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Dimensionality and boundary conditions in Hadamard regularization

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Abstract

In this thesis, we analyze the role of boundaries and spacetime dimensionality in the context of Hadamard regularization. We start by considering the constant density star model, whose field equation reduces to a time independent Schrödinger-like equation with a potential having a jump discontinuity. Due to the jump, which can also be expected in slightly more realistic but still simple models, there are more types of boundary conditions that one can enforce. Renormalization is what determines if a choice is physically meaningful or not, and the most general renormalization method that can be applied to a wide variety of models is the Hadamard subtraction, which relies on the universal parametrix of the Feynman propagator near coincidence limit. We attempt this renormalization procedure to flat spacetime models whose equations of motion are formally analogous to the homogeneous star case, with custom potentials. First, we apply it to the well-known case of a real massless scalar in two-dimensional flat spacetime in a Dirichlet box, and find out that it works. Then, we enhance the spacetime dimensionality to three and work with a Dirichlet spherical cavity, in which case the divergences differ from those obtained from the Hadamard parametrix. This hints to the fact that Dirichlet boundaries may not be physically meaningful in four-dimensional models. After that, we consider another variation of the first model where the Dirichlet box is substituted with a step potential with a jump discontinuity. In this case, we find that the Hadamard parametrix holds, hinting that, in two-dimensional spacetime models, the Hadamard method may work independently of the choice of boundary conditions. When possible, components of the renormalized energy-momentum tensor of the studied models are also found.

Contents

Chapter 1

Introduction and motivation

In this first chapter, we are going to give a motivational introduction to the work that is presented in the rest of the thesis. We first talk about the general issue of quantizing gravity, then present the framework of quantum field theory in curved spacetime background and semiclassical gravity. After that, we motivate the study of compact objects in flat spacetime, being in formal analogy with the case of a homogeneous compact star, which is the natural laboratory where quantum gravitational effects may play a crucial role.

1.1 Quantum gravity and compact stars

One of the main problems in theoretical physics today is the search for a theory of quantum gravity. The theories that currently best describe our universe are the Standard Model of particle physics and General Relativity. Both of them have limitations, and cannot correctly describe some existing phenomena like dark matter and neutrino oscillations, but so far they are the best we have to make predictions about experiments and explain observations. There is one further problem about this situation: the Standard Model speaks the language of quantum field theory, which in turn is framed in quantum mechanical terms, but the usual quantization procedure we use for gauge theories breaks down when General Relativity is brought into the picture. In fact, General Relativity is a gauge theory with respect to diffeomorphism invariance, and one needs to fix the gauge when quantizing, because the quantization process consists in assigning transition amplitudes between physically distinct classical states. However, the only solutions we know to Einstein field equations contain isometries, but these make the gauge fixing ineffective[1], since mathematically equivalent descriptions of the same physical state can be confused with two distinct physical configurations having the same mathematical description. Therefore, the known ways to quantize gauge theories do not work for the currently known solutions of General Relativity, and the search for a mathematically consistent way to unite gravity with quantum field theory must go on.

One way to investigate about quantum gravity would be to solve Einstein equations in absence of isometries, so that nothing prevents gauge fixing and the quantum theory can be built coherently. However, solving the Einstein equations without isometries is extremely hard, and analytical tools are of little use. The closest coherent framework to a full quantum gravity theory that we have at our disposal is Quantum Field Theory in

[1]See [30] for a thorough investigation about this issue.

curved background. Here, we assume to have a fixed non-dynamical classical background consisting of a lorentzian manifold with metric tensor $g_{\mu\nu}$ and energy-momentum tensor $T_{\mu\nu}$ satisfying Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu},$$

and dynamical quantum fields living on this background. These give a contribution to the (renormalized) energy-momentum tensor such that the induced modification of the metric tensor is assumed to be negligible with respect to the original one. In this regime, which we may call *perturbational*, one can build a fully functional quantum field theory, and compute observables such as scattering amplitudes and decay rates. However, there is no a priori guarantee, even in the vacuum state[2], that the contribution of the quantum fluctuations to the energy-momentum tensor induces a modification to the metric tensor and to the topology of the spacetime manifold which is negligible. This is something that should be verified a posteriori and goes by the name of *backreaction problem*. Indeed, the Einstein equations in the presence of a quantum field $\hat{\phi}$ in the state $|\psi\rangle$, become

$$\bar{R}_{\mu\nu} - \frac{1}{2} \bar{R} \bar{g}_{\mu\nu} = \frac{8\pi G}{c^4} \left(T_{\mu\nu} + \langle\psi| \hat{T}^{\phi}_{\mu\nu} |\psi\rangle_{ren} \right),$$

where the bars on the left-hand side are present to mark quantum-corrected geometrical quantities and distinguish them from the classical non-barred ones. The quantum correction given by the field $\hat{\phi}$ to the energy-momentum tensor plays a crucial role because, even if it is small with respect to the classical source, the modifications on the metric tensor and on the topology of the manifold may be drastic due to the non-linearity of the left-hand side. Furthermore, the metric also enters the calculation of the correction term, and if it changes drastically, then also the correction does. Therefore, one could think of iterative numerical methods that reach a steady state that solves the semiclassical version of the Einstein equations. This is the core of backreaction, and research on this path is still ongoing. In some interesting cases, the first backreaction iteration may reveal some underlying property of a full quantum gravity theory, and this motivates the interest in finding the renormalized energy-momentum tensor in models where gravity starts to couple strongly with the quantum fields.

One may look for a natural laboratory where quantum gravitational effects start becoming important, and one is given by collapsing astrophysical objects. The classical picture about gravitational collapse is given by the Oppenheimer-Snyder model [35]. In this model, the stellar collapse ends with the formation of a black hole, meaning that both an event horizon and a singularity are present. Black holes are relatively simpler than stars and other astrophysical objects thanks to the no-hair theorem [34]. So, if the end state is a black hole, one can hope for simpler calculations, and this is why results about black holes are more present in the literature than those about stellar models. However, when quantum fields are added to the background, phenomena like Hawking evaporation come into play [13]. It is worth noting that, in the presence of Hawking radiation, the resulting spacetime manifold drastically differs from the classical stationary solution. In the classical case, the black hole is eternal and stays there forever, while with quantum fields present it vanishes into thermal radiation. This shows that the very presence of quantum fields can backreact and produce results that are very different from the classical

[2]One may recall that the notion of vacuum in curved spacetime is ambiguous, but here we stay general for the purpose of the argument we are making about backreaction.

picture. For a collapsing star, the backreaction may become important at scales that are still far from the Planck scale, and therefore quantum gravity could be studied in these macroscopic systems (see [21] for an example). It is therefore very interesting to study such phenomena, which are still not fully understood[3], using the framework of QFT in curved background. When applied to black hole physics, this approach is also called *black hole perturbation theory* (BHPT for short), and many interesting results have been obtained (see Section 1 of [18]).

In what follows, we provide some theoretical notions about quantum field theories in spherically symmetric backgrounds. These will help us deal with spacetime models that contain compact spherical objects. For simplicity, we will replace the homogeneous star model with flat spacetime models having a potential that approximates the effective one given by the constant density star. The goal that one needs to keep in mind is to find the renormalized energy-momentum tensor, since it is the core object that provides information about the quantum properties of the matter distribution. Some of such models of simple compact objects are then studied throughout the rest of this thesis, and some interesting results about renormalization close to the boundary are presented.

1.2 Spherically symmetric spacetimes

A spherically symmetric lorentzian manifold $\mathcal{M}$ can be written by definition as a cartesian product $\mathcal{M} = \mathcal{M}_2 \times \mathbb{S}^2$, and the metric tensor can be decomposed as a direct sum

$$\boldsymbol{g} = \boldsymbol{\gamma} \oplus \bar{\boldsymbol{\Omega}},$$

where $\boldsymbol{\gamma}$ and $\bar{\boldsymbol{\Omega}}$ are the metric tensors on $\mathcal{M}_2$ and $\mathbb{S}^2$, respectively. The line element, therefore, reads

$$ds^2 = \gamma_{AB}(x^C)dx^A dx^B + \rho^2(x^C)\Omega_{ab}dx^a dx^b, \tag{1.1}$$

with

$$\Omega_{ab}dx^a dx^b = d\theta^2 + \sin^2\theta d\varphi^2,$$

where the upper case indices $A, B \in \{0, 1\}$ label coordinates in $\mathcal{M}_2$ and lower case ones $a, b \in \{\theta, \varphi\}$ label angular coordinates in $\mathbb{S}^2$. Spherical symmetry ensures that the functions γ_{AB} and ρ do not depend on angular coordinates.

If the spacetime is also static, we can further decompose the metric $\boldsymbol{\gamma}$ into a temporal and a spacial part, and we can label the coordinates x^0 and x^1 as t and r respectively. The line element, therefore, becomes

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + \rho^2(r)\Omega_{ab}dx^a dx^b. \tag{1.2}$$

One can introduce the tortoise coordinate $r_* = r_*(r)$ as follows

$$dr_* = \left(\frac{h(r)}{f(r)}\right)^{\frac{1}{2}} dr, \tag{1.3}$$

so that the line element in the new coordinate chart becomes (as $r = r(r_*)$)

$$ds^2 = -f(r_*)dt^2 + f(r_*)dr_*^2 + r(r_*)(d\theta^2 + \sin^2\theta d\varphi^2). \tag{1.4}$$

[3]See [31] for a review about quantum effects in stellar collapse models.

1.2.1 D'Alembert operator

In general, the action of the D'Alembert operator in an arbitrary metric manifold on a smooth function $\phi \in C^2(\mathcal{M}, \mathbb{C})$ is given by the well-known formula

$$\Box\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\partial_\nu\phi\right).$$

If we plug the form of the metric (1.1) inside the above formula, we get

$$\Box\phi = \Box_{(2)}\phi + \frac{2}{\rho}(\partial_A\rho)\gamma^{AB}\partial_B\phi + \frac{1}{\rho^2}\Delta_{(2)}\phi, \tag{1.5}$$

with

$$\Box_{(2)}\phi = \frac{1}{\sqrt{-\gamma}}\partial_A\left(\sqrt{-\gamma}\gamma^{AB}\partial_B\phi\right), \tag{1.6a}$$

$$\Delta_{(2)}\phi = \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\phi\right) + \frac{1}{\sin^2\theta}\partial^2_\varphi\phi. \tag{1.6b}$$

We can notice that the operator $\Delta_{(2)}$ acting on functions defined on $\mathbb{S}^2$ admits spherical harmonics $Y_{\ell m}(\theta,\varphi)$ as eigenfunctions (see appendix A.3 for further details about spherical harmonics):

$$\Delta_{(2)}Y_{\ell m} = -\ell(\ell+1)Y_{\ell m}. \tag{1.7}$$

The spherical harmonics form a complete orthonormal set for the Hilbert space of square integrable functions on the sphere $L^2(\mathbb{S}^2)$. Therefore, the function ϕ can be decomposed in a mode sum:

$$\phi(x^A,\theta,\varphi) = \sum_{\ell=0}^{+\infty}\sum_{\ell=-m}^{m} v_{\ell m}(x^A)Y_{\ell m}(\theta,\varphi).$$

The above decomposition in spherical harmonics is a general result for fields defined on spherically symmetric spacetimes.

If the spacetime is also static, using $\{x^0 = t, x^1 = r\}$ coordinates, the matrix of the metric tensor components γ^{AB} becomes diagonal. One can then completely separate the time variable from the spacial ones. In this way, the temporal equation becomes that of a harmonic oscillator, and one can introduce the Fourier basis $e^{-i\omega t}$ to express ϕ as a superposition of monochromatic components:

$$\phi(t,r,\theta,\varphi) = \int\frac{d\omega}{2\pi}e^{-i\omega t}\sum_{\ell=0}^{+\infty}\sum_{\ell=-m}^{m} v_{\ell m}(\omega;r)Y_{\ell m}(\theta,\varphi). \tag{1.8}$$

1.3 Quantum field theory in static spherically symmetric background

As anticipated, we now revise the basic concepts of Quantum Field Theory in curved background. In particular, for the purpose of this thesis, we will only consider a real massless scalar field[4] Φ on a D-dimensional spacetime manifold $\mathcal{M}$, with mostly-plus

[4] Fields with higher spin, of course, have computational complications, but have a similar treatment. See Section 2.1 of [18].

signature (timelike squared line elements ds^2 are negative), minimally coupled with gravity (the usual coupling constant ξ is zero). We will follow Section 2.1 of [18] with the cosmological constant Λ set to zero. The theory is fully specified by the Einstein-Hilbert action functional S:

$$S[\boldsymbol{g}, \Phi] = \frac{1}{16\pi G} \int d^D x \sqrt{-g} R + \int d^D x \sqrt{-g} \mathcal{L}_m,$$

where we indicated the metric tensor with a bold $\boldsymbol{g}$ to avoid using the coordinate-dependent notation $g_{\mu\nu}$, since the action is a scalar quantity, and $g = \det \boldsymbol{g}$. The matter Lagrangian is defined as

$$\mathcal{L}_m = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi$$

In the action, additional terms must be taken into account if the spacetime manifold has boundaries. We will talk about the boundary issues later.

Classically, the equations of motion for this theory are given by the Einstein equations and the massless Klein-Gordon equation in curved spacetime[5]:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.9a}$$

$$g^{\mu\nu} \nabla_\mu \nabla_\nu \Phi = 0, \tag{1.9b}$$

where the energy-momentum tensor $T_{\mu\nu}$ is the one associated with the matter lagrangian containing the field Φ:

$$T_{\mu\nu} = \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi.$$

The set of the above equations (1.9) form a system of non-linear partial differential equations describing the evolution of the field Φ and the metric $\boldsymbol{g}$. A particular solution to the above system is called *background solution*, and is given by the triple $(\mathcal{M}, \boldsymbol{g}^{BG}, \Phi^{BG})$. We therefore have a classical spacetime manifold $\mathcal{M}$ on which we are going to define perturbations of the dynamical fields $\boldsymbol{g}$ and Φ. By using ϵ as a formal parameter that keeps track of the perturbative order, we define

$$\boldsymbol{g} = \boldsymbol{g}^{BG} + \epsilon \boldsymbol{h}$$
$$\Phi = \Phi^{BG} + \epsilon \phi.$$

Now, by keeping in mind that $\boldsymbol{g}^{BG}$ and Φ^{BG} solve the equations of motion, we need to find the dynamical equations for $\boldsymbol{h}$ and ϕ, and for this purpose one can plug the above expansions in the equations (1.9), keep terms up to first order in ϵ and simplify by using the background equations. After this procedure, one gets the linearized equations for $\boldsymbol{h}$, which describe gravitational waves, and again a Klein-Gordon equation for ϕ (since the equation itself is linear in the field):

$$\Box \phi = 0.$$

We will not focus on the gravitational wave sector, but rather on the scalar perturbations ϕ solving the Klein-Gordon wave equation above. One way to physically justify this

[5] We will use the convention $c = 1$ from now on.

setup[6] is by noting that the expectation value of many observables $\hat{O}$ satisfy classical equations to a good approximation, and so the quantum state $|\psi\rangle$ must be such that

$$\langle\psi|\,\hat{O}\,|\psi\rangle = O_{cl} + \langle\psi|\,\hat{o}\,|\psi\rangle \simeq O_{cl},$$

with O_{cl} being the classical solution. We then assume that the quantum correction $\langle\psi|\,\hat{o}\,|\psi\rangle$ is small with respect to O_{cl}, and will be the only dynamical quantity we will use in the theory, leaving O_{cl} as a fixed background quantity.

If we are in a static spherically symmetric spacetime with coordinates $\{t, r, \theta, \varphi\}$ and line element given by (1.2) with $\rho(r) = r$ (being r the areal radius), we can express $\phi(t, r, \theta, \varphi)$ as in (1.8), where we reduced the problem to finding the function $v(\omega; r)$ with arbitrary but fixed frequency ω. We can conveniently view (1.8) as a linear combination of the following modes:

$$\phi_{\ell m}(\omega; t, r, \theta, \varphi) = e^{-i\omega t}\frac{u_{\ell m}(\omega; r)}{r} Y_{\ell m}(\theta, \varphi), \tag{1.10}$$

where we cast $v_{\ell m}(\omega; r) = u_{\ell m}(\omega; r)/r$ for convenience. Now, if we plug the above mode into the field equation and use the tortoise coordinate r_* defined by (1.3), so that the line element becomes (1.4), we get the *master equation* for $u_{\ell m}(\omega; r_*)$:

$$\boxed{\frac{d^2 u_{\ell m}}{dr_*^2} = \left[V_{\ell m}(r_*) - \omega^2\right] u_{\ell m},} \tag{1.11}$$

with

$$\boxed{V_{\ell m}(r_*) = \frac{\partial^2_{r_*} r}{r(r_*)} + \frac{\ell(\ell+1)}{r^2(r_*)} f(r_*).} \tag{1.12}$$

Proof. *To get the master equation, we work with the tortoise coordinate r_*, and line element given by* (1.4)*. In the language of the previous section, we have $\sqrt{-\gamma} = f$, $\gamma^{AB} = diag(-1/f, 1/f)^{AB}$. Therefore, in the d'alembertian* (1.5)*, we have the following terms:*

$$\begin{aligned}
&\Box_{(2)}\phi_{\ell m} = -\frac{1}{f}\partial_t^2\phi_{\ell m} + \frac{1}{f}\partial^2_{r_*}\phi_{\ell m} = \frac{1}{f}\omega^2\phi_{\ell m} + \frac{1}{f}\partial_{r_*}\phi_{\ell m},\\
&\frac{2}{r}\partial_{r_*} r\frac{1}{f}\partial_{r_*}\phi_{\ell m},\\
&\frac{1}{r^2}\Delta_{(2)}\phi_{\ell m} = -\frac{\ell(\ell+1)}{r^2}\phi_{\ell m},
\end{aligned}$$

where we used the expansions (1.6) *and the spherical harmonic eigenvalue relation* (1.7)*. By putting everything into* (1.5) *and equating to zero, we get, after multiplying everything by f:*

$$\omega^2\phi_{\ell m} + \left(\partial^2_{r_*}\phi_{\ell m} + 2\frac{\partial_{r_*} r}{r}\partial_{r_*}\phi_{\ell m}\right) - \frac{\ell(\ell+1)}{r^2} f\phi_{\ell m} = 0. \tag{1.13}$$

Now, if we expand using (1.10)*, we can cancel out the $e^{-i\omega t} Y_{\ell m}(\theta, \varphi)$ factor, and get an equation for $u_{\ell m}$. However, we need to compute the following derivatives:*

$$\begin{aligned}
&\partial_{r_*}\phi_{\ell m} = \frac{\partial_{r_*} u_{\ell m}}{r} - \frac{u_{\ell m}}{r^2}\partial_{r_*} r,\\
&\partial^2_{r_*}\phi_{\ell m} = \frac{\partial_{r_*} u_{\ell m}}{r} - \frac{1}{r^2}\partial_{r_*} u_{\ell m}\partial_{r_*} r - \frac{1}{r^2}\partial_{r_*} u_{\ell m}\partial_{r_*} r + 2\frac{u_{\ell m}}{r^3}(\partial_{r_*} r)^2 - \frac{u_{\ell m}}{r^2}\partial^2_{r_*} r,
\end{aligned}$$

[6] See Section 4 of [30].

so that the term in the parentheses in Eq. (1.13) *becomes*

$$\begin{aligned}\partial^2_{r_*}\phi_{\ell m}+2\frac{\partial_{r_*}r}{r}\partial_{r_*}\phi_{\ell m} &= \\ &= \frac{\partial^2_{r_*}u_{\ell m}}{r} - \cancel{\frac{1}{r^2}\partial_{r_*}u_{\ell m}\partial_{r_*}r} \\ &\quad - \cancel{\frac{1}{r^2}\partial_{r_*}u_{\ell m}\partial_{r_*}r} + \cancel{2\frac{u_{\ell m}}{r^3}(\partial_{r_*}r)^2} - \frac{u_{\ell m}}{r^2}\partial^2_{r_*}r + \cancel{\frac{2}{r^2}\partial_{r_*}r\partial_{r_*}u_{\ell m}} - \cancel{2\frac{u_{\ell m}}{r^3}(\partial_{r_*}r)^2} \\ &= \frac{\partial^2_{r_*}u_{\ell m}}{r} - \frac{u_{\ell m}}{r^2}\partial^2_{r_*}r.\end{aligned}$$

Equation (1.13)*, then, becomes*

$$\frac{\omega^2 u_{\ell m}}{r} + \frac{\partial^2_{r_*}u_{\ell m}}{r} - \frac{u_{\ell m}}{r^2}\partial^2_{r_*}r - \frac{\ell(\ell+1)}{r^2}f\frac{u_{\ell m}}{r} = 0,$$

and after multiplying everything by r *and moving all terms except the second derivative on the right, we get*

$$\partial^2_{r_*}u_{\ell m} = \left[\left(\frac{\partial^2_{r_*}r}{r} + \frac{\ell(\ell+1)}{r^2}f\right) - \omega^2\right]u_{\ell m},$$

which is the master equation we wanted to prove.

■

1.4 Constant density compact stars

In the paper [27], the master equation (1.11) appears in Eq. 7, and is applied in the case of a constant density Schwarzschild star. This model is a particular solution of Einstein equations that saturates the Buchdahl bound. To understand what this bound says, we assume to have an isotropic perfect fluid star with energy-momentum tensor $T^\mu_{\ \nu} = \mathrm{diag}(-\rho, p, p, p)^\mu_{\ \nu}$ on a static spherically symmetric spacetime with metric

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

with $\rho > 0$ and $\partial_r\rho \leq 0$. The Buchdahl bound says that the metric above, satisfying the Einstein equations, is regular only if

$$R \geq \frac{9}{4}GM,$$

where R is the radius of the star, and M is its mass.

The constant density star saturates the Buchdahl bound when $\rho = \frac{3M}{4\pi R^3}$. Even if this model is unrealistic, it is useful because it is analytitcally simple and therefore well suited for QFT calculations. For this model, the metric components $f(r)$ and $h(r)$ read

$$f(r) = \left(\frac{3}{2}\sqrt{1-\frac{2GM}{R}} - \frac{1}{2}\sqrt{1-\frac{2GMr^2}{R^3}}\right)^2 \tag{1.14a}$$

$$h(r) = \left(1-\frac{2GMr^2}{R^3}\right)^{-1}. \tag{1.14b}$$

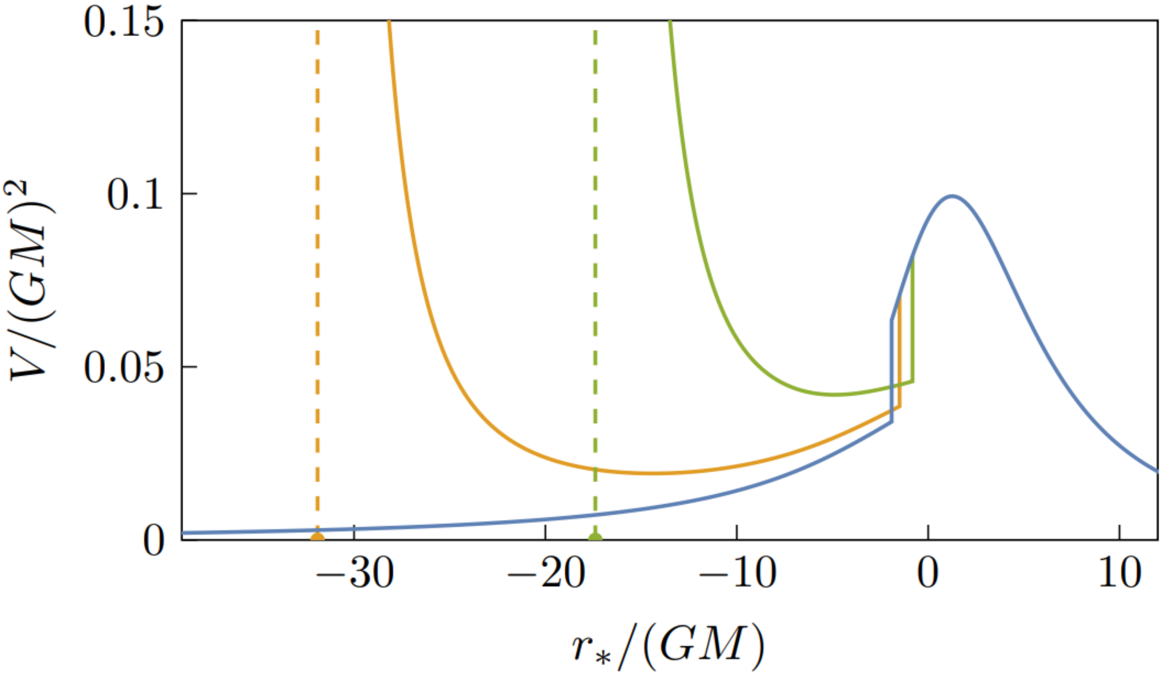

Figure 1.1: Plot taken from [27], Fig. 1, representing the potential (1.12) for $\ell = 1$, for values of $R/(GM) = 9/4$ (blue), 2.3 (orange) and 2.4 (green). The dashed lines mark the value of the tortoise coordinate r_* where $r = 0$. We notice a jump discontinuity at the surface of the star.

When $r = R$, we can easily detect that the two expressions match with the exterior vacuum Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

We can apply the master equation (1.11) (Eq. 7 in the paper [27]) with potential given by (1.12), where the tortoise coordinate is defined in terms of (1.3). By plotting the potential with $\ell = 1$ as in Fig. 1.1, we can see that it has a step discontinuity on the surface of the star. Since the potential enters in the master equation, which has to be solved, one needs to impose boundary conditions on $u_{\ell m}$ at $r = R$. Therefore, the question arises: which boundary conditions do we impose? One can think of a few options:

- Dirichlet: $\phi(r = R) = 0$;
- Neumann: $\partial_r\phi(r = R) = 0$;
- smoothness:

$$\begin{cases} \phi(r = R^-) = \phi(r = R^+) \\ \partial_r\phi(r = R^-) = \partial_r\phi(r = R^+) \end{cases}.$$

Each boundary condition amounts to a specific physical choice about the nature of the star surface and must be carefully considered. This is the issue we are going to address in this thesis, by analyzing some of the possible options.

1.5 Content and motivation

As we said before, the constant density star is not very realistic, but nonetheless its analytical simplicity allows QFT calculations to be performed. The next step one can think of is to find a more realistic model which still retains analytical manipulability. One core feature of the homogeneous star model is that its potential has a step discontinuity on the star surface, as we explored in the previous section. This step discontinuity is the price to pay for analytical simplicity, and encodes the lack of realism of the model. As we engineer slightly more realistic models, we may still encounter a discontinuity in the potential, and the need to choose suitable and physically meaningful boundary conditions persists.

We need to keep in mind that our goal is to find the renormalized energy-momentum tensor in those more realistic models. In the case of a homogeneous star, in [27], the renormalized energy-momentum tensor is found via the conformal anomaly, by exploiting the fact that the spacetime is conformally flat. This may not be the case for more realistic models, and therefore we need a more generally valid procedure in order to compute the renormalized energy-momentum tensor. The tool we need is the Hadamard renormalization procedure, discussed algorithmically in [17]. Reviews of the main renormalization methods that are used in curved spacetime are presented in Chapter 3 of [19] and Chapter 6 of [2], and include point-splitting, adiabatic regularization, proper time regularization, dimensional regularization and cutoff regularization. The Hadamard method is a particular case of point-splitting, which is based on a core mathematical property of the propagator, that is its universal, state-independent, covariant and local singularity structure, called *Hadamard parametrix.* The advantage of this method is that it is very general, it does not rely on any particular symmetry of the background spacetime, it does not depend on the particular quantum state (as long as it is of the Hadamard type), and the singular terms are purely geometric and local. We refer to Appendix D for further discussion about the mathematical properties of Hadamard states. Furthermore, once an approximate local form of the propagator is known up to second order in spacetime coordinates, the energy-momentum tensor can be found as simply as applying a formula.

Thanks to all of these advantages, one may hope that the Hadamard renormalization procedure can be applied to more general models of stellar collapse to find the renormalized energy-momentum tensor. However, these models often contain discontinuities or boundaries due to the presence of the compact object, and it is not clear whether the Hadamard procedure correctly applies to any spacetime with boundaries or discontinuities, nor if, in absence of a working renormalization prescription, the modeled physical objects are defined at all.

In this thesis, we are going to investigate the problem of renormalization from the Hadamard point of view, and apply it to models that are inspired by the constant density star, which contain boundaries or discontinuities. Indeed, we expect the Hadamard procedure to carry out fine if the potential associated to the master equation is smooth, but it is quite hard to even find the normal modes in this case. Therefore, we are going to attempt the Hadamard procedure to toy models that contain boundaries and discontinuities, and we will see that the number of spacetime dimensions plays an important role in the behaviour of the divergences near the boundary. We should, more precisely, talk about *Hadamard regularization* before one performs the subtraction that removes divergences. In particular, we will see that in some of the models that we are going to consider, we will just stop at analyzing the divergent part, and the renormalization step is not performed.

We will consider massless real scalar field theories in flat spacetime, having an equation of motion that is formally analogous to the master equation (1.11). In fact, in flat spacetime we have that the frequency ω is defined without issues, and by factorizing the (monochromatic component of the) field $\phi_\omega(t, \boldsymbol{x}) = e^{-i\omega t} u_\omega(\boldsymbol{x})$, the Klein-Gordon equation with potential given by $U(\boldsymbol{x})$

$$(\Box - U(\boldsymbol{x}))\phi = 0$$

becomes formally analogous to the master equation (1.11)

$$\nabla^2_{\boldsymbol{x}} u_\omega(\boldsymbol{x}) = \left[U(\boldsymbol{x}) - \omega^2\right] u_\omega(\boldsymbol{x}),$$

with the spacial coordinate analogy $\boldsymbol{x} \leftrightarrow r_*$.

In particular, we will analyze flat spacetime massless real scalar quantum field theories in the following scenarios:

- one spacial dimension with Dirichlet boundary conditions at $x = 0, L$ with $L > 0$;
- three spacial dimensions in spherical symmetry with Dirichlet boundary at the surface of a sphere of radius $R > 0$;
- one spacial dimension with step potential $U(x) = U_0 \Theta(|x| - L)$, with Θ being the Heaviside step function, and smoothness conditions.

The reason why we consider 2D models is because of their simplicity, but also to highlight the role that dimensionality plays in the UV divergences of the energy-momentum tensor.

In all of the three cases above, we compute the normal modes, the divergent part of the Feynman propagator $G^F(x, y) = i\langle T\hat{\phi}(x)\hat{\phi}(y)\rangle$ at coincidence limit near the boundary (and, if possible, in other regions of the spacetime), and when the mathematical problem can be analytically solved, also the renormalized energy-momentum tensor at some points. At the end of the journey, the hope is to have an idea of the role of boundaries in Hadamard regularization, to lay down the foundations for further research about quantum field theory in stellar models and gravitational collapse.

Chapter 2

Real massless scalar in a box in 1+1 dimensional Minkowski spacetime

The first model that we are going to consider is a real massless scalar field in 1+1-dimensional flat spacetime with Dirichlet boundary conditions at two finite edges. We will see that in this case the divergences of the propagator are of the Hadamard type, and we will be able to subtract them and obtain the renormalized energy-momentum tensor. In order to gain confidence with the Hadamard renormalization procedure, we will also attempt a naive renormalization approach by explicitly introducing an exponential damping regulator, subtract the divergent term and then take the limit where the regulator approaches 1. Since this model is pretty simple, we will do all the calculations explicitly, in order to gain confidence with all the steps that are needed to get to the final goal of renormalizing the energy-momentum tensor.

2.1 Classical theory

Let us then start with a real scalar field in one infinite temporal dimension $t \in (-\infty, +\infty)$ and one bounded spacial dimension $x \in [0, L]$, with $L > 0$ and Minkowski metric

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.1}$$

such that Dirichlet boundary conditions hold at the spacial boundary: $\phi(t, x = 0) = \phi(t, x = L) = 0$. The Klein-Gordon massless action will then be

$$S = \int_{-\infty}^{+\infty} dt \int_0^L dx \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right). \tag{2.2}$$

By varying the action with respect to ϕ, we obtain the Klein-Gordon equation of motion

$$\boxed{\Box \phi = 0,} \tag{2.3}$$

where $\Box = -\partial_t^2 + \partial_x^2$.

Proof. *The lagrangian is*

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi.$$

The derivatives are

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0,$$
$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -g^{\mu\nu}\partial_\nu \phi,$$
$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -g^{\mu\nu}\partial_\mu\partial_\nu \phi = -\Box\phi = (\partial_t^2 - \partial_x^2)\phi.$$

The Euler-Lagrange equations of motion for this theory reduce to a single equation:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial \mathcal{L}}{\partial \phi},$$

and if we substitute the derivatives calculated previously we get

$$\Box\phi = 0.$$

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The above equation is linear, and a complete set of solutions satisfying the Dirichlet boundary conditions is given by

$$u_n(t,x) = N_n \sin(k_n x) e^{-ik_n t},$$
$$u_n^*(t,x) = N_n^* \sin(k_n x) e^{ik_n t}$$

with $n \in \{1, 2, \ldots\} \equiv \mathbb{N}^+$, N_n being a normalization factor to be determined, and

$$k_n = \frac{\pi n}{L}. \tag{2.4}$$

Proof. *We now give a full derivation for the solutions, by taking a more mathematical approach. The equation we need to solve is*

$$(-\partial_t^2 + \partial_x^2)\phi(t,x) = 0.$$

The above equation is manifestly linear, and therefore we look for a basis of the linear space of solutions $\mathcal{A}$ (with complex coefficients). We proceed by separating variables, thanks to the derivatives not being mixed. Assume $\phi(t,x) = \alpha(t)\beta(x)$, so the equation of motion becomes:

$$\ddot{\alpha}(t)\beta(x) = \alpha(t)\beta''(x).$$

By dividing both sides by $\alpha(t)\beta(x)$ (assuming $\phi(t,x) \neq 0$ for a moment), we get

$$\frac{\ddot{\alpha}(t)}{\alpha(t)} = \frac{\beta''(t)}{\beta(t)}.$$

If we inspect the above equation, we see that the two sides depend on two disjunct sets of variables, and therefore they must both separately be equal to a constant, say $-k^2 < 0$, with $k > 0$. We can then avoid the division by $\alpha(t)\beta(x)$ step, which was only useful to realize we could split the PDE into two ODEs, and directly write:

$$\begin{cases} \ddot{\alpha} + k^2\alpha = 0 \\ \beta'' + k^2\beta = 0. \end{cases}$$

We then have two harmonic oscillators, and therefore the solutions can be written in terms of the two bases:

$$\alpha(t) \in \text{span}\{e^{ikt}, e^{-ikt}\}_{k>0}, \qquad \beta(x) \in \text{span}\{\cos(kx), \sin(kx)\}_{k>0}, \tag{2.5}$$

where we chose the trigonometric basis for the spacial part since it is convenient when imposing boundary conditions. The actual space of solutions $\mathcal{A}$ is not given by the tensor product of the two separate solution spaces, because we need to recall that k must be the same for both factors to satisfy the equation of motion. Therefore, the basis for the linear space $\mathcal{A}$ is given by

$$\phi(t,x) \in \text{span}\{e^{-ikt}\cos(kx), e^{-ikt}\sin(kx), e^{ikt}\cos(kx), e^{ikt}\sin(kx)\}_{k>0},$$

with the additional constraint that linear coefficients must be such that the field is real-valued. The general solution is then a linear combination of the above modes with constrained coefficients:

$$\phi(t,x) = \sum_k \left[e^{-ikt}(a_k \sin(kx) + b_k \cos(kx)) + e^{ikt}(a_k^* \sin(kx) + b_k^* \cos(kx))\right],$$

where the equality $\phi(t,x) = \phi^*(t,x)$ is manifest.

Now, we need to impose Dirichlet boundary conditions:

$$\begin{cases} \phi(t,0) = 0 \\ \phi(t,L) = 0. \end{cases}$$

We may also impose the same conditions on the spacial derivatives, but they would just reduce to the above equations, since taking the spacial derivative only amounts to an extra factor of $\pm k$ in any mode. If we plug $x = 0$ in the general solution, we get

$$\phi(t,0) = \sum_k \left(e^{-ikt}b_k + e^{ikt}b_k^*\right),$$

so we immediately have $b_k = 0$. The updated general solution now only has sine factors:

$$\phi(t,x) = \sum_k \sin(kx)\left(a_k e^{-ikt} + a_k^* e^{ikt}\right).$$

We still need to enforce $\phi(t, x = L) = 0 \quad \forall t \in \mathbb{R}$. This is achieved by imposing

$$\sin(kL) = 0,$$

meaning that

$$k = \frac{n\pi}{L},$$

with $n \in \{1, 2, \ldots\} \equiv \mathbb{N}^+$ (we exclude non-positive integers because we are in the case $k > 0$).

Considering only the spacial part, this means that any function $f \in L^2([0,L])$ satisfying Dirichlet boundary conditions at $x = 0, L$ can be expressed as a linear combination of sine modes $\{\sin(k_n x)\}_{n=1}^{+\infty}$. Let us define for a moment the standard inner product in $L^2([0,L], \mathbb{R})$

$$\langle v_1, v_2 \rangle = \int_0^L v_1(x) v_2(x),$$

and compute the orthonormality relation of the sine modes. We therefore need to compute the following integral:

$$I_{nm} = \int_0^L dx \sin(k_n x)\sin(k_m x).$$

We can use the Werner formula $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$ *to get*

$$I_{nm} = \frac{1}{2}\int_0^L dx\,(\cos[(k_n - k_m)x] - \cos[(k_n + k_m)x])$$

Now, assume $n \neq m$ *and recall that* $k_n = \frac{\pi n}{L}$, *so*

$$\begin{aligned} I_{nm} &= \frac{1}{2}\left(\frac{1}{k_n - k_m}\sin[(k_n - k_m)x] - \frac{1}{k_n + k_m}\sin[(k_n + k_m)x]\right)\Bigg|_0^L \\ &= \frac{L}{2\pi}\left(\frac{1}{n-m}\sin[\pi(n-m)] - \frac{1}{n+m}\sin[\pi(n+m)]\right) = 0, \end{aligned}$$

since $\sin(p\pi) = 0\ \forall p \in \mathbb{Z}$. *Whereas, if* $n = m$, *we have:*

$$I_{nn} = \frac{1}{2}\int_0^L dx\,[1 - \cos(2k_n x)] = \frac{L}{2} - \frac{1}{2k_n}\sin(2n\pi) = \frac{L}{2}$$

We can therefore claim that

$$\int_0^L dx\,\sin(k_n x)\sin(k_m x) = \frac{L}{2}\delta_{nm}. \tag{2.6}$$

The orthonormal basis of the space $L^2([0,L],\mathbb{R})$ *with Dirichlet boundary, and equipped with the standard inner product is then given by*

$$v_n(x) = \sqrt{\frac{2}{L}}\sin(k_n x).$$

Now, take an arbitrary test function $f \in L^2([0,L],\mathbb{R})$ *such that* $f(0) = f(L) = 0$. *We can expand* f *onto the complete basis* $\{v_n\}_{n=1}^{+\infty}$ *as follows*

$$\begin{aligned} f(x) &= \sum_{n=1}^{+\infty}\langle f, v_n\rangle v_n(x) \\ &= \sum_{n=1}^{+\infty}\left[\int_0^L dy f(y) v_n(y)\right] v_n(x) \\ &= \int_0^L \left[\sum_{n=1}^{+\infty} v_n(x) v_n(y)\right] f(y) dy. \end{aligned}$$

The above equality chain implies that the object inside the square brackets in the last expression acts as a Dirac delta distribution on the test function[1] f, *and therefore we can write down the following distributional identity (after substituting the expressions for* v_n *and multiplying both sides by* $L/2$*), which is a completeness relation:*

$$\sum_{n=1}^{+\infty}\sin(k_n x)\sin(k_n y) = \frac{L}{2}\delta(x-y) \tag{2.7}$$

Going back to the equation of motion, the final general solution is therefore given by the following linear combination

$$\phi(t,x) = \sum_{n=1}^{+\infty}\left[a_n u_n(t,x) + a_n^* u_n^*(t,x)\right],$$

[1]See Appendix C for a primer on theory of distributions.

