}&
\end{aligned}$$

The net effect, apart from an irrelevant multiplicative constant (also possibly infinite) coming from the normalization of the smearing distribution, is the possibility of substituting the functional Dirac delta with an arbitrary functional of the gauge-fixing function:

$$\delta[f(h;\bar{g})] \quad \to \quad e^{-S_{gf}[f(h;\bar{g})]} \equiv e^{-S_{gf}[h;\bar{g}]} \tag{2.2.73}$$

Or equivalently adding an arbitrary gauge-fixing term to the action containing the gauge-fixing function. In the thesis we will consider exclusively gauge-fixing functions of the type $f_\mu(h;\bar{g})$, i.e. constructed to be a 1-form tensor field.

3. Ghost term: Also the Faddeev-Popov determinant can be rewritten in exponential form, obtaining another term which can be formally added to the gauge-fixed action. First we expand using the functional chain rule, in explicit notation, the functional derivative inside the Faddeev-Popov determinant:

$$\begin{aligned}\frac{\delta f_\mu(h^{(\xi)};\bar g)(x)}{\delta\xi^\nu(y)}\bigg|_{\xi=0} &= \int d^Dz\ \frac{\delta f_\mu(h;\bar g)(x)}{\delta h_{\alpha\beta}(z)}\frac{\delta h^{(\xi)}_{\alpha\beta}(z)}{\delta\xi^\nu(y)}\bigg|_{\xi=0} = \\ &= \int d^Dz\ \frac{\delta f_\mu(h;\bar g)(x)}{\delta h_{\alpha\beta}(z)}\frac{\delta}{\delta\xi^\nu(y)}\left(h_{\alpha\beta}(z)+\delta_\xi h_{\alpha\beta}+O(\xi^2)\right)\Big|_{\xi=0} = \\ &= \int d^Dz\ \frac{\delta f_\mu(h;\bar g)(x)}{\delta h_{\alpha\beta}(z)}\frac{\delta}{\delta\xi^\nu(y)}\left(\delta_\xi h_{\alpha\beta}\right) = \\ &= \int d^Dz\ \frac{\delta f_\mu(h;\bar g)(x)}{\delta h_{\alpha\beta}(z)}\left(\partial_\nu g_{\alpha\beta}(z)+g_{\nu\beta}(z)\partial_{(z)\alpha}+g_{\nu\alpha}(z)\partial_{(z)\beta}\right)\delta(z-y)\end{aligned}$$

In particular, we remark that, due to the evaluation in $\xi=0$, only the first order form of the true gauge variation of the metric fluctuation is necessary. Performing the z-integration, the final result is given by a certain differential operator, depending on the gauge-fixing function:

$$\mathcal{G}_{\mu\nu}(x,y) = \int d^Dz\ \frac{\delta f_\mu(h;\bar g)(x)}{\delta h_{\alpha\beta}(z)}\left(\partial_\nu g_{\alpha\beta}(z)+g_{\nu\beta}(z)\partial_{(z)\alpha}+g_{\nu\alpha}(z)\partial_{(z)\beta}\right)\delta(z-y) \quad (2.2.74)$$

The operator between parenthesis acting on the Dirac delta $\delta(z-y)$ is essentially a Lie derivative of the metric from which the vector field has been removed; indeed, when the operator acts linearly on a vector field A^μ, this takes the place of the missing vector field and the parenthesis gives again a Lie derivative of the metric[4]:

$$\int d^Dy\mathcal{G}_{\beta\gamma}(x,y)A^\gamma(y) = \int d^Dz\,\frac{\delta f_\beta(h;\bar g)(x)}{\delta h_{\mu\nu}(z)}\,\pounds_A g_{\mu\nu}(z) \quad (2.2.75)$$

Introducing two Grassmann-odd fields, the ghost and antighost fields c^μ and $\bar c_\mu$, the Faddeev-Popov determinant can be thus rewritten as the result of the following gaussian path integral:

$$\begin{aligned}\Delta[h;\bar g] = \mathrm{Det}\left[\mathcal{G}_{\mu\nu}(x,y)\right] &= \int \mathcal{D}c\mathcal{D}\bar c\, e^{+\int d^Dxd^Dy\sqrt{\bar g(x)}\bar c_\alpha(x)\bar g^{\alpha\beta}(x)\mathcal{G}_{\beta\gamma}(x,y)c^\gamma(y)} \\ &= \int \mathcal{D}c\mathcal{D}\bar c\, e^{-S_{gh}[h,c,\bar c;\bar g]}\end{aligned} \quad (2.2.76)$$

The $+$ sign in the gaussian exponent is chosen such that, for the specific choices of gauge-fixing considered, the ghost kinetic term will have the correct sign[5]. Using the property (2.2.75) considering the ghost field as vector field, the explicit expression for the ghost action term is:

$$\begin{aligned}S_{gh}[h,c,\bar c;\bar g] &= -\int d^Dx\sqrt{\bar g}\,\bar c_\alpha\bar g^{\alpha\beta}\mathcal{G}_{\beta\gamma}\bullet c^\gamma = \\ &= -\int d^Dx\sqrt{\bar g}\,\bar c_\alpha\bar g^{\alpha\beta}\frac{\delta f_\beta(h;\bar g)}{\delta h_{\mu\nu}}\bullet\pounds_c(\bar g_{\mu\nu}+h_{\mu\nu})\end{aligned} \quad (2.2.77)$$

[4]The operator can be moved on the right hand side of the Dirac delta by noticing that $\partial_\nu g_{\alpha\beta}(z)\delta(z-y) = \delta(z-y)\partial_\nu g_{\alpha\beta}(y)$, and integrating by parts within the y-integration (where there is no $\sqrt{\bar g}$ factor) to move the ordinary derivatives: $\partial_{(z)\mu}\delta(z-y) = -\partial_{(y)\mu}\delta(z-y) \overset{\text{i.b.p}}{=} \delta(z-y)\partial_{(y)\mu}$.

[5]Namely $-\int d^Dx\sqrt{\bar g}\bar c_\mu\bar\Box c^\mu$. We recall however from the theory of Grassmannian-graded objects that the integral is defined as a Berezin integral and would be defined for any sign used in the exponent.

where we introduced the symbol $\bullet$ representing a multiplication with an implicit spacetime integration.
The ghost and antighost fields are formally defined to be respectively a vector and a 1-form field; their variation rules under an infinitesimal gauge transformation are defined accordingly:

$$\begin{cases} \delta_\xi c^\mu = \pounds_\xi c^\mu = \xi^\lambda \partial_\lambda c^\mu - \partial_\lambda \xi^\mu c^\lambda \\ \qquad\qquad\quad = \xi^\lambda \nabla_\lambda c^\mu - \nabla_\lambda \xi^\mu c^\lambda \\ \delta_\xi \bar{c}_\mu = \pounds_\xi \bar{c}_\mu = \xi^\lambda \partial_\lambda \bar{c}_\mu + \partial_\mu \xi^\lambda \bar{c}_\lambda \\ \qquad\qquad\quad = \xi^\lambda \nabla_\lambda \bar{c}_\mu + \nabla_\mu \xi^\lambda \bar{c}_\lambda \end{cases} \tag{2.2.78}$$

and in the background field method the true and background gauge transformations can be defined in the same way (since we do not apply the background split also to the ghost and antighost fields):

$$\begin{cases} \delta_\xi c^\mu = \bar{\delta}_\xi c^\mu = \pounds_\xi c^\mu = \xi^\lambda \partial_\lambda c^\mu - \partial_\lambda \xi^\mu c^\lambda \\ \qquad\qquad\qquad\qquad = \xi^\lambda \bar{\nabla}_\lambda c^\mu - \bar{\nabla}_\lambda \xi^\mu c^\lambda \\ \delta_\xi \bar{c}_\mu = \bar{\delta}_\xi \bar{c}_\mu = \pounds_\xi \bar{c}_\mu = \xi^\lambda \partial_\lambda \bar{c}_\mu + \partial_\mu \xi^\lambda \bar{c}_\lambda \\ \qquad\qquad\qquad\qquad = \xi^\lambda \bar{\nabla}_\lambda \bar{c}_\mu + \bar{\nabla}_\mu \xi^\lambda \bar{c}_\lambda \end{cases} \tag{2.2.79}$$

4. Final result: In conclusion, substituting in the Faddeev-Popov path integral (2.2.70) the functional Dirac delta according to (2.2.73) and the Faddeev-Popov determinant according to (2.2.76), the final result of the procedure is:

$$Z[\bar{g}] = \int \mathcal{D}\mu[h;\bar{g}] \mathcal{D}c \mathcal{D}\bar{c}\, e^{-S[h,c,\bar{c};\bar{g}]} \tag{2.2.80}$$

with the gauge-fixed action:

$$S[h,c,\bar{c};\bar{g}] = S_{EH}[h;\bar{g}] + S_{gf}[h;\bar{g}] + S_{gh}[h,c,\bar{c};\bar{g}] \tag{2.2.81}$$

We remark that at this stage the gauge-fixing is still generic in two ways: firstly, the form of the gauge-fixing term is arbitrary, secondly the gauge-fixing function is also arbitrary (apart from the condition of breaking symmetry under true gauge transformations). So, strictly speaking, the gauge-fixing will be complete only after choosing a specific form for the gauge-fixing term and the gauge-fixing function.
We also remark the important fact that for each choice of those gauge-fixing details, the gauge-fixed path integral is still formally equal to the naive path integral divided by the volume of the gauge orbit:

$$Z_{S_{gf1},f_1}[\bar{g}] = \frac{Z_{EH}[\bar{g}]}{\mathrm{vol}\mathcal{O}} = Z_{S_{gf2},f_2}[\bar{g}] \quad , \quad \forall S_{gf1}, S_{gf2} \;\; \forall f_1, f_2 \tag{2.2.82}$$

representing the fact that the gauge-fixing procedure does not change physics. In fact, quantum physical observables computed with the gauge-fixed path integral are expected to be independent of the gauge-fixing details and to have the same value regardless of the gauge-fixing used.

5. Nakanishi-Lautrup field with noise: The gauge-fixed path integral resulting from the Faddeev-Popov method can be rewritten by interpreting the gauge-fixing term as the result of an additional path integration over a Nakanishi-Lautrup auxiliary field. This way of rewriting the gauge-fixed theory allows to express BRST symmetry, reviewed in the next subsection, in its off-shell form; moreover, it will be also useful to generalize the standard form for the gauge-fixing term in order to switch to the non-standard one used in the third part of the thesis. Consider a Nakanishi-Lautrup auxiliary field b_μ, defined to be a 1-form field, and introduce an additional path integration in the Faddeev-Popov gauge-fixed path integral (2.2.80):

$$Z[\bar{g}] = \int \mathcal{D}\mu[h;\bar{g}]\mathcal{D}c\mathcal{D}\bar{c}\mathcal{D}b\, e^{-S[h,c,\bar{c};b;\bar{g}]} \tag{2.2.83}$$

The action is now defined as:

$$S[h,c,\bar{c};b;\bar{g}] = S_{EH}[h;\bar{g}] + S_{gf}[h;b;\bar{g}] + S_{NL}[b;\bar{g}] + S_{gh}[h,c,\bar{c};\bar{g}] \tag{2.2.84}$$

The so called off-shell gauge-fixing term is defined to be linear in the Nakanishi-Lautrup field:

$$S_{gf}[h;b;\bar{g}] = \int d^D x \sqrt{\bar{g}}\, b_\mu \bar{g}^{\mu\nu} f_\nu(h;\bar{g}) \tag{2.2.85}$$

The arbitrariness in the choice of a specific form for the gauge-fixing term is now moved to the choice of an additional action term for the Nakanishi-Lautrup field, which is an algebraic functional (being the Nakanishi-Lautrup field just an auxiliary field introduced for convenience, a kinetic term is not required). The associated traditional, or on-shell, gauge-fixing term is obtained by integrating over the Nakanishi-Lautrup field:

$$e^{-S_{gf}[h;\bar{g}]} = \int \mathcal{D}b\, e^{-S_{gf}[h;b;\bar{g}] - S_{NL}[b;\bar{g}]} \tag{2.2.86}$$

which typically corresponds to formally substituting in the action the on-shell expression for the Nakanishi-Lautrup field imposed by its equation of motion (which is an algebraic condition due to the absence of derivative terms).
It is also interesting to formally think the action term for the Nakanishi-Lautrup field as the result of another path integration over a second auxiliary field, a noise field n_μ, also defined to be a 1-form field:

$$Z[\bar{g}] = \int \mathcal{D}\mu[h;\bar{g}]\mathcal{D}c\mathcal{D}\bar{c}\mathcal{D}b\mathcal{D}n\, e^{-S[h,c,\bar{c};b,n;\bar{g}]} \tag{2.2.87}$$

The action is now defined as:

$$S[h,c,\bar{c};\bar{g};b,n] = S_{EH}[h;\bar{g}] + S_{gf}[h;b;\bar{g}] + S_{noise}[b,n;\bar{g}] + S_{gh}[h,c,\bar{c};\bar{g}] \tag{2.2.88}$$

The action term for the noise field is defined such that the action term for the Nakanishi-Lautrup field is obtained by integrating over the noise field:

$$e^{-S_{NL}[b;\bar{g}]} = \int \mathcal{D}n\, e^{-S_{noise}[b,n;\bar{g}]} \tag{2.2.89}$$

This additional operation will be also useful to generalize the standard form for the gauge-fixing term and switch to the non-standard one used in the third part of the thesis.

2.2.3 Multiplet notation

We now introduce a convenient notation to handle together the three dynamical fields present in the gauge-fixed theory, i.e. metric fluctuation and ghosts, and write compactly the generating functionals describing the associated quantum theory [6]. The formulas are a generalization of the standard ones reviewed in subsection 1.1.1 for the scalar theory to the case of multiple fields, possibly Grassmann-even and Grassmann-odd, and in presence of a classical curved metric, in this case the background metric $\bar{g}_{\mu\nu}$.

Consider the Faddeev-Popov gauge-fixed path integral (2.2.80) with a generic gauge-fixed action (2.2.81) and introduce a source field for the metric fluctuation, the ghost and the antighost; we group them in the following multiplets:

$$\phi^i \equiv \begin{pmatrix} h_{\mu\nu} \\ c^\mu \\ -\bar{c}_\mu \end{pmatrix} \qquad \phi^{+i} \equiv \begin{pmatrix} h_{\mu\nu} & -c^\mu & \bar{c}_\mu \end{pmatrix} \tag{2.2.90}$$

$$J_i \equiv \begin{pmatrix} t^{\mu\nu} \\ \bar{\eta}_\mu \\ \eta^\mu \end{pmatrix} \qquad J_i^+ \equiv \begin{pmatrix} t^{\mu\nu} & \bar{\eta}_\mu & \eta_\mu \end{pmatrix} \tag{2.2.91}$$

$t^{\mu\nu}$ is Grassmann-even, while $\bar{\eta}_\mu$, and η^μ are Grassmann-odd. The "adjunction" operation is defined such that, as far as fields are concerned, Grassmann-even fields do not change sign in the adjoint row vector while Grassmann-odd fields do; as far as sources are concerned, both Grassmann-even and Grassmann-odd sources do not change sign.
We introduce also the following shorthand notation for left and right derivatives of functionals taken with respect to multiplets and including also $1/\sqrt{\bar{g}}$ factors:

$$A^{\overrightarrow{(m)}}_{M_1\ldots M_m}(x_1\ldots x_m) \equiv \frac{1}{\sqrt{\bar{g}(x_1)}}\cdots\frac{1}{\sqrt{\bar{g}(x_n)}}\frac{\delta}{\delta M_1^+(x_1)}\cdots\frac{\delta}{\delta M_m^+(x_m)}A \tag{2.2.92}$$

$$A^{\overleftarrow{(n)}}_{N_1\ldots N_n}(x_1\ldots x_n) \equiv \frac{1}{\sqrt{\bar{g}(x_1)}}\cdots\frac{1}{\sqrt{\bar{g}(x_n)}}A\frac{\overleftarrow{\delta}}{\delta N_1(x_1)}\cdots\frac{\overleftarrow{\delta}}{\delta N_n(x_n)} \tag{2.2.93}$$

where, conventionally, left and right derivatives are taken with respect to multiplets and adjoint multiplets, respectively. We use a special notation for the second derivative composed of one left and one right derivative:

$$A^{(2)}_{MN}(x,y) \equiv \frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}\frac{\delta}{\delta M^+(x)}A\frac{\overleftarrow{\delta}}{\delta N(y)} \tag{2.2.94}$$

Adopting these notations, the quantum generating functionals and their properties are written as:

1. Path integral: The path integral with sources, generator of correlation functions, is written as:

$$Z[J;\bar{g}] = \int \mathcal{D}\mu[\phi]\, e^{-S[\phi;\bar{g}]-S_{source}[\phi;J;\bar{g}]} \tag{2.2.95}$$

with the source term:

$$S_{source}[\phi;J;\bar{g}] = -\int d^Dx\sqrt{\bar{g}}\,J_i^+\phi^i = -\int d^Dx\sqrt{\bar{g}}\,\left(t^{\mu\nu}h_{\mu\nu}+\bar{\eta}_\mu c^\mu+\bar{c}_\mu\bar{\eta}^\mu\right) \tag{2.2.96}$$

Correlation functions of fields ϕ^i are obtained by taking left derivatives with respect to sources J_i^+ of the path integral and appropriately multiplying by $1/\sqrt{\bar{g}}$ factors to account for the measure in the source term:

$$\begin{aligned}\left\langle \phi^{(\theta)i_1}(x_1)\dots\phi^{(\theta)i_n}(x_n)\right\rangle &= \frac{1}{Z}\int \mathcal{D}\mu[\phi]\,\phi^{(\theta)i_1}(x_1)\dots\phi^{(\theta)i_n}(x_n)\,e^{-S[\phi;\bar{g}]} \\ &= \frac{1}{Z}Z^{\overrightarrow{(n)}}_{J_{i_1}\dots J_{i_n}}(x_1\dots x_n)\Big|_{J=0}\end{aligned} \tag{2.2.97}$$

or without the normalization $1/Z$ for the unnormalized one $\left\langle \phi^{(\theta)i_1}(x_1)\dots\phi^{(\theta)i_n}(x_n)\right\rangle_u$. According to the definition of the adjoint multiplet, correlation functions of fields ϕ^{+i} are similarly obtained by taking right derivatives with respect to sources J_i.

2. Path integral logarithm: The path integral logarithm, generator of connected correlation function, is:

$$W[J;\bar{g}] = \log Z[J;\bar{g}] \tag{2.2.98}$$

Connected correlation functions of fields ϕ^i and ϕ^{+i} are obtained with the same rules of normal ones, for instance:

$$\left\langle \phi^{(\theta)i_1}(x_1)\dots\phi^{(\theta)i_n}(x_n)\right\rangle_c = W^{\overrightarrow{(n)}}_{J_{i_1}\dots J_{i_n}}(x_1\dots x_n)\Big|_{J=0} \tag{2.2.99}$$

Normalized correlation functions can be expressed as sums of connected ones. In the following we will need in particular the expression for the 2-point correlation function $\langle\phi^i\phi^{+j}\rangle$ in terms of connected ones:

$$\left\langle \phi^i(x)\phi^{+j}(y)\right\rangle = \left\langle \phi^i(x)\phi^{+j}(y)\right\rangle_c + \left\langle \phi^i(x)\right\rangle_c\left\langle \phi^{+j}(y)\right\rangle_c \tag{2.2.100}$$

deriving from the relation between second derivatives:

$$\frac{1}{Z}Z^{(2)}_{J_iJ_j}(x,y) = W^{(2)}_{J_iJ_j}(x,y) + W^{\overrightarrow{(1)}}_{J_i}(x)W^{\overleftarrow{(1)}}_{J_j}(y) \tag{2.2.101}$$

which follows immediately from the derivative rule of the logarithm:

$$\frac{\delta W\overleftarrow{\delta}}{\delta J_i^+(x)\delta J_j(y))} = \frac{\delta}{\delta J_i^+(x)}\left(\frac{1}{Z}\frac{Z\overleftarrow{\delta}}{\delta J_j(y))}\right) = \frac{1}{Z}\frac{\delta Z\overleftarrow{\delta}}{\delta J_i^+(x)\delta J_j(y))} - \frac{1}{Z^2}\frac{\delta Z}{\delta J_i^+(x)}\frac{Z\overleftarrow{\delta}}{\delta J_j(y))}$$

and appropriately introducing $1/\sqrt{\bar{g}}$ factors.

3. Effective action: The effective action, generator of 1PI vertices, is given by the Legendre transform:

$$\begin{aligned}\Gamma[\Phi;\bar{g}] &= \sup_J\left\{\int d^Dx\sqrt{\bar{g}}\,J_i^+\Phi^i - W[J;\bar{g}]\right\} \\ &= \int d^Dx\sqrt{\bar{g}}\,J_i^+\Phi^i - W[J;\bar{g}]\end{aligned} \tag{2.2.102}$$

In the second it is understood that the multiplet of sources is expressed as $J = J(\Phi)$ by inverting the relations given by imposing the extremality condition of the argument of the

Legendre transform:

$$\begin{aligned} \Phi^i(x) = \left\langle \phi^i(x) \right\rangle_J &= \frac{1}{\sqrt{\bar{g}(x)}} \frac{\delta W[J;\bar{g}]}{\delta J_i^+(x)} \\ \Phi^{+i}(x) = \left\langle \phi^{+i}(x) \right\rangle_J &= \frac{1}{\sqrt{\bar{g}(x)}} \frac{W \overleftarrow{\delta} [J;\bar{g}]}{\delta J_i(x)} \end{aligned} \tag{2.2.103}$$

which is given by the average fields in presence of sources. Similarly, we can express the path integral logarithm via the inverse Legendre transform:

$$\begin{aligned} W[J;\bar{g}] &= \sup_{\Phi} \left\{ \int d^D x \sqrt{\bar{g}}\, J_i^+ \Phi^i - \Gamma[\Phi;\bar{g}] \right\} \\ &= \int d^D x \sqrt{\bar{g}}\, J_i^+ \Phi^i - \Gamma[\Phi;\bar{g}] \end{aligned} \tag{2.2.104}$$

where now in the second it is understood that the multiplet of fields is expressed as $\Phi = \Phi(J)$ by inverting the relations:

$$\begin{aligned} J_i(x) &= \frac{1}{\sqrt{\bar{g}(x)}} \frac{\delta \Gamma[\Phi;\bar{g}]}{\delta \Phi^{+i}(x)} \\ J_i^+(x) &= \frac{1}{\sqrt{\bar{g}(x)}} \frac{\Gamma \overleftarrow{\delta} [\Phi;\bar{g}]}{\delta \Phi^i(x)} \end{aligned} \tag{2.2.105}$$

In the following we will make explicit those dependencies when needed. The effective action can be also defined as solution of the integro-differential equation:

$$e^{-\Gamma[\Phi;\bar{g}]} = \int \mathcal{D}\mu[\phi] \, \exp\left\{ -S[\phi;\bar{g}] + \int d^D x \frac{\Gamma \overleftarrow{\delta} [\Phi;\bar{g}]}{\delta \Phi^i} \left(\phi^i - \Phi^i \right) \right\} \tag{2.2.106}$$

which follows from equating the definition of the path integral logarithm (2.2.98) and its expression given by the inverse Legendre transform (2.2.104), computed in $J = J(\Phi)$ given by (2.2.105):

$$\begin{aligned} e^{W[J(\Phi);\bar{g}]} &= \int \mathcal{D}\mu[\phi] \, \exp\left\{ -S[\phi;\bar{g}] + \int d^D x \frac{\Gamma \overleftarrow{\delta} [\Phi;\bar{g}]}{\delta \Phi^i} \phi^i \right\} \\ &= \exp\left\{ -\Gamma[\Phi;\bar{g}] + \int d^D x \frac{\Gamma \overleftarrow{\delta} [\Phi;\bar{g}]}{\delta \Phi^i} \Phi^i \right\} \end{aligned}$$

The second derivative of the effective action is the inverse operator of the second derivative of the path integral logarithm:

$$W^{(2)}_{J_i J_j}(x,y) = \Gamma^{(2)-1}_{\Phi^i \Phi^j}(x,y) \tag{2.2.107}$$

in the following sense:

$$\begin{aligned} W^{(2)}_{J_i J_k}(x,\cdot) \bullet_{\bar{g}} \Gamma^{(2)}_{\Phi^k \Phi^j}(\cdot,y) &= \frac{\delta^i_j \delta(x-y)}{\sqrt{\bar{g}(y)}} \\ \Gamma^{(2)}_{\Phi^i \Phi^k}(x,\cdot) \bullet_{\bar{g}} W^{(2)}_{J_k J_j}(\cdot,y) &= \frac{\delta^j_i \delta(x-y)}{\sqrt{\bar{g}(y)}} \end{aligned} \tag{2.2.108}$$

where the symbol $\bullet_{\bar{g}}$ represents an implicit spacetime integration with a factor $\sqrt{\bar{g}}$ in the measure. The first inverse rule, for instance, from expressing a delta as derivative of the multiplet of fields with respect to itself and then interpreting it as function of the multiplet of sources $\Phi = \Phi(J)$:

$$\delta^i_j \delta(x-y) = \frac{\Phi^i(x)\overleftarrow{\delta}}{\delta\Phi^j(y)} = \int d^D z \frac{\Phi^i(J)(x)\overleftarrow{\delta}}{\delta J_k(z)} \frac{J_k(\Phi)(z)\overleftarrow{\delta}}{\delta\Phi^j(y)} =$$
$$= \int d^D z \frac{1}{\sqrt{\bar{g}(x)}} \frac{\delta W \overleftarrow{\delta}}{\delta J_i(x)\delta J_k(z)} \frac{1}{\sqrt{\bar{g}(z)}} \frac{\delta\Gamma\overleftarrow{\delta}}{\delta\Phi^k(z)\delta\Phi^j(y)}$$

and appropriately introducing $1/\sqrt{\bar{g}}$ factors.

2.2.4 Off-shell BRST symmetry

Consider the Faddeev-Popov gauge-fixed action (2.2.81) and assume that an appropriate gauge-fixing function is chosen; as a result of the gauge-fixing procedure, as desired, the complete action loses the local symmetry under true gauge transformations:

$$\delta_\xi S[h, c, \bar{c}; \bar{g}] \neq 0 \tag{2.2.109}$$

However, it is still characterized, for any specific gauge-fixing function chosen, by a "residual" 1-parameter rigid symmetry with a grassmannian anticommuting number θ as parameter, the BRST symmetry (Becchi-Rouet-Stora-Tyutin) [24][25][26]:

$$\delta_\theta S[h, c, \bar{c}; \bar{g}] = 0 \tag{2.2.110}$$

Before presenting the explicit form of the BRST transformation rules for the fields in the specific case of the pure Einstein-Hilbert theory, we mention that the presence of a BRST symmetry, with its specific structure properties, is a fundamental feature of all gauge theories gauge-fixed and quantized according to the Faddeev-Popov method.
In order to sketch the general form of those structure properties, we recall from the theory of Grassmann-graded objects that a variation δ_θ dependent on anticommuting parameter θ, such that $\theta^2 = 0$, can be expressed in general in terms of a Slavnov variation s:

$$\delta_\theta = \theta s \tag{2.2.111}$$

which must be a variation operator changing the Grassmann character of a graded object from commuting to anticommuting and vice versa, so that conversely the variation δ_θ does not, thanks to the multiplication for θ. In the following we are going to use in particular the two properties:

$$s(AB) = (sA)B \pm A(sB) \tag{2.2.112}$$

$$s^2(AB) = (s^2A)B + A(s^2B) \tag{2.2.113}$$

The first has a $+$ or $-$ sign respectively if the first object is Grassmann-even or odd and follows from the fact that in both cases we must have $\delta_\theta(AB) = (\delta_\theta A)B + A(\delta_\theta B)$; the

second holds whichever is the Grassmann character of the two fields and follows from the first by a direct computation.
Considering a generic gauge theory, the characteristic structure properties of BRST symmetry are:

1. The number of ghosts and antighosts fields, c_a and $\bar{c}_a$, introduced is equal to the number of local functions, ξ_a, featuring in the gauge transformation associated to the gauge symmetry which has been gauge-fixed.

2. The BRST variations of the original dynamical field variables of the non-gauge-fixed theory and the ghosts are not related to the gauge-fixing and are given by:

2.A The BRST variation of the dynamical field variables, ϕ_i, is equal to a gauge variation where gauge functions are given by $\xi_a = \theta c_a$:

$$\delta_\theta \phi_i = \delta_{\xi=\theta c}\phi_i$$

2.B The BRST variation of the ghosts fields is such that the second BRST Slavnov variation of the dynamical fields is vanishing:

$$s^2\phi_i = 0$$

along with the one for their own BRST Slavnov variation:

$$s^2 c_a = 0$$

3. The BRST variations of the antighosts and the Nakanishi-Lautrup fields, if present, are instead related to the gauge-fixing:

3.A <u>Off-shell BRST symmetry</u>: In presence of Nakanishi-Lautrup fields, b_a, as many as the ghosts and antighosts, the BRST variation rules are simply:

$$\begin{cases} \delta_\theta \bar{c}_a = \theta b_a \\ \delta_\theta b_a = \ 0 \end{cases}$$

The correspondent second BRST Slavnov variations are trivially vanishing independently of those of the dynamical fields and ghosts:

$$\begin{cases} s^2\bar{c}_a & = & 0 \\ s^2 b_a & = & 0 \end{cases}$$

The BRST Slavnov variation is nilpotent, i.e. the second variation of all elementary fields is identically vanishing.

3.B <u>On-shell BRST symmetry</u>: In absence of Nakanishi-Lautrup fields, the transformation rule of the antighosts is typically given by the on-shell expression for the Nakanishi-Lautrup fields:

$$\delta_\theta \bar{c}_a = \theta b_a\Big|_{on-shell}$$

It depends on the chosen form for the gauge-fixing term and possibly also on the chosen gauge-fixing function; the second BRST Slavnov variation may be vanishing only upon using the classical equations of motion of the theory:

$$s^2\bar{c}_a \overset{\text{e.o.m.}}{=} 0$$

The BRST Slavnov variation is nilpotent, but possibly only upon using the classical equations of motion of the theory.

Off-shell BRST symmetry at the classical level

Consider the generic gauge-fixed action in presence of a Nakanishi-Lautrup field (2.2.84). The off-shell BRST transformations of the elementary fields and the Nakanishi-Lautrup field are:

$$\begin{cases} h_{\mu\nu}(x) & \rightarrow & h^{(\theta)}_{\mu\nu}(x) & = h_{\mu\nu}(x) + \delta_\theta h_{\mu\nu}(x) \\ \bar{g}_{\mu\nu}(x) & \rightarrow & h^{(\theta)}_{\mu\nu}(x) & = \bar{g}_{\mu\nu}(x) + \delta_\theta \bar{g}_{\mu\nu}(x) \\ c^\mu(x) & \rightarrow & c^{(\theta)\mu}(x) & = c^\mu(x) + \delta_\theta c^\mu(x) \\ \bar{c}_\mu(x) & \rightarrow & \bar{c}^{(\theta)}_\mu(x) & = \bar{c}_\mu(x) + \delta_\theta \bar{c}_\mu(x) \\ b_\mu(x) & \rightarrow & b^{(\theta)}_\mu(x) & = b_\mu(x) + \delta_\theta b_\mu(x) \end{cases} \tag{2.2.114}$$

where the off-shell BRST variations are defined as:

$$\begin{cases} \delta_\theta h_{\mu\nu} = \theta \pounds_c(\bar{g}_{\mu\nu} + h_{\mu\nu}) \\ \qquad = \theta\left(c^\lambda\partial_\lambda\bar{g}_{\mu\nu} + \partial_\mu c^\lambda\bar{g}_{\lambda\nu} + \partial_\nu c^\lambda\bar{g}_{\lambda\mu}\right) + \theta\left(c^\lambda\partial_\lambda h_{\mu\nu} + \partial_\mu c^\lambda h_{\lambda\nu} + \partial_\nu c^\lambda h_{\lambda\mu}\right) \\ \qquad = \theta\left(\bar{\nabla}_\mu c^\lambda\bar{g}_{\lambda\nu} + \bar{\nabla}_\nu c^\lambda\bar{g}_{\lambda\mu}\right) + \theta\left(c^\lambda\bar{\nabla}_\lambda h_{\mu\nu} + \bar{\nabla}_\mu c^\lambda h_{\lambda\nu} + \bar{\nabla}_\nu c^\lambda h_{\lambda\mu}\right) \\ \delta_\theta\bar{g}_{\mu\nu} = 0 \\ \delta_\theta c^\mu = \theta c^\lambda\partial_\lambda c^\mu = \theta c^\lambda\bar{\nabla}_\lambda c^\mu \\ \delta_\theta\bar{c}_\mu = \theta b_\mu \\ \delta_\theta b_\mu = 0 \end{cases} \tag{2.2.115}$$

The correspondent off-shell BRST Slavnov variations are:

$$\begin{cases} s h_{\mu\nu} = \pounds_c(\bar{g}_{\mu\nu} + h_{\mu\nu}) \\ \qquad = \left(c^\lambda\partial_\lambda\bar{g}_{\mu\nu} + \partial_\mu c^\lambda\bar{g}_{\lambda\nu} + \partial_\nu c^\lambda\bar{g}_{\lambda\mu}\right) + \left(c^\lambda\partial_\lambda h_{\mu\nu} + \partial_\mu c^\lambda h_{\lambda\nu} + \partial_\nu c^\lambda h_{\lambda\mu}\right) \\ \qquad = \left(\bar{\nabla}_\mu c^\lambda\bar{g}_{\lambda\nu} + \bar{\nabla}_\nu c^\lambda\bar{g}_{\lambda\mu}\right) + \left(c^\lambda\bar{\nabla}_\lambda h_{\mu\nu} + \bar{\nabla}_\mu c^\lambda h_{\lambda\nu} + \bar{\nabla}_\nu c^\lambda h_{\lambda\mu}\right) \\ s\bar{g}_{\mu\nu} = 0 \\ s c^\mu = c^\lambda\partial_\lambda c^\mu = c^\lambda\bar{\nabla}_\lambda c^\mu \\ s\bar{c}_\mu = b_\mu \\ s b_\mu = 0 \end{cases} \tag{2.2.116}$$

As as in the case of true and background gauge transformations, (2.2.9) and (2.2.12), we wrote the Lie derivative both in terms of partial derivatives and covariant derivatives compatible with the background; this can be done also for the BRST variation rule of the ghost field, in fact, due to its anticommuting nature, the partial derivative can be substituted with any covariant derivative ∇_μ associated to a symmetric connection $\Gamma^\lambda_{\mu\nu}$ (as the metric connection which we are considering):

$$c^\lambda \nabla_\lambda c^\mu = c^\lambda \partial_\lambda c^\mu + \underbrace{\Gamma^\mu_{\rho\lambda}}_{\substack{\text{sym.}\\ \lambda\leftrightarrow\rho}} \underbrace{c^\lambda c^\rho}_{\substack{\text{antisym.}\\ \lambda\leftrightarrow\rho}} = c^\lambda \partial_\lambda c^\mu$$

Inspecting the BRST variation rules, we can recognize the characteristic structure properties of a BRST symmetry anticipated above, in particular:

1. The BRST variations of the metric fields, and in particular of the metric fluctuation, are equal to true gauge variations with vector field given by $\xi^\mu = \theta c^\mu$:

$$\begin{cases} \delta_\theta h_{\mu\nu} = \delta_{\xi=\theta c} h_{\mu\nu} \\ \delta_\theta \bar{g}_{\mu\nu} = \delta_{\xi=\theta c} \bar{g}_{\mu\nu} \end{cases} \tag{2.2.117}$$

This can be easily seen by inspecting the BRST variations and taking the parameter θ, which is just a number, through the various derivatives to multiply directly the ghosts.

2. The BRST Slavnov variation is nilpotent on elementary fields:

$$\begin{cases} s^2 h_{\mu\nu} &= 0 \\ s^2 \bar{g}_{\mu\nu} &= 0 \\ s^2 c^\mu &= 0 \\ s^2 \bar{c}_\mu &= 0 \\ s^2 b_\mu &= 0 \end{cases} \tag{2.2.118}$$

Nilpotency on the ghost and the metric fluctuation is shown in appendix B.1. In the case of the metric fluctuation it follows directly from the BRST variation rule of the ghost, while in the case of the ghost it follows directly from its anticommuting nature; in particular, as anticipated above, it appears that the BRST variation of the ghost is precisely the one guaranteeing those conditions, since nilpotent on its own and, as seen in appendix B.1, such that:

$$s^2 h_{\mu\nu} = 0 \qquad \Longleftrightarrow \qquad s c^\mu = c^\lambda \partial_\lambda c^\mu \tag{2.2.119}$$

Nilpotency on the background metric is clearly trivial. In the case of the antighost and the Nakanishi-Lautrup field it follows independently of the others from their simple BRST variation rules, in particular:

$$s^2 \bar{c}_\mu = s b_\mu = 0 \tag{2.2.120}$$

From the nilpotency of the variation on elementary fields it follows also that any polynomial function of the fields has also a vanishing second BRST Slavnov variation:

$$s^2 F(h, c, \bar{c}; b; \bar{g}) = 0 \tag{2.2.121}$$

since from property (2.2.113) we have that if two objects have vanishing second Slavnov variations $s^2A = s^2B = 0$, also the product does $s^2(AB) = 0$, and similarly for an arbitrary power (also if the function contains derivatives of the fields, since BRST Slavnov variations simply go through). Therefore, the BRST Slavnov operator s is a nilpotent linear operator acting in the space of elementary fields and their functions, where it thus defines a cohomology structure. The kernel of s, $\ker(s)$, consisting of BRST-closed elements, i.e. with zero BRST Slavnov variation $sA = 0$, contains the image of s, $\mathrm{im}(s) \subset \ker(s)$, consisting of BRST-exact elements, i.e. expressible as BRST Slavnov variation of others $sA = B$. The space is partitioned in the union of $\mathrm{im}(s)$, i.e. BRST-exact elements, $\ker(s) \setminus \mathrm{im}(s)$, i.e. BRST-closed but not exact elements, and $\overline{\ker}(s)$, i.e. non-BRST-closed elements. BRST-closed elements can be grouped in equivalence classes, according to the equivalence relation stating that two elements are equivalent if differing by a BRST-exact term:

$$A \sim B \quad : \quad A - B \in \mathrm{im}(s) \tag{2.2.122}$$

and those form the cohomology group of s:

$$H(s) = {}^{\ker(s)}\Big/_{\mathrm{im}(s)} = {}^{\ker(s)}\Big/_{\sim} \tag{2.2.123}$$

The invariance of the action (2.2.84) under a BRST variation is a manifest consequence of those structure properties. From the property of coinciding with a true gauge variation as far as the metric fields are concerned, we have immediately that the BRST variation of the original, non-gauge-fixed Einstein-Hilbert action is zero:

$$\delta_\theta S_{EH}[h; \bar{g}] = 0 \tag{2.2.124}$$

From the invariance of the Nakanishi-Lautrup field and the background metric we have also:

$$\delta_\theta S_{NL}[h; b; \bar{g}] = 0 \tag{2.2.125}$$

Consider now the gauge-fixing term and the ghost term. Using the BRST Slavnov variation of the antighost field, the first can be rewritten as:

$$S_{gf}[h, b; \bar{g}] = \int d^D x \sqrt{\bar{g}}\, b_\mu \bar{g}^{\mu\nu} f_\nu(h; \bar{g}) = \int d^D x \sqrt{\bar{g}}\, s\bar{c}_\mu \bar{g}^{\mu\nu} f_\nu(h; \bar{g}) \tag{2.2.126}$$

The ghost term can be similarly rewritten by noticing that the BRST variation of the generic gauge-fixing function can be expressed as:

$$\delta_\theta f_\mu(h; \bar{g}) = \frac{\delta f_\mu(h; \bar{g})}{\delta h_{\alpha\beta}} \bullet \delta_\theta h_{\alpha\beta} = \theta \frac{\delta f_\mu(h; \bar{g})}{\delta h_{\alpha\beta}} \bullet \pounds_c(\bar{g}_{\alpha\beta} + h_{\alpha\beta}) = \theta \mathcal{G}_{\mu\nu} c^\nu \tag{2.2.127}$$

In the last equality we recognized, recalling property (2.2.75), the action of the ghost operator (2.2.74); the parameter θ can be commuted with the functional derivative of the gauge-fixing function since the latter is a Grassmann-even object, involving only the metric fields. So, the associated Slavnov variation is:

$$s f_\mu(h; \bar{g}) = \frac{\delta f_\mu(h; \bar{g})}{\delta h_{\alpha\beta}} \bullet s h_{\alpha\beta} = \frac{\delta f_\mu(h; \bar{g})}{\delta h_{\alpha\beta}} \bullet \pounds_c(\bar{g}_{\alpha\beta} + h_{\alpha\beta}) = \mathcal{G}_{\mu\nu} \bullet c^\nu \tag{2.2.128}$$

And the ghost term can be thus rewritten as:

$$S_{gh}[h,c,\bar{c};\bar{g}] = -\int d^Dx\sqrt{\bar{g}}\,\bar{c}_\alpha\bar{g}^{\alpha\beta}\mathcal{G}_{\beta\gamma}\bullet c^\gamma = -\int d^Dx\sqrt{\bar{g}}\,\bar{c}_\mu\bar{g}^{\mu\nu}sf_\nu(h;\bar{g}) \tag{2.2.129}$$

Summing the two terms we obtain:

$$\begin{aligned}S_{gf}[h,b;\bar{g}] + S_{gh}[h,c,\bar{c};\bar{g}] &= \int d^Dx\sqrt{\bar{g}}\,\left(s\bar{c}_\mu\bar{g}^{\mu\nu}f_\nu(h;\bar{g}) - \bar{c}_\mu\bar{g}^{\mu\nu}sf_\nu(h;\bar{g})\right) = \\ &= s\int d^Dx\sqrt{\bar{g}}\,\bar{c}_\mu\bar{g}^{\mu\nu}f_\nu(h;\bar{g})\end{aligned}$$

where we notice the use of property (2.2.112) and the fact that the background metric, and the related objects, namely its inverse and determinant, are BRST-invariant and therefore transparent for the BRST variation. We recognize that the result is a BRST-exact term:

$$S_{gf}[h,b;\bar{g}] + S_{gh}[h,c,\bar{c};\bar{g}] = s\int d^Dx\sqrt{\bar{g}}\,\bar{c}_\mu\bar{g}^{\mu\nu}f_\nu(h;\bar{g}) \equiv sS_{BRST}[h,\bar{c};\bar{g}]$$

and therefore manifestly BRST-invariant due to the nilpotency property:

$$\delta_\theta\left(S_{gf}[h;\bar{g}] + S_{gh}[h,c,\bar{c};\bar{g}]\right) = 0 \tag{2.2.130}$$

In conclusion, the complete action:

$$S[h,c,\bar{c},b;\bar{g}] = S_{EH}[h;\bar{g}] + S_{NL}[b;\bar{g}] + sS_{BRST}[h,\bar{c};\bar{g}] \tag{2.2.131}$$

is also manifestly BRST-invariant:

$$\delta_\theta S[h,c,\bar{c};b;\bar{g}] = 0 \tag{2.2.132}$$

In particular we make the important remark that the gauge-fixing term and the ghost term, which constitute the part of the action behaving non-trivially under a BRST transformation, are not separately BRST-invariant, but their sum is BRST-exact, therefore their BRST variations compensate and precisely eliminate each other. Crucially, this holds for any specific choice of the gauge-fixing function. Due to the invariance of the Nakanishi-Lautrup field and the background metric, the symmetry is clearly also present independently of the specific Nakanishi-Lautrup term chosen. Therefore we can conclude that: *the gauge-fixed action* (2.2.84) *enjoys a BRST symmetry under* (2.2.115) *regardless of the gauge-fixing details.*
As we will see in subsection 2.3.2, this holds also for the on-shell version, although the specific form of the variations may depend on the gauge-fixing function.

Off-shell BRST symmetry at the quantum level

We conclude the subsection by briefly discussing some aspects of BRST symmetry at the quantum level which will be of interest in the following. Consider the Faddeev-Popov gauge-fixed path integral in presence of a Nakanishi-Lautrup field (2.2.83) and introduce source terms for the elementary fields; in multiplet notation:

$$Z[J;\bar{g}] = \int \mathcal{D}\mu[\phi]\mathcal{D}b\, e^{-S[\phi;b;\bar{g}]+\int d^Dx\sqrt{\bar{g}}\,J_i^+\phi^i} \tag{2.2.133}$$

As discussed for a scalar theory in subsection 1.1.2, a symmetry for the classical action is preserved at the quantum level if non-anomalous, i.e. also the path integral measure enjoys the symmetry. It can be seen that the diffeomorphism-invariant measure $\mathcal{D}\mu[h;\bar{g}]$, which we are formally employing, can indeed be constructed such that the total measure, complete also of the ghosts and Nakanishi-Lautrup sector, is BRST-invariant [21]:

$$\mathcal{D}\mu[h;\bar{g}]\mathcal{D}c\mathcal{D}\bar{c}\mathcal{D}b \quad \to \quad \mathcal{D}\mu[h^{(\theta)};\bar{g}^{(\theta)}]\mathcal{D}c^{(\theta)}\mathcal{D}\bar{c}^{(\theta)}\mathcal{D}b^{(\theta)} = \mathcal{D}\mu[h;\bar{g}]\mathcal{D}c\mathcal{D}\bar{c}\mathcal{D}b \tag{2.2.134}$$

In the following we will always assume that the formal path integral measure considered is constructed in such a way and thus that BRST symmetry is non-anomalous. Therefore, as seen in subsection 1.1.2 for a scalar theory with a non-anomalous symmetry, we have that BRST symmetry also holds at the level of correlation functions and generates an associated Ward-Takahashi equation. Indeed, changing variables from the fields to the BRST transformed fields and then using the BRST-invariance of the action and the measure, the path integral can be also written as:

$$\begin{aligned} Z[J;\bar{g}] &= \int \mathcal{D}\mu[\phi]\mathcal{D}b\, e^{-S[\phi;b;\bar{g}]+\int d^Dx\sqrt{\bar{g}}\,J_i^+\phi^i} = \\ &= \int \mathcal{D}\mu[\phi^{(\theta)}]\mathcal{D}b^{(\theta)}\, e^{-S[\phi^{(\theta)};b^{(\theta)};\bar{g}^{(\theta)}]+\int d^Dx\sqrt{\bar{g}}\,J_i^+\phi^{(\theta)i}} = \\ &= \int \mathcal{D}\mu[\phi]\mathcal{D}b\, e^{-S[\phi;b;\bar{g}]+\int d^Dx\sqrt{\bar{g}}\,J_i^+\phi^{(\theta)i}} \end{aligned} \tag{2.2.135}$$

where in the last the BRST transformed fields are now regarded as functions of the non-transformed ones. Comparing the first and last expression and taking arbitrary functionals derivatives with respect to the sources, it follows that generic correlation functions of BRST transformed and non-transformed fields are equal, i.e. BRST symmetry holds also at the level of quantum correlation functions:

$$\left\langle \phi^{(\theta)i_1}(x_1)\dots\phi^{(\theta)i_N}(x_N)\right\rangle = \left\langle \phi^{i_1}(x_1)\dots\phi^{i_N}(x_N)\right\rangle \tag{2.2.136}$$

Writing the BRST transformed fields in terms of BRST variations:

$$\begin{aligned} Z[J;\bar{g}] &= \int \mathcal{D}\mu[\phi]\mathcal{D}b\, e^{\int d^Dx\sqrt{\bar{g}}\,J_i^+\delta_\theta\phi^i}\, e^{-S[\phi;b;\bar{g}]+\int d^Dx\sqrt{\bar{g}}\,J_i^+\phi^i} = \\ &= Z[J;\bar{g}]\left\langle 1+\int d^Dx\sqrt{\bar{g}}\,J_i^+\delta_\theta\phi^i+\cdots\right\rangle_J = \\ &= Z[J;\bar{g}] + Z[J;\bar{g}]\left\langle \int d^Dx\sqrt{\bar{g}}\,J_i^+\delta_\theta\phi^i\right\rangle_J + \cdots \end{aligned}$$

we obtain the condition:

$$\left\langle \int d^Dx\sqrt{\bar{g}}\,J_i^+\delta_\theta\phi^i\right\rangle_J = 0 \tag{2.2.137}$$

which corresponds to the Ward-Takahashi equation associated to BRST symmetry, explicitly:

$$\begin{aligned} 0 &= \int d^D x \sqrt{\bar{g}}\, J_i^+ \left\langle \delta_\theta \phi^i \right\rangle_J \\ &= \int d^D x \sqrt{\bar{g}} \left(t^{\mu\nu} \left\langle \delta_\theta h_{\mu\nu} \right\rangle_J + \bar{\eta}_\mu \left\langle \delta_\theta c^\mu \right\rangle_J - \eta^\mu \left\langle \delta_\theta \bar{c}_\mu \right\rangle_J \right) \\ &= \theta \int d^D x \sqrt{\bar{g}} \left(t^{\mu\nu} \left\langle s h_{\mu\nu} \right\rangle_J - \bar{\eta}_\mu \left\langle s c^\mu \right\rangle_J + \eta^\mu \left\langle s \bar{c}_\mu \right\rangle_J \right) \end{aligned} \tag{2.2.138}$$

By taking arbitrary derivatives with respect to the sources and then setting them to zero, the equation can be used to generate the series of identities between correlation functions representing the constraints in which BRST symmetry is encoded at the quantum level; in particular, by taking N functional derivatives $\delta/\delta J_{i_1}^+(x_1) \ldots \delta/\delta J_{i_N}^+(x_N)$ and setting the sources to zero, one obtains the Ward-Takahashi identity:

$$\begin{aligned} 0 &= \sum_{n=1}^{N} \left\langle \phi^{i_1}(x_1) \ldots \delta_\theta \phi^{i_n}(x_n) \ldots \phi^{i_N}(x_N) \right\rangle \\ &= \left\langle \delta_\theta \left(\phi^{i_1}(x_1) \ldots \phi^{i_n}(x_n) \ldots \phi^{i_N}(x_N) \right) \right\rangle \\ &= \delta_\theta \left\langle \phi^{i_1}(x_1) \ldots \phi^{i_n}(x_n) \ldots \phi^{i_N}(x_N) \right\rangle \end{aligned} \tag{2.2.139}$$

The third expression follows from the assumed BRST-invariance of the path integral measure and can be seen as the infinitesimal form of (2.2.136). From the second it follows that the average of the BRST variation of a function of fields, i.e. a BRST-exact object, is vanishing:

$$\left\langle \delta_\theta F(h, c, \bar{c}; b; \bar{g}) \right\rangle = 0 \tag{2.2.140}$$

Finally, we mention that BRST symmetry plays a key role in quantum gauge theories in the identification of physical observables and physical states (in the operatorial formalism) [27]. Here we limit to mention that, in the path integral formalism, a quantum physical observable O is defined to be BRST-invariant, i.e. BRST-closed:

$$s\mathsf{O} = 0 \tag{2.2.141}$$

Moreover, observables in the same cohomology class, i.e. differing for a BRST-exact term, should be considered physically equivalent, since considering another observable O' such that:

$$\mathsf{O}' = \mathsf{O} + sX \tag{2.2.142}$$

we have from (2.2.140), that their average, i.e. quantum expectation value, is the same:

$$\langle \mathsf{O}' \rangle = \langle \mathsf{O} \rangle + \langle sX \rangle = \langle \mathsf{O} \rangle \tag{2.2.143}$$

So, observables should be identified with the cohomology classes of the BRST Slavnov variation, clearly excluded the one of BRST-exact elements since equivalent to zero.

