ALMA MATER STUDIORUM · UNIVERSITY OF BOLOGNA

School of Science
Department of Physics and Astronomy
Master Degree in Physics

The Cosmological Constant Problem in a Minimal Length Scenario

Supervisor:
Prof. Roberto Casadio

Co-supervisor:
Dr. Tommaso Bambagiotti

Submitted by:
Matteo Falzone

Academic Year 2024/2025

So I’m picking up the pieces now, where to begin?
The hardest part of ending is starting again.

Linkin Park, *Waiting for the End*

Abstract

The cosmological constant problem represents one of the most important unresolved issues in modern theoretical physics. It consists in the enormous numerical discrepancy between the measured value of vacuum energy density and the one estimated within the framework of quantum field theory (QFT).

Following the idea that, regardless of the choice of a specific model, the quantum nature of gravity should give rise to a fundamental minimal length scale, this thesis aims to study the phenomenological consequences of introducing such a minimal length within the framework of QFT in curved spacetimes, specifically in relation to the cosmological constant problem. In particular, the mathematical apparatus related to the point-splitting technique to regularize quadratic functions of quantum fields was employed.

The results achieved do not appear to differ significantly from those obtained through the typical flat spacetime approach. Nevertheless, the proposed approach could suggest new perspectives on the problem of divergences in quantum field theory.

Contents

Introduction

Current observational data, primarily from Type Ia supernovae and the cosmic microwave background, suggest that our universe is undergoing accelerated expansion. Such a cosmic dynamic is consistent with a vacuum-dominated universe model.

Contrary to common intuition, the *vacuum* in this context does not represent the complete absence of entities or physical properties. Rather, it refers to the ground state upon which the entire framework of quantum field theory is built, what has been described as "a stormy sea of quantum fluctuations" [46]. This vacuum possesses its own energy density which, at the theoretical level, is estimated to be vastly greater (by about 120 orders of magnitude) than what cosmological observations appear to indicate.

To reconcile this enormous discrepancy, Einstein's field equations of general relativity would need to include a cosmological constant term whose value almost perfectly cancels out the vacuum energy contribution. The extreme improbability of such a precise fine-tuning has led to this issue being famously known as the *cosmological constant problem*.

In recent decades, the cosmological constant problem has revealed numerous facets, and an equally wide range of theoretical approaches have been developed to address it. Nevertheless, none of these has proven entirely satisfactory.

One might argue that, until we achieve a robust understanding of the quantum properties of gravity, any attempt to resolve the problem will remain fundamentally incomplete. In this regard, however, modern approaches to quantum gravity seem to converge, almost unanimously, on the idea that nature admits a fundamental minimal length scale. This scale arises as a consequence of the Heisenberg uncertainty principle, combined with the gravitational field's response to it.

All of this may suggest that introducing such a minimal length scale into the framework of quantum field theory in curved spacetimes could shed light on the cosmological constant problem, as an effective manifestation of quantum gravity. This is the path we have followed in the present thesis.

The first chapter begins by outlining the logical path, accompanied by a historical perspective, that led to the formulation of the cosmological constant problem. Subsequently, we present several approaches to the problem that still hold relevance today, in particular: supersymmetry, the anthropic principle, and the quintessence model for dark energy.

The second chapter develops the concept of a minimal length scale and its connection to the generalized uncertainty principle. In particular, various thought experiments are presented that seem to support the validity of these concepts, mixing quantum and gravitational phenomena. We then show how the idea of a minimal length can emerge naturally from certain well-structured quantum gravity theories, in particular string theory and loop quantum gravity.

In the third and final chapter, we present our attempt to introduce the concept of a minimal length at a phenomenological level within the context of quantum field theory in curved spacetimes. To this end, we rely on the framework of the point-separation regularization method for the quantization of quadratic fields. This is followed by a discussion of the results obtained and their connection to the cosmological constant problem.

Conventions The metric and curvature conventions adopted are those of Misner, Thorne and Wheeler [28]. In particular, the metric signature is $(-+++)$.

The covariant derivative of a tensor quantity is expressed as:

$$\nabla_\mu X^\nu \equiv X^\nu{}_{;\mu}. \tag{1}$$

Natural units $c = \hbar = 1$ are used, so that the Planck length and Planck mass are given by:

$$l_P = \sqrt{G}, \quad m_P = 1/\sqrt{G}. \tag{2}$$

In some sections of the thesis, we introduce the reduced Planck mass:

$$M_P = \frac{1}{\sqrt{8\pi G}}. \tag{3}$$

Chapter 1

The cosmological constant problem

Vanitas vanitatum et omnia vanitas.

1.1 The cosmological constant

Einstein completed his formulation of general relativity between 1915 and 1916 by introducing the field equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{1.1}$$

Initially, motivated by the assumption that the velocities of stars are negligible compared to the speed of light, he tried to apply his theory to the entire universe, aiming for a static cosmological model.

Since, on very large scales ($\gtrsim 100\,\mathrm{Mpc}$), the universe is considered spatially homogeneous and isotropic, its geometry can be described by the Friedmann–Lemaître– Robertson–Walker (FLRW) metric:

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right), \tag{1.2}$$

where $a(t)$ is the cosmic scale factor and k (that one can always rescale to: -1, 0, or $+1$; depending on the spatial topology) is the curvature constant.

The adoption of this metric reduces the Einstein equations to the Friedmann's:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{1.3}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{1.4}$$

A static universe ($\dot{a} = 0$) requires a positive curvature ($k = +1$) to be consistent with a positive energy density ρ, which must be in any case appropriately tuned. However, such conditions cannot lead to a vanishing $\ddot{a}$, unless a negative pressure p is considered (but pressure is known to be non-negative for usual forms of matter).

To overcome these difficulties, Einstein introduced the cosmological constant[1] Λ in his equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{1.5}$$

This new term modifies the Friedmann equations, that become:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \tag{1.6}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}, \tag{1.7}$$

allowing for a static solution associated to non-negative values of ρ, p and Λ. In any case, this configuration is far from being non-problematic, since it is unstable to fluctuations of these parameters.

Eventually, Hubble's discovery that the universe is expanding removed the empirical need for a static model and Einstein withdrew the cosmological constant[2].

1.2 The problem

Whether the cosmological constant should appear in the field equations of general relativity is a subtle issue. In fact, anything that contributes to the energy density of the vacuum $\langle\rho\rangle$ acts just like a cosmological constant.

[1] For a historical account of the introduction of the cosmological constant, see [44, 4, 26].

[2] Actually, Einstein found the presence of the cosmological constant disappointing, especially after de Sitter's 1917 discovery of an expanding cosmological model as a vacuum solution to the modified field equations, which showed that matter was not necessary to generate inertia; contradicting Mach's principle.

Lorentz invariance requires that, in locally inertial coordinate systems, the energy-momentum tensor of the vacuum be proportional to the Minkowski metric $\eta_{\mu\nu}$. In general coordinate systems, this condition translates to (see [43]):

$$\langle T_{\mu\nu} \rangle = -\langle \rho \rangle g_{\mu\nu}, \tag{1.8}$$

where the minus sign arises from adopting the $(-, +, +, +)$ metric convention.

This is equivalent to considering the presence, in Einstein equations, of an effective cosmological constant:

$$\Lambda_{\text{eff}} = \Lambda + 8\pi G \langle \rho \rangle, \tag{1.9}$$

or, alternatively, an effective vacuum density:

$$\rho_V = \langle \rho \rangle + \frac{\Lambda}{8\pi G} = \frac{\Lambda_{\text{eff}}}{8\pi G}. \tag{1.10}$$

Observational data (see [4]) from Type Ia supernovae and the CMB suggest the actual existence of a positive cosmological constant Λ_{eff}, which drives an accelerated expansion of the universe, and provide an experimental upper bound on it (or ρ_V).

The present estimation of the Hubble parameter is:

$$\left(\frac{\dot{a}}{a} \right)_{\text{now}} \equiv H_0 \simeq 70 \text{ Km/sec/Mpc}, \tag{1.11}$$

and, since the effects of curvature are negligible and the vacuum energy dominates the expansion's dynamics, the first Friedmann equation yields to:

$$|\Lambda_{\text{eff}}| \lesssim H_0^2, \tag{1.12}$$

that is:

$$|\rho_V| \lesssim 10^{-29} \, \text{g/cm}^3 \approx 10^{-48} \, \text{GeV}^4. \tag{1.13}$$

Quantum field theory (QFT) allows one to estimate the vacuum energy density by summing the zero-point energies of all normal modes of a scalar field with mass m, up to an ultraviolet momentum cut-off $M \gg m$ (see, e.g., [44]):

$$\langle \rho \rangle = \int_0^M \frac{4\pi k^2 dk}{(2\pi)^3} \frac{1}{2} \sqrt{k^2 + m^2} \simeq \frac{M^4}{16\pi^2}. \tag{1.14}$$

The nature of the problem becomes apparent. By imposing a sharp cut-off at the Planck scale $M \simeq (8\pi G)^{-1/2} \equiv M_P$ (where M_P is the reduced Planck mass; assuming the validity of ordinary QFT up to this order of magnitude), the vacuum energy density is found to be[3]:

$$\langle \rho \rangle \approx 2 \times 10^{72}\,\text{GeV}^4. \tag{1.15}$$

As shown above, $|\rho_V| = |\langle \rho \rangle + \frac{\Lambda}{8\pi G}|$ is less than about $10^{-48}\,\text{GeV}^4$, which means that the two terms in the sum must cancel to better than 120 decimal places. Such an extreme fine-tuning is utterly unreasonable, leading some authors to label this result as "the largest discrepancy between theory and experiment in all of science" [1] and "the worst theoretical prediction in the history of physics" [19].

The approach to calculating the vacuum energy density discussed so far dates back to Pauli [10]. He originally introduced an ultraviolet cutoff of the order of the electron mass m_e, realizing that such a choice would imply a cosmological horizon with a radius smaller than the distance between the Earth and the Moon:

$$r_H \lesssim 1/H_{vac} \sim M_P/m_e^2 \sim 10^6\,\text{Km}. \tag{1.16}$$

In this context, shifting the cutoff to the Planck scale makes the prediction for the Hubble radius even more dramatic: the universe would be unable to grow beyond the Planck scale itself.

It is worth noting that some recent arguments pointed out that the usage of a cutoff to regularize the vacuum energy density should be avoided since it breaks the Lorentz invariance (for details, see [26]). Indeed, it is believed that this choice underlies the incorrect equation of state that arises when evaluating the vacuum expectation value of the full energy-momentum tensor of a free scalar field (namely, $p = 1/3\,\rho$)[4].

A procedure which preserves this symmetry, such as *dimensional regularization* (followed by applying the *modified minimal subtraction* ($\overline{\text{MS}}$) renormalization scheme), appears much more suitable and leads to a different expression

[3]In QFT the value of the vacuum energy density has no observational consequences and can be simply neglected through a *normal ordering* procedure. However, this is not the case when gravity is introduced in the picture, since every source of energy, the vacuum included, influences the geometry of spacetime.

[4]For the theory to be self-consistent, one would expect the typical relation for a vacuum-dominated universe, $p = -\rho$, to hold.

for the zero-point energy density[5]:

$$\langle \rho \rangle \simeq \frac{m^4}{64\pi^2} \ln\left(\frac{m^2}{\mu^2}\right), \tag{1.17}$$

where m denotes the mass of the field and μ the renormalization scale.

Regardless of the exact value of the parameter μ, reasonable choices yields computed vacuum energy densities that are significantly lower than the previously estimated $10^{72}\,\mathrm{GeV}^4$. Nevertheless, the discrepancy with the experimental data remains enormous and the *cosmological constant problem* (or *vacuum catastrophe*) persists.

1.3 Approaches to the problem

Several theoretical approaches[6] have been pursued to address the cosmological constant problem, but none has yet proven fully satisfactory. However, each has offered valuable insights into different aspects of the issue.

Some of the most intriguing and historically relevant ideas are briefly reviewed below.

1.3.1 Supersymmetry and supergravity

A physical parameter is "naturally small"[7] only if setting it exactly to zero accentuates the symmetry of an underlying theory. In such cases, its vanishing is not a coincidence, but a consequence of symmetry.

The typical example is the photon mass (experimentally constrained to $m_\gamma^2 \lesssim \mathcal{O}(10^{-50})\,\mathrm{GeV}^2$), which is understood to be exactly zero as a direct consequence of the $U(1)$ gauge symmetry of quantum electrodynamics (QED), together with Lorentz invariance.

By analogy, one may speculate that a symmetry could exist to suppress the effective cosmological constant by many orders of magnitude, potentially explaining its extreme smallness. Supersymmetry (SUSY) appears to be a natural candidate.

The case can be reviewed through the free Wess-Zumino model, whose lagrangian is given by:

[5] This latter result holds within the more rigorous framework where the curvature of spacetime is considered (see [26]).

[6] For detailed overviews, see [4, 26, 30, 45, 44]

[7] The concept of *naturalness* is well exposed in [30].

$$\mathcal{L}_{\rm WZ} = -\partial_\mu \Phi^* \partial^\mu \Phi - m_B \Phi^* \Phi - \frac{1}{2} \bar{\Psi}_M \left(i\gamma^\mu \partial_\mu - m_F \right) \Psi_M, \tag{1.18}$$

where Φ denotes a complex scalar field and Ψ_M is a Majorana spinor satisfying the condition $\Psi_M = \Psi_M^c$, with $\Psi^c \equiv C\bar{\Psi}^T$ and C is the charge conjugation operator. The parameters m_B and m_F represent, respectively, the masses of the bosonic and fermionic fields.

The contribution of a free field of spin s to the vacuum energy density is given by:

$$\langle \rho \rangle = \frac{1}{2}(-1)^{2s}(2s+1) \int \frac{d^3k}{(2\pi)^3} \sqrt{k^2 + m^2}, \tag{1.19}$$

and, since a complex scalar field is equivalent to two real scalar fields, assuming equal masses, $m_B = m_F$, the bosonic and fermionic contributions exactly cancel, resulting in a vanishing vacuum energy density.

More generally, any theory ensuring for each mass m an equal number of bosonic and fermionic degrees of freedom yields a net vacuum energy contribution of zero. Supersymmetry is precisely such a symmetry.

The spin-1/2 supersymmetry generators (supercharges) Q_α satisfy the anticommutation relation:

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu, \tag{1.20}$$

remembering that $P_\mu \equiv -i\partial_\mu$ is the four-momentum operator and $(\sigma^\mu)_{\alpha\dot{\beta}}$ are the Pauli matrices including the identity.

If supersymmetry is unbroken, then the vacuum state $|0\rangle$ satisfies:

$$Q_\alpha |0\rangle = 0, \tag{1.21}$$

implying:

$$\langle 0 | P_0 | 0 \rangle = 0, \tag{1.22}$$

i.e., the vacuum expectation value of the hamiltonian $\langle 0|H|0\rangle$ (the vacuum energy) is zero.

The same result can be found following a different path. In supersymmetric theories the scalar-field potential (depending on ϕ^i and their complex conjugate $\bar{\phi}^i$) is derived from a (holomorphic) superpotential $W(\phi^i)$. In the Wess-Zumino models of spin-0 and spin-1/2 fields, for example, the scalar potential is given by:

$$V(\phi^i, \bar{\phi}^i) = \sum_i \left| \frac{\partial W}{\partial \phi^i} \right|^2 . \tag{1.23}$$

One can show that, in such a theory, SUSY is unbroken only for values of ϕ^i such that $\partial W/\partial \phi^i = 0$, implying $V(\phi^i, \bar{\phi}^i) = 0$.

Perturbative quantum effects do not change this conclusion, because with boson-fermion symmetry, the fermion loops cancel the boson ones in the Feynman diagrams.

However, each boson (fermion) of the standard model of particle physics is not observed to have a super-symmetric partner of the same mass, meaning that, in the real world, supersymmetry must be broken (at least up to the TeV scale, which again, if taken as a cut-off, induces a large vacuum energy).