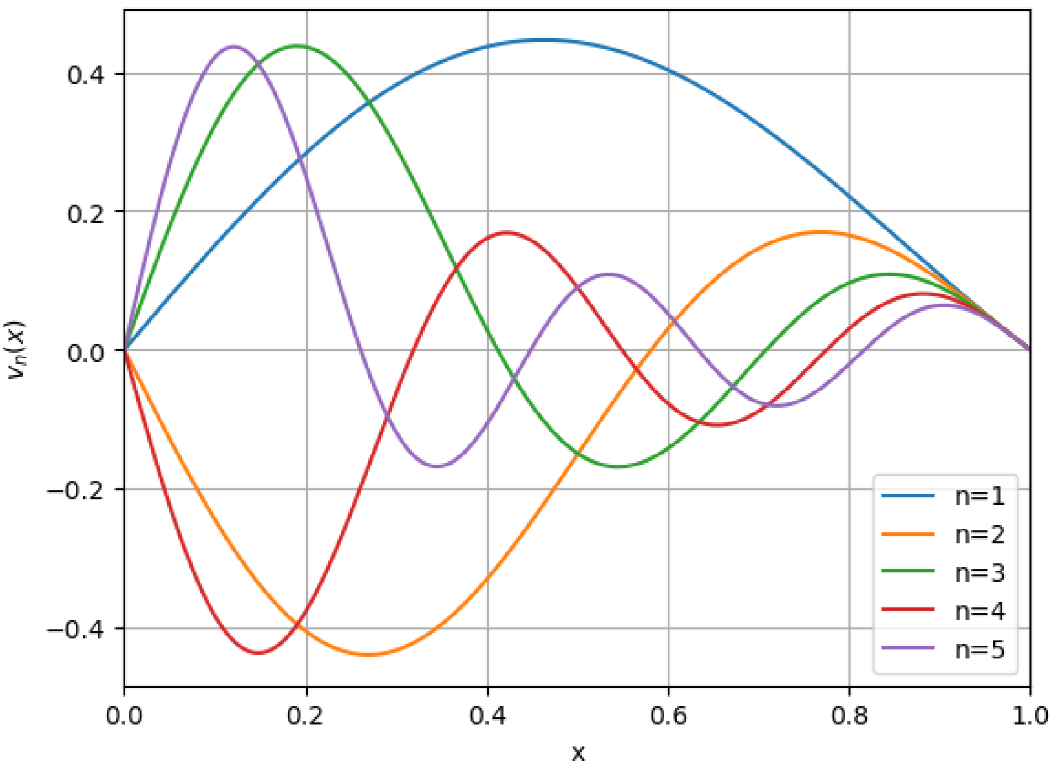

Figure 2.1: Spacial part $v_n(x)$ of the normal modes $u_n(t,x)$ for $n = 1, 2, 3, 4, 5$ and $L = 1$.

where the modes u_n and u_n^* are defined by

$$u_n(t,x) = N_n e^{-ikt} \sin(kx),$$

with N_n being a normalization factor to be determined later.

We may also have chosen a non-negative separation constant, but in this case the field solution would be identically vanishing after imposing Dirichlet boundary conditions. Indeed, if the separation constant was 0, the spacial solution would be $\beta(x) = c_0 + c_1 x$, so $\beta(0) = 0$ yields $c_0 = 0$ and then $\beta(L) = 0$ yields $c_1 = 0$. Whereas, if the separation constant was positive, the spacial part would have been $\beta(x) = d_+ \cosh(kx) + d_- \sinh(kx)$, so $\beta(0) = 0$ yields $d_+ = 0$ and then $\beta(L) = 0$ yields $d_- = 0$ (since $k > 0$ and so $\sinh(kL) > 0$).

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By setting

$$N_n = \frac{1}{\sqrt{k_n L}},$$

the modes above become:

$$u_n(t,x) = \frac{1}{\sqrt{k_n L}} \sin(k_n x) e^{-ik_n t}, \tag{2.8a}$$

$$u_n^*(t,x) = \frac{1}{\sqrt{k_n L}} \sin(k_n x) e^{ik_n t}. \tag{2.8b}$$

By denoting $u_n(t,x) = e^{-ik_n t} v_n(x)$, we can see the plot of the spacial part v_n in Figure 2.1.

The modes u_n are orthonormal with respect to the Klein-Gordon inner product

$$\langle \phi_1, \phi_2 \rangle = -i \int_0^L dx \left(\phi_1 \partial_t \phi_2^* - \phi_2^* \partial_t \phi_1 \right), \tag{2.9}$$

yielding the orthonormality relations

$$\langle u_n, u_m \rangle = \delta_{nm}, \tag{2.10a}$$
$$\langle u_n^*, u_m^* \rangle = -\delta_{nm}, \tag{2.10b}$$
$$\langle u_n, u_m^* \rangle = 0, \tag{2.10c}$$

for all $n, m \in \mathbb{N}^+$.

Proof. Let us denote $u_n \equiv u_n^-$ and $u_n^* \equiv u_n^+$ so that (assuming $N_n \in \mathbb{N}$):

$$u_n^\pm(t,x) = N_n \sin(k_n x) e^{\pm i k_n t}$$

We have, by direct calculation (with $\sigma_1, \sigma_2 \in \{+,-\}$):

$$\begin{aligned}\langle u_n^{\sigma_1}, u_m^{\sigma_2} \rangle = -i \int_0^L & dx \big(N_n N_m \sin(k_n x) \sin(k_m x)(\sigma_2 i k_m) e^{i(\sigma_1 k_n - \sigma_2 k_m)t} \\ & - N_n N_m \sin(k_n x) \sin(k_m x)(\sigma_1 i k_n) e^{i(\sigma_1 k_n - \sigma_2 k_m)t} \big) \\ = N_n N_m (\sigma_1 k_n + \sigma_2 k_m) & e^{i(\sigma_1 k_n - \sigma_2 k_m)t} \int_0^L dx \sin(k_n x) \sin(k_m x).\end{aligned}$$

By using (2.6) in the inner product, we have

$$\langle u_n^{\sigma_1}, u_m^{\sigma_2} \rangle = N_n^2 (\sigma_1 + \sigma_2) k_n e^{i(\sigma_1 - \sigma_2) k_n t} \frac{L}{2} \delta_{nm}.$$

Now, for $\langle u_n, u_m \rangle$ and $\langle u_n^*, u_m^* \rangle$, we have $\sigma_1 = \sigma_2$ and therefore:

$$\langle u_n, u_m \rangle = \langle u_n^*, u_m^* \rangle = \sigma_1 N_n^2 k_n L \delta_{nm},$$

so we obtain

$$N_n = \frac{1}{\sqrt{k_n L}},$$

and based on the sign of $\sigma_1 = \sigma_2$ we get a plus for $\langle u_n, u_m \rangle$ and a minus $\langle u_n^*, u_m^* \rangle$. Whereas, for $\langle u_n, u_m^* \rangle$, we have $\sigma_1 = -\sigma_2$ and therefore $\sigma_1 + \sigma_2 = 0$, making the inner product vanish.

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From now on, we shall not specify the domain of the index n for the modes, keeping in mind it is $\mathbb{N}^+$. In addition, these modes have definite frequency, being eigenfunctions of the Schrödinger operator $i\partial_t$:

$$i\partial_t u_n = k_n u_n, \tag{2.11a}$$
$$i\partial_t u_n^* = -k_n u_n^*. \tag{2.11b}$$

So, we see that

- $u_n(t,x)$ are positive-norm and positive-frequency;
- $u_n^*(t,x)$ are negative-norm and negative-frequency.

2.2 Field quantization

The general solution to the field equation is given by

$$\phi(t,x) = \sum_{n=1}^{+\infty} \left[a_n^- u_n(t,x) + a_n^+ u_n^*(t,x) \right], \tag{2.12}$$

with the constraint that $(a_n^-)^* = a_n^+$ to ensure that ϕ is real. To quantize, we need the canonical conjugate momentum, which is given by

$$\Pi(t,x) = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \partial_t \phi = \sum_{n=1}^{+\infty} \left[a_n^- \partial_t u_n(t,x) + a_n^+ \partial_t u_n^*(t,x) \right]. \tag{2.13}$$

By promoting the coefficients a_n^-, a_n^+ to operators $\hat{a}_n^-, \hat{a}_n^+$ with $(\hat{a}_n^-)^\dagger = \hat{a}_n^+$, we obtain the quantum versions of the field ϕ and its canonical conjugate momentum Π:

$$\boxed{\begin{aligned} \hat{\phi}(t,x) &= \sum_{n=1}^{+\infty} \frac{\sin(k_n x)}{\sqrt{k_n L}} \left(\hat{a}_n^- e^{-ik_n t} + \hat{a}_n^+ e^{ik_n t} \right), && \text{(2.14a)} \\ \hat{\Pi}(t,x) &= \sum_{n=1}^{+\infty} i \sqrt{\frac{k_n}{L}} \sin(k_n x) \left(\hat{a}_n^+ e^{ik_n t} - \hat{a}_n^- e^{-ik_n t} \right). && \text{(2.14b)} \end{aligned}}$$

If we impose the algebra of creation-annihilation operators as follows

$$\left[\hat{a}_n^-, \hat{a}_m^- \right] = 0, \tag{2.15a}$$
$$\left[\hat{a}_n^+, \hat{a}_m^+ \right] = 0, \tag{2.15b}$$
$$\left[\hat{a}_n^-, \hat{a}_m^+ \right] = \delta_{nm}, \tag{2.15c}$$

we can verify that $\hat{\phi}$ and $\hat{\Pi}$ satisfy the equal-time canonical commutation relations

$$\left[\hat{\phi}(t,x), \hat{\phi}(t,y) \right] = 0 \tag{2.16a}$$
$$\left[\hat{\Pi}(t,x), \hat{\Pi}(t,y) \right] = 0 \tag{2.16b}$$
$$\left[\hat{\phi}(t,x), \hat{\Pi}(t,y) \right] = i\delta(x-y) \tag{2.16c}$$

PROOF. *By direct calculation, using the relations above, we have (factors with vanishing commutators are immediately dropped)*

$$\begin{aligned} \left[\hat{\phi}(t,x), \hat{\phi}(t,y) \right] &= \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{\sin(k_n x)\sin(k_m y)}{L\sqrt{k_n k_m}} \left[\hat{a}_n^- e^{-ik_n t} + \hat{a}_n^+ e^{ik_n t}, \hat{a}_m^- e^{-ik_m t} + \hat{a}_m^+ e^{ik_m t} \right] \\ &= \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{\sin(k_n x)\sin(k_m y)}{L\sqrt{k_n k_m}} \left(\left[\hat{a}_n^-, \hat{a}_m^+ \right] e^{-i(k_n - k_m)t} + \left[\hat{a}_n^+, \hat{a}_m^- \right] e^{i(k_n - k_m)t} \right) \\ &= \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{\sin(k_n x)\sin(k_m y)}{L\sqrt{k_n k_m}} \left(\delta_{nm} e^{-i(k_n - k_m)t} - \delta_{nm} e^{i(k_n - k_m)t} \right) \\ &= \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{\sin(k_n x)\sin(k_n y)}{k_n L} (\delta_{nm} - \delta_{nm}) = 0. \end{aligned}$$

$$\begin{aligned}
&\left[\hat{\Pi}(t,x),\hat{\Pi}(t,y)\right] = \\
&\quad = -\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{\sqrt{k_n k_m}}{L}\sin(k_n x)\sin(k_m y)\left[\hat{a}_n^+ e^{ik_n t} - \hat{a}_n^- e^{-ik_n t}, \hat{a}_m^+ e^{ik_m t} - \hat{a}_m^- e^{-ik_m t}\right] \\
&\quad = -\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{\sqrt{k_n k_m}}{L}\sin(k_n x)\sin(k_m y)\left(-\left[\hat{a}_n^+,\hat{a}_m^-\right]e^{i(k_n-k_m)t} - \left[\hat{a}_n^-,\hat{a}_m^+\right]e^{-i(k_n-k_m)t}\right) \\
&\quad = \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{\sqrt{k_n k_m}}{L}\sin(k_n x)\sin(k_m y)\left(-\delta_{nm}e^{i(k_n-k_m)t} + \delta_{nm}e^{-i(k_n-k_m)t}\right) \\
&\quad = \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{k_n}{L}\sin(k_n x)\sin(k_n y)k_n L(-\delta_{nm}+\delta_{nm}) = 0.
\end{aligned}$$

$$\begin{aligned}
\left[\hat{\phi}(t,x),\hat{\Pi}(t,y)\right] &= \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{i}{L}\sqrt{\frac{k_m}{k_n}}\sin(k_n x)\sin(k_m y)\left[\hat{a}_n^- e^{-ik_n t} + \hat{a}_n^+ e^{ik_n t}, \hat{a}_m^+ e^{ik_m t} - \hat{a}_m^- e^{-ik_m t}\right] \\
&= \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{i}{L}\sqrt{\frac{k_m}{k_n}}\sin(k_n x)\sin(k_m y)\left(\left[\hat{a}_n^-,\hat{a}_m^+\right]e^{-i(k_n-k_m)t} - \left[\hat{a}_n^+,\hat{a}_m^-\right]e^{i(k_n-k_m)t}\right) \\
&= \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{i}{L}\sqrt{\frac{k_m}{k_n}}\sin(k_n x)\sin(k_m y)\left(\delta_{nm}e^{-i(k_n-k_m)t} + \delta_{nm}e^{i(k_n-k_m)t}\right) \\
&= \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{i}{L}\sin(k_n x)\sin(k_n y)\cdot 2\delta_{nm} \\
&= \frac{2i}{L}\sum_{n=1}^{+\infty}\sin(k_n x)\sin(k_n y).
\end{aligned}$$

We can now employ (2.7) *and see that*

$$\left[\hat{\phi}(t,x),\hat{\Pi}(t,y)\right] = \frac{2i}{L}\frac{L}{2}\delta(x-y) = i\delta(x-y).$$

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Recall that the state space of the quantum field theory is called Fock space, and can be built starting from the vacuum $|0\rangle$, which is the state that is annihilated by all the annihilation operators:

$$\hat{a}_n^- |0\rangle = 0.$$

The excited states are built by acting on the vacuum with creation operators $\hat{a}_n^+$. In what follows, we will often use the following identity:

$$\langle \hat{a}_n^- \hat{a}_m^+ \rangle \equiv \langle 0| \hat{a}_n^- \hat{a}_m^+ |0\rangle = \delta_{nm}. \tag{2.17}$$

Proof. *We have*

$$\langle 0| \hat{a}_n^- \hat{a}_m^+ |0\rangle = \langle 0| [\hat{a}_n^-, \hat{a}_m^+] + \hat{a}_m^+ \hat{a}_n^- |0\rangle = \langle 0| \delta_{nm} |0\rangle - \cancel{\langle 0| \hat{a}_m^+ \hat{a}_n^- |0\rangle} = \delta_{nm},$$

since the vacuum state is normalized $\langle 0|0\rangle = 1$.

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2.3 Energy-momentum tensor

The core quantity we are interested in is the energy-momentum tensor vacuum expectation value $\langle 0|\hat{T}_{\mu\nu}|0\rangle \equiv \langle\hat{T}_{\mu\nu}\rangle$. To compute it, we use the standard formula (see e.g. Eq. 2.26 of [2])

$$\hat{T}_{\mu\nu} = \partial_\mu\hat{\phi}\partial_\nu\hat{\phi} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\hat{\phi}\partial_\beta\hat{\phi}. \tag{2.18}$$

By a straightforward calculation, we obtain

$$\langle\hat{T}_{tt}\rangle = \langle\hat{T}_{xx}\rangle = \frac{\pi}{2L^2}\sum_{n=1}^{+\infty} n, \tag{2.19a}$$

$$\langle\hat{T}_{tx}\rangle = -\langle\hat{T}_{xt}\rangle = -i\frac{\pi}{2L^2}\sum_{n=1}^{+\infty} n\sin(2k_n x). \tag{2.19b}$$

PROOF. *We can use the definitions of the field operator and its conjugate momentum* (2.14). *First, we need the derivative of* $\hat{\phi}$ *with respect to* x*:*

$$\partial_x\hat{\phi}(t,x) = \sum_{n=1}^{+\infty}\sqrt{\frac{k_n}{L}}\cos(k_n x)\left(\hat{a}_n^- e^{-ik_n t} + \hat{a}_n^+ e^{ik_n t}\right).$$

Let us compute the expectation values of the squares of the differentiated operators:

$$\begin{aligned}
\langle\hat{\Pi}^2\rangle &= -\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{\sqrt{k_n k_m}}{L}\sin(k_n x)\sin(k_m x)\left\langle\left(\hat{a}_n^+ e^{ik_n t} - \hat{a}_n^- e^{-ik_n t}\right)\left(\hat{a}_m^+ e^{ik_m t} - \hat{a}_m^- e^{-ik_m t}\right)\right\rangle \\
&= -\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{\sqrt{k_n k_m}}{L}e^{-i(k_n-k_m)t}\sin(k_n x)\sin(k_m x)\langle -\hat{a}_n^-\hat{a}_m^+\rangle \\
&= \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{\sqrt{k_n k_m}}{L}e^{-i(k_n-k_m)t}\sin(k_n x)\sin(k_m x)\delta_{nm} \\
&= \sum_{n=1}^{+\infty}\frac{k_n}{L}\sin^2(k_n x),
\end{aligned}$$

$$\begin{aligned}
\langle(\partial_x\hat{\phi})^2\rangle &= \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{\sqrt{k_n k_m}}{L}\cos(k_n x)\cos(k_m x)\left\langle\left(\hat{a}_n^- e^{-ik_n t} + \hat{a}_n^+ e^{ik_n t}\right)\left(\hat{a}_m^- e^{-ik_m t} + \hat{a}_m^+ e^{ik_m t}\right)\right\rangle \\
&= \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{\sqrt{k_n k_m}}{L}e^{-i(k_n-k_m)t}\cos(k_n x)\cos(k_m x)\delta_{nm} \\
&= \sum_{n=1}^{+\infty}\frac{k_n}{L}\cos^2(k_n x).
\end{aligned}$$

Now, we can compute the components of the vacuum expectation value of the energy-momentum *tensor:*

$$\begin{aligned}\langle \hat{T}_{tt}\rangle &= \langle \hat{\Pi}^2\rangle - \frac{1}{2}(-1)\left[-\langle \hat{\Pi}^2\rangle + \langle (\partial_x\hat{\phi})^2\rangle\right] = \frac{1}{2}\left[\langle \hat{\Pi}^2\rangle + \langle (\partial_x\hat{\phi})^2\rangle\right]\\ &= \frac{1}{2}\sum_{n=1}^{+\infty}\frac{k_n}{L}\left[\sin^2(k_n x) + \cos^2(k_n x)\right] = \frac{1}{2}\sum_{n=1}^{+\infty}\frac{k_n}{L} = \frac{\pi}{2L^2}\sum_{n=1}^{+\infty} n,\\ \langle \hat{T}_{xx}\rangle &= \langle (\partial_x\hat{\phi})^2\rangle - \frac{1}{2}\left[-\langle \hat{\Pi}^2\rangle + \langle (\partial_x\hat{\phi})^2\rangle\right] = \frac{1}{2}\left[\langle \hat{\Pi}^2\rangle + \langle (\partial_x\hat{\phi})^2\rangle\right] = \frac{\pi}{2L^2}\sum_{n=1}^{+\infty} n,\end{aligned}$$

$$\begin{aligned}\langle \hat{T}_{tx}\rangle &= \langle \hat{\Pi}\partial_x\hat{\phi}\rangle =\\ &= i\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{\sqrt{k_n k_m}}{L}\sin(k_n x)\cos(k_m x)\left\langle\left(\hat{a}_n^+ e^{ik_n t} - \hat{a}_n^- e^{-ik_n t}\right)\left(\hat{a}_m^- e^{-ik_m t} + \hat{a}_m^+ e^{ik_m t}\right)\right\rangle\\ &= -i\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{\sqrt{k_n k_m}}{L}\sin(k_n x)\cos(k_m x)e^{-i(k_n-k_m)t}\underbrace{\langle \hat{a}_n^-\hat{a}_m^+\rangle}_{\delta_{nm}}\\ &= -i\sum_{n=1}^{+\infty}\frac{k_n}{L}\sin(k_n x)\cos(k_n x)\\ &= -i\frac{\pi}{2L^2}\sum_{n=1}^{+\infty} n\sin(2k_n x),\end{aligned}$$

$$\begin{aligned}\langle \hat{T}_{xt}\rangle &= \langle \hat{\Pi}\partial_x\hat{\phi}\rangle =\\ &= i\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{\sqrt{k_n k_m}}{L}\cos(k_n x)\sin(k_m x)\left\langle\left(\hat{a}_n^- e^{-ik_n t} + \hat{a}_n^+ e^{ik_n t}\right)\left(\hat{a}_m^+ e^{ik_m t} - \hat{a}_m^- e^{-ik_m t}\right)\right\rangle\\ &= i\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{\sqrt{k_n k_m}}{L}\cos(k_n x)\sin(k_m x)e^{-i(k_n-k_m)t}\underbrace{\langle \hat{a}_n^-\hat{a}_m^+\rangle}_{\delta_{nm}}\\ &= i\sum_{n=1}^{+\infty}\frac{k_n}{L}\sin(k_n x)\cos(k_n x)\\ &= i\frac{\pi}{2L^2}\sum_{n=1}^{+\infty} n\sin(2k_n x).\end{aligned}$$

■

We notice a few things:

- the diagonal components include a quadratically divergent series;
- the off-diagonal components contain an oscillating series, which does not converge;
- the off-diagonal components are not equal, as one would expect from the energy-momentum tensor being symmetric. This is due to the operator ordering ambiguity when computing the product of non-commuting operators.

These three issues are all solved by renormalization. We will apply both a simple regularization method and the Hadamard method to this case, to get a taste on how the latter works. We expect to get the same result from both methods, being the energy-momentum tensor an observable quantity.

2.4 Simple renormalization

Since this is a simple case, we can just introduce a regulator by hand inside both series, make them converge, and then recover the original case in some limit[2]. The regulating factor we will use is

$$e^{-\alpha k_n} = e^{-\alpha \frac{\pi}{L} n}, \tag{2.20}$$

which is an exponentially damping factor. The series is convergent for $\alpha > 0$. We recover the divergence in the limit $\alpha \to 0$, where the exponential regulating factor equals 1.

A common and simple renormalization prescription (found in Birrel-Davies [2], chapter 3) is given by subtracting from the regularized vacuum expectation value of the energy-momentum tensor the one it has in the limit $L \to +\infty$:

$$\langle : \hat{T}_{\mu\nu} : \rangle = \langle \hat{T}_{\mu\nu} \rangle_{reg} - \lim_{L \to +\infty} \langle \hat{T}_{\mu\nu} \rangle_{reg}, \tag{2.21}$$

and the renormalized energy-momentum tensor will be

$$\langle \hat{T}_{\mu\nu} \rangle_{ren} = \lim_{\alpha \to 0} \langle : \hat{T}_{\mu\nu} : \rangle \tag{2.22}$$

2.4.1 Diagonal terms

By introducing the regulator by hand in the diagonal terms (2.19a), we get (no sum over $\mu = t, x$)

$$\langle \hat{T}_{\mu\mu} \rangle_{reg} = \frac{\pi}{2L^2} \sum_{n=1}^{+\infty} n e^{-\alpha \frac{\pi}{L} n},$$

which evaluates to

$$\langle \hat{T}_{\mu\mu} \rangle_{reg} = \frac{\pi}{2L^2} \frac{e^{\frac{\pi\alpha}{L}}}{(e^{\frac{\pi\alpha}{L}} - 1)^2}.$$

Proof. We know that the geometric series, with $|r| < 1$, converges to ([32]):

$$\sum_{n=0}^{+\infty} r^n = \frac{1}{1-r}.$$

Now, we set $r = e^{-\alpha \frac{\pi}{L}}$ which is positive and less than 1 for all $\alpha > 0$. We have

$$\sum_{n=0}^{+\infty} e^{-\alpha \frac{\pi n}{L}} = \frac{1}{1 - e^{-\alpha \frac{\pi}{L}}}.$$

For $\alpha > 0$, we can apply the operator $\left(-\frac{L}{\pi}\frac{\partial}{\partial \alpha}\right)$, and get

$$-\frac{L}{\pi}\frac{\partial}{\partial \alpha} \sum_{n=0}^{+\infty} e^{-\alpha \frac{\pi n}{L}} = -\frac{L}{\pi}\frac{\partial}{\partial \alpha} \frac{1}{1 - e^{-\alpha \frac{\pi}{L}}}.$$

[2] More rigorously, we will treat the two expressions as distributions, being limits of a family of smooth functions as the regulator tends to 1. See Section C.3 for the mathematical meaning of regularization.

Since the series is absolutely convergent (all terms are positive), we can swap the derivative operator and the infinite sum, to get

$$\cancel{\left(-\frac{L}{\pi}\right)} \sum_{n=0}^{+\infty} \cancel{\left(-\frac{\pi}{L}\right)} n e^{-\alpha\frac{\pi n}{L}} = \cancel{\left(-\frac{L}{\pi}\right)} \frac{\cancel{\left(-\frac{\pi}{L}\right)} e^{-\alpha\frac{\pi}{L}}}{(1-e^{-\alpha\frac{\pi}{L}})^2}.$$

Now we notice that the start of the summation that we need to compute is not $n = 0$ but rather $n = 1$, however this fact does not change the result, since the term associated to $n = 0$ is vanishing due to the n factor in front of the exponential. Therefore, we get our final result upon multiplying by the prefactor $\frac{\pi}{2L^2}$ and collecting $e^{-2\alpha\frac{\pi}{L}}$ in the denominator:

$$\langle \hat{T}_{\mu\mu} \rangle_{reg} = \frac{\pi}{2L^2} \sum_{n=0}^{+\infty} n e^{-\alpha\frac{\pi n}{L}} = \frac{\pi}{2L^2} \frac{e^{-\alpha\frac{\pi}{L}}}{e^{-2\alpha\frac{\pi}{L}}(e^{\alpha\frac{\pi}{L}}-1)^2} = \frac{\pi}{2L^2} \frac{e^{\frac{\pi\alpha}{L}}}{(e^{\frac{\pi\alpha}{L}}-1)^2}.$$

■

To make the divergence manifest, we now expand the previous expression in power series of α around 0^+, and get

$$\langle \hat{T}_{\mu\mu} \rangle_{reg} = \frac{1}{2\pi\alpha^2} - \frac{\pi}{24L^2} + \mathcal{O}\left(\frac{\alpha}{L^3}\right).$$

PROOF. *The power series is just a Taylor series expansion around $\alpha = 0^+$. Let us call $z = \alpha\frac{\pi}{L}$ and expand around $z = 0^+$. The numerator is:*

$$N = e^z = 1 + z + \frac{z^2}{2} + \mathcal{O}(z^3),$$

while the denominator is

$$\begin{aligned} D = (e^z - 1)^2 &= \left(z + \frac{z^2}{2} + \frac{z^3}{6} + \mathcal{O}(z^4)\right)^2 \\ &= z^2 + z^3 + \frac{z^4}{3} + \frac{z^4}{4} + \mathcal{O}(z^5) \\ &= z^2\left(1 + z + \frac{7}{12}z^2 + \mathcal{O}(z^3)\right). \end{aligned}$$

Now, we use the following standard Taylor expansion

$$\frac{1}{1 + az + bz^2 + \mathcal{O}(z^3)} = 1 - az + (a^2 - b)z^2 + \mathcal{O}(z^3),$$

with $a = 1$ and $b = 7/12$ and we get

$$\begin{aligned} \frac{N}{D} &= z^{-2}\left(1 + z + \frac{z^2}{2} + \mathcal{O}(z^3)\right)\left(1 - z + \frac{5}{12}z^2 + \mathcal{O}(z^3)\right) \\ &= z^{-2}\left(1 \cancel{+z} \cancel{-z} - z^2 + \frac{z^2}{2} + \frac{5}{12}z^2 + \mathcal{O}(z^3)\right) \\ &= z^{-2}\left(1 - \frac{z^2}{12} + \mathcal{O}(z^3)\right) = \frac{1}{z^2} - \frac{1}{12} + \mathcal{O}(z). \end{aligned}$$

Now, substitute everything in the original formula:

$$\begin{aligned}\langle \hat{T}_{\mu\mu} \rangle_{reg} &= \frac{\pi}{2L^2}\frac{e^z}{(e^z-1)^2} = \\ &= \frac{\pi}{2L^2}\left(\frac{1}{z^2} - \frac{1}{12} + \mathcal{O}(z)\right) \\ &= \frac{\pi}{2L^2}\left(\frac{L^2}{\pi^2\alpha^2} - \frac{1}{12} + \mathcal{O}\left(\frac{\alpha}{L}\right)\right) \\ &= \frac{1}{2\pi\alpha^2} - \frac{\pi}{24L^2} + \mathcal{O}\left(\frac{\alpha}{L^3}\right).\end{aligned}$$

■

The renormalization prescription simply reduces to discarding the divergent term, since it is the only one that survives in the limit $L \to +\infty$, and therefore gets subtracted according to prescription (2.21). By taking the limit $\alpha \to 0$ after the subtraction, we get that the renormalized diagonal components of the vacuum expectation value of the energy-momentum tensor are equal to

$$\boxed{\langle \hat{T}_{tt} \rangle_{ren} = \langle \hat{T}_{xx} \rangle_{ren} = -\frac{\pi}{24L^2}} \tag{2.23}$$

2.4.2 Off-diagonal terms

By introducing the regulator in the off-diagonal terms (2.19b), we get (with $\mu \neq \nu$)

$$\langle \hat{T}_{\mu\nu} \rangle_{reg} = \pm i\frac{\pi}{2L^2}\sum_{n=1}^{+\infty} n e^{-\alpha\frac{\pi}{L}n}\sin\left(\frac{\pi n}{L}x\right), \tag{2.24}$$

which evaluates to

$$\langle \hat{T}_{\mu\nu} \rangle_{reg} = \pm i\frac{\pi}{4L^2}\frac{\sin x \sin\left(\frac{\pi}{L}\alpha\right)}{\left[\cos\left(\frac{\pi}{L}\alpha\right) - \cos x\right]^2}.$$

Proof. *We have*

$$\begin{aligned} S = \sum_{n=1}^{+\infty} n e^{-\alpha\frac{\pi}{L}n}\sin\left(\frac{\pi n}{L}x\right) &= \operatorname{Im}\left[\sum_{n=1}^{+\infty} n e^{-\alpha\frac{\pi}{L}n} e^{i\frac{\pi n}{L}x}\right] \\ = \operatorname{Im}\left[-\frac{L}{\pi}\sum_{n=1}^{+\infty}\left(-\frac{\pi n}{L}e^{-\frac{\pi n}{L}(\alpha - ix)}\right)\right] &= \operatorname{Im}\left[-\frac{L}{\pi}\frac{\partial}{\partial\alpha}\sum_{n=1}^{+\infty} e^{-\frac{\pi n}{L}(\alpha - ix)}\right].\end{aligned}$$

By setting $z \equiv -\frac{\pi}{L}(\alpha - ix)$, we get

$$S = -\frac{L}{\pi}\frac{\partial}{\partial\alpha}\operatorname{Im}\left[\sum_{n=1}^{+\infty} e^{nz}\right] \equiv -\frac{L}{\pi}\frac{\partial}{\partial\alpha}I(\alpha).$$

The above is a geometric series with argument e^z. In this case, we have z complex, but since e^z is holomorphic in all the complex plane, we can use the analytical extension of the geometric series formula, provided $|e^z| < 1$. Indeed,

$$|e^z| = |e^{-\frac{\pi}{L}(\alpha - ix)}| = e^{-\frac{\pi}{L}\alpha} < 1 \quad \forall \alpha > 0.$$

Let us denote $e^z = a - ib$, with

$$a = e^{-\frac{\pi}{L}\alpha}\cos x,$$
$$b = e^{-\frac{\pi}{L}\alpha}\sin x.$$

We then have

$$\begin{aligned}
I(\alpha) &= \operatorname{Im}\frac{1}{(1-a)+ib} = \operatorname{Im}\left[\frac{1}{(1-a)+ib}\cdot\frac{(1-a)-ib}{(1-a)-ib}\right] = \operatorname{Im}\frac{(1-a)-ib}{(1-a)^2+b^2} = -\frac{b}{(1-a)^2+b^2}\\
&= -\frac{e^{-\frac{\pi}{L}\alpha}\sin x}{(1-e^{-\frac{\pi}{L}\alpha}\cos x)^2 + e^{-2\frac{\pi}{L}\alpha}\sin^2 x} = \frac{e^{-\frac{\pi}{L}\alpha}\sin x}{1+e^{-2\frac{\pi}{L}\alpha} - 2e^{-\frac{\pi}{L}\alpha}\cos x}\\
&= -\frac{\cancel{e^{-\frac{\pi}{L}\alpha}}\sin x}{\cancel{e^{-\frac{\pi}{L}\alpha}}(e^{\frac{\pi}{L}\alpha}+e^{-\frac{\pi}{L}\alpha} - 2\cos x)} = -\frac{1}{2}\frac{\sin x}{\cos\left(\frac{\pi}{L}\alpha\right) - \cos x}.
\end{aligned}$$

Now, we need to differentiate and multiply by $-L/\pi$:

$$S = \frac{L}{2\pi}\frac{\partial}{\partial\alpha}\frac{\sin x}{\cos\left(\frac{\pi}{L}\alpha\right) - \cos x} = \frac{\cancel{L}\sin x}{2\cancel{\pi}}(-1)\frac{-\cancel{\frac{\pi}{L}}\sin\left(\frac{\pi}{L}\alpha\right)}{\left(\cos\left(\frac{\pi}{L}\alpha\right) - \cos x\right)^2}.$$

Finally, we need to multiply by $\pm\frac{i\pi}{2L^2}$ to get $\langle\hat{T}_{\mu\nu}\rangle_{reg}$:

$$\langle\hat{T}_{\mu\nu}\rangle_{reg} = \pm\frac{i\pi}{2L^2}S = \pm i\frac{\pi}{4L^2}\frac{\sin x\sin\left(\frac{\pi}{L}\alpha\right)}{\left[\cos\left(\frac{\pi}{L}\alpha\right) - \cos x\right]^2}$$

■

From this expression, we can conclude

$$\langle\hat{T}_{\mu\nu}\rangle_{reg} = \mathcal{O}\left(\frac{\alpha}{L^3}\right).$$

PROOF. *Assume $x \neq 0$ and $x \neq L$, since in both cases all the terms in the infinite sum would vanish, and the result would trivially be 0. By keeping x fixed and taking $\alpha < \frac{L}{\pi}x$, we can expand around $\alpha = 0^+$ and get*

$$\begin{aligned}
\langle\hat{T}_{\mu\nu}\rangle_{reg} &= \pm\frac{i\pi}{4L^2}\frac{\sin x\left(\frac{\pi}{L}\alpha + \mathcal{O}\left(\frac{\alpha^2}{L^2}\right)\right)}{\left[1-\cos x + \mathcal{O}\left(\frac{\alpha}{L}\right)\right]^2}\\
&= \pm\frac{i\pi}{4L^2}\left[\frac{\sin x}{(1-\cos x)^2} + \mathcal{O}\left(\frac{\alpha}{L}\right)\right]\left(\frac{\pi}{L}\alpha + \mathcal{O}\left(\frac{\alpha^2}{L^2}\right)\right) = \mathcal{O}\left(\frac{\alpha}{L^3}\right)
\end{aligned}$$

■

If we send $L \to +\infty$ in the above, it vanishes, meaning that there is nothing to subtract, according to the prescription (2.21). In fact, we also see that there are no divergent terms in $\alpha \to 0$. In addition, there is no finite part at all, since the leading term is of order α. Therefore, when we send $\alpha \to 0$, we get that the off-diagonal components of the energy-momentum tensor vanish, and the symmetry of the EMT is recovered.

$$\boxed{\langle\hat{T}_{tx}\rangle_{ren} = \langle\hat{T}_{xt}\rangle_{ren} = 0.} \tag{2.25}$$

It is interesting to notice that the series was not divergent, and the quantum ordering ambiguity has been solved just by regularization. We can interpret the result distributionally, as the limit of the sequence of well-defined functions parameterized by α, with expression given by (2.24).

2.5 Hadamard renormalization

We are now working with a simple model, where the spacetime is flat and the number of spacetime dimensions is 2. However, when dealing with general static and spherically symmetric spacetimes, there is no obvious and quick way to add a regulator by hand, subtract the divergences and compute the renormalized energy-momentum tensor. As discussed in the introductory chapter, the Hadamard renormalization procedure is a systematic way to do this, which only pre-requires that the Feynman propagator expression of the theory is known, or at least a second-order approximation of it near the coincidence limit. We will give a schematic presentation of how to apply the Hadamard procedure, and then employ it to compute the renormalized energy-momentum tensor of our model, to indeed verify that the result is the same one we got with the naive (but working, just for this case) approach.

2.5.1 General algorithmic procedure

In order to properly perform the Hadamard renormalization, we follow algorithmically the procedure outlined by Decanini-Folacci in [17], in the case $D = 2$. The general outline of the procedure, in an arbitrary spacetime of dimensionality D with field equation of motion $(\Box - m^2 - \xi R)\phi = 0$ is as follows[3].

- Compute the Feynman propagator $G^F(x, x')$ of the theory, or at least an approximation near coincidence limit at second order in spacetime variables.

- Compute the following quantity (Eq. 32 in Decanini-Folacci)

$$\alpha_D = \begin{cases} \frac{1}{2\pi} & \text{if } D = 2 \\ \frac{\Gamma\left(\frac{D}{2}-1\right)}{(2\pi)^{\frac{D}{2}}} & \text{if } D \neq 2 \end{cases} . \tag{2.26}$$

- Write down the Hadamard singularity structure expression G^F_{sing} (or *parametrix*) of the propagator, which is given in terms of the Synge world function, defined as half the geodesic distance squared:

$$\sigma(x, x') = \frac{1}{2}\tau(x, x')^2;$$

  the specific form of the Hadamard parametrix depends on the local geometry only, since it encodes the ultraviolet divergences, that dominate short distances. We will not report all the details here, but they are well explained in a self-contained fashion in the paper by Decanini and Folacci [17].