2.3 Faddeev-Popov quantization with standard gauge-fixing

In this section we discuss the gauge-fixed quantum theory obtained via the Faddeev-Popov method by choosing the standard gauge-fixing term and gauge-fixing function typically used in various practical applications, namely a gauge-fixing term quadratic in the gauge-fixing function and the de Donder gauge-fixing function. This theory, in particular, is the one used in the standard FRG construction for a QEG theory, presented in section 2.4. After having introduced the gauge-fixing term, we discuss immediately the associated on-shell BRST symmetry, to emphasize that it is present for any specific gauge-fixing function chosen; at the end we present the gauge-fixing function considered and give the complete gauge-fixed action.

2.3.1 Standard gauge-fixing term and generating functionals

Gauge-fixing term

Consider the Faddeev-Popov gauge-fixed action (2.2.81). The standard form typically chosen for the gauge-fixing term in the Faddeev-Popov gauge-fixed action is a quadratic term in the gauge-fixing function:

$$S_{gf}[h;\bar{g}] = -\frac{1}{\kappa^2}\int d^D x\sqrt{\bar{g}}\,\frac{1}{\alpha}\,f_\mu(h;\bar{g})\bar{g}^{\mu\nu}f_\nu(h;\bar{g}) \tag{2.3.1}$$

This type of gauge-fixing term is used in several practical applications, especially in combination with the de Donder gauge-fixing function, analyzed in subsection 2.3.3.
α is a gauge-fixing parameter, i.e an arbitrary real number which parametrizes a class of possible gauge-fixing choices within the form (2.3.1); according to the remark made in subsection 2.2.2, since it is an "external" object introduced in the theory via the gauge-fixing sector, it should appear only in objects related to the gauge-fixing and quantum physical observables should instead not depend on it.
The main feature of this type of gauge-fixing term is that it is invariant under the background gauge transformations (2.2.12). Indeed, as noticed in subsection 2.2.1 under the definition, background gauge transformations can be interpreted as induced by a change of coordinates; therefore, as long as the gauge-fixing function is constructed out of the metric fields so that it is a proper tensor, namely a 1-form field, we have that it transforms covariantly under a background gauge transformation:

$$\bar{\delta}_\xi f_\mu(h;\bar{g}) = \pounds_\xi f_\mu(h;\bar{g}) \tag{2.3.2}$$

And the integrand in the gauge-fixing term is a proper scalar, so that the former is an action term written in proper covariant language; therefore, with a properly constructed gauge-fixing function, while the gauge-fixing term breaks by construction invariance under true gauge transformations, it is invariant under the background gauge transformations:

$$\bar{\delta}_\xi S_{gf}[h;\bar{g}] = 0 \tag{2.3.3}$$

Similarly, for the ghost term (2.2.77), taking the ghost and antighost to transform covariantly as a vector and a 1-form field as in (2.2.79):

$$\bar{\delta}_\xi S_{gh}[h, c, \bar{c}; \bar{g}] = 0 \tag{2.3.4}$$

So, the standard gauge-fixed action (still with a generic, properly constructed, gauge-fixing function):

$$S[h, c, \bar{c}; \bar{g}] = S_{EH}[h; \bar{g}] + S_{gf}[h; \bar{g}] + S_{gh}[h, c, \bar{c}; \bar{g}] \tag{2.3.5}$$

with the standard gauge-fixing term (2.3.1), which we will consider in the remaining of this part, is invariant under background gauge transformations:

$$\bar{\delta}_\xi S[h, c, \bar{c}; \bar{g}] = 0 \tag{2.3.6}$$

According to the general discussion on BRST symmetry made in subsection 2.2.3, the action will have also a rigid on-shell BRST symmetry, discussed in the next subsection.

If a Nakanishi-Lautrup field is introduced, the Nakanishi-Lautrup action term which enforces off-shell the gauge-fixing term (2.3.1) is quadratic:

$$S_{NL}[b; \bar{g}] = \int d^D x \sqrt{\bar{g}}\, \frac{\kappa^2 \alpha}{4} b_\mu \bar{g}^{\mu\nu} b_\nu \tag{2.3.7}$$

Indeed, performing the resulting gaussian integration over the Nakanishi-Lautrup field we obtain:

$$\int \mathcal{D}b\, e^{\int d^D x \sqrt{\bar{g}} \left(-\frac{\kappa^2 \alpha}{4} b_\mu \bar{g}^{\mu\nu} b_\nu - b_\mu \bar{g}^{\mu\nu} f_\nu(h; \bar{g}) \right)} = e^{+\int d^D x \sqrt{\bar{g}}\, \frac{1}{\kappa^2 \alpha} f_\mu(h; \bar{g}) \bar{g}^{\mu\nu} f_\nu(h; \bar{g})} = e^{-S_{gf}[h; \bar{g}]}$$

As mentioned above, we obtain the same result by formally substituting in the action the on-shell expression for the Nakanishi-Lautrup field imposed by its equations of motion:

$$0 = \frac{1}{\sqrt{\bar{g}}} \frac{\delta S}{\delta b_\mu}[h, c, \bar{c}; b; \bar{g}] = \sqrt{\bar{g}}\, \bar{g}^{\mu\nu} \left(f_\nu(h; \bar{g}) + \frac{\kappa^2 \alpha}{2} b_\nu \right) \quad \Longrightarrow \quad b_\mu|_{on-shell} = -\frac{2}{\kappa^2 \alpha} f_\mu(h; \bar{g}) \tag{2.3.8}$$

Indeed:

$$\left(S_{gf}[h; b; \bar{g}] + S_{NL}[b; \bar{g}] \right) \overset{\text{e.o.m.}}{=} -\int d^D x \sqrt{\bar{g}}\, \frac{1}{\kappa^2 \alpha} f_\mu(h; \bar{g}) \bar{g}^{\mu\nu} f_\nu(h; \bar{g}) = S_{gf}[h; \bar{g}]$$

Finally, if a noise field is introduced, the noise distribution necessary to obtain a quadratic Nakanishi-Lautrup term is gaussian, i.e. also the noise term must be quadratic:

$$S_{noise}[b, n; \bar{g}] = \int d^D x \sqrt{\bar{g}} \left(\frac{1}{\kappa^2 \alpha} n_\mu \bar{g}^{\mu\nu} n_\nu - i b_\mu \bar{g}^{\mu\nu} n_\nu \right) \tag{2.3.9}$$

Indeed, performing the resulting gaussian integration over the noise field we obtain again:

$$\int \mathcal{D}b\, e^{\int d^D x \sqrt{\bar{g}} \left(-\frac{1}{\kappa^2 \alpha} n_\mu \bar{g}^{\mu\nu} n_\nu + i b_\mu \bar{g}^{\mu\nu} n_\nu \right)} = e^{-\int d^D x \sqrt{\bar{g}}\, \frac{\kappa^2 \alpha}{4} b_\mu \bar{g}^{\mu\nu} b_\nu} = e^{-S_{NL}[b; \bar{g}]}$$

Additional sources and generating functionals

At the quantum level the theory is described by the generating functionals introduced in subsection 2.2.3 computed with the gauge-fixed action (2.3.5). For later convenience, it is useful to modify them by introducing additional sources for two non-elementary fields, namely the BRST Slavnov variations of the metric fluctuation and the ghost. So, we define an additional set of multiplets:

$$\psi^i \equiv \begin{pmatrix} sh_{\mu\nu} \\ sc^\mu \end{pmatrix} \qquad \psi^{+i} \equiv \begin{pmatrix} -sh_{\mu\nu} & sc^\mu \end{pmatrix} \tag{2.3.10}$$

$$K_i \equiv \begin{pmatrix} k^{\mu\nu} \\ l_\mu \end{pmatrix} \qquad K_i^+ \equiv \begin{pmatrix} k^{\mu\nu} & l_\mu \end{pmatrix} \tag{2.3.11}$$

l_μ and $k^{\mu\nu}$ are respectively Grassmann-even and Grassmann-odd. The total source term is now:

$$S_{source}[\phi; J; K; \bar{g}] = S_{source-J}[\phi; J; \bar{g}] + S_{source-K}[\phi; K; \bar{g}] \tag{2.3.12}$$

with:

$$S_{source-J}[\phi; J; \bar{g}] = -\int d^Dx\sqrt{\bar{g}}\, J_i^+\phi^i = -\int d^Dx\sqrt{\bar{g}}\, \left(t^{\mu\nu}h_{\mu\nu} + \bar{\eta}_\mu c^\mu + \bar{c}_\mu \bar{\eta}^\mu\right) \tag{2.3.13}$$

$$S_{source-K}[\phi; K; \bar{g}] = -\int d^Dx\sqrt{\bar{g}}\, K_i^+\psi^i = -\int d^Dx\sqrt{\bar{g}}\, \left(k^{\mu\nu}sh_{\mu\nu} + l_\mu sc^\mu\right) \tag{2.3.14}$$

The expressions introduced in subsection 2.2.3 are now dependent on the additional sources:

$$Z[J; K; \bar{g}] = \int \mathcal{D}\mu[\phi]\, e^{-S[\phi;\bar{g}]-S_{source}[\phi;J;K;\bar{g}]} \tag{2.3.15}$$

$$W[J; K; \bar{g}] = \log Z[J; K; \bar{g}] \tag{2.3.16}$$

$$\begin{aligned} \Gamma[\Phi; K; \bar{g}] &= \sup_J \left\{ \int d^Dx\sqrt{\bar{g}}\, J_i^+\Phi^i - W[J; K; \bar{g}] \right\} \\ &= \int d^Dx\sqrt{\bar{g}}\, J_i^+\Phi^i - W[J; K; \bar{g}] \end{aligned} \tag{2.3.17}$$

$$\begin{aligned} W[J; K; \bar{g}] &= \sup_\Phi \left\{ \int d^Dx\sqrt{\bar{g}}\, J_i^+\Phi^i - \Gamma[\Phi; K; \bar{g}] \right\} \\ &= \int d^Dx\sqrt{\bar{g}}\, J_i^+\Phi^i - \Gamma[\Phi; K; \bar{g}] \end{aligned} \tag{2.3.18}$$

In particular, the Legendre transform connecting the path integral logarithm and the effective action is still done with respect to the elementary fields and sources only, while the additional sources remain spectators. The relation between fields and sources in the transformation becomes dependent on the additional sources $J = J(\Phi; K)$, $\Phi = \Phi(J; K)$:

$$
\begin{aligned}
\Phi^i(x) = \ \left\langle \phi^i(x) \right\rangle_{JK} \ &= \frac{1}{\sqrt{\bar{g}(x)}} \frac{\delta W[J;K;\bar{g}]}{\delta J_i^+(x)} \\
\Phi^{+i}(x) = \left\langle \phi^{+i}(x) \right\rangle_{JK} &= \frac{1}{\sqrt{\bar{g}(x)}} \frac{W \overleftarrow{\delta} [J;K;\bar{g}]}{\delta J_i(x)}
\end{aligned}
\tag{2.3.19}
$$

$$
\begin{aligned}
J_i(x) &= \frac{1}{\sqrt{\bar{g}(x)}} \frac{\delta \Gamma[\Phi;K;\bar{g}]}{\delta \Phi^{+i}(x)} \\
J_i^+(x) &= \frac{1}{\sqrt{\bar{g}(x)}} \frac{\Gamma \overleftarrow{\delta} [\Phi;K;\bar{g}]}{\delta \Phi^i(x)}
\end{aligned}
\tag{2.3.20}
$$

The integro-differential equation for the effective action now reads:

$$
\begin{aligned}
e^{-\Gamma[\Phi;K;\bar{g}]} = \int \mathcal{D}\mu[\phi] \ \exp\Bigg\{ &-S[\phi;\bar{g}] - S_{source-K}[\phi;K;\bar{g}] \\
&+ \int d^D x \frac{\Gamma \overleftarrow{\delta} [\Phi;K;\bar{g}]}{\delta \Phi^i} \left(\phi^i - \Phi^i \right) \Bigg\}
\end{aligned}
\tag{2.3.21}
$$

Thanks to the additional sources we can now express correlation functions involving the non-elementary fields as derivatives of the generating functionals; in particular, we can express the average with sources of non-elementary fields as:

$$
\begin{aligned}
\Psi^i(x) = \ \left\langle \psi^i(x) \right\rangle_{JK} \ &= \frac{1}{\sqrt{\bar{g}(x)}} \frac{\delta W[J;K;\bar{g}]}{\delta K_i^+(x)} &= -\frac{1}{\sqrt{\bar{g}(x)}} \frac{\delta \Gamma[\Phi;K;\bar{g}]}{\delta K_i^+(x)} \\
\Psi^{+i}(x) = \left\langle \psi^{+i}(x) \right\rangle_{JK} &= \frac{1}{\sqrt{\bar{g}(x)}} \frac{W \overleftarrow{\delta} [J;K;\bar{g}]}{\delta K_i(x)} &= -\frac{1}{\sqrt{\bar{g}(x)}} \frac{\Gamma \overleftarrow{\delta} [\Phi;K;\bar{g}]}{\delta K_i(x)}
\end{aligned}
\tag{2.3.22}
$$

The expressions in terms of the effective action follows from the fact that the first derivatives with respect to the additional sources of the path integral logarithm and the effective action are equal up to a minus sign:

$$
W^{\overrightarrow{(1)}}_{K_i}(x) = -\Gamma^{\overrightarrow{(1)}}_{K_i}(x) \qquad , \qquad W^{\overleftarrow{(1)}}_{K_i}(x) = -\Gamma^{\overleftarrow{(1)}}_{K_i}(x)
\tag{2.3.23}
$$

where, to match the functional dependence, it is understood that one has to use the relations $J = J(\Phi; K)$, $\Phi = \Phi(J; K)$ after having taken the derivative. Those relations are a consequence of the properties of the Legendre transform; the first relation, for instance, follows from (2.3.17), recalling that $J = J(\Phi; K)$ according to (2.3.19):

$$
\begin{aligned}
\frac{\delta \Gamma}{\delta K_i^+(x)} = &\int d^D y \cancel{\sqrt{\bar{g}(y)} \frac{\delta J_j^+(\Phi;K)(y)}{\delta K_i^+(x)}} \Phi^j(y) \\
&- \left(\frac{\delta W}{\delta K_i^+(x)} \Bigg|_{J(\Phi;K)} + \int d^D y \, \cancel{\frac{\delta J_j^+(\Phi;K)(y)}{\delta K_i^+(x)} \frac{\delta W}{\delta J_j^+(y)}} \right)
\end{aligned}
$$

In the following we will need also the relations between the matrices of second derivatives of the path integral, the path integral logarithm and the effective action. In presence of additional sources (2.2.101) immediately generalizes to:

$$\frac{1}{Z} Z^{(2)}_{MN}(x,y) = W^{(2)}_{MN}(x,y) + W^{\overrightarrow{(1)}}_{M}(x) W^{\overleftarrow{(1)}}_{N}(y) \tag{2.3.24}$$

with $M, N \in \{J_i, K_j\}$. Taking another derivative with respect to the additional sources in (2.3.23), we obtain immediately the relation between KK blocks:

$$W^{(2)}_{K_i K_j}(x,y) = -\Gamma^{(2)}_{K_i, K_j}(x,y) \tag{2.3.25}$$

Taking instead another right or left derivative with respect to the sources for elementary fields in (2.3.23) and using again the properties of the Legendre transform, one finds the relation between KJ and JK blocks:

$$W^{(2)}_{K_i J_j}(x,y) = -\Gamma^{(2)}_{K_i \Phi^k}(x,\cdot) \bullet_{\bar{g}} \Gamma^{(2)-1}_{\Phi^k \Phi^j}(\cdot, y) \tag{2.3.26}$$

$$W^{(2)}_{J_i K_j}(x,y) = -\Gamma^{(2)-1}_{\Phi^i \Phi^k}(x,\cdot) \bullet_{\bar{g}} \Gamma^{(2)}_{\Phi^k K_j}(\cdot, y) \tag{2.3.27}$$

The first relation, for instance, follows from:

$$\begin{aligned} \frac{\delta W \overleftarrow{\delta}}{\delta K_i^+(x) \delta J_j^+(y)} &= -\frac{\delta \Gamma}{\delta K_i^+(x)}\bigg|_{\Phi=\Phi(J;K)} \frac{\overleftarrow{\delta}}{\delta J_j^+(y)} = -\int d^D z \frac{\delta \Gamma \overleftarrow{\delta}}{\delta K_i^+(x) \delta \Phi^k(z)} \frac{\Phi^k(z) \overleftarrow{\delta}}{\delta J_j^+(y)} = \\ &= -\int d^D z \frac{\delta \Gamma \overleftarrow{\delta}}{\delta K_i^+(x) \delta \Phi^k(z)} \frac{1}{\sqrt{\bar{g}(z)}} \frac{\delta W \overleftarrow{\delta}}{\delta J_k^+(z) \delta J_j^+(y)} \end{aligned}$$

after introducing appropriate $1/\sqrt{\bar{g}}$ factors and using identity (2.2.107):

$$W^{(2)}_{J_i J_j}(x,y) = \Gamma^{(2)-1}_{\Phi^i \Phi^j}(x,y) \tag{2.3.28}$$

which holds also in presence of additional sources and gives itself the relation between JJ blocks of the two matrices of second derivatives.

We conclude the subsection by stressing the important fact that if the gauge-fixed action is background gauge-invariant, the generating functionals inherit background gauge symmetry as an explicit symmetry, in particular:

$$\bar{\delta}_\xi W[J;K;\bar{g}] \ , \ \bar{\delta}_\xi \Gamma[\Phi;K;\bar{g}] = 0 \tag{2.3.29}$$

since the measure is diffeomorphism-invariant by assumption and also the source terms, assuming that all sources transform covariantly according to their tensorial character in a background gauge transformation.

2.3.2 Standard on-shell BRST symmetry and Zinn-Justin equation

On-shell BRST symmetry at the classical level

Consider the gauge-fixed action (2.3.5); the on-shell BRST variations of the elementary fields are defined as:

$$\begin{cases} \delta_\theta h_{\mu\nu} = \theta \pounds_c(\bar{g}_{\mu\nu} + h_{\mu\nu}) \\ \qquad\quad = \theta\big(c^\lambda \partial_\lambda \bar{g}_{\mu\nu} + \partial_\mu c^\lambda \bar{g}_{\lambda\nu} + \partial_\nu c^\lambda \bar{g}_{\lambda\mu}\big) + \theta\left(c^\lambda \partial_\lambda h_{\mu\nu} + \partial_\mu c^\lambda h_{\lambda\nu} + \partial_\nu c^\lambda h_{\lambda\mu}\right) \\ \qquad\quad = \theta\big(\bar{\nabla}_\mu c^\lambda \bar{g}_{\lambda\nu} + \bar{\nabla}_\nu c^\lambda \bar{g}_{\lambda\mu}\big) + \theta\left(c^\lambda \bar{\nabla}_\lambda h_{\mu\nu} + \bar{\nabla}_\mu c^\lambda h_{\lambda\nu} + \bar{\nabla}_\nu c^\lambda h_{\lambda\mu}\right) \\ \delta_\theta \bar{g}_{\mu\nu} = \qquad 0 \\ \delta_\theta c^\mu = \qquad \theta c^\lambda \partial_\lambda c^\mu \quad = \quad \theta c^\lambda \bar{\nabla}_\lambda c^\mu \\ \delta_\theta \bar{c}_\mu = -\theta \dfrac{2}{\kappa^2 \alpha} f_\mu(h;\bar{g}) \end{cases} \tag{2.3.30}$$

The correspondent on-shell BRST Slavnov variations are:

$$\begin{cases} s h_{\mu\nu} = \pounds_c(\bar{g}_{\mu\nu} + h_{\mu\nu}) \\ \qquad = \big(c^\lambda \partial_\lambda \bar{g}_{\mu\nu} + \partial_\mu c^\lambda \bar{g}_{\lambda\nu} + \partial_\nu c^\lambda \bar{g}_{\lambda\mu}\big) + \big(c^\lambda \partial_\lambda h_{\mu\nu} + \partial_\mu c^\lambda h_{\lambda\nu} + \partial_\nu c^\lambda h_{\lambda\mu}\big) \\ \qquad = \big(\bar{\nabla}_\mu c^\lambda \bar{g}_{\lambda\nu} + \bar{\nabla}_\nu c^\lambda \bar{g}_{\lambda\mu}\big) + \big(c^\lambda \bar{\nabla}_\lambda h_{\mu\nu} + \bar{\nabla}_\mu c^\lambda h_{\lambda\nu} + \bar{\nabla}_\nu c^\lambda h_{\lambda\mu}\big) \\ s\bar{g}_{\mu\nu} = \qquad 0 \\ s c^\mu = \qquad c^\lambda \partial_\lambda c^\mu \quad = \quad \theta c^\lambda \bar{\nabla}_\lambda c^\mu \\ s\bar{c}_\mu = -\dfrac{2}{\kappa^2 \alpha} f_\mu(h;\bar{g}) \end{cases} \tag{2.3.31}$$

In accordance with the general remarks on BRST symmetry made at the beginning of subsection 2.2.3, the sector of the transformation unrelated to the gauge-fixing, i.e. metric fields and ghost, is the same as in the off-shell transformation (2.2.115), while the one related to the gauge-fixing, i.e. the antighost, is equal to that of the off-shell transformation (2.2.115) upon substituting the on-shell form of the Nakanishi-Lautrup field (2.3.8) which enforces the standard gauge-fixing term (2.3.1).

As anticipated, the BRST variation of the antighost depends explicitly on the gauge-fixing function. Moreover, the nilpotency of the BRST Slavnov variation now holds only upon using the classical equations of motion:

$$\begin{cases} s^2 h_{\mu\nu} &= 0 \\ s^2 \bar{g}_{\mu\nu} &= 0 \\ s^2 c^\mu &= 0 \\ s^2 \bar{c}_\mu &\overset{\text{e.o.m.}}{=} 0 \end{cases} \tag{2.3.32}$$

since the second BRST Slavnov variation of the antighost is vanishing only using its equations of motion:

$$0 = \frac{\delta S}{\delta \bar{c}_\mu}[h, c, \bar{c}; \bar{g}] = -\sqrt{\bar{g}}\, \bar{g}^{\mu\beta} \mathcal{G}_{\beta\gamma} \bullet c^\gamma \qquad \Longrightarrow \qquad \mathcal{G}_{\mu\nu} \bullet c^\nu = 0 \tag{2.3.33}$$

indeed, from identity (2.2.128) we have:

$$s^2 \bar{c}_\mu = -\frac{2}{\kappa^2 \alpha} s f_\mu(h; \bar{g}) = -\frac{2}{\kappa^2 \alpha} \mathcal{G}_{\mu\nu} \bullet c^\nu \overset{\text{e.o.m.}}{=} 0$$

From (2.2.113) it follows that any polynomial function of the fields has also a vanishing second BRST Slavnov variation, possibly after imposing the equations of motion of the antighost, if present:

$$s^2 F[h, c, \bar{c}; b; \bar{g}] \overset{\text{e.o.m.}}{=} 0 \tag{2.3.34}$$

One can verify that the gauge-fixed action (2.3.5) is indeed invariant under the BRST variations (2.3.30) by a direct computation. Indeed, the Einstein-Hilbert action is BRST-invariant; the BRST Slavnov variation of the gauge-fixing term and the ghost term gives, respectively:

$$\begin{aligned} sS_{gf}[h; \bar{g}] &= s \int d^D x \sqrt{\bar{g}} \left(-\frac{1}{\kappa^2 \alpha}\, f_\mu(h; \bar{g}) \bar{g}^{\mu\nu} f_\nu(h; \bar{g}) \right) = \\ &= \int d^D x \sqrt{\bar{g}} \left(-\frac{2}{\kappa^2 \alpha}\, f_\mu(h; \bar{g}) \bar{g}^{\mu\nu} s f_\nu(h; \bar{g}) \right) = \int d^D x \sqrt{\bar{g}}\, \left(s\bar{c}_\mu \bar{g}^{\mu\nu} s f_\nu(h; \bar{g}) \right) \end{aligned}$$

and:

$$\begin{aligned} sS_{gf}[h; \bar{g}] &= s \int d^D x \sqrt{\bar{g}}\, \left(-\bar{c}_\mu \bar{g}^{\mu\nu} s f_\nu(h; \bar{g}) \right) = \\ &= \int d^D x \sqrt{\bar{g}}\, \left(-s\bar{c}_\mu \bar{g}^{\mu\nu} s f_\nu(h; \bar{g}) \right) = \int d^D x \sqrt{\bar{g}}\, \left(-s\bar{c}_\mu \bar{g}^{\mu\nu} s f_\nu(h; \bar{g}) \right) \end{aligned}$$

In particular, we notice the use of $s^2 f_\nu(h; \bar{g}) = 0$, which is actually valid also off-shell, since the first variation of the gauge-fixing function, which contains only the metric fields, introduces only ghosts and no antighost. The sum of the two variations is zero:

$$s \left(S_{gf}[h; \bar{g}] + S_{gh}[h, c, \bar{c}; \bar{g}] \right) = \int d^D x \sqrt{\bar{g}}\, \left(s\bar{c}_\mu \bar{g}^{\mu\nu} s f_\nu(h; \bar{g}) - s\bar{c}_\mu \bar{g}^{\mu\nu} s f_\nu(h; \bar{g}) \right) = 0$$

Therefore:

$$\delta_\theta \left(S_{gf}[h; \bar{g}] + S_{gh}[h, c, \bar{c}; \bar{g}] \right) = 0 \tag{2.3.35}$$

and in conclusion:

$$\delta_\theta S[h, c, \bar{c}; \bar{g}] = 0 \tag{2.3.36}$$

Similarly to the off-shell case, the standard gauge-fixing term and the ghost term are not separately BRST-invariant, but their BRST variations compensate and precisely eliminate each other and this holds for any specific choice of the gauge-fixing function. Therefore we can again conclude that the gauge-fixed action (2.3.5) enjoys a BRST symmetry under

(2.3.30) regardless of the gauge-fixing details; in this case however the BRST variations depend explicitly on the gauge-fixing function.

On-shell BRST symmetry at the quantum level

We conclude the subsection by deriving the explicit form of the Ward-Takahashi equation describing BRST symmetry at the quantum level for the standard gauge-fixed theory. With respect to the result in subsection 2.2.3, we need to consider in principle the non-elementary fields and the additional sources introduced in the quantization; so, (2.2.137) now gives:

$$\left\langle \int d^D x \sqrt{\bar{g}} \left(J_i^+ \delta_\theta \phi^i + K_i^+ \delta_\theta \psi^i \right) \right\rangle_{JK} = 0 \tag{2.3.37}$$

However, we have clearly $\delta_\theta \psi^i = 0$ due to BRST nilpotency, since the non-elementary fields are BRST Slavnov variations, therefore we have again:

$$\begin{aligned} 0 &= \int d^D x \sqrt{\bar{g}}\, J_i^+ \left\langle \delta_\theta \phi^i \right\rangle_{JK} \\ &= \int d^D x \sqrt{\bar{g}} \left(t^{\mu\nu} \left\langle \delta_\theta h_{\mu\nu} \right\rangle_{JK} + \bar{\eta}_\mu \left\langle \delta_\theta c^\mu \right\rangle_{JK} - \eta^\mu \left\langle \delta_\theta \bar{c}_\mu \right\rangle_{JK} \right) \\ &= \theta \int d^D x \sqrt{\bar{g}} \left(t^{\mu\nu} \left\langle s h_{\mu\nu} \right\rangle_{JK} - \bar{\eta}_\mu \left\langle s c^\mu \right\rangle_{JK} + \eta^\mu \left\langle s \bar{c}_\mu \right\rangle_{JK} \right) \end{aligned} \tag{2.3.38}$$

and substituting the explicit expression for the BRST Slavnov variation of the antighost field in the last form of the equation, the Ward-Takahashi equation associated to BRST symmetry in the standard gauge-fixed theory can be written as:

$$\int d^D x \sqrt{\bar{g}} \left(t^{\mu\nu} \left\langle s h_{\mu\nu} \right\rangle_{JK} - \bar{\eta}_\mu \left\langle s c^\mu \right\rangle_{JK} - \frac{2}{\kappa^2 \alpha} \eta^\mu \left\langle f_\mu(h;\bar{g}) \right\rangle_{JK} \right) = 0 \tag{2.3.39}$$

As seen at the end of subsection 2.2.3, this equation can be used to generate identities between correlation functions. Moreover, it can be recast as a functional equation for the effective action, which in the literature takes then the name of Zinn-Justin equation [28]. Such equation is obtained expressing the Ward-Takahashi equation in the variables of the effective action by evaluating the sources with the expressions given by the Legendre transform, $J = J(\Phi, K)$, and writing them explicitly in terms of derivatives of the effective action using (2.3.20), together with the averages of non-elementary fields, thanks to the additional sources specifically introduced, using (2.3.22). Performed the substitutions, we obtain:

$$\Sigma\left[\Gamma\right] = - \int d^D x \left(\frac{1}{\sqrt{\bar{g}}} \frac{\delta \Gamma}{\delta \mathsf{h}_{\mu\nu}} \frac{\delta \Gamma}{\delta k^{\mu\nu}} + \frac{1}{\sqrt{\bar{g}}} \frac{\delta \Gamma}{\delta \mathsf{c}^\mu} \frac{\delta \Gamma}{\delta l_\mu} + \frac{2}{\kappa^2 \alpha} \frac{\delta \Gamma}{\delta \bar{\mathsf{c}}_\mu} \left\langle f_\mu(h;\bar{g}) \right\rangle_{JK} \right) = 0 \tag{2.3.40}$$

It can be seen that using the quantum equations of motion of the antighost $\langle 1/\sqrt{\bar{g}} \delta S / \delta \bar{c}_\mu \rangle = 0$ and defining:

$$\Gamma'[\Phi; K; \bar{g}] = \Gamma[\Phi; K; \bar{g}] - S_{gf}[\mathsf{h}; \bar{g}] \tag{2.3.41}$$

also the average of the gauge-fixing function can be rewritten in terms of the effective action, and the Zinn-Justin equation for the standard gauge-fixed theory can be expressed as [5][6]:

$$\Sigma[\Gamma] = -\int d^D x \frac{1}{\sqrt{\bar{g}}}\left(\frac{\delta\Gamma'}{\delta \mathsf{h}_{\mu\nu}}\frac{\delta\Gamma'}{\delta k^{\mu\nu}} + \frac{\delta\Gamma'}{\delta \mathsf{c}^{\mu}}\frac{\delta\Gamma'}{\delta l_{\mu}}\right) = 0 \tag{2.3.42}$$

The equation is quadratic and linear in the derivatives of the effective action and represents the constraint imposed by BRST symmetry on its functional form. If additional non-BRST-invariant terms are added to the action, the Ward-Takahashi equation is modified with additional terms (since on top of the BRST variation of the source term, it appears also the one of the additional non-BRST-invariant terms) and one can rewrite it as in (2.3.39), or equivalently rewrite the Zinn-Justin equation as in (2.3.42), but with a non-zero right hand side, representing that BRST symmetry has been explicitly broken by the additional terms and that it is recovered only in the limit in which the additional terms are removed and the original equations satisfied.

2.3.3 Standard de Donder gauge-fixing function

We now give a specific form for the gauge-fixing function to consider in the standard gauge-fixed theory obtained in subsection 2.3.1, namely the so called de Donder gauge-fixing function:

$$f_\mu(h;\bar{g}) = \bar{\nabla}^\nu h_{\nu\mu} - \frac{1}{2}\bar{\nabla}_\mu h = \bar{K}^{\alpha\beta}{}_{\mu\nu}\bar{\nabla}^\nu h_{\alpha\beta} \tag{2.3.43}$$

which enforces the so called background covariant de Donder gauge-fixing condition (one can also consider generalizations of this condition where the 1/2 is substituted by an arbitrary real parameter, known as generalized harmonic gauge conditions [5]). Clearly, it correctly breaks the invariance under true gauge transformations (2.2.9) and is covariant under the background gauge transformations (2.2.12). Substituting in (2.3.1), integrating by parts and recognizing the structure tensor $\bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma}$ (2.2.39) we obtain the de Donder gauge-fixing term:

$$S_{gf}[h;\bar{g}] = \frac{1}{\kappa^2}\int d^D x\sqrt{\bar{g}}\,\frac{1}{2\alpha}\,h_{\alpha\beta}\bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\rho\bar{\nabla}^\sigma h_{\mu\nu} \tag{2.3.44}$$

which corresponds precisely to the piece $S_{EH,2-deD}[h;\bar{g}]$ in the quadratic term of the background expansion of the Einstein-Hilbert action (2.2.31) deformed with the gauge-fixing parameter. The functional derivative in the ghost term (2.2.77) gives:

$$\frac{\delta f_\alpha(h;\bar{g})(x)}{\delta h_{\mu\nu}(y)} = \bar{K}^{\mu\nu}{}_{\alpha\beta}(x)\bar{\nabla}^\beta_{(x)}\delta(x-y) \tag{2.3.45}$$

[6] Notice the different conventions used, namely: $\kappa = 1/\sqrt{32\pi G}$ instead of $\kappa = \sqrt{32\pi G}$; $\eta^\mu\bar{c}_\mu = -\bar{c}_\mu\eta^\mu$ as source term for the antighost instead of $\bar{c}_\mu\eta^\mu$; and an additional factor 1/2 in the form of the standard quadratic gauge-fixing term (2.3.1), which ultimately results in an additional factor $\sqrt{2}$ inside the gauge-fixing function when a specific form is specified .

therefore the ghost term is given by[7]:

$$S_{gh}[h,c,\bar{c};\bar{g}] = -\int d^Dx\sqrt{\bar{g}}\,\bar{c}_\lambda \bar{g}^{\lambda\rho}\bar{K}^{\alpha\beta}{}_{\rho\sigma}\bar{\nabla}^\sigma \pounds_c(\bar{g}_{\mu\nu}+h_{\mu\nu}) \tag{2.3.46}$$

The kinetic ghost term can be made explicit by noticing that using the definition of the structure tensor $\bar{K}^{\mu\nu}{}_{\rho\sigma}$ (2.2.38) and writing the Lie derivative of the background metric as $\pounds_c\bar{g}_{\mu\nu} = \bar{\nabla}_\mu c^\alpha \bar{g}_{\alpha\nu} + \bar{\nabla}_\nu c^\alpha \bar{g}_{\alpha\mu}$, one has:

$$\begin{aligned}\bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\sigma \pounds_c\bar{g}_{\mu\nu} &= \bar{\nabla}^\mu\bar{\nabla}_\mu c^\sigma \bar{g}_{\sigma\rho} + \left[\bar{\nabla}^\nu,\bar{\nabla}_\rho\right]c^\alpha\bar{g}_{\alpha\nu} = \\ &= \bar{\Box}c^\sigma\bar{g}_{\sigma\rho} + \bar{R}^\alpha{}_\beta{}^\nu{}_\rho c^\beta \bar{g}_{\alpha\nu}\end{aligned}$$

where we have used the rule (1.3.25) to commute the background covariant derivatives (torsion-less). Therefore:

$$\bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\sigma \pounds_c\bar{g}_{\mu\nu} = \bar{g}_{\rho\sigma}\bar{\Box}c^\sigma + \bar{R}_{\rho\sigma}c^\sigma \tag{2.3.47}$$

and the de Donder ghost term can be rewritten as:

$$S_{gh}[h,c,\bar{c};\bar{g}] = S_{gh-kin}[c,\bar{c};\bar{g}] + S_{gh-int}[h,c,\bar{c};\bar{g}] \tag{2.3.48}$$

with:

$$S_{gh-kin}[c,\bar{c};\bar{g}] = \int d^Dx\sqrt{\bar{g}}\,\bar{c}_\mu\left(-\bar{\Box}\right)c^\mu \tag{2.3.49}$$

$$S_{gh-int}[h,c,\bar{c};\bar{g}] = \int d^Dx\sqrt{\bar{g}}\,\left[-\bar{c}_\lambda\bar{g}^{\lambda\rho}\left(\bar{R}_{\rho\sigma}c^\sigma + \bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\sigma \pounds_c h_{\mu\nu}\right)\right] \tag{2.3.50}$$

We write also the complete de Donder gauge-fixed action, making also explicit the background expansion of the Einstein-Hilbert action and the Lie derivative in the hcc-interaction term:

$$\begin{aligned}&S[h,c,\bar{c};\bar{g}] = S_{EH}[h;\bar{g}] + S_{gf}[h;\bar{g}] + S_{gh}[h,c,\bar{c};\bar{g}] = \\ &= \int d^Dx\sqrt{\bar{g}}\left\{\frac{1}{\kappa^2}\left[\frac{1}{2}h_{\alpha\beta}\left(-\bar{K}^{\alpha\beta,\mu\nu}\bar{\Box} + \left(1+\frac{1}{\alpha}\right)\bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\rho\bar{\nabla}^\sigma + \bar{O}_2^{\alpha\beta,\mu\nu}\right)h_{\mu\nu}\right.\right. \\ &\qquad\left. + 2\left(-\bar{R}+2\Lambda\right) + 2\left(\bar{G}_{\mu\nu}+\Lambda\bar{g}_{\mu\nu}\right)h_{\mu\nu} + \sum_{n=3}^{\infty}\frac{1}{n!}\bar{O}_n h^n\right] \\ &\qquad + \bar{c}_\mu(-\bar{\Box})c^\mu \\ &\qquad\left. - \bar{c}_\lambda\bar{g}^{\lambda\rho}\left[\bar{R}_{\rho\sigma}c^\sigma + \bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\sigma\left(c^\tau\bar{\nabla}_\tau h_{\mu\nu} + \bar{\nabla}_\mu c^\tau h_{\tau\nu} + \bar{\nabla}_\nu c^\tau h_{\tau\mu}\right)\right]\right\}\end{aligned} \tag{2.3.51}$$

In particular we notice that thanks to the $-$ sign introduced in the ghost operator (2.2.74) in paragraph 2.2.2, the kinetic ghost term has the "right"[5] sign, as the one for the metric fluctuation.

[7]The covariant derivative multiplying the Dirac delta in (2.3.45) is implicitly in the form acting on rank-$(0,2)$ tensors, and can be moved on the right hand side by noticing that $\Gamma^\lambda_{\mu\nu}(x)\delta(x-y) = \delta(x-y)\Gamma^\lambda_{\mu\nu}(y)$, and integrating by parts in the y-integration in (2.2.77) (where there is no $\sqrt{g}$ factor) to move the ordinary derivatives as noted in 4: $\partial_{(x)\mu}\delta(x-y) = -\partial_{(y)\mu}\delta(x-y) \overset{\text{i.b.p}}{=} \delta(x-y)\partial_{(y)\mu}$.

2.4 Standard FRG flow

In this section we apply the essential concepts in FRG theory reviewed in section 1.2 to the standard gauge-fixed theory in order to briefly present the standard construction of an FRG flow in a QEG theory. In particular, we present the standard Wetterich-Morris equation and discuss how the construction is background gauge-invariant but incompatible with BRST symmetry, which is broken in the regularized theory.

2.4.1 Standard FRG regularization

Regularization

Consider the path integral (2.3.15) describing the quantum theory stemming from the standard gauge-fixed action (2.3.51). According to the preliminary analysis made in subsection 2.1.2 (taking now into account also the presence of ghosts), the contributions from the various field configurations to the path integral can be naturally ordered in the background field method by rewriting the integrated fields in generalized momentum space, i.e. expanded in the basis of eigenfunctions $u_p(x)$ of the negative background laplacian, and the integration measure in terms of the corresponding Fourier weights $\prod_p d\mu[\tilde{h}_{\mu\nu}(p), \tilde{c}^\mu(p), \tilde{\bar{c}}_\mu(p)]$.The integrated weights are now ordered by the value of their generalized momentum, i.e. the eigenvalue p^2 of the corresponding eigenfunction of the negative background laplacian, from IR modes, i.e. low p^2, to UV modes, i.e. high p^2.
As seen in subsection 2.1.2, the FRG coarse-graining can be now implemented similarly to the scalar theory in subsection 1.2.1 by IR-regulating the path integral by manually adding to the gauge-fixed action a regulator term suppressing the integration of IR modes and leaving untouched UV ones with respect to a floating scale k^2, for both the metric fluctuation and the ghosts. The standard regulator has the form:

$$\Delta S_{(k)}[h, c, \bar{c}; \bar{g}] = \Delta S_{gr(k)}[h; \bar{g}] + \Delta S_{gh(k)}[c, \bar{c}; \bar{g}] \tag{2.4.1}$$

with a quadratic regulator term for the metric fluctuation and one for the ghosts:

$$\Delta S_{gr(k)}[h; \bar{g}] = \frac{1}{2\kappa^2} \int d^D x \sqrt{\bar{g}}\, h_{\alpha\beta} \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)} \left(-\bar{\Box}\right) h_{\mu\nu} \tag{2.4.2}$$

$$\Delta S_{gh(k)}[c, \bar{c}; \bar{g}] = \int d^D x \sqrt{\bar{g}}\, \bar{c}_\mu \mathcal{R}_{gh(k)} \left(-\bar{\Box}\right) c^\mu \tag{2.4.3}$$

The regulators $\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)} \left(-\bar{\Box}\right)$, $\mathcal{R}_{gh(k)} \left(-\bar{\Box}\right)$ depend on the background metric; in particular, we assume that the regulator for the metric fluctuation has a factorizable tensorial structure, possibly k-dependent, contained in a generic structure tensor constructed with the background metric:

$$\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)} \left(-\bar{\Box}\right) \equiv \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)} \mathcal{R}_{gr(k)} \left(-\bar{\Box}\right) \tag{2.4.4}$$

Similarly to the regulators for the scalar theory in subsection 1.2.1, the functional form of the regulators $\mathcal{R}_{gr(k)} \left(-\bar{\Box}\right)$, $\mathcal{R}_{gh(k)} \left(-\bar{\Box}\right)$ in generalized momentum space, i.e. $\mathcal{R}_{gr(k)} \left(p^2\right)$,

$\mathcal{R}_{gh(k)}(p^2)$, must satisfy properties (1.2.6) and show the qualitative behavior depicted in figure 1:

$$\begin{cases} \textbf{1.} \quad \mathcal{R}_{gr/gh(k)}(p^2) & \to \quad > 0 & , & \dfrac{p^2}{k^2} \to 0 \\ \textbf{2.} \quad \mathcal{R}_{gr/gh(k)}(p^2) & \to \quad 0 & , & \dfrac{p^2}{k^2} \to \infty \\ \textbf{3.} \quad \mathcal{R}_{gr/gh(k)}(p^2) & \to \quad \infty & , & k^2 \to k_{UV}^2 \to \infty \\ \textbf{4.} \quad \mathcal{R}_{gr/gh(k)}(p^2) & \to \quad 0 & , & k^2 \to 0 \end{cases} \tag{2.4.5}$$

where k_{UV} is some large UV scale. Thanks to the first two properties, the regulators implement the desired coarse-graining: in the path integral, Fourier weights with eigenvalue p^2 are suppressed if $p^2 \lesssim k^2$ (property **1.**) and left untouched if $p^2 \gtrsim k^2$ (property **2.**). The second two properties are required to set the limits of the flow of the effective average action, as seen below. In order to guarantee those properties, we consider regulators with the standard functional structure as in subsection 1.2.1:

$$\mathcal{R}_{gr/gh(k)}(p^2) = k^2 \mathcal{R}_0\left(\frac{p^2}{k^2}\right) \tag{2.4.6}$$

with $\mathcal{R}_0(x)$ is a standard dimensionless and positive shape function which interpolates between $\mathcal{R}_0(0) = 1$ and $\mathcal{R}_0(\infty) = 0$:

$$\mathcal{R}_0(x) \begin{cases} \to 1 & , \quad x \to 0 \\ \to 0 & , \quad x \to \infty \end{cases} \tag{2.4.7}$$

Adding the regulator term to the action we define the regulated action:

$$\tilde{S}_{(k)}[h, c, \bar{c}; \bar{g}] = S[h, c, \bar{c}; \bar{g}] + \Delta S_{(k)}[h, c, \bar{c}; \bar{g}] \tag{2.4.8}$$

At the quantum level, the regularized theory is described by the regularized generating functionals constructed with the regulated action. Similarly to the case of the scalar theory in subsection 1.2.1, in order to be properly defined, they are also implicitly regulated with k_{UV} as sharp UV cut-off on the values of the generalized momenta considered in the measure, $\prod_{0 \leq |p| \leq |k_{UV}|} d\mu[\tilde{h}_{\mu\nu}(p), \tilde{c}^{\mu}(p), \tilde{\bar{c}}_{\mu}(p)]$. The regulated expressions correspondent to those in subsection 2.3.1 are:

$$\tilde{Z}_{(k)}[J; K; \bar{g}] = \int \mathcal{D}\mu[\phi]\, e^{-\tilde{S}_{(k)}[\phi;\bar{g}] - S_{source}[\phi;J;K;\bar{g}]} \tag{2.4.9}$$

$$\tilde{W}_{(k)}[J; K; \bar{g}] = \log \tilde{Z}_{(k)}[J; K; \bar{g}] \tag{2.4.10}$$

$$\begin{aligned} \tilde{\Gamma}_{(k)}[\Phi; K; \bar{g}] &= \sup_J \left\{ \int d^D x \sqrt{\bar{g}}\, J_i^+ \Phi^i - \tilde{W}_{(k)}[J; K; \bar{g}] \right\} \\ &= \int d^D x \sqrt{\bar{g}}\, J_i^+ \Phi^i - \tilde{W}_{(k)}[J; K; \bar{g}] \end{aligned} \tag{2.4.11}$$

$$\begin{aligned}\tilde{W}_{(k)}[J;K;\bar{g}] &= \sup_{\Phi}\left\{\int d^Dx\sqrt{\bar{g}}\,J_i^+\Phi^i - \tilde{\Gamma}_{(k)}[\Phi;K;\bar{g}]\right\}\\ &= \int d^Dx\sqrt{\bar{g}}\,J_i^+\Phi^i - \tilde{\Gamma}_{(k)}[\Phi;K;\bar{g}]\end{aligned} \tag{2.4.12}$$

Now all averages are k-dependent, as well as fields and sources related in the Legendre transform, $J = J_{(k)}(\Phi, K)$, $\Phi = \Phi_{(k)}(J, K)$:

$$\begin{aligned}\Phi^i(x) &= \left\langle \phi^i(x)\right\rangle_{JK} = \frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta \tilde{W}_{(k)}[J;K;\bar{g}]}{\delta J_i^+(x)}\\ \Phi^{+i}(x) &= \left\langle \phi^{+i}(x)\right\rangle_{JK} = \frac{1}{\sqrt{\bar{g}(x)}}\frac{\tilde{W}_{(k)}\overleftarrow{\delta}[J;K;\bar{g}]}{\delta J_i(x)}\end{aligned} \tag{2.4.13}$$

$$\begin{aligned}J_i(x) &= \frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta\tilde{\Gamma}_{(k)}[\Phi;K;\bar{g}]}{\delta\Phi^{+i}(x)}\\ J_i^+(x) &= \frac{1}{\sqrt{\bar{g}(x)}}\frac{\tilde{\Gamma}_{(k)}\overleftarrow{\delta}[\Phi;K;\bar{g}]}{\delta\Phi^i(x)}\end{aligned} \tag{2.4.14}$$

$$\begin{aligned}\Psi^i(x) &= \left\langle \psi^i(x)\right\rangle_{JK} = \frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta \tilde{W}_{(k)}[J;K;\bar{g}]}{\delta K_i^+(x)} = -\frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta\tilde{\Gamma}_{(k)}[\Phi;K;\bar{g}]}{\delta K_i^+(x)}\\ \Psi^{+i}(x) &= \left\langle \psi^{+i}(x)\right\rangle_{JK} = \frac{1}{\sqrt{\bar{g}(x)}}\frac{\tilde{W}_{(k)}\overleftarrow{\delta}[J;K;\bar{g}]}{\delta K_i(x)} = -\frac{1}{\sqrt{\bar{g}(x)}}\frac{\tilde{\Gamma}_{(k)}\overleftarrow{\delta}[\Phi;K;\bar{g}]}{\delta K_i(x)}\end{aligned} \tag{2.4.15}$$

The integro-differential equation for the regulated effective action is:

$$\begin{aligned}e^{-\tilde{\Gamma}_{(k)}[\Phi;K;\bar{g}]} = \int \mathcal{D}\mu[\phi]\,\exp\Bigg\{&-\tilde{S}_{(k)}[\phi;\bar{g}] - S_{source(K)}[\phi;K;\bar{g}]\\ &+\int d^Dx\frac{\tilde{\Gamma}_{(k)}\overleftarrow{\delta}[\Phi;K;\bar{g}]}{\delta\Phi^i}\left(\phi^i-\Phi^i\right)\Bigg\}\end{aligned} \tag{2.4.16}$$

Effective average action

The formal definition, i.e. based on the above generating functionals, of the effective average action describing the scale-dependent theory interpolating between the classical and quantum regime is:

$$\Gamma_{(k)}[\Phi;K;\bar{g}] \equiv \tilde{\Gamma}_{(k)}[\Phi;K;\bar{g}] - \Delta S_{(k)}[\Phi;\bar{g}] \tag{2.4.17}$$

The second couple of required properties for the regulators in (2.4.5) allows to formally fix the limits of its flow:

$$\Gamma_{(k\to 0)}[\Phi;K;\bar{g}] = \Gamma[\Phi;K;\bar{g}] \tag{2.4.18}$$

$$\Gamma_{(k\to k_{UV}\to\infty)}[\Phi;K;\bar{g}] = S[\Phi;\bar{g}] + S_{source(K)}[\Phi;K;\bar{g}] + \cdots \tag{2.4.19}$$

In particular, for $k \to 0$, the effective average action tends to the unregulated quantum effective action since $\Delta S_{(k\to 0)} \to 0$ (property **4.**):

$$\Gamma_{(k\to 0)}[\Phi;K;\bar{g}] = \tilde{\Gamma}_{(k\to 0)}[\Phi;K;\bar{g}] - \Delta S_{(k\to 0)}[\Phi;\bar{g}] = \Gamma[\Phi;K;\bar{g}] - 0$$

For $k \to k_{UV} \to \infty$, the effective average action is expected to approximately tend to the unregulated classic action (plus the source term for non-elementary fields, if those are not set to zero), with corrective terms related to the reconstruction problem. Exactly as seen for the scalar theory in subsection 1.2.1, this can be deduced from the integro-differential equation (2.4.16) rewritten for the effective average action:

$$\begin{aligned} e^{-\Gamma_{(k)}[\Phi;K;\bar{g}]} = \int \mathcal{D}\mu[\phi] \exp\Bigg\{ & - S[\phi;\bar{g}] - S_{source(K)}[\phi;K;\bar{g}] \\ & - \Delta S_{(k)}[\phi;\bar{g}] + \Delta S_{(k)}[\Phi;\bar{g}] \\ & + \int d^D x \frac{\Delta S_{(k)} \overleftarrow{\delta} [\Phi;\bar{g}]}{\delta \Phi^i} \left(\phi^i - \Phi^i\right) \\ & + \int d^D x \frac{\Gamma_{(k)} \overleftarrow{\delta} [\Phi;K;\bar{g}]}{\delta \Phi^i} \left(\phi^i - \Phi^i\right) \Bigg\} \end{aligned} \tag{2.4.20}$$

Indeed, due to the quadratic nature of the regulator terms, we have again that the three terms in the second and third line of the equation combine to give precisely the regulator term itself computed in the difference between the integrated and average fields $\phi - \Phi$:

$$-\Delta S_{(k)}[\phi;\bar{g}] + \Delta S_{(k)}[\Phi;\bar{g}] + \int d^D x \frac{\Delta S_{(k)} \overleftarrow{\delta} [\Phi;\bar{g}]}{\delta \Phi^i} \left(\phi^i - \Phi^i\right) = -\Delta S_{(k)}[\phi - \Phi;\bar{g}] \tag{2.4.21}$$

Therefore, since $\mathcal{R}_{gr/gh(k\to k_{UV}\to\infty)} \to \infty$ (property **3.**), one recognizes the functional equivalent of the gaussian limit representation of a Dirac delta:

$$e^{-\Delta S_{(k)}[\phi-\Phi;\bar{g}]} \overset{k\to k_{UV}\to\infty}{\longrightarrow} \sim \delta[h - \mathsf{h}]\delta[c - \mathsf{c}]\delta[\bar{c} - \bar{\mathsf{c}}] \tag{2.4.22}$$

and therefore in the integro-differential equation, in the limit $k \to k_{UV} \to \infty$, we obtain as dominant term the integrand computed in $\phi = \Phi$, i.e. $\exp(-S[\Phi;\bar{g}] - S_{source-K}[\Phi;K;\bar{g}])$, from which it follows (2.4.19).