In any case, to properly discuss the cosmological constant problem, gravity must be brought into the picture; and in curved spacetime, the global transformations of ordinary supersymmetry are promoted to the gauge transformations of supergravity (SUGRA).

Within this framework, the Hamiltonian and supercharges assume different roles than in flat spacetime, yet it remains possible to express the vacuum energy in terms of a scalar field potential $V(\phi^i, \bar{\phi}^i)$, which now takes the form:

$$V(\phi^i, \bar{\phi}^j) = e^{K/M_P^2} \left[(D_i W)(\mathcal{G}^{-1})^{i\bar{j}} (D_{\bar{j}} \overline{W}) - 3 M_P^{-2} |W|^2 \right] , \tag{1.24}$$

where $K(\phi^i, \bar{\phi}^j)$ is the real-valued Kähler potential and $(\mathcal{G}^{-1})^{i\bar{j}}$ represents the inverse of the Kähler metric:

$$\mathcal{G}_{i\bar{j}} \equiv \partial^2 K / \partial \phi^i \partial \bar{\phi}^j . \tag{1.25}$$

Furthermore, $D_i W$ is the Kähler derivative:

$$D_i W \equiv \frac{\partial W}{\partial \phi^i} + \frac{1}{M_P^2} \frac{\partial K}{\partial \phi^i} W. \tag{1.26}$$

By expressing the (inverse) Kähler metric in the canonical form, $(\mathcal{G}^{-1})^{i\bar{j}} = \delta^{i\bar{j}}$, in the limit $M_P \to \infty$ (or, equivalently, $G \to 0$), the first term in square brackets reduces to the flat-space result, which is non-negative. However, when gravity is included, an additional second non-positive term arises.

Supersymmetry is unbroken when $D_i W = 0$, which implies that the effective cosmological constant is non-positive.

In fact, it is possible to consider a scenario in which supersymmetry is broken in such a manner that the two aforementioned terms cancel to an extraordinary degree of accuracy, requiring a new unexplained fine-tuning, replacing the original fine-tuning at the heart of the cosmological constant problem.

1.3.2 Anthropic principle

The anthropic principle states that certain parameters characterizing the observed universe, such as the cosmological constant, may not be strictly determined by fundamental physical laws alone, but are also constrained by the (rather trivial) requirement that intelligent observers can only exist under conditions compatible with their own existence.

For this tautology to be usable, alternative conditions (more or less compatible with the emergence of intelligent life forms) must exist "elsewhere" in the universe (or in the multiverse): either in space, time, or branches of its quantum wavefunction.

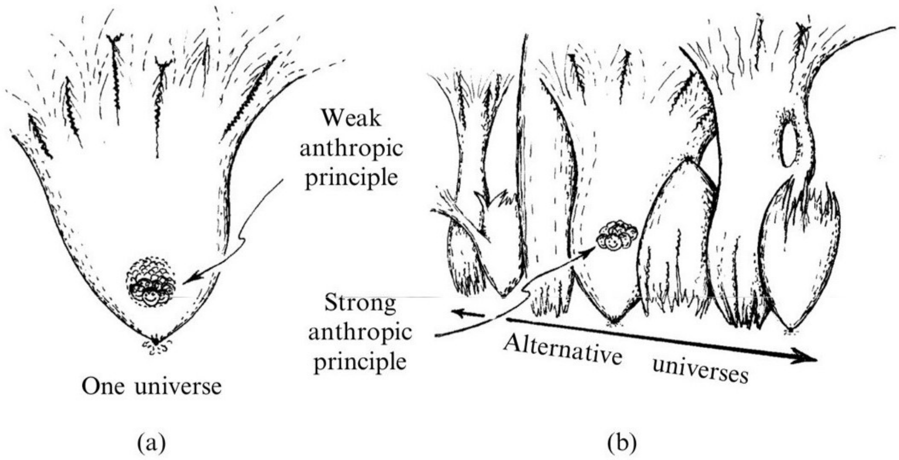

Figure 1.1: Anthropic principle. (a) Weak form: sentient beings must find themselves in a spatio-temporal location in the universe at which the conditions are suitable for sentient life. (b) Strong form: rather than considering just one universe we envisage an ensemble of possible universes, among which the fundamental constants of nature may vary. Sentient beings must find themselves to be located in a universe where the constants of nature (in addition to the spacetime location) are congenial. Image from Penrose [35].

Several inflationary scenarios, quantum cosmologies and string theory predict the existence of such different domains, characterized by significantly varying physical properties: including multiple vacua (each associated with distinct vacuum energies) and different values for the fundamental "constants" of nature.

In the light of what has been discussed so far, one is faced with the task of estimating quantitatively the likelihood of observing any specific value of Λ within such a variegate landscape.

The most recognizable anthropic constraint on the vacuum energy density is that it must not be so large to preclude the formation of galaxies.

Overdense regions in the cosmic fluid cease to undergo gravitational collapse once the cosmological constant begins to dominate the energy density of the universe. If this transition occurs before the epoch of galaxy formation, the resulting universe would lack galaxies, and consequently stars, planets and, presumably, intelligent life.

Thus, in terms of density parameters, one must have $\Omega_\Lambda(z_{\text{gal}}) \leq \Omega_{\text{Matter}}(z_{\text{gal}})$, i.e.:

$$\frac{\Omega_{\Lambda 0}}{\Omega_{\text{M}0}} \leq a_{\text{gal}}^{-3} = (1 + z_{\text{gal}})^3 \sim 125, \tag{1.27}$$

where the redshift of formation of the first galaxies is taken to be $z_{\text{gal}} \sim 4$. This demonstrates that the cosmological constant could, in principle, exceed its observed value and still remain consistent with galaxy formation. Nonetheless, it is evident that among the possible universes capable of supporting intelligent life, those characterized by values of the ratio $\Omega_{\Lambda 0}/\Omega_{\text{M}0} \sim 1$ tend to contain a greater number of galaxies and, consequently, potential observers, with respect to the ones where $\Omega_{\Lambda 0}/\Omega_{\text{M}0} \sim 100$.

Up to this point, one might ask what is the most probable value of Ω_Λ (or ρ_V), i.e. what is the value that would be experienced by the largest number of observers.

Quantitatively, one could define the probability measure for ρ_V:

$$d\mathcal{P}(\rho_V) = \nu(\rho_V)\mathcal{P}_*(\rho_V)\, d\rho_V \tag{1.28}$$

where $\mathcal{P}_*(\rho_V)\, d\rho_V$ is the *a priori* probability measure for the vacuum energy density, and $\nu(\rho_V)$ is the average number of galaxies which form at the specified value of ρ_V (it is often assumed to be proportional to the number of baryons in the universe).

The precise form of $\mathcal{P}_*$ is a subtle issue. Some authors (e.g. Weinberg [42]) argue that it is natural to assume the latter to be approximately constant in the narrow range of vacuum energy densities for which $\nu(\rho_V)$ is supposed to be non-vanishing[8]. Others, based on arguments from quantum cosmology, affirm that this assumption may not necessarily be true.

In any case, the conclusion is that observing $\Omega_{\Lambda 0}$ to be of the same order of magnitude as Ω_{M0} is not improbable at all. In particular, certain calculations (see [25]) based on the so-called *spherical infall model* [15] indicate that the probability of observing a big bang characterized by a vacuum energy density sufficiently large to yield a present-day value of $\Omega_{\Lambda 0} \leq 0.7$ is approximately 5% to 12%.

Although this latest result may seem reassuring, calculations based on anthropic reasoning are still quite uncertain and should be treated with caution, given the large number of assumptions made about poorly known physical conditions.

1.3.3 Quintessence

The presence of a cosmological constant is not necessarily the only explanation for the accelerated expansion of the universe. Indeed, current observational data may be equally well described by a form of *dark energy* that does not cluster gravitationally on small scales and therefore does not contribute to the formation of structures (hence, it does not contribute to Ω_M).

Regardless of its underlying nature, it is common in theoretical modeling to describe dark energy through an effective equation of state $p = \omega\rho$[9]. In the case of a true cosmological constant, $\omega = -1$.

The simplest model that satisfies the aforementioned conditions is a single slowly-rolling homogeneous scalar field $\phi(t)$ called *quintessence*. In an expanding universe, and in the presence of a potential $V(\phi)$, the equation of motion for such a field, minimally coupled to gravity, is given by:

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0. \tag{1.29}$$

where H is the Hubble parameter and $V' \equiv \partial_\phi V$.

The associated energy density and pressure are respectively $\rho = \dot{\phi}^2/2 + V$ and $p = \dot{\phi}^2/2 - V$, implying an equation of state parameter:

[8] This range is characterized by values of ρ_V much smaller than what particle physics suggests, as in the case of our universe, which is considered to be typical among the ones that could host intelligent life (following the so-called *principle of mediocrity* [41]).

[9] For the universe to undergo accelerated expansion, the *strong energy condition* must be violated, i.e. $\omega < -1/3$.

$$\omega = \frac{p}{\rho} = \frac{\dot{\phi}^2/2 - V}{\dot{\phi}^2/2 + V}. \tag{1.30}$$

When the field is slowly-varying and $\dot{\phi}^2 \ll V(\phi)$, we have $\omega \sim -1$, and the scalar field potential acts like a cosmological constant.

One might wonder whether replacing a constant parameter Λ with a dynamical field could relax the fine-tuning that inevitably accompanies the cosmological constant. Indeed, *tracker models*[10] can be constructed in which the quintessence field evolves alongside matter or radiation, eventually becoming the dominant component at late times, driving the accelerated expansion of the universe observed today. In these models, quintessence's energy density exhibits weak dependence on the field's initial conditions.

However, the ultimate value ρ_ϕ, which must be compared with observational data, remains highly sensitive to the specific parameters of the potential.

Furthermore, quintessence models introduce new "naturalness" problems, as the requirement for slow-roll dynamics demands the (effective) mass of the field fluctuations to be $m_\phi \sim H_0 \sim 10^{-33}\,\text{eV}$ (an extremely small value compared to typical particle physics scales, unless protected by some kind of symmetry).

1.4 Weinberg no-go theorem

A popular class of proposed resolutions to the cosmological constant problem involves the introduction of additional fields to cancel the vacuum energy (a mechanism known as *self-adjustment* or *self-tuning*). However, Weinberg proposed a no-go theorem that shows, under very general conditions, that such approaches cannot succeed without fine-tuning[11]. Weinberg's argument is as follows (see [44, 31]).

Consider a theory characterized by a spacetime metric $g_{\mu\nu}$ and a set of self-adjusting matter fields ϕ_i. The dynamics is described by a general Lagrangian

[10] A classic example is given by the potential:

$$V(\phi) = \frac{M^{4+\alpha}}{\phi^\alpha}, \quad \alpha > 0. \tag{1.31}$$

[11] The theorem does not directly apply to the case of quintessence described in the previous section. It only constrains mechanisms that attempt to cancel vacuum energy in stationary configurations. Quintessence, on the other hand, describes a dynamical universe in which dark energy is an evolving component, not a constant to be canceled. In this sense, quintessence "bypasses" the theorem.

density:

$$\mathcal{L}(g_{\mu\nu}, \phi_i). \tag{1.32}$$

One assumes that the vacuum is translationally invariant, so that:

$$g_{\mu\nu} = \text{constant}, \quad \phi_i = \text{constant}. \tag{1.33}$$

The residual symmetry implies invariance under the general linear group $GL(4)$, corresponding to coordinate transformations of the form:

$$x^\mu \to (M^{-1})^\mu{}_\nu x^\nu, \tag{1.34}$$

where $M^\mu{}_\nu$ is a constant matrix.

Under this transformation, the metric transforms as:

$$g_{\mu\nu} \to g_{\alpha\beta} M^\alpha{}_\mu M^\beta{}_\nu, \tag{1.35}$$

and the Lagrangian as:

$$\mathcal{L} \to (\det M)\, \mathcal{L}. \tag{1.36}$$

To linear order, these transformations are:

$$\delta g_{\mu\nu} = \delta M_{\mu\nu} + \delta M_{\nu\mu}, \quad \delta\mathcal{L} = \mathrm{Tr}(\delta M)\, \mathcal{L}. \tag{1.37}$$

Since the fields are constant, we can compute the variation of the Lagrangian as:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial g_{\mu\nu}}\delta g_{\mu\nu} \tag{1.38}$$

and vacuum field equations are:

$$\frac{\partial\mathcal{L}}{\partial\phi_i} = 0, \quad \frac{\partial\mathcal{L}}{\partial g_{\mu\nu}} = 0. \tag{1.39}$$

First, suppose these equations hold independently. Then, using the $GL(4)$ variation and setting $\partial\mathcal{L}/\partial\phi_i = 0$, one finds:

$$\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}}(\delta M_{\mu\nu} + \delta M_{\nu\mu}) = \mathrm{Tr}(\delta M)\, \mathcal{L}. \tag{1.40}$$

This implies:

$$\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}} = \frac{1}{2} g^{\mu\nu}\mathcal{L}. \tag{1.41}$$

This differential equation is solved by:

$$\mathcal{L} = \sqrt{-g}\, V(\phi_i), \tag{1.42}$$

where $V(\phi_i)$ is a potential depending on the self-tuning fields.

The remaining equation $\partial\mathcal{L}/\partial g_{\mu\nu} = 0$ leads to:

$$V(\phi_i) = 0, \tag{1.43}$$

thus, requiring a fine-tuned cancellation of the potential.

Now consider the second case, where the field equations are not independent. Weinberg assumes the existence of a relation of the form:

$$2g^{\mu\nu}\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} = \sum_i f_i(\phi)\frac{\partial\mathcal{L}}{\partial\phi_i}, \tag{1.44}$$

which implies a scaling symmetry:

$$\delta_\epsilon g_{\mu\nu} = 2\epsilon g_{\mu\nu}, \quad \delta_\epsilon\phi_i = -\epsilon f_i(\phi). \tag{1.45}$$

One can define new field variables $\tilde{\phi}_i$ such that:

$$\delta_\epsilon\tilde{\phi}_0 = -\epsilon, \quad \delta_\epsilon\tilde{\phi}_i = 0 \quad (i \neq 0). \tag{1.46}$$

The Lagrangian must then be of the form:

$$\mathcal{L} = \mathcal{L}\left(e^{2\tilde{\phi}_0} g_{\mu\nu}, \tilde{\phi}_i\right). \tag{1.47}$$

Returning to $GL(4)$ invariance, we compute:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\tilde{\phi}_0}\delta\tilde{\phi}_0 + \frac{\partial\mathcal{L}}{\partial\tilde{\phi}_i}\delta\tilde{\phi}_i + \frac{\partial\mathcal{L}}{\partial g_{\mu\nu}}\delta g_{\mu\nu}. \tag{1.48}$$

Since $\tilde{\phi}_0$ is a scalar under $GL(4)$, it does not transform ($\delta\tilde{\phi}_0 = 0$). Thus, the independent field equations are now:

$$\frac{\partial\mathcal{L}}{\partial\tilde{\phi}_i} = 0, \quad \frac{\partial\mathcal{L}}{\partial g_{\mu\nu}} = 0. \tag{1.49}$$

Proceeding as before, the form of the Lagrangian becomes:

$$\mathcal{L} = \sqrt{-g}\, e^{4\tilde{\phi}_0}\, V(\tilde{\phi}_i). \tag{1.50}$$

Applying the field equation for the metric, one obtains:

$$e^{4\tilde{\phi}_0} V(\tilde{\phi}_i) = 0. \tag{1.51}$$

This condition admits two possibilities:

- $V(\tilde{\phi}_i) = 0$, which again represents fine-tuning;
- $e^{\tilde{\phi}_0} \to 0$, which drives all mass scales to zero.