- Compute the renormalized propagator $W(x, x')$ (Eq. 85 in Decanini-Folacci):

$$W(x, x') = \frac{2}{i\alpha_D}[G^F(x, x') - G^F_{sing}(x, x')]. \tag{2.27}$$

[3] m is the bare mass of the scalar field and R is the Ricci scalar. ξ is the coupling constant between the field and the Ricci scalar.

- Next, compute the following limits:

$$w(x) = \lim_{x' \to x} W(x, x'), \tag{2.28a}$$

$$w_{\mu\nu}(x) = \lim_{x' \to x} \nabla_\mu \nabla_\nu W(x, x'). \tag{2.28b}$$

- Finally, compute the renormalized vacuum expectation value of the energy-momentum tensor:

$$\langle \hat{T}_{\mu\nu} \rangle_{ren} = \frac{\alpha_D}{2} \left[-w_{\mu\nu} + \frac{1}{2}(1-2\xi)\nabla_\mu \nabla_\nu w + \frac{1}{2}\left(2\xi - \frac{1}{2}\right) g_{\mu\nu} \Box w + \xi R_{\mu\nu} w - g_{\mu\nu} v_1 \right] + \Theta_{\mu\nu}, \tag{2.29}$$

where $R_{\mu\nu}$ is the Ricci tensor, v_1 and $\Theta_{\mu\nu}$ depend on the spacetime dimension, the geometry and the parameters of the theory, and can be computed by using the formulas reported in Decanini-Folacci, Section III.

2.5.2 Computing the Feynman propagator

First, we need the Feynman propagator

$$G^F(t, x; t', x') = i\langle T\hat{\phi}(t, x)\hat{\phi}(t', x')\rangle = i\langle \hat{\phi}(t, x)\hat{\phi}(t', x')\rangle \Theta(t-t') + i\langle \hat{\phi}(t', x')\hat{\phi}(t, x)\rangle \Theta(t'-t).$$

By direct calculation, we get

$$\langle \hat{\phi}(t, x)\hat{\phi}(t', x')\rangle = \sum_{n=1}^{+\infty} \frac{1}{\pi n} \sin(k_n x) \sin(k_n x') e^{-ik_n(t-t')}, \tag{2.30}$$

$$\langle \hat{\phi}(t', x')\hat{\phi}(t, x)\rangle = \sum_{n=1}^{+\infty} \frac{1}{\pi n} \sin(k_n x) \sin(k_n x') e^{ik_n(t-t')}. \tag{2.31}$$

We can then easily put the Feynman propagator in a compact form:

$$G^F(t, x; t', x') = \sum_{n=1}^{+\infty} \frac{i}{\pi n} \sin(k_n x) \sin(k_n x') e^{-ik_n|t-t'|}, \tag{2.32}$$

where we notice the appearance of an absolute value.

PROOF. *Let us compute the quantity* $\langle \hat{\phi}(t, x)\hat{\phi}(t', x')\rangle$. *We have, by using* (2.14a)*:*

$$\begin{aligned}\langle \hat{\phi}(t, x)\hat{\phi}(t', x')\rangle &= \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} \frac{1}{L\sqrt{k_n k_m}} \sin(k_n x) \sin\left(k_m x'\right) \\ &\quad \left\langle \hat{a}_n^- \hat{a}_m^- e^{-ik_n t} e^{-ik_m t'} + \hat{a}_n^- \hat{a}_m^+ e^{-ik_n t} e^{ik_m t'} + \hat{a}_n^+ \hat{a}_m^- e^{ik_n t} e^{-ik_m t'} + \hat{a}_n^+ \hat{a}_m^+ e^{ik_n t} e^{ik_m t'} \right\rangle \\ &= \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} \frac{1}{L\sqrt{k_n k_m}} \sin(k_n x) \sin\left(k_m x'\right) e^{-ik_n t} e^{ik_m t'} \delta_{nm} \\ &= \sum_{n=1}^{+\infty} \frac{1}{k_n L} \sin(k_n x) \sin\left(k_m x'\right) e^{-ik_n(t'-t)}\end{aligned}$$

The Feynman propagator trivially follows from this calculation, by noticing $k_n L = \pi n$, *multiplying by* i *and inserting the absolute value for time ordering.*

■

2.5.3 Hadamard parametrix

After acknowledging that in our case we have $\alpha_D = \frac{1}{2\pi}$, we now need to write down the Hadamard parametrix, which is reported in Eq. 86 of Decanini-Folacci:

$$G^F_{sing}(t,x;t',x') = \frac{i}{4\pi} V(t,x,t',x') \log[\sigma(t,x,t',x') + i0^+]. \tag{2.33}$$

Here, $V(x,x')$ is given in Section III A of Decanini-Folacci, in Eqs. 87-91. Since in our case $R_{\mu\nu} = 0$, $R = 0$ and $m = 0$, we then have:

$$\begin{aligned} v_0 &= -1, \\ v_{0\ a} &= 0, \\ v_{0\ ab} &= -\frac{1}{12} R g_{ab} = 0, \\ v_1 &= -\frac{1}{2} m^2 - \frac{1}{2}\left(\xi - \frac{1}{6}\right) R = 0. \end{aligned}$$

This means that

$$\begin{aligned} V_0 &= v_0 - v_{0\ a} \nabla^a \sigma + \frac{1}{2!} v_{0\ ab} \nabla^a \sigma \nabla^b \sigma + \mathcal{O}(\sigma^{\frac{3}{2}}) = -1 + \mathcal{O}(\sigma^{\frac{3}{2}}), \\ V_1 &= v_1 + \mathcal{O}(\sigma^{\frac{1}{2}}) = \mathcal{O}(\sigma^{\frac{1}{2}}), \end{aligned}$$

and therefore

$$V(t,x;t',x') = V_0 + V_1 + \mathcal{O}(\sigma^{\frac{3}{2}}) = -1 + \mathcal{O}(\sigma^{\frac{3}{2}}).$$

The Synge world function, in this flat case, reduces to

$$\sigma(t,x;t',x') = \frac{1}{2}[-(t-t')^2 + (x-x')^2]. \tag{2.34}$$

Therefore, the Hadamard parametrix of the propagator singularity is

$$G^F_{sing}(t,x;t',x') = \frac{-i}{4\pi} \log\left[\frac{-(t-t')^2 + (x-x')^2}{2} + i0^+\right].$$

Therefore, by putting everything together, the renormalized propagator $W(t,x;t',x')$ now becomes:

$$W(t,x;t',x') = \sum_{n=1}^{+\infty} \frac{4}{n} \sin(k_n x) \sin(k_n x') e^{-ik_n|t-t'|} + \log\left[\frac{(x-x')^2 - (t-t')^2}{2} + i0^+\right]. \tag{2.35}$$

2.5.4 Renormalizing the energy-momentum tensor

In order to compute the renormalized vacuum expectation value of the energy-momentum tensor, we need to apply (2.29). In our case, the formula reduces to

$$\langle \hat{T}_{\mu\nu} \rangle_{ren} = \frac{1}{4\pi}\left(-w_{\mu\nu} + \frac{1}{2}\partial_\mu \partial_\nu w - \frac{1}{4} g_{\mu\nu} \Box w\right). \tag{2.36}$$

PROOF. Indeed, in the flat and massless case, covariant derivatives become partial derivatives, $R_{\mu\nu} = 0$, $v_1 = 0$ as seen before and we are free to choose $\xi = 1/2$ so that the second term inside the brackets vanishes. In addition, we have, by Eq. 92 of Decanini-Folacci:

$$\Theta^{M^2}_{\mu\nu} = \frac{\log M^2}{4\pi}[-(1/2)m^2 g_{\mu\nu}] = 0.$$

■

We need to compute the following limits

$$\begin{aligned} w(t,x) &= \lim_{(t',x')\to(t,x)} W(t,x;t',x'), \\ w_{\mu\nu}(t,x) &= \lim_{(t',x')\to(t,x)} \partial_\mu \partial_\nu W(t,x;t',x'). \end{aligned}$$

The results, in our case, are:

$$w(t,x) = 2\ln\left[\frac{L\sqrt{2}}{\pi}\sin\left(\frac{\pi x}{L}\right)\right], \tag{2.38a}$$

$$w_{tt}(t,x) = w_{xx}(t,x) = \frac{\pi^2}{6L^2}\left[1 - 3\csc^2\left(\frac{\pi x}{L}\right)\right], \tag{2.38b}$$

$$w_{tx}(t,x) = w_{xt}(t,x) = 0. \tag{2.38c}$$

We notice that w and $w_{\mu\nu}$ do not depend on time, therefore we will write $w = w(x)$ and $w_{\mu\nu} = w_{\mu\nu}(x)$.

PROOF. First, we compute the derivatives of $W(t,x,t',x')$:

$$W(t,x;t',x') = \sum_{n=1}^{+\infty} \frac{4}{n}\sin(k_n x)\sin(k_n x')e^{-ik_n|t-t'|} + \log\left[\frac{(x-x')^2-(t-t')^2}{2} + i0^+\right],$$

$$\partial_x W(t,x;t',x') = \sum_{n=1}^{+\infty} \frac{4}{n}k_n\cos(k_n x)\sin(k_n x')e^{-ik_n|t-t'|} + \frac{2(x-x')}{(x-x')^2-(t-t')^2+i0^+},$$

$$\partial_t W(t,x;t',x') = \sum_{n=1}^{+\infty} \frac{4}{n}ik_n\sin(k_n x)\sin(k_n x')e^{-ik_n|t-t'|}\,\mathrm{sgn}(t'-t) - \frac{2(t-t')}{(x-x')^2-(t-t')^2+i0^+},$$

$$\partial_t\partial_x W(t,x;t',x') = \partial_x\partial_t W(t,x;t',x') = \sum_{n=1}^{+\infty} \frac{4i}{n}k_n^2\cos(k_n x)\sin(k_n x')e^{-ik_n|t-t'|}\,\mathrm{sgn}(t'-t) + \frac{4(t-t')(x-x')}{[(x-x')^2-(t-t')^2+i0^+]^2},$$

$$\partial_x^2 W(t,x;t',x') = -\sum_{n=1}^{+\infty} \frac{4}{n}k_n^2\sin(k_n x)\sin(k_n x')e^{-ik_n|t-t'|} - \frac{2}{(x-x')^2-(t-t')^2}.$$

We also need the second time derivative of $e^{-ik_n|t-t'|}$ in order to compute $\partial_t^2 W(t,x;t',x')$.

$$\begin{aligned} \partial_t e^{-ik_n|t-t'|} &= -ik_n e^{-ik_n|t-t'|}\,\mathrm{sgn}(t-t'), \\ \partial_t^2 e^{-ik_n|t-t'|} &= (-ik_n)^2 e^{-ik_n|t-t'|}\,\mathrm{sgn}^2(t-t') - ik_n e^{-ik_n|t-t'|}\cdot 2\delta(t-t') \\ &= -k_n e^{-ik_n|t-t'|}[k_n\,\mathrm{sgn}^2(t-t') - 2i\delta(t-t')], \end{aligned}$$

where we used the following distributional identities[4]*:*

$$\begin{aligned}\partial_z|z| &= \operatorname{sgn} z = 2\Theta(z) - 1\\ \partial_z\Theta(z) &= \delta(z).\end{aligned}$$

Therefore, we get the second time derivative of W*:*

$$\begin{aligned}\partial_t^2 W(t,x;t',x') = &-\sum_{n=1}^{+\infty}\frac{4}{n}k_n e^{-ik_n|t-t'|}[k_n \operatorname{sgn}^2(t-t') - 2i\delta(t-t')]\sin(k_n x)\sin(k_n x')\\ &- 2\frac{(x-x')^2+(t-t')^2}{[(x-x')^2-(t-t')^2]^2}.\end{aligned}$$

Now, we need to take the limits. Since the expression is regularized, the limit can be taken in any direction, with a small caveat about the second time derivative, where we should avoid setting $t - t' = 0$ *exactly, because the Dirac delta can give issues. In this latter case, we will first perform the limit in time, assuming that* t' *is very close to* t *but never equal, and only after we take the limit in the space direction. When* $t' \to t$ *but not exactly equal, we have* $\operatorname{sgn}^2(t'-t) = 1$*,* $\delta(t'-t) = 0$ *and* $e^{-ik_n|t'-t|} = 1$*, so*

$$\lim_{t'\to t}\partial_t^2 W(t,x;t',x') = -\sum_{n=1}^{+\infty}\frac{4}{n}k_n^2\sin(k_n x)\sin(k_n x') - \frac{2}{(x-x')^2}.$$

We notice that the above limit is equal to the same limit taken on the second spacial derivative:

$$\lim_{t'\to t}\partial_x^2 W(t,x;t',x') = -\sum_{n=1}^{+\infty}\frac{4}{n}k_n^2\sin(k_n x)\sin(k_n x') - \frac{2}{(x-x')^2} = \lim_{t'\to t}\partial_t^2 W(t,x;t',x').$$

Therefore, we will have the same expression for w_{tt} *and* w_{xx}*, which is obtained by taking the limit* $x' \to x$*. Let us parameterize* $x' = x + \epsilon$*, so that the limit becomes* $\epsilon \to 0^+$ *(we are free to choose the direction of the limit). We then have, by applying Werner trigonometric identity involving the product of two sine functions:*

$$\begin{aligned}w_{tt}(t,x) = w_{xx}(t,x) &= \lim_{\epsilon\to 0^+}\left[-\sum_{n=1}^{+\infty}\frac{4}{n}k_n^2\sin(k_n x)\sin(k_n x') - \frac{2}{\epsilon^2}\right]\\ &= \lim_{\epsilon\to 0^+}\left[-\frac{2\pi^2}{L^2}\sum_{n=1}^{+\infty} n\left(\cos\left(\frac{2\pi x}{L}n\right) - \cos\left(\frac{\pi\epsilon}{L}n\right)\right) - \frac{2}{\epsilon^2}\right]\\ &= \lim_{\epsilon\to 0^+}\left[\frac{2\pi^2}{L^2}\left(\sum_{n=1}^{+\infty} n\cos\left(\frac{2\pi x}{L}n\right) - \sum_{n=1}^{+\infty} n\cos\left(\frac{\pi\epsilon}{L}n\right)\right) - \frac{2}{\epsilon^2}\right]\end{aligned}$$

Now we use the distributional identity (C.9a) *twice, and get*

$$\begin{aligned}w_{tt}(t,x) = w_{xx}(t,x) &= \lim_{\epsilon\to 0^+}\left[\frac{2\pi^2}{L^2}\left(\frac{1}{2\left(\cos\frac{2\pi x}{L} - 1\right)} - \frac{1}{2\left(\cos\frac{\pi\epsilon}{L} - 1\right)}\right) - \frac{2}{\epsilon^2}\right]\\ &= \lim_{\epsilon\to 0^+}\left[\frac{\pi^2}{2L^2}\left(\frac{1}{\sin^2\frac{\pi\epsilon}{2L}} - \frac{1}{\sin^2\frac{\pi x}{L}}\right) - \frac{2}{\epsilon^2}\right],\end{aligned}$$

where we also used the trigonometric identity $1 - \cos\alpha = 2\sin^2(\alpha/2)$*. Now, we employ the following expansion:*

$$\frac{1}{\sin^2\alpha} = \frac{1}{\alpha^2} + \frac{1}{3} + \mathcal{O}(\alpha),$$

[4]See Section C.2 for the definition of distributional derivatives.

so that we get

$$w_{tt}(t,x) = w_{xx}(t,x) = \lim_{\epsilon\to 0^+}\left[\frac{\pi^2}{2L^2}\left(\frac{4L^2}{\pi^2\epsilon^2}+\frac{1}{3}+\mathcal{O}(\epsilon)-\csc^2\frac{\pi x}{L}\right)-\frac{2}{\epsilon^2}\right]$$
$$= \lim_{\epsilon\to 0^+}\left[\frac{2}{\epsilon^2}+\frac{\pi^2}{6L^2}-\frac{\pi^2}{2L^2}\csc^2\frac{\pi x}{L}-\frac{2}{\epsilon^2}+\mathcal{O}(\epsilon)\right]$$
$$= \frac{\pi^2}{6L^2}\left[1-3\csc^2\left(\frac{\pi x}{L}\right)\right].$$

Then, we compute $w_{tx} = w_{xt}$. *To do this, we first send* $t' \to t$ *in* $\partial_t\partial_x W = \partial_x\partial_t W$:

$$\lim_{t'\to t^\pm}\left[\sum_{n=1}^{+\infty}\frac{4i}{n}k_n^2\cos(k_n x)\sin(k_n x')e^{-ik_n|t'-t|}\,\mathrm{sgn}(t'-t)+\frac{4(t-t')(x-x')}{[(x-x')^2-(t-t')^2+i0^+]^2}\right]=$$
$$\pm\frac{4i\pi^2}{L^2}\sum_{n=1}^{+\infty}n\cos(k_n x)\sin(k_n x').$$

We can now set $x' = x+\epsilon$ *and send* $\epsilon \to 0$ *after applying Werner trigonometric formula:*

$$w_{tx}(t,x) = w_{xt}(t,x) = \pm\lim_{\epsilon\to 0^+}\frac{2i\pi^2}{L^2}\left[\sum_{n=1}^{+\infty}n\sin(2k_n x)+\sum_{n=1}^{+\infty}n\sin(k_n\epsilon)\right]=0.$$

To get the final equality above, we used (C.9b). *We then see that the off-diagonal components* $w_{tx} = w_{xt}$ *vanish, but this is somehow expected since the infinite sum is multiplied by* i *and therefore it must vanish for the off-diagonal components of the energy-momentum tensor to be real.*

We are now left with computing w, *for which we just need to take the coincidence limit of* $W(t,x;t',x')$. *We will use* (C.11) *in what follows.*

$$w(t,x) = \lim_{x'\to x}\lim_{t'\to t}\left[\sum_{n=1}^{+\infty}\frac{4}{n}\sin(k_n x)\sin(k_n x')e^{-ik_n|t'-t|}+\log\left(\frac{(x-x')^2-(t-t')^2}{2}+i0^+\right)\right]$$
$$= \lim_{x'\to x}\left[\sum_{n=1}^{+\infty}\frac{4}{n}\sin(k_n x)\sin(k_n x')+\log\left(\frac{(x-x')^2}{2}\right)\right]$$
$$= \lim_{\epsilon\to 0^+}\left[\sum_{n=1}^{+\infty}\frac{2}{n}\cos\left(\epsilon\frac{\pi}{L}n\right)-\sum_{n=1}^{+\infty}\frac{2}{n}\cos\left(\frac{2\pi x}{L}n\right)+\log\left(\frac{\epsilon^2}{2}\right)\right]$$
$$= \lim_{\epsilon\to 0^+}\left[-2\log\left|2\sin\left(\frac{\epsilon\pi}{2L}\right)\right|+2\log\left|2\sin\left(\frac{\pi x}{L}\right)\right|+2\log\left|\frac{\epsilon}{\sqrt{2}}\right|\right]$$
$$= \lim_{\epsilon\to 0^+}\left[-2\log\left|\frac{\sin\left(\frac{\epsilon\pi}{2L}\right)}{\frac{\epsilon\pi}{2L}}\frac{\pi\sqrt{2}}{2L}\right|+2\log\left|2\sin\left(\frac{\pi x}{L}\right)\right|\right]$$
$$= 2\log\left|\frac{2L\sin\left(\frac{\pi x}{L}\right)}{\pi\sqrt{2}}\right| = 2\log\left|\frac{L\sqrt{2}}{\pi}\sin\left(\frac{\pi x}{L}\right)\right|,$$

where we used the standard limit $\lim_{\alpha\to 0}\frac{\sin\alpha}{\alpha}=1$, *and the fact that we can bring the limit inside the* log *due to regularity of* log *for positive argument.*

■

By plugging everything inside (2.36), we obtain

$$\langle\hat{T}_{tt}\rangle_{ren} = \langle\hat{T}_{xx}\rangle_{ren} = -\frac{\pi}{24L^2}, \tag{2.39}$$
$$\langle\hat{T}_{tx}\rangle_{ren} = \langle\hat{T}_{xt}\rangle_{ren} = 0. \tag{2.40}$$

Proof. *First, we compute the derivatives of* w*:*

$$\begin{aligned}
\partial_t w(x) &= \partial_t \partial_x w(x) = \partial_x \partial_t w(x) = \partial_t^2 w(x) = 0, \\
\partial_x w(x) &= \frac{2}{\frac{L\sqrt{2}}{\pi}\sin\left(\frac{\pi x}{L}\right)} \frac{L\sqrt{2}}{\pi} \cos\left(\frac{\pi x}{L}\right) \frac{\pi}{L} = \frac{2\pi}{L} \cot\left(\frac{\pi x}{L}\right), \\
\partial_x^2 w(x) &= \Box w(x) = \frac{2\pi^2}{L^2 \sin^2\left(\frac{\pi x}{L}\right)} = \frac{2\pi^2}{L^2} \csc^2\left(\frac{\pi x}{L}\right).
\end{aligned}$$

Now, we have

$$\begin{aligned}
\langle \hat{T}_{tt} \rangle_{ren} &= \frac{1}{4\pi}\left[-w_{tt} + \frac{1}{2}\partial_t^2 w - \frac{1}{4} g_{tt} \Box w\right] = \frac{1}{4\pi}\left[-w_{xx} + \frac{1}{4}\partial_x^2 w\right] \\
&= \frac{1}{4\pi}\left(-\frac{\pi^2}{6L^2}\left[1 - 3\csc^2\left(\frac{\pi x}{L}\right)\right] + \frac{1}{4}\frac{2\pi^2}{L^2}\csc^2\left(\frac{\pi x}{L}\right)\right) \\
&= -\frac{\pi}{24L^2} + \cancel{\frac{\pi}{8L^2}\csc^2\left(\frac{\pi x}{L}\right)} + \cancel{\frac{\pi}{8L^2}\csc^2\left(\frac{\pi x}{L}\right)} = -\frac{\pi}{24L^2},
\end{aligned}$$

$$\langle \hat{T}_{tx} \rangle_{ren} = \langle \hat{T}_{xt} \rangle_{ren} = \frac{1}{4\pi}\left[-w_{tx} + \frac{1}{2}\partial_t \partial_x w - \frac{1}{4} g_{tx} \Box w\right] = 0,$$

$$\begin{aligned}
\langle \hat{T}_{xx} \rangle_{ren} &= \frac{1}{4\pi}\left[-w_{xx} + \frac{1}{2}\partial_x^2 w - \frac{1}{4} g_{xx} \Box w\right] = \frac{1}{4\pi}\left[-w_{xx} + \frac{1}{2}\partial_x^2 w - \frac{1}{4}\partial_x^2 w\right] \\
&= \frac{1}{4\pi}\left[-w_{xx} + \frac{1}{4}\partial_x^2 w\right] = \langle \hat{T}_{tt} \rangle_{ren} = -\frac{\pi}{24L^2}.
\end{aligned}$$

■

The above are exactly the values we obtained with the simple renormalization method in the previous paragraph. This means that we correctly performed the Hadamard renormalization procedure, and we now gained confidence with a general tool for renormalizing the energy-momentum tensor.

We will see that things get much more complicated in scenarios that involve four spacetime dimensions, even with the highly symmetric background, and in those where there is an additional potential term in the equation of motion. In this simple case, we can notice that the Hadamard parametrix correctly captures the divergence of the propagator in the coincidence limit everywhere, but we will see, in the next chapter, that in the four dimensional theory the Hadamard parametrix does not hold near the boundary.

2.6 Casimir effect

The natural step, once the energy momentum tensor has been renormalized, is to talk about the Casimir effect. This phenomenon has been widely discussed in literature, and can be summarized as the force between two surfaces that is due to the presence of a field in vacuum state between them. In our case, the spacetime is flat, is two dimensional and the two "surfaces" are just the two points $x = 0$ and $x = L$. The attractive force between those two points can be computed from the Casimir energy, which is just the total energy present between the two points. We can easily write down the Casimir energy by integrating the energy density over the spacial domain:

$$E_C(L) = \int_0^L dx \rho(x) = \int_0^L dx \langle \hat{T}_{tt} \rangle_{ren} = -\frac{\pi}{24L}.$$

The Casimir force is given by the spacial gradient of the Casimir energy. In an inertial reference frame where $x = 0$ and $x = L$ points have constant velocity, each point will feel a force of

$$F = -\partial_L E_C(L) = -\partial_L \left(-\frac{\pi}{24L} \right) = -\frac{\pi}{24L^2},$$

which is attractive. We therefore conclude that in the presence of a massless real scalar field, two points where the field vanishes attract each other with a net force equivalent to F.

2.7 Further research directions

There are several natural ways to extend this simple model or make useful variations on it, by keeping in mind the primary objective of studying the collapse of a star or another compact object:

- add a constant mass term and compute the renormalized energy-momentum tensor;
- use a step potential instead of Dirichlet boundaries to model the compact object (in the homogeneous star model, the effective potential has a step discontinuity at the edge of the star);
- make the distance between the two Dirichlet boundaries vary with time and see what happens to the energy-momentum tensor.

Actually, the first two points are connected, in the sense that adding a step potential effectively means that the field has mass outside the compact object, and no mass inside. In the next chapter, we will study the four dimensional version of this problem and see that Dirichlet boundary can be problematic with Hadamard renormalization.

Chapter 3

Real massless scalar in a spherical cavity in 1+3 dimensional Minkowski spacetime

In order to proceed with our understanding of the role of boundaries in Hadamard regularization, it is useful to study models of increasing complexity. As a first simple model, we considered a 1+1 dimensional Minkowski quantum field theory in a spacial Dirichlet box. The next step is to take 3 spacial dimensions, and consider a spherical spacial Dirichlet boundary of radius R, while still retaining flat geometry. This is still too simple as a model of a star, but in this way we can investigate the role that dimensionality plays in the Hadamard renormalization procedure (however, in this case, we will see that a problem arises during the regularization step, that is the divergent terms do not match the universal Hadamard parametrix). A nice reference for this model is [15], where an explicit (integral) expression for the renormalized energy density is given. In this thesis, however, we want to present an alternative calculation, which makes use of the heat kernel method and fractional laplacian formalism. Other useful references are [5, 8], which treat the problem of Casimir energy in spherical geometry.

3.1 Classical theory

Consider a four-dimensional lorentzian metric manifold with signature $(-+++)$. We employ the usual time coordinate $t \in \mathbb{R}$, alongside with spacial spherical coordinates $r \in [0, R]$ with $R > 0$, $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi)$.

Metric

The metric is

$$g_{\mu\nu}(t, r, \theta, \varphi) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix},$$

with inverse given by

$$g^{\mu\nu}(t,r,\theta,\varphi) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}.$$

Action

Let us now write down the action of our theory. Since we are in a spherical cavity with radius R, the action is

$$S = \int_{\mathcal{M}} d^4x\sqrt{|g|}\mathcal{L} = \int_{-\infty}^{+\infty} dt \int_0^R dr \int_0^\pi r d\theta \int_0^{2\pi} r\sin\theta d\varphi \left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right), \tag{3.1}$$

with lagrangian density given by

$$\mathcal{L} = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \tag{3.2}$$

Equation of motion

The Euler-Lagrange equation yields the equation of motion for the field:

$$\boxed{\Box\phi = 0,} \tag{3.3}$$

Proof. Euler-Lagrange equation for scalar fields reads (see e.g. eq. 2.3 in [23])

$$\frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}.$$

We have

$$\frac{\partial\mathcal{L}}{\partial\phi} = 0$$
$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = -g^{\mu\nu}\partial_\nu\phi = -\partial^\mu\phi,$$

so

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = -\partial_\mu\partial^\mu\phi = -\Box\phi = 0.$$

■

In components, the Klein-Gordon equation above becomes:

$$-r^2\sin\theta\partial_t^2\phi + \sin\theta 2r\partial_r\phi + \sin\theta r^2\partial_r^2\phi + \cos\theta\partial_\theta\phi + \sin\theta\partial_\theta^2\phi + \frac{1}{\sin^2\theta}\partial_\varphi^2\phi = 0 \tag{3.4}$$

Boundary condition

When solving the equation of motion, we need to impose the Dirichlet boundary condition at $r = R$, that is $\forall t \in \mathbb{R}, \forall\theta \in [0,\pi], \forall\varphi \in [0,2\pi)$,

$$\boxed{\phi(t,R,\theta,\varphi) = 0.} \tag{3.5}$$

3.1.1 Normal modes

Since the equation of motion is linear in the field, we can look for a complete orthonormal basis of solutions which spans the whole linear space of the field's physical dynamical trajectories. To achieve this goal, we need to manipulate Eq. (3.4) and define an inner product.

Solving the equation of motion

First, we make the ansatz

$$\phi(t, r, \theta, \varphi) = A(t)B(r)C(\theta)D(\varphi).$$

Then, by substituting the above ansatz into (3.4) and dividing by $\phi = ABCD$ we get

$$-\frac{A''(t)}{A(t)} + \frac{B''(r)}{B(r)} + \frac{2}{r}\frac{B'(r)}{B(r)} + \frac{1}{r^2}\frac{C''(\theta)}{C(\theta)} + \frac{1}{r^2}\cot\theta\frac{C'(\theta)}{C(\theta)} + \frac{1}{r^2}\csc^2\theta\frac{D''(\varphi)}{D(\varphi)} = 0.$$

Now, multiply by r^2 and separate the variables (t, r) from (θ, φ):

$$r^2\left[-\frac{A''(t)}{A(t)} + \frac{B''(r)}{B(r)} + \frac{2}{r}\frac{B'(r)}{B(r)}\right] = -\left[\frac{C''(\theta)}{C(\theta)} + \cot\theta\frac{C'(\theta)}{C(\theta)} + \csc^2\theta\frac{D''(\varphi)}{D(\varphi)}\right].$$

Since the two sides depend on disjoint sets of variables, we can equate both of them to a constant simultaneously, which we call $\ell(\ell+1)$:

$$-\frac{A''(t)}{A(t)} + \frac{B''(r)}{B(r)} + \frac{2}{r}\frac{B'(r)}{B(r)} = \frac{\ell(\ell+1)}{r^2} \tag{3.6a}$$

$$\frac{C''(\theta)}{C(\theta)} + \cot\theta\frac{C'(\theta)}{C(\theta)} + \csc^2\theta\frac{D''(\varphi)}{D(\varphi)} = -\ell(\ell+1) \tag{3.6b}$$

Eq. (3.6b) above is just the spherical harmonics equation, and the general solution is a linear combination of spherical harmonics, which are discussed in Appendix A.3.

Consider equation (3.6a), containing temporal and radial components. We can again separate variables and call the separation variable $-\omega^2 = -k^2$ (we will use both interchangeably):

$$\begin{cases} \dfrac{A''(t)}{A(t)} = -\omega^2 \\ \dfrac{B''(r)}{B(r)} + \dfrac{2}{r}\dfrac{B'(r)}{B(r)} - \dfrac{\ell(\ell+1)}{r^2} = -\omega^2 \end{cases}.$$

We can solve them independently and get the general solutions

$$A(t) = a_1 e^{i\omega t} + a_2 e^{-i\omega t}, \tag{3.7}$$

$$B(r) = b_1 j_\ell(\omega r) + b_2 y_\ell(\omega r), \tag{3.8}$$

where j_ℓ and y_ℓ are the spherical Bessel functions of first and second kind, respectively (see Appendix B). Since $y_\ell(\omega r)$ diverges for $r = 0$, we manually set $b_2 = 0$ to get a regular solution at $r = 0$.

Recall that we need to impose the Dirichlet boundary condition on the field at $r = R$, meaning that $B(R) = 0$. This means that we want

$$b_1 j_\ell(\omega R) = 0,$$

which implies that ω is quantized

$$\boxed{\omega_{\ell n} = \frac{\alpha_{\ell n}}{R},} \tag{3.9}$$

where $\alpha_{\ell n}$ is the n^{th} zero of the spherical Bessel function j_ℓ. This, in turn, means that also the frequencies of the temporal harmonic oscillator become quantized.

There will then be as many modes as the possible values that the triplet of quantum numbers (n, ℓ, m) can take. We know from the angular part that $\ell \in \{0, 1, 2, \dots\}$ and $m \in \{-\ell, \dots, \ell\}$. Furthermore, the zeros of spherical Bessel functions of integer order are countably infinite (and unbounded), i.e. $n \in \{1, 2, 3, \dots\}$. Our modes will look like

$$\begin{aligned}\tilde{u}_{n\ell m}(t, r, \theta, \varphi) &= e^{-i\omega_{\ell n} t} j_\ell(\omega_{\ell n} r) Y_{\ell m}(\theta, \varphi)\\ \tilde{u}^*_{n\ell m}(t, r, \theta, \varphi) &= e^{i\omega_{\ell n} t} j_\ell(\omega_{\ell n} r) Y^*_{\ell m}(\theta, \varphi).\end{aligned}$$

Klein-Gordon inner product

In a scalar field theory, one can introduce the Klein-Gordon (non positive-definite) inner product. Given two solutions ϕ_1 and ϕ_2 and a spacelike hypersurface Σ_t with constant t, we define their inner product as (see e.g. Eq. 2.9 in [2])

$$\langle \phi_1, \phi_2 \rangle = -i \int_{\Sigma_t} \sqrt{|g|} (\phi_1 \partial_0 \phi_2^* - \phi_2^* \partial_0 \phi_1) d^3x, \tag{3.10}$$

and it does not depend on the value of t defining the hypersurface Σ_t.

Proof. First, let us prove that the properties of indefinite inner products are satisfied. We need to show conjugate symmetry and linearity in the first argument. Indeed, linearity is proven by

$$\begin{aligned}\langle a\phi_a + b\phi_b, \phi_2 \rangle &= -i \int_{\Sigma_t} \sqrt{|g|} [a(\phi_a \partial_0 \phi_2^* - \phi_2^* \partial_0 \phi_a) + b(\phi_b \partial_0 \phi_2^* - \phi_2^* \partial_0 \phi_b)] d^3x\\ &= a\langle \phi_a, \phi_2 \rangle + b\langle \phi_b, \phi_2 \rangle,\end{aligned}$$

while conjugate symmetry follows from

$$\begin{aligned}\langle \phi_1, \phi_2 \rangle^* &= i \int_{\Sigma_t} \sqrt{|g|} (\phi_1^* \partial_0 \phi_2 - \phi_2 \partial_0 \phi_1^*) d^3x\\ &= -i \int_{\Sigma_t} \sqrt{|g|} (\phi_2 \partial_0 \phi_1^* - \phi_1^* \partial_0 \phi_2) d^3x = \langle \phi_2, \phi_1 \rangle.\end{aligned}$$

We now need to prove independence from t. Let us define the quantity

$$J_\mu(\phi_1, \phi_2) = -i(\phi_1 \partial_\mu \phi_2^* - \phi_2^* \partial_\mu \phi_1). \tag{3.11}$$

By using the Leibniz rule, we can compute the four-divergence of J^μ (changing the sign to the temporal component since we are raising the index with signature $-+++$):

$$\begin{aligned}\partial_\mu J^\mu(\phi_1, \phi_2) = -i(&\cancel{-\partial_0\phi_1\partial_0\phi_2^*} - \phi_1\partial_0^2\phi_2^* \cancel{+\partial_0\phi_2^*\partial_0\phi_1} + \phi_2^*\partial_0^2\phi_1\\ &\cancel{+\partial_1\phi_1\partial_1\phi_2^*} + \phi_1\partial_1^2\phi_2^* \cancel{-\partial_1\phi_2^*\partial_1\phi_1} - \phi_2^*\partial_1^2\phi_1\\ &\cancel{+\partial_2\phi_1\partial_2\phi_2^*} + \phi_1\partial_2^2\phi_2^* \cancel{-\partial_2\phi_2^*\partial_2\phi_1} - \phi_2^*\partial_2^2\phi_1\\ &\cancel{+\partial_3\phi_1\partial_3\phi_2^*} + \phi_1\partial_3^2\phi_2^* \cancel{-\partial_3\phi_2^*\partial_3\phi_1} - \phi_2^*\partial_3^2\phi_1),\end{aligned}$$

which turns out to be vanishing:

$$\partial_\mu J^\mu(\phi_1, \phi_2) = -i(\phi_1 \Box \phi_2^* - \phi_2^* \Box \phi_1) = 0,$$

since both ϕ_1 and ϕ_2^* solve the Klein-Gordon equation $\Box\phi = 0$. This means by definition that J^μ is a conserved quantity. Now let us define the following hypersurfaces using spacial spherical coordinates and $t_1 < t_2$:

$$\begin{aligned}
\Sigma_{t_1,R} &= \{(t_1, r, \theta, \varphi) | 0 \le r \le R, 0 \le \theta \le \pi, 0 \le \varphi \le 2\pi\}; \\
\Sigma_{t_2,R} &= \{(t_2, r, \theta, \varphi) | 0 \le r \le R, 0 \le \theta \le \pi, 0 \le \varphi \le 2\pi\}; \\
V_R &= \{(t, R, \theta, \varphi) | t \in [t_1, t_2], 0 \le \theta \le \pi, 0 \le \varphi \le 2\pi\}; \\
\partial U_R &= \Sigma_{t_1,R} \cup \Sigma_{t_2,R} \cup V_R.
\end{aligned}$$

We notice that ∂U is a closed hypersurface which is the boundary of a hypercylinder from t_1 to t_2. We can use Stokes' theorem (see e.g. Appendix E of Carroll's book [12]) which states that, being n^μ the unit vector normal to the hypersurface at each point,

$$\oint_{\partial U} J_\mu n^\mu d^3x = \int_U \partial_\mu J^\mu d^4x = 0,$$

where the last equality follows from J^μ being a conserved quantity. The left-hand side can be written as (being $n^\mu = (1, 0, 0, 0)$ and m^μ orthogonal to V_R)

$$\left(\int_{\Sigma_{t_1,R}} - \int_{\Sigma_{t_2,R}} \right) J_\mu n^\mu d^3x + \int_{V_R} J_\mu m^\mu d^3x = 0,$$

but the integral over V_R vanishes because the fields ϕ_1 and ϕ_2^* vanish at $r = R$ due to the Dirichlet boundary condition. By the fact that $J_\mu n^\mu = J_0$ in the coordinates we have chosen, and since it is a scalar quantity, we have proven the following equality:

$$\int_{\Sigma_{t_1,R}} J_0 d^3x = \int_{\Sigma_{t_2,R}} J_0 d^3x,$$

and this equivalence holds for any values of t_1 and t_2.

■

To be more explicit, the Klein-Gordon inner product in our case looks like

$$\boxed{\langle \phi_1, \phi_2 \rangle = -i \int_0^R dr \int_0^\pi r d\theta \int_0^{2\pi} r \sin\theta d\varphi \left(\phi_1 \partial_t \phi_2^* - \phi_2^* \partial_t \phi_1 \right).} \tag{3.12}$$

Orthonormalization

Our aim is to compute the norm of $\tilde{u}_{n\ell m}$ and $\tilde{u}^*_{n\ell m}$ to normalize the modes, and also to check that distinct modes are orthogonal. We have the following relations:

$$\langle \tilde{u}_{n\ell m}, \tilde{u}_{n'\ell'm'} \rangle = \alpha_{\ell n} R^2 j^2_{\ell\pm1}(\alpha_{\ell n}) \delta_{nn'} \delta_{\ell\ell'} \delta_{mm'} \tag{3.13a}$$

$$\langle \tilde{u}^*_{n\ell m}, \tilde{u}^*_{n'\ell'm'} \rangle = -\alpha_{\ell n} R^2 j^2_{\ell\pm1}(\alpha_{\ell n}) \delta_{nn'} \delta_{\ell\ell'} \delta_{mm'} \tag{3.13b}$$

$$\langle \tilde{u}_{n\ell m}, \tilde{u}^*_{n'\ell'm'} \rangle = 0 \tag{3.13c}$$

Proof. *First, we note that*

$$\partial_t \tilde{u}_{n\ell m}(t,r,\theta,\varphi) = -i\omega_{\ell n}\tilde{u}_{n\ell m}(t,r,\theta,\varphi),$$
$$\partial_t \tilde{u}^*_{n\ell m}(t,r,\theta,\varphi) = +i\omega_{\ell n}\tilde{u}^*_{n\ell m}(t,r,\theta,\varphi).$$

Using the Klein-Gordon inner product (3.12)*, we have, for the first relation,*

$$\begin{aligned}
\langle \tilde{u}_{n\ell m}, \tilde{u}_{n'\ell' m'}\rangle &= -i\int_0^R dr \int_0^\pi r d\theta \int_0^{2\pi} r\sin\theta d\varphi (\tilde{u}_{n\ell m}\partial_t \tilde{u}^*_{n'\ell' m'} - \tilde{u}^*_{n'\ell' m'}\partial_t \tilde{u}_{n\ell m}) \\
&= -i\int_0^R dr \int_0^\pi r d\theta \int_0^{2\pi} r\sin\theta d\varphi (i\omega_{\ell' n'} + i\omega_{\ell n}) j_\ell(\omega_{\ell n} r) j_{\ell'}(\omega_{\ell' n'} r)\cdot \\
&\qquad \cdot Y_{\ell m}(\theta,\varphi) Y^*_{\ell' m'}(\theta,\varphi) e^{-i(\omega_{\ell n}-\omega_{\ell' n'})t} \\
&= (\omega_{\ell n} + \omega_{\ell' n'}) e^{-i(\omega_{\ell n}-\omega_{\ell' n'})t} \left[\int_0^R r^2 dr j_\ell(\omega_{\ell n} r) j_{\ell'}(\omega_{\ell' n'} r)\cdot \right. \\
&\qquad \left. \cdot \int_0^\pi d\theta \int_0^{2\pi} \sin\theta d\varphi Y_{\ell m}(\theta,\varphi) Y^*_{\ell' m'}(\theta,\varphi)\right] \\
&\equiv (\omega_{\ell n} + \omega_{\ell' n'}) e^{-i(\omega_{\ell n}-\omega_{\ell' n'})t} N,
\end{aligned}$$

where we are left with evaluating the integral

$$N = \int_0^R r^2 dr j_\ell(\omega_{\ell n} r) j_{\ell'}(\omega_{\ell' n'} r) \int_0^\pi d\theta \int_0^{2\pi} \sin\theta d\varphi Y_{\ell m}(\theta,\varphi) Y^*_{\ell' m'}(\theta,\varphi).$$

Similarly, for the second relation, we have

$$\begin{aligned}
\langle \tilde{u}^*_{n\ell m}, \tilde{u}^*_{n'\ell' m'}\rangle &= -i\int_0^R dr \int_0^\pi r d\theta \int_0^{2\pi} r\sin\theta d\varphi (\tilde{u}^*_{n\ell m}\partial_t \tilde{u}_{n'\ell' m'} - \tilde{u}_{n'\ell' m'}\partial_t \tilde{u}^*_{n\ell m}) \\
&= -i\int_0^R dr \int_0^\pi r d\theta \int_0^{2\pi} r\sin\theta d\varphi (-i\omega_{\ell' n'} - i\omega_{\ell n}) j_\ell(\omega_{\ell n} r) j_{\ell'}(\omega_{\ell' n'} r)\cdot \\
&\qquad \cdot Y^*_{\ell m}(\theta,\varphi) Y_{\ell' m'}(\theta,\varphi) e^{+i(\omega_{\ell n}-\omega_{\ell' n'})t} \\
&= -(\omega_{\ell n} + \omega_{\ell' n'}) e^{+i(\omega_{\ell n}-\omega_{\ell' n'})t} N.
\end{aligned}$$

Let us now evaluate N*:*

$$\begin{aligned}
N &= \left[\int_0^R r^2 dr j_\ell(\omega_{\ell n} r) j_{\ell'}(\omega_{\ell' n'} r)\right]\cdot\left[\int_0^\pi d\theta \int_0^{2\pi} \sin\theta d\varphi Y_{\ell m}(\theta,\varphi) Y^*_{\ell' m'}(\theta,\varphi)\right] \\
&= \int_0^R r^2 dr j_\ell(\omega_{\ell n} r) j_\ell(\omega_{\ell n'} r)\delta_{\ell\ell'}\delta_{mm'} \\
&= \frac{R^3}{2} j^2_{\ell\pm 1}(\alpha_{\ell n})\delta_{nn'}\delta_{\ell\ell'}\delta_{mm'},
\end{aligned}$$

where we have used the orthonormality of the spherical harmonics (A.13) *and the orthogonality of spherical Bessel functions in finite domain* (B.6) *in succession. Therefore, we have*

$$\langle \tilde{u}_{n\ell m}, \tilde{u}_{n'\ell' m'}\rangle = \omega_{\ell n} R^3 j^2_{\ell\pm 1}(\alpha_{\ell n})\delta_{nn'}\delta_{\ell\ell'}\delta_{mm'} = \alpha_{\ell n} R^2 j^2_{\ell\pm 1}(\alpha_{\ell n})\delta_{nn'}\delta_{\ell\ell'}\delta_{mm'},$$
$$\langle \tilde{u}^*_{n\ell m}, \tilde{u}^*_{n'\ell' m'}\rangle = -\omega_{\ell n} R^3 j^2_{\ell\pm 1}(\alpha_{\ell n})\delta_{nn'}\delta_{\ell\ell'}\delta_{mm'} = -\alpha_{\ell n} R^2 j^2_{\ell\pm 1}(\alpha_{\ell n})\delta_{nn'}\delta_{\ell\ell'}\delta_{mm'},$$

since the factors of 2 *simplify, and* $\omega_{\ell n} = \omega_{\ell' n'} = \frac{\alpha_{\ell n}}{R}$.

For the third relation, we have

$$\begin{aligned}
\langle \tilde{u}_{n\ell m}, \tilde{u}^*_{n'\ell' m'} \rangle &= -i \int_0^R dr \int_0^\pi r d\theta \int_0^{2\pi} r \sin\theta d\varphi (\tilde{u}_{n\ell m} \partial_t \tilde{u}_{n'\ell' m'} - \tilde{u}_{n'\ell' m'} \partial_t \tilde{u}_{n\ell m}) \\
&= -i \int_0^R dr \int_0^\pi r d\theta \int_0^{2\pi} r \sin\theta d\varphi (-i\omega_{\ell' n'} + i\omega_{\ell n}) j_\ell(\omega_{\ell n} r) j_{\ell'}(\omega_{\ell' n'} r) \cdot \\
&\qquad\qquad \cdot Y_{\ell m}(\theta,\varphi) Y_{\ell' m'}(\theta,\varphi) e^{-i(\omega_{\ell n} + \omega_{\ell' n'})t} \\
&= (\omega_{\ell n} - \omega_{\ell' n'}) e^{-i(\omega_{\ell n} + \omega_{\ell' n'})t} \int_0^R r^2 dr j_\ell(\omega_{\ell n} r) j_{\ell'}(\omega_{\ell' n'} r) \int_0^{2\pi} d\varphi e^{i(m+m')\varphi} \cdot \\
&\quad \cdot \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} \sqrt{\frac{2\ell'+1}{4\pi} \frac{(\ell'-m')!}{(\ell'+m')!}} \int_0^\pi d\theta \sin\theta P_{\ell m}(\cos\theta) P_{\ell' m'}(\cos\theta),
\end{aligned}$$

where we used the definition of spherical harmonics in terms of associated Legendre polynomials (A.9)*. The integral in* φ *is just a Kronecker delta* $2\pi\delta_{m,-m'}$*, and we can set* $m' = -m$ *everywhere, including*

$$P_{\ell' m'}(\cos\theta) = P_{\ell',-m}(\cos\theta) = (-1)^m \frac{(\ell'-m)!}{(\ell'+m)!} P_{\ell' m}(\cos\theta),$$

where we used (A.8)*. The relation becomes*

$$\begin{aligned}
\langle \tilde{u}_{n\ell m}, \tilde{u}^*_{n'\ell' m'} \rangle &= (\omega_{\ell n} - \omega_{\ell' n'}) e^{-i(\omega_{\ell n} + \omega_{\ell' n'})t} \left[\int_0^R r^2 dr j_\ell(\omega_{\ell n} r) j_{\ell'}(\omega_{\ell' n'} r) \right] 2\pi \delta_{mm'} \cdot \\
&\quad \cdot (-1)^m \frac{(\ell'-m)!}{(\ell'+m)!} \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} \sqrt{\frac{2\ell'+1}{4\pi} \frac{(\ell'-m)!}{(\ell'+m)!}} \cdot \\
&\quad \cdot \int_0^\pi d\theta \sin\theta P_{\ell m}(\cos\theta) P_{\ell' m}(\cos\theta) \\
&= (\omega_{\ell n} - \omega_{\ell n'}) e^{-i(\omega_{\ell n} + \omega_{\ell n'})t} \left[\int_0^R r^2 dr j_\ell(\omega_{\ell n} r) j_\ell(\omega_{\ell n'} r) \right] \cancel{2\pi} \delta_{m,-m'} \cdot \\
&\quad \cdot (-1)^m \frac{(\ell-m)!}{(\ell+m)!} \frac{\cancel{2\ell+1}}{\cancel{4\pi}} \frac{\cancel{(\ell-m)!}}{\cancel{(\ell+m)!}} \frac{\cancel{2}}{\cancel{2\ell+1}} \frac{\cancel{(\ell+m)!}}{\cancel{(\ell-m)!}} \delta_{\ell\ell'} \\
&= (-1)^m (\omega_{\ell n} - \omega_{\ell n'}) e^{-i(\omega_{\ell n} + \omega_{\ell n'})t} \left[\int_0^R r^2 dr j_\ell(\omega_{\ell n} r) j_\ell(\omega_{\ell n'} r) \right] \frac{(\ell-m)!}{(\ell+m)!} \delta_{\ell\ell'} \delta_{m,-m'} \\
&= (-1)^m (\cancel{\omega_{\ell n}} - \cancel{\omega_{\ell n}}) e^{-2i\omega_{\ell n} t} \frac{R^3}{2} j^2_{\ell\pm1}(\alpha_{\ell n}) \frac{(\ell-m)!}{(\ell+m)!} \delta_{nn'} \delta_{\ell\ell'} \delta_{m,-m'} \\
&= 0,
\end{aligned}$$

where we used the orthogonality relation of associated Legendre polynomials (A.7) *and also the orthogonality relation of spherical Bessel functions in a finite domain* (B.6)*.*

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The orthonormal modes are then given by dividing $\tilde{u}$ and $\tilde{u}^*$ by the norm, which is the square root of the prefactor of the Kronecker deltas in (3.13a), and also in (3.13b) with a negative sign:

$$\sqrt{\alpha_{\ell n}} R j_{\ell\pm1}(\alpha_{\ell n}) = \sqrt{\frac{\pi}{2}} R J_{\ell+\frac{1}{2}\pm1}(\alpha_{\ell n}) \equiv \sqrt{\frac{\pi}{2}} R B_{\ell n},$$

where we used the identity (B.3) and defined

$$\boxed{B_{\ell n} = J_{\ell+\frac{1}{2}\pm1}(\alpha_{\ell n}).} \tag{3.14}$$

Orthonormal modes

We then get the following orthonormal modes, whose radial components (denoted by $v_{\ell n}(r)$ such that $u_{n\ell m}(t,r,\theta,\varphi) = e^{-i\omega_{n\ell}t}v_{\ell n}(r)Y_\ell^m(\theta,\varphi)$) are plotted in Figure **??**:

$$\boxed{\begin{aligned} u_{n\ell m}(t,r,\theta,\varphi) &= \sqrt{\frac{2}{\pi}}\frac{1}{RB_{\ell n}}j_\ell(\omega_{n\ell}r)Y_\ell^m(\theta,\varphi)e^{-i\omega_{n\ell}t}, \quad (3.15a)\\ u^*_{n\ell m}(t,r,\theta,\varphi) &= \sqrt{\frac{2}{\pi}}\frac{1}{RB_{\ell n}}j_\ell(\omega_{n\ell}r)Y_\ell^{m*}(\theta,\varphi)e^{i\omega_{n\ell}t}. \quad (3.15b)\end{aligned}}$$

The normal modes satisfy the orthonormality relations:

$$\langle u_{n\ell m}, u_{n'\ell'm'}\rangle = \delta_{nn'}\delta_{\ell\ell'}\delta_{mm'} \tag{3.16}$$
$$\langle u^*_{n\ell m}, u^*_{n'\ell'm'}\rangle = -\delta_{nn'}\delta_{\ell\ell'}\delta_{mm'} \tag{3.17}$$
$$\langle u_{n\ell m}, u^*_{n'\ell'm'}\rangle = 0. \tag{3.18}$$

Frequency

The orthonormal modes (3.15) are eigenfunctions of the frequency operator $i\partial_t$, with definite positive or negative frequency eigenvalue:

$$i\partial_t u_{n\ell m} = \omega_{n\ell}u_{n\ell m}, \tag{3.19}$$
$$i\partial_t u^*_{n\ell m} = -\omega_{n\ell}u_{n\ell m}. \tag{3.20}$$

Therefore, we have that

- $u_{n\ell m}(t,r,\theta,\phi)$ is **positive norm** and **positive frequency**;
- $u^*_{n\ell m}(t,r,\theta,\phi)$ is **negative norm** and **negative frequency**.

The plots of the frequencies and the spacing between them are reported in Figure 3.2

Classical field general solution

The general solution to the Klein-Gordon equation (3.3) is a linear combination of the modes (3.15) with complex coefficients $a^-_{n\ell m}, a^+_{n\ell m} \in \mathbb{C}$:

$$\boxed{\phi(t,r,\theta,\varphi) = \sum_{n=1}^{+\infty}\sum_{\ell=0}^{+\infty}\sum_{m=-\ell}^{\ell}\left[a^-_{n\ell m}u_{n\ell m}(t,r,\theta,\varphi) + a^+_{n\ell m}u^*_{n\ell m}(t,r,\theta,\varphi)\right],} \tag{3.21}$$

where $(a^-_{n\ell m})^* = a^+_{n\ell m}$ to make the field real-valued.

Classical canonical momentum field

The canonical momentum field is given by

$$\Pi(t,r,\theta,\varphi) = \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}, \tag{3.22}$$

so that we have the standard equal-time Poisson bracket:

$$\{\phi(t,r,\theta,\varphi), \Pi(t,r',\theta',\varphi')\} = \frac{1}{\sqrt{|g|}}\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi'). \tag{3.23}$$

We are now ready for quantizing the field.

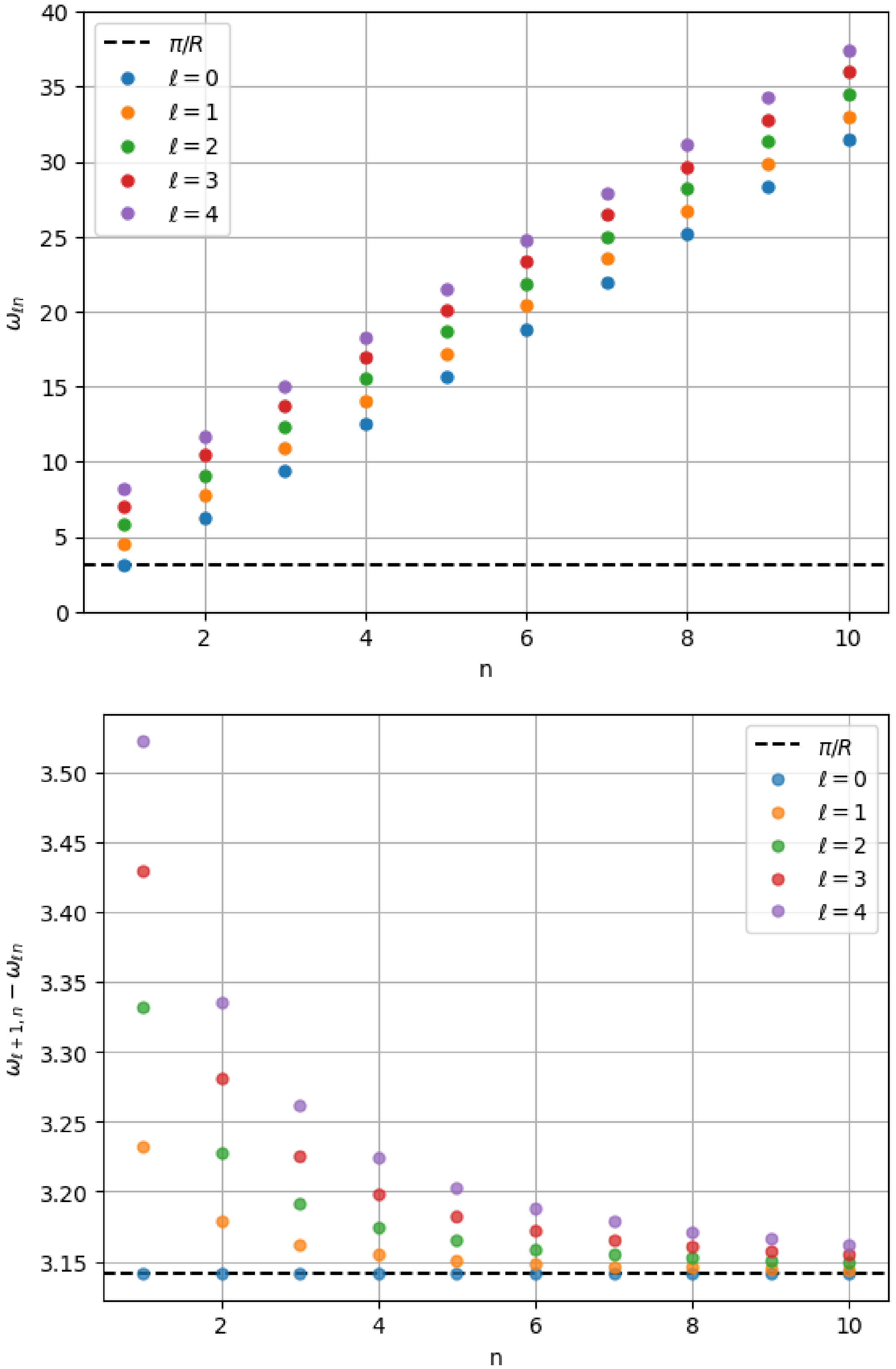

Figure 3.1: Plot of the frequency (energy) eigenvalues (above), and spacing between them (below) for $\ell = 0, 1, 2, 3, 4$ and $R = 1.0$.

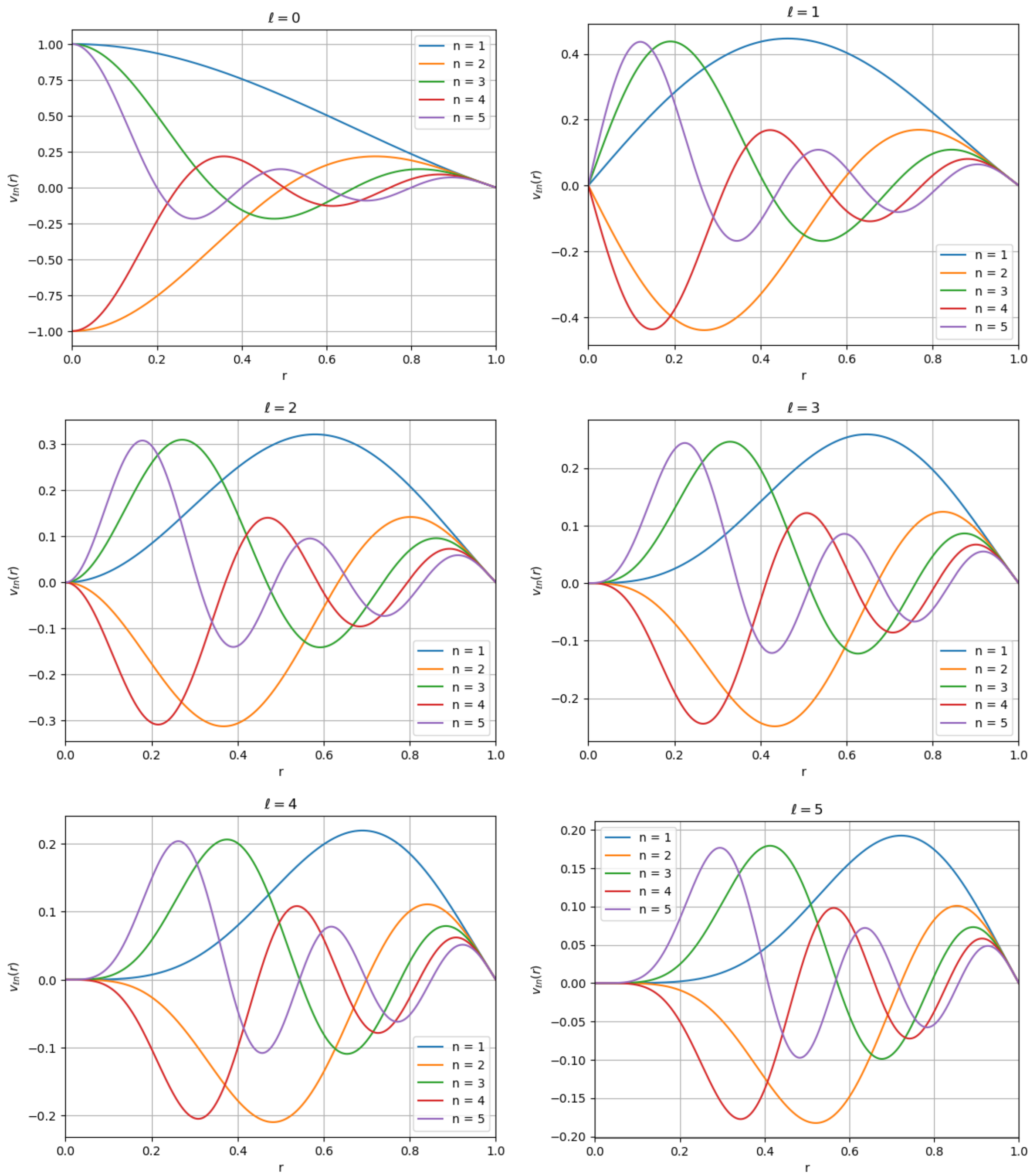

Figure 3.2: Plot of the radial part $v_{\ell n}(r)$ of the normal modes $u_{n\ell m}(t, r, \theta, \varphi) = e^{-i\omega_{n\ell}t} v_{\ell n}(r) Y_\ell^m(\theta, \varphi)$ for $\ell = 0, 1, 2, 3, 4, 5$ and $R = 1.0$.

3.2 Quantization

We promote the mode coefficients in the general solution to creation and annihilation operator pairs, and write down the quantized field and canonical momentum.

$$\boxed{\begin{aligned}\hat{\phi}(t,r,\theta,\varphi) &= \sum_{n=1}^{+\infty}\sum_{\ell=0}^{+\infty}\sum_{m=-\ell}^{+\ell}\left[\hat{a}^-_{n\ell m}u_{n\ell m}(t,r,\theta,\varphi)+\hat{a}^+_{n\ell m}u^*_{n\ell m}(t,r,\theta,\varphi)\right] \\ \hat{\Pi}(t,r,\theta,\varphi) &= \sum_{n=1}^{+\infty}\sum_{\ell=0}^{+\infty}\sum_{m=-\ell}^{+\ell}\left[\hat{a}^-_{n\ell m}\partial_t u_{n\ell m}(t,r,\theta,\varphi)+\hat{a}^+_{n\ell m}\partial_t u^*_{n\ell m}(t,r,\theta,\varphi)\right]\end{aligned}} \quad \begin{aligned}(3.24a)\\(3.24b)\end{aligned}$$

By imposing the standard algebra for creation-annihilation operator pairs, we get the equal time canonical commutation relations with the right prefactors:

$$\begin{aligned}\left[\hat{\phi}(t,r,\theta,\varphi),\hat{\phi}(t,r',\theta',\varphi')\right] &= 0 \\ \left[\hat{\Pi}(t,r,\theta,\varphi),\hat{\Pi}(t,r',\theta',\varphi')\right] &= 0 \\ \left[\hat{\phi}(t,r,\theta,\varphi),\hat{\Pi}(t,r',\theta',\varphi')\right] &= \frac{i}{\sqrt{|g|}}\delta(r-r')\delta(\theta-\theta')\delta(\varphi-\varphi')\end{aligned}$$

3.3 Feynman propagator

In order to study the energy-momentum tensor, it is useful to analyze the propagator, in particular its behaviour near coincidence limit. We define the Feynman propagator as in eq. 21 of Decanini-Folacci [17]:

$$\begin{aligned}G_F(x_1,x_2) &= i\langle 0|\,T\hat{\phi}(x_1)\hat{\phi}(x_2)\,|0\rangle \\ &= i\left[\Theta(t_1-t_2)G_+(x_1,x_2)+\Theta(t_2-t_1)G_-(x_2,x_1)\right],\end{aligned}$$

where

$$\begin{aligned}G_+(x_1,x_2) &= \langle 0|\,\hat{\phi}(x_1)\hat{\phi}(x_2)\,|0\rangle \\ G_-(x_1,x_2) &= \langle 0|\,\hat{\phi}(x_2)\hat{\phi}(x_1)\,|0\rangle\,,\end{aligned}$$

and $x_i = (t_i, r_i, \theta_i, \varphi_i)$. By plugging (3.24a) inside the above expressions we get (after some algebra, done with Wolfram Mathematica)

$$\begin{aligned}G_+(x_1,x_2) &= \frac{2}{\pi R^2}\sum_{n=1}^{+\infty}\sum_{\ell=0}^{+\infty}\sum_{m=-\ell}^{+\ell}\frac{1}{B^2_{\ell n}}e^{-i\omega_{\ell n}(t_1-t_2)}j_\ell(\omega_{\ell n}r_1)j_\ell(\omega_{\ell n}r_2)Y^m_\ell(\theta_1,\varphi_1)Y^{m*}_\ell(\theta_2,\varphi_2) \\ G_-(x_1,x_2) &= \frac{2}{\pi R^2}\sum_{n=1}^{+\infty}\sum_{\ell=0}^{+\infty}\sum_{m=-\ell}^{+\ell}\frac{1}{B^2_{\ell n}}e^{-i\omega_{\ell n}(t_2-t_1)}j_\ell(\omega_{\ell n}r_1)j_\ell(\omega_{\ell n}r_2)Y^m_\ell(\theta_1,\varphi_1)Y^{m*}_\ell(\theta_2,\varphi_2),\end{aligned}$$

which differ just by the sign of the time oscillating exponential. Therefore, the time ordering is achieved just by using an absolute value, and the Feynman propagator will be

$$G_F(x_1,x_2) = \frac{2i}{\pi R^2}\sum_{n=1}^{+\infty}\sum_{\ell=0}^{+\infty}\sum_{m=-\ell}^{\ell}\frac{1}{B^2_{\ell n}}e^{-i\omega_{\ell n}|t_1-t_2|}j_\ell(\omega_{\ell n}r_1)j_\ell(\omega_{\ell n}r_2)Y^m_\ell(\theta_1,\varphi_1)Y^{m*}_\ell(\theta_2,\varphi_2). \tag{3.25}$$

By using the spherical harmonic addition theorem (A.14), we obtain (with γ defined in (A.15))

$$G_F(x_1, x_2) = \frac{i}{2\pi^2 R^2} \sum_{n=1}^{+\infty} \sum_{\ell=0}^{+\infty} \frac{2\ell+1}{B_{\ell n}^2} e^{-i\omega_{\ell n}|t_1-t_2|} j_\ell(\omega_{\ell n} r_1) j_\ell(\omega_{\ell n} r_2) P_\ell(\cos\gamma). \tag{3.26}$$

Coincidence limit

Since we are interested in the behaviour near coincidence $x_2 \to x_1$, we now choose a purely timelike spacetime path to perform the limit (we are assuming that the finite result after renormalization does not depend on the renormalization scheme that is chosen, and therefore we can freely choose the spacetime path along which to perform the limit):

$$\begin{aligned} &|t_1 - t_2| = \epsilon > 0 \\ &r_2 = r_1 \equiv r \\ &\theta_2 = \theta_1 \equiv \theta \\ &\varphi_2 = \varphi_1 \equiv \varphi, \end{aligned}$$

so we are interested in the limit $\epsilon \to 0^+$.

The Feynman propagator in this spacetime path simplifies to

$$G_F(x_1, x_2) = \frac{i}{2\pi^2 R^2} \sum_{n=1}^{+\infty} \sum_{\ell=0}^{+\infty} \frac{2\ell+1}{B_{\ell n}^2} e^{-i\omega_{\ell n}\epsilon} j_\ell^2(\omega_{\ell n} r), \tag{3.27}$$

since $P_\ell(1) = 1$ as stated in (A.1), which ensures spherical symmetry.

Wick rotation

In order to work with this expression, it is useful to perform a Wick rotation, which amounts to substituting

$$\epsilon \to -i\beta. \tag{3.28}$$

PROOF. *The path integral is*

$$Z[0] = \int \mathcal{D}\phi e^{iS[\phi]},$$

with action defined as in (3.1) *(and lagrangian given by* (3.2)*):*

$$S[\phi] = \int d^4x \sqrt{|g|}\mathcal{L} = \int dt \int d^3x \sqrt{|g_{(3)}|} \left[\frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\vec{\nabla}\phi)^2\right].$$

The Wick rotation is so that $\partial_t = -i\partial_\beta$, and $dt = -i\beta$, so we have

$$iS[\phi] \to i\int (-id\beta) \left[-\frac{1}{2}(\partial_\beta\phi)^2 - \frac{1}{2}(\vec{\nabla}\phi)^2\right] = -S_E[\phi],$$

and the euclidean action $S_E[\phi]$ is manifestly positive definite, so that the euclidean path integral is convergent:

$$Z_E[0] = \int \mathcal{D}\phi e^{-S_E[\phi]}.$$

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In addition, we have

$$G_E(x_1, x_2) = -iG_F(x_1^E, x_2^E),$$

where $x^E = (\beta, \vec{x})$. We then obtain the euclidean propagator

$$\boxed{G_E(x_1, x_2) = \frac{1}{2\pi^2 R^2} \sum_{n=1}^{+\infty} \sum_{\ell=0}^{+\infty} \frac{2\ell+1}{J^2_{\ell-\frac{1}{2}}(\alpha_{\ell n})} e^{-\beta \frac{\alpha_{\ell n}}{R}} j_\ell^2 \left(\frac{\alpha_{\ell n}}{R} r \right).} \tag{3.29}$$

It is not clear if the above expression can be put in a closed form, but for our purposes we can focus on the behaviour of the propagator in two particular cases:

- near the boundary $r \simeq R^-$
- at the center $r = 0$

3.3.1 Near-boundary behaviour

To analyze the behaviour of the propagator near the boundary $r \simeq R^-$, it is useful to set

$$\begin{aligned} x &\equiv \frac{r}{R} \simeq 1^- \\ y &\equiv 1 - x = 1 - \frac{r}{R} \simeq 0^+, \end{aligned}$$

so that $r/R = 1 - y$ and $y \simeq 0^+$. By substituting in the argument of the spherical Bessel function in the propagator (3.29), and Taylor expanding, we obtain

$$\begin{aligned} j_\ell \left(\alpha_{\ell n} \frac{r}{R} \right) &= j_\ell (\alpha_{\ell n} x) = \sqrt{\frac{\pi}{2\alpha_{\ell n} x}} J_{\ell+\frac{1}{2}} (\alpha_{\ell n}(1-y)) = \sqrt{\frac{\pi}{2\alpha_{\ell n} x}} J_{\ell+\frac{1}{2}} (\alpha_{\ell n} - y\alpha_{\ell n}) \\ &= \sqrt{\frac{\pi}{2\alpha_{\ell n} x}} \left[\cancel{J_{\ell+\frac{1}{2}}(\alpha_{\ell n})} - y\alpha_{\ell n} J'_{\ell+\frac{1}{2}}(\alpha_{\ell n}) + \mathcal{O}(y^2) \right] \\ &= -(1-x) \sqrt{\frac{\pi}{2\alpha_{\ell n} x}} \alpha_{\ell n} J_{\ell-\frac{1}{2}}(\alpha_{\ell n}) + \mathcal{O}(y^2) \\ &= \frac{x-1}{\sqrt{x}} \sqrt{\frac{\pi \alpha_{\ell n}}{2}} J_{\ell-\frac{1}{2}}(\alpha_{\ell n}) + \mathcal{O}(y^2), \end{aligned}$$

where we used the relations (B.3), (B.1c) and the fact that $\alpha_{\ell n}$ is a zero of $J_{\ell+\frac{1}{2}}$. By plugging the above into (3.29), we get a very nice simplification

$$\begin{aligned} G_E(\beta, x) &= \frac{-1}{2\pi^2 R^2} \sum_{n=1}^{+\infty} \sum_{\ell=0}^{+\infty} \frac{2\ell+1}{\cancel{J^2_{\ell-\frac{1}{2}}(\alpha_{\ell n})}} e^{-\beta \frac{\alpha_{\ell n}}{R}} \frac{(1-x)^2}{x} \frac{\pi}{2} \alpha_{\ell n} \cancel{J^2_{\ell-\frac{1}{2}}(\alpha_{\ell n})} + \mathcal{O}(y^3) \\ &= \frac{-1}{4\pi R^2} \frac{(1-x)^2}{x} \sum_{n=1}^{+\infty} \sum_{\ell=0}^{+\infty} (2\ell+1) \alpha_{\ell n} e^{-\beta \frac{\alpha_{\ell n}}{R}} + \mathcal{O}(y^3) \\ &= \frac{-1}{4\pi R^2} \frac{(1-x)^2}{x} (-R) \frac{\partial}{\partial \beta} \sum_{n=1}^{+\infty} \sum_{\ell=0}^{+\infty} (2\ell+1) e^{-\beta \frac{\alpha_{\ell n}}{R}} + \mathcal{O}(y^3), \end{aligned}$$

and so we write the euclidean propagator as follows:

$$\boxed{G_E(\beta, x) = \frac{1}{4\pi R} \frac{(1-x)^2}{x} \frac{\partial}{\partial \beta} \sum_{n=1}^{+\infty} \sum_{\ell=0}^{+\infty} (2\ell+1) e^{-\beta \frac{\alpha_{\ell n}}{R}} + \mathcal{O}(y^3).} \tag{3.30}$$

We now focus on the double series, which we call $\tilde{K}(\beta)$:

$$\tilde{K}(\beta) \equiv \sum_{n=1}^{+\infty}\sum_{\ell=0}^{+\infty}(2\ell+1)e^{-\beta\frac{\alpha_{\ell n}}{R}}. \tag{3.31}$$

Heat kernel interpretation of the double series

Let $\mathcal{M} = \mathbb{B}^3(R)$ be the manifold of a 3D ball of radius $R > 0$, and consider the pseudo-laplacian[1] operator $\sqrt{-\nabla^2}$. By solving the Klein-Gordon equation (in Section 3.1.1) we have found that the solutions obey

$$(-\partial_t^2 + \nabla^2)\phi(t, r, \theta, \varphi) = 0,$$

therefore

$$-\nabla^2\phi(t, r, \theta, \varphi) = -\partial_t^2\phi(t, r, \theta, \varphi). \tag{3.32}$$

We have also found that a complete set of solutions (i.e. a basis for the linear space of solutions) is given by the positive and negative frequency modes $u_{n\ell m}(t, r, \theta, \varphi)$ and $u^*_{n\ell m}(t, r, \theta, \varphi)$. In addition, we know that these modes admit a time-space factorization as follows:

$$u_{n\ell m}(t, r, \theta, \varphi) = e^{-i\omega_{\ell n}t}v_{n\ell m}(r, \theta, \varphi)$$
$$u^*_{n\ell m}(t, r, \theta, \varphi) = e^{i\omega_{\ell n}t}v^*_{n\ell m}(r, \theta, \varphi).$$

By plugging the above factorization into (3.32), we obtain (after cancelling out the extra factor $e^{\pm i\omega_{\ell n}t}$):

$$-\nabla^2 v_{n\ell m}(r, \theta, \varphi) = \omega_{\ell n}^2 v_{n\ell m}(r, \theta, \varphi)$$
$$-\nabla^2 v^*_{n\ell m}(r, \theta, \varphi) = \omega_{\ell n}^2 v^*_{n\ell m}(r, \theta, \varphi).$$

The second equation is actually the first one in disguise, in fact (see (3.13b) without time exponential factor):

$$v^*_{n\ell m}(r, \theta, \varphi) = \sqrt{\frac{2}{\pi}}\frac{1}{RB_{\ell n}}j_\ell(\omega_{\ell n})Y_\ell^{m*}(\theta, \varphi)$$

and $Y_\ell^{m*}(\theta, \varphi) = (-1)^m Y_\ell^{-m}(\theta, \varphi)$, therefore the second equation above just becomes

$$-\nabla^2 v_{n,\ell,-m}(r, \theta, \varphi)\cancel{(-1)^m} = \omega_{\ell n}^2 v_{n,\ell,-m}(r, \theta, \varphi)\cancel{(-1)^m},$$

and since m runs from $-\ell$ to ℓ, it contains the exact same information of the first one.

The relation

$$-\nabla^2 v_{n\ell m}(r, \theta, \varphi) = \omega_{\ell n}^2 v_{n\ell m}(r, \theta, \varphi) \tag{3.33}$$

is just an eigenvalue equation telling us that $v_{n\ell m}$ are eigenfunctions of the operator $-\nabla^2$ with eigenvalues $\omega_{\ell n}^2$, and we can also claim that the set of all $v_{n\ell m}$ is complete because we can build any solution to the spacial part of the Klein-Gordon equation with them. The full spectrum of the operator $-\nabla^2$ therefore consists of the set of all $\omega_{\ell n}^2$, each with degeneracy $2\ell + 1$.

[1]See Section E.3 for an introduction to fractional laplacians.

Let us now consider the operator $\sqrt{-\nabla^2}$. Its spectrum will consist of the square root of the eigenvalues of $-\nabla^2$, which are simply the values $\omega_{\ell n}$, each with degeneracy $2\ell+1$. We then consider the operator

$$e^{-\beta\sqrt{-\nabla^2}}.$$

Obviously, its spectrum is made of the values

$$\lambda_{\ell n} = e^{-\beta\omega_{\ell n}} = e^{-\beta\frac{\alpha_{\ell n}}{R}},$$

each one having degeneracy $2\ell + 1$ as well. If we compute the trace of the operator $e^{-\beta\sqrt{-\nabla^2}}$, we get

$$\operatorname{Tr} e^{-\beta\sqrt{-\nabla^2}} = \sum_{n=1}^{+\infty}\sum_{\ell=0}^{+\infty}(2\ell+1)e^{-\beta\frac{\alpha_{\ell n}}{R}},$$

which is exactly equal to $\tilde{K}(\beta)$, defined in (3.31). This is called *heat kernel*, since it can be interpreted as a function that solves the heat diffusion equation, as described in [11] and in Appendix E.

Heat kernel expansion for small βs

Since we are interested in the coincidence limit of the propagator ($\beta \to 0^+$), we would like to perform an expansion of $\tilde{K}(\beta)$ for small values of β. This is called the *heat kernel expansion* or *Schwinger-DeWitt expansion*. In general, for a manifold of dimension n and a pseudo-laplacian operator Δ of integer order m, we have the following expansion of the heat trace (see [14]):

$$\operatorname{Tr} e^{-\beta\Delta} \simeq \sum_{k=0}^{+\infty} A_k \beta^{\frac{k-n}{m}}. \tag{3.34}$$

In our case, our operator $\sqrt{-\nabla^2}$ is of order $m = 1$, since $-\nabla^2$ is of order 2. The dimension is $n = 3$, therefore our expansion looks like

$$\tilde{K}(\beta) \simeq \sum_{k=0}^{+\infty} A_k \beta^{k-3}. \tag{3.35}$$

We now need the coefficients A_k. To compute those, we consider an easier and well-known version of the problem: we still retain the 3-dimensional ball with Dirichlet boundary conditions at $r = R$, but we take the laplacian operator $-\nabla^2$ instead of its square root, and its heat kernel $K(\beta)$. Equation 1.1 in Bordag et al. paper [4] tells us that the heat kernel $K(\beta)$ (the paper uses $t \leftrightarrow \beta$) can be expanded for small values of β as ($D = 3$):

$$K(\beta) \simeq (4\pi\beta)^{-\frac{3}{2}} \sum_{k=0,\frac{1}{2},1,\dots}^{+\infty} B_k\beta^k = \sum_{k=0}^{+\infty} C_k \beta^{\frac{k-3}{2}}, \tag{3.36}$$

which matches the general form (3.35) with $m = 2$, $n = 3$, $\Delta = -\nabla^2$. In the above, while we retain the same notation of Bordag et al. for the coefficients B_k, we have defined for later convenience the coefficients C_k which relate to the former ones as:

$$C_k = \frac{B_{k/2}}{(4\pi)^{\frac{3}{2}}}. \tag{3.37}$$

Bordag et al. paper [4] also computes the coefficients B_k in the case of a 3-dimensional ball with radius R and Dirichlet boundary conditions for the laplacian operator $-\nabla^2+m^2$. It states that the heat kernel can be factorized in a massless part and a massive exponentially damping factor $K(\beta) = K_{m=0}(\beta)e^{-m^2\beta}$. Appendix B of Bordag et al. paper [4] reports the coefficients B_k of $K_{m=0}(\beta)$, which are the ones we will need in order to compute A_k in our case.

Subordination formula and heat kernel coefficients

What we now need is a way to relate A_k to the B_k coefficients. It is here that the subordination formula comes into play. Stinga's paper [24] reports, at page 7, the subordination formula for the semigroup[2] $U(y,x) = e^{-y\sqrt{-\nabla^2}}u(x)$ (defined in Eq. 3 of the paper), which reads:

$$U(y) = \frac{y^{2s}}{4^s\Gamma(s)} \int_0^{+\infty} e^{-\frac{y^2}{4t}} e^{-t(-\nabla^2)} \frac{u}{t^{1+s}} dt.$$

If we set $s = \frac{1}{2}$, $u(x) = 1$, and use β in place of y, we get

$$e^{-\beta\sqrt{-\nabla^2}} = \frac{\beta}{2\sqrt{\pi}} \int_0^{+\infty} e^{-\frac{\beta^2}{4t}} e^{-t(-\nabla^2)} \frac{dt}{t^{\frac{3}{2}}}. \tag{3.38}$$

Recall that our goal is to compute the heat kernel coefficients A_j of the fractional laplacian $\sqrt{-\nabla^2}$, defined in (3.35), in terms of the B_k coefficients, or equivalently the C_k ones, defined in (3.36). We have, for $j \in \{0,1,2,3\}$:

$$\boxed{A_j = \frac{C_j}{2^{j-3}\sqrt{\pi}}\Gamma\left(\frac{4-j}{2}\right) = \frac{B_{j/2}}{2^j\pi^2}\Gamma\left(\frac{4-j}{2}\right)} \tag{3.39}$$

Proof. Let us start by setting the notation:

$$K_{\sqrt{-\nabla^2}}(\beta) = \operatorname{Tr} e^{-\beta\sqrt{-\nabla^2}} \simeq_{\beta\to 0^+} \sum_{j=0}^{+\infty} A_j \beta^{j-3} \tag{3.40a}$$

$$K_{-\nabla^2}(t) = \operatorname{Tr} e^{-t(-\nabla^2)} \simeq_{t\to 0^+} \sum_{j=0}^{+\infty} C_j t^{\frac{j-3}{2}}. \tag{3.40b}$$

Now we use the subordination formula (3.38)*, which in this notation reads*

$$K_{\sqrt{-\nabla^2}}(\beta) = \frac{\beta}{2\sqrt{\pi}} \int_0^{+\infty} K_{-\nabla^2}(t) e^{-\frac{\beta^2}{4t}} t^{-\frac{3}{2}} dt.$$

Before being able to plug the heat kernel expansions inside the subordination formula, we need to deal with the fact that the integral goes from 0 *to* $+\infty$*, which could give problems because the expansion is only valid for small arguments.*

[2]The subordination formula relates the heat trace of the negative laplacian operator to the heat trace of the fractional laplacian. This formula lies within the framework of the *semigroup approach* (see e.g. [25]), which is a method of dealing with irreversible time evolution, like the one happening in heat diffusion. This is discussed in Appendix E, alongside with a basic introduction to subordination formulas. See also [9] for further reference.

We can split the integral on a positive value of t which we denote $t_0 > 0$:

$$\begin{aligned}
K_{\sqrt{-\nabla^2}}(\beta) &= \frac{\beta}{2\sqrt{\pi}} \int_0^{+\infty} \operatorname{Tr} e^{-t(-\nabla^2)} e^{-\frac{\beta^2}{4t}} t^{-\frac{3}{2}} dt \\
&= \frac{\beta}{2\sqrt{\pi}} \int_0^{t_0} \operatorname{Tr} e^{-t(-\nabla^2)} e^{-\frac{\beta^2}{4t}} t^{-\frac{3}{2}} dt + \frac{\beta}{2\sqrt{\pi}} \int_{t_0}^{+\infty} \operatorname{Tr} e^{-t(-\nabla^2)} e^{-\frac{\beta^2}{4t}} t^{-\frac{3}{2}} dt \\
&= I_<(\beta) + I_>(\beta)
\end{aligned}$$

Since $e^{-t_0\lambda_n} \le e^{-t\lambda_n}$, we have (for $t_0 > 0$)

$$\operatorname{Tr} e^{-t(-\nabla^2)} \le \operatorname{Tr} e^{-t_0(-\nabla^2)} \equiv M(t_0) < +\infty,$$

where we assumed that the heat trace converges. In fact, we can use Weyl's law [28] to prove it, which states that the number density of the eigenvalues of the laplacian in a bounded Dirichlet domain in $\mathbb{R}^n$, for high eigenvalues λ, is proportional to $\lambda^{n/2}$, which in our case gives $N(\lambda) = N_0\lambda^{\frac{3}{2}}$. By approximating the trace sum of the eigenvalues with an integral, we have

$$\operatorname{Tr} e^{-t_0(-\nabla^2)} \sim \int_0^{+\infty} e^{-t_0\lambda} dN(\lambda) = \int_0^{+\infty} C e^{-t_0\lambda} \lambda^{\frac{1}{2}} d\lambda = \frac{C\sqrt{\pi}}{2t_0^{\frac{3}{2}}},$$

where C is a purely geometrical constant. We can now bound the second integral $I_>(\beta)$ above:

$$\begin{aligned}
I_>(\beta) &= \frac{\beta}{2\sqrt{\pi}} \int_{t_0}^{+\infty} \operatorname{Tr} e^{-t(-\nabla^2)} e^{-\frac{\beta^2}{4t}} t^{-\frac{3}{2}} dt \\
&\le \frac{\beta}{2\sqrt{\pi}} \int_{t_0}^{+\infty} \operatorname{Tr} e^{-t_0(-\nabla^2)} e^{-\frac{\beta^2}{4t}} t^{-\frac{3}{2}} dt \\
&= M(t_0) \frac{\beta}{2\sqrt{\pi}} \int_{t_0}^{+\infty} e^{-\frac{\beta^2}{4t}} t^{-\frac{3}{2}} dt \\
&= M(t_0) \operatorname{erf}\left(\frac{\beta}{2\sqrt{t_0}}\right),
\end{aligned}$$

which goes as $\mathcal{O}(\beta)$ for $\beta \to 0^+$. Therefore, since we are interested in this limit, we can safely assume $I_>(\beta) = \mathcal{O}(\beta)$ and proceed with the calculation of the heat kernel coefficients.

We have thus cast the subordination formula into the following relation:

$$K_{\sqrt{-\nabla^2}}(\beta) = \frac{\beta}{2\sqrt{\pi}} \int_0^{t_0} K_{-\nabla^2}(t) e^{-\frac{\beta^2}{4t}} t^{-\frac{3}{2}} dt + \mathcal{O}(\beta).$$

For $t_0 > 0$ sufficiently small, we can plug the expansion (3.40b) *inside the above relation, and get*

$$K_{\sqrt{-\nabla^2}}(\beta) = \frac{\beta}{2\sqrt{\pi}} \sum_{k=0}^{+\infty} C_k \int_0^{t_0} t^{\frac{k-3}{2}} e^{-\frac{\beta^2}{4t}} t^{-\frac{3}{2}} dt + \mathcal{O}(\beta).$$

Now, we make the substitution

$$\begin{aligned}
u &= \frac{\beta^2}{4t}, \\
t &= \frac{\beta^2}{4u}, \\
dt &= -\frac{\beta^2}{4u^2} du,
\end{aligned}$$

and we get

$$\begin{aligned}
K_{\sqrt{-\nabla^2}}(\beta) &= \frac{\beta}{2\sqrt{\pi}} \sum_{k=0}^{+\infty} C_k \int_{+\infty}^{\frac{\beta^2}{4t_0}} \left(\frac{\beta^2}{4u}\right)^{\frac{k}{2}-3} e^{-u} \left(-\frac{\beta^2}{4u^2}\right) du + \mathcal{O}(\beta) \\
&= \sum_{k=0}^{+\infty} C_k \frac{\beta}{2\sqrt{\pi}} \left(\frac{\beta}{2}\right)^{k-4} \int_{\frac{\beta^2}{4t_0}}^{+\infty} e^{-u} u^{1-\frac{k}{2}} du \\
&\simeq_{\beta\to 0} \sum_{k=0}^{+\infty} \frac{C_k}{2^{k-3}\sqrt{\pi}} \Gamma\left(\frac{4-k}{2}\right) \beta^{k-3},
\end{aligned}$$

from which we can immediately read off the coefficients A_k.

We can indeed check with the literature that our derivation is correct. In fact, Gorbar, in his paper [6], states, in Eq. 58, that the relation between the heat kernel coefficients of the square root of the laplacian and the ones without the square root is

$$E_m^{\sqrt{-\nabla^2}} = 2\frac{\Gamma(3-m)}{\Gamma\left(\frac{3-m}{2}\right)} E_m^{-\nabla^2},$$

and we can reduce to the above equation by using the gamma duplication formula [33]:

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z).$$

By setting $z = \frac{3-k}{2}$ we get

$$\Gamma\left(\frac{4-k}{2}\right) = 2^{k-2}\sqrt{\pi}\frac{\Gamma(3-k)}{\Gamma\left(\frac{3-k}{2}\right)},$$

so that

$$A_k = C_k \frac{2^{k-2}\sqrt{\pi}}{2^{k-3}\sqrt{\pi}} \frac{\Gamma(3-k)}{\Gamma\left(\frac{3-k}{2}\right)} = 2\frac{\Gamma(3-k)}{\Gamma\left(\frac{3-k}{2}\right)} C_k,$$

which is precisely the result stated by Gorbar [6].

■

Let us notice that we cannot go beyond $j = 3$ because, for higher values, the gamma function has singularities, and therefore we can conclude nothing about the higher order coefficients, and we need to settle with the terms up to the power β^0.

Divergence of the propagator near the boundary

We are now able to compute the divergent behaviour of the euclidean propagator near the boundary, thanks to the heat kernel expansion for small β values. We have, by (3.35)

$$\tilde{K}(\beta) = \sum_{k=0}^{+\infty} A_k \beta^{k-3},$$

and we know, from (3.39), that

$$A_k = \frac{B_{k/2}}{2^k \pi^2} \Gamma\left(\frac{4-k}{2}\right).$$

We now need the values of $B_{k/2}$ for the 3-dimensional Dirichlet ball of radius R, which we can read from Bordag et al. paper [4] in appendix B:

$$\begin{aligned} B_0 &= \frac{4}{3}\pi R^3, \\ B_{1/2} &= -2\pi^{3/2}R^2, \\ B_1 &= \frac{8\pi R}{3}, \\ B_{3/2} &= -\frac{1}{6}\pi^{3/2}. \end{aligned}$$

The corresponding A_k are:

$$\begin{aligned} A_0 &= \frac{4R^3}{3\pi}, \\ A_1 &= -\frac{R^2}{2}, \\ A_2 &= \frac{2R}{3\pi}, \\ A_3 &= -\frac{1}{48}, \end{aligned}$$

and we finally get

$$\boxed{\tilde{K}(\beta) \simeq_{\beta\to 0^+} \frac{4R^3}{3\pi}\frac{1}{\beta^3} - \frac{R^2}{2}\frac{1}{\beta^2} + \frac{2R}{3\pi}\frac{1}{\beta} - \frac{1}{48} + \mathcal{O}(\beta).} \tag{3.41}$$

Now we need to go back to Eq. (3.30) to finally compute the divergent part of the propagator near the boundary:

$$G_E(\beta, x) = \frac{1}{4\pi R}\frac{(1-x)^2}{x}\frac{\partial}{\partial\beta}\tilde{K}(\beta) + \mathcal{O}(y^3),$$

where $\tilde{K}(\beta)$ is defined in (3.31). By differentiating the asymptotic heat-kernel expansion (3.41) with respect to β, we get

$$\frac{\partial\tilde{K}}{\partial\beta}(\beta) = -\frac{4R^3}{\pi}\frac{1}{\beta^4} + R^2\frac{1}{\beta^3} - \frac{2R}{3\pi}\frac{1}{\beta^2} + \mathcal{O}(1).$$

So the propagator becomes

$$G_E(\beta, x) = \frac{(1-x)^2}{x}\left[\frac{R^2}{\pi^2}\frac{1}{\beta^4} - \frac{R}{4\pi}\frac{1}{\beta^3} + \frac{1}{6\pi^2}\frac{1}{\beta^2} + \mathcal{O}(1)\right],$$

which we can Wick rotate back to lorentzian time

$$\boxed{G_F(t, x) = iG_E(it, x) = \frac{(1-x)^2}{x}\left[\frac{R^2}{\pi^2}\frac{i}{t^4} + \frac{R}{4\pi}\frac{1}{t^3} - \frac{1}{6\pi^2}\frac{i}{t^2} + \mathcal{O}(1)\right].} \tag{3.42}$$

3.3.2 Behaviour at $r = 0$

Now we proceed to analyze the euclidean propagator $G_E(x_1, x_2)$ at $r = 0$. Let us start from expression (3.29): if we set $r = 0$ we get a huge simplification due to the fact that the only spherical Bessel function that survives when the argument is 0 is j_0 and it equals 1, while all the others vanish:

$$
\begin{aligned}
j_0(0) &= 1, \\
j_{\ell>0}(0) &= 0,
\end{aligned}
$$

and the fact that the zeros of the spherical Bessel function j_0 are just multiples of π:

$$\alpha_{0n} = n\pi.$$

We end up with

$$G_E(\beta, r = 0) = \frac{1}{2\pi^2 R^2} \sum_{n=1}^{+\infty} \frac{e^{-\beta \frac{\alpha_{0n}}{R}}}{J^2_{-\frac{1}{2}}(\alpha_{0n})}.$$

Now we use relations (B.1a), (B.3) and (B.4b) to write

$$J^2_{-\frac{1}{2}}(\alpha_{0n}) = J^2_{\frac{3}{2}}(\alpha_{0n}) = \frac{2\alpha_{0n}}{\pi} j_1^2(\alpha_{0n}) = 2n j_1^2(n\pi) = 2\cancel{n}\frac{\cos^2(n\pi)}{n^{\cancel{2}}\pi^2} = \frac{2}{n\pi^2}.$$

The propagator reduces to

$$G_E(\beta, r = 0) = \frac{1}{4R^2} \sum_{n=1}^{+\infty} n e^{-\beta \frac{n\pi}{R}} = \frac{1}{4R^2} \frac{e^{\frac{\pi\beta}{R}}}{\left(e^{\frac{\pi\beta}{R}} - 1\right)^2} = \frac{1}{4\pi^2\beta^2} - \frac{1}{48R^2} + \frac{\pi^2\beta^2}{960R^4} + \mathcal{O}(\beta^4), \tag{3.43}$$

where we performed similar steps as in Section 2.4.1. We can now Wick rotate back to lorentzian time

$$\boxed{G_F(t, r = 0) = iG_E(it, r = 0) = \frac{i}{4R^2} \frac{e^{\frac{i\pi t}{R}}}{\left(e^{\frac{i\pi t}{R}} - 1\right)^2} = -\frac{i}{4\pi^2 t^2} - \frac{i}{48R^2} - \frac{i\pi^2 t^2}{960R^4} + \mathcal{O}(t^4).} \tag{3.44}$$

3.3.3 Hadamard singular part

In the absence of boundaries, the propagator presents a well characterized singular behaviour at coincidence limit. The main reference we use for this is the Decanini-Folacci paper [17]. In Section 2.5 we outline the algorithmic procedure to follow in order to renormalize the energy-momentum tensor. In the first steps, the Hadamard parametrix is required in order to perform the subtraction. Let us then check the Hadamard singular part of our propagator, which is reported in Eq. 101 of Decanini-Folacci paper:

$$G^F_{sing}(x_1, x_2) = \frac{i}{8\pi^2} \left(\frac{U(x_1, x_2)}{\sigma(x_1, x_2) + i0^+} + V(x_1, x_2) \ln[\sigma(x_1, x_2) + i0^+] \right). \tag{3.45}$$

To compute $U(x_1, x_2)$ and $V(x_1, x_2)$, we look at Section III C of the paper [17], and since the curvature tensors, scalar and the mass m are all vanishing in our model, we have $U = 1$ and $V = 0$, so

$$G^F_{sing}(x_1, x_2) = \frac{i}{8\pi^2} \frac{1}{\sigma(x_1, x_2) + i0^+}.$$

We now analyze the Synge's world function $\sigma(x_1, x_2)$ and impose that the time splitting is purely temporal:

$$\sigma(x_1, x_2) = \frac{1}{2}[-(t_2 - t_1)^2] = -\frac{t^2}{2},$$

so the singular part becomes

$$G^F_{sing}(t) = -\frac{i}{4\pi^2 t^2}. \tag{3.46}$$

Behaviour at the center We can immediately check that the renormalized propagator (3.44) is indeed finite in the coincidence limit, since the singular part (3.46) exactly matches the divergence appearing in (3.44). This is consistent with the fact that we are far away from the boundary, and the Hadamard UV divergence is purely local, so it is not affected by the presence of the boundary as long as we are far away from it. In the next section we will exploit this result to compute the renormalized energy density at $r = 0$.

Behaviour near boundary Near the Dirichlet boundary, things get quite complicated because the local presence of boundary affects the UV divergence. The way the divergence gets distorted depends purely on the geometry of the boundary, as one can deduce by the very fact that the heat kernel expansion we performed to derive the near-boundary behaviour relies solely on the geometry of the boundary. A nice interpretation of this issue is presented in a paper by McAvity and Osborn [3]. Here it is showed that the additional UV divergent contribution comes from the short length geodesic paths linking two points near the boundary which undergo reflection.

Behaviour near the center The exact expression of the propagator near time coincidence limit at the center $r = 0$ could be computed analytically due to huge simplifications. One could also attempt to compute the approximate expression near the center, at $r \neq 0$, but this is not an easy task because there is no obvious analytical way to proceed. Numerically, one could engineer a mode-by-mode subtraction method and hope that the terms of the series decay to zero in a convenient way so that one can truncate the sum and still get a good numerical result. It seems, however, that there is no obvious way to proceed, since the orthonormal basis on which the bare propagator is expanded is not suitable for expanding the Hadamard singular part, because the latter does not satisfy the Dirichlet boundary condition at $r = R$. We therefore leave this problem open for further research.

3.4 Energy-momentum tensor

Eq. (71) of Decanini-Folacci [17] gives us the prescription for computing the energy-momentum tensor starting from the renormalized propagator:

$$\langle 0|\,\hat{T}_{\mu\nu}\,|0\rangle_{ren} = \frac{\alpha_D}{2}\left[-w_{\mu\nu} + \frac{1}{2}(1-2\xi)w_{;\mu\nu} + \frac{1}{2}\left(2\xi - \frac{1}{2}\right)g_{\mu\nu}\Box w + \xi R_{\mu\nu}w - g_{\mu\nu}v_1\right], \tag{3.47}$$

where (Eqs. 75, 85 and 32 of Decanini-folacci [17])

$$\begin{aligned} w(x) &= \lim_{x_2\to x_1\equiv x} W(x_1,x_2),\\ w_{\mu\nu}(x) &= \lim_{x_2\to x_1\equiv x} \nabla_\mu\nabla_\nu W(x_1,x_2),\\ W(x_1,x_2) &= \frac{2}{i\alpha_D}[G^F(x_1,x_2) - G^F_{sing}(x_1,x_2)],\\ \alpha_D &= \frac{1}{8\pi^2}. \end{aligned}$$

3.4.1 $r = 0$

Our goal is now to compute the renormalized energy density at $r = 0$, that is the tt component of the energy-momentum tensor. We make the following considerations:

- since we are in flat spacetime, we are free to choose any value of ξ, which couples the field ϕ to the Ricci scalar, which vanishes identically;
- our expression of the propagator (3.44) is already in the spacial coincidence limit, the only variable which is not point split is the time $t = t_2 - t_1$;
- therefore, we are only able to compute the time derivatives of our propagator, not the spacial ones;
- if we set $\mu = t$ and $\nu = t$ in (3.47), the only term that contains spacial derivatives is the one containing $\Box w$, but if we choose $\xi = 1/4$ we can make it vanishing;
- since the only derivatives that we will take are temporal, the fact that we already are in the spacial coincidence limit does not spoil the procedure;
- $R_{\mu\nu} = 0$ identically;
- $v_1 = 0$;
- $\alpha_D = \frac{1}{4\pi^2}$, since $D = 4$, by Eq. 32 in Decanini-Folacci paper [17];
- $\Theta_{tt} = 0$.

We therefore reduce to

$$\rho_{ren}(r=0) = \frac{1}{8\pi^2}\left[-w_{tt} + \frac{1}{4}\partial_t^2 w\right]. \tag{3.48}$$

So, we are left with computing w_{tt} and $w_{,tt}$. We have

$$
\begin{aligned}
W(t_1,t_2,r=0) &= \frac{8\pi^2}{i}\left(-\frac{i}{48R^2}-\frac{i\pi^2(t_2-t_1)^2}{960R^4}\right)+\mathcal{O}((t_2-t_1)^4),\\
w(t,r=0) &= -\frac{\pi^2}{6R^2},\\
w_{,tt}(t,r=0) &= 0,\\
w_{tt}(t,r=0) &= \lim_{t_2\to t_1\equiv t}\partial_{t_1}\partial_{t_1}\left(-\frac{\pi^4}{120R^4}(t_2-t_1)^2+\mathcal{O}((t_2-t_1)^4)\right)=-\frac{\pi^4}{60R^4}.
\end{aligned}
$$

Finally, Eq. (3.48) becomes

$$
\boxed{\rho(r=0)=\frac{\pi^2}{480R^4}.} \tag{3.49}
$$

3.4.2 $r \neq 0$ away from boundary

The energy-momentum tensor for a real massless scalar field reads

$$
\hat{T}_{\mu\nu}=\frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\hat{\phi}\partial_\beta\hat{\phi}-\partial_\mu\hat{\phi}\partial_\nu\hat{\phi}.
$$

Calculations (done in Wolfram Mathematica) yield:

$$
\begin{aligned}
\langle\hat{T}_{tt}\rangle &= -\mathcal{T},\\
\langle\hat{T}_{rr}\rangle &= -15\mathcal{T},\\
\langle\hat{T}_{\theta\theta}\rangle &= r^2\mathcal{T},\\
\langle\hat{T}_{\varphi\varphi}\rangle &= r^2\sin^2\theta\mathcal{T},
\end{aligned}
$$

and all the off-diagonal components are zero. Here, $\mathcal{T}$ is defined as

$$
\begin{aligned}
\mathcal{T} &= \frac{1}{4\pi R^4}\sum_{n=1}^{+\infty}\sum_{\ell=0}^{+\infty}\sum_{m=-\ell}^{\ell}\frac{\alpha_{\ell n}^2}{B_{n\ell}^2}|Y_\ell^m(\theta,\varphi)|^2\left[j_{\ell-1}(\omega_{n\ell}r)-j_{\ell+1}(\omega_{n\ell}r)\right]^2\\
&= \frac{1}{16\pi^2R^4}\sum_{n=1}^{+\infty}\sum_{\ell=0}^{+\infty}\frac{\alpha_{\ell n}^2}{J_{\ell-\frac{1}{2}}^2(\alpha_{\ell n})}(2\ell+1)\left[j_{\ell-1}(\omega_{n\ell}r)-j_{\ell+1}(\omega_{n\ell}r)\right]^2,
\end{aligned}
$$

where we used the spherical harmonic addition theorem (A.14). Note that the above is not renormalized and therefore yields a divergent result. This is where the Hadamard renormalization procedure comes into play. However, for the Hadamard procedure to work properly, one has to be able to perform the subtraction needed for the calculation of the limits (2.28). While in the two-dimensional model we were able to put the infinite discrete sum in a closed form and perform the subtraction explicitly, here there is no known closed form for the infinite sum, and therefore we cannot proceed with the calculation of w and $w_{\mu\nu}$ in general. We leave this problem open for further investigation, perhaps using numerical methods to deal with the finite part after subtraction.

3.4.3 Near the boundary

As previously stated, near the boundary things get more complicated since the divergence of the propagator does not match the Hadamard parametrix. One could then proceed by subtracting all divergent terms by hand, but it is not clear what the physical meaning of this subtraction could be. This means that the physical significance of the Dirichlet boundary is not clear, and other alternatives should be considered.

3.5 Final considerations, remarks, and further research paths

In this chapter, we studied the Feynman propagator in a spherical Dirichlet cavity, and attempted to compute the renormalized energy-momentum tensor. Here are some remarks.

- The crucial step is taking the coincidence limit, but while in the 2-dimensional model we first took the time coincidence limit and then the spacial one, in this case we proceeded the other way around. Indeed, when working with trigonometric functions, there are prosthaphaeresis and Werner formulas that lead to huge simplifications, but in the case of spherical Bessel functions there are no equivalent formulas that can help.

- Therefore, we first took the spacial coincidence limit, Wick rotated to euclidean time, and then worked with a decaying time exponential which works as a regulator to make the series convergent.

- In spacial coincidence, the information about the time separation is preserved, and one is able to compute time derivatives (after Wick rotating back). It turned out that to apply the Hadamard renormalization procedure at $r = 0$ on $\langle \hat{T}_{tt} \rangle$, this information was sufficient, and we obtained the renormalized energy density at $r = 0$.

- To compute the other diagonal components of the renormalized energy-momentum tensor, one should instead take the coincidence limit spacially and try to simplify the propagator in such a way that the subtraction is possible and gives a closed expression. For the off-diagonal components, two coordinates should be non-coincident in the limiting process, since there are two distinct derivatives to take.

- To compute the renormalized energy-momentum tensor in the bulk, one can still use the Hadamard procedure, since the singular parametrix holds correctly in the bulk region (away from boundary). However, handling the mode sum and performing the subtraction is not easy, and one can think of using numerical methods.

- We used the heat kernel formalism and obtained the divergent expression of the Feynman propagator near the boundary. This divergence does not match the Hadamard parametrix. In the two-dimensional case, however, this issue was not present. One can conjecture an explaination tied to the geodesic reflection phenomenon described in [3]. This is another research path that one can take to further investigate the issue.

- It is not obvious how to proceed in order to compute the renormalized energy-momentum tensor close to the boundary. The naive subtraction does not have clear physical significance, since it is different from the mere Hadamard subtraction, which, physically, in this case, just means subtracting the infinite Minkowski spacetime contribution. This suggests another type of boundary conditions should be used to physically model the compact object.

The next natural step is to substitute the Dirichlet boundary with a step potential, which is what we will do in one spacial dimension in the next chapter.

Chapter 4

Real massless scalar in a 1+1 dimensional Minkowski spacetime with step potential

As anticipated, we now turn to a model which is somewhat closer to the model of a static Schwarzschild star, but still not quite, since we are still going to work with flat spacetime and two dimensions. We saw that the Hadamard renormalization procedure was successful in the two dimensional model discussed in Chapter 2, even near the boundary, but unsuccessful near the spherical Dirichlet boundary of the previous chapter. Since the goal of this thesis is to get an idea of how to deal with boundaries of compact objects, it is useful to verify if the presence of a step discontinuity spoils the Hadamard parametrix or not. To do this, we will work in two dimensions for simplicity and compute the Feynman propagator of a two dimensional flat spacetime model with a potential having a step discontinuity, as discussed in the introduction. Some tricks that will be used in solving the equation of motion and imposing the matching conditions are inspired by [29].

4.1 Solving the equation of motion

Let us now consider the Minkowski metric $g_{\mu\nu} = \mathrm{diag}(-1, 1)$, and the action

$$S = \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dx \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} U(x) \phi^2 \right), \tag{4.1}$$

where the potential $U(x)$ is placed in the ϕ^2 term as an effective position-dependent mass term. In order to resemble the static Schwarzschild star effective potential, which has a step discontinuity at $r = R$, we introduce the simplest step potential, that is 0 inside the compact object, and $U_0 > 0$ outside:

$$U(x) = U_0 \Theta(|x| - L), \tag{4.2}$$

with $U_0 > 0$, $L > 0$ and Θ being the Heaviside step function. The plot is shown in Figure 4.1 One may ask why we did not set the potential outside to be zero, so that we recover massless Minkowski modes at infinity. The reason is that in this way one should have the potential inside to be less than zero, giving a negative effective mass squared, leading to tachyonic instabilities. We therefore pay the price of adding an effective mass outside in order to keep the model as simple as possible while having a step discontinuity.

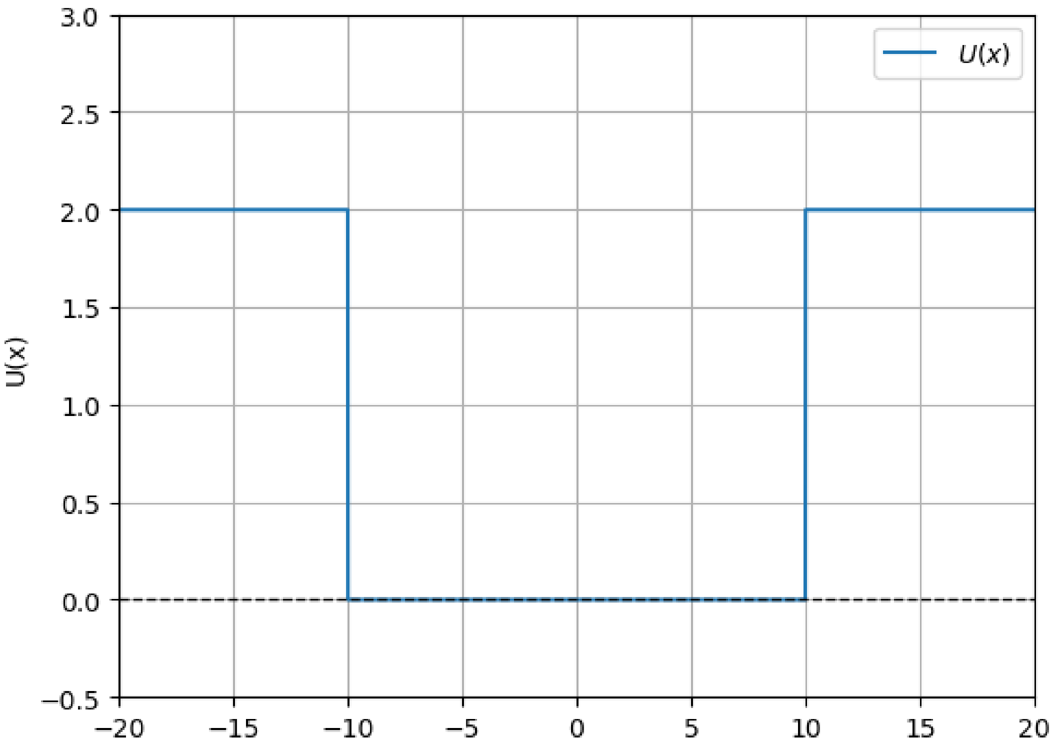

Figure 4.1: Plot of the step potential $U(x)$ with $L = 10.0$.

The Euler-Lagrange equation of motion yields:

$$(\Box - U(x))\phi = 0. \tag{4.3}$$

Proof. *We have*

$$\frac{\partial \mathcal{L}}{\partial \phi} = -U(x)\phi$$
$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -\partial^\mu \phi.$$

So that

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = -\Box\phi + U(x)\phi,$$

and equating the above to zero yields

$$(\Box - U(x))\phi = 0.$$

■

We then have two distinct cases locally:

- $|x| < L$, with equation of motion $\Box\phi = 0$;
- $|x| > L$, with equation of motion $(\Box - U_0)\phi = 0$.

Of course, there are two disconnected regions that fall into the second case: $x < -L$ and $x > L$. Let us then proceed to analyze the two above cases.

4.1.1 Case $|x| < L$

In this subdomain, $U(x) = 0$, and the equation of motion (4.3) reduces to

$$\Box\phi = 0. \tag{4.4}$$

The above is easily solved via separation of variables, yielding the following independent modes (with $k > 0$ and definite parity):

$$u^{in}_{k,\pm}(t,x) = A^{\pm}_k e^{-ikt}(e^{ikx} \pm e^{-ikx}) \tag{4.5a}$$
$$u^{in*}_{k,\pm}(t,x) = A^{\pm *}_k e^{ikt}(e^{-ikx} \pm e^{ikx}). \tag{4.5b}$$

Proof. *First, expand the box operator. The equation of motion becomes*

$$(-\partial_t^2 + \partial_x^2)\phi(t,x) = 0.$$

Now factorize $\phi(t,x)$ into $\phi(t,x) = \alpha(t)\beta(x)$ and substitute. After dividing by $\alpha\beta$ and using $-k^2$ as negative separating constant (with $k > 0$, see Section 2.1 for the full discussion on the reason of this choice), we get two harmonic oscillators with frequency k. The positive time frequency solution is given by e^{-ikt} multiplied by a general linear combination of the two spacial modes $e^{\pm ikx}$:

$$u^{in}_k(t,x) = e^{-ikt}(A_k e^{ikx} + \tilde{A}_k e^{-ikx}).$$

We now separate the positive and negative parity parts in the spacial sector, by noting that any one-variable function can be written as a sum of a positive and negative parity functions:

$$f(x) = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x)).$$

We can then require u^{in}_k to have definite parity by imposing $u^{in}_{k,\pm}(-x) = \pm u^{in}_{k,\pm}(x)$, where we added the $\pm$ label to keep track of the parity. This constraint translates to:

$$e^{-ikt}(A^{\pm}_k e^{-ikx} + \tilde{A}^{\pm}_k e^{ikx}) = \pm e^{-ikt}(A^{\pm}_k e^{ikx} + \tilde{A}^{\pm}_k e^{-ikx}),$$

which is satisfied if and only if $\tilde{A}^{\pm}_k = \pm A^{\pm}_k$. By substituting $\tilde{A}^{\pm}$ in the solution, we get to the result we wanted to prove.

■

4.1.2 Case $|x| > L$

In this subdomain, $U(x) = U_0 > 0$, and the equation of motion (4.3) reduces to

$$(\Box - U_0)\phi = 0. \tag{4.6}$$

When separating variables, we have an effective squared mass $m^2 = U_0$, and therefore the solutions behave as massive field modes. By defining $\omega_k \equiv \sqrt{|k^2 - U_0|}$ and $\sigma_\pm(x)$ as

$$\sigma_\pm(x) = \begin{cases} 1 & \text{if} + \\ \operatorname{sgn} x & \text{if} - \end{cases} = \begin{cases} 1 & \text{if } x > 0 \\ \pm 1 & \text{if } x < 0 \end{cases}, \tag{4.7}$$

we have two cases:

- $k > \sqrt{U_0}$:

$$u^{out}_{k,\pm}(t,x) = e^{-i\omega_k t}(B^{\pm}_k e^{i\omega_k(|x|-L)} + C^{\pm}_k e^{-i\omega_k(|x|-L)})\sigma_\pm(x), \tag{4.8a}$$

$$u^{out*}_{k,\pm}(t,x) = e^{i\omega_k t}(B^{\pm*}_k e^{-i\omega_k(|x|-L)} + C^{\pm*}_k e^{i\omega_k(|x|-L)})\sigma_\pm(x), \tag{4.8b}$$

- $k < \sqrt{U_0}$:

$$u^{out}_{k,\pm}(t,x) = D^{\pm}_k e^{-ikt} e^{-\omega_k(|x|-L)}\sigma_\pm(x) \tag{4.9a}$$

$$u^{out*}_{k,\pm}(t,x) = D^{\pm*}_k e^{ikt} e^{-\omega_k(|x|-L)}\sigma_\pm(x). \tag{4.9b}$$

PROOF. *Let us write down the equation of motion after expanding the box operator:*

$$(-\partial_t^2 + \partial_x^2 - U_0)\phi(t,x) = 0.$$

Now, factorize $\phi(t,x) = \alpha(t)\beta(x)$, substitute in the equation of motion and divide everything by $\alpha\beta$ (see 2.1 for clarity about this division being legitimate and the separating constant $-k^2$ with $k > 0$), so that

$$\frac{\ddot{\alpha}}{\alpha} = \frac{\beta''}{\beta} - U_0 = -k^2.$$

We then have

$$\ddot{\alpha} + k^2\alpha = 0$$
$$\beta'' + (k^2 - U_0)\beta = 0.$$

The solution of the first equation is a general linear combination of e^{-ikt} and e^{ikt}, but we choose to build positive frequency modes and then also consider their complex conjugate, as usual. Now, to solve the second equation, we define $\omega_k = \sqrt{|k^2 - U_0|}$ so that there are three cases:

- *$k^2 - U_0 > 0$: the spacial solution is a general linear combination of $\{e^{-i\omega_k(|x|-L)}, e^{i\omega_k(|x|-L)}\}$, where we chose to pull out an additional factor of $e^{\pm i\omega_k L}$ for later convenience when enforcing matching conditions. In general, since we want to build a solution on the domain $|x| > L$, which has two disconnected patches, we should assign them different coefficients, so we write*

$$u^{out}_k(t,x) = e^{-ikt}(B_k e^{i\omega_k(|x|-L)} + C_k e^{-i\omega_k(|x|-L)}) \qquad \text{if } x > L$$
$$u^{out}_k(t,x) = e^{-ikt}(\tilde{B}_k e^{i\omega_k(|x|-L)} + \tilde{C}_k e^{-i\omega_k(|x|-L)}) \qquad \text{if } x < L,$$

  but imposing definite parity $u^{out}_{k,\pm}(t,-x) = \pm u^{out}_{k,\pm}(t,x)$, we get $\tilde{B}^{\pm}_k = \pm B^{\pm}_k$ and $\tilde{C}^{\pm}_k = \pm C^{\pm}_k$. To account for these identities, we just need to substitute and insert the function $\sigma_\pm(x)$ as a factor, so that we end up with the expression we wanted to prove.

- *$k^2 - U_0 < 0$: the spacial solution in this case is a general linear combination of $e^{-\omega_k(|x|-L)}$ and $e^{\omega_k(|x|-L)}$, however, we set the coefficient of the growing exponential to zero by hand because we want normalizable solutions. In this case we also have two distinct coefficients for the two disconnected patches of the domain, but upon imposing definite parity we get that the coefficient $\tilde{D}^{\pm}_k$ of the region $x < -L$ is equal to $\pm D^{\pm}_k$, with $D^{\pm}_k$ being the coefficient of the region $x > L$. To account for this, we need again to employ the function $\sigma_\pm(x)$, and putting all the factors together we end up with the expression we wanted to prove.*

- *$k^2 = U_0$: in this case we reduce to a trivial linear solution, which becomes identically zero if we want the field to vanish at spacial infinity. We can therefore discard this case and work with the other two.*

■

4.2 Matching conditions

We want the solution to be continuous and have continuous first spacial derivative everywhere, which is the stricter achievable requirement for a field. From now on, we need to work in the two separate cases: $k > \sqrt{U_0}$ and $k < \sqrt{U_0}$. The points where we can have discontinuities are $x = \pm L$, so it suffices to enforce the following conditions:

$$\begin{aligned}
&\phi(t, L^-) = \phi(t, L^+) \\
&\partial_x \phi(t, L^-) = \partial_x \phi(t, L^+) \\
&\phi(t, -L^-) = \phi(t, -L^+) \\
&\partial_x \phi(t, -L^-) = \partial_x \phi(t, -L^+).
\end{aligned}$$

Since the general solution of the field ϕ will be written in terms of the independent modes, we need to enforce the matching conditions mode by mode. However, since each mode has definite parity, we do not need to deal with the point $x = -L$ as long as we enforce the matching conditions at $x = L$.

PROOF. *We want to prove that, by enforcing the matching conditions at $x = L$, they hold automatically at $x = -L$. Let us then assume $u_\pm(-x) = \pm u_\pm(x)$ and also assume*

$$\begin{aligned}
u_\pm(t, L^+) &= u_\pm(t, L^-), \\
\partial_x u_\pm(t, L^+) &= \partial_x u_\pm(t, L^-).
\end{aligned}$$

We then have

$$\begin{aligned}
u_\pm(t, -L^-) &= \pm u_\pm(t, L^+) = \pm u_\pm(t, L^-) = u_\pm(t, -L^+) \\
\partial_x u_\pm(t, -L^-) &= \pm \partial_x u_\pm(t, L^+) = \pm \partial_x u_\pm(t, L^-) = \partial_x u_\pm(t, -L^+).
\end{aligned}$$

We therefore see by the two chains of equality that the matching conditions are enforced at $x = -L$ too.

■

4.2.1 Case $k > \sqrt{U_0}$

Let us start with the case $k > \sqrt{U_0}$. By matching (4.5a) and (4.8a) at $x = L$, we get the following expressions for the coefficients $B_k^\pm$ and $C_k^\pm$ as a function of $A_k^\pm$:

$$B_k^+ = A_k^+ \left[\cos(kL) + i \frac{k}{\omega_k} \sin(kL) \right], \tag{4.10a}$$

$$C_k^+ = A_k^+ \left[\cos(kL) - i \frac{k}{\omega_k} \sin(kL) \right], \tag{4.10b}$$

$$B_k^- = A_k^- \left[\frac{k}{\omega_k} \cos(kL) + i \sin(kL) \right], \tag{4.10c}$$

$$C_k^- = -A_k^- \left[\frac{k}{\omega_k} \cos(kL) - i \sin(kL) \right]. \tag{4.10d}$$

PROOF. *What we want to enforce is*

$$\begin{cases} u^{in}_{k,\pm}(t,L) = u^{out}_{k,\pm}(t,L) \\ \partial_x u^{in}_{k,\pm}(t,L) = \partial_x u^{out}_{k,\pm}(t,L) \end{cases},$$

$$\begin{cases} A_k^{\pm}(e^{ikL} \pm e^{-ikL}) = (C_k^{\pm} + B_k^{\pm})\cancel{\sigma_{\pm}(L)} \\ \cancel{i}kA_k^{\pm}(e^{ikL} \mp e^{-ikL}) = -\cancel{i}\omega_k \cancel{\operatorname{sgn} L}(C_k^{\pm} - B_k^{\pm})\cancel{\sigma_{\pm}(L)} \end{cases},$$

$$\begin{cases} A_k^{\pm}(e^{ikL} \pm e^{-ikL}) = C_k^{\pm} + B_k^{\pm} \\ -\frac{k}{\omega_k}A_k^{\pm}(e^{ikL} \mp e^{-ikL}) = C_k^{\pm} - B_k^{\pm} \end{cases}.$$

We can then split the two cases of $\pm$*:*

- *case* $+$*:*

$$\begin{cases} B_k^+ + C_k^+ = 2A_k^+ \cos(kL) & (I) \\ B_k^+ - C_k^+ = 2iA_k^+ \frac{k}{\omega_k} \sin(kL) & (II) \end{cases};$$

- *case* $-$*:*

$$\begin{cases} B_k^- + C_k^- = 2iA_k^- \sin(kL) & (I) \\ B_k^- - C_k^- = 2A_k^+ \frac{k}{\omega_k} \cos(kL) & (II) \end{cases}.$$

In both cases, we find $B_k^{\pm}$ *and* $C_k^{\pm}$ *by doing*

$$\frac{(I) \pm (II)}{2},$$

which gives exactly the results we aimed for.

■

If we substitute back into the expressions (4.5a),(4.8a), we get

$$u^{in,>}_{k,+}(t,x) = 2A_k^+ e^{-ikt}\cos(kx), \tag{4.11a}$$

$$u^{out,>}_{k,+}(t,x) = 2A_k^+ \left(\cos(kL)\cos[\omega_k(|x|-L)] - \frac{k}{\omega_k}\sin(kL)\sin[\omega_k(|x|-L)]\right), \tag{4.11b}$$

$$u^{in,>}_{k,-}(t,x) = 2iA_k^- e^{-ikt}\sin(kx), \tag{4.11c}$$

$$u^{out,>}_{k,-}(t,x) = 2iA_k^- \operatorname{sgn} x \cdot \cdot \left(\frac{k}{\omega_k}\cos(kL)\sin[\omega_k(|x|-L)] + \sin(kL)\cos[\omega_k(|x|-L)]\right). \tag{4.11d}$$

4.2.2 Case $k < \sqrt{U_0}$

Now we turn to the case $k < \sqrt{U_0}$. By matching (4.5a) and (4.9a) at $x = L$ we get the following expression for the coefficients $D_k^{\pm}$ as a function of $A_k^{\pm}$:

$$D_k^+ = 2A_k^+ \cos(kL), \tag{4.12a}$$

$$D_k^- = 2iA_k^- \sin(kL), \tag{4.12b}$$

and the quantization conditions on k:

$$\sin(k^+L) = \frac{\omega_k^+}{k^+}\cos(k^+L), \tag{4.13a}$$

$$\cos(k^-L) = -\frac{\omega_k^-}{k^-}\sin(k^-L). \tag{4.13b}$$

Proof. *The matching conditions we want to enforce are*

$$\begin{cases} u_{k,\pm}^{in}(t,L) = u_{k,\pm}^{out,<}(t,L) \\ \partial_x u_{k,\pm}^{in}(t,L) = \partial_x u_{k,\pm}^{out,<}(t,L) \end{cases},$$

$$\begin{cases} A_k^\pm(e^{ikL} \pm e^{-ikL}) = D_k^\pm \\ ikA_k^\pm(e^{ikL} \mp e^{-ikL}) = -D_k^\pm\omega_k \end{cases}.$$

We can now apply the substitution $D_k^\pm = 2A_k^\pm\cos(kL)$*, so that the second equation becomes:*

$$ik\cancel{A_k^\pm}(e^{ikL} \mp e^{-ikL}) = -\omega_k\cancel{A_k^\pm}(e^{ikL} \pm e^{-ikL}).$$

We can now split into the two cases of $\pm$. *We have*

- *case* $+$*:*

$$\begin{cases} D_k^+ = 2A_k^+\cos(k^+L) \\ \cancel{-2}k^+\sin(k^+L) = \cancel{-2}\omega_k^+\cos(k^+L) \end{cases};$$

- *case* $-$*:*

$$\begin{cases} D_k^- = 2iA_k^-\sin(k^-L) \\ \cancel{2i}k^-\cos(k^-L) = -\cancel{2i}\omega_k^-\cos(k^-L) \end{cases}.$$

We can immediately read off the relations we wanted to prove from the above systems.

■

The quantization conditions are present because there is only a single hyperbolic oscillator mode entering the solution, that is the decaying exponential, while there is no growing exponential mode which carries an additional coefficient. This implies that there is one less constant to determine, while the matching conditions are still two. Therefore, the condition that would have constrained the coefficient of the growing exponential mode, constrains k instead, making it discrete (quantized) in the low energy sector $0 < k < \sqrt{U_0}$. This is consistent with the general theory, which tells us that the spectrum is discrete inside the convex region of a one-dimensional potential, and continuous outside. We can cast the quantization conditions in a more convenient form that helps us visualize the solutions (see fig. 4.2):

$$\cot(k^+L) = \frac{k^+}{\omega_k^+}, \tag{4.14a}$$

$$\tan(k^-L) = -\frac{k^-}{\omega_k^-}. \tag{4.14b}$$

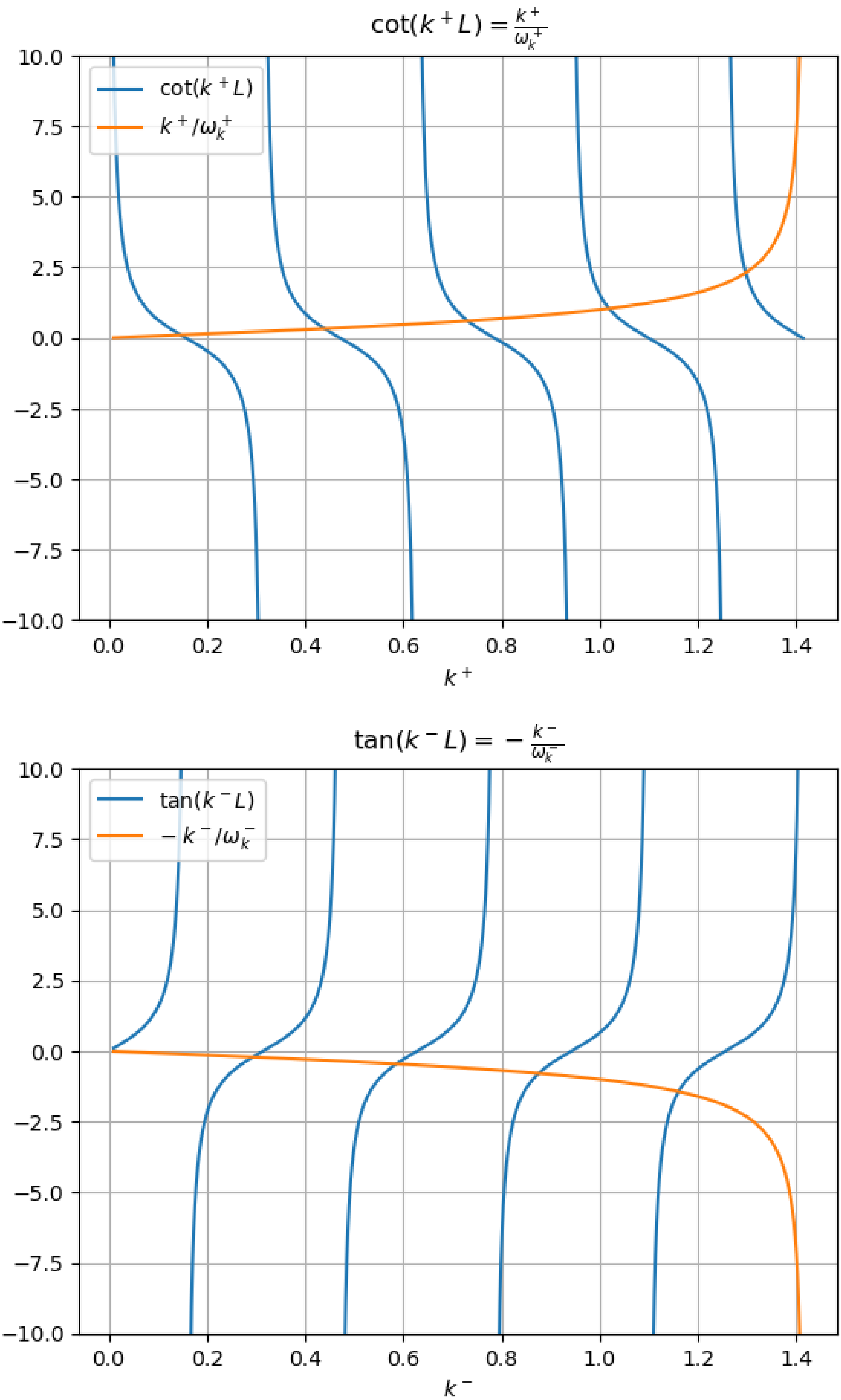

Figure 4.2: Plots of the transcendental quantization conditions for k^+ (above) and k^- (below), with $U_0 = 2.0$ and $L = 10.0$. The intersection points are the allowed values for k^+ and k^-.

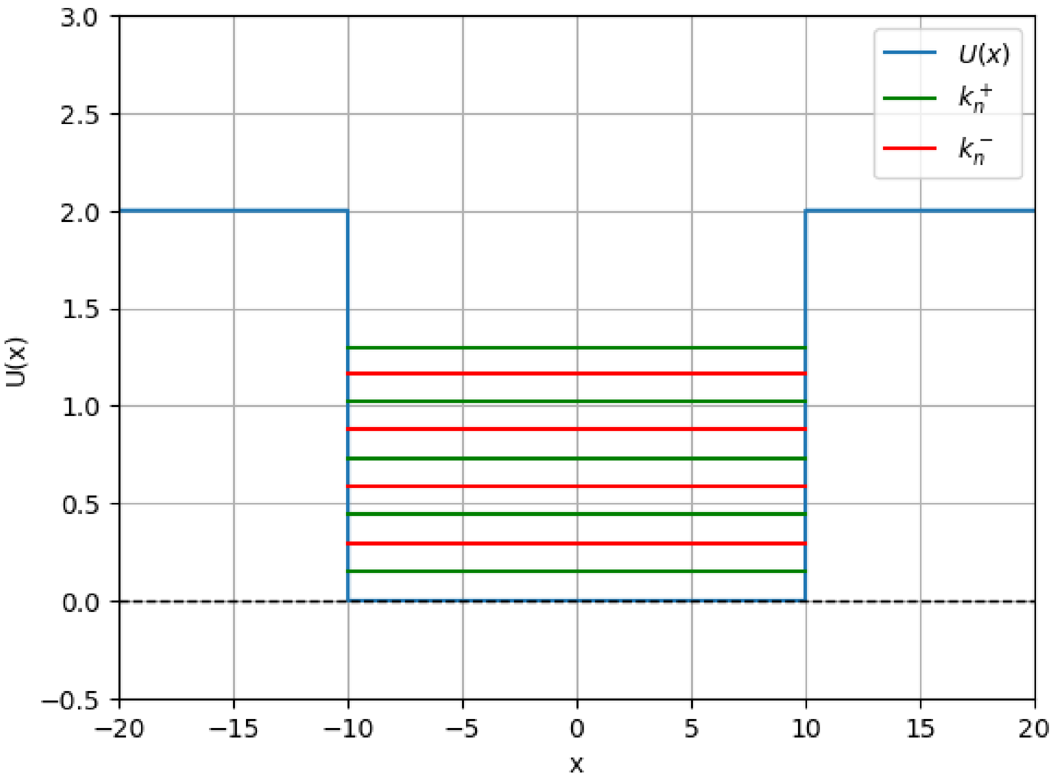

Figure 4.3: Plot of the eigenvalues $k_n^\pm$ with $U_0 = 2.0$ and $L = 10.0$.

We can see that the number of allowed values for $k^\pm$ is finite, and we will call this number $N_0^\pm$. We will also label the solutions $k_n^\pm$ with a discrete index $n \in \{1, \dots, N_0^\pm\}$ so that $0 < k_n^\pm < \sqrt{U_0}$ and $k_n^\pm < k_{n+1}^\pm$. We will also relabel $\omega_k^\pm$ into $\omega_n^\pm$ and $A_k^\pm$ into $A_n^\pm$ when dealing with the low energy sector. The plot of the finite eigenvalues alongside with the potential is shown in Figure 4.3.

Let us finally write down the low energy modes (4.5a):

$$u_{n,+}^{in,<}(t,x) = 2A_n^+ e^{-ik_n^+ t} \cos(k_n^+ x), \tag{4.15a}$$

$$u_{n,+}^{out,<}(t,x) = 2A_n^+ e^{-ik_n^+ t} \cos(k_n^+ L) e^{-\omega_n^+(|x|-L)}, \tag{4.15b}$$

$$u_{n,-}^{in,<}(t,x) = 2iA_n^- e^{-ik_n^- t} \sin(k_n^- x), \tag{4.15c}$$

$$u_{n,-}^{out,<}(t,x) = 2iA_n^- e^{-ik_n^- t} \sin(k_n^- L) e^{-\omega_n^-(|x|-L)} \operatorname{sgn} x, \tag{4.15d}$$

and the transcendental quantization conditions

$$\sin(k_n^+ L) = \frac{\omega_n^+}{k_n^+} \cos(k_n^+ L), \tag{4.16a}$$

$$\cos(k_n^- L) = -\frac{\omega_n^-}{k_n^-} \sin(k_n^- L), \tag{4.16b}$$

with $0 < k_n^\pm < \sqrt{U_0}$ and $n \in \{1, \dots, N_0^\pm\}$.

4.3 Normalization

We now introduce the Klein-Gordon inner product and compute the normalization factor of the modes that we found in the previous section. We define

$$\langle u_1, u_2 \rangle = -i \int_{-\infty}^{+\infty} dx (u_1 \partial_t u_2^* - u_2^* \partial_t u_1). \tag{4.17}$$

In the case where the spacetime is static, the modes can be factorized as

$$u_i(t, x) = e^{-ik_i t} f_i(x),$$

so that the Klein-Gordon inner product reduces to

$$\langle u_1, u_2 \rangle = (k_1 + k_2) \int_{-\infty}^{+\infty} dx f_1(x) f_2^*(x). \tag{4.18}$$

Thanks to the way we built our modes, we can avoid checking if they are orthogonal, since we used orthogonal bases when constructing solutions. We are then left with computing the coefficients $A_n^\pm$ and $A_k^\pm$ of discrete and continuous modes respectively. In our case, the spacial part of the modes is defined piecewise:

$$f(x) = \begin{cases} f^{in}(x) & \text{if } |x| \leq L \\ f^{out}(x) & \text{if } |x| > L. \end{cases}$$

In addition, the modes have definite parity, so that when they are squared they become even functions $f^2(-x) = f^2(x)$ and the integral becomes

$$\int_{-\infty}^{+\infty} f^2(x) dx = 2 \left(\int_0^L + \int_L^{+\infty} \right) f^2(x) dx = 2 \left(\int_0^L f^{in}(x)^2 dx + \int_L^{+\infty} f^{out}(x)^2 dx \right).$$

We will use the above relation in the calculation of the squared norm of our modes.

4.3.1 Discrete modes

We need to compute the coefficients $A_n^\pm$ of (4.15) by imposing normalization of the modes. We get

$$|A_n^\pm| = \sqrt{\frac{\omega_n^\pm}{8k_n^\pm(1 + \omega_n^\pm L)}} \tag{4.19}$$

Proof. *We have*

$$\begin{aligned} \langle u_{n,+}^<, u_{n,+}^< \rangle &= 16 k_n^+ |A_n^+|^2 \left(\int_0^L \cos^2(k_n^+ x) dx + \int_L^{+\infty} e^{-2\omega_n^+(x-L)} \cos^2(k_n L) dx \right) \\ &= 16 k_n^+ |A_n^+|^2 \left(\int_0^L \frac{1 + \cos(2k_n^+ x)}{2} dx + \frac{\cos^2(k_n^+ L)}{2\omega_n^+} \right) \\ &= 8 k_n^+ |A_n^+|^2 \left(L + \frac{\sin(2k_n^+ L)}{2k_n^+} + \frac{\cos^2(k_n^+ L)}{\omega_n^+} \right) \\ &= 8 k_n^+ |A_n^+|^2 \left(L + \frac{1}{k_n^+} \sin(k_n^+ L) \cos(k_n^+ L) + \frac{1}{\omega_n^+} \cos^2(k_n^+ L) \right). \end{aligned}$$

Now, we use the transcendental quantization condition (4.16a) *by substituting* $\cos(k_n^+) = \frac{k_n^+}{\omega_n^+}\sin(k_n^+ L)$*:*

$$\begin{aligned}\langle u^<_{n,+}, u^<_{n,+}\rangle &= 8k_n^+|A_n^+|^2\left(L + \frac{1}{\omega_n^+}\sin^2(k_n^+ L) + \frac{1}{\omega_n^+}\cos^2(k_n^+ L)\right)\\ &= 8k_n^+|A_n^+|^2\left(L + \frac{1}{\omega_n^+}\right) = |A_n^+|^2 \cdot 8k_n^+\frac{\omega_n^+ L + 1}{\omega_n^+}.\end{aligned}$$

If we impose that the above expression equals 1*, we can invert for* $|A_n^+|$ *and get the final result.*

Now, we do the same with the odd one:

$$\begin{aligned}\langle u^<_{n,-}, u^<_{n,-}\rangle &= 16k_n^-|A_n^-|^2\left(\int_0^L \sin^2(k_n^- x)dx + \int_L^{+\infty} e^{-2\omega_n^-(x-L)}\sin^2(k_n L)dx\right)\\ &= 8k_n^-|A_n^-|^2\left(L - \frac{\sin(2k_n^- L)}{2k_n^-} + \frac{\sin^2(k_n^- L)}{\omega_n^-}\right)\\ &= 8k_n^-|A_n^-|^2\left(L - \frac{1}{k_n^-}\sin(k_n^- L)\cos(k_n^- L) + \frac{1}{\omega_n^-}\sin^2(k_n^- L)\right).\end{aligned}$$

We now employ the transcendental quantization condition (4.16b) *by substituting* $\sin(k_n^- L) = -\frac{k_n^-}{\omega_n^-}\cos(k_n^- L)$*, and get*

$$\begin{aligned}\langle u^<_{n,-}, u^<_{n,-}\rangle &= 8k_n^-|A_n^-|^2\left(L + \frac{1}{\omega_n^-}\cos^2(k_n^- L) + \frac{1}{\omega_n^-}\sin^2(k_n^- L)\right)\\ &= 8k_n^-|A_n^-|^2\left(L + \frac{1}{\omega_n^-}\right) = |A_n^-|^2 \cdot 8k_n^-\frac{\omega_n^- L + 1}{\omega_n^-}.\end{aligned}$$

By inverting the above relation for $|A_n^-|$ *after imposing it equals* 1*, we get the final result.*

■

Of course, the normalization coefficients $A_n^\pm$ are defined up to an arbitrary phase factor, and we can choose them so that the factor i in the odd sector gets cancelled and the final expressions for the modes are real. The plots of the spacial part of the even and odd normal modes for the discrete case are shown in Figure 4.4.

4.3.2 Continuous modes

For the continuous modes, the normalization factors for the even and odd ones are, respectively,

$$|A_k^+| = \left[4\omega_k\left(\cos^2(kL) + \frac{k^2}{\omega_k^2}\sin^2(kL)\right)\right]^{-\frac{1}{2}} \tag{4.20a}$$

$$|A_k^-| = \left[4\omega_k\left(\sin^2(kL) + \frac{k^2}{\omega_k^2}\cos^2(kL)\right)\right]^{-\frac{1}{2}} \tag{4.20b}$$

PROOF. *Since the product of two functions having the same parity is an even function, we can take double the integral from* 0 *to* $+\infty$ *instead of the integral over all* $\mathbb{R}$*. We have*

$$\begin{aligned}\langle u^>_{k,+}, u^>_{q,+}\rangle = 8A_k^{+*}A_q^+\cdot(k+q)\Bigg[&\int_0^L \cos(kx)\cos(qx)dx +\\ &\int_L^{+\infty}\left(\cos(kL)\cos[\omega_k(x-L)] - \frac{k}{\omega_k}\sin(kL)\sin[\omega_k(x-L)]\right)\\ &\left(\cos(qL)\cos[\omega_q(x-L)] - \frac{q}{\omega_q}\sin(qL)\sin[\omega_q(x-L)]\right)dx\Bigg].\end{aligned}$$

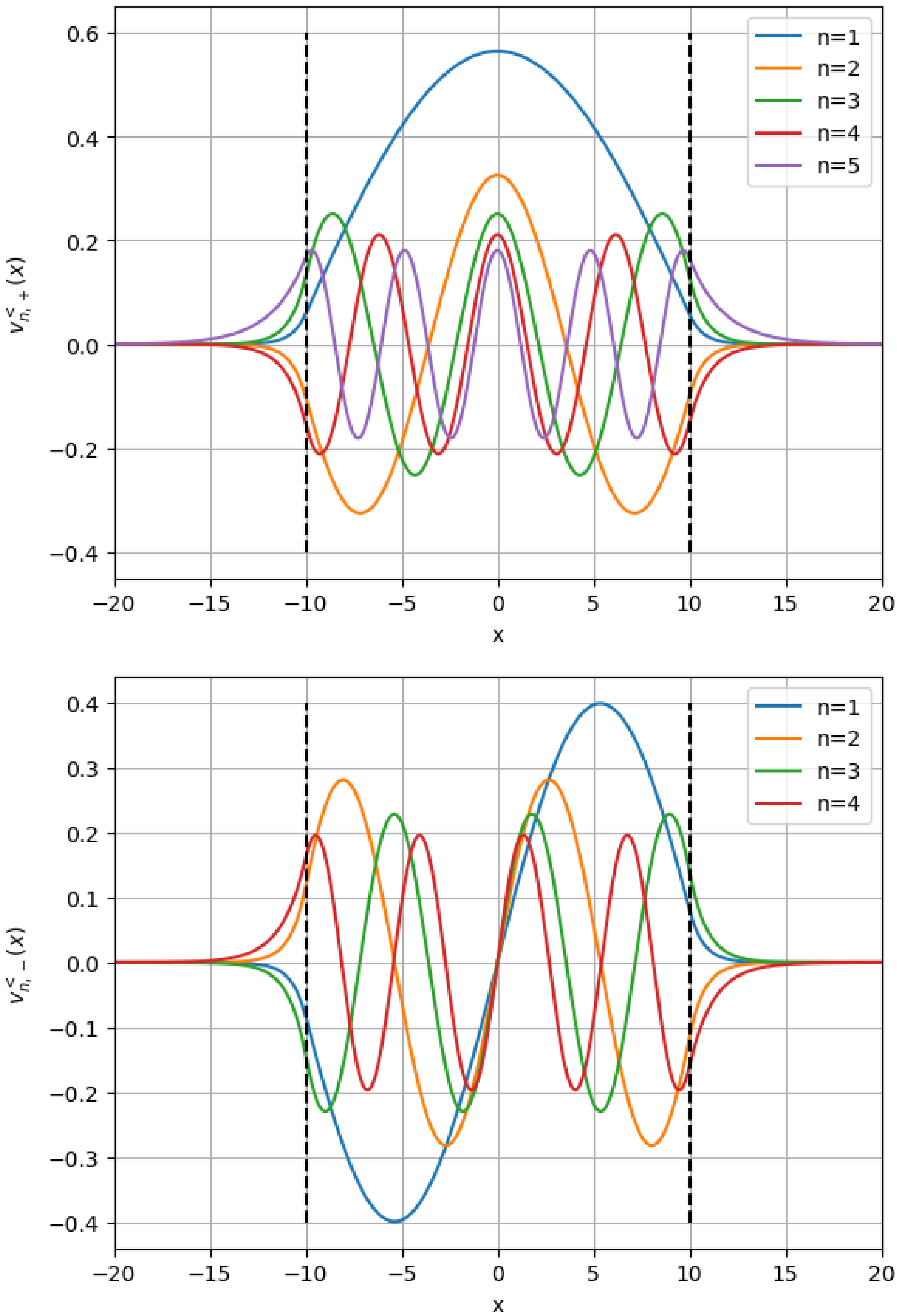

Figure 4.4: Plots of the spacial component $v^<_{n,\pm}(x)$ of the discrete normal modes $u^<_{n,\pm}(t,x) = e^{-ik^\pm_n t} v^<_{n,\pm}(x)$ in the even (above) and odd (below) cases, with $U_0 = 2.0$ and $L = 10.0$.

By substituting $y = x - L$ in the second integral, we have

$$\langle u^{>}_{k,+}, u^{>}_{q,+}\rangle = 8A_k^{+*}A_q^{+} \cdot (k+q)\Bigg[\int_0^L \cos(kx)\cos(qx)dx + \int_0^{+\infty}\left(\cos(kL)\cos(\omega_k y) - \frac{k}{\omega_k}\sin(kL)\sin(\omega_k y)\right)\left(\cos(qL)\cos(\omega_q y) - \frac{q}{\omega_q}\sin(qL)\sin(\omega_q y)\right)dy\Bigg].$$

Our goal is to compute the coefficient of the delta distribution that comes out due to the spectrum being continuous. We already know that when $k \neq q$ the modes are orthogonal, so we only care about $k = q$. In this case, we have a singularity due to the expected presence of a delta distribution, and we therefore only care about divergent contributions in the calculation. Let us then discard all the finite terms, getting:

$$\begin{aligned}
\langle u^{>}_{k,+}, u^{>}_{q,+}\rangle &\simeq 8A_k^{+*}A_q^{+} \cdot (k+q)\Bigg[\int_0^{+\infty}\left(\cos(kL)\cos(\omega_k y) - \frac{k}{\omega_k}\sin(kL)\sin(\omega_k y)\right)\left(\cos(qL)\cos(\omega_q y) - \frac{q}{\omega_q}\sin(qL)\sin(\omega_q y)\right)dy\Bigg] \\
&= 8A_k^{+*}A_q^{+} \cdot (k+q)\int_0^{+\infty} dx \cdot \Bigg[\cos(kL)\cos(qL)\cos(\omega_k x)\cos(\omega_q x) - \frac{k}{\omega_k}\sin(kL)\cos(qL)\sin(\omega_k x)\cos(\omega_q x) \\
&\quad - \frac{q}{\omega_q}\sin(qL)\cos(kL)\sin(\omega_q x)\cos(\omega_k x) + \frac{kq}{\omega_k\omega_q}\sin(kL)\sin(qL)\sin(\omega_k x)\sin(\omega_q x)\Bigg]
\end{aligned}$$

Now, we use Eqs. (C.7)*. We have that $\omega_k + \omega_q \neq 0$ always (since both $\omega_k, \omega_q > 0$) so that we can neglect $\delta(\omega_k + \omega_q)$ which is* 0 *and $P\frac{1}{\omega_k+\omega_q}$ which is finite (non-divergent terms can be neglected). We then have*

$$\langle u^{>}_{k,+}, u^{>}_{q,+}\rangle \simeq 8A_k^{+*}A_q^{+} \cdot (k+q)\Bigg[\frac{\pi}{2}\delta(\omega_k - \omega_q)\left(\cos(kL)\cos(qL) + \frac{kq}{\omega_k\omega_q}\sin(kL)\sin(qL)\right) - P\frac{1}{\omega_k - \omega_q}\left(\frac{k}{\omega_k} + \frac{q}{\omega_q}\right)\cdot\frac{1}{2}\sin[(k-q)L]\Bigg].$$

Now, we see that the coefficient of the Cauchy principal value includes a $\sin[(k-q)L]$*, which goes to zero when $\omega_k = \omega_q$ since in this case we also have $k = q$ and can therefore be neglected. In the other term, we can equate $k = q$ and $\omega_k = \omega_q$ but we would like to cast the Dirac delta in terms of $k - q$. We can use the following standard relation*

$$\delta(g(k)) = \sum_i \frac{\delta(k - k_i)}{|g'(k_i)|}, \tag{4.21}$$

where k_i are the zeros of $g(k)$ (see [10] for details about this relation). In our case $g(k) = \sqrt{U_0 - k^2} - \sqrt{U_0 - q^2}$. Since $k > 0$, its only zero is at $k = q$. The derivative evaualted at $k = q$ is $g'(q) = -2k/(2\sqrt{U_0 - k^2}) = -q/\sqrt{U_0 - q^2} = -q/\omega_q$. Therefore, we have

$$\delta(\omega_k - \omega_q) = \frac{\omega_q}{q}\delta(k - q).$$

By substituting above, we get

$$\begin{aligned}
\langle u^{>}_{k,+}, u^{>}_{q,+}\rangle &\simeq 8|A_q^{+}|^2 \cdot 2k\left[\frac{\pi}{2}\frac{\omega_k}{k}\delta(k-q)\left(\cos^2(kL) + \frac{k^2}{\omega_k^2}\sin^2(kL)\right)\right] \\
&= |A_q^{+}|^2 \cdot 8\pi\omega_k\left(\cos^2(kL) + \frac{k^2}{\omega_k^2}\sin^2(kL)\right)\delta(k-q) = 2\pi\delta(k-q),
\end{aligned}$$

where we imposed the right orthonormality expression in the last equality. We then have

$$|A_k^+| = \left[4\omega_k \left(\cos^2(kL) + \frac{k^2}{\omega_k^2}\sin^2(kL)\right)\right]^{-\frac{1}{2}}.$$

Now, let us compute the normalization coefficient A_k^-. We have

$$\begin{aligned}\langle u^>_{k,-}, u^>_{q,-}\rangle = 8A_k^{-*}A_q^- \cdot (k+q)\Bigg[\int_0^L \sin(kx)\sin(qx)dx + \\ \int_0^{+\infty}\left(\frac{k}{\omega_k}\cos(kL)\sin(\omega_k y) + \sin(kL)\cos(\omega_k y)\right) \\ \left(\frac{q}{\omega_q}\cos(qL)\sin(\omega_q y) + \sin(qL)\cos(\omega_q y)\right)dy\Bigg],\end{aligned}$$

where we already performed the substitution $y = x - L$. Now, as before, we can discard the finite contribution given by the first integral, and write

$$\begin{aligned}\langle u^>_{k,-}, u^>_{q,-}\rangle \simeq 8A_k^{-*}A_q^- \cdot (k+q)\int_0^{+\infty} dx \cdot \\ \Bigg[\frac{kq}{\omega_k\omega_q}\cos(kL)\cos(qL)\sin(\omega_k x)\sin(\omega_q x) + \frac{k}{\omega_k}\cos(kL)\sin(qL)\sin(\omega_k x)\cos(\omega_q x) \\ + \frac{q}{\omega_q}\cos(qL)\sin(kL)\sin(\omega_q x)\cos(\omega_k x) + \sin(kL)\sin(qL)\cos(\omega_k x)\cos(\omega_q x)\Bigg].\end{aligned}$$

Again, by neglecting terms proportional to $P\frac{1}{\omega_k+\omega_q}$ and $\delta(\omega_k + \omega_q)$, we have

$$\begin{aligned}\langle u^>_{k,-}, u^>_{q,-}\rangle \simeq 8A_k^{-*}A_q^- \cdot (k+q)\Bigg[\frac{\pi}{2}\delta(\omega_k - \omega_q)\left(\frac{kq}{\omega_k\omega_q}\cos(kL)\cos(qL) + \sin(kL)\sin(qL)\right) \\ + P\frac{1}{\omega_k - \omega_q}\left(\frac{k}{\omega_k} + \frac{q}{\omega_q}\right)\cdot\frac{1}{2}\sin[(k-q)L]\Bigg].\end{aligned}$$

We see that the term proportional to $P\frac{1}{\omega_k-\omega_q}$ vanishes when $k = q$, and so we are left with

$$\begin{aligned}\langle u^>_{k,-}, u^>_{q,-}\rangle &\simeq 8|A_q^-|^2 \cdot 2k\left[\frac{\pi}{2}\frac{\omega_k}{k}\delta(k-q)\left(\frac{k^2}{\omega_k^2}\cos^2(kL) + \sin^2(kL)\right)\right] \\ &= |A_q^-|^2 \cdot 8\pi\omega_k\left(\frac{k^2}{\omega_k^2}\cos^2(kL) + \sin^2(kL)\right)\delta(k-q) = 2\pi\delta(k-q),\end{aligned}$$

where we used again (4.21)*. By imposing the last equality, we have*

$$|A_k^-| = \left[4\omega_k\left(\sin^2(kL) + \frac{k^2}{\omega_k^2}\cos^2(kL)\right)\right]^{-\frac{1}{2}}.$$

■

The plots of the spacial components of the even and odd normal modes in the continuous spectrum are shown in Figure 4.5.