Symmetries and theory space

Assuming that the regulators are constructed with a proper tensorial structure, the regulator term (2.4.1) is invariant under background gauge transformations (2.2.12):

$$\bar{\delta}_\xi \Delta S_{(k)}[h,c,\bar{c};\bar{g}] = 0 \tag{2.4.23}$$

Therefore, the background gauge-invariance of the unregulated theory is preserved for any value of the FRG scale k, at the classical level in the regulated action:

$$\bar{\delta}_\xi \tilde{S}_{(k)}[h,c,\bar{c};\bar{g}] = 0 \tag{2.4.24}$$

and as an explicit symmetry at the quantum level in the regulated functionals:

$$\bar{\delta}_\xi \tilde{W}_{(k)}[J;K;\bar{g}] \ , \ \bar{\delta}_\xi \tilde{\Gamma}_{(k)}[\Phi;K;\bar{g}] \ , \ \bar{\delta}_\xi \Gamma_{(k)}[\Phi;K;\bar{g}] = 0 \tag{2.4.25}$$

As far as BRST symmetry is concerned, instead, the quadratic regulator term (2.4.1) is clearly not invariant under the BRST transformations (2.3.30):

$$\delta_\theta \Delta S_{(k)}[h,c,\bar{c};\bar{g}] \neq 0 \tag{2.4.26}$$

Therefore, the BRST invariance of the unregulated theory is spoiled:

$$\delta_\theta \tilde{S}_{(k)}[h,c,\bar{c};\bar{g}] \neq 0 \tag{2.4.27}$$

and recovered only in the limit in which the regulators vanish:

$$\delta_\theta \tilde{S}_{(k\to 0)}[h,c,\bar{c};\bar{g}] = \delta_\theta S[h,c,\bar{c};\bar{g}] = 0 \tag{2.4.28}$$

At the quantum level this appears in the fact that the Ward-Takahashi equation (2.3.39) is modified by the regulator term, which, being non-BRST-invariant, must be taken into account inside (2.3.37) together with the source terms:

$$\left\langle \int d^D x \sqrt{\bar{g}} \left(J_i^+ \delta_\theta \phi^i + K_i^+ \delta_\theta \psi^i \right) - \delta_\theta \Delta S_{(k)}[\phi;\bar{g}] \right\rangle_{JK} = 0 \tag{2.4.29}$$

Therefore (2.3.39) is substituted by the modified Ward-Takahashi equation, valid for any value of the FRG scale k:

$$\int d^D x \sqrt{\bar{g}} \left(t^{\mu\nu} \langle s h_{\mu\nu} \rangle_{JK} - \bar{\eta}_\mu \langle s c^\mu \rangle_{JK} - \frac{2}{\kappa^2 \alpha} \eta^\mu \langle f_\mu(h;\bar{g}) \rangle_{JK} \right) = \left\langle \delta_\theta \Delta S_{(k)}[\phi;\bar{g}] \right\rangle_{JK} \tag{2.4.30}$$

and BRST symmetry is recovered only in the limit in which the regulators vanish and the original identity is again satisfied.
Equivalently, from the point of view of the effective average action, it can be seen that the modified Ward-Takahashi equation can be rewritten as a modified Zinn-Justin equation which substitutes (2.3.42), valid for any value of the FRG scale k [5][6]:

$$\Sigma\left[\Gamma_{(k)}\right] = -\int d^D x \frac{1}{\sqrt{\bar{g}}} \left(\frac{\delta \Gamma'_{(k)}}{\delta \mathsf{h}_{\mu\nu}} \frac{\delta \Gamma'_{(k)}}{\delta k^{\mu\nu}} + \frac{\delta \Gamma'_{(k)}}{\delta \mathsf{c}^\mu} \frac{\delta \Gamma'_{(k)}}{\delta l_\mu} \right) = Y_{(k)} \tag{2.4.31}$$

where:

$$\Gamma'_{(k)}[\Phi;K;\bar{g}] = \Gamma_{(k)}[\Phi;K;\bar{g}] - S_g f[\mathsf{h};\bar{g}] \tag{2.4.32}$$

and $Y_{(k)}$ is a functional trace, dependent on the regulators and derivatives of the effective average action, characterized by:

$$Y_{(k)} \ \to \ 0 \quad , \quad k \to 0 \tag{2.4.33}$$

so that again BRST symmetry is again recovered only in the limit in which the regulators vanish and the effective average action satisfies the original Zinn-Justin equation:

$$\Sigma\left[\Gamma_{(k\to 0)}\right] = \Sigma\left[\Gamma\right] = 0 \tag{2.4.34}$$

According to those symmetry properties, the theory space, i.e. the space spanned by all operators which can appear inside the effective average action, is given by background gauge-invariant operators, i.e. functionals with a proper scalar integrand, which also satisfy the modified Zinn-Justin equation and thus belong to the k-dependent hypersurface $\tilde{\Sigma}_{(k)} \equiv \Sigma - Y_{(k)} = 0$:

$$\mathcal{T} = \left\{ A[\Phi; K; \bar{g}] \quad : \quad \bar{\delta}_\xi A[\Phi; K; \bar{g}] = 0 \quad , \quad \tilde{\Sigma}_{(k)}[A[\Phi; K; \bar{g}]] = 0 \right\} \tag{2.4.35}$$

Inside the space of background gauge-invariant operators it is contained also the hypersurface of BRST-invariant operators, $\Sigma = 0$, to which the unregulated effective action belongs. The hypersurface $\tilde{\Sigma}_{(k)} = 0$ tends $\Sigma = 0$ in the limit $k \to 0$.

2.4.2 Standard Wetterich-Morris equation

In this subsection we briefly present the standard Wetterich-Morris equation describing the flow of the effective average action (2.4.17). The equation is derived by expressing the derivative of the effective average action with respect to the FRG time $t = \log k$ starting from the formal definition (2.4.17) in terms of the regulated generating functionals.
Exactly as seen for the scalar theory in subsection 1.2.2, one starts by taking the time derivative of the regulated effective action (2.4.11) to find its flow equation. Similarly, we have the equalities:

$$\begin{aligned} \partial_t \tilde{\Gamma}_{(k)}[\Phi; K; \bar{g}] = -\partial_t \tilde{W}_{(k)}[J; K; \bar{g}] &= -\frac{\partial_t \tilde{Z}_{(k)}[J; K; \bar{g}]}{\tilde{Z}_{(k)}[J; K; \bar{g}]} \\ &= \left\langle \partial_t \Delta S_{(k)}[\phi; \bar{g}] \right\rangle_{JK} \end{aligned} \tag{2.4.36}$$

where it is understood that the time derivatives of source-dependent objects are computed in $J = J_{(k)}(\Phi, K)$. In particular, the first equality comes from the Legendre transform (2.4.11) recalling that $J = J_{(k)}(\Phi, K)$ according to (2.4.13):

$$\partial_t \tilde{\Gamma}_{(k)} = \int d^D x \cancel{\sqrt{\bar{g}}\, \partial_t J^+_{(k)i} \Phi^i} - \left(\partial_t \tilde{W}_{(k)}|_{J_{(k)}} + \int d^D x \, \cancel{\partial_t J^+_{(k)i} \frac{\delta \tilde{W}_{(k)}}{\delta J^+_i}} \right)$$

The others follows immediately by the definition of the regulated path integral. So:

$$\partial_t \tilde{\Gamma}_{(k)}[\Phi; K; \bar{g}] = \left\langle \partial_t \Delta S_{(k)}[\phi; \bar{g}] \right\rangle_{JK} \tag{2.4.37}$$

Substituting the explicit expression of the regulator (2.4.1), the right hand side is given by:

$$\partial_t \tilde{\Gamma}_{(k)}[\Phi; K; \bar{g}] = \int d^D x \sqrt{\bar{g}} \left[\frac{1}{2\kappa^2} \left\langle h_{\alpha\beta} \partial_t \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)} \left(-\bar{\Box}\right) h_{\mu\nu} \right\rangle_{JK} + \left\langle \bar{c}_\mu \partial_t R_{gh(k)} \left(-\bar{\Box}\right) c^\mu \right\rangle_{JK} \right] \tag{2.4.38}$$

Introducing an additional integration in order to formally express the objects on which the regulators act in a different spacetime variable, we can then collect the regulators:

$$\begin{aligned}
&\partial_t \tilde{\Gamma}_{(k)}[\Phi; K; \bar{g}] = \\
&\quad = \int d^D x d^D y \sqrt{\bar{g}(x)}\, \delta(x-y) \Bigg[\\
&\qquad \frac{1}{2\kappa^2} \partial_t \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)} \left(-\bar{\Box}_{(y)}\right) \langle h_{\alpha\beta}(x) h_{\mu\nu}(y) \rangle_{JK} + \delta^\mu_\nu \partial_t R_{gh(k)} \left(-\bar{\Box}_{(y)}\right) \langle \bar{c}_\mu(x) c^\nu(y) \rangle_{JK} \Bigg] \\
&\quad = \mathrm{Tr}_{\bar{g}} \left[\frac{1}{2\kappa^2} \partial_t \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)} \left(-\bar{\Box}\right) \langle h_{\alpha\beta} \otimes h_{\mu\nu} \rangle_{JK} + \delta^\mu_\nu \partial_t R_{gh(k)} \left(-\bar{\Box}\right) \langle \bar{c}_\mu \otimes c^\nu \rangle_{JK} \right]
\end{aligned} \tag{2.4.39}$$

where in the last expression $\mathrm{Tr}_{\bar{g}}$ denotes the functional trace with a factor $\sqrt{\bar{g}}$ in the measure and it is understood that the regulators act on the second of the two terms in the direct products. Finally, the flow equation for the regulated effective action is obtained by expressing the 2-point correlation functions (derivatives of the regulated path integral) in terms of connected correlation functions (derivatives of the regulated path integral logarithm) using relation (2.3.24) :

$$\frac{1}{\tilde{Z}_{(k)}} \tilde{Z}^{(2)}_{(k)MN}(x,y) = \tilde{W}^{(2)}_{(k)MN}(x,y) + \tilde{W}^{\overrightarrow{(1)}}_{(k)M}(x) \tilde{W}^{\overleftarrow{(1)}}_{(k)N}(y) \tag{2.4.40}$$

The second piece gives a product of average fields $\langle h_{\alpha\beta}(x) \rangle_{JK} \langle h_{\mu\nu}(y) \rangle_{JK} = \mathsf{h}_{\alpha\beta}(x) \mathsf{h}_{\mu\nu}(y)$ and $\langle \bar{c}_\mu(x) \rangle_{JK} \langle c^\mu(y) \rangle_{JK} = \bar{\mathsf{c}}_\mu(x) \mathsf{c}^\mu(y)$, which traced gives back precisely the derivative of the regulator term (2.4.1) computed in the average fields $\partial_t \Delta S_{(k)}[\Phi; \bar{g}]$:

$$\mathrm{Tr}_{\bar{g}} \left[\frac{1}{2\kappa^2} \partial_t \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)} \left(-\bar{\Box}\right) \mathsf{h}_{\alpha\beta} \otimes \mathsf{h}_{\mu\nu} + \partial_t R_{gh(k)} \left(-\bar{\Box}\right) \bar{\mathsf{c}}_\mu \otimes \mathsf{c}^\mu \right] = \partial_t \Delta S_{(k)}[\Phi; \bar{g}] \tag{2.4.41}$$

The first is related to the second derivatives of the regulated effective action according to the relations (2.3.28), (2.3.25), (2.3.26),(2.3.27). In particular, since there is no correlation function of non-elementary fields, we need only the first:

$$\tilde{W}^{(2)}_{(k)J_i J_j}(x,y) = \tilde{\Gamma}^{(2)-1}_{(k)\Phi^i \Phi^j}(x,y) \tag{2.4.42}$$

Appropriately antisymmetrizing the ghost sector and using identity (2.4.40) and (2.4.42), one obtains the standard flow equation for the regulated effective action [5][6]:

$$\begin{aligned}
\partial_t \tilde{\Gamma}_{(k)}[\Phi; K; \bar{g}] = \frac{1}{2} \mathrm{Tr}_{\bar{g}} \Bigg\{ & \frac{1}{\kappa^2} \partial_t \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)} \left(-\bar{\Box}\right) \tilde{\Gamma}^{(2)-1}_{(k)\mathsf{h}_{\alpha\beta}\mathsf{h}_{\mu\nu}} \\
& - \delta^\mu_\nu \partial_t \mathcal{R}_{gh(k)} \left(-\bar{\Box}\right) \left[\tilde{\Gamma}^{(2)-1}_{(k)\bar{\mathsf{c}}_\mu \mathsf{c}^\nu} - \tilde{\Gamma}^{(2)-1}_{(k)\mathsf{c}^\mu \bar{\mathsf{c}}_\nu} \right] \Bigg\} \\
+ \partial_t \Delta S_{(k)}[\Phi; \bar{g}] &
\end{aligned} \tag{2.4.43}$$

where it is understood that regulators act on the second implicit spacetime argument of the inverted second derivatives of the regulated effective action.

Using the formal definition (2.4.17) to write the regulated effective action as $\tilde{\Gamma}_{(k)} = \Gamma_{(k)} + \Delta S_{(k)}$, and the relations[8]:

$$\tilde{\Gamma}^{(2)}_{(k)\mathsf{h}_{\alpha\beta}\mathsf{h}_{\mu\nu}}(x,y) = \Gamma^{(2)}_{(k)\mathsf{h}_{\alpha\beta}\mathsf{h}_{\mu\nu}}(x,y) + \frac{1}{\kappa^2}\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}_{(x)}\right)\delta(x-y) \tag{2.4.44}$$

$$\tilde{\Gamma}^{(2)}_{(k)\bar{\mathsf{c}}_\mu\,\mathsf{c}^\nu}(x,y) = \Gamma^{(2)}_{(k)\bar{\mathsf{c}}_\mu\,\mathsf{c}^\nu}(x,y) \quad + \delta^\mu_\nu\mathcal{R}_{gh(k)}\left(-\bar{\Box}_{(x)}\right)\delta(x-y) \tag{2.4.45}$$

one obtains the standard Wetterich-Morris equation for the effective average action:

$$\begin{aligned}\partial_t\Gamma_{(k)}[\Phi;K;\bar{g}] = \frac{1}{2}\,\mathrm{Tr}_{\bar{g}}\Bigg\{ &\frac{1}{\kappa^2}\partial_t\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}\right)\left(\;\Gamma^{(2)}_{(k)\mathsf{h}_{\alpha\beta}\mathsf{h}_{\mu\nu}} + \frac{1}{\kappa^2}\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}\right)\right)^{-1} \\ &- \delta^\mu_\nu\partial_t\mathcal{R}_{gh(k)}\left(-\bar{\Box}\right)\Bigg[\left(\Gamma^{(2)}_{(k)\bar{\mathsf{c}}_\mu\,\mathsf{c}^\nu} + \delta^\mu_\nu\mathcal{R}_{gh(k)}\left(-\bar{\Box}\right)\right)^{-1} \\ &- \left(\Gamma^{(2)}_{(k)\mathsf{c}^\nu\,\bar{\mathsf{c}}_\mu} - \delta^\mu_\nu\mathcal{R}_{gh(k)}\left(-\bar{\Box}\right)\right)^{-1}\Bigg]\Bigg\}\end{aligned} \tag{2.4.46}$$

In particular, we notice that the term $\partial_t\Delta S_{(k)}[\Phi;\bar{g}]$ in (2.4.43) is precisely canceled. We also notice that, being the regulator term (2.4.1) quadratic and composed of separated terms for the metric fluctuation and the ghosts, its matrix of second derivatives $\Delta S^{(2)}_{(k)\Phi^i\Phi^j}$ is field-independent and block-diagonal, with the two blocks given by the regulators appearing inside the round brackets. We make the following remarks:

1. The effective average action can be now defined as solution of the Wetterich-Morris equation (2.4.46), in the place of the formal definition (2.4.17), and the latter as the fundamental object defining the quantum theory, in the place of the generating functionals: according to the limits (2.4.18), (2.4.19), given the classical theory described by the gauge-fixed action S, the solution of the equation describes a trajectory in theory space which leads to the quantum theory described by the effective action Γ, i.e. $\sim S \xrightarrow{\infty\to k} \Gamma_{(k)} \xrightarrow{k\to 0} \Gamma$. In particular, the classical action does not enter in the derivation of the equation, and we can now lift the considered de Donder gauge-fixed Einstein-Hilbert action to any other gravitational action with an appropriate gauge-fixing, formally serving as initial condition of the flow.

[8] In the formula we dot not write the $1/\sqrt{\bar{g}}$ factor which should multiply the regulators, according to the definition of the symbol $A^{(2)}_{MN}$ for the second derivative. In fact, adopting the compact notation where second derivatives $A^{(2)}_{MN}$ and traces $\mathrm{Tr}_{\bar{g}}$ contain the appropriate $\sqrt{\bar{g}}$ factors, one can verify that it is effectively correct to write and manipulate operators as in flat spacetime where the $\sqrt{\bar{g}}$ factors are absent, as we will do in the following.

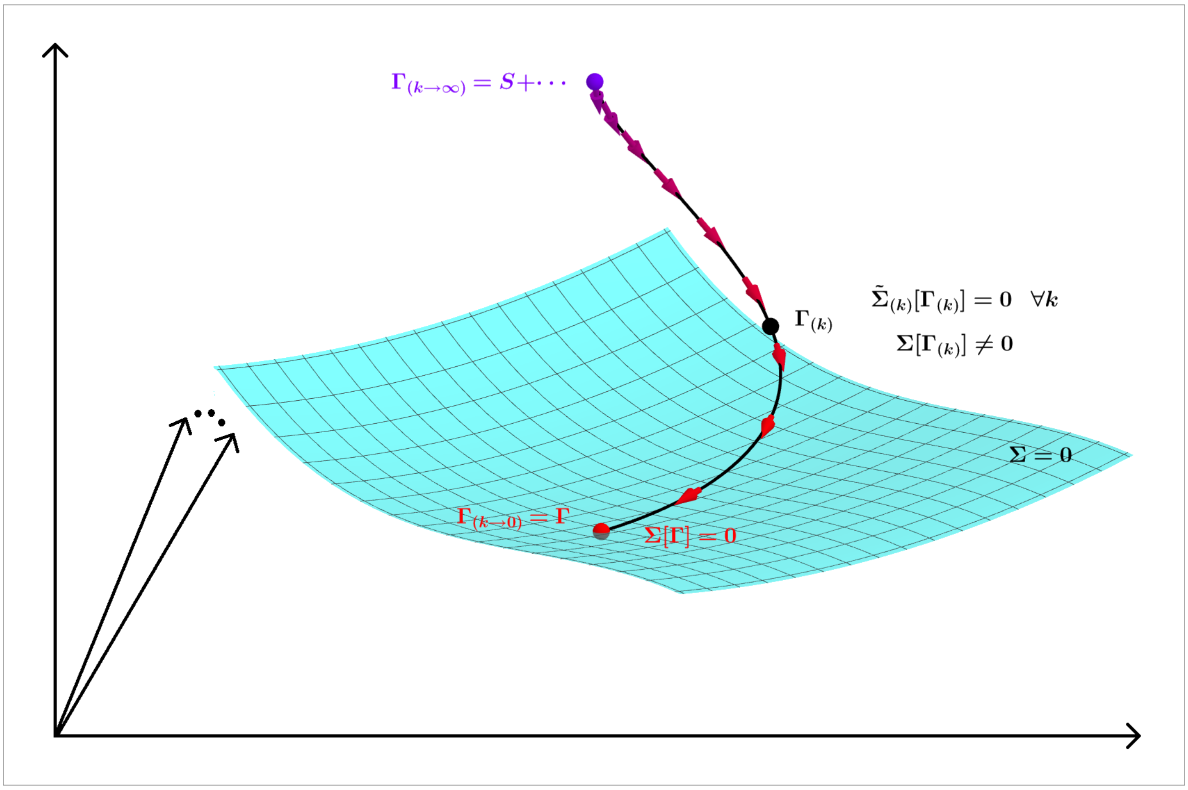

Figure 4: *Pictorial representation of the FRG flow trajectory in theory space described by the Wetterich-Morris equation for the effective average action $\Gamma_{(k)}$, which unfolds from the gauge-fixed action S (plus counter terms) to the quantum effective action Γ. The flow is constrained on a k-dependent hypersurface $\tilde{\Sigma}_k = 0$ (not depicted) embedded in the space of background gauge-invariant operators, but is forced out from the hypersurface of BRST-invariant operators $\Sigma = 0$, due to the BRST symmetry breaking caused by the regularization procedure, with the exception of the last point, i.e. the quantum effective action, where the regulators disappear and BRST symmetry is recovered.*

2. The implicit UV cut-off k_{UV}, deriving from the formal definition (2.4.17), can be safely removed by letting $k_{UV} \to \infty$: formally expressing the traced operator in generalized momentum space and the trace as a sum over generalized momenta, the derivatives of the regulators have, thanks to properties (2.4.5), the qualitative behavior depicted in figure 1; therefore, the dominant contributions to the sum come only from a narrow band of generalized momenta centered around k and those from the UV region are suppressed.

3. The Wetterich-Morris equation (2.4.46) is an exact functional differential equation with two separated sectors, i.e. one for the metric fluctuation and one for the ghosts, both with a 1-loop structure and second order derivatives, similarly to (1.2.33): both are given by the loop formed by tracing the exact regulated propagator, i.e. $\tilde{\Gamma}^{(2)-1}_{(k)\Phi^i\Phi^i} = \tilde{W}^{(2)}_{(k)J_iJ_i} = \langle \phi^i \phi^{+i} \rangle_{JK}$, with an insertion of the correspondent regulator. The equation contains only second derivatives due to the quadratic nature of the regulator (2.4.1), which makes appear inside $\left\langle \partial_t \Delta S_{(k)}[\phi; \bar{g}] \right\rangle_{JK}$ in (2.4.37) only 2-point correlation functions.

4. According to the symmetry properties of the regularized theory stemming from the formal definition of the effective average action (2.4.17), the Wetterich-Morris equation (2.4.46) is expected to generate only background gauge invariant terms satisfying the constraint imposed by the modified Zinn-Justin equation, $\tilde{\Sigma}_k = 0$, so that the trajectory is constrained on the k-dependent hypersurface $\tilde{\Sigma}_k = 0$ along the full flow from the UV to the IR, $\infty \to k \to 0$. In particular, the equation is expected to generate non-BRST-invariant terms which forces the flow outside from the hypersurface $\Sigma = 0$, in which it arrives only in the last point, where the hypersurface $\tilde{\Sigma}_k = 0$ tends to $\Sigma = 0$, and specifically in correspondence of the effective action (figure 4).

2.4.3 Einstein-Hilbert truncation and beta functions

We conclude this second part of the thesis by giving a brief outline of the computation which allows to write the Wetterich-Morris equation (2.4.43) in component form in the Einstein-Hilbert truncation, in which the running couplings are the Newton's and cosmological constants [5]. Firstly, we rewrite the equation performing a first truncation which allows to neglect the dependence on the additional sources, namely we fix the dependence of the effective average action on the additional sources to be linear and given by the source term (2.3.14) appearing in the path integral:

$$\Gamma_{(k)}[\Phi; K; \bar{g}] = \Gamma_{(k)}[\Phi; \bar{g}] + S_{source-K}[\Phi; K; \bar{g}] \tag{2.4.47}$$

which we refer to as linear-K truncation. We rewrite the equation also removing the antisymmetrization in the ghost sector and separating the two traces:

$$\begin{aligned} \partial_t \Gamma_{(k)}[\Phi; \bar{g}] = \frac{1}{2} \mathrm{Tr}_{\bar{g}} \left[\partial_t \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)} \left(-\bar{\Box}\right) \left(\kappa^2 \Gamma^{(2)}_{(k)\mathsf{h}_{\alpha\beta}\mathsf{h}_{\mu\nu}} + \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)} \left(-\bar{\Box}\right) \right)^{-1} \right] \\ - \mathrm{Tr}_{\bar{g}} \left[\delta^\mu_\nu \partial_t \mathcal{R}_{gh(k)} \left(-\bar{\Box}\right) \left(\Gamma^{(2)}_{(k)\bar{\mathsf{c}}_\mu \mathsf{c}^\nu} + \delta^\mu_\nu \mathcal{R}_{gh(k)} \left(-\bar{\Box}\right) \right)^{-1} \right] \end{aligned} \tag{2.4.48}$$

Being the source term scale-independent, there is no term dependent on the additional sources in the left hand side, therefore we can neglect their contribution also in the right hand side, considering directly derivatives of $\Gamma_{(k)}\left[\Phi; \bar{g}\right]$. The Einstein-Hilbert truncation is given by the following ansatz for the effective average action:

$$\Gamma_{(k)}[\mathsf{h}, \mathsf{c}, \bar{\mathsf{c}}; \bar{g}] = \Gamma_{EH(k)}[g = \mathsf{h} + \bar{g}] + \Gamma_{gf(k)}[\mathsf{h}; \bar{g}] + \Gamma_{gh}[\mathsf{h}, \mathsf{c}, \bar{\mathsf{c}}; \bar{g}] \tag{2.4.49}$$

where the first two scale-dependent terms are given by the Einstein-Hilbert action and the de Donder gauge-fixing term (2.3.44) with the Newton's and cosmological constants promoted to running couplings, while the third scale-independent term is given by the de Donder ghost term (2.3.46):

$$\Gamma_{EH(k)}[g] = S_{EH}[g]\Big|_{\substack{G \to G_{(k)} \\ \Lambda \to \Lambda_{(k)}}} \tag{2.4.50}$$

$$\Gamma_{gf(k)}[\mathsf{h}; \bar{g}] = S_{gf}[\mathsf{h}; \bar{g}]\big|_{G \to G_{(k)}} \tag{2.4.51}$$

$$\Gamma_{gh}[\mathsf{h}, \mathsf{c}, \bar{\mathsf{c}}; \bar{g}] = S_{gh}[\mathsf{h}, \mathsf{c}, \bar{\mathsf{c}}; \bar{g}] \tag{2.4.52}$$

In particular, according to this ansatz, we are neglecting a possible evolution in the ghost sector. Substituting the ansatz in the equation, we have then:

$$\begin{aligned}&\partial_t \left(\Gamma_{EH(k)}[g = \mathsf{h} + \bar{g}] + \Gamma_{gf(k)}[\mathsf{h}; \bar{g}]\right) = \\ &\qquad = \frac{1}{2}\,\mathrm{Tr}_{\bar{g}}\left[\partial_t \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}\right)\left(\kappa^2\left(\Gamma_{EH(k)} + \Gamma_{gf(k)}\right)^{(2)}_{\mathsf{h}_{\alpha\beta}\mathsf{h}_{\mu\nu}} + \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}\right)\right)^{-1}\right] \\ &\qquad\quad - \mathrm{Tr}_{\bar{g}}\left[\delta^\mu_\nu \partial_t \mathcal{R}_{gh(k)}\left(-\bar{\Box}\right)\left(S^{(2)}_{gh\,\bar{\mathsf{c}}_\mu\,\mathsf{c}^\nu} + \delta^\mu_\nu \mathcal{R}_{gh(k)}\left(-\bar{\Box}\right)\right)^{-1}\right]\end{aligned} \tag{2.4.53}$$

The Einstein-Hilbert truncation is a particular example of single-metric truncation [5], i.e. the non-gauge-fixing-related sector of the ansatz, in this case the running Einstein-Hilbert action, is taken to be a functional dependent only on the full metric $g_{\mu\nu} = \bar{g}_{\mu\nu} + \mathsf{h}_{\mu\nu}$, i.e. only on the sum of the background metric and the metric fluctuation and not on the two independently. Therefore, one can project the equation on the subspace of single-metric functionals according to the projection rule:

$$A[\mathsf{h}; \bar{g}] \quad \to \quad A[g] \equiv A[\mathsf{h}; \bar{g}]\Big|_{\substack{\mathsf{h}=0\\ \bar{g}=g}} \tag{2.4.54}$$

which in the right hand side of the equation must be applied after having computed the various derivatives:

$$\begin{aligned}&\partial_t \Gamma_{EH(k)}[g] = \\ &\quad = \frac{1}{2}\,\mathrm{Tr}_{g}\left[\partial_t \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\Box\right)\left(\kappa^2\left(\Gamma_{EH(k)} + \Gamma_{gf(k)}\right)^{(2)}_{\mathsf{h}_{\alpha\beta}\mathsf{h}_{\mu\nu}}\Big|_{\substack{\mathsf{h}=0\\ \bar{g}=g}} + \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\Box\right)\right)^{-1}\right] \\ &\qquad - \mathrm{Tr}_{g}\left[\delta^\mu_\nu \partial_t \mathcal{R}_{gh(k)}\left(-\Box\right)\left(S^{(2)}_{gh\,\bar{\mathsf{c}}_\mu\,\mathsf{c}^\nu}\Big|_{\substack{\mathsf{h}=0\\ \bar{g}=g}} + \delta^\mu_\nu \mathcal{R}_{gh(k)}\left(-\Box\right)\right)^{-1}\right]\end{aligned} \tag{2.4.55}$$

In particular, the de Donder gauge-fixing term is canceled in the left hand side of the equation, which thus reads:

$$\mathrm{LHS}_{(k)}(R) = \partial_t \Gamma_{EH(k)}[g] = \frac{2}{\kappa^2}\int d^D x \sqrt{g}\left[-R\partial_t Z_{N(k)} + 2\partial_t (Z_{N(k)}\Lambda_{(k)})\right] \tag{2.4.56}$$

where the dimensionless running coupling $Z_{N(k)}$ is defined as:

$$Z_{N(k)} = \frac{G}{G_{(k)}} \tag{2.4.57}$$

The objective is now find the contributions in the right hand side at order $O(1)$ and $O(R)$, i.e. the terms proportional to $\int d^D x\sqrt{g}$ and $\int d^D x\sqrt{g}\,R$, in order to project the equation and read off the expressions for the derivatives $\partial_t Z_{N(k)}$, $\partial_t(Z_{N(k)}\Lambda_{(k)})$. For this purpose one can use an arbitrary family of metrics which is general enough to identify the terms $\int d^D x\sqrt{g}$ and $\int d^D x\sqrt{g}\,R$ and to distinguish them from the higher-order ones.

According to [5], we consider a maximally symmetric space, whose curvature tensors are characterized by the relations:

$$R_{\alpha\beta\mu\nu} = \frac{1}{D(D-1)} \left(g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu}\right) R \tag{2.4.58}$$

$$R_{\mu\nu} = \frac{1}{D} g_{\mu\nu} R \tag{2.4.59}$$

The right hand side receives a contribution from the metric fluctuation trace sector and one from the ghosts trace sector:

$$\mathrm{RHS}_{(k)}(R) = \mathrm{RHS}_{gr(k)}(R) + \mathrm{RHS}_{gh(k)}(R) \tag{2.4.60}$$

As far as the metric fluctuation trace sector is considered, the second derivative $\left(\Gamma_{EH(k)} + \Gamma_{gf(k)}\right)^{(2)}_{\mathsf{h}_{\alpha\beta}\mathsf{h}_{\mu\nu}} |_{\substack{\mathsf{h}=0 \\ \bar{g}=g}}$ receives a contribution only from the quadratic part of the running Einstein-Hilbert action, given by (2.2.31) with running couplings, with the de Donder term multiplied by $(1+1/\alpha)$ according to the de Donder gauge-fixing; in particular, we choose the gauge-fixing parameter to be $\alpha = -1$ to cancel the de Donder term:

$$\Gamma_{EH,2(k)}[g] + \Gamma_{gf(k)}[\mathsf{h};\bar{g}] = \frac{Z_{N(k)}}{\kappa^2} \int d^D x \sqrt{\bar{g}}\, \frac{1}{2} h_{\alpha\beta} \left(-\bar{K}^{\alpha\beta,\mu\nu} \bar{\Box} + \bar{O}_2^{\alpha\beta,\mu\nu}\right) h_{\mu\nu} \tag{2.4.61}$$

and the second functional derivative gives[9]:

$$\kappa^2 \left(\Gamma_{EH(k)} + \Gamma_{gf(k)}\right)^{(2)}_{\mathsf{h}_{\alpha\beta}\mathsf{h}_{\mu\nu}} \bigg|_{\substack{\mathsf{h}=0 \\ \bar{g}=g}} = Z_{N(k)} \left(-K^{\alpha\beta,\mu\nu} \Box + O_2^{\alpha\beta,\mu\nu}\right) \tag{2.4.62}$$

However, in order to easily perform the inversion and compute the traces, it is convenient to partially diagonalize the quadratic operator by rewriting the metric fluctuation in terms of its trace and traceless part:

$$\mathring{\mathsf{h}}_{\mu\nu} = \mathsf{h}_{\mu\nu} - \frac{1}{D} \mathsf{h} \bar{g}_{\mu\nu} \tag{2.4.63}$$

One can see that in general the quadratic term is rewritten as:

$$\begin{aligned}
&\left[\Gamma_{EH(k)}[g] + \Gamma_{gf(k)}[\mathsf{h};\bar{g}]\right]_{h^2-term} = \\
&\quad = \frac{Z_{N(k)}}{\kappa^2} \int d^D x \sqrt{\bar{g}} \left[\frac{1}{2} \mathring{\mathsf{h}}_{\mu\nu} \left(-\bar{\Box} - 2\Lambda_{(k)} + \bar{R}\right) \mathring{\mathsf{h}}^{\mu\nu} \right. \\
&\qquad\qquad - \frac{D-2}{4D} \mathsf{h} \left(-\bar{\Box} - 2\Lambda_{(k)} + \frac{D-4}{D} \bar{R}\right) \mathsf{h} \\
&\qquad\qquad \left. + \bar{R}_{\mu\nu} \mathring{\mathsf{h}}^{\nu\rho} \mathring{\mathsf{h}}^{\mu}{}_{\rho} - \bar{R}_{\alpha\beta\mu\nu} \mathring{\mathsf{h}}^{\alpha\mu} \mathring{\mathsf{h}}^{\beta\nu} + \frac{D-4}{D} h \bar{R}_{\mu\nu} \mathring{\mathsf{h}}^{\mu\nu} \right]
\end{aligned} \tag{2.4.64}$$

[9] As noticed in 8, here and in the following we do not write the $\sqrt{g}$ factors working in compact notation.

and in a maximally symmetric spacetime it becomes:

$$\begin{aligned}\left[\Gamma_{EH(k)}[g]+\Gamma_{gf(k)}[\mathsf{h};\bar{g}]\right]_{\mathsf{h}^2-term} = \\ = \frac{Z_{N(k)}}{\kappa^2}\int d^D x\sqrt{\bar{g}}\,\frac{1}{2}\Bigg[\mathring{\mathsf{h}}_{\mu\nu}\left(-\bar{\Box}-2\Lambda_{(k)}+\frac{(D-4)(D+1)}{D(D-1)}\bar{R}\right)\mathring{\mathsf{h}}^{\mu\nu} \\ -\frac{D-2}{2D}\mathsf{h}\left(-\bar{\Box}-2\Lambda_{(k)}+\frac{D-4}{D}\bar{R}\right)\mathsf{h}\Bigg]\end{aligned} \tag{2.4.65}$$

The trace in the right hand side of the flow equation can be now rewritten in a sum of a trace over traceless symmetric metric fluctuations, i.e.traceless symmetric tensors, $\mathrm{Tr}_{g\,T}$ and one over the traces, i.e. scalars, $\mathrm{Tr}_{g\,S}$, respectively containing the operators:

$$\kappa^2\left(\Gamma_{EH(k)}+\Gamma_{gf(k)}\right)^{(2)}_{\mathring{\mathsf{h}}_{\alpha\beta}\mathring{\mathsf{h}}_{\mu\nu}}\Big|_{\substack{\mathsf{h}=0\\ \bar{g}=g}} = Z_{N(k)}\left(-\Box-2\Lambda_{(k)}+\frac{(D-4)(D+1)}{D(D-1)}R\right)\delta_T^{\alpha\beta,\mu\nu} \tag{2.4.66}$$

$$\kappa^2\left(\Gamma_{EH(k)}+\Gamma_{gf(k)}\right)^{(2)}_{\mathsf{hh}}\Big|_{\substack{\mathsf{h}=0\\ \bar{g}=g}} = -Z_{N(k)}\frac{D-2}{2D}\left(-\Box-2\Lambda_{(k)}+\frac{D-4}{D}R\right) \tag{2.4.67}$$

where in the first $\delta_T^{\alpha\beta,\mu\nu}$ is the identity in the space of traceless symmetric tensors. We now specify the regulator; as anticipated in subsection 2.4.1 we consider the structure:

$$\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(p^2\right) = \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\mathcal{R}_{gr(k)}\left(p^2\right) \tag{2.4.68}$$

with the regulating part given by a standard regulator:

$$\mathcal{R}_{gr(k)}\left(p^2\right) = k^2\mathcal{R}_0\left(\frac{p^2}{k^2}\right) \tag{2.4.69}$$

We pick the tensorial prefactor to be:

$$\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)} = Z_{N(k)}\left[\left(\delta^{\alpha\beta,\mu\nu}-\frac{\bar{g}^{\alpha\beta}\bar{g}^{\mu\nu}}{D}\right)-\frac{D-2}{2}\frac{\bar{g}^{\alpha\beta}\bar{g}^{\mu\nu}}{D}\right] \tag{2.4.70}$$

where in particular $P_T^{\alpha\beta,\mu\nu}=\delta^{\alpha\beta,\mu\nu}-\bar{g}^{\alpha\beta}\bar{g}^{\mu\nu}/D$ and $P_S^{\alpha\beta,\mu\nu}=\bar{g}^{\alpha\beta}\bar{g}^{\mu\nu}/D$ are respectively the two projectors in the spaces of traceless symmetric tensors and traces. Thanks to this choice the regulator is rewritten in terms of the traceless metric fluctuation and its trace as:

$$\Delta S_{gr(k)}[\mathsf{h};\bar{g}] = \frac{Z_{N(k)}}{2\kappa^2}\int d^D x\sqrt{\bar{g}}\left[\mathring{\mathsf{h}}_{\mu\nu}\mathcal{R}_{gr(k)}\left(-\bar{\Box}\right)\mathring{\mathsf{h}}^{\mu\nu}-\frac{D-2}{2D}h\mathcal{R}_{gr(k)}\left(-\bar{\Box}\right)h\right] \tag{2.4.71}$$

In particular, the two sectors have the same multiplicative prefactors as the second derivatives in (2.4.66). Therefore, inside the traces $\mathrm{Tr}_{g\,T}$ and $\mathrm{Tr}_{g\,S}$ there appear respectively the regulators $Z_{N(k)}\mathcal{R}_{gr(k)}\left(-\Box\right)\delta_T^{\alpha\beta,\mu\nu}$ and $-(D-2)/(2D)Z_{N(k)}\mathcal{R}_{gr(k)}\left(-\Box\right)$ which sum correctly with the second derivatives in (2.4.66). Finally, splitting also the derivative

$\partial_t \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}(-\Box)$ in the two sectors, the contribution of the metric fluctuation sector in the right hand side of the flow equation is rewritten as:

$$\begin{aligned}\mathrm{RHS}_{gr(k)}(R) = \mathrm{Tr}_{g\,T}\Bigg[&\frac{1}{2Z_{N(k)}}\partial_t\left(Z_{N(k)}\mathcal{R}_{gr(k)}(-\Box)\right)\cdot\\&\cdot\left(-\Box+\mathcal{R}_{gr(k)}(-\Box)-2\Lambda_{(k)}+\frac{(D-4)(D+1)}{D(D-1)}R\right)^{-1}\Bigg]\\+\mathrm{Tr}_{g\,S}\Bigg[&\frac{1}{2Z_{N(k)}}\partial_t\left(Z_{N(k)}\mathcal{R}_{gr(k)}(-\Box)\right)\cdot\\&\cdot\left(-\Box+\mathcal{R}_{gr(k)}(-\Box)-2\Lambda_{(k)}+\frac{D-4}{D}R\right)^{-1}\Bigg]\end{aligned} \tag{2.4.72}$$

In the traces it is understood an identity in the corresponding space, giving the discrete part of the trace, equal to number of independent components of the objects in the space, $\mathrm{tr}_T(\mathbb{1}) = \delta_{T\mu\nu}^{\mu\nu} = D(D+1)/2-1$ for traceless symmetric tensors and simply $\mathrm{tr}_S(\mathbb{1}) = 1$ for scalars.
As far as the ghosts trace sector is considered, the second derivative $S^{(2)}_{gh\,\bar{\mathsf{c}}_\mu\,\mathsf{c}^\nu}\big|_{\substack{\mathsf{h}=0\\ \bar{g}=g}}$ receives a contribution only from the quadratic part of the de Donder ghost term:

$$[S_{gh}[\mathsf{h},\mathsf{c},\bar{\mathsf{c}};\bar{g}]]_{\bar{\mathsf{c}}\mathsf{c}-term} = \int d^Dx\sqrt{\bar{g}}\,\bar{\mathsf{c}}_\mu\left(-\bar{\Box}\delta^\mu_\nu - \bar{R}^\mu_\nu\right)\mathsf{c}^\nu \tag{2.4.73}$$

which in a maximally symmetric spacetime reads:

$$[S_{gh}[\mathsf{h},\mathsf{c},\bar{\mathsf{c}};\bar{g}]]_{\bar{\mathsf{c}}\mathsf{c}-term} = \int d^Dx\sqrt{\bar{g}}\,\bar{\mathsf{c}}_\mu\left(-\bar{\Box}-\frac{\bar{R}}{D}\right)\mathsf{c}^\mu \tag{2.4.74}$$

Therefore:

$$S^{(2)}_{gh\,\bar{\mathsf{c}}_\mu\,\mathsf{c}^\nu}\Big|_{\substack{\mathsf{h}=0\\ \bar{g}=g}} = \left(-\Box-\frac{R}{D}\right)\delta^\mu_\nu \tag{2.4.75}$$

As regulator we consider the one with the same shape function used for the metric fluctuation:

$$\mathcal{R}_{gh(k)}\left(p^2\right) = k^2\mathcal{R}_0\left(\frac{p^2}{k^2}\right) \tag{2.4.76}$$

and the contribution of the ghosts sector in the right hand side of the flow equation is rewritten as:

$$\mathrm{RHS}_{gh(k)}(R) = -\mathrm{Tr}_{g\,V}\left[\partial_t\mathcal{R}_{gh(k)}(-\Box)\left(-\Box+\mathcal{R}_{gh(k)}(-\Box)-\frac{R}{D}\right)^{-1}\right] \tag{2.4.77}$$

where it is again understood an identity, in this case in the space of vectors, which gives the discrete part of the trace $\mathrm{tr}_T(\mathbb{1}) = \delta^\mu_\mu = D$.

