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A CRITICAL LOOK AT SINGULARITIES

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Abstract

The focus of the present thesis is, quite simply, spacetime singularities. We begin by providing a historical account of the idea of spacetime singularities; we look at works by Friedmann, Lemaitre, and Oppenheimer and Snyder. Next, we expose the framework introduced by Raychaudhuri, based on the notion of geodesic congruence. Then, we present the 1965 theorem by Roger Penrose and discuss some improvements suggested by the author himself in a paper where he collaborated with Hawking, this concludes the first part of the thesis where we lay down what we attempt to deconstruct in the second part. The second part opens with the discussion of the 2023 paper by Kerr, in which he contests the common interpretation of the singularity theorems. He provides examples of incomplete geodesics that do not end up at singularities, a link that seems to be taken for granted in the theorems. In the last part, we endeavour to find examples analogous to those of Kerr; investigating to that end regular metrics, the Hayward metric in particular.

Contents

Chapter 1

Introduction

In this study, we endeavoured to provide a critical account of the singularity theorems. Starting from the origins of the notion of spacetime singularities, we built our way towards the theorems and then took a step back and reflected on the possible drawbacks of the conclusions of these theorems, hoping also to make a miniature contribution to the field ourselves.

In Chapter 2 we expose the fundamental cornerstones in the history of the study of spacetime singularities to see how this idea came into being, evolved and assumed its mature form. The first appearance of singularities in Einstein's theory is found in the work by Karl Schwarzschild the year that the former published his results, believing that the reader is familiar with this result, we begin our story with the developments that followed it. The first figure we mention is Friedmann. In his paper that appeared seven years after Einstein's advent of GR (general relativity), Friedmann was in search of deriving some known solutions from certain general assumptions. Supposing that the energy-momentum is that of dust and looking for solutions of a particular form (which will later harbor his name among others who made their contribution to the study, see FLRW (Friedmann-Lemaitre-Robertson-Walker) metric), his analysis lead him to consider what he called the 'creation time', by which he meant the point in time in the past at which the scale factor vanished. The conclusion he reached is that the 'creation time' should be finite, thus implying that a finite time back in the history of our universe a singularity should be exhibited. Lemaître worked on another sort of singularity, namely the one appearing in the Schwarzschild metric, he showed that this singularity was not physical, being merely a product of the choice of coordinates. To follow with, we discuss Oppenheimer and Snyder's work, which in a way reversed the situation considered by Friedmann: instead of looking at an expanding spacetime, we look at a contracting (portion of) spacetime. This produces a singularity in the future. The cases treated by Friedmann and Oppenheimer & Snyder, respectively, being the inverse in some sense of one another, efforts to explain them in a single framework was natural, the man who provided it was Raychaudhuri. In the second part of the historical background we expose his ideas based on the concept of geodesic congruence. The theorem that he put forward can be considered as the first singularity theorem.

The cardinal work in the study of spacetime singularities is without a doubt the 1965 paper by Roger Penrose. It deserves a lot of attention; accordingly, we dedicated Chapter 3 of the present thesis to its discussion. We begin with the point of departure of Penrose and give a résumé of the 1965 paper. Opening a brief parenthesis on the notion of trapped surfaces which is of fundamental importance to the theorems, we then follow with some developments of the results provided a few years later in a paper where Penrose collaborated with Hawking.

After having laid down the theorems, we move into the second part of the thesis where we direct a critical gaze onto the subject, as echoed in the title. Our point of departure is the 2023 paper [11] by Roy Kerr entitled *Do black holes have singularities?* The 'crux' of his argument, as he puts it, is that the incompleteness of geodesics does not necessarily imply singularities in the physical sense, as seems to be taken for granted in the theorems. He provides several counter-examples to illustrate his point. In Chapter 4, we give a summary of his paper and then discuss the essential points. Additionally, we provide a lengthy appendix where we re-calculate the important results in the paper and furnish discussions where we feel it is necessary.

Inspired by Kerr's perspective we endeavoured to find examples of incomplete geodesics in spacetimes that, by definition, do not exhibit any singularities. Metrics describing such spacetimes are called regular and a famous kind is the one introduced by Hayward. We begin Chapter 5 by introducing regular metrics, then we expose the Hayward metric providing a summary of his 2006 paper. Finally, in accordance with the initial motivation, we investigate the null geodesics in Hayward spacetime, hoping to find counter-examples to the predictions of the singularity theorems.

Chapter 2

Historical background

2.1 Some early catastrophes

In order to appreciate the Penrose singularity theorem one needs to have some knowledge of the historical developments that had paved the way to it. To that end, we will begin our story by relating some of the notable catastrophes (that is to say, examples of singular behaviour) exhibited in post-Einsteinian GR.

2.1.1 Friedmann

Friedmann builds on previous work by Einstein and de Sitter. To give an idea of the background I directly quote the paper [2]: "Einstein obtains the so-called cylindrical world, in which space has constant, time-independent curvature, where the curvature radius is connected to the total mass of matter present in space; de Sitter obtains a spherical world in which not only space, but in a certain sense also the world can be addressed as a world of constant curvature." He then sets down the goal of his treatment: "the derivation of the cylindrical and spherical worlds (as special cases) from some general assumptions, and secondly the proof of the possibility of a world whose space curvature" is time-dependent.

The assumptions talked about here come in two species. Firstly, those on the sort of relation the gravitational potential obeys and the prescriptions on the energy-momentum. Namely, the Einstein equation,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G_N}{c^4}T_{\mu\nu}, \tag{2.1}$$

and the assumption that the energy-momentum be that of dust,

$$T_{\mu\nu} = \begin{cases} \rho & \text{if} \quad \mu = \nu = 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

The second class of assumptions are geometrical in nature. In particular, Friedmann sought solutions of the form

$$ds^2 = -Fdt^2 + a^2(t)\left(d\chi^2 + \sin^2\chi d\Omega^2\right). \tag{2.3}$$

The purely spatial component of the Einstein equations under these assumptions gives

$$\Lambda = 2\frac{\ddot{a}}{a} + \frac{1}{a^2}\left(1 + \dot{a}^2\right), \tag{2.4}$$

whereas the purely temporal one gives

$$\kappa\rho + \Lambda = \frac{3}{a^2}\left(1 + \dot{a}^2\right), \tag{2.5}$$

where $\kappa = \frac{8\pi G_N}{c^4}$. The integral of the former gives

$$a(\dot{a}^2 + 1) = A + \frac{\Lambda}{3}a^3, \tag{2.6}$$

where A is an integration constant. To see this multiply the equation in question by a^2 and integrate it:

$$\int da\left(\Lambda a^2\right) = \int da\left[2\ddot{a}a + (\dot{a}^2 + 1)\right] = \int dt\frac{da}{dt}\left[2\ddot{a}a + (\dot{a}^2 + 1)\right] \tag{2.7}$$

$$\frac{\Lambda}{3}a^3 + A' = \int dt(c\dot{a})\left[(\dot{a}^2 + 1) + 2\ddot{a}a\right] = c\int dt\left[\dot{a}(\dot{a}^2 + 1) + a(2\dot{a}\ddot{a})\right] \tag{2.8}$$

$$\frac{\Lambda}{3}a^3 + A' = c\int dt\frac{d}{d(ct)}\left[a(\dot{a}^2 + 1)\right] = a(\dot{a}^2 + 1) + A'' \tag{2.9}$$

$$\frac{\Lambda}{3}a^3 + (A' - A'') = a(\dot{a}^2 + 1) \tag{2.10}$$

$$\frac{\Lambda}{3}a^3 + A = a(\dot{a}^2 + 1) \tag{2.11}$$

where we have noted that dot represents derivation not with respect to t but ct, also A' and A'' are integration constants.

From the latter Einstein equation, we can read off the constant A:

$$A = \kappa\rho\frac{a^3}{3}. \tag{2.12}$$

Friedmann manipulated the equation we obtained by integrating to find an expression for time. The line of reasoning is as follows. The equation in question can be rewritten as

$$\frac{1}{c^2}\left(\frac{da}{dt}\right)^2 = \frac{A - a + \frac{\Lambda}{3}a^3}{a}. \tag{2.13}$$

Infinitesimal time then is related to infinitesimal scale factor via

$$dt = \frac{1}{c}\sqrt{\frac{a}{A - a + \frac{\Lambda}{3}a^3}}da. \tag{2.14}$$

Integrating,

$$t = \frac{1}{c}\int da\sqrt{\frac{a}{A - a + \frac{\Lambda}{3}a^3}}. \tag{2.15}$$

From this definition he went towards what he called ‘creation time’, time elapsed since the creation of the world. Naturally, the lower bound of the integral should be zero and the upper one the present value of the scale factor:

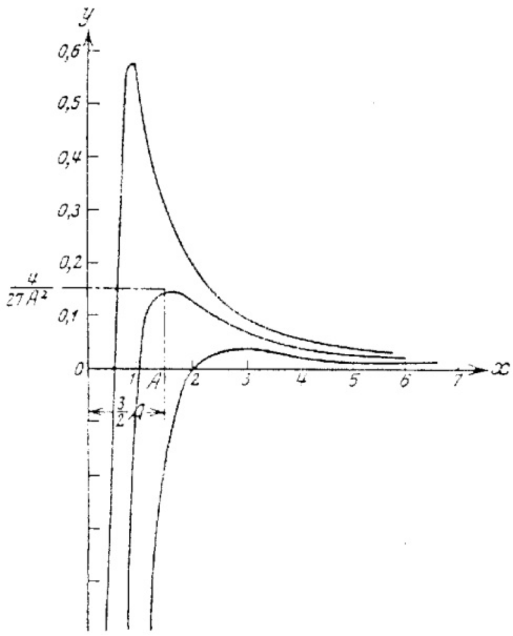

Figure 2.1: The plot of $y = \frac{\Lambda}{3}$ against $x = a$, representing the roots of Eq. (2.17). The middle curve intersects the x-axis at A (figure from [2]).

$$t' = \frac{1}{c} \int_0^{a_0} da \sqrt{\frac{a}{A - a + \frac{\Lambda}{3} a^3}}. \tag{2.16}$$

We will now make some very remarkable observations based on this definition. To better understand the behaviour of the integral, Friedmann investigated the roots of the denominator. Setting $x = a$ and $y = \frac{\Lambda}{3}$, the roots of the denominator,

$$yx^3 - x + A = 0, \tag{2.17}$$

are reflected in Figure 2.1.

Let us stress what this plot tells us physically. We're in a plane whose coordinates are the scale factor ($x = a$) and the cosmological constant ($y = \frac{\Lambda}{3}$), therefore the points constituting the curve are those combinations of the scale factor and the cosmological constant that make the integrand singular- and hence the creation time infinite. On which curve we are is determined by the energy density as it is sits inside A (note that A is the parameter of this family of curves). Now, focus on one of the curves (in order to see the dependence of the physical quantities of interest to A, we'll follow with the generic case $A = A$, that is to say the curve that intersects the x-axis at A).

We observe in Figure 2.2 that when $y = \frac{\Lambda}{3}$ is negative there's only one value of the scale factor that allows for an infinite creation time, when $y \in (0, \frac{4}{27A^2})$ there are two. Finally, for $y > \frac{4}{27A^2}$ there are no roots, so there's no way for the integrand to be singular.

It can be shown that when

$$\Lambda < \frac{\kappa\rho}{2}, \tag{2.18}$$

the creation time is finite. This means that some finite time into the past, there has to be an instant at which the scale factor (that is to say, the size of the universe) is vanishing with non-zero energy content, i.e., a singularity!

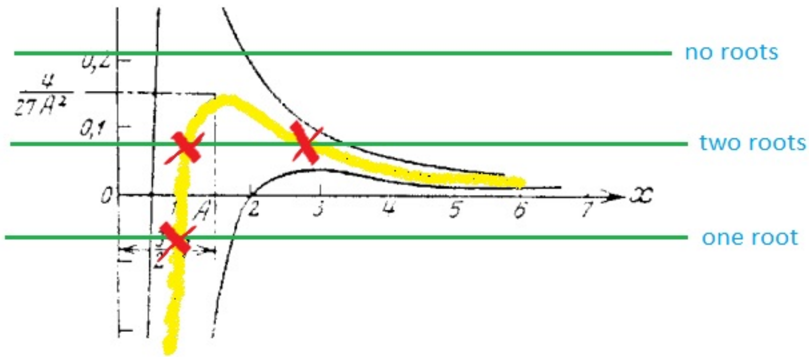

Figure 2.2: In the picture is betrayed the relationship between the interval in which the value of the cosmological constant sits (since $y = \frac{\Lambda}{3}$) and the number of roots of Eq. (2.17).

2.1.2 Lemaître

Friedmann's work, as we saw, concerned the initial singularity from which the world seems to have emerged, Lemaître focused on another kind of singularity- what was then called the Schwarzschild singularity- and endeavoured to resolve the contradiction it appeared to present with respect to Friedmann's conclusions.

The part of Lemaître's 1933 paper [3] that has direct relevance to our treatment is section 11. He begins this section by stating the above mentioned contradiction[1]: "The equations of the Friedmann universe admit, for a non-vanishing mass, solutions where the radius of the universe tends to zero. This seems to be in opposition to the generally held result that a given mass cannot have a radius less than $\frac{2G_N m}{c^2}$."

We remind ourselves that Schwarzschild had worked out the vacuum solution to the Einstein equations in the presence of spherical symmetry and vanishing cosmological constant. The solution reads

$$ds^2 = -\left(1 - \frac{\alpha}{r}\right) c^2 dt^2 + \left(1 - \frac{\alpha}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{2.19}$$

where $\alpha = \frac{2G_N m}{c^2}$. The inclusion of the cosmological constant- as laid out in the paper- furnishes

$$ds^2 = -\left(1 - \frac{\alpha}{r} - \frac{\Lambda}{3}r^2\right) c^2 dt^2 + \left(1 - \frac{\alpha}{r} - \frac{\Lambda}{3}r^2\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{2.20}$$

Lemaître continues[2]: "We propose to show that the singularity of the field isn't real and results simply from what we wanted to employ as coordinates for which the field is static." And he does so. We're not going to present here the calculations undertaken to reach the desired end, let us merely give the outline of what he did. Lemaître started with Eq. (2.20) and proceeded with a change of coordinates arriving at an intermediate

[1]Les équations de l'univers de Friedmann admettent pour une masse non nulle, des solutions où le rayon de l'univers tend vers zéro. Ceci est en opposition avec le résultat généralement admis qu'une masse donnée ne peut avoir un rayon plus petit que $\frac{2Km}{c^2}$.

[2]Nous nous proposons de montrer que la singularité du champ n'est pas réelle et provient simplement de ce qu'on a voulu employer des coordonnées pour lesquelles le champ est statique.

form for the metric. But note that the $r = \alpha$ singularity isn't present unless $\Lambda = 0$, to that end he sent $\Lambda \to 0$ in the expression he found. The metric he obtained was

$$ds^2 = -c^2dt^2 + \frac{\alpha}{r}d\chi^2 + r^2d\Omega^2 \quad \text{with} \quad r = \left[\frac{3}{2}\sqrt{\alpha}\,(t-\chi)\right]^{\frac{2}{3}}, \tag{2.21}$$

where the $r = \alpha$ 'singularity', as one can attest, is eliminated.

2.1.3 Oppenheimer & Snyder

Friedmann's account demonstrated that an expanding spacetime necessitates a singularity in the past. Lemaître then, clarified (that is to say, resolved) the seeming contradiction of Friedmann's results with Schwarzschild's picture of a spherically symmetric mass distribution. Motivated (as they clearly express in the first paragraph of their paper [4]) by recent research on neutron stars- which showed that stars with sufficiently high mass are doomed to undergo ceaseless gravitational collapse- Oppenheimer and Snyder investigated the continued gravitational contraction of a sphere of 'dust' for which- thanks to Lemaître- the $r = \alpha$ singularity is ruled out but the $r = 0$ one still persists. So, in a way, what Oppenheimer and Snyder did was to reverse the situation treated by Friedmann: instead of looking at an initially expanding spacetime (as did Friedmann) they considered an initially contracting (portion of) spacetime and arrived at a singularity, at $r = 0$, in the future (analogous to the past singularity of Friedmann).

The physical object we're interested in is a star at the end of its life. There, the thermonuclear pressure ceases, so we can model the star as a sphere of pressureless dust. Setting $p = 0$ in the energy-momentum expression for a perfect fluid,

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right)U_\mu U_\nu + pg_{\mu\nu}, \tag{2.22}$$

the corresponding energy-momentum is

$$T_{\mu\nu} = \rho(\tau)U_\mu U_\nu, \tag{2.23}$$

where τ is the proper time of the comoving surfaces undergoing gravitational collapse (what 'comoving' means will become clear in a moment). The fact that the density is only a function of τ is due to assuming the homogeneity of the density distribution.

In the absence of thermonuclear reactions, evidently, the star will start to contract.

A spherically symmetric, contracting spacetime (with a homogeneous matter distribution) admits the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric and the exterior of such a spacetime, by Birkhoff's theorem, admits the Schwarzschild one. They ought to match on the surface of the star. This is the guideline we'll follow. The above-mentioned FLRW metric,

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-\kappa r^2} + r^2d\Omega^2\right), \tag{2.24}$$

is really Eq. (2.3) cast in its modern form, where κ measures the constant curvature of the spatial part of the metric and takes the values $\kappa = 0, \pm1$ depending on whether the geometry is flat, open or closed. Actually in the present case (as we talk about contracting spheres) the geometry is closed so we take $\kappa = 1$, Eq. (2.3) is then recovered:

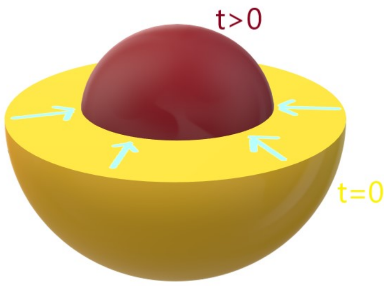

Figure 2.3: A contracting star.

$$ds_{int}^2 = -d\tau^2 + a^2(\tau)\left(d\chi^2 + sin\chi^2 d\Omega^2\right) \quad \text{with} \quad \chi = sin^{-1}r, \tag{2.25}$$

where, like in the rest of this section, we have set $c = 1$. We have seen in the first section that a metric ansatz like that in Eq. (2.25) along with the assumption of dust (as we have here) lead Friedmann to the conclusion that $a = 0$ will be reached in finite time, let us make it all more quantitative now. In proceeding, the business is facilitated if we introduce a conformal time coordinate $\eta \in [0, \pi]$ via $d\eta = d\tau/a$. The effect of the scale factor in the denominator is such that $\eta = 0$ coincides with the zero proper time and thus represents the starting moment of the collapse, $\eta = \pi$ on the other hand points to the end of the collapse. In terms of it, the scale factor and the proper time read

$$a = \frac{a_{max}}{2}(1 + cos\eta), \tag{2.26}$$

$$\tau = \frac{a_{max}}{2}(\eta + sin\eta). \tag{2.27}$$

We make the observation that at $\eta = \pi$, i.e. when the collapse comes to an end, $a = 0$ and $\tau = \frac{\pi a_{max}}{2}$. So each collapsing surface reaches the singularity in (the same) finite proper time. Figure 2.4 is illustrative.

In order to infer what will happen for an outside observer, we shall match the interior and the exterior metrics on the boundary of the collapsing body. Take the outermost surface of the body with initial radius R_0 and proper time-dependent radius $R(\tau)$, corresponding to that surface there should be a particular value of the parameter χ, call it χ_0. Rearranging the terms in the exterior metric, along the surface, the following equality must hold:

$$-d\tau^2 + a^2(\tau)sin^2\chi_0 d\Omega^2 = -\left[\left(1 - \frac{2M}{R}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{2M}{R}\right)^{-1}\left(\frac{dR}{d\tau}\right)^2\right] d\tau^2 + R^2(\tau)d\Omega^2. \tag{2.28}$$

This furnishes two equalities at once:

$$R(\tau) = a(\tau)sin\chi_0, \tag{2.29}$$

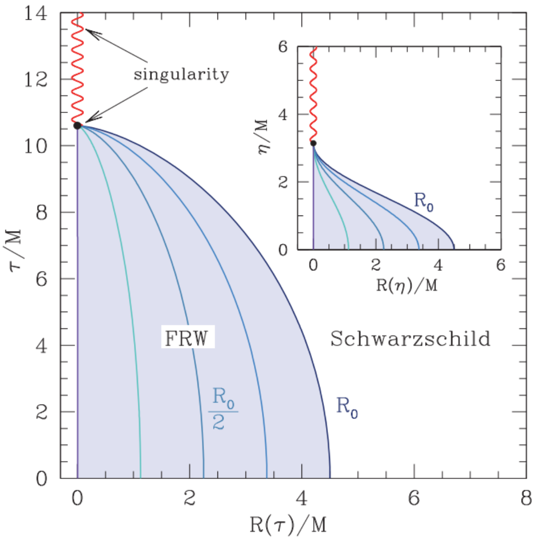

Figure 2.4: The figure shows how each layer of the collapsing star reaches the singularity in the same finite proper time (figure from [5]).

$$1 = \left(1 - \frac{2M}{R}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{2M}{R}\right)^{-1}\left(\frac{dR}{d\tau}\right)^2. \tag{2.30}$$

The interpretation of the former is trivial. The latter can be solved to obtain

$$\dot{t} = \frac{\sqrt{\dot{R}^2 + \left(1 - \frac{2M}{R}\right)}}{1 - \frac{2M}{R}}. \tag{2.31}$$

For R $\sim 2M$, this leads to

$$dt = \frac{dR}{1 - \frac{2M}{R}}. \tag{2.32}$$

And finally, integration gives

$$R(\tau) \approx 2M\left(1 + e^{-\frac{t}{\tau}}\right). \tag{2.33}$$

So, although the surface reaches the singularity in finite proper time, for an outside observer whose time coordinate is the one appearing in the Schwarzschild metric, one has $R \to 2M$ as $t \to \infty$, in accordance with expectation.

Earlier, we found the proper time to reach the singularity to be $\tau = \frac{\pi a_{max}}{2}$. Let us make this explicit by inferring a_{max}. The maximum value the scale factor can have is at the moment the collapse starts and the minimum is when it reaches the Schwarzschild radius. We claim that it is legitimate to express the maximum of the scale factor in the form

$$a_{max} = R_0\left(\frac{R_0}{2M}\right)^{\frac{1}{2}} = \left(\frac{R_0^3}{2M}\right)^{\frac{1}{2}}. \tag{2.34}$$

So in terms of the initial radius and the mass of the collapsing star, the proper time to reach the singularity reads

$$\tau = \frac{\pi}{2}\left(\frac{R_0^3}{2M}\right)^{\frac{1}{2}}. \tag{2.35}$$

2.2 The intermediate step

We have seen in the preceding chapter that an initially expanding spacetime leads to a singularity in the past (Friedmann) and an initially contracting one leads to a singularity in the future (Oppenheimer and Snyder). Following these finds, naturally, an all-encompassing formulation of singularities was called for. The first of such endeavours came in 1955 by Raychaudhuri in his seminal paper [6]. This was to be the bridge to the 1965 theorem.

2.2.1 The Raychaudhuri equation

The key idea of Raychaudhuri's work is geodesic congruence. We consider a metric manifold $(M, \boldsymbol{g})$ and a timelike vector field U^μ,

$$g_{\mu\nu}U^\mu U^\nu = -1, \tag{2.36}$$

which, naturally, is tangent to a congruence of timelike geodesics,

$$\nabla_{\vec{U}}\vec{U} = 0. \tag{2.37}$$

The points along each geodesic can be parametrized by some parameter τ. Furthermore, one could define yet another family of curves by connecting the points along different geodesics that have the same value for the parameter τ. Similarly, these curves will induce a vector field tangent to them, call it X^μ (which will go under the name of displacement vector). Parametrizing the latter by x, we get a coordinate grid (τ, x). By construction, X^μ is invariant under the flow of U^μ, and vice versa. The consequence of this is that their Lie bracket vanishes.

$$0 = [\vec{U}, \vec{X}] = \pounds_{\vec{U}}\vec{X} = \nabla_{\vec{U}}\vec{X} - \nabla_{\vec{X}}\vec{U}. \tag{2.38}$$

At this point we define the gradient velocity tensor:

$$B_{\mu\nu} \equiv \nabla_\mu U_\nu. \tag{2.39}$$

Look at what the vanishing of the Lie bracket implies:

$$0 = (\nabla_{\vec{U}}\vec{X})^\alpha - (\nabla_{\vec{X}}\vec{U})^\alpha \Longrightarrow (\nabla_{\vec{U}}\vec{X})^\alpha = (\nabla_{\vec{X}}\vec{U})^\alpha = X^\mu \nabla_\mu U^\alpha = X^\mu B^\alpha_\mu. \tag{2.40}$$

Therefore, the gradient velocity communicates the failure of the displacement vector to be parallelly transported along the geodesics generated by U^μ.

To see explicitly what is encapsulated in $B_{\mu\nu}$, we decompose it as

$$B_{\mu\nu} = \frac{1}{3}\theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}, \tag{2.41}$$

where

$$h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu \tag{2.42}$$

is the spatial metric. We will now scrutinize what each term purports.

1. **Expansion:**

   The expansion is defined as

   $$\theta \equiv h^{\mu\nu} B_{\mu\nu}. \tag{2.43}$$

   In order to understand the physical meaning of it, take a look at the following equality (which can be shown to hold):

   $$U^\alpha \nabla_\alpha X^\mu = \frac{1}{3}\theta X^\mu. \tag{2.44}$$

   We read off from this equation that the expansion θ is the proportionality factor precisely measuring the rate of change of the displacement vector X^μ along geodesics to which is tangent the vector field U^μ. Note that as spacetime is four dimensional, there are three directions orthogonal to U^μ and hence three vector fields X_i^μ, this is why we have the factor of $\frac{1}{3}$ there. We can therefore say that θ measures how the volume of a body moving along a geodesic in a congruence changes.

2. **Shear:**

   The shear is defined as

   $$\sigma_{\mu\nu} \equiv B_{(\mu\nu)} - \frac{1}{3}\theta h_{\mu\nu}. \tag{2.45}$$

   Obviously it is symmetric, what isn't obvious to see is that it is traceless. So when we diagonalize it, its three eigenvalues add up to zero: $\sum_i \sigma_i = 0$. What this points at physically is that a body centered along a geodesic in the congruence (and traveling along it) will undergo a deformation in its shape with the volume kept unhindered.

3. **Twist:**

   The twist is simply the anti-symmetric part of the gradient velocity tensor:

   $$\omega_{\mu\nu} \equiv B_{[\mu\nu]}. \tag{2.46}$$

   It's no surprise that twist (rotation) is described by an anti-symmetric matrix, indeed, in physics we often encounter quantities related to rotation being described by anti-symmetric matrices. Although a more profound answer could be given, to satiate the curiosity let us point to the fact that the special orthogonal group (to which the concept of rotation is unmistakably related) admits a Lie algebra consisting of anti-symmetric matrices.

   A thing of paramount importance concerning the twist that should be noted is that it spoils hypersurface orthogonality (which is at the heart of most of the analysis we carry out here). It should be intuitive enough why twist should lead to hypersurfaces not being orthogonal any longer, if, on the other hand, a firmer grasp is hankered for, one can consult Frobenius' theorem.

The equation that Raychaudhuri laid down regards the evolution of congruences, in particular, it tells about the time evolution of the expansion θ. Bearing in mind that U^μ is a timelike vector (the curves generated by which are parametrized by τ), it shouldn't be hard to convince oneself that

$$\frac{d\theta}{d\tau} = U^\alpha \nabla_\alpha B^\mu_\mu. \tag{2.47}$$

Before looking at the contraction let us find out what $U^\alpha \nabla_\alpha B_{\mu\nu}$ is:

$$\begin{aligned} U^\alpha \nabla_\alpha B_{\mu\nu} &= U^\alpha \nabla_\alpha \nabla_\mu U_\nu && (2.48)\\ &= U^\alpha \nabla_\mu \nabla_\alpha U_\nu + R_{\alpha\mu\nu}{}^{\beta} U^\alpha U_\beta && (2.49)\\ &= \nabla_\mu \left(U^\alpha \nabla_\alpha U_\nu\right) - (\nabla_\mu U^\alpha)(\nabla_\alpha U_\nu) + R_{\alpha\mu\nu}{}^{\beta} U^\alpha U_\beta && (2.50)\\ &= -B_\mu{}^{\alpha} B_{\alpha\nu} + R_{\alpha\mu\nu}{}^{\beta} U^\alpha U_\beta. && (2.51)\end{aligned}$$

Contraction therefore yields

$$U^\alpha \nabla_\alpha B^\mu_\mu = \frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} - \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}U^\mu U^\nu. \tag{2.52}$$

This is the massive Raychaudhuri equation. It's called massive as timelike geodesics are those along which objects with non-vanishing mass travel. How about the massless case then? Now we consider a null vector field K^μ,

$$g_{\mu\nu} K^\mu K^\nu = 0, \tag{2.53}$$

tangent to a congruence of lightlike geodesics. Observe that, in the present case, the condition that the displacement vector is orthogonal to the vector field along the geodesics,

$$g_{\mu\nu} X^\mu K^\nu = 0, \tag{2.54}$$

does not rule out the possibility that X^μ is proportional to K^μ. What is the consequence of this? To understand this we shall conjure up the dimensions that we have ignored in following with the simplified picture of a 2-grid (the coordinate grid (τ, x) that we evoked in our treatment). At bottom, X^μ lives on the hypersurfaces; owing to the fact that the world we inhabit is (3+1)-dimensional, we are treating 3-dimensional hypersurfaces stacked along the direction of time, consequently, there are (potentially) three linearly independent X^μ (X^μ_i with $i = 1, 2, 3$). Note that, in the null case, the orthogonality condition allows for

$$\vec{X}_1 - \vec{X}_2 = b\vec{K}, \tag{2.55}$$

where b is some constant. So two of the spatial directions are left linearly independent. This means that the vector space orthogonal to the geodesics is 2-dimensional. This is in parallel with what naive SR (special relativity) intuition would lead one to think: as a body speeds up the length along the direction of motion is contracted, in the limit $v \longrightarrow c$ the dimension in question is totally contracted and one is left with two spatial dimensions.

We define the spatial metric and gradient velocity as

$$\hat{h}_{\mu\nu} = g_{\mu\nu} + bK_\mu K_\nu, \tag{2.56}$$

$$\hat{B}_{\mu\nu} = b\nabla_\mu K_\nu, \tag{2.57}$$

where b is a constant that distinguishes elements in the same equivalence class.

The components of gradient velocity are defined analogously to the timelike case. In terms of them we lay down the massless Raychaudhuri equation:

$$\frac{d\hat{\theta}}{d\lambda} = -\frac{1}{2}\hat{\theta}^2 - \hat{\sigma}_{\mu\nu}\hat{\sigma}^{\mu\nu} - \hat{\omega}_{\mu\nu}\hat{\omega}^{\mu\nu} - R_{\mu\nu}K^\mu K^\nu, \tag{2.58}$$

where λ is the parameter along the null geodesics and the factor $1/2$ that replaces the former $1/3$ is due to the contraction of the spatial dimension along the direction of motion we talked about above.

2.2.2 Energy conditions

Ultimately, of course, Raychaudhuri's equation will enable us to make some physical predictions. We observe that all the objects in the Raychaudhuri equation except one are purely geometric objects of our own making, that is to say, we cannot introduce through them any conditions the need for which arises from physical concerns. The physically significant object we're speaking of is $R_{\mu\nu}$, its nature being dictated by the Einstein equation. More precisely, the latter tells us in what way $R_{\mu\nu}$ is related to $T_{\mu\nu}$, so any condtion we might want to impose on the energy-momentum will be echoed in the Ricci curvature (for instance, we might want to keep away from negative energies). In this section, we will see how physical concerns enter into Raychaudhuri's equation.

In the Einstein equation take $\Lambda = 0$,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G_N}{c^4}T_{\mu\nu}. \tag{2.59}$$

Contracting with $Y^\mu Y^\nu$, where Y^μ is a timelike or null vector, we get

$$R_{\mu\nu}Y^\mu Y^\nu = \frac{8\pi G_N}{c^4}T_{\mu\nu}Y^\mu Y^\nu + \frac{1}{2}Rg_{\mu\nu}Y^\mu Y^\nu. \tag{2.60}$$

We'll now contract the Einstein equation itself, this will furnish a relation that enables us to write down the above in a nicer form.

$$\frac{8\pi G_N}{c^4}T_{\mu\nu}g^{\mu\nu} = R_{\mu\nu}g^{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}g^{\mu\nu} \tag{2.61}$$

$$\frac{8\pi G_N}{c^4}T = R - 2R = -R, \tag{2.62}$$

where we have used the fact that $g_{\mu\nu}g^{\mu\nu} = 4$. Plugging this into the above relation,

$$R_{\mu\nu}Y^\mu Y^\nu = \frac{8\pi G_N}{c^4}T_{\mu\nu}Y^\mu Y^\nu + \frac{1}{2}\left(-\frac{8\pi G_N}{c^4}T\right) g_{\mu\nu}Y^\mu Y^\nu \tag{2.63}$$

$$= \frac{8\pi G_N}{c^4}\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right) Y^\mu Y^\nu. \tag{2.64}$$

If Y^μ is timelike we have

$$R_{\mu\nu}Y^\mu Y^\nu = \frac{8\pi G_N}{c^4}\left(T_{\mu\nu}Y^\mu Y^\nu + \frac{1}{2}T\right). \tag{2.65}$$

If it is null, then

$$R_{\mu\nu}Y^\mu Y^\nu = \frac{8\pi G_N}{c^4}T_{\mu\nu}Y^\mu Y^\nu. \tag{2.66}$$

Raychaudhuri worked with a perfect fluid,

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) Y_\mu Y_\nu + p g_{\mu\nu}. \tag{2.67}$$

Following is a list of commonly employed energy conditions:

1. **Dominant energy condition (DEC):** For any timelike or null vector field Y^μ, the flux of energy $J^\mu = T^\mu{}_\nu Y^\nu$ is timelike or null.

2. **Weak energy condition (WEC):** For any timelike vector field U^μ,

$$T_{\mu\nu}U^\mu U^\nu \geq 0. \tag{2.68}$$

3. **Null energy condition (NEC):** For any null vector field K^μ,

$$T_{\mu\nu}K^\mu K^\nu \geq 0. \tag{2.69}$$

4. **Strong energy condition (SEC):** For any timelike vector field U^μ,

$$T_{\mu\nu}U^\mu U^\nu + \frac{1}{2}T \geq 0. \tag{2.70}$$

Using Eq. (2.65) (resp. Eq. (2.66)), the energy conditions that we have laid down can be expressed in geometrical terms. In particular, the SEC for timelike fields and the NEC for lightlike fields both lead to

$$R_{\mu\nu}Y^\mu Y^\nu \geq 0. \tag{2.71}$$

This is called the timelike (resp. null) convergence condition. Why is it called so? Elucidation of this question will be the subject matter of the next section.

2.2.3 The focusing theorem

We begin by evoking the massive Raychaudhuri equation (the result for the massless case will be trivially inferable).

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} - \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}U^\mu U^\nu. \tag{2.72}$$

Noting that the contraction of the symmetric and anti-symmetric bits alike are non-negative, we get

$$\frac{d\theta}{d\tau} + \frac{1}{3}\theta^2 = -\sigma_{\mu\nu}\sigma^{\mu\nu} - \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}U^\mu U^\nu \leq -R_{\mu\nu}U^\mu U^\nu. \tag{2.73}$$

Under the so-called convergence condition then, we have

$$\frac{d\theta}{d\tau} + \frac{1}{3}\theta^2 \leq 0, \tag{2.74}$$

which can be put into the form

$$0 \geq \frac{1}{\theta^2}\left(\frac{d\theta}{d\tau} + \frac{1}{3}\theta^2\right) = -\frac{d\theta^{-1}}{d\tau} + \frac{1}{3}. \tag{2.75}$$

Integrating it,

$$0 \geq \int_0^\tau d\tau' \left(-\frac{d\theta^{-1}}{d\tau'} + \frac{1}{3}\right) \tag{2.76}$$

$$= -\left.\theta^{-1}\right|_0^\tau + \left.\frac{1}{3}\tau'\right|_0^\tau \tag{2.77}$$

$$= -\left(\theta^{-1} - \theta_0^{-1}\right) + \frac{1}{3}\tau, \tag{2.78}$$

we get the bound

$$\frac{1}{\theta} \geq \frac{1}{\theta_0} + \frac{\tau}{3}. \tag{2.79}$$

Having seen how the sign of the expansion relates to the behaviour of congruences, we can now interpret the inequality. It furnishes a critical time $\tau_c = -\frac{3}{\theta_0}$,

$$0 = \frac{1}{\theta_0} + \frac{\tau_c}{3} \Longrightarrow \tau_c = -\frac{3}{\theta_0}, \tag{2.80}$$

that defines the latest possible time before which an initially focusing congruence should collapse to a point. Or, respectively, the earliest possible time in the past after which there should be a point from which emerges what is to become, at time zero, an expanding congruence. In Figure 2.5 is reflected the equal sign in the inequality (where, for the sake of being illustrative, we took $\theta_0 = \pm 1$).