The second possibility is unphysical. Since physical masses scale as $e^{\tilde{\phi}_0}$, the limit $\tilde{\phi}_0 \to -\infty$ corresponds to a scale-invariant universe with vanishing masses, in clear contradiction with observational reality.

It is worth noting that the validity of vacuum translational invariance depends largely on the assumption of a flat, static spacetime. However, in a more realistic context, where gravity is dynamic and quantum effects come into play, this assumption is less certain. Therefore, while Weinberg no-go theorem is a significant result, it may not exclude all possible mechanisms for the dynamical relaxation of the cosmological constant, especially in scenarios where the vacuum is not invariant.

1.5 Towards a quantum gravity-based approach

The cosmological constant problem represents one of the most complex issues in contemporary physics, highlighting a deep discrepancy between the predictions of quantum field theory and cosmological observations. Although several theoretical approaches to the problem have been proposed, none have proven free from fundamental difficulties.

This challenge emphasizes the urgency of developing a more suitable theoretical framework capable of integrating gravitational and quantum phenomena, and naturally explaining the observed value of the cosmological constant.

Although there is still no unanimous consensus on the correct path toward formulating a theory of quantum gravity, it is noteworthy that numerous models suggest the existence of a minimum measurable length scale.

Beyond the question of which model of quantum gravity is most promising, one may therefore ask what implications the introduction of such a minimal length could have within the framework of quantum field theory. In particular, it is worth considering whether it could offer new perspectives on the cosmological constant problem and, optimistically, help to alleviate some of the challenges it poses.

The following chapters will be dedicated to this topic: first, the scenario of minimal length will be introduced, followed by its phenomenological implementation in quantum field theory on curved spacetime.

Chapter 2

Quantum gravity and minimal length

What we observe is not nature itself, but nature exposed to our method of questioning.

Werner Heisenberg

2.1 Thought experiments

Numerous thought experiments suggest that any theory attempting to describe the quantum nature of gravity must incorporate a minimal measurable length, presumably of the order of the Planck scale.

Several examples are presented, illustrating how the inclusion of gravitational effects in a quantum setting leads to a modification of the Heisenberg uncertainty relations, giving rise to what is referred to as the *generalized uncertainty principle* (GUP).

2.1.1 Heisenberg microscope

The well-known Heisenberg microscope *Gedankenexperiment* [17] offers an explicit illustration of the uncertainty principle in quantum mechanics.

Schematically, a light beam of frequency ω is used to illuminate an electron and the scattered photons pass through the objective of a microscope before reaching the observer's eye. Classical optics limits the accuracy of position measurements introducing an uncertainty along the direction transverse to the line of sight, the x-axis, given by:

$$\Delta x \gtrsim \frac{1}{\omega \sin \delta\theta}, \tag{2.1}$$

where $\delta\theta$ is the opening angle of the cone of light entering the objective after interacting with the electron. The photon used to measure the particle's position transfers momentum to it during the scattering process. Since the observer cannot determine the photon's incoming direction more precisely than the angle $\delta\theta$, the uncertainty in the electron's momentum along the x-direction is given by:

$$\Delta p_x \gtrsim \omega \sin \delta\theta. \tag{2.2}$$

The two uncertainties combine to yield the Heisenberg uncertainty relation:

$$\Delta x \, \Delta p_x \gtrsim 1. \tag{2.3}$$

(From this point on, Δp_x will be denoted simply by Δp.)

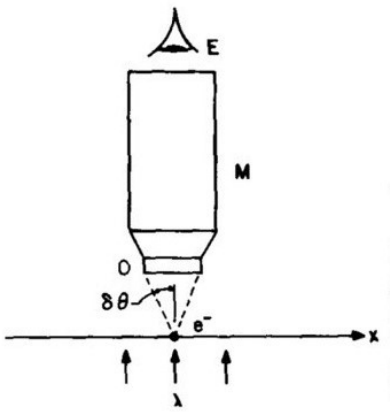

Figure 2.1: Heisenberg microscope. Light of frequency λ bounces off the electron, enters the objective O of the microscope and enters the eye E of the observer. Image from Shankar [39].

The case in which gravity is taken into account in the context of Heisenberg's microscope was carefully analyzed by Mead [27] using both a Newtonian approximation and the complete framework of general relativity.

Nevertheless, the Newtonian approximation alone is sufficient to obtain significant results.

In addition to the uncertainty due to the unknown scattering direction of the photon (yielding $\Delta x \gtrsim 1/\Delta p$), an additional contribution arises from the gravitational interaction between the photon and the observed particle. The particle is attracted toward the photon with acceleration $l_P^2\,\omega/r^2$, where ω is the photon's energy and r the size of the interaction region. Over a time $\sim r$, this leads to a displacement $\sim l_P^2\,\omega$ of the electron in the (unknown) photon's direction. Projected on the x-axis, this implies an additional uncertainty:

$$\Delta x \gtrsim l_P^2\,\Delta p. \tag{2.4}$$

Combining both pieces of uncertainty, one gets:

$$\Delta x \gtrsim \max\left(\frac{1}{\Delta p}, l_P^2\,\Delta p\right) \gtrsim l_P. \tag{2.5}$$

The physical idea behind this relation is quite straightforward. To get better spatial resolution, photons with higher energy are needed, as suggested by the usual Heisenberg uncertainty principle. However, increasing the photon's energy also makes its gravitational pull on the particle stronger, adding more disturbance. If the energy is too high, gravity affects the particle so much that it cancels out the benefit of using high-energy photons to improve precision.

Assuming that the uncertainties (2.3) and (2.4) add linearly, one obtains:

$$\Delta x \gtrsim \frac{1}{\Delta p} + l_P^2\,\Delta p, \tag{2.6}$$

that is invariant under the replacement:

$$l_P\,\Delta p \leftrightarrow \frac{1}{l_P\,\Delta p}. \tag{2.7}$$

Relations such as (2.6) are examples of the aforementioned GUP.

The standard microscope argument demonstrates a fundamental limit resulting from the non-commutativity of position and momentum operators in quantum mechanics. The appearance of a generalized uncertainty relation naturally suggests that some modification of quantum mechanics may be required to account for it (the details will be discussed in a later section).

The fully relativistic treatment is not presented here[1]. Nonetheless, a heuristic argument involving general relativity is outlined.

A fundamental aspect of Einstein's theory of gravity is that black holes form when the energy density in a region becomes too high. According to Thorne *hoop conjecture* [40], if energy ω is compressed into a region with circumference $R \leq 4\pi G\omega$, a black hole will form. Although this conjecture is

[1] For further details, the reader is referred to Mead's original work [27] or to appropriate review articles (see e.g. [13, 20]).

unproven, analytical and numerical evidence supports it.

Considering a particle with energy ω, quantum mechanics imposes a limit on how precisely the particle can be localized: its size R cannot be smaller than its Compton wavelength:

$$R \geq \lambda_C \sim \frac{1}{\omega}. \tag{2.8}$$

This means that increasing the particle's energy allows it to be confined in a smaller region. However, if the size R falls below the threshold given by the hoop conjecture, the particle would gravitationally collapse into a black hole with radius:

$$R_{\mathrm{BH}} = 2G\omega. \tag{2.9}$$

Because the black hole radius increases linearly with energy while the Compton wavelength decreases inversely with energy, these two limits overlap, setting a fundamental minimal length scale:

$$R_{\min} \sim \sqrt{G}, \tag{2.10}$$

which corresponds, again, to the Planck length l_P.

2.1.2 Clock synchronization

The synchronization of a clock with another, taken as standard reference, can be performed through photon exchange.

Two primary sources of uncertainty affect the clock reading. First, the Heisenberg time-energy uncertainty relation limits the precision in the emission or absorption of photons, restricting the accuracy to $\Delta T_{\mathrm{QM}} \sim 1/\Delta\omega$. Second, the gravitational interaction between the photon and the clock introduces an additional uncertainty.

Assuming a strong interaction between photon and clock within a region of radius r lasting for a time r (with the clock considered stationary), the proper duration of the interaction as measured by the clock is given by:

$$T = \sqrt{g_{00}}\, r, \tag{2.11}$$

where g_{00} is the time-time component of the gravitational field generated by the photon and experienced by the clock[2]:

[2]The deviation from the Schwarzschild metric, represented by the factor 4, is explained

$$g_{00} = 1 - \frac{4l_P^2\omega}{r}. \tag{2.12}$$

Since the metric depends on the energy of the photon, the error on ω propagates into T by:

$$\Delta T_{\mathrm{GR}} \sim \left|\frac{\partial T}{\partial \omega}\right| \Delta\omega, \tag{2.13}$$

that is:

$$\Delta T_{\mathrm{GR}} \sim \frac{2l_P^2}{\sqrt{1 - 4l_P^2\omega/r}}\Delta\omega \gtrsim 2l_P^2\,\Delta\omega, \tag{2.14}$$

resulting in the overall uncertainty in the reading of the clock:

$$\Delta T \gtrsim \max\left(\frac{1}{\Delta\omega}, l_P^2\,\Delta\omega\right) \gtrsim l_P. \tag{2.15}$$

It is important to note that the uncertainty relations described so far represent low-energy approximations of what a complete theory of quantum gravity would predict. They all consist of two contributions: the first arises from the standard Heisenberg uncertainty principle, while the second stems from the dynamical response of spacetime to quantum fluctuations. However, this modification should, in turn, generate a new uncertainty in the gravitational field itself, an effect that recursively introduces further uncertainty. This chain of corrections reflects the inherent non-linearity of gravitational interactions.

2.1.3 Black hole horizon

What follows is a thought experiment proposed by Maggiore [24], aimed at deriving an uncertainty relation based on the measurement of a black hole's radius.

In classical general relativity, the radius of the event horizon cannot be measured; it can only be inferred from other parameters of the black hole (namely, its mass, charge and angular momentum), but not directly verified. However, this is no longer the case once quantum effects are taken into account. In fact, Hawking radiation would allow for a direct assessment of the black hole's area, and thus enable a validation of the relationship between its radius and the other aforementioned parameters.

Now, consider sending a photon with energy ω from an asymptotic region

in [27].

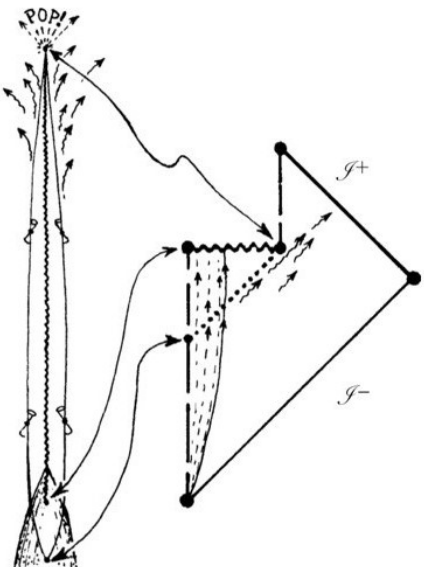

Figure 2.2: Hawking black hole evaporation. A black hole forms through calssical gravitational collapse. Then, over an extremely long period, it loses mass at a very slow rate, through Hawking radiation. Image from Penrose [35].

toward an *extremal* black hole (one with vanishing surface gravity, and therefore no thermal radiation emission). The black hole's mass will increase by the amount ω, and it will subsequently re-emit this energy to return to a stable extremal state (assumed to occur through a single photon, that is eventually measured).

By repeating the process many times, the observer would eventually be able to "see" the black hole.

As in the preceding cases, the evaluation of the horizon radius R_H is affected by the usual uncertainty arising from the photon's finite wavelength, which limits the observer's knowledge of the photon's point of origin and leads again to the Heisenberg inequality (2.3), where, this time, Δp represent the uncertainty in the final momentum of the black hole.

During the emission process, the mass of the black hole decreases from $M + \omega$ to M, and the corresponding horizon radius must adjust accordingly. If the energy of the emitted photon is known only within an uncertainty Δp, this uncertainty propagates into the precision with which the horizon radius can be studied:

$$\Delta R_H \sim \left| \frac{\partial R_H}{\partial M} \right| \Delta p. \tag{2.16}$$

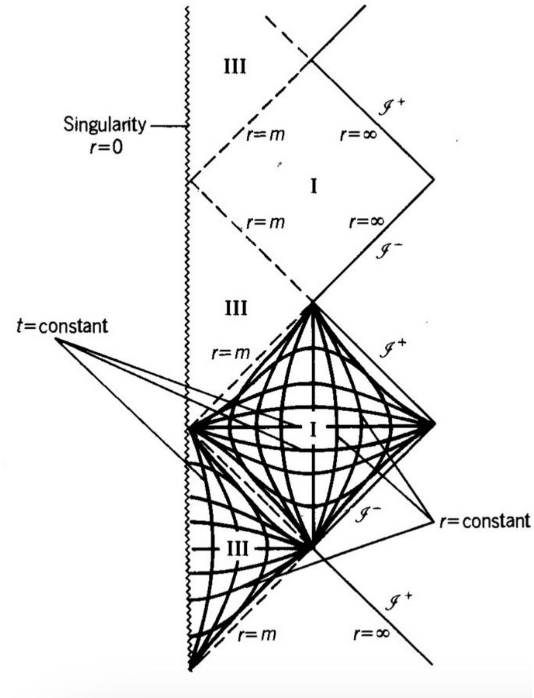

Figure 2.3: Reissner-Nordström metric. Penrose diagram for the case $Q^2 = GM^2$. Image from d'Inverno [8].

Considering a Reissner-Nordström black hole with charge Q (the extension to the Kerr-Newman's case is quite straightforward), whose outer horizon's radius is given by:

$$R_H = GM\left[1 + \left(1 - \frac{Q^2}{GM^2}\right)^{1/2}\right], \tag{2.17}$$

and assuming that no naked singularities occur in nature (i.e., $M^2G \leq Q^2$, where the equality holds for the extremal case), it follows that:

$$\Delta R_H \gtrsim l_P^2\,\Delta p. \tag{2.18}$$

In his article, Maggiore combines linearly the two sources of uncertainties (using Δx in place of ΔR_H), and obtains:

$$\Delta x \gtrsim \frac{1}{\Delta p} + \alpha\, l_P^2\,\Delta p, \tag{2.19}$$

which represents a GUP. Where the dimensionless constant α is introduced to facilitate comparison with predictions from various quantum gravity models (e.g., string theory).

2.2 Quantum gravity models

So far, a series of thought experiments has been presented which, independently of the specific model, provide arguments supporting the existence of a minimal measurable length in the context of quantum gravity.

In this section, by contrast, it is shown how the idea of a minimal length can emerge from specific theories.

2.2.1 Quantized conformal fluctuations

The formulation of quantum gravity developed by Padmanabhan [32] is based on the path integral:

$$K = \int \mathcal{D}g_{\mu\nu}\, e^{iS[g_{\mu\nu}]}, \tag{2.20}$$

where $S[g_{\mu\nu}]$ is the Einstein-Hilbert action:

$$S = \frac{1}{16\pi G} \int \sqrt{-g}\, d^4x\, R. \tag{2.21}$$

For simplicity, consider a conformally flat metric of the form:

$$g_{\mu\nu}(x) = \left[1 + \phi(x)^2\right]\, \eta_{\mu\nu}, \tag{2.22}$$

that reduces the path integral to:

$$\int \mathcal{D}\phi\, \exp\left(\frac{i}{l_P^2} \int d^4x\, \eta^{\mu\nu} \phi_{;\mu} \phi_{;\nu}\right), \tag{2.23}$$

which presents a quadratic action and can be evaluated in closed form. This allowed Padmanabhan to write the probability amplitude for a measurement of the conformal fluctuation to give a value ϕ:

$$\mathcal{A}(\phi) = \left(\frac{l}{l_P}\right)^{1/4} \exp\left(-\frac{l^2}{l_P^2}\phi^2\right), \tag{2.24}$$

where l is the resolution of the measuring apparatus. If the region over which one measures is very large ($l \gg l_P$), then the probability amplitude $\mathcal{A}$ will have a sharp peak at flat spacetime (i.e., $\phi = 0$).

The distribution leads to an uncertainty relation of the form:

$$\Delta\phi\, l \gtrsim l_P. \tag{2.25}$$

In other words, as the scale l decreases (and resolution increases), the fluctuations of the conformal factor ϕ become stronger.

To proceed, consider the coincidence limit of the vacuum expectation value of the proper interval between two nearby events:

$$\langle 0| \, ds^2 \, |0\rangle = \lim_{x \to y} l^2(x, y) = \lim_{x \to y} \left[1 + \langle \phi(x)\phi(y)\rangle\right] l_0^2(x, y), \tag{2.26}$$

where $l_0^2(x, y)$ denotes the proper interval computed with respect to the flat background metric $\eta_{\mu\nu}$.

Given that the action of the field ϕ corresponds to that of a free massless scalar field with an inverted sign, one obtains the Green's function:

$$\langle \phi(x)\phi(y)\rangle \sim \frac{l_P^2}{l_0^2(x, y)}, \tag{2.27}$$

which suggests, once inserted in (2.26), that the Planck length l_P sets a lower bound for the proper interval between any two events.

2.2.2 String theory

String theory[3] is one of the leading candidates for a theory of quantum gravity.

For the purposes of this discussion, it is more than sufficient to observe that a *string* is represented by a two-dimensional surface, known as the *worldsheet*, which is swept out in a higher-dimensional spacetime. For internal consistency, supersymmetric string theory requires a total of nine spatial dimensions, implying the existence of six additional spatial dimensions beyond the familiar three.

In this section, the total number of spacetime dimensions is denoted by D (Greek indices range from 0 to $D - 1$).

The worldsheet is denoted by X^ν and is parameterized by two dimensionless coordinates: τ, representing the time-like direction, and σ, typically ranging from 0 to 2π. The physical state of a string can be described as a combination of its discrete excitations and the motion of its center of mass.

Due to conformal invariance, the worldsheet acquires a complex structure and can thus be treated as a Riemann surface, with complex coordinates labeled z and $\overline{z}$. Scattering amplitudes in string theory are computed as sums over such Riemann surfaces.

[3]For a comprehensive and pedagogic introduction to string theory, see [36].

String theory already contains a minimum length as can be seen from a simple (albeit rather formal) argument (see [21]). The classical relativistic string dynamics is encoded inside the Polyakov action:

$$\mathcal{S} \propto \int d\tau d\sigma \sqrt{-h}\, h^{ab}\, \eta_{\mu\nu}\, \partial_a X^{\mu}(\tau,\sigma)\, \partial_b X^{\nu}(\tau,\sigma), \tag{2.28}$$

where h^{ab} is the inverse of the worldsheet metric and h denotes its determinant.

The corresponding quantum theory can be expressed in the path integral formalism. In discretized form:

$$Z_{\text{string}} \sim \int \mathcal{D}X \, \exp\left\{ i\frac{1}{\alpha\, l_P^2} \sum \epsilon^2 \left[\left(\frac{\delta_\tau X}{\epsilon}\right)^2 + \left(\frac{\delta_\sigma X}{\epsilon}\right)^2 \right] \right\}, \tag{2.29}$$

where the constant α that appears in the exponent depends on the particular model that is considered. It can be observed that, since the worldsheet is two-dimensional, the ϵ-factors cancel out, resulting in a spacetime distance:

$$\langle (\delta x)^2 \rangle \sim \alpha\, l_P^2$$

which is independent of ϵ. This is in contrast with the case of a point particle, where the *worldline* is one-dimensional: in that case, the corresponding expression for the spacetime interval maintains a dependence on ϵ, and no intrinsic length scale emerges in the same manner.

Furthermore, several rigorous calculations of high-energy string scattering (e.g., [14]) indicate modifications to the Heisenberg uncertainty principle, highlighting a linear relationship between the string's longitudinal spread and the probe energy E:

$$\Delta x^{\nu} \Delta p^{\nu} \gtrsim 1 + l_s E, \tag{2.30}$$

where l_s is the *string scale*, which, in most versions of string theory, is close to the Planck scale.

2.2.3 Loop quantum gravity

Loop quantum gravity[4] (LQG) is a non-perturbative approach to the quantization of gravity, formulated through a specific choice of canonical variables known as the Ashtekar variables.

[4]The standard reference for LQG is [38].

The formalism arises from a Hamiltonian treatment of general relativity, employing the $3+1$ ADM decomposition of spacetime into spacelike hypersurfaces. In this context, the spatial metric is supplemented by the extrinsic curvature K_{ab}, which encodes how the geometry evolves along the foliation. To facilitate quantization, the spatial metric h_{ab} is reformulated using a triad (or *dreibein*) basis E_i^a:

$$h^{ab} = E_i^a E_j^b \delta^{ij} \tag{2.31}$$

and the dynamical variables are replaced by the densitized triad $\tilde{E}_i^a = \sqrt{h}\, E_i^a$ and the su(2) connection A_a^i. These variables form a canonical pair and, in the quantum setting, are promoted to operators satisfying the commutation relations:

$$\left[A_a^j(x), \tilde{E}_i^b(y)\right] = i\beta\, \delta_a^b \delta_j^i\, \delta^3(x-y), \tag{2.32}$$

with β denoting the Barbero-Immirzi parameter.

The quantized theory admits various representations, among which the *loop* (from which the name loop quantum gravity derives) and *spin network* representations are of central importance. Spin networks, in particular, provide a basis of quantum states $|\psi_s\rangle$ defined on graphs whose edges are labeled by irreducible su(2) representations.

In LQG the area of a two-dimensional surface Σ becomes an operator A_Σ acting on spin network states. Its eigenvalues are discrete and depend on the su(2) half-integer representation labels j_I associated with edges intersecting the surface:

$$A_\Sigma \,|\psi_s\rangle = 8\pi l_P^2 \beta \sum_I \sqrt{j_I(j_I+1)}\, |\psi_s\rangle\,. \tag{2.33}$$

As a result, area is quantized (as shown in [37]), and the theory predicts the existence of a minimal nonzero area, proportional to the square of the Planck length and scaled by the Barbero–Immirzi parameter.

This discrete spectrum of geometric operators implies a fundamental minimal length scale within the framework of LQG.

A note on canonical quantization We believe it is worth mentioning, among the notable approaches, the canonical quantization program of gravity[5], from which LQG itself originates. The presence of a minimal length

[5] For a complete discussion, see [22].

does not appear to be, in itself, a structural feature of the theory. Nonetheless, a minimal scale seems to emerge dynamically in the study of certain mini-superspace models of gravitational collapse (see, e.g., [23]), in which the (simplified) Wheeler–DeWitt equation appears to suggest a bounce mechanism once the star reaches Planck-scale dimensions.

2.3 Non-commutative geometry

Non-commutative geometry (for a detailed discussion, see [18]) emerges both as a modification of quantum mechanics and quantum field theory in certain approaches to quantum gravity, and as a distinct class of theoretical models. The idea is strongly motivated by developments in string theory. Furthermore, there are indications that LQG may give rise to a specific type of non-commutative spacetime known as κ-Poincaré geometry.

The key principle of non-commutative geometry is that, upon quantization, the spacetime coordinates x^ν become Hermitian operators $\hat{x}^\nu$ that do not commute:

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{2.34}$$

where $\theta^{\mu\nu}$ is a real, antisymmetric tensor of dimension length squared, known as the *Poisson tensor*. This tensor introduces a fundamental minimal area scale, similar to the role of $\hbar$ in quantum mechanics. The resulting uncertainty relation among spatial coordinates implies:

$$\Delta x^\mu \Delta x^\nu \gtrsim \frac{1}{2}|\theta^{\mu\nu}|. \tag{2.35}$$

The elements of $\theta^{\mu\nu}$ are free parameters to be constrained experimentally, even though, in the light of what has been discussed so far, one would expect them to be of the order of the Planck length squared. Notably, $\theta^{\mu\nu}$ defines a preferred frame, breaking Lorentz invariance.

Quantization in non-commutative geometry extends beyond coordinates to functions on spacetime. Using Weyl quantization, a mapping W is defined from classical functions $f(x)$ to Hermitian operators:

$$\hat{f} = W(f) = \frac{1}{(2\pi)^4} \int d^4k \, e^{-ik_\mu \hat{x}^\mu} \tilde{f}(k), \tag{2.36}$$

where $\tilde{f}(k)$ is the Fourier transform of $f(x)$.

One can then introduce a product of functions known as the *Moyal–Weyl* (or $\star$) product [29]:

$$f(x) \star g(x) = \exp\left(\frac{i}{2}\frac{\partial}{\partial x^\mu}\theta^{\mu\nu}\frac{\partial}{\partial y^\nu}\right) f(x)g(y)\bigg|_{x\to y}, \tag{2.37}$$

which implies:

$$W(f \star g)(x) = W(f) \cdot W(g) = \hat{f} \cdot \hat{g}, \tag{2.38}$$

establishing a homomorphism between the algebras of functions and Hermitian operators.

2.4 Modified commutation relations and GUP

In the previous sections, it was shown, through the presentation of different thought experiments, how extensions of Heisenberg's uncertainty principle can emerge dynamically.

Indeed, in quantum mechanics, uncertainty relations are not dynamical; rather, they come from the kinematical structure of the theory. In particular, the fact that the position and momentum operators do not commute leads to an uncertainty relation between them. This applies to any pair of non-commuting observables: they cannot be measured simultaneously with arbitrary precision. This raises the question of whether generalized uncertainty relations can also be derived from the kinematical structure of the theory.

In this regard, a widely studied approach to incorporate a minimal length scale in quantum mechanics and quantum field theory is based on the implementation of modified commutation relations between position and momentum operators. These modifications may extend beyond the canonical algebra and can imply a non-commutative geometry in position (as seen in the previous section) and/or momentum space.

To illustrate the mechanism, consider the standard quantization of the variables $\mathbf{k} = (\omega, k^i)$, where k^i are the spatial components, and $\mathbf{x} = (t, x^i)$. These satisfy the canonical commutation relations:

$$[x^\nu, x^\kappa] = 0, \quad [x^\nu, k_\kappa] = i\delta^\nu_\kappa, \quad [k_\nu, k_\kappa] = 0. \tag{2.39}$$

Now define a new set of momentum variables $\mathbf{p} = (E, p^i) = f(\mathbf{k})$, where f is an invertible function. The inverse function $\mathbf{k} = f^{-1}(\mathbf{p})$ is thus well-defined.

We can express these transformations as:

$$p_\mu = h_\mu^\alpha(\mathbf{k})\, k_\alpha \tag{2.40}$$

and:

$$k_\alpha = h_\alpha^\mu(\mathbf{p})\, p_\mu. \tag{2.41}$$

Under this change of variables, the commutation relations become:

$$[x^\nu, x^\kappa] = 0, \quad [x^\nu, p_\kappa] = i\frac{\partial f_\kappa}{\partial k_\nu}, \quad [p_\nu, p_\kappa] = 0. \tag{2.42}$$

This directly leads to a modified uncertainty relation:

$$\Delta x_i \Delta p_i \geq \frac{1}{2}\left\langle \frac{\partial f_i}{\partial k_i} \right\rangle. \tag{2.43}$$

To make this concrete, consider a simple deformation:

$$p_i \approx k_i \left(1 + \frac{\alpha k^2}{m_P^2}\right), \tag{2.44}$$

where α is a dimensionless constant and $k^2 = |\vec{k}|^2$ (the same notation holds for p). Inverting this relation yields:

$$k_i \approx p_i \left(1 - \frac{\alpha p^2}{m_P^2}\right). \tag{2.45}$$

From this, the Jacobian becomes:

$$\frac{\partial f_i}{\partial k^j} \approx \delta_{ij}\left(1 + \frac{\alpha p^2}{m_P^2}\right) + \frac{2\alpha\, p_i p_j}{m_P^2}. \tag{2.46}$$

Taking the expectation value, one obtains a generalized uncertainty relation:

$$\Delta x_i \Delta p_i \geq \frac{1}{2}\left(1 + \frac{\alpha\langle p^2\rangle}{m_P^2} + \frac{2\alpha\langle p_i^2\rangle}{m_P^2}\right). \tag{2.47}$$

Rewriting this inequality explicitly for Δx^i (remember, $\Delta A^2 = \langle A^2\rangle - \langle A\rangle^2$), one finds:

$$\Delta x_i \geq \frac{1}{2}\left(\frac{1}{\Delta p_i} + \frac{3\alpha \Delta p_i}{m_P^2}\right), \tag{2.48}$$

consistently with what was obtained through the thought experiments previously examined.

The inequality (2.48) carries operational meaning only if $\mathbf{p}$ is interpreted as a physical momentum (it is represented by a Hermitian operator). To distinguish the physical quantity $\mathbf{p}$ from $\mathbf{k}$ (which satisfies the canonical commutation relations with $\mathbf{x}$), the latter is sometimes called the *pseudo-momentum* or the *wave vector*.

Now, consider the issue of Lorentz invariance. Without additional assumptions beyond the commutation relations, the transformation properties of the quantities remain undetermined. They could, in principle, transform arbitrarily, implying a possible violation of Lorentz invariance.

If the latter is imposed, further questions arise: how is it preserved, what is the geometry of the corresponding phase space and, most importantly, how can one identify physically meaningful coordinates on this space?

No widely accepted picture has emerged.

Nonetheless, assume that the phase space is a trivial fiber bundle $\mathcal{S} = \mathcal{M} \otimes \mathcal{P}$, where $\mathcal{M}$ denotes spacetime and $\mathcal{P}$ denotes momentum space. Elements of this space are of the form $(\mathbf{x}, \mathbf{p})$, with $\mathbf{x} \in \mathcal{M}$ and $\mathbf{p} \in \mathcal{P}$.

One can further consider $\mathbf{p}$ as a coordinate on $\mathcal{P}$ that transforms under standard Lorentz transformations and let $\mathbf{k}$ be an alternative coordinate system on $\mathcal{P}$, related to $\mathbf{p}$ through the function f.

Under a Lorentz transformation Λ, the momentum transforms as $\mathbf{p}' = \Lambda\mathbf{p}$ and the transformation of $\mathbf{k}$ follows as:

$$\mathbf{k}' = f(\mathbf{p}') = f(\Lambda\mathbf{p}) = f(\Lambda f^{-1}(\mathbf{k})), \tag{2.49}$$

which defines a modified Lorentz transformation $\mathbf{k}' = \tilde{\Lambda}(\mathbf{k})$ in the k-coordinates.

Importantly, one can choose the function f such that it maps infinite values of $\mathbf{p}$ (in either the spatial or temporal components, or both) to finite values of $\mathbf{k}$ (possibly of the order of the Planck scale). The corresponding Lorentz transformation in k-space then preserves the Planck scale, all without introducing a preferred frame of reference. This construction forms the basis for *deformations of special relativity*.

In any case, the choice of how to fix the transformation behavior is not unique. An alternative to the approach outlined above treats $\mathbf{k}$ as transforming conventionally and interprets $\mathbf{p}$ as the physical momentum (or $\mathbf{x}$ as 'pseudo-coordinates').

This variety of conventions is one of the main reasons why the literature on modified commutation relations can be difficult to navigate.

2.5 Vacuum energy and minimal length

Returning to the cosmological constant problem, the calculation of the vacuum energy density has been carried out in a minimal length scenario [6] (see also [3]) by employing the modified commutation relations:

$$[x_i, p_j] = i\delta_{ij}(1 + \beta p^2). \tag{2.50}$$

When performing the integration over the normal modes, one must take into account that the measure in momentum space now takes the following form:

$$\frac{d^3\vec{p}}{(1 + \beta p^2)^3}, \tag{2.51}$$

and the vacuum energy density becomes:

$$\langle \rho \rangle = \int \frac{4\pi p^2}{(2\pi)^3} \frac{dp}{(1 + \beta p^2)^3} \frac{1}{2} \sqrt{p^2 + m^2}. \tag{2.52}$$

Since the integrand behaves as $\mathcal{O}\left(\frac{1}{p^3}\right)$ at large momentum, the integral is convergent and does not require a momentum cutoff. In fact, (2.52) can be evaluated exactly for any value of m. For simplicity, in the massless case, the integral reduces to:

$$\langle \rho \rangle = \frac{1}{16\pi^2 \beta^2}. \tag{2.53}$$

If one takes β to be of the order of the Planck scale, the result still yields a GUP-modified vacuum energy that is, again, approximately 120 orders of magnitude larger than the observed vacuum energy.

Thus, although the GUP functional cutoff factor (2.51) makes $\langle \rho \rangle$ finite through justified physical assumptions, it still predicts a value that is vastly too large, and therefore fails to resolve the vacuum catastrophe.

In any case, addressing the cosmological constant problem requires taking into account the curvature of spacetime. To this end, the aim of the next chapter is to heuristically adapt quantum field theory in curved spacetime to a framework that incorporates the existence of a minimal length scale.

Chapter 3

Covariant point-separation renormalization and minimal length

Cosmologists are often in error but never in doubt.

Lev Landau

Guide to the chapter The goal of this chapter is to derive an expression for the energy density of a massive scalar field in a de Sitter background, taking into account the presence of a minimal spacetime interval.

In quantum field theory in curved spacetime, the vacuum expectation value (VEV) of the energy-momentum tensor diverges, since it is constructed from the product of two field operators evaluated at the same spacetime point. To address this issue, several renormalization approaches have been developed. Among them, the *covariant geodesic point separation method* stands out; it involves replacing a quadratic operator with the product of two field operators evaluated at nearby spacetime points. In this way, a finite result is obtained, which can be expressed in terms of the *Hadamard function*. The latter is written in terms of the biscalar of geodetic interval (also known as the *Synge world function*), which provides a measure of the square of the geodesic distance between the separated points.

Through the Synge biscalar, the Hadamard function explicitly reveals the structure of the divergences that affect the Feynman propagator in the limit where the previously separated points are brought together.

Such divergences, which are typically removed through appropriate regularization techniques, may not arise in a scenario with a fundamental minimal length, where the coincidence limit loses its physical meaning. In this context, a minimal length scale emerging from the quantum nature of gravity appears to provide a natural regularization mechanism for the ultraviolet divergences that afflict quantum field theory.

3.1 The point-separated energy-momentum tensor

The action functional for a scalar field in a curved background is:

$$S[\phi] = -\frac{1}{2}\int d^4x\sqrt{g}\left(g^{\mu\nu}\nabla_\mu\phi\,\nabla_\nu\phi + \xi R\phi^2 + m^2\phi^2\right), \tag{3.1}$$

where:

- $g = -\det(g_{\mu\nu})$;
- R is the Ricci scalar;
- m is the mass of the scalar field;
- ξ is a coupling constant that governs the interaction between the field and the curvature[1].

From this action one obtains the equations of motion for the scalar field:

$$0 = \frac{\delta S}{\delta\phi} = -\sqrt{g}\left(\Box - \xi R - m^2\right)\phi, \tag{3.2}$$

where $\Box = \nabla^\mu\nabla_\mu$.

The classical stress tensor is defined as:

$$\begin{aligned}T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} = &\frac{1}{2}(1-2\xi)\left\{\phi^{;\mu},\phi^{;\nu}\right\} + \frac{1}{2}\left(2\xi - \frac{1}{2}\right)g^{\mu\nu}\left\{\phi_{;\alpha},\phi^{;\alpha}\right\}\\ &-\xi\left\{\phi^{;\mu\nu},\phi\right\} + g^{\mu\nu}\left\{\phi_{;\alpha}{}^{\alpha},\phi\right\}\\ &+\xi\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)\{\phi,\phi\} - \frac{1}{2}m^2 g^{\mu\nu}\{\phi,\phi\},\end{aligned} \tag{3.3}$$

[1]The value of ξ depends on the physical context: for a minimally coupled scalar field, $\xi = 0$, while for a conformally coupled scalar field in four dimensions, $\xi = \frac{1}{6}$.

with $\{\,,\}$ denoting the anti-commutator.

$T_{\mu\nu}$ is symmetric and covariantly conserved, i.e.,

$$\nabla_\mu T^{\mu\nu} = 0, \tag{3.4}$$

as a consequence of the field equations. Moreover, when the scalar field is massless ($m = 0$) and conformally coupled ($\xi = \frac{1}{6}$), the tensor is also traceless:

$$T^\mu_{\ \mu} = 0. \tag{3.5}$$

Going from classical to quantum theory, the scalar field ϕ is promoted to a quantum operator $\hat{\phi}$. The stress-energy tensor then involves products of field operators (or their derivatives) evaluated at the same spacetime point, which become ill-defined when taking vacuum expectation values.

To circumvent this difficulty, one can employ the *point-separation*[2] procedure, which consists in replacing one of the field operators $\hat{\phi}(x)$, appearing in quadratic terms, with $\hat{\phi}(x')$, where x' is a point close to x.

The finite quantity:

$$G^{(1)}(x, x') \equiv \langle 0| \{\hat{\phi}(x), \hat{\phi}(x')\} |0\rangle \tag{3.6}$$

takes the name of *Hadamard elementary function*[3] and allows one to write the divergent VEV of the energy momentum tensor, after some manipulations, as:

$$\begin{aligned}
\langle \hat{T}^{\mu\nu} \rangle_{\rm div} = \lim_{x'\to x} \frac{1}{2} &\left(\frac{1}{2} - \xi\right) \left(G^{(1)\,;\mu'\nu} + G^{(1)\,;\mu\nu'}\right) + \left(\xi - \frac{1}{4}\right) g^{\mu\nu} G^{(1)\,\alpha'}_{\ ;\alpha} \\
&- \frac{1}{2}\xi \left(G^{(1)\,;\mu\nu} + G^{(1)\,;\mu'\nu'}\right) + \frac{1}{8}\xi\, g^{\mu\nu} \left(G^{(1)\,\alpha}_{\ ;\alpha} + G^{(1)\,\alpha'}_{\ ;\alpha'}\right) \\
&+ \frac{3}{4}\xi\, g^{\mu\nu} \left(\xi R + m^2\right) G^{(1)} + \frac{1}{2}\xi \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right) G^{(1)} \\
&- \frac{1}{4} m^2 g^{\mu\nu} G^{(1)},
\end{aligned} \tag{3.7}$$

where primed Greek indices denote covariant derivatives acting on fields dependent on x'.

The latter expression is purely formal: terms such as $G^{(1)\,;\mu\nu} + G^{(1)\,;\mu'\nu'}$ are meaningless, since each term transforms differently as a bitensor, and cannot be added.

[2] Our discussion follows that of Christensen [7].

[3] A regularization technique based on the Hadamard function is presented in the appendix.

3.2 The Schwinger-DeWitt method

The Hadamard function can be studied starting from the Feynman Green's function:

$$G(x,x') = \bar{G}(x,x') - \frac{i}{2} G^{(1)}(x,x'), \tag{3.8}$$

where $\bar{G}$ is the principal value part.

To compute the Feynman Green's function in curved spacetime, it is useful to employ DeWitt's generalization of the Schwinger proper-time method[4].

One considers an abstract Hilbert space equipped with a set of formal operators $\hat{x}^\mu$ and $\hat{p}_\nu$, satisfying the canonical commutation relations:

$$\begin{aligned} [\hat{x}^\mu, \hat{x}^\nu] &= 0, \\ [\hat{p}_\mu, \hat{p}_\nu] &= 0, \\ [\hat{x}^\mu, \hat{p}_\nu] &= i\delta^\mu_\nu. \end{aligned} \tag{3.9}$$

These operators act on a basis of eigenvectors $|x\rangle$, normalized such that:

$$\langle x|x'\rangle = \langle x|\mathbb{I}|x'\rangle = \delta(x,x'). \tag{3.10}$$

Consider the quantum theory of a non-minimally coupled scalar field ϕ. The field equation can be written in terms of an operator F:

$$F\phi(x) = \sqrt{g}\left(\Box - m^2 - \xi R\right)\phi(x) = 0. \tag{3.11}$$

Within the Schwinger formalism, the operator F and the associated Green's function are interpreted as matrix elements in the abstract Hilbert space:

$$G(x,x') = \langle x|\mathbf{G}|x'\rangle, \quad F(x,x') = \langle x|\mathbf{F}|x'\rangle \tag{3.12}$$

and satisfy the equation:

$$\mathbf{F}\mathbf{G} = -\mathbb{I}. \tag{3.13}$$

One may assign to the operator $\mathbf{F}$ an infinitesimal "positive" imaginary part. Consequently, the following operator identity holds:

[4]See [11] for further details.

$$g^{1/4}\mathbf{G}\, g^{1/4} = -\frac{1}{g^{-1/4}\mathbf{F}\, g^{-1/4} + i0^+} = i\int_0^\infty ds\, \exp\left\{i\, g^{-1/4}\mathbf{F} g^{-1/4}\, s\right\}. \quad (3.14)$$

This leads to the proper-time representation of the Feynman propagator:

$$G(x, x') = i\int_0^\infty g^{-1/4}(x)\langle x, s|x', 0\rangle g^{-1/4}(x')\, ds, \quad (3.15)$$

where the *kernel* $\langle x, s|x', 0\rangle = \langle x| \exp\left\{i\, g^{-1/4}\mathbf{F} g^{-1/4}\, s\right\} |x'\rangle$ denotes the probability amplitude for a fictitious particle to propagate, in a "proper-time" interval s, from point x to point x' on a hypersurface having the number of dimensions of the original spacetime. It satisfies the Schrödinger-like equation:

$$i\frac{\partial}{\partial s}\langle x, s|x', 0\rangle = F_x\langle x, s|x', 0\rangle. \quad (3.16)$$

In the case where the spacetime points x and x' are close, a WKB expansion can be written:

$$\langle x, s|x', 0\rangle \sim \frac{iD^{1/2}(x, x')}{(4\pi s)^2} \exp\left[\frac{i\sigma(x, x')}{2s} - im^2 s\right] \Omega(x, x', s), \quad (3.17)$$

where:

- $\sigma(x, x') \equiv \frac{1}{2}\tau(x, x')^2$ is the Synge's world function, equal to half the squared geodesic distance between x and x'. From geodesic theory $\sigma(x, x') = \frac{1}{2}\sigma^{;\mu}\sigma_{;\mu}$;
- $D(x, x') \equiv -\det(-\sigma_{;\mu\nu'})$ is the Van Vleck–Morette determinant, which satisfies the identity $D^{-1}\left(D\, \sigma^{;\mu}\right)_{;\mu} = 4$;
- $\Omega(x, x', s)$ encodes the higher-order contributions and admits an asymptotic expansion in powers of s:

$$\Omega(x, x', s) \sim \sum_{n=0}^{\infty} a_n(x, x')(is)^n. \quad (3.18)$$

$\langle x, s|x', 0\rangle$ reduces to $\delta(x, x')$ as $s \to 0$ and $\Omega(x, x', 0) = a_0(x, x) = 1$ for all x and x'.

Substituting (3.17) and (3.18) in (3.16), one gets a set of recursion relations for the a_n:

$$a_{0;\mu}\sigma^{;\mu} = 0, \tag{3.19}$$

which is trivially satisfied since $a_0 = 1$, and:

$$\sigma^{;\mu}a_{n+1\,;\mu} + (n+1)a_{n+1} = \Delta^{-1/2}\left(\Delta^{1/2}a_n\right)_{;\mu}{}^{\mu} - \xi R a_n, \tag{3.20}$$

where $\Delta(x, x') = g^{-1/2}(x)\, D(x, x')\, g^{-1/2}(x')$.

Substituting (3.17) into (3.15) and exchanging the summation in (3.18) with the integration sign, one obtains the following form for the Feynman Green's function:

$$G(x, x') = \frac{\Delta^{1/2}}{(4\pi)^2}\sum_{n=0}^{\infty} a_n \left(-\frac{\partial}{\partial m^2}\right)^n \int_0^{\infty} \frac{1}{s^2}\exp\left[-i\left(m^2 s - \frac{\sigma}{2s}\right)\right] ds. \tag{3.21}$$

Since:

$$\frac{1}{(4\pi)^2}\int_0^{\infty} \frac{1}{s^2}\exp\left[-i\left(m^2 s - \frac{\sigma}{2s}\right)\right] ds = -\frac{m^2}{8\pi}\frac{H_1^{(2)}(\sqrt{-2m^2\sigma})}{(\sqrt{-2m^2\sigma})}, \tag{3.22}$$

where $H_1^{(2)}$ is the Hankel function of the second kind of order one, and the following identities hold:

$$\frac{1}{\sigma + i0^+} = \frac{1}{\sigma} - i\pi\delta(\sigma), \quad \log(\sigma + i0^+) = \log|\sigma| + i\pi\theta(-\sigma), \tag{3.23}$$

where:

$$\theta(-\sigma) = \begin{cases} 1, & \sigma < 0 \\ 0, & \sigma > 0 \end{cases} \tag{3.24}$$

one can expand the Hadamard function as:

$$
\begin{aligned}
G^{(1)}(x,x') \sim \frac{\Delta^{1/2}}{4\pi^2} \Bigg\{ & \frac{1}{\sigma} + m^2 \left(\gamma + \frac{1}{2}\log|m^2\sigma/2|\right)\left(1 + \frac{1}{4}m^2\sigma + \dots\right) \\
& - \frac{1}{2}m^2 - \frac{5}{16}m^4\sigma - \dots \\
& - a_1 \left[\left(\gamma + \frac{1}{2}\log|m^2\sigma/2|\right)\left(1 + \frac{1}{2}m^2\sigma + \dots\right) - \frac{1}{2}m^2\sigma - \dots\right] \\
& + \left(\frac{1}{2}a_1^2 + a_2\right)\sigma \left[\left(\gamma + \frac{1}{2}\log|m^2\sigma/2|\right)\left(\frac{1}{2} + \frac{1}{8}m^2\sigma + \dots\right) - \frac{1}{4} - \dots\right] \\
& + \dots + \frac{1}{2m^2}\left[\frac{1}{2}a_1^2 + a_2 + O(\sigma)\right] \\
& + \frac{1}{2m^4}\left[\frac{1}{6}a_1^3 + a_1 a_2 + a_3 + O(\sigma)\right] + \dots \Bigg\}, \qquad (3.25)
\end{aligned}
$$

with γ denoting the Euler-Mascheroni constant.

The latter equation includes only those terms which contribute to the divergences and some finite terms in $\langle \hat{T}^{\mu\nu} \rangle_{\text{div}}$. In particular, the divergences in $G^{(1)}(x,x')$ appear as σ^{-1} and $\log|\frac{1}{2}\, m^2\sigma|$ terms which blow up when $\sigma \to 0$ as $x' \to x$.

3.3 The covariant expansion of the energy momentum tensor

The expression that defines the vacuum expectation value of the energy momentum tensor (3.7) is purely formal and presents certain inconsistencies at the practical level. Specifically, the left-hand side of the equation represents a tensorial quantity defined at the point x, while the right-hand side involves bitensors[5] that depend on both x and x', and therefore transform differently under coordinate changes associated with each point.

To make the two sides consistent, it is necessary to rewrite the bitensors by expanding them in terms of functions defined at x and the tangent vector $\sigma^\mu \equiv \sigma^{;\mu}$, e.g., for the two-indices-object:

$$
T_{\alpha\beta'} = t_{\alpha\beta}(x) + t_{\alpha\beta\rho}(x)\sigma^\rho + \dots \qquad (3.26)
$$

Such an expression is ill-defined as well, as it once again involves elements on either side of the equation that transform differently under coordinate changes.

[5]More details on bitensors are given in the appendix.

It is therefore necessary to replace the bitensor with a tensor that depends only on the point x, obtained from the original bitensor by means of parallel transport implemented through the *parallel propagator* $g_{\beta}{}^{\gamma'}$:

$$\bar{T}_{\alpha\beta} = g_{\beta}{}^{\gamma'} T_{\alpha\gamma'} = t_{\alpha\beta} + t_{\alpha\beta\rho}\sigma^{\rho} + \frac{1}{2} t_{\alpha\beta\rho\lambda}\sigma^{\rho}\sigma^{\lambda} + \dots \tag{3.27}$$

The t-coefficients can be obtained through the $x' \to x$ limit of the bitensor, $g_{\mu}{}^{\nu'}$ and their derivatives.

An example of covariant expansion, among the ones useful to compute the VEV of the energy-momentum tensor, is given by:

$$\begin{aligned}\bar{\sigma}_{\alpha\beta} = g_{\beta}{}^{\rho'}\sigma_{;\,\alpha\rho'} = -g_{\alpha\beta} - \frac{1}{6} R_{\alpha\mu\beta\nu}\sigma^{\mu}\sigma^{\nu} + \frac{1}{12} R_{\alpha\mu\beta\nu;\sigma}\sigma^{\mu}\sigma^{\nu}\sigma^{\sigma} \\ - \left(\frac{1}{40} R_{\alpha\mu\beta\nu;\sigma\tau} + \frac{7}{360} R^{\kappa}{}_{\mu\alpha\nu} R_{\kappa\sigma\beta\tau}\right)\sigma^{\mu}\sigma^{\nu}\sigma^{\sigma}\sigma^{\tau} + \dots\end{aligned} \tag{3.28}$$

Substituting the series for σ, Δ, a_1 and a_2 in the formula for the Hadamard's $G^{(1)}(x, x')$ (3.25) one gets:

$$\begin{aligned}4\pi^2 G^{(1)}(x, x') = & \frac{2}{(\sigma^{\rho}\sigma_{\rho})} + \left[m^2 - \left(\frac{1}{6} - \xi\right) R\right]\left[\gamma + \frac{1}{2}\ln\left|\frac{1}{4} m^2 (\sigma^{\rho}\sigma_{\rho})\right|\right] - \frac{1}{2} m^2 \\ & + \frac{1}{6} R_{\alpha\beta}\frac{\sigma^{\alpha}\sigma^{\beta}}{(\sigma^{\rho}\sigma_{\rho})} + \frac{1}{2m^2}\left[\frac{1}{2}\left(\frac{1}{6} - \xi\right)^2 R^2 - \frac{1}{180} R^{\rho\tau} R_{\rho\tau}\right. \\ & \left. + \frac{1}{180} R^{\rho\tau\kappa l} R_{\rho\tau\kappa l} + \frac{1}{6}\left(\frac{1}{5} - \xi\right) R_{;\rho}{}^{\rho}\right] + O\left(1/m^4\right).\end{aligned} \tag{3.29}$$

Differentiating (3.25) and inserting the due expansions, one finally obtains, after a series of long and tedious steps, $\langle \hat{T}^{\mu\nu} \rangle_{\text{div}}$. For the purpose of this work, it is sufficient to present the divergent terms up to logarithmic order[6]:

$$\left\langle \hat{T}^{\mu\nu} \right\rangle_{\text{quartic}} = \frac{1}{2\pi^2}\frac{1}{(\sigma^{\rho}\sigma_{\rho})^2}\left[g^{\mu\nu} - 4\frac{\sigma^{\mu}\sigma^{\nu}}{(\sigma^{\rho}\sigma_{\rho})}\right], \tag{3.30}$$

[6] Other terms can be found in [7].

$$
\begin{aligned}
\left\langle \hat{T}^{\mu\nu} \right\rangle_{\text{quadratic}} &= \frac{1}{4\pi^2}\frac{1}{(\sigma^\rho\sigma_\rho)}\left(\left\{\frac{2}{3}R^{(\mu}{}_{\alpha}\frac{\sigma^{\nu)}\sigma^\alpha}{(\sigma^\rho\sigma_\rho)} - \frac{2}{3}R_{\alpha\beta}\frac{\sigma^\alpha\sigma^\beta\sigma^\mu\sigma^\nu}{(\sigma^\rho\sigma_\rho)^2}\right.\right.\\
&\left. -\frac{1}{2}m^2\left[g^{\mu\nu} - 2\frac{\sigma^\mu\sigma^\nu}{(\sigma^\rho\sigma_\rho)}\right]\right\}\\
&- \left(\frac{1}{6}-\xi\right)\left\{R^{\mu\nu} - \frac{1}{2}R\left[g^{\mu\nu} - 2\frac{\sigma^\mu\sigma^\nu}{(\sigma^\rho\sigma_\rho)}\right]\right.\\
&\left.\left. -2R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\frac{\sigma^\alpha\sigma^\beta}{(\sigma^\rho\sigma_\rho)} + 2R_{\alpha\beta}\frac{\sigma^\alpha\sigma^\beta}{(\sigma^\rho\sigma_\rho)}g^{\mu\nu}\right\}\right), \qquad (3.31)
\end{aligned}
$$

$$
\begin{aligned}
\left\langle \hat{T}^{\mu\nu} \right\rangle_{\text{logarithmic}} &= \frac{1}{4\pi^2}\left\{\left[\frac{1}{60}\left(R^{\rho\mu\tau\nu}R_{\rho\tau} - \frac{1}{4}R^{\rho\tau}R_{\rho\tau}g^{\mu\nu}\right) - \frac{1}{180}R\left(R^{\mu\nu} - \frac{1}{4}Rg^{\mu\nu}\right)\right.\right.\\
&\left. +\frac{1}{120}R^{\mu\nu}_{;\rho}{}^{\rho} - \frac{1}{360}R^{;\mu\nu} - \frac{1}{720}R_{;\rho}{}^{\rho}g^{\mu\nu} - \frac{1}{8}m^4g^{\mu\nu}\right]\\
&-\frac{1}{2}\left(\frac{1}{6}-\xi\right)\left[m^2\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right)\right]\\
&\left.-\frac{1}{4}\left(\frac{1}{6}-\xi\right)^2\left[-2R\left(R^{\mu\nu} - \frac{1}{4}Rg^{\mu\nu}\right) + 2R^{;\mu\nu} - 2R_{;\rho}{}^{\rho}g^{\mu\nu}\right]\right\}\\
&\times\left[\gamma + \frac{1}{2}\ln\left|\frac{1}{4}m^2\left(\sigma^\rho\sigma_\rho\right)\right|\right] \qquad (3.32)
\end{aligned}
$$

3.4 Minimum length in de Sitter spacetime

Within the mathematical framework introduced in this chapter, Synge's world function $\sigma(x, x')$ appears to be the most suitable structure for heuristically introducing the concept of a minimal length scale, identified in this section by the parameter ϵ.

To address the problem of the cosmological constant's value, we proceed considering the theory of a free massive scalar field in a fixed de Sitter spacetime, which serves as a model for both the inflationary phase of the early universe and the present-day accelerated expansion.

The stress-energy tensor $\langle \hat{T}^{\mu\nu} \rangle_{\text{div}}$ for a generic spacetime, derived from the Hadamard function, presents a highly complex and lengthy expression and no immediate or physically transparent results can be extracted until some simplifying assumptions are introduced.

Whatever the assumed value of the minimum length scale, it must lie well below the range accessible to current experimental investigations. Therefore, in

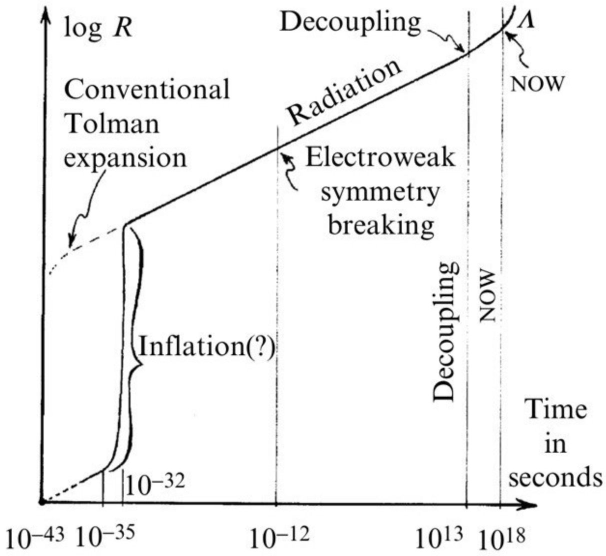

Figure 3.1: A commonly described history of the universe, as a logarithmic plot, including an inflationary phase. Here $\log R(t)$ (with $R(t)$ denoting the scale factor) is plotted against $\log t$. Image from Penrose [35].

light of this "smallness", it is reasonable to initially focus on the most divergent contribution (the quartic) of the stress-energy tensor in order to extract some preliminary insights.

By adopting the de Sitter metric in its FLRW form:

$$ds^2 = -dt^2 + e^{2Ht}\left(dr^2 + r^2 d\Omega^2\right) \tag{3.33}$$

and choosing an appropriate timelike geodesic tangent vector:

$$\sigma^\mu = (\epsilon, 0, 0, 0), \tag{3.34}$$

such that $\sigma^\alpha \sigma_\alpha = -\epsilon^2$ (remember, with ϵ of the order of the minimum length), one obtains, through the covariant point-splitting method, a diagonal form for the mixed-component stress-energy tensor:

$$\langle \hat{T}^\mu{}_\nu \rangle_{\text{quartic}} = \frac{3}{2\pi^2\epsilon^4} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}. \tag{3.35}$$

Equating the previously derived $\langle \hat{T}^0{}_0 \rangle_{\text{quartic}}$ component of the stress-energy tensor with (minus) the vacuum energy density $\langle \rho \rangle$, related to the cosmological constant, we find:

$$\frac{3}{2\pi^2\epsilon^4} = \frac{\Lambda}{8\pi l_P^2}. \tag{3.36}$$

When we insert the currently observed value of the cosmological constant $\Lambda \sim 10^{-122}\, l_P^{-2}$, the resulting minimum length scale turns out to be highly unphysical (indeed, $\epsilon \sim 10^{30}\, l_P \sim 10\,\mu\text{m}$; the size of a red blood cell). Shifting perspective, imposing a minimum length of the order of the Planck scale leads back to the same discrepancy between theory and observation found in flat spacetime (indeed, the resulting energy-momentum tensor is nearly identical to that obtained in Minkowski space through the imposition of an ultraviolet cutoff in momentum space).

One might ask whether retaining higher-order divergent terms, such as those up to the logarithmic order, could remedy this pathology by allowing the tuning of additional parameters that would now come into play (namely, the mass of the scalar field m and the additional mass scale μ that can always be introduced into the argument of the logarithmic term).

The computation of the divergent terms below the quartic order is already greatly simplified by the fact that de Sitter space is a constant curvature spacetime, characterized by the Riemann tensor $R_{\alpha\beta\mu\nu} = \frac{\Lambda}{3}(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu})$, but it can be further shortened if we impose the condition $\xi = \frac{1}{6}$, which corresponds to conformal coupling, although we consider a nonzero mass m. This choice is justified by the fact that the terms multiplying the now vanishing factor $(1/6 - \xi)$ are independent of the parameters m and ϵ. Moreover, they consist only of geometric contributions, with the cosmological constant Λ being the only physical parameter involved. Since we are interested in the regime of a very small cosmological constant, the simplification appears physically motivated.

What we finally obtain is the following result:

$$\begin{aligned}\langle \hat{T}^{\mu}{}_{\nu}\rangle_{\text{div}} \simeq \frac{1}{2\pi^2\epsilon^4}\begin{pmatrix}-3 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} - \frac{m^2}{8\pi^2\epsilon^2}\begin{pmatrix}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\\ - \frac{m^4}{64\pi^2}\delta^{\mu}{}_{\nu}\ln\left(\frac{m^2}{\mu^2}\right).\end{aligned} \tag{3.37}$$

We grasp how no reasonable choices for the parameters m, ϵ and μ yield significant progress on the cosmological constant problem.

However, the new terms identified in our analysis open the door to further considerations, which we develop in the following sections.

3.5 Discussion

The approach we have pursued so far, based on the introduction of a minimal observable length scale within the Schwinger-DeWitt/Hadamard renormalization framework, does not appear to lead to satisfactory results. On the contrary, it seems to be affected by some inconsistencies and ambiguities that go well beyond the mere failure to achieve the desired outcomes.

Nonetheless, it offers the opportunity to view certain results from standard approaches to quantum field theory in curved spacetime from a new perspective.

3.5.1 Direction dependence

The outcome exhibits a strong sensitivity to the choice of the separation vector used in the point-splitting procedure. For instance, adopting a spacelike geodesic vector of the form:

$$\sigma^\mu = \left(0, e^{-Ht}\epsilon, 0, 0\right), \tag{3.38}$$

defined such that $\sigma_\mu \sigma^\mu = \epsilon^2$, yields a different result from the original expression obtained using a timelike geodesic vector. In particular, the components $\langle \hat{T}^0{}_0 \rangle$ and $\langle \hat{T}^1{}_1 \rangle$ become interchanged. For instance, the quartic divergent term becomes:

$$\langle \hat{T}^\mu{}_\nu \rangle_{\text{quartic}} = \frac{3}{2\pi^2\epsilon^4} \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}. \tag{3.39}$$

This alternative choice leads to a problematic negative energy density and appears particularly troubling since, from a physical perspective, the use of a spacelike separation vector might appear even more natural, especially considering that, in a well-posed initial value formulation, one typically specifies the stress-energy tensor on a spacelike Cauchy surface.

A possible interpretation of this directional preference is that the minimal length should not be regarded as a geometric property of spacetime, but rather as a scale that characterizes scattering processes. In this regard, Casadio and Kuntz [5], using the Schwinger-Keldysh formalism, showed that a minimal length emerges as the vacuum expectation value of the metric, promoted to a quantum operator, but only when considering in-out amplitudes:

$$l_{\text{in-out}}(x, y) = \sqrt{\langle 0_{\text{out}}|\, ds^2 \,|0_{\text{in}}\rangle}, \tag{3.40}$$

and not in-in ones. Indeed, even though this is not explicitly shown in our derivation, which remains essentially formal, the Green's functions employed are in fact associated with in-out amplitudes. Therefore, the physical meaningfulness of considering spacetime points separated by spacelike intervals, within the point-splitting framework, becomes questionable. This feature does not represent a real issue in the usual renormalization schemes, since these problematic direction-dependent terms are eventually subtracted (before taking the limit $\epsilon \to 0$). This is not the case in the framework we are adopting, which relies on introducing a physical ultraviolet cut-off via fixing ϵ as a minimum length scale.

(In order to deal with physically-meaningful results, the following considerations are stated considering the original timelike vector $\sigma^\mu = (\epsilon, 0, 0, 0)$.)

3.5.2 ***Wrong*** equation of state and flat spacetime similarities

The energy momentum tensor we obtained fails to satisfy the expected vacuum equation of state $p = -\rho$, typical of a vacuum-dominated universe. In the leading divergent term, one instead finds the relation $p = \frac{1}{3}\rho$, which corresponds to the behavior of a radiation-dominated universe.

In the absence of an introduced mass scale μ, one might attempt to eliminate the directional dependence by imposing a specific relation between the cutoff parameter ϵ and the mass m, such as $\epsilon^4 = 64/m^4$. This tuning enforces the condition $p = -\rho$, effectively restoring isotropy. However, this constraint appears ad hoc and lacks a compelling physical motivation. Moreover, even if such a relation is assumed, the resulting vacuum energy density takes the form

$$\langle \rho \rangle \simeq \frac{1}{2\pi^2\epsilon^4}\left(\ln 64 + 1\right) \sim \frac{1}{\epsilon^4}, \tag{3.41}$$

which, when compared with the observed value of the cosmological constant, implies again a minimal length scale on the order of tens of micrometers.

The straightforward approach of estimating the vacuum energy density of a scalar field in flat spacetime by imposing an ultraviolet cut-off in momentum space yields (at the leading order) a result proportional to M^4, where M is the cutoff scale with the dimension of a mass (equivalently, the inverse of a length in natural units where $\hbar = c = 1$). Furthermore, in the same approach, one finds that the pressure satisfies the equation of state $p = \frac{1}{3}\rho$.

These results corresponds, in every respect, to what we have found in the quartically-divergent term of $\langle T^{\mu}{}_{\nu}\rangle_{\text{div}}$.

Indeed, such a coincidence of results is not surprising. Since we are dealing with the ultraviolet behavior of quantum field theory, any local analysis (regardless of whether a minimal length framework is used) that neglects spacetime curvature effects should necessarily yield correct and consistent results.

In the flat spacetime case, the *wrong* equation of state relating energy density and pressure stems from the usage of a procedure, the one of introducing a sharp cutoff, that breaks the Lorentz invariance. This is not what happens in our curved spacetime approach, which is, for its part, fully covariant. Nevertheless, similarly, a non-vanishing value of ϵ selects a "preferred direction in spacetime". In any case, the connection between these interpretations, if any, remains unclear.

Going back to the flat spacetime case, as mentioned in the first chapter, some authors suggest that one should treat the regularization of the vacuum energy density in a way that preserves Lorentz symmetry. Dimensional regularization leads to a different expression for the zero-point energy density:

$$\langle \rho \rangle \simeq \frac{m^4}{64\pi^2} \ln\left(\frac{m^2}{\mu^2}\right), \tag{3.42}$$

where m denotes the mass of the field and μ the renormalization scale. The identical leading-order result is obtainable in curved spacetime by applying, again, dimensional regularization.

We recognize that the logarithmic divergent term stemming from our minimum length computation takes exactly the same form.

Even if these results could be reconciled, the cosmological constant problem would remain fundamentally unresolved; alleviated by several orders of magnitude, but far from healed.

Conclusions

The various approaches devised in recent decades to tackle the cosmological constant problem, although enlightening in clarifying the many facets of the issue, have not proven to be fully conclusive.

It seems reasonable to suppose that the current difficulties in proposing a satisfactory solution may stem from our still profound lack of understanding regarding the quantum nature of spacetime. In fact, the cosmological constant problem arises precisely at the intersection of the two fundamentally incompatible theories that currently define our understanding of the universe: quantum field theory, which allows us to estimate the vacuum energy density, and general relativity, which provides a model for the current expansion of the universe, entirely incompatible with what is suggested by the former.

To date, the most well-known approaches to the problem of quantum gravity are string theory and loop quantum gravity, but neither is supported by empirical evidence. These theories, and several others, although quite diverse, seem to share a common feature: the emergence of a minimal measurable length scale. This idea appears to be supported by several thought experiments based on both quantum and gravitational phenomena. Indeed, both impose epistemic limits related to measurement processes: the former through Heisenberg's uncertainty principle and the latter via the concept of event horizons in black hole physics.

Based on this, one might hypothesize that, regardless of the particular model of quantum gravity from which it arises, a minimal length scale, introduced phenomenologically within the framework of quantum field theory, could shed new light on the cosmological constant problem. To this end, we proceeded as follows.

We focused on the vacuum expectation value of the energy-momentum tensor associated with a free massive scalar field defined on a de Sitter spacetime. This energy-momentum tensor has a quadratic structure in the fields that define it and is therefore divergent in quantum theory. To circumvent this issue, various regularization techniques can be employed, but the one most suited to our purposes is the so-called covariant point-splitting method. In practice, this involves replacing the squares of fields evaluated at the same spacetime point with the product of two fields defined at nearby, but non-coincident, points.

This prescription allows us to work within the Feynman Green function formalism. These functions display their divergences explicitly when written in terms of the Hadamard function, which depends on the bitensor of the geodesic interval through terms like $1/\sigma$ and $\log(\sigma/\lambda^2)$. When we restore the coincidence limit, the biscalar σ goes to zero and the Hadamard function diverges. Typical regularization methods rely on the ad hoc subtraction of these divergent structures, which are computable via the covariant Schwinger-DeWitt expansion. However, if one considers the existence of a minimal length, the coincidence limit loses its meaning, and the Green functions would remain finite.

As attractive as this idea may be, the results we obtained, by inserting a minimal length into the divergent structure of the energy-momentum tensor, do not seem to alleviate the cosmological constant problem. On the contrary, new ambiguities seem to emerge. The results we obtained do not differ significantly from those of computing the vacuum energy density in flat spacetime using an ultraviolet momentum-space cutoff (although our method might provide a physical justification for the latter). In retrospect, this coincidence should not surprise us: the equivalence principle in general relativity reduces any local analysis, such as the one we employed, to one carried out in Minkowski space.

These results, physically consistent with what is already known in the literature, hold if one uses a timelike minimal geodesic interval. If instead a spacelike interval is chosen, one must contend with a problematic negative energy density. Indeed, although choosing a spatial interval might seem more natural (in light of an initial-value formulation in general relativity), it would be imprudent to dismiss the choice of a temporal interval as meaningless. It is worth noting that our mathematical construction is based on Green functions, and thus on the propagation of dynamical fields, which cannot occur superluminally without violating causality. Therefore, it may be more appropriate to interpret a possible minimal measurable length as a scale associated with scattering processes, rather than as a geometrical property of space.

Returning to the similarity of the results obtained in flat spacetime and through a minimal length approach: in both cases, the energy-momentum tensors exhibit the relation between energy density and pressure $p = \frac{1}{3}\rho$, which does not correspond to the vacuum equation of state, but rather to that of radiation. Some authors argue that, in the Minkowski case, this inconsistency is due to the improper imposition of a sharp cutoff in momentum space, which breaks Lorentz symmetry. This does not occur in our approach, yet the choice of a specific geodesic interval on which to define a minimal length scale still selects a preferred direction in spacetime. It remains unclear whether these two symmetry breakings are somehow related.

Ultimately, we must acknowledge that the theoretical framework developed in this thesis does not seem suitable for resolving the cosmological constant problem (nevertheless, we cannot entirely rule out that the considerations presented here might lead to new perspectives on the interpretation of divergences in quantum field theory).

After all, we cannot exempt the standard approaches to calculating the vacuum energy density, which underlie the cosmological constant problem, from a similar critique. These methodologies are based on purely local reasoning that entirely neglects the global geometry[7] of spacetime and its dynamics. Given that Weinberg's no-go theorem itself relies on the same oversimplifications, one might question whether the entire framework in which the cosmological constant problem is posed, including the concept of vacuum, needs to be fundamentally reimagined (assuming we can even speak of a *problem* in the absence of solid theoretical foundations that clearly define it).

In this regard, to conclude, we believe it is worth mentioning some modern perspectives on QFT in curved spacetime that differ significantly from our discussion thus far.

Recent studies have strongly criticized the construction of a quantum field theory on a fixed background spacetime, as we did with de Sitter space, arguing that it is necessary to explore the mutual dynamical interaction between matter and the spacetime background (the backreaction problem). In this regard, Becker and Reuter [2] (later taken up by Ferrero and Percacci [12]) have devised a non-perturbative, background-independent scheme in which quantum theory is regularized via a sequence of quasi-physical approximating systems with a finite number of degrees of freedom. Each of these approximations interacts autonomously with the gravitational field. In the continuum limit concerning these degrees of freedom, a preferred spacetime metric appears to emerge spontaneously. Within this framework, the vacuum contribution to the

[7]It is even plausible that the topology of the universe influences the quantum systems it contains by imposing boundary conditions (see, e.g., [33]).

cosmological constant would dissolve precisely due to the effect of backreaction.

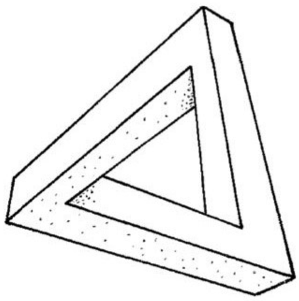

Figure 3.2: Penrose tribar, an *impossible object*. Locally, there is nothing impossible about what the drawing represents, but the complete picture tells a very different story. It serves as a powerful metaphor for the essence of our conclusions. Image from Penrose [35].

Appendix A

Hadamard regularization

The Hadamard regularization method[1] is a general and powerful technique for regularizing the expectation values of quadratic operators such as $\langle \phi^2 \rangle$ and the energy-momentum tensor $\langle T_{\mu\nu} \rangle$. It operates at the level of the Hadamard Green function, defined as:

$$G^{(1)}(x, x') \equiv \frac{1}{2} \langle 0 | \{\phi(x), \phi(x')\} | 0 \rangle \,, \tag{A.1}$$

which is singular in the coincidence limit $x' \to x$.

In four spacetime dimensions, the Green function admits a formal Hadamard series expansion of the form

$$G^{(1)}(x, x') = \frac{\Delta^{1/2}(x, x')}{(4\pi)^2} \left[\frac{2}{\sigma(x, x')} + v(x, x') \ln \left(\frac{\sigma(x, x')}{\lambda^2} \right) + w(x, x') \right], \tag{A.2}$$

where:

- $\sigma(x, x')$ is the Synge world function (i.e., half the squared geodesic distance between x and x');
- $\Delta(x, x')$ is the Van Vleck-Morette determinant;
- $\lambda > 0$ is a renormalization length scale;
- $v(x, x')$ and $w(x, x')$ are smooth biscalar functions admitting power series expansions in σ:

$$v(x, x') = \sum_{n=0}^{\infty} v_n(x, x') \sigma^n, \tag{A.3}$$

[1] For more information on this and other regularization techniques, see [34] .

$$w(x,x') = \sum_{n=0}^{\infty} w_n(x,x')\sigma^n. \tag{A.4}$$

The coefficients v_n and w_n of the expansions are determined by recursive differential relations[2]:

$$\begin{aligned}
&v_0 + v_{0,\mu}\sigma^{,\mu} = (1/6)R - \Delta^{-1/2}\left(\Delta^{1/2}\right)_{,\mu}{}^{\mu}, \\
&v_n + \frac{v_{n,\mu}\sigma^{,\mu}}{n+1} = -\frac{1}{2n(n+1)}\left(\Delta^{-1/2}\left(\Delta^{1/2}v_{n-1}\right)_{,\mu}{}^{\mu} - (1/6)Rv_{n-1}\right) \\
&w_n + \frac{w_{n,\mu}\sigma^{,\mu}}{n+1} = -\frac{1}{2n(n+1)}\left(\Delta^{-1/2}\left(\Delta^{1/2}w_{n-1}\right)_{,\mu}{}^{\mu}\right. \\
&\left.-(1/6)Rw_{n-1}\right) - \frac{v_n}{n+1} \\
&-\frac{1}{2n^2(n+1)}\left(\Delta^{-1/2}\left(\Delta^{1/2}v_{n-1}\right)_{,\mu}{}^{\mu}\right. \\
&\left.-(1/6)Rv_{n-1}\right).
\end{aligned} \tag{A.5}$$

All v_n are uniquely determined, whereas w_0 remains arbitrary and depends on the global boundary conditions on the Green function.

It is natural to split of the Hadamard series into a *locally determined* part $G^L(x,x')$, obtained by setting $w_0 = 0$, and a boundary-condition-dependent remainder.

The regularized Green function is then defined as:

$$G^{(1)}_{\text{reg}}(x,x') \equiv G^{(1)}(x,x') - G^L(x,x') \tag{A.6}$$

and the regularized expectation value of ϕ^2 is obtained in the coincidence limit:

$$\langle \phi^2(x)\rangle_{\text{reg}} = \lim_{x'\to x} G^{(1)}_{\text{reg}}(x,x'). \tag{A.7}$$

Since the subtraction is performed prior to taking the limit, this procedure is an example of *point-splitting regularization*.

To compute the regularized energy-momentum tensor, one starts from the classical expression for a scalar field. Applying the same point-splitting procedure to its quantum version leads to a finite but generally non-conserved tensor $\langle T^{(0)}_{\mu\nu}\rangle_{\text{reg}}$. As shown by Wald, conservation can be restored by adding a

[2]For practical use, see [9].

local geometric correction:

$$\langle T_{\mu\nu} \rangle_{\text{reg}} = \langle T^{(0)}_{\mu\nu} \rangle_{\text{reg}} - \frac{1}{2(4\pi)^2} g_{\mu\nu} v_1(x, x). \tag{A.8}$$

This corrected tensor satisfies:

- $\nabla^\mu \langle T_{\mu\nu} \rangle_{\text{reg}} = 0$ (covariant conservation);
- For conformally invariant fields: $\langle T^\mu{}_\mu \rangle_{\text{reg}} = a_2(x)/(4\pi)^2$, where $a_2(x) = -2v_1(x, x)$[3] is the Schwinger–DeWitt coefficient associated with the trace anomaly.

In summary, Hadamard regularization provides a consistent local method to obtain finite, conserved, and geometrically meaningful expressions for the vacuum expectation values of quantum fields in curved spacetime.

[3] There exists a correlation between the Hadamard coefficients and the Schwinger-DeWitt coefficients. Several connections among the various renormalization techniques in curved spacetime are rigorously presented in [16].

Appendix B

Bitensors

A bitensor is a tensorial object that depends on two points in a manifold:

$$T_{\alpha_1 \ldots \alpha_n \beta'_1 \ldots \beta'_m}(x, x'). \tag{B.1}$$

It transforms like the product of two tensors, one at each spacetime point:

$$A_{\alpha \ldots}(x) B_{\beta \ldots}(x'). \tag{B.2}$$

In our discussion we have introduced the biscalars: σ, Δ and a_n. Another important bitensor is $g^{\mu}{}_{\nu'}$, i.e, the *bivector of parallel displacement.* This object, when acting on a vector $A^{\nu'}$, gives the vector $\bar{A}^{\mu}$ which is obtained by parallel transport of the first along the geodesic connecting x and x':

$$\bar{A}^{\mu} = g^{\mu}{}_{\nu'} A^{\nu'}. \tag{B.3}$$

We can find the properties of $g^{\mu}{}_{\nu'}$ by studying its action on $\sigma^{;\nu'}$, which is tangent to the geodesic at x', has magnitude equal to the geodesic distance between x and x', and is oriented in the $x \to x'$ direction:

$$-\sigma^{;\mu} = g^{\mu}{}_{\nu'} \sigma^{;\nu'}. \tag{B.4}$$

The minus sign comes from the fact that $\sigma^{;\mu}$ is oriented in the $x' \to x$ direction.

The coincidence limit of a bitensor is expressed adopting Synge's bracket notation:

$$[T_{\alpha \ldots \beta' \ldots}] = \lim_{x' \to x} T_{\alpha \ldots \beta' \ldots}. \tag{B.5}$$

We now focus on the coincidence limits of $\sigma(x, x')$ and its derivatives. As

x' approaches x, the length of the geodesic goes to zero by definition:

$$[\sigma] = 0 \tag{B.6}$$

and:

$$[\sigma_{;\mu}] = 0. \tag{B.7}$$

We have seen that:

$$\sigma = \frac{1}{2}\sigma^{;\mu}\sigma_{;\mu} \tag{B.8}$$

holds, such that, differentiating, one finds the relations:

$$\sigma_{;\mu} = \sigma^{;\rho}\sigma_{;\rho\mu}, \tag{B.9}$$

$$\sigma_{;\mu\nu} = \sigma^{;\rho}{}_{\nu}\sigma_{;\rho\mu} + \sigma^{;\rho}\sigma_{;\rho\mu\nu}, \tag{B.10}$$

$$\begin{aligned} \sigma_{;\mu\nu\sigma} =& \sigma^{;\rho}{}_{\nu\sigma}\sigma_{;\rho\mu} + \sigma^{;\rho}{}_{\nu}\sigma_{;\rho\mu\sigma} + \sigma^{;\rho}{}_{\sigma}\sigma_{;\rho\mu\nu} \\ & + \sigma^{;\rho}\sigma_{;\rho\mu\nu\sigma}, \end{aligned} \tag{B.11}$$

and so forth. Up to this point, one can evaluate the coincidence limits:

$$[\sigma_{;\mu\nu}] = g_{\mu\nu}, \tag{B.12}$$

$$[\sigma_{;\mu\nu\sigma}] = 0, \tag{B.13}$$

$$[\sigma_{;\mu\nu\sigma\tau}] = S_{\mu\nu\sigma\tau} \equiv -\frac{1}{3}\left(R_{\mu\sigma\nu\tau} + R_{\mu\tau\nu\sigma}\right), \tag{B.14}$$

$$[\sigma_{;\mu\nu\sigma\tau\rho}] = \frac{3}{4}\left(S_{\mu\nu\sigma\tau;\rho} + S_{\mu\nu\tau\rho;\sigma} + S_{\mu\nu\rho\sigma;\tau}\right). \tag{B.15}$$

From these latter expansions it is possible to compute the coincidence limits of the Van Vleck-Morette determinant, the bivector of parallel displacement, their derivatives and, thus, the Scwhinger-DeWitt coefficients. Here we just present some of the main results:

$$[\Delta^{1/2}] = 1, \tag{B.16}$$

$$\left[\Delta^{1/2};\alpha\right] = 0, \tag{B.17}$$

$$\left[\Delta^{1/2};\alpha\beta\right] = \frac{1}{6}R_{\alpha\beta}, \tag{B.18}$$

$$\left[\Delta^{1/2};\alpha\beta\gamma\right] = \frac{1}{12}\left(R_{\alpha\beta;\gamma} + R_{\alpha\gamma;\beta} + R_{B\gamma;\alpha}\right); \tag{B.19}$$

$$[g^{\mu}{}_{\nu'}] = \delta^{\mu}{}_{\nu'}, \tag{B.20}$$

$$[g^{\mu}{}_{\nu';\alpha}] = 0, \tag{B.21}$$

$$[g^{\mu}{}_{\nu';\alpha\beta}] = -\frac{1}{2}R^{\mu}{}_{\nu\alpha\beta}, \tag{B.22}$$

$$[g^{\mu}{}_{\nu';\alpha\beta\ldots}]\sigma^{;\alpha}\sigma^{;\beta}\cdots = 0; \tag{B.23}$$

$$[a_0] = a_0(x, x') = 1, \tag{B.24}$$

$$[a_1] = \left(\frac{1}{6} - \xi\right) R, \tag{B.25}$$

$$[a_{1;\mu}] = \frac{1}{2}\left(\frac{1}{6} - \xi\right) R_{;\mu}, \tag{B.26}$$

$$[a_{1;\mu\nu}] = \left(\frac{1}{20} - \frac{1}{3}\xi\right) R_{;\mu\nu} + \frac{1}{60}R_{\mu\nu;\rho}{}^{\rho} + \frac{1}{90}R^{\rho\tau}R_{\rho\kappa\tau\nu} - \frac{1}{45}R_{\mu\rho}R^{\rho}{}_{\nu} + \frac{1}{90}R^{\rho\kappa\tau}{}_{\mu}R_{\rho\kappa\tau\nu} \tag{B.27}$$

$$[a_2] = -\frac{1}{180}R^{\rho\tau}R_{\rho\tau} + \frac{1}{180}R^{\rho\tau\kappa\iota}R_{\rho\tau\kappa\iota} \quad \frac{1}{6}\left(\frac{1}{5} - \xi\right) R_{;\rho}{}^{\rho} + \frac{1}{2}\left(\frac{1}{6} - \xi\right)^2 R^2. \tag{B.28}$$

As we have seen in the construction of the energy-momentum tensor through the Schwinger-DeWitt method, for practical purposess, it is useful to expand bitensors in the form:

$$T_{\alpha\beta'} = t_{\alpha\beta}(x) + t_{\alpha\beta\rho}(x)\sigma^{\rho} + \frac{1}{2!}t_{\alpha\beta\rho\sigma}(x)\sigma^{\rho}\sigma^{\sigma} + \ldots \tag{B.29}$$

t-coefficients can be obtained by exploiting coincidence limits:

$$t_{\alpha\beta} = [T_{\alpha\beta'}], \tag{B.30}$$

$$t_{\alpha\beta\mu} = [T_{\alpha\beta';\mu}] - t_{\alpha\beta;\mu}, \tag{B.31}$$

$$t_{\alpha\beta\mu\nu} = [T_{\alpha\beta';\mu\nu}] - t_{\alpha\beta;\mu\nu} - t_{\alpha\beta\mu;\nu} + [g_{\alpha\beta';\mu\nu}]\,\text{terms}, \tag{B.32}$$

these latter terms have no importance since they do not contribute in the expansion, because of property (B.23).

Bibliography

[1] R. J. Adler, B. Casey, and O. C. Jacob. "Vacuum catastrophe: An elementary exposition of the cosmological constant problem". In: *American Journal of Physics* 63.7 (1995), pp. 620–626. ISSN: 0002-9505. DOI: 10.1119/1.17850.

[2] M. Becker and M. Reuter. "Background Independent Field Quantization with Sequences of Gravity-Coupled Approximants". In: *Physical Review D* 102.12 (2020), p. 125001. DOI: 10.1103/PhysRevD.102.125001. arXiv: 2008.09430 [gr-qc].

[3] Michael Bishop et al. "Comments on the cosmological constant in generalized uncertainty models". In: *arXiv* 2209.03172 (2022). URL: https://ar5iv.labs.arxiv.org/html/2209.03172.

[4] S. M. Carroll. "The Cosmological Constant". In: *Living Reviews in Relativity* 4.1 (2001). Submitted April 2000, published Feb 2001. DOI: 10.12942/lrr-2001-1.

[5] R. Casadio and I. Kuntz. "Revisiting the minimum length in the Schwinger–Keldysh formalism". In: *Eur. Phys. J. C* 80.10 (2020), p. 958. DOI: 10.1140/epjc/s10052-020-08535-1. arXiv: 2006.08450 [hep-th].

[6] L. N. Chang et al. "The Effect of the Minimal Length Uncertainty Relation on the Density of States and the Cosmological Constant Problem". In: (2002). Submitted 4 January 2002; revised version available as v3 on 24 September 2009. arXiv: hep-th/0201017 [hep-th].

[7] S. M. Christensen. "Vacuum expectation value of the stress tensor in an arbitrary curved background: The covariant point-separation method". In: *Physical Review D* 14.10 (1976), pp. 2490–2501. DOI: 10.1103/PhysRevD.14.2490.

[8] Ray d'Inverno. *Introducing Einstein's Relativity.* U.S.A.: Oxford University Press, 1990.

[9] Y. Décanini and A. Folacci. "Hadamard renormalization of the stress-energy tensor for a quantized scalar field in a general spacetime of arbitrary dimension". In: *Physical Review D* 78.4 (2008), p. 044025. DOI: 10.1103/PhysRevD.78.044025.

[10] C. P. Enz and A. Thellung. "Nullpunktsenergie und Anordnung nicht vertauschbarer Faktoren im Hamiltonoperator". In: *Helvetica Physica Acta* 33.8 (1960), pp. 839–848.

[11] G. Esposito, A. Y. Kamenshchik, and G. Pollifrone. *Euclidean Quantum Gravity on Manifolds with Boundary*. Vol. 85. Fundamental Theories of Physics. Dordrecht: Springer Science+Business Media Dordrecht, 1997, p. 322. ISBN: 978-0-7923-4472-8. DOI: 10.1007/978-94-011-5806-0.

[12] R. Ferrero and R. Percacci. "The cosmological constant problem and the effective potential of a gravity-coupled scalar". In: *arXiv preprint* (2024). Revised version (v2) submitted 24 Sep 2024. arXiv: 2404.12357 [hep-th].

[13] L. J. Garay. "Quantum gravity and minimum length". In: *International Journal of Modern Physics A* 10 (1995). arXiv preprint, first submitted 2 March 1994 (v1, withdrawn), revised 9 May 1995 (v2), pp. 145–166. DOI: 10.1142/S0217751X95000085. arXiv: gr-qc/9403008 [gr-qc]. URL: https://arxiv.org/abs/gr-qc/9403008v2.

[14] D. J. Gross and P. F. Mende. "String theory beyond the Planck scale". In: *Nuclear Physics B* 303.3 (1988), pp. 407–454. DOI: 10.1016/0550-3213(88)90390-2.

[15] J. E. Gunn and J. R. Gott. "On the Infall of Matter Into Clusters of Galaxies and Some Effects on Their Evolution". In: *The Astrophysical Journal* 176 (1972), pp. 1–19. DOI: 10.1086/151605.

[16] T. P. Hack and V. Moretti. "On the Stress-Energy Tensor of Quantum Fields in Curved Spacetimes — Comparison of Different Regularization Schemes and Symmetry of the Hadamard/Seeley-DeWitt Coefficients". In: *Journal of Physics A: Mathematical and Theoretical* 45.37 (2012), p. 374019. DOI: 10.1088/1751-8113/45/37/374019. arXiv: 1202.5107 [gr-qc].

[17] W. Heisenberg. "Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik". German. In: *Zeitschrift für Physik* 43 (1927), pp. 172–198. DOI: 10.1007/BF01397280. URL: https://link.springer.com/article/10.1007/BF01397280.

[18] I. Hinchliffe, N. Kersting, and Y. L. Ma. “Review of the Phenomenology of Noncommutative Geometry”. In: *International Journal of Modern Physics A* 19.02 (2004), pp. 179–204. DOI: 10.1142/S0217751X04017094. arXiv: hep-ph/0205040 [hep-ph].

[19] M. P. Hobson, G. P. Efstathiou, and A. N. Lasenby. *General Relativity: An Introduction for Physicists*. Reprint. Cambridge University Press, 2006, p. 187. ISBN: 978-0-521-82951-9.

[20] S. Hossenfelder. “Minimal Length Scale Scenarios for Quantum Gravity”. In: *Living Reviews in Relativity* 16.2 (2012). revised version 2 on 30 Apr 2013. DOI: 10.12942/lrr-2013-2. arXiv: 1203.6191 [gr-qc]. URL: https://arxiv.org/abs/1203.6191.

[21] M. Kato. “Particle theories with minimum observable length and open string theory”. In: *Physics Letters B* 245.1 (1990), pp. 43–47. DOI: 10.1016/0370-2693(90)90162-Y. URL: https://doi.org/10.1016/0370-2693(90)90162-Y.

[22] Claus Kiefer. *Quantum Gravity*. 3rd ed. International Series of Monographs on Physics, Vol.155. Oxford: Oxford University Press, 2012, p. 408. ISBN: 978-0199585205.

[23] Claus Kiefer and Hamid Mohaddes. *Quantum Theory of the Lemaître Model for Gravitational Collapse*. arXiv preprint arXiv:2412.12947. Invited contribution to the Proceedings of the Lemaître Conference 2024 at Specola Vaticana. 2024.

[24] M. Maggiore. “A generalized uncertainty principle in quantum gravity”. In: *Physics Letters B* 304.1 (1993), pp. 65–69. DOI: 10.1016/0370-2693(93)91401-8. URL: https://doi.org/10.1016/0370-2693(93)91401-8.

[25] H. Martel, P. R. Shapiro, and S. Weinberg. “Likely Values of the Cosmological Constant”. In: *The Astrophysical Journal* 492 (1998), pp. 29–40. DOI: 10.1086/305016.

[26] J. Martin. “Everything You Always Wanted to Know About The Cosmological Constant Problem (But Were Afraid To Ask)”. In: *Comptes Rendus Physique* 13.6–7 (2012), pp. 566–665. DOI: 10.1016/j.crhy.2012.04.008. arXiv: 1205.3365 [astro-ph.CO]. URL: https://arxiv.org/abs/1205.3365v1.

[27] C. A. Mead. "Possible Connection Between Gravitation and Fundamental Length". In: *Physical Review* 135.5B (Aug. 1964), B849–B862.

[28] C. W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation*. San Francisco: W. H. Freeman, 1973. ISBN: 0-7167-0344-0.

[29] J. E. Moyal. "Quantum mechanics as a statistical theory". In: *Proceedings of the Cambridge Philosophical Society* 45 (1949), pp. 99–124.

[30] S. Nobbenhuis. "The Cosmological Constant Problem, an Inspiration for New Physics". arXiv:gr-qc/0609011v1. Ph.D. Thesis. Utrecht University, 2006. URL: `https://arxiv.org/abs/gr-qc/0609011v1`.

[31] A. Padilla. *Lectures on the Cosmological Constant Problem*. Prepared for the X Mexican School on Gravitation and Mathematical Physics. 2015. arXiv: `1502.05296 [hep-th]`.

[32] T. Padmanabhan. "Planck length as the lower bound to all physical length scales". In: *General Relativity and Gravitation* 17 (1985), pp. 215–221. DOI: `10.1007/BF00760244`. URL: `https://doi.org/10.1007/BF00760244`.

[33] Evangelos Achilleas Paraskevas and Leandros Perivolaropoulos. "Probing the Universe's Topology through a Quantum System?" In: *Physical Review D* 112.4 (Aug. 2025), p. 043524. DOI: `10.1103/7p89-q82n`.

[34] L. E. Parker and D. J. Toms. *Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity*. Cambridge Monographs on Mathematical Physics. Cambridge; New York: Cambridge University Press, 2009, p. 472. ISBN: 9780521877879.

[35] Roger Penrose. *The Road to Reality: A Complete Guide to the Laws of the Universe*. London: Jonathan Cape, 2004. ISBN: 0-224-04447-8.

[36] J. G. Polchinski. *String Theory. Volume 1: An Introduction to the Bosonic String*. Cambridge Monographs on Mathematical Physics. Cambridge; New York: Cambridge University Press, 1998. ISBN: 0-521-63303-6.

[37] C. Rovelli. "A generally covariant quantum field theory and a prediction on quantum measurements of geometry". In: *Nuclear Physics B* 405 (1993), pp. 797–816. DOI: `10.1016/0550-3213(93)90567-9`.

[38] C. Rovelli. *Quantum Gravity*. Cambridge Monographs on Mathematical Physics. Cambridge: Cambridge University Press, 2007. ISBN: 978-0521715966.

[39] R. Shankar. *Principles of Quantum Mechanics*. 2nd ed. New York, NY: Springer, 1994. ISBN: 0306447908.

[40] K. S. Thorne. In: *Magic Without Magic: John Archibald Wheeler*. Ed. by John A. Klauder. San Francisco: W. H. Freeman, 1972.

[41] A. Vilenkin. In: *Cosmological Constant and the Evolution of the Universe*. Ed. by K. Sato et al. Tokyo: Universal Academy Press, 1996.

[42] S. Weinberg. In: *Critical Dialogs in Cosmology*. Ed. by N. Turok. Singapore: World Scientific, 1997.

[43] S. Weinberg. *Cosmology*. Oxford: Oxford University Press, 2008. ISBN: 9780198526827.

[44] S. Weinberg. "The cosmological constant problem". In: *Reviews of Modern Physics* 61.1 (1989), pp. 1–23. DOI: `10.1103/RevModPhys.61.1`.

[45] S. Weinberg. "The Cosmological Constant Problems". In: *arXiv preprint astro-ph/0005265* (2000). Talk given at Dark Matter 2000, February 2000. DOI: `10.48550/arXiv.astro-ph/0005265`. URL: `https://arxiv.org/abs/astro-ph/0005265`.

[46] A. Zee. *Quantum Field Theory in a Nutshell*. 2nd. In a Nutshell. Princeton, NJ: Princeton University Press, 2010. ISBN: 978-0691140346.