The final step to extract the beta functions consists in using the operatorial Taylor formula for the inverse operator:

$$(1+\mathcal{O})^{-1} = \sum_{n=0}^{\infty}(-1)^n\mathcal{O}^n \tag{2.4.78}$$

to expand in powers of the Ricci scalar the arguments of the traces in the right hand side of the flow equation:

$$\mathrm{RHS}_{(k)}(R) = \mathrm{Tr}_g\left[W_1(-\square)\right] + \mathrm{Tr}_g\left[W_2(-\square)R\right] + O(R^2) \tag{2.4.79}$$

and then evaluating the traces using heat kernel methods, which allow to compute traces of generic functions of the covariant laplacian. In particular, up to order $O(R)$, one has the expansions [5]:

$$\mathrm{Tr}_g\left[W(-\square)\right] = \frac{\mathrm{tr}(\mathbb{1})}{(4\pi)^{D/2}}\left\{Q_{D/2}[W]\int d^Dx\sqrt{g} + \frac{1}{6}Q_{D/2-1}[W]\int d^Dx\sqrt{g}\,R\right\} + O(R^2) \tag{2.4.80}$$

$$\mathrm{Tr}_g\left[W(-\square)R\right] = \frac{\mathrm{tr}(\mathbb{1})}{(4\pi)^{D/2}}Q_{D/2}[W]\int d^Dx\sqrt{g}R + O(R^2) \tag{2.4.81}$$

where the Q-functionals are given by:

$$Q_0[W] = W(0) \tag{2.4.82}$$

$$Q_{n>0}[W] = \frac{1}{\Gamma(n)}\int_0^\infty dz\, z^{n-1}W(z) \tag{2.4.83}$$

$$Q_{n<0}[W] = \left(-\frac{d}{dz}\right)^{|n|} W(z)\bigg|_{z=0} \tag{2.4.84}$$

where $\Gamma(n)$ is Euler's gamma function. Finally, the derivatives $\partial_t Z_{N(k)}$, $\partial_t(Z_{N(k)}\Lambda_{(k)})$ are found by equating left and right hand side of the flow equation:

$$\partial_t(Z_{N(k)}\Lambda_{(k)}) = +\frac{\kappa^2}{4}\,\mathrm{RHS}_{gr(k)}(R)|_{O(1)} + \frac{\kappa^2}{4}\,\mathrm{RHS}_{gh(k)}(R)|_{O(1)} \tag{2.4.85}$$

$$\partial_t Z_{N(k)} = -\frac{\kappa^2}{2}\,\mathrm{RHS}_{gr(k)}(R)|_{O(R)} - \frac{\kappa^2}{2}\,\mathrm{RHS}_{gh(k)}(R)|_{O(R)} \tag{2.4.86}$$

We limit to present the result of the computation, which can be expressed for a generic regulator shape $\mathcal{R}_0(x)$ in terms of the threshold functions:

$$\Phi_n^p(w) = \frac{1}{\Gamma(n)}\int_0^\infty dx\, x^{n-1}\frac{\mathcal{R}_0(x) - x\mathcal{R}_0'(x)}{[x+\mathcal{R}_0(x)+w]^p} \tag{2.4.87}$$

$$\tilde{\Phi}_n^p(w) = \frac{1}{\Gamma(n)}\int_0^\infty dx\, x^{n-1}\frac{\mathcal{R}_0(x)}{[x+\mathcal{R}_0(x)+w]^p} \tag{2.4.88}$$

In particular one has:

$$\begin{aligned}\mathrm{RHS}_{gr(k)}(R)|_{O(1)} &= \\ &= \frac{1}{4}\frac{1}{(4\pi)^{D/2}}\,k^D\left[2D(D+1)\Phi^1_{D/2}\left(-\tfrac{2\Lambda_{(k)}}{k^2}\right) - D(D+1)\eta_N\tilde{\Phi}^1_{D/2}\left(-\tfrac{2\Lambda_{(k)}}{k^2}\right)\right]\end{aligned} \tag{2.4.89}$$

$$\mathrm{RHS}_{gh(k)}(R)|_{O(1)} = -\frac{2D}{(4\pi)^{D/2}}k^D\Phi^1_{D/2}(0) \tag{2.4.90}$$

and:

$$\begin{aligned}
&\mathrm{RHS}_{gr(k)}(R)|_{O(R)} = \\
&= \frac{\kappa^2}{12}\frac{1}{(4\pi)^{D/2}}\, k^{D-2}\Big\{ D(D+1)\left[\Phi^1_{D/2-1}\left(-\tfrac{2\Lambda_{(k)}}{k^2}\right) - \frac{1}{2}\eta_N \tilde{\Phi}^1_{D/2-1}\left(-\tfrac{2\Lambda_{(k)}}{k^2}\right)\right] \\
&\qquad - 6D(D-1)\left[\Phi^2_{D/2}\left(-\tfrac{2\Lambda_{(k)}}{k^2}\right) - \frac{1}{2}\eta_N \tilde{\Phi}^2_{D/2}\left(-\tfrac{2\Lambda_{(k)}}{k^2}\right)\right]\Big\}
\end{aligned} \tag{2.4.91}$$

$$\mathrm{RHS}_{gh(k)}(R)|_{O(R)} = -\frac{2D}{(4\pi)^{D/2}} k^{D-2}\left(\frac{1}{6}\Phi^1_{D/2-1}(0) + \frac{1}{D}\Phi^2_{D/2}(0)\right) \tag{2.4.92}$$

where η_N, the anomalous dimension of the running Newton's constant, is defined as:

$$\eta_N = -\partial_t \log Z_{N(k)} \tag{2.4.93}$$

In particular the terms coming from the metric fluctuation trace sector of the right hand side of the flow equation, which contains the running cosmological constant, give threshold functions computed in $-2\Lambda_{(k)}/k^2$, while the terms coming from the ghosts trace sector, which does not contain the running cosmological constant, give the threshold functions computed in zero. Finally, the derivatives $\partial_t Z_{N(k)}$, $\partial_t(Z_{N(k)}\Lambda_{(k)})$ in the Einstein-Hilbert truncation are given by [5][6]:

$$\begin{cases}
\partial_t(Z_{N(k)}\Lambda_{(k)}) = \dfrac{\kappa^2}{16}\dfrac{1}{(4\pi)^{D/2}}\, k^{D}\Big[2D(D+1)\Phi^1_{D/2}\left(-\tfrac{2\Lambda_{(k)}}{k^2}\right) - D(D+1)\eta_N\tilde{\Phi}^1_{D/2}\left(-\tfrac{2\Lambda_{(k)}}{k^2}\right) \\
\qquad - 8D\Phi^1_{D/2}(0)\Big] \\
\partial_t Z_{N(k)} = -\dfrac{\kappa^2}{24}\dfrac{1}{(4\pi)^{D/2}}\, k^{D-2}\Big\{ D(D+1)\left[\Phi^1_{D/2-1}\left(-\tfrac{2\Lambda_{(k)}}{k^2}\right) - \dfrac{1}{2}\eta_N \tilde{\Phi}^1_{D/2-1}\left(-\tfrac{2\Lambda_{(k)}}{k^2}\right)\right] \\
\qquad - 6D(D-1)\left[\Phi^2_{D/2}\left(-\tfrac{2\Lambda_{(k)}}{k^2}\right) - \dfrac{1}{2}\eta_N \tilde{\Phi}^2_{D/2}\left(-\tfrac{2\Lambda_{(k)}}{k^2}\right)\right] \\
\qquad - 4D\Phi^1_{D/2-1}(0) - 24\Phi^2_{D/2}(0)\Big\}
\end{cases} \tag{2.4.94}$$

For instance, using the Litim optimized regulator [29]:

$$\mathcal{R}^{opt}_{(k)}(p^2) = (k^2 - p^2)\theta(k^2 - p^2) \tag{2.4.95}$$

for which:

$$\mathcal{R}^{opt}_0(x) = (1-x)\theta(1-x) \tag{2.4.96}$$

$$\mathcal{R}^{opt\,\prime}_0(x) = -\theta(1-x) - (1-x)\delta(1-x) \tag{2.4.97}$$

where $\theta(x)$ is the Heaviside step function, one has:

$$\Phi^p_n(w) = \frac{1}{\Gamma(n+1)}\frac{1}{(1+w)^p} \tag{2.4.98}$$

$$\tilde{\Phi}_n^p(w) = \frac{1}{\Gamma(n+2)} \frac{1}{(1+w)^p} \tag{2.4.99}$$

The beta functions for the Newton's and cosmological constant can be now derived from (2.4.94) and are typically expressed in terms of their dimensionless expressions:

$$g_{(k)} = k^{D-2} G_{(k)} \tag{2.4.100}$$

$$\lambda_{(k)} = k^{-2} \Lambda_{(k)} \tag{2.4.101}$$

Taking the derivatives of these expressions and using the definition of the anomalous dimension, one finds:

$$\partial_t g_{(k)} = \left[D - 2 + \eta_N(g_{(k)}, \lambda_{(k)})\right] g_{(k)} \tag{2.4.102}$$

$$\partial_t \lambda_{(k)} = -\left[2 - \eta_N(g_{(k)}, \lambda_{(k)})\right] \lambda_{(k)} + \frac{32\pi}{\kappa^2} k^{-D} g_{(k)} \partial_t (Z_{N(k)} \Lambda_{(k)}) \tag{2.4.103}$$

Where $\partial_t(Z_{N(k)}\Lambda_{(k)})$ can be substituted directly from (2.4.94) while the anomalous dimension can be expressed as $\eta_N = \eta_N(g_{(k)}, \lambda_{(k)})$ using the equation for $\partial_t Z_{N(k)}$, which can be rewritten as:

$$\eta_N(k) = g_{(k)} B_1(\lambda_{(k)}) + \eta_N(k) g_{(k)} B_2(\lambda_{(k)}) \tag{2.4.104}$$

with:

$$\begin{aligned} B_1(\lambda_{(k)}) = \frac{1}{3}(4\pi)^{1-D/2} \Big[& D(D+1)\Phi^1_{D/2-1}\left(-2\lambda_{(k)}\right) - 6D(D-1)\Phi^2_{D/2}\left(-2\lambda_{(k)}\right) \\ & - 4D\Phi^1_{D/2-1}(0) - 24\Phi^2_{D/2}(0) \Big] \end{aligned} \tag{2.4.105}$$

$$B_2(\lambda_{(k)}) = -\frac{1}{6}(4\pi)^{1-D/2} \Big[D(D+1)\tilde{\Phi}^1_{D/2-1}\left(-2\lambda_{(k)}\right) - 6D(D-1)\tilde{\Phi}^2_{D/2}\left(-2\lambda_{(k)}\right) \Big] \tag{2.4.106}$$

So, the anomalous dimension has no explicit k-dependence and is given by the expression:

$$\eta_N(g_{(k)}, \lambda_{(k)}) = \frac{g_{(k)} B_1(\lambda_{(k)})}{1 - g_{(k)} B_2(\lambda_{(k)})} \tag{2.4.107}$$

Substituting one finds the autonomous system of FRG equations for the dimensionless running Newton's and cosmological constants:

$$\begin{cases} \partial_t g_{(k)} = \beta_g(g_{(k)}, \lambda_{(k)}) \\ \partial_t \lambda_{(k)} = \beta_\lambda(g_{(k)}, \lambda_{(k)}) \end{cases} \tag{2.4.108}$$

with the beta functions given by:

$$\beta_g(g_{(k)}, \lambda_{(k)}) = \left[D - 2 + \eta_N(g_{(k)}, \lambda_{(k)})\right] g_{(k)} \tag{2.4.109}$$

$$\begin{aligned}\beta_\lambda(g_{(k)}, \lambda_{(k)}) = &- \left[2 - \eta_N(g_{(k)}, \lambda_{(k)})\right] \lambda_{(k)} \\ &+ \frac{1}{2}(4\pi)^{1-D/2} g_{(k)} \Big[2D(D+1)\Phi^1_{D/2}\left(-2\lambda_{(k)}\right) \\ &\qquad - D(D+1)\eta_N(g_{(k)}, \lambda_{(k)})\tilde{\Phi}^1_{D/2}\left(-2\lambda_{(k)}\right) \\ &\qquad - 8D\Phi^1_{D/2}(0)\Big]\end{aligned} \tag{2.4.110}$$

Part 3
BRST-invariant FRG flow in Quantum Einstein Gravity

The third part of the thesis is devoted to the presentation of the main results of the work: the development of a formalism to implement FRG methods in a QEG theory preserving explicitly BRST symmetry and its use in the construction of a BRST-compatible Wetterich-Morris equation.
The part is divided in three sections. In the first section we briefly present the idea at the base of the formalism, discussing how the central problem of introducing quadratic regulator terms in the gauge-fixed action without breaking the explicit BRST symmetry can be solved by combining the regularization and gauge-fixing procedures in a single step. In the second section we implement the idea thanks to a non-standard choice for the gauge-fixing term and the gauge-fixing function which allows to introduce quadratic terms in the gauge-fixed action without breaking the explicit BRST symmetry. In the third section we use the gauge-fixing structure constructed in the second section as a template to regulate the theory in an explicitly BRST-invariant manner and we derive the Wetterich-Morris equation describing the FRG flow of the theory, proving its compatibility with the Zinn-Justin equation representing the constraint imposed by BRST symmetry and presenting its component form within the Einstein-Hilbert truncation.

3.1 Premises

3.1.1 Intuitive idea and outline

The reason for which the standard FRG flow for a QEG theory fails to explicitly preserve BRST symmetry is that the standard quadratic regulator term is not BRST-invariant. Therefore, conceptually, the central objective to achieve in order to construct a BRST-invariant FRG flow is finding a way to introduce in the gauge-fixed action quadratic regulator terms for the metric fluctuation and the ghosts without breaking the explicit BRST symmetry.
The method which we are going to use to obtain this result is inspired from the one introduced and developed in [6] to construct a BRST-invariant FRG flow for the Yang-Mills theory. Our work will share the same backbone structure and will consist in a generalization of the formalism to theories of gravity. The method is based on the observation regarding BRST symmetry made in subsection 2.2.3: the gauge-fixed action

enjoys BRST symmetry independently of the specific choice for the gauge-fixing term and the gauge-fixing function, since in any case the sum of the gauge-fixing term and the ghost term is BRST-invariant. This suggests the key idea at the base of the method. As we saw in subsection 2.4.1, the standard procedure consists in first gauge-fixing and then regulating the action by manually adding a quadratic regulator term:

$$\begin{gathered} S_{EH}[h;\bar{g}] \\ \downarrow \\ S[h,c,\bar{c};\bar{g}] = S_{EH}[h;\bar{g}] + S_{gf}[h;\bar{g}] + S_{gh}[h,c,\bar{c};\bar{g}] \\ \downarrow \\ \tilde{S}_{(k)}[h,c,\bar{c};\bar{g}] = S[h,c,\bar{c};\bar{g}] + \Delta S_{(k)}[h,c,\bar{c};\bar{g}] \end{gathered} \tag{3.1.1}$$

In this way the explicit BRST symmetry is inevitably broken, since quadratic terms in the metric fluctuation and ghosts cannot be, alone, BRST-invariant:

$$\delta_\theta \Delta S_{(k)}[h,c,\bar{c};\bar{g}] \neq 0 \qquad \Longrightarrow \qquad \delta_\theta \tilde{S}_{(k)}[h,c,\bar{c};\bar{g}] \neq 0 \tag{3.1.2}$$

Consider instead, intuitively, the possibility of gauge-fixing and regulating the theory in a single step, by introducing the quadratic regulator terms directly as part of the gauge-fixing sector, thanks to a suitable choice for the form of the gauge-fixing term and the gauge-fixing function:

$$\begin{gathered} S_{EH}[h;\bar{g}] \\ \downarrow \\ \tilde{S}_{(k)}[h,c,\bar{c};\bar{g}] = S_{EH}[h;\bar{g}] + \underbrace{S_{gf(k)}[h;\bar{g}] + S_{gh(k)}[h,c,\bar{c};\bar{g}]}_{\cdots + \Delta S_{(k)}[h,c,\bar{c};\bar{g}] + \cdots} \end{gathered} \tag{3.1.3}$$

In this way we are effectively introducing the necessary regulator terms to construct an FRG flow, but now without breaking the explicit BRST symmetry of the theory, since, technically, we are only performing a particular gauge-fixing and therefore the sum of the terms introduced is BRST-invariant by construction:

$$\delta_\theta \left(S_{gf(k)}[h;\bar{g}] + S_{gh(k)}[h,c,\bar{c};\bar{g}] \right) = 0 \qquad \Longrightarrow \qquad \delta_\theta \tilde{S}_{(k)}[h,c,\bar{c};\bar{g}] = 0 \tag{3.1.4}$$

In particular, there will be introduced, automatically, also the additional non-standard terms necessary to balance the BRST-symmetry-breaking of the standard quadratic regulator terms and recover BRST symmetry.

As far as the practical implementation of this regulating-gauge-fixing procedure is concerned, following [6], it is interesting to separate the process in two conceptual steps. The quadratic regulator terms can be seen as quadratic mass terms where the mass parameters have been substituted by regulators. Therefore, the procedure outlined above is also a way to introduce in the gauge-fixed action quadratic mass terms for the metric fluctuation and the ghosts as part of the gauge-fixing sector, and therefore without affecting BRST symmetry, which is an interesting result on its own. In light of this observation, we will articulate the process in the following way:

1. Massive gauge-fixing: Firstly, in section 3.2, we will make use of a non-standard gauge-fixing to introduce in the action quadratic mass terms for the metric fluctuation and the ghosts:

$$S_{m_{gr}}[h;\bar{g}] = \frac{1}{\kappa^2}\int d^D x\sqrt{\bar{g}}\,\frac{m_{gr}^2}{2}\,h_{\alpha\beta}\bar{M}^{\alpha\beta,\mu\nu}h_{\mu\nu} \tag{3.1.5}$$

$$S_{m_{gh}}[c,\bar{c};\bar{g}] = \int d^D x\sqrt{\bar{g}}\,m_{gh}^2\bar{c}_\mu c^\mu \tag{3.1.6}$$

as part of the gauge-fixing sector, and therefore without affecting BRST symmetry. $\bar{M}^{\alpha\beta,\mu\nu}$ will be a generic structure tensor symmetric under the exchanges $\alpha \leftrightarrow \beta, \mu \leftrightarrow \nu, \{\alpha\beta\} \leftrightarrow \{\mu\nu\}$.

2. Promotion of mass parameters to regulators: Secondly, in section 3.3, we will use the gauge-fixing structure constructed as a template to regulate the theory in an explicitly BRST-invariant manner, by promoting the mass parameters to regulators:

$$m_{gr}^2 \quad \rightarrow \quad \mathcal{R}_{gr(k)}\left(-\bar{\Box}\right) \tag{3.1.7}$$

$$m_{gh}^2 \quad \rightarrow \quad \mathcal{R}_{gh(k)}\left(-\bar{\Box}\right) \tag{3.1.8}$$

At this point we will apply the standard FRG techniques to derive the Wetterich-Morris equation describing the FRG flow of the BRST-symmetrically-regulated theory and discuss its properties.

3.2 Faddeev-Popov quantization with massive gauge-fixing

In this section we cover the first of the two steps outlined above, presenting how the desired result can be achieved via a non-standard gauge-fixing, namely linear in the gauge-fixing function (instead of quadratic, as the standard one) and a gauge-fixing function quadratic in the metric fluctuation (instead of linear, as the standard one). We will repeat the same steps of section 2.3 in order to highlight the differences with the standard gauge-fixing.

3.2.1 Linear gauge-fixing term

Consider the general result for the gauge-fixed action (2.2.81) resulting from the Faddeev-Popov method. According to the strategy, we need to engineer a gauge-fixing term and a gauge-fixing function such that the gauge-fixed action contains, among the various terms introduced in the gauge-fixing sector, quadratic mass terms of the type (3.1.5) and (3.1.6). We have two essential difficulties to overcome:

1. The mass term for the ghosts must be originated via the ghost term, whose dependence on the gauge-fixing function, contrary to the gauge-fixing term, is not arbitrary,

but necessarily linear in its functional derivative, as given by (2.2.77). A quadratic antighost-ghost term can be thus obtained with a gauge-fixing function containing a linear piece in the metric fluctuation and proportional to m_{gh}^2; however, due to the Lie derivative $\pounds_c(\bar{g}_{\mu\nu}+h_{\mu\nu})$, the originated terms may not contain the correct contraction $\bar{c}_\mu c^\mu$, due to the presence of derivatives and contractions with the background metric. Consider for instance the de Donder gauge-fixing function (2.3.43) seen in subsection 2.3.3, a term with the correct contraction is indeed generated, i.e the kinetic term for the ghosts $-\bar{c}_\mu \bar{\Box} c^\mu$, but it contains also derivatives acting on the ghost.

2. The mass term for the metric fluctuation must be originated via the gauge-fixing term and, given its arbitrary dependence on the gauge-fixing function, it can be obtained in various ways; however, due to the necessary linear piece in the gauge-fixing function for the ghosts mass term, we could incur in the generation of unwanted terms proportional to mixed products of mass parameters. Consider for instance the standard gauge-fixing term (2.3.1), given the quadratic dependence on the gauge-fixing function, a mass term for the metric fluctuation can be obtained with a gauge-fixing function containing a linear piece in the metric fluctuation and proportional to m_{gr}, but, due to the also necessary linear piece for the ghosts mass term, computing the square we obtain mixed terms $m_{gr}m_{gh}^2$.

Following [6], the first problem can be solved by considering a linear piece in the gauge-fixing function of the type $m_{gh}^2\mathcal{K}h$, where $\mathcal{K}$ is a suitable operator engineered to remove unwanted derivatives of the ghost; we will give in subsection 3.2.3 the specific expression. The second problem can be instead conveniently solved by considering a non-standard gauge-fixing term linear in the gauge-fixing function, so that, accommodating separately in the function the necessary contributions to obtain the two mass terms, we are safe from any kind of mixing between the two when computing the gauge-fixing term; as a consequence of this choice, the piece for the metric fluctuation mass term in the gauge-fixing function must now be quadratic. Schematically, as far as the generation of the desired mass terms is concerned, the gauge fixing structure which we would like to implement should have the form:

$$\begin{gathered} f(h;\bar{g}) \sim m_{gr}^2 h^2 + m_{gh}^2 \mathcal{K} h + \cdots \\ \downarrow \\ S_{gf}[h;\bar{g}] \sim \int f(h;\bar{g}) \sim \int m_{gr}^2 h^2 + \cdots \\ S_{gh}[h,c,\bar{c};\bar{g}] \sim \int \bar{c}\,\frac{\delta f(h;\bar{g})}{\delta h}\,\pounds_c(\bar{g}+h) \sim \int m_{gh}^2 \bar{c}c + \cdots \end{gathered} \tag{3.2.1}$$

The dots represent additional terms independent on the mass parameters which must be also introduced to have appropriate gauge-fixing terms surviving in the limit in which the mass parameters go to zero; otherwise, the regulated theory obtained by promoting the mass parameters to regulators will have as classical starting point for the FRG flow, i.e. the unregulated action, a non-gauge-fixed action. In subsection 3.2.3 we will see the precise form of the gauge-fixing function and compute the various terms. In the rest of this subsection we instead see how to formally introduce a linear gauge-fixing term.

Gauge-fixing term

The key point in the construction of a linear gauge-fixing term, given that the gauge-fixing function must have a free low index, is that we need to introduce an external field to contract with the function: $S_{gf}[h;\bar{g}] = \int d^D x\sqrt{\bar{g}}\,(\dots)^\mu f_\mu(h;\bar{g})$. Formally, this can can achieved by considering the generic gauge-fixed action in presence of a Nakanishi-Lautrup field and a noise field (2.2.88), where the gauge-fixing term (2.2.85) is by definition linear in gauge-fixing function thanks to the Nakanishi-Lautrup field, and picking a Fourier noise distribution, i.e. a noise term of the type:

$$S_{noise}[b,n;\bar{g}] = -i\int d^D x\sqrt{\bar{g}}\,(b_\mu - v_\mu)\bar{g}^{\mu\nu}n_\nu \tag{3.2.2}$$

where v_μ is an external 1-form field. In this way the noise integration gives the functional Fourier integral representation of the functional Dirac delta:

$$e^{-S_{NL}[b;\bar{g}]} = \int \mathcal{D}n\, e^{-S_{noise}[b,n;\bar{g}]} = \int \mathcal{D}n\, e^{i\int d^D x\sqrt{\bar{g}}\,(b_\mu - v_\mu)} = \delta[b_\mu - v_\mu]$$

and the Nakanishi-Lautrup term is formally given by the expression:

$$e^{-S_{NL}[b;\bar{g}]} = \delta[b_\mu - v_\mu] \tag{3.2.3}$$

The integration over the Nakanishi-Lautrup field sets as on-shell condition:

$$b_\mu|_{on-shell} = v_\mu \tag{3.2.4}$$

and the gauge-fixing term, given by:

$$e^{-S_{gf}[h;\bar{g}]} = \int \mathcal{D}b\, e^{-S_{gf}[h;b;\bar{g}]-S_{NL}[b;\bar{g}]} = \int \mathcal{D}b\,\delta[b_\mu - v_\mu]\, e^{-S_{gf}[h;b;\bar{g}]} = e^{-\int d^D x\sqrt{\bar{g}}\,v_\mu\bar{g}^{\mu\nu}n_\nu f_\nu(h;\bar{g})}$$

is linear in the gauge-fixing function:

$$S_{gf}[h;v;\bar{g}] = \int d^D x\sqrt{\bar{g}}\,v_\mu\bar{g}^{\mu\nu}f_\nu(h;v;\bar{g}) \tag{3.2.5}$$

In particular, we added the external field also between the arguments of the gauge-fixing function since we will allow for it to have a parametric dependence on the former. The ghost term acquires also a potential dependence on the external field:

$$S_{gh}[h,c,\bar{c};v;\bar{g}] = -\int d^D x\sqrt{\bar{g}}\,\bar{c}_\alpha\bar{g}^{\alpha\beta}\frac{\delta f_\beta(h;v;\bar{g})}{\delta h_{\mu\nu}}\bullet \pounds_c(\bar{g}_{\mu\nu}+h_{\mu\nu}) \tag{3.2.6}$$

And the complete gauge-fixed action (still with a generic, properly constructed, gauge-fixing function) is:

$$S[h,c,\bar{c};v;\bar{g}] = S_{EH}[h;\bar{g}] + S_{gf}[h;v;\bar{g}] + S_{gh}[h,c,\bar{c};v;\bar{g}] \tag{3.2.7}$$

Intuitively, the external field plays in this gauge-fixing a similar role to the one of the parameter α in the standard gauge-fixing: it is an external object introduced in the gauge-fixing sector, and thus expected to not affect physical observables quantities, which parametrizes a class of possible gauge-fixing choices, with the crucial difference of being a field and not a simple real number. Moreover, due to the on-shell condition (3.2.4), the external field can be also regarded as an "on-shell Nakanishi-Lautrup field", which has been "promoted" from auxiliary field to a field appearing also in the action.

Similar to the standard gauge-fixing term, considering a gauge-fixing function constructed as a proper tensor and the external field to transform covariantly as a 1-form field under background gauge transformations (2.2.12):

$$\bar{\delta}_\xi v_\mu = \pounds_\xi v_\mu \tag{3.2.8}$$

The gauge-fixing term and the ghost term are background gauge-invariant, and thus the complete gauge-fixed action:

$$\bar{\delta}_\xi S[h, c, \bar{c}; v; \bar{g}] = 0 \tag{3.2.9}$$

Additional sources and generating functionals

At the quantum level the theory is described by the generating functionals introduced in subsection 2.2.3 computed with the gauge-fixed action (3.2.7). In particular, the dependence of the latter on the external field will be inherited by all functionals. With respect to standard gauge-fixed theory in subsection 2.3.1, we also introduce for later convenience further two additional sources for the non-elementary fields:

$$H_{\mu\nu} = \frac{v_\alpha}{v^2}\bar{g}^{\alpha\beta}\bar{c}_\beta s h_{\mu\nu} \tag{3.2.10}$$

$$\Omega_{\mu\nu} = \frac{v_\alpha}{v^2}\bar{g}^{\alpha\beta}\bar{c}_\beta h_{\mu\nu} \tag{3.2.11}$$

which are respectively Grassmann-even and odd; in particular, we notice that the corresponding BRST Slavnov variations are given by:

$$sH_{\mu\nu} = sh_{\mu\nu} \tag{3.2.12}$$

$$s\Omega_{\mu\nu} = h_{\mu\nu} - H_{\mu\nu} \tag{3.2.13}$$

which imply in particular that the metric fluctuation and the field $H_{\mu\nu}$ belong to the same BRST cohomology class, i.e. their difference is a BRST exact term. So, the multiples of non-elementary fields and correspondent sources are now redefined as:

$$\psi^i \equiv \begin{pmatrix} sh_{\mu\nu} \\ sc^\mu \\ H_{\mu\nu} \\ \Omega_{\mu\nu} \end{pmatrix} \qquad \psi^{+i} \equiv \begin{pmatrix} -sh_{\mu\nu} & c^\mu & H_{\mu\nu} & -\Omega_{\mu\nu} \end{pmatrix} \tag{3.2.14}$$

$$K_i \equiv \begin{pmatrix} k^{\mu\nu} \\ l_\mu \\ m^{\mu\nu} \\ n^{\mu\nu} \end{pmatrix} \qquad K_i^+ \equiv \begin{pmatrix} k^{\mu\nu} & l_\mu & m^{\mu\nu} & n^{\mu\nu} \end{pmatrix} \tag{3.2.15}$$

$m^{\mu\nu}$ and $n^{\mu\nu}$ are respectively Grassmann-even and Grassmann-odd. The total source term, now dependent on the external field is:

$$S_{source}[\phi; J; K; v; \bar{g}] = S_{source-J}[\phi; J; \bar{g}] + S_{source-K}[\phi; K; v; \bar{g}] \tag{3.2.16}$$

with:

$$S_{source-J}[\phi;J;\bar{g}] = -\int d^D x\sqrt{\bar{g}}\,J_i^+\phi^i = -\int d^D x\sqrt{\bar{g}}\,\left(t^{\mu\nu}h_{\mu\nu} + \bar{\eta}_\mu c^\mu + \bar{c}_\mu\bar{\eta}^\mu\right) \quad (3.2.17)$$

$$\begin{aligned} S_{source-K}[\phi;K;v;\bar{g}] = -\int d^D x\sqrt{\bar{g}}\,K_i^+\psi^i = -\int d^D x\sqrt{\bar{g}}\,\big(&k^{\mu\nu}sh_{\mu\nu} + l_\mu sc^\mu \\ &+ m^{\mu\nu}H_{\mu\nu} + n^{\mu\nu}\Omega_{\mu\nu}\big) \end{aligned} \quad (3.2.18)$$

Given those redefinitions, the generating functionals read as subsection 2.3.1 with the extra dependence on the external field. We write explicitly the main formulas for an easier reference:

$$Z[J;K;v;\bar{g}] = \int \mathcal{D}\mu[\phi]\,e^{-S[\phi;v;\bar{g}]-S_{source}[\phi;J;K;v;\bar{g}]} \quad (3.2.19)$$

$$W[J;K;v;\bar{g}] = \log Z[J;K;v;\bar{g}] \quad (3.2.20)$$

$$\begin{aligned} \Gamma[\Phi;K;v;\bar{g}] &= \sup_J\left\{\int d^D x\sqrt{\bar{g}}\,J_i^+\Phi^i - W[J;K;v;\bar{g}]\right\} \\ &= \int d^D x\sqrt{\bar{g}}\,J_i^+\Phi^i - W[J;K;v;\bar{g}] \end{aligned} \quad (3.2.21)$$

$$\begin{aligned} W[J;K;v;\bar{g}] &= \sup_\Phi\left\{\int d^D x\sqrt{\bar{g}}\,J_i^+\Phi^i - \Gamma[\Phi;K;v;\bar{g}]\right\} \\ &= \int d^D x\sqrt{\bar{g}}\,J_i^+\Phi^i - \Gamma[\Phi;K;v;\bar{g}] \end{aligned} \quad (3.2.22)$$

The relations between fields and sources in the Legendre transform and the average non-elementary fields are:

$$\begin{aligned} \Phi^i(x) &= \left\langle\phi^i(x)\right\rangle_{JK} = \frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta W[J;K;v;\bar{g}]}{\delta J_i^+(x)} \\ \Phi^{+i}(x) &= \left\langle\phi^{+i}(x)\right\rangle_{JK} = \frac{1}{\sqrt{\bar{g}(x)}}\frac{W\overleftarrow{\delta}[J;K;v;\bar{g}]}{\delta J_i(x)} \end{aligned} \quad (3.2.23)$$

$$\begin{aligned} J_i(x) &= \frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta\Gamma[\Phi;K;v;\bar{g}]}{\delta\Phi^{+i}(x)} \\ J_i^+(x) &= \frac{1}{\sqrt{\bar{g}(x)}}\frac{\Gamma\overleftarrow{\delta}[\Phi;K;v;\bar{g}]}{\delta\Phi^i(x)} \end{aligned} \quad (3.2.24)$$

$$\begin{aligned} \Psi^i(x) &= \left\langle\psi^i(x)\right\rangle_{JK} = \frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta W[J;K;v;\bar{g}]}{\delta K_i^+(x)} = -\frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta\Gamma[\Phi;K;v;\bar{g}]}{\delta K_i^+(x)} \\ \Psi^{+i}(x) &= \left\langle\psi^{+i}(x)\right\rangle_{JK} = \frac{1}{\sqrt{\bar{g}(x)}}\frac{W\overleftarrow{\delta}[J;K;v;\bar{g}]}{\delta K_i(x)} = -\frac{1}{\sqrt{\bar{g}(x)}}\frac{\Gamma\overleftarrow{\delta}[\Phi;K;v;\bar{g}]}{\delta K_i(x)} \end{aligned} \quad (3.2.25)$$

The integro-differential equation for the effective action is:

$$e^{-\Gamma[\Phi;K;v;\bar{g}]} = \int \mathcal{D}\mu[\phi] \, \exp\left\{ -S[\phi;\bar{g}] - S_{source-K}[\phi;K;v;\bar{g}] \right. \\ \left. + \int d^D x \frac{\Gamma \overleftarrow{\delta} [\Phi;K;v;\bar{g}]}{\delta \Phi^i} \left(\phi^i - \Phi^i\right) \right\} \tag{3.2.26}$$

We recall also the relations between the matrices of second derivatives of the path integral, the path integral logarithm and the effective action

$$\frac{1}{Z} Z^{(2)}_{MN}(x,y) = W^{(2)}_{MN}(x,y) + W^{\overrightarrow{(1)}}_{M}(x) W^{\overleftarrow{(1)}}_{N}(y) \tag{3.2.27}$$

$$W^{(2)}_{J_i J_j}(x,y) = \Gamma^{(2)-1}_{\Phi^i \Phi^j}(x,y) \tag{3.2.28}$$

$$W^{(2)}_{K_i K_j}(x,y) = -\Gamma^{(2)}_{K_i, K_j}(x,y) \tag{3.2.29}$$

$$W^{(2)}_{K_i J_j}(x,y) = -\Gamma^{(2)}_{K_i \Phi^k}(x,\cdot) \bullet_{\bar{g}} \Gamma^{(2)-1}_{\Phi^k \Phi^j}(\cdot,y) \tag{3.2.30}$$

$$W^{(2)}_{J_i K_j}(x,y) = -\Gamma^{(2)-1}_{\Phi^i \Phi^k}(x,\cdot) \bullet_{\bar{g}} \Gamma^{(2)}_{\Phi^k K_j}(\cdot,y) \tag{3.2.31}$$

Finally, as in the standard case, if the gauge-fixed action is background gauge-invariant, the generating functionals inherit background gauge symmetry as an explicit symmetry, in particular:

$$\bar{\delta}_\xi W[J;K;v;\bar{g}] \; , \; \bar{\delta}_\xi \Gamma[\Phi;K;v;\bar{g}] = 0 \tag{3.2.32}$$

3.2.2 On-shell BRST symmetry and Zinn-Justin equation

On-shell BRST symmetry at the classical level

In this subsection we present the on-shell version of BRST variations and their properties for the linear gauge-fixed theory. Consider the gauge-fixed action (3.2.7); the on-shell BRST variations of the elementary fields and the external field are defined as:

$$\begin{cases} \delta_\theta h_{\mu\nu} = \theta \pounds_c (\bar{g}_{\mu\nu} + h_{\mu\nu}) \\ \qquad = \theta\big(c^\lambda \partial_\lambda \bar{g}_{\mu\nu} + \partial_\mu c^\lambda \bar{g}_{\lambda\nu} + \partial_\nu c^\lambda \bar{g}_{\lambda\mu}\big) + \theta\left(c^\lambda \partial_\lambda h_{\mu\nu} + \partial_\mu c^\lambda h_{\lambda\nu} + \partial_\nu c^\lambda h_{\lambda\mu}\right) \\ \qquad = \theta\big(\bar{\nabla}_\mu c^\lambda \bar{g}_{\lambda\nu} + \bar{\nabla}_\nu c^\lambda \bar{g}_{\lambda\mu}\big) + \theta\left(c^\lambda \bar{\nabla}_\lambda h_{\mu\nu} + \bar{\nabla}_\mu c^\lambda h_{\lambda\nu} + \bar{\nabla}_\nu c^\lambda h_{\lambda\mu}\right) \\ \delta_\theta \bar{g}_{\mu\nu} = \quad 0 \\ \delta_\theta c^\mu = \theta c^\lambda \partial_\lambda c^\mu \; = \theta c^\lambda \bar{\nabla}_\lambda c^\mu \\ \delta_\theta \bar{c}_\mu = \quad \theta v_\mu \\ \delta_\theta v_\mu = \quad 0 \end{cases} \tag{3.2.33}$$

The correspondent on-shell BRST Slavnov variations are:

$$\begin{cases} sh_{\mu\nu} = \pounds_c(\bar{g}_{\mu\nu} + h_{\mu\nu}) \\ \qquad = \left(c^\lambda\partial_\lambda\bar{g}_{\mu\nu} + \partial_\mu c^\lambda\bar{g}_{\lambda\nu} + \partial_\nu c^\lambda\bar{g}_{\lambda\mu}\right) + \left(c^\lambda\partial_\lambda h_{\mu\nu} + \partial_\mu c^\lambda h_{\lambda\nu} + \partial_\nu c^\lambda h_{\lambda\mu}\right) \\ \qquad = \left(\bar{\nabla}_\mu c^\lambda\bar{g}_{\lambda\nu} + \bar{\nabla}_\nu c^\lambda\bar{g}_{\lambda\mu}\right) + \left(c^\lambda\bar{\nabla}_\lambda h_{\mu\nu} + \bar{\nabla}_\mu c^\lambda h_{\lambda\nu} + \bar{\nabla}_\nu c^\lambda h_{\lambda\mu}\right) \\ s\bar{g}_{\mu\nu} = \quad 0 \\ sc^\mu = c^\lambda\partial_\lambda c^\mu \ = \theta c^\lambda\bar{\nabla}_\lambda c^\mu \\ s\bar{c}_\mu = \quad v_\mu \\ sv_\mu = \quad 0 \end{cases} \tag{3.2.34}$$

In accordance with the general remarks on BRST symmetry made at the beginning of subsection 2.2.3, the sector of the transformation unrelated to the gauge-fixing, i.e. metric fields and ghost, is the same as in the off-shell transformation (2.2.115), while the one related to the gauge-fixing, i.e. antighost and external field, is equal to that of the off-shell transformation (2.2.115) upon substituting the on-shell form of the Nakanishi-Lautrup field (3.2.4) which enforces the linear gauge-fixing term (3.2.5).

Due to the peculiarity of the chosen action for the Nakanishi-Lautrup field and the resulting simple on-shell condition, we notice in particular that the appearance of the variations is essentially equal to the off-shell version, with the role of the Nakanishi-Lautrup field taken by the external field; moreover, the BRST variation of the antighost does not depend on the gauge-fixing function, contrary to the case of the on-shell BRST variations for the standard gauge-fixing term (2.3.30). As a result, the structure properties of on-shell BRST symmetry typically related to the gauge-fixing choice are in this case essentially equal to the off-shell version, with the external field as "on-shell" Nakanishi-Lautrup field. Indeed, the BRST Slavnov variations are nilpotent without needing to use the classical equations of motion:

$$\begin{cases} s^2 h_{\mu\nu} = 0 \\ s^2\bar{g}_{\mu\nu} = 0 \\ s^2 c^\mu = 0 \\ s^2\bar{c}_\mu = 0 \\ s^2 v_\mu = 0 \end{cases} \tag{3.2.35}$$

In particular, the equations of motion of the antighost are not required to have the nilpotency of the correspondent second BRST Slavnov variation since:

$$s^2\bar{c}_\mu = sv_\mu = 0 \tag{3.2.36}$$

From (2.2.113) it follows that any polynomial function of the elementary fields and the external field has also a vanishing second BRST Slavnov variation:

$$s^2 F(h, c, \bar{c}; v; \bar{g}) = 0 \tag{3.2.37}$$

As in the off-shell case, the action can be written in an explicitly BRST-invariant form. Indeed, the Einstein-Hilbert action is BRST-invariant; the gauge-fixing term and the ghost term can be rewritten respectively as:

$$S_{gf}[h,v;\bar{g}] = \int d^D x \sqrt{\bar{g}}\, v_\mu \bar{g}^{\mu\nu} f_\nu(h;v;\bar{g}) = \int d^D x \sqrt{\bar{g}}\, s\bar{c}_\mu \bar{g}^{\mu\nu} f_\nu(h;v;\bar{g}) \tag{3.2.38}$$

and:

$$S_{gh}[h,c,\bar{c};v;\bar{g}] = -\int d^D x \sqrt{\bar{g}}\, \bar{c}_\alpha \bar{g}^{\alpha\beta} \mathcal{G}_{\beta\gamma} \bullet c^\gamma = -\int d^D x \sqrt{\bar{g}}\, \bar{c}_\mu \bar{g}^{\mu\nu} s f_\nu(h;v;\bar{g}) \tag{3.2.39}$$

The sum is a BRST-exact term:

$$S_{gf}[h,v;\bar{g}] + S_{gh}[h,c,\bar{c},v;\bar{g}] = s\int d^D x \sqrt{\bar{g}}\, \bar{c}_\mu \bar{g}^{\mu\nu} f_\nu(h;v;\bar{g}) \equiv sS_{BRST}[h,\bar{c};v;\bar{g}]$$

so, manifestly BRST-invariant:

$$\delta_\theta \left(S_{gf}[h;v;\bar{g}] + S_{gh}[h,c,\bar{c};v;\bar{g}]\right) = 0 \tag{3.2.40}$$

Finally, the gauge-fixed action can be then rewritten in the explicitly BRST-invariant form:

$$S[h,c,\bar{c},b;\bar{g}] = S_{EH}[h;\bar{g}] + sS_{BRST}[h,\bar{c};v;\bar{g}] \tag{3.2.41}$$

and:

$$\delta_\theta S[h,c,\bar{c};v;\bar{g}] = 0 \tag{3.2.42}$$

As in the general off-shell version of the symmetry, the linear gauge-fixing term and the ghost term are not separately BRST-invariant, but their sum is BRST-exact, therefore their BRST variations compensate and precisely eliminate each other, and this holds for any specific choice of the gauge-fixing function. Therefore we can again conclude that the gauge-fixed action (3.2.7) enjoys a BRST symmetry under (3.2.33) regardless of the gauge-fixing details; therefore, we can indeed proceed as explained in subsection 3.1.1 to introduce in the gauge-fixed action mass terms (and then regulator terms) without breaking the explicit BRST symmetry, thanks to a suitable choice of the gauge-fixing function.

On-shell BRST symmetry at the quantum level

We conclude the subsection by deriving the explicit form of the Ward-Takahashi equation and the Zinn-Justin equation describing BRST symmetry at the quantum level for the linear gauge-fixed theory, as done for the standard gauge-fixed theory in subsection 2.3.2. In the derivation of the Ward-Takahashi equation the only difference is that now we need to take care of the additional source terms for the non-elementary fields (3.2.10) and (3.2.11) that we chose to introduce, since those have non-vanishing BRST Slavnov variations; so, (2.3.37) now gives:

$$\left\langle \int d^D x \sqrt{\bar{g}} \left(J_i^+ \delta_\theta \phi^i + K_i^+ \delta_\theta \psi^i\right) \right\rangle_{JK} = 0 \tag{3.2.43}$$

with $\delta_\theta \psi^i \neq 0$; in particular, we have:

$$\begin{aligned} 0 &= \int d^D x \sqrt{\bar{g}} \left(J_i^+ \left\langle \delta_\theta \phi^i \right\rangle_{JK} + K_i^+ \left\langle \delta_\theta \psi^i \right\rangle_{JK} \right) \\ &= \int d^D x \sqrt{\bar{g}} \Big(t^{\mu\nu} \left\langle \delta_\theta h_{\mu\nu} \right\rangle_{JK} + \bar{\eta}_\mu \left\langle \delta_\theta c^\mu \right\rangle_{JK} - \eta^\mu \left\langle \delta_\theta \bar{c}_\mu \right\rangle_{JK} + \\ &\qquad\qquad + m^{\mu\nu} \left\langle \delta_\theta H_{\mu\nu} \right\rangle_{JK} + n^{\mu\nu} \left\langle \delta_\theta \Omega_{\mu\nu} \right\rangle_{JK} \Big) \\ &= \theta \int d^D x \sqrt{\bar{g}} \Big(t^{\mu\nu} \left\langle s h_{\mu\nu} \right\rangle_{JK} - \bar{\eta}_\mu \left\langle s c^\mu \right\rangle_{JK} + \eta^\mu \left\langle s \bar{c}_\mu \right\rangle_{JK} + \\ &\qquad\qquad + m^{\mu\nu} \left\langle s H_{\mu\nu} \right\rangle_{JK} - n^{\mu\nu} \left\langle s \Omega_{\mu\nu} \right\rangle_{JK} \Big) \end{aligned} \tag{3.2.44}$$

and substituting the BRST Slavnov variation of the antighost and those of the two new non-elementary fields, (3.2.12) and (3.2.13), in the last form of the equation, the Ward-Takahashi equation associated to BRST symmetry in the linear gauge-fixed theory can be written as:

$$\begin{aligned} \int d^D x \sqrt{\bar{g}} \Big(& t^{\mu\nu} \left\langle s h_{\mu\nu} \right\rangle_{JK} - \bar{\eta}_\mu \left\langle s c^\mu \right\rangle_{JK} + \eta^\mu v_\mu + \\ & + m^{\mu\nu} \left\langle s h_{\mu\nu} \right\rangle_{JK} - n^{\mu\nu} \left\langle h_{\mu\nu} \right\rangle_{JK} + n^{\mu\nu} \left\langle H_{\mu\nu} \right\rangle_{JK} \Big) = 0 \end{aligned} \tag{3.2.45}$$

where we have used that $\left\langle v_\mu \right\rangle_{JK} = v_\mu \left\langle 1 \right\rangle_{JK} = v_\mu$, since the external field is a constant with respect to the integrated fields in the path integral.
As in subsection 2.3.2, the associated Zinn-Justin equation is obtained expressing the equation in the variables of the effective action by evaluating the sources with the expressions given by the Legendre transform, $J = J(\Phi, K)$, and writing them explicitly in terms of derivatives of the effective action using (3.2.24), together with the averages of non-elementary fields using (3.2.25). The Zinn-Justin equation associated to BRST symmetry in the linear gauge-fixed theory is:

$$\begin{aligned} \Sigma[\Gamma] = \int d^D x \Big(& -\frac{1}{\sqrt{\bar{g}}} \frac{\delta \Gamma}{\delta \mathsf{h}_{\mu\nu}} \frac{\delta \Gamma}{\delta k^{\mu\nu}} - \frac{1}{\sqrt{\bar{g}}} \frac{\delta \Gamma}{\delta \mathsf{c}^\mu} \frac{\delta \Gamma}{\delta l_\mu} + v_\mu \frac{\delta \Gamma}{\delta \bar{\mathsf{c}}_\mu} + \\ & - m^{\mu\nu} \frac{1}{\sqrt{\bar{g}}} \frac{\delta \Gamma}{\delta k^{\mu\nu}} - n^{\mu\nu} \mathsf{h}_{\mu\nu} - n^{\mu\nu} \frac{1}{\sqrt{\bar{g}}} \frac{\delta \Gamma}{\delta m^{\mu\nu}} \Big) = 0 \end{aligned} \tag{3.2.46}$$

As (2.3.42), it is quadratic and linear in the derivatives of the effective action. In this case the third term does not depend on the gauge-fixing function, substituted by the external field as a consequence of the particular form of the on-shell BRST variations for the linear gauge-fixed theory; the additional three terms are just due to the source terms for the non-elementary fields (3.2.10) and (3.2.11) that we chose to introduce in the quantization.

