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Multipole expansion of the electromagnetic field with applications

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Sommario

Nel presente lavoro si determina il campo elettromagnetico generato da una generica distribuzione di carica e corrente, utilizzando le equazioni di Maxwell nel gauge di Lorenz. Questo semplifica il problema a equazioni d'onda non omogenee che sono risolte con soluzioni ritardate. Mediante l'espansione in multipoli si ottengono le espressioni dei campi elettrico e magnetico, distinguendo tra regime di campo vicino e lontano, e si analizza la potenza irradiata. I risultati vengono applicati al caso di una carica puntiforme in moto generico, poi circolare periodico. Infine, si ricavano le forze e i momenti torcenti su una distribuzione di carica e corrente puntiforme immersa in un campo elettromagnetico esterno, con applicazioni alla polarizzazione elettrica, alla precessione di Larmor e all'esperimento di Stern-Gerlach in ottica classica.

Abstract

In this work, the electromagnetic field generated by a distribution of charge and current is determined, using Maxwell's equations in the Lorenz gauge. This simplifies the problem to inhomogeneous wave equations, which are solved through retarded solutions. By employing a multipole expansion, expressions for the electric and magnetic fields are derived, distinguishing between the near and far field regimes, and the radiated power is analyzed. The results are then applied to the case of a point charge moving along a general trajectory and a periodic circular trajectory. Finally, the forces and torques exerted on a point charge and current distribution in an external electromagnetic field are derived, with applications to electric polarization, Larmor precession, and a classical interpretation of the Stern-Gerlach experiment.

Contents

Introduction

During the 19th century, the scientific study of electric and magnetic phenomena truly began to take shape. Although certain electrostatic and magnetic effects—such as the attractive property of rubbed amber (from which electricity takes its name) or the persistent force between samples of magnetite—had been known since Ancient Greek times (7th century BCE), it was in the 19th century that a systematic and quantitative approach to these phenomena emerged.

Pioneering figures such as Coulomb, who formulated the law of electrostatic attraction between charges; Biot and Savart, who conducted early quantitative studies in magnetism; and Ampère, who investigated current-induced magnetic effects, made significant strides in understanding these forces. Ørsted and Faraday experimentally demonstrated that a magnetic field—and thus a force—arises around a current-carrying wire, revealing for the first time a fundamental connection between electricity and magnetism. Ampère later confirmed this link by observing interactions between parallel current-carrying wires.

In the latter half of the century, James Clerk Maxwell unified the results of these experiments into a single theoretical framework through the formulation of four equations. Together with Lorentz and Joule laws, these equations describe all known electric and magnetic phenomena, linking electric and magnetic fields and explaining their interactions with electric charges. This marked the birth of the concept of the electromagnetic field, a unified entity encompassing electric and magnetic components. Einstein later reinforced this perspective by showing, through his theory of special relativity, that electricity and magnetism are manifestations of the same physical reality, observed differently in different reference frames. Today, the electromagnetic field is considered a fundamental entity that permeates all space, intrinsically coupled to the presence and motion of electric charges.

This work presents a framework for computing the solution of Maxwell's equations for the electromagnetic field generated by an arbitrary distribution of charge and current. The analysis employs the multipole expansion formalism as an approximation technique to obtain analytically tractable expressions. The results so obtained are then applied to various configurations and used to explain relevant electromagnetic phenomena.

The study begins by expressing Maxwell's equations in a more convenient form by casting the electric and magnetic fields in terms of their potentials and exploiting their gauge symmetry. In particular, by adopting the Lorenz gauge, the equations reduce to inhomogeneous wave equations, whose solutions are presented. For the purpose of describing the electromagnetic potentials, only the retarded solution is considered. To simplify its computation the multipole expansion is employed, in virtue of which the expressions become controlled by key coefficients, called multipoles, with clear physical interpretations. The approximation adopts simplified forms within distinct physical regimes, notably the near-field and far-field regimes, which are carefully discussed along with the radiation regime being a particular case of the second one.

After deriving the expressions for the electric and magnetic fields in both regimes, the analysis focuses on evaluating the radiated power within the radiation regime. The general result is applied first to an accelerated point charge moving along an arbitrary trajectory, and subsequently to the specific case of a charge in periodic circular motion, drawing an analogy with Rutherford's atomic model.

Finally, the interaction between a pointlike charge and current distribution and an external electromagnetic field is analyzed. General expressions for the resulting forces and torques acting on the distribution are derived. These results are then applied to explain relevant electromagnetic phenomena, such as electric polarization and Larmor precession, by modeling molecules and atoms as electric and magnetic dipoles, respectively. Additionally, a partial classical interpretation of the Stern–Gerlach experiment is proposed, in which atoms possessing a magnetic dipole moment traverse an inhomogeneous magnetic field, resulting in a deflection of their trajectories.

Chapter 1

Electromagnetism equations

1.1 Foundations of electromagnetism

Classical electromagnetism is the theory that describes electric and magnetic phenomena in nature, based on the *Maxwell equations*:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi\rho, \qquad \text{(Electric Gauss law)} \tag{1.1a}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = \boldsymbol{0}, \qquad \text{(Magnetic Gauss law)} \tag{1.1b}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \qquad \text{(Faraday - Neumann - Lenz law)} \tag{1.1c}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}. \qquad \text{(Ampère - Maxwell law)} \tag{1.1d}$$

The electric field $\boldsymbol{E} = \boldsymbol{E}(\boldsymbol{x}, t)$ and magnetic induction field $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{x}, t)$ are generated by electric charges $\rho = \rho(\boldsymbol{x}, t)$ and electric currents $\boldsymbol{J} = \boldsymbol{J}(\boldsymbol{x}, t)$ and bound to each other according to (1.1). From Gauss and Ampère - Maxwell laws one can derive the *continuity equation*

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0 \tag{1.2}$$

in which is enclosed the conservation of the electric charge. From Magnetic Gauss and Faraday - Neumann - Lenz laws one can deduce that a scalar field $\Phi = \Phi(\boldsymbol{x}, t)$ and a vector field $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{x}, t)$ may be defined to derive the electric and magnetic induction fields

$$\boldsymbol{E} = -\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \tag{1.3a}$$

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \tag{1.3b}$$

As it will be stressed, the choice of the *potentials* Φ and $\boldsymbol{A}$ is not unique, and this has important consequences.

The potential formulation of electromagnetism is often more convinient, and it leads to a more managable form of the inhomogeneous Maxwell equations (1.1a),(1.1c) which were not exploited to formulate (1.3) and so are indipendent. Those combined give

$$\boldsymbol{\nabla}^2\Phi + \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\boldsymbol{A} = -4\pi\rho \tag{1.4a}$$

$$\boldsymbol{\nabla}^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \boldsymbol{\nabla}\left[\frac{1}{c}\frac{\partial \Phi}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{A}\right] = -\frac{4\pi}{c}\boldsymbol{J} \tag{1.4b}$$

Finding a solution of these equations is equivalent to solve for $\boldsymbol{E}$ and $\boldsymbol{B}$ in Maxwell equations.

1.2 Gauge invariance and Lorenz gauge

The theory of electromagnetism benefits from an important symmetry, *gauge invariance*. Under a simultaneous trasformation G of Φ and $\boldsymbol{A}$, depending on an arbitrary scalar field $\chi = \chi(\boldsymbol{x}, t)$,

$$\Phi \xrightarrow{\mathrm{G}} \Phi' = \Phi + \frac{1}{c}\frac{\partial \chi}{\partial t} \tag{1.5a}$$

$$\boldsymbol{A} \xrightarrow{\mathrm{G}} \boldsymbol{A}' = \boldsymbol{A} - \boldsymbol{\nabla}\chi \tag{1.5b}$$

the resulting fields $\boldsymbol{E}' = -\boldsymbol{\nabla}\Phi' - \frac{\partial \boldsymbol{A}'}{\partial t}$ and $\boldsymbol{B}' = \boldsymbol{\nabla} \times \boldsymbol{A}'$ coincide with $\boldsymbol{E}$ and $\boldsymbol{B}$ for every choice of χ. The fields Φ and $\boldsymbol{A}$ chosen to determine $\boldsymbol{E}$ and $\boldsymbol{B}$ define a *gauge choice*.

It is possible to formulate a constraint $\Gamma[\Phi, \boldsymbol{A}] = 0$ in which Γ is a scalar field that may depend on finitely many derivatives of Φ and $\boldsymbol{A}$. The constraint is called a *gauge fixing condition*. In general not all the $\Gamma[\Phi, \boldsymbol{A}]$ are acceptable to define a gauge fixing condition. Indeed one must satisfy the condition that for every pair Φ, $\boldsymbol{A}$ there exist corresponding Φ', $\boldsymbol{A}'$ after a gauge trasformation (1.5) satisfying $\Gamma[\Phi', \boldsymbol{A}'] = 0$. Those which satisfy this condition are said to be *admissible*.

A noteworthy gauge fixing condition is the *Lorenz gauge*:

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{A} = 0 \tag{1.6}$$

which is an admissible gauge fixing condition. There is a residual gauge invariance respecting the Lorenz gauge that corresponds to all the gauge trasformations such that the scalar field χ satisfies the D'Alembert equation

$$\boldsymbol{\nabla}^2\chi - \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2} = 0 \tag{1.7}$$

Proof: Let Φ', $\boldsymbol{A}'$ be the potentials satisfying the Lorenz gauge after a gauge transformation (1.5). Thus

$$\begin{aligned}\frac{1}{c}\frac{\partial \Phi'}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{A}' &= \frac{1}{c}\frac{\partial}{\partial t}\left(\Phi + \frac{1}{c}\frac{\partial \chi}{\partial t}\right) + \boldsymbol{\nabla}\cdot(\boldsymbol{A} - \boldsymbol{\nabla}\chi) \\ &= -\left(\boldsymbol{\nabla}^2\chi - \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2}\right) + \frac{1}{c}\frac{\partial \Phi}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{A} = 0\end{aligned}$$

is satisfied for every choice of χ provided

$$\boldsymbol{\nabla}^2\chi - \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2} = \frac{1}{c}\frac{\partial \Phi}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{A} \tag{1.8}$$

holds. It can be shown that the former equation always has solutions using the Green's function method, thereby making the Lorenz gauge admissible. If it is required for any pair Φ,$\boldsymbol{A}$ to meet the Lorenz gauge (1.6) too, from (1.8) follows that (1.7) must be satisfied in order to perform an additional gauge transformation.

The Lorenz gauge (1.6) is useful in this case because using it in (1.4) yields

$$\boldsymbol{\nabla}^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -4\pi\rho, \tag{1.9a}$$

$$\boldsymbol{\nabla}^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = -\frac{4\pi}{c}\boldsymbol{J}, \tag{1.9b}$$

a more symmetric casting of the equations of the electromagnetic field potentials.

Chapter 2

Wave equation and multipole expansion

2.1 The scalar and vector wave equation

Let $\psi = \psi(\boldsymbol{x}, t)$ be a scalar field, the *inhomogeneous scalar wave equation* is

$$\boldsymbol{\nabla}^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -4\pi f, \tag{2.1}$$

where $f = f(\boldsymbol{x}, t)$ is the *scalar source*. If the source is absent f is identically zero (2.1) becomes the *homogeneous scalar wave equation*, also referred to as *D'Alambert equation*

$$\boldsymbol{\nabla}^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0. \tag{2.2}$$

The inhomogeneous scalar wave equation admits the standard *scalar advanced* and *retarded solutions*:

$$\psi_f^{\pm}(\boldsymbol{x}, t) = \int d^3\boldsymbol{x}' \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} f\left(\boldsymbol{x}', t \pm \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{c}\right) \tag{2.3}$$

In the course of the discussion only the retarded solution will be taken into account because of its physical relevance in studying retarded potentials of the electromagnetic field. The principle of causality states that every event can be influenced only by past events; therefore, the retarded solution that depends on the source evaluated at a future event must be discarded. The time shift in the evaluation of the source, due to the finiteness of the speed of light, leads to the field being computed after a time interval of $|\boldsymbol{x} - \boldsymbol{x}'|/c$, which corresponds to the time it takes for light to travel from the source to the observation point.

Similarly to what it is done for scalar fields, for a vector field $\boldsymbol{\psi} = \boldsymbol{\psi}(\boldsymbol{x}, t)$ it is defined the inhomogeneous vector wave equation by the substitution $\psi \mapsto \boldsymbol{\psi}, f \mapsto \boldsymbol{f}$ in (2.1). $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x}, t)$ is the *vector source* and (2.3) holds for each component of $\boldsymbol{\psi}$ and $\boldsymbol{f}$. Solving (2.1) and its vector version provides the solution of (1.9) as they are of the same form.

2.2 Multipole expansion

It is clear that finding a solution of the form (2.3) may be a very difficult task for a generic source f. In this case it comes in helpful the multipole expansion of the retarded solution which gives an approximated evaluation of the latter. As will be seen, the multipole expansion provides a very convenient framework to work in, as it allows all the $\boldsymbol{x}$-dependent terms to be taken outside the integrals appearing in the solution (2.3), so that the computation is reduced to standard time depending coefficients, the multipole moments, in which the dependence of the solution on the source f is encoded.

For the multipole expansion to be valid it is necessary that the following requirements are fulfilled. There is a lenght scale d such that

$$f(\boldsymbol{x}', t) \approx 0, \quad |\boldsymbol{x}'| \gg d \tag{2.4a}$$

and that

$$|f(\boldsymbol{x}', t)| \gg \left| \frac{d}{c} \frac{\partial f}{\partial t}(\boldsymbol{x}', t) \right| \gg \left| \left(\frac{d}{c}\right)^2 \frac{\partial^2 f}{\partial t^2}(\boldsymbol{x}', t) \right| \gg ..., \qquad |\boldsymbol{x}'| \lesssim d \tag{2.4b}$$

at every time instant t, besides

$$|\boldsymbol{x}| \gg d. \tag{2.5}$$

Conditions (2.4) means that there exists a region of the space of linear size of d in which the source is confined. Outside of this region the source-if not entirely absent-is still negligible. Furthermore the magnitude of the source inside the region is much bigger than the variation that undergoes the source in the time taken for light to travel the region, and this happens at every order of derivation. Mathematically speaking (2.4b) implies that the Taylor expansion of f (chosen to be as smooth as required) around $t' = t + \frac{d}{c}$ in the limit of $t' \to t$

$$f(\boldsymbol{x}', t') \approx \sum_{n=0}^{N} \frac{1}{n!}\left(\frac{d}{c}\right)^n \frac{\partial^n f}{\partial t^n}(\boldsymbol{x}', t) \tag{2.6}$$

rapidly converges and is approximately equal to its first term $f(\boldsymbol{x}', t)$. The last condition (2.5) states that in order to carry the analysis the observation point where the wave is computed must be far from the source. When the (2.4), (2.5) conditions are met the wave retarded solution (2.3) admits the *multipole expansion*

$$\begin{aligned} \psi_f(\boldsymbol{x}, t) \approx \frac{1}{|\boldsymbol{x}|}\Bigg[f^{(0)}(t_-) + \frac{\boldsymbol{f}^{(1)}(t_-)\cdot\hat{\boldsymbol{x}}}{|\boldsymbol{x}|} + \frac{\dot{\boldsymbol{f}}^{(1)}(t_-)\cdot\hat{\boldsymbol{x}}}{c} + \frac{\ddot{f}'^{(2)}(t_-)}{3c^2} \\ + \frac{\boldsymbol{f}^{(2)}(t_-)\cdot\cdot\,\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}}{|\boldsymbol{x}|^2} + \frac{\dot{\boldsymbol{f}}^{(2)}(t_-)\cdot\cdot\,\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}}{c|\boldsymbol{x}|} + \frac{\ddot{\boldsymbol{f}}^{(2)}(t_-)\cdot\cdot\,\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}}{3c^2} \Bigg]_{t_- = t - \frac{|\boldsymbol{x}|}{c}} \end{aligned} \tag{2.7}$$

in which

$$f^{(0)}(t) = \int d^3\boldsymbol{x}'\, f(\boldsymbol{x}', t), \tag{2.8a}$$

$$\boldsymbol{f}^{(1)}(t) = \int d^3\boldsymbol{x}'\, f(\boldsymbol{x}', t)\boldsymbol{x}', \tag{2.8b}$$

$$f'^{(2)}(t) = \int d^3\boldsymbol{x}'\, f(\boldsymbol{x}', t)\frac{|\boldsymbol{x}'|^2}{2}, \tag{2.8c}$$

$$\boldsymbol{f}^{(2)}(t) = \int d^3\boldsymbol{x}'\, f(\boldsymbol{x}', t)\frac{3\boldsymbol{x}'\boldsymbol{x}' - \boldsymbol{x}'^2\mathbf{1}}{2}, \tag{2.8d}$$

are the *multipole moments*. As one can see, the solution now depends on the position $\boldsymbol{x}$ in a simple form: a sum of powers of the reciprocal of $|\boldsymbol{x}|$. It is the contribution of the multipole moments that shapes the wave, based on the statistical-like moments of the source scalar field f. Indeed, the moments in equation (2.8) are analogous to statistical moments in their mathematical structure—except that, in electromagnetism, the source f can be negative and vector-valued, and therefore does not represent a probability distribution. See fig.2.1 for a pictorial representation of the problem.