The results of this chapter might be summarized in the following theorem due to Raychaudhuri and Komar which can be regarded as the first singularity theorem (initially, Raychaudhuri's account lacked the imposition of energy conditions; it was Komar who implemented them).

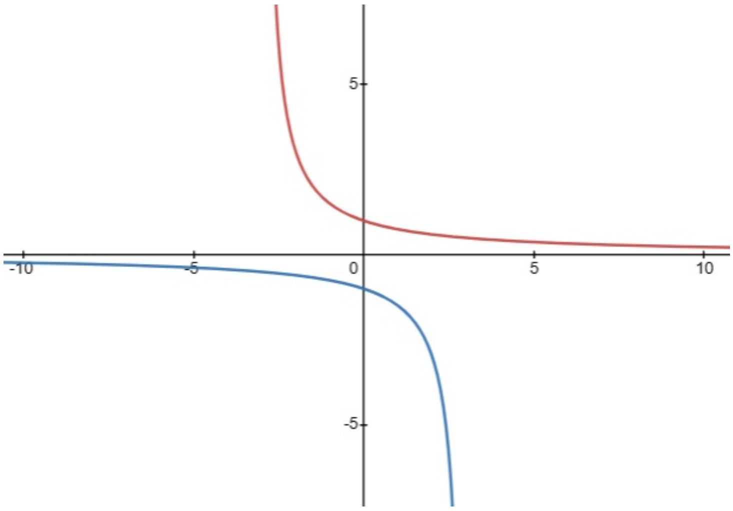

Figure 2.5: The plot of expansion θ against time τ reflecting the equal sign in the inequality. The red curve shows the case with $\theta_0 = 1$, whereas the blue one that with $\theta_0 = -1$

Theorem 1 *Assume $\Lambda = 0$ and a perfect fluid energy-momentum,*

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) U_\mu U_\nu + p g_{\mu\nu} \quad \textit{with} \quad U_\mu U^\mu = -1, \tag{2.81}$$

where U^μ is geodesic and irrotational. If the expansion is positive (resp. negative) at an instant of time and the strong energy condition holds, then the energy density ρ of the fluid diverges in the finite past (resp. future) along every integral curve of U^μ.

An analogous statement can be made for the null case, though we're not going to provide it here.

Chapter 3

The singularity theorems

With the explanation it provides, in geometrical language, regarding the existence of singularities and how the latter relates to the energy content of the universe, the Raychaudhuri-Komar theorem is already quite beautiful. It fails however to capture some crucial situations. Observe, for instance, that the convergence condition is satisfied trivially in the case of flat spacetime but obviously, contrary to what the Rachaudhuri-Komar theorem would tempt us to think, it exhibits no singularities. Some other elements should be taken into consideration in order to avoid such pitfalls. We will find remedy in the works by Penrose et al. We will taste both the bliss of being on firm grounds and the ecstasy of sweet mathematical language.

3.1 Enter Penrose

The point of departure of Penrose's 1965 paper [7] is spherical symmetry, more specifically, its plausible connection to the emergence of singularities: "The question has been raised as to whether this singularity is, in fact, simply a property of the high symmetry assumed." Accordingly, his discussion revolves around collapse "without assumptions of symmetry". One might be curious about why Penrose had suspected the singularities to be resulting from spherical symmetry. This is because in Newtonian gravity the singularity at $r = 0$ created by the collapse of a spherically symmetric mass distribution can be evaded by "perturbing away from spherical symmetry" (see p. 211 of [10]). Before proceeding into the corpus of the treatment, he briefly mentions the conclusion reached: "It will be shown that, after a certain critical condition has been fulfilled, deviations from spherical symmetry cannot prevent spacetime singularities from arising." It is pointed out that if one insist on the impossibility of the existence of a singularity, then one needs to come to terms with one of the following:

- Negative local energy occurs.
- Einstein's equations are violated.
- The spacetime manifold is incomplete.
- The concept of spacetime loses its meaning at very high curvatures (possibly due to quantum phenomena).

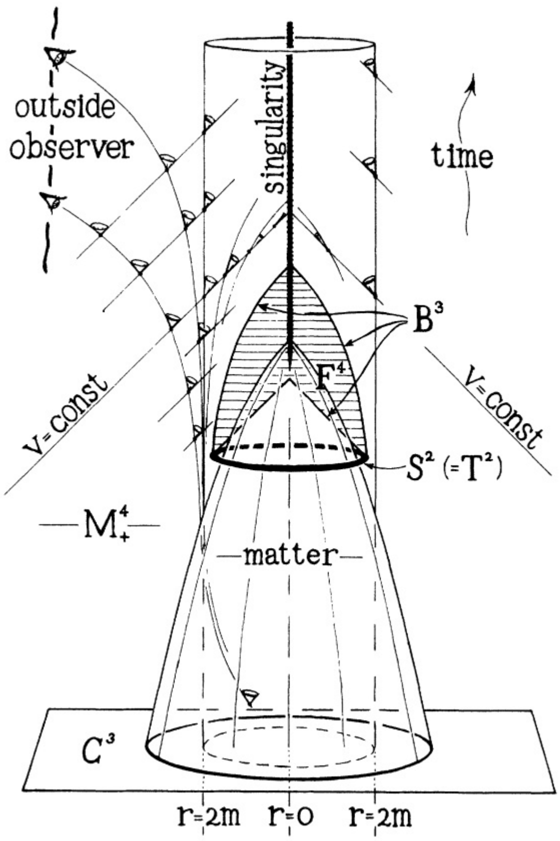

Figure 3.1: Figure from Penrose's 1965 paper [7] elucidating the spacetime geometry of gravitational collapse. Explanation for it is found in the text that follows.

Ironically, Penrose's treatment is based on a spherically symmetric model. It is asserted, however, that the conclusions should hold the same in the asymmetrical case alike. The analysis will be carried out through Figure 3.1, where one spatial dimension is suppressed.

C^3 is the spherical body undergoing gravitational collapse. If we look at the light cones, we see that light (or anything) emanated from those that reside within the critical value of the radius fails to reach an outside observer, in other words, it is trapped. From this we forge the concept of a trapped surface. Penrose's definition is as follows:

Definition: A *trapped surface* is a closed, spacelike 2-surface with the property that a system of null geodesics meeting it orthogonally converge locally in future directions.

We will talk about trapped surfaces in more detail later on. What is done in the remainder of the paper is, essentially, to demonstrate that the existence of a trapped surface necessitates the existence of a singularity.

In the figure is illustrated one of such trapped surfaces, S^2. The triangular cone on top of it represents the null geodesics meeting S^2 and converging locally in future directions, as stated in the definition. The timelike geodesics passing through S^2 on the other hand, will be in the shaded 4-dimensional volume F^4, its boundary being B^3.

Now let A be the area of a given layer of the spherical body under collapse. How does A evolve along a geodesic? Considering a null geodesic, let λ be the parameter along it and K^μ the tangent vector that generates the curve (geodesic). As we have seen during the discussion of the Raychaudhuri equation, the evolution of some quantity Q along the geodesic may be expressed as

$$\frac{dQ}{d\lambda} = K^\mu \nabla_\mu Q. \tag{3.1}$$

The second derivative likewise reads

$$\frac{d^2Q}{d\lambda^2} = \frac{d}{d\lambda}\left(\frac{dQ}{d\lambda}\right) = K^\nu \nabla_\nu \left(K^\mu \nabla_\mu Q\right). \tag{3.2}$$

Now consider

$$\frac{d^2A}{d\lambda^2} = K^\nu \nabla_\nu \left(K^\mu \nabla_\mu A\right), \tag{3.3}$$

why this object is of interest to us will become apparent in a minute. It can be shown that

$$\frac{d^2A}{d\lambda^2} = -A\left(\frac{1}{2}\hat{\theta}^2 + \hat{\sigma}_{\mu\nu}\hat{\sigma}^{\mu\nu} + R_{\mu\nu}K^\mu K^\nu\right), \tag{3.4}$$

where we have employed the massless Raychaudhuri equation (see Eq. 2.58) and set the antisymmetric term equal to zero in order to spare hypersurface orthogonality as the latter will be a requirement in Penrose's theorem (recall that twist spoils hypersurface orthogonality).

The first two terms inside the parenthesis are positive, then the convergence condition guarantees us that

$$\frac{d^2A}{d\lambda^2} \leq 0. \tag{3.5}$$

This means that the function in question is concave down, so the area A inevitably reaches the value zero within a finite value of the parameter λ, ergo a singularity.

A geodesic (such as the one we treated in the above analysis) that cannot be extended beyond a certain value of the affine parameter is said to be incomplete. *Geodesic incompleteness* is how one refers to a singularity in modern language. Indeed, we now lay down Penrose's version of the singularity theorem which is expressed in terms of this new notion [7]:

Theorem 2 *If the spacetime contains a non-compact Cauchy hypersurface* C^3 *and a closed future-trapped surface, and if the convergence condition holds for null* K^μ *then there are future incomplete null geodesics.*

3.2 Trapped surfaces

As they are of central importance in modern singularity theorems, we find it worthwhile at this point to utter a few more words on trapped surfaces. We begin by pointing out more explicitly what it is that A measures. A is the area of a given contracting 2-sphere;

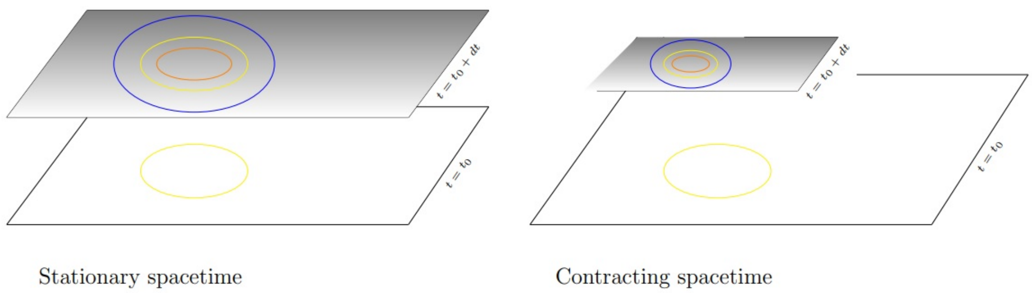

Figure 3.2: One spatial dimension being suppressed, the yellow circle represents the sphere from which the in-going (in orange) and out-going (in blue) signals are emitted. For a stationary spacetime the behaviour is as would be expected, for a contracting one if the contraction is fierce enough (as would be in the case of a trapped surface) the out-going wavefront (as well as, of course, the in-going one) remains always inside the sphere from which the signal is emanated in the first place (figure from [1]).

a layer, that is to say, of the spherical body undergoing gravitational collapse. Naturally, it is given by

$$A = \int \sqrt{g_{\theta\theta} g_{\phi\phi}} d\theta d\phi. \tag{3.6}$$

Hearkening back to the objects appearing in the Raychaudhuri equation, the evolution of A is precisely what the expansion θ introduced there communicates. Take a spherical layer and imagine two light rays one directed outwards and the other inwards, let the corresponding tangent vectors be $K^\mu_\pm$. The expansions $\theta_\pm \equiv \nabla_\mu K^\mu_\pm$ then reflect the rates of the change of the volume of the two spheres the boundaries of which are the respective wavefronts of these two rays. In this language, a trapped surface then is a surface for which the expansions $\theta_\pm$ are negative so that the wavefronts lie always within the the original layer from which the signals were emanated. See Figure 3.2 for a firmer understanding [1].

Observe that we already arrived at this result in the context of the focusing theorem (see Figure 2.5). We have discussed here (the name is self-explicatory) a future-trapped surface, an analogous definition can be made for a past-trapped surface through the demand that the expansions be positive instead of negative (this case is represented by the other curve in Figure 2.5).

One might wonder what the limiting case here is. That is to say, at what point do we go from a surface that is not trapped to one that is? Such a surface would naturally be the outermost of all trapped surfaces, it is quite appropriately called a *marginally outer trapped surface (MOTS)*. It shouldn't be surprising to learn that for a MOTS the expansion vanishes identically.

3.3 Enter Hawking

The following year saw the appearance of a variety of singularity theorems (see pp. 2-3 of [8]), or more justly, various improvements of the 1965 result. In 1969 Hawking and Penrose wrote a paper [8] together (published in 1970) establishing a more generalized form of the singularity theorem, incorporating the works of the past few years (carried out for the most part by Hawking himself).

They followed with two fundamental assumptions:

1. There are no closed timelike curves.

2. The SEC holds.

Along with these, there's also the assumption of the so-called generic (or generality) condition which actually is an important element in the theorem. We will talk about it in due time. The first one is a mere causal statement. What's demanded is that a traveller (to which is associated a timelike curve) cannot turn back to the point where they started from, excluding the possibility of time travel. We're already familiar with the second condition.

Let us go directly into the theorem [8].

Theorem 3 *No spacetime M can satisfy all of the following three requirements together:*

1. M contains no closed timelike curves.

2. Every inextendible causal geodesic in M contains a pair of conjugate points.

3. There exists a future- (or past-) trapped set $S \subset M$.

We shall talk about each of these conditions one by one.

1. *The first condition*

   We already talked about what a closed timelike curve means.

2. *The second condition*

   For the second condition, we start with the definition of "inextendible" [10] (we give the definition for a future-inextendible curve, it should be easy enough to see how the arguments apply in the past case).

   Definition: A curve λ is said to be *future-inextendible* if it has no future endpoint.

   This calls for the following definition:

   Definition: We say that $p \in M$ is a *future endpoint* of λ if for every neigbourhood O of p, there exists a t_0 such that $\lambda(t) \in O$ for all $t > t_0$.

   The lesson to draw here is that an inextendible curve might be a curve that *i)* "runs off to infinity", *ii)* "runs around and around forever" or *iii)* "runs into a singularity". It cannot, for instance, be a curve that stops at a point because it isn't defined to go further [10]. So whatever the statement of the second condition will be, it will be made on causal geodesics that either "run off to infinity" or "run into a singularity" (as option *ii)* is excluded due to the first condition).

Next comes the notion of conjugate points. For a profound understanding we encourage the reader to look at section 9.3 of [10]. We will content ourselves with laying down a proposition that suffices to make our point (this is proposition 9.3.2 of [10]).

Proposition: Let a manifold M satisfy the timelike generic condition, $R_{\mu\nu\rho\sigma}U^\rho U^\sigma \neq 0$ at at least one point, and suppose the SEC holds for all timelike U^μ. Then every complete timelike geodesic contains a pair of conjugate points.

Observe that the the difference between the two options (namely "run off to infinity" and "run into a singularity") we have pointed out for an inxtendible curve can be precisely qualified through the notion of geodesic completeness we had introduced earlier. The above proposition then, demands us to select, among the inextendible curves, those that are complete, i.e., the ones that "run off to infinity". Thus, the second condition can be rephrased as:

All causal geodesics are complete in M.

The contrary of this statement would be:

There exists an incomplete causal geodesic in M.

We will need this latter in a short while.

3. *The third condition*

To understand what the last one purports, we obviously need to know what a trapped set is.

Definition: A *future-trapped (resp. past-trapped) set* is a non-empty achronal closed set for which $E^+(S)$ (resp. $E^-(S)$) is compact.

First, what is an achronal set?

Definition: A set is *achronal* if it contains no pair of points which can be linked with a timelike curve such that one of the points constitutes the past endpoint and the other the future endpoint.

So, in an achronal set, there can be found no two spacetime points such that one lies either to the past or to the future of the other. But this is what happens with a spacelike surface, which the Cauchy hypersurfaces referred to in Theorem 2 are. To get a better physical grasp, one can refer to Figure 3.1. The trapped surface S^2 there, representing a layer of the collapsing body within the critical radius at some instant of time, might constitute an example of our S here. Let us move on to $E^+(S)$.

$E^+(S)$ is defined to be

$$E^+(S) \equiv J^+(S) - I^+(S), \tag{3.7}$$

where $J^+(S)$ is the set of points in the future of S that can be linked with the points of S by a causal curve (so either timelike or null), whereas $I^+(S)$ is the set of points

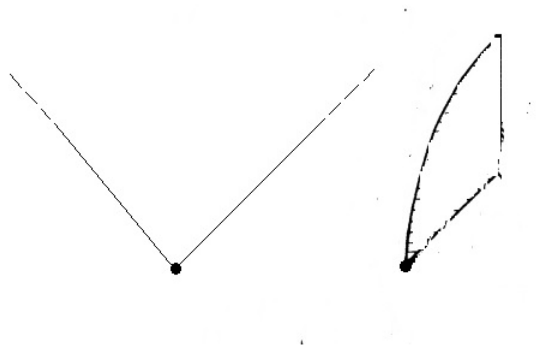

Figure 3.3: Light cones in flat (left) and contracting (right) spacetime geometries are contrasted.

that can be linked with a timelike curve (see pp. 5-6 of [8]). In Figure 3.1, one can make the identifications: $J^+(S) = F^4 \cup B^3$ & $I^+(S) = F^4$. This leads us to $E^+(S) = B^3$, which is obviously compact. Thus a trapped surface is an example of a trapped set.

Let us interpret this theorem. It is obvious that the first condition is absolutely indispensable, as otherwise we're led to nightmarish causal paradoxes. Then if the third condition is satisfied for a spacetime (we saw that an example is a trapped surface, we'll see the other prospects in a little while), then the second condition cannot possibly hold. In other words, geodesic completeness cannot be in place.

This is a good point to stop and ponder about what link there is between the compactness of $E^+(S)$ and singularities (as the latter is implied by geodesic incompleteness). As we can view $E^+(S)$ as a subset of Euclidean space, its compactness purports closedness and boundedness. It is the latter that will enable us to establish the above-mentioned link. Consider again Figure 3.1, $E^+(S) = B^3$ is really the boundary of the union of light cones emanated from S^2. We now look at Figure 3.3, taking one of such light cones and contrasting it with a light cone in flat geometry, we observe that geodesic incompleteness comes into play when the area subtended by the light cone is bounded.

As we are readily confident about the link between geodesic incompleteness and existence of singularities, it might be helpful to see the theorem stated in terms of geodesic incompleteness. To that end, in order to understand the physical meaning of the singularity theorem by Hawking and Penrose, it is, I believe, a better idea to look at what they lay out as a corollary of their theorem (as it is more overtly instructive).