3.2.3 Quadratic massive gauge-fixing function

We now present the precise form of the gauge-fixing function sketched in (3.2.1) and we give the full expression of the gauge-fixed action. According to the discussion in subsection

3.2.1, the gauge-fixing function must contain a quadratic and a linear term, therefore the most general structure is:

$$f_\lambda(h; v; \bar{g}) = h_{\alpha\beta} \bullet \mathcal{Q}_\lambda{}^{\alpha\beta,\mu\nu} \bullet h_{\mu\nu} + \mathcal{L}_\lambda{}^{\mu\nu} \bullet h_{\mu\nu} \tag{3.2.47}$$

where the two operators for the quadratic and linear term are respectively dependent on three and two spacetime points, $\mathcal{Q}_\lambda{}^{\alpha\beta,\mu\nu} = \mathcal{Q}_\lambda{}^{\alpha\beta,\mu\nu}(x, y, z)$ and $\mathcal{L}_\lambda{}^{\mu\nu} = \mathcal{L}_\lambda{}^{\mu\nu}(x, y)$; as usual $\bullet$ represent in compact notation an understood integration, so that the gauge-fixing function is dependent on a single spacetime point. Substituting the ansatz for the gauge-fixing function in the gauge-fixing and the ghost term, we obtain respectively the general expressions:

$$\begin{aligned} S_{gf}[h; v; \bar{g}] &= \int d^D x \sqrt{\bar{g}}\, v_\rho \bar{g}^{\rho\sigma} f_\sigma(h; v; \bar{g}) = \\ &= \int d^D x \sqrt{\bar{g}}\, v_\rho \bar{g}^{\rho\sigma} \left(h_{\alpha\beta} \bullet \mathcal{Q}_\sigma{}^{\alpha\beta,\mu\nu} \bullet h_{\mu\nu} + \mathcal{L}_\sigma{}^{\mu\nu} \bullet h_{\mu\nu} \right) \end{aligned} \tag{3.2.48}$$

and:

$$\begin{aligned} S_{gh}[h, c, \bar{c}; v; \bar{g}] &= -\int d^D x \sqrt{\bar{g}}\, \bar{c}_\rho \bar{g}^{\rho\sigma} \frac{\delta f_\sigma(h; v; \bar{g})}{\delta h_{\mu\nu}} \bullet \pounds_c(\bar{g}_{\mu\nu} + h_{\mu\nu}) = \\ &= -\int d^D x \sqrt{\bar{g}}\, \bar{c}_\rho \bar{g}^{\rho\sigma} \left(h_{\alpha\beta} \bullet \mathcal{Q}_\sigma{}^{\alpha\beta,\mu\nu} + \mathcal{Q}_\sigma{}^{\mu\nu,\alpha\beta} \bullet h_{\alpha\beta} + \mathcal{L}_\sigma{}^{\mu\nu} \right) \bullet \pounds_c(\bar{g}_{\mu\nu} + h_{\mu\nu}) \end{aligned} \tag{3.2.49}$$

Taking inspiration from [6], we consider specific operators $\mathcal{Q}_\lambda{}^{\alpha\beta,\mu\nu}$ and $\mathcal{L}_\lambda{}^{\mu\nu}$, which are capable of introducing the desired mass terms and also reproducing the standard terms of the de Donder gauge-fixing; in this way the regulated theory obtained by promoting the mass parameters to regulators will have as classical starting point for the FRG flow, i.e. the unregulated action, a properly gauge-fixed action similar to the standard one (there will be additional terms dependent on the external field). Such operators are:

$$\begin{aligned} \mathcal{Q}_\lambda{}^{\alpha\beta,\mu\nu}(x, y, z) &\equiv \delta(x-y)\delta(y-z)\mathcal{Q}_\lambda{}^{\alpha\beta,\mu\nu} \\ &= \delta(x-y)\delta(y-z)\cdot \\ &\quad \cdot \frac{1}{\kappa^2}\frac{v_\lambda(z)}{2v^2(z)} \left(\frac{1}{\alpha} \bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma}(z) \bar{\nabla}^\rho_{(z)} \bar{\nabla}^\sigma_{(z)} + m_{gr}^2 \bar{M}^{\alpha\beta,\mu\nu}(z) \right) \end{aligned} \tag{3.2.50}$$

$$\mathcal{L}_\lambda{}^{\mu\nu}(x, y) = \left(\delta(x-y) - m_{gh}^2 \sqrt{\bar{g}(y)}\, \bar{\Box}^{-1}(y, x) \right) \bar{K}^{\mu\nu}{}_{\lambda\rho}(y) \bar{\nabla}^\rho_{(y)} \tag{3.2.51}$$

where: **1.** α is an arbitrary parameter as in the standard gauge-fixing term; **2.** $\bar{K}^{\mu\nu}{}_{\lambda\rho}$ and $\bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma}$ are the structure tensors (2.2.38) and (2.2.39) defined in subsection 2.2.1 and appearing in the quadratic term of the Einstein-Hilbert action; **3.** the square of the external field is defined as:

$$v^2 \equiv \bar{g}^{\mu\nu} v_\mu v_\nu \tag{3.2.52}$$

4. $\bar{M}^{\alpha\beta,\mu\nu}$ is a generic structure tensor, symmetric under the exchanges $\alpha \leftrightarrow \beta, \mu \leftrightarrow \nu, \{\alpha\beta\} \leftrightarrow \{\mu\nu\}$, which gives the structure of the mass term for the metric fluctuation; we will not need to write an explicit expression, since our focus is not the construction of a mass term meaningful per se, but rather finalized to be promoted to a generic FRG regulator with a generic tensorial structure[10]. **5.** $\bar{\Box}^{-1}$ formally indicates the Green function of the background laplacian, defined by the condition:

$$\bar{\Box}_{(x)}\bar{\Box}^{-1}(x,y) = \frac{1}{\sqrt{\bar{g}(x)}}\delta(x-y) \tag{3.2.53}$$

which in particular implies:

$$\int d^D y \sqrt{\bar{g}(y)}\, \bar{\Box}_{(x)}\bar{\Box}^{-1}(x,y)A(y) = A(x) \tag{3.2.54}$$

$$\int d^D y \sqrt{\bar{g}(y)}\, \bar{\Box}^{-1}(y,x)\bar{\Box}_{(y)}A(y) = A(x) \tag{3.2.55}$$

The first follows directly from the defining property, while the second from integrating by parts the two covariant derivatives of the background laplacian and then applying the defining property.
According to this choice, ${\mathcal{Q}_\lambda}^{\alpha\beta,\mu\nu}$ is a local operator since proportional to two Dirac deltas and therefore, intuitively, different from zero only for $x = y = z$; ${\mathcal{L}_\lambda}^{\mu\nu}$ contains instead a local piece proportional to a Dirac delta as well as a non-local one proportional to the Green function $\bar{\Box}^{-1}$, which is in general different from zero also for $x \neq y$. So, substituting the operators ${\mathcal{Q}_\lambda}^{\alpha\beta,\mu\nu}$ and ${\mathcal{L}_\lambda}^{\mu\nu}$ in the general ansatz, the resulting gauge-fixing function:

$$\begin{aligned} f_\lambda(h;v;\bar{g}) = \frac{1}{\kappa^2}\frac{v_\lambda}{2v^2}\left(\frac{1}{\alpha}\, h_{\alpha\beta}\bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\rho\bar{\nabla}^\sigma h_{\mu\nu} + m_{gr}^2 h_{\alpha\beta}\bar{M}^{\alpha\beta,\mu\nu}h_{\mu\nu}\right) \\ + \left(\bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\sigma h_{\mu\nu} - m_{gh}^2\sqrt{\bar{g}}\,\bar{\Box}^{-1} \bullet \bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\sigma h_{\mu\nu}\right) \end{aligned} \tag{3.2.56}$$

contains a quadratic term made of two local pieces and a linear term made of a local piece and a non-local one containing a spacetime integration, which we wrote in compact notation. Consider the quadratic term; the two pieces are engineered to originate, inside the gauge-fixing term, respectively the de Donder gauge-fixing term (2.3.44) found in subsection 2.3.3 and the desired mass term for the metric fluctuation. Indeed, substituting the explicit expression for the gauge-fixing function, the gauge-fixing term (3.2.5) is:

$$\begin{aligned} S_{gf}[h;v;\bar{g}] &= \int d^D x\sqrt{\bar{g}}\, v_\rho \bar{g}^{\rho\sigma}\left(h_{\alpha\beta}{\mathcal{Q}_\sigma}^{\alpha\beta,\mu\nu}h_{\mu\nu} + {\mathcal{L}_\sigma}^{\mu\nu}\bullet h_{\mu\nu}\right) = \\ &= \int d^D x\sqrt{\bar{g}}\left[\frac{1}{\kappa^2}\frac{1}{2\alpha}h_{\alpha\beta}\bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\rho\bar{\nabla}^\sigma h_{\mu\nu}\right. \end{aligned}$$

[10] Moreover, constructing a meaningful mass term for the metric fluctuation, i.e constructing a theory for a massive spin-2 graviton propagating in a background spacetime, possibly curved, is historically a non-trivial problem, in particular due to the task of making propagate only the correct number of degrees of freedom. The correspondent research line is known as massive gravity, see for instance the early work [30] and the recent review [31].

$$+\frac{1}{\kappa^2}\frac{m_{gr}^2}{2}h_{\alpha\beta}\bar{M}^{\alpha\beta,\mu\nu}h_{\mu\nu}$$
$$+v_\lambda\bar{g}^{\lambda\rho}\left(1-m_{gh}^2\sqrt{\bar{g}}\,\bar{\Box}^{-1}\right)\bullet\bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\sigma h_{\mu\nu}\Bigg] \tag{3.2.57}$$

The first two terms are respectively the de Donder gauge-fixing term and the mass term for the metric fluctuation; in particular, the multiplication for v_σ/v^2 in the quadratic term of the gauge-fixing function is introduced precisely to cancel the one by $v_\rho\bar{g}^{\rho\sigma}$ in the linear gauge-fixing term. The gauge-fixing term receives also a third non-local contribution, dependent on the ghost mass parameter and the external field, given by the product of the latter with the linear term of the gauge-fixing function. So, the gauge-fixing term can be written as:

$$S_{gf}[h;v;\bar{g}]=S_{gf-deD}[h;\bar{g}]+S_{gf-m_{gr}}[h;\bar{g}]+S_{gf-v}[h;v;\bar{g}] \tag{3.2.58}$$

with:

$$S_{gf-deD}[h;\bar{g}]=\frac{1}{\kappa^2}\int d^Dx\sqrt{\bar{g}}\,\frac{1}{2\alpha}h_{\alpha\beta}\bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\rho\bar{\nabla}^\sigma h_{\mu\nu} \tag{3.2.59}$$

$$S_{gf-m_{gr}}[h;\bar{g}]=\frac{1}{\kappa^2}\int d^Dx\sqrt{\bar{g}}\,\frac{1}{\kappa^2}\frac{m_{gr}^2}{2}h_{\alpha\beta}\bar{M}^{\alpha\beta,\mu\nu}h_{\mu\nu} \tag{3.2.60}$$

$$S_{gf-v}[h;v;\bar{g}]=\int d^Dx\sqrt{\bar{g}}\,v_\lambda\bar{g}^{\lambda\rho}\left(1-m_{gh}^2\sqrt{\bar{g}}\,\bar{\Box}^{-1}\right)\bullet\bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\sigma h_{\mu\nu} \tag{3.2.61}$$

Consider now the linear term of the gauge-fixing function; the two pieces are engineered to originate, inside the ghost term, respectively the de Donder ghost term (2.3.46) found in subsection 2.3.3 and the desired mass term for the ghosts. Indeed, substituting the explicit expression for the gauge-fixing function, the ghost term (3.2.6) is:

$$\begin{aligned}
&S_{gh}[h,c,\bar{c};v;\bar{g}]=\\
&\quad=-\int d^Dx\sqrt{\bar{g}}\,\bar{c}_\rho\bar{g}^{\rho\sigma}\left(h_{\alpha\beta}\mathcal{Q}_\sigma{}^{\alpha\beta,\mu\nu}+\mathcal{Q}_\sigma{}^{\mu\nu,\alpha\beta}h_{\alpha\beta}+\mathcal{L}_\sigma{}^{\mu\nu}\bullet\right)\pounds_c(\bar{g}_{\mu\nu}+h_{\mu\nu})=\\
&\quad=\int d^Dx\sqrt{\bar{g}}\,\Bigg\{\\
&\qquad-\frac{1}{\kappa^2}\frac{v_\lambda}{v^2}\bar{g}^{\lambda\tau}\bar{c}_\tau\left[\frac{1}{2\alpha}\bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma}\left(\bar{\nabla}^\rho\bar{\nabla}^\sigma h_{\mu\nu}+h_{\mu\nu}\bar{\nabla}^\rho\bar{\nabla}^\sigma\right)+m_{gr}^2\bar{M}^{\mu\nu,\alpha\beta}h_{\alpha\beta}\right]\pounds_c(\bar{g}_{\mu\nu}+h_{\mu\nu})\Bigg\}\\
&\qquad-\bar{c}_\lambda\bar{g}^{\lambda\rho}\left(1-m_{gh}^2\sqrt{\bar{g}}\,\bar{\Box}^{-1}\right)\bullet\bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\sigma\pounds_c(\bar{g}_{\mu\nu}+h_{\mu\nu})
\end{aligned} \tag{3.2.62}$$

The first piece obtained by distributing the product in the second term is the standard de Donder ghost term, sum of the kinetic ghost term and the de Donder interaction term as shown in subsection 2.3.3 using identity (2.3.47); the second is the sum of a mass term for the ghosts and a non-local modification of the de Donder interaction term dependent on the ghost mass parameter. Indeed, using again identity (2.3.47), we have:

$$
\begin{aligned}
&S_{gh-m_{gh}\times deD}[h,c,\bar{c};\bar{g}] = \\
&\quad = \int d^D x\sqrt{\bar{g}}\, m_{gh}^2 \bar{c}_\lambda \bar{g}^{\lambda\rho}\sqrt{\bar{g}}\,\bar{\square}^{-1}\bullet \bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\sigma \pounds_c(\bar{g}_{\mu\nu}+h_{\mu\nu}) = \\
&\quad = \int d^D x\sqrt{\bar{g}}\,\left\{\left[m_{gh}^2\bar{c}_\lambda c^\lambda\right] + \left[m_{gh}^2\bar{c}_\lambda\bar{g}^{\lambda\rho}\sqrt{\bar{g}}\,\bar{\square}^{-1}\bullet\left(\bar{R}_{\rho\tau}c^\tau + \bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\sigma\pounds_c h_{\mu\nu}\right)\right]\right\} = \\
&\quad = S_{gh-m_{gh}}[c,\bar{c};\bar{g}] + S_{gh-m_{gh}-int}[h,c,\bar{c};\bar{g}]
\end{aligned}
\tag{3.2.63}
$$

where we have used property (3.2.55) of the green function $\bar{\square}^{-1}$ to write:

$$
\begin{aligned}
m_{gh}^2\bar{c}_\lambda\bar{g}^{\lambda\rho}\sqrt{\bar{g}}\,\bar{\square}^{-1}\bullet\bar{g}_{\rho\tau}\bar{\square}c^\tau &= m_{gh}^2\bar{c}_\lambda(x)\bar{g}^{\lambda\rho}(x)\int d^D y\sqrt{\bar{g}(y)}\,\bar{\square}^{-1}(y,x)\bar{g}_{\rho\tau}(y)\bar{\square}_{(y)}c^\tau(y) = \\
&= m_{gh}^2\bar{c}_\lambda(x)\bar{g}^{\lambda\rho}(x)\bar{g}_{\rho\tau}(x)c^\tau(x) = \\
&= m_{gh}^2\bar{c}_\lambda(x)c^\lambda(x)
\end{aligned}
$$

In particular, the non-local operator $-\sqrt{\bar{g}}\,\bar{\square}^{-1}\bullet\bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\sigma$ is the explicit form of the operator $\mathcal{K}$ mentioned at the beginning of subsection 3.2.1; it allows to obtain the mass term for the ghosts with the correct contraction of ghost and antighost and without derivatives and contractions with the background metric: the correct contraction of ghost and antighost is obtained thanks to the derivative structure $\bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\sigma$ as in the de Donder kinetic ghost term, $-\bar{c}_\lambda\bar{\square}c^\lambda$; the background covariant laplacian is removed thanks to the Green function $\sqrt{\bar{g}}\,\bar{\square}^{-1}\bullet$. Finally, the ghost term receives also a second local contribution, dependent on the metric fluctuation mass parameter and the external field, coming from the quadratic term of the gauge-fixing function; in particular, in (3.2.62), we used the symmetry properties of the structure tensors $\bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma}$ and $\bar{M}^{\alpha\beta,\mu\nu}$, namely $\bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma} = \bar{D}^{\mu\nu,\alpha\beta}{}_{\sigma\rho}$ and $\bar{M}^{\alpha\beta,\mu\nu} = \bar{M}^{\mu\nu,\alpha\beta}$, to group together the terms coming from $h_{\alpha\beta}\mathcal{Q}_\sigma{}^{\alpha\beta,\mu\nu}$ and $\mathcal{Q}_\sigma{}^{\mu\nu,\alpha\beta}h_{\alpha\beta}$; expanding the Lie derivatives, one obtains an $hhc\bar{c}$-interaction term dependent on the metric fluctuation mass parameter and several higher-derivative $hhc\bar{c}$-interaction terms. So, the ghost term can be written as:

$$
\begin{aligned}
S_{gh}[h,c,\bar{c};v;\bar{g}] = S_{gh-kin}[c,\bar{c};\bar{g}] &+ S_{gh-m_{gh}}[c,\bar{c};\bar{g}] \\
&+ S_{gh-deD-int}[h,c,\bar{c};\bar{g}] + S_{gh-m_{gh}-int}[h,c,\bar{c};\bar{g}] + S_{gh-v}[h,c,\bar{c};v;\bar{g}]
\end{aligned}
\tag{3.2.64}
$$

with:

$$
S_{gh-kin}[c,\bar{c};\bar{g}] = \int d^D x\sqrt{\bar{g}}\,\bar{c}_\mu\left(-\bar{\square}\right)c^\mu \tag{3.2.65}
$$

$$
S_{gh-m_{gh}}[c,\bar{c};\bar{g}] = \int d^D x\sqrt{\bar{g}}\, m_{gh}^2\bar{c}_\mu c^\mu \tag{3.2.66}
$$

$$
S_{gh-deD-int}[h,c,\bar{c};\bar{g}] = \int d^D x\sqrt{\bar{g}}\,\left[-\bar{c}_\lambda\bar{g}^{\lambda\rho}\left(\bar{R}_{\rho\tau}c^\tau + \bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\sigma\pounds_c h_{\mu\nu}\right)\right] \tag{3.2.67}
$$

$$
S_{gh-m_{gh}-int}[h,c,\bar{c};\bar{g}] = \int d^D x\sqrt{\bar{g}}\, m_{gh}^2\bar{c}_\lambda\bar{g}^{\lambda\rho}\sqrt{\bar{g}}\,\bar{\square}^{-1}\bullet\left(\bar{R}_{\rho\tau}c^\tau + \bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\sigma\pounds_c h_{\mu\nu}\right) \tag{3.2.68}
$$

$$
\begin{aligned}
&S_{gh-v}[h,c,\bar{c};v;\bar{g}] = \\
&\quad = \int d^Dx\sqrt{\bar{g}}\left\{-\frac{1}{\kappa^2}\frac{v_\lambda}{v^2}\bar{g}^{\lambda\tau}\bar{c}_\tau\left[\frac{1}{2\alpha}\bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma}\left(\bar{\nabla}^\rho\bar{\nabla}^\sigma h_{\mu\nu}+h_{\mu\nu}\bar{\nabla}^\rho\bar{\nabla}^\sigma\right)\right.\right. \\
&\qquad\qquad\left.\left.+m_{gr}^2\bar{M}^{\mu\nu,\alpha\beta}h_{\alpha\beta}\right]\pounds_c(\bar{g}_{\mu\nu}+h_{\mu\nu})\right\}
\end{aligned}
\tag{3.2.69}
$$

For completeness, we write also the complete action with all the terms together and the Lie derivatives written explicitly:

$$
\begin{aligned}
&S[h,c,\bar{c};v;\bar{g}] = S_{EH}[h;\bar{g}]+S_{gf}[h;v;\bar{g}]+S_{gh}[h,c,\bar{c};v;\bar{g}] = \\
&\quad = \int d^Dx\sqrt{\bar{g}}\Bigg\{ \\
&\qquad \frac{1}{\kappa^2}\left[\frac{1}{2}h_{\alpha\beta}\left(-\bar{K}^{\alpha\beta,\mu\nu}\bar{\Box}+\left(1+\frac{1}{\alpha}\right)\bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\rho\bar{\nabla}^\sigma+\bar{O}_2^{\alpha\beta,\mu\nu}+m_{gr}^2\bar{M}^{\alpha\beta,\mu\nu}\right)h_{\mu\nu}\right. \\
&\qquad\qquad\left.+2\left(-\bar{R}+2\Lambda\right)+2\left(\bar{G}_{\mu\nu}+\Lambda\bar{g}_{\mu\nu}\right)h_{\mu\nu}+\sum_{n=3}^{\infty}\frac{1}{n!}\bar{O}_nh^n\right] \\
&\qquad +\bar{c}_\mu(-\bar{\Box}+m_{gh}^2)c^\mu \\
&\qquad +v_\lambda\bar{g}^{\lambda\rho}\left(1-m_{gh}^2\sqrt{\bar{g}}\,\bar{\Box}^{-1}\right)\bullet\bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\sigma h_{\mu\nu} \\
&\qquad -\bar{c}_\lambda\bar{g}^{\lambda\rho}\left(1-m_{gh}^2\sqrt{\bar{g}}\,\bar{\Box}^{-1}\right)\bullet\left[\bar{R}_{\rho\sigma}c^\sigma+\bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\sigma\left(c^\tau\bar{\nabla}_\tau h_{\mu\nu}+\bar{\nabla}_\mu c^\tau h_{\tau\nu}+\bar{\nabla}_\nu c^\tau h_{\tau\mu}\right)\right] \\
&\qquad -\frac{1}{\kappa^2}\frac{v_\lambda}{v^2}\bar{g}^{\lambda\zeta}\bar{c}_\zeta\left[\frac{1}{2\alpha}\bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma}\left(\bar{\nabla}^\rho\bar{\nabla}^\sigma h_{\mu\nu}+h_{\mu\nu}\bar{\nabla}^\rho\bar{\nabla}^\sigma\right)+m_{gr}^2\bar{M}^{\mu\nu,\alpha\beta}h_{\alpha\beta}\right]\cdot \\
&\qquad\qquad\cdot\left(\bar{\nabla}_\mu c^\tau\bar{g}_{\tau\nu}+\bar{\nabla}_\nu c^\tau\bar{g}_{\tau\mu}+c^\tau\bar{\nabla}_\tau h_{\mu\nu}+\bar{\nabla}_\mu c^\tau h_{\tau\nu}+\bar{\nabla}_\nu c^\tau h_{\tau\mu}\right)\Bigg\}
\end{aligned}
\tag{3.2.70}
$$

To summarize, using the linear gauge-fixing term constructed in subsection 3.2.1 in combination with the gauge-fixing function presented in this subsection: **1.** we obtain the desired mass terms for the metric fluctuation and ghosts and we also recover the standard de Donder gauge-fixing and ghost terms; in particular thanks to the quadratic part of the gauge-fixing function we obtain, inside the gauge-fixing term, the de Donder gauge fixing term and the mass term for the metric fluctuation, while thanks to the linear part we obtain, inside the ghost term, the de Donder ghost term and the mass term for the ghosts; **2.** in addition, respectively from the linear and quadratic part of the gauge-fixing function inside the gauge-fixing and ghost term, we obtain additional $hc\bar{c}$- and $hhc\bar{c}$-interaction terms dependent on the mass parameters and the external field; the sum off all the terms introduced in the gauge-fixing sector is BRST-invariant by construction under the transformations presented in subsection 3.2.2; **3.** the price to pay to perform such construction is represented by the non-local and higher-derivatives pieces contained in the additional interaction terms.

3.3 BRST-invariant FRG flow

In this section we cover the second of the two steps outlined at the end of subsection 3.1.1, using the gauge-fixing structure constructed in the previous section as a template to regulate the theory in an explicit BRST-invariant manner by promoting the mass parameters to regulators. Obtained the regularized theory, we follow the standard procedure to derive the Wetterich-Morris equation describing the FRG flow of the theory and we prove its compatibility with the constraint imposed by BRST symmetry, i.e the Zinn-Justin equation; finally we present the equation in component form within the Einstein-Hilbert truncation and a particular regularization scheme, and we compare the result with the one for the standard theory.

3.3.1 BRST-invariant FRG regularization

Regularization

According to the strategy described in subsection 3.1.1, we now regularize the theory directly via the gauge-fixing procedure by considering the gauge-fixing structure introduced in the previous section and promote the mass parameters to regulators. In particular, taking inspiration from [6], we choose to perform the promotion according to the following substitution rules:

$$m_{gr}^2 \bar{M}^{\alpha\beta,\mu\nu} \quad \to \quad \mathcal{R}_{gr(k)}^{\alpha\beta,\mu\nu}\left(-\bar{\Box}\right) \tag{3.3.1}$$

$$m_{gh}^2 \quad \to \quad -r_{gh(k)}\left(-\bar{\Box}\right)\bar{\Box} \tag{3.3.2}$$

where $\mathcal{R}_{gr(k)}^{\alpha\beta,\mu\nu}\left(-\bar{\Box}\right)$ and $r_{gh(k)}\left(-\bar{\Box}\right)$ are two operators with functional dependence still to be specified, at this stage. The gauge-fixed theory described by (3.2.70) is reinterpreted as a regulated-gauge-fixed theory:

$$\begin{aligned} S[h,c,\bar{c};v;\bar{g}] &= S_{EH}[h;\bar{g}] + S_{gf}[h;v;\bar{g}] + S_{gh}[h,c,\bar{c};v;\bar{g}] \\ &\downarrow \\ \tilde{S}_{(k)}[h,c,\bar{c};v;\bar{g}] &= S_{EH}[h;\bar{g}] + S_{gf(k)}[h;v;\bar{g}] + S_{gh(k)}[h,c,\bar{c};v;\bar{g}] \\ &\equiv S_0[h,c,\bar{c};v;\bar{g}] + \Delta S_{(k)}[h,c,\bar{c};v;\bar{g}] \end{aligned} \tag{3.3.3}$$

with the unregulated action $S_0[\phi;v;\bar{g}]$ and the regulator term $\Delta S_{(k)}[\phi;v;\bar{g}]$. According to the promotion rules (3.3.1) and (3.3.2) the gauge-fixing function (3.2.47) is promoted to a regulating-gauge-fixing function:

$$f_{(k)\lambda}(h;v;\bar{g}) = h_{\alpha\beta}\mathcal{Q}_{(k)\lambda}{}^{\alpha\beta,\mu\nu}h_{\mu\nu} + \mathcal{L}_{(k)\lambda}{}^{\mu\nu}h_{\mu\nu} \tag{3.3.4}$$

where the regulating-gauge-fixing operators $\mathcal{Q}_{(k)\lambda}{}^{\alpha\beta,\mu\nu}$ and $\mathcal{L}_{(k)\lambda}{}^{\mu\nu}$ are given by:

$$\begin{aligned} \mathcal{Q}_{(k)\lambda}{}^{\alpha\beta,\mu\nu}(x,y,z) &\equiv \delta(x-y)\delta(y-z)\mathcal{Q}_{(k)\lambda}{}^{\alpha\beta,\mu\nu} \\ &= \delta(x-y)\delta(y-z)\cdot \\ &\quad \cdot \frac{1}{\kappa^2}\frac{v_\lambda(z)}{2v^2(z)}\left(\frac{1}{\alpha}\bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma}(z)\bar{\nabla}^\rho_{(z)}\bar{\nabla}^\sigma_{(z)} + \mathcal{R}_{gr(k)}^{\alpha\beta,\mu\nu}\left(-\bar{\Box}_{(z)}\right)\right) \end{aligned} \tag{3.3.5}$$

$$\begin{aligned}\mathcal{L}_{(k)\lambda}{}^{\mu\nu}(x,y) &\equiv \delta(x-y)\mathcal{L}_{(k)\lambda}{}^{\mu\nu} \\ &= \delta(x-y)\left(1 + r_{gh(k)}\left(-\bar{\Box}_{(y)}\right)\right)\bar{K}^{\mu\nu}{}_{\lambda\rho}(y)\bar{\nabla}^{\rho}_{(y)}\end{aligned} \tag{3.3.6}$$

In particular, expressing the regulator for the ghosts in the (non-restrictive) form (3.3.2), i.e. with a factorized background laplacian, allows to eliminate the Green function $\bar{\Box}^{-1}$, since from the defining property (3.2.53) we have $m^2_{gh}\sqrt{\bar{g}}\,\bar{\Box}^{-1} \to -r_{gh(k)}\left(-\bar{\Box}\right)\bar{\Box}\sqrt{\bar{g}}\,\bar{\Box}^{-1} = -r_{gh(k)}\left(-\bar{\Box}\right)$[11]. So, the explicit expression for the regulating-gauge-fixing function is:

$$\begin{aligned}f_{(k)\lambda}(h;v;\bar{g}) = \frac{1}{\kappa^2}\frac{v_\lambda}{2v^2}&\left(\frac{1}{\alpha}h_{\alpha\beta}\bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma}\bar{\nabla}^{\rho}\bar{\nabla}^{\sigma}h_{\mu\nu} + h_{\alpha\beta}\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}\right)h_{\mu\nu}\right) \\ &+\left(\bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^{\sigma}h_{\mu\nu} + r_{gh(k)}\left(-\bar{\Box}\right)\bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^{\sigma}h_{\mu\nu}\right)\end{aligned} \tag{3.3.7}$$

In particular, the regulating-gauge-fixing operators and thus the regulating-gauge-fixing itself are divided in a regulator-independent sector and a regulator-dependent sector (corresponding in the massive gauge-fixing in section 3.2 to the mass-independent sector and the mass-dependent sector):

$$\mathcal{Q}_{(k)\lambda}{}^{\alpha\beta,\mu\nu} = \mathcal{Q}_{0\lambda}{}^{\alpha\beta,\mu\nu} + \mathcal{Q}_{\mathcal{R}(k)\lambda}{}^{\alpha\beta,\mu\nu} \tag{3.3.8}$$

$$\mathcal{L}_{(k)\lambda}{}^{\mu\nu} = \mathcal{L}_{0\lambda}{}^{\mu\nu} + \mathcal{L}_{\mathcal{R}(k)\lambda}{}^{\mu\nu} \tag{3.3.9}$$

$$f_{(k)\lambda}(h;v;\bar{g}) = f_{0\lambda}(h;v;\bar{g}) + f_{\mathcal{R}(k)\lambda}(h;v;\bar{g}) \tag{3.3.10}$$

Clearly, due to the linear dependence on the gauge-fixing function of both the gauge-fixing term (thanks to our particular choice of gauge-fixing) and the ghost term (in general), the terms entering $S_0[\phi;v;\bar{g}]$ and $\Delta S_{(k)}[\phi;v;\bar{g}]$ are originated in the gauge-fixing sector by the corresponding parts of the gauge-fixing function, i.e. the regulator-independent one and the regulator-dependent one, without mixing. In particular, for the gauge-fixing term (3.2.5) we have:

$$S_{gf(k)}[h;v;\bar{g}] = S_{gf,0}[h;v;\bar{g}] + \Delta S_{gf(k)}[h;v;\bar{g}] \tag{3.3.11}$$

where the two contributions to $S_0[\phi;v;\bar{g}]$ and $\Delta S_{(k)}[\phi;v;\bar{g}]$ are respectively given by:

$$S_{gf,0}[h;v;\bar{g}] = \int d^Dx\sqrt{\bar{g}}\,v_\mu\bar{g}^{\mu\nu}f_{0\nu}(h;v;\bar{g}) \tag{3.3.12}$$

$$\Delta S_{gf(k)}[h;v;\bar{g}] = \int d^Dx\sqrt{\bar{g}}\,v_\mu\bar{g}^{\mu\nu}f_{\mathcal{R}(k)\nu}(h;v;\bar{g}) \tag{3.3.13}$$

Similarly, for the ghost term (3.2.6) we have:

$$S_{gh(k)}[h,c,\bar{c};v;\bar{g}] = S_{gh,0}[h,c,\bar{c};v;\bar{g}] + \Delta S_{gh(k)}[h,c,\bar{c};v;\bar{g}] \tag{3.3.14}$$

[11]This choice is essentially for writing convenience, since the terms appearing in the gauge-fixing sector are in this way apparently local; in fact, as we will see later, the ghost regulator $r_{gh(k)}\left(-\bar{\Box}\right)$ will be required to depend on the background laplacian according to an inverse relation in order to generate a proper FRG regulator term for the ghosts satisfying the standard properties.

where the two contributions to $S_0[\phi; v; \bar{g}]$ and $\Delta S_{(k)}[\phi; v; \bar{g}]$ are respectively given by:

$$S_{gh,0}[h, c, \bar{c}; v; \bar{g}] = -\int d^D x \sqrt{\bar{g}}\, \bar{c}_\alpha \bar{g}^{\alpha\beta} \frac{\delta f_{0\beta}(h; v; \bar{g})}{\delta h_{\mu\nu}} \bullet \pounds_c(\bar{g}_{\mu\nu} + h_{\mu\nu}) \tag{3.3.15}$$

$$\Delta S_{gh(k)}[h, c, \bar{c}; v; \bar{g}] = -\int d^D x \sqrt{\bar{g}}\, \bar{c}_\alpha \bar{g}^{\alpha\beta} \frac{f_{\mathcal{R}(k)\beta}(h; v; \bar{g})}{\delta h_{\mu\nu}} \bullet \pounds_c(\bar{g}_{\mu\nu} + h_{\mu\nu}) \tag{3.3.16}$$

We can now compute the explicit expression for the various terms. As far as the unregulated action $S_0[\phi; v; \bar{g}]$ is concerned, the contribution from the gauge-fixing term:

$$\begin{aligned} S_{gf,0}[h; v; \bar{g}] &= \int d^D x \sqrt{\bar{g}}\, v_\rho \bar{g}^{\rho\sigma} \left(h_{\alpha\beta} \mathcal{Q}_{0\sigma}{}^{\alpha\beta,\mu\nu} h_{\mu\nu} + \mathcal{L}_{0\sigma}{}^{\mu\nu} h_{\mu\nu} \right) = \\ &= \int d^D x \sqrt{\bar{g}} \left(\frac{1}{\kappa^2} \frac{1}{2\alpha} h_{\alpha\beta} \bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma} \bar{\nabla}^\rho \bar{\nabla}^\sigma h_{\mu\nu} \right. \\ &\qquad \left. + v_\lambda \bar{g}^{\lambda\rho} \bar{K}^{\mu\nu}{}_{\rho\sigma} \bar{\nabla}^\sigma h_{\mu\nu} \right) \end{aligned} \tag{3.3.17}$$

contains the standard de Donder gauge-fixing term and a residual linear term dependent on the external field:

$$S_{gf,0}[h; v; \bar{g}] = S_{gf-deD}[h; \bar{g}] + S_{gf-v}[h; v; \bar{g}] \tag{3.3.18}$$

i.e. :

$$S_{gf-deD}[h; \bar{g}] = -\frac{1}{\kappa^2} \int d^D x \sqrt{\bar{g}}\, \frac{1}{2\alpha} h_{\alpha\beta} \bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma} \bar{\nabla}^\rho \bar{\nabla}^\sigma h_{\mu\nu} \tag{3.3.19}$$

$$S_{gf-v}[h; v; \bar{g}] = \int d^D x \sqrt{\bar{g}}\, v_\lambda \bar{g}^{\lambda\rho} \bar{K}^{\mu\nu}{}_{\rho\sigma} \bar{\nabla}^\sigma h_{\mu\nu} \tag{3.3.20}$$

The contribution from the ghost term:

$$\begin{aligned} &S_{gh,0}[h, c, \bar{c}; v; \bar{g}] = \\ &\quad = -\int d^D x \sqrt{\bar{g}}\, \bar{c}_\rho \bar{g}^{\rho\sigma} \left(h_{\alpha\beta} \mathcal{Q}_{0\sigma}{}^{\alpha\beta,\mu\nu} + \mathcal{Q}_{0\sigma}{}^{\mu\nu,\alpha\beta} h_{\alpha\beta} + \mathcal{L}_{0\sigma}{}^{\mu\nu} \right) \pounds_c(\bar{g}_{\mu\nu} + h_{\mu\nu}) = \\ &\quad = \int d^D x \sqrt{\bar{g}} \left[-\frac{1}{\kappa^2} \frac{v_\lambda}{v^2} \bar{g}^{\lambda\tau} \bar{c}_\tau \frac{1}{2\alpha} \bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma} \left(\bar{\nabla}^\rho \bar{\nabla}^\sigma h_{\mu\nu} + h_{\mu\nu} \bar{\nabla}^\rho \bar{\nabla}^\sigma \right) \pounds_c(\bar{g}_{\mu\nu} + h_{\mu\nu}) \right. \\ &\qquad\qquad \left. - \bar{c}_\lambda \bar{g}^{\lambda\rho} \bar{K}^{\mu\nu}{}_{\rho\sigma} \bar{\nabla}^\sigma \pounds_c(\bar{g}_{\mu\nu} + h_{\mu\nu}) \right] \end{aligned} \tag{3.3.21}$$

contains the standard de Donder ghost term, sum of the kinetic ghost term and the de Donder interaction term, and a residual higher-derivative $hhc\bar{c}$-interaction term dependent on the external field:

$$S_{gh,0}[h, c, \bar{c}; v; \bar{g}] = S_{gh-kin}[c, \bar{c}; \bar{g}] + S_{gh-deD-int}[h, c, \bar{c}; \bar{g}] + S_{gh-v}[h, c, \bar{c}; v; \bar{g}] \tag{3.3.22}$$

i.e. :

$$S_{gh-kin}[c,\bar{c};\bar{g}] = \int d^Dx\sqrt{\bar{g}}\,\bar{c}_\mu\left(-\bar{\Box}\right)c^\mu \tag{3.3.23}$$

$$S_{gh-deD-int}[h,c,\bar{c};\bar{g}] = \int d^Dx\sqrt{\bar{g}}\,\left[-\bar{c}_\lambda\bar{g}^{\lambda\rho}\left(\bar{R}_{\rho\tau}c^\tau + \bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\sigma\pounds_c h_{\mu\nu}\right)\right] \tag{3.3.24}$$

$$\begin{aligned} &S_{gh-v}[h,c,\bar{c};v;\bar{g}] = \\ &= -\int d^Dx\sqrt{\bar{g}}\,\frac{1}{\kappa^2}\frac{v_\lambda}{v^2}\bar{g}^{\lambda\tau}\bar{c}_\tau\frac{1}{2\alpha}\bar{D}^{\alpha\beta,\mu\nu}{}_{\rho\sigma}\left(\bar{\nabla}^\rho\bar{\nabla}^\sigma h_{\mu\nu} + h_{\mu\nu}\bar{\nabla}^\rho\bar{\nabla}^\sigma\right)\pounds_c(\bar{g}_{\mu\nu}+h_{\mu\nu}) \end{aligned} \tag{3.3.25}$$

Summing the two contributions to the Einstein-Hilbert action, the unregulated action is:

$$S_0[h,c,\bar{c};v;\bar{g}] = S_{EH}[h;\bar{g}] + S_{gf,0}[h;v;\bar{g}] + S_{gh,0}[h,c,\bar{c};v;\bar{g}] \tag{3.3.26}$$

Clearly, since it is the result of the gauge-fixing without the contribution of the masses - regulators, it coincides with a massive gauge-fixed Einstein-Hilbert action (3.2.70) where mass parameters have been set to zero. Essentially, it is the standard de Donder gauge-fixed action with the additional terms dependent on the external field, and we will thus refer to it as modified de Donder gauge-fixed action.
As far as the regulator term $\Delta S_{(k)}[\phi;v;\bar{g}]$ is concerned, the contribution from the gauge-fixing term:

$$\begin{aligned} \Delta S_{gf(k)}[h;v;\bar{g}] &= \int d^Dx\sqrt{\bar{g}}\,v_\rho\bar{g}^{\rho\sigma}\left(h_{\alpha\beta}\mathcal{Q}_{\mathcal{R}(k)\sigma}{}^{\alpha\beta,\mu\nu}h_{\mu\nu} + \mathcal{L}_{\mathcal{R}(k)\sigma}{}^{\mu\nu}h_{\mu\nu}\right) = \\ &= \int d^Dx\sqrt{\bar{g}}\,\left(\frac{1}{2\kappa^2}h_{\alpha\beta}\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}\right)h_{\mu\nu}\right. \\ &\qquad\left. + v_\lambda\bar{g}^{\lambda\rho}r_{gh(k)}\left(-\bar{\Box}\right)\bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\sigma h_{\mu\nu}\right) \end{aligned} \tag{3.3.27}$$

contains a quadratic regulator term for the metric fluctuation and an additional linear in the metric fluctuation also dependent on the external field:

$$\Delta S_{gf(k)}[h;v;\bar{g}] = \Delta S_{gf-hh(k)}[h;\bar{g}] + \Delta S_{gf-v(k)}[h;v;\bar{g}] \tag{3.3.28}$$

i.e. :

$$\Delta S_{gf-hh(k)}[h;\bar{g}] = \frac{1}{2\kappa^2}\int d^Dx\sqrt{\bar{g}}\,h_{\alpha\beta}\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}\right)h_{\mu\nu} \tag{3.3.29}$$

$$\Delta S_{gf-v(k)}[h;v;\bar{g}] = \int d^Dx\sqrt{\bar{g}}\,v_\lambda\bar{g}^{\lambda\rho}r_{gh(k)}\left(-\bar{\Box}\right)\bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\sigma h_{\mu\nu} \tag{3.3.30}$$

The contribution from the ghost term:

$$\begin{aligned} &\Delta S_{gh(k)}[h,c,\bar{c};v;\bar{g}] = \\ &= -\int d^Dx\sqrt{\bar{g}}\,\bar{c}_\rho\bar{g}^{\rho\sigma}\left(h_{\alpha\beta}\mathcal{Q}_{\mathcal{R}(k)\sigma}{}^{\alpha\beta,\mu\nu} + \mathcal{Q}_{\mathcal{R}(k)\sigma}{}^{\mu\nu,\alpha\beta}h_{\alpha\beta} + \mathcal{L}_{\mathcal{R}(k)\sigma}{}^{\mu\nu}\right)\pounds_c(\bar{g}_{\mu\nu}+h_{\mu\nu}) = \\ &= \int d^Dx\sqrt{\bar{g}}\,\left[-\frac{1}{\kappa^2}\frac{v_\lambda}{2v^2}\bar{g}^{\lambda\tau}\bar{c}_\tau\left(h_{\alpha\beta}\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}\right) + \mathcal{R}^{\mu\nu,\alpha\beta}_{gr(k)}\left(-\bar{\Box}\right)h_{\alpha\beta}\right)\pounds_c(\bar{g}_{\mu\nu}+h_{\mu\nu})\right. \end{aligned}$$

$$- \bar{c}_\lambda \bar{g}^{\lambda\rho} r_{gh(k)} \left(-\bar{\Box}\right) \bar{K}^{\mu\nu}{}_{\rho\sigma} \bar{\nabla}^\sigma \pounds_c (\bar{g}_{\mu\nu} + h_{\mu\nu}) \Big] \tag{3.3.31}$$

contains a regulator-dependent version of the standard de Donder ghost term and a mixed $hhc\bar{c}$ regulator term also dependent in the external field. Similarly to how was done to reveal the mass term for the ghosts in the massive gauge-fixing, we rewrite the former using identity (2.3.47), in this case grouping the two quadratic pieces antighost-ghost:

$$\begin{aligned}
&\Delta S_{gh-r_{gh}\times deD(k)}[h, c, \bar{c}; \bar{g}] = \\
&\quad = - \int d^D x \sqrt{\bar{g}}\, \bar{c}_\lambda \bar{g}^{\lambda\rho} r_{gh(k)} \left(-\bar{\Box}\right) \bar{K}^{\mu\nu}{}_{\rho\sigma} \bar{\nabla}^\sigma \pounds_c (\bar{g}_{\mu\nu} + h_{\mu\nu}) = \\
&\quad = \int d^D x \sqrt{\bar{g}} \Big\{ \left[-\bar{c}_\lambda r_{gh(k)} \left(-\bar{\Box}\right) \left(\bar{\Box}\, \delta^\lambda_\tau + \bar{R}^\lambda{}_\tau \right) c^\tau \right] \\
&\qquad\qquad + \left[-\bar{c}_\lambda \bar{g}^{\lambda\rho} r_{gh(k)} \left(-\bar{\Box}\right) \bar{K}^{\mu\nu}{}_{\rho\sigma} \bar{\nabla}^\sigma \pounds_c h_{\mu\nu} \right] \Big\} = \\
&\quad = \Delta S_{gh-\bar{c}c(k)}[c, \bar{c}; \bar{g}] + \Delta S_{gh-h\bar{c}c(k)}[h, c, \bar{c}; \bar{g}]
\end{aligned} \tag{3.3.32}$$

We thus recognize a quadratic regulator term for the ghosts, in which also the piece with the Ricci tensor is considered, and an additional mixed $hc\bar{c}$ regulator term corresponding to a regulator-dependent version of the de Donder $hc\bar{c}$-interaction term. Regarding the mixed $hhc\bar{c}$ regulator term, we notice that the two terms $h_{\alpha\beta}\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}\right)$ and $\mathcal{R}^{\mu\nu,\alpha\beta}_{gr(k)}\left(-\bar{\Box}\right) h_{\alpha\beta}$, coming from the contractions $h_{\alpha\beta}\mathcal{Q}_{\mathcal{R}(k)\sigma}{}^{\alpha\beta,\mu\nu}$ and $\mathcal{Q}_{\mathcal{R}(k)\sigma}{}^{\mu\nu,\alpha\beta} h_{\alpha\beta}$, are not equal due to the operatorial nature of the regulator and its different position (contrary to the corresponding terms $m^2_{gr}\bar{M}^{\alpha\beta,\mu\nu}$ in the massive gauge-fixing). So, the contribution from the ghost term to the total regulator term is given by:

$$\Delta S_{gh(k)}[h, c, \bar{c}; v; \bar{g}] = \Delta S_{gh-\bar{c}c(k)}[c, \bar{c}; \bar{g}] + \Delta S_{gh-hc\bar{c}(k)}[h, c, \bar{c}; \bar{g}] + \Delta S_{gh-v(k)}[h, c, \bar{c}; v; \bar{g}] \tag{3.3.33}$$

with:

$$\Delta S_{gh-\bar{c}c(k)}[c, \bar{c}; \bar{g}] = \int d^D x \sqrt{\bar{g}}\, \bar{c}_\mu \left[-r_{gh(k)} \left(-\bar{\Box}\right) \left(\bar{\Box}\, \delta^\mu_\nu + \bar{R}^\mu{}_\nu \right) \right] c^\nu \tag{3.3.34}$$

$$\Delta S_{gh-hc\bar{c}(k)}[h, c, \bar{c}; \bar{g}] = - \int d^D x \sqrt{\bar{g}}\, \bar{c}_\lambda \bar{g}^{\lambda\rho} r_{gh(k)} \left(-\bar{\Box}\right) \bar{K}^{\mu\nu}{}_{\rho\sigma} \bar{\nabla}^\sigma \pounds_c h_{\mu\nu} \tag{3.3.35}$$

$$\begin{aligned}
&\Delta S_{gh-v(k)}[h, c, \bar{c}; v; \bar{g}] = \\
&\quad = - \int d^D x \sqrt{\bar{g}}\, \frac{1}{\kappa^2} \frac{v_\lambda}{2v^2} \bar{g}^{\lambda\tau} \bar{c}_\tau \left(h_{\alpha\beta} \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)} \left(-\bar{\Box}\right) + \mathcal{R}^{\mu\nu,\alpha\beta}_{gr(k)} \left(-\bar{\Box}\right) h_{\alpha\beta} \right) \pounds_c (\bar{g}_{\mu\nu} + h_{\mu\nu})
\end{aligned} \tag{3.3.36}$$

Summing the two contributions the total regulator term is:

$$\Delta S_{(k)}[h, c, \bar{c}; v; \bar{g}] = \Delta S_{gf(k)}[h; v; \bar{g}] + \Delta S_{gh(k)}[h, c, \bar{c}; v; \bar{g}] \tag{3.3.37}$$

As the standard regulator term (2.4.1), it contains the two desired quadratic regulator terms for the metric fluctuation and ghosts, but in addition also non-quadratic terms, namely a linear term in the metric fluctuation and two mixed $hc\bar{c}$ and $hhc\bar{c}$ terms. In particular, the quadratic terms should be responsible of implementing a correct FRG coarse-graining and therefore they should contain regulators with functional form $\mathcal{R}_{gr(k)}(p^2)$, $\mathcal{R}_{gh(k)}(p^2)$ satisfying the properties (2.4.5):

$$\begin{cases} \textbf{1.} \quad \mathcal{R}_{gr/gh(k)}(p^2) \quad \to \quad >0 \quad , \quad \frac{p^2}{k^2} \to 0 \\ \textbf{2.} \quad \mathcal{R}_{gr/gh(k)}(p^2) \quad \to \quad 0 \quad , \quad \frac{p^2}{k^2} \to \infty \\ \textbf{3.} \quad \mathcal{R}_{gr/gh(k)}(p^2) \quad \to \quad \infty \quad , \quad k^2 \to k_{UV}^2 \to \infty \\ \textbf{4.} \quad \mathcal{R}_{gr/gh(k)}(p^2) \quad \to \quad 0 \quad , \quad k^2 \to 0 \end{cases} \tag{3.3.38}$$

where k_{UV} is some large UV scale. The quadratic regulator term for the metric fluctuation has the same form of the standard one and the operator $\mathcal{R}_{gr(k)}^{\alpha\beta,\mu\nu}\left(-\bar{\Box}\right)$, introduced with the substitution rule (3.3.1), is already the corresponding regulator; as we will see in the following, it will be sufficient to consider the standard form:

$$\mathcal{R}_{gr(k)}^{\alpha\beta,\mu\nu}\left(-\bar{\Box}\right) = \mathcal{R}_{gr(k)}^{\alpha\beta,\mu\nu}\mathcal{R}_{gr(k)}\left(-\bar{\Box}\right) \tag{3.3.39}$$

Instead, the quadratic regulator term for the ghosts, differently from the standard one, has a dependence on the Ricci tensor and the corresponding regulator is related to the operator $r_{gr(k)}\left(-\bar{\Box}\right)$ introduced with the substitution rule (3.3.2) by:

$$\mathcal{R}_{gh(k)}{}^{\mu}{}_{\nu}\left(-\bar{\Box}\right) = -r_{gh(k)}\left(-\bar{\Box}\right)\left(\bar{\Box}\,\delta^{\mu}_{\nu} + \bar{R}^{\mu}{}_{\nu}\right) \tag{3.3.40}$$

For now we leave the functional form of the operator $r_{gr(k)}\left(-\bar{\Box}\right)$ unspecified, assuming that it is such that the properties (3.3.38) are satisfied; later we will give a specific expression useful to simplify computations.
Finally, the regulated-gauge-fixed action is:

$$\tilde{S}_{(k)}[h, c, \bar{c}; v; \bar{g}] = S_0[h, c, \bar{c}; v; \bar{g}] + \Delta S_{(k)}[h, c, \bar{c}; v; \bar{g}] \tag{3.3.41}$$