Proof: The Fourier coefficients of the solution of the scalar wave equation (2.3) are

$$\psi_{f\omega}(\boldsymbol{x}) = \int d^3\boldsymbol{x}' \, \frac{e^{ik_\omega|\boldsymbol{x}-\boldsymbol{x}'|}}{|\boldsymbol{x}-\boldsymbol{x}'|} f_\omega(\boldsymbol{x}') \tag{2.9}$$

where f_ω is the Fourier transform of frequency ω of the source field f and $k_\omega = \frac{\omega}{c}$ the wave vector. The kernel integral in the equation may be expanded around $|\boldsymbol{x}|$, with step $\boldsymbol{x}'$:

$$\begin{aligned}\frac{e^{ik_\omega|\boldsymbol{x}-\boldsymbol{x}'|}}{|\boldsymbol{x}-\boldsymbol{x}'|} = \frac{e^{ik_\omega|\boldsymbol{x}|}}{|\boldsymbol{x}|}\Big\{&1 + (1 - ik_\omega|\boldsymbol{x}|)\hat{\boldsymbol{x}}\cdot\frac{\boldsymbol{x}'}{|\boldsymbol{x}|} - \frac{1}{6}(k_\omega|\boldsymbol{x}|)^2\Big(\frac{|\boldsymbol{x}'|}{|\boldsymbol{x}|}\Big)^2 + \\ &(1 - ik_\omega|\boldsymbol{x}| - \frac{1}{3}(k_\omega|\boldsymbol{x}|)^2)\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}\cdot\cdot\frac{3\boldsymbol{x}'\boldsymbol{x}' - \boldsymbol{x}'^2\mathbf{1}}{2|\boldsymbol{x}|^2} + \mathcal{O}\Big(\frac{|\boldsymbol{x}'|^3}{|\boldsymbol{x}|^3}\Big)\Big\}\end{aligned} \tag{2.10}$$

This expansion is the key for the multipole formula and the proof resides in it. To show it, let $F(r)$ be a smooth function in $r \geq 0$. Using Taylor's expansion

$$F(|\boldsymbol{x}-\boldsymbol{x}'|) = F(|\boldsymbol{x}|) - \boldsymbol{x}'\cdot\boldsymbol{\nabla}F(|\boldsymbol{x}|) + \frac{1}{2}\boldsymbol{x}'\boldsymbol{x}'\cdot\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}F(|\boldsymbol{x}|) + \mathcal{O}\Big(\frac{|\boldsymbol{x}'|^3}{|\boldsymbol{x}|^3}\Big) \tag{2.11}$$

Now if

$$F(r) = \frac{e^{ik_\omega r}}{r},$$

substituting it in the previous formula gives (2.10).

From now on the terms in the third order or higher of the expansion will be neglected.

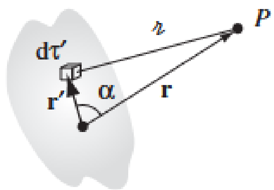


Figure 2.1: A graphic representation of the problem, the grey shade represents a generic charge and current distribution volume, in the text $\boldsymbol{x} \equiv \boldsymbol{r}$, $\boldsymbol{x}' \equiv \boldsymbol{r}'$, $|\boldsymbol{x}-\boldsymbol{x}'| \equiv r$, $d^3\boldsymbol{x}' \equiv d\tau'$. The observation point is denoted by P. ([2], p.152)

The approximation is valid in the case

$$|\boldsymbol{x}| \gg |\boldsymbol{x}'| \tag{2.12a}$$

$$|k_\omega \boldsymbol{x}'| \ll 1 \tag{2.12b}$$

These two conditions are sufficient to grant the validity of the latest approximation.

Let d be a lenght scale such that

$$f_\omega(\boldsymbol{x}') \approx 0, \quad |\boldsymbol{x}'| \gg d \tag{2.13a}$$

and

$$|k_\omega d| \ll 1 \tag{2.13b}$$

for every frequency ω. This last condition has a precise physical meaning. Defining the wavelenght with λ, as $k \sim 1/\lambda$ by (2.13b) follows that $d \ll \lambda$, which will be stressed ahead. Furthermore suppose that

$$|\boldsymbol{x}| \gg d \tag{2.14}$$

So the integration in (2.9) is restricted to the values such that $|\boldsymbol{x}'| \lesssim d$ by the first condition which together with the second one are sufficient to grant (2.12).

It can be shown that (2.13) are equal to (2.4). The equivalence of the first hypothesis is obvious since the relation between f and its coefficients f_ω in the Fourier transform formula. To prove that (2.13b) is the same as requiring (2.4b), an euristic argument based on equivalence of orders of magnitude will be used. Fix $\boldsymbol{x}'$, suppose that $f(\boldsymbol{x}', t) \not\approx 0$ for some t in the neighborhood of $\bar{t}$ of breadth Δt and $f_\omega(\boldsymbol{x}') \not\approx 0$ for some ω in the neighborhood of $\bar{\omega}$ of breadth $\Delta\omega$. It is known from Fourier theory that

$$\Delta\omega\Delta t \sim 2\pi \tag{2.15}$$

and from an analysis based on this result follows

$$
\begin{aligned}
\left|\left(\frac{d}{c}\right)^n \frac{\partial^n f}{\partial t^n}(\boldsymbol{x}', t)\right| &= \left|\left(\frac{d}{c}\right)^n \frac{\partial^n}{\partial t^n} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t} f_\omega(\boldsymbol{x}')\right| \\
&= \left|\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t} \left(\frac{-i\omega d}{c}\right)^n f_\omega(\boldsymbol{x}')\right| \\
&\sim \frac{\Delta\omega}{2\pi} \left(\frac{\bar{\omega} d}{c}\right)^n |f_{\bar{\omega}}(\boldsymbol{x}')| \\
&= \frac{\Delta\omega}{2\pi} (k_{\bar{\omega}} d)^n \left|\int_{-\infty}^{+\infty} dt\, e^{i\bar{\omega} t} f(\boldsymbol{x}', t)\right| \\
&\sim \frac{\Delta\omega \Delta t}{2\pi} (k_{\bar{\omega}} d)^n |f(\boldsymbol{x}', \bar{t})| \\
&\sim (k_\omega d)^n |f(\boldsymbol{x}', t)|
\end{aligned}
$$

and the equivalence of (2.4b) and (2.13b).

The last thing to do is to reconstruct the solution in its multipole expansion. Using the integral kernel expansion (2.10) and taking out the integrals all the terms which do not depend on the measure

$$
\begin{aligned}
\psi_{f\omega}(\boldsymbol{x}) = \frac{e^{ik_\omega |\boldsymbol{x}|}}{|\boldsymbol{x}|} \Big\{ f_\omega^{(0)} + (1 - ik_\omega |\boldsymbol{x}|) \hat{\boldsymbol{x}} \cdot \frac{\boldsymbol{f}_\omega^{(1)}}{|\boldsymbol{x}|} - \frac{1}{3} (k_\omega |\boldsymbol{x}|)^2 \frac{f_\omega'^{(2)}}{|\boldsymbol{x}|^2} \\
+ (1 - ik_\omega |\boldsymbol{x}| - \frac{1}{3} (k_\omega |\boldsymbol{x}|)^2) \hat{\boldsymbol{x}}\hat{\boldsymbol{x}} \cdot\cdot \frac{\boldsymbol{f}_\omega^{(2)}}{|\boldsymbol{x}|^2} \Big\}
\end{aligned}
\tag{2.16}
$$

where

$$ f_\omega^{(0)} = \int d^3\boldsymbol{x}'\, f_\omega(\boldsymbol{x}'), \tag{2.17a} $$

$$ \boldsymbol{f}_\omega^{(1)} = \int d^3\boldsymbol{x}'\, f_\omega(\boldsymbol{x}') \boldsymbol{x}', \tag{2.17b} $$

$$ f_\omega'^{(2)} = \int d^3\boldsymbol{x}'\, f_\omega(\boldsymbol{x}') \frac{|\boldsymbol{x}'|^2}{2}, \tag{2.17c} $$

$$ \boldsymbol{f}_\omega^{(2)} = \int d^3\boldsymbol{x}'\, f_\omega(\boldsymbol{x}') \frac{3\boldsymbol{x}'\boldsymbol{x}' - \boldsymbol{x}'^2 \mathbf{1}}{2}, \tag{2.17d} $$

are the *frequency representation multipole moments* of the source f. Using the Fourier transform to get back to the time representation of the solution of the wave function and

(2.16)

$$\psi_f(\boldsymbol{x}, t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t} \psi_{f\omega}(\boldsymbol{x}) \tag{2.18}$$

one notices that (2.17) are the fourier coefficients of (2.8) and by performing a transform so that $-i\omega \mapsto \frac{d}{dt}$ from (2.18) one gets (2.7).

A remark should be made regarding the use of the conditions that enabled the approximation. Because of (2.4a) the integration was restricted only in a well delimited region of the space. This request with (2.5) made possible the expansion (2.10), the fundamental computation at the root of the multipole expansion. Recall that retarded solutions are considered in (2.3). The Fourier formulation was crucial to displace the dipendence of the time variable on $\boldsymbol{x}'$, namely the integration variable, and (2.4b) to remove it from all the other factors containing $|\boldsymbol{x} - \boldsymbol{x}'|$.

The results obtained for the scalar field can be generalized to the vector wave field solution.

Chapter 3

Multipole expansion for the electromagnetic field

3.1 Electromagnetic formulation

In the last chapter an approximation of the solution of (2.1) was found. Recalling that the latter is of the same form of (1.9), the results are valid for Φ, $\boldsymbol{A}$ with the respective sources ρ, $\boldsymbol{J}$. In the electromagnetic formulation the multipole expansion conditions (2.4) are expressed in the following form.

There exists a lenght scale d (which is the scale of the region of localization of the charges and currents) such that

$$\rho(\boldsymbol{x}, t) \approx 0, \quad \boldsymbol{J}(\boldsymbol{x}, t) \approx \boldsymbol{0}, \quad |\boldsymbol{x}| \gg d \tag{3.1a}$$

$$|\rho(\boldsymbol{x}, t)| \gg \left| \frac{d}{c} \frac{\partial \rho}{\partial t}(\boldsymbol{x}, t) \right| \gg \left| \left(\frac{d}{c}\right)^2 \frac{\partial^2 \rho}{\partial t^2}(\boldsymbol{x}, t) \right| \gg ..., \tag{3.1b}$$

$$|\boldsymbol{J}(\boldsymbol{x}, t)| \gg \left| \frac{d}{c} \frac{\partial \boldsymbol{J}}{\partial t}(\boldsymbol{x}, t) \right| \gg \left| \left(\frac{d}{c}\right)^2 \frac{\partial^2 \boldsymbol{J}}{\partial t^2}(\boldsymbol{x}, t) \right| \gg ..., \qquad |\boldsymbol{x}| \lesssim d \tag{3.1c}$$

at every time t.

The multipole expansion allows to cast the fields Φ,$\boldsymbol{A}$ in terms of the multipole moments found in (2.8). However the constraint given by the continuity equation (1.2) puts the sources of the electromagnetic field in a relationship such that the only multipole moments needed to express the expansion of the potentials are

$$q(t) = \int d^3\boldsymbol{x}\, \rho(\boldsymbol{x}, t), \tag{3.2a}$$

$$\boldsymbol{p}(t) = \int d^3\boldsymbol{x}\, \rho(\boldsymbol{x}, t)\boldsymbol{x}, \tag{3.2b}$$

$$\boldsymbol{Q}(t) = \int d^3\boldsymbol{x}\, \rho(\boldsymbol{x}, t)\frac{3\boldsymbol{x}\boldsymbol{x} - \boldsymbol{x}^2\mathbf{1}}{2}, \tag{3.2c}$$

$$\boldsymbol{m}(t) = \int d^3\boldsymbol{x}\, \frac{1}{2c}\boldsymbol{x}\times\boldsymbol{j}(\boldsymbol{x}, t). \tag{3.2d}$$

$q(t)$ is the *electric charge*, it is a scalar that can be shown to be indipendent from the time t. $\boldsymbol{p}(t)$ and $\boldsymbol{m}(t)$ are both vectors called respectively *electric dipole moment* and *magnetic dipole moment*. $\boldsymbol{Q}(t)$ is the *electric quadrupole moment* and it is a traceless symmetric dyad.

The multipole expansion is given provided the observation point $\boldsymbol{x}$ in which the potentials are computed is such that

$$|\boldsymbol{x}| \gg d \tag{3.3}$$

as required in (2.5).

Then the retarded gauge potentials Φ,$\boldsymbol{A}$ in the Lorenz gauge admit the expansion

$$\begin{aligned}\Phi(\boldsymbol{x}, t) = \Bigg[&\frac{q}{|\boldsymbol{x}|} + \frac{\boldsymbol{p}(t_-)\cdot\hat{\boldsymbol{x}}}{|\boldsymbol{x}|^2} + \frac{\dot{\boldsymbol{p}}(t_-)\cdot\hat{\boldsymbol{x}}}{c|\boldsymbol{x}|} \\ &+ \frac{\boldsymbol{Q}(t_-)\cdot\cdot\,\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}}{|\boldsymbol{x}|^3} + \frac{\dot{\boldsymbol{Q}}(t_-)\cdot\cdot\,\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}}{c|\boldsymbol{x}|^2} + \frac{\ddot{\boldsymbol{Q}}(t_-)\cdot\cdot\,\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}}{3c^2|\boldsymbol{x}|}\Bigg]_{t_-=t-\frac{|\boldsymbol{x}|}{c}}\end{aligned} \tag{3.4a}$$

$$\begin{aligned}\boldsymbol{A}(\boldsymbol{x}, t) = \Bigg[&\frac{\dot{\boldsymbol{p}}(t_-)}{c|\boldsymbol{x}|} + \frac{\dot{\boldsymbol{Q}}(t_-)\cdot\hat{\boldsymbol{x}}}{3c|\boldsymbol{x}|^2} + \frac{\ddot{\boldsymbol{Q}}(t_-)\cdot\hat{\boldsymbol{x}}}{3c^2|\boldsymbol{x}|} \\ &+ \frac{\boldsymbol{m}(t_-)\times\hat{\boldsymbol{x}}}{|\boldsymbol{x}|^2} + \frac{\dot{\boldsymbol{m}}(t_-)\times\hat{\boldsymbol{x}}}{c|\boldsymbol{x}|}\Bigg]_{t_-=t-\frac{|\boldsymbol{x}|}{c}}\end{aligned} \tag{3.4b}$$

up to a gauge transformation (1.5) that preserves the Lorenz gauge (1.6).

Proof: Because of the Lorenz gauge the potentials Φ, $\boldsymbol{A}$ satisfy inhomogeneous wave equations with ρ and $\boldsymbol{J}$ as sources (1.9) and the conditions for a multipole expansion are met in (3.1), so the fields can be expressed as

$$\Phi(\boldsymbol{x},t) = \frac{1}{|\boldsymbol{x}|}\Bigg[\rho^{(0)}(t_-) + \frac{\boldsymbol{\rho}^{(1)}(t_-)\cdot\hat{\boldsymbol{x}}}{|\boldsymbol{x}|} + \frac{\dot{\boldsymbol{\rho}}^{(1)}(t_-)\cdot\hat{\boldsymbol{x}}}{c} + \frac{\ddot{\rho}'^{(2)}(t_-)}{3c^2} + \frac{\boldsymbol{\rho}^{(2)}(t_-)\cdot\cdot\,\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}}{|\boldsymbol{x}|^2} + \frac{\dot{\boldsymbol{\rho}}^{(2)}(t_-)\cdot\cdot\,\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}}{c|\boldsymbol{x}|} + \frac{\ddot{\boldsymbol{\rho}}^{(2)}(t_-)\cdot\cdot\,\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}}{3c^2}\Bigg]_{t_-=t-\frac{|\boldsymbol{x}|}{c}} \tag{3.5a}$$

$$\boldsymbol{A}(\boldsymbol{x},t) = \frac{1}{c|\boldsymbol{x}|}\Bigg[\boldsymbol{J}^{(0)}(t_-) + \frac{\boldsymbol{J}^{(1)}(t_-)\cdot\hat{\boldsymbol{x}}}{|\boldsymbol{x}|} + \frac{\dot{\boldsymbol{J}}^{(1)}(t_-)\cdot\hat{\boldsymbol{x}}}{c}\Bigg]_{t_-=t-\frac{|\boldsymbol{x}|}{c}} \tag{3.5b}$$

where

$$\rho^{(0)}(t) = \int d^3\boldsymbol{x}'\,\rho(\boldsymbol{x}',t), \tag{3.6a}$$

$$\boldsymbol{\rho}^{(1)}(t) = \int d^3\boldsymbol{x}'\,\rho(\boldsymbol{x}',t)\boldsymbol{x}', \tag{3.6b}$$

$$\rho'^{(2)}(t) = \int d^3\boldsymbol{x}'\,\rho(\boldsymbol{x}',t)\frac{|\boldsymbol{x}'|^2}{2}, \tag{3.6c}$$

$$\boldsymbol{\rho}^{(2)}(t) = \int d^3\boldsymbol{x}'\,\rho(\boldsymbol{x}',t)\frac{3\boldsymbol{x}'\boldsymbol{x}' - \boldsymbol{x}'^2\mathbf{1}}{2}, \tag{3.6d}$$

$$\boldsymbol{J}^{(0)}(t) = \int d^3\boldsymbol{x}'\,\boldsymbol{J}(\boldsymbol{x}',t), \tag{3.6e}$$

$$\boldsymbol{J}^{(1)}(t) = \int d^3\boldsymbol{x}'\,\boldsymbol{J}(\boldsymbol{x}',t)\boldsymbol{x}', \tag{3.6f}$$

The terms in ρ were already introduced in (2.8) under the label of f, whereas $\boldsymbol{J}^{(0)}$ and $\boldsymbol{J}^{(1)}$ are, respectively, a vector and a dyadic function. One may notice that the terms in $\boldsymbol{A}$ were truncated at first order, rather than at second order as in the case of Φ. The reason for this choice is that, as will be seen, in many cases the first-order terms of the magnetic vector potential are of comparable relevance to—or even less significant than—the second-order terms of the electric potential.

Some of the multipole moments in (3.2) are directly identifiable with

$$\rho^{(0)}(t) = q(t) \tag{3.7a}$$

$$\boldsymbol{\rho}^{(1)}(t) = \boldsymbol{p}(t) \tag{3.7b}$$

$$\boldsymbol{\rho}^{(2)}(t) = \boldsymbol{Q}(t) \tag{3.7c}$$

Now it can be seen that $q(t)$ is time indipendent, indeed

$$\dot{q}(t) = \int d^3\boldsymbol{x}\, \frac{\partial \rho}{\partial t}(\boldsymbol{x}, t) = -\int d^3\boldsymbol{x}\, \boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{J}(\boldsymbol{x}, t) = -\oint d^2\boldsymbol{x} \boldsymbol{\cdot} \boldsymbol{J}(\boldsymbol{x}, t) = 0 \tag{3.8}$$

where the continuity equation (1.2) and Gauss's theorem are applied. Notice that the term representing the surface flux over the boundary of the volume is, from here on, always zero whenever the integral involves the source ρ or a component of $\boldsymbol{J}$. This is because the charge and current distributions are confined to a finite region of space, and as the surface of integration is taken to infinity (the volume integral being performed over the entire Euclidean space), the sources vanish identically.

Using again the continuity equation,

$$\begin{aligned}
\boldsymbol{J}^{(0)}(t) &= \int d^3\boldsymbol{x} \boldsymbol{J}(\boldsymbol{x}, t) \\
&= -\int d^3\boldsymbol{x}\, \boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{J}(\boldsymbol{x}, t)\boldsymbol{x} \\
&= \int d^3\boldsymbol{x}\, \frac{\partial \rho}{\partial t}(\boldsymbol{x}, t)\boldsymbol{x} \\
&= \frac{d}{dt}\int d^3\boldsymbol{x}\, \rho(\boldsymbol{x}, t)\boldsymbol{x} \\
&= \dot{\boldsymbol{p}}(t),
\end{aligned} \tag{3.9}$$

where in the second step an integration by parts was performed. This same approach is used in many of the computations below.

Consider now $\boldsymbol{J}^{(1)}$. It can be decomposed in the standard way for a matrix

$$\boldsymbol{J}^{(1)} = \boldsymbol{J}^{(1)ts} + \boldsymbol{J}^{(1)as} + \frac{1}{3}J^{(1)t}\mathbf{1} \tag{3.10}$$

in which

$$J^{(1)t} = tr(\boldsymbol{J}^{(1)}), \tag{3.11a}$$

$$\boldsymbol{J}^{(1)ts} = \frac{\boldsymbol{J}^{(1)} + \boldsymbol{J}^{(1)T}}{2} - tr(\boldsymbol{J}^{(1)}), \tag{3.11b}$$

$$\boldsymbol{J}^{(1)as} = \frac{\boldsymbol{J}^{(1)} - \boldsymbol{J}^{(1)T}}{2}. \tag{3.11c}$$

All three terms are expressible in virtue of a multipole moment always thanks to (1.2)

$$\begin{aligned}
J^{(1)t}(t) &= \int d^3\boldsymbol{x}\, \boldsymbol{J}(\boldsymbol{x},t)\boldsymbol{\cdot}\boldsymbol{x} \\
&= -\int d^3\boldsymbol{x}\, \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{J}(\boldsymbol{x},t)\frac{\boldsymbol{x}^2}{2} \\
&= \int d^3\boldsymbol{x}\, \frac{\partial\rho}{\partial t}(\boldsymbol{x},t)\frac{\boldsymbol{x}^2}{2} \\
&= \frac{d}{dt}\int d^3\boldsymbol{x}\, \rho(\boldsymbol{x},t)\frac{\boldsymbol{x}^2}{2} \\
&= \dot{\rho}^{(2)\prime}(t)
\end{aligned} \tag{3.12a}$$

$$\begin{aligned}
\boldsymbol{J}^{(1)ts}(t) &= \int d^3\boldsymbol{x} \left[\frac{\boldsymbol{J}(\boldsymbol{x},t)\boldsymbol{x} + \boldsymbol{J}(\boldsymbol{x},t)\boldsymbol{x}}{2} - \frac{1}{3}\boldsymbol{J}(\boldsymbol{x},t)\boldsymbol{\cdot}\boldsymbol{x}\mathbf{1}\right] \\
&= -\frac{1}{3}\int d^3\boldsymbol{x}\, \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{J}(\boldsymbol{x},t)\frac{3\boldsymbol{x}\boldsymbol{x} - \boldsymbol{x}^2\mathbf{1}}{2} \\
&= \frac{1}{3}\int d^3\boldsymbol{x}\, \frac{\partial\rho}{\partial t}(\boldsymbol{x},t)\frac{3\boldsymbol{x}\boldsymbol{x} - \boldsymbol{x}^2\mathbf{1}}{2} \\
&= \frac{1}{3}\frac{d}{dt}\int d^3\boldsymbol{x}\, \rho(\boldsymbol{x},t)\frac{3\boldsymbol{x}\boldsymbol{x} - \boldsymbol{x}^2\mathbf{1}}{2} \\
&= \frac{1}{3}\dot{Q}(t)
\end{aligned} \tag{3.12b}$$

and introducing the Hodge star operator $*\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{w} = -\boldsymbol{v}\times\boldsymbol{w}$, and the wedge product $\boldsymbol{v}\wedge\boldsymbol{w} = \boldsymbol{v}\boldsymbol{w} - \boldsymbol{w}\boldsymbol{v}$,

$$\begin{aligned}
\boldsymbol{J}^{(1)as}(t) &= \int d^3\boldsymbol{x} \left(\frac{\boldsymbol{J}(\boldsymbol{x},t)\boldsymbol{x} - \boldsymbol{J}(\boldsymbol{x},t)\boldsymbol{x}}{2}\right) \\
&= -\int d^3\boldsymbol{x} \left(\frac{\boldsymbol{x}\wedge\boldsymbol{J}(\boldsymbol{x},t)}{2}\right) \\
&= -c\int d^3\boldsymbol{x}\, \frac{1}{2c} * (\boldsymbol{x}\times\boldsymbol{J}(\boldsymbol{x},t)) \\
&= -c * \int d^3\boldsymbol{x}\, \frac{1}{2c}\, \boldsymbol{x}\times\boldsymbol{J}(\boldsymbol{x},t) \\
&= -c *\boldsymbol{m}(t).
\end{aligned} \tag{3.12c}$$

Using (3.7),(3.9), (3.10) and (3.12) into (3.6) one readily gets (3.4), less than terms in $\ddot{\rho}^{\prime(2)}(t)$. Those latter are gauge artifacts which can be removed with an appropriate choice of χ (cfr. (1.5))

$$\chi(\boldsymbol{x},t) = \frac{\dot{\rho}'^{(2)}(t-\frac{|\boldsymbol{x}|}{c})}{3c|\boldsymbol{x}|} \tag{3.13}$$

Performing a gauge transformation with this scalar function yields to potentials that satisfy the Lorenz gauge (1.6) as χ is an outgoing sferical wave solving D'Alembert equation.

Under the same assumptions (2.4), which give the multipole expansion of the potentials, the electric and magnetic induction fields are obtained

$$\begin{aligned}
\boldsymbol{E}(\boldsymbol{x},t) = \Bigg[&\frac{q\hat{\boldsymbol{x}}}{|\boldsymbol{x}|^2} + \frac{3\boldsymbol{p}(t_-)\cdot\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - \boldsymbol{p}(t_-)}{|\boldsymbol{x}|^3} + \frac{3\dot{\boldsymbol{p}}(t_-)\cdot\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - \dot{\boldsymbol{p}}(t_-)}{c|\boldsymbol{x}|^2} \\
&+ \frac{\hat{\boldsymbol{x}}\times(\hat{\boldsymbol{x}}\times\ddot{\boldsymbol{p}}(t_-))}{c^2|\boldsymbol{x}|} + \frac{5\boldsymbol{Q}(t_-)\cdot\cdot\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - 2\boldsymbol{Q}(t_-)\cdot\hat{\boldsymbol{x}}}{|\boldsymbol{x}|^4} \\
&+ \frac{5\dot{\boldsymbol{Q}}(t_-)\cdot\cdot\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - 2\dot{\boldsymbol{Q}}(t_-)\cdot\hat{\boldsymbol{x}}}{c|\boldsymbol{x}|^3} + \frac{\hat{\boldsymbol{x}}\times(\hat{\boldsymbol{x}}\times\ddot{\boldsymbol{Q}}(t_-)\cdot\hat{\boldsymbol{x}})}{c^2|\boldsymbol{x}|^2} \\
&+ \frac{\hat{\boldsymbol{x}}\times(\hat{\boldsymbol{x}}\times\dddot{\boldsymbol{Q}}(t_-)\cdot\hat{\boldsymbol{x}})}{3c^3|\boldsymbol{x}|} + \frac{\hat{\boldsymbol{x}}\times\dot{\boldsymbol{m}}(t_-)}{c|\boldsymbol{x}|^2} + \frac{\hat{\boldsymbol{x}}\times\ddot{\boldsymbol{m}}(t_-)}{c^2|\boldsymbol{x}|}\Bigg]_{t_-=t-\frac{|\boldsymbol{x}|}{c}}
\end{aligned} \tag{3.14a}$$

$$\begin{aligned}
\boldsymbol{B}(\boldsymbol{x},t) = \Bigg[&-\frac{\hat{\boldsymbol{x}}\times\dot{\boldsymbol{p}}(t_-)}{c|\boldsymbol{x}|^2} - \frac{\hat{\boldsymbol{x}}\times\ddot{\boldsymbol{p}}(t_-)}{c^2|\boldsymbol{x}|} - \frac{\hat{\boldsymbol{x}}\times\dot{\boldsymbol{Q}}(t_-)\cdot\hat{\boldsymbol{x}}}{c|\boldsymbol{x}|^3} \\
&- \frac{\hat{\boldsymbol{x}}\times\ddot{\boldsymbol{Q}}(t_-)\cdot\hat{\boldsymbol{x}}}{c^2|\boldsymbol{x}|^2} - \frac{\hat{\boldsymbol{x}}\times\dddot{\boldsymbol{Q}}(t_-)\cdot\hat{\boldsymbol{x}}}{3c^3|\boldsymbol{x}|} + \frac{3\boldsymbol{m}(t_-)\cdot\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - \boldsymbol{m}(t_-)}{|\boldsymbol{x}|^3} \\
&+ \frac{3\dot{\boldsymbol{m}}(t_-)\cdot\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - \dot{\boldsymbol{m}}(t_-)}{c|\boldsymbol{x}|^2} + \frac{\hat{\boldsymbol{x}}\times(\hat{\boldsymbol{x}}\times\ddot{\boldsymbol{m}}(t_-))}{c^2|\boldsymbol{x}|}\Bigg]_{t_-=t-\frac{|\boldsymbol{x}|}{c}}
\end{aligned} \tag{3.14b}$$

in the case (3.3) holds.

Proof: The equations are derived after a computation from the standard relations (1.3) knowing the expressions of the potentials Φ, $\boldsymbol{A}$ from the previous result (3.4).

3.2 Near and far field regime

In the theory discussed above, only the length scale d—which characterizes the size of the region where the charge and current are localized, as defined in (3.1a)—was considered. In addition to (3.1a), equations (3.1b) and (3.1c) also impose a condition on d: namely, that the relative time variations of ρ and $\boldsymbol{J}$ over the time interval d/c—the time it takes light to travel across the localized source region—are negligible.

There is another important length scale that fundamentally divides the theory and enables several further approximations: $|\boldsymbol{x}|/c$, where $\boldsymbol{x}$ is the observation point.

Quasi static regime

The *near field regime* is set when the condition

$$|\rho(\boldsymbol{x}', t)| \gg \left|\frac{|\boldsymbol{x}|}{c}\frac{\partial \rho}{\partial t}(\boldsymbol{x}', t)\right| \gg \left|\left(\frac{|\boldsymbol{x}|}{c}\right)^2\frac{\partial^2 \rho}{\partial t^2}(\boldsymbol{x}', t)\right| \gg \dots, \tag{3.15a}$$

$$|\boldsymbol{J}(\boldsymbol{x}', t)| \gg \left|\frac{|\boldsymbol{x}|}{c}\frac{\partial \boldsymbol{J}}{\partial t}(\boldsymbol{x}', t)\right| \gg \left|\left(\frac{|\boldsymbol{x}|}{c}\right)^2\frac{\partial^2 \boldsymbol{J}}{\partial t^2}(\boldsymbol{x}', t)\right| \gg \dots, \qquad |\boldsymbol{x}'| \lesssim d \tag{3.15b}$$

is satisfied at the time t. In the near field regime the multipole expansion of the electric and magnetic induction fields are

$$\boldsymbol{E}(\boldsymbol{x}, t) = \frac{q\hat{\boldsymbol{x}}}{|\boldsymbol{x}|^2} + \frac{3\boldsymbol{p}(t)\cdot\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - \boldsymbol{p}(t)}{|\boldsymbol{x}|^3} + \frac{5\boldsymbol{Q}(t)\cdot\cdot\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - 2\boldsymbol{Q}(t)\cdot\hat{\boldsymbol{x}}}{|\boldsymbol{x}|^4} \tag{3.16a}$$

$$\boldsymbol{B}(\boldsymbol{x}, t) = \frac{3\boldsymbol{m}(t)\cdot\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - \boldsymbol{m}(t)}{|\boldsymbol{x}|^3} \tag{3.16b}$$

Proof: Let $\mu(t)$ be a moment or the component of the higher order moments in (3.2). Generally it can be expressed as

$$\mu(t) = \int d^3\boldsymbol{x}\, w(\boldsymbol{x}, t) P(\boldsymbol{x}) \tag{3.17}$$

with w being either ρ or a component of $\boldsymbol{J}$ and $P(\boldsymbol{x})$ a homogeneous polynomial of k-th degree. As

$$\frac{d^n\mu}{dt^n}(t) = \int d^3\boldsymbol{x}\, \frac{\partial^n w}{\partial t^n}(\boldsymbol{x}, t) P(\boldsymbol{x}) \tag{3.18}$$

the order of magnitude of the n-th time derivative is estimated to be

$$\left|\frac{d^n\mu}{dt^n}(t)\right| \sim d^{k+3} \max_{\boldsymbol{x}} \left|\frac{\partial^n w}{\partial t^n}(\boldsymbol{x}, t)\right| \tag{3.19}$$

Using the near field regime conditions (3.15),

$$|\mu(t)| \gg \left|\frac{|\boldsymbol{x}|}{c}\frac{d\mu}{dt}(t)\right| \gg \left|\left(\frac{|\boldsymbol{x}|}{c}\right)^2 \frac{d^2\mu}{dt^2}(t)\right| \gg \dots \tag{3.20}$$

In this last equation, the time t is arbitrary, so it must also hold at t_-. Moreover, the strong inequalities imply that the higher the time derivative, the smaller its contribution. Thus, in (3.14), all terms involving time derivatives of order higher than one in each multipole moment are negligible. Discarding these higher-order terms and applying the convenient substitution $t_- \mapsto t$ are therefore justified, leading to (3.16).

Notice that because of the approximation made possible by (3.15), in the equations (3.16) the fields are computed in t instead of the usual t_-. Furthermore all the time derivatives dropped out the solutions, and for those reasons the near field regime is also referred to as the *quasi static regime*.

Radiation regime

The *far field regime* is set when the condition

$$|\rho(\boldsymbol{x}', t)| \ll \left|\frac{|\boldsymbol{x}|}{c}\frac{\partial\rho}{\partial t}(\boldsymbol{x}', t)\right| \ll \left|\left(\frac{|\boldsymbol{x}|}{c}\right)^2 \frac{\partial^2\rho}{\partial t^2}(\boldsymbol{x}', t)\right| \ll \dots, \tag{3.21a}$$

$$|\boldsymbol{J}(\boldsymbol{x}', t)| \ll \left|\frac{|\boldsymbol{x}|}{c}\frac{\partial\boldsymbol{J}}{\partial t}(\boldsymbol{x}', t)\right| \ll \left|\left(\frac{|\boldsymbol{x}|}{c}\right)^2 \frac{\partial^2\boldsymbol{J}}{\partial t^2}(\boldsymbol{x}', t)\right| \ll \dots, \qquad |\boldsymbol{x}'| \lesssim d \tag{3.21b}$$

is satisfied at the time t. In the far field regime the multipole expansion of the electric and magnetic induction fields are

$$\boldsymbol{E}(\boldsymbol{x},t) = \left[\frac{q\hat{\boldsymbol{x}}}{|\boldsymbol{x}|^2} + \frac{\hat{\boldsymbol{x}}\times(\hat{\boldsymbol{x}}\times\ddot{\boldsymbol{p}}(t_-))}{c^2|\boldsymbol{x}|} + \frac{\hat{\boldsymbol{x}}\times(\hat{\boldsymbol{x}}\times\dddot{\boldsymbol{Q}}(t_-)\cdot\hat{\boldsymbol{x}})}{3c^3|\boldsymbol{x}|} + \frac{\hat{\boldsymbol{x}}\times\ddot{\boldsymbol{m}}(t_-)}{c^2|\boldsymbol{x}|}\right]_{t_-=t-\frac{|\boldsymbol{x}|}{c}} \tag{3.22a}$$

$$\boldsymbol{B}(\boldsymbol{x},t) = \left[-\frac{\hat{\boldsymbol{x}}\times\ddot{\boldsymbol{p}}(t_-)}{c^2|\boldsymbol{x}|} - \frac{\hat{\boldsymbol{x}}\times\dddot{\boldsymbol{Q}}(t_-)\cdot\hat{\boldsymbol{x}}}{3c^3|\boldsymbol{x}|} + \frac{\hat{\boldsymbol{x}}\times(\hat{\boldsymbol{x}}\times\ddot{\boldsymbol{m}}(t_-))}{c^2|\boldsymbol{x}|}\right]_{t_-=t-\frac{|\boldsymbol{x}|}{c}} \tag{3.22b}$$

Proof: To show this, the reasoning is similar to that used in the proof of (3.16), but now with (3.21) providing the opposite result

$$|\mu(t)| \ll \left|\frac{|\boldsymbol{x}|}{c}\frac{d\mu}{dt}(t)\right| \ll \left|\left(\frac{|\boldsymbol{x}|}{c}\right)^2\frac{d^2\mu}{dt^2}(t)\right| \ll \dots \tag{3.23}$$

which is valid at every time t, so at $t_- = t - |\boldsymbol{x}|/c$. Then all and only the highest terms in time derivation in (3.14) of each moment survive leading to (3.22).

For large enough $|\boldsymbol{x}|$ the first term, the Coulombic term, in (3.22a) drops out because it is the only $\mathcal{O}(1/|\boldsymbol{x}|^2)$. This particular condition in the far field regime leads to the *radiation regime*. Indeed as it is known from electromagnetic theory, in the radiation regime $|\boldsymbol{E}| \sim |\boldsymbol{B}| \sim 1/|\boldsymbol{x}|$. This last one is a general property of the electric and magnetic induction fields regardless of Maxwell equations (1.1) (with which they are always compatible too), in addition to

$$\boldsymbol{B}(\boldsymbol{x},t) = \hat{\boldsymbol{x}}\times\boldsymbol{E}(\boldsymbol{x},t) \tag{3.24a}$$

$$\boldsymbol{B}(\boldsymbol{x},t)\cdot\hat{\boldsymbol{x}} = \boldsymbol{E}(\boldsymbol{x},t)\cdot\hat{\boldsymbol{x}} = 0 \tag{3.24b}$$

always in the radiation regime, which characterizes the electric field, magnetic field, and observation point as forming an orthogonal triad.

The difference between the near and far field regime is that in the first case the source remains about the same in the time that it takes the light to reach the observation point, instead of the second one when it undergoes a appreciable change in magnitude. In physics the time it takes for a wave to undergo the highest excursion possible is the period τ. The period of the source is related to the *wavelenght* λ of the resulting electromagnetic field according to $\lambda \sim c\tau$. So a new lenght scale has been found to tell whether regime is set. The conditions (3.1) can be recast as $\lambda \gg d$ thinking of d as the distance of the observation point under which the quasi-static regime is always established. Even if (3.3) is still valid, this does not tell anything about the relationship between $|\boldsymbol{x}|$ and λ depending on which the regimes are distinguished. Indeed, if $|\boldsymbol{x}| \ll \lambda$ the source does not appreciably change in the time the wave reaches the observation point, so it is the case of the quasi-static regime. Viceversa $|\boldsymbol{x}| \gg \lambda$ is to attribute to the far field regime.

Follows an analysis based on the order of magnitude of the fields given by each multipole moment which will be denoted with a subscript related to the moment generating the field.

Let $r = |\boldsymbol{x}|$, in quasi-static regime:

$$E_q \sim \frac{q}{r^2} \sim \frac{q_0}{r^2}, \qquad B_q \sim 0 \tag{3.25a}$$

$$E_p \sim \frac{p}{r^3} \sim \frac{q_0 d}{r^3}, \qquad B_p \sim 0 \tag{3.25b}$$

$$E_Q \sim \frac{Q}{r^4} \sim \frac{q_0 d^2}{r^4}, \qquad B_Q \sim 0 \tag{3.25c}$$

$$E_m \sim 0, \qquad B_m \sim \frac{m}{r^3} \sim \frac{dqd}{cr^3\tau} \sim \frac{q_0 d^2}{\lambda r^3} \tag{3.25d}$$

and using $\lambda \gg r \gg d$, (3.25) make possible the comparison

$$E_m \ll E_Q \sim E_p \frac{d}{r} \ll E_p \sim E_q \frac{d}{r} \ll E_q \tag{3.26a}$$

$$B_Q \sim B_p \sim B_q \sim 0 \ll B_m \sim E_Q \frac{r}{\lambda} \ll E_Q \tag{3.26b}$$

As expected the multipoles of the higher order of expansion give a lower contribute to the magnitude of the electric field in (3.26a). (3.26b) shows that the only moment generating the magnetic induction field is the magnetic dipole, and in this regime the resulting magnitude is lower than each one of the terms of the electric field. Now it is clear why the vector potential expansion (3.5b) was arrested to the first order.

In the radiation regime:

$$E_q \sim B_q \sim 0 \tag{3.27a}$$

$$E_p \sim B_p \sim \frac{\ddot{p}}{c^2 r} \sim \frac{p}{c^2\tau^2 r} \sim \frac{q_0 d}{\lambda^2 r} \tag{3.27b}$$

$$E_Q \sim B_Q \sim \frac{\dddot{Q}}{c^3 r} \sim \frac{Q}{c^3\tau^3 r} \sim \frac{q_0 d^2}{\lambda^3 r} \tag{3.27c}$$

$$E_m \sim B_m \sim \frac{\ddot{p}}{c^2 r} \sim \frac{p}{c^2\tau^2 r} \sim \frac{q_0 d^2}{\lambda^3 r} \tag{3.27d}$$

and using $r \gg \lambda \gg d$, (3.27) make possible the comparison

$$E_m \sim B_m \sim E_Q \sim B_Q \sim E_p \frac{d}{\lambda} \sim B_p \frac{d}{\lambda} \ll E_p \sim B_p \tag{3.28}$$

Once again it is verified that the magnetic dipole contribute is weak and a first order expansion of the vector potential is not sloppy. The leading contribuite of the electric charge q in the near field regime is now absent, the higher one due to the electric dipole p.

Notice that only for few appropriate geometric spacial configurations of the charge both $q = q_0 = 0$. In general if $q = 0$, it might be that $q_0 \neq 0$, as for the physical electric dipole which consists in two equal magnitude point charges separated by a non zero distance. The argument is valid even for the electric dipole contribute itself in (3.25), (3.27), indeed there are cases in which the geometry of the charge distribution gives exactly an electric dipole of order 0. This is why quadrupoles and magnetic dipoles are taken into account even if they give much lower contribution to the total magnitude. There are problems in nuclear physics which involve nuclear configurations such that the quadrupole interaction is non zero and the corresponding values and signs reflect the nature of the forces bewteen neutrons and protons and the shape of the nucleus itself. Higher order moments are almost always so small that their incorporation is required only in high precision computations [5].

Chapter 4

Radiated power in radiation regime

4.1 Differential and total radiated power

The electromagnetic field carries energy, which flow is associated to *Poynting vector* of electromagnetic energy current density

$$\boldsymbol{S}(\boldsymbol{x},t) = \frac{c}{4\pi}\boldsymbol{E}(\boldsymbol{x},t)\times\boldsymbol{B}(\boldsymbol{x},t) \tag{4.1}$$

The energy δW radiated by the charge and current distribution to the observation point $\boldsymbol{x}$ through a solid angle $\delta\Omega$ along the direction $\hat{\boldsymbol{x}}$ in a time interval δt is given by

$$\delta W = \boldsymbol{S}(\boldsymbol{x},t)\cdot\hat{\boldsymbol{x}}|\boldsymbol{x}|^2\delta\Omega\delta t \tag{4.2}$$

Using (4.2) follows the definition of the power radiated to the point $\boldsymbol{x}$ along $\hat{\boldsymbol{x}}$ per unit solid angle

$$\frac{dP}{d\Omega}(\boldsymbol{x},t) = \boldsymbol{S}(\boldsymbol{x},t)\cdot\hat{\boldsymbol{x}}|\boldsymbol{x}|^2 \tag{4.3}$$

called *differential radiated power*.

In the radiation regime the differential radiated power is related to the magnitude of the electric and magnetic induction fields by

$$\frac{dP}{d\Omega}(\boldsymbol{x},t) = \frac{c}{4\pi}(|\boldsymbol{x}|\boldsymbol{E}(\boldsymbol{x},t))^2 = \frac{c}{4\pi}(|\boldsymbol{x}|\boldsymbol{B}(\boldsymbol{x},t))^2 \tag{4.4}$$

For the conservation of energy to hold it must be that the fields decrease as $\sim 1/|\boldsymbol{x}|$ at long distance from the source, a general property of the electromagnetic field that was introduced in the last chapter.

Proof: The radiation regime relations (3.24) for $\boldsymbol{E}$ and $\boldsymbol{B}$ exploited in (4.1) yield

$$\begin{aligned} \boldsymbol{S} &= \frac{c}{4\pi}\boldsymbol{E}\times(\hat{\boldsymbol{x}}\times\boldsymbol{E}) \\ &= \frac{c}{4\pi}\Big[\hat{\boldsymbol{x}}(\boldsymbol{E}\cdot\boldsymbol{E}) - \boldsymbol{E}(\hat{\boldsymbol{x}}\cdot\boldsymbol{E})\Big] \\ &= \frac{c}{4\pi}\hat{\boldsymbol{x}}|\boldsymbol{E}|^2 \end{aligned} \tag{4.5}$$

in which in the second step is used the useful vector identity

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b}). \tag{4.6}$$

Using the result in (4.3) gives the first equivalence in (4.4). To show the second equivalence one recalls (3.24) and the computation is straightforward.

Notice that in this proof only the orthogonality of the electric and magnetic induction field was used. Satisfying the conservation of energy means that (4.4) must not have explicit dependence on the distance $|\boldsymbol{x}|$. This is true only if $|\boldsymbol{E}| \sim |\boldsymbol{B}| \sim 1/|\boldsymbol{x}|$.

Then the differential radiation power in terms of multipole moments is given by

$$\frac{dP}{d\Omega}(\boldsymbol{x},t) = \frac{1}{4\pi c^3}\left| -\hat{\boldsymbol{x}}\times\ddot{\boldsymbol{p}}(t_-) - \frac{\hat{\boldsymbol{x}}\times\dddot{\boldsymbol{Q}}(t_-)\cdot\hat{\boldsymbol{x}}}{3c} + \hat{\boldsymbol{x}}\times(\hat{\boldsymbol{x}}\times\ddot{\boldsymbol{m}}(t_-))\right|^2_{t_-=t-\frac{|\boldsymbol{x}|}{c}} \tag{4.7}$$

Proof: The expression follows immediately by (4.4) knowing $\boldsymbol{B}$ in the radiation regime is determined as in (3.22b).

An integration over the unit sphere solid angle of the differential radiation power leads to the *total radiated power*

$$P(\boldsymbol{x},t) = \int d\Omega\,\frac{dP}{d\Omega}(\boldsymbol{x},t) \tag{4.8}$$

in which formally the sum of $d\Omega = d^2\boldsymbol{x}/|\boldsymbol{x}|^2$ is a first species surface integral over the domain identified by a unit vector applied in the origin sweeping in all the directions, so that $\int d\Omega = 4\pi$. An explicit computation of the former integral gives

$$P(|\boldsymbol{x}|,t) = \frac{2}{3c^3}\left|\ddot{\boldsymbol{p}}(t_-)^2 + \frac{1}{30c^2}\dddot{\boldsymbol{Q}}(t_-)^2 + \ddot{\boldsymbol{m}}(t_-)^2\right|_{t_-=t-\frac{|\boldsymbol{x}|}{c}} \tag{4.9}$$

Notice that the total radiated power depends spatially only on the distance from the charge and current distribuition, and that is only caused by the implicit dependence $t = t(|\boldsymbol{x}|)$. This result is in agreement with the conservation of energy.

Proof: The development of the square (4.7) with the useful identities for generic vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$:

$$|\boldsymbol{a}\times\boldsymbol{b}| = |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a}\cdot\boldsymbol{b})^2 \tag{4.10a}$$

$$\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{a}\times\boldsymbol{c} = |\boldsymbol{a}|^2\boldsymbol{b}\cdot\boldsymbol{c} - (\boldsymbol{a}\cdot\boldsymbol{b})(\boldsymbol{a}\cdot\boldsymbol{c}) \tag{4.10b}$$

furnishes

$$\begin{aligned}\frac{dP}{d\Omega}(\boldsymbol{x},t) = \frac{1}{4\pi c^3}\Big\{&\ddot{\boldsymbol{p}}(t_-)^2 - (\ddot{\boldsymbol{p}}(t_-)\cdot\hat{\boldsymbol{x}})^2 + \frac{1}{9c^2}\Big[(\dddot{\boldsymbol{Q}}(t_-)\cdot\hat{\boldsymbol{x}})^2 - (\dddot{\boldsymbol{Q}}(t_-)\cdot\cdot\,\hat{\boldsymbol{x}}\hat{\boldsymbol{x}})^2\Big]\\ &+ \ddot{\boldsymbol{m}}(t_-)^2 - (\ddot{\boldsymbol{m}}(t_-)\cdot\hat{\boldsymbol{x}})^2 + 2\ddot{\boldsymbol{p}}(t_-)\times\ddot{\boldsymbol{m}}(t_-)\cdot\hat{\boldsymbol{x}}\\ &+ \frac{2}{3c}\Big[\ddot{\boldsymbol{p}}(t_-)\cdot\dddot{\boldsymbol{Q}}(t_-)\cdot\hat{\boldsymbol{x}} - \ddot{\boldsymbol{p}}(t_-)\cdot\hat{\boldsymbol{x}}\dddot{\boldsymbol{Q}}(t_-)\cdot\cdot\,\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}\Big]\\ &+ \frac{2}{3c}\ddot{\boldsymbol{m}}(t_-)\cdot\hat{\boldsymbol{x}}\times\dddot{\boldsymbol{Q}}(t_-)\cdot\hat{\boldsymbol{x}}\Big\}_{t_-=t-\frac{|\boldsymbol{x}|}{c}}\end{aligned} \tag{4.11}$$

Inserting this last equation in (4.8) leads to the calculation of integrals of the form

$$\int d\Omega\, \hat{\boldsymbol{x}}\hat{\boldsymbol{x}}...\hat{\boldsymbol{x}}\cdot\cdot\, ...\, \cdot\boldsymbol{v}_1\boldsymbol{v}_2...\boldsymbol{v}_n \tag{4.12}$$

with n up to four. Its general solution is written in terms of the general isotropic tensor I of n-th order contracted with n vectors $\boldsymbol{v}_1\boldsymbol{v}_2...\boldsymbol{v}_n$, and a real constant κ_n,

$$\int d\Omega\, \hat{\boldsymbol{x}}\hat{\boldsymbol{x}}...\hat{\boldsymbol{x}} = \kappa_n I \tag{4.13}$$

The general isotropic tensor has the property of remaining unchanged under any proper orthogonal transformation and here is employed to write the solution of the integral because of the rotation symmetry of $d\Omega$ when the integration is performed over all the unit sphere. It should be symmetric under the exchange of any two indices as one can deduce from its construction. The isotropic tensor of odd rank can be written only in terms of the Levi-Civita symbol, but that would give an antisymmetric tensor in contraddiction with the former remark. This means the tensor is identically 0 for an odd number of vectors. In even dimension I can be written as a sum of products of kronecker delta's, with all possible pairings. In the current formalism Kronecker delta's translate as a sum of scalar products as follows

$$\int d\Omega\, \hat{\boldsymbol{x}}\cdot\boldsymbol{v}_1 = 0 \tag{4.14a}$$

$$\int d\Omega\, \hat{\boldsymbol{x}}\hat{\boldsymbol{x}}\cdot\cdot\boldsymbol{v}_1\boldsymbol{v}_2 = \kappa_2 \boldsymbol{v}_1\cdot\boldsymbol{v}_2 \tag{4.14b}$$

$$\int d\Omega\, \hat{\boldsymbol{x}}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}\cdot\cdot\cdot\boldsymbol{v}_1\boldsymbol{v}_2\boldsymbol{v}_3 = 0 \tag{4.14c}$$

$$\int d\Omega\, \hat{\boldsymbol{x}}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}\cdot\cdot\cdot\cdot\boldsymbol{v}_1\boldsymbol{v}_2\boldsymbol{v}_3\boldsymbol{v}_4 = \kappa_4[\boldsymbol{v}_1\cdot\boldsymbol{v}_2\boldsymbol{v}_3\cdot\boldsymbol{v}_4 + \boldsymbol{v}_1\cdot\boldsymbol{v}_3\boldsymbol{v}_2\cdot\boldsymbol{v}_4 + \boldsymbol{v}_1\cdot\boldsymbol{v}_4\boldsymbol{v}_2\cdot\boldsymbol{v}_3] \tag{4.14d}$$

To compute the remaining constants κ_2, κ_4 put $\boldsymbol{v}_1 = \boldsymbol{v}_2 = ... = \boldsymbol{v}_n = \boldsymbol{e}_3$ the versor of the third axis in cartesian coordinates. In polar coordinates $\hat{\boldsymbol{x}}\cdot\boldsymbol{e}_3 = \cos\theta$ and the solid angle integration is

$$\int d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \tag{4.15}$$

So (4.14b) becomes

$$\begin{aligned}\kappa_2 = \int d\Omega\, \hat{\boldsymbol{x}}\hat{\boldsymbol{x}}\cdot\cdot\boldsymbol{e}_3\boldsymbol{e}_3 &= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta(\cos\theta)^2 \\ &= -2\pi \int_{+1}^{-1} du\, u^2 = \frac{4}{3}\pi\end{aligned} \tag{4.16}$$

upon $u = \cos\theta$ substitution, so that follows $\kappa_2 = 4\pi/3$. Then the computation of (4.14d) with the same strategy is

$$3\kappa_4 = \int d\Omega\, \hat{\boldsymbol{x}}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}\cdots\cdot\boldsymbol{v}_1\boldsymbol{v}_2\boldsymbol{v}_3\boldsymbol{v}_4 = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta(\cos\theta)^4 \\ = -2\pi \int_{+1}^{-1} du\, u^4 = \frac{4}{5}\pi \tag{4.17}$$

which gives $\kappa_4 = 4\pi/15$.

Then one is able to compute the integrals in the terms of (4.11) of the form of (4.14) and get (4.9).

4.2 Accelerated charged point particle

In this section results from previous analysis of radiated power will be exploited to tackle the problem of an accelerating charged particle energy loss. In the first decades of the XX century physicists tried to find a model to describe the inside structure of an atom. Eventually, in 1911, Ernest Rutherford came up with the planetary model, according to which electrons orbit around the proton as planets do around the sun. However, in classic electromagnetism, accelerated charges radiate power and this leads to a consistency problem in Rutherford's model, because that would mean that atoms are unstable and in less than nanoseconds electrons would fall on protons. The goal of this section is to show how to obtain the relations, already known at the time of Rutherford and that led to the rejection of his model, that furnish the differential and total radiated power of a point charge, called Larmor's formulae.

Let q_0 be the charge of a point particle moving along the trajectory parameterized by $\boldsymbol{x}(t)$. For this system the electric charge and current distributions associated are

$$\rho(\boldsymbol{x}, t) = q_0\delta(\boldsymbol{x} - \boldsymbol{x}(t)) \tag{4.18a}$$

$$\boldsymbol{J}(\boldsymbol{x}, t) = q_0\delta(\boldsymbol{x} - \boldsymbol{x}(t))\dot{\boldsymbol{x}}(t) \tag{4.18b}$$

The multipole moments of the charge (from now on it will be used to refer to the point particle) then are

$$q = q_0 \tag{4.19a}$$

$$\boldsymbol{p}(t) = q_0 \boldsymbol{x}(t) \tag{4.19b}$$

$$\boldsymbol{Q}(t) = q_0 \frac{3\boldsymbol{x}(t)\boldsymbol{x}(t) - \boldsymbol{x}(t)^2 \mathbf{1}}{2} \tag{4.19c}$$

$$\boldsymbol{m}(t) = \frac{q_0}{2c} \boldsymbol{x}(t) \times \dot{\boldsymbol{x}}(t) \tag{4.19d}$$

Proof: The statement follows directly using (4.18) with the properties of the Dirac delta function in the expressions of the multipole moments of a generic charge and current distribution (3.2).

According to (3.1a) the trajectory of the charge must be confined inside a region of linear size d around the origin of the reference frame. Again (3.1b),(3.1c) force the strong inequality $d/c \ll \tau$, which with the usual relation between period and wavelenght $\lambda \sim c\tau$ gives

$$d \ll \lambda \tag{4.20}$$

In this case τ is a time scale that characterizes the particle motion. About this more can be said, as the former strong inequality has an important physical interpretation. Since it was hypothesized that the motion has the lenght scale of about d and time scale of τ the particle velocity magnitude must be of order $\dot{x} \sim d/\tau$, so from (4.20) follows

$$\frac{\dot{x}}{c} \ll 1, \tag{4.21}$$

which says that the charge motion can not be relativistic.

In order to proceed with the multipole expansion by (3.3) one has that

$$|\boldsymbol{x}(t)| \ll |\boldsymbol{x}| \tag{4.22}$$

must be satisfied.

The following analysis will be carried in the radiation regime which is recalled to verify when $q \approx 0$ and $\lambda \ll |\boldsymbol{x}|$ that implies (4.22) as the charge trajectory is confined in a region of size d obeying (4.20). Thus the first term to consider in order of magnitude contribution is the electric dipole moment one since $\boldsymbol{p}(t) \neq \mathbf{0}$. In the radiation regime the electric field and magnetic induction field of the charge are

$$\boldsymbol{E}(\boldsymbol{x}, t) = \frac{q_0}{c^2} \frac{\hat{\boldsymbol{x}} \times (\hat{\boldsymbol{x}} \times \ddot{\boldsymbol{x}}(t_-))}{|\boldsymbol{x}|} \tag{4.23a}$$

$$\boldsymbol{B}(\boldsymbol{x}, t) = -\frac{q_0}{c^2} \frac{\hat{\boldsymbol{x}} \times \ddot{\boldsymbol{x}}(t_-)}{|\boldsymbol{x}|} \tag{4.23b}$$

where $t_- = t - |\boldsymbol{x}|/c$ as from now on.

Proof: In the radiation regime the leading term is given by the electric dipole as seen in (3.28). So the equations of the electric and magnetic induction fields follows straightly by inserting (4.19) in (3.22a), (3.22b), neglecting the coulombic term and all the others that do not depend on the electric dipole moment.

The differential and total radiated power of the charge in radiation regime are

$$\frac{dP}{d\Omega}(\boldsymbol{x}, t) = \frac{q_0^2}{4\pi c^3} (\hat{\boldsymbol{x}} \times \ddot{\boldsymbol{x}}(t_-))^2 \tag{4.24a}$$

$$P(|\boldsymbol{x}|, t) = \frac{2q_0^2}{3c^3} \ddot{\boldsymbol{x}}(t_-)^2 \tag{4.24b}$$

known as Larmor's formulae.

Proof: (4.24a) follows by inserting into (4.4) both (4.23a) or (4.23b). To show (4.24b) one uses (4.19) in (4.9) keeping only the electric dipole term.

4.3 Non relativistic circular accelerator

In this section, carrying on the former analysis in radiation regime, a model of a point particle moving on a circular periodic trajectory with constant velocity will be provided. Using this to explain the planetary model theorized by Rutherford one finds that in this latter, in order to keep the orbit circular, the energy conservation law must be violated because of the electron energy loss.

Let the charge q_0 be in motion on an orbit of radius r_0 with constant angular velocity ω. In the multipole formalism that means $r_0 \sim d$ and

$$r_0 \ll \frac{c}{\omega} \ll |\boldsymbol{x}| \tag{4.25}$$

from the results of the last section and the relationship between the period τ and ω. The trajectory of the charge is given by

$$\boldsymbol{x}(t) = r_0[\boldsymbol{e}_1 \cos(\omega t) + \boldsymbol{e}_2 \sin(\omega t)] \tag{4.26}$$

Again, proceeding considering only the electric dipole moment contribuition one has

$$\boldsymbol{E}(\boldsymbol{x}, t) = \frac{q_0\omega^2}{c^2} \frac{(\hat{\boldsymbol{x}} \times \boldsymbol{x}(t_-)) \times \hat{\boldsymbol{x}}}{|\boldsymbol{x}|} \tag{4.27a}$$

$$\boldsymbol{B}(\boldsymbol{x}, t) = \frac{q_0\omega^2}{c^2} \frac{\hat{\boldsymbol{x}} \times \boldsymbol{x}(t_-)}{|\boldsymbol{x}|} \tag{4.27b}$$

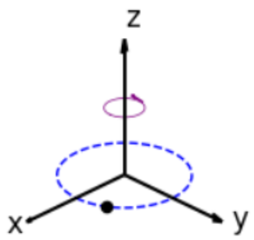


Figure 4.1: Schematic representation of the particle trajectory.

Proof: The particle trajectory (4.26) is such that

$$\ddot{\boldsymbol{x}}(t) = -\omega^2 \boldsymbol{x}(t) \tag{4.28}$$

Exploiting this last equation in (4.23a), (4.23b), respectively (4.27a) and (4.27b) are shown.

Using Larmor's formulae, differential and total radiated power of small circular accelerator are

$$\frac{dP}{d\Omega}(\boldsymbol{x}, t) = \frac{(q_0\omega^2)^2}{4\pi c^3}(\hat{\boldsymbol{x}} \times \boldsymbol{x}(t_-))^2 \tag{4.29a}$$

$$P(|\boldsymbol{x}|, t) = \frac{2(q_0\omega^2)^2}{3c^3}\boldsymbol{x}(t_-)^2 \tag{4.29b}$$

Proof: As seen the trajectory is (4.26), which inserted in Larmor's formulae for a general trajectory (4.24a),(4.24b) immediately gives the wanted result.

Spherical coordinates formulation

The relations derived for the small circular accelerator may be expressed in spherical coordinates to obtain a more manageable form of Larmor's formulae, due to the trajectory's symmetry with respect to the polar axis $\boldsymbol{e}_3$.

The transformation is such that

$$\begin{cases} \hat{\boldsymbol{e}}_1 = \sin\theta\cos\phi\,\hat{\boldsymbol{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{e}}_2 = \sin\theta\sin\phi\,\hat{\boldsymbol{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{e}}_3 = \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases} \tag{4.30}$$

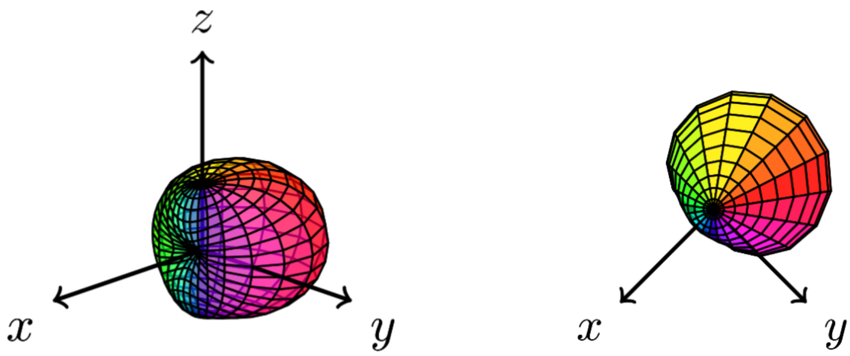
with the standard relation

$$\boldsymbol{x} = r\,\hat{\boldsymbol{r}} \tag{4.31}$$

In terms of spherical coordinates the electric and magnetic induction fields are

$$\boldsymbol{E}(r, \theta, \phi, t) = \frac{q_0\omega^2 r_0}{c^2 r}\Big[\cos(\omega t_- - \phi)\cos\theta\hat{\boldsymbol{\theta}} + \sin(\omega t_- - \phi)\hat{\boldsymbol{\phi}}\Big] \tag{4.32a}$$

$$\boldsymbol{B}(r, \theta, \phi, t) = \frac{q_0\omega^2 r_0}{c^2 r}\Big[-\sin(\omega t_- - \phi)\hat{\boldsymbol{\theta}} + \cos(\omega t_- - \phi)\cos\theta\hat{\boldsymbol{\phi}}\Big] \tag{4.32b}$$



(a) Side view (b) Top view

Figure 4.2: Visualization of the differential radiated power by the plot of the function (4.33a), with $t_- = 0$. The distance of the intersection of the line of direction specified by the spherical angles θ, ϕ with the surface of the plot from the origin is proportional to $1/\sin\theta \, dP/d\theta d\phi \, (r, \theta, \phi, t)$

The differential and total radiated power are

$$\frac{1}{\sin\theta}\frac{dP}{d\theta d\phi}(r,\theta,\phi,t) = P_0 \frac{1 - \sin^2\theta\cos(\omega t_- - \phi)}{2} \tag{4.33a}$$

$$P(r,t) = \frac{4\pi}{3} P_0 \tag{4.33b}$$

in which $t_- = t - r/c$ and it is defined P_0 the intrinsic power scale of the small circular accelerator

$$P_0 = \frac{(q_0 r_0 \omega^2)^2}{2\pi c^3} \tag{4.34}$$

Proof: The relations follow respectively by using the spherical coordinates (4.30) into (4.27a),(4.27b),(4.29a) and (4.29b).

Chapter 5

Charge distribution in an external electromagnetic field

5.1 Pointlike charge and current distributions

In this section a mathematical modelization of a pointlike charge and current distribution will be provided. The structure of the distribution around the origin $\mathbf{0}$ may be probed using a generic test function χ_l based on the lenght scale l such that

$$\int d^3\boldsymbol{x}\,\chi_l(\boldsymbol{x}) = 1, \tag{5.1a}$$

$$\chi_l(\boldsymbol{x}) \approx l^{-3}, \qquad \text{if } |\boldsymbol{x}| \leq l, \tag{5.1b}$$

$$\chi_l(\boldsymbol{x}) \to 0, \qquad \text{if } |\boldsymbol{x}| \gg l. \tag{5.1c}$$

The analysis is made through the averaged forms of charge ρ and the current $\boldsymbol{J}$:

$$\bar{\rho}(\mathbf{0}, t) = \int d^3\boldsymbol{x}\,\rho(\boldsymbol{x}, t)\chi_l(\boldsymbol{x}) \tag{5.2a}$$

$$\bar{\boldsymbol{J}}(\mathbf{0}, t) = \int d^3\boldsymbol{x}\,\boldsymbol{J}(\boldsymbol{x}, t)\chi_l(\boldsymbol{x}) \tag{5.2b}$$

Now suppose that the charge and current distribution is concentrated in a region of space of linear size d near $\mathbf{0}$ and that $d \ll l$. Then when the distribution appreciably differs from 0, $\chi_l(\boldsymbol{x}) \approx l^{-3}$, thus making accurate the Taylor expansion in $\boldsymbol{x}$ around $\mathbf{0}$,

$$\chi_l(\boldsymbol{x}) = \chi_l(\mathbf{0}) + \boldsymbol{x}\cdot\boldsymbol{\nabla}\chi_l(\mathbf{0}) + \frac{1}{2}\boldsymbol{x}\boldsymbol{x}\cdot\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\chi_l(\mathbf{0}) + \mathcal{O}(|\boldsymbol{x}|^3), \quad \text{when } |\boldsymbol{x}| \leq d \tag{5.3}$$

This expansion leads to the approximate equations of the sources for a pointlike distibuition

$$\bar{\rho}(\mathbf{0},t) = q\chi_l(\mathbf{0}) + \boldsymbol{p}(t)\cdot\boldsymbol{\nabla}\chi_l(\mathbf{0}) + \frac{1}{3}\boldsymbol{Q}(t)\cdot\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\chi_l(\mathbf{0}) + \frac{1}{3}Q'(t)\boldsymbol{\nabla}^2\chi_l(\mathbf{0}) \tag{5.4a}$$

$$\bar{\boldsymbol{J}}(\mathbf{0},t) = \dot{\boldsymbol{p}}(t)\chi_l(\mathbf{0}) + \frac{1}{3}\dot{\boldsymbol{Q}}(t)\cdot\boldsymbol{\nabla}\chi_l(\mathbf{0}) + \frac{1}{3}\dot{Q}'(t)\boldsymbol{\nabla}\chi_l(\mathbf{0}) + c\boldsymbol{m}(t)\times\boldsymbol{\nabla}\chi_l(\mathbf{0}) \tag{5.4b}$$

in which the multipole moments are defined as in (3.2) and it is defined the *scalar electric quadrupole moment*

$$Q'(t) = \int d^3\boldsymbol{x}\,\rho(\boldsymbol{x},t)\frac{|\boldsymbol{x}|^2}{2} \tag{5.5}$$

Proof: Recall the useful notations (3.6) and observe that (5.5) is the same as (3.6c). Expanding according to (5.3) the expressions of the means (5.2) respecively to the second and to the first order, one finds

$$\begin{aligned}\bar{\rho}(\mathbf{0},t) &= \int d^3\boldsymbol{x}\,\rho(\boldsymbol{x},t)\Big[\chi_l(\mathbf{0}) + \boldsymbol{x}\cdot\boldsymbol{\nabla}\chi_l(\mathbf{0}) + \frac{1}{2}\boldsymbol{x}\boldsymbol{x}\cdot\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\chi_l(\mathbf{0})\Big] \\ &= \rho^{(0)}(t)\chi_l(\mathbf{0}) + \boldsymbol{\rho}^{(1)}(t)\cdot\boldsymbol{\nabla}\,\chi_l(\mathbf{0}) + \frac{1}{3}\rho'^{(2)}(t)\boldsymbol{\nabla}^2\chi_l(\mathbf{0}) \\ &\qquad + \frac{1}{3}\boldsymbol{\rho}^{(2)}(t)\cdot\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\chi_l(\mathbf{0})\end{aligned} \tag{5.6a}$$

$$\begin{aligned}\bar{\boldsymbol{J}}(\mathbf{0},t) &= \int d^3\boldsymbol{x}\,\boldsymbol{J}(\boldsymbol{x},t)\Big[\chi_l(\mathbf{0}) + \boldsymbol{x}\cdot\boldsymbol{\nabla}\chi_l(\mathbf{0})\Big] \\ &= \boldsymbol{J}^{(0)}(t)\chi_l(\mathbf{0}) + \boldsymbol{J}^{(1)}(t)\cdot\boldsymbol{\nabla}\chi_l(\mathbf{0})\end{aligned} \tag{5.6b}$$

From this equations using the relations between the terms labelled by ρ, $\boldsymbol{J}$ and the multipole moments found in the proof of (3.4), (5.4) are shown.

Knowing that $d \ll l$, the charge and current distributions ρ and $\boldsymbol{J}$ can be approximated by

$$\rho(\boldsymbol{x},t) = q\delta(\boldsymbol{x}) - \boldsymbol{p}(t)\cdot\boldsymbol{\nabla}\delta(\boldsymbol{x}) + \frac{1}{3}\boldsymbol{Q}(t)\cdot\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\delta(\boldsymbol{x}) + \frac{1}{3}Q'(t)\boldsymbol{\nabla}^2\delta(\boldsymbol{x}) \tag{5.7a}$$

$$\boldsymbol{J}(\boldsymbol{x},t) = \dot{\boldsymbol{p}}(t)\delta(\boldsymbol{x}) - \frac{1}{3}\dot{\boldsymbol{Q}}(t)\cdot\boldsymbol{\nabla}\delta(\boldsymbol{x}) - \frac{1}{3}\dot{Q}'(t)\boldsymbol{\nabla}\delta(\boldsymbol{x}) - c\boldsymbol{m}(t)\times\boldsymbol{\nabla}\delta(\boldsymbol{x}) \tag{5.7b}$$

Proof: Inserting (5.7) into (5.2) yields (5.4) using basic properties of Dirac's delta function $\delta(\boldsymbol{x})$ and its first and second order derivatives. This shows that the relations are valid in order to obtain an expansion of the averages of the sources probed by χ_l.

From (5.7) one notices that considering only the first term of ρ and $\boldsymbol{J}$ leads to the case of a point particle (cf. eqs. (4.18)). Thus to portray a pointlike physical system of charges and currents it is needed an expansion to an higher order that does that no important interaction is overlooked.

5.2 Forces and torques acting on a pointlike distribution

Now it will be tackled the problem of a charge and current distribution immersed in an external electromagnetic field. Suppose that the sources that generate the external field are far enough that their presence does not interfer directly on the charge and current distribution considered here. Then assume that the distribution is localized in a region of linear size d way smaller than the order of magnitude of the external field wavelenght λ_e. Now taking $l = \lambda_e$, one recovers the assumption

$$d \ll \lambda_e \sim l \tag{5.8}$$

that made possible the expansions of ρ, J (5.7) in the framework of last section. Again, the identification of the scale l with λ_e is not random, but it is made for the physical meaning of the latter as the space region in which a wave undergoes a full excursion (similarly to what was done to distinguish the near from the far field regimes). So the strong inequality (5.8) assures that the external field is approximable as constant in the region where the charge and current distribution is localized.

The force exerted by the external electromagnetic field E_e, B_e, according to Lorentz's law is given by

$$\boldsymbol{F}(t) = \int d^3\boldsymbol{x}\Big(\rho\boldsymbol{E}_e + \frac{1}{c}\boldsymbol{J}\times\boldsymbol{B}_e\Big) \tag{5.9}$$

Under the previous assumptions, the force exerted on the pointlike distribution is

$$\begin{aligned}\boldsymbol{F}(t) &= q\boldsymbol{E}_e(\boldsymbol{0},t) + \boldsymbol{p}(t)\cdot\boldsymbol{\nabla}\boldsymbol{E}_e(\boldsymbol{0},t) + \frac{1}{c}\dot{\boldsymbol{p}}(t)\times\boldsymbol{B}_e(\boldsymbol{0},t)\\ &\quad + \frac{1}{3}\boldsymbol{Q}(t)\cdot\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{E}_e(\boldsymbol{0},t) + \frac{1}{3c}\dot{\boldsymbol{Q}}(t)\cdot\times\boldsymbol{\nabla}\boldsymbol{B}_e(\boldsymbol{0},t) + \frac{1}{3c^2}\frac{d}{dt}\Big[Q'(t)\frac{\partial\boldsymbol{E}_e}{\partial t}(\boldsymbol{0},t)\Big]\\ &\quad + \boldsymbol{m}(t)\cdot\boldsymbol{\nabla}\boldsymbol{B}_e(\boldsymbol{0},t) + \frac{1}{c}\boldsymbol{m}(t)\times\frac{\partial\boldsymbol{E}_e}{\partial t}(\boldsymbol{0},t)\end{aligned} \tag{5.10}$$

where it was introduced the notation $\boldsymbol{K}\cdot\times\boldsymbol{\nabla}\boldsymbol{H} \equiv (\boldsymbol{K}\cdot\boldsymbol{\nabla})\times\boldsymbol{H}$ for generic dyads $\boldsymbol{K},\boldsymbol{H}$.

Proof: Inserting (5.7) in (5.10) one gets

$$\begin{aligned}\boldsymbol{F}(t) &= \int d^3\boldsymbol{x}\,\Big[\Big(q\delta(\boldsymbol{x}) - \boldsymbol{p}(t)\cdot\boldsymbol{\nabla}\delta(\boldsymbol{x}) + \frac{1}{3}\boldsymbol{Q}(t)\cdot\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\delta(\boldsymbol{x})\\ &\qquad + \frac{1}{3}Q'(t)\boldsymbol{\nabla}^2\delta(\boldsymbol{x})\Big)\boldsymbol{E}_e(\boldsymbol{x},t) + \frac{1}{c}\Big(\dot{\boldsymbol{p}}(t)\delta(\boldsymbol{x}) - \frac{1}{3}\dot{\boldsymbol{Q}}(t)\cdot\boldsymbol{\nabla}\delta(\boldsymbol{x})\\ &\qquad - \frac{1}{3}\dot{Q}'(t)\boldsymbol{\nabla}\delta(\boldsymbol{x}) - c\boldsymbol{m}(t)\times\boldsymbol{\nabla}\delta(\boldsymbol{x})\Big)\times\boldsymbol{B}_e(\boldsymbol{x},t)\Big]\\ &= q\boldsymbol{E}_e(\boldsymbol{0},t) + \boldsymbol{p}(t)\cdot\boldsymbol{\nabla}\boldsymbol{E}_e(\boldsymbol{0},t) + \frac{1}{3}\boldsymbol{Q}(t)\cdot\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{E}_e(\boldsymbol{0},t)\\ &\qquad + \frac{1}{3}Q'(t)\boldsymbol{\nabla}^2\boldsymbol{E}_e(\boldsymbol{0},t) + \frac{1}{c}\Big(\dot{\boldsymbol{p}}(t)\times\boldsymbol{B}_e(\boldsymbol{0},t) + \frac{1}{3}\dot{\boldsymbol{Q}}(t)\cdot\times\boldsymbol{\nabla}\boldsymbol{B}_e(\boldsymbol{0},t)\\ &\qquad + \frac{1}{3}\dot{Q}'(t)\boldsymbol{\nabla}\times\boldsymbol{B}_e(\boldsymbol{0},t) + (c\boldsymbol{m}(t)\times\boldsymbol{\nabla})\times\boldsymbol{B}_e(\boldsymbol{0},t)\Big)\end{aligned} \tag{5.11}$$

by the basic properties of $\delta(\boldsymbol{x})$,$\boldsymbol{\nabla}\delta(\boldsymbol{x})$ and $\boldsymbol{\nabla}^2\delta(\boldsymbol{x})$. Maxwell equations (1.1) when the source is absent can be manipulated to get D'Alembert equation

$$\boldsymbol{\nabla}^2\boldsymbol{E}_e - \frac{1}{c^2}\frac{\partial^2\boldsymbol{E}_e}{\partial t^2} = 0 \tag{5.12}$$

and

$$\boldsymbol{\nabla}\times\boldsymbol{B}_e = \frac{1}{c}\frac{\partial\boldsymbol{E}_e}{\partial t} \tag{5.13}$$

which are valid in $\boldsymbol{x} = \mathbf{0}$ since the sources of the external field are supposed to be far away from $\mathbf{0}$. Moreover,

$$\begin{aligned}(\boldsymbol{m}(t)\times\boldsymbol{\nabla})\times\boldsymbol{B}_e(\boldsymbol{x},t) &= \boldsymbol{\nabla}\boldsymbol{B}_e(\boldsymbol{x},t)\cdot\boldsymbol{m}(t) - \boldsymbol{m}(t)\boldsymbol{\nabla}\cdot\boldsymbol{B}_e(\boldsymbol{x},t)\\ &= \boldsymbol{\nabla}\boldsymbol{B}_e(\boldsymbol{x},t)\cdot\boldsymbol{m}(t)\\ &= \boldsymbol{\nabla}\boldsymbol{B}_e(\boldsymbol{x},t)\cdot\boldsymbol{m}(t) - \boldsymbol{m}(t)\cdot\boldsymbol{\nabla}\boldsymbol{B}_e\,(\boldsymbol{x},t) + \boldsymbol{m}(t)\cdot\boldsymbol{\nabla}\boldsymbol{B}_e\,(\boldsymbol{x},t)\\ &= \boldsymbol{m}(t)\times(\boldsymbol{\nabla}\times\boldsymbol{B}_e(\boldsymbol{x},t)) + \boldsymbol{m}(t)\cdot\boldsymbol{\nabla}\boldsymbol{B}_e\,(\boldsymbol{x},t)\\ &= \boldsymbol{m}(t)\times\frac{1}{c}\frac{\partial\boldsymbol{E}_e}{\partial t}(\mathbf{0},t) + \boldsymbol{m}(t)\cdot\boldsymbol{\nabla}\boldsymbol{B}_e\,(\mathbf{0},t)\end{aligned} \tag{5.14}$$

Substituting these into (5.11) shows (5.10).

From Lorentz's law follows the definition of the moment of the forces

$$\boldsymbol{\tau}(t) = \int d^3\boldsymbol{x}\ \boldsymbol{x}\times\left(\rho\boldsymbol{E}_e + \frac{1}{c}\boldsymbol{J}\times\boldsymbol{B}_e\right) \tag{5.15}$$

The computation of the moment of the forces exerted on the distribution yields

$$\begin{aligned}\boldsymbol{\tau}(t) = \boldsymbol{p}(t)\times\boldsymbol{E}_e(\mathbf{0},t) + \frac{2}{3}\boldsymbol{Q}(t)\cdot\times\boldsymbol{\nabla}\boldsymbol{E}_e(\mathbf{0},t) - \frac{2}{3c}Q'(t)\frac{\partial\boldsymbol{B}_e}{\partial t}\\ + \frac{1}{3c}\dot{\boldsymbol{Q}}(t)\cdot\boldsymbol{B}_e(\mathbf{0},t) + \frac{1}{3c}\dot{Q}'(t)\boldsymbol{B}_e(\mathbf{0},t) + \boldsymbol{m}(t)\times\boldsymbol{B}_e(\mathbf{0},t)\end{aligned} \tag{5.16}$$

Proof: Inserting (5.7) in (5.15) one gets

$$\begin{aligned}\boldsymbol{\tau}(t) = \int d^3\boldsymbol{x}\ \boldsymbol{x}\times\Big[\Big(&q\delta(\boldsymbol{x}) - \boldsymbol{p}(t)\cdot\boldsymbol{\nabla}\delta(\boldsymbol{x}) + \frac{1}{3}\boldsymbol{Q}(t)\cdot\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\delta(\boldsymbol{x})\\ &+ \frac{1}{3}Q'(t)\boldsymbol{\nabla}^2\delta(\boldsymbol{x})\Big)\boldsymbol{E}_e(\boldsymbol{x},t) + \frac{1}{c}\Big(\dot{\boldsymbol{p}}(t)\delta(\boldsymbol{x}) - \frac{1}{3}\dot{\boldsymbol{Q}}(t)\cdot\boldsymbol{\nabla}\delta(\boldsymbol{x})\\ &- \frac{1}{3}\dot{Q}'(t)\boldsymbol{\nabla}\delta(\boldsymbol{x}) - c\boldsymbol{m}(t)\times\boldsymbol{\nabla}\delta(\boldsymbol{x})\Big)\times\boldsymbol{B}_e(\boldsymbol{x},t)\Big]\end{aligned} \tag{5.17}$$

Using the property of Dirac's delta function

$$\int d^3\boldsymbol{x}\, f(\boldsymbol{x})\delta(\boldsymbol{x}) = f(\mathbf{0}) \tag{5.18}$$

in the terms in which $\delta(\boldsymbol{x})$ is not differentiated gives

$$\int d^3\boldsymbol{x}\ \boldsymbol{x}\times q\delta(\boldsymbol{x})\boldsymbol{E}_e = 0, \quad \int d^3\boldsymbol{x}\ \boldsymbol{x}\times\frac{1}{c}\Big(\dot{\boldsymbol{p}}(t)\delta(\boldsymbol{x})\times\boldsymbol{B}_e\Big) = 0 \tag{5.19}$$

Let $\boldsymbol{v}$, $\boldsymbol{F}(\boldsymbol{x})$ be respectively a constant vector and a vector field, than the useful identity obtained by an integration by parts holds

$$\int d^3\boldsymbol{x}\,(\boldsymbol{v}\cdot\boldsymbol{\nabla})\delta(\boldsymbol{x})\,\boldsymbol{x}\times\boldsymbol{F}(\boldsymbol{x}) = -\boldsymbol{v}\times\boldsymbol{F}(\boldsymbol{0}) \tag{5.20}$$

so that

$$\int d^3\boldsymbol{x}\ \boldsymbol{x}\times\Big[\boldsymbol{p}(t)\cdot\boldsymbol{\nabla}\delta(\boldsymbol{x})\boldsymbol{E}_e\Big] = -\boldsymbol{p}(t)\times\boldsymbol{E}_e(\boldsymbol{0},t) \tag{5.21}$$

Let $\boldsymbol{A}$, $\boldsymbol{F}(\boldsymbol{x})$ be respectively a constant symmetric dyad and a vector field, than the useful identity obtained by a double integration by parts holds

$$\int d^3\boldsymbol{x}\,(\boldsymbol{A}\cdot\cdot\boldsymbol{\nabla}\boldsymbol{\nabla})\delta(\boldsymbol{x})\,\boldsymbol{x}\times\boldsymbol{F}(\boldsymbol{x}) = 2\boldsymbol{A}\cdot\times\boldsymbol{\nabla}\boldsymbol{F}(\boldsymbol{0}) \tag{5.22}$$

and taking $\boldsymbol{A} = \boldsymbol{1}$

$$\int d^3\boldsymbol{x}\,\boldsymbol{\nabla}^2\delta(\boldsymbol{x})\,\boldsymbol{x}\times\boldsymbol{F}(\boldsymbol{x}) = 2\boldsymbol{\nabla}\times\boldsymbol{F}(\boldsymbol{0}) \tag{5.23}$$

which give

$$\int d^3\boldsymbol{x}\ \boldsymbol{x}\times\left[\frac{1}{3}\boldsymbol{Q}(t)\cdot\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\delta(\boldsymbol{x})\boldsymbol{E}_e\right] = \frac{2}{3}\boldsymbol{Q}(t)\cdot\times\boldsymbol{\nabla}\boldsymbol{E}_e(\boldsymbol{0},t) \tag{5.24a}$$

$$\int d^3\boldsymbol{x}\ \boldsymbol{x}\times\left[\frac{1}{3}Q'(t)\boldsymbol{\nabla}^2\delta(\boldsymbol{x})\boldsymbol{E}_e\right] = \frac{2}{3}Q'(t)\boldsymbol{\nabla}\times\boldsymbol{E}_e(\boldsymbol{0},t) \tag{5.24b}$$

Let $\boldsymbol{A}$, $\boldsymbol{F}(\boldsymbol{x})$ be respectively a constant symmetric traceless dyad and a vector field, than the useful identity obtained by an integration by parts and the availment of (4.6) holds

$$\int d^3\boldsymbol{x}\ \boldsymbol{x}\times\Big[(\boldsymbol{A}\cdot\boldsymbol{\nabla})\delta(\boldsymbol{x})\times\boldsymbol{F}(\boldsymbol{x})\Big] = -\boldsymbol{A}\cdot\boldsymbol{F}(\boldsymbol{0}) \tag{5.25}$$

and taking $\boldsymbol{A} = \boldsymbol{1}$

$$\int d^3\boldsymbol{x}\ \boldsymbol{x}\times\Big[\boldsymbol{\nabla}\delta(\boldsymbol{x})\times\boldsymbol{F}(\boldsymbol{x})\Big] = -\boldsymbol{F}(\boldsymbol{0}) \tag{5.26}$$

which give

$$\int d^3\boldsymbol{x}\ \boldsymbol{x}\times\Big[-\frac{1}{3c}\dot{\boldsymbol{Q}}(t)\cdot\boldsymbol{\nabla}\ \delta(\boldsymbol{x})\times\boldsymbol{B}_e\Big]=\frac{1}{3c}\dot{\boldsymbol{Q}}(t)\cdot\boldsymbol{B}_e\,(\boldsymbol{0},t) \tag{5.27a}$$

$$\int d^3\boldsymbol{x}\ \boldsymbol{x}\times\Big[-\frac{1}{3c}\dot{Q}'(t)\boldsymbol{\nabla}\delta(\boldsymbol{x})\times\boldsymbol{B}_e\Big]=\frac{1}{3c}\dot{Q}'(t)\boldsymbol{B}_e(\boldsymbol{0},t) \tag{5.27b}$$

And lastly, let $\boldsymbol{v}$, $\boldsymbol{F}(\boldsymbol{x})$ be respectively a constant vector and a solenoidal vector field. The useful identity obtained by (4.6) and an integration by parts holds

$$\int d^3\boldsymbol{x}\ \boldsymbol{x}\times\Big[(\boldsymbol{v}\times\boldsymbol{\nabla})\delta(\boldsymbol{x})\times\boldsymbol{F}(\boldsymbol{x})\Big]=-\boldsymbol{v}\times\boldsymbol{F}(\boldsymbol{0}) \tag{5.28}$$

giving

$$\int d^3\boldsymbol{x}\ \boldsymbol{x}\times\Big[\Big(-c\boldsymbol{m}(t)\times\boldsymbol{\nabla}\delta(\boldsymbol{x})\Big)\times\boldsymbol{B}_e\Big]=c\boldsymbol{m}(t)\times\boldsymbol{B}_e(\boldsymbol{0},t) \tag{5.29}$$

Faraday Neumann Lenz law (1.1c) and all the calculation done inserted in (5.17) yield (5.16).

Electric polarization

In matter atoms and molecules often behave as electric dipoles when immersed in an external homogeneous electromagnetic field. Since the leading multipole if non absent is the electric dipole, neglecting the lower order terms the moment of the forces (5.16) reduces to

$$\boldsymbol{\tau}=\boldsymbol{p}\times\boldsymbol{E}_e \tag{5.30}$$

which is exerted until it vanishes, that is when $\boldsymbol{p}\parallel\boldsymbol{E}_e$.

The condition of stable equilibrium is established when $\boldsymbol{p},\boldsymbol{E}_e$ are aligned.

Proof: From the definition, the potential energy U of a force $\boldsymbol{F}$ is

$$\boldsymbol{F}=-\boldsymbol{\nabla}U \tag{5.31}$$

In this case, consider (5.10) only in its electric dipole-electric field coupling terms. Since the electric dipole $\boldsymbol{p}$ does not depend on the position, the force expression reduces to

$$\begin{aligned} \boldsymbol{F} &= \boldsymbol{p} \cdot \boldsymbol{\nabla} \boldsymbol{E}_e \\ &= -\boldsymbol{\nabla}(-\boldsymbol{p} \cdot \boldsymbol{E}_e) \end{aligned} \tag{5.32}$$

by which one concludes that

$$U = -\boldsymbol{p} \cdot \boldsymbol{E}_e \tag{5.33}$$

For geometrical considerations, a stable equilibrium, corresponding to a minimum of the potential energy U, occurs when the dipole moment $\boldsymbol{p}$ is aligned with the electric field $\boldsymbol{E}_e$ at any given time t.

A neutral charge distribution containing an electric dipole generates an electric field (cf. eqs.(3.16)) that is

$$\boldsymbol{E}(\boldsymbol{x}) = \frac{3\boldsymbol{p} \cdot \hat{\boldsymbol{x}} \hat{\boldsymbol{x}} - \boldsymbol{p}}{|\boldsymbol{x}|^3} \tag{5.34}$$

Choosing $\boldsymbol{p} \parallel \boldsymbol{e}_3$ and observing the field along the polar axis $\boldsymbol{x} \parallel \boldsymbol{e}_3$,

$$\boldsymbol{E} \parallel \boldsymbol{e}_3 \tag{5.35}$$

Carrying the analysis for a physical dipole (well representing atoms and molecules), that is composed of two separate point charges of the same magnitude but different sign, one finds that in the space between those latter along $\boldsymbol{e}_3$ the field is opposite to

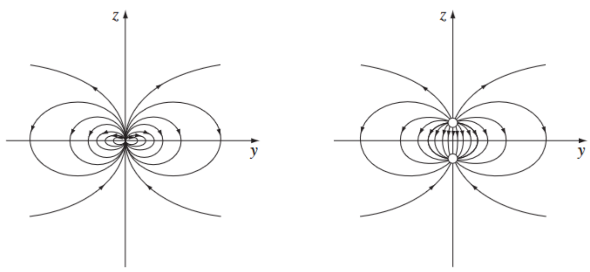


Figure 5.1: Comparison between a "pure" dipole (left), generated by a point-like charge distribution, and a physical dipole (right), generated by two charged point particles of different sign. The electric field lines between the charges is oriented in the opposite verse with respect to the rest of the z axis. ([2], p.159)

$\boldsymbol{E}$ (see fig.5.1). Thus the electric field $\boldsymbol{E}_i$ generated where there is the neutral charge distribution immersed in external electromagnetic field is anti-aligned with respect to $\boldsymbol{E}_e$. Knowing that the electric field satisfies the superposition principle, one concludes that the resulting electric field

$$\boldsymbol{E}_{tot} = \boldsymbol{E}_e + \boldsymbol{E}_i = |\boldsymbol{E}_e|\boldsymbol{e}_3 - |\boldsymbol{E}_i|\boldsymbol{e}_3 \tag{5.36}$$

is lower than the original external field $\boldsymbol{E}_e$. So the types of materials that behave as explained, called *dielectrics*, have the property to screen the external electromagnetic field, and this phenomenon is known as *electric polarization*.

Larmor precession

Now consider a charge and current distribution which only relevant multipole contribution is given by magnetic dipole. By (5.16) the moment of the forces on the distribution is

$$\boldsymbol{\tau} = \boldsymbol{m} \times \boldsymbol{B}_e \tag{5.37}$$

Suppose that the charge and current distribution belongs to a body of mass $M = \int d^3\boldsymbol{x}\, \rho_M(\boldsymbol{x}, t)$. From classical mechanics

$$\boldsymbol{\tau} = \frac{d\boldsymbol{L}}{dt} \tag{5.38}$$

where $\boldsymbol{L}$ is the angular momentum of the body, given by

$$\boldsymbol{L}(t) = \int d^3\boldsymbol{x}\, \boldsymbol{x} \times \rho_M(\boldsymbol{x}, t)\boldsymbol{v}(\boldsymbol{x}, t) \tag{5.39}$$

expressed in terms of the density of mass ρ_M and the velocity vector field $\boldsymbol{v}$.

If the density of mass is a function that is proportional to the charge distribution for some constant which will be denoted as κ,

$$\rho_M = \kappa\rho \tag{5.40}$$

than the relation between the angular momentum and the magnetic dipole of the body

$$\boldsymbol{m} = \frac{1}{2\kappa c}\boldsymbol{L} \tag{5.41}$$

leads to the differential equation

$$\frac{d\boldsymbol{m}}{dt}(t) = \frac{1}{2\kappa c}\boldsymbol{m}(t)\times\boldsymbol{B}_e(\boldsymbol{0},t) \tag{5.42}$$

Proof: From the definition (3.2)

$$\begin{aligned}\boldsymbol{m}(t) &= \int d^3\boldsymbol{x}\,\frac{1}{2c}\boldsymbol{x}\times\boldsymbol{J}(\boldsymbol{x},t)\\ &= \int d^3\boldsymbol{x}\,\frac{1}{2c}\boldsymbol{x}\times\rho(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)\\ &= \int d^3\boldsymbol{x}\,\frac{1}{2\kappa c}\boldsymbol{x}\times\rho_M(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)\\ &= \frac{1}{2\kappa c}\boldsymbol{L}(t)\end{aligned} \tag{5.43}$$

Substituting this relation into (5.38), and coupling it with (5.37) through the $\boldsymbol{\tau}$ dependence one finds (5.42).

(5.42) describes the time evolution of the magnetic dipole of a charge and current distribution immersed in an external electromagnetic field under the assumptions stressed at the start of this section. To solve it will be assumed that

$$\boldsymbol{B}_e(\boldsymbol{0},t) = B_0\boldsymbol{e}_3, \qquad B_0 = \omega(-2\kappa c) \tag{5.44}$$

that is the choice of a frame reference and $\boldsymbol{B}_e$ is time indipendent. It was defined ω, called *Larmor frequency*. It can also be defined

$$\boldsymbol{\omega} = -\frac{1}{2\kappa c}\boldsymbol{B}_e \tag{5.45}$$

the *vector Larmor frequency*. Now (5.42) can be solved decoupling all the vectors in their components parallel and orthogonal with respect to $\boldsymbol{e}_3$. For a generic vector $\boldsymbol{a}$ the decomposition reads

$$\boldsymbol{a} = a_{\parallel}\boldsymbol{e}_3 + \boldsymbol{a}_{\perp}, \tag{5.46}$$

which components are

$$a_{\parallel} = \boldsymbol{a}\boldsymbol{\cdot}\boldsymbol{e}_3, \qquad \boldsymbol{a}_{\perp} = \boldsymbol{a} - \boldsymbol{a}\boldsymbol{\cdot}\boldsymbol{e}_3\boldsymbol{e}_3 = \boldsymbol{e}_3\times(\boldsymbol{a}\times\boldsymbol{e}_3). \tag{5.47}$$

Thus (5.42) for $\boldsymbol{m}$ decoupled in its components is

$$\frac{dm_{\parallel}}{dt} = 0 \tag{5.48a}$$

$$\frac{d\boldsymbol{m}_{\perp}}{dt} = -\boldsymbol{m}_{\perp}\times\omega\boldsymbol{e}_3 \tag{5.48b}$$

Proof: First, from its definition the vector Larmor frequency can be decomposed only in its parallel component,

$$\boldsymbol{\omega} = -\frac{1}{2\kappa c}\boldsymbol{B}_e = \omega\boldsymbol{e}_3 \tag{5.49}$$

Than, the parallel component of (5.42) if found by performing a scalar product with $\boldsymbol{e}_3$ as from construction

$$\frac{dm_{\parallel}}{dt} = \frac{d\boldsymbol{m}}{dt}\boldsymbol{\cdot}\boldsymbol{e}_3 = (\boldsymbol{m}\times\boldsymbol{\omega})\boldsymbol{\cdot}\boldsymbol{e}_3 = 0 \tag{5.50}$$

Instead, the orthogonal component is given by

$$\begin{aligned}\frac{d\boldsymbol{m}_{\perp}}{dt} &= \boldsymbol{e}_3\times\left[\frac{d\boldsymbol{m}}{dt}\times\boldsymbol{e}_3\right] = \boldsymbol{e}_3\times\left[-(\boldsymbol{m}\times\boldsymbol{\omega})\times\boldsymbol{e}_3\right]\\ &= \omega\boldsymbol{e}_3\times(\boldsymbol{m} - \boldsymbol{m}\boldsymbol{e}_3\boldsymbol{\cdot}\boldsymbol{e}_3) = \omega\boldsymbol{e}_3\times\boldsymbol{m}_{\perp}\end{aligned} \tag{5.51}$$

Given the initial conditions,

$$m_{\parallel}(0) = m_{\parallel 0}, \qquad \boldsymbol{m}_{\perp}(0) = \boldsymbol{m}_{\perp 0} \tag{5.52}$$

by direct substitution one verifies that

$$m_{\parallel}(t) = m_{\parallel 0} \tag{5.53a}$$

$$\boldsymbol{m}_{\perp}(t) = \cos(\omega t)\boldsymbol{m}_{\perp 0} + \sin(\omega t)\boldsymbol{e}_3\times\boldsymbol{m}_{\perp 0} \tag{5.53b}$$

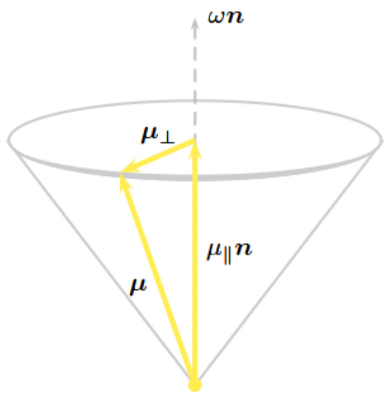


Figure 5.2: Larmor precession of the components of $\boldsymbol{\mu} \equiv \boldsymbol{m}$ around the normal $\boldsymbol{n}$ (in the text $\boldsymbol{e}_3$). [17]

is solution of (5.48). See the figure 5.2 for a pictorial representation.

Thus it was found that the component of the magnetic dipole parallel to the direction of the external magnetic induction field stays constant, conversely the orthogonal spins around the axis of the same direction with period $\tau = 2\pi/\omega$. This phenomenon goes under the name of *Larmor precession*.

5.3 Classical theory of the Stern and Gerlach experiment

The Stern-Gerlach experiment, first performed in 1922 by Otto Stern and Walther Gerlach, is an important experiment which led to the hypothesis of the existence of an intrinsic angular momentum (spin) of electrons and its quantization.

From a classical point of view electrons have angular momentum due to the revolution around the nucleus, and that generates a magnetic dipole according to (5.41) identifying

$$\kappa = \frac{m}{(-e)}, \tag{5.54}$$

that is the ratio of proportionality between mass ρ_M and charge ρ densities of a body made of particles of mass m and charge $-e$. It was observed that magnetic dipoles interact with external magnetic induction fields. To determine whether this interaction accurately describes atomic behavior, one can generate an external magnetic field and observe how atoms respond upon passing through it. This is precisely the principle underlying the Stern–Gerlach experiment. In this experiment, a collimated beam of unbound atoms is let pass through a region with a non-uniform magnetic field $\boldsymbol{B}$, generated by a specially shaped magnet. The positions at which the atoms subsequently arrive on a detection screen are then recorded and analyzed. The magnet is designed by exploiting geometric symmetries so that, in the region traversed by the atomic beam, the magnetic field is primarily directed along the upward vertical axis and exhibits a spatial gradient along this same direction. A reference frame is fixed such that the axis $\boldsymbol{e}_3$ identifies it. Thus the magnetic field is approximately given by

$$\boldsymbol{B}(\boldsymbol{x}) = B(z)\boldsymbol{e}_3 \tag{5.55}$$

The charges passing throught the field are subject to a force (cf. eq.(5.10))

$$\boldsymbol{F} = \boldsymbol{m} \cdot \boldsymbol{\nabla} \boldsymbol{B} \tag{5.56}$$

and a torque

$$\boldsymbol{\tau} = \boldsymbol{m} \times \boldsymbol{B} \tag{5.57}$$

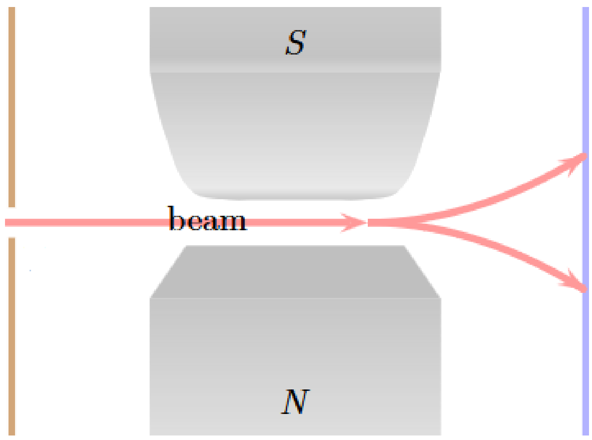


Figure 5.3: A schematic representation of the experimental setup of Stern-Gerlach experiment. The magnetic field generated by the magnet interacts with the magnetic dipole of atoms deflecting the trajectory. [17]

According to classical mechanics the linear momentum of a massive body obeys

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F} \tag{5.58}$$

which in this case yields the differential equation

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{m} \cdot \boldsymbol{\nabla} \boldsymbol{B} \tag{5.59}$$

that if solved gives the trajectory of the charge. The moment of the forces was seen to be following

$$\frac{d\boldsymbol{m}}{dt} = -\frac{e}{2mc} \boldsymbol{m} \times \boldsymbol{B} \tag{5.60}$$

that in the last section was solved to give the Larmor precession of $\boldsymbol{m}$ around $\boldsymbol{B}$ (cf. eq.(5.53)). Now it should be taken into account the position dependence of the external magnetic field. The coupling of this last equation with (5.59) makes finding the solution harder. Because of the non uniformity of the magnetic field the vector Larmor frequency (5.45) is not constant anymore. However, if the relative variation of the Larmor frequency caused by the deflection of the atom's trajectory according to (5.59) in the istantaneous period of precession is small, the frequency position dependence can be neglected and ω taken to be constant. The solution of (5.60) with the former hypothesis is provided by (5.53). Using the same convention of vector decomposition around the polar axis $\boldsymbol{e}_3$ employed in the last section (5.46), one finds

$$p_{\parallel}(t) = p_{0\parallel} + m_{0\parallel} \int_0^t dt' \, \frac{dB(z')}{dz} \tag{5.61a}$$

$$\boldsymbol{p}_{\perp}(t) = \boldsymbol{p}_{0\perp} \tag{5.61b}$$

where are given the initial conditions

$$p_{\parallel}(0) = p_{0\parallel}, \qquad \boldsymbol{p}_{\perp}(0) = \boldsymbol{p}_{0\perp} \tag{5.62}$$

Proof: The approximation of constant Larmor frequency is essential to grant the same result that in the previous chapter led to Larmor precession. However, notice that the result

$$m_{\parallel}(t) = m_{0\parallel} \tag{5.63}$$

which is exploited later in this proof, remains valid regardless of the approximation of constant Larmor frequency (see proof of (5.48)).

Now (5.59) should be decomposed in its orthogonal and parallel components. Using the equation for the force exerted (5.56) according to the semplification of the magnetic field (5.55),

$$\frac{dp_{\parallel}}{dt} = \frac{d\boldsymbol{p}}{dt}\boldsymbol{\cdot}\boldsymbol{e}_3 = \boldsymbol{m}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{e}_3 = m_{0\parallel}\frac{dB(z)}{dz} \tag{5.64}$$

in which $d\boldsymbol{B}/dz$ is computed in the position of the charge along the z axis. The orthogonal component is instead given by

$$\begin{aligned}\frac{d\boldsymbol{p}_{\perp}}{dt} = \boldsymbol{e}_3\times\left(\frac{d\boldsymbol{p}}{dt}\times\boldsymbol{e}_3\right) &= \boldsymbol{e}_3\times\Big(\boldsymbol{m}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{B}\times\boldsymbol{e}_3\Big) \\ &= \boldsymbol{e}_3\times\Big(\boldsymbol{m}\boldsymbol{\cdot}\boldsymbol{\nabla}B(z)\boldsymbol{e}_3\times\boldsymbol{e}_3\Big) = \boldsymbol{0}\end{aligned} \tag{5.65}$$

Now it is trivial to perform an integration over time in the equations obtained with initial conditions (5.62) to get (5.61).

This means that the atoms of the beam are deflected only in the direction $\boldsymbol{e}_3$, while the orthogonal component of initial momentum remains unchanged. The deflection is expected to be small for high magnitudes of $\boldsymbol{p}_{0\perp}$. Indeed one verifies that an another time integration of the orthogonal component in (5.61) and the knowledge of the distance of the screen from the starting point of the trajectory yield the flight time, which is smaller for higher values of the initial momentum. The longer the flight time, the greater the deflection, as the charge remains under the influence of the magnet for a longer duration (an observation that is corroborated by the integral in (5.61)).

Inspecting the parallel component one notices that the position on the screen reached by each atom varies based on the value of $m_{0\parallel}$. Indeed every atom is prepared so that $p_{0\parallel}$ is small (ideally zero), the contribute given by the integral over the gradient of the magnetic field is almost the same for all atoms as they undergo near paths with approximately the same value of $\boldsymbol{p}_{\perp}$. The relevant contribute is given by the continuous value of $m_{0\parallel}$ (proportional to the angular momentum parallel component according to

(5.41)) which can also be negative so that the deflection happens in both verses.

Notice that the analysis was carried only for the magnetic dipole of the electrons. An important remark needs to be done on why the contribution of the angular momentum given by an eventual nucleus rotation was not taken in consideration. For a proton κ is the opposite of (5.54) due to the charge, with m being the mass of the proton $\sim 1TeV$ that is around three orders of magnitude bigger than the mass of the electron $\sim 1MeV$. So inspecting (5.41) it is found that the magnitude of the moment of magnetic dipole of an electron is more than 10^3 times greater than the one of the proton, making this latter negligible.

In conclusion from the former quantitative analysis it is expected that on the screen a continuous stripe of atoms approximately centered in the atoms initial position on the vertical axis and displaced along it is observed. This reveals partially true. In reality atoms reach only the ends of the stripe, but not the middle positions. In light of quantum mechanics theory this is explained with the quantization of the total angular momentum, also determined by the spin and not only the orbital angular momentum as hypothesized. Indeed it is observed that even when the orbital angular momentum should be zero the deflection is present, but a further treatment is outside of this scope. Even if the classical approach to this problem does not provide a full explanation of the observations, it is fundamental to understand what happens physically predicting in the right way the presence and direction of the deflection due to the interaction between moving atoms which possess a magnetic dipole and the external magnetic field.

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