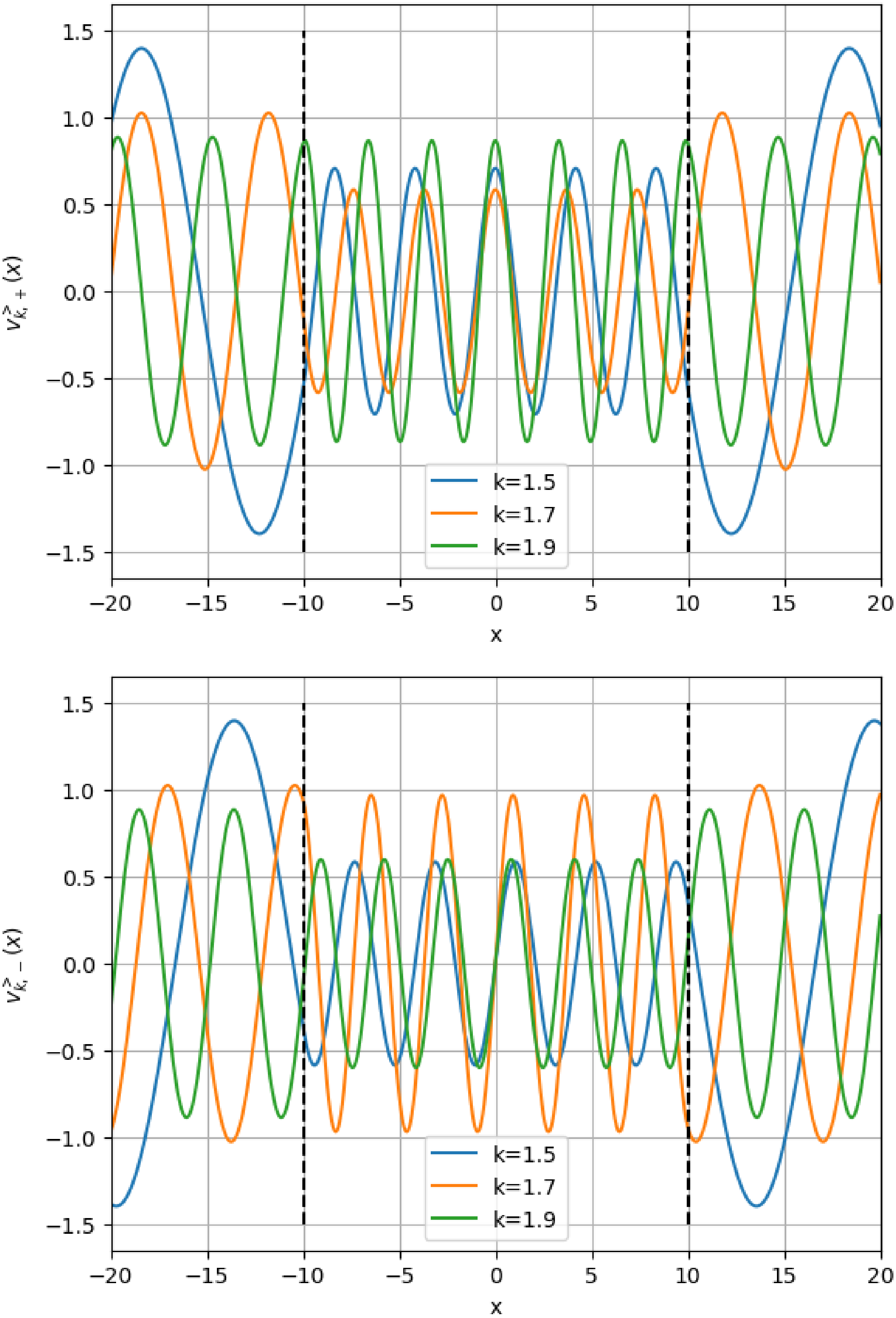

Figure 4.5: Plots of the spacial part $v^{>}_{k,\pm}(x)$ of the continuous family of normal modes $u^{>}_{k,\pm} = e^{-ikt} v^{>}_{k,\pm}(x)$ in the even (above) and odd (below) cases, with $U_0 = 2.0$ and $L = 10.0$.

4.4 Quantization

Let us now write the general solution of the field equation, which is a linear combination of the orthonormal even and odd modes:

$$
\begin{aligned}
\phi(t,x) &= \phi_+(t,x) + \phi_-(t,x) = \sum_{s=+,-} \phi_s(t,x), \\
\phi_\pm(t,x) &= \phi_\pm^<(t,x) + \phi_\pm^>(t,x), \\
\phi_\pm^<(t,x) &= \sum_{n=1}^{N_0^\pm} \left[a_{\pm,n} u_{\pm,n}^<(t,x) + a_{\pm,n}^* u_{\pm,n}^{<*}(t,x) \right], \\
\phi_\pm^>(t,x) &= \int_{\sqrt{U_0}}^{+\infty} \frac{dk}{2\pi} \left[b_{\pm,k} u_{\pm,k}^>(t,x) + b_{\pm,k}^* u_{\pm,k}^{>*}(t,x) \right].
\end{aligned}
$$

By promoting the coefficients to creation-annihilation operator pairs, we obtain the canonical commutation relations

$$
\begin{aligned}
\left[\hat\phi(t,x), \hat\phi(t,y)\right] &= 0, \\
\left[\hat\Pi(t,x), \hat\Pi(t,y)\right] &= 0, \\
\left[\hat\phi(t,x), \hat\Pi(t,y)\right] &= i\delta(x-y).
\end{aligned}
$$

The algebra of the creation-annihilation operator pairs is given by the set of all possible commutators between them, the only nonvanishing ones being

$$
\begin{aligned}
\left[\hat a_{s,n}, \hat a^\dagger_{s',n'}\right] &= \delta_{ss'}\delta_{nn'}, \\
\left[\hat b_{s,k}, \hat b^\dagger_{s',k'}\right] &= \delta_{ss'} \cdot 2\pi\delta(k-k').
\end{aligned}
$$

One can then proceed with the usual Fock space construction, by assuming the vacuum $|0\rangle$ to be the state that is annihilated by all the annihilation operators, and by creating all the excited states by acting with creation (bosonic) operators on the vacuum. We will make use of the following identities:

$$
\langle \hat a_{n,s} \hat a^\dagger_{m,s'} \rangle = \left\langle \left[\hat a_{n,s}, \hat a^\dagger_{m,s'}\right] \right\rangle - \cancel{\langle \hat a^\dagger_{m,s'} \hat a_{n,s} \rangle} = \delta_{ss'}\delta_{nm} \tag{4.22a}
$$

$$
\langle \hat b_{k,s} \hat b^\dagger_{q,s'} \rangle = \left\langle \left[\hat b_{k,s}, \hat b^\dagger_{q,s'}\right] \right\rangle - \cancel{\langle \hat b^\dagger_{q,s'} \hat b_{k,s} \rangle} = 2\pi\delta_{ss'}\delta(k-q) \tag{4.22b}
$$

4.5 Propagator inside

Let us write down the explicit expression of the field operator inside, by using (4.5):

$$\begin{aligned}\hat{\phi}(t,x) &= \sum_{n=1}^{N_0^+} 2A_n^+ \cos(k_n^+ x)\left(\hat{a}_{n,+}e^{-ik_n^+ t} + \hat{a}_{n,+}^\dagger e^{ik_n^+ t}\right)\\ &+ \sum_{n=1}^{N_0^-} 2A_n^- \sin(k_n^+ x)\left(\hat{a}_{n,-}e^{-ik_n^- t} + \hat{a}_{n,-}^\dagger e^{ik_n^- t}\right)\\ &+ \int_{\sqrt{U_0}}^{+\infty} \frac{dk}{2\pi} 2A_k^+ \cos(kx)\left(\hat{b}_{k,+}e^{-ikt} + \hat{b}_{k,+}^\dagger e^{ikt}\right)\\ &+ \int_{\sqrt{U_0}}^{+\infty} \frac{dk}{2\pi} 2A_k^- \sin(kx)\left(\hat{b}_{k,-}e^{-ikt} + \hat{b}_{k,-}^\dagger e^{ikt}\right).\end{aligned}$$

To write down the propagator, one needs to compute $\langle \mathrm{T}\hat{\phi}(x)\hat{\phi}(y)\rangle$, but the only terms which do not get annihilated are those proportional to $\langle \hat{a}_{n,+}\hat{a}_{n,+}^\dagger\rangle$, $\langle \hat{a}_{n,-}\hat{a}_{n,-}^\dagger\rangle$, $\langle \hat{b}_{n,+}\hat{b}_{n,+}^\dagger\rangle$, $\langle \hat{b}_{n,-}\hat{b}_{n,-}^\dagger\rangle$. We have

$$\begin{aligned}\langle \mathrm{T}\hat{\phi}(t,x)\hat{\phi}(t',y)\rangle &= \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} 4A_n^+A_m^+ \cos(k_n^+ x)\cos(k_m^+ y)e^{\mp i(k_n^+ t - k_m^+ t')}\langle \hat{a}_{n,+}\hat{a}_{m,+}^\dagger\rangle\\ &+ \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} 4A_n^-A_m^- \cos(k_n^- x)\cos(k_m^- y)e^{\mp i(k_n^- t - k_m^- t')}\langle \hat{a}_{n,-}\hat{a}_{m,-}^\dagger\rangle\\ &+ \int_{\sqrt{U_0}}^{+\infty}\int_{\sqrt{U_0}}^{+\infty} \frac{dkdq}{(2\pi)^2} 4A_k^+A_q^+ \cos(kx)\cos(qy)e^{\mp i(kt-qt')}\langle \hat{b}_{k,+}\hat{b}_{q,+}^\dagger\rangle\\ &+ \int_{\sqrt{U_0}}^{+\infty}\int_{\sqrt{U_0}}^{+\infty} \frac{dkdq}{(2\pi)^2} 4A_k^-A_q^- \cos(kx)\cos(qy)e^{\mp i(kt-qt')}\langle \hat{b}_{k,-}\hat{b}_{q,-}^\dagger\rangle,\end{aligned}$$

where the sign $\mp$ stands for $\mathrm{sgn}(t'-t)$. Now, we can use (4.22) and get

$$\begin{aligned}\langle \mathrm{T}\hat{\phi}(t,x)\hat{\phi}(t',y)\rangle &= \sum_{n=1}^{N_0^+} 4|A_n^+|^2 \cos(k_n^+ x)\cos(k_n^+ y)e^{-ik_n^+|t-t'|}\\ &+ \sum_{n=1}^{N_0^-} 4|A_n^-|^2 \sin(k_n^- x)\sin(k_n^- y)e^{-ik_n^-|t-t'|}\\ &+ \int_{\sqrt{U_0}}^{+\infty} \frac{dk}{2\pi} 4|A_k^+|^2 \cos(kx)\cos(ky)e^{-ik|t-t'|}\\ &+ \int_{\sqrt{U_0}}^{+\infty} \frac{dk}{2\pi} 4|A_k^-|^2 \sin(kx)\sin(ky)e^{-ik|t-t'|}.\end{aligned}$$

What we want to do now is to check if the divergence of the propagator is of the Hadamard form, and since we are in two dimensional flat spacetime, we aim at the parametrix (2.5.3):

$$G^F_{sing}(t,x,t',y) = -\frac{i}{4\pi}\log\left[\frac{-(t-t')^2+(x-y)^2}{2} + i0^+\right].$$

The above expression is singular at coincidence limit $t' \to t$, $y \to x$. We will take the limit spacially, by first setting $t' = t$ and then letting $y \to x$. We have

$$G^F_{sing}(t,x,t,y) \simeq -\frac{i}{4\pi}\log[(x-y)^2] = -\frac{i}{2\pi}\log(x-y), \tag{4.23}$$

where we discarded a finite term coming from the fixed factor of 2 of the denominator inside the logarithm. Indeed, we are only interested in the divergent contribution when y approaches x. From the expression of $\langle \mathrm{T}\hat{\phi}(t,x)\hat{\phi}(t',y)\rangle$ above, we only care about the two integrals, since the two sums are finite and we are only interested in terms with ultraviolet divergences. By substituting the expressions of $A_k^{\pm}$ and setting $t' = t$, we end up with the following integrals:

$$\begin{aligned}
\langle \mathrm{T}\hat{\phi}(t,x)\hat{\phi}(t,y)\rangle =& \int_{\sqrt{U_0}}^{+\infty}\frac{dk}{2\pi}\left[\frac{(k^2-U_0)\cos^2(kL)+k^2\sin^2(kL)}{\sqrt{k^2-U_0}}\right]^{-1}\cos(kx)\cos(ky)\\
&+\int_{\sqrt{U_0}}^{+\infty}\frac{dk}{2\pi}\left[\frac{(k^2-U_0)\sin^2(kL)+k^2\cos^2(kL)}{\sqrt{k^2-U_0}}\right]^{-1}\sin(kx)\sin(ky)\\
&\simeq \int_{\sqrt{U_0}}^{+\infty}\frac{dk}{2\pi}\frac{1}{k}\cos(kx)\cos(ky)+\int_{\sqrt{U_0}}^{+\infty}\frac{dk}{2\pi}\frac{1}{k}\sin(kx)\sin(ky)+\mathcal{O}(1),
\end{aligned}$$

PROOF. *We need to manipulate the coefficient. Let us denote $f(x) = \sin^2(x)$ or $\cos^2(x)$ interchangeably so that in any case $0 \leq f(x) \leq 1$. The factors inside the brackets in the above integrals can both be put in the same form:*

$$\frac{\sqrt{k^2-U_0}}{k^2-U_0 f(kL)} = \frac{k\sqrt{1-\frac{U_0}{k^2}}}{k^2\left(1-\frac{U_0}{k^2}f(kL)\right)} = \frac{1}{k}\frac{\sqrt{1-\frac{U_0}{k^2}}}{1-\frac{U_0}{k^2}f(kL)}$$

We have, by the fact that $0 \leq f(x) \leq 1$:

$$\frac{\sqrt{1-\frac{U_0}{k^2}}}{1-\frac{U_0}{k^2}f(kL)} \leq \frac{\sqrt{1-\frac{U_0}{k^2}}}{1-\frac{U_0}{k^2}} = \frac{1}{\sqrt{1-\frac{U_0}{k^2}}} = 1+\frac{U_0}{2k^2}+\mathcal{O}(k^{-4}),$$

and

$$\frac{\sqrt{1-\frac{U_0}{k^2}}}{1-\frac{U_0}{k^2}f(kL)} \geq \sqrt{1-\frac{U_0}{k^2}} = 1-\frac{U_0}{2k^2}+\mathcal{O}(k^{-4}).$$

Therefore, we have

$$\frac{\sqrt{1-\frac{U_0}{k^2}}}{1-\frac{U_0}{k^2}f(kL)} = 1+\mathcal{O}(k^{-2})$$

We can therefore conclude that the prefactor yields

$$|A_k^{\pm}| = \frac{1}{k}(1+\mathcal{O}(k^{-2})) = \frac{1}{k}+\mathcal{O}(k^{-3}).$$

The divergence, therefore, only comes from the $1/k$ term, while the $\mathcal{O}(k^{-3})$ term gives a finite contribution upon integration, which we denote with $\mathcal{O}(1)$.

■

By performing the integrals, we get

$$\langle \mathrm{T}\hat{\phi}(t,x)\hat{\phi}(t,y)\rangle = -\frac{1}{2\pi}\,\mathrm{Ci}\left[\sqrt{U_0}(x-y)\right] + \mathcal{O}(1),$$

where Ci is the cosine integral function, defined in [36], at Chapter 6.

Proof. By using Werner formulas, we get

$$\begin{aligned}
\int_{\sqrt{U_0}}^{+\infty}\frac{dk}{2\pi}\frac{1}{k}\cos(kx)\cos(ky) &= \frac{1}{2}\left[\int_{\sqrt{U_0}}^{+\infty}\frac{dk}{2\pi}\frac{\cos[k(x-y)]}{k} + \int_{\sqrt{U_0}}^{+\infty}\frac{dk}{2\pi}\frac{\cos[k(x+y)]}{k}\right] \\
&= \frac{1}{4\pi}\left(-\,\mathrm{Ci}\left[\sqrt{U_0}(x-y)\right] - \mathrm{Ci}\left[\sqrt{U_0}(x+y)\right]\right) \\
\int_{\sqrt{U_0}}^{+\infty}\frac{dk}{2\pi}\frac{1}{k}\sin(kx)\sin(ky) &= \frac{1}{2}\left[\int_{\sqrt{U_0}}^{+\infty}\frac{dk}{2\pi}\frac{\cos[k(x-y)]}{k} - \int_{\sqrt{U_0}}^{+\infty}\frac{dk}{2\pi}\frac{\cos[k(x+y)]}{k}\right] \\
&= \frac{1}{4\pi}\left(-\,\mathrm{Ci}\left[\sqrt{U_0}(x-y)\right] + \mathrm{Ci}\left[\sqrt{U_0}(x+y)\right]\right).
\end{aligned}$$

By summing the above two integrals, we get the final result.

■

The two-point function $\langle \mathrm{T}\hat{\phi}(t,x)\hat{\phi}(t,y)\rangle$ is related to the propagator via an i factor (see eq. 21 of [17]), and yields

$$G^F(t,x,t,y) = i\langle \mathrm{T}\hat{\phi}(t,x)\hat{\phi}(t,y)\rangle = -\frac{i}{2\pi}\,\mathrm{Ci}\left[\sqrt{U_0}(x-y)\right] + \mathcal{O}(1)$$

By using the series expansion of the cosine integral function (see 6.6.6 in DLMF [36]) we can write (after absorbing the Euler constant and the power series into $\mathcal{O}(1)$)

$$G^F(t,x,t,y) = -\frac{i}{2\pi}\log\left[\sqrt{U_0}(x-y)\right] + \mathcal{O}(1) = -\frac{i}{2\pi}\log(x-y) + \mathcal{O}(1),$$

where we also absorbed the term proportional to $\log\sqrt{U_0}$ in the finite remainder.

We therefore see that the divergent contribution in the coincidence limit exactly matches the Hadamard parametrix (4.23) everywhere.

4.6 Final considerations and remarks

In this chapter, we introduced a step potential in 1+1-dimensional flat spacetime to model a compact object. We obtained a singularity structure of the propagator inside the object that matches the Hadamard parametrix, even close to the boundary of the object, that we saw gives rise to problems when the spacial dimensionality is three. This is also what happened in the case of a Dirichlet boundary, and it probably hints to a general rule that in 1+1 dimensions the singular part of the propagator near the boundary is of Hadamard type, and numerical work can be done starting from this conclusion.

It is worth to notice that we were able to detect the type of UV divergence in this case because we had a one-dimensional integral whose result could be written in a closed form in terms of the Ci special function. When one extends the problem to three spacial dimensions, a sum over ℓ appears from the spherical harmonic addition theorem, and we saw in the previous chapter that the calculations become more involved, especially near the boundary. In the Dirichlet case, we were able to perform a Taylor expansion of the spherical Bessel functions near the boundary, and the terms that came out simplified with the complicated prefactor exactly. Our problem, then, reduced to just computing a heat trace of a fractional laplacian operator. If, instead of imposing Dirichlet boundary conditions, we impose matching conditions, the Taylor expansion gives additional zero-th order terms, which were absent in the Dirichlet case, and calculations become even more complicated. To get a grasp about the nature of the leading singularity, one can try to engineer a numerical method that fits the divergent power law near coincidence, but this is left for future works.

Chapter 5

Remarks and conclusions

The aim of this thesis was to better understand the role that boundaries and dimensionality play in the divergences of the energy-momentum tensor, in order to lay the foundations for further research and numerical work on computing the finite part of it and investigating the quantum properties of the matter that constitutes strongly gravitationally coupled objects. In particular, we considered the problem of a homogeneous static and spherically symmetric star near the Buchdahl limit, and saw that the equation that governs the dynamics of a real massless scalar on this background can be reduced to a time-independent Schrödinger-like equation with an effective potential that contains a jump discontinuity. Inspired by this fact, we then proceeded to study simpler models in flat spacetime, by keeping in mind that the question one needs to answer is: what boundary conditions does one need to enforce in order to have a physically consistent model of a compact object? Indeed, the end goal of finding the energy-momentum tensor must go through the process of renormalization, and this is where physical conditions play a crucial role.

Motivated by the ambition of studying QFT on backgrounds which more and more realistically model the existing stars, we focused on the most general renormalization procedure that is available for QFT on curved spacetime: the Hadamard renormalization. This relies on the fact that, in the absence of boundaries and topological singularities, the singular structure of the propagator at the coincidence limit is only dependent on the local geometry, and no global information like the quantum state or far away sources enter it. By having the expression of the Feynman propagator at least to second order in spacetime coordinates near the coincidence limit, one can directly subtract the singularities and find the renormalized energy-momentum tensor by just performing derivatives and limits. It is clear that, if the procedure is successful, one can interpret the final result in physical terms and work with it, but in this thesis it is shown that boundaries can spoil the Hadamard parametrix. Therefore, the question arises whether quantities which diverge more than the Hadamard parametrix have any physical meaning at all.

To begin with, we applied the Hadamard procedure to a well-known case of a real massless scalar in two spacetime dimensions, enclosed in a Dirichlet box. Here, we computed the renormalized energy-momentum tensor in two different ways: by explicitly introducing a regulator in the formula and manually subtracting the divergent part, and by applying the Hadamard procedure. We showed that the Feynman propagator has the same singularity as the Hadamard parametrix prescribes, and therefore the renormalization procedure is successful. In both cases, we found the same components of the (vacuum expectation value of the) renormalized energy-momentum tensor.

Then, we proceeded to extend the spacial dimensionality from one to three, by retaining spherical symmetry. Here, the calculations were much more involved, since we had to deal with spherical Bessel functions and zeros. By using the heat kernel formalism, we managed to obtain the expression of the time-split Feynman propagator, and we observed that, near the boundary, the Hadamard parametrix was not matched. We therefore have one first interesting result: by increasing the number of spacial dimensions from one to three, Dirichlet boundaries spoil the Hadamard renormalization procedure near the boundary. In the bulk region, away from the Dirichlet boundary, we saw that the Feynman propagator had the Hadamard form locally, and we managed to find the renormalized energy density at the center of the spherical cavity. Proximate to the center, one can think of performing numerical calculations, since the divergent part is known and can easily be subtracted, leaving the computer working only with finite expressions. However, the issue near the boundary remains, and it is not clear whether one can work with Dirichlet boundaries at all in this case, since the physical meaning of mathematical expressions is only assigned after renormalization.

After that, we enhanced the first two-dimensional model by substituting the Dirichlet boundary with a simple step potential, and enforcing continuity of the field and its spacial derivative across the jump. Here, we were able to compute normal modes and find the singular part of the Feynman propagator everywhere. We saw that also in this case the Hadamard parametrix was matched inside the object, and the road for numerical calculations to find the renormalized energy-momentum tensor is open, even close to the boundary. Another interesting result seems to stem out of our work: two-dimensional models seem to keep the Hadamard singular part unspoiled, but this has to be checked in the general case, maybe studying geodesic reflection.

The path to a full understanding of the quantum properties of matter in compact stars is still long, but the hope is that this thesis can constitute a small step in that direction, and lay the foundations for further research about this topic, which is located inside the bigger picture of understanding quantum effects in gravity.

Appendix A

Legendre polynomials and spherical harmonics

For reference and clarity, in this appendix we will state some facts regarding Legendre polynomials and spherical harmonics which are used throughout this dissertation. The references are [38, 37, 39].

A.1 Legendre polynomials

Legendre polynomials can be defined via the Rodrigues' formula as ($\ell \in \{0, 1, 2, \ldots\}$)

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell.$$

The set of all Legendre polynomials $\{P_\ell(x)\}_{\ell=0}^{+\infty}$ forms a complete orthogonal basis for the space $L^2([-1, 1], \mathbb{R}, w(x) = 1)$. We have

$$P_\ell(1) = 1, \tag{A.1}$$

and

$$\int_{-1}^{1} P_\ell(x) P_{\ell'}(x) dx = \frac{2}{2\ell + 1} \delta_{\ell\ell'} \tag{A.2}$$

A.2 Associated Legendre polynomials

Associated Legendre polynomials $P_\ell^m(x)$ can be defined via the following formula (with $\ell \in \{0, 1, 2, \ldots\}$ and $m \in \{0, \ldots, \ell\}$)

$$P_\ell^m(x) = (-1)^m (1 - x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_\ell(x), \tag{A.3}$$

where $P_\ell(x)$ are the unassociated Legendre polynomials discussed in the previous section. Legendre polynomials for negative m are defined by

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell - m)!}{(\ell + m)!} P_\ell^m(x). \tag{A.4}$$

They satisfy the following orthogonality relation

$$\int_{-1}^{1} P_\ell^m(x) P_{\ell'}^m(x) dx = \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!} \delta_{\ell\ell'} \tag{A.5}$$

We also use the convention

$$P_{\ell m}(x) = (-1)^m P_\ell^m(x), \tag{A.6}$$

and the orthogonality relation stays the same due to $(-1)^{2m}$ appearing on both sides due to index lowering:

$$\int_{-1}^{1} P_{\ell m}(x) P_{\ell' m}(x) dx = \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!} \delta_{\ell\ell'}. \tag{A.7}$$

The definition for negative m (A.6) with lower index stays the same (due to a factor $(-1)^m = (-1)^{-m}$ appearing on both sides when lowering the index):

$$P_{\ell,-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_{\ell m}(x). \tag{A.8}$$

A.3 Spherical harmonics

Spherical harmonics are defined as follows:

$$Y_\ell^m(\theta, \varphi) = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\theta) e^{im\varphi}. \tag{A.9}$$

They are orthonormal:

$$\int_0^\pi d\theta \int_0^{2\pi} \sin\theta d\varphi Y_\ell^m(\theta, \varphi) Y_{\ell'}^{m'*}(\theta, \varphi) = \delta_{\ell\ell'} \delta_{mm'}. \tag{A.10}$$

The complex conjugation gives

$$Y_\ell^{m*}(\theta, \varphi) = (-1)^m Y_\ell^{-m}(\theta, \varphi). \tag{A.11}$$

By lowering the m index in the associated Legendre polynomial, we can also lower the index of the spherical harmonics:

$$Y_{\ell m}(\theta, \varphi) = (-1)^m Y_\ell^m(\theta, \varphi), \tag{A.12}$$

and the orthonormality relation is the same (a factor of $(-1)^{m+m'}$ appears, but it equals 1 due to $m = m'$ from the Kronecker delta):

$$\int_0^\pi d\theta \int_0^{2\pi} \sin\theta d\varphi Y_{\ell m}(\theta, \varphi) Y_{\ell' m'}^*(\theta, \varphi) = \delta_{\ell\ell'} \delta_{mm'}. \tag{A.13}$$

There is the so called *spherical harmonic addition theorem* which states

$$\sum_{m=-\ell}^{+\ell} Y_\ell^m(\theta_1, \varphi_1) Y_\ell^{m*}(\theta_2, \varphi_2) = \frac{2\ell+1}{4\pi} P_\ell(\cos\gamma), \tag{A.14}$$

where

$$\cos\gamma = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\varphi_1 - \varphi_2) \tag{A.15}$$

Appendix B

Bessel functions

The definitions and properties of Bessel functions can be found in chapter 10 of NIST DLMF (Digital Library of Mathematical Functions) [36]. Here we report some useful identities and properties that are used throughout this dissertation.

B.1 Bessel function of the first kind

We denote the Bessel function of the first kind of order ν evaluated on z as $J_\nu(z)$, and the n^{th} zero of J_ν as $j_{\nu,n}$. We always consider $\nu \in \mathbb{R}$, so, as stated by [36] in 10.21, $J_\nu(z)$ has a countably infinite number of positive real simple zeros, indexed by $n \in \{1, 2, 3, \ldots\}$.

B.1.1 Recurrece relations

The following recurrence relations are useful (DLMF [36] 10.6.1-2):

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z) \tag{B.1a}$$

$$J_{\nu-1}(z) - J_{\nu+1}(z) = 2J'_\nu(z) \tag{B.1b}$$

$$J'_\nu(z) = J_{\nu-1}(z) - \frac{\nu}{z} J_\nu(z) \tag{B.1c}$$

$$J'_\nu(z) = -J_{\nu+1}(z) + \frac{\nu}{z} J_\nu(z) \tag{B.1d}$$

B.1.2 Orthogonality in finite domain

The following orthogonality relation holds (DLMF [36] 10.22.37):

$$\int_0^1 t J_\nu(j_{\nu,n} t) J_\nu(j_{\nu,n'} t) dt = \frac{1}{2} [J'_\nu(j_{\nu,n})]^2 \delta_{nn'} \tag{B.2}$$

B.2 Spherical Bessel function of the first kind

The spherical Bessel function of the first kind of order ℓ evaluated on z is denoted by $j_\ell(z)$. We assume $\ell \in \mathbb{N} \cup \{0\}$.

B.2.1 Relation with Bessel function of the first kind

The following identity between Bessel functions and spherical Bessel functions of the first kind holds (DLMF [36] 10.47.3):

$$j_\ell(z) = \sqrt{\frac{\pi}{2z}} J_{\ell+\frac{1}{2}}(z). \tag{B.3}$$

B.2.2 Expressions for few spherical Bessel functions of first kind

We have:

$$j_0(z) = \frac{\sin z}{z}, \tag{B.4a}$$

$$j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}, \tag{B.4b}$$

$$j_2(z) = \left(\frac{3}{z^3} - \frac{1}{z}\right)\sin z - \frac{3}{z^2}\cos z. \tag{B.4c}$$

B.2.3 Zeros

The zeros of the spherical Bessel function of the first kind of order ℓ, denoted by $\alpha_{\ell n}$ with $n \in \{1, 2, \ldots\}$, are thus equal to those of the Bessel function of the first kind of order $\ell + \frac{1}{2}$:

$$\alpha_{\ell n} = j_{\ell, n+\frac{1}{2}}. \tag{B.5}$$

B.2.4 Orthogonality for finite domain

The following orthogonality relation holds (with $R > 0$):

$$\int_0^R r^2 dr j_\ell\left(\alpha_{\ell n}\frac{r}{R}\right) j_\ell\left(\alpha_{\ell n'}\frac{r}{R}\right) = \frac{R^3}{2} j^2_{\ell\pm1}(\alpha_{\ell n})\delta_{nn'} \tag{B.6}$$

PROOF. *We want to evaluate*

$$I = \int_0^R r^2 dr j_\ell\left(\alpha_{\ell n}\frac{r}{R}\right) j_\ell\left(\alpha_{\ell n'}\frac{r}{R}\right).$$

We make the following change of variable

$$t = \frac{r}{R},$$

$$dt = \frac{1}{R}dr,$$

so the integral becomes

$$I = R^3 \int_0^1 t^2 dt j_\ell\left(\alpha_{\ell n} t\right) j_\ell\left(\alpha_{\ell n'} t\right).$$

Now, we use (B.3) *twice and get*

$$I = \frac{\pi R^3}{2\sqrt{\alpha_{\ell n}\alpha_{\ell n'}}} \int_0^1 t dt J_{\ell+\frac{1}{2}}(\alpha_{\ell n} t) J_{\ell+\frac{1}{2}}(\alpha_{\ell n'} t).$$

Using (B.2), *we obtain*

$$I = \frac{R^3}{2}\frac{\pi}{2\alpha_{\ell n}}[J'_{\ell+\frac{1}{2}}(\alpha_{\ell n})]^2\delta_{nn'},$$

and employing (B.1c) *and* (B.1d) *we have*

$$\begin{aligned} I &= \frac{R^3}{2}\frac{\pi}{2\alpha_{\ell n}}\left[J_{\ell-\frac{1}{2}}(\alpha_{\ell n}) - \cancel{\frac{\ell+\frac{1}{2}}{\alpha_{\ell n}}J_{\ell+\frac{1}{2}}(\alpha_{\ell n})}\right]^2\delta_{nn'} \\ &= \frac{R^3}{2}\frac{\pi}{2\alpha_{\ell n}}\left[-J_{\ell+\frac{3}{2}}(\alpha_{\ell n}) + \cancel{\frac{\ell+\frac{1}{2}}{\alpha_{\ell n}}J_{\ell+\frac{1}{2}}(\alpha_{\ell n})}\right]^2\delta_{nn'}, \end{aligned}$$

where we recognize that $\alpha_{\ell n} = j_{\ell,n+\frac{1}{2}}$ *and so we cancel the terms proportional to Bessel functions evaluated on their zeros. We thus have found that*

$$I = \frac{R^3}{2}\frac{\pi}{2\alpha_{\ell n}}J^2_{\ell-\frac{1}{2}}(\alpha_{\ell n})\delta_{nn'} = \frac{R^3}{2}\frac{\pi}{2\alpha_{\ell n}}J^2_{\ell+\frac{3}{2}}(\alpha_{\ell n})\delta_{nn'},$$

and by using (B.3) *we finally have*

$$I = \frac{R^3}{2}j^2_{\ell-1}(\alpha_{\ell n})\delta_{nn'} = \frac{R^3}{2}J^2_{\ell+1}(\alpha_{\ell n})\delta_{nn'}.$$

■

Appendix C

Theory of distributions

Many quantities in quantum field theory need to be understood in the distributional sense, and this is sometimes crucial to not lose the mathematical sense on which the actual theory relies on. In this appendix, we are going to give a brief introduction to distribution theory. A good reference is [10], and some topics have been taken from [29]. We also provide an introduction to microlocal analysis, wave-front sets and products of distributions, for which good references are [16, 20].

C.1 Test functions spaces

The main idea is that a distribution $T(x)$ should be viewed as an integral kernel which must be smeared onto a function $f(x)$ like $\int T(x)f(x)dx$, yielding a finite result. Distributions are not functions, since they can be undefined pointwise. A distribution is well defined if and only if its smearing on all of the functions within a certain space, called *test functions space* is well defined. Of course, there are multiple choices that are possible for test functions. The core idea of test functions is that they vanish at the boundary of their domain. Here we present two of the most commonly used in physics. The formal definition of distributions does not rely on integration, being only later connected to it via a notational trick by analogy with a specific example. Let us proceed step by step.

Definition 1 (Compactly Supported Functions). *The space* $\mathcal{D}(\mathbb{R}^n) \equiv C_c^\infty(\mathbb{R}^n)$ *consists of smooth real or complex valued functions with compact support with respect to the standard topology of* $\mathbb{R}^n$.

Being compactly supported and regular, the integral of these functions inside their domain is finite.

Definition 2 (Schwartz Space). *The* Schwartz space $\mathcal{S}(\mathbb{R}^n)$ *is the set of all real or complex valued functions* $f \in C^\infty(\mathbb{R}^n)$ *such that for every pair of multi-indices* α, β,

$$\sup_{x\in\mathbb{R}^n} |x^\alpha \partial_\beta f(x)| < \infty.$$

These are infinitely differentiable functions that, along with all their derivatives, decay faster than any power of $1/|x|$ at infinity. Functions belonging to the Schwartz space are those who have a definite Fourier transform. Schwartz space can be informally thought of as the extension of compactly supported functions space with domain extended at infinity. Of course a compactly supported function is also a Schwartz function.

C.2 Distributions

Now that we have in mind test functions spaces, we can define what is a distribution.

Definition 3 (Distribution). *Given a test function space on* $\mathbb{R}^n$*, denoted by* $\mathcal{D}(\mathbb{R}^n)$*, a distribution* T *on* $\mathbb{R}^n$ *is a continuous linear functional on* $\mathcal{D}(\mathbb{R}^n)$*:*

$$T : \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}.$$

The space of distributions is denoted by $\mathcal{D}'(\mathbb{R}^n)$*. The action of a distribution* $T \in \mathcal{D}'(\mathbb{R}^n)$ *on a test function* $f \in \mathcal{D}(\mathbb{R}^n)$ *as a linear functional is denoted by* $T[f] \in \mathbb{R}$*. If* $\mathcal{D}(\mathbb{R}^n)$ *is the Schwartz space, the distribution is said to be* tempered.

Every locally integrable function $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ defines a tempered distribution T_g on the Schwartz space by

$$T_g[f] = \int_{\mathbb{R}^n} f(x)g(x)dx, \quad f \in \mathcal{D}(\mathbb{R}^n).$$

We keep this example in mind when defining distributions that are not functions, and we will set up a notational trick that makes us write proper distributions as "functions" of the position, and their action on test functions as a smearing integration, where the distribution acts like an integral kernel.

We now define two of the most commonly used distributions: the Dirac delta and the Heaviside step function.

Definition 4 (Dirac Delta). *Given a test function space* $\mathcal{D}(\mathbb{R}^n)$*, the* Dirac delta δ *is the distribution defined by*

$$\delta[f] = f(0), \quad f \in \mathcal{D}(\mathbb{R}).$$

By analogy with the case of distributions defined by locally integrable functions, we can define the notation $\delta(x)$ *to represent the distribution, where the following integral smearing action is intended:*

$$\delta[f] = \int_{\mathbb{R}^n} \delta(x)f(x)d^n x = f(0).$$

Consequently, we can interpret $\delta(x)$ *informally as a "function" which is zero everywhere except at* $x = 0$*, where it equals infinity. However, this functional definition is not sufficient to specify the distributional property of the Dirac delta, and we also need the above integral action.*

From now on, we will use the two notations for distributions interchangeably.

Definition 5 (Heaviside Function). *The* Heaviside step function Θ *is the function defined on* $\mathbb{R}$

$$\Theta(x) = \begin{cases} 0 & x < 0, \\ 1 & x > 0. \end{cases}$$

Θ *is locally integrable, and it acts on test functions via* $\Theta[f] = \int_0^\infty f(x)dx$*.*

Now, we define the concept of distributional derivative, which relies on the analogy with integration by parts.

Definition 6 (Distributional derivative). *Let $\mathcal{D}(\mathbb{R}^n)$ be a test function space and let $T \in \mathcal{D}'(\mathbb{R}^n)$ be a distribution. The distributional derivative of T in the direction x_j is itself a distribution, defined via its action on an arbitrary test function $f \in \mathcal{D}(\mathbb{R}^n)$:*

$$(\partial_j T)[f] = -T[\partial_j f].$$

This definition can be justified by analogy with integration by parts, keeping in mind that test functions vanish at the boundary. Therefore, we have:

$$(\partial_j T)[f] = \int_{\mathbb{R}^n} (\partial_j T)(x) f(x) dx = \cancel{T(x) f(x)|_{\partial \mathbb{R}_j}} - \int_{\mathbb{R}^n} T(x)(\partial_j f)(x) dx = -T[\partial_j f].$$

With this definition, we show that the distributional derivative of the Heaviside step function is the Dirac delta in $\mathbb{R}$.

Theorem 1. *In the distributional sense,*

$$\frac{d}{dx}\Theta(x) = \delta(x).$$

PROOF. *Let $f \in \mathcal{D}(\mathbb{R})$. Then*

$$\frac{d\Theta}{dx}[f] = -\Theta\left[\frac{df}{dx}\right] = -\int_0^\infty f'(x) dx = f(0) - \cancel{f(+\infty)} = f(0),$$

where we used the property that test functions vanish at infinity. Since this holds for all test functions, we have that the distributional derivative of Θ behaves as a Dirac delta, and therefore equals it.

■

Definition 7 (Fourier transform of distributions). *Consider the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of test functions and let $f \in \mathcal{S}(\mathbb{R}^n)$, so that its Fourier transform is well defined. Let $T \in \mathcal{S}'(\mathbb{R}^n)$ be a (tempered) distribution. We define the Fourier transform $\mathcal{F}T$ of T as a distribution via its action on the arbitrary test function f:*

$$(\mathcal{F}T)[f] = T[\mathcal{F}f].$$

This definition reduces to the ordinary Fourier transform if T happens to be an ordinary function, and defines a continuous automorphism in $\mathcal{S}'(\mathbb{R}^n)$.

C.3 Regularizing distributions

In quantum field theory, one often applies a regularization procedure to some seemingly not well defined quantity to make it well defined and convergent. Then, the regularized quantity is carried along in calculations, and only at the end one recovers the original quantity in some limit. To those who are unfamiliar with distributions, this procedure may seem rather obscure, leaving some second thoughts about the legitimacy of the calculation. However, this procedure, called *regularization*, can be given a precise mathematical meaning using distributions. We state the following theorem without proof (see [10], Chapter 5), which clarifies what happens.

Theorem 2 (Mollifier regularization theorem). *Let $T \in \mathcal{D}'(\mathbb{R}^n)$. Then, there exists a family $\{\varphi_\epsilon\}_{\epsilon>0}$ with $\varphi_\epsilon \in C^\infty(\mathbb{R}^n)$ such that*

$$\lim_{\epsilon\to 0^+} \int_{\mathbb{R}^n} \varphi_\epsilon(x) f(x) dx = T[f] \quad \forall f \in \mathcal{D}(\mathbb{R}^n).$$

We can make an example by instantiating the above theorem for the Dirac delta distribution.

Theorem 3. *Let $g : \mathbb{R} \to \mathbb{R}$ be a function of $y \in \mathbb{R}$ such that $g(y) \geq 0$ and*

$$\int_{-\infty}^{+\infty} g(y) dy = 1.$$

Then, define the mollifier*:*

$$\varphi_\epsilon(x) \equiv \epsilon^{-1} g\left(\frac{x}{\epsilon}\right). \tag{C.1}$$

Then, we have, distributionally,

$$\lim_{\epsilon\to 0^+} \varphi_\epsilon(x) = \delta(x)$$

Proof. We have $\varphi_\epsilon(x) \geq 0$ since $\epsilon^{-1} > 0$ and $g(x/\epsilon) \geq 0$. Then, if we consider the substitution $x = \epsilon y$, we have

$$\int_{-\infty}^{+\infty} dx \varphi_\epsilon(x) = \int_{-\infty}^{+\infty} e^{-1} g\left(\frac{x}{\epsilon}\right) = \int_{-\infty}^{+\infty} g(y) dy = 1.$$

Now, pick an arbitrary test function $f(x)$. Then,

$$\int_{-\infty}^{+\infty} \varphi_\epsilon(x) f(x) dx = \int_{-\infty}^{+\infty} \epsilon^{-1} g\left(\frac{x}{\epsilon}\right) f(x) dx = \int_{-\infty}^{+\infty} g(y) f(\epsilon y) dy \xrightarrow[\epsilon\to 0^+]{} \int_{-\infty}^{\infty} g(y) f(0) dy = f(0).$$

This means that φ_ϵ becomes δ when $\epsilon \to 0^+$.

■

If we choose the function

$$g(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}},$$

we have

$$\varphi_\epsilon(x) = \frac{1}{\epsilon\sqrt{2\pi}} e^{-\frac{x^2}{2\epsilon^2}},$$

which is a normalized gaussian function with standard deviation $\sigma = \epsilon$, and by the above theorem we have that the Dirac delta can be viewed as the limit of the gaussian distribution with standard deviation approaching 0. Alternatively, one can prove

$$\lim_{\epsilon\to 0^+} \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2} = \delta(x) \tag{C.2}$$

Proof. Let us exploit the previous theorem with

$$g(y) = \frac{1}{\pi} \frac{1}{1 + y^2} \geq 0,$$

and

$$\int_{-\infty}^{+\infty} g(y) dy = 1.$$

Indeed, by going on the complex plane and closing the contour with a semicircle C_R^+ of radius $R > 0$ on the upper half plane, we have, by Cauchy residue theorem (see [26]):

$$\int_{-\infty}^{+\infty} \frac{1}{1+y^2} dy = \lim_{R\to\infty} \left(\oint_{\mathbb{R}\cup C_R^+} - \not{\int}_{C_R^+} \right) \frac{1}{1+z^2} dz =$$
$$= 2\pi i \operatorname{Res} \left. \frac{1}{(z+i)(z-i)} \right|_{z=i} = 2\pi i \left. \frac{1}{z+i} \right|_{z=i} = \pi,$$

and so $1/\pi$ is the right normalization factor. We then have, in accordance with the previous theorem,

$$\varphi_\epsilon(x) = \frac{1}{\pi\epsilon} \frac{1}{1 + \frac{x^2}{\epsilon^2}} = \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2},$$

and the assertion follows by applying the previous theorem.

■

C.4 Cauchy principal value and Sokhotski-Plemelj theorem

Definition 8 (Cauchy principal value). *We define the distribution $P\frac{1}{x}$, called* Cauchy principal value *of $1/x$, as a distribution. Its action on an arbitrary test function f is*

$$\int_{-\infty}^{+\infty} P\frac{1}{x} f(x) = \lim_{\epsilon\to 0^+} \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{+\infty} \right) \frac{f(x)}{x} dx.$$

The above definition is well posed.

Proof. We have

$$\int_{-\infty}^{-\epsilon} \frac{f(x)}{x} dx + \int_{\epsilon}^{+\infty} \frac{f(x)}{x} dx = \int_{\epsilon}^{+\infty} \frac{f(-x)}{-x} dx + \int_{\epsilon}^{+\infty} \frac{f(x)}{x} dx = \int_{\epsilon}^{+\infty} \frac{f(x) - f(-x)}{x} dx.$$

For $\epsilon \to 0^+$, the integrand becomes $f'(0)$, which is well defined.

■

Theorem 4. *The Cauchy principal value is approximated by the following family of smooth functions, with $\epsilon \to 0^+$:*

$$P\frac{1}{x} = \lim_{\epsilon\to 0^+} \frac{x}{x^2 + \epsilon^2} \tag{C.3}$$

Proof. *Let $f(x)$ be a test function. Then,*

$$\int_{-\infty}^{+\infty} \frac{x}{x^2+\epsilon^2} f(x)dx = \int_{-\epsilon}^{+\epsilon} \frac{x}{x^2+\epsilon^2} f(x)dx + \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{+\infty}\right)\left[\frac{x}{x^2+\epsilon^2} - \frac{1}{x}\right] f(x)dx$$
$$+ \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{+\infty}\right)\frac{1}{x} f(x)dx.$$

Now, we want to show that the first two terms are $\mathcal{O}(\epsilon)$ so that we are left with what we wanted to prove. To do this, we substitute $y = x/\epsilon$. Let us start with the first term

$$\int_{-\epsilon}^{+\epsilon} \frac{x}{x^2+\epsilon^2} f(x)dx = \int_{-1}^{1} \frac{y}{y^2+1} f(\epsilon y)dy = \underbrace{\int_{-1}^{1} \frac{ydy}{y^2+1}}_{=0} f(0) + \mathcal{O}(\epsilon) = \mathcal{O}(\epsilon),$$

where we used the fact that odd integrals vanish on symmetric domains. Now, let us deal with the second term

$$\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{+\infty}\right)\left[\frac{x}{x^2+\epsilon^2} - \frac{1}{x}\right] f(x)dx = \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{+\infty}\right)\frac{-\epsilon^2}{x(x^2+\epsilon^2)} f(x)dx =$$
$$= -\left(\int_{-\infty}^{-1} + \int_{1}^{+\infty}\right)\frac{1}{y(y^2+1)} f(\epsilon y)dy = -\underbrace{\left(\int_{-\infty}^{-1} + \int_{1}^{+\infty}\right)\frac{1}{y(y^2+1)}dy}_{=0}\, f(0) + \mathcal{O}(\epsilon) = \mathcal{O}(\epsilon),$$

where again we made the symmetric odd integral vanish. We therefore have

$$\int_{-\infty}^{+\infty} \frac{x}{x^2+\epsilon^2} f(x)dx = \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{+\infty}\right)\frac{1}{x} f(x)dx + \mathcal{O}(\epsilon),$$

which concludes the proof upon taking the limit $\epsilon \to 0^+$.

■

Theorem 5 (Sokhotski-Plemelj theorem). *We define the distribution*

$$\frac{1}{x \pm i0^+}$$

via its action on a test function $f(x)$:

$$\int_{-\infty}^{+\infty} \frac{1}{x \pm i0^+} f(x)dx = \lim_{\epsilon \to 0^+} \int_{-\infty}^{+\infty} \frac{1}{x \pm i\epsilon} f(x)dx.$$

We have the following distributional identity

$$\frac{1}{x \pm i0^+} = P\frac{1}{x} \mp i\pi\delta(x) \tag{C.4}$$

Proof. *Let $f(x)$ be a test function. We then have*

$$\int_{-\infty}^{+\infty} \frac{1}{x \pm i\epsilon} f(x)dx = \int_{-\infty}^{+\infty} \frac{x \mp i\epsilon}{x^2+\epsilon^2} f(x)dx = \int_{-\infty}^{+\infty} \frac{x}{x^2+\epsilon^2} f(x)dx \mp i\pi \int_{-\infty}^{+\infty} \frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2} f(x)dx$$

Now, we use eqs. (C.3) *and* (C.2) *to take the limit* $\epsilon \to 0^+$ *and write*

$$\lim_{\epsilon \to 0^+} \int_{-\infty}^{+\infty} \frac{1}{x \pm i\epsilon} f(x)dx = \int_{-\infty}^{+\infty} P\frac{1}{x} f(x)dx \mp i\pi \int_{-\infty}^{+\infty} \delta(x)f(x)dx = \int_{-\infty}^{+\infty} \left[P\frac{1}{x} \mp i\pi\delta(x)\right] f(x)dx,$$

which shows us that the distribution $\frac{1}{x \pm i0^+}$ *acts on an arbitrary test function as* $P\frac{1}{x} \mp i\pi\delta(x)$*, and therefore equals it.*

■

C.5 Products of distributions and microlocal analysis

In order to define the product of two distributions, we need to be careful about what happens with the singularities of the factors. Indeed, the product of two distribution can be ill-defined, and there is a precise condition for when it is well-defined.

Example 1. *Consider the Schwartz space* $\mathcal{S}(\mathbb{R})$ *and the distributions* $\delta(x)$ *and* $P\frac{1}{x}$ *on it. Then, the product* $\delta(x) \cdot P\frac{1}{x}$ *is ill-defined.*

PROOF. *We will compute the action on a test function* $f \in \mathcal{S}(\mathbb{R})$ *by regularizing one of the two distributions and show that we have a different result if we instead regularize the other one.*

Let us start by regularizing the Dirac delta using an even smooth mollifier $\varphi_\epsilon(x)$ (C.1)*. We then have*

$$T_A = \left(P\frac{1}{x}\right)[\varphi_\epsilon(x)f(x)] = P\int_{\mathbb{R}} \frac{\varphi_\epsilon(x)f(x)}{x}dx.$$

Now, since f *is smooth, we can write* $f(x) = f(0) + xg(x)$ *with* g *smooth, and get*

$$T_A = f(0) \cdot P\int_{\mathbb{R}} \frac{\varphi_\epsilon(x)}{x}dx + \int_{\mathbb{R}} \varphi_\epsilon(x)g(x)dx = f'(0).$$

We now have that the first integral vanishes because it is an odd integral over a symmetric domain (recall that we assumed φ_ϵ *to be even, and* $\frac{1}{x}$ *is odd), and the second integral gives* $g(0)$ *after taking* $\epsilon \to 0^+$*, which is equal to* $f'(0)$ *(locally, only at* $x = 0$*).*

Now, let us compute the test action on f *by swapping the role of the two distributions. We regularize* $P\frac{1}{x}$ *via* (C.3) *(which is odd), and get*

$$T_B = \delta\left[\frac{x}{x^2 + \epsilon^2} f(x)\right] = \int_{\mathbb{R}} \delta(x)\frac{x}{x^2 + \epsilon^2} f(x) = \frac{0}{\epsilon^2} f(0) = 0.$$

We see that the two actions T_A *and* T_B *coincide if and only if* $f'(0) = 0$*, but this must hold for an arbitrary test function* f*, and of course there are functions whose derivative at* $x = 0$ *is not zero. Therefore, the product is ill defined.*

■

However, one can indeed define the product between a proper distribution and a smooth function (which is itself a distribution), and the result is a distribution. We can then ask ourselves if there are cases where two proper distributions can be multiplied without any trouble. The answer is yes, and there is a condition for when this can be done, called *Hörmander condition.* To state it, we first need to define another concept, the *wave-front set* $WF(T)$ of a distribution T, which is in turn based on the concept of *microlocal smoothness.*

Definition 9 (Microlocal smoothness). *Let $T \in \mathcal{S}'(\mathbb{R}^n)$ be a tempered distribution over $\mathbb{R}^n$, let $x_0, \xi_0 \in \mathbb{R}^n$ and $\xi_0 \neq 0$. Then, T is* microlocally smooth *at (x_0, ξ_0) if there exists*

- *a compactly supported cutoff $\chi \in C_c^\infty(\mathbb{R}^n)$ with $\chi(x_0) \neq 0$,*
- *a conic neighbourhood V of ξ_0*

such that the Fourier transform $\mathcal{F}[\chi T] \in \mathcal{S}'(\mathbb{R}^n)$ is a Schwartz function in V (decays faster than any power at infinity in V):

$$\forall N \in \mathbb{N}, \quad \sup_{\xi \in V} (1 + |\xi|)^N |\mathcal{F}[\chi T](\xi)| < \infty.$$

Definition 10 (Wave-front set). *Let $T \in \mathcal{S}'(\mathbb{R}^n)$ be a tempered distribution over $\mathbb{R}^n$. The* wave-front set *of T is the set*

$$WF(T) = \{(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) | T \text{ is not microlocally smooth at } (x_0, \xi_0)\}. \quad \text{(C.5)}$$

The wave-front set can be informally thought as the set of positions and momenta directions where a distribution is singular. Of course, smooth functions have empty wavefront sets. As an example, let us compute the wave-front set of the delta distribution and the Cauchy principal value.

Example 2. *The wave-front set of the Dirac delta distribution is*

$$WF(\delta) = \{(0, \xi) | \xi \in \mathbb{R}^n \setminus \{0\}\}. \quad \text{(C.6)}$$

Therefore, the Dirac delta is singular at $x_0 = 0$ in all momentum directions.

PROOF. *Choose $x_0 \neq 0$, and compactly supported χ such that $\chi(0) = 0$ but $\chi(x_0) \neq 0$. Then, $\chi\delta = 0$ everywhere, and therefore also its Fourier transform $\mathcal{F}[\chi\delta] = 0$ everywhere. Therefore, it decays faster than every power in any cone. This means that δ is microlocally smooth at all (x_0, ξ_0) with $x_0 \neq 0$.*

Now pick $x_0 = 0$. Then, for any $\chi(0) \neq 0$ we have $\chi\delta = \chi(0)\delta$. Its Fourier transform is the constant function $\mathcal{F}[\chi\delta] = \chi(0)$ everywhere, and therefore it does not decay at all in any cone. Therefore, δ is not microlocally smooth at $(0, \xi_0)$ for any $\xi_0 \in \mathbb{R}^n \setminus \{0\}$.

■

Example 3. *We also give, without proof, the wave-front set of the Cauchy principal value $P\frac{1}{x}$:*

$$WF\left(P\frac{1}{x}\right) = \{(0, \xi) | \xi \neq 0\}.$$

We are now ready to state the condition (which we will not prove) of when the product of two distributions is well defined.

Proposition 1 (Hörmander condition). *Let $T_1, T_2 \in \mathcal{S}'(\mathbb{R}^n)$ be two tempered distributions, with respective wave-front sets $WF(T_1)$ and $WF(T_2)$. Then, $T_1 T_2$ is well-defined if and only if the composite wave-front set $WF(T_1) \oplus WF(T_2) = \{(x, \xi_1 + \xi_2) | (x, \xi_1) \in WF(T_1), (x, \xi_2) \in WF(T_2)\}$ does not contain an element of the form $(x, 0)$.*

Of course, since smooth functions have empty wave-front sets, they can always be multiplied correctly. Let us now recall our initial example of product of distributions.

Example 4. *We can check that the product of* δ *and* $P\frac{1}{x}$ *does not satisfy the above condition. Indeed they have the same singular support* $x = 0$ *in position space, and for every non-zero momentum* ξ_0 *of the Dirac delta there is a corresponding* $-\xi_0$ *of the Cauchy principal value such that the sum of momenta is zero. Therefore, the product of these two distributions is not defined, as we expected.*

We will see that the product of distributions plays a crucial role in understanding renormalization in quantum field theory, in the next appendix.

C.6 Useful formulae

In this section we collect some useful formulas that are used throughout the dissertation, and rely on distribution theory or complex analysis to make mathematical sense.

C.6.1 Trigonometric integrals

Let us prove the following distributional identities:

$$\int_0^{+\infty} \cos(px)\cos(qx)dx = \frac{\pi}{2}\left[\delta(p-q)+\delta(p+q)\right] \tag{C.7a}$$

$$\int_0^{+\infty} \sin(px)\sin(qx)dx = \frac{\pi}{2}\left[\delta(p-q)-\delta(p+q)\right] \tag{C.7b}$$

$$\int_0^{+\infty} \sin(px)\cos(qx)dx = \frac{1}{2}\left[P\frac{1}{p+q}+P\frac{1}{p-q}\right] \tag{C.7c}$$

$$\int_0^{+\infty} \sin(px)\cos(qx)dx = \frac{1}{2}\left[P\frac{1}{p+q}-P\frac{1}{p-q}\right] \tag{C.7d}$$

Proof. *Distributionally, we have*

$$\int_0^{+\infty} e^{ikx} = \lim_{\epsilon\to 0^+}\int_0^{+\infty} e^{ikx}e^{-\epsilon x}dx = \lim_{\epsilon\to 0^+}\left.\frac{e^{x(ik-\epsilon)}}{ik-\epsilon}\right|_0^{+\infty} = -\frac{1}{i}\frac{1}{k+i0^+} = \frac{i}{k+i0^+}.$$

We now use the Sokhotski-Plemelj theorem (C.4) *to write*

$$\int_0^{+\infty} e^{ikx} = iP\frac{1}{k}+\pi\delta(k),$$

and by taking the real and imaginary parts we get the following identities

$$\int_0^{+\infty} \cos(kx)dx = \pi\delta(k), \tag{C.8a}$$

$$\int_0^{+\infty} \sin(kx)dx = P\frac{1}{k}. \tag{C.8b}$$

We can now apply Werner trigonometric formulas to compute the integrals one by one. In what follows, we will use the above relations without mention, with $k = p \pm q$.

$$\int_0^{+\infty} \cos(px)\cos(qx)dx = \frac{1}{2}\int_0^{+\infty}(\cos[(p-q)x] + \cos[(p+q)x])dx = \frac{\pi}{2}[\delta(p-q) + \delta(p+q)],$$
$$\int_0^{+\infty} \sin(px)\sin(qx)dx = \frac{1}{2}\int_0^{+\infty}(\cos[(p-q)x] - \cos[(p+q)x])dx = \frac{\pi}{2}[\delta(p-q) - \delta(p+q)],$$
$$\int_0^{+\infty} \sin(px)\cos(qx)dx = \frac{1}{2}\int_0^{+\infty}(\sin[(p-q)x] + \sin[(p+q)x])dx = \frac{1}{2}\left[P\frac{1}{p-q} + P\frac{1}{p+q}\right],$$
$$\int_0^{+\infty} \cos(px)\sin(qx)dx = \frac{1}{2}\int_0^{+\infty}(\sin[(p-q)x] - \sin[(p+q)x])dx = \frac{1}{2}\left[P\frac{1}{p-q} - P\frac{1}{p+q}\right]. \blacksquare$$

C.6.2 Infinite sums

We want to prove the following distributional[1] identities

$$\sum_{n=1}^{+\infty} n\cos(n\theta) = \frac{1}{2(\cos\theta - 1)}, \tag{C.9a}$$
$$\sum_{n=1}^{+\infty} n\sin(n\theta) = 0. \tag{C.9b}$$

Proof. *The identities we want to prove are the real and imaginary parts of the following series:*

$$S = \sum_{n=0}^{+\infty} ne^{in\theta} = \sum_{n=0}^{+\infty}\left(-i\frac{\partial}{\partial\theta}\right)e^{in\theta},$$

where we changed the start value of n in the summation to 0 for convenience, since the term with $n = 0$ is vanishing. Now, to manipulate this expression distributionally, introduce the regulating factor $e^{-\alpha n}$ term by term, which makes the series convergent for $\alpha > 0$. The distributional value of the series is recovered in the limit $\alpha \to 0^+$.

$$S = \lim_{\alpha\to 0^+}\left(-i\frac{\partial}{\partial\alpha}\right)\sum_{n=0}^{+\infty} e^{n(-\alpha+i\theta)} = \lim_{\alpha\to 0^+}\left(-i\frac{\partial}{\partial\alpha}\right)\frac{1}{1-e^{-\alpha+i\theta}},$$

where we swapped the derivative with the sum due to the series being absolutely convergent. Indeed, if we set $z = -\alpha + i\theta$ we have $|e^z| = e^{-\alpha} < 1\ \forall\alpha > 0$, and we are inside the convergence radius of the geometric series. Now, compute the derivative and then restore the value $\alpha = 0$, since the regulator has done its job.

$$S = \lim_{\alpha\to 0}\left(-i(-1)\frac{-e^{-\alpha+i\theta}}{(1-e^{-\alpha+i\theta})^2}\right) = \frac{e^{i\theta}}{(1-e^{i\theta})^2}.$$

[1] Note that the series are manifestly not convergent in the classical sense, and can be made sense of only distributionally.

Now, let us denote $a \equiv \cos\theta$ *and* $b \equiv \sin\theta$ *so that* $e^{i\theta} = a + ib$*. We then have:*

$$\begin{aligned} S &= \frac{a+ib}{[(1-a)-ib]^2} = \frac{a+ib}{(1-a)^2 - b^2 - 2ib(1-a)} \\ &= \frac{a+ib}{[\cancel{1} - 2a + a^2 \cancel{-1} + a^2] - 2ib(1-a)} \\ &= \frac{a+ib}{2a(a-1) + 2ib(a-1)} = \frac{\cancel{a+ib}}{2(a-1)\cancel{(a+ib)}} = \frac{1}{2(a-1)}, \end{aligned}$$

and by restoring $a = \cos\theta$ *we get the final result:*

$$S = \sum_{n=1}^{+\infty} n e^{in\theta} = \frac{1}{2(\cos\theta - 1)}.$$

Upon taking the real and imaginary parts of the above expressions, we immediately obtain the identities we wanted to prove.

■

Let us now consider another identity, to prove which we will make use of the logarithmic power series in complex analysis (see [26], eq. 2.29), which states:

$$\log(1+z) = \sum_{n=1}^{+\infty} (-1)^n \frac{z^n}{n}. \tag{C.10}$$

The infinite sum distributional identity we want to prove is the following:

$$\sum_{n=1}^{+\infty} \frac{\cos(n\theta)}{n} = -\log\left|2\sin\frac{\theta}{2}\right|. \tag{C.11}$$

Proof. *Let us split the cosine in imaginary exponentials and introduce a regulator* $e^{-\alpha n}$ *term by term:*

$$S = \sum_{n=1}^{+\infty} \frac{\cos(n\theta)}{n} = \lim_{\alpha\to 0^+} \sum_{n=1}^{+\infty} e^{-\alpha n} \frac{e^{in\theta} + e^{-in\theta}}{2n} = \lim_{\alpha\to 0^+} \frac{1}{2}\left[\sum_{n=1}^{+\infty} \frac{e^{in\theta-\alpha n}}{n} + \sum_{n=1}^{+\infty} \frac{e^{-in\theta-\alpha n}}{n}\right].$$

Now, we can use (C.10)*, but flipping the sign in fromt of* z*, so that we get:*

$$-\log(1-z) = \sum_{n=1}^{+\infty} \frac{z^n}{n}.$$

Then, by setting $z \equiv e^{-\alpha \pm i\theta}$*, we can write*

$$S = -\lim_{\alpha\to 0^+} \frac{1}{2}\left[\log(1 - e^{i\theta-\alpha}) + \log(1 - e^{-i\theta-\alpha})\right].$$

We can now apply the property $\log ab = \log a + \log b$*, which still holds for complex arguments (as*

[26] states in sec. 2.7), and write:

$$\begin{aligned}
S &= -\lim_{\alpha\to 0^+}\frac{1}{2}\log\left[1-e^{-i\theta-\alpha}-e^{i\theta-\alpha}+e^{-2\alpha}\right]\\
&= -\lim_{\alpha\to 0^+}\frac{1}{2}\log\left[1-e^{-\alpha}(e^{i\theta}+e^{-i\theta})+e^{-2\alpha}\right]\\
&= -\lim_{\alpha\to 0^+}\frac{1}{2}\log\left[1-2e^{-\alpha}\cos\theta+e^{-2\alpha}\right]\\
&= -\frac{1}{2}\log\left[2(1-\cos\theta)\right] = -\frac{1}{2}\log\left(4\sin^2\frac{\theta}{2}\right)\\
&= -\log\left(4\sin^2\frac{\theta}{2}\right)^{\frac{1}{2}} = -\log\left|2\sin\frac{\theta}{2}\right|.
\end{aligned}$$

■

Appendix D

Distributions in QFT, Hadamard states and renormalization

In quantum field theory, distributions play a crucial role. Indeed, there is an axiomatic formulation of quantum field theory by Wightman [1], which states that field operators are not ordinary operator-valued functions, but rather operator valued distributions. Furthermore, since all relevant quantities (such as the propagator, and the energy-momentum tensor, or higher order correlation functions) are constructed in terms of the field and its derivatives, the theory of distributions discussed in the previous chapter turns very useful.

D.1 Fields as operator-valued distributions

Let us consider the quantum state $\hat{\phi}(x)\,|0\rangle$ in standard four-dimensional Minkowski real scalar field theory, where $\hat{\phi}(x)$ is the field operator evaluated at spacetime point x, and $|0\rangle$ is the standard Minkowski vacuum state belonging to the usual Fock space $\mathcal{H}$. We have

$$\left\|\hat{\phi}(x)\,|0\rangle\right\|^2 = \langle 0|\,\hat{\phi}(x)\hat{\phi}(x)\,|0\rangle = -iG_F(x,x) = \infty,$$

since the bare propagator is divergent at coincidence limit. We therefore see that $\hat{\phi}(x)$ is not a good operator, since when it acts on a normalized state such as the vacuum, it yields a state with undefined norm. This means that, in order to hope to make sense of the new state, one has to try smearing it against a test function in the Schwartz space (or a compactly supported functions space). It turns out that the result of the smearing process is a genuine element of the original state space $\mathcal{H}$, having finite norm. The element $\hat{\phi}(x)$, therefore, is not an operator-valued function, since when evaluated at a point does not behave as an operator, but rather an operator-valued distribution. Indeed, when acting on a state, it gives as a result an element of an extended space of states $\mathcal{H}' \supset \mathcal{H}$. This extended space is equipped with a distributional inner product, in the sense that inner products (brakets) between any two states of $\mathcal{H}'$ are distributions. We will see that the propagator is one of those.

In Definition 6, we defined the distributional derivative, and saw that it is indeed a well-defined distribution. We can therefore claim that the conjugate momentum $\hat{\Pi}(x) = \partial_t\hat{\phi}(x)$ and spacial derivatives of the field $\partial_j\hat{\phi}(x)$ are themselves distributions. We can now start to build observables in terms of distributions.

D.2 Propagator as a distribution

In ordinary Minkowski QFT, the propagator in momentum space is a distribution that looks like

$$\tilde{G}_F(p) = \frac{i}{p^2 - m^2 + i0^+} = P\frac{i}{p^2 - m^2} + \pi\delta(p^2 - m^2),$$

where we used the Sokhotski-Plemelj theorem (C.4). Upon Fourier transforming to configuration space, one obtains

$$G_F(x-y) = \mathcal{F}^{-1}\left[\tilde{G}_F\right](x-y),$$

which is a tempered distribution in the variable $x - y$. Microlocally, the singular support of G_F lies on the light cone $(x-y)^2 = 0$, and the corresponding wavefront set encodes the directions in momentum space generating this cone. Thus G_F is a perfectly good distribution as long as one stays off the diagonal $x = y$, which is a one-dimensional manifold. There, at coincidence, a problem arises. To define objects such as $\hat{\phi}(x)^2$ one would need to restrict the bi-distribution $G^F(x,y) = \langle\hat{\phi}(x)\hat{\phi}(y)\rangle$ to the diagonal $x = y$. Hörmander's pullback theorem tells us that this restriction is possible only if

$$WF\left(G^F\right) \cap N^*(\text{Diag}) = \varnothing,$$

where $N^*(\text{Diag}) = \{(x,k,x,-k)\}$ is the conormal bundle of the diagonal (see [22], Theorem 8.2.4). For Hadamard two-point functions, however, the wavefront set is precisely (Radzikowski condition, see [7])

$$WF(G^F) = \{(x,k_x,y,-k_y)|(x,k_x) \sim (y,k_y), k_x \in \overline{V}_+\}, \tag{D.1}$$

where $(x,k_x) \sim (y,k_y)$ means that x and y are joined by a null geodesic and k_x, k_y are cotangent covectors at x and y respectively, obtained by parallel transport along the geodesic. $\overline{V}_+$ is the closed future light cone in cotangent space. The above set does indeed intersect $N^*(\text{Diag})$ and therefore the raw product $\hat{\phi}(x)\hat{\phi}(x)$ is not defined as a distribution. This is the microlocal statement of the ultraviolet divergence at coincidence. More informally, another way to look at this issue is that, when multiplying $\hat{\phi}(x)$ by itself, the Hörmander criterion is violated and the result is more singular than allowed to form a distribution. To resolve this issue, one needs renormalization, which involves the subtraction of a universal Hadamard parametrix $G^F_{sing}(x,y)$ that has the same singular wavefront set. The difference $G^F - G^F_{sing}$ is smooth near the diagonal $x = y$, and its restriction to $x = y$ is well defined. This yields the renormalized Wick square $:\hat{\phi}^2:(x)$ and, more generally, provides the microlocal foundation of renormalization in curved spacetime QFT.

D.3 Hadamard states and renormalization

In the paper [17], which is used in this thesis as a reference for Hadamard renormalization, it is stated that the renormalization procedure only works for Hadamard states. One can think of a Hadamard state being the analogue of Minkowski vacuum in general curved spacetime. More precisely, a Hadamard state $|H\rangle$ is a quantum state of the QFT such that the expectation value $\langle H|\,\hat{\phi}(x)\hat{\phi}(y)\,|H\rangle$ has the same singular structure as the

Minkowski vacuum locally. This is consistent with local flatness in general relativity, since we expect the short-distance ultraviolet divergences (those which we subtract by renormalization) to be purely local. We therefore require physical states to have the same universal Hadamard singular structure, so that the counterterm is state-independent and purely local, and after renormalization the state-dependent information is preserved and smooth. Of course, this discourse applies to the renormalization of all expectation values quadratic in field and its derivatives. Microlocally, the definition of a Hadamard state $|H\rangle$ is given in terms of the wave-front set of the two-point function of H, and is the Radzikowski condition (D.1). By subtracting a distribution with the same wave-front set, we obtain an object whose wave-front set is empty, and therefore we have a smooth function. This is exactly what $W(x, y)$ is in Decanini-Folacci paper [17], and from this we can directly build the renormalized energy-momentum tensor.

Appendix E

Heat kernel expansion, semigroups and fractional laplacian

In this appendix, we want to give a basic introduction to the theory of heat kernel expansion, starting from the heat equation and its solution. Let us start from the definition of an elliptic operator.

E.1 Heat semigroup and heat kernel

Definition 11 (Elliptic operators). *Let $\mathcal{M}$ be a smooth Riemannian manifold of dimension D endowed with euclidean metric $\boldsymbol{g}$. An operator L acting on smooth functions on $\mathcal{M}$ is called a* laplacian elliptic operator *if, in local coordinates, it takes the form*

$$L = -g^{\mu\nu}\nabla_\mu\nabla_\nu + V(x),$$

where $V(x)$ is a potential term. Such elliptic operators L are self-adjoint and non-negative, and their spectrum is discrete and unbounded

$$L\phi_j = \lambda_j\phi_j, \qquad 0 \leq \lambda_1 \leq \lambda_2 \leq \dots,$$

with eigenfunctions $\{\phi_j\}$ forming an orthonormal basis of $L^2(\mathcal{M})$.

Now, we introduce an abstract tool, which is the positive exponential map of the operator L, and which will turn out to be useful later.

Definition 12 (Heat semigroup). *Let L be an elliptic operator with non-negative eigenvalues $\{\lambda_j\}_{j=1}^{+\infty}$ and $t > 0$. Let also $f \in L^2(\mathcal{M})$. We then define the* heat semigroup *as an operator acting on $L^2(\mathcal{M})$ like*

$$e^{-tL}f = \sum_{j=1}^{+\infty} e^{-t\lambda_j}\langle f, \phi_j\rangle\phi_j, \tag{E.1}$$

where $\langle f, \phi_j\rangle$ is the standard L^2 inner product.

The name *heat semigroup* is related to the fact that, after it acts on an arbitrary function $f \in L^2(\mathcal{M})$, it satisfies the heat equation. Indeed, define

$$u(t, x) = (e^{-tL}f)(x),$$

then, $u(t,x)$ satisfies the heat equation with initial conditions and $t > 0$

$$\begin{cases} (\partial_t + L)u(t,x) = 0, \\ u(0,x) = f(x). \end{cases} \tag{E.2}$$

This is because we have $\partial_t(e^{-tL}f(x)) = -Le^{-tL}f$, and $e^{-0\cdot L} = 1$. In addition, the name *semigroup* stems from the fact that the elements e^{-tL} with $t > 0$ form a semigroup. We are not interested in the negative values of t, since the spectral sum (E.1) would be divergent, being the eigenvalues non-negative. Since the solution to the heat equation is unique, the heat semigroup is a powerful abstract tool to represent it.

Proposition 2 (Heat kernel). *By definition, the operator e^{-tL} has a smooth integral kernel $K(t,x,y)$ such that*[1]

$$(e^{-tL}f)(x) = \int_{\mathcal{M}} K(t,x,y)f(y)dy.$$

PROOF. *Indeed, from the spectral representation* (E.1), *we have*

$$(e^{-tL}f)(x) = \sum_{j=1}^{+\infty} e^{-t\lambda_j}\langle f,\phi_j\rangle\phi_j(x) = \int_{\mathcal{M}} \underbrace{\left(\sum_{j=1}^{+\infty} e^{-t\lambda_j}\phi_j(x)\phi_j^*(y)\right)}_{K(t,x,y)} f(y)dy = \int_{\mathcal{M}} K(t,x,y)f(y).$$

■

It follows that the integral kernel $K(t,x,y)$ solves the heat equation with initial conditions

$$\begin{cases} (\partial_t + L_x)K(t,x,y) = 0 \\ K(0,x,y) = \delta(x-y) \end{cases}.$$

PROOF. *We already know that, for an arbitrary $f \in L^2(\mathcal{M})$, the expression $u(t,x) = e^{-tL}f$ solves the Cauchy problem* (E.2). *Let us then expand the heat semigroup in its integral representation. We get*

$$\begin{cases} (\partial_t + L_x)\int_{\mathcal{M}} K(t,x,y)f(y)dy = \int_{\mathcal{M}} \left[(\partial_t + L_x)K(t,x,y)\right]f(y)dy = 0 \\ \int_{\mathcal{M}} K(0,x,y)f(y)dy = f(x) \end{cases},$$

and since these equations need to hold for an arbitrary f, we can conclude that

$$\begin{cases} (\partial_t + L_x)K(t,x,y) = 0 \\ K(0,x,y) = \delta(x-y) \end{cases}.$$

■

---

[1]In this appendix, we work with the convention $dx = \sqrt{|\det \boldsymbol{g}|}d^n x$.

E.2 Heat trace and heat kernel expansion

We now get closer to the heat kernel expansion, and define the *heat trace* as the trace of the heat semigroup

$$K(t) = \operatorname{Tr} e^{-tL} = \sum_{j=1}^{+\infty} e^{-t\lambda_j},$$

or equivalently, in terms of the kernel,

$$K(t) = \operatorname{Tr} e^{-tL} = \int_{\mathcal{M}} K(t, x, x) dx.$$

We therefore get to the very interesting equality

$$\boxed{\sum_{j=1}^{+\infty} e^{-t\lambda_j} = \int_{\mathcal{M}} K(t, x, x) dx,} \tag{E.3}$$

which connects spectral data (eigenvalues of the operator L) with the geometry of the manifold (the kernel of the diagonal).

We now make contact with Vassilevich [11], in particular Eq. 1.13, where we read that the heat kernel admits an asymptotic expansion (i.e. its radius of convergence is zero, and therefore fails if we add corrections higher than a certain order and then truncate):

$$K(t, x, y) \simeq \frac{e^{-\frac{\sigma(x,y)}{2t}}}{(4\pi t)^{D/2}} \sum_{k=0}^{+\infty} a_k(x, y) t^k, \tag{E.4}$$

where $\sigma(x, y)$ is the Synge world function, which corresponds to half the square of the geodesic distance between x and y, and $a_k(x, y)$ are called the *heat kernel coefficients.*

Sometimes, it is easier to work with the heat trace expansion, instead of expanding the full kernel. We therefore can take the trace of (E.4) to get

$$K(t) = \int_{\mathcal{M}} K(t, x, x) dx \simeq (4\pi t)^{-D/2} \sum_{k=0}^{+\infty} \underbrace{\left(\int_{\mathcal{M}} a_k(x, x) dx \right)}_{A_k} t^k,$$

and we have the *heat trace coefficients* A_k, which differ from the heat kernel coefficients $a_k(x, y)$ by a trace action.

E.3 Fractional laplacian and subordination formula

Let us now define the fractional laplacian L^α and state the subordination formula that relates the heat semigroup of L^α to the one of L. We reference [24] for this section.

Definition 13 (Fractional laplacian). *Let L be an elliptic operator on a smooth manifold $\mathcal{M}$ with eigenvalues $\lambda_j \geq 0$ and eigenfunctions given by ϕ_j. For any $\alpha > 0$, we define the fractional power of L by raising the eigenvalues to the same power:*

$$L^\alpha \phi_j = \lambda_j^\alpha \phi_j.$$

This fully determines the action of L on any arbitrary $f \in L^2(\mathcal{M})$, since the operator is linear and the eigenfunctions span the whole functional space.

It is worth mentioning that for non-integer values of α, the operator is non-local since the value of $L^\alpha f(x)$ for a given $x \in \mathcal{M}$ depends on the value of f at other points which are far from x.

At page 5 of [24], we have the integral representation of the action of L^α on a function f, if $0 < \alpha < 1$, and we get the *subordination formula*:

$$L^\alpha f = \frac{1}{\Gamma(-\alpha)} \int_0^{+\infty} (e^{-tL} f - f) \frac{dt}{t^{1+\alpha}},$$

which relates the operator L^α with L, hence the name *subordination*.

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