At the quantum level, the regularized theory is described by the regularized generating functionals constructed with the regulated-gauge-fixed action. As in the standard case in subsection 2.4.1, they are also implicitly regulated with k_{UV} as sharp UV cut-off on the values of the generalized momenta considered in the measure, $\prod_{0\leq|p|\leq|k_{UV}|} d\mu[h_{(p)\mu\nu}, \bar{c}_{(p)\mu}, c^{\mu}_{(p)}]$. The regulated expressions correspondent to those in subsection 3.2.1 are:

$$\tilde{Z}_{(k)}[J; K; v; \bar{g}] = \int \mathcal{D}\mu[\phi]\, e^{-\tilde{S}_{(k)}[\phi;v;\bar{g}] - S_{source}[\phi;J;K;v;\bar{g}]} \tag{3.3.42}$$

$$\tilde{W}_{(k)}[J; K; v; \bar{g}] = \log \tilde{Z}_{(k)}[J; K; v; \bar{g}] \tag{3.3.43}$$

$$\begin{aligned}\tilde{\Gamma}_{(k)}[\Phi;K;v;\bar{g}] &= \sup_J \left\{ \int d^D x \sqrt{\bar{g}}\, J_i^+ \Phi^i - \tilde{W}_{(k)}[J;K;v;\bar{g}] \right\} \\ &= \int d^D x \sqrt{\bar{g}}\, J_i^+ \Phi^i - \tilde{W}_{(k)}[J;K;v;\bar{g}]\end{aligned} \tag{3.3.44}$$

$$\begin{aligned}\tilde{W}_{(k)}[J;K;v;\bar{g}] &= \sup_\Phi \left\{ \int d^D x \sqrt{\bar{g}}\, J_i^+ \Phi^i - \tilde{\Gamma}_{(k)}[\Phi;K;v;\bar{g}] \right\} \\ &= \int d^D x \sqrt{\bar{g}}\, J_i^+ \Phi^i - \tilde{\Gamma}_{(k)}[\Phi;K;v;\bar{g}]\end{aligned} \tag{3.3.45}$$

Now all averages are k-dependent, as well as fields and sources related in the Legendre transform, $J = J_{(k)}(\Phi, K)$, $\Phi = \Phi_{(k)}(J, K)$:

$$\begin{aligned}\Phi^i(x) = \left\langle \phi^i(x) \right\rangle_{JK} &= \frac{1}{\sqrt{\bar{g}(x)}} \frac{\delta \tilde{W}_{(k)}[J;K;v;\bar{g}]}{\delta J_i^+(x)} \\ \Phi^{+i}(x) = \left\langle \phi^{+i}(x) \right\rangle_{JK} &= \frac{1}{\sqrt{\bar{g}(x)}} \frac{\tilde{W}_{(k)} \overleftarrow{\delta} [J;K;v;\bar{g}]}{\delta J_i(x)}\end{aligned} \tag{3.3.46}$$

$$\begin{aligned}J_i(x) &= \frac{1}{\sqrt{\bar{g}(x)}} \frac{\delta \tilde{\Gamma}_{(k)}[\Phi;K;v;\bar{g}]}{\delta \Phi^{+i}(x)} \\ J_i^+(x) &= \frac{1}{\sqrt{\bar{g}(x)}} \frac{\tilde{\Gamma}_{(k)} \overleftarrow{\delta} [\Phi;K;v;\bar{g}]}{\delta \Phi^i(x)}\end{aligned} \tag{3.3.47}$$

$$\begin{aligned}\Psi^i(x) = \left\langle \psi^i(x) \right\rangle_{JK} &= \frac{1}{\sqrt{\bar{g}(x)}} \frac{\delta \tilde{W}_{(k)}[J;K;v;\bar{g}]}{\delta K_i^+(x)} = -\frac{1}{\sqrt{\bar{g}(x)}} \frac{\delta \tilde{\Gamma}_{(k)}[\Phi;K;v;\bar{g}]}{\delta K_i^+(x)} \\ \Psi^{+i}(x) = \left\langle \psi^{+i}(x) \right\rangle_{JK} &= \frac{1}{\sqrt{\bar{g}(x)}} \frac{\tilde{W}_{(k)} \overleftarrow{\delta} [J;K;v;\bar{g}]}{\delta K_i(x)} = -\frac{1}{\sqrt{\bar{g}(x)}} \frac{\tilde{\Gamma}_{(k)} \overleftarrow{\delta} [\Phi;K;v;\bar{g}]}{\delta K_i(x)}\end{aligned} \tag{3.3.48}$$

The integro-differential equation for the regulated effective action is:

$$\begin{aligned}e^{-\tilde{\Gamma}_{(k)}[\Phi;K;v;\bar{g}]} = \int \mathcal{D}\mu[\phi]\, \exp\Bigg\{ &-\tilde{S}_{(k)}[\phi;\bar{g}] - S_{source-K}[\phi;K;v;\bar{g}] \\ &+ \int d^D x \frac{\tilde{\Gamma}_{(k)} \overleftarrow{\delta} [\Phi;K;v;\bar{g}]}{\delta \Phi^i} \left(\phi^i - \Phi^i \right) \Bigg\}\end{aligned} \tag{3.3.49}$$

<u>*Effective average action*</u>

The formal definition, i.e. based on the above generating functionals, of the effective average action describing the scale-dependent theory interpolating between the classical and quantum regime is:

$$\Gamma_{(k)}[\Phi;K;v;\bar{g}] \equiv \tilde{\Gamma}_{(k)}[\Phi;K;v;\bar{g}] - \Delta S_{(k)}[\Phi;v;\bar{g}] \tag{3.3.50}$$

As in the standard case, the second couple of required properties for the regulators in (3.3.38) allows to formally fix the limits of its flow:

$$\Gamma_{(k\to 0)}[\Phi;K;v;\bar{g}] = \Gamma_0[\Phi;K;v;\bar{g}] \tag{3.3.51}$$

$$\Gamma_{(k\to k_{UV}\to\infty)}[\Phi;K;v;\bar{g}] = S_0[\Phi;v;\bar{g}] + S_{source-K}[\Phi;K;v;\bar{g}] + \cdots \tag{3.3.52}$$

In particular, for $k \to 0$, the effective average action tends to the unregulated quantum effective action, i.e. the effective action for the modified de Donder gauge-fixed theory, since $\Delta S_{(k\to 0)} \to 0$ (property **4.**):

$$\Gamma_{(k\to 0)}[\Phi;K;v;\bar{g}] = \tilde{\Gamma}_{(k\to 0)}[\Phi;K;v;\bar{g}] - \Delta S_{(k\to 0)}[\Phi;v;\bar{g}] = \Gamma_0[\Phi;K;v;\bar{g}] - 0$$

For $k \to k_{UV} \to \infty$, the effective average action is expected to approximately tend to the unregulated classic action, i.e. the modified de Donder gauge-fixed action (plus the source term for non-elementary fields, if those are not set to zero), with corrective terms related to the reconstruction problem. In principle, this follows similarly to the standard case from the integro-differential equation (3.3.49) rewritten for the effective average action:

$$\begin{aligned} e^{-\Gamma_{(k)}[\Phi;K;v;\bar{g}]} = \int \mathcal{D}\mu[\phi] \exp\Bigg\{ & - S_0[\phi;\bar{g}] - S_{source-K}[\phi;K;v;\bar{g}] \\ & - \Delta S_{(k)}[\phi;v;\bar{g}] + \Delta S_{(k)}[\Phi;v;\bar{g}] \\ & + \int d^D x \frac{\Delta S_{(k)} \overleftarrow{\delta} [\Phi;v;\bar{g}]}{\delta \Phi^i} \left(\phi^i - \Phi^i\right) \\ & + \int d^D x \frac{\Gamma_{(k)} \overleftarrow{\delta} [\Phi;K;v;\bar{g}]}{\delta \Phi^i} \left(\phi^i - \Phi^i\right) \Bigg\} \end{aligned} \tag{3.3.53}$$

Since due to the limit $\mathcal{R}_{gr/gh(k\to k_{UV}\to\infty)} \to \infty$ (property **3.**) we expect a divergent contribution in the exponent:

$$-\Delta S_{(k)}[\phi;v;\bar{g}] + \Delta S_{(k)}[\Phi;v;\bar{g}] + \int d^D x \frac{\Delta S_{(k)} \overleftarrow{\delta} [\Phi;v;\bar{g}]}{\delta \Phi} (\phi - \Phi) \tag{3.3.54}$$

However, due to the non-quadratic mixed pieces now contained in the regulator term (3.3.37), we do not have the simple expression (2.4.21) featuring the regulator term computed in the difference $\phi - \Phi$. Therefore, the limit should be treated more precisely via a saddle-point approximation, in order to properly set the reconstruction problem; see for instance [6] for the analogous treatment in the Yang-Mills theory.

Symmetries and theory space

Similarly to the standard case, assuming that the regulators are constructed with a proper tensorial structure, the regulated-gauge-fixed action is background gauge-invariant:

$$\bar{\delta}_\xi \tilde{S}_{(k)}[h, c, \bar{c}; v; \bar{g}] = 0 \tag{3.3.55}$$

and at the quantum level, the regulated functionals are explicitly background gauge-invariant:

$$\bar{\delta}_\xi W_{(k)}[J;K;v;\bar{g}] \ , \ \bar{\delta}_\xi \Gamma_{(k)}[\Phi;K;v;\bar{g}] \ , \ \bar{\delta}_\xi \Gamma_{(k)}[\Phi;K;v;\bar{g}] = 0 \tag{3.3.56}$$

Contrary to the standard case, BRST symmetry is now always manifestly preserved by construction, since as in the general case with a generic gauge-fixing function (3.2.40), we have:

$$\delta_\theta \left(S_{gf(k)}[h;v;\bar{g}] + S_{gh(k)}[h,c,\bar{c};v;\bar{g}]\right) = 0 \tag{3.3.57}$$

and therefore:

$$\delta_\theta \tilde{S}_{(k)}[h,c,\bar{c};v;\bar{g}] = 0 \tag{3.3.58}$$

Moreover, expressing the terms entering $S_0[\phi;v;\bar{g}]$ and $\Delta S_{(k)}[\phi;v;\bar{g}]$ as in (3.3.12)-(3.3.16) and summing them accordingly, we can recognize the same explicit BRST-invariant structure of the total action, just with the complete gauge-fixing function substituted respectively by its regulator independent and its regulator-dependent parts; consequently, we have also:

$$\delta_\theta \left(S_{0,gf}[h;v;\bar{g}] + S_{0,gh}[h,c,\bar{c};v;\bar{g}]\right) = 0 \tag{3.3.59}$$

$$\delta_\theta \left(\Delta S_{gf(k)}[h;v;\bar{g}] + \Delta S_{gh(k)}[h,c,\bar{c};v;\bar{g}]\right) = 0 \tag{3.3.60}$$

and therefore, individually:

$$\delta_\theta S_0[h,c,\bar{c};v;\bar{g}] = 0 \tag{3.3.61}$$

$$\delta_\theta \Delta S_{(k)}[h,c,\bar{c};v;\bar{g}] = 0 \tag{3.3.62}$$

At the quantum level, this appears in the fact that the regulated effective action satisfies by construction exactly the same Zinn-Justin equation of the unregulated theory (3.2.46) for any value of the FRG scale k. Indeed, having performed technically just a particular gauge-fixing without introducing by hand the regulators, the Ward-Takahashi equation (3.2.45) is by definition unchanged:

$$\begin{aligned}\int d^Dx\sqrt{\bar{g}}\Big(t^{\mu\nu}\left\langle sh_{\mu\nu}\right\rangle_{JK} - \bar{\eta}_\mu \left\langle sc^\mu\right\rangle_{JK} + \eta^\mu v_\mu + \\ + m^{\mu\nu}\left\langle sh_{\mu\nu}\right\rangle_{JK} - n^{\mu\nu}\left\langle h_{\mu\nu}\right\rangle_{JK} + n^{\mu\nu}\left\langle H_{\mu\nu}\right\rangle_{JK}\Big) = 0\end{aligned} \tag{3.3.63}$$

apart from the fact that averages are now obtained from the regulated functionals, and thus k-dependent. Therefore, substituting sources and averages according to the regulated expressions (3.3.47) and (3.3.48), we also arrive at the same Zinn-Justin equation for the regulated effective action:

$$\begin{aligned}\Sigma\left[\tilde{\Gamma}_{(k)}\right] = \int d^Dx\Big(-\frac{1}{\sqrt{\bar{g}}}\frac{\delta\tilde{\Gamma}_{(k)}}{\delta \mathsf{h}_{\mu\nu}}\frac{\delta\tilde{\Gamma}_{(k)}}{\delta k^{\mu\nu}} - \frac{1}{\sqrt{\bar{g}}}\frac{\delta\tilde{\Gamma}_{(k)}}{\delta \mathsf{c}^\mu}\frac{\delta\tilde{\Gamma}_{(k)}}{\delta l_\mu} + v_\mu \frac{\delta\tilde{\Gamma}_{(k)}}{\delta \bar{\mathsf{c}}_{\ \mu}} + \\ - m^{\mu\nu}\frac{1}{\sqrt{\bar{g}}}\frac{\delta\tilde{\Gamma}_{(k)}}{\delta k^{\mu\nu}} - n^{\mu\nu}\mathsf{h}_{\mu\nu} - n^{\mu\nu}\frac{1}{\sqrt{\bar{g}}}\frac{\delta\tilde{\Gamma}_{(k)}}{\delta m^{\mu\nu}}\Big) = 0\end{aligned} \tag{3.3.64}$$

and it is satisfied by construction for any value of the FRG scale k:

$$\Sigma\left[\tilde{\Gamma}_{(k)}\right]=0 \quad , \quad \forall k \tag{3.3.65}$$

We do not derive an explicit form for the equivalent constraint satisfied for any value of the FRG scale k by the effective average action, since in the following we will mainly focus for technical convenience on the regulated effective action. We implicitly indicate it also as $\Sigma\left[\Gamma_{(k)}\right]=0$.

According to those symmetry properties, the theory space is now given by background gauge-invariant operators lying on the fixed hypersurface of BRST-invariance operators $\Sigma=0$:

$$\mathcal{T}=\left\{A[\Phi;K;\bar{g}] \quad : \quad \bar{\delta}_\xi A[\Phi;K;\bar{g}]=0 \quad , \quad \Sigma[A[\Phi;K;\bar{g}]]=0\right\} \tag{3.3.66}$$

3.3.2 BRST-compatible Wetterich-Morris equation

In this subsection we present the Wetterich-Morris equation describing the flow of the effective average action between in the BRST-symmetrically-regulated theory. Actually, due to the higher algebraic complexity with respect to the standard case, we will focus on the flow of the regulated effective action, which is the only one for which we derive an explicit flow equation.
As in the standard case we have the following equalities between the derivatives with respect to the FRG time $t=\log k$ of the regulated generating functionals:

$$\begin{aligned}\partial_t\tilde{\Gamma}_{(k)}[\Phi;K;v;\bar{g}]=-\partial_t\tilde{W}_{(k)}[J;K;v;\bar{g}]&=-\frac{\partial_t\tilde{Z}_{(k)}[J;Kv;\bar{g}]}{\tilde{Z}_{(k)}[J;K;v;\bar{g}]}\\&=\left\langle\partial_t\Delta S_{(k)}[\phi;v;\bar{g}]\right\rangle_{JK}\end{aligned} \tag{3.3.67}$$

with time derivatives of source-dependent objects computed in $J=J_{(k)}(\Phi,K)$; in particular:

$$\partial_t\tilde{\Gamma}_{(k)}[\Phi;K;v;\bar{g}]=\left\langle\partial_t\Delta S_{(k)}[\phi;v;\bar{g}]\right\rangle_{JK} \tag{3.3.68}$$

In order to compute the average in the right hand side, we rewrite from the previous subsection the explicit result for the regulator term; for convenience, in the ghost contribution (3.3.31) we keep together the two pieces coming from the regulator-dependent version of the de Donder ghost term and we separate the two pieces in the term dependent on the external field, renaming indices in the second such that both pieces contain the regulator for the metric fluctuation with the same index structure:

$$\begin{aligned}\Delta S_{(k)}[\phi;v;\bar{g}]=\int d^Dx\sqrt{\bar{g}}\Bigg[&\frac{1}{2\kappa^2}h_{\alpha\beta}\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}\right)h_{\mu\nu}\\&+v_\lambda\bar{g}^{\lambda\rho}r_{gh(k)}\left(-\bar{\Box}\right)\bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\sigma h_{\mu\nu}\\&-\bar{c}_\lambda\bar{g}^{\lambda\rho}r_{gh(k)}\left(-\bar{\Box}\right)\bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\sigma\pounds_c(\bar{g}_{\mu\nu}+h_{\mu\nu})\\&-\frac{1}{\kappa^2}\frac{v_\lambda}{2v^2}\bar{g}^{\lambda\tau}\bar{c}_\tau h_{\alpha\beta}\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}\right)\pounds_c(\bar{g}_{\mu\nu}+h_{\mu\nu})\\&-\frac{1}{\kappa^2}\frac{v_\lambda}{2v^2}\bar{g}^{\lambda\tau}\bar{c}_\tau\pounds_c(\bar{g}_{\alpha\beta}+h_{\alpha\beta})\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}\right)h_{\mu\nu}\Bigg]\end{aligned} \tag{3.3.69}$$

In the last piece we also commuted the Lie derivative, Grassmann-odd, with the regulator and the metric fluctuation, Grassmann-even, to bring it on the left side of the regulator. It follows that:

$$\begin{aligned}\partial_t\tilde{\Gamma}_{(k)}[\Phi;K;v;\bar{g}] = \int d^Dx\sqrt{\bar{g}}\Bigg[&\frac{1}{2\kappa^2}\left\langle h_{\alpha\beta}\partial_t\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}\right)h_{\mu\nu}\right\rangle_{JK}\\ &+\left\langle v_\lambda\bar{g}^{\lambda\rho}\partial_t r_{gh(k)}\left(-\bar{\Box}\right)\bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\sigma h_{\mu\nu}\right\rangle_{JK}\\ &-\left\langle\bar{c}_\lambda\bar{g}^{\lambda\rho}\partial_t r_{gh(k)}\left(-\bar{\Box}\right)\bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\sigma\pounds_c(\bar{g}_{\mu\nu}+h_{\mu\nu})\right\rangle_{JK}\\ &-\frac{1}{2\kappa^2}\left\langle\frac{v_\lambda}{v^2}\bar{g}^{\lambda\tau}\bar{c}_\tau h_{\alpha\beta}\partial_t\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}\right)\pounds_c(\bar{g}_{\mu\nu}+h_{\mu\nu})\right\rangle_{JK}\\ &-\frac{1}{2\kappa^2}\left\langle\frac{v_\lambda}{v^2}\bar{g}^{\lambda\tau}\bar{c}_\tau\pounds_c(\bar{g}_{\alpha\beta}+h_{\alpha\beta})\partial_t\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}\right)h_{\mu\nu}\right\rangle_{JK}\Bigg]\end{aligned} \tag{3.3.70}$$

We again introduce an additional integration in order to formally express the objects on which the regulators act in a different spacetime variable, and then collect the regulators:

$$\begin{aligned}&\partial_t\tilde{\Gamma}_{(k)}[\Phi;K;v;\bar{g}] =\\ &\quad=\int d^Dxd^Dy\sqrt{\bar{g}(x)}\,\delta(x-y)\Bigg[\\ &\qquad\frac{1}{2\kappa^2}\partial_t\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}_{(y)}\right)\Bigg(\langle h_{\alpha\beta}(x)h_{\mu\nu}(y)\rangle_{JK}\\ &\qquad\qquad-\left\langle\frac{v_\lambda(x)}{v^2(x)}\bar{g}^{\lambda\tau}(x)\bar{c}_\tau(x)h_{\alpha\beta}(x)\pounds_c(\bar{g}_{\mu\nu}+h_{\mu\nu})(y)\right\rangle_{JK}\\ &\qquad\qquad-\left\langle\frac{v_\lambda(x)}{v^2(x)}\bar{g}^{\lambda\tau}(x)\bar{c}_\tau(x)\pounds_c(\bar{g}_{\alpha\beta}+h_{\alpha\beta})(x)h_{\mu\nu}(y)\right\rangle_{JK}\Bigg)\\ &\qquad+\bar{g}^{\lambda\rho}(y)\partial_t r_{gh(k)}\left(-\bar{\Box}_{(y)}\right)\bar{K}^{\mu\nu}{}_{\rho\sigma}(y)\bar{\nabla}^\sigma_{(y)}\Big(\langle v_\lambda(x)h_{\mu\nu}(y)\rangle_{JK}\\ &\qquad\qquad-\langle\bar{c}_\lambda(x)\pounds_c(\bar{g}_{\mu\nu}+h_{\mu\nu})(y)\rangle_{JK}\Big)\Bigg]\end{aligned} \tag{3.3.71}$$

With respect to the standard case, due to the non-standard non-quadratic regulator terms introduced in the regulating-gauge-fixing procedure, we have the crucial difference that not all averages are 2-point correlation functions of elementary fields. Taking inspiration from [6], we choose to recover a situation where only 2-point correlation functions, possibly of non-elementary fields, are present; in this way, we will obtain a Wetterich-Morris equation containing the inverse matrix of second derivatives of the regulated effective action, similarly to the standard case, and additional derivatives at most of the second order.

We achieve this result by making use of the additional sources for non-elementary field combinations introduced in the quantization. Indeed, we recognize that the two combinations evaluated in x in the second and third term of the first round bracket correspond precisely to the non-elementary fields $H_{\mu\nu}$ and $\Omega_{\mu\nu}$, (3.2.10) and (3.2.11), introduced in the previous subsection, while the Lie derivatives correspond by definition to the BRST Slavnov variation of the metric fluctuation; therefore:

$$
\begin{aligned}
&\partial_t \tilde{\Gamma}_{(k)}[\Phi; K; v; \bar{g}] = \\
&= \int d^D x d^D y \sqrt{\bar{g}(x)}\, \delta(x-y) \Bigg[\\
&\quad \frac{1}{2\kappa^2} \partial_t \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)} \left(-\bar{\Box}_{(y)}\right) \Big(\langle h_{\alpha\beta}(x) h_{\mu\nu}(y) \rangle_{JK} - \langle \Omega_{\alpha\beta}(x) s h_{\mu\nu}(y) \rangle_{JK} - \langle H_{\alpha\beta}(x) h_{\mu\nu}(y) \rangle_{JK} \Big) \\
&\quad + \bar{g}^{\lambda\rho}(y) \partial_t r_{gh(k)} \left(-\bar{\Box}_{(y)}\right) \bar{K}^{\mu\nu}{}_{\rho\sigma}(y) \bar{\nabla}^{\sigma}_{(y)} \Big(v_\lambda(x) \langle h_{\mu\nu}(y) \rangle_{JK} - \langle \bar{c}_\lambda(x) s h_{\mu\nu}(y) \rangle_{JK} \Big) \Bigg] \\
&= \mathrm{Tr}_{\bar{g}} \Bigg[\frac{1}{2\kappa^2} \partial_t \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)} \left(-\bar{\Box}\right) \Big(\langle h_{\alpha\beta} \otimes h_{\mu\nu} \rangle_{JK} - \langle \Omega_{\alpha\beta} \otimes s h_{\mu\nu} \rangle_{JK} - \langle H_{\alpha\beta} \otimes h_{\mu\nu} \rangle_{JK} \Big) \\
&\qquad + \bar{g}^{\lambda\rho} \partial_t r_{gh(k)} \left(-\bar{\Box}\right) \bar{K}^{\mu\nu}{}_{\rho\sigma} \bar{\nabla}^{\sigma} \Big(v_\lambda \otimes \langle h_{\mu\nu} \rangle_{JK} - \langle \bar{c}_\lambda \otimes s h_{\mu\nu} \rangle_{JK} \Big) \Bigg]
\end{aligned}
\tag{3.3.72}
$$

with the regulators acting on the second of the two terms in the direct products. Now all averages correspond to 2-point correlation functions (apart from $\langle h_{\mu\nu} \rangle_{JK} = \mathsf{h}_{\mu\nu}$ which is an average field). Proceeding now as in the standard case, we use (3.2.27):

$$
\frac{1}{\tilde{Z}_{(k)}} \tilde{Z}^{(2)}_{(k)MN}(x,y) = \tilde{W}^{(2)}_{(k)MN}(x,y) + \tilde{W}^{\overrightarrow{(1)}}_{(k)M}(x) \tilde{W}^{\overleftarrow{(1)}}_{(k)N}(y) \tag{3.3.73}
$$

In this case, since also correlation functions of non-elementary fields are involved, we need all the relations (3.2.28)-(3.2.31) to rewrite the first term in terms of derivatives of the regulated effective action:

$$
\tilde{W}^{(2)}_{(k)J_i J_j}(x,y) = \tilde{\Gamma}^{(2)-1}_{(k)\Phi^i \Phi^j}(x,y) \tag{3.3.74}
$$

$$
\tilde{W}^{(2)}_{(k)K_i K_j}(x,y) = -\tilde{\Gamma}^{(2)}_{(k)K_i K_j}(x,y) \tag{3.3.75}
$$

$$
\tilde{W}^{(2)}_{(k)K_i J_j}(x,y) = -\tilde{\Gamma}^{(2)}_{(k)K_i \Phi^k}(x,\cdot) \bullet_{\bar{g}} \tilde{\Gamma}^{(2)-1}_{(k)\Phi^k \Phi^j}(\cdot,y) \tag{3.3.76}
$$

$$
\tilde{W}^{(2)}_{(k)J_i K_j}(x,y) = -\tilde{\Gamma}^{(2)-1}_{(k)\Phi^i \Phi^k}(x,\cdot) \bullet_{\bar{g}} \tilde{\Gamma}^{(2)}_{(k)\Phi^k K_j}(\cdot,y) \tag{3.3.77}
$$

As shown in appendix C.1, according to those relations the four 2-point correlation functions appearing in (3.3.72) give:

$$
\langle h_{\alpha\beta}(x) h_{\mu\nu}(y) \rangle_{JK} = \tilde{\Gamma}^{(2)-1}_{(k)\mathsf{h}_{\alpha\beta}\mathsf{h}_{\mu\nu}}(x,y) + \mathsf{h}_{\alpha\beta}(x)\mathsf{h}_{\mu\nu}(y) \tag{3.3.78}
$$

$$
\langle \Omega_{\alpha\beta}(x) s h_{\mu\nu}(y) \rangle_{JK} = \tilde{\Gamma}^{(2)}_{(k)n^{\alpha\beta}k^{\mu\nu}}(x,y) - \tilde{\Gamma}^{\overrightarrow{(1)}}_{(k)n^{\alpha\beta}}(x) \tilde{\Gamma}^{\overleftarrow{(1)}}_{(k)k^{\mu\nu}}(y) \tag{3.3.79}
$$

$$
\langle H_{\alpha\beta}(x) h_{\mu\nu}(y) \rangle_{JK} = -\tilde{\Gamma}^{(2)}_{(k)m^{\alpha\beta}\Phi^i}(x,\cdot) \bullet_{\bar{g}} \tilde{\Gamma}^{(2)-1}_{(k)\Phi^i \mathsf{h}_{\mu\nu}}(\cdot,y) - \tilde{\Gamma}^{\overrightarrow{(1)}}_{(k)m^{\mu\nu}}(x) \mathsf{h}_{\mu\nu}(y) \tag{3.3.80}
$$

$$
\langle \bar{c}_\lambda(x) s h_{\mu\nu}(y) \rangle_{JK} = -\tilde{\Gamma}^{(2)-1}_{(k)(-\bar{\mathsf{c}}_\lambda)\Phi^i}(x,\cdot) \bullet_{\bar{g}} \tilde{\Gamma}^{(2)}_{(k)\Phi^i k^{\mu\nu}}(\cdot,y) + \bar{\mathsf{c}}_\lambda(x) \tilde{\Gamma}^{\overleftarrow{(1)}}_{(k)k^{\mu\nu}}(y) \tag{3.3.81}
$$

The term $v_\lambda(x)\left\langle h_{\mu\nu}(y)\right\rangle_{JK} = v_\lambda(x)\mathsf{h}_{\mu\nu}(y)$ in (3.3.72) and the product of average fields $\left\langle h_{\alpha\beta}(x)\right\rangle_{JK}\left\langle h_{\mu\nu}(y)\right\rangle_{JK} = \mathsf{h}_{\alpha\beta}(x)\mathsf{h}_{\mu\nu}(y)$ in (3.3.78) once traced gives back the derivative of the gauge fixing contribution to the regulator term (3.3.27) computed in the average fields $\partial_t\Delta S_{gf(k)}[\mathsf{h};v;\bar{g}]$:

$$\mathrm{Tr}_{\bar{g}}\left[\frac{1}{2\kappa^2}\partial_t\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}\right)\mathsf{h}_{\alpha\beta}\otimes\mathsf{h}_{\mu\nu}+\bar{g}^{\lambda\rho}\partial_t r_{gh(k)}\left(-\bar{\Box}\right)\bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^{\sigma}v_\lambda\otimes\mathsf{h}_{\mu\nu}\right] = \partial_t\Delta S_{gf(k)}[\mathsf{h};v;\bar{g}] \tag{3.3.82}$$

Substituting the results in (3.3.72), one obtains the BRST-compatible flow equation for the regulated effective action:

$$\boxed{\begin{aligned}
&\partial_t\tilde{\Gamma}_{(k)}\left[\Phi;K;v;\bar{g}\right] =\\
&= \mathrm{Tr}_{\bar{g}}\Bigg[\frac{1}{2\kappa^2}\partial_t\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}\right)\Bigg(\tilde{\Gamma}^{(2)-1}_{(k)\mathsf{h}_{\alpha\beta}\mathsf{h}_{\mu\nu}}+\tilde{\Gamma}^{(2)}_{(k)m^{\alpha\beta}\Phi^i}\bullet_{\bar{g}}\tilde{\Gamma}^{(2)-1}_{(k)\Phi^i\mathsf{h}_{\mu\nu}}-\tilde{\Gamma}^{(2)}_{(k)n^{\alpha\beta}k^{\mu\nu}}\\
&\qquad\qquad+\overrightarrow{\tilde{\Gamma}}^{(1)}_{(k)m^{\alpha\beta}}\otimes\mathsf{h}_{\mu\nu}+\overrightarrow{\tilde{\Gamma}}^{(1)}_{(k)n^{\alpha\beta}}\otimes\overleftarrow{\tilde{\Gamma}}^{(1)}_{(k)k^{\mu\nu}}\Bigg)\\
&\qquad+\bar{g}^{\lambda\rho}\partial_t r_{gh(k)}\left(-\bar{\Box}\right)\bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^{\sigma}\Bigg(\tilde{\Gamma}^{(2)-1}_{(k)\bar{\mathsf{c}}_\lambda\Phi^i}\bullet_{\bar{g}}\tilde{\Gamma}^{(2)}_{(k)\Phi^i k^{\mu\nu}}-\bar{\mathsf{c}}_\lambda\otimes\overleftarrow{\tilde{\Gamma}}^{(1)}_{(k)k^{\mu\nu}}\Bigg)\Bigg]\\
&\quad+\partial_t\Delta S_{gf(k)}[\mathsf{h};v;\bar{g}]
\end{aligned}} \tag{3.3.83}$$

where it is understood that regulators act on the second implicit spacetime argument of the various objects inside the round brackets. Differently from the standard flow equation (2.4.43), only the derivative of the gauge-fixing piece $\partial_t\Delta S_{gf(k)}[\mathsf{h};v;\bar{g}]$ appears, not the derivative of the full regulator term $\partial_t\Delta S_{(k)}[\Phi;v;\bar{g}]$.

Using the formal definition (3.3.50) to write the regulated effective action as $\tilde{\Gamma}_{(k)} = \Gamma_{(k)}+\Delta S_{(k)}$, one obtains the BRST-compatible Wetterich-Morris equation for the effective average action, implicitly given by:

$$\partial_t\Gamma_{(k)}[\Phi;K;v;\bar{g}] = \partial_t\tilde{\Gamma}_{(k)}[\Phi;K;v;\bar{g}]-\partial_t\Delta S_{(k)}[\Phi;v;\bar{g}] \tag{3.3.84}$$

Contrary to the standard Wetterich-Morris equation (2.4.46), we have a left over term $-\partial_t\Delta S_{gh(k)}[\Phi;v;\bar{g}]$, due to absence in (3.3.83) of the derivative of the full regulator term $\partial_t\Delta S_{(k)}[\Phi;v;\bar{g}]$. Moreover, rewriting the pieces coming from (3.3.83) in the right hand side in terms of derivatives of the effective average action, in this case we do not obtain simply terms of the form $(\tilde{\Gamma}^{(2)}_{(k)\Phi^i\Phi^j}+\mathcal{R}(-\bar{\Box}))^{-1}$; in particular, the first and second derivatives of the regulated effective action with respect to the external sources can be immediately translated in those of the effective average action, since $\overrightarrow{\tilde{\Gamma}}^{(1)}_{(k)K_i} = \overrightarrow{\Gamma}^{(1)}_{(k)K_i}$, $\overleftarrow{\tilde{\Gamma}}^{(1)}_{(k)K_i} = \overleftarrow{\Gamma}^{(1)}_{(k)K_i}$; instead; the inverse matrix of second derivatives generate terms $(\tilde{\Gamma}^{(2)}_{(k)\Phi^i\Phi^j}+\Delta S^{(2)}_{(k)\Phi^i\Phi^j})^{-1}$, where the matrix of second derivatives of the regulator term $\Delta S^{(2)}_{(k)\Phi^i\Phi^j}$ is now field-dependent and non-block-diagonal, due to the non-quadratic and mixed $hc\bar{c}$ and $hhc\bar{c}$ terms. We do not give the explicit form of the equation, focusing instead on the flow equation (3.3.83). We make the following remarks:

1. Similarly to the standard construction, the regulated effective average action can be now defined as solution of the flow equation (3.3.83), in the place of the formal definition (3.3.50), and the latter as the fundamental object defining the quantum theory, in the place of the generating functionals, along with an initial condition. In particular, in this case we can consider as classic gauge-fixed action serving as initial condition any other gravitational action than the Einstein-Hilbert one, but not with an arbitrary gauge-fixing, since due to the regulating-gauge-fixing procedure, the unregulated action is always gauge-fixed with the modified de Donder gauge-fixing, by construction.

2. As in the standard construction, the implicit UV cut-off k_{UV}, deriving from the formal definition (3.3.50), can be safely removed by letting $k_{UV} \to \infty$ thanks to the presence of the derivatives of the regulators in the trace.

3. Similarly to the standard flow equation (2.4.43), the flow equation (3.3.83) is an exact functional differential equation with two sectors, featuring respectively the two regulators. However, in this case in each sector there are both functional derivatives of the regulated effective action with respect to the metric fluctuation and the ghosts, and also with respect to the additional sources. Thanks to the introduction of the latter, the equation contains as (2.4.43) at most second derivatives. In particular, derivatives with respect to the additional sources appear only as first and second order derivatives, derivatives with respect to the fields appear in the inverse matrices of second derivatives of the regulated effective action, similarly to (2.4.43), although in this case also non-diagonal element are present. As far as the structure is concerned there are three kinds of terms:

- 1-loop terms given by the trace of the operators:

$$\tilde{\Gamma}^{(2)-1}_{(k)\mathsf{h}_{\alpha\beta}\mathsf{h}_{\mu\nu}} \quad , \quad \tilde{\Gamma}^{(2)}_{(k)m^{\alpha\beta}\Phi^i} \bullet_{\bar{g}} \tilde{\Gamma}^{(2)-1}_{(k)\Phi^i\mathsf{h}_{\mu\nu}} \quad , \quad \tilde{\Gamma}^{(2)-1}_{(k)\bar{\mathsf{c}}_\lambda\Phi^i} \bullet_{\bar{g}} \tilde{\Gamma}^{(2)}_{(k)\Phi^i k^{\mu\nu}}$$

  multiplied by the correspondent regulator. The first is equal to that in (2.4.43), i.e. a loop of the exact metric fluctuation propagator with a regulator insertion. The second and third are each one a sum of three loops formed by exact propagators of both the metric fluctuation and ghosts with the insertion of a regulator and an additional operator. We present the explicit form in the linear-K truncation in subsection 3.3.4.

- Traces of the operators with a direct product:

$$\tilde{\Gamma}^{\overrightarrow{(1)}}_{(k)m^{\alpha\beta}} \otimes \mathsf{h}_{\mu\nu} \quad , \quad \tilde{\Gamma}^{\overrightarrow{(1)}}_{(k)n^{\alpha\beta}} \otimes \tilde{\Gamma}^{\overleftarrow{(1)}}_{(k)k^{\mu\nu}} \quad , \quad \bar{\mathsf{c}}_\lambda \otimes \tilde{\Gamma}^{\overleftarrow{(1)}}_{(k)k^{\mu\nu}}$$

  multiplied by the correspondent regulator. Those assume a simple form adopting the linear-K truncation.

- Trace of a second order derivative of the regulated effective action with respect to the additional sources.

$$\tilde{\Gamma}^{(2)}_{(k)n^{\alpha\beta}k^{\mu\nu}}$$

  multiplied by the correspondent regulator.

4. According to the symmetry properties of the regularized theory stemming from the formal definition of the effective average action (3.3.50), the Wetterich-Morris equation (3.3.84) is expected to generate only background gauge invariant terms satisfying the constraint imposed by the Zinn-Justin equation, $\Sigma = 0$, so that the trajectory is constrained on the fixed hypersurface of BRST-invariant operators along the full flow from the UV to the IR, $\infty \to k \to 0$, contrary to the one satisfying the standard Wetterich-Morris equation (figure 5).
In particular, we expect that the flow equation (3.3.83) is BRST-compatible, i.e. compatible with the constraint imposed by the Zinn-Justin equation (3.3.64). This means that the evolution in FRG time described by the flow equation must preserve the value of the constraint, so that if the trajectory solution of the equation satisfies the constraint at some FRG scale k_0, then it does so for any other value of the FRG scale k:

$$\Sigma[\tilde{\Gamma}_{(k_0)}] = 0 \quad \Longrightarrow \quad \Sigma[\tilde{\Gamma}_{(k)}] = 0 \quad \forall k$$

This property represents the BRST-invariance of the regularized theory assuming the flow equation as fundamental concept, in the place of the formal definition in terms of the regulated generating functionals. It is proved in the next subsection.

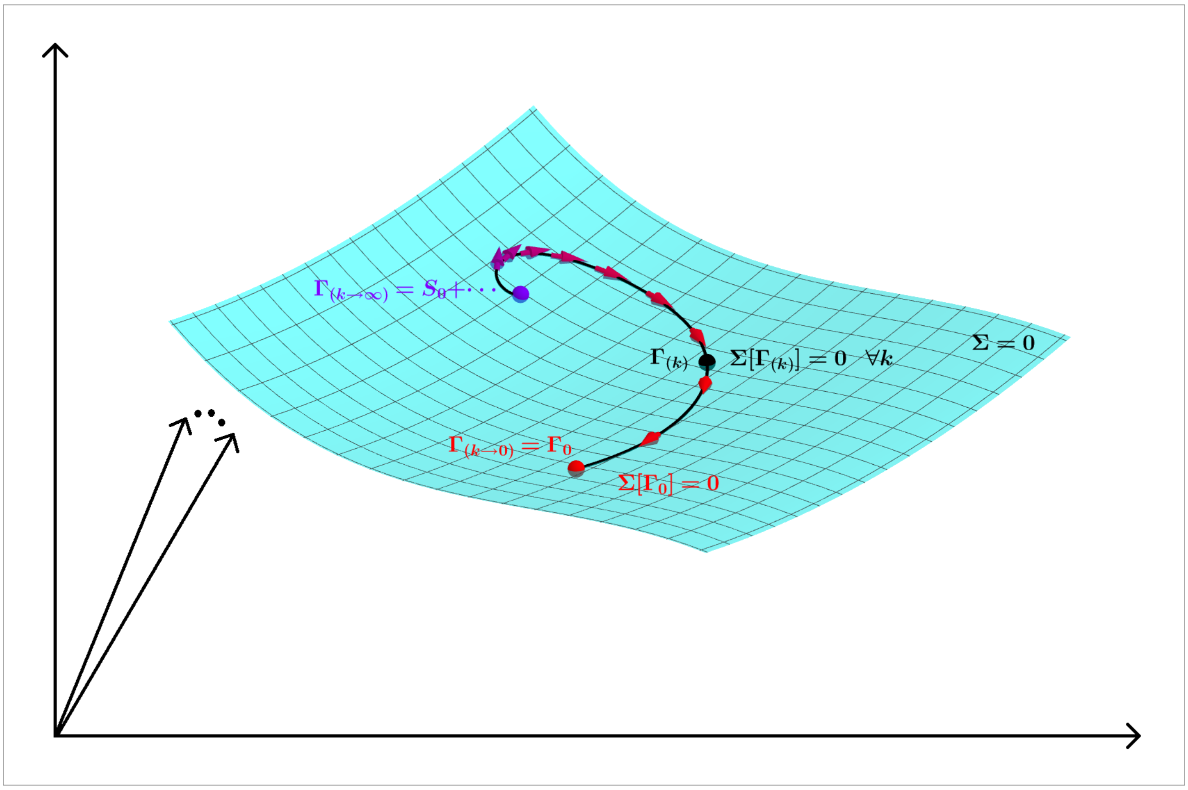

Figure 5: *Pictorial representation of the FRG flow trajectory in theory space described by the BRST-compatible Wetterich-Morris equation for the effective average action* $\Gamma_{(k)}$*, which unfolds from the modified de Donder gauge-fixed action* S_0 *(plus counter terms) to the quantum effective action* Γ_0*. The flow is constrained on the hypersurface of BRST-invariant operators* $\Sigma = 0$ *embedded in the space of background gauge-invariant operators.*

3.3.3 BRST-compatibility proof

In this subsection we prove the BRST-compatibility property of the flow equation (3.3.83). As anticipated in the previous subsection, this corresponds to proving that if the trajectory solution of the equation satisfies the constraint imposed by BRST symmetry, i.e. the Zinn-Justin equation (3.3.64), at some FRG scale k_0, then it satisfies the constraint also for any other value of the FRG scale k:

$$\Sigma[\tilde{\Gamma}_{(k_0)}] = 0 \quad \Longrightarrow \quad \Sigma[\tilde{\Gamma}_{(k)}] = 0 \quad \forall k \tag{3.3.85}$$

In order to prove this property, it is convenient to focus on the regulated path integral and the associated flow equation, instead of considering directly the regulated effective action and the flow equation (3.3.83). The proof goes as follows:

1. BRST symmetry constraint generator: Consider initially the formal definition of the regulated path integral in terms of the functional integral (3.3.42). The constraint imposed on the quantum theory by BRST symmetry is represented by the Ward-Takahashi equation (3.3.63), which can be written in terms of derivatives of the regulated effective action, i.e. the Zinn-Justin equation (3.3.64). The identity can be written also in terms of derivatives of the regulated path integral; in particular, rewriting first the identity in terms of unnormalized averages, i.e. multiplying both on the left and the right by the regulated path integral:

$$\int d^D x \sqrt{\bar{g}} \Big(t^{\mu\nu} \left\langle s h_{\mu\nu} \right\rangle_{JK,u} - \bar{\eta}_\mu \left\langle s c^\mu \right\rangle_{JK,u} + \eta^\mu v_\mu \tilde{Z}_{(k)} + \\ + m^{\mu\nu} \left\langle s h_{\mu\nu} \right\rangle_{JK,u} - n^{\mu\nu} \left\langle h_{\mu\nu} \right\rangle_{JK,u} + n^{\mu\nu} \left\langle H_{\mu\nu} \right\rangle_{JK,u} \Big) = 0 \tag{3.3.86}$$

and writing the unnormalized averages as derivatives of the regulated path integral with respect to the corresponding sources, we obtain:

$$\Xi[\tilde{Z}_{(k)}] = \int d^D x \Big(t^{\mu\nu} \frac{\delta \tilde{Z}_{(k)}}{\delta k^{\mu\nu}} - \bar{\eta}_\mu \frac{\delta \tilde{Z}_{(k)}}{\delta l_\mu} + \sqrt{\bar{g}}\, \eta^\mu v_\mu \tilde{Z}_{(k)} \\ + m^{\mu\nu} \frac{\delta \tilde{Z}_{(k)}}{\delta k^{\mu\nu}} - n^{\mu\nu} \frac{\delta \tilde{Z}_{(k)}}{\delta t^{\mu\nu}} + n^{\mu\nu} \frac{\delta \tilde{Z}_{(k)}}{\delta m^{\mu\nu}} \Big) = 0 \tag{3.3.87}$$

In particular, contrary to the case of the Zinn-Justin equation (3.3.64), now the sources appearing in the expression are not rewritten as derivatives, since they are already the natural variables of the regulated path integral; therefore, while the left hand side of the Zinn-Justin equation (3.3.64) is non-linear in the regulated effective action, in this case the expression which is constrained to be zero by BRST symmetry is given by a linear functional operator, containing up to first order derivatives in the sources, acting on the regulated path integral:

$$\Xi[\tilde{Z}_{(k)}] = \int d^D x \Big(t^{\mu\nu} \frac{\delta}{\delta k^{\mu\nu}} - \bar{\eta}_\mu \frac{\delta}{\delta l_\mu} + \sqrt{\bar{g}}\, \eta^\mu v_\mu \\ + m^{\mu\nu} \frac{\delta}{\delta k^{\mu\nu}} - n^{\mu\nu} \frac{\delta}{\delta t^{\mu\nu}} + n^{\mu\nu} \frac{\delta}{\delta m^{\mu\nu}} \Big) \tilde{Z}_{(k)}[J; K; v; \bar{g}] \tag{3.3.88}$$

The operator is the generator of the BRST symmetry constraint $\mathcal{S}$:

$$\Xi[\tilde{Z}_{(k)}] = \mathcal{S}\tilde{Z}_{(k)} \tag{3.3.89}$$

which can be expressed as:

$$\mathcal{S} = \int d^Dx\sqrt{\bar{g}(x)}\,\mathcal{S}(x) \tag{3.3.90}$$

where $\mathcal{S}(x)$ is the non-integrated form of the generator:

$$\begin{aligned}\mathcal{S}(x) = \frac{1}{\sqrt{\bar{g}(x)}}\Bigg(&t^{\mu\nu}(x)\frac{\delta}{\delta k^{\mu\nu}(x)} - \bar{\eta}_\mu(x)\frac{\delta}{\delta l_\mu(x)} + \sqrt{\bar{g}(x)}\eta^\mu(x)v_\mu(x)\\ &+ m^{\mu\nu}(x)\frac{\delta}{\delta k^{\mu\nu}(x)} - n^{\mu\nu}(x)\frac{\delta}{\delta t^{\mu\nu}(x)} + n^{\mu\nu}(x)\frac{\delta}{\delta m^{\mu\nu}(x)}\Bigg)\end{aligned} \tag{3.3.91}$$

Finally, we notice also that, since the left member of the identity (3.3.86) is just the one which leads to the Zinn-Justin equation (3.3.64) multiplied by the regulated path integral, we have the formal relation:

$$\Xi[\tilde{Z}_{(k)}] = \tilde{Z}_{(k)}\Sigma[\tilde{\Gamma}_{(k)}] \tag{3.3.92}$$

2. FRG flow generator: Consider again the formal definition of the regulated path integral in terms of the functional integral (3.3.42), and the associated flow equation. According to the relation (3.3.67) between derivatives of the various generating functionals, the flow equation is formally given by:

$$\partial_t\tilde{Z}_{(k)}[J;K;v;\bar{g}] = -\left\langle \Delta S_{(k)}[\phi;v;\bar{g}]\right\rangle_{JK,u} \tag{3.3.93}$$

The unnormalized average on the right hand side can be expressed, in explicit notation, by considering equation (3.3.72) and substituting normalized correlation functions with unnormalized ones:

$$\begin{aligned}&\partial_t\tilde{Z}_{(k)}[J;K;v;\bar{g}] =\\ &= \int d^Dx d^Dy\sqrt{\bar{g}(x)}\,\delta(x-y)\Bigg[\\ &\quad \frac{1}{2\kappa^2}\partial_t\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}_{(y)}\right)\Big(-\left\langle h_{\alpha\beta}(x)h_{\mu\nu}(y)\right\rangle_{JK,u} + \left\langle \Omega_{\alpha\beta}(x)sh_{\mu\nu}(y)\right\rangle_{JK,u}\\ &\qquad\qquad + \left\langle H_{\alpha\beta}(x)h_{\mu\nu}(y)\right\rangle_{JK,u}\Big)\\ &\quad + \bar{g}^{\lambda\rho}(y)\partial_t r_{gh(k)}\left(-\bar{\Box}_{(y)}\right)\bar{K}^{\mu\nu}{}_{\rho\sigma}(y)\bar{\nabla}^\sigma_{(y)}\Big(-v_\lambda(x)\left\langle h_{\mu\nu}(y)\right\rangle_{JK,u} + \left\langle \bar{c}_\lambda(x)sh_{\mu\nu}(y)\right\rangle_{JK,u}\Big)\Bigg]\end{aligned} \tag{3.3.94}$$

Rewriting the unnormalized correlation functions of the various fields as derivatives of the regulated path integral with respect to the corresponding sources (recalling in particular that for the antighost there is a minus coming from the definition of the correspondent source term), the flow equation for the regulated path integral can be thus expressed in the form:

$$\begin{aligned}
&\partial_t \tilde{Z}_{(k)}[J;K;v;\bar{g}] = \\
&= \int d^D x d^D y \sqrt{\bar{g}(x)}\, \delta(x-y) \frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}} \Bigg[\\
&\quad \frac{1}{2\kappa^2} \partial_t \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)} \left(-\bar{\Box}_{(y)}\right) \left(- \frac{\delta}{\delta t^{\alpha\beta}(x)} \frac{\delta \tilde{Z}_{(k)}}{\delta t^{\mu\nu}(y)} + \frac{\delta}{\delta n^{\alpha\beta}(x)} \frac{\delta \tilde{Z}_{(k)}}{\delta k^{\mu\nu}(y)} + \frac{\delta}{\delta m^{\alpha\beta}(x)} \frac{\delta \tilde{Z}_{(k)}}{\delta t^{\mu\nu}(y)} \right) \\
&\quad + \bar{g}^{\lambda\rho}(y) \partial_t r_{gh(k)} \left(-\bar{\Box}_{(y)}\right) \bar{K}^{\mu\nu}{}_{\rho\sigma}(y) \bar{\nabla}^{\sigma}_{(y)} \left(- \sqrt{\bar{g}(x)} v_\lambda(x) \frac{\delta \tilde{Z}_{(k)}}{\delta t^{\mu\nu}(y)} - \frac{\delta}{\delta \eta^{\rho}(x)} \frac{\delta \tilde{Z}_{(k)}}{\delta k^{\mu\nu}(y)} \right) \Bigg].
\end{aligned} \tag{3.3.95}$$

where it is understood that all derivatives are left derivatives acting on the regulated path integral on the right, in the order in which their are written (which is relevant in particular for the second couple in the first bracket and the third couple in the second bracket, which are taken with respect to Grassmann-odd sources and are therefore anticommuting). Similarly to the previous point, while the right hand side of the flow equation (3.3.83) is non-linear in the regulated effective action, in this case the right hand side of the flow equation is given by a linear functional operator, containing up to second order derivatives in the sources, acting on the regulated path integral:

$$\begin{aligned}
&\partial_t \tilde{Z}_{(k)}[J;K;v;\bar{g}] = \\
&= \int d^D x d^D y \sqrt{\bar{g}(x)}\, \delta(x-y) \frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}} \Bigg[\\
&\quad \frac{1}{2\kappa^2} \partial_t \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)} \left(-\bar{\Box}_{(y)}\right) \left(- \frac{\delta}{\delta t^{\alpha\beta}(x)} \frac{\delta}{\delta t^{\mu\nu}(y)} + \frac{\delta}{\delta n^{\alpha\beta}(x)} \frac{\delta}{\delta k^{\mu\nu}(y)} + \frac{\delta}{\delta m^{\alpha\beta}(x)} \frac{\delta}{\delta t^{\mu\nu}(y)} \right) \\
&\quad + \bar{g}^{\lambda\rho}(y) \partial_t r_{gh(k)} \left(-\bar{\Box}_{(y)}\right) \bar{K}^{\mu\nu}{}_{\rho\sigma}(y) \bar{\nabla}^{\sigma}_{(y)} \left(- \sqrt{\bar{g}(x)} v_\lambda(x) \frac{\delta}{\delta t^{\mu\nu}(y)} - \frac{\delta}{\delta \eta^{\rho}(x)} \frac{\delta}{\delta k^{\mu\nu}(y)} \right) \Bigg] \cdot \\
&\qquad \cdot \tilde{Z}_{(k)}[J;K;v;\bar{g}]
\end{aligned} \tag{3.3.96}$$

The operator is the generator of the FRG flow $\mathcal{F}$ for the regulated path integral:

$$\partial_t \tilde{Z}_{(k)} = \mathcal{F} \tilde{Z}_{(k)} \tag{3.3.97}$$

which can be expressed as:

$$\mathcal{F} = \mathrm{Tr}_{\bar{g}} \left[\mathcal{F}(x,y) \right] \tag{3.3.98}$$

where $\mathcal{F}(x,y)$ is the non-integrated version of the operator:

$$\mathcal{F}(x,y) = \\ = \frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}\Bigg[\\ \frac{1}{2\kappa^2}\partial_t\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}_{(y)}\right)\left(-\frac{\delta}{\delta t^{\alpha\beta}(x)}\frac{\delta}{\delta t^{\mu\nu}(y)}+\frac{\delta}{\delta n^{\alpha\beta}(x)}\frac{\delta}{\delta k^{\mu\nu}(y)}+\frac{\delta}{\delta m^{\alpha\beta}(x)}\frac{\delta}{\delta t^{\mu\nu}(y)}\right) \\ +\bar{g}^{\lambda\rho}(y)\partial_t r_{gh(k)}\left(-\bar{\Box}_{(y)}\right)\bar{K}^{\mu\nu}{}_{\rho\sigma}(y)\bar{\nabla}^{\sigma}_{(y)}\left(-\sqrt{\bar{g}(x)}v_\lambda(x)\frac{\delta}{\delta t^{\mu\nu}(y)}-\frac{\delta}{\delta\eta^\lambda(x)}\frac{\delta}{\delta k^{\mu\nu}(y)}\right)\Bigg] \tag{3.3.99}$$

Similarly to the case of the regulated effective action, at this point the regulated path integral can be defined as solution of this flow equation.

3. Compatibility: Obtained the two generators $\mathcal{F}$ and $\mathcal{S}$, it is immediate to prove that the compatibility condition (3.3.85) is equivalent to the commutativity of the two:

$$[\mathcal{F},\mathcal{S}] = 0 \tag{3.3.100}$$

where the commutator is explicitly given by:

$$[\mathcal{F},\mathcal{S}] = \int d^Dx d^Dy d^Dz \sqrt{\bar{g}(x)}\delta(x-y)\sqrt{\bar{g}(z)}\,[\mathcal{F}(x,y),\mathcal{S}(z)] \tag{3.3.101}$$

Indeed, consider the regulated path integral defined as solution of the correspondent flow equation; given a point $\tilde{Z}_{(t)}$ on the trajectory, at some given FRG time t, the immediately following point $\tilde{Z}_{(t+dt)}$, at an infinitesimal distance in FRG time dt, can be expressed as:

$$\tilde{Z}_{(t+dt)} = \tilde{Z}_{(t)} + dt\mathcal{F}\tilde{Z}_{(t)} \tag{3.3.102}$$

If we now apply the generator of the BRST symmetry constraint, assuming the commutation rule (3.3.100), we obtain:

$$\mathcal{S}\tilde{Z}_{(t+dt)} = \mathcal{S}\tilde{Z}_{(t)} + dt\mathcal{S}\mathcal{F}\tilde{Z}_{(t)} = \mathcal{S}\tilde{Z}_{(t)} + dt\mathcal{F}\mathcal{S}\tilde{Z}_{(t)} \tag{3.3.103}$$

In particular, we can deduce the implication:

$$\mathcal{S}\tilde{Z}_{(t)} = \Xi[\tilde{Z}_{(t)}] = 0 \quad\Longrightarrow\quad \mathcal{S}\tilde{Z}_{(t+dt)} = \Xi[\tilde{Z}_{(t+dt)}] = 0 \tag{3.3.104}$$

From (3.3.92), the implication is equivalent to:

$$\Sigma[\tilde{\Gamma}_{(t)}] = 0 \quad\Longrightarrow\quad \Sigma[\tilde{\Gamma}_{(t+dt)}] = 0 \tag{3.3.105}$$

which, by iteration, can be extended to the case of points at an arbitrary finite distance along the FRG trajectory, finally implying the compatibility condition (3.3.85). In appendix C.2 we prove that the commutation rule (3.3.100) is indeed verified; in particular, the result follows from a precise cancellation of the additional terms arising from commuting the various functional derivatives inside the two generators.

3.3.4 Einstein-Hilbert truncation and beta functions

We conclude the thesis by applying the standard Einstein-Hilbert truncation scheme to the flow equation (3.3.83) in order to find the beta functions for the gravitational and cosmological constants predicted by the theory and compare the results with those stemming from the standard Wetterich-Morris equation (2.4.46), presented in subsection 2.4.3.

Linear-K truncation

Before specifying the full truncation scheme, we show how the flow equation (3.3.83) appears operating, as a preliminary step, a linear-K truncation, projecting it on the subspace of operators with a linear dependence on the additional sources given by the source term appearing in the path integral (3.2.18):

$$\tilde{\Gamma}_{(k)}[\Phi; K; v; \bar{g}] = \tilde{\Gamma}_{(k)}[\Phi; v; \bar{g}] + S_{source-K}[\Phi; K; v; \bar{g}] \tag{3.3.106}$$

In particular, according to the ansatz, the first derivative of the regulated effective action is simply given by the vector of non-elementary fields, while the second is clearly vanishing:

$$\tilde{\Gamma}^{\overrightarrow{(1)}}_{(k)K_i}(x) = -\Psi^i(x) \quad , \quad \tilde{\Gamma}^{\overleftarrow{(1)}}_{(k)K_i}(x) = -\Psi^{+i}(x) \tag{3.3.107}$$

$$\tilde{\Gamma}^{(2)}_{(k)K_iK_j}(x,y) = 0 \tag{3.3.108}$$

Adopting the truncation in the flow equation (3.3.83), we can thus set:

$$\tilde{\Gamma}^{\overrightarrow{(1)}}_{(k)m^{\alpha\beta}} \otimes \mathsf{h}_{\mu\nu} = -H_{\alpha\beta} \otimes h_{\mu\nu} = -\frac{v_\rho}{v^2}\bar{g}^{\rho\lambda}\bar{c}_\lambda \pounds_c\left(\bar{g}_{\alpha\beta} + h_{\alpha\beta}\right) \otimes h_{\mu\nu} \tag{3.3.109}$$

$$\tilde{\Gamma}^{\overrightarrow{(1)}}_{(k)n^{\alpha\beta}} \otimes \tilde{\Gamma}^{\overleftarrow{(1)}}_{(k)k^{\mu\nu}} = -\Omega_{\alpha\beta} \otimes sh_{\mu\nu} = -\frac{v_\rho}{v^2}\bar{g}^{\rho\lambda}\bar{c}_\lambda h_{\alpha\beta} \otimes \pounds_c\left(\bar{g}_{\mu\nu} + h_{\mu\nu}\right) \tag{3.3.110}$$

$$\bar{\mathsf{c}}_\lambda \otimes \tilde{\Gamma}^{\overleftarrow{(1)}}_{(k)k^{\mu\nu}} = \bar{\mathsf{c}}_\lambda \otimes sh_{\mu\nu} = \bar{\mathsf{c}}_\lambda \otimes \pounds_c\left(\bar{g}_{\mu\nu} + h_{\mu\nu}\right) \tag{3.3.111}$$

$$\tilde{\Gamma}^{(2)}_{(k)n^{\alpha\beta}k^{\mu\nu}} = 0 \tag{3.3.112}$$

Given the first three equalities, one can recognize that the three terms given by tracing the direct products now reconstruct precisely the derivative of the ghost piece (3.3.31) of the regulator term computed in the average fields $\partial_t \Delta S_{gh(k)}[\mathsf{h}, \mathsf{c}, \bar{\mathsf{c}}; v; \bar{g}]$:

$$\begin{aligned}\mathrm{Tr}_{\bar{g}}\Bigg[&\frac{1}{2\kappa^2}\partial_t \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}\right)\left(\tilde{\Gamma}^{\overrightarrow{(1)}}_{(k)m^{\alpha\beta}} \otimes \mathsf{h}_{\mu\nu} + \tilde{\Gamma}^{\overrightarrow{(1)}}_{(k)n^{\alpha\beta}} \otimes \tilde{\Gamma}^{\overleftarrow{(1)}}_{(k)k^{\mu\nu}}\right) \\ &- \bar{g}^{\lambda\rho}\partial_t r_{gh(k)}\left(-\bar{\Box}\right)\bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^\sigma\left(\bar{\mathsf{c}}_\lambda \otimes \tilde{\Gamma}^{\overleftarrow{(1)}}_{(k)k^{\mu\nu}}\right)\Bigg] = \partial_t \Delta S_{gh(k)}[\mathsf{h}, \mathsf{c}, \bar{\mathsf{c}}; v; \bar{g}]\end{aligned} \tag{3.3.113}$$

Therefore, the flow equation for the regulated effective action within the linear-K truncation can be expressed as:

$$
\begin{aligned}
\partial_t \tilde{\Gamma}_{(k)}[\Phi; v; \bar{g}] = \mathrm{Tr}_{\bar{g}} \Bigg[& \frac{1}{2\kappa^2} \partial_t \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)} \left(-\bar{\Box}\right) \left(\tilde{\Gamma}^{(2)-1}_{(k)\mathsf{h}_{\alpha\beta}\mathsf{h}_{\mu\nu}} + \tilde{\Gamma}^{(2)}_{(k)m^{\alpha\beta}\Phi^i} \bullet_{\bar{g}} \tilde{\Gamma}^{(2)-1}_{(k)\Phi^i\mathsf{h}_{\mu\nu}} \right) \\
& + \bar{g}^{\lambda\rho} \partial_t r_{gh(k)} \left(-\bar{\Box}\right) \bar{K}^{\mu\nu}{}_{\rho\sigma} \bar{\nabla}^{\sigma} \left(\tilde{\Gamma}^{(2)-1}_{(k)\bar{\mathsf{c}}_\lambda \Phi^i} \bullet_{\bar{g}} \tilde{\Gamma}^{(2)}_{(k)\Phi^i k^{\mu\nu}} \right) \Bigg] \\
& + \partial_t \Delta S_{(k)}[\Phi; v; \bar{g}]
\end{aligned}
\tag{3.3.114}
$$

In particular we notice that the flow equation is now structurally similar to the standard one (2.4.43), in the sense that only loop terms are present and also the derivative of the full regulator term $\partial_t \Delta S_{(k)}[\Phi; v; \bar{g}]$ appears. Making the sum and the integration explicit, the two product terms are given by:

$$
\begin{aligned}
\tilde{\Gamma}^{(2)}_{(k)m^{\alpha\beta}\Phi^i}(x,\cdot) \bullet_{\bar{g}} \tilde{\Gamma}^{(2)-1}_{(k)\Phi^i\mathsf{h}_{\mu\nu}}(\cdot, y) = \int d^D z \sqrt{\bar{g}(z)} \Big(& \tilde{\Gamma}^{(2)}_{(k)m^{\alpha\beta}\mathsf{h}_{\rho\sigma}}(x,z) \tilde{\Gamma}^{(2)-1}_{(k)\mathsf{h}_{\rho\sigma}\mathsf{h}_{\mu\nu}}(z,y) \\
& + \tilde{\Gamma}^{(2)}_{(k)m^{\alpha\beta}\mathsf{c}^\rho}(x,z) \tilde{\Gamma}^{(2)-1}_{(k)(-\mathsf{c}^\rho)\mathsf{h}_{\mu\nu}}(z,y) \\
& + \tilde{\Gamma}^{(2)}_{(k)m^{\alpha\beta}(-\bar{\mathsf{c}}_\rho)}(x,z) \tilde{\Gamma}^{(2)-1}_{(k)\bar{\mathsf{c}}_\rho\mathsf{h}_{\mu\nu}}(z,y) \Big)
\end{aligned}
\tag{3.3.115}
$$

$$
\begin{aligned}
\tilde{\Gamma}^{(2)-1}_{(k)\bar{\mathsf{c}}_\lambda\Phi^i}(x,\cdot) \bullet_{\bar{g}} \tilde{\Gamma}^{(2)}_{(k)\Phi^i k^{\mu\nu}}(\cdot, y) = \int d^D z \sqrt{\bar{g}(z)} \Big(& \tilde{\Gamma}^{(2)-1}_{(k)\bar{\mathsf{c}}_\lambda\mathsf{h}_{\alpha\beta}}(x,z) \tilde{\Gamma}^{(2)}_{(k)\mathsf{h}_{\alpha\beta}k^{\mu\nu}}(z,y) \\
& + \tilde{\Gamma}^{(2)-1}_{(k)\bar{\mathsf{c}}_\lambda\mathsf{c}^\alpha}(x,z) \tilde{\Gamma}^{(2)}_{(k)(-\mathsf{c}^\alpha)k^{\mu\nu}}(z,y) \\
& + \tilde{\Gamma}^{(2)-1}_{(k)\bar{\mathsf{c}}_\lambda(-\bar{\mathsf{c}}_\alpha)}(x,z) \tilde{\Gamma}^{(2)}_{(k)\bar{\mathsf{c}}_\alpha k^{\mu\nu}}(z,y) \Big)
\end{aligned}
\tag{3.3.116}
$$

Within the linear-K truncation, according to (3.3.109), the operators acting on the various components of the inverse matrix of second derivatives of the regulated effective action are:

$$
\tilde{\Gamma}^{(2)}_{(k)m^{\alpha\beta}\Phi^i}(x,z) = \begin{pmatrix} -\frac{v_\gamma}{v^2} \bar{g}^{\gamma\delta} \bar{c}_\delta X^{\rho\sigma}{}_{\alpha\beta}(\bar{\nabla}) \\ -\frac{v_\gamma}{v^2} \bar{g}^{\gamma\delta} \bar{c}_\delta Y_{\rho\alpha\beta}(\bar{\nabla}) \\ -\frac{v_\lambda}{v^2} \bar{g}^{\lambda\rho} s h_{\alpha\beta} \end{pmatrix}_{(x)} \frac{\delta(x-z)}{\sqrt{\bar{g}(z)}}
\tag{3.3.117}
$$

$$
\tilde{\Gamma}^{(2)}_{(k)\Phi^i k^{\mu\nu}}(z,y) = \begin{pmatrix} X^{\alpha\beta}{}_{\mu\nu}(\bar{\nabla}) \\ Y_{\alpha\mu\nu}(\bar{\nabla}) \\ 0 \end{pmatrix}_{(y)} \frac{\delta(z-y)}{\sqrt{\bar{g}(z)}}
\tag{3.3.118}
$$

with:

$$
\begin{aligned}
X^{\alpha\beta}{}_{\mu\nu}(x,y) &\equiv X^{\alpha\beta}{}_{\mu\nu}(\bar{\nabla}_{(x)})\delta(x-y) \\
&= \frac{\delta}{\delta h_{\alpha\beta}(y)}(sh_{\mu\nu}(x)) = \\
&= \left(\delta^{\alpha\beta}_{\mu\nu}c^{\lambda}\bar{\nabla}_{\lambda} + \delta^{\alpha\beta}_{\lambda\nu}\bar{\nabla}_{\mu}c^{\lambda} + \delta^{\alpha\beta}_{\lambda\mu}\bar{\nabla}_{\nu}c^{\lambda}\right)_{(x)} \delta(x-y)
\end{aligned}
\tag{3.3.119}
$$

$$
\begin{aligned}
Y_{\alpha\mu\nu}(x,y) &\equiv Y_{\alpha\mu\nu}(\bar{\nabla}_{(x)})\delta(x-y) \\
&= \frac{\delta}{\delta c^{\alpha}(y)}(sh_{\mu\nu}(x)) = \\
&= \left((\bar{g}_{\alpha\nu} + h_{\alpha\nu})\bar{\nabla}_{\mu} + (\bar{g}_{\alpha\mu} + h_{\alpha\mu})\bar{\nabla}_{\nu} + \bar{\nabla}_{\alpha}h_{\lambda\mu}\right)_{(x)} \delta(x-y)
\end{aligned}
\tag{3.3.120}
$$

Integrating by parts within the z-integrations[12], we can move those operators on the matrix elements of the inverse matrix of second derivatives of the regulated effective action and the explicit form of the equation in the linear-K truncation is, in compact notation:

$$
\begin{aligned}
&\partial_t\tilde{\Gamma}_{(k)}\big[\Phi; v; \bar{g}\big] = \\
&= \mathrm{Tr}_{\bar{g}}\Bigg\{ \frac{1}{2\kappa^2}\partial_t\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}\right)\Bigg[\tilde{\Gamma}^{(2)-1}_{(k)\mathsf{h}_{\alpha\beta}\mathsf{h}_{\mu\nu}} \\
&\qquad - \frac{v_{\gamma}}{v^2}\bar{g}^{\gamma\delta}\bar{c}_{\delta}\left(\delta^{\rho\sigma}_{\alpha\beta}c^{\lambda}\bar{\nabla}_{\lambda} + \delta^{\rho\sigma}_{\lambda\beta}\bar{\nabla}_{\alpha}c^{\lambda} + \delta^{\rho\sigma}_{\lambda\alpha}\bar{\nabla}_{\beta}c^{\lambda}\right)\tilde{\Gamma}^{(2)-1}_{(k)\mathsf{h}_{\rho\sigma}\mathsf{h}_{\mu\nu}} \\
&\qquad + \frac{v_{\gamma}}{v^2}\bar{g}^{\gamma\delta}\bar{c}_{\delta}\left((\bar{g}_{\rho\alpha} + h_{\rho\alpha})\bar{\nabla}_{\beta} + (\bar{g}_{\rho\beta} + h_{\rho\beta})\bar{\nabla}_{\alpha} + \bar{\nabla}_{\rho}h_{\alpha\beta}\right)\tilde{\Gamma}^{(2)-1}_{(k)\mathsf{c}^{\rho}\mathsf{h}_{\mu\nu}} \\
&\qquad - \frac{v_{\lambda}}{v^2}\bar{g}^{\lambda\rho}\pounds_{c}(\bar{g}_{\alpha\beta} + h_{\alpha\beta})\tilde{\Gamma}^{(2)-1}_{(k)\bar{\mathsf{c}}_{\rho}\mathsf{h}_{\mu\nu}}\Bigg] \\
&\quad + \bar{g}^{\lambda\rho}\partial_t r_{gh(k)}\left(-\bar{\Box}\right)\bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^{\sigma}\Bigg[\left(\delta^{\alpha\beta}_{\mu\nu}c^{\tau}\bar{\nabla}_{\tau} + \delta^{\alpha\beta}_{\tau\nu}\bar{\nabla}_{\mu}c^{\tau} + \delta^{\alpha\beta}_{\tau\mu}\bar{\nabla}_{\nu}c^{\tau}\right)\tilde{\Gamma}^{(2)-1}_{(k)\bar{\mathsf{c}}_{\lambda}\mathsf{h}_{\alpha\beta}} \\
&\qquad + \left((\bar{g}_{\alpha\nu} + h_{\alpha\nu})\bar{\nabla}_{\mu} + (\bar{g}_{\alpha\mu} + h_{\alpha\mu})\bar{\nabla}_{\nu} + \bar{\nabla}_{\alpha}h_{\lambda\mu}\right)\tilde{\Gamma}^{(2)-1}_{(k)\bar{\mathsf{c}}_{\lambda}\mathsf{c}^{\alpha}}\Bigg]\Bigg\} \\
&\quad + \partial_t\Delta S_{(k)}[\Phi; v; \bar{g}]
\end{aligned}
\tag{3.3.121}
$$

where it is understood that the field-dependent differential operators in the round brackets in the metric fluctuation trace sector act on the first spacetime argument of the following operators and those in the ghosts trace sector on the second.
At this point, similarly to the case of the standard equation, since on the left hand side we do not have anymore terms depending on the additional sources, we can neglect their contribution also on the right hand side, considering directly derivatives of $\tilde{\Gamma}_{(k)}\big[\Phi; v; \bar{g}\big]$.

[12]Similarly to computation sketched in 7.

Einstein-Hilbert truncation

Taking inspiration from the Einstein-Hilbert truncation for the standard equation, the proper Einstein-Hilbert truncation for the BRST-symmetrically-regulated theory is given by:

$$\begin{aligned}\tilde{\Gamma}_{(k)}[\mathsf{h},\mathsf{c},\bar{\mathsf{c}};v;\bar{g}] = \Gamma_{EH(k)}[g=\mathsf{h}+\bar{g}] + \Gamma_{gf,0(k)}[\mathsf{h};v;\bar{g}] + \Gamma_{gh,0(k)}[\mathsf{h},\mathsf{c},\bar{\mathsf{c}};v;\bar{g}] \\ + \Delta S_{(k)}[\mathsf{h},\mathsf{c},\bar{\mathsf{c}};v;\bar{g}]\end{aligned} \tag{3.3.122}$$

where the various terms are obtained by promoting the Newton's and cosmological constants to running couplings in the modified de Donder gauge-fixed action (3.3.26) (with in addition the regulator term, since we are considering the regulated effective action in the place of the effective average action):

$$\Gamma_{EH(k)}[g] = S_{EH}[g]\Big|_{\substack{G\to G_{(k)}\\ \Lambda\to\Lambda_{(k)}}} \tag{3.3.123}$$

$$\Gamma_{gf,0(k)}[\mathsf{h};v;\bar{g}] = S_{gf,0}[\mathsf{h};v;\bar{g}]\Big|_{G\to G_{(k)}} = \tag{3.3.124}$$

$$= S_{gf,0-deD}[\mathsf{h};\bar{g}]\Big|_{G\to G_{(k)}} + S_{gf,0-v}[\mathsf{h};v;\bar{g}] \tag{3.3.125}$$

$$\Gamma_{gh,0(k)}[\mathsf{h},\mathsf{c},\bar{\mathsf{c}};v;\bar{g}] = S_{gh,0}[\mathsf{h},\mathsf{c},\bar{\mathsf{c}};v;\bar{g}]\Big|_{G\to G_{(k)}} = \tag{3.3.126}$$

$$= S_{gh,0-deD}[\mathsf{h},\mathsf{c},\bar{\mathsf{c}};\bar{g}] + S_{gh,0-v}[\mathsf{h},\mathsf{c},\bar{\mathsf{c}};v;\bar{g}]\Big|_{G\to G_{(k)}} \tag{3.3.127}$$

Therefore, inserting the ansatz in the equation, we have in the left hand side a contribution from the Einstein-Hilbert part, the standard de Donder gauge-fixing part and the $hhc\bar{c}$-interaction term dependent on the external field of the modified de Donder ghost term; in particular the latter corresponds to a ghosts-dependent running term. However, we are interested in deriving the beta functions for the Newton's and cosmological constants, which can be extracted from the first two alone, without needing to consider the latter. Therefore we can disregard the third term in the left hand side along with all the terms in the right hand side which produce ghost-dependent operators. The result is effectively analogous to the Einstein-Hilbert truncation for the standard equation, i.e. a single-metric truncation with running Einstein-Hilbert action and de Donder gauge-fixing term and the scale-independent de Donder ghost term:

$$\begin{aligned}&\partial_t\left(\Gamma_{EH(k)}[g=\mathsf{h}+\bar{g}] + \Gamma_{gf,0-deD(k)}[\mathsf{h};\bar{g}]\right) = \\ &= \frac{1}{2}\,\mathrm{Tr}_{\bar{g}}\left[\partial_t \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}\right)\,\left(\kappa^2\left(\Gamma_{EH(k)}+\Gamma_{gf,0-deD(k)}\right)^{(2)}_{\mathsf{h}_{\alpha\beta}\mathsf{h}_{\mu\nu}} + \kappa^2\Delta S^{(2)}_{(k)\mathsf{h}_{\alpha\beta}\mathsf{h}_{\mu\nu}}\right)^{-1}\right] \\ &\quad + \mathrm{Tr}_{\bar{g}}\left[\bar{g}^{\lambda\rho}\partial_t r_{gh(k)}\left(-\bar{\Box}\right)\bar{K}^{\mu\nu}{}_{\rho\sigma}\bar{\nabla}^{\sigma}\left(\bar{g}_{\alpha\nu}\bar{\nabla}_{\mu}+\bar{g}_{\alpha\mu}\bar{\nabla}_{\nu}\right)\left(S^{(2)}_{gh,0-deD\,\bar{\mathsf{c}}_{\lambda}\,\mathsf{c}^{\alpha}} + \Delta S^{(2)}_{(k)\bar{\mathsf{c}}_{\lambda}\,\mathsf{c}^{\alpha}}\right)^{-1}\right]\end{aligned} \tag{3.3.128}$$

In particular, we notice that the metric fluctuation trace sector has got the same structure of the standard one, while the one relative to ghosts has got a different structure as a result of the construction. At this point we can perform the single metric truncation according to (2.4.54):

$$\begin{aligned}
&\partial_t \Gamma_{EH(k)}[g] = \\
&= \frac{1}{2}\,\mathrm{Tr}_g\left[\partial_t \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\Box\right)\left(\kappa^2\left(\Gamma_{EH(k)}+\Gamma_{gf,0-deD(k)}\right)^{(2)}_{\mathsf{h}_{\alpha\beta}\mathsf{h}_{\mu\nu}}\Big|_{\substack{\mathsf{h}=0\\ \bar{g}=g}}\right.\right.\\
&\qquad\qquad\left.\left.+\kappa^2\Delta S^{(2)}_{(k)\mathsf{h}_{\alpha\beta}\mathsf{h}_{\mu\nu}}\Big|_{\substack{\mathsf{h}=0\\ \bar{g}=g}}\right)^{-1}\right]\\
&\quad+\mathrm{Tr}_g\left[g^{\lambda\rho}\partial_t r_{gh(k)}\left(-\Box\right)K^{\mu\nu}{}_{\rho\sigma}\nabla^\sigma\left(g_{\alpha\nu}\nabla_\mu+g_{\alpha\mu}\nabla_\nu\right)\left(S^{(2)}_{gh,0-deD\,\bar{\mathsf{c}}_\lambda\,\mathsf{c}^\alpha}\Big|_{\substack{\mathsf{h}=0\\ \bar{g}=g}}\right.\right.\\
&\qquad\qquad\left.\left.+\Delta S^{(2)}_{(k)\bar{\mathsf{c}}_\lambda\,\mathsf{c}^\alpha}\Big|_{\substack{\mathsf{h}=0\\ \bar{g}=g}}\right)^{-1}\right]
\end{aligned} \tag{3.3.129}$$

The differential operator acting on the inverse operator in the ghosts trace sector gives:

$$g^{\lambda\rho}K^{\mu\nu}{}_{\rho\sigma}\nabla^\sigma\left(g_{\alpha\nu}\nabla_\mu+g_{\alpha\mu}\nabla_\nu\right)=\delta^\lambda_\alpha\Box+\left[\nabla^\lambda,\nabla_\alpha\right] \tag{3.3.130}$$

The two derivatives of the regulator give instead[13]:

$$\Delta S^{(2)}_{(k)\mathsf{h}_{\alpha\beta}\mathsf{h}_{\mu\nu}}\Big|_{\substack{\mathsf{h}=0\\ \bar{g}=g}}=\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\Box\right) \tag{3.3.131}$$

$$\Delta S^{(2)}_{(k)\bar{\mathsf{c}}_\lambda\,\mathsf{c}^\alpha}\Big|_{\substack{\mathsf{h}=0\\ \bar{g}=g}}=-r_{gh(k)}\left(-\bar{\Box}\right)\left(\bar{\Box}\,\delta^\lambda_\alpha+\bar{R}^\lambda{}_\alpha\right) \tag{3.3.132}$$

Therefore the final simplified form of the flow equation in the Einstein-Hilbert truncation is:

$$\begin{aligned}
&\partial_t \Gamma_{EH(k)}[g] = \\
&= \frac{1}{2}\,\mathrm{Tr}_g\left[\partial_t \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\Box\right)\left(\kappa^2\left(\Gamma_{EH(k)}+\Gamma_{gf,0-deD(k)}\right)^{(2)}_{\mathsf{h}_{\alpha\beta}\mathsf{h}_{\mu\nu}}\Big|_{\substack{\mathsf{h}=0\\ \bar{g}=g}}+\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\Box\right)\right)^{-1}\right]\\
&\quad+\mathrm{Tr}_g\left[\partial_t r_{gh(k)}\left(-\Box\right)\left(\delta^\lambda_\alpha\Box+\left[\nabla^\lambda,\nabla_\alpha\right]\right)\left(S^{(2)}_{gh,0-deD\,\bar{\mathsf{c}}_\lambda\,\mathsf{c}^\alpha}\Big|_{\substack{\mathsf{h}=0\\ \bar{g}=g}}\right.\right.\\
&\qquad\qquad\left.\left.-r_{gh(k)}\left(-\bar{\Box}\right)\left(\bar{\Box}\,\delta^\lambda_\alpha+\bar{R}^\lambda{}_\alpha\right)\right)^{-1}\right]
\end{aligned} \tag{3.3.133}$$

At this point, within the truncation and assumptions made, we have lost completely the

[13]At this point we suppress the $\sqrt{g}$ factors in the expressions working in compact notation, as noticed in 9; in particular we will write $\Box\Box^{-1}=\Box^{-1}\Box=\mathbb{1}$.

dependence of the external field, which consequently will not appear in the final form of the equation in component form. The left hand side of the equation is given as in the standard case by:

$$\mathrm{LHS}_{(k)}(R) = \partial_t \Gamma_{EH(k)}[g] = \frac{2}{\kappa^2}\int d^Dx\sqrt{g}\left[-R\partial_t Z_{N(k)} + 2\partial_t(Z_{N(k)}\Lambda_{(k)})\right] \tag{3.3.134}$$

with the dimensionless running constant $Z_{N(k)} = G/G_{(k)}$. The right hand side is given by the two trace sectors:

$$\mathrm{RHS}_{(k)}(R) = \mathrm{RHS}^{std}_{gr(k)}(R) + \mathrm{RHS}_{gh(k)}(R) \tag{3.3.135}$$

and can be evaluated in a maximally symmetric spacetime characterized by (2.4.58). In particular, the trace sector of metric fluctuation is exactly equal as in the standard case and adopting the same regularization scheme, namely:

$$\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(p^2\right) = \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\mathcal{R}_{gr(k)}\left(p^2\right) \tag{3.3.136}$$

with:

$$\mathcal{R}_{gr(k)}\left(p^2\right) = k^2\mathcal{R}_0\left(\frac{p^2}{k^2}\right) \tag{3.3.137}$$

$$\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)} = Z_{N(k)}\left[\left(\delta^{\alpha\beta,\mu\nu} - \frac{\bar{g}^{\alpha\beta}\bar{g}^{\mu\nu}}{D}\right) - \frac{D-2}{2}\frac{\bar{g}^{\alpha\beta}\bar{g}^{\mu\nu}}{D}\right] \tag{3.3.138}$$

yields exactly the result presented in subsection 2.4.3. Therefore, we need only to evaluate the ghosts trace sector. As in the standard case the second derivative receives a contribution only from the quadratic part of the de Donder ghost term:

$$[S_{gh,0-deD}[\mathsf{h},\mathsf{c},\bar{\mathsf{c}};\bar{g}]]_{\bar{\mathsf{c}}\mathsf{c}-term} = \int d^Dx\sqrt{\bar{g}}\,\bar{\mathsf{c}}_\mu\left(-\bar{\Box}\delta^\mu_\nu - \bar{R}^\mu_\nu\right)\mathsf{c}^\nu \tag{3.3.139}$$

which in a maximally symmetric spacetime reads:

$$[S_{gh,0-deD}[\mathsf{h},\mathsf{c},\bar{\mathsf{c}};\bar{g}]]_{\bar{\mathsf{c}}\mathsf{c}-term} = \int d^Dx\sqrt{\bar{g}}\,\bar{\mathsf{c}}_\mu\left(-\bar{\Box} - \frac{\bar{R}}{D}\right)\mathsf{c}^\mu \tag{3.3.140}$$

and:

$$S^{(2)}_{gh,0-deD\,\bar{\mathsf{c}}_\lambda\,\mathsf{c}^\alpha}\Big|_{\substack{\mathsf{h}=0\\ \bar{g}=g}} = \left(-\Box - \frac{R}{D}\right)\delta^\mu_\nu \tag{3.3.141}$$

So, the ghosts trace sector of the right hand side of the flow equation is:

$$\begin{aligned}\mathrm{RHS}_{gh(k)}(R) &= -\mathrm{Tr}_{g\,V}\left[\partial_t r_{gh(k)}\left(-\Box\right)\Box\left(\left(1 + r_{gh(k)}\left(-\Box\right)\right)\left(\Box + \frac{R}{D}\right)\right)^{-1}\right] = \\ &= -\mathrm{Tr}_{g\,V}\left[\partial_t r_{gh(k)}\left(-\Box\right)\Box\left(\Box + \frac{R}{D}\right)^{-1}\left(1 + r_{gh(k)}\left(-\Box\right)\right)^{-1}\right]\end{aligned} \tag{3.3.142}$$

where it is understood an identity in the space of vectors giving the discrete part of the trace $\mathrm{tr}_T(\mathbb{1}) = \delta^\mu_\mu = D$; we notice also that the various operators inside the trace are all functions of the laplacian and can be thus arbitrarily commuted between themselves.
We now specify the regulator; in particular, we exploit the freedom allowed in the FRG framework to simplify the inverse operator $(\Box + R/D)^{-1}$, choosing a regulator $r_{gh(k)}\left(-\Box\right) \propto (\Box + R/D)$, i.e. characterized by the following form:

$$r_{gh(k)}\left(p^2\right) = -\frac{\mathcal{R}_{(k)}\left(p^2\right)}{p^4}\left(-p^2 + \frac{R}{D}\right) \tag{3.3.143}$$

where $\mathcal{R}_{(k)}\left(p^2\right)$ is a standard regulator with the same shape function used for the regulator for the metric fluctuation:

$$\mathcal{R}_{(k)}\left(p^2\right) = k^2 \mathcal{R}_0\left(\frac{p^2}{k^2}\right) \tag{3.3.144}$$

The factor p^{-4} is required to have the correct dimensionality, since $r_{gh(k)}\left(p^2\right)$ must be adimensional. According to (3.3.40) (in a maximally symmetric spacetime), this choice corresponds to a regulator

$$\mathcal{R}_{gh(k)}\left(p^2\right) = \frac{\mathcal{R}_{(k)}\left(p^2\right)}{p^4}\left(-p^2 + \frac{R}{D}\right)^2 \tag{3.3.145}$$

which inherits the shape, and thus the required properties (3.3.38), of a standard regulator, just "smeared" with by the factor $(R/D\, 1/p^2 - 1)$. Adopting this regularization scheme, the trace in the right hand side is rewritten as:

$$\begin{aligned}
\mathrm{RHS}_{gh(k)}(R) &= \mathrm{Tr}_{g\,V}\left\{\partial_t \mathcal{R}_{(k)}\left(-\Box\right)\Box^{-1}\left[1 - \mathcal{R}_{(k)}\left(-\Box\right)\Box^{-2}\left(\Box + \frac{R}{D}\right)\right]^{-1}\right\} = \\
&= \mathrm{Tr}_{g\,V}\left\{\partial_t \mathcal{R}_{(k)}\left(-\Box\right)\Box\left[\Box^2 - \mathcal{R}_{(k)}\left(-\Box\right)\left(\Box + \frac{R}{D}\right)\right]^{-1}\right\} = \\
&= \mathrm{Tr}_{g\,V}\left\{\partial_t \mathcal{R}_{(k)}\left(-\Box\right)\Box\left(\Box^2 - \mathcal{R}_{(k)}\left(-\Box\right)\Box\right)^{-1} \cdot \right. \\
&\qquad \left. \cdot\left[1 - \left(\Box^2 - \mathcal{R}_{(k)}\left(-\Box\right)\Box\right)^{-1}\mathcal{R}_{(k)}\left(-\Box\right)\frac{R}{D}\right]^{-1}\right\}
\end{aligned} \tag{3.3.146}$$

Using now the operatorial Taylor expansion $(1 + \mathcal{O})^{-1} = \sum_{n=0}^{\infty}(-1)^n \mathcal{O}^n$ we identify the $O(1)$ and $O(R)$ terms:

$$\begin{aligned}
\mathrm{RHS}_{gh(k)}(R) = &- \mathrm{Tr}_{g\,V}\left[\partial_t \mathcal{R}_{(k)}\left(-\Box\right)\left(-\Box + \mathcal{R}_{(k)}\left(-\Box\right)\right)^{-1}\right] \\
&- \mathrm{Tr}_{g\,V}\left[\partial_t \mathcal{R}_{(k)}\left(-\Box\right)\mathcal{R}_{(k)}\left(-\Box\right)\left(-\Box\right)^{-1}\left(-\Box + \mathcal{R}_{(k)}\left(-\Box\right)\right)^{-2}\frac{R}{D}\right] + O(R^2)
\end{aligned} \tag{3.3.147}$$

Writing the regulator and its derivative in terms of the shape function:

$$\partial_t \mathcal{R}_{(k)} \left(-\Box\right) = 2k^2 \left[\mathcal{R}_0 \left(-\frac{\Box}{k^2}\right) + \frac{\Box}{k^2}\mathcal{R}_0' \left(-\frac{\Box}{k^2}\right)\right] \tag{3.3.148}$$

with $\mathcal{R}_0'(x) = d\mathcal{R}_0(x)/dx$, the result can be written as:

$$\mathrm{RHS}_{gh(k)}(R) = -2\mathrm{Tr}_{g\,V}[W_1\left(-\Box\right)] - \frac{2}{Dk^2}\mathrm{Tr}_{g\,V}[W_2\left(-\Box\right) R] + O(R^2) \tag{3.3.149}$$

where the two functions are:

$$W_1(z) = \frac{\mathcal{R}_0\left(\frac{z}{k^2}\right) - \frac{z}{k^2}\mathcal{R}_0'\left(\frac{z}{k^2}\right)}{\frac{z}{k^2} + \mathcal{R}_0\left(\frac{z}{k^2}\right)} \tag{3.3.150}$$

$$W_2(z) = \frac{\mathcal{R}_0\left(\frac{z}{k^2}\right)\left(\mathcal{R}_0\left(\frac{z}{k^2}\right)\frac{z}{k^2} - \mathcal{R}_0'\left(\frac{z}{k^2}\right)\right)}{\frac{z}{k^2}\left(\frac{z}{k^2} + \mathcal{R}_0\left(\frac{z}{k^2}\right)\right)^2} \tag{3.3.151}$$

The corresponding Q-functionals (2.4.82) are:

$$Q_{n>0}[W_1] = k^{2n}\Phi^1_n(0) \tag{3.3.152}$$

$$Q_{n>0}[W_2] = k^{2n}\Theta^2_n \tag{3.3.153}$$

with the standard threshold function $\Phi^p_n(w)$ (2.4.87) for $p = 2$ computed in $w = 0$, and the coefficients:

$$\Theta^2_n = \frac{1}{\Gamma(n)}\int_0^\infty dx\, x^{n-1}\frac{\mathcal{R}_0\left(x\right)\left(\mathcal{R}_0\left(x\right) - x\mathcal{R}_0'\left(x\right)\right)}{x\left(x + \mathcal{R}_0\left(x\right)\right)^2} \tag{3.3.154}$$

According to the heat kernel expansions (2.4.80) and (2.4.81) we have:

$$\begin{aligned}\mathrm{RHS}_{gh(k)}(R) = &-\frac{2D}{(4\pi)^{D/2}}k^D\Phi^1_{D/2}(0)\int d^Dx\sqrt{g} \\ &-\frac{2D}{(4\pi)^{D/2}}k^{D-2}\left(\frac{1}{6}\Phi^1_{D/2-1}(0) + \frac{1}{D}\Theta^2_{D/2}\right)\int d^Dx\sqrt{g}\,R + O(R^2)\end{aligned} \tag{3.3.155}$$

and:

$$\mathrm{RHS}_{gh(k)}(R)|_{O(1)} = -\frac{2D}{(4\pi)^{D/2}}k^D\Phi^1_{D/2}(0) \tag{3.3.156}$$

$$\mathrm{RHS}_{gh(k)}(R)|_{O(R)} = -\frac{2D}{(4\pi)^{D/2}}k^{D-2}\left(\frac{1}{6}\Phi^1_{D/2-1}(0) + \frac{1}{D}\Theta^2_{D/2}\right) \tag{3.3.157}$$

In particular, the $O(1)$ contribution is equal to the standard one (2.4.90), while the $O(R)$ one differs from (2.4.92) according to the substitution of the threshold function $\Phi^p_n(w)$ (2.4.87) for $p = 2$ computed in $w = 0$ with the coefficient $\Theta^2_{D/2}$:

$$\Phi^2_{D/2}(0) \quad \rightarrow \quad \Theta^2_{D/2} \tag{3.3.158}$$

In conclusion, equating left and right hand side of the flow equation:

$$\partial_t(Z_{N(k)}\Lambda_{(k)}) = +\frac{\kappa^2}{4}\,\mathrm{RHS}^{std}_{gr(k)}(R)|_{O(1)} + \frac{\kappa^2}{4}\,\mathrm{RHS}^{std}_{gh(k)}(R)|_{O(1)} \tag{3.3.159}$$

$$\partial_t Z_{N(k)} = -\frac{\kappa^2}{2}\,\mathrm{RHS}^{std}_{gr(k)}(R)|_{O(R)} - \frac{\kappa^2}{2}\,\mathrm{RHS}_{gh(k)}(R)|_{O(R)} \tag{3.3.160}$$

we recover the standard expression for the derivative $\partial_t(Z_{N(k)}\Lambda_{(k)})$ and an expression with the standard functional form, but corrected according to the substitution (3.3.158), for the derivative $\partial_t Z_{N(k)}$:

$$\begin{cases} \partial_t(Z_{N(k)}\Lambda_{(k)}) = \dfrac{\kappa^2}{16}\dfrac{1}{(4\pi)^{D/2}}\,k^D\Big[2D(D+1)\Phi^1_{D/2}\left(-\frac{2\Lambda_{(k)}}{k^2}\right) - D(D+1)\eta_N\tilde{\Phi}^1_{D/2}\left(-\frac{2\Lambda_{(k)}}{k^2}\right) \\ \qquad\qquad - 8D\Phi^1_{D/2}(0)\Big] \\ \partial_t Z_{N(k)} = -\dfrac{\kappa^2}{24}\dfrac{1}{(4\pi)^{D/2}}\,k^{D-2}\Big\{D(D+1)\left[\Phi^1_{D/2-1}\left(-\frac{2\Lambda_{(k)}}{k^2}\right) - \frac{1}{2}\eta_N\tilde{\Phi}^1_{D/2-1}\left(-\frac{2\Lambda_{(k)}}{k^2}\right)\right] \\ \qquad\qquad - 6D(D-1)\left[\Phi^2_{D/2}\left(-\frac{2\Lambda_{(k)}}{k^2}\right) - \frac{1}{2}\eta_N\tilde{\Phi}^2_{D/2}\left(-\frac{2\Lambda_{(k)}}{k^2}\right)\right] \\ \qquad\qquad - 4D\Phi^1_{D/2-1}(0) - 24\Theta^2_{D/2}\Big\} \end{cases} \tag{3.3.161}$$

Therefore, also the anomalous dimension has the standard form in terms of the dimensionless Newton's and cosmological constant:

$$\eta_N(g_{(k)},\lambda_{(k)}) = \frac{g_{(k)}B_1(\lambda_{(k)})}{1-g_{(k)}B_2(\lambda_{(k)})} \tag{3.3.162}$$

up to the substitution (3.3.158) in the function $B_1(\lambda_{(k)})$:

$$B_1(\lambda_{(k)}) = \frac{1}{3}(4\pi)^{1-D/2}\Big[D(D+1)\Phi^1_{D/2-1}\left(-2\lambda_{(k)}\right) - 6D(D-1)\Phi^2_{D/2}\left(-2\lambda_{(k)}\right) \\ - 4D\Phi^1_{D/2-1}(0) - 24\Theta^2_{D/2}\Big] \tag{3.3.163}$$

$$B_2(\lambda_{(k)}) = -\frac{1}{6}(4\pi)^{1-D/2}\Big[D(D+1)\tilde{\Phi}^1_{D/2-1}\left(-2\lambda_{(k)}\right) - 6D(D-1)\tilde{\Phi}^2_{D/2}\left(-2\lambda_{(k)}\right)\Big] \tag{3.3.164}$$

and consequently we also have formally equal system of FRG equations for the dimensionless running Newton's and cosmological constants:

$$\begin{cases} \partial_t g_{(k)} = \beta_g(g_{(k)},\lambda_{(k)}) \\ \partial_t \lambda_{(k)} = \beta_\lambda(g_{(k)},\lambda_{(k)}) \end{cases} \tag{3.3.165}$$

with the beta functions differing implicitly due to the change in the anomalous dimension:

$$\beta_g(g_{(k)}, \lambda_{(k)}) = \left[D - 2 + \eta_N(g_{(k)}, \lambda_{(k)})\right] g_{(k)} \tag{3.3.166}$$

$$\begin{aligned}\beta_\lambda(g_{(k)}, \lambda_{(k)}) = &- \left[2 - \eta_N(g_{(k)}, \lambda_{(k)})\right] \lambda_{(k)} \\ &+ \frac{1}{2}(4\pi)^{1-D/2} g_{(k)} \Big[2D(D+1)\Phi^1_{D/2}\left(-2\lambda_{(k)}\right) \\ &\qquad - D(D+1)\eta_N(g_{(k)}, \lambda_{(k)})\tilde{\Phi}^1_{D/2}\left(-2\lambda_{(k)}\right) \\ &\qquad - 8D\Phi^1_{D/2}(0)\Big]\end{aligned} \tag{3.3.167}$$

For instance using Litim optimized regulator (2.4.95), the coefficients Θ^2_n are given by (with $n > 1$):

$$\Theta^2_n = \frac{1}{\Gamma(n)} \int_0^1 dx\, x^{n-1} \frac{1-x}{x} = \frac{1}{\Gamma(n)n(n-1)} = \frac{1}{(n-1)\Gamma(n+1)} \tag{3.3.168}$$

Therefore, the correction in the equations with respect to the standard case, given by (2.4.98), is minor:

$$\Phi^2_{D/2}(0) = \frac{1}{\Gamma\left(\frac{D}{2}+1\right)} \quad \to \quad \Theta^2_{D/2} = \frac{1}{\left(\frac{D}{2}-1\right)\Gamma\left(\frac{D}{2}+1\right)} \tag{3.3.169}$$

and moreover exactly equal in $D = 4$ dimensions, when the extra prefactor is 1. We conclude that within the Einstein-Hilbert truncation and the regularization scheme employed for the ghosts, the BRST-compatible Wetterich-Morris equation reproduces (almost) the same results for the beta functions of the couplings of the standard equation. In particular, employing the Litim optimized regulator, we have in $D = 4$ dimensions precisely the same beta functions, and therefore the same fixed points of the FRG flow [5].