Theorem 4 *A spacetime M cannot satisfy causal geodesic completeness if, together with Einstein's equations, the following four conditions hold:*

1. *M contains no closed timelike curves.*
2. *The SEC is satisfied at every point.*
3. *The generality condition, $Y_{[\mu}R_{\nu]\rho\sigma[\gamma}Y_{\eta]}Y^{\rho}Y^{\sigma} = 0$, is satisfied for every causal geodesic.*
4. *M contains either*
   - *a trapped surface,*

   or
   - *a point p for which the expansion of all the null geodesics through p changes sign to the past of p,*

   or
   - *a compact spacelike hypersurface.*

The so-called generality condition serves to avoid some exceptional situations. We will not go into any further detail regarding its nature. We have explained above how a trapped surface (note that we have treated the case of a future-trapped surface as we looked at gravitational collapse, yet, it should be trivial enough to see that the above analysis could have been done for a past-trapped surface, which, as we know, has to do with the initial Big Bang singularity) is an example of a trapped set, there are some other species, however, that also do the job (appearing in the fourth condition). On the matter of the other two alternatives that make up an example of a trapped set, curious reader is encouraged to look at pp. 15-16 of [8].

Chapter 4

Kerr's perspective on singularities

4.1 Outline of Kerr's paper

4.1.1 History

Kerr starts by outlining the history of singularity theorems. It is seen that the point of attack will be to demonstrate that the equivalence of FALLs (light rays of finite affine length) to singularities (which seems to be taken for granted in the singularity theorems) is fallacious. He briefly explains how his treatment will be able to show this point.

4.1.2 Affine parameters

It is uttered that the reason for the above-mentioned fallacy results from confusing geodesic distance with affine distance, he says that this is "the crux of the argument".

Geodesic distance s is defined through

$$\frac{ds}{dt} = \sqrt{g_{\mu\nu} dx^\mu dx^\nu}. \tag{4.1}$$

This definition does not work for light rays, $ds^2 = 0$, and we need to replace it with affine distance a satisfying a second order equation:

$$\frac{d^2 x^\mu}{da^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{da} \frac{dx^\beta}{da} = 0. \tag{4.2}$$

He then shows how this equation leads to the boundedness of the affine parameter which is the key ingredient of singularity theorems. As this conclusion is reached without the introduction of any "physics", he concludes that the result has nothing to do with the physical concept of a singularity.

4.1.3 Shwarzschild and Eddington

Kerr calculates the two families of PNVs (principal null vectors) for the Eddington-Finkelstein (EF) extension of the Schwarzschild metric. He makes the observation that "The second set of PNVs are asymptotic to the event horizon as $t_- \to -\infty$ for a BH [black hole] and as $t_+ \to +\infty$ for a WH [white hole]. In both cases the affine parameter r is necessarily bounded as the PNV approaches the appropriate horizon". He points out that this constitutes a contradiction to the basic argument that all singularity theorems are based on.

4.1.4 The Kruskal extension of Schwarzschild

In this section, the author shows that the Kruskal metric is an analytic extension of Eddington, rather than Schwarzschild. He asserts that Kruskal has no physical significance. He argues that that the infinite curvature at the centre of Eddington (to which Kruskal is an extension) has to do with the claimed physics of the star and that the affine parameters are irrelevant.

4.1.5 The Kerr metric

The Kerr metric exhibits a ring singularity. In a sense, it is this singularity that "generates" the Kerr metric. The author points out the possibility of replacing this singularity by an actual rotating body like a neutron star.

When the PNVs are analyzed, it is seen that the fast geodesics continue straight through both horizons but the slow ones are asymptotic to the one or the other horizon depending on the limit considered. Indeed, one can calculate the affine length of the slow geodesic between the horizons, we get a finite quantity: $2\sqrt{m^2 - a^2}$. This is a demonstration of how FALLs do not necessarily imply actual singularities. A similar conclusion is reached for the fast geodesics as well since they become slow geodesics on the other side of the star, thus having finite affine distance due to the arguments presented above.

4.1.6 Conclusion

To summarize, what has been shown is that boundedness of affine parameters has nothing to do with singularities. To be more explicit, it has been shown that FALLs can exist asymptotic to event horizons; the assertion that FALLs lead to singularities is therefore false.

4.2 The significance of Kerr's ideas

Kerr's paper opens with the definition of a singularity in his own terms: "a region or place where the metric or curvature tensor is either unbounded or not suitably differentiable". Then, he lays down his thesis: "the existence of a FALL by itself is not an example of this". In effect, what he will be doing in the remainder is merely to provide examples to support this thesis.

Let us investigate how he attacks the singularity theorems. We recall that, essentially, what the singularity theorems imply is that the existence of a trapped surface leads to FALLs, which is how singularities are referred to in the language of these theorems. Kerr points out to a link taken for granted in this reasoning that, as far as his argument goes, isn't on firm grounds: "It was then decreed, without proof, that these [FALLs] must end in actual points where the metric is singular in some unspecified way." It is true that there is no such proof but one must mention that "the founding fathers" of these theorems were well aware of this drawback. Penrose and Hawking, in their 1970 paper [8] state:

“However, one cannot conclude, on the basis of the corollary, that such a singularity need necessarily be of the ‘infinite curvature’ type.” Nonetheless, this didn’t hold back those who embrace the idea of singularities to interpret the theorems such that the singularity implied is of the ‘infinite curvature’ type. We believe that examples furnished by Kerr will convince the reader that such a link is not altogether obvious, and that it is, most probably, fallacious.

In the appendix, we discussed in detail the cases that constitute counter-examples to the above-mentioned interpretation of the singularity theorems. We, therefore, do not feel the need to expose them here.

4.3 Recalculating Kerr’s results

As Kerr’s paper is of fundamental importance for our study we wanted to be reassured of the results that he provides. To that end, we ventured at a recalculation of the essential points that he either arrives at or asserts without giving any proof. We gathered them all in the appendix. Here’s a list of the questions addressed:

1. Boundedness of the affine parameter and how it has no real link to singularities
2. The two families of PNVs of the EF extension of the Schwarzschild metric
3. Discussion regarding the Kruskal extension
4. Showing that k in (10a) is a PNV
5. Showing that r is an affine parameter for the Kerr metric
6. PNVs tangent to the central ring in the Kerr metric
7. PNVs asymptotic to the event horizons in the Kerr metric
8. Incomplete geodesics in Kerr spacetime that do not end up at singularities

Also, in the very beginning of the appendix, we laid down the Christoffel symbols that we computed by hand, which served us in A.5.

It is important to note that in hassling with these calculations we came across three typos in the paper. They are to be found in A.2, A.4 and A.5, respectively.

Chapter 5

Extending the analysis to regular metrics

Now, we will look at some counter-examples ourselves. We find it worthwhile to investigate regular metrics which - by definition - do not harbour any singularities, to see if geodesic incompleteness is exhibited. If we manage to find such examples, then we're on the same lines as Kerr.

5.1 Singularity-free gravitational collapse

Below, we expose the idea of regular metrics as they are introduced in [12].

5.1.1 Singularity regularization in effective geometries

Penrose's theeorem relies on the following assumptions:

1. The WEC is staisfied.
2. The field equations hold.
3. Global hyperbolicity holds.
4. Pseudo-Riemannian geometry provides an adequate description of spacetime.

The framework of the paper consists of hypothesizing that $\{3{,}4\}$ hold but that $\{1{,}2\}$ shall be relaxed.

More explicitly, the geometries that will be discussed satisfy the following properties:

- Global hyperbolicity.
- Geodesic completeness.
- Asymptotic flatness.
- Existence of a closed trapped surface.
- Finiteness of curvature invariants.

Such geometries must violate at least one of {1,2}.

The classification of these geometries will come through the expansion,

$$\theta^{(X)} = \frac{1}{\sqrt{h}}\mathcal{L}_X\sqrt{h} = h^{ab}\nabla_a X_b, \quad X \in \{l, k\} \tag{5.1}$$

where h_{ab} is the induced metric on S^2 and l & k stand for outgoing and ingoing geodesics respectively.

A *trapped surface* is defined by

$$\theta^{(k)} < 0 \quad \& \quad \theta^{(l)} < 0, \tag{5.2}$$

whereas a *focusing point* is characterized by $\theta^{(l)} \to -\infty$.

What one needs to do in order to make the spacetime geodesically complete is to modify its geometry in the vicinity of the focusing point. This will be done either by creating a defocusing point or by displacing the focusing point to infinite affine distance.

This produces four regular classes in the spherically symmetric case:

1. **Evansecent horizon:** $\theta^{(l)}$ changes sign, $\theta^{(k)}$ stays negative. The inner and outer horizons merge in finite time.

2. **Hidden wormhole:** The singularity is changed by a (global or local) minimum radius hypersurface. Both expansions change signs.

3. **Everlasting horizon:** The limiting case of **1**, the two horizons never merge.

4. **Asymptotic hidden wormhole:** The limiting case of **2**, the minimum radius hypersurface is reached in infinite affine time.

Classes {1,3} are simply connected, {2,4} are non-simply connected.

5.1.2 Geodesically complete alternatives to static black holes

One works with the spherical metric

$$ds^2 = -F(r)dt^2 + F(r)^{-1}dr^2 + \rho^2(r)d\Omega^2 \tag{5.3}$$

in the EF form:

$$ds^2 = -F(r)dv^2 + 2dvdr + \rho^2(r)d\Omega^2. \tag{5.4}$$

The line element has a trapping horizon at $r = r_H$ whenever $F(r_H) = 0$. The expansions read

$$\theta^{(l)} = 2F(r)\frac{\partial_r\rho(r)}{\rho(r)} \quad \& \quad \theta^{(k)} = -2\frac{\partial_r\rho(r)}{\rho(r)}. \tag{5.5}$$

Note that in order to avoid curvature singularities either $F(r)$ has an even number of zeros (leading to a symmetric horizon structure that encloses a regular core) or $\rho(r)$ has a minimum (ensuring a bounce rather than a collapse).

The Schwarzschild metric satisfies neither of these, however, the following deformation of it satisfies both:

$$F(r) = 1 - \frac{2M\rho^2(r)}{\rho^3(r) + 2Ml_1^2} \quad \& \quad \rho^2(r) = r^2 + l_2^2. \tag{5.6}$$

The geometries suggested by this construction is such that

- $l_1 \neq 0 \Leftrightarrow$ simply connected $\Leftrightarrow$ introduction of an inner horizon,
- $l_2 \neq 0 \Leftrightarrow$ non-simply connected $\Leftrightarrow$ introduction of a wormhole throat.

Whether or not there's an outer horizon depends on the value of l_1 with respect to M in the former case and the value of l_2 with respect to M in the latter case. There's also the case of "mixed" geometries. Each scenario is treated in detail in the paper, I won't note them down. In particular, some scenarios resemble those resulting from the RN metric, we will evoke them when we discuss the latter later on.

5.1.3 Dynamical geometries

The static nature of the solutions explored, besides being non-physical, causes further problems such as the existence of Cauchy horizons, that are generically not present in time-dependent situations.

These geometries describe the formation of inner/outer horizons and wormhole throats that evolve dynamically. Possible dynamical behaviours are treated in this section, we will not mention them here.

5.2 Hayward's metric

We will concentrate on one of the species of spacetimes we considered above, those that are simply connected. The metric that describes them is due to Hayward.

In his 2006 paper [13] he introduces spacetimes "which describe the formation of a (locally defined) black hole from an initial vacuum region, its quiessence as a static region, and its subsequent evaporation to a vacuum region". In particular, he provides models for "the collapse and evaportaion phases, using Vaidya-like regions with ingoing or outgoing radiation".

In this picture, one considers the spherically symmetric metric

$$ds^2 = r^2 d\Omega^2 + \frac{dr^2}{F(r)} - F(r)dt^2, \tag{5.7}$$

with $F(r)$ determining the gravitational trapping as illustrated in Figure 5.1.

He introduces a *minimal model* for $F(r)$:

$$F(r) = 1 - \frac{2mr^2}{r^3 + 2l^2m}. \tag{5.8}$$

We encountered this form before, in the discussion of regular black holes. This produces three cases, depending on the respective values of l and m: *i)* a regular spacetime with the same causal structure as flat spacetime *ii)* a regular extreme black hole with degenerate Killing horizon *iii)* regular non-extreme black hole with both outer and inner Killing horizons.

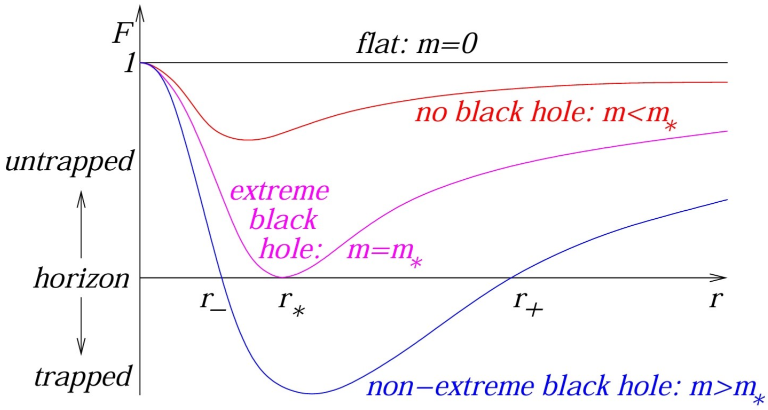

Figure 5.1: $F(r)$ determines the trapping horizons (figure from [13]).

He then expresses the metric in advanced time,

$$ds^2 = r^2 d\Omega^2 + 2dvdr - Fdv^2 \quad \text{with} \quad v = t + \int \frac{dr}{F(r)}. \tag{5.9}$$

With this modification, most of the energy-momentum tensor components retain their form, there is now, however, an additional independent component, corresponding to radially ingoing energy flux.

$$G_v^r = \frac{2r^4 m'}{(r^3 + 2l^2 m)^2}. \tag{5.10}$$

Hayward remarks that this describes pure radiation, recovering the Vaidya solutions at large radius for $l = 0$. There is, nonetheless, a key difference: "in the Vaidya solutions, the ingoing radiation creates a central singularity, but in these models, the centre remains regular".

Noting that for a metric g in the form we laid out above, the Einstein tensor has the cosmological-constant form,

$$G \sim -\Lambda g \quad \text{as} \quad r \to 0 \quad \text{with} \quad \Lambda = 3/l^2, \tag{5.11}$$

this amounts to saying that the effective cosmological constant protects the core.

In the paper a model for the formation and evaporation of a static black hole region is provided, where one can observe the appearance and disappearance of trapping horizons, marked by the transition times. Without going into the details of this model, we will content ourselves with saying that "these horizons join smoothly at the transitions and therefore unite as a single smooth trapping horizon enclosing a compact region of trapped surfaces".

He also models the outgoing Hawking radiation, the idea relies on pair creation: "pair creation of ingoing particles with negative energy and outgoing particles with positive energy, locally conserving energy".

The Penrose diagram for Hayward's picture (given in Figure 5.2) is quite illustrative; where, as can be attested trivially, there are no singularities, and no event horizon.

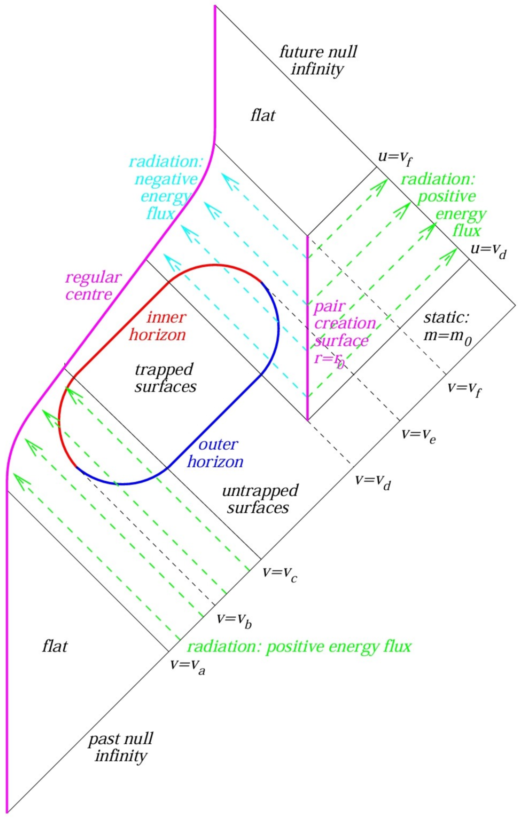

Figure 5.2: Penrose diagram of formation and evaporation of a regular black hole in the given models (figure from [13]).

To conclude, we quote here an entire paragraph from the end of the paper, which we feel is quite elucidative: "A trapping horizon with both inner and outer sections typically develops in numerical simulations of binary black-hole coalescence, in analytical examples of gravitational collapse such as Oppenheimer-Snyder collapse and according to general arguments. A key point here is that the inner horizon never reaches the centre, where a singularity would form. This is compatible with the classical singularity theorems, which make assumptions that are already not satisfied by a Bardeen black hole, such as the strong energy condition. The negative-energy nature of ingoing Hawking radiation shows that such theorems do not apply to a black hole that might someday begin to evaporate."

5.3 Null geodesics in Hayward spacetime

The Hayward metric is

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2 d\Omega^2, \tag{5.12}$$

where

$$F(r) = 1 - \frac{2mr^2}{r^3 + 2l^2 m}. \tag{5.13}$$

The Lagrangian, given by

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu, \tag{5.14}$$

therefore reads

$$\mathcal{L} = \frac{1}{2} \left(-F(r)\dot{t}^2 + \frac{\dot{t}^2}{F(r)} + r^2\dot{\theta}^2 + r^2 \sin^2\theta \dot{\phi}^2 \right). \tag{5.15}$$

In this setting, we will analyze the null geodesics in Hayward spacetime with $L = 0$ and $L \neq 0$.

5.3.1 The case with $L = 0$

We begin by noting that for radial motion the Lagrangian is reduced to

$$\mathcal{L} = \frac{1}{2} \left(-F(r)\dot{t}^2 + \frac{\dot{t}^2}{F(r)} \right). \tag{5.16}$$

Since the metric is stationary, there is a conserved quantity, the energy E, associated with the Killing vector ∂_t:

$$p_t = \frac{\partial \mathcal{L}}{\partial \dot{t}} = -F(r)\dot{t}. \tag{5.17}$$

Defining $E := -p_t$,

$$\dot{t} = \frac{E}{F(r)}. \tag{5.18}$$

Imposing the null condition, $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0$, and using this identity, we get:

$$-\frac{E^2}{F(r)} + \frac{\dot{r}^2}{F(r)} = 0. \tag{5.19}$$

This gives:

$$\dot{r} = \pm E. \tag{5.20}$$

Thus, the radial coordinate r changes linearly with the affine parameter λ, the minus sign corresponding to ingoing and the plus sign corresponding to outgoing radial geodesics.

To determine whether the radial null geodesic reaches $r = 0$ in finite affine length, we integrate:

$$\Delta\lambda = \int d\lambda = \pm \int_{r_0}^{0} \frac{dr}{E} = \mp \frac{r_0}{E}. \tag{5.21}$$

The fact that the affine length of the trajectory is finite indicates that a pathology might be exhibited.

One needs to check the behaviour of the Kretschmann scalar,

$$K = R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho}, \tag{5.22}$$

to see if there's a singularity of infinite curvature type.

The Kretschmann scalar for both Schwarzschild and Hayward can be found in literature [14]:

$$K_{Schw} = \frac{48m^2}{r^6}. \tag{5.23}$$

$$K_{Hay} = \frac{48m^2\left(r^{12} - 8l^2mr^9 + 72l^4m^2r^6 - 16l^9m^3r^3 + 32l^8m^4\right)}{\left(r^3 + 2ml^2\right)^6}. \tag{5.24}$$

Note that if we set $l = 0$ in the latter, we recover the former. We observe that as $r \to 0$ this scalar diverges for Schwarzschild, whereas it remains finite for Hayward: $K_{Hay} = \frac{24}{l^4}$.

5.3.2 The case with $L \neq 0$

In dealing with the case $L \neq 0$, we will restrict the motion to the equatorial plane ($\theta = \frac{\pi}{2}$), so the Lagrangian reads:

$$\mathcal{L} = \frac{1}{2}\left(-F(r)\dot{t}^2 + \frac{\dot{r}^2}{F(r)} + r^2\dot{\phi}^2\right). \tag{5.25}$$

There are two conserved quantities:

$$\text{Energy:} \quad \frac{\partial \mathcal{L}}{\partial \dot{t}} = -F(r)\dot{t} = -E \Longrightarrow \dot{t} = \frac{E}{F(r)}, \tag{5.26}$$

$$\text{Angular momentum:} \quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = r^2\dot{\phi} = L \Longrightarrow \dot{\phi} = \frac{L}{r^2}. \tag{5.27}$$

Substituting these into the Lagrangian and imposing the null condition:

$$-E^2 + \dot{r}^2 + F(r)\frac{L^2}{r^2} = 0. \tag{5.28}$$

This can be put into the following form:

$$\dot{r}^2 = E^2 - V_{eff}(r) \quad \text{with} \quad V_{eff} = F(r)\frac{L^2}{r^2}. \tag{5.29}$$

We can now analyze the behaviour of the affine parameter λ. From the above relation:

$$\dot{r}^2 = E^2 - F(r)\frac{L^2}{r^2} \Longrightarrow \frac{dr}{d\lambda} = \sqrt{E^2 - F(r)\frac{L^2}{r^2}}. \tag{5.30}$$

The affine distance is therefore given by:

$$\Delta\lambda = \int_{r_1}^{r_2} \frac{dr}{\sqrt{E^2 - F(r)\frac{L^2}{r^2}}}. \tag{5.31}$$

If the result is finite as $r \to 0$, the null geodesic reaches the center in finite affine parameter, which may indicate geodesic completeness. However, in regular spacetimes, like Hayward, we expect this integral to diverge as $r \to 0$.

For Schwarzschild:

$$\Delta\lambda = \int_{r_0}^{0} \frac{dr}{\sqrt{E^2 + \frac{2mL^2}{r^3}}}. \tag{5.32}$$

This integral is hard to deal with analytically. But investigating the behaviour near $r = 0$ enables us the reach a conclusion. Near $r = 0$, the second term in the square root dominates, so that

$$\Delta\lambda \sim \int_{r_0}^{0} \frac{dr}{\sqrt{\frac{2mL^2}{r^3}}} = \frac{1}{\sqrt{2mL^2}} \int_{r_0}^{0} r^{\frac{3}{2}} dr. \tag{5.33}$$

As the integral is convergent, the affine distance to the center is finite, indicating a potential singularity which can be attested through the value of the Kretschmann invariant at the terminal point.

For Hayward:

For Hayward as well, we look at the behaviour near $r = 0$. The function $F(r)$ can be simplified:

$$F(r) = 1 - \frac{2mr^2}{r^3 + 2ml^2} \approx 1 - \frac{2mr^2}{2ml^2} = 1 - \frac{r^2}{l^2}. \tag{5.34}$$

So the integrand becomes:

$$\frac{1}{\sqrt{E^2 - F(r)\frac{L^2}{r^2}}} = \frac{1}{\sqrt{E^2 - \left(1 - \frac{r^2}{l^2}\right)\frac{L^2}{r^2}}} = \frac{1}{\sqrt{E^2 - \left(\frac{L^2}{r^2} - \frac{L^2}{l^2}\right)}}. \tag{5.35}$$

Defining $A =: E^2 + \frac{L^2}{l^2}$, the affine distance reads:

$$\Delta\lambda = \int_{r_0}^{0} \frac{dr}{\sqrt{A - \frac{L^2}{r^2}}}. \tag{5.36}$$

Near $r = 0$, the second term inside the square root dominates, so that

$$\Delta\lambda \sim \int_{r_0}^{0} \sqrt{-\frac{r^2}{L^2}} dr. \tag{5.37}$$

The minus sign inside the square root reflects the fact that a non-vanishing L will in general cause the trajectory to bounce at a minimum $r > 0$.

The Kretschmann scalar is independent of L, so the expression that we laid down before holds also for $L \neq 0$.

Chapter 6

Conclusion

Let us take a moment to recap and to reflect on what we have encountered throughout this study.

Providing a historical account of the pathologies exhibited in post-Einsteinian general relativity, we began by tracing the evolution of the idea of singularities. Initially vague and physically ambiguous, singularities became more mathematically precise as the theory matured. The treatment of these pathologies in a more organized framework came first through the work of Raychaudhuri, whose analysis of geodesic congruences laid the groundwork for a more systematic understanding of gravitational focusing. This foundational insight was later elevated by Penrose, whose singularity theorem introduced a remarkably elegant and general formalism grounded in causal structure and trapped surfaces. Investigating the key elements of these theorems, such as energy conditions and the notion of global hyperbolicity, we built a firm understanding of how they function and what assumptions they rely upon. The key lesson to draw was that the existence of a trapped surface generically leads to incomplete geodesics; the latter are then interpreted as indicators of singularities in the language of these theorems.

To follow with, we turned to Kerr's recent ideas contesting the equivalence of geodesic incompleteness to the actual existence of singularities. His critique highlights an often-overlooked distinction: the mere finiteness of the affine parameter along a geodesic does not, by itself, entail divergent curvature or ill-defined geometry. After having investigated (in the appendix) the examples that he provides to illustrate this point—particularly in the context of Kerr and Schwarzschild spacetimes—we endeavoured to reinforce the argument by examining examples ourselves.

We figured that it could be a good idea to work on regular spacetimes that—by definition—do not harbour any curvature singularities, and to investigate whether geodesic incompleteness can still be exhibited in such contexts. This would allow us to isolate and critically evaluate the physical meaning of geodesic incompleteness itself. First, we presented in a general setting a class of regular metrics especially devised to avoid singularities, often by violating certain energy conditions or modifying the core behavior of the gravitational field. Then, we concentrated our study on one specific and well-studied member of this class: the Hayward metric.

Introducing the Hayward metric, we moved on to investigate the behavior of null geodesics in Hayward spacetime. Our analysis has been fruitful. In particular, we examined the cases of vanishing and non-vanishing angular momentum. In both cases—though there are some subtleties in the non-vanishing case—the affine length of the trajectory of interest is found to be finite. This raises the natural question of whether such incompleteness signals the presence of a physical singularity.

In order to attest whether or not there is a genuine singularity in these scenarios, we examined the Kretschmann scalar, a curvature invariant often used as a diagnostic for physical singularities. In the Hayward spacetime, we found this quantity to remain finite at the center, indicating the absence of curvature divergence. This stands in sharp contrast with the Schwarzschild metric, where the same scalar diverges at $r = 0$, confirming the presence of a curvature singularity. This means that, indeed, in Hayward spacetime there exist geodesics with finite affine length which do not terminate at singularities. This was precisely the sort of result that we were after: an example demonstrating that the presence of incomplete geodesics does not necessarily entail a breakdown of spacetime in a physical sense.

Thus, the broader implication of our work is that the standard interpretation of the singularity theorems—namely, that geodesic incompleteness is synonymous with physical singularities—may be too hasty. Our study supports the view that incompleteness should be understood as a geometric indicator that must be interpreted in the context of the specific spacetime under consideration.

Appendix A

Kerr's paper

Christoffel symbols of the Kerr metric

As they will be of great use in the calculations to follow, we lay down the Christoffel components that we calculated by hand.

With the Kerr metric given as

$$\begin{aligned} ds^2 = -dt^2 + \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) + \left(r^2 + a^2\right)\sin^2\theta d\phi^2 \\ + \frac{2Mr}{\Sigma}\left(a\sin^2\theta d\phi - dt\right)^2, \end{aligned}$$

the results are as follows:

$$
\begin{aligned}
&\Gamma^t_{tt} = \Gamma^t_{t\phi} = \Gamma^t_{rr} = \Gamma^t_{\phi\phi} = \Gamma^r_{tr} = \Gamma^r_{r\phi} = \Gamma^\theta_{tr} = \Gamma^\theta_{r\phi} = \Gamma^\phi_{tt} = \Gamma^\phi_{t\phi} = \Gamma^\phi_{rr} = \Gamma^\phi_{\phi\phi} = 0\\
&\Gamma^t_{tr} = -\frac{M}{\Delta\Sigma}\left(r^2+a^2\right)\left(1-\frac{2r^2}{\Sigma}\right)\\
&\Gamma^t_{r\phi} = \frac{Ma\sin^2\theta}{\Delta\Sigma}\left[\left(r^2+a^2\right)\left(1-\frac{2r^2}{\Sigma}\right)-2r^2\right]\\
&\Gamma^r_{tt} = -\frac{M\Delta}{\Sigma^2}\left(1-\frac{2r^2}{\Sigma}\right)\\
&\Gamma^r_{t\phi} = \frac{\Delta Ma\sin^2\theta}{\Sigma^2}\left(1-\frac{2r^2}{\Sigma}\right)\\
&\Gamma^r_{rr} = \frac{1}{\Sigma\Delta}\Big[r\left(\Delta-\Sigma\right)+\Sigma M\Big] = \frac{1}{\Sigma\Delta}\Big[\sin^2\theta a^2 r + M\left(a^2\cos^2\theta - r^2\right)\Big]\\
&\Gamma^r_{\phi\phi} = -\frac{\Delta\sin^2\theta}{\Sigma}\left[r+\frac{Ma^2\sin^2\theta}{\Sigma}\left(1-\frac{2r^2}{\Sigma}\right)\right]\\
&\Gamma^\theta_{tt} = -\frac{2Mra^2}{\Sigma^3}\cos\theta\sin\theta\\
&\Gamma^\theta_{t\phi} = \frac{2Mra\sin\theta\cos\theta}{\Sigma^3}\left(r^2+a^2\right)\\
&\Gamma^\theta_{rr} = \frac{a^2}{\Sigma\Delta}\cos\theta\sin\theta\\
&\Gamma^\theta_{\phi\phi} = -\frac{\sin\theta\cos\theta}{\Sigma^3}\Big[\left(r^2+a^2\right)\left(r^4+a^4\cos^4\theta+2r^2a^2\cos^2\theta+4Mra^2\sin^2\theta\right)-2Mra^4\sin^4\theta\Big]\\
&\Gamma^\phi_{tr} = -\frac{Ma}{\Sigma\Delta}\left(1-\frac{2r^2}{\Sigma}\right)\\
&\Gamma^\phi_{r\phi} = \frac{1}{\Sigma^2\Delta}\left\{\Sigma^2 r + M\Big[\Sigma\left(a^2\sin^2\theta-2r^2\right)-2r^2a^2\sin^2\theta\Big]\right\}
\end{aligned}
$$

A.1 Boundedness of the affine parameter and how it has no real link to singularities

Kerr starts off by noting that for a geodesic the acceleration is proportional to the velocity:

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = \lambda(t)\frac{dx^\mu}{dt}. \tag{A.1}$$

An affine parameter is a parameter a that leads to a vanishing RHS,

$$\frac{d^2x^\mu}{da^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{da}\frac{dx^\beta}{da} = 0. \tag{A.2}$$

The meaning of this is that the tangent vector $\frac{dx^\mu}{da}$ is parallelly propagated along the ray. The affine parameter a is related to the non-affine parameter t via:

$$\frac{d^2a}{dt^2} = \lambda\frac{da}{dt}. \tag{A.3}$$

The general solution to this equation should be of the form:

$$a = Aa_0 + C, \tag{A.4}$$

where a_0 is a particular solution and A and C are constants. Furthermore, if λ is a constant then the particular solution should read:

$$a_0 = e^{\lambda t}. \tag{A.5}$$

We observe that $a(t)$, the affine parameter, is bounded at either $t = +\infty$ or $t = -\infty$. This result (the boundedness of the affine parameter) has nothing to do with the physical concept of a singularity.

A.2 The two families of PNVs of the EF extension of the Schwarzschild metric

The Schwarzschild metric reads

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt_S^2 + \left(1 - \frac{2m}{r}\right) dr^2 + r^2 d\Omega^2. \tag{A.6}$$

The two EF extensions come through the following re-definitions of the time coordinate:

$$t_- = t_S - 2m \log|r - 2m| \quad \text{and} \quad t_+ = t_S + 2m \log|r - 2m|. \tag{A.7}$$

So the two time coordinates are related to one another via

$$t_+ - = t_- + 4m \log|r - 2m|. \tag{A.8}$$

The EF extensions of the Schwarzschild metric can be put into the so-called Kerr-Schild form

$$ds^2 = ds_{0\pm}^2 + \frac{2m}{r} (k_{\pm\mu} dx^\mu)^2 \quad \text{with} \quad ds_{0\pm}^2 = dr^2 + r^2 d\Omega^2 - dt_\pm^2. \tag{A.9}$$

The $k_\pm$ here are the ingoing/outgoing PNVs:

$$k_\pm = k_{\pm\mu} dx^\mu = \pm dr - dt_\pm. \tag{A.10}$$

There's also a second set of PNVs:

$$k_\pm^* = k_{\pm\mu}^* dx^\mu = \pm \frac{r - 2m}{r + 2m} dr - dt_\pm. \tag{A.11}$$

This is calculated using the transformation formula involving t_+ and t_- we presented above.

Tracking the respective signs of the radial and the time coordinates of the PNV in question, it is a simple matter to observe that for a BH both $\boldsymbol{k}_-$ and $\boldsymbol{k}_-^*$ point inwards inside the event horizon, the former does so also outside but the latter points outwards in that case.

The relevance of all this to our discussion is understood if we look at the behaviour of the second set of PNVs. "They are asymptotic to the event horizon as $t_- \longrightarrow -\infty$ for a black hole and as $t_+ \longrightarrow +\infty$ for a white hole. In both cases the affine parameter is necessarily bounded as the PNV approaches the appropriate horizon." Voilà an instance of a geodesic the affine parameter along which is bounded, which nonetheless does not end at a singularity!

There's a question mark regarding the passage from Eq. A.10 to Eq. A.11. The transformation does not quite seem to work. Below I lay down the relevant calculations.

To get Eq. A.11 we should simply be plugging Eq. A.8 into Eq. A.10. Let us work with k_+, k_- should be analogous:

$$\begin{aligned} k_+ &= dr - dt_+ = dr - d\left(t_- + 4m\log|r-2m|\right) \\ &= dr - dt_- - 4m\frac{1}{|r-2m|}dr \\ &= -dt_- + \left(1 - \frac{4m}{|r-2m|}\right)dr \end{aligned}$$

For $r < 2m$,

$$\begin{aligned} k_+ &= -dt_- + \left(1 - \frac{4m}{2m-r}\right)dr \\ &= -dt_- + \left(\frac{2m-r-4m}{2m-r}\right)dr \\ &= -dt_- + \left(\frac{-r-2m}{-r+2m}\right)dr \\ &= -dt_- + \frac{r+2m}{r-2m}dr. \end{aligned}$$

For $r > 2m$,

$$\begin{aligned} k_+ &= -dt_- + \left(1 - \frac{4m}{r-2m}\right)dr \\ &= -dt_- + \left(\frac{r-2m-4m}{r-2m}\right)dr \\ &= -dt_- + \frac{r-6m}{r-2m}dr. \end{aligned}$$

Likewise for k_- we get,

$$\begin{aligned} k_- &= -dt_- - \frac{r+2m}{r-2m}dr \quad \textbf{when} \quad r < 2m \\ k_- &= -dt_- - \frac{r-6m}{r-2m}dr \quad \textbf{when} \quad r > 2m. \end{aligned}$$

Kerr asserts that there is a correspondence between $k_\pm$ and $k^*_\mp$: "Since the second PNV, $k^*_\pm$ say, in one coordinate system is the first one in the other, $k_\mp$."
The results that are relevant in the above analysis seem to be those for $r < 2m$.

Thus, the correspondence should be between the following pairs:

$$\begin{aligned} k_+ &= -dt_- + \frac{r+2m}{r-2m}dr \quad \text{corresponds to} \quad k^*_- = -dt_- - \frac{r-2m}{r+2m}dr, \\ k_- &= -dt_+ - \frac{r+2m}{r-2m}dr \quad \text{corresponds to} \quad k^*_+ = -dt_+ + \frac{r-2m}{r+2m}dr. \end{aligned}$$

As can be seen, however, they do not match. We reach the conclusion that this is a typo in Kerr's paper.

A.3 Discussion regarding the Kruskal extension

Let us comment on Section 4 of the paper. In this section, Kerr critiques the conventional use of Kruskal-Szekeres coordinates to interpret the Schwarzschild solution. He argues that Kruskal coordinates are not a physical extension of Schwarzschild spacetime, but rather a mathematical reformulation of the Eddington-Finkelstein coordinates.

We begin by writing down the (ingoing) EF metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2\, dv\, dr + r^2 d\Omega^2. \tag{A.12}$$

Since this form of the metric is already nonsingular at the horizon, the Kruskal construction does not remove any physical singularity—it merely repackages the same geometry in a different coordinate system.

The Kruskal metric reads:

$$ds^2 = \frac{32M^3}{r} e^{-r/(2M)} dU dV + r^2 d\Omega^2. \tag{A.13}$$

The Kruskal extension includes additional regions (e.g., white holes and a second asymptotically flat universe) that are not realized in realistic gravitational collapse. Thus, it does not correspond to the physical formation of black holes, which originate from collapsing matter, not from eternal white holes.

Moreover, while Kruskal extends the spacetime smoothly across $r = 2M$, it does not remove or explain the central singularity at $r = 0$. Kerr emphasizes that along ingoing null geodesics, the affine parameter remains finite as $r \to 0$, just as in EF coordinates. This demonstrates that the Kruskal transformation does not address the issue of geodesic incompleteness.

A.4 Showing that k in (10a) is a PNV

When we attempted to show that k in Eq. (10a) in the Kerr paper is a PNV, we failed. We reasoned that this should be due to a typo regarding the sign of the ϕ-component of the vector in question. Below are presented some calculations aiming at an understanding of what is it that does not work.

To begin with, we need to be reassured that when $a = 0$ Eq. (10b) of Kerr's paper is the Minkowski metric and that (10a) is the Schwarzschild one. We did the calculations and found the former to be in accordance quite trivially, whereas for the latter in order to arrive at the desired end one needs to make the assumption that $2m \ll r$. So there's nothing problematic here.

The obvious point of attack is the ϕ-component of k. In order to track the effect of the sign in front of it we labelled it with $\pm$ instead of the $+$ in the paper.

$$k = dr + a\sin^2\theta d\phi + dt \longrightarrow k = dr \pm a\sin^2\theta d\phi + dt. \tag{A.14}$$

The covariant form of the said PNV reads:

$$(k_\mu) = \begin{pmatrix} 1 & 1 & 0 & \pm a\sin^2\theta d\phi \end{pmatrix}. \tag{A.15}$$

We're interested in calculating $k_\mu k^\mu = k_\mu k_\nu g^{\mu\nu}$, so we'll have to calculate explicitly the inverse metric. Fortunately, not all the components are of use to us, so we'll just infer the components that we need through the identity $\delta_\alpha^\beta = g_{\alpha\gamma}g^{\gamma\beta}$.

From (10a) and (10b) (with the expression for k modified),

$$ds^2 = ds_0^2 + \frac{2mr}{\Sigma}k^2; \quad k = dr \pm a\sin^2\theta d\phi + dt, \tag{A.16}$$

$$ds_0^2 = dr^2 + \Sigma d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2 + 2a\sin^2\theta d\phi dt - dt^2, \tag{A.17}$$

we read off the following metric components that are relevant:

$$g_{00} = -1 + \frac{2mr}{\Sigma}, \quad g_{01} = g_{10} = \frac{2mr}{\Sigma}, \quad g_{11} = 1 + \frac{2mr}{\Sigma}, \tag{A.18}$$

$$g_{03} = g_{30} = a\sin^2\theta(1 \pm 1), \quad g_{13} = g_{31} = \pm a\sin^2\theta, \quad g_{33} = \sin^2\theta\left[a^2\sin^2\theta + (r^2 + a^2)\right]. \tag{A.19}$$

All in all, the Kerr metric in matrix form reads

$$(g_{\mu\nu}) = \begin{pmatrix} -1 + \frac{2mr}{\Sigma} & \frac{2mr}{\Sigma} & 0 & a\sin^2\theta(1 \pm 1) \\ \frac{2mr}{\Sigma} & 1 + \frac{2mr}{\Sigma} & 0 & \pm a\sin^2\theta \\ 0 & 0 & \Sigma & 0 \\ a\sin^2\theta(1 \pm 1) & \pm a\sin^2\theta & 0 & \sin^2\theta\left[a^2\sin^2\theta + (r^2 + a^2)\right] \end{pmatrix} \tag{A.20}$$

In order to infer the components of the inverse metric we'll look at the identity $\delta_\alpha^\beta = g_{\alpha\gamma}g^{\gamma\beta}$. In matrix form, this reads:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} g_{00} & g_{01} & 0 & g_{03} \\ g_{01} & g_{11} & 0 & g_{13} \\ 0 & 0 & g_{22} & 0 \\ g_{03} & g_{13} & 0 & g_{33} \end{pmatrix} \begin{pmatrix} g^{00} & g^{01} & 0 & g^{03} \\ g^{01} & g^{11} & 0 & g^{13} \\ 0 & 0 & g^{22} & 0 \\ g^{03} & g^{13} & 0 & g^{33} \end{pmatrix} \tag{A.21}$$

This gives us a series a series of equations from which we could "in theory" derive the inverse metric components. Unfortunately, it is virtually impossible to isolate them, one needs further conditions to simplify the equations.

So instead of calculating the inverse metric and looking at $k_\mu k_\nu g^{\mu\nu}$ in order to see whether or not k_μ is null, we'll follow a less hands-on approach. By virtue of the Kerr-Schild form, we can actually assert that if a vector is null with respect to the Minkowski metric then it is null also with respect to the full metric. We'll, therefore, just look at $k_\mu k_\nu \eta^{\mu\nu}$.

Minkowski metric in Papapetrou form (Minkowski in spherical coordinates in the presence of rotation), as we already laid down above, reads:

$$ds_0^2 = dr^2 + \Sigma d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2 + 2a\sin^2\theta d\phi dt - dt^2. \tag{A.22}$$

In matrix form:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & a\sin^2\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ a\sin^2\theta & 0 & 0 & (r^2+a^2)\sin^2\theta \end{pmatrix}. \tag{A.23}$$

Its inverse is:

$$g_{\mu\nu} = \begin{pmatrix} \frac{-\Delta\sin^2\theta}{a^2\sin^2\theta+\Delta\sin^2\theta} & 0 & 0 & \frac{a\sin^2\theta}{a^2\sin^2\theta+\Delta\sin^2\theta} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\Sigma} & 0 \\ \frac{a\sin^2\theta}{a^2\sin^2\theta+\Delta\sin^2\theta} & 0 & 0 & \frac{1}{a^2\sin^2\theta+\Delta\sin^2\theta} \end{pmatrix}, \tag{A.24}$$

where $\Delta = r^2 + a^2$.

To see what ϕ-component in k_μ would lead to $k_\mu k_\nu \eta^{\mu\nu} = 0$, let us write k_μ in the form:

$$k = dt + dr + \xi d\phi. \tag{A.25}$$

We can plunge in:

$$\begin{aligned} k_\mu k_\nu \eta^{\mu\nu} &= k_0^2\eta^{00} + k_1^2\eta^{11} + k_3^2\eta^{33} + 2k_0k_3^2\eta^{03} \\ &= \frac{-\Delta\sin^2\theta}{a^2\sin^2\theta + \Delta\sin^2\theta} + 1 + \xi^2\frac{1}{a^2\sin^2\theta + \Delta\sin^2\theta} + 2\xi\frac{a\sin^2\theta}{a^2\sin^2\theta + \Delta\sin^2\theta}. \end{aligned}$$

Multiplying by $a^2\sin^2\theta + \Delta\sin^2\theta$,

$$\begin{aligned} (a^2\sin^2\theta + \Delta\sin^2\theta)k_\mu k_\nu \eta^{\mu\nu} &= -\Delta\sin^2\theta \\ &+ (a^2\sin^2\theta + \Delta\sin^2\theta) + \xi^2 + 2a\sin^2\theta\xi \\ &= \xi^2 + 2a\sin^2\theta\xi + a^2\sin^4\theta. \end{aligned}$$

It is a trivial matter to see that $\xi = -a\sin^2\theta$ makes this expression vanish:

$$\begin{aligned} \xi^2 + 2a\sin^2\theta\xi + a^2\sin^4\theta &= (-a\sin^2\theta)^2 + 2a\sin^2\theta(-a\sin^2\theta) + a^2\sin^4\theta \\ &= 0. \end{aligned}$$

Thus we conclude that , the inverse metric given as above, the vector

$$k = dt + dr - a\sin^2\theta d\phi \tag{A.26}$$

is a null vector with respect to the Papapetrou form of the Minkowski metric, and as a consequence, by virtue of the Kerr-Schild form, also with respect to the Kerr metric.

So the expression for the PNV k given in Eq. (10a) of Kerr's paper,

$$k = dt + dr + a\sin^2\theta d\phi, \tag{A.27}$$

does indeed exhibit a typo. The correct expression is:

$$k = dt + dr - a\sin^2\theta d\phi. \tag{A.28}$$

A.5 Showing that r is an affine parameter for the Kerr metric

The Kerr metric reads:

$$\begin{aligned} ds^2 = -dt^2 + \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) + \left(r^2 + a^2\right)\sin^2\theta d\phi^2 \\ + \frac{2Mr}{\Sigma}\left(a\sin^2\theta d\phi - dt\right)^2. \end{aligned}$$

The Kerr-Schild form is manifest here. Indeed the first line is just Minkowski in spherical coordinates when $a = 0$ and $2M \ll r$ and from the second line we read off the PNV $k = dt - a\sin^2\theta d\phi$ (we will provide later the calculations regarding whether or not this is an actual PNV).

The non-vanishing metric components can be laid down as:

$$g_{tt} = -\left(1 - \frac{2Mr}{\Sigma}\right), \quad g_{rr} = \frac{\Sigma}{\Delta}, \quad g_{t\phi} = -\frac{2Mr}{\Sigma}a\sin^2\theta, \tag{A.29}$$

$$g_{\theta\theta} = \Sigma, \quad g_{\phi\phi} = \left(r^2 + a^2 + \frac{2Mr}{\Sigma}a^2\sin^2\theta\right)\sin^2\theta. \tag{A.30}$$

It has the matrix representation:

$$g_{\mu\nu} = \begin{pmatrix} g_{tt} & 0 & 0 & g_{t\phi} \\ 0 & \frac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ g_{t\phi} & 0 & 0 & g_{\phi\phi} \end{pmatrix}. \tag{A.31}$$

The matrix representation of its inverse is analogous:

$$g^{\mu\nu} = \begin{pmatrix} g^{tt} & 0 & 0 & g^{t\phi} \\ 0 & \frac{\Delta}{\Sigma} & 0 & 0 \\ 0 & 0 & \frac{1}{\Sigma} & 0 \\ g^{t\phi} & 0 & 0 & g^{\phi\phi} \end{pmatrix}. \tag{A.32}$$

Its components read:

$$g^{tt} = -\frac{1}{\Delta}\left(r^2 + a^2 + \frac{2Mr}{\Sigma}a^2\sin^2\theta\right), \tag{A.33}$$

$$g^{t\phi} = -\frac{2Mr}{\Sigma\Delta}a, \quad g^{\phi\phi} = \frac{\Delta - a^2\sin^2\theta}{\Sigma\Delta\sin^2\theta}. \tag{A.34}$$

Now, we're going to look at the geodesic equation,

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0, \tag{A.35}$$

and see if it remains intact when we set $\lambda = r$.

There are two pieces of information we need in order two solve this mystery: the Christoffel symbols and the quantities $\frac{dx^\mu}{dr}$. The Christoffel symbols are laid down in the beginning of the appendix, as for the latter we lay down:

$$\frac{dt}{dr} = -\frac{r^2+a^2}{\Delta} \quad \& \quad \frac{d\phi}{dr} = -\frac{a}{\Delta}. \tag{A.36}$$

<u>**t-component of the geodesic equation:**</u>

$$\begin{aligned}
&\frac{d^2t}{dr^2} + \Gamma^t_{\alpha\beta}\frac{dx^\alpha}{dr}\frac{dx^\beta}{dr} \\
&= \frac{d^2t}{dr^2} + \Gamma^t_{tt}\left(\frac{dt}{dr}\right)^2 + 2\Gamma^t_{tr}\frac{dt}{dr} + 2\Gamma^t_{t\theta}\frac{dt}{dr}\frac{d\theta}{dr} + 2\Gamma^t_{t\phi}\frac{dt}{dr}\frac{d\phi}{dr} + \Gamma^t_{rr} \\
&+ 2\Gamma^t_{r\theta}\frac{d\theta}{dr} + 2\Gamma^t_{r\phi}\frac{d\phi}{dr} + \Gamma^t_{\theta\theta}\left(\frac{d\theta}{dr}\right)^2 + 2\Gamma^t_{\theta\phi}\frac{d\theta}{dr}\frac{d\phi}{dr} + \Gamma^t_{\phi\phi}\left(\frac{d\phi}{dr}\right)^2
\end{aligned}$$

Before proceeding, let us deal with the second derivative.

$$\begin{aligned}
\frac{d^2t}{dr^2} &= \frac{d}{dr}\left(\frac{dt}{dr}\right) = \frac{d}{dr}\left(-\frac{r^2+a^2}{\Delta}\right) \\
&= -\left(\frac{\frac{d}{dr}\left(r^2+a^2\right)\Delta - \left(r^2+a^2\right)\frac{d\Delta}{dr}}{\Delta^2}\right) \\
&= -\left(\frac{2r\Delta - \left(r^2+a^2\right)(2r-2M)}{\Delta^2}\right) \\
&= -\left(\frac{2r(r^2-2Mr+a^2) - \left(r^2+a^2\right)(2r-2M)}{\Delta^2}\right) \\
&= -\left(\frac{(2r^3-4Mr^2+2a^2r) - (2r^3-2Mr^2+2a^2r-2Ma^2)}{\Delta^2}\right) \\
&= -\left(\frac{-2Mr^2+2Ma^2}{\Delta^2}\right) \\
&= \frac{2M}{\Delta^2}\left(r^2-a^2\right).
\end{aligned}$$

Now we can turn back to the geodesic equation:

$$\frac{d^2t}{dr^2} + \Gamma^t_{\alpha\beta}\frac{dx^\alpha}{dr}\frac{dx^\beta}{dr}$$
$$= \frac{d^2t}{dr^2} + \Gamma^t_{tt}\left(\frac{dt}{dr}\right)^2 + 2\Gamma^t_{tr}\frac{dt}{dr} + \Gamma^t_{rr}$$
$$+ 2\Gamma^t_{t\theta}\frac{dt}{dr}\frac{d\theta}{dr} + 2\Gamma^t_{t\phi}\frac{dt}{dr}\frac{d\phi}{dr} + 2\Gamma^t_{r\theta}\frac{d\theta}{dr} + 2\Gamma^t_{r\phi}\frac{d\phi}{dr}$$
$$+ \Gamma^t_{\theta\theta}\left(\frac{d\theta}{dr}\right)^2 + 2\Gamma^t_{\theta\phi}\frac{d\theta}{dr}\frac{d\phi}{dr} + \Gamma^t_{\phi\phi}\left(\frac{d\phi}{dr}\right)^2$$

In the calculations to follow we make two assumptions, it is worthwhile to note them:

1. $\frac{d\theta}{dr} = 0$ due to the axial symmetry exhibited by the Kerr metric.
2. The metric (and the inverse metric) components do not have explicit $t-$ or $\phi-$dependence. This means that $\partial_t g_{\mu\nu}$ and $\partial_\phi g_{\mu\nu}$ all vanish.

If a mismatch happens to appear, we may search for the problem in these assumptions. The first assumption simplifies the business quite a lot:

$$= \frac{d^2t}{dr^2} + \Gamma^t_{tt}\left(\frac{dt}{dr}\right)^2 + 2\Gamma^t_{tr}\frac{dt}{dr} + \Gamma^t_{rr}$$
$$+ 2\Gamma^t_{t\phi}\frac{dt}{dr}\frac{d\phi}{dr} + 2\Gamma^t_{r\phi}\frac{d\phi}{dr} + \Gamma^t_{\phi\phi}\left(\frac{d\phi}{dr}\right)^2.$$

Putting in the relevant expressions already figured out:

$$= \frac{d}{dr}\left(-\frac{r^2+a^2}{\Delta}\right) + \Gamma^t_{tt}\left(-\frac{r^2+a^2}{\Delta}\right)^2 + 2\Gamma^t_{tr}\left(-\frac{r^2+a^2}{\Delta}\right)$$
$$+ 2\Gamma^t_{t\phi}\left(-\frac{r^2+a^2}{\Delta}\right)\left(-\frac{a}{\Delta}\right) + 2\Gamma^t_{r\phi}\left(-\frac{a}{\Delta}\right) + \Gamma^t_{\phi\phi}\left(-\frac{a}{\Delta}\right)^2$$
$$= \left[\frac{2M}{\Delta^2}\left(r^2-a^2\right)\right] + 0 + 2\left[-\frac{M}{\Delta\Sigma}\left(r^2+a^2\right)\left(1-\frac{2r^2}{\Sigma}\right)\right]\left(-\frac{r^2+a^2}{\Delta}\right) + 0 + 2\Gamma^t_{r\phi}\left(-\frac{a}{\Delta}\right) + 0$$
$$= \frac{2M}{\Delta^2}\left[\left(r^2-a^2\right) + \frac{(r^2+a^2)^2}{\Sigma}\left(1-\frac{2r^2}{\Sigma}\right)\right]$$
$$+ 2\left\{\frac{Ma\sin^2\theta}{\Delta\Sigma}\left[\left(r^2+a^2\right)\left(1-\frac{2r^2}{\Sigma}\right) - 2r^2\right]\right\}\left(-\frac{a}{\Delta}\right)$$
$$= \frac{2M}{\Delta^2}\left\{\left[\left(r^2-a^2\right) + \frac{(r^2+a^2)^2}{\Sigma}\left(1-\frac{2r^2}{\Sigma}\right)\right] - \frac{a^2\sin^2\theta}{\Sigma}\left[\left(r^2+a^2\right)\left(1-\frac{2r^2}{\Sigma}\right) - 2r^2\right]\right\}$$
$$= \frac{2M}{\Delta^2}\left\{\left(r^2-a^2\right) + \frac{(r^2+a^2)}{\Sigma}\left(1-\frac{2r^2}{\Sigma}\right)\left[\left(r^2+a^2\right) - a^2\sin^2\theta\right] + \frac{2r^2a^2\sin^2\theta}{\Sigma}\right\}$$
$$= \frac{2M}{\Delta^2}\left[\left(r^2-a^2\right) + \frac{(r^2+a^2)}{\Sigma}\left(1-\frac{2r^2}{\Sigma}\right)\left(r^2+a^2\cos^2\theta\right) + \frac{2r^2a^2\sin^2\theta}{\Sigma}\right]$$
$$= \frac{2M}{\Delta^2}\left[\left(r^2-a^2\right) + \left(r^2+a^2\right)\left(1-\frac{2r^2}{\Sigma}\right) + \frac{2r^2a^2\sin^2\theta}{\Sigma}\right]$$

To manipulate this expression, we'll consider:

$$
\begin{aligned}
&(r^2 - a^2)\left(1 - \frac{2r^2}{\Sigma}\right) = r^2 - \frac{2r^4}{\Sigma} - a^2 + \frac{2r^2a^2}{\Sigma} \\
&= (r^2 - a^2) - \frac{2r^4}{\Sigma} + \frac{2r^2a^2}{\Sigma}(\cos^2\theta + \sin^2\theta) \\
&= (r^2 - a^2) + \frac{2r^2a^2\sin^2\theta}{\Sigma} + \frac{2r^2}{\Sigma}\left(a^2\cos^2\theta - r^2\right) \\
&= (r^2 - a^2) + \frac{2r^2a^2\sin^2\theta}{\Sigma} + \frac{2r^2}{\Sigma}\left(\Sigma - 2r^2\right) \\
&= (r^2 - a^2) + \frac{2r^2a^2\sin^2\theta}{\Sigma} + 2r^2\left(1 - \frac{2r^2}{\Sigma}\right)
\end{aligned}
$$

We notice that the first two terms appear in the expression we were investigating, isolating them:

$$
\begin{aligned}
(r^2 - a^2) + \frac{2r^2a^2\sin^2\theta}{\Sigma} &= (r^2 - a^2)\left(1 - \frac{2r^2}{\Sigma}\right) - 2r^2\left(1 - \frac{2r^2}{\Sigma}\right) \\
&= -(r^2 + a^2)\left(1 - \frac{2r^2}{\Sigma}\right).
\end{aligned}
$$

We can plug this into the expression for the geodesic equation:

$$
\begin{aligned}
&\frac{d^2t}{dr^2} + \Gamma^t_{\alpha\beta}\frac{dx^\alpha}{dr}\frac{dx^\beta}{dr} \\
&= \frac{2M}{\Delta^2}\left[(r^2 - a^2) + \frac{2r^2a^2\sin^2\theta}{\Sigma} + (r^2 + a^2)\left(1 - \frac{2r^2}{\Sigma}\right)\right] \\
&= \frac{2M}{\Delta^2}\left[-(r^2 + a^2)\left(1 - \frac{2r^2}{\Sigma}\right) + (r^2 + a^2)\left(1 - \frac{2r^2}{\Sigma}\right)\right] \\
&= 0.
\end{aligned}
$$

Splendid! So the $t-$component of the geodesic equation does indeed hold.

<u>r**-component of the geodesic equation:**</u>

$$\begin{aligned}
&\frac{d^2r}{dr^2} + \Gamma^r_{\alpha\beta}\frac{dx^\alpha}{dr}\frac{dx^\beta}{dr} \\
&= \Gamma^r_{\alpha\beta}\frac{dx^\alpha}{dr}\frac{dx^\beta}{dr} \\
&= \Gamma^r_{tt}\left(\frac{dt}{dr}\right)^2 + 2\Gamma^r_{tr}\frac{dt}{dr} + 2\Gamma^r_{t\theta}\frac{dt}{dr}\frac{d\theta}{dr} + 2\Gamma^r_{t\phi}\frac{dt}{dr}\frac{d\phi}{dr} + \Gamma^r_{rr} + 2\Gamma^r_{r\theta}\frac{d\theta}{dr} \\
&+ 2\Gamma^r_{r\phi}\frac{d\phi}{dr} + \Gamma^r_{\theta\theta}\left(\frac{d\theta}{dr}\right)^2 + 2\Gamma^r_{\theta\phi}\frac{d\theta}{dr}\frac{d\phi}{dr} + \Gamma^r_{\phi\phi}\left(\frac{d\phi}{dr}\right)^2 \\
&= \Gamma^r_{tt}\left(\frac{dt}{dr}\right)^2 + 2\Gamma^r_{tr}\frac{dt}{dr} + 2\Gamma^r_{t\phi}\frac{dt}{dr}\frac{d\phi}{dr} + \Gamma^r_{rr} + 2\Gamma^r_{r\phi}\frac{d\phi}{dr} + \Gamma^r_{\phi\phi}\left(\frac{d\phi}{dr}\right)^2
\end{aligned}$$

Plugging in the expressions for the Christoffel symbols and the derivative terms:

$$
\begin{aligned}
&\frac{d^2r}{dr^2} + \Gamma^r_{\alpha\beta}\frac{dx^\alpha}{dr}\frac{dx^\beta}{dr} \\
&= \Gamma^r_{tt}\left(\frac{dt}{dr}\right)^2 + 2\Gamma^r_{tr}\frac{dt}{dr} + 2\Gamma^r_{t\phi}\frac{dt}{dr}\frac{d\phi}{dr} + \Gamma^r_{rr} + 2\Gamma^r_{r\phi}\frac{d\phi}{dr} + \Gamma^r_{\phi\phi}\left(\frac{d\phi}{dr}\right)^2 \\
&= \left[-\frac{M\Delta}{\Sigma^2}\left(1-\frac{2r^2}{\Sigma}\right)\right]\left(-\frac{r^2+a^2}{\Delta}\right)^2 + 2\left[\frac{\Delta M a\sin^2\theta}{\Sigma^2}\left(1-\frac{2r^2}{\Sigma}\right)\right]\left(-\frac{r^2+a^2}{\Delta}\right)\left(-\frac{a}{\Delta}\right) \\
&+ \left\{\frac{1}{\Sigma\Delta}\left[\sin^2\theta a^2 r + M(a^2\cos^2\theta - r^2)\right]\right\} + \left\{-\frac{\Delta\sin^2\theta}{\Sigma}\left[r + \frac{Ma^2\sin^2\theta}{\Sigma}\left(1-\frac{2r^2}{\Sigma}\right)\right]\right\}\left(-\frac{a}{\Delta}\right)^2 \\
&= -\frac{M(r^2+a^2)^2}{\Sigma^2\Delta}\left(1-\frac{2r^2}{\Sigma}\right) + \frac{2Ma^2\sin^2\theta(r^2+a^2)}{\Sigma^2\Delta}\left(1-\frac{2r^2}{\Sigma}\right) \\
&+ \frac{1}{\Sigma\Delta}\left[\sin^2\theta a^2 r + M(a^2\cos^2\theta - r^2)\right] - \frac{a^2\sin^2\theta}{\Sigma\Delta}\left[r + \frac{Ma^2\sin^2\theta}{\Sigma}\left(1-\frac{2r^2}{\Sigma}\right)\right] \\
&= \frac{1}{\Sigma\Delta}\Bigg\{\frac{M}{\Sigma}(r^2+a^2)\left(1-\frac{2r^2}{\Sigma}\right)\left[-(r^2+a^2) + 2a^2\sin^2\theta\right] \\
&+ \left[\sin^2\theta a^2 r + M(a^2\cos^2\theta - r^2) - a^2\sin^2\theta r - \frac{M}{\Sigma}a^4\sin^4\theta\left(1-\frac{2r^2}{\Sigma}\right)\right]\Bigg\} \\
&= \frac{1}{\Sigma\Delta}\Bigg\{\frac{M}{\Sigma}(r^2+a^2)\left(1-\frac{2r^2}{\Sigma}\right)\left[-(r^2+a^2) + 2a^2\sin^2\theta\right] \\
&+ \left[M(a^2\cos^2\theta - r^2) - \frac{M}{\Sigma}a^4\sin^4\theta\left(1-\frac{2r^2}{\Sigma}\right)\right]\Bigg\} \\
&= \frac{M}{\Sigma^2\Delta}\Bigg\{(r^2+a^2)\left(1-\frac{2r^2}{\Sigma}\right)\left[-(r^2+a^2) + 2a^2\sin^2\theta\right] \\
&+ \left[\Sigma(a^2\cos^2\theta - r^2) - a^4\sin^4\theta\left(1-\frac{2r^2}{\Sigma}\right)\right]\Bigg\} \\
&= \frac{M}{\Sigma^2\Delta}\Bigg\{(r^2+a^2)\left(1-\frac{2r^2}{\Sigma}\right)\left[-r^2 - a^2(1-\sin^2\theta) + a^2\sin^2\theta\right] \\
&+ \left[\Sigma(a^2\cos^2\theta - r^2) - a^4\sin^4\theta\left(1-\frac{2r^2}{\Sigma}\right)\right]\Bigg\} \\
&= \frac{M}{\Sigma^2\Delta}\Bigg\{(r^2+a^2)\left(1-\frac{2r^2}{\Sigma}\right)\left[-r^2 - a^2(\cos^2\theta - \sin^2\theta)\right] \\
&+ \left[\Sigma(a^2\cos^2\theta - r^2) - a^4\sin^4\theta\left(1-\frac{2r^2}{\Sigma}\right)\right]\Bigg\} \\
&= \frac{M}{\Sigma^3\Delta}\Bigg\{(r^2+a^2)(\Sigma - 2r^2)\Big[-r^2 - a^2(\cos^2\theta - \sin^2\theta)\Big] \\
&+ \Sigma^2(a^2\cos^2\theta - r^2) - \Sigma a^4\sin^4\theta + 2r^2a^4\sin^4\theta\Bigg\}
\end{aligned}
$$

We may try to see if plugging in the definition of Σ is helpful or not:

$$= \frac{M}{\Sigma^3\Delta}\Big\{(r^2+a^2)(a^2\cos^2\theta - r^2)\Big[-r^2 - a^2(\cos^2\theta - \sin^2\theta)\Big]$$
$$+ (r^2 + a^2\cos^2\theta)^2(a^2\cos^2\theta - r^2) - (r^2 + a^2\cos^2\theta)a^4\sin^4\theta + 2r^2a^4\sin^4\theta\Big\}$$
$$= \frac{M}{\Sigma^3\Delta}\Big\{(r^2a^2\cos^2\theta - r^4 + a^4\cos^2\theta - r^2a^2)\Big[-r^2 - a^2(\cos^2\theta - \sin^2\theta)\Big]$$
$$+ (r^4 + 2r^2a^2\cos^2\theta + a^4\cos^4\theta)(a^2\cos^2\theta - r^2) - (r^2 + a^2\cos^2\theta)a^4\sin^4\theta + 2r^2a^4\sin^4\theta\Big\}$$
$$= \frac{M}{\Sigma^3\Delta}\Big[-(-r^4 - r^2a^2\sin^2\theta + a^4\cos^2\theta)(r^2 + a^2\cos 2\theta)$$
$$+ (r^4a^2\cos^2\theta - r^6 + 2r^2a^4\cos^4\theta - 2r^4a^2\cos^2\theta + a^6\cos^6\theta - r^2a^4\cos^4\theta)$$
$$- r^2a^4\sin^4\theta - a^6\cos^2\theta\sin^4\theta + 2r^2a^4\sin^4\theta\Big]$$
$$= \frac{M}{\Sigma^3\Delta}\Big\{-\Big(-r^6 - r^4a^2\cos 2\theta - r^4a^2\sin^2\theta - r^2a^4\sin^2\theta\cos 2\theta + r^2a^4\cos^2\theta + a^6\cos^2\theta\cos 2\theta\Big)$$
$$+ \Big[(r^4a^2\cos^2\theta - r^6 + 2r^2a^4\cos^4\theta - 2r^4a^2\cos^2\theta + a^6\cos^6\theta - r^2a^4\cos^4\theta)$$
$$- r^2a^4\sin^4\theta - a^6\cos^2\theta\sin^4\theta + 2r^2a^4\sin^4\theta\Big]\Big\}$$
$$= \frac{M}{\Sigma^3\Delta}\Big\{-\Big[-r^6 - r^4a^2(\cos^2\theta - \sin^2\theta) - r^4a^2\sin^2\theta - r^2a^4\sin^2\theta(\cos^2\theta - \sin^2\theta)$$
$$+ r^2a^4\cos^2\theta + a^6\cos^2\theta(\cos^2\theta - \sin^2\theta)\Big] + \Big[(r^4a^2\cos^2\theta - r^6 + 2r^2a^4\cos^4\theta - 2r^4a^2\cos^2\theta$$
$$+ a^6\cos^6\theta - r^2a^4\cos^4\theta) - r^2a^4\sin^4\theta - a^6\cos^2\theta\sin^4\theta\Big] + 2r^2a^4\sin^4\theta\Big\}$$
$$= \frac{M}{\Sigma^3\Delta}\Big\{\Big[r^6 + r^4a^2(\cos^2\theta - \sin^2\theta + \sin^2\theta) + r^2a^4(\sin^2\theta\cos^2\theta - \sin^4\theta - \cos^2\theta)$$
$$-a^6(\cos^4\theta - \cos^2\theta\sin^2\theta)\Big] + \Big[-r^6 + r^4a^2(\cos^2\theta - 2\cos^2\theta)$$
$$+ r^2a^4(2\cos^4\theta - \cos^4\theta - \sin^4\theta) + a^6(\cos^6\theta - \cos^2\theta\sin^4\theta)\Big] + 2r^2a^4\sin^4\theta\Big\}$$
$$= \frac{M}{\Sigma^3\Delta}\Big\{\Big[r^6 + r^4a^2\cos^2\theta + r^2a^4(\sin^2\theta\cos^2\theta - \sin^4\theta - \cos^2\theta) - a^6(\cos^4\theta - \cos^2\theta\sin^2\theta)\Big]$$
$$+ \Big[-r^6 - r^4a^2\cos^2\theta + r^2a^4(\cos^4\theta - \sin^4\theta) + a^6(\cos^6\theta - \cos^2\theta\sin^4\theta)\Big] + 2r^2a^4\sin^4\theta\Big\}$$
$$= \frac{M}{\Sigma^3\Delta}\Big[r^2a^4(\sin^2\theta\cos^2\theta - \sin^4\theta - \cos^2\theta + \cos^4\theta - \sin^4\theta)$$
$$+ a^6(-\cos^4\theta + \cos^2\theta\sin^2\theta + \cos^6\theta - \cos^2\theta\sin^4\theta) + 2r^2a^4\sin^4\theta\Big]$$

It might be reasonable to employ some identities in order to get rid of the higher order trigonometric functions:

$$= \frac{M}{\Sigma^3\Delta}\Big\{r^2a^4\Big[\sin^2\theta\cos^2\theta + (1-\sin^2\theta)(1+\sin^2\theta) - 1 - \cos^2\theta + (\cos^2\theta - 1)(\cos^2\theta + 1) + 1$$
$$+ (1-\sin^2\theta)(1+\sin^2\theta) - 1 + 2\sin^4\theta\Big] + a^6\Big[\cos^4\theta(\cos^2\theta - 1) + \cos^2\theta\sin^2\theta(1-\sin^2\theta)\Big]\Big\}$$
$$= \frac{M}{\Sigma^3\Delta}\Big\{r^2a^4\Big[\sin^2\theta\cos^2\theta + (\cos^2\theta)(1+\sin^2\theta) - 1 - \cos^2\theta + (-\sin^2\theta)(\cos^2\theta + 1) + 1$$
$$+ (\cos^2\theta)(1+\sin^2\theta) - 1 + 2\sin^4\theta\Big] + a^6\Big[\cos^4\theta(-\sin^2\theta) + \cos^2\theta\sin^2\theta(\cos^2\theta)\Big]\Big\}$$
$$= \frac{M}{\Sigma^3\Delta}\Big\{r^2a^4\Big[\sin^2\theta\cos^2\theta + \cos^2\theta + \cos^2\theta\sin^2\theta - 1 - \cos^2\theta - \sin^2\theta\cos^2\theta - \sin^2\theta + 1$$
$$+ \cos^2\theta + \cos^2\theta\sin^2\theta - 1 + 2\sin^4\theta\Big] + a^6\Big[-\cos^4\theta\sin^2\theta + \cos^4\theta\sin^2\theta\Big]\Big\}$$
$$= \frac{M}{\Sigma^3\Delta}\Big\{r^2a^4\Big[\sin^2\theta\cos^2\theta(1+1-1+1) + \cos^2\theta(1-1+1) + (-1+1-1) - \sin^2\theta + 2\sin^4\theta\Big]\Big\}$$
$$= \frac{M}{\Sigma^3\Delta}\Big[r^2a^4\Big(2\sin^2\theta\cos^2\theta + \cos^2\theta - 1 - \sin^2\theta + 2\sin^4\theta\Big)\Big]$$
$$= \frac{M}{\Sigma^3\Delta}\Big\{r^2a^4\Big[2\sin^2\theta\cos^2\theta - (1-\cos^2\theta) - \sin^2\theta + 2\sin^4\theta\Big]\Big\}$$
$$= \frac{M}{\Sigma^3\Delta}\Big[r^2a^4\Big(2\sin^2\theta\cos^2\theta - 2\sin^2\theta + 2\sin^4\theta\Big)\Big]$$
$$= \frac{M}{\Sigma^3\Delta}\Big\{r^2a^4\Big[2\sin^2\theta\Big(\cos^2\theta - 1\Big) + 2\sin^4\theta\Big]\Big\}$$
$$= \frac{M}{\Sigma^3\Delta}\Big[r^2a^4\Big(-2\sin^4\theta + 2\sin^4\theta\Big)\Big]$$
$$= 0.$$

θ-component of the geodesic equation:

$$\frac{d^2\theta}{dr^2} + \Gamma^\theta_{\alpha\beta}\frac{dx^\alpha}{dr}\frac{dx^\beta}{dr}$$
$$= \frac{d^2\theta}{dr^2} + \Gamma^\theta_{tt}\left(\frac{dt}{dr}\right)^2 + 2\Gamma^\theta_{tr}\frac{dt}{dr} + 2\Gamma^\theta_{t\theta}\frac{dt}{dr}\frac{d\theta}{dr} + 2\Gamma^\theta_{t\phi}\frac{dt}{dr}\frac{d\phi}{dr} + \Gamma^\theta_{rr} + 2\Gamma^\theta_{r\theta}\frac{d\theta}{dr}$$
$$+ 2\Gamma^\theta_{r\phi}\frac{d\phi}{dr} + \Gamma^\theta_{\theta\theta}\left(\frac{d\theta}{dr}\right)^2 + 2\Gamma^\theta_{\theta\phi}\frac{d\theta}{dr}\frac{d\phi}{dr} + \Gamma^\theta_{\phi\phi}\left(\frac{d\phi}{dr}\right)^2$$
$$= \Gamma^\theta_{tt}\left(\frac{dt}{dr}\right)^2 + 2\Gamma^\theta_{tr}\frac{dt}{dr} + 2\Gamma^\theta_{t\phi}\frac{dt}{dr}\frac{d\phi}{dr} + \Gamma^\theta_{rr} + 2\Gamma^\theta_{r\phi}\frac{d\phi}{dr} + \Gamma^\theta_{\phi\phi}\left(\frac{d\phi}{dr}\right)^2$$

Plugging in the expressions for the Christoffel symbols and the derivative terms:

$$
\begin{aligned}
&\frac{d^2\theta}{dr^2} + \Gamma^{\theta}_{\alpha\beta}\frac{dx^\alpha}{dr}\frac{dx^\beta}{dr} \\
&= \Gamma^{\theta}_{tt}\left(\frac{dt}{dr}\right)^2 + 2\Gamma^{\theta}_{tr}\frac{dt}{dr} + 2\Gamma^{\theta}_{t\phi}\frac{dt}{dr}\frac{d\phi}{dr} + \Gamma^{\theta}_{rr} + 2\Gamma^{\theta}_{r\phi}\frac{d\phi}{dr} + \Gamma^{\theta}_{\phi\phi}\left(\frac{d\phi}{dr}\right)^2 \\
&= \left(-\frac{2Mra^2}{\Sigma^3}\cos\theta\sin\theta\right)\left(-\frac{r^2+a^2}{\Delta}\right)^2 + 0 + 2\left[\frac{2Mra\sin\theta\cos\theta}{\Sigma^3}\left(r^2+a^2\right)\right]\left(-\frac{r^2+a^2}{\Delta}\right)\left(-\frac{a}{\Delta}\right) \\
&+ \left(\frac{a^2}{\Sigma\Delta}\cos\theta\sin\theta\right) + 0 + \Bigg\{ -\frac{\sin\theta\cos\theta}{\Sigma^3}\Big[\big(r^2+a^2\big)(r^4 + a^4\cos^4\theta + 2r^2a^2\cos^2\theta + 4Mra^2\sin^2\theta) \\
&- 2Mra^4\sin^4\theta\Big]\Bigg\}\left(-\frac{a}{\Delta}\right)^2 \\
&= \frac{a^2\sin\theta\cos\theta}{\Sigma^3\Delta^2}\Big[-2Mr(r^2+a^2)^2 + 4Mr(r^2+a^2)^2 + \Sigma^2\Delta \\
&- \big(r^2+a^2\big)(r^4 + a^4\cos^4\theta + 2r^2a^2\cos^2\theta + 4Mra^2\sin^2\theta) + 2Mra^4\sin^4\theta\Big] \\
&= \frac{a^2\sin\theta\cos\theta}{\Sigma^3\Delta^2}\Big[2Mr(r^2+a^2)^2 + (r^2 + a^2\cos^2\theta)^2(r^2 - 2Mr + a^2) \\
&- \big(r^2+a^2\big)(r^4 + a^4\cos^4\theta + 2r^2a^2\cos^2\theta + 4Mra^2\sin^2\theta) + 2Mra^4\sin^4\theta\Big] \\
&= \frac{a^2\sin\theta\cos\theta}{\Sigma^3\Delta^2}\Bigg\{2Mr\left[\left(r^2+a^2\right)^2 - \left(r^2 + a^2\cos^2\theta\right)^2 - 2a^2\sin^2\theta\left(r^2+a^2\right) + a^4\sin^4\theta\right] \\
&+ \left(r^2 + a^2\cos^2\theta\right)^2(r^2+a^2) - \left(r^2+a^2\right)\left(r^4 + a^4\cos^4\theta + 2r^2a^2\cos^2\theta\right)\Bigg\} \\
&= \frac{a^2\sin\theta\cos\theta}{\Sigma^3\Delta^2}\Bigg\{2Mr\Big[\left(r^4 + a^4 + 2a^2r^2\right) - \left(r^4 + a^4\cos^4\theta + 2r^2a^2\cos^2\theta\right) - 2a^2\sin^2\theta\left(r^2+a^2\right) \\
&+ a^4\sin^4\theta\Big] + \left(r^2 + a^2\cos^2\theta\right)^2(r^2+a^2) - \left(r^2+a^2\right)\left(r^2 + a^2\cos^2\theta\right)^2\Bigg\} \\
&= \frac{2Mra^2\sin\theta\cos\theta}{\Sigma^3\Delta^2}\Big[\left(r^4 + a^4 + 2a^2r^2\right) - \left(r^4 + a^4\cos^4\theta + 2r^2a^2\cos^2\theta\right) \\
&- 2a^2\sin^2\theta\left(r^2+a^2\right) + a^4\sin^4\theta\Big] \\
&= \frac{2Mra^2\sin\theta\cos\theta}{\Sigma^3\Delta^2}\left[r^4\Big(1-1\Big) + 2r^2a^2\Big(1 - \cos^2\theta - \sin^2\theta\Big) + a^4\Big(1 - \cos^4\theta - 2\sin^2\theta + \sin^4\theta\Big)\right] \\
&= \frac{2Mra^2\sin\theta\cos\theta}{\Sigma^3\Delta^2}\left[a^4\Big(1 - \cos^4\theta - 2\sin^2\theta + \sin^4\theta\Big)\right]
\end{aligned}
$$

After this point a simple employment of a trig identity leads us to the desired end:

$$
\begin{aligned}
&= \frac{2Mra^2\sin\theta\cos\theta}{\Sigma^3\Delta^2}\left[a^4\Big(1-\cos^4\theta-2\sin^2\theta+\sin^4\theta\Big)\right]\\
&= \frac{2Mra^2\sin\theta\cos\theta}{\Sigma^3\Delta^2}\left\{a^4\Big[(1-2\sin^2\theta)-(\cos^4\theta-\sin^4\theta)\Big]\right\}\\
&= \frac{2Mra^2\sin\theta\cos\theta}{\Sigma^3\Delta^2}\left\{a^4\Big[(1-2\sin^2\theta)-(\cos^2\theta-\sin^2\theta)(\cos^2\theta+\sin^2\theta)\Big]\right\}\\
&= \frac{2Mra^2\sin\theta\cos\theta}{\Sigma^3\Delta^2}\left\{a^4\Big[(1-2\sin^2\theta)-(\cos^2\theta-\sin^2\theta)\Big]\right\}\\
&= \frac{2Mra^2\sin\theta\cos\theta}{\Sigma^3\Delta^2}\left[a^4\Big(\cos 2\theta-\cos 2\theta\Big)\right]\\
&= 0.
\end{aligned}
$$

ϕ-component of the geodesic equation:

$$
\begin{aligned}
&\frac{d^2\phi}{dr^2}+\Gamma^{\phi}_{\alpha\beta}\frac{dx^\alpha}{dr}\frac{dx^\beta}{dr}\\
&= \frac{d^2\phi}{dr^2}+\Gamma^{\phi}_{tt}\left(\frac{dt}{dr}\right)^2+2\Gamma^{\phi}_{tr}\frac{dt}{dr}+2\Gamma^{\phi}_{t\theta}\frac{dt}{dr}\frac{d\theta}{dr}+2\Gamma^{\phi}_{t\phi}\frac{dt}{dr}\frac{d\phi}{dr}+\Gamma^{\phi}_{rr}+2\Gamma^{\phi}_{r\theta}\frac{d\theta}{dr}\\
&+2\Gamma^{\phi}_{r\phi}\frac{d\phi}{dr}+\Gamma^{\phi}_{\theta\theta}\left(\frac{d\theta}{dr}\right)^2+2\Gamma^{\phi}_{\theta\phi}\frac{d\theta}{dr}\frac{d\phi}{dr}+\Gamma^{\phi}_{\phi\phi}\left(\frac{d\phi}{dr}\right)^2\\
&= \Gamma^{\phi}_{tt}\left(\frac{dt}{dr}\right)^2+2\Gamma^{\phi}_{tr}\frac{dt}{dr}+2\Gamma^{\phi}_{t\phi}\frac{dt}{dr}\frac{d\phi}{dr}+\Gamma^{\phi}_{rr}+2\Gamma^{\phi}_{r\phi}\frac{d\phi}{dr}+\Gamma^{\phi}_{\phi\phi}\left(\frac{d\phi}{dr}\right)^2
\end{aligned}
$$

Plugging in the expressions for the Christoffel symbols and the derivative terms:

$$
\begin{aligned}
&\frac{d^2\phi}{dr^2}+\Gamma^{\phi}_{\alpha\beta}\frac{dx^\alpha}{dr}\frac{dx^\beta}{dr}\\
&=\Gamma^{\phi}_{tt}\left(\frac{dt}{dr}\right)^2+2\Gamma^{\phi}_{tr}\frac{dt}{dr}+2\Gamma^{\phi}_{t\phi}\frac{dt}{dr}\frac{d\phi}{dr}+\Gamma^{\phi}_{rr}+2\Gamma^{\phi}_{r\phi}\frac{d\phi}{dr}+\Gamma^{\phi}_{\phi\phi}\left(\frac{d\phi}{dr}\right)^2\\
&=0+2\left[-\frac{Ma}{\Sigma\Delta}\left(1-\frac{2r^2}{\Sigma}\right)\right]\left(-\frac{r^2+a^2}{\Delta}\right)+0+0\\
&+2\left\{\frac{1}{\Sigma^2\Delta}\left\{\Sigma^2 r+M\Big[\Sigma\left(a^2\sin^2\theta-2r^2\right)-2r^2a^2\sin^2\theta\Big]\right\}\right\}\left(-\frac{a}{\Delta}\right)+0\\
&=\frac{2Ma\left(r^2+a^2\right)}{\Sigma^2\Delta^2}\left(\Sigma-2r^2\right)-\frac{2a}{\Sigma^2\Delta^2}\left\{\Sigma^2 r+M\Big[\Sigma\left(a^2\sin^2\theta-2r^2\right)-2r^2a^2\sin^2\theta\Big]\right\}\\
&=\frac{2a}{\Sigma^2\Delta^2}\left\{M\left(r^2+a^2\right)\left(\Sigma-2r^2\right)-\left\{\Sigma^2 r+M\Big[\Sigma\left(a^2\sin^2\theta-2r^2\right)-2r^2a^2\sin^2\theta\Big]\right\}\right\}\\
&=\frac{2a}{\Sigma^2\Delta^2}\Big[M\left(r^2+a^2\right)\left(\Sigma-2r^2\right)-\Sigma^2 r-M\Sigma\left(a^2\sin^2\theta-2r^2\right)+2Mr^2a^2\sin^2\theta\Big]\\
&=\frac{2a}{\Sigma^2\Delta^2}\Big\{M\left(r^2+a^2\right)\left[(r^2+a^2\cos^2\theta))-2r^2\right]-(r^2+a^2\cos^2\theta)^2 r\\
&-M(r^2+a^2\cos^2\theta)\left(a^2\sin^2\theta-2r^2\right)+2Mr^2a^2\sin^2\theta\Big\}\\
&=\frac{2a}{\Sigma^2\Delta^2}\Big[M\left(r^2+a^2\right)\left(a^2\cos^2\theta-r^2\right)-(r^4+a^4\cos^4\theta+2r^2a^2\cos^2\theta)r\\
&-M(r^2a^2\sin^2\theta-2r^4+a^4\cos^2\theta\sin^2\theta-2r^2a^2\cos^2\theta)+2Mr^2a^2\sin^2\theta\Big]\\
&=\frac{2a}{\Sigma^2\Delta^2}\Big\{-(r^4+a^4\cos^4\theta+2r^2a^2\cos^2\theta)r+M\Big[r^4(-1+2)\\
&+r^2a^2(\cos^2\theta-\cos^2\theta-\sin^2\theta+2\cos^2\theta+2\sin^2\theta)+a^4(\cos^2\theta-\cos^2\theta\sin^2\theta)\Big]\Big\}
\end{aligned}
$$

It seems wiser to keep Σ implicit:

$$
\begin{aligned}
&=\frac{2a}{\Sigma^2\Delta^2}\left\{-\Sigma^2 r+M\Big[r^4+r^2a^2(2\cos^2\theta+\sin^2\theta)+a^4\cos^4\theta\Big]\right\}\\
&=\frac{2a}{\Sigma^2\Delta^2}\left\{-\Sigma^2 r+M\Big[(r^4+2r^2a^2\cos^2\theta+a^4\cos^4\theta)+r^2a^2\sin^2\theta\Big]\right\}\\
&=\frac{2a}{\Sigma^2\Delta^2}\Big[-\Sigma^2 r+M\Big(\Sigma^2+r^2a^2\sin^2\theta\Big)\Big]\\
&=\frac{2a}{\Sigma^2\Delta^2}\Big[\Sigma^2\Big(M-r\Big)+Mr^2a^2\sin^2\theta\Big]
\end{aligned}
$$

It is evident that this isn't correct. We must get a vanishing result, like with the other components of the geodesic equation. We are unable to resolve this issue for the moment being.

A note aside:

There's another point we want to touch upon. We had pointed out in the beginning that the Kerr paper and the file on which we based the above analysis (for now we will call it the "The KS file" alluding to the fact that it is entitled "The Kerr Solution") do not seem to be in accord with one another regarding the Kerr metric and the PNV. Let us elaborate on that.

We recall that they were given as:

The Kerr paper

$$\begin{aligned} ds^2 &= -dt^2 + dr^2 + \Sigma d\theta^2 + (r^2+a^2)\sin^2\theta d\phi^2 + 2a\sin^2\theta d\phi dt \\ &+ \frac{2mr}{\Sigma}\left(dr - a\sin^2\theta d\phi + dt\right)^2 \end{aligned}$$

Note: Recall that we found out that in order for k to be a PNV the sign of the ϕ-component should be modified, thus we have a minus sign above.

The KS file

$$\begin{aligned} ds^2 &= -dt^2 + \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) + \left(r^2+a^2\right)\sin^2\theta d\phi^2 \\ &+ \frac{2Mr}{\Sigma}\left(a\sin^2\theta d\phi - dt\right)^2. \end{aligned}$$

The discrepancy starts with the absence of Δ in the former. This isn't such a big deal as $\Delta \approx \Sigma$ for $r \longrightarrow \infty$ and $a \longrightarrow 0$, so both are asymptotically Minkowski.

Ignoring this discrepancy, the first lines are identical. So, if both are Kerr, then so must be the second lines. Are they?

$$\frac{2Mr}{\Sigma}\left(a\sin^2\theta d\phi - dt\right)^2 \quad \& \quad \frac{2mr}{\Sigma}\left(dr - a\sin^2\theta d\phi + dt\right)^2 \tag{A.37}$$

The one from the Kerr paper contains the term in the KS file but also includes some extra terms:

$$\begin{aligned} \frac{2mr}{\Sigma}\left(dr - a\sin^2\theta d\phi + dt\right)^2 &= \frac{2mr}{\Sigma}\left[dr + \left(dt - a\sin^2\theta d\phi\right)\right]^2 \\ &= \frac{2Mr}{\Sigma}\left(a\sin^2\theta d\phi - dt\right)^2 \\ &+ 2a\sin^2\theta\left(1 - \frac{2Mr}{\Sigma}\right)d\phi dt + \frac{4Mr}{\Sigma}dr dt. \end{aligned}$$

We are yet to provide an explanation for this discrepancy.

A.6 PNVs tangent to the central ring in the Kerr metric

In the Kerr spacetime, the ring singularity lies at $r = 0$, $\theta = \pi/2$, which corresponds in Kerr-Schild coordinates to:

$$x^2 + y^2 = a^2, \quad z = 0. \tag{A.38}$$

The Kerr metric admits the PNV (this vector, expressed in Kerr-Schild coordinates, can be found in any standard text):

$$k^a = \left(1, \frac{rx + ay}{r^2 + a^2}, \frac{ry - ax}{r^2 + a^2}, \frac{z}{r}\right). \tag{A.39}$$

Now, if we set $z = 0$ (so that $\theta = \pi/2$) and then take $r \to 0$:

$$k^x \to \frac{ay}{a^2}, \quad k^y \to -\frac{ax}{a^2}. \tag{A.40}$$

This direction is tangent to the circle $x^2 + y^2 = a^2$, showing that the PNV becomes tangent to the ring in the equatorial plane.

A.7 PNVs asymptotic to the event horizons in the Kerr metric

We begin by casting the PNVs of Kerr in the following form:

$$k^\mu \partial_\mu = \left(\frac{r^2 + a^2}{\Delta}\right) \partial_t \pm \partial_r + \left(\frac{a}{\Delta}\right) \partial_\phi, \tag{A.41}$$

where the sign of ∂_r determines whether the PNV is ingoing (-) or outgoing (+). The above form suggests:

$$\frac{dt}{dr} = \frac{k^t}{k^r} = \frac{(r^2 + a^2)/\Delta}{\pm 1} = \pm \frac{r^2 + a^2}{\Delta}. \tag{A.42}$$

This expression simplifies with the respective approximations for $r \to r_+$ and $r \to r_-$.

For outgoing:

$$\frac{dt}{dr} \approx \frac{r_+^2 + a^2}{(r - r_+)(r_+ - r_-)} = A \frac{1}{r - r_+}. \tag{A.43}$$

For ingoing:

$$\frac{dt}{dr} \approx -\frac{r_-^2 + a^2}{(r - r_-)(r_- - r_+)} = B \frac{1}{r - r_-}. \tag{A.44}$$

In order to see how we got the respective approximations for Δ, one needs but to set $r = r_\pm + \epsilon$. We should also note that A and B are positive constants. Now we integrate from some r_0 between the horizons to respectively r_+ and r_-. Noting the sign of the denominator of the integrand in each case, we get:

For outgoing:

$$t \sim \int_{r_0}^{r+} \frac{1}{r - r_+} dr = -\infty. \tag{A.45}$$

For ingoing:

$$t \sim \int_{r_0}^{r-} \frac{1}{r - r_-} dr = +\infty. \tag{A.46}$$

So, indeed, as Kerr claims, the PNVs in question are asymptotic to the outer horizon as $t \to -\infty$ and to the inner horizon as $t \to +\infty$.

A.8 Incomplete geodesics in Kerr spacetime that do not end up at singularities

In the original coordinates, the Kerr metric reads

$$ds^2 = ds_0^2 + \frac{2mr}{\Sigma} k^2, \quad k = dr + a \sin^2\theta d\phi + dt, \tag{A.47}$$

where

$$ds_0^2 = dr^2 + \Sigma d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2 + 2a\sin^2\theta d\phi dt - dt^2 \tag{A.48}$$

with

$$\Sigma = r^2 + a^2\cos^2\theta. \tag{A.49}$$

With the transformation

$$x + iy = (r + ia)e^{i\phi}\sin\theta, \quad z = r\cos\theta, \tag{A.50}$$

we get the Kerr-Schild form of the metric:

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 + \frac{2mr^3}{r^4 + a^2z^2}\left[dt + \frac{z}{r}dz + \frac{r}{r^2+a^2}(xdx + ydy) + \frac{a}{r^2+a^2}(xdy - ydx)\right]^2. \tag{A.51}$$

As can be seen clearly from the first version of the metric, a singularity is exhibited when $r = 0$ and $\theta = \frac{\pi}{2}$. This is the ring singularity. “There is no singularity problem when the ring is replaced by an appropriate star.”

The axial PNVs are found through:

$$ds^2 = -dt^2 + dr^2 + \frac{2mr}{r^2 + a^2}(dr + dt^2) = 0. \tag{A.52}$$

This yields

$$\frac{dr}{dt} = -1 \tag{A.53}$$

for the incoming geodesics and

$$\frac{dr}{dt} = \frac{r^2 - 2mr + a^2}{r^2 + 2mr + a^2} \tag{A.54}$$

for the outgoing ones.

So, inside the horizon also the outgoing PNV points inwards. In particular, an outgoing light ray in this region would approach the inner horizon as $t \to +\infty$ and the outer horizon $t \to -\infty$, this is quite easy to intuit.

The distance between the two horizons is $2\sqrt{m^2 - a^2}$. r being an affine parameter, the affine distance of this trajectory is finite, yet we do not end up at a singularity.

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