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A progressive derivation of Pure $\mathcal{N} = 1$, $\mathcal{D} = 4$ Supergravity

Relatore:
Fiorenzo Bastianelli

Presentata da:
Cesare Sabattini Manginella

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I would like to thank my family and my supervisor
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Abstract

La Supergravità pura $\mathcal{N} = 1$, $\mathcal{D} = 4$ è una moderna teoria di campo di gauge della Supersimmetria, elaborata da Daniel Z. Freedman, Peter van Nieuwenhuizen e Sergio Ferrara nel 1976. Nella sua versione massimale, ha rappresentato una delle più promettenti proposte teoriche in termini di unificazione delle interazioni fondamentali del secolo scorso, precorrendo la Teoria delle Stringhe.

Si vuole qui ricostruirne progressivamente il formalismo, introducendo per gradi le conoscenze teoriche necessarie, quali gruppi di simmetria spazio-temporale, teoria quantistica di campi scalare e fermionico, Supersimmetria, interpretazione della Relatività Generale come teoria di gauge, formalismo tetrardico, teoria di Rarita Schwinger e derivazione del Lagrangiano di Supergravità pura a partire dal modello supersimmetrico di Wess-Zumino.

Contents

Introduction

Supergravity (often abbreviated as SUGRA) represents the very first promising mirage of *Unification Theory* of the four fundamental forces under a unique gauged symmetry group.

Originally formulated in 1976 by Daniel Zissel Freedman, Sergio Ferrara and Peter van Nieuwenhuizen, in its pure form, it is the theory of gauged unbroken $\mathcal{N} = 1$ Supersymmetry (SUSY).

Therefore, to approach SUGRA, it's essential to comprehend SUSY, which is fundamentally a generalisation of a continuous *global* symmetry group to both fermionic and bosonic generators in a super-Lie algebra.

Gauging a symmetry substantially means promoting its parameters to functions on the group domain; for instance, the Standard Model is based on the $SU(3) \otimes SU(2) \otimes U(1)$ gauged symmetry group, under which the electroweak and strong forces are unified.

Supersimmetry attempts to connect internal and space-time symmetries whithin a non trivially factorisable group, overcoming the limitations of the *Coleman-Mandula* theorem. Since an algebra has an irreducible representation on particle states, bosons and fermions are grouped in multiplets.

Therefore, we will first need to achieve two preliminary goals: first comprehending some basic *representation theory* of the Poincarè group and the *Wigner classification*, to then probe Quantum Field Theory and the Lorentz invariant Lagrangian formalism of the Canonical Quantization, and apply it to spinors in Special Relativity, deriving the renowned Dirac equation.

At this point, we will be ready to delve into Supersymmetry, its algebra representation on particle states, and understand how superpartners arise.

Then we will be almost halfway there: indeed, SUGRA requires the spinorial parameter of a SUSY transformation to be graded to a space-time function, and we will see that a naive Minkowskian domain is not suitable for such purpose: for this reason, the second part of the Supersymmetry chapter will be pinpointed on the derivation of the Wess-Zumino model in the more fitting *superfield formalism*.

Next, we will introduce the gauge formulation of General Relativity, and its *tetrads* (or *vierbein*) formalism, exploring how the *graviton* emerges as a first order perturbative tensorial field of the Minkowski metric, and completing the SUGRA multiplet, by quantizing the vector-spinor Rarita-Schwinger field and identify the *gravitino*.

At this point, we will be prepared enough to tackle the final boss: deriving the pure SUGRA Lagrangian as a natural term arising when trying to achieve local SUSY invariance of the Wess-Zumino action, and covariantize it in the vierbein formalism.

I know this may sound like a painful endeavour, and I cannot deny that during the last months I really had to pledge myself to both gather it and to make this dissertation as flowing as possible.

The funniest part is that not only SUGRA has no direct experimental support (no graviton and gravitino have ever been observed): most of its predictions are wrong. For instance, it foresees a ridicolously high value of the cosmological constant.

So why should we care about SUGRA at all? First, a delightful answer for everyone: it represents the low energy limit ($\leq 10^{19}\ eV$) of Superstring Theory, which nowadays is one of the most accredited unification theories in theoretical physics. Secondly, I will let someone much more qualified than me rejoin:

> It is more important to have beauty in one's equations than to have them fit experiment.
>
> — *Paul A. M. Dirac* [2]

Chapter 1

Space-Time Symmetry Groups

In this chapter, we aim to get to grips with some representation formalism, by studying the Poincaré group $ISO(1,3)$, being the defining symmetry group of a relativistic theory.

First, we will infer the *structure constants* of its Lie algebra, on an abstract level, to then examine its *particle representation* in the context of Wigner's classification, and introduce the concept of *spin*.

Thereafter, we will focus on irreducible representations of the double cover of its Lorentz subgroup $SO(1,3)$, algebraically deducing the *Weyl spinor*, to then construct the *Dirac Spinor* Ψ, instead transforming in a reducible one, by studying the Clifford algebra, and making sure to spell out the connection among these two structures in the chiral basis.

All the statements will be properly demonstrated, but we want to reassure the reader that a more discursive treatement will follow alongside, since the key purpose of this chapter is understanding the captivating power of the representation theory of $ISO(1,3)$ and its $SO(1,3)$ subgroup.

1.1 Poincaré group $ISO(1,3)$

In its natural representation, $ISO(1,3)$ is the group of isomorphisms on Minkowski space-time $\mathbb{R}^{1,3}$, defined as the external semiproduct of the Lorentz group $O(1,3)$ and the translations group $\mathbb{R}^4$:

$$\begin{gathered} ISO(1,3) \equiv O(1,3) \ltimes \mathbb{R}^4 \Longleftrightarrow ISO(1,3) = \left\{ g_\alpha \in GL(1,3) :\, g_\alpha^{-1} \eta g_\alpha = \eta \right\} \Longleftrightarrow \\ \Longleftrightarrow \left\{ (\Lambda, a) \in ISO(1,3) \Longleftrightarrow (\Lambda, a) \cdot (\Lambda', a') = (\Lambda\Lambda', a + \Lambda a') \ , \forall\, \Lambda \in O(1,3) \ , \, a \in \mathbb{R}^4 \right\} \end{gathered} \tag{1.1}$$

where $\cdot$ is the $ISO(1,3)$ group composition, η is the Minkowskian metric tensor and $GL(1,3)$ is the group of linear space-time transformations.

1.1.1 $ISO(1,3)$ algebra

$ISO(1,3)$ is a Lie group:

$$\left[\begin{cases} g_\alpha \cdot g_\beta \in ISO(1,3) \\ \exists\; g_\alpha^{-1} \in ISO(1,3) \\ \exists\; I \in ISO(1,3) : g_\alpha^{-1} \cdot g_\alpha = I \end{cases} \vee \quad \begin{cases} g_\alpha \cdot g_\beta \\ g_\alpha^{-1} \end{cases} \text{are diffeomorphisms} \right] \forall\; g_\alpha, g_\beta \in ISO(1,3) \tag{1.2}$$

of dimension $d(ISO(1,3)) = d((O(1,3))) + d(\mathbb{R}^4) = 10$.

Therefore, there exists a basis of 10 generators for its tangent space (i.e. its Lie algebra $\mathfrak{iso}(1,3)$): since the factors of a semiproduct forming a Lie group are also Lie groups, the resulting basis of generators will be given by the union of those of the two subgroups.

The difference of a product and a semiproduct physically lies in the non commutativity of the generators of the subgroups between each others, resulting in non trivial structure constants. We now want to compute the commutation relations between the generators, directly from Equation 1.1.

Lorentz algebra $\mathfrak{o}(1,3)$

Given $\Lambda \in SO(1,3)$, $\omega \in GL(1,3)$ and $\epsilon \in \mathbb{R}^+$, we can Taylor expand the Lorentz transformation near the identity:

$$\Lambda = e^{\epsilon\omega^{\mu}{}_{\nu}} \simeq \delta^{\mu}{}_{\nu} + \epsilon\omega^{\mu}{}_{\nu} + \mathcal{O}(\epsilon^2) \quad \epsilon \ll 1 \tag{1.3}$$

and, seeing as $(\Lambda, \mathbf{1}) \in ISO(1,3)$, Λ is an isomorphism, and consequently:

$$\begin{aligned} &\Lambda^{\mu}{}_{\rho}\eta_{\mu\nu}\Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma} \Longrightarrow \left(\delta^{\mu}{}_{\rho} + \epsilon\omega^{\mu}{}_{\rho}\right)\eta_{\mu\nu}\left(\delta^{\nu}{}_{\sigma} + \epsilon\omega^{\nu}{}_{\sigma}\right) = \eta_{\rho\sigma} \Longleftrightarrow \\ &\eta_{\rho\sigma} + \epsilon\omega_{\rho\sigma} + \epsilon\omega_{\sigma\rho} + \epsilon^2 m^T_{\rho_\nu} m^{\nu}_{\sigma} \simeq \eta_{\rho\sigma} + \epsilon\omega_{\rho\sigma} + \epsilon\omega_{\sigma\rho} = \eta_{\rho\sigma} \Longrightarrow \omega_{\rho\sigma} = -\omega_{\sigma\rho} \end{aligned} \tag{1.4}$$

Ergo an element of $\mathfrak{o}(1,3)$ is a real antisymmetric 4×4 matrix, which can be expressed as a linear combination of six real antisymmetric generators M^{AB}:

$$\begin{cases} \omega^{\mu}{}_{\nu} = \frac{1}{2}\left(\Omega_{AB}M^{AB}\right)^{\mu}{}_{\nu} \\ M^{AB} = -M^{BA} \end{cases} \quad , \quad A, B, \mu, \nu \in [0,4] \cap \mathbb{N} \tag{1.5}$$

In this notation, A and B are the *representation indices*, expressing ω on the basis of the algebra, and μ, ν are the usual space-time indices.

A possible set of generators is:

$$(M^{AB})^{\mu}{}_{\nu} = \eta^{A\mu}\delta^{B}{}_{\nu} - \eta^{B\mu}\delta^{A}{}_{\nu} \tag{1.6}$$

since:

$$(M^{BA})^{\mu}{}_{\nu} = \eta^{B\mu}\delta^{A}{}_{\nu} - \eta^{A\mu}\delta^{B}{}_{\nu} = -\left(\eta^{A\mu}\delta^{B}{}_{\nu} - \eta^{B\mu}\delta^{A}{}_{\nu}\right) = -M^{AB} \tag{1.7}$$

which can be used to derive the representation-independent commutation relations of the Lorentz algebra:

$$\begin{aligned} &\left[M^{AB}, M^{CD}\right] = (M^{AB})^{\mu}{}_{\nu}(M^{CD})^{\nu}{}_{\sigma} - (M^{CD})^{\nu}{}_{\sigma}(M^{AB})^{\mu}{}_{\nu} = \\ &= \left(\eta^{A\mu}\delta^{B}{}_{\nu} - \eta^{B\mu}\delta^{A}{}_{\nu}\right)\left(\eta^{C\nu}\delta^{D}{}_{\sigma} - \eta^{D\nu}\delta^{C}{}_{\sigma}\right) - \left(\eta^{C\mu}\delta^{D}{}_{\nu} - \eta^{D\mu}\delta^{C}{}_{\nu}\right)\left(\eta^{A\nu}\delta^{B}{}_{\sigma} - \eta^{B\nu}\delta^{A}{}_{\sigma}\right) = \\ &= \eta^{BC}\left(\eta^{A\mu}\delta^{D}{}_{\sigma} - \eta^{D\mu}\delta^{A}{}_{\sigma}\right) - \eta^{BD}\left(\eta^{A\mu}\delta^{C}{}_{\sigma} - \eta^{C\mu}\delta^{A}{}_{\sigma}\right) - \eta^{AC}\left(\eta^{B\mu}\delta^{D}{}_{\sigma} - \eta^{D\mu}\delta^{B}{}_{\sigma}\right) + \eta^{AD}\left(\eta^{B\mu}\delta^{C}{}_{\sigma} - \eta^{C\mu}\delta^{B}{}_{\sigma}\right) = \\ &= \eta^{BC}(M^{AD})^{\mu}{}_{\sigma} - \eta^{BD}(M^{AC})^{\mu}{}_{\sigma} - \eta^{AC}(M^{BD})^{\mu}{}_{\sigma} + \eta^{AD}(M^{BC})^{\mu}{}_{\sigma} \end{aligned}$$

from which we obtain the defining algebra of $SO(1,3)$:

$$\left[M^{AB}, M^{CD}\right] = \eta^{BC}M^{AD} - \eta^{BD}M^{AC} - \eta^{AC}M^{BD} + \eta^{AD}M^{BC} \tag{1.8}$$

Translations algebra $\mathfrak{R}^4$

A pure translation $T(a)$ in $\mathbb{R}^{1,3}$ is defined as:

$$T(a) \in GL(1,3) : x^{\mu} \Longrightarrow x'^{\mu} = x^{\mu} + a^{\mu} \tag{1.9}$$

and it is naturally connected to the identity, given that, for small a^μ:

$$T(a)\, x^\mu = x^\mu + a^\mu \simeq x^\mu + a^\nu \partial_\nu x^\mu = (\delta^\mu{}_\nu + a^\nu \partial_\nu) x^\mu \simeq e^{a^\nu \partial_\nu} x^\nu \tag{1.10}$$

We have just found the directional derivatives on the basis of the Minkowski space to generate the Lie algebra of $\mathfrak{R}(1,3)$, so, since they form a coordinate basis at any point, they commute:

$$[P_\mu, P_\nu] = 0 \tag{1.11}$$

Poincarè algebra $\mathfrak{iso}(1,3)$

All that remains is calculating the relations between the generators of the Lorentz transformations and the translations.