Conclusion

The main result of the thesis is the generalization to quantum theories of gravity of the formalism introduced in [6] to implement an FRG framework in a manifestly BRST-invariant manner in a quantum non-abelian gauge theory. This was achieved by combining the regularization and gauge-fixing procedures in a single step, and in particular by employing a non-standard gauge-fixing choice capable of introducing in the gauge-fixed action quadratic mass terms for the metric fluctuation and the ghosts, which, promoting the mass parameters to FRG regulators with suitable properties, allowed to regularize the theory preserving explicitly BRST symmetry, and thus to obtain a Wetterich-Morris equation compatible with the constraint imposed by BRST symmetry, i.e. the Zinn-Justin equation, satisfied by the effective average action for all values of the FRG scale. Specifically, the gauge-fixing required the introduction of a linear gauge-fixing term and a quadratic gauge-fixing function, contrary to the typical standard choices involving a quadratic gauge-fixing term and a linear gauge-fixing function; in turn, this required the introduction of an external field and resulted in the emergence in the gauge-fixed action of non-standard interaction terms dependent on it. Using this gauge-fixing structure to perform an FRG regularization, this produced an unregulated action equipped with a standard gauge-fixing plus additional terms dependent on the external field, as well as a regulator term containing the standard quadratic regulators for the metric fluctuation and the ghosts plus additional non-quadratic mixed terms. Due to the presence of the latter, it was necessary to introduce additional sources, in order to write the Wetterich-Morris equation, which as a consequence resulted to be more complicated than the standard one; nevertheless, it was shown that within the Einstein-Hilbert truncation and with a suitable regularization scheme, the equation in component form has the same structure of the standard one.

This work may lead to several possible future developments. In particular, at the practical level, an immediate further study could be an analysis of the dependence of the theory on the regularization scheme, in particular in the ghost sector, which shows a non-standard behavior; moreover, it could be of interest also considering more general truncation schemes, possibly non-linear in the additional sources. At a more formal level, another research line could concern the study of the FRG flow of quantum gravitational observables, i.e BRST-invariant functionals of the fields, by means of an analogous BRST-compatible equation, and their physical interpretation.

Appendix

A Background field method

A.1 Background expansion of curvature tensors

Under the linear background split:

$$g_{\mu\nu}(x) = g_{\mu\nu}(x) + h_{\mu\nu}(x) \tag{A.1}$$

we have the following background expansions:

- Inverse metric: the expansion follows from the matrix Taylor expansion formula for $(\mathbb{1} + \mathrm{M})^{-1} = \sum_{n=0}^{\infty} (-1)^n \mathrm{M}^n$:

$$\begin{aligned} g^{\mu\nu} &= \left[\left(\mathbb{1} + \bar{\mathrm{g}}^{-1}\mathrm{h} \right)^{-1} \bar{\mathrm{g}}^{-1} \right]^{\mu\nu} = \\ &= \left[\sum_{n=0}^{\infty} (-1)^n (\bar{\mathrm{g}}^{-1}\mathrm{h})^n \bar{\mathrm{g}}^{-1} \right]^{\mu\nu} = \\ &= \sum_{n=0}^{\infty} (-1)^n (h^n)^{\mu}{}_{\lambda}\, \bar{g}^{\lambda\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu} + h^{\mu\lambda} h_{\lambda}{}^{\nu} + O(h^3) \end{aligned} \tag{A.2}$$

- Square root metric determinant: the expansion follows from the matrix identity $\det e^{\mathrm{M}} = e^{\mathrm{tr M}}$ and the matrix Taylor expansion formula for $\log(\mathbb{1} + \mathrm{M}) = \sum_{n=1}^{\infty} (-1)^n / n \mathrm{M}^n$:

$$\begin{aligned} \sqrt{g} &= \sqrt{\bar{g}} \sqrt{\det(\mathbb{1} + \bar{\mathrm{g}}^{-1}\mathrm{h})} = \\ &= \sqrt{\bar{g}} \exp\left\{ \frac{1}{2} \mathrm{tr} \left[\log(\mathbb{1} + \bar{\mathrm{g}}^{-1}\mathrm{h}) \right] \right\} = \\ &= \sqrt{\bar{g}} \exp\left\{ \frac{1}{2} \mathrm{tr} \left[\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (\bar{\mathrm{g}}^{-1}\mathrm{h})^n \right] \right\} = \\ &= \sqrt{\bar{g}} \exp\left(\frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (h^n)^{\mu}{}_{\mu} \right) = \sqrt{\bar{g}} \left(1 + \frac{1}{2} h - \frac{1}{4} h_{\mu\nu} h^{\mu\nu} + \frac{1}{8} h^2 + O(h^3) \right) \end{aligned} \tag{A.3}$$

- Metric connection: the expansion follows from the definition $\Gamma^{\lambda}_{\mu\nu} = g^{\lambda\rho}/2\,(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$ after trading the partial derivative for background covariant derivatives:

$$\Gamma^{\alpha}_{\mu\nu} = \bar{\Gamma}^{\alpha}_{\mu\nu} + \delta\Gamma^{\alpha}_{\mu\nu} + O(h^2) \tag{A.4}$$

$$\delta\Gamma^{\alpha}_{\mu\nu} = \frac{\bar{g}^{\alpha\beta}}{2}\left(\bar{\nabla}_\mu h_{\nu\beta} + \bar{\nabla}_\nu h_{\mu\beta} - \bar{\nabla}_\beta h_{\mu\nu}\right)$$

- Riemann tensor: the expansion follows from the definition $R^{\alpha}{}_{\mu\beta\nu} = \partial_\beta \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\beta} + \Gamma^{\lambda}_{\beta\mu}\Gamma^{\alpha}_{\lambda\nu} - \Gamma^{\lambda}_{\nu\mu}\Gamma^{\alpha}_{\lambda\beta}$ after recognizing the background covariant derivatives of $\delta\Gamma^{\alpha}_{\mu\nu}$:

$$R^{\alpha}{}_{\mu\beta\nu} = \bar{R}^{\alpha}{}_{\mu\beta\nu} + \delta R^{\alpha}{}_{\mu\beta\nu} + O(h^2) \tag{A.5}$$

$$\begin{aligned}\delta R^{\alpha}{}_{\mu\beta\nu} &= \bar{\nabla}_\beta \delta\Gamma^{\alpha}_{\mu\nu} - \bar{\nabla}_\nu \delta\Gamma^{\alpha}_{\mu\beta} = \\ &= \frac{1}{2}\left(-\bar{\nabla}_\beta\bar{\nabla}^\alpha h_{\mu\nu} - \bar{\nabla}_\nu\bar{\nabla}_\mu h^{\alpha}{}_{\beta} + \bar{\nabla}_\nu\bar{\nabla}^\alpha h_{\mu\beta} + \bar{\nabla}_\beta\bar{\nabla}_\mu h^{\alpha}{}_{\nu} + [\bar{\nabla}_\beta, \bar{\nabla}_\nu] h^{\alpha}{}_{\mu}\right)\end{aligned}$$

- Ricci tensor: the expansion follows from the definition $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$:

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \delta R_{\mu\nu} + O(h^2) \tag{A.6}$$

$$\delta R_{\mu\nu} = \delta R^{\alpha}{}_{\mu\alpha\nu} = \frac{1}{2}\left(-\bar{\Box} h_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_\nu h + \bar{\nabla}_\alpha\bar{\nabla}_\mu h^{\alpha}{}_{\nu} + \bar{\nabla}_\alpha\bar{\nabla}_\nu h^{\alpha}{}_{\mu}\right)$$

- Ricci scalar: the expansion follows from the definition $R = g^{\mu\nu}R_{\mu\nu}$:

$$R = \bar{R} + \delta R + O(h^2) \tag{A.7}$$

$$\delta R = \bar{g}^{\mu\nu}\delta R_{\mu\nu} + \bar{R}_{\mu\nu}\delta\bar{g}^{\mu\nu} = -\bar{\Box} h + \bar{\nabla}_\mu\bar{\nabla}_\nu h^{\mu\nu} - \bar{R}_{\mu\nu}h^{\mu\nu}$$

- Einstein tensor: the expansion follows from the definition $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu}$:

$$G_{\mu\nu} = \bar{G}_{\mu\nu} + \delta G_{\mu\nu} + O(h^2) \tag{A.8}$$

$$\begin{aligned}\delta G_{\mu\nu} &= \delta R_{\mu\nu} - \frac{1}{2}\left(\delta R \bar{g}_{\mu\nu} + \bar{R}\delta g_{\mu\nu}\right) = \\ &= \frac{1}{2}\left(-\bar{\Box} h_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_\nu h + \bar{\nabla}_\alpha\bar{\nabla}_\mu h^{\alpha}{}_{\nu} + \bar{\nabla}_\alpha\bar{\nabla}_\nu h^{\alpha}{}_{\mu} + \bar{g}_{\mu\nu}\bar{\Box} h - \bar{g}_{\mu\nu}\bar{\nabla}_\alpha\bar{\nabla}_\beta h^{\alpha\beta}\right) \\ &\quad + \frac{1}{2}\bar{g}_{\mu\nu}\bar{R}_{\alpha\beta}h^{\alpha\beta} - \frac{1}{2}\bar{R}h_{\mu\nu}\end{aligned}$$

A.2 Second order background expansion of the Einstein-Hilbert action

The Taylor terms in the background expansion of the Einstein-Hilbert action can be computed according to (2.2.17):

$$S_{EH,n}[h;\bar{g}] = \frac{1}{n!}\delta^n S_{EH}[g]\Big|_{g_{\mu\nu}=\bar{g}_{\mu\nu},\, \delta g_{\mu\nu}=h_{\mu\nu}} \tag{A.9}$$

To find the zeroth, first and second order term are sufficient the variations introduced in subsection (1.3.2):

$$\delta\sqrt{g} = -\frac{1}{2}\sqrt{g}g_{\mu\nu}\delta g^{\mu\nu} \tag{A.10}$$

$$\delta\left(\sqrt{g}g^{\mu\nu}\right) = \sqrt{g}\left(\delta g^{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\delta g^{\alpha\beta}\right) \tag{A.11}$$

$$\delta\left(\sqrt{g}R\right) = \delta\left(\sqrt{g}g^{\mu\nu}\right)R_{\mu\nu} + \sqrt{g}g^{\mu\nu}\delta R_{\mu\nu} = \sqrt{g}G_{\mu\nu}\delta g^{\mu\nu} + \sqrt{g}\,\nabla_\lambda X^\lambda \tag{A.12}$$

We have:

- Zeroth order term: the zeroth order term is given by the Einstein-Hilbert action computed in the background metric:

$$S_{EH,0}[h;\bar{g}] = S_{EH}[g=\bar{g}] = \frac{2}{\kappa^2}\int d^Dx\sqrt{\bar{g}}\left(-\bar{R}+2\Lambda\right) \tag{A.13}$$

- First order term: assuming that the boundary term originating from $\sqrt{g}g^{\mu\nu}\delta R_{\mu\nu}$ vanishes, the first variation of the Einstein-Hilbert action is:

$$\begin{aligned}\delta S_{EH}[g] &= \frac{2}{\kappa^2}\int d^Dx\left[-\delta\left(\sqrt{g}R\right)+2\Lambda\delta\sqrt{g}\right] = \\ &= \frac{2}{\kappa^2}\int d^Dx\left[-\delta\left(\sqrt{g}g^{\mu\nu}\right)R_{\mu\nu}+2\Lambda\delta\sqrt{g}\right] = \\ &= -\frac{2}{\kappa^2}\int d^Dx\sqrt{g}\left(G_{\mu\nu}+\Lambda g_{\mu\nu}\right)\delta g^{\mu\nu}\end{aligned} \tag{A.14}$$

  From appendix A.1 we have:

$$\delta g^{\mu\nu} = -h^{\mu\nu} \tag{A.15}$$

  It follows that:

$$S_{EH,1}[h;\bar{g}] = \delta S_{EH}[g]\Big|_{g_{\mu\nu}=\bar{g}_{\mu\nu},\, \delta g_{\mu\nu}=h_{\mu\nu}} = \frac{2}{\kappa^2}\int d^Dx\sqrt{\bar{g}}\,(\bar{G}_{\mu\nu}+\Lambda\bar{g}_{\mu\nu})h^{\mu\nu} \tag{A.16}$$

- Second order term: starting from the expression of the first variation, the second variation of the Einstein-Hilbert action is:

$$
\begin{aligned}
\delta^2 S_{EH}[g] &= \frac{2}{\kappa^2}\int d^D x \left[-\delta\left(\delta\left(\sqrt{g}g^{\mu\nu}\right)R_{\mu\nu}\right) + 2\Lambda\delta^2\sqrt{g}\right] = \\
&= \frac{2}{\kappa^2}\int d^D x \left[-\delta^2\left(\sqrt{g}g^{\mu\nu}\right)R_{\mu\nu} - \delta\left(\sqrt{g}g^{\mu\nu}\right)\delta R_{\mu\nu} + 2\Lambda\delta^2\sqrt{g}\right]
\end{aligned}
\tag{A.17}
$$

In particular, if the boundary term originating from $\sqrt{g}g^{\mu\nu}\delta R_{\mu\nu}$ vanishes, the second variation of the Ricci tensor does not contribute to the second variation of the Einstein-Hilbert action. From appendix A.1 we have:

$$\delta R_{\mu\nu} = \frac{1}{2}\left(-\bar{\Box}h_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_\nu h + \bar{\nabla}_\alpha\bar{\nabla}_\mu h^\alpha{}_\nu + \bar{\nabla}_\alpha\bar{\nabla}_\nu h^\alpha{}_\mu\right) \tag{A.18}$$

$$\frac{1}{2}\delta^2\sqrt{g} = \sqrt{\bar{g}}\left(-\frac{1}{4}h_{\mu\nu}h^{\mu\nu} + \frac{1}{8}h^2\right) \tag{A.19}$$

$$\delta\left(\sqrt{g}g^{\mu\nu}\right) = \sqrt{\bar{g}}\left(-h^{\mu\nu} + \frac{1}{2}h\bar{g}^{\mu\nu}\right) \tag{A.20}$$

$$\frac{1}{2}\delta^2\left(\sqrt{g}g^{\mu\nu}\right) = \sqrt{\bar{g}}\left(h^{\mu\lambda}h_\lambda{}^\nu - \frac{1}{2}hh^{\mu\nu} + \frac{1}{8}h^2\bar{g}^{\mu\nu} - \frac{1}{4}h_{\alpha\beta}h^{\alpha\beta}\bar{g}^{\mu\nu}\right) \tag{A.21}$$

One finds:

$$
\begin{aligned}
S_{EH,2}[h;\bar{g}] &= \frac{1}{2}\delta^2 S_{EH}[g]\Big|_{g_{\mu\nu}=\bar{g}_{\mu\nu},\,\delta g_{\mu\nu}=h_{\mu\nu}} = \\
&= \frac{1}{\kappa^2}\int d^D x\sqrt{\bar{g}}\Bigg[-\frac{1}{2}\left(h^{\mu\nu}\bar{\Box}h_{\mu\nu} - \frac{1}{2}\,h\bar{\Box}h\right) \\
&\qquad + \left(h^{\mu\nu} - \frac{1}{2}h\bar{g}^{\mu\nu}\right)\bar{\nabla}_\alpha\bar{\nabla}_\mu\left(h^\alpha{}_\nu - \frac{1}{2}h\delta^\alpha_\nu\right) \\
&\qquad - 2\bar{R}_{\mu\nu}\left(h^{\mu\lambda}h_\lambda{}^\nu - \frac{1}{2}hh^{\mu\nu}\right) \\
&\qquad + \frac{1}{2}\left(\bar{R} - 2\Lambda\right)\left(h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}\,h^2\right)\Bigg]
\end{aligned}
\tag{A.22}
$$

The term in the second line can be rewritten using the rule (1.3.25) to commute the background covariant derivatives (torsion-less) acting on $h^\alpha{}_\nu$ (being h a scalar, we have $\bar{\nabla}_\alpha\bar{\nabla}_\mu h = \bar{\nabla}_\mu\bar{\nabla}_\alpha h$) and integrating by parts:

$$
\begin{aligned}
&\left(h^{\mu\nu} - \frac{1}{2}h\bar{g}^{\mu\nu}\right)\bar{\nabla}_\alpha\bar{\nabla}_\mu\left(h^\alpha{}_\nu - \frac{1}{2}h\delta^\alpha_\nu\right) = \\
&\quad = \left(h^{\mu\nu} - \frac{1}{2}h\bar{g}^{\mu\nu}\right)\left[\bar{\nabla}_\mu\bar{\nabla}_\alpha\left(h^\alpha{}_\nu - \frac{1}{2}h\delta^\alpha_\nu\right) - \bar{R}_{\alpha\mu\beta\nu}h^{\alpha\beta} + \bar{R}_{\mu\alpha}h^\alpha{}_\nu\right] \\
&\quad \overset{\text{i.b.p}}{\to} -\left(\bar{\nabla}^\mu h_{\mu\nu} - \frac{1}{2}\bar{\nabla}_\nu h\right)^2 - \bar{R}_{\alpha\mu\beta\nu}h^{\alpha\beta}h^{\mu\nu} + \bar{R}_{\mu\nu}h^{\mu\lambda}h_\lambda{}^\nu
\end{aligned}
\tag{A.23}
$$

Finally, one finds:

$$
\begin{aligned}
S_{EH,2}[h;\bar{g}] = {} & \frac{1}{2}\delta^2 S_{EH}[g]\Big|_{g_{\mu\nu}=\bar{g}_{\mu\nu},\, \delta g_{\mu\nu}=h_{\mu\nu}} = \\
= {} & \frac{1}{\kappa^2}\int d^D x\sqrt{\bar{g}}\Big[-\frac{1}{2}\left(h^{\mu\nu}\bar{\Box}h_{\mu\nu}-\frac{1}{2}\,h\bar{\Box}h\right) \\
& \qquad -\left(\bar{\nabla}^{\mu}h_{\mu\nu}-\frac{1}{2}\,\bar{\nabla}_{\nu}h\right)^2 \\
& \qquad -\bar{R}_{\alpha\mu\beta\nu}h^{\alpha\beta}h^{\mu\nu} \\
& \qquad -\bar{R}_{\mu\nu}\left(h^{\mu\lambda}h_{\lambda}{}^{\nu}-hh^{\mu\nu}\right) \\
& \qquad +\frac{1}{2}\,(\bar{R}-2\Lambda)\left(h^{\mu\nu}h_{\mu\nu}-\frac{1}{2}\,h^2\right)\Big]
\end{aligned}
\tag{A.24}
$$

B BRST symmetry

B.1 Nilpotency of BRST Slavnov variations of the metric and the ghost

The nilpotency of the BRST Slavnov variation of the ghost follows from its anticommuting nature as Grassmann-odd field:

$$
\begin{aligned}
s^2c^{\mu} &= s(c^{\nu}\partial_{\nu}c^{\mu}) = \\
&= c^{\lambda}\partial_{\lambda}c^{\nu}\partial_{\nu}c^{\mu}-c^{\nu}\partial_{\nu}(c^{\lambda}\partial_{\lambda}c^{\mu}) = \\
&= c^{\lambda}\partial_{\lambda}c^{\nu}\partial_{\nu}c^{\mu}-c^{\nu}\partial_{\nu}c^{\lambda}\partial_{\lambda}c^{\mu}-\underbrace{c^{\nu}c^{\alpha}}_{\substack{\text{antisym.}\\ \nu\leftrightarrow\alpha}}\ \underbrace{\partial_{\nu}\partial_{\alpha}c^{\mu}}_{\substack{\text{sym.}\\ \nu\leftrightarrow\alpha}}=0
\end{aligned}
\tag{B.1}
$$

In order to prove the nilpotency of the BRST Slavnov variation of the metric fluctuation it is convenient to rewrite the variation in terms of the full metric:

$$
sh_{\mu\nu}=\pounds_c(\bar{g}_{\mu\nu}+h_{\mu\nu})=sg_{\mu\nu} \tag{B.2}
$$

and to express the Lie derivative in terms of ordinary derivatives:

$$
\pounds_c g_{\mu\nu}=c^{\alpha}\partial_{\alpha}g_{\mu\nu}+\partial_{\mu}c^{\alpha}g_{\alpha\nu}+\partial_{\nu}c^{\alpha}g_{\alpha\mu} \tag{B.3}
$$

The second BRST Slavnov variation gives:

$$
\begin{aligned}
s^2h_{\mu\nu} &= s\left(\pounds_c g_{\mu\nu}\right) = \\
&= s\left(c^{\alpha}\partial_{\alpha}g_{\mu\nu}+\partial_{\mu}c^{\alpha}g_{\alpha\nu}+\partial_{\nu}c^{\alpha}g_{\alpha\mu}\right) = \\
&= \underbrace{sc^{\alpha}\partial_{\alpha}g_{\mu\nu}}_{\boldsymbol{A}}+\underbrace{\partial_{\mu}(sc^{\alpha})g_{\alpha\nu}}_{\boldsymbol{B}}+\underbrace{\partial_{\nu}(sc^{\alpha})g_{\alpha\mu}}_{\boldsymbol{C}}-c^{\alpha}\partial_{\alpha}(sg_{\mu\nu})-\partial_{\mu}c^{\alpha}sg_{\alpha\nu}-\partial_{\nu}c^{\alpha}sg_{\alpha\mu}
\end{aligned}
\tag{B.4}
$$

We leave the first three terms untouched and rewrite the second three as:

$$\begin{aligned}
-c^\alpha\partial_\alpha(sg_{\mu\nu}) =& -c^\alpha\partial_\alpha(c^\beta\partial_\beta g_{\mu\nu}+\partial_\mu c^\beta g_{\beta\nu}+\partial_\nu c^\beta g_{\beta\mu}) = \\
=& -\underbrace{c^\alpha\partial_\alpha c^\beta\partial_\beta g_{\mu\nu}}_{\boldsymbol{A}} - \underbrace{c^\alpha c^\beta}_{\substack{\text{antisym.}\\ \alpha\leftrightarrow\beta}}\underbrace{\partial_\alpha\partial_\beta g_{\mu\nu}}_{\substack{\text{sym.}\\ \alpha\leftrightarrow\beta}} - \underbrace{c^\alpha\partial_\alpha\partial_\mu c^\beta g_{\beta\nu}}_{\boldsymbol{B}} \\
& -\underbrace{c^\alpha\partial_\mu c^\beta\partial_\alpha g_{\beta\nu}}_{\blacksquare} - \underbrace{c^\alpha\partial_\alpha\partial_\nu c^\beta g_{\beta\mu}}_{\boldsymbol{C}} - \underbrace{c^\alpha\partial_\nu c^\beta\partial_\alpha g_{\beta\mu}}_{\blacktriangle}
\end{aligned} \tag{B.5}$$

$$\begin{aligned}
-\partial_\mu c^\alpha s g_{\alpha\nu} =& -\partial_\mu c^\alpha(c^\beta\partial_\beta g_{\alpha\nu}+\partial_\alpha c^\beta g_{\beta\nu}+\partial_\nu c^\beta g_{\beta\alpha}) = \\
=& -\underbrace{\partial_\mu c^\alpha c^\beta\partial_\beta g_{\alpha\nu}}_{\blacksquare} - \underbrace{\partial_\mu c^\alpha\partial_\alpha c^\beta g_{\beta\nu}}_{\boldsymbol{B}} - \underbrace{\partial_\mu c^\alpha\partial_\nu c^\beta g_{\beta\alpha}}_{\bullet}
\end{aligned} \tag{B.6}$$

$$\begin{aligned}
-\partial_\nu c^\alpha s g_{\alpha\mu} =& -\partial_\nu c^\alpha(c^\beta\partial_\beta g_{\alpha\mu}+\partial_\alpha c^\beta g_{\beta\mu}+\partial_\mu c^\beta g_{\beta\alpha}) = \\
=& -\underbrace{\partial_\nu c^\alpha c^\beta\partial_\beta g_{\alpha\mu}}_{\blacktriangle} - \underbrace{\partial_\nu c^\alpha\partial_\alpha c^\beta g_{\beta\mu}}_{\boldsymbol{C}} - \underbrace{\partial_\nu c^\alpha\partial_\mu c^\beta g_{\beta\alpha}}_{\bullet}
\end{aligned} \tag{B.7}$$

The couples of terms marked with the same geometric symbol cancel after anticommuting, in one of the two, the ghost and the ghost derivative and renaming indices ($\blacksquare$, $\blacktriangle$) or the two ghost derivatives and using the symmetry of the metric ($\bullet$). The terms marked with a letter can be combined with the corresponding ones above:

$$\begin{aligned}
s^2h_{\mu\nu} =& \underbrace{[sc^\alpha - c^\beta\partial_\beta c^\alpha]\partial_\alpha g_{\mu\nu}}_{\boldsymbol{A}} + \underbrace{[\partial_\mu(sc^\alpha)-\partial_\mu(c^\beta\partial_\beta c^\alpha)]g_{\alpha\nu}}_{\boldsymbol{B}} + \underbrace{[\partial_\nu(sc^\alpha)-\partial_\nu(c^\beta\partial_\beta c^\alpha)]g_{\alpha\mu}}_{\boldsymbol{C}} = \\
=& \, 0
\end{aligned} \tag{B.8}$$

and, for each letter, the corresponding sum is vanishing due to the form of the BRST Slavnov variation of the ghost:

$$s^2h_{\mu\nu} = 0 \quad \Longleftrightarrow \quad sc^\mu = c^\nu\partial_\nu c^\mu \tag{B.9}$$

C BRST-invariant FRG flow

C.1 BRST-compatible Wetterich-Morris equation

Using relations (3.3.73)-(3.3.77):

$$\frac{1}{\tilde{Z}_{(k)}}\tilde{Z}^{(2)}_{(k)MN}(x,y) = \tilde{W}^{(2)}_{(k)MN}(x,y) + \tilde{W}^{\overrightarrow{(1)}}_{(k)M}(x)\tilde{W}^{\overleftarrow{(1)}}_{(k)N}(y) \tag{C.1}$$

$$\tilde{W}^{(2)}_{(k)J_iJ_j}(x,y) = \tilde{\Gamma}^{(2)-1}_{(k)\Phi^i\Phi^j}(x,y) \tag{C.2}$$

$$\tilde{W}^{(2)}_{(k)K_iK_j}(x,y) = -\tilde{\Gamma}^{(2)}_{(k)K_iK_j}(x,y) \tag{C.3}$$

$$\tilde{W}^{(2)}_{(k)K_iJ_j}(x,y) = -\tilde{\Gamma}^{(2)}_{(k)K_i\Phi^k}(x,\cdot) \bullet_{\bar{g}} \tilde{\Gamma}^{(2)-1}_{(k)\Phi^k\Phi^j}(\cdot,y) \tag{C.4}$$

$$\tilde{W}^{(2)}_{(k)J_iK_j}(x,y) = -\tilde{\Gamma}^{(2)-1}_{(k)\Phi^i\Phi^k}(x,\cdot) \bullet_{\bar{g}} \tilde{\Gamma}^{(2)}_{(k)\Phi^kK_j}(\cdot,y) \tag{C.5}$$

and taking into account the Grassmann character of the various sources, the 2-point correlation functions appearing in (3.3.72):

$$\begin{aligned}
&\partial_t\tilde{\Gamma}_{(k)}[\Phi;K;v;\bar{g}] = \\
&= \int d^Dxd^Dy\sqrt{\bar{g}(x)}\,\delta(x-y)\Big[\\
&\quad \frac{1}{2\kappa^2}\partial_t\mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}_{(y)}\right)\Big(\langle h_{\alpha\beta}(x)h_{\mu\nu}(y)\rangle_{JK} - \langle\Omega_{\alpha\beta}(x)sh_{\mu\nu}(y)\rangle_{JK} - \langle H_{\alpha\beta}(x)h_{\mu\nu}(y)\rangle_{JK}\Big) \\
&\quad + \bar{g}^{\lambda\rho}(y)\partial_t r_{gh(k)}\left(-\bar{\Box}_{(y)}\right)\bar{K}^{\mu\nu}{}_{\rho\sigma}(y)\bar{\nabla}^{\sigma}_{(y)}\Big(v_\lambda(x)\,\langle h_{\mu\nu}(y)\rangle_{JK} - \langle\bar{c}_\lambda(x)sh_{\mu\nu}(y)\rangle_{JK}\Big)\Big]
\end{aligned}$$

can be expressed as:

$$\begin{aligned}
\langle h_{\alpha\beta}(x)h_{\mu\nu}(y)\rangle_{JK} &= \frac{1}{\tilde{Z}_{(k)}}\frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}\frac{\delta}{\delta t^{\alpha\beta}(x)}\frac{\delta\tilde{Z}_{(k)}}{\delta t^{\mu\nu}(y)} = \\
&= \frac{1}{\tilde{Z}_{(k)}}\frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}\frac{\delta}{\delta t^{\alpha\beta}(x)}\tilde{Z}_{(k)}\frac{\overleftarrow{\delta}}{\delta t^{\mu\nu}(y)} = \\
&= \frac{1}{\tilde{Z}_{(k)}}\tilde{Z}^{(2)}_{(k)t^{\alpha\beta}t^{\mu\nu}}(x,y) = \\
&= \tilde{W}^{(2)}_{(k)t^{\alpha\beta}t^{\mu\nu}}(x,y) + \tilde{W}^{\overrightarrow{(1)}}_{(k)t^{\alpha\beta}}(x)\tilde{W}^{\overleftarrow{(1)}}_{(k)t^{\mu\nu}}(y) = \\
&= \tilde{\Gamma}^{(2)-1}_{(k)\mathsf{h}_{\alpha\beta}\mathsf{h}_{\mu\nu}}(x,y) + \mathsf{h}_{\alpha\beta}(x)\mathsf{h}_{\mu\nu}(y)
\end{aligned} \tag{C.6}$$

$$\begin{aligned}
\langle\Omega_{\alpha\beta}(x)sh_{\mu\nu}(y)\rangle_{JK} &= \frac{1}{\tilde{Z}_{(k)}}\frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}\frac{\delta}{\delta n^{\alpha\beta}(x)}\frac{\delta\tilde{Z}_{(k)}}{\delta k^{\mu\nu}(y)} = \\
&= -\frac{1}{\tilde{Z}_{(k)}}\frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}\frac{\delta}{\delta n^{\alpha\beta}(x)}\tilde{Z}_{(k)}\frac{\overleftarrow{\delta}}{\delta k^{\mu\nu}(y)} = \\
&= -\frac{1}{\tilde{Z}_{(k)}}\tilde{Z}^{(2)}_{(k)n^{\alpha\beta}k^{\mu\nu}}(x,y) = \\
&= -\tilde{W}^{(2)}_{(k)n^{\alpha\beta}k^{\mu\nu}}(x,y) - \tilde{W}^{\overrightarrow{(1)}}_{(k)n^{\alpha\beta}}(x)\tilde{W}^{\overleftarrow{(1)}}_{(k)k^{\mu\nu}}(y) = \\
&= \tilde{\Gamma}^{(2)}_{(k)n^{\alpha\beta}k^{\mu\nu}}(x,y) - \tilde{\Gamma}^{\overrightarrow{(1)}}_{(k)n^{\alpha\beta}}(x)\tilde{\Gamma}^{\overleftarrow{(1)}}_{(k)k^{\mu\nu}}(y)
\end{aligned} \tag{C.7}$$

$$
\begin{aligned}
\langle H_{\alpha\beta}(x)h_{\mu\nu}(y)\rangle_{JK} &= \frac{1}{\tilde{Z}_{(k)}}\frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}\frac{\delta}{\delta m^{\alpha\beta}(x)}\frac{\delta\tilde{Z}_{(k)}}{\delta t^{\mu\nu}(y)} = \\
&= \frac{1}{\tilde{Z}_{(k)}}\frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}\frac{\delta}{\delta m^{\alpha\beta}(x)}\tilde{Z}_{(k)}\frac{\overleftarrow{\delta}}{\delta t^{\mu\nu}(y)} = \\
&= \frac{1}{\tilde{Z}_{(k)}}\tilde{Z}^{(2)}_{(k)m^{\alpha\beta}t^{\mu\nu}}(x,y) = \\
&= \tilde{W}^{(2)}_{(k)m^{\alpha\beta}t^{\mu\nu}}(x,y) + \tilde{W}^{\overrightarrow{(1)}}_{(k)m^{\alpha\beta}}(x)\tilde{W}^{\overleftarrow{(1)}}_{(k)t^{\mu\nu}}(y) = \\
&= -\tilde{\Gamma}^{(2)}_{(k)m^{\alpha\beta}\Phi^i}(x,\cdot)\bullet_{\bar{g}}\tilde{\Gamma}^{(2)-1}_{(k)\Phi^i\mathsf{h}_{\mu\nu}}(\cdot,y) - \tilde{\Gamma}^{\overrightarrow{(1)}}_{(k)m^{\mu\nu}}(x)\mathsf{h}_{\mu\nu}(y)
\end{aligned}
\tag{C.8}
$$

$$
\begin{aligned}
\langle \bar{c}_\lambda(x)sh_{\mu\nu}(y)\rangle_{JK} &= \frac{1}{\tilde{Z}_{(k)}}\frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}\frac{-\delta}{\delta\eta^{\lambda}(x)}\frac{\delta\tilde{Z}_{(k)}}{\delta k^{\mu\nu}(y)} = \\
&= \frac{1}{\tilde{Z}_{(k)}}\frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}\frac{\delta}{\delta\eta^{\lambda}(x)}\tilde{Z}_{(k)}\frac{\overleftarrow{\delta}}{\delta k^{\mu\nu}(y)} = \\
&= \frac{1}{\tilde{Z}_{(k)}}\tilde{Z}^{(2)}_{(k)\eta^{\lambda}k^{\mu\nu}}(x,y) = \\
&= \tilde{W}^{(2)}_{(k)\eta^{\lambda}k^{\mu\nu}}(x,y) + \tilde{W}^{\overrightarrow{(1)}}_{(k)\eta^{\lambda}}(x)\tilde{W}^{\overleftarrow{(1)}}_{(k)t^{\mu\nu}}(y) = \\
&= -\tilde{\Gamma}^{(2)-1}_{(k)\bar{\mathsf{c}}_{\lambda}\Phi^i}(x,\cdot)\bullet_{\bar{g}}\tilde{\Gamma}^{(2)}_{(k)\Phi^i k^{\mu\nu}}(\cdot,y) + \bar{\mathsf{c}}_{\lambda}(x)\tilde{\Gamma}^{\overleftarrow{(1)}}_{(k)k^{\mu\nu}}(y)
\end{aligned}
\tag{C.9}
$$

C.2 BRST-compatibility proof

The commutator of the two operators (3.3.90) and (3.3.98):

$$
[\mathcal{F},\mathcal{S}] = \int d^Dx d^Dy d^Dz \sqrt{\bar{g}(x)}\delta(x-y)\sqrt{\bar{g}(z)}\,[\mathcal{F}(x,y),\mathcal{S}(z)]
\tag{C.10}
$$

follows from the one of the non-integrated operators:

$$
\begin{aligned}
\mathcal{S}(z) = \frac{1}{\sqrt{\bar{g}(z)}}\Bigg(& t^{\mu\nu}(z)\frac{\delta}{\delta k^{\mu\nu}(z)} - \bar{\eta}_{\mu}(z)\frac{\delta}{\delta l_{\mu}(z)} + \sqrt{\bar{g}(z)}\eta^{\mu}(z)v_{\mu}(z) \\
&+ m^{\mu\nu}(z)\frac{\delta}{\delta k^{\mu\nu}(z)} - n^{\mu\nu}(z)\frac{\delta}{\delta t^{\mu\nu}(z)} + n^{\mu\nu}(z)\frac{\delta}{\delta m^{\mu\nu}(z)}\Bigg)
\end{aligned}
\tag{C.11}
$$

$$
\begin{aligned}
&\mathcal{F}(x,y) = \\
&\quad = \frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}\Bigg[\\
&\qquad \frac{1}{2\kappa^2}\partial_t \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}_{(y)}\right)\left(-\frac{\delta}{\delta t^{\alpha\beta}(x)}\frac{\delta}{\delta t^{\mu\nu}(y)} + \frac{\delta}{\delta n^{\alpha\beta}(x)}\frac{\delta}{\delta k^{\mu\nu}(y)} + \frac{\delta}{\delta m^{\alpha\beta}(x)}\frac{\delta}{\delta t^{\mu\nu}(y)}\right) \\
&\qquad + \bar{g}^{\lambda\rho}(y)\partial_t r_{gh(k)}\left(-\bar{\Box}_{(y)}\right)\bar{K}^{\mu\nu}{}_{\rho\sigma}(y)\bar{\nabla}^{\sigma}_{(y)}\left(-\sqrt{\bar{g}(x)}v_\lambda(x)\frac{\delta}{\delta t^{\mu\nu}(y)} - \frac{\delta}{\delta\eta^\lambda(x)}\frac{\delta}{\delta k^{\mu\nu}(y)}\right)\Bigg]
\end{aligned}
\tag{C.12}
$$

By linearity, we can evaluate the commutator by computing the commutators of the various terms inside $\mathcal{F}(x,y)$ with $\mathcal{S}(z)$, individually:

$$
\begin{aligned}
[\mathcal{F},\mathcal{S}] = \int d^Dx d^Dy d^Dz\, \delta(x-y)\frac{\sqrt{\bar{g}(z)}}{\sqrt{\bar{g}(y)}}\Bigg\{ \\
\frac{1}{2\kappa^2}\partial_t \mathcal{R}^{\alpha\beta,\mu\nu}_{gr(k)}\left(-\bar{\Box}_{(y)}\right)\Bigg(\left[-\frac{\delta}{\delta t^{\alpha\beta}(x)}\frac{\delta}{\delta t^{\mu\nu}(y)},\mathcal{S}(z)\right] \\
+\left[\frac{\delta}{\delta n^{\alpha\beta}(x)}\frac{\delta}{\delta k^{\mu\nu}(y)},\mathcal{S}(z)\right] \\
+\left[\frac{\delta}{\delta m^{\alpha\beta}(x)}\frac{\delta}{\delta t^{\mu\nu}(y)},\mathcal{S}(z)\right]\Bigg) \\
+\bar{g}^{\lambda\rho}(y)\partial_t r_{gh(k)}\left(-\bar{\Box}_{(y)}\right)\bar{K}^{\mu\nu}{}_{\rho\sigma}(y)\bar{\nabla}^{\sigma}_{(y)}\Bigg(\left[-\sqrt{\bar{g}(x)}v_\lambda(x)\frac{\delta}{\delta t^{\mu\nu}(y)},\mathcal{S}(z)\right] \\
+\left[-\frac{\delta}{\delta\eta^\lambda(x)}\frac{\delta}{\delta k^{\mu\nu}(y)},\mathcal{S}(z)\right]\Bigg)\Bigg\}
\end{aligned}
\tag{C.13}
$$

In particular, the various functional derivatives inside $\mathcal{F}(x,y)$ pass through $\mathcal{S}(z)$ according to the derivative product rule and their Grassmann character, generating an additional term if the corresponding source is present:

$$\frac{\delta}{\delta t^{\mu\nu}(x)}\mathcal{S}(z) = \frac{\delta}{\delta k^{\mu\nu}(z)}\delta(z-x) + \mathcal{S}(z)\frac{\delta}{\delta t^{\mu\nu}(x)} \tag{C.14}$$

$$\frac{\delta}{\delta k^{\mu\nu}(x)}\mathcal{S}(z) = -\mathcal{S}(z)\frac{\delta}{\delta k^{\mu\nu}(x)} \tag{C.15}$$

$$\frac{\delta}{\delta n^{\alpha\beta}(x)}\mathcal{S}(z) = \left(-\frac{\delta}{\delta t^{\alpha\beta}(z)} + \frac{\delta}{\delta m^{\alpha\beta}(z)}\right)\delta(z-x) - \mathcal{S}(z)\frac{\delta}{\delta n^{\alpha\beta}(x)} \tag{C.16}$$

$$\frac{\delta}{\delta m^{\alpha\beta}(x)}\mathcal{S}(z) = \frac{\delta}{\delta k^{\alpha\beta}(z)}\delta(z-x) + \mathcal{S}(z)\frac{\delta}{\delta m^{\alpha\beta}(x)} \tag{C.17}$$

$$\frac{\delta}{\delta\eta^\lambda(x)}\mathcal{S}(z) = \sqrt{\bar{g}(z)}v_\lambda(z)\delta(z-x) - \mathcal{S}(z)\frac{\delta}{\delta\eta^\lambda(x)} \tag{C.18}$$

In the metric fluctuation sector of $\mathcal{F}(x,y)$, we have thus the relations:

$$
\begin{aligned}
-\frac{\delta}{\delta t^{\alpha\beta}(x)}\frac{\delta}{\delta t^{\mu\nu}(y)}\mathcal{S}(z) =& -\frac{\delta}{\delta t^{\alpha\beta}(x)}\left(\frac{\delta}{\delta k^{\mu\nu}(z)}\delta(z-y)+\mathcal{S}(z)\frac{\delta}{\delta t^{\mu\nu}(y)}\right) = \\
=& -\frac{\delta}{\delta t^{\alpha\beta}(x)}\frac{\delta}{\delta k^{\mu\nu}(z)}\delta(z-y) \\
& -\left(\frac{\delta}{\delta k^{\alpha\beta}(z)}\delta(z-x)+\mathcal{S}(z)\frac{\delta}{\delta t^{\alpha\beta}(x)}\right)\frac{\delta}{\delta t^{\mu\nu}(y)} = \\
=& -\mathcal{S}(z)\frac{\delta}{\delta t^{\alpha\beta}(x)}\frac{\delta}{\delta t^{\mu\nu}(y)} \\
& -\frac{\delta}{\delta t^{\alpha\beta}(x)}\frac{\delta}{\delta k^{\mu\nu}(z)}\delta(z-y)-\frac{\delta}{\delta k^{\alpha\beta}(z)}\frac{\delta}{\delta t^{\mu\nu}(y)}\delta(z-x)
\end{aligned}
\tag{C.19}
$$

$$
\begin{aligned}
\frac{\delta}{\delta n^{\alpha\beta}(x)}\frac{\delta}{\delta k^{\mu\nu}(y)}\mathcal{S}(z) =& -\frac{\delta}{\delta n^{\alpha\beta}(x)}\mathcal{S}(z)\frac{\delta}{\delta k^{\mu\nu}(y)} = \\
=& -\left[\left(-\frac{\delta}{\delta t^{\alpha\beta}(z)}+\frac{\delta}{\delta m^{\alpha\beta}(z)}\right)\delta(z-x)-\mathcal{S}(z)\frac{\delta}{\delta n^{\alpha\beta}(x)}\right]\frac{\delta}{\delta k^{\mu\nu}(y)} = \\
=& \mathcal{S}(z)\frac{\delta}{\delta n^{\alpha\beta}(x)}\frac{\delta}{\delta k^{\mu\nu}(y)} \\
& +\frac{\delta}{\delta t^{\alpha\beta}(z)}\frac{\delta}{\delta k^{\mu\nu}(y)}\delta(z-x)-\frac{\delta}{\delta m^{\alpha\beta}(z)}\frac{\delta}{\delta k^{\mu\nu}(y)}\delta(z-x)
\end{aligned}
\tag{C.20}
$$

$$
\begin{aligned}
\frac{\delta}{\delta m^{\alpha\beta}(x)}\frac{\delta}{\delta t^{\mu\nu}(y)}\mathcal{S}(z) =& \frac{\delta}{\delta m^{\alpha\beta}(x)}\left(\frac{\delta}{\delta k^{\mu\nu}(z)}\delta(z-y)+\mathcal{S}(z)\frac{\delta}{\delta t^{\mu\nu}(y)}\right) = \\
=& \frac{\delta}{\delta m^{\alpha\beta}(x)}\frac{\delta}{\delta k^{\mu\nu}(z)}\delta(z-y) \\
& +(c)\frac{\delta}{\delta t^{\mu\nu}(y)} = \\
=& \mathcal{S}(z)\frac{\delta}{\delta m^{\alpha\beta}(x)}\frac{\delta}{\delta t^{\mu\nu}(y)} \\
& +\frac{\delta}{\delta m^{\alpha\beta}(x)}\frac{\delta}{\delta k^{\mu\nu}(z)}\delta(z-y)+\frac{\delta}{\delta k^{\alpha\beta}(z)}\frac{\delta}{\delta t^{\mu\nu}(y)}\delta(z-x)
\end{aligned}
\tag{C.21}
$$

while in the ghosts sector of $\mathcal{F}(x,y)$:

$$
\begin{aligned}
-\sqrt{\bar{g}(x)}v_\lambda(x)\frac{\delta}{\delta t^{\mu\nu}(y)}\mathcal{S}(z) =& -\sqrt{\bar{g}(x)}v_\lambda(x)\left(\frac{\delta}{\delta k^{\mu\nu}(z)}\delta(z-y)+\mathcal{S}(z)\frac{\delta}{\delta t^{\mu\nu}(y)}\right) = \\
=& -\sqrt{\bar{g}(x)}v_\lambda(x)\mathcal{S}(z)\frac{\delta}{\delta t^{\mu\nu}(y)}-\sqrt{\bar{g}(x)}v_\lambda(x)\frac{\delta}{\delta k^{\mu\nu}(z)}\delta(z-y)
\end{aligned}
\tag{C.22}
$$

$$\begin{aligned}
-\frac{\delta}{\delta\eta^\lambda(x)}\frac{\delta}{\delta k^{\mu\nu}(y)}\mathcal{S}(z) =&\frac{\delta}{\delta\eta^\lambda(x)}\mathcal{S}(z)\frac{\delta}{\delta k^{\mu\nu}(y)} = \\
=&\left(\sqrt{\bar{g}(z)}v_\lambda(z)\delta(z-x)-\mathcal{S}(z)\frac{\delta}{\delta\eta^\lambda(x)}\right)\frac{\delta}{\delta k^{\mu\nu}(y)} = \\
=&-\mathcal{S}(z)\frac{\delta}{\delta\eta^\lambda(x)}\frac{\delta}{\delta k^{\mu\nu}(y)}+\sqrt{\bar{g}(z)}v_\lambda(z)\frac{\delta}{\delta k^{\mu\nu}(y)}\delta(z-x)
\end{aligned} \tag{C.23}$$

We have then the commutation rules:

$$\left[-\frac{\delta}{\delta t^{\alpha\beta}(x)}\frac{\delta}{\delta t^{\mu\nu}(y)},\mathcal{S}(z)\right] = -\underbrace{\frac{\delta}{\delta t^{\alpha\beta}(x)}\frac{\delta}{\delta k^{\mu\nu}(z)}\delta(z-y)}_{\boldsymbol{A}}-\underbrace{\frac{\delta}{\delta k^{\alpha\beta}(z)}\frac{\delta}{\delta t^{\mu\nu}(y)}\delta(z-x)}_{\boldsymbol{B}} \tag{C.24}$$

$$\left[\frac{\delta}{\delta n^{\alpha\beta}(x)}\frac{\delta}{\delta k^{\mu\nu}(y)},\mathcal{S}(z)\right] = +\underbrace{\frac{\delta}{\delta t^{\alpha\beta}(z)}\frac{\delta}{\delta k^{\mu\nu}(y)}\delta(z-x)}_{\boldsymbol{A}}-\underbrace{\frac{\delta}{\delta m^{\alpha\beta}(z)}\frac{\delta}{\delta k^{\mu\nu}(y)}\delta(z-x)}_{\boldsymbol{C}} \tag{C.25}$$

$$\left[\frac{\delta}{\delta m^{\alpha\beta}(x)}\frac{\delta}{\delta t^{\mu\nu}(y)},\mathcal{S}(z)\right] = +\underbrace{\frac{\delta}{\delta m^{\alpha\beta}(x)}\frac{\delta}{\delta k^{\mu\nu}(z)}\delta(z-y)}_{\boldsymbol{C}}+\underbrace{\frac{\delta}{\delta k^{\alpha\beta}(z)}\frac{\delta}{\delta t^{\mu\nu}(y)}\delta(z-x)}_{\boldsymbol{B}} \tag{C.26}$$

$$\left[-\sqrt{\bar{g}(x)}v_\lambda(x)\frac{\delta}{\delta t^{\mu\nu}(y)},\mathcal{S}(z)\right] = \underbrace{-\sqrt{\bar{g}(x)}v_\lambda(x)\frac{\delta}{\delta k^{\mu\nu}(z)}\delta(z-y)}_{\boldsymbol{D}} \tag{C.27}$$

$$\left[-\frac{\delta}{\delta\eta^\lambda(x)}\frac{\delta}{\delta k^{\mu\nu}(y)},\mathcal{S}(z)\right] = \underbrace{+\sqrt{\bar{g}(z)}v_\lambda(z)\frac{\delta}{\delta k^{\mu\nu}(y)}\delta(z-x)}_{\boldsymbol{D}} \tag{C.28}$$

Inserting the results in the total commutator, the terms marked with the same letter cancel against each other, possibly after integrating the Dirac deltas, and the total commutator is vanishing:

$$[\mathcal{F},\mathcal{S}] = \int d^Dx d^Dy d^Dz\sqrt{\bar{g}(x)}\delta(x-y)\sqrt{\bar{g}(z)}\left[\mathcal{F}(x,y),\mathcal{S}(z)\right] = 0 \tag{C.29}$$

In particular, we have an independent cancellation in the metric fluctuation and ghosts sector.

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