Alma Mater Studiorum · Università di Bologna

Department of Physics and Astronomy "Augusto Righi"
Bachelor's Degree in Physics

The Noether Theorem and its Applications

Supervisor:
Prof. Roberto Balbinot

Submitted by:
Stefano Doria

Academic Year 2024/2025

Contents

Abstract

In questo lavoro si è discussa la teoria relativistica dei campi e, in particolare, quali siano le conseguenze della presenza di una simmetria in un sistema.

Innanzitutto si è costruito il framework necessario allo sviluppo di questa teoria richiamando concetti di relatività ristretta, in particolare i concetti di scalare, quadrivettore e quadritensore sotto Lorentz.

Si è quindi proceduto introducendo la teoria relativistica dei campi. Ciò significa costruire il formalismo lagrangiano che può poi essere usato per descrivere la dinamica di una campo.

Si sono discusse due teorie di campo: la teoria relativa a campi scalari, trattando in particolare la teoria di Klein-Gordon, e la teoria relativa al campo elettromagnetico.

Prossimo passaggio è stato presentare il *teorema di Noether*, che associa ad ogni simmetria continua di un sistema delle correnti che sono conservate e delle cariche che sono costanti del moto.

Si è infine applicato il teorema di Noether a due simmetrie. La prima di queste è stata la simmetria traslazionale, presente in ogni sistema. Si è osservato che tale simmetria è legata alla conservazione dell'energia e della quantità di moto. La seconda è la simmetria di scala, che è invece presente solo quando una teoria è priva una scala dimensionale caratteristica.

Introduction

A fundamental aspect in modern physics is the study of symmetries. Symmetries are often seen as the basic features that then determine the objects and the theories we use.

A fundamental result regarding symmetries is the *Noether theorem*, which allows us to define conserved quantities when a system has a continuous symmetry. By using the *Noether theorem*, we can see how fundamental conservation laws, like that of energy or of momentum, are the result of the presence of a symmetry.

It's important to note that the theories used to described physical systems are field theories. Even objects classically considered as particles have a field associated to them. This works also the other way around, meaning objects classically considered as fields have a corresponding particle.

Our aim in this work will be, starting from some basics concepts of special relativity, to work our way up to relativistic field theory and the *Noether theorem*, ending then by discussing a couple of uses this theorem.

Chapter 1

Special relativity

The special theory of relativity is built upon two postulates, which are based on experimental evidence:

1 Relativity principle: physical laws have the same form in every inertial frame of reference, as it was for Galilean relativity.

2 Light speed invariance: the speed of light in vacuum has the same value in all inertial frames.

These postulates affect the framework used to describe physical phenomena. Central to this framework is the notion of an event, which is characterised by happening at a certain point in space and a certain moment in time. This means that one can associate a quadruple of real numbers (ct, x, y, z) to any event, where c is the light speed in vacuum, t is the time of the event and $\mathbf{x} = (x, y, z)$ are the spatial coordinates of the point where the event happens. We shall write these quadruples as x^μ, with $\mu = 0, 1, 2, 3$ and

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z. \tag{1.1}$$

The 4-dimensional space to which these belong is called Minkowski spacetime and we shall write it as $\mathbf{M}$.

We want to see how x^μ transforms under a change from one inertial frame to another. This transformation has to be linear, so that, if $\frac{d^2\mathbf{x}}{dt^2} = 0$ in one inertial frame, it does so in all inertial frames. Then, x^μ will transform as $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ [1], with $\Lambda^\mu{}_\nu$ being a 4x4 matrix.

From the two postulates of special relativity one can obtain that the quadratic form

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{1.2}$$

is invariant under a transformation between inertial frames.

Eq. 1.2 can also be written as

$$ds^2 = dx_\mu dx^\mu = \eta_{\mu\nu} dx^\nu dx^\mu \tag{1.3}$$

[1] We use Einstein notation, meaning that summation over repeated indices is implicit.

where $x_\mu = (ct, -x, -y, -z)$ and $\eta_{\mu\nu}$ is called metric tensor, a 4x4 matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

in every inertial frame. The metric tensor $\eta_{\mu\nu}$ has the property that, for any vector x^ν, $x_\mu = \eta_{\mu\nu}x^\nu$, that is, it lowers the index of the vector. The inverse metric tensor $\eta^{\mu\nu}$, which is a matrix of the same form as $\eta_{\mu\nu}$, raises the index of a vector instead, meaning $x^\mu = \eta^{\mu\nu}x_\nu$.

From the invariance under a transformation between inertial frames of Eq. 1.3 we obtain that the matrix $\Lambda^\mu{}_\nu$ must satisfy

$$\eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = \eta_{\alpha\beta}. \tag{1.4}$$

We can note that, due to $\eta_{\mu\nu}$ being symmetrical, Eq. 1.4 imposes 10 constraints on the 16 components of $\Lambda^\mu{}_\nu$, leaving 6 free parameters: the relative rotation angles around three axes and the three components of the relative velocity between two inertial frames.

Transformations of the type

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{1.5}$$

with $\Lambda^\mu{}_\nu$ satisfying Eq. 1.4 are called Lorentz transformations and they form the Lorentz group O(3,1), that means $det(\Lambda^\mu{}_\nu) = \pm 1$ and $\Lambda^0{}_0 \geq 1$ or $\Lambda^0{}_0 \leq -1$. An important subgroup of Lorentz transformations are the special Lorentz transformations. These transformations belong to the group $SO^+(3,1)$, meaning that $det(\Lambda^\mu{}_\nu)$=1 and $\Lambda^0{}_0 \geq 1$.

Finally, we note that, to have the most possible generic transformation between inertial frames, we should also consider a translation, so that

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu. \tag{1.6}$$

These are called Poincaré transformations.

1.1 4-tensors

We can now define different kinds of objects based on how they transform under a special Lorentz transformation.

An object $\phi(x)$ [2] such that $\phi'(x') = \phi(x)$ is called a scalar under Lorentz.

A generic quadruple $A^\mu(x) = (A^0(x), A^1(x), A^2(x), A^3(x))$ that transforms as

$$A'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu}A^\nu(x) = \Lambda^\mu{}_\nu A^\nu(x) \tag{1.7}$$

is called a contravariant vector. The prototype of such an object is x^μ, which represents a point in **M**. Contravariant vectors are characterised by having a raised index and they are regarded as rank $(1, 0)$ tensors.

[2] We shall express that an object f is dependent on x^μ by writing it as $f(x)$

Another possible type of 4-vector is a quadruple $A_\mu(x) = (A_0(x), A_1(x), A_2(x), A_3(x))$ that transforms as

$$A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x) = \Lambda_\mu{}^\nu A_\nu(x), \tag{1.8}$$

where by $\Lambda_\mu{}^\nu$ we mean $(\Lambda^{-1})^t{}_\mu{}^\nu$. Such objects are called covariant vectors and an example of them is the quadrigradient of a scalar

$$\partial_\mu \phi(x) = \frac{\partial \phi}{\partial x^\mu}(x) = (\frac{1}{c}\frac{\partial \phi}{\partial t}(x), \frac{\partial \phi}{\partial x}(x), \frac{\partial \phi}{\partial y}(x), \frac{\partial \phi}{\partial z}(x)).$$

Covariant vectors are characterised by having a lowered index and they are regarded as rank $(0, 1)$ tensors.

We can also have a rank (m, n) tensor, which will have m raised indices and n lowered indices. A 4-tensor of this kind is written as $T^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n}(x)$ and transforms as

$$T'^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n}(x') = \Lambda^{\mu_1}_{\alpha_1} \ldots \Lambda^{\mu_m}_{\alpha_m} \Lambda_{\nu_1}^{\beta_1} \ldots \Lambda_{\nu_n}^{\beta_n} T^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_n}(x) \tag{1.9}$$

We note that notions of symmetric and antisymmetric tensors are valid only if the indices we swap are of the same kind.

We observe now some important properties of 4-tensors:

1 The linear combination of tensors of the same rank is a tensor of the same rank.

  Example: $aA^\mu{}_\nu + bB^\mu{}_\nu = T^\mu{}_\nu$.

2 The product of a rank (m, n) tensor and a rank (m', n') tensor is a rank $(m + m', n + n')$ tensor.

  Example: $A^\mu{}_\gamma B^\nu = T^\mu{}_\gamma{}^\nu$.

3 Given a rank (m, n) tensor, we can contract two indices of different kind and we obtain a rank $(m - 1, n - 1)$ tensor.

  Example: $A^\mu{}_{\nu\mu}{}^\alpha = A^0{}_{\nu 0}{}^\alpha + A^1{}_{\nu 1}{}^\alpha + A^2{}_{\nu 2}{}^\alpha + A^3{}_{\nu 3}{}^\alpha = T_\nu{}^\alpha$.

4 By differentiating a rank (m, n) tensor with $\partial_\mu = \frac{\partial}{\partial x^\mu}$, we obtain a rank $(m, n + 1)$ tensor.

  Example: $\partial_\mu T^\alpha_\beta$.

Lastly, we note that by multiplying a rank (m, n) tensor T by $g_{\mu\nu}$ and then contracting a raised index of T with and index of $g_{\mu\nu}$ we are effectively just lowering an index of T. In the same way, we can raise a lowered index using $g^{\mu\nu}$.

Examples: $g_{\mu\nu} T^\nu{}_{\gamma\lambda} = T_{\mu\gamma\lambda}, \quad g^{\mu\nu} T^\gamma_{\nu\lambda} = T^{\gamma\mu}{}_\lambda$.

Chapter 2

Relativistic field theory

One fundamental type of object we encounter in physics are fields, which are functions of spacetime. The most basic example of a field is the electromagnetic field, represented by the four-potential $A^\mu(x) = (\phi(x), \mathbf{A}(x))$ where $\phi(x)$ and $\mathbf{A}$(x) are the electrostatic and the vector potential respectively. The strong and weak interactions can be described by fields too. Moreover, we now know that it's possible to associate a field to any particle and vice-versa. For example, there is a field associated to the pion π^0 and there is a particle, the photon, associated to the electromagnetic field.

Fields can be of different kinds, such as:

- Scalar fields $\phi(x)$.

  Example: particles of spin 0, like the pion π^0 and the pair of pions π^+ and π^-.

- Vector fields V^μ.

  Example: particles of spin 1, like the photon, to which corresponds the electromagnetic four-potential $A^\mu(x)$.

- Spinor fields $\psi(x)$.

  Example: particles of spin $\frac{1}{2}$, like quarks and leptons.

We now want to see how fields evolve in the spacetime. To do so we shall assume that the dynamics of a generic field $\phi(x)$ [1] is described by an action $S[\phi]$ of the form

$$S[\phi] = \int dx^0 \; L = \int d^4x \; \mathcal{L} \tag{2.1}$$

where L is the Lagrangian of the field ϕ and $\mathcal{L}$ is the Lagrangian density [2], which could be a function of $x^\mu, \phi(x), \partial_\mu \phi(x), \partial^2_\mu \phi(x), \ldots$. We will then use the *principle of stationary action* to obtain the equations of motion that govern how $\phi(x)$ evolves.

[1] Here $\phi(x)$ is not necessarily a scalar field.

[2] We will often refer to $\mathcal{L}$ simply as Lagrangian.

The action S must satisfy two important properties: it must be a scalar under Poincaré transformations and it must be real. Reality directly translates to $\mathcal{L}$. The same happens for the second property because, by

$$d^4x' = |det\Lambda| d^4x = d^4x \tag{2.2}$$

d^4x is a scalar under Poincaré (due to Λ being the Jacobian matrix of a generic Poincaré transformation), which means that, for S to be a scalar, $\mathcal{L}$ has to be one.

Requiring invariance under Poincaré transformations means requiring invariance under translations and under Lorentz transformations. Since, under a generic transformation, the way a field transforms is given by $\frac{\partial x'^\mu}{\partial x^\nu}$ or $\frac{\partial x^\nu}{\partial x'^\mu}$, any field is invariant under translations, that is $\phi'(x') = \phi(x)$. This means that requiring invariance under translations reduces to requiring that $\mathcal{L}$ has no explicit dependence on x^μ.

Moreover, we assume that $\mathcal{L}$ has no dependence on derivatives of ϕ higher than the first order derivative. This is done to avoid equations of motion with derivatives higher than the second order derivative, which would make the equations much more complex.

2.1 Euler-Lagrange equation

We are now ready to use the *principle of stationary action* to obtain the equations of motion of the field $\phi(x)$ described by the Lagrangian $\mathcal{L}(\phi(x), \partial_\mu\phi(x))$, The *principle of stationary action* states that, given the action

$$S = \int_\Omega d^4x\ \mathcal{L}(\phi(x), \partial_\mu\phi(x)), \tag{2.3}$$

where Ω is a volume in spacetime, the configuration taken by the field $\phi(x)$ is such that $\delta S = 0$ for an arbitrary infinitesimal variation of $\phi(x)$

$$\phi(x) \to \phi(x) + \delta\phi(x) \quad \text{with} \quad \delta\phi(x) = 0 \ \text{ on } \ \partial\Omega, \tag{2.4}$$

where $\partial\Omega$ is the contour of Ω.

This variation has no effect on the integration volume or on the coordinates x^μ, so we have

$$\begin{aligned} \delta S = \delta \int_\Omega d^4x\ \mathcal{L}(\phi(x), \partial_\mu\phi(x)) = \int_\Omega d^4x\ \delta\mathcal{L}(\phi(x), \partial_\mu\phi(x)) = \\ = \int_\Omega d^4x\ (\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)). \end{aligned} \tag{2.5}$$

Since x^μ does not change we have

$$\delta(\partial_\mu\phi) = \partial_\mu(\phi(x) + \delta\phi(x)) - \partial_\mu\phi(x) = \partial_\mu\delta\phi(x) \tag{2.6}$$

which means that Eq. 2.6 becomes

$$\delta S = \int_\Omega d^4x\ (\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\delta\phi). \tag{2.7}$$

Integrating by parts the second term of Eq. 2.7 and then using *Gauss's theorem* we obtain

$$\delta S = \int_{\Omega} d^4x \, (\frac{\partial \mathcal{L}}{\partial \phi} - \partial_{\mu}(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}))\delta\phi \, + \int_{\partial\Omega} d\sigma_{\mu} \, \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}\delta\phi, \tag{2.8}$$

where the last term vanishes because $\delta\phi = 0$ on $\partial\Omega$ by Eq. 2.4.

Since $\delta\phi$ is arbitrary, we have that

$$\delta S = \int_{\Omega} d^4x \, (\frac{\partial \mathcal{L}}{\partial \phi} - \partial_{\mu}(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}))\delta\phi = 0 \, \rightarrow \, \frac{\partial \mathcal{L}}{\partial \phi} - \partial_{\mu}(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}) = 0, \tag{2.9}$$

In general ϕ may have multiple components ϕ_i, with $i = 1, 2, ..., N$, so we have

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_{\mu}(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi_i)}) = 0 \quad \text{with} \quad i = 1, 2, ...N. \tag{2.10}$$

These are known as *Euler-Lagrange equations* and solving them gives the equations of motion for the field $\phi(x)$.

We see here the importance of requiring that $\mathcal{L}$ is real. It's thanks to this requirement that we have N equations for the N components ϕ_i of the field. If $\mathcal{L}$ was allowed to have an imaginary part, it would result in having $2N$ independent equations, which would make the problem overdetermined.

It's important to note that there are multiple choices of $\mathcal{L}$ that give the same equations of motion. In fact, given the Lagrangian $\mathcal{L}$ of a certain field $\phi(x)$, the following transformations are possible leaving the equations of motion unchanged:

- We can multiply $\mathcal{L}$ by a constant α

$$\mathcal{L} \, \rightarrow \, \mathcal{L}' = \alpha\mathcal{L}. \tag{2.11}$$

  Doing this is equivalent to multiplying the *Euler-Lagrange equation* by a constant.

- Given a generic function of $\phi(x)$ $\Lambda^{\mu}(\phi)$, we can add it to $\mathcal{L}$

$$\mathcal{L} \, \rightarrow \, \mathcal{L}' = \mathcal{L} + \partial_{\mu}\Lambda^{\mu}(\phi). \tag{2.12}$$

  By doing so, S' becomes

$$S' = S + \int_{\Omega} d^4x \, \partial_{\mu}\Lambda^{\mu}(\phi). \tag{2.13}$$

  Since the second term of Eq. 2.13 vanishes when we study a variation $\delta\phi$ of the type described in Eq. 2.6, we conclude that adding $\Lambda^{\mu}(\phi)$ to $\mathcal{L}$ leaves the equations of motions unchanged.

Chapter 3

Klein-Gordon Lagrangian

The simplest field we can study is a real scalar field $\phi(x)$. Such fields represent particles with spin 0 and with no electric charge, like the pion π^0 or the Higgs boson.

A general real scalar field $\phi(x)$ is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - F(\phi), \tag{3.1}$$

where $F(\phi)$ is a potential for ϕ. The term $\frac{1}{2}\partial_\mu\phi\partial^\mu\phi$ is referred to as the kinetic term, because its form mirrors that of the kinetic term in the Lagrangian of a particle.

We shall briefly check if this Lagrangian satisfies the requirements that we determined in the previous section. The requirement of reality is obviously satisfied thanks to ϕ being real. Lorentz invariance is also satisfied since ϕ and $\partial_\mu\phi\partial^\mu\phi$ are scalars under Lorentz.

Using the *Euler-Lagrange equation* 2.10, we can now obtain the equation of motion of $\phi(x)$. First we calculate the partial derivatives of $\mathcal{L}$ with respect to ϕ and $\partial_\mu\phi$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi \quad ; \quad \frac{\partial\mathcal{L}}{\partial\phi} = -\frac{dF}{d\phi}. \tag{3.2}$$

Then, by inserting these into Eq. 2.10, we obtain

$$\Box\phi + \frac{dF}{d\phi} = 0, \tag{3.3}$$

where $\Box = \partial_\mu\partial^\mu$. We have thus obtained the equation of motion of a generic real scalar field $\phi(x)$ with potential $F(\phi)$.

3.1 Real Klein-Gordon theory

We will now discuss the Klein-Gordon theory, which is the most basic theory for a real scalar field. This theory describes particles associated to fields with the Lagrangian

$$\mathcal{L}_{KG} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2, \tag{3.4}$$

known as the Klein-Gordon Lagrangian. The parameter μ can also be written as $\frac{mc}{\hbar}$, where m is a parameter associated to ϕ with the dimensions of a mass, c is the light speed in vacuum and $\hbar$ is the reduced Planck constant.

Using Eq. 3.3 with $F(\phi) = \frac{1}{2}\mu^2\phi^2$ we obtain that

$$(\Box + \mu^2)\phi(x) = 0 \tag{3.5}$$

is the equation of motion of a field described by $\mathcal{L}_{KG}$. Eq. 3.5 is known as the *Klein-Gordon equation.*

The term $\frac{1}{2}\mu^2\phi^2$ in Eq. 3.4 is generally referred to as the mass term. To see why that is let's find a plane wave solution to the *Klein-Gordon equation*

$$\phi(x) = e^{-ik_\alpha x^\alpha} \quad with \quad k^\alpha = (k^0, \mathbf{k}) = (\frac{\omega}{c}, \mathbf{k}), \tag{3.6}$$

where k^α has the dimensions of the inverse of a length. By inserting Eq. 3.6 into Eq. 3.5 we obtain that k^μ must satisfy the relation

$$-k_\mu k^\mu + \mu^2 = 0. \tag{3.7}$$

With a few simple algebraic passages and by substituting $\mu = \frac{mc}{\hbar}$ and $k^\mu = (\frac{\omega}{c}, \mathbf{k})$, we obtain

$$(\frac{\hbar\omega}{c})^2 = |\hbar\mathbf{k}|^2 + m^2c^2. \tag{3.8}$$

Finally, if we use the *Planck-Einstein relation* $E = \hbar\omega$ and the *de Broglie relation* $\mathbf{p} = \hbar\mathbf{k}$ [1] known for quantum mechanics, Eq. 3.8 becomes

$$\frac{E^2}{c^2} = |\mathbf{p}|^2 + m^2c^2, \tag{3.9}$$

which is the *energy-momentum relation* for a particle of mass m. We can thus conclude that terms $\sim \phi^2$ in $\mathcal{L}_{KG}$ are related to the mass of the particle described by ϕ and that a Lagrangian like that in Eq. 3.4 describes a particle of mass m.

Moreover, we note that the *Klein-Gordon equation* is linear, which is due to the presence only of terms $\sim \phi^2$ in $\mathcal{L}_{KG}$. This means that we can use the superposition principle for solutions of the *Klein-Gordon equation*: if we have multiple solutions, their sum is also a solution. This highlights a fundamental aspect of the Klein-Gordon theory, which is that it describes non interacting particles. To have a theory describing interacting particles we should add terms $\sim \phi^3$ or $\sim \phi^4$ to $\mathcal{L}_{KG}$, leading to non-linear equations of motion.

3.2 Complex Klein-Gordon theory

We can also consider the case of a complex scalar field $\phi(x)$, which corresponds to a particle with spin 0 and with an electric charge. Such a particle is described by a complex version of the Klein-Gordon Lagrangian

$$\mathcal{L}_{KG} = \partial_\mu\phi^\star\partial^\mu\phi - \mu^2\phi^\star\phi. \tag{3.10}$$

[1] $\mathbf{p}$ is the linear momentum.

An example of a particle described by this theory is the couple particle-antiparticle of the pions π^+ π^-.

We shall briefly check the properties we require of the Lagrangian of a field. $\mathcal{L}_{KG}$ is Lorentz invariant due to ϕ and $\partial_\mu\phi^\star\partial^\mu\phi$ being scalars under Lorentz. The requirement of reality is satisfied too because the combinations $\partial_\mu\phi^\star\partial^\mu\phi$ and $\phi^\star\phi$ are real.

By calculating the partial derivatives of $\mathcal{L}_{KG}$ with respect to ϕ and $\partial_\mu\phi$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi^\star \quad ; \quad \frac{\partial\mathcal{L}}{\partial\phi} = -\mu^2\phi^\star \tag{3.11}$$

and $\phi^\star$ and $\partial_\mu\phi^\star$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\star)} = \partial^\mu\phi \quad ; \quad \frac{\partial\mathcal{L}}{\partial\phi} = -\mu^2\phi \tag{3.12}$$

and then inserting them into the *Euler-Lagrange equation* 2.10, we obtain the equations of motion for $\phi(x)$

$$(\Box + \mu^2)\phi^\star(x) = 0 \tag{3.13}$$

and $\phi^\star(x)$

$$(\Box + \mu^2)\phi(x) = 0. \tag{3.14}$$

We note that, like what we had seen while treating real fields, particles described by $\mathcal{L}_{KG}$ have mass m and are non-interacting. To have a theory for interacting particles we should add a term $\sim \phi^{\star 2}\phi^2$ to the Lagrangian.

Chapter 4

Lagrangian of the electromagnetic field

Another type of field we encounter are vector fields. We will now study a field of this type, the electromagnetic field, which is represented by the four-potential $A^\mu(x) = (\phi(x), \mathbf{A}(x))$.

Before proceeding with the construction of the Lagrangian $\mathcal{L}_{em}$, it's important to make a couple considerations on the electromagnetic field. First, we know that the components of electric field $\mathbf{E}(x)$ and of the magnetic field $\mathbf{B}(x)$ can be arranged in the so-called electromagnetic tensor $F^{\mu\nu}(x)$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \tag{4.1}$$

which is antisymmetric. It's possible to obtain $F^{\mu\nu}$ from A^μ through

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{4.2}$$

We note that $F^{\mu\nu}$, A^μ, $\mathbf{E}$ and $\mathbf{B}$ are all real quantities. We also know that the four-potential A^μ that determines a certain $F^{\mu\nu}$ is not unique. In fact, if we transform a certain $A^\mu(x)$ like

$$A'^\mu(x) = A^\mu(x) - \partial^\mu \chi(x), \tag{4.3}$$

where $\chi(x)$ is an arbitrary real scalar function of x^μ, the corresponding $F^{\mu\nu}$ obtained through Eq. 4.2 does not change. Transformations like Eq. 4.3 are called gauge transformations. Since $F^{\mu\nu}$ appears directly in the expression of the force acting on a charged particle while A^μ does not, we expect the equations used to describe a system to be invariant under gauge transformations.

We are now ready to start building the Lagrangian $\mathcal{L}_{em}$ of the electromagnetic field. We'll start by considering the case with no electric charge density and no electric current density. Thanks to the previous considerations, we know that $\mathcal{L}_{em}$ must be invariant under gauge transformations. To satisfy this requirement, we will use $F^{\mu\nu}$, which contains the first order derivatives of the field A^μ and is invariant under gauge transformations, to build the Lagrangian. Since $\mathcal{L}_{em}$ must also be a scalar under Lorentz, we will need scalar quantities obtained from $F^{\mu\nu}$. The only two objects of this kind are

$$F^{\mu\nu}F_{\mu\nu} = -2(|\mathbf{E}|^2 - |\mathbf{B}|^2), \tag{4.4}$$

$$\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} = -8\ \mathbf{E}\cdot\mathbf{B}\ ^1. \tag{4.5}$$

It can easily be seen that

$$\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} = 4\partial_\mu(\varepsilon^{\mu\nu\alpha\beta}A_\nu\partial_\alpha A_\beta), \tag{4.6}$$

which means, thanks to what we have seen in Sec. 2.1, that $\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$ gives no contribution to the equations of motion. We are then left only with $F^{\mu\nu}F_{\mu\nu}$, which satisfies also the condition of reality since $F^{\mu\nu}$ is real. The Lagrangian $\mathcal{L}_{em}$ of the electromagnetic field is then

$$\mathcal{L}_{em} = -\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu}. \tag{4.7}$$

The derivative of $\mathcal{L}_{em}$ with respect to the field A_ν vanishes, while the derivative with respect to the first order derivatives of the field $\partial_\mu A_\nu$ are given by

$$\frac{\partial\mathcal{L}_{em}}{\partial(\partial_\mu A_\nu)} = \frac{\partial\mathcal{L}_{em}}{\partial F_{\alpha\beta}}\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{8\pi}F^{\alpha\beta}(\delta^\mu{}_\alpha\delta^\nu{}_\beta - \delta^\mu{}_\beta\delta^\nu{}_\alpha) = -\frac{1}{4\pi}F^{\mu\nu}. \tag{4.8}$$

Substituting the derivatives of $\mathcal{L}_{em}$ into the *Euler-Lagrange equations* 2.10, we obtain the equations of motion for A^μ

$$\partial_\mu F^{\mu\nu} = 0. \tag{4.9}$$

These are the inhomogeneous Maxwell equations when there are no electric charge density and no electric current density. We can also write Eq. 4.9 in terms of the field A^μ as

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0. \tag{4.10}$$

The homogeneous Maxwell equations, which can be written as

$$\partial_\mu(\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}) = 0, \tag{4.11}$$

are automatically satisfied by an electromagnetic tensor $F^{\mu\nu}$ obtained through Eq. 4.2.

Let's now show that the Maxwell equations can be obtained from a Lagrangian $\mathcal{L}$ also when there are an electric charge density $\rho(x)$ and an electric current density $\mathbf{J}(x)$. We know that we can define the four-vector $J^\mu(x) = (c\rho(x), \mathbf{J}(x))$ which is called four-current, where c is the light speed in vacuum. We now have to add to $\mathcal{L}_{em}$ a term $\mathcal{L}_{int}$ expressing the coupling between A^μ and J^μ

$$\mathcal{L} = \mathcal{L}_{em} + \mathcal{L}_{int} = -\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} - \frac{1}{c}J_\mu A^\mu. \tag{4.12}$$

[1] $\varepsilon^{\mu\nu\alpha\beta}$ is a tensor antisymmetric in any two indices. It's completely defined by the property that $\varepsilon^{\mu\nu\alpha\beta} = 1$ if (μ,ν,α,β) is an even permutation of $(0,1,2,3)$, $\varepsilon^{\mu\nu\alpha\beta} = -1$ if (μ,ν,α,β) is an odd permutation of $(0,1,2,3)$ and $\varepsilon^{\mu\nu\alpha\beta} = 0$ if any two indices are equal.

The new term $\mathcal{L}_{int} = -\frac{1}{c}J_\mu A^\mu$ is clearly a scalar under Lorentz. Reality is also immediate since J^μ and A^μ are real. However, this term is not invariant under a gauge transformation. In fact, if we perform a transformation like Eq. 4.3, the variation of $\mathcal{L}$ is

$$\delta\mathcal{L} = \mathcal{L}' - \mathcal{L} = \frac{1}{c}J^\mu \partial_\mu \chi. \tag{4.13}$$

Exploiting the fact that J^μ satisfies the continuity equation $\partial_\mu J^\mu = 0$, we have

$$\delta\mathcal{L} = \frac{1}{c}J^\mu \partial_\mu \chi = \frac{1}{c}J^\mu \partial_\mu \chi + \frac{1}{c}\chi \partial_\mu J^\mu = \frac{1}{c}\partial_\mu (J^\mu \chi). \tag{4.14}$$

We know, thanks to what we have seen in Sec. 2.1, that such a variation does not change the equations of motion. This means that the Lagrangian in Eq. 4.12 is physically acceptable, since the equations of motion we obtain from it are invariant under a gauge transformation.

We will now obtain the equations of motion in presence of a J^μ. We note that the derivative of $\mathcal{L}$ with respect to $\partial_\mu A_\nu$ is equal to that of $\mathcal{L}_{em}$ given by Eq. 4.8. The derivative of $\mathcal{L}$ with respect to A_ν is

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = -\frac{1}{c}J^\nu. \tag{4.15}$$

Using the *Euler-Lagrange equations* 2.10 we obtain the equations of motion

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}J^\nu, \tag{4.16}$$

which are the inhomogeneous Maxwell equations.

We are left with an important remark to be made on $\mathcal{L}$. The field A^μ appears in the Lagrangian through $F^{\mu\nu}$, which is given by Eq. 4.2, or together with J^μ. This means that there are no terms with only the field A^μ itself in $\mathcal{L}$. This is because scalars obtained directly from A^μ alone would not be gauge invariant and would make the equations of motion not gauge invariant. Since, like what we have seen with the Klein-Gordon Lagrangian in Ch. 3, terms $\sim A^\mu A_\mu$ would be related to the mass of the field and such terms are not allowed because they would break gauge invariance, the electromagnetic field A^μ is massless.

Chapter 5

Noether theorem, conserved currents and charges

We will now consider the concept of symmetry and its fundamental role in field theory.

First, let's define what we mean by symmetry. We say that a system is invariant under a certain transformations if the equations of motion of the system remain unchanged after performing one of these transformations. This means that the system has a symmetry with respect to these transformations.

One important type of transformation are continuous transformations. These transformations are characterised by the fact that we can obtain them from the identity transformation through a series of infinitesimal transformations. When a system is invariant under a certain continuous transformation we shall say that the system has a continuous symmetry. We will now see how the presence of a continuous symmetry for a system implies that there are conserved currents and constants of motions. This result is known as the *Noether theorem.*

First, let's consider a generic field $\phi(x)$ and the associated Lagrangian $\mathcal{L}(\phi, \partial_\mu \phi)$. Let's now consider an infinitesimal transformation of the field $\phi(x)$ and the coordinates x^μ

$$\begin{cases} x'^\mu = x^\mu + \delta x^\mu \\ \phi'(x') = \phi(x) + \delta\phi(x). \end{cases} \tag{5.1}$$

By operating at the first infinitesimal order, the total variation $\delta\phi(x)$ of the field can be expressed as the sum of two terms

$$\delta\phi(x) = \phi'(x') - \phi(x) = \phi'(x + \delta x) - \phi(x) = \phi'(x) + \partial_\mu \phi' \delta x^\mu - \phi(x) = \\ = \delta_0 \phi(x) + \partial_\mu \phi \delta x^\mu, \tag{5.2}$$

where in the last passage we have put $\partial_\mu \phi \delta x^\mu = \partial_\mu \phi' \delta x^\mu$, which is justified since $\partial_\mu \phi' \delta x^\mu - \partial_\mu \phi \delta x^\mu \sim |\delta x^\mu|^2$ and we are operating at the first infinitesimal order, and $\delta_0 \phi(x) = \phi'(x) - \phi(x)$. The term $\delta_0 \phi(x)$ is connected to the change of just the form of the field $\phi(x)$ caused by the transformation, while $\partial_\mu \phi \delta x^\mu$ is due to the transformation of the coordinates. We note that we can formally write Eq. 5.2 as

$$\delta = \delta_0 + \delta x^\mu \partial_\mu. \tag{5.3}$$

We want to study transformations which depend continuously on a certain set of parameters ω^a, with $a = 1, 2, ..., n$. For such transformations we can write δx^μ and $\delta_0\phi$ as

$$\delta x^\mu = \sum_a \omega^a \bar{\delta}_a x^\mu, \qquad \delta_0\phi = \sum_a \omega^a \bar{\delta}_a \phi, \tag{5.4}$$

where $\bar{\delta}_a x^\mu$ and $\bar{\delta}_a \phi$ are finite quantities. We note that if $\omega^a \to 0$ this transformation becomes the identity transformation.

We explained before that, if a transformation is a symmetry of a system, the equations of motions remain unchanged by performing such transformation. If a system is described by the action S, saying that the equations of motion don't change means that the variation of the action is

$$\delta S = \int d^4x \; \partial_\mu \Lambda^\mu(\phi), \tag{5.5}$$

as we have seen in Sec. 2.1.

The total variation of S is given by

$$\delta S = \int d^4x \; \delta\mathcal{L}(\phi, \partial_\mu\phi) + \int \delta d^4x \; \mathcal{L}(\phi, \partial_\mu\phi). \tag{5.6}$$

We'll first deal with $\delta\mathcal{L}(\phi, \partial_\mu\phi)$. The variation of the Lagrangian can be written as

$$\delta\mathcal{L}(\phi, \partial_\mu\phi) = \frac{\partial\mathcal{L}}{\partial\phi}\delta_0\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_0(\partial_\mu\phi) + (\partial_\mu\mathcal{L})\delta x^\mu, \tag{5.7}$$

where here $\partial_\mu\mathcal{L}$ are used to signify the total derivatives of $\mathcal{L}$ with respect to the coordinates x^μ. Since δ_0 represents a variation only of the form of a field and not of the coordinates, we have that

$$\delta_0(\partial_\mu\phi) = \partial_\mu(\phi + \delta_0\phi) - \partial_\mu\phi = \partial_\mu(\delta_0\phi). \tag{5.8}$$

Using this result we can rewrite the second term of Eq. 5.7

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_0(\partial_\mu\phi) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta_0\phi) = \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_0\phi) - \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)})\delta_0\phi \tag{5.9}$$

By substituting Eq. 5.9 into Eq. 5.7 we obtain

$$\delta\mathcal{L}(\phi, \partial_\mu\phi) = (\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)})\delta_0\phi + \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_0\phi) + (\partial_\mu\mathcal{L})\delta x^\mu. \tag{5.10}$$

We will now study δd^4x. In general we have that

$$d^4x' = |detJ(x)|d^4x, \tag{5.11}$$

where $J(x) = \frac{\delta x'^\mu}{\delta x^\nu}$ is the Jacobian matrix of the transformation $x' = x'(x)$. The transformation we are considering now is $x'^\mu = x^\mu + \delta x^\mu$, as expressed in Eq. 5.1, which means that in our case

$$J(x) = \delta^\mu{}_\nu + \partial_\nu(\delta x^\mu), \tag{5.12}$$

where ${\delta^{\mu}}_{\nu}$ is the Kronecker delta [1]. The determinant of a generic matrix A can be obtained using the formula

$$detA = e^{Tr\ lnA}. \tag{5.13}$$

Since we are operating at the first order of δx^{μ}, we have that

$$Tr\ lnJ(x) = Tr\ ln({\delta^{\mu}}_{\nu} + \partial_{\nu}(\delta x^{\mu})) = Tr(\partial_{\nu}(\delta x^{\mu})) = \partial_{\mu}(\delta x^{\mu}) \tag{5.14}$$

and

$$|detJ(x)| = |e^{Tr\ lnJ(x)}| = |e^{\partial_{\mu}(\delta x^{\mu})}| = |1 + \partial_{\mu}(\delta x^{\mu})| = 1 + \partial_{\mu}(\delta x^{\mu}), \tag{5.15}$$

where in the last passage we exloited the fact that we are considering infinitesimal transformations. Then, by substituting Eq. 5.15 into Eq. 5.11, we have that

$$d^4x' = (1 + \partial_{\mu}(\delta x^{\mu}))d^4x. \tag{5.16}$$

Finally, we obtain that

$$\delta d^4x = d^4x' - d^4x = d^4x\ \partial_{\mu}(\delta x^{\mu}). \tag{5.17}$$

We can now use Eq. 5.17 and Eq. 5.10 to rewrite Eq. 5.6 expressing δS as

$$\begin{aligned}\delta S = \int d^4x \left[(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_{\mu}\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)})\delta_0\phi + \partial_{\mu}(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}\delta_0\phi) + (\partial_{\mu}\mathcal{L})\delta x^{\mu} + \partial_{\mu}(\delta x^{\mu})\mathcal{L}\right] = \\ = \int d^4x \left[(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_{\mu}\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)})\delta_0\phi + \partial_{\mu}(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}\delta_0\phi) + \partial_{\mu}(\delta x^{\mu}\mathcal{L})\right] = \\ = \int d^4x \left[(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_{\mu}\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)})\delta_0\phi + \partial_{\mu}(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}\delta_0\phi + \mathcal{L}\delta x^{\mu})\right].\end{aligned} \tag{5.18}$$

For the equations of motion to be invariant, δS is required to satisfy Eq. 5.5. For simplicity, we will first consider the case $\delta S = 0$, and then we will add to the final result the contribution of $\int d^4x\ \partial_{\mu}\Lambda^{\mu}(\phi)$.

Using Eq. 5.18, the condition $\delta S = 0$ becomes

$$\int d^4x \left[(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_{\mu}\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)})\delta_0\phi + \partial_{\mu}(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}\delta_0\phi + \mathcal{L}\delta x^{\mu})\right] = 0. \tag{5.19}$$

Due to the arbitrariness of δx^{μ} and $\delta_0\phi$, the integrand must vanish for Eq. 5.19 to hold

$$(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_{\mu}\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)})\delta_0\phi + \partial_{\mu}(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}\delta_0\phi + \mathcal{L}\delta x^{\mu}) = 0. \tag{5.20}$$

The first term vanishes due to the *Euler-Lagrange equations* 2.10, which a field is required to satisfy. We are then left with

$$\partial_{\mu}(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}\delta_0\phi + \mathcal{L}\delta x^{\mu}) = 0. \tag{5.21}$$

[1] The Kronecker delta ${\delta^{\mu}}_{\nu}$ is characterised by being $= 1$ if $\mu = \nu$ and $= 0$ if $\mu \neq \nu$.

By substituting Eq. 5.4 into Eq. 5.21 we have

$$\sum_a \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \bar{\delta}_a \phi + \mathcal{L} \bar{\delta}_a x^\mu\right) \omega^a = 0. \tag{5.22}$$

Since the parameters ω^a are independent, we obtain the n equations

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \bar{\delta}_a \phi + \mathcal{L} \bar{\delta}_a x^\mu\right) = 0. \tag{5.23}$$

We can now define the n currents

$$j_a^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \bar{\delta}_a \phi + \mathcal{L} \bar{\delta}_a x^\mu. \tag{5.24}$$

Thanks to Eq. 5.23, we have that these currents satisfy the continuity equations

$$\partial_\mu j_a^\mu = 0 \tag{5.25}$$

and therefore are conserved. We note that we have as many currents as we have parameters for the transformation we are considering.

As said before, we could also have $\delta S = \int d^4x \ \partial_\mu \Lambda^\mu(\phi)$. By writing Λ^μ as $\Lambda^\mu = \sum_a \omega^a \bar{\delta}_a \Lambda^\mu$, where $\bar{\delta}_a \Lambda^\mu$ are finite quantities, the n conserved currents j_a^μ become

$$j_a^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \bar{\delta}_a \phi + \mathcal{L} \bar{\delta}_a x^\mu - \bar{\delta}_a \Lambda^\mu. \tag{5.26}$$

We have just proven the *Noether theorem*, which states that, if a system is invariant under a transformation which depends continuously on n parameters, it's possible to find n four-currents, known as *Noether currents*, which are conserved.

It's also possible to define the n quantities, called *Noether charges*,

$$Q_a(\sigma) = \int_\sigma d\sigma_\mu j_a^\mu, \tag{5.27}$$

where σ is a space-like hypersurface [2] that extends to spatial infinity and covers all space in the Minkowski spacetime $\mathbf{M}$.

If we assume, as it's usually done, that ϕ and $\partial_\mu \phi$ vanish rapidly enough when going to spatial infinity, we can use *Gauss's theorem* to prove that Q_a does not depend on σ. In fact, given any two spacetime hypersurfaces σ_1 and σ_2 that cover all space and the time-like [3] hypersurface σ_∞ located at spatial infinity that connects σ_1 and σ_2, by *Gauss's theorem* we have that

$$\int_\Omega d^4x \ \partial_\mu j_a^\mu = \int_{\sigma_1} d\sigma_\mu j_a^\mu - \int_{\sigma_2} d\sigma_\mu j_a^\mu + \int_{\sigma_\infty} d\sigma_\mu j_a^\mu, \tag{5.28}$$

[2] A space-like hypersurface σ in the Minkowski spacetime $\mathbf{M}$ is a hypersurface such that, if $x^\mu, y^\mu \in \sigma$ and $x^\mu \neq y^\mu$, $(x^\mu - y^\mu)(x_\mu - y_\mu) < 0$

[3] A time-like hypersurface σ in the Minkowski spacetime $\mathbf{M}$ is a hypersurface such that, if $x^\mu, y^\mu \in \sigma$ and $x^\mu \neq y^\mu$, $(x^\mu - y^\mu)(x_\mu - y_\mu) > 0$

where Ω is the volume enclosed by σ_1, σ_2 and σ_∞. By Eq. 5.25 and thanks to the assumptions just made on ϕ and $\partial_\mu \phi$, the integrals over Ω and σ_∞ vanish and we obtain that

$$\int_{\sigma_1} d\sigma_\mu j_a^\mu = \int_{\sigma_2} d\sigma_\mu j_a^\mu, \tag{5.29}$$

which proves that the Q_a do not depend on the choice of σ.

One particular choice we can make is that of a surface for which the time t is constant. That means that the surface element is $d\sigma_\mu = (d^3x, 0, 0, 0)$ and the charges Q_a are given by

$$Q_a(t) = \int d^3x \; j_a^0(\mathbf{x}, t), \tag{5.30}$$

where we are integrating over all $\mathbf{R^3}$ at a fixed time t. Having proven before that the Q_a do not depend on the surface chosen and since, by Eq. 5.30, choosing a surface means choosing a time, we conclude that the Q_a do not depend on time, proving that they are constants of motion. From Eq. 5.30, we can explicitly write an expression for the Q_a using Eq. 5.24

$$Q_a = \int d^3x \; (\frac{1}{c}\frac{\partial \mathcal{L}}{\partial t}\bar{\delta}_a \phi + c\mathcal{L}\bar{\delta}_a t). \tag{5.31}$$

We are now able to give an interpretation of the j_a^μ. We see that, thanks to Eq. 5.30, we can interpret the $j_a^0(\mathbf{x}, t)$ as the density of the $Q_a(t)$. To see what interpretation we may give to the other components of the j_a^μ, we integrate the continuity equations Eq. 5.25 over a finite 3-dimensional volume V

$$0 = \int_V d^3x \; \partial_\mu j_a^\mu = \frac{1}{c}\frac{\partial}{\partial t}\int_V d^3x \; j_a^0 + \int_V d^3x \; \nabla \cdot \mathbf{j}_a, \tag{5.32}$$

where $\mathbf{j}_a = (j_a^1, j_a^2, j_a^3)$. By using *Gauss's theorem* in 3 dimensions on the second term we obtain

$$\frac{1}{c}\frac{\partial Q_a}{\partial t} = -\int_{\partial V} d\sigma \cdot \mathbf{j}_a, \tag{5.33}$$

where ∂V is the contour of V. From Eq. 5.33 we see that the variation of $\frac{Q_a}{c}$ inside a volume V is equal to the flux of $\mathbf{j}_a$ through the surface ∂V enclosing said volume. We conclude then that the $\mathbf{j}_a$ can be interpreted as the flux density of the $\frac{Q_a}{c}$.

We can prove that the conserved currents j_a^μ are not uniquely defined. In fact, given an antisymmetric tensor $X^{\nu\mu}(x)$, we can transform any one of the *Noether currents* by adding to it $\partial_\nu X^{\nu\mu}(x)$

$$j'^\mu = j^\mu + \partial_\nu X^{\nu\mu}, \qquad X^{\nu\mu} = -X^{\mu\nu} \tag{5.34}$$

without altering the property that the current is conserved and without changing the corresponding *Noether charge*. We can first prove that j'^μ still satisfies the continuity equation Eq 5.25

$$\partial_\mu j'^\mu = \partial_\mu(j^\mu + \partial_\nu X^{\nu\mu}) = \partial_\mu j^\mu + \partial_\mu \partial_\nu X^{\nu\mu} = \partial_\mu j^\mu = 0, \tag{5.35}$$

where in the second to last passage we have used that $\partial_\mu\partial_\nu X^{\nu\mu} = 0$ since $X^{\nu\mu}$ is antisymmetric and $\partial_\mu\partial_\nu$ is symmetric. Let's now call Q' the charge associated to j'^μ and Q the one associated to j^μ. We now want to prove that $Q' = Q$, which reduces to proving that $\int_\sigma d\sigma_\mu\ \partial_\nu X^{\nu\mu} = 0$ over any space-like hypersurface σ covering all space. If we assume that $\partial_\nu X^{\nu\mu}$ vanishes rapidly enough at spatial infinity, with the same reasoning used for Eq. 5.28 and Eq. 5.29 we can prove that the value of $\int_\sigma d\sigma_\mu\ \partial_\nu X^{\nu\mu}$ does not depend on the choice of σ. We then choose a surface σ at fixed time so that $d\sigma_\mu = (d^3x, 0, 0, 0)$. We now have that

$$\int_\sigma d\sigma_\mu\ \partial_\nu X^{\nu\mu} = \int d^3x\ \partial_\nu X^{\nu 0}, \tag{5.36}$$

integrating over all $\mathbf{R^3}$. Since $X^{00} = 0$ due to $X^{\nu\mu}$ being antisymmetric and by using *Gauss's theorem* in 3 dimensions, we obtain

$$\int d^3x\ \partial_\nu X^{\nu 0} = \int d^3x\ \partial_i X^{i0} = \int_\infty d\sigma_i\ X^{i0}, \tag{5.37}$$

where $i = 1, 2, 3$ and where we are integrating over a surface in $\mathbf{R^3}$ located at infinity. If we assume that X^{i0} vanishes rapidly enough at spatial infinity, we have that Eq. 5.37, and therefore $\int_\sigma d\sigma_\mu\ \partial_\nu X^{\nu\mu}$, vanishes, proving that $Q' = Q$. We have thus proven that we are free to transform any *Noether current* by adding to it $\partial_\nu X^{\nu\mu}$, with $X^{\nu\mu}(x)$ being an antisymmetric tensor. Quantities like $X^{\nu\mu}(x)$ are sometimes called superpotentials.

We end our brief general study of symmetries with an important remark. Based on what is involved in their corresponding transformations, symmetries can be divided into two categories:

- Spacetime symmetries, whose corresponding transformations involve a change in the coordinates.

- Internal symmetries, whose corresponding transformations act only on the fields, not on the coordinates.

Chapter 6

Translational symmetry

We will now consider a continuous spacetime symmetry, the translational symmetry.

The corresponding infinitesimal transformation for the coordinates x^μ is a spacetime translation by an infinitesimal vector ε^μ

$$x'^\mu = x^\mu + \varepsilon^\mu. \tag{6.1}$$

It's possible to write the variation of x^μ as

$$\delta x^\mu = \varepsilon^\mu = \delta^\mu{}_\nu \varepsilon^\nu. \tag{6.2}$$

We see that we can identify ε^μ as the four parameters of an infinitesimal translation and, using $\delta x^\mu = \varepsilon^\nu \bar{\delta}_\nu x^\mu$ from Eq. 5.4, we have that

$$\bar{\delta}_\nu x^\mu = \delta^\mu{}_\nu. \tag{6.3}$$

As previously discussed, since the transformation of a field depends only on $\frac{\partial x'^\mu}{\partial x^\nu}$ and $\frac{\partial x^\nu}{\partial x'^\mu}$, any generic field $\phi(x)$ is invariant under translations, meaning

$$\phi'(x') = \phi(x) \quad \rightarrow \quad \delta\phi(x) = 0. \tag{6.4}$$

Since $\delta\phi(x) = \delta_0\phi(x) + \partial_\mu\phi\delta x^\mu$ by Eq. 5.2 and using Eq. 6.2, we have that

$$\delta_0\phi(x) = -\partial_\nu\phi\delta x^\nu = -\partial_\nu\phi\ \varepsilon^\nu. \tag{6.5}$$

We can write $\delta_0\phi = \varepsilon^\nu\bar{\delta}_\nu\phi$ thanks to Eq. 5.4, which means that

$$\bar{\delta}_\nu\phi = -\partial_\nu\phi. \tag{6.6}$$

We now ask ourselves which systems show a translational symmetry. We remind that, in Ch. 2, we required a generic Lagrangian $\mathcal{L}$ to be invariant under Poincaré transformations, which include translations. Moreover, since the Jacobian matrix of a translation is $J(x) = \delta^\mu{}_\nu$, we have that $d^4x' = |detJ(x)|d^4x = d^4x$. From these two facts we obtain that, under translations, the variation of the action is zero for any system. We conclude then that any system is invariant under translations. This means that the

Noether currents related to the translational symmetry will always be conserved and the corresponding *Noether charges* will always be constants of motion.

We obtain the n *Noether currents* $j^\mu{}_\nu(x)$ by substituting the expressions for $\bar{\delta_\nu} x^\mu$ from Eq. 6.3 and for $\bar{\delta_\nu}\phi$ from Eq. 6.6 into Eq. 5.24

$$j^\mu{}_\nu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\bar{\delta_\nu}\phi + \mathcal{L}\bar{\delta_\nu}x^\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\nu\phi + \mathcal{L}\delta^\mu{}_\nu. \tag{6.7}$$

We note that the $j^\mu{}_\nu(x)$ appear arranged in the form of a rank $(1,1)$ tensor. We can now define the rank $(2,0)$ tensor $\Theta^{\mu\nu}(x)$, called the canonical energy-momentum tensor,

$$\Theta^{\mu\nu}(x) = -j^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial^\nu\phi - \mathcal{L}\eta^{\mu\nu} \text{ [1]} \tag{6.8}$$

with the corresponding continuity equation

$$\partial_\mu \Theta^{\mu\nu} = 0. \tag{6.9}$$

The *Noether charges* associated with translations are given by

$$Q^\nu = cP^\nu = \int_\sigma d\sigma_\mu \Theta^{\mu\nu}, \tag{6.10}$$

where σ is a space-like hypersurface covering all space. We can then choose as σ a surface at fixed time so that the surface element is $d\sigma_\mu = (d^3x, 0, 0, 0)$. With this choice, the cP^ν become

$$cP^\nu = \int d^3x \; \Theta^{0\nu}, \tag{6.11}$$

where we are integrating over all $\mathbf{R^3}$. Using Eq 6.8, we can write the components of P^ν as

$$\begin{aligned} cP^0 &= \int d^3x \; \Theta^{00} = \int d^3x \; (\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}\partial^0\phi - \mathcal{L}), \\ cP^i &= \int d^3x \; \Theta^{0i} = \int d^3x \; \frac{\partial \mathcal{L}}{\partial(\partial_i \phi)}\partial^0\phi, \end{aligned} \tag{6.12}$$

with $i = 1, 2, 3$. The four *Noether charges* cP^ν we just obtained correspond to the four components of the four-momentum $P^\nu = (\frac{E}{c}, P_x, P_y, P_z) = (\frac{E}{c}, \mathbf{P})$ times the light speed in vacuum c. We can thus say that, since any system is invariant under translations, the four-momentum P^ν of a system is always conserved.

From Eq. 6.12 we see that we can interpret Θ^{00} as the density of E and Θ^{0i} as the density of the i-th component of $c\mathbf{P}$. For the interpretation of the other elements of $\Theta^{\mu\nu}$, we can integrate Eq. 6.9 over a finite volume V in $\mathbf{R^3}$ with contour ∂V

$$\frac{1}{c}\frac{\partial}{\partial t}\int_V d^3x \; \Theta^{0\nu} = -\int_V d^3x \; \partial_i\Theta^{i\nu}. \tag{6.13}$$

[1] Raising an index of $\delta^\mu{}_\nu$ means doing $\eta^{\nu\alpha}\delta^\mu{}_\alpha$, which, given the properties of the Kronecker delta and thanks to $\eta^{\mu\nu}$ being symmetrical, is equal to $\eta^{\mu\nu}$.

Then, by using *Gauss's theorem* in 3 dimensions and remembering Eq. 6.11, we obtain

$$\frac{\partial P^\nu}{\partial t} = \frac{1}{c}\frac{\partial}{\partial t}\int_V d^3x\ \Theta^{0\nu} = -\int_{\partial V} d\sigma_i \Theta^{i\nu}. \tag{6.14}$$

This constitutes a continuity equation for the components of the four-momentum, meaning we can interpret each $\Theta^{i\nu}$ as the flux density of P^ν in the i-th direction. In particular, the Θ^{ij}, which represent the flux density of the P^j in the i-th direction, are organised in the so-called stress tensor [2].

It's important to note that the canonical energy-momentum tensor $\Theta^{\mu\nu}$ is not unique. In fact, we have seen in Ch. 5 that we are free to transform any Noether current by adding to it $\partial_\nu X^{\nu\mu}$, provided that $X^{\nu\mu}(x)$ is an antisymmetric tensor. This means that, if $G^{\nu\rho\mu}$ is a tensor antisymmetric with respect to ρ and μ, we are free to add $\partial_\rho G^{\nu\rho\mu}$ to $\Theta^{\mu\nu}$

$$\Theta'^{\mu\nu} = \Theta^{\mu\nu} + \partial_\rho G^{\nu\rho\mu}, \qquad G^{\nu\rho\mu} = -G^{\nu\mu\rho} \tag{6.15}$$

without changing the corresponding *Noether charges* cP^μ and still satisfying the continuity equation $\partial_\mu \Theta'^{\mu\nu} = 0$. This ambiguity can be exploited to obtain a symmetrical energy-momentum tensor, which is a property that the $\Theta^{\mu\nu}$ obtained through the *Noether theorem* does not generally satisfy outright. For example, $\Theta^{\mu\nu}$ is symmetrical for scalar fields but not for vector fields.

We will now show how the need for a symmetrical energy-momentum tensor arises through some simple arguments. First we show that we must have $\Theta^{i0} = \Theta^{0i}$. Since Θ^{i0} is the flux density of $\frac{E}{c}$ in the i-th direction, it is equal to the density of $\frac{E}{c}$ times the flux velocity in the i-th direction. Using the mass–energy equivalence $E = mc^2$, we have that the density of $\frac{E}{c}$ is equal to that of mc. Now, we have that Θ^{i0} is equal to the density of mc times the flux velocity in the i-th direction, which clearly corresponds to the i-th component of the momentum times c $cP^i = \Theta^{0i}$. We have thus shown that $\Theta^{i0} = \Theta^{0i}$. We now consider the components Θ^{ij}. To prove that we must have $\Theta^{ij} = \Theta^{ji}$, we'll study an infinitesimal cube with side length dL. The mass inside such a cube, thanks to the mass-energy equivalence, is $dM = c^2\Theta^{00}dL^3$. Since the moment of inertia I of a cube is proportional to ML^2 we have that

$$dI \propto dM\ dL^2 = \Theta^{00} dL^6. \tag{6.16}$$

We proceed by centering the cube in the origin of a reference frame with its faces perpendicular to the axes of the frame. We can now say that $\Theta^{ij}dL^2$ is the j-th component dF^j of the force exerted by the field on the face of the cube perpendicular to the i axis. Considering any even permutation (i, j, k) of $(1, 2, 3)$, we can obtain that the k-th component $d\tau^k$ of the torque exerted on the cube is equal to $(\Theta^{ij} - \Theta^{ji})dL^3$

$$d\tau^k = (\Theta^{ij}dL^2)\cdot(\frac{dL}{2}) + (-\Theta^{ij}dL^2)\cdot(-\frac{dL}{2}) - (\Theta^{ji}dL^2)\cdot(\frac{dL}{2}) - (-\Theta^{ji}dL^2)\cdot(-\frac{dL}{2}) = \\ = (\Theta^{ij} - \Theta^{ji})dL^3. \tag{6.17}$$

[2] The 3x3 matrix called stress tensor is a tensor under rotations in $\mathbf{R^3}$, not under Lorentz.

Since the components of the angular acceleration α^k are given by $\frac{\tau^k}{I}$, using Eq. 6.16 and Eq. 6.17 we have that

$$\alpha^k \propto \frac{1}{dL^2}. \tag{6.18}$$

For a cube with infinitesimal side length dL that would mean having an infinite angular acceleration. Since this can't be possible, we see from Eq. 6.17 that the only way to avoid it is to have $\Theta^{ij} = \Theta^{ji}$. We have thus completed the proof that $\Theta^{\mu\nu} = \Theta^{\nu\mu}$, meaning that the energy-momentum tensor has to be symmetrical. We note that the symmetry of the spatial components Θ^{ij} has been obtained only with the use of classical arguments, without involving relativity. We instead used relativistic concepts to prove the symmetry of the mixed components Θ^{0i} and Θ^{i0} by exploiting the mass-energy equivalence.

Since, as said before, the canonical energy-momentum tensor $\Theta^{\mu\nu}$ is generally not symmetrical and we have proven that a physically sound energy-momentum tensor has to be symmetrical, $\Theta^{\mu\nu}$ may have no direct physical meaning. In general, we will need to symmetrize $\Theta^{\mu\nu}$ through a transformation like Eq 6.15 in order to obtain a symmetrical energy-momentum tensor $T^{\mu\nu}$ which is physically acceptable.

6.1 Energy-momentum tensor of a real scalar field

Let's consider a real scalar field $\phi(x)$ described by the Lagrangian $\mathcal{L}$ in Eq. 3.1. Using Eq. 6.8 we see that the corresponding canonical energy-momentum tensor $\Theta^{\mu\nu}(x)$ is

$$\Theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial^\nu \phi - \mathcal{L}\eta^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \frac{1}{2}\partial_\alpha \phi \partial^\alpha \phi \; \eta^{\mu\nu} + F(\phi)\eta^{\mu\nu}. \tag{6.19}$$

Since this tensor is symmetrical, it's already a physically acceptable energy-momentum tensor. Thanks to Eq. 6.12, we can obtain expressions for the energy E and the momentum $\mathbf{P}$ of the system

$$E = \int d^3x \left[(\partial^0 \phi)^2 - \frac{1}{2}\partial_\alpha \phi \partial^\alpha \phi + F(\phi) \right] = \int d^3x \left[\frac{1}{2}(\partial^0 \phi)^2 + \frac{1}{2}(\nabla \phi)^2 + F(\phi) \right], \tag{6.20}$$

$$\mathbf{P} = -\frac{1}{c}\int d^3x \; \partial^0 \phi \nabla \phi \; ^3. \tag{6.21}$$

If the form of $\mathcal{L}$ is that of the Klein-Gordon Lagrangian $\mathcal{L}_{KG}$ given by Eq. 3.4, the energy momentum tensor is

$$\Theta^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \frac{1}{2}\partial_\alpha \phi \partial^\alpha \phi \; \eta^{\mu\nu} + \frac{1}{2}\mu^2 \phi^2 \eta^{\mu\nu}. \tag{6.22}$$

Seeking to observe Eq. 6.20 and Eq. 5.21 in this case, we notice that the expression of the momentum doesn't change by specifying $F(\phi)$, while the energy becomes

$$E = \int d^3x \; \left[\frac{1}{2}(\partial^0 \phi)^2 + \frac{1}{2}(\nabla \phi)^2 + \frac{1}{2}\mu^2 \phi^2 \right]. \tag{6.23}$$

[3] The $-$ sign is due to the fact that $\partial^i = -\partial_i$ and $\nabla = (\partial_1, \partial_2, \partial_3)$.

6.2 Energy-momentum tensor of the electromagnetic field

We'll now obtain the energy-momentum tensor of the electromagnetic field $A^\mu(x)$. Substituting the Lagrangian of the field $\mathcal{L}_{em}$ given by Eq. 4.7 into Eq. 6.7, we obtain the canonical energy-momentum tensor $\Theta^{\mu\nu}_{em}(x)$

$$\Theta^{\mu\nu}_{em} = \frac{\partial \mathcal{L}_{em}}{\partial(\partial_\mu A_\rho)}\partial^\nu A_\rho - \mathcal{L}_{em}\eta^{\mu\nu} = -\frac{1}{4\pi}F^{\mu\rho}\partial^\nu A_\rho + \frac{1}{16\pi}F^{\rho\sigma}F_{\rho\sigma}\eta^{\mu\nu}. \tag{6.24}$$

Contrary to what we have seen with scalar fields, this canonical energy-momentum tensor $\Theta^{\mu\nu}_{em}$ is not symmetrical. Moreover, it's not invariant under a gauge transformation. To obtain an energy momentum tensor $T^{\mu\nu}_{em}(x)$ which is symmetrical and gauge invariant, we'll transform $\Theta^{\mu\nu}_{em}$ using an appropriate tensor $G^{\mu\rho\nu}_{em}$ like in Eq. 6.15. In this case, the tensor $G^{\mu\rho\nu}_{em}$ is given by

$$G^{\mu\rho\nu}_{em} = -\frac{1}{4\pi}F^{\rho\mu}A^\nu. \tag{6.25}$$

Since $F^{\rho\mu}$ is antisymmetric, $G^{\mu\rho\nu}_{em}$ satisfies the requirement of being antisymmetric in ρ and μ. Through the transformation shown in Eq. 6.15 we obtain $T^{\mu\nu}_{em}$

$$\begin{aligned} T^{\mu\nu}_{em} = \Theta^{\mu\nu}_{em} - \frac{1}{4\pi}\partial_\rho(F^{\rho\mu}A^\nu) = \\ = -\frac{1}{4\pi}F^{\mu\rho}(\partial^\nu A_\rho - \partial_\rho A^\nu) - \frac{1}{4\pi}A^\nu\partial_\rho F^{\rho\mu} + \frac{1}{16\pi}F^{\rho\sigma}F_{\rho\sigma}\eta^{\mu\nu} = \\ = -\frac{1}{4\pi}F^{\mu\rho}F^\nu{}_\rho + \frac{1}{16\pi}F^{\rho\sigma}F_{\rho\sigma}\eta^{\mu\nu}, \end{aligned} \tag{6.26}$$

where in the last passage we exploited the inhomogeneous Maxwell equations for when $J^\mu = 0$, which can be written as $\partial_\rho F^{\rho\mu} = 0$. The energy-momentum tensor $T^{\mu\nu}_{em}$ just obtained is clearly symmetrical. $T^{\mu\nu}_{em}$ is also invariant under a gauge transformation since only the electromagnetic tensor, which is gauge invariant, appears in Eq. 6.26. We can thus say that $T^{\mu\nu}_{em}$ is a physically acceptable energy-momentum tensor for the electromagnetic field.

We briefly note that the trace of $T^{\mu\nu}_{em}$ vanishes

$$T^{\mu}_{em,\ \mu} = -\frac{1}{4\pi}F^{\mu\rho}F_{\mu\rho} + \frac{1}{16\pi}F^{\rho\sigma}F_{\rho\sigma}\delta^\mu{}_\mu = -\frac{1}{4\pi}F^{\mu\rho}F_{\mu\rho} + \frac{1}{4\pi}F^{\rho\sigma}F_{\rho\sigma} = 0. \tag{6.27}$$

We can now see what the components of $T^{\mu\nu}_{em}$ represent by using the relations $F^{0i} = -E^i$ and $F^{ij} = -\varepsilon^{ijk}B^k$:

- T^{00}_{em} is the energy density u of the electromagnetic field

$$\begin{aligned} T^{00}_{em} = -\frac{1}{4\pi}F^{0\rho}F^0{}_\rho + \frac{1}{16\pi}F^{\rho\sigma}F_{\rho\sigma} = \\ = -\frac{1}{4\pi}F^{0i}F^0{}_i + \frac{1}{16\pi}F^{\rho\sigma}F_{\rho\sigma} = \frac{1}{4\pi}F^{0i}F^{0i} + \frac{1}{16\pi}F^{\rho\sigma}F_{\rho\sigma} = \\ = \frac{1}{4\pi}|\mathbf{E}|^2 - \frac{1}{8\pi}(|\mathbf{E}|^2 - |\mathbf{B}|^2) = \frac{1}{4\pi}(|\mathbf{E}|^2 + |\mathbf{B}|^2) = u. \end{aligned} \tag{6.28}$$

- The $T^{0i}_{em} = T^{i0}_{em}$ are the components of $\frac{\mathbf{S}}{c}$, where $\mathbf{S}$ is the Poynting vector and c is the light speed in vacuum

$$T^{0i}_{em} = T^{i0}_{em} = -\frac{1}{4\pi}F^{0\rho}F^{i}{}_{\rho} = -\frac{1}{4\pi}F^{0j}F^{i}{}_{j} = \frac{1}{4\pi}F^{0j}F^{ij} = \\ = \frac{1}{4\pi}\varepsilon^{ijk}E^{j}B^{k} = \frac{1}{4\pi}(\mathbf{E}\times\mathbf{B})^{i} = \frac{S^{i}}{c}. \quad (6.29)$$

- The T^{ij}_{em} are the elements of the Maxwell stress tensor.

Chapter 7

Scale symmetry

The last topic we will discuss is another spacetime symmetry, the scaling symmetry. The corresponding transformation is of the type

$$x'^{\mu} = \lambda x^{\mu}, \tag{7.1}$$

where λ is a real parameter. The group of transformations made up by Poincaré transformations and scale transformations is the Weyl group. We note that, if $ds^2 = dx_{\mu}dx^{\mu} = 0$ [1], ds^2 is invariant under a scale transformation.

If a generic field $\phi(x)$ has dimensions $[\text{length}]^n$, under a scale transformation it transforms like

$$\phi'(x') = \lambda^n \phi(x) \tag{7.2}$$

Let's now consider an infinitesimal scale transformation. By writing $\lambda = 1 - \varepsilon$, where ε is an infinitesimal real parameter, the transformation of the coordinates is given by

$$x'^{\mu} = (1 - \varepsilon)x^{\mu}, \tag{7.3}$$

which means that the variation δx^{μ} is

$$\delta x^{\mu} = x'^{\mu} - x^{\mu} = -\varepsilon x^{\mu}. \tag{7.4}$$

Since clearly a scale transformation is a transformation depending continuously on the one parameter ε, using Eq. 5.4 we can write δx^{μ} as

$$\delta x^{\mu} = \varepsilon \bar{\delta} x^{\mu}. \tag{7.5}$$

Putting Eq. 7.4 and Eq. 7.5 together, we get

$$\bar{\delta} x^{\mu} = -x^{\mu}. \tag{7.6}$$

Regarding the field ϕ, we can write Eq. 7.2 as

$$\phi'(x) = \lambda^n \phi(\lambda^{-1} x) \tag{7.7}$$

[1] A $ds^2 = dx_{\mu}dx^{\mu}$ such that $ds^2 = 0$ is referred to as a light-like interval.

Then, if we consider an infinitesimal transformation with $\lambda = 1 - \varepsilon$ and operating at the first infinitesimal order, we get

$$\phi'(x) = (1 - n\varepsilon)\phi(x + x\varepsilon) = \phi(x) - n\varepsilon\phi(x) + \varepsilon x^\mu \partial_\mu \phi, \tag{7.8}$$

from which we get that

$$\delta_0\phi = \phi'(x) - \phi(x) = \varepsilon(-n\phi + x^\mu \partial_\mu \phi). \tag{7.9}$$

Using Eq. 5.4, we can write $\delta_0\phi$ also as

$$\delta_0\phi = \varepsilon\bar{\delta}\phi. \tag{7.10}$$

Putting Eq. 7.4 and Eq. 7.5 together, we obtain

$$\bar{\delta}\phi = -n\phi + x^\mu \partial_\mu \phi. \tag{7.11}$$

Since, by a scale transformation, d^4x results simply multiplied by a constant, for a system to be invariant under scale transformations we just need the variation of the Lagrangian to be zero or to be of the form $\partial_\mu \Lambda^\mu(\phi)$.

Let's suppose that the Lagrangian of the system we are studying is scale invariant, meaning it's variation under scale transformations is zero. We can substitute $\bar{\delta}x^\mu$ from Eq. 7.6 and $\bar{\delta}\phi$ from Eq. 7.11 into Eq. 5.24 to obtain the conserved *Noether current* j^μ

$$\begin{aligned} j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\bar{\delta}\phi + \mathcal{L}\bar{\delta}x^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot (-n\phi + x_\nu \partial^\nu \phi) - \mathcal{L}x^\mu = \\ = (\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial^\nu \phi - \mathcal{L}\eta^{\mu\nu})x_\nu - n\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\phi = \Theta^{\mu\nu}x_\nu - n\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\phi, \end{aligned} \tag{7.12}$$

where $\Theta^{\mu\nu}$ is the canonical energy-momentum tensor of the field ϕ. The current j^μ satisfies the continuity equation

$$\partial_\mu j^\mu = 0. \tag{7.13}$$

Let's now consider a real scalar field $\phi(x)$ with the Lagrangian $\mathcal{L}$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - F(\phi). \tag{7.14}$$

Such a field has dimensions $\propto$[length]$^{-1}$. Using Eq. 7.12 with $n = -1$, we obtain the *Noether current* j^μ related to scale transformations for ϕ

$$j^\mu = \Theta^{\mu\nu}x_\nu + \phi\partial^\mu \phi. \tag{7.15}$$

We thus have that $\partial_\mu j^\mu$ is

$$\partial_\mu j^\mu = \partial_\mu(\Theta^{\mu\nu}x_\nu) + \partial_\mu(\phi\partial^\mu\phi) = \Theta^\mu{}_\mu + \partial_\mu\phi\partial^\mu\phi + \phi\,\Box\phi, \tag{7.16}$$

where in the last passage we used $\partial_\mu x_\nu = \eta_{\mu\nu}$ and the continuity equation for $\Theta^{\mu\nu}$ Eq. 6.9. Using Eq. 6.19, we can see that $\Theta^\mu{}_\mu$ is

$$\Theta^\mu{}_\mu = -\partial_\mu\phi\partial^\mu\phi + 4F(\phi). \tag{7.17}$$

Substituting Eq. 7.17 into Eq. 7.16 we obtain

$$\partial_\mu j^\mu = \phi\ \Box\phi + 4F(\phi), \tag{7.18}$$

If our real scalar field ϕ is described by the Klein-Gordon Lagrangian $\mathcal{L}_{KG}$ in Eq. 3.4, Eq. 7.18 becomes

$$\partial_\mu j^\mu = \phi\ \Box\phi + 2\mu^2\phi^2 = \mu^2\phi^2, \tag{7.19}$$

where in the last passage we exploited the *Klein-Gordon equation* Eq. 3.5, which is the equation of motion of the field. Having seen that, in this case, $\partial_\mu j^\mu \neq 0$, we can conclude that the current j^μ corresponding to scale transformations is not conserved for a field described by the Klein-Gordon Lagrangian, meaning that the Klein-Gordon theory is not scale invariant.

To have a scale invariant theory for real scalar field we can consider the so-called ϕ^4 theory. In this theory the Lagrangian $\mathcal{L}$ describing the field ϕ is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{g}{4}\phi^4. \tag{7.20}$$

Using the *Euler-Lagrange equation* 2.10 with this Lagrangian, we obtain the equations of motion

$$\Box\phi = -g\phi^3. \tag{7.21}$$

Taking into account Eq. 7.20 and Eq. 7.21, Eq. 7.18 becomes a continuity equation for j^μ

$$\partial_\mu j^\mu = 0. \tag{7.22}$$

This means that the current j^μ is conserved, which leads us to conclude that the ϕ^4 theory is scale invariant.

To see why the ϕ^4 theory is scale invariant and the Klein-Gordon theory is not, it is useful to reason using natural units, which means setting $c = \hbar = 1$, with c being the light speed in vacuum and $\hbar$ being the reduced Planck constant. In these units, ϕ has exactly dimensions [length]$^{-1}$. The ϕ^4 theory is scale invariant because the only constant g that appears in its Lagrangian is dimensionless, meaning there is no characteristic dimensional scale in this theory. The constant μ in the Klein-Gordon theory has instead the dimensions of a [length]$^{-1}$=[mass], which means that μ constitutes a characteristic dimensional scale for the field and therefore the theory can't be scale invariant. Using the same reasoning, if we observe the Lagrangian of the electromagnetic field $\mathcal{L}_{em}$ in Eq. 4.7, we notice that there are no characteristic dimensional scales and we can conclude that electromagnetism is scale invariant.

Bibliography

As reference for the writing of this work, the following sources were used:

V. Barone: *Relatività. Princìpi e applicazioni*. Bollati Boringhieri, 2004.

L.H. Ryder: *Quantum Field Theory*. Cambridge University Press, 1996.

P. Ramond: *Field Theory: A Modern Primer*. Routledge, 2 edition, 2001.