If we tried to proceed directly we would reach a formal contradiction, lying in the incompatibility of the representations we used in the previous demonstrations: although the structure constants of an algebra are completely independent from its representation, we must be careful to be coherent when connecting the subalgebras.

Indeed, for the Lorentz transformations, we have operated on a tensor field on $\mathbb{R}^{1,3}$, while for the translations we've acted directly on space time, via differential operators, which is perfectly fine, up until we want to merge the approaches.

Of course, we can choose either one of these representations and express all the generators accordingly. For instance, we may decide to adopt the differential one:

$$M^{\mu\nu} = x^\mu \partial^\nu - x^\nu \partial^\mu \tag{1.12}$$

and compute the commutation relations as follows:

$$\begin{aligned}
&[M^{\mu\nu}, P^\rho] = (x^\mu \partial^\nu - x^\nu \partial^\mu)\, \partial^\rho - \partial^\rho\, (x^\mu \partial^\nu - x^\nu \partial^\mu) = \\
&= x^\mu \partial^\nu \partial^\rho - x^\nu \partial^\mu \partial^\rho - \partial^\nu \partial^\rho x^\mu - x^\mu \partial^\rho \partial^\nu + \partial^\mu \partial^\rho x^\nu + x^\nu \partial^\rho \partial^\mu = \\
&= \partial^\mu \partial^\rho x^\nu - \partial^\nu \partial^\rho x^\mu = \partial^\mu \eta^{\rho\nu} - \partial^\nu \eta^{\rho\mu} = P^\mu \eta^{\rho\nu} - P^\nu \eta^{\rho\mu}
\end{aligned} \tag{1.13}$$

We finally collected them all, and in the meantime, we got our hands dirty with some representation theory! For a reason that will be clarified in chapter 2, since a generator is defined up to a multiplicative scalar, we can rewrite them as:

$$\begin{cases} [P_\mu, P_\nu] = 0 \\ [M_{\mu\nu}, P_\rho] = i(\eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu) \\ [M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\sigma} M_{\nu\rho}) \end{cases} \tag{1.14}$$

1.1.2 Irreducible Particle Representation of $ISO(1,3)$

$ISO(1,3)$ is the defining space-time symmetry group of any relativistic *quantum field theory*: therefore, we need to find a suitable representation on particle states.

From the Quantum Mechanics course, we recall the **Wigner Theorem**:

> **Wigner's Theorem:**
> Symmetries in Quantum Mechanics are implemented as (anti)unitary operators on a Hilbert space.

In Groups Theory, a Casimir is defined as an operator commuting with all the generators of the algebra, and thus invariant under the action of the group, and it can be used to label a representation.

For $ISO(1,3)$, there are two of such transformations:

$$\begin{cases} C_1 = P_\mu P^\mu \\ C_2 = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma} \epsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma} \equiv W_\mu W^\mu \end{cases} \tag{1.15}$$

From our backing knowledges of the covariant formalism of Special Relativity, we may be tempted to argue that C_1 and C_2 are simply 4-scalars, and thus necessarily invariant: but this is not obvious at all! In general, an operator being invariant in a certain representation doesn't extend to any other: thus, to make sure that it is actually a Casimir, one must verify its commutativity with the generators.

With a little algebra, this is rapidly achieved:

$$(i) \quad [P^\mu P_\mu, P_\mu] = P^\mu [P_\mu, P_\mu] + [P^\mu, P_\mu] P_\mu = 0$$

$$[M^{\mu\nu}, P^\rho P_\rho] = [M^{\mu\nu}, P^\rho] P_\rho + P_\rho [M^{\mu\nu}, P^\rho] = i(\eta^{\nu\rho} P^\mu - \eta^{\mu\rho} P^\nu) P_\rho + i P_\rho (\eta^{\nu\rho} P^\mu - \eta^{\mu\rho} P^\nu) =$$

$$= i\left((\eta^{\nu\rho} P^\mu - \eta^{\mu\rho} P^\nu) P_\rho + P_\rho (\eta^{\rho\nu} P^\mu - \eta^{\rho\mu} P^\nu)\right) = i\left(P^\nu P^\mu - P^\mu P^\nu + P^\nu P^\mu - P^\mu P^\nu\right) = 0$$

$$(ii) \quad \frac{1}{4}\left[\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma} \epsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma}, P^\alpha\right] \propto \epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma}\left[P_\nu M_{\rho\sigma} P^\nu M^{\rho\sigma}, P^\alpha\right] =$$

$$= \epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma}\left(P^\nu M^{\rho\sigma}\left[P_\nu M_{\rho\sigma}, P^\alpha\right] + \left[P_\nu M_{\rho\sigma}, P^\alpha\right] P^\nu M^{\rho\sigma}\right) =$$

$$= \epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma}\left(P^\nu M^{\rho\sigma}\left(P_\nu\left[M_{\rho\sigma}, P^\alpha\right] + \left[M_{\rho\sigma}, P^\alpha\right] P_\nu\right) + \left(P_\nu\left[M_{\rho\sigma}, P^\alpha\right] + \left[M_{\rho\sigma}, P^\alpha\right] P_\nu\right) P^\nu M^{\rho\sigma}\right) = 0$$

$$(iii) \quad \frac{1}{4}\left[\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma} \epsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma}, M^{\alpha\beta}\right] \propto \epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma}\left[P_\nu M_{\rho\sigma} P^\nu M^{\rho\sigma}, M^{\alpha\beta}\right] =$$

$$= \epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma}\left(P^\nu M^{\rho\sigma}\left[P_\nu M_{\rho\sigma}, M^{\alpha\beta}\right] + \left[P_\nu M_{\rho\sigma}, M^{\alpha\beta}\right] P^\nu M^{\rho\sigma}\right) =$$

$$= \epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma}\left(P^\nu M^{\rho\sigma}\left(P_\nu\left[M_{\rho\sigma}, M^{\alpha\beta}\right] + \left[M_{\rho\sigma}, M^{\alpha\beta}\right] P_\nu\right) + \left(P_\nu\left[M_{\rho\sigma}, M^{\alpha\beta}\right] + \left[M_{\rho\sigma}, M^{\alpha\beta}\right] P_\nu\right) P^\nu M^{\rho\sigma}\right) = 0$$

Now let's try to understand the physical meaning of the Casimirs.

From QM, we know that a translation on a wave function on the coordinate space is implemented by exponentiating the momentum operator $\boldsymbol{p} = -i\hbar\nabla$, and in the previous section we demonstrated that the partial derivatives are the generators of $\mathbb{R}^{1,3}$.

Similiarly, the time translations generator is the Hamiltonian $\boldsymbol{H}$, which, via the correspondence principle, is the observable associated to the energy of a system: therefore, it is logically straightforward to identify P^μ with the 4-momentum of a particle. Now that we know what P^μ is, the physical meaning of W^μ (namely, the *Pauli-Lubanski* vector) will soon be clarified.

In the context of the *Wigner's classification*, we can construct irreducible representations on particle states, viz *multiplets*, labeled by the eigenvalues of the casimirs.

For this purpose, we can choose to a convenient frame, where the Casimirs are easily computed, expressing the Lorentz transformation in the boosts and rotations Lie basis:

$$\begin{cases} J_i = \frac{1}{2}\epsilon_{ijk}M^{jk} \quad j,k = 1,2,3 \\ K_i = M_{0i} \quad i = 0,1,2,3 \end{cases} \tag{1.16}$$

and then discern the analysis in terms of the particle mass.

Massive case

If the mass m of a particle is not null we can always boost to a frame such that:

$$\begin{cases} P^\mu = (m,0,0,0) \\ W^\mu = \left(0, -m\vec{J}\right) \end{cases} \implies \begin{cases} C_1 = m^2 \\ C_2 = -m^2 J^2 \end{cases} \tag{1.17}$$

Since J^i are spatial rotation generators, we can interpret j as the eigenvalue of the module of J^2, i.e. the spin of the particle, and j_3 as its projection on the z axis.

From the angular momenta algebra, we deduce that a massive multiplet of the Poincaré group is composed by $(2j+1)$ massive particle states, labeled as:

$$\{\, |\, m, j, j_3 \,\rangle \,\} \quad , \quad j_3 \leq |j| \tag{1.18}$$

Maseless case

If $m = 0$, we can opt for a frame such that:

$$\begin{cases} P^\mu = (E,0,0,E) \\ W^\mu = M_{12}P^\mu \end{cases} \implies \begin{cases} C_1 = 0 \\ C_2 = 0 \end{cases} \tag{1.19}$$

Since both Casimirs are null for any maseless particle, how can we distinguish them?

Well, if we notice that in this coordinates the Pauli-Lubanski vector is just the phase shifted 4-momentum, we can replace the second casimir with the parameter of such $U(1)$ transformation, namely the *helicity*.

Before jumping to hasty conclusions, we have to remember that a maseless particle always carries two opposite helicity states, and this aspect cannot be ignored, since it has fundamental implications on the predictions of the Standard Model. Hence, the multiplet will be composed of:

$$\{\, |\, E, h \,\rangle \,, |\, E, -h \,\rangle \,\} \tag{1.20}$$

In this section we discussed the importance of the spin in the representation of a particle.

The next question we will attempt to answer is: what are the geometrical involvements telling whole from halved spin particles? We will demonstrate that fermions naturally arise in a specific representation of the Lorentz group.

1.2 $Spin(1,3)$ group

Noticeably, the Lorentz group has spinorial representations. What does this mean?

First, we note that there exists a bijection between a generic $x^\mu \in \mathbb{R}^{1,3}$ and a 2×2 hermitian matrix $X \in GL(2,\mathcal{C})$:

$$\left\{x_\mu \longrightarrow X = x_\mu \sigma^\mu = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \quad , \quad \sigma^\mu = \left(\mathbf{1}, \sigma^i\right)\right\} \Longrightarrow x^\mu x_\mu = det\,[X] \tag{1.21}$$

where σ^i are the Pauli matrices. Then we consider the action of the group of unitary matrices $SL\,(2,\mathcal{C})$ on X:

$$X \xrightarrow{SL(2,\mathcal{C})} X' = SXS^\dagger \Longrightarrow det\left[SXS^\dagger\right] = det^2\,[S]\,det\,[X] = \det\,[X] \tag{1.22}$$

meaning S is an isomorphism on $GL(2,\mathbb{C})$, and thus a Lorentz transformation on $\mathbb{R}^{1,3}$ (indeed, it has 6 degrees of freedom): it directly follows that a representation of $SL(2,\mathbb{C})$ is also a valid for $SO(1,3)$ up to global properties.

To be precise, defining $Spin(1,3)$ such that:

$$SO(1,3) \simeq Spin(1,3)/\mathbb{Z}_2 \tag{1.23}$$

the spinorial representation of the Lorentz groups are those that are double valued representations, though single valued as reps of $SL(2,\mathbb{C})$, meaning that the actual isomorphism is:

$$SL(2,\mathcal{C}) \simeq Spin(1,3) \tag{1.24}$$

In terms of representation, the lowest dimensional non trivial vector space which $SL(2,\mathbb{C})$ acts upon, is certainly $\mathbb{C}^2$.

A *left hand Weyl spinor* is a two component complex vector transforming in an irreducible representation of $Spin(1,3)$:

$$\psi_\alpha \in \left(\frac{1}{2}, 0\right) : \psi'_\alpha = S_\alpha^{\ \beta}\psi_\beta \quad , \quad S \in SL\,(2,\mathbb{C}) \tag{1.25}$$

From the group theoretical representation theory, we can borrow the following result:

Given a representation of a group, its complex conjugate, transforming by S^*, is another representation, independent from the other one if:

$$\nexists\, C \ : \ S^* = CSC^\dagger \tag{1.26}$$

In our case, this condition is satisfied, and we may hence construct another independent irreducible representation of $Spin(1,3)$, as its action on a *right handed Weyl spinor*:

$$(\psi_\alpha)^\dagger \equiv \bar{\psi}_{\dot{\alpha}} \in \left(0, \frac{1}{2}\right) : \bar{\psi}'_{\dot{\alpha}} = (S^*)_{\dot{\alpha}}^{\ \dot{\beta}}\,\bar{\psi}_{\dot{\beta}} \tag{1.27}$$

1.2.1 Algebraic structures from spinors

Before moving on, it is suitable to acquaint ourselves with some spinorial algebra, and build systematic methods to construct tensors from them.

Scalars

First, we define the following invariant tensors under $SL(2,\mathbb{C})$:

$$\begin{cases} \epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{cases} \quad : \quad \begin{cases} \epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \mathbf{1}^{\alpha}{}_{\gamma} \\ \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon_{\dot{\beta}\dot{\gamma}} = \mathbf{1}^{\dot{\alpha}}{}_{\dot{\gamma}} \end{cases} \tag{1.28}$$

A scalar, belonging to the $(0,0)$ representation of the $Spin(1,3)$ group, can be written as a product of left *or* right spinors:

$$\begin{cases} \psi, \chi \in \left(\frac{1}{2}, 0\right) \Longrightarrow \psi\chi = \epsilon^{\alpha\beta}\psi_\alpha\chi_\beta = \psi_\beta \epsilon^{T\,\beta\alpha}\chi_\alpha = \chi_1\psi_2 - \chi_2\psi_1 \\ \bar{\psi}, \bar{\chi} \in \left(0, \frac{1}{2}\right) \Longrightarrow \bar{\psi}\bar{\chi} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\alpha}}\bar{\chi}_{\dot{\beta}} = \bar{\chi}_2\bar{\psi}_1 - \bar{\chi}_1\bar{\psi}_2 \end{cases} \Longleftrightarrow (\psi\chi)^\dagger = \bar{\psi}\bar{\chi}$$

which is indeed a scalar under the action of $SO(2,\mathbb{C})$, since:

$$\psi\chi \longrightarrow S_\alpha{}^\gamma S_\beta{}^\delta \epsilon^{\alpha\beta}\psi_\delta\chi_\gamma = S_\alpha{}^\gamma S^{T\,\delta}{}_\beta\, \epsilon^{T\,\beta\alpha}\psi_\delta\chi_\gamma = S_\alpha{}^\gamma S^{T\,\delta\alpha}\psi_\delta\chi_\gamma = *$$

$$S^{T\,\delta}{}_\beta\,\epsilon^{T\,\beta\alpha} = \begin{pmatrix} S_{11} & S_{21} \\ S_{12} & S_{22} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} S_{21} & -S_{11} \\ S_{22} & -S_{12} \end{pmatrix} = S^{T\,\delta\alpha}$$

$$* = S^{T\gamma}{}_\alpha S^{\alpha\delta}\psi_\delta\chi_\gamma = \begin{pmatrix} S_{11} & S_{21} \\ S_{12} & S_{22} \end{pmatrix}\begin{pmatrix} S_{21} & S_{22} \\ -S_{11} & -S_{12} \end{pmatrix}\psi\chi = \begin{pmatrix} 0 & det\,[S] \\ det\,[S] & 0 \end{pmatrix}\psi\chi =$$

$$= det\,[S]\,\epsilon^{\gamma\delta}\psi_\delta\chi_\gamma = \psi\chi$$

So, in analogy with the metric tensor, one can use the ϵ matrices to rise and lower the indices of the spinor, coherently with their anticommutation properties:

$$\begin{cases} \psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta \\ \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta \end{cases} \qquad \begin{cases} \psi^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\psi_{\dot{\beta}} \\ \bar{\psi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}} \end{cases} \tag{1.29}$$

Vectors

Vectors belong to the $\left(\frac{1}{2}, \frac{1}{2}\right)$ of $Spin(1,3)$ and can be explicitly built from left *and* right Weyl spinors:

$$\begin{cases} \psi\sigma^\mu\bar{\chi} = \psi^\alpha \left(\sigma^\mu\right)_{\alpha\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} \\ (\sigma^\mu)_{\alpha\dot{\alpha}} = \left(\mathbf{1}, \sigma^i\right)_{\alpha\dot{\alpha}} \end{cases} \tag{1.30}$$

which transforms as a vector under a Lorentz transformation, which can be verified by contracting it with the transformed of a 4-vector x^μ under the isomorphism 1.21:

$$\psi X\bar{\chi} = \psi^\alpha X_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = \epsilon^{\alpha\beta}\psi_\beta X_{\alpha\dot{\alpha}}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\chi}_{\dot{\beta}} \longrightarrow \epsilon^{\alpha\beta}\left[S_\beta{}^\gamma\psi_\gamma\right]\left[S_\alpha{}^\delta X_{\delta\dot{\delta}}\left(S^*\right)_{\dot{\alpha}}^{\dot{\delta}}\right]\epsilon^{\dot{\alpha}\dot{\beta}}\left[\left(S^*\right)_{\dot{\beta}}^{\dot{\gamma}}\bar{\chi}_{\dot{\gamma}}\right] =$$

$$= \left[\psi^\beta\left(S^{-1}\right)_\beta^{\ \alpha}\right]\left[S_\alpha{}^\delta X_{\delta\dot{\delta}}\left(S^*\right)_{\dot{\alpha}}^{\dot{\delta}}\right]\left[\bar{\chi}^{\dot{\beta}}\left(S^{*\,-1}\right)_{\dot{\beta}}^{\ \dot{\alpha}}\right] = \psi X\bar{\chi}$$

One may symmetrically proceed in the same way to construct a vector from another set of matrices:

$$\begin{cases} \left(\bar{\sigma}^\mu\right)^{\dot{\alpha}\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\left(\sigma^\mu\right)_{\beta\dot{\beta}} = \left(\mathbf{1}, -\sigma^i\right)^{\dot{\alpha}\alpha} \\ \bar{\chi}_{\dot{\alpha}}\left(\bar{\sigma}^\mu\right)^{\dot{\alpha}\alpha}\psi_\alpha = -\psi^\alpha\left(\sigma^\mu\right)_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} \end{cases} \tag{1.31}$$

which does not form an independent representation with respect to the first one, since in can be written as a linear combination, as shown, and thus the condition in Equation 1.26 does not hold.

1.2.2 Generators of $SO(1,3)$

We now need to clarify the connection between the same Lorentz transformation Λ in the vectorial and in the spinorial representations of $SO(1,3)$. To do this, we may define the antisymmetric product of sigma matrices as:

$$\left(\sigma^{\mu\nu}\right)_\alpha^{\ \beta} = \frac{i}{4}\left(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu\right)_\alpha^{\ \beta} \tag{1.32}$$

One can indeed demonstrate them to obey the Lorentz algebra, by computing their commutors:

$$[\sigma^{\mu\nu}, \sigma^{\rho\tau}] = i(\eta^{\mu\rho}\sigma^{\nu\tau} - \eta^{\nu\rho}\sigma^{\mu\tau} + \eta^{\nu\tau}\sigma^{\mu\rho} - \eta^{\mu\tau}\sigma^{\nu\rho}) \tag{1.33}$$

The conjugate generators are given by:

$$\left(\bar{\sigma}^{\mu\nu}\right)^{\dot{\alpha}}_{\ \dot{\beta}} = \frac{i}{4}\left(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu\right)^{\dot{\alpha}}_{\ \dot{\beta}} \tag{1.34}$$

Therefore, the Weyl spinors transform accordingly as:

$$\begin{cases} \psi_\alpha \xrightarrow{Spin(1,3)} exp\left\{-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right\}_\alpha^{\ \beta}\psi_\beta \\ \bar{\psi}_{\dot{\alpha}} \xrightarrow{Spin(1,3)} exp\left\{-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}\right\}^{\dot{\alpha}}_{\ \dot{\beta}}\bar{\psi}^{\dot{\beta}} \end{cases} \tag{1.35}$$

1.3 Dirac Spinors

A fundamental geometrical structure to describe massive fermions is the *Dirac spinor* Ψ, which is a four component complex vector transforming in a **reducible** representation of the $Spin(1,3)$ group.

Before deriving it, it's essential to introduce the *Clifford Algebra* $Cl(1,3)$ in four dimensions:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1} \tag{1.36}$$

where γ^μ are 4×4 matrices and $\eta^{\mu\nu}$ is the Minkowski metric.

In a general representation, we can construct a new set of matrices $\{S^{\mu\nu}\}$:

$$S^{\mu\nu} = \frac{1}{4}\left[\gamma^\mu, \gamma^\nu\right] = \begin{cases} 0 & \mu = \nu \\ \frac{1}{2}\gamma^\mu\gamma^\nu & \mu \neq \nu \end{cases} \tag{1.37}$$

and verify that they form a representation of the Lorentz algebra, by evaluating their commutation relations:

$$[S^{\mu\nu}, S^{\rho\sigma}] = \frac{1}{2}\left[S^{\mu\nu}, \gamma^\rho\gamma^\sigma\right] = \frac{1}{2}\left\{\gamma^\rho\left[S^{\mu\nu}, \gamma^\sigma\right] + \left[S^{\mu\nu}, \gamma^\rho\right]\gamma^\sigma\right\} = *$$

$$\left[S^{ij}, \gamma^k\right] = \frac{1}{2}\left(\gamma^i\gamma^j\gamma^k + \gamma^i\gamma^k\gamma^j - \gamma^i\gamma^k\gamma^j - \gamma^k\gamma^i\gamma^j - \gamma^i\gamma^k\gamma^j + \gamma^i\gamma^k\gamma^j\right) =$$

$$= \frac{1}{2}\left(\gamma^i\left\{\gamma^j, \gamma^k\right\} - \left\{\gamma^k, \gamma^i\right\}\gamma^j\right) = \frac{1}{2}\left(\gamma^i(2\eta^j k) - (2\eta^{ki})\gamma^j\right) = \gamma^i\eta^{jk} - \eta^{ki}\gamma^j$$

$$* = \frac{1}{2}\left\{\gamma^\rho\left(\gamma^\mu\eta^{\nu\sigma} - \eta^{\sigma\mu}\gamma^\nu\right) + \left(\gamma^\mu\eta^{\nu\rho} - \eta^{\rho\mu}\gamma^\nu\right)\gamma^\sigma\right\} =$$

$$= \frac{1}{2}\gamma^\rho\gamma^\mu\eta^{\nu\sigma} - \frac{1}{2}\gamma^\rho\gamma^\nu\eta^{\sigma\mu} + \frac{1}{2}\gamma^\mu\gamma^\sigma\eta^{\nu\rho} - \frac{1}{2}\gamma^\nu\gamma^\sigma\eta^{\rho\mu} =$$

$$= \frac{1}{2}\left(2S^{\rho\mu} + \eta^{\rho\mu}\right)\eta^{\nu\sigma} - \frac{1}{2}\left(2S^{\rho\nu} + \eta^{\rho\nu}\right)\eta^{\sigma\mu} + \frac{1}{2}\left(2S^{\mu\sigma} + \eta^{\mu\sigma}\right)\eta^{\nu\rho} - \frac{1}{2}\left(2S^{\nu\sigma} + \eta^{\nu\sigma}\right)\eta^{\rho\mu} =$$

$$= S^{\rho\mu}\eta^{\nu\sigma} - S^{\rho\nu}\eta^{\sigma\mu} + S^{\mu\sigma}\eta^{\nu\rho} - S^{\nu\sigma}\eta^{\rho\mu}$$

The Dirac spinor transforms in the spinorial representation of $SO(1,3)$ as:

$$\begin{cases} \Psi^\alpha(x) \longrightarrow S\left[\Lambda\right]^\alpha_{\ \beta}\Psi^\beta(\Lambda^{-1}x) \\ \Lambda = \exp\left\{\frac{1}{2}\Omega_{\mu\nu}M^{\mu\nu}\right\} \\ S\left[\Lambda\right] = \exp\left\{\frac{1}{2}\Omega_{\mu\nu}S^{\mu\nu}\right\} \end{cases} \tag{1.38}$$

where $M^{\mu\nu}$ are the generators of the Lorentz group acting on the coordinate space, while $S^{\mu\nu}$ are those acting on the spinor space. It's important to notice that, albeit the basis of $SO(1,3)$ is different, the same choice of representation coordinates $\Omega_{\mu\nu}$ assures to make the same Lorentz transformation in both spaces.

Now one may ask how are Weyl spinors related to Dirac spinors, and the answer lies in the question, since there exists a particular basis of $Cl(1,3)$ where the link becomes explicit: the *chiral* (or Weyl) representation:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad , \quad \sigma^\mu = \left(\mathbf{1}, \sigma^i\right) \quad , \quad \bar{\sigma}^\mu = \left(\mathbf{1}, -\sigma^i\right) \tag{1.39}$$

It is straightforward to prove them to verify the defining commutation relation of the algebra:

$$\gamma^\mu\gamma^\nu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{pmatrix} = \begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu \end{pmatrix} \Longrightarrow \{\gamma^\mu,\gamma^\nu\} = \begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu \end{pmatrix} \Longrightarrow$$

$$\Longrightarrow \begin{cases} \{\gamma^0,\gamma^0\} = \begin{pmatrix} 2\mathbf{1} & 0 \\ 0 & 2\mathbf{1} \end{pmatrix} = 2\mathbf{1} \\ \{\gamma^0,\gamma^i\} = \begin{pmatrix} -\mathbf{1}\sigma^i + \sigma^i\mathbf{1} & 0 \\ 0 & \mathbf{1}\sigma^i + \bar{\sigma}^i\mathbf{1} \end{pmatrix} = 0 \\ \{\gamma^i,\gamma^j\} = -\begin{pmatrix} \{\sigma^i,\sigma^j\} & 0 \\ 0 & \{\sigma^i,\sigma^j\} \end{pmatrix} = -2\delta^{ij}\mathbf{1} \end{cases} \Longrightarrow \{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}$$

In this representation, the $S[\Lambda]$ matrices are block diagonal:

$$[\gamma^\mu,\gamma^\nu] = \begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu \end{pmatrix} \Longrightarrow$$

$$S^{\mu\nu} = \frac{1}{4}[\gamma^\mu,\gamma^\nu] = \begin{cases} \frac{1}{4}[\gamma^0,\gamma^i] = \begin{pmatrix} -2\sigma^i\mathbf{1} & 0 \\ 0 & 2\sigma^i\mathbf{1} \end{pmatrix} & \text{boosts} \\ S^{ij} = -\frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} & \text{rotations} \end{cases}$$

and thus also finite Lorentz transformations are block diagonal:

$$S[\Lambda] = \begin{cases} \begin{pmatrix} e^{-\frac{i}{2}\omega_{ij}\sigma^k} & 0 \\ 0 & e^{-\frac{i}{2}\omega_{ij}\sigma^k} \end{pmatrix} & \text{rotations} \\ \begin{pmatrix} e^{-\frac{i}{2}\omega_{0i}\sigma^i} & 0 \\ 0 & e^{\frac{i}{2}\omega_{0i}\sigma^i} \end{pmatrix} & \text{boosts} \end{cases} = \begin{pmatrix} e^{-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}} & 0 \\ 0 & e^{-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}} \end{pmatrix} \tag{1.40}$$

The diagonal terms are exactly the Lorentz transformations of the left and right Weyl spinors, proving that the Dirac representation is indeed reducible, and decomposes into the chiral ones:

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} \Longrightarrow \begin{cases} \psi_\alpha \longrightarrow \exp\left\{-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right\}_\alpha^{\ \beta}\psi_\beta \\ \bar{\psi}_{\dot{\alpha}} \longrightarrow \exp\left\{-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}\right\}^{\dot{\alpha}}_{\ \dot{\beta}}\bar{\psi}^{\dot{\beta}} \end{cases} \Longleftrightarrow \Psi \in \left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right) \tag{1.41}$$

Before moving on to the next chapter, it's appropriate to mention the *Majorana spinor*, as it will be adopted when deriving SUGRA Lagrangian in chapter 6.

A Majorana spinor ψ is a real 4-component spinor, satisfying the following reality condition:

$$\psi^C \equiv C\bar{\psi}^T = iC\gamma^{0T}\psi^* = \psi \quad , \quad \begin{cases} C^T = -C \\ \gamma_\mu^T = -C\gamma_\mu C^{-1} \end{cases} \tag{1.42}$$

We shall not dive any further in it, for this is more than enough for our purposes.

Now, backed by the acquired knowledge, it's time to move on and explore the foundations of Quantum Field Theory.

Chapter 2

Essential Quantum Field Theory

Quantum Field Theory a modern theoretical framework describing the dynamics of relativistic particles, and an inevitable step to gather the foundational formalism of the Standard Model.

There are two main formulations of QFT: the *path integral* and the *canonical quantization* one. While the first one has the advantage of being manifestly Lorentz covariant, it requires additional mathematical tools to be properly defined, and which would exceed the purposes of this dissertation.

Hence, we'll opt for the canonical quantization, although this will require some cautions along the way: the issue with such an Hamiltonian-based formalism is that it requires to compute a conjugate momentum, by separating the previously on equal footing spacetime coordinates.

From our backing knowledges of Quantum Mechanics, we know that the canonical quantization is the standard method to jump from the classical to a quantum theory, consisting in promoting dynamical variables to operators, and imposing quantum conditions on their commutators.

Likewise, in QFT one defines a *quantum field* ϕ to be an **operator-valued function** of spacetime coordinates, and the *momentum field* π its conjugate dynamical variable.

To consistently include QM, for the scalar field analogous commutation relations must hold:

$$\begin{cases} [\phi(\mathbf{x}), \phi(\mathbf{y})] = 0 \\ [\pi(\mathbf{x}), \pi(\mathbf{y})] = 0 \\ [\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \end{cases} \tag{2.1}$$

The goal of this chapter is to understand how Lorentz invariant Lagrangian densities for the fields can be constructed: not only this will allow us to derive the equations of motion, but, crucially, their Hamiltonian, which will exhibit how particles actually emerge from a quantized field.

Especially, we will describe how to expand free fields in terms of creation and annihilation operators, and highlighting the differences between bosonic and fermionic quantizations, to build particle states consistently with the *Spin-Statistics* theorem.

2.1 Scalar Field

Before getting to spinor fields, it's a good idea to start from the basics and have a glimpse to the scalar one for, although its algebraic properties might look kind of trivial, from a purely physical perspective its description leads to decisive involvements.

2.1.1 Plausibility proof of the Klein Gordon Lagrangian

The dynamics of a scalar field is described by the Klein-Gordon Lagrangian:

$$\mathcal{L}_{KG} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\pi^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 \tag{2.2}$$

We now want to justify its form.

In the first chapter, we began fiddling with some symmetries, by introducing the *Wigner's Theorem*, which essentially states that such a transformation must be implemented as a unitary, or antiunitary, operator on an Hilbert space. Maybe it's me, but this doesn't sound like a first principle: first, it specifically refers to the ket representation, and second, the Analitical Mechanics course oriented me to think to a symmetry as a pure geometrical feature of a manifold.

Quantum Field Theory is built upon a Lagrangian formalism, which defines a symmetry of a field to be a transformation leaving the action S invariant, allowing to invoke the renowned *Noether's Theorem*, stating that any infinitesimal continuous symmetry of a field gives rise to a conserved current:

$$\begin{cases}\delta\phi^i(x) = \epsilon^A\Delta_A\ \phi^i(x)\\ \epsilon^A \ll 1\end{cases} \quad : \quad S(\phi) = S(\phi') \Longrightarrow \exists\ J^\mu_A = -\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^i)}\Delta_A\phi^i + K^\mu_A \quad : \quad \partial_\mu J^\mu_A = 0 \tag{2.3}$$

where Δ_A denote the generators of the symmetry group, and K^μ_A is a total derivative.

Energy-momentum tensor

The conserved current associated to space-time translations is the energy-momentum tensor $T^{\mu\nu}$

$$\mathcal{L}(\phi^i + \epsilon^\nu\partial_\nu\phi^i) = \mathcal{L}(\phi^i) + \epsilon^\nu\partial_\nu\mathcal{L} \Longrightarrow (j^\mu)^\nu \equiv T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)}\partial^\nu\phi^i - \eta^{\mu\nu}\mathcal{L} \tag{2.4}$$

What's peculiar of this tensor is that its null 4-divergence manifests the energy-momentum conservation laws, providing a rather simple plausibility argument of the Klein Gordon Lagrangian. Hence, let's exhibit the $\mathcal{S}_{KG}$ invariance under the action of $\mathbb{R}^4$:

$$\mathcal{L}_{KG}(\phi + \epsilon^\nu\partial_\nu\phi) = \frac{1}{2}\partial^\mu(\phi + \epsilon^\nu\partial_\nu\phi)\partial_\mu(\phi + \epsilon^\nu\partial_\nu\phi) - \frac{1}{2}m^2(\phi + \epsilon^\nu\partial_\nu\phi)^2 =$$

$$= \frac{1}{2}\left(\partial^\mu\phi + \epsilon^\nu\partial_\nu\partial^\mu\phi\right)\left(\partial_\mu\phi + \epsilon^\nu\partial_\nu\partial_\mu\phi\right) - \frac{1}{2}m^2\left(\psi^2 + (\epsilon^\nu\partial_\nu\phi)^2 + 2\epsilon^\nu(\partial_\nu\phi)\phi\right) =$$

$$= \frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi + \epsilon^\nu\partial_\nu\partial^\mu\phi\partial_\mu\phi + +\epsilon^\nu\partial_\nu\partial^\mu\phi\partial_\mu\phi + \epsilon^\nu\partial_\nu\partial^\mu\phi\epsilon^\nu\partial_\nu\partial_\mu\phi\right) - \frac{1}{2}m^2\left(\psi^2 + (\epsilon^\nu\partial_\nu\phi)^2 + 2\epsilon^\nu(\partial_\nu\phi)\phi\right) \overset{(1)}{\simeq}$$

$$\overset{(1)}{\simeq} \frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi + \epsilon^\nu\partial_\nu\partial^\mu\phi\partial_\mu\phi + \epsilon^\nu\partial_\nu\partial^\mu\phi\partial_\mu\phi\right) - \frac{1}{2}m^2\left(\phi^2 + 2\epsilon^\nu\partial_\nu\phi\phi\right) =$$

$$= \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 + \epsilon^\nu\left(\partial_\nu\partial^\mu\phi\partial_\mu\phi - \partial_\nu m^2\phi^2\right) = \mathcal{L}_{KG} + \epsilon^\nu\partial_\nu\left(\partial^\mu\phi\partial_\mu\phi - m^2\phi^2\right)$$

Since the Lagrangian varies by a total derivative, the action is invariant under translations, which implies that we can derive the energy-momentum tensor from Equation 2.4.

$$T^{\mu\nu}_{KG} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}_{KG} \tag{2.5}$$

and exhibit the resulting conservation laws, by separating the components of $\partial_\mu T^{\mu\nu} = 0$.

- $\nu = 0$ (**Energy** conservation):

  To prove the consistency of the energy conservation law, we first write the Hamiltonian density as the Legendre transform of the Lagrangian one:

$$\mathcal{H} = \dot{\phi}^2 - \partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \tag{2.6}$$

  to then focus on the first equation provided by the null divergence:

$$\partial_\mu T^{\mu 0} = \partial_0 T^{00} + \partial_i T^{i0} = *$$

$$T^{00} = \partial^0\phi\partial^0\phi - \frac{1}{2}\partial_i\phi\partial^i\phi - \frac{1}{2}m^2\phi^2 = \partial^0\phi\partial^0\phi - \frac{1}{2}\partial_i\phi\partial^i\phi - \frac{1}{2}m^2\phi^2 =$$

$$= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2$$

$$T^{0i} = \partial^0\partial^i\phi - \eta^{0i}\mathcal{L} = \dot{\phi}\vec{\nabla}\phi$$

$$* = \partial_0\left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2\right] - \vec{\nabla}\cdot\left[\dot{\phi}\vec{\nabla}\phi\right] = \frac{\partial}{\partial t}\mathcal{H} - \vec{\nabla}\cdot\vec{S} = 0$$

- $\nu = i$ (**Momentum** conservation):

$$\partial_\mu T^{\mu i} = \partial_0 T^{0i} + \partial_j T^{ji} = *$$

$$T^{0i} = \partial^0\phi\partial^i\phi - \eta^{0i}\mathcal{L} = \dot{\phi}\cdot\vec{\nabla}\phi \Longrightarrow \partial_0 T^{0i} = \partial_0\left[\dot{\phi}\vec{\nabla}\phi\right] = \ddot{\phi}\partial^i\phi + \dot{\phi}\partial^i\dot{\phi}$$

$$T^{ij} = \partial^i\phi\partial^j\phi - \eta^{ij}\left[\frac{1}{2}\partial_k\phi\partial^k\phi - \frac{1}{2}m^2\phi^2\right]$$

$$\partial_j(\partial^j\phi\partial^i\phi) = (\partial_j\partial^j\phi)\partial^i\phi + \partial^j\phi\partial_j\partial^i\phi$$

$$\partial_j\eta^{ij}\mathcal{L} = -\eta^{ij}\partial_j\mathcal{L} = -\partial^i\mathcal{L}$$

$$* = \partial_0 T^{0i} + \partial_j T^{ji} = \ddot{\phi}\vec{\nabla}\phi + \dot{\phi}\vec{\nabla}\dot{\phi} + (\Box\phi)\vec{\nabla}\phi + \vec{\nabla}\phi\partial_j\partial^i\phi - \partial^i\mathcal{L} =$$

$$= \frac{\partial}{\partial t}\vec{p} + \vec{\nabla}\cdot\pi = 0$$

The Lagrangian is indeed physically consistent, so we shall derive the dynamics of the field from the Euler-Lagrange equations:

$$\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\right) - \frac{\delta\mathcal{L}}{\delta\phi} = \partial_\mu\partial_\mu\phi + m^2\phi \equiv \Box\phi + m^2\phi = 0 \tag{2.7}$$

which is the renowned Klein-Gordon equation.

2.1.2 Scalar field quantization

To evaluate canonical quantization, we can show that Equation 2.7 could've been obtained by quantizing the relativistic energy-momentum relation on an Hilbert space diagonalised with respect to the coordinate operator:

$$\begin{cases} E \longrightarrow \mathbf{E} = -i\partial_t \\ p \longrightarrow \mathbf{p} = -i\vec{\nabla} \end{cases} \Longrightarrow -\partial_0^2\phi + \vec{\nabla}^2\phi + m^2\phi = \Box\phi + m^2\phi = 0 \tag{2.8}$$

Creation and annihilation operators

Directly inserting the Fourier transform of the scalar field in the Klein-Gordon equation, we find:

$$\phi(\vec{x},t) = \int \frac{dx^3}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}}\phi(\vec{p},t) \Longrightarrow \left[\partial_0^2 + (p^2+m^2)\right]\phi(\vec{p},t) = 0 \tag{2.9}$$

whose general solution to be is given by an infinite linear superposition of simple harmonic oscillators of frequency $\omega = \sqrt{p^2+m^2}$.

From the backing knowledges of non-relativistic QM, we know how to algebraically solve its associated eigenvalue problem: in QFT this can be analogously achieved by introducing the *creation and annihilation* (or *ladder*) operators of the fields:

$$\begin{cases} a \equiv \sqrt{\frac{\omega}{2}}\phi + \frac{i}{2\omega}\pi \\ a^\dagger \equiv \sqrt{\frac{\omega}{2}}\phi - \frac{i}{2\omega}\pi \end{cases} \Longleftrightarrow \begin{cases} \phi = \sqrt{\frac{1}{2\omega}}(a+a^\dagger) \\ \pi = -i\sqrt{\frac{\omega}{2}}(a-a^\dagger) \end{cases} \tag{2.10}$$

allowing us to rewrite the Fourier expansion 2.9 as:

$$\phi(\vec{x},t) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega}}\left(a_{\vec{p}}\,e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger\,e^{-i\vec{p}\cdot\vec{x}}\right) \tag{2.11}$$

uniquely defining its dynamical conjugate:

$$\pi(\vec{x},t) = \int \frac{d^3\vec{p}}{(2\pi)^3}\sqrt{\frac{\omega}{2}}\left(-i\omega a_{\vec{p}}\,e^{i\vec{p}\cdot\vec{x}} + i\omega a_{\vec{p}}^\dagger\,e^{-i\vec{p}\cdot\vec{x}}\right) \tag{2.12}$$

From commutation relations between the fields in Equation 2.1, we compute the creation and annihilation operators' ones:

$$(i) \qquad [\phi(\vec{x}),\phi(\vec{y})] = 0 \Longleftrightarrow e^{i(\vec{p}\cdot x+\vec{q}\cdot\vec{y})}\left[a_{\vec{p}},a_{\vec{q}}\right] + e^{i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})}\left[a_{\vec{p}},a_{\vec{q}}^\dagger\right] + \text{ c.c.} = 0$$

$$(ii) \qquad [\pi(\vec{x}),\pi(\vec{y})] = 0 \Longleftrightarrow e^{i(\vec{p}\cdot x+\vec{q}\cdot\vec{y})}\left[a_{\vec{p}},a_{\vec{q}}\right] - e^{i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})}\left[a_{\vec{p}},a_{\vec{q}}^\dagger\right] + \text{ c.c.} = 0$$

$$(iii) \qquad [\phi(\vec{x}),\pi(\vec{y})] = -i\omega\int\frac{d^3p\ d^3q}{(2\pi)^6}\frac{1}{\sqrt{2\omega}}\sqrt{\frac{\omega}{2}}\left[\left(a_{\vec{p}}\,e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}\,e^{-i\vec{p}\cdot\vec{x}}\right),\left(a_{\vec{q}}\,e^{\vec{q}\cdot\vec{y}} + a_{\vec{q}}^\dagger\,e^{-i\vec{q}\cdot\vec{y}}\right)\right] =$$

$$= \frac{i}{2}\int\frac{d^3p\,d^3q}{(2\pi)^6}\left(e^{i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})}\left[a_{\vec{p}},a_{\vec{q}}^\dagger\right] + e^{-i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})}\left[a_{\vec{p}}^\dagger,a_{\vec{q}}\right]\right) \overset{2.1}{=} i\delta(\vec{x}-\vec{y}) = i\int\frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \Longleftrightarrow$$

$$\Longleftrightarrow \int\frac{d^3q}{(2\pi)^3}\left[a_{\vec{p}},a_{\vec{q}}^\dagger\right]e^{i\vec{p}\cdot(\vec{x}-\vec{y})} = e^{i\vec{p}\cdot\vec{x}} \Longleftrightarrow \int\frac{d^3q}{(2\pi)^3}\left[a_{\vec{p}},a_{\vec{q}}^\dagger\right] = 1 = \int d^3q\ \delta^{(3)}(\vec{p}-\vec{q})$$

From the sum of (i) and (ii), along with the summability condition, and from the last one, we obtain:

$$\begin{cases} [a_{\vec{p}}, a(\vec{q})] = \left[a^\dagger_{\vec{p}}, a^\dagger(\vec{q})\right] = 0 \\ \left[a_{\vec{p}}, a^\dagger(\vec{q})\right] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}) \end{cases} \tag{2.13}$$

Hamiltonian operator

The last step is to rewrite the Hamiltonian in terms of the ladder operators, by integrating its density $\mathcal{H}$ over the spacial domain [1]

$$H = \frac{1}{2}\int d^3x \left(\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right) = \frac{1}{2}\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6}\left[-\frac{\sqrt{\omega_{\vec{p}}\,\omega_{\vec{q}}}}{2}\left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}}\right)\left(a_{\vec{q}}\, e^{i\vec{q}\cdot\vec{x}} - a^\dagger_{\vec{q}}\, e^{-i\vec{q}\cdot\vec{x}}\right) - \right.$$

$$\left.\frac{1}{2\sqrt{\omega_{\vec{p}}\,\omega_{\vec{q}}}}\left(i\vec{p}\, a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}}\right)\left(i\vec{q}\, a_{\vec{q}}\, e^{i\vec{q}\cdot\vec{x}} - i\vec{q}\, a^\dagger_{\vec{q}}\, e^{-i\vec{q}\cdot\vec{x}}\right) + \frac{m^2}{2\sqrt{\omega_{\vec{p}}\,\omega_{\vec{q}}}}\left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}}\right)\left(a_{\vec{q}}\, e^{i\vec{q}\cdot\vec{x}} + a^\dagger_{\vec{q}}\, e^{-i\vec{q}\cdot\vec{x}}\right)\right] =$$

$$= \frac{1}{2}\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6}\left[-\frac{\sqrt{\omega_{\vec{p}}\,\omega_{\vec{q}}}}{2}\left(a_{\vec{p}}\, a_{\vec{q}}\, e^{i\vec{x}\cdot(\vec{p}+\vec{q})} - a_{\vec{p}}\, a^\dagger_{\vec{q}}\, e^{i\vec{x}\cdot(\vec{p}-\vec{q})} - a^\dagger_{\vec{p}}\, a_{\vec{q}}\, e^{-i\vec{x}\cdot(\vec{p}-\vec{q})} + a^\dagger_{\vec{p}}\, a^\dagger_{\vec{q}}\, e^{-i\vec{x}\cdot(\vec{p}+\vec{q})}\right) - \right.$$

$$-\frac{i}{2\sqrt{\omega_{\vec{p}}\,\omega_{\vec{q}}}}\left(\vec{p}\cdot\vec{q}\, a_{\vec{p}}\, a_{\vec{q}}\, e^{i\vec{x}\cdot(\vec{p}+\vec{q})} - \vec{p}\cdot\vec{q}\, a_{\vec{p}}\, a^\dagger_{\vec{q}} e^{i\vec{x}\cdot(\vec{p}-\vec{q})} + \vec{p}\cdot\vec{q}\, a^\dagger_{\vec{p}}\, a_{\vec{q}}\, e^{-i\vec{x}\cdot(\vec{p}-\vec{q})} - \vec{p}\cdot\vec{q}\, a^\dagger_{\vec{p}}\, a^\dagger_{\vec{q}}\, e^{-i\vec{x}\cdot(\vec{p}+\vec{q})}\right) +$$

$$\left. + \frac{m^2}{2\sqrt{\omega_{\vec{p}}\,\omega_{\vec{q}}}}\left(a_{\vec{p}}\, a_{\vec{q}}\, e^{i\vec{x}\cdot(\vec{p}+\vec{q})} + a_{\vec{p}}\, a^\dagger_{\vec{q}} e^{i\vec{x}\cdot(\vec{p}-\vec{q})} + a^\dagger_{\vec{p}}\, a_{\vec{q}}\, e^{-i\vec{x}\cdot(\vec{p}-\vec{q})} + a^\dagger_{\vec{p}}\, a^\dagger_{\vec{q}}\, e^{-i\vec{x}\cdot(\vec{p}+\vec{q})}\right)\right] =$$

$$= \frac{1}{4}\int \frac{d^3p\, d^3q}{(2\pi)^3}\left[\sqrt{\omega_{\vec{p}}\,\omega_{\vec{q}}}\left(a_{\vec{p}}\, a_{\vec{q}}\delta^{(3)}(\vec{p}+\vec{q}) - a_{\vec{p}}\, a^\dagger_{\vec{q}}\,\delta^{(3)}(\vec{p}-\vec{q}) + a^\dagger_{\vec{p}}\, a_{\vec{q}}\delta^{(3)}(\vec{p}-\vec{q}) - a^\dagger_{\vec{p}}\, a^\dagger_{\vec{q}}\,\delta^{(3)}(\vec{p}+\vec{q})\right) - \right.$$

$$-\frac{i}{\sqrt{\omega_{\vec{p}}\,\omega_{\vec{q}}}}\left(\vec{p}\cdot\vec{q}\, a_{\vec{p}}\, a_{\vec{q}}\,\delta^{(3)}(\vec{p}+\vec{q}) - \vec{p}\cdot\vec{q}\, a_{\vec{p}}\, a^\dagger_{\vec{q}}\delta^{(3)}(\vec{p}-\vec{q}) - \vec{p}\cdot\vec{q}\, a^\dagger_{\vec{p}}\, a_{\vec{q}}\,\delta^{(3)}(\vec{p}-\vec{q}) + \vec{p}\cdot\vec{q}\,\delta^{(3)}\,(\vec{p}+\vec{q})\right) +$$

$$\left. + \frac{m^2}{\sqrt{\omega_{\vec{p}}\,\omega_{\vec{q}}}}\left(a_{\vec{p}}\, a_{\vec{q}}\,\delta^{(3)}(\vec{p}+\vec{q}) + a_{\vec{p}}\, a^\dagger_{\vec{q}}\,\delta^{(3)}(\vec{p}-\vec{q}) - a^\dagger_{\vec{p}}\, a_{\vec{q}}\,\delta^{(3)}(\vec{p}-\vec{q}) - a^\dagger_{\vec{p}}\, a^\dagger_{\vec{q}}\,\delta^{(3)}(\vec{p}+\vec{q})\right)\right] =$$

$$= \frac{1}{4}\int \frac{d^3p}{(2\pi)^3}\left[\sqrt{\omega_{\vec{p}}\,\omega_{-\vec{p}}}\left(a_{\vec{p}}\, a_{\vec{p}} - a_{\vec{p}}\, a^\dagger_{\vec{p}} + a^\dagger_{\vec{p}}\, a_{-\vec{p}} - a^\dagger_{\vec{p}}\, a^\dagger_{-\vec{p}}\right) - \frac{ip^2}{\sqrt{\omega_{\vec{p}}\,\omega_{-\vec{p}}}}\left(a_{\vec{p}}\, a_{-\vec{p}} - a_{\vec{p}}\, a^\dagger_{\vec{p}} - a^\dagger_{\vec{p}}\, a_{\vec{p}} + a^\dagger_{\vec{p}}\, a^\dagger_{-\vec{p}}\right) + \right.$$

$$\left. + \frac{m^2}{\sqrt{\omega_{\vec{p}}\,\omega_{-\vec{p}}}}\left(a_{\vec{p}}\, a_{-\vec{p}} + a_{\vec{p}}\, a^\dagger_{\vec{p}} - a^\dagger_{\vec{p}}\, a_{\vec{p}} - a^\dagger_{\vec{p}}\, a^\dagger_{-\vec{p}}\right)\right] =$$

$$= \frac{1}{4}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\omega_{\vec{p}}}\left[\left(-\omega^2_{\vec{p}} + p^2 + m^2\right)\left(a_{\vec{p}}\, a_{-\vec{p}} + a^\dagger_{\vec{p}}\, a^\dagger_{-\vec{p}}\right) + \left(\omega^2_{\vec{p}} + p^2 + m^2\right)\left(a_{\vec{p}}\, a^\dagger_{\vec{p}} + a^\dagger_{\vec{p}}\, a_{\vec{p}}\right)\right] =$$

$$= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\omega_{\vec{p}}\left(a_{\vec{p}}\, a^\dagger_{\vec{p}} + a^\dagger_{\vec{p}}\, a_{\vec{p}}\right) = \int \frac{d^3p}{(2\pi)^3}\omega_{\vec{p}}\left(a^\dagger_{\vec{p}}\, a_{\vec{p}} + \frac{1}{2}(2\pi)^3\delta^{(3)}(0)\right)$$

This divergence arises since we're trying to compute the energy of a vacuum state in the whole space.

Neglecting further discussions about *renormalization*, for our purposes, we can simply redifine the Hamiltonian as the energy difference between the excited states and the vacuum, by imposing the zero energy to be

[1] We reassure that the following calculation is not necessary to understand the physical involvements of the canonical quantization, although we suggest to go through it to anyone whose interested in developing some familiarity with the algebraic side.

null, so that:

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}}\, a^\dagger_{\vec{p}} a_{\vec{p}} \tag{2.14}$$

We can then easily compute the commutation relations of the ladder operators with the Hamiltonian, simply noticing that, from Equation 2.13, if $\vec{q} \neq \vec{p}$, the commutator between the latter vanishes, and hence we can reduce to an integration of a Dirac delta:

$$[a_{\vec{p}}, H] = a_{\vec{p}} \left(\int \frac{d^3q}{(2\pi)^3} \omega_{\vec{q}}\, a^\dagger_{\vec{q}} a_{\vec{q}} \right) - \left(\int \frac{d^3q}{(2\pi)^3} \omega_{\vec{q}}\, a^\dagger_{\vec{q}} a_{\vec{q}} \right) a_{\vec{p}} =$$

$$= a_{\vec{p}} \left[\int \frac{d^3q}{(2\pi)^3} \omega_{\vec{q}}\, a^\dagger_{\vec{q}} a_{\vec{q}}\, \delta\left(\vec{p} - \vec{q}\right) \right] - \left[\int \frac{d^3q}{(2\pi)^3} \omega_{\vec{q}}\, a^\dagger_{\vec{q}} a_{\vec{q}}\, \delta\left(\vec{p} - \vec{q}\right) \right] a_{\vec{p}} =$$

$$= a_{\vec{p}} \left[-\frac{1}{(2\pi)^3} \omega_{\vec{p}}\, a^\dagger_{\vec{p}} a_{\vec{p}} \right] - \left[-\frac{1}{(2\pi)^3} \omega_{\vec{p}}\, a^\dagger_{\vec{p}} a_{\vec{p}} \right] a_{\vec{p}} =$$

$$= -\frac{1}{(2\pi)^3} \omega_{\vec{p}} \left[a_{\vec{p}} a^\dagger_{\vec{p}} a_{\vec{p}} - a^\dagger_{\vec{p}} a_{\vec{p}} a_{\vec{p}} \right] = -\frac{1}{(2\pi)^3} \omega_{\vec{p}} \left[a_{\vec{p}}, a^\dagger_{\vec{p}} \right] a_{\vec{p}} = -\omega_{\vec{p}}\, a_{\vec{p}}$$

with an analogous calculation for $a^\dagger_{\vec{p}}$, leading to:

$$\begin{cases} [a_{\vec{p}}, H] = -\omega_{\vec{p}}\, a_{\vec{p}} \\ \left[a^\dagger_{\vec{p}}, H\right] = \omega_{\vec{p}}\, a^\dagger_{\vec{p}} \end{cases} \tag{2.15}$$

We finally proved that particles in QFT arise as excitations of fields:

$$|\,\vec{p}\rangle = a^\dagger_{\vec{p}}\,|\,0\,\rangle \Longrightarrow \begin{cases} H|\,p\,\rangle = \omega_{\vec{p}}|\,\vec{p}\rangle \\ \omega^2_{\vec{p}} = \vec{p}^2 + m^2 \end{cases} \tag{2.16}$$

2.1.3 Relativistic Normalization

We've stressed a lot the fact that canonical quantization is not manifestly Lorentz covariant, and all the previous derivations are accomplished in the non-relativistic limit.

Can we directly extend this formalism to relativistic particles? Let's figure this out.

The scalar product of two particles is given by:

$$\left\langle \vec{p}\,\middle|\,\vec{q}\right\rangle = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \tag{2.17}$$

$\vec{p}$ and $\vec{q}$ are 3-vectors, so the result is not a Lorentz scalar. Therefore, we need to define a Lorentz invariant metric.

From the relativistic energy-momentum relation:

$$p_\mu p^\mu = p_0^2 - \vec{p}^2 = m^2 = E^2 - \vec{p}^2 \Longleftrightarrow p_0 = \pm\sqrt{E} \tag{2.18}$$

The signed value of p_0 is Lorentz invariant, and so it is the following measure:

$$\int d^4p\ \delta(p_0^2 - \vec{p}^2 - m^2) = \int \frac{d^3p}{2E_{\vec{p}}} \tag{2.19}$$

Therefore, this can be taken as the relativistic normalized measure in the momentum space, allowing to derive the relativistically normalised state $|\, p \,\rangle$:

$$|\, p \,\rangle = \sqrt{2E_{\vec{p}}}\, |\, \vec{p} \,\rangle = \sqrt{2E_{\vec{p}}}\, a^{\dagger}_{\vec{p}}\, |\, \vec{p} \,\rangle \tag{2.20}$$

The scalar field is the simplest prototype we could've analyzed to understand the dynamics in QFT, yet it's effective to understand what's required to approach the next step: the Dirac spinor field.

2.2 The Dirac Field

As anticipated in section 1.3, a Dirac spinor is a 4 component complex object transforming in a reducible representation of $Spin(1,3)$:

$$\Psi \in \left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right) \tag{2.21}$$

Now it is time to connect its algebra with the QFT formalism, and we will by studying its dynamics.

2.2.1 Plausibility proof of the Dirac Lagrangian

The dynamics of a spinor field Ψ is ruled by the *Dirac action*:

$$S_D = \int d^4x \, \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi \tag{2.22}$$

In the first place, as we did for the scalar field, since we don't like to take result for granted[2], we want to give a plausibility argument for this Lagrangian, by proving that it is Lorentz invariant. If one wanted to push further, another argument is proving the energy conservation consistency, by computing the energy-momentum tensor as we did in section 2.1.1 for the scalar field; since the Lorentz invariance proof is already not prosaic, we will omit this and encourage the reader to attemt it autonomously.

For our purposes, we will work in the chiral representation, and verify that each component of the Lagrangian is indeed invariant under the action of $Spin(1,3)$.

We know from Equation 1.38 a spinor to transform in a general representation as:

$$\Psi^\alpha(x) \to S\left[\Lambda\right]^\alpha_{\ \beta} \Psi^\beta\left(\Lambda^{-1}x\right) \tag{2.23}$$

implying its conjugate to transform as:

$$\left(\Psi^*\right)^T(x) = \Psi^{\dagger\,\alpha}(x) \longrightarrow \Psi^{\dagger\,\beta}\left(\Lambda^{-1}x\right) S\left[\Lambda\right]^{\dagger\ \alpha}_{\ \beta} \tag{2.24}$$

Therefore their product, transforms as:

$$\Psi^{\dagger\,\alpha}(x)\Psi_\alpha(x) \longrightarrow \Psi^{\dagger\,\beta}\left(\Lambda^{-1}x\right) S\left[\Lambda\right]^{\dagger\ \alpha}_{\ \beta} S\left[\Lambda\right]_\alpha^{\ \beta} \Psi_\beta\left(\Lambda^{-1}x\right) \tag{2.25}$$

But, unlike rotations, there's no way that a boost is in general a unitary transformation, meaning that this quantity is not a Lorentz scalar. Anyway, we can solve this issue rather easily, by leveraging the easily verifiable relation $\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger$:

$$S^{\dagger\,\mu\nu} = \frac{1}{4}\left([\gamma^\nu, \gamma^\mu]\right)^\dagger = \frac{1}{4}\left[\gamma^{\dagger\,\nu}, \gamma^{\dagger\,\mu}\right] = -\gamma^0 S^{\mu\nu}\gamma^0 \Longrightarrow S\left[\Lambda\right]^\dagger = \gamma^0 S\left[\Lambda\right]^{-1}\gamma^0 \tag{2.26}$$

and define the *Dirac adjoint* as:

$$\bar{\Psi}(x) \equiv \Psi^\dagger(x)\gamma^0 \longrightarrow \Psi^\dagger\gamma^0 S\left[\Lambda\right]^{-1}\gamma^0\gamma^0 = \Psi^\dagger\gamma^0 S\left[\Lambda\right]^{-1} \tag{2.27}$$

[2]...and to start to get comfortable with some spinor algebra.

Now we can perform the whole calculation, by expanding the transformations to linear order:

$$\bar{\Psi}(x)^\dagger \left(i\gamma^\mu \partial_\mu - m\right)\Psi(x) \xrightarrow{Spin(1,3)} \Psi^\dagger S\left[\Lambda\right]^\dagger \gamma^0 \left(i\gamma^\mu \Lambda_\mu{}^\nu \partial_\nu - m\right) S\left[\Lambda\right]\Psi = *$$

$$\bar{\Psi}\Psi = \psi^\dagger \gamma^0 \psi(x) \longrightarrow \Psi^\dagger S[\Lambda]^\dagger \gamma^0 S[\Lambda]\Psi = \Psi^\dagger \gamma^0 \Psi = \bar{\Psi}\Psi$$

$$* = -m\Psi^\dagger S[\Lambda]^\dagger \gamma^0 S[\Lambda]\Psi + i\Psi^\dagger S[\Lambda]^\dagger \gamma^0 \gamma^\mu \Lambda_\mu{}^\nu \partial_\nu S[\Lambda]\Psi \overset{(2.27)}{=} -m\bar{\Psi}\Psi + i\bar{\Psi}S[\Lambda]^{-1}\gamma^\mu \Lambda_\mu{}^\nu \partial_\nu S[\Lambda]\Psi = *$$

$$S[\Lambda]^{-1}\gamma^\mu S[\Lambda] \simeq \left(1 + \frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right)\gamma^\mu \left(1 - \frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right) = \gamma^\mu - \frac{1}{2}\gamma^\mu \Omega_{\rho\sigma}S^{\rho\sigma} + \frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\gamma^\mu + \frac{1}{4}\Omega_{\rho\sigma}S^{\rho\sigma}\gamma^\mu \Omega_{\rho\sigma}S^{\rho\sigma} =$$

$$= \gamma^\mu - \Omega_{\rho\sigma}\frac{1}{2}\left[\gamma^\mu, S^{\rho\sigma}\right] + \frac{1}{4}\Omega_{\rho\sigma}S^{\rho\sigma}\gamma^\mu \Omega_{\rho\sigma}S^{\rho\sigma} = \gamma^\mu - \frac{1}{2}\Omega_{\rho\sigma}\left(\eta^{\sigma\mu}\gamma^\rho - \eta^{\rho\nu}\gamma^\sigma\right) + \frac{1}{4}\Omega_{\rho\sigma}S^{\rho\sigma}\gamma^\mu \Omega_{\rho\sigma}S^{\rho\sigma} =$$

$$= \gamma^\mu + \frac{1}{2}\Omega_{\rho\sigma}\left(\eta^{\rho\mu}\delta^\sigma{}_\nu - \eta^{\sigma\mu}\delta^\rho{}_\nu\right)\gamma^\nu + \frac{1}{4}\Omega_{\rho\sigma}S^{\rho\sigma}\gamma^\mu \Omega_{\rho\sigma}S^{\rho\sigma} \simeq \gamma^\mu + \frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\gamma^\mu \simeq \Lambda^\mu{}_\nu \gamma^\nu$$

$$* = -m\bar{\Psi}\Psi + i\bar{\Psi}S\left[\Lambda\right]^{-1} S\left[\Lambda\right]\gamma^\mu S\left[\Lambda\right]^{-1}\partial_\nu S\left[\Lambda\right]\Psi = \bar{\Psi}\left(i\gamma^\mu \partial_\mu - m\right)\Psi$$

Varying the action with respect to $\bar{\Psi}$, one directly gets the renowned Dirac equation:

$$\left(i\gamma^\mu \partial_\mu - m\right)\Psi \equiv \left(\not{\partial} - m\right)\Psi = 0 \tag{2.28}$$

2.2.2 Fermionic Quantization

To retrace what we did for the scalar case, we need to quantize the theory, meaning to expand the fields in terms of the creation and annihilation operators and compute their quantum relations with the Hamiltonian operator, to give rise to particle states.

This represents a fundamental passage: from the QM and the Statistical Mechanics course, we learnt that bosons and fermions are diametrically different: the firsts obey the Bose-Einstein statistic, have an integer spin and a composite system must be described by a symmetric wavefunction on an Hilbert space, while the seconds are described by the Fermi-Dirac statistic, have half-integer spin and obey the Pauli exclusion principle.

We need to change something in the canonical quantization, something that may reflect the antisymmetry of a fermionic wavefunction. Directly from 2.13 we observe that it is the ladder operators algebra to determine the bosonic nature of the scalar fields:

$$a^\dagger_{\vec{p}} a^\dagger_{\vec{q}} \,|\,0\,\rangle = a^\dagger_{\vec{p}} a^\dagger_{\vec{q}} \,|\,0\,\rangle \Longleftrightarrow |\,\vec{p}, \vec{q}\,\rangle = |\,\vec{q}, \vec{p}\,\rangle \tag{2.29}$$

In this section we will show that, in order to achieve a fermionic binary system, one can directly substitute the commutation with analogous anticommutation relations[3]:

$$\begin{cases} \left\{\Psi_\alpha\left(\vec{x}\right), \Psi_\beta\left(\vec{y}\right)\right\} = \left\{\Psi^\dagger_\alpha\left(\vec{x}\right), \Psi^\dagger_\beta\left(\vec{y}\right)\right\} = 0 \\ \left\{\Psi_\alpha\left(\vec{x}\right), \Psi^\dagger_\beta\left(\vec{y}\right)\right\} = \delta_{\alpha,\beta}\delta^{(3)}\left(\vec{x} - \vec{y}\right) \end{cases} \tag{2.30}$$

[3]Note that the conjugate dynamical variable of Ψ is $i\Psi^\dagger$.

Monocromatic solutions of the Dirac equation

Before quantizing, we will show the free Dirac field to be a linear superposition of positive ad negative energy states, by solving the Dirac equation using the plane wave ansatz:

$$\Psi = u_{\vec{p}}e^{-i\vec{p}\cdot x} + v_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} \tag{2.31}$$

Equation 2.28 then becomes:

$$\left(\gamma^{\mu}p_{\mu} - m\right)\left(u_{\vec{p}}e^{-i\vec{p}\cdot x} + v_{\vec{p}}e^{i\vec{p}\cdot\vec{x}}\right) = 0 \tag{2.32}$$

which we can solve term by term, by leveraging the linearity:

$$\left(\gamma^{\mu}p_{\mu} - m\right)u_{\vec{p}} = \begin{pmatrix} -m & p_{\mu}\sigma^{\mu} \\ p_{\mu}\bar{\sigma}^{\mu} & -m \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0 \Longleftrightarrow \begin{cases} p_{\mu}\sigma^{\mu}u_2 = mu_1 \\ p_{\mu}\bar{\sigma}^{\mu}u_1 = mu_2 \end{cases} \Longrightarrow$$

$$\Longrightarrow \left(p\cdot\sigma\right)\left(p\cdot\bar{\sigma}\right)u_1\cdot u_2 = m^2 u_1\cdot u_2$$

A rapid calculation shows that $\left(p\cdot\sigma\right)\left(p\cdot\bar{\sigma}\right) = m^2$, meaning that these equations are not independent; we can hence choose either one and try to impose a second ansatz $u_1 = \left(p\cdot\sigma\right)\tilde{\xi}$, where $\tilde{\xi}$ is a constant spinor:

$$p_{\mu}\bar{\sigma}^{\mu}u_1 = \left(p\cdot\bar{\sigma}\right)\left(p\cdot\sigma\right)\tilde{\xi} = -mu_2 \Longrightarrow u_2 = mu_1 \Longrightarrow u_{\vec{p}} = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar{\sigma}}\xi \end{pmatrix} \tag{2.33}$$

In a similiar fashion, one finds:

$$v_{\vec{p}} = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta \\ -\sqrt{p\cdot\bar{\sigma}}\eta \end{pmatrix} \tag{2.34}$$

where η is another constant spinor.

We proved Ψ to be a linear superposition of two spinors oscillating with positive ($u_{\vec{p}}$) and negative ($v_{\vec{p}}$) frequencies. As foreseeable in the maseless limit, these define positive and negative energy states[4], corresponding to two a particle-antiparticle couple.

Quantization of the Dirac field

Being a monocromatic Dirac spinor a linear superposition of two positive and negative energy states, a generic one can be Fourier expanded as:

$$\begin{cases} \Psi\left(\vec{x}\right) = \sum_{s=1}^{2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[b_{\vec{p}}^{s}u_{\vec{p}}^{s}e^{i\vec{p}\cdot\vec{x}} + c_{\vec{p}}^{\dagger\,s}v_{\vec{p}}^{s}e^{-i\vec{p}\cdot\vec{x}}\right] \\ \Psi^{\dagger}\left(\vec{x}\right) = \sum_{s=1}^{2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[b_{\vec{p}}^{s\,\dagger}u_{\vec{p}}^{s\,\dagger}e^{-i\vec{p}\cdot\vec{x}} + c_{\vec{p}}^{s}v_{\vec{p}}^{s\,\dagger}e^{i\vec{p}\cdot\vec{x}}\right] \end{cases} \tag{2.35}$$

where:

- s labels the helicity states.
- $b_{\vec{p}}$ and $c_{\vec{p}}$ are respectively the annihilation operators for a spinor and an antispinor.
- $b_{\vec{p}}^{\dagger}$ and $c_{\vec{p}}^{\dagger}$ are the creation operators for a spinor and an anti-spinor.

[4]This statement can be properly demonstrated from the energy-momentum tensor of the field.

We now want to prove the fermionic algebra of the ladder operators:

$$\left\{b^r_{\vec{p}}, b^{s\,\dagger}_{\vec{q}}\right\} = \left\{c^r_{\vec{p}}, c^{s\,\dagger}_{\vec{q}}\right\} = (2\pi)^3\,\delta^{rs}\delta^{(3)}\left(\vec{p}-\vec{q}\right) \tag{2.36}$$

to be equivalent to the fields' ones in 2.30.

$$\left\{\Psi\left(\vec{x}\right), \Psi^\dagger\left(\vec{y}\right)\right\} = \sum_{r,s}^{2}\int \frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}}\left[\left\{b^r_{\vec{p}}, b^{s\,\dagger}_{\vec{q}}\right\} u^r_{\vec{p}}u^{s\,\dagger}_{\vec{q}}\;e^{i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})} + \left\{c^{r\,\dagger}_{\vec{p}}, c^s_{\vec{q}}\right\} v^r_{\vec{p}}v^{s\,\dagger}_{\vec{q}}e^{-i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})}\right] =$$

$$\overset{2.36}{=} \sum_{r,s}^{2}\int \frac{d^3p\,d^3q}{(2\pi)^3}\frac{1}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}}\delta^{rs}\left[u^r_{\vec{p}}u^{s\,\dagger}_{\vec{q}}\;e^{i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})} + v^r_{\vec{p}}v^{s\,\dagger}_{\vec{q}}e^{-i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})}\right]\delta^{(3)}\left(\vec{p}-\vec{q}\right) =$$

$$= \sum_{s}^{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left[u^s_{\vec{p}}u^s_{\vec{q}}\gamma^0\;e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + v^s_{\vec{p}}v^s_{\vec{q}}\gamma^0 e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\right] =$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left[\left(p_0\gamma^0 + p_i\gamma^i + m\right)\gamma^0 + \left(p_0\gamma^0 - p_i\gamma^i - m\right)\gamma^0\right]e^{i\vec{p}\cdot(\vec{x}-\vec{y})} =$$

$$= \int \frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot(\vec{x}-\vec{y})} = \delta^{(3)}\left(\vec{x}-\vec{y}\right)$$

Hamiltonian operator

As we did for the scalar field, the next step is writing the Hamiltonian operator in terms of the creation and the annihilation operators, to then derive its quantum relations with the ladder operators.

$$(i)\quad \mathcal{H} = \pi\dot{\Psi} - \mathcal{L} = \Psi^\dagger \partial_0 \Psi - \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi = \bar{\Psi}\left(-i\gamma^i\partial_i + m\right)\Psi$$

$$(ii)\quad \left(-i\gamma^i\partial_i + m\right)\sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[b^s_{\vec{p}}u^s_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} + c^{\dagger\, s}_{\vec{p}}v^s_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right] =$$

$$= \sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[b^s_{\vec{p}}(-\gamma^i p_i + m)u^s_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} + c^{\dagger\, s}_{\vec{p}}(\gamma^i p_i + m)v^s_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right] =$$

$$= \sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3}\frac{\gamma^0}{\sqrt{2E_{\vec{p}}}}\left[b^s_{\vec{p}}p_0u^s_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} - c^{\dagger\, s}_{\vec{p}}p_0v^s_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right] = \sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_{\vec{p}}}{2}}\gamma^0\left[b^s_{\vec{p}}u^s_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} - c^{\dagger\, s}_{\vec{p}}v^s_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right]$$

$$(iii)\quad H = \left\{\sum_{r=1}^{2}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{q}}}}\left[b^{r\,\dagger}_{\vec{q}}u^{r\,\dagger}_{\vec{q}}e^{-i\vec{q}\cdot\vec{x}} + c^r_{\vec{q}}v^{r\,\dagger}_{\vec{q}}e^{i\vec{q}\cdot\vec{x}}\right]\right\}\cdot\left\{\sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_{\vec{p}}}{2}}\gamma^0\left[b^s_{\vec{p}}u^s_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} - c^{\dagger\, s}_{\vec{p}}v^s_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right]\right\} =$$

$$= \sum_{r,s=1}^{2}\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6}\sqrt{\frac{E_{\vec{p}}}{4E_{\vec{q}}}}\left[b^{r\,\dagger}_{\vec{q}}u^{r\,\dagger}_{\vec{q}}e^{-i\vec{q}\cdot\vec{x}} + c^r_{\vec{q}}v^{r\,\dagger}_{\vec{q}}e^{i\vec{q}\cdot\vec{x}}\right]\cdot\left[b^s_{\vec{p}}u^s_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} - c^{\dagger\, s}_{\vec{p}}v^s_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right] = *$$

$$\int \frac{d^3x}{(2\pi)^3}\left[b^{r\,\dagger}_{\vec{q}}u^{r\,\dagger}_{\vec{q}}e^{-i\vec{q}\cdot\vec{x}} + c^r_{\vec{q}}v^{r\,\dagger}_{\vec{q}}e^{i\vec{q}\cdot\vec{x}}\right]\cdot\left[b^s_{\vec{p}}u^s_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} - c^{\dagger\, s}_{\vec{p}}v^s_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right] =$$

$$= \int \frac{d^3x}{(2\pi)^3}\left[b^{r\,\dagger}_{\vec{q}}u^{r\,\dagger}_{\vec{q}}b^s_{\vec{p}}u^s_{\vec{p}}e^{i(\vec{p}-\vec{q})\cdot\vec{x}} - b^{r\,\dagger}_{\vec{q}}u^{r\,\dagger}_{\vec{q}}c^{\dagger\, s}_{\vec{p}}v^s_{\vec{p}}e^{-i(\vec{p}+\vec{q})\cdot\vec{x}} + c^r_{\vec{q}}v^{r\,\dagger}_{\vec{q}}b^s_{\vec{p}}u^s_{\vec{p}}e^{i(\vec{p}+\vec{q})\cdot\vec{x}} - c^r_{\vec{q}}v^{r\,\dagger}_{\vec{q}}c^{\dagger\, s}_{\vec{p}}v^s_{\vec{p}}e^{-i(\vec{p}-\vec{q})\cdot\vec{x}}\right] =$$

$$= b^{r\,\dagger}_{\vec{q}}u^{r\,\dagger}_{\vec{q}}b^s_{\vec{p}}u^s_{\vec{p}}\delta^{(3)}(\vec{p}-\vec{q}) - b^{r\,\dagger}_{\vec{q}}u^{r\,\dagger}_{\vec{q}}c^{\dagger\, s}_{\vec{p}}v^s_{\vec{p}}\delta^{(3)}(\vec{p}+\vec{q}) + c^r_{\vec{q}}v^{r\,\dagger}_{\vec{q}}b^s_{\vec{p}}u^s_{\vec{p}}\delta^{(3)}(\vec{p}+\vec{q}) - c^r_{\vec{q}}v^{r\,\dagger}_{\vec{q}}c^{\dagger\, s}_{\vec{p}}v^s_{\vec{p}}\delta^{(3)}(\vec{p}-\vec{q}) =$$

$$= b^{r\,\dagger}_{\vec{p}}b^s_{\vec{p}}u^{r\,\dagger}_{\vec{p}}u^s_{\vec{p}} - b^{r\,\dagger}_{-\vec{p}}u^{r\,\dagger}_{-\vec{p}}c^{\dagger\, s}_{\vec{p}}v^s_{\vec{p}} + c^r_{-\vec{p}}v^{r\,\dagger}_{-\vec{p}}b^s_{\vec{p}}u^s_{\vec{p}} - c^r_{\vec{p}}v^{r\,\dagger}_{\vec{p}}c^{\dagger\, s}_{\vec{p}}v^s_{\vec{p}}$$

$$* = \sum_{r,s=1}^{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\left[b^{r\,\dagger}_{\vec{p}}b^s_{\vec{p}}u^{r\,\dagger}_{\vec{p}}u^s_{\vec{p}} - b^{r\,\dagger}_{-\vec{p}}u^{r\,\dagger}_{-\vec{p}}c^{\dagger\, s}_{\vec{p}}v^s_{\vec{p}} + c^r_{-\vec{p}}v^{r\,\dagger}_{-\vec{p}}b^s_{\vec{p}}u^s_{\vec{p}} - c^r_{\vec{p}}v^{r\,\dagger}_{\vec{p}}c^{\dagger\, s}_{\vec{p}}v^s_{\vec{p}}\right] \overset{2.33}{=}$$

$$= \int \frac{d^3p}{(2\pi)^3}E_{\vec{p}}\left(b^{s\,\dagger}_{\vec{p}}b^s_{\vec{p}} - c^s_{\vec{p}}c^{\dagger\, s}_{\vec{p}}\right) \overset{2.36}{=} \int \frac{d^3p}{(2\pi)^3}E_{\vec{p}}\left(b^{s\,\dagger}_{\vec{p}}b^s_{\vec{p}} + c^{\dagger\, s}_{\vec{p}}c^s_{\vec{p}} - (2\pi)^3\delta^{(3)}(0)\right)$$

Analogously to the scalar case, we can neglect further discussions about renormalization and redifine the Hamiltonian as:

$$H = \int \frac{d^3p}{(2\pi)^3}E_{\vec{p}}\left(b^{s\,\dagger}_{\vec{p}}b^s_{\vec{p}} + c^{s\,\dagger}_{\vec{p}}c^s_{\vec{p}}\right) \tag{2.37}$$

With similiar calculations to 2.1.2, the commutation relations between the ladder operators and H are straightforwardly obtained:

$$\begin{cases} \left[H, b^r_{\vec{p}}\right] = -E_{\vec{p}} b^r_{\vec{p}} \\ \left[H, c^r_{\vec{p}}\right] = -E_{\vec{p}} c^r_{\vec{p}} \\ \left[H, b^{r\,\dagger}_{\vec{p}}\right] = E_{\vec{p}} b^{r\,\dagger}_{\vec{p}} \\ \left[H, c^{r\,\dagger}_{\vec{p}}\right] = E_{\vec{p}} c^{r\,\dagger}_{\vec{p}} \end{cases} \tag{2.38}$$

As required a multiple fermionic state satisfies the spin-statistics theorem:

$$b^{r_1\,\dagger}_{\vec{p}_1} b^{r_2\,\dagger}_{\vec{p}_2} |\, 0 \,\rangle = |\, \vec{p}_1, r_1; \vec{p}_2, r_2 \,\rangle = -b^{r_2\,\dagger}_{\vec{p}_2} b^{r_1\,\dagger}_{\vec{p}_1} |\, 0 \,\rangle = -|\, \vec{p}_2, r_2; \vec{p}_1, r_1 \,\rangle \tag{2.39}$$

Chapter 3

Supersymmetry

3.1 A brief introduction

To start off, we shall recall the main differences between photons and fermions, which we discussed in the previous chapters, to understand why Supersymmetry thinks, to say the least, out of the box, or, to be more precise, doesn't even perceive the box as a limitation:

1. **Transformation properties**: a generic fermion belongs to a spinorial representation of $Spin(1,3)$, while a boson to a tensorial one:
$$\begin{cases} |f\rangle \in (n,0)\oplus(0,n) & n\in\mathbb{N}/2 \\ |b\rangle \in \left(\frac{m}{2},\frac{m}{2}\right) & m\in\mathbb{N} \end{cases} \tag{3.1}$$
as a matter of fact, since the helicity of a particle is given by the sum of the left and the right hand chirality components, its value is integral for bosons and semi-integral for fermions.

2. **Spin-statistics theorem**: multiple fermionic and bosonic systems are respectively described by anti-symmetric and symmetric states on an Hilbert space, and are thus described by the Fermi-Dirac and the Bose-Einstein statistics:
$$f_{FD}(\epsilon) = \frac{1}{\exp\left\{\frac{\epsilon}{k_B T}\right\}+1} \quad , \quad f_{BE}(\epsilon) = \frac{1}{\exp\left\{\frac{\epsilon}{k_B T}\right\}-1} \tag{3.2}$$
resulting in diametrically different properties, most of which are known from the Statistical Mechanics course.

3. **Quantization**: fermions and bosons respectively emerge as excitations of spinorial and gauge tensorial fields, and the quantum relations between creation and annihilation operators in the two cases are:
$$\begin{cases} [a_{\vec{p}}, a_{\vec{q}}] = \left[a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}\right] = 0 \\ \left[a_{\vec{p}}, a^\dagger_{\vec{q}}\right] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}) \end{cases} \quad \text{for bosons}$$
$$\begin{cases} \{a_{\vec{p}}, a_{\vec{q}}\} = \left\{a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}\right\} = 0 \\ \left\{a_{\vec{p}}, a^\dagger_{\vec{q}}\right\} = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}) \end{cases} \quad \text{for fermions} \tag{3.3}$$

4. **Physical role**: In the Standard Model, bosons are force carriers: the photon γ $(s=1)$ mediates the electromagnetic interaction, the gluon g $(s=1)$ the strong force, and so on and so forth. On the other hand, fermions, such as the electron $(s=\frac{1}{2})$ and the quarks $(s=\frac{1}{2})$ form matter, and interact through bosons.

And if this is not enough to discourage anyone to search for a symmetry in such context, we shall mention that there exists a so called *no-go theorem*, namely the **Coleman-Mandula Theorem**, which apparently definitely shuts down any attempt of *unification*: at the core, it states that space-time ($ISO(1,3)$) and internal ($U(1), SU(3), SU(2)_L$,...) symmetry groups can combine only trivially, meaning as a direct factorisation, essentially implies that the structure constant relating generators of different subgroups of the resulting Lie one are necessarily null.

Such a group would simply be represented, on particle states, as its action on a multiplet formed by those of the subgroups, meaning that no transformation could ever make ones turn into each others.

Supersymmetry is a brilliant stratagem to overcome this limitation: if a standard Lie group isn't a possibility, why not combining bosonic and fermionic generators to form a *Lie supergroup*? The resulting *superalgebra* would hence be defined by commutation and anticommutation relations.

The free $N = 1$ SUSY algebra implements the *Poincarè supergroup*, and is formed by a spinor-antispinor pair, namely the *supercharges*, and the generators of the Poincarè group, plus the generator of the $U(1)$ internal symmetry, that we will discuss in a bit.

The first part of this chapter will indeed be pinpointed on the representation theory of the SUSY algebra.

But, by the end of it, one may ask: if a Lie group is a differentiable manifold, what the heck is a Lie supergroup? Unsurprisingly, a *supermanifold*, which will be the focus of the second half.

3.2 $N = 1$ SUSY algebra

We hereby introduce $N = 1$ SUSY algebra, in a generic representation. The Poincarè supergroup is implemented by the ensuing generators:

$$\underbrace{Q_\alpha, \bar{Q}_{\dot{\alpha}}}_{\text{supercharges}} \quad , \quad \underbrace{P_\mu, M_{\mu\nu}}_{\text{ISO(1,3)}} \quad , \quad \underbrace{M}_{U(1)} \tag{3.4}$$

and its defining superalgebra is formed by the following commutation and anticommutation relations:

- ***Poincaré algebra***:

  Recalling from Equation 1.14:

$$\begin{cases} [P_\mu, P_\nu] = 0 \\ [M_{\mu\nu}, P_\rho] = i(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu) \\ [M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho}) \end{cases} \tag{3.5}$$

  as proven in subsection 1.1.1.

- ***Lorentz transformations of supercharges***:

  Supercharges must equivalently transform under the action of the Lorentz group as spinors and as fermionic operators acting on an Hilbert space.

$$\begin{cases} Q_\alpha \Longrightarrow \exp\left\{-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right\}_\alpha^{\ \beta} Q_\beta & \text{spinor} \\ Q_\alpha \Longrightarrow \exp\left\{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right\} Q_\alpha \, \exp\left\{-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right\} & \text{operator} \end{cases} \Longrightarrow$$

$$\Longrightarrow \exp\left\{-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right\}_\alpha^{\ \beta} Q_\beta = \exp\left\{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right\} Q_\alpha \, \exp\left\{-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right\} \Longrightarrow$$

$$\Longrightarrow \left(\mathbf{1} - \frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right) Q_\alpha = \left(\mathbf{1} + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right) Q_\alpha \left(\mathbf{1} - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right) \Longrightarrow$$

$$\Longrightarrow -\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}Q_\alpha \simeq -\frac{i}{2}Q_\alpha \, \omega_{\mu\nu}M^{\mu\nu} + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}Q_\alpha \Longrightarrow (\sigma^{\mu\nu})_\alpha^{\ \beta}Q_\beta = [M_{\mu\nu}, Q_\alpha]$$

- ***Supercharges translations***:

$$[Q_\alpha, P^\mu] = 0 \tag{3.6}$$

  For the Lorentz covariance, the right hand side of this equation must be of type $c(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{Q}^{\dot{\alpha}}$, with $c \in \mathbb{C}$. Then, for the Jacobi identity:

$$[P^\mu, [P^\nu, Q_\alpha]] + [P^\nu, [Q_\alpha, P^\mu]] + [Q_\alpha, [P^\mu, P^\nu]] = [P^\mu, [P^\nu, Q_\alpha]] + [P^\nu, [Q_\alpha, P^\mu]] =$$

$$= -c(\sigma^\nu)_{\alpha\dot{\alpha}} \left[P^\mu, \bar{Q}^{\dot{\alpha}}\right] + c(\sigma^\mu)_{\alpha\dot{\alpha}} \left[P^\nu, \bar{Q}^{\dot{\alpha}}\right] = |c|^2 \left(\sigma^\nu\bar{\sigma}^\mu - \sigma^\mu\bar{\sigma}^\nu\right)_\alpha^{\ \beta} Q_\beta = 0 \Longrightarrow c = 0$$

- ***Supercharges anticommutation***:

$$\{Q_\alpha, Q_\beta\} = 0 \tag{3.7}$$

  With a similiar argument to the previous point, for the Lorentz covariance:

$$\{Q_\alpha, Q_\beta\} = c(\sigma^{\mu\nu})_{\alpha\beta}M_{\mu\nu} \Longrightarrow [P^\mu, \{Q_\alpha, Q_\beta\}] = c(\sigma^{\mu\nu})_{\alpha\beta} \left[P^\mu, M_{\mu\nu}\right] = 0 \Longrightarrow c = 0$$

since, for the Poincaré algebra $[P^\mu, M_{\mu\nu}] \neq 0$.

- ***(Anti)Supercharges anticommutation***:

$$\left\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\right\} = 2\left(\sigma^\mu\right)_{\alpha\dot{\alpha}} P_\mu \tag{3.8}$$

- ***R-symmetry***:

$$\begin{cases} [R, Q_\alpha] = -Q_\alpha \\ [R, Q_{\dot{\alpha}}] = -Q_{\dot{\alpha}} \end{cases} \tag{3.9}$$

Supercharges must equivalently transform under the action of $U(1)$ as spinors and as fermionic operators on an Hilbert space, so, similiarly to the second point:

$$\begin{cases} Q_\alpha \Longrightarrow e^{-i\omega R} Q_\alpha & \text{spinor} \\ Q_\alpha \Longrightarrow e^{i\omega R} Q_\alpha e^{-i\omega R} & \text{operator} \end{cases} \Longrightarrow$$

$$e^{-i\omega R} Q_\alpha e^{i\omega R} Q_\alpha e^{-i\omega R} \Longrightarrow$$

$$\Longrightarrow (1 - i\omega R)\, Q_\alpha = (1 + i\omega R)\, Q_\alpha\, (1 - i\omega R) \Longrightarrow$$

$$\Longrightarrow RQ_\alpha \overset{(1)}{\simeq} Q_\alpha R + RQ_\alpha = -Q_\alpha \Longrightarrow [R, Q_\alpha] = -Q_\alpha$$

3.2.1 Fermionic and Bosonic degrees of freedom

Before diving into the representation theory of the SUSY algebra on particle states, we shall point out a simple, yet crucial, implication of the relations we derived in the previous section: since a fermionic generator carries a semi-integral spin, the state obtained applying it to a s spin one will definitely have $s \pm \frac{1}{2}$ spin, for the algebra of $SU(2)$.

Therefore, one may define the following operator:

$$(-1)^F \quad : \quad \begin{cases} (-1)^F |\, b\,\rangle = |\, b\,\rangle \\ (-1)^F |\, f\,\rangle = -|\, f\,\rangle \end{cases} \tag{3.10}$$

and apply it to the supercharge operator Q_α:

$$(-1)^F Q_\alpha = -Q_\alpha (-1)^F \Longrightarrow \left\{(-1)^F, Q_\alpha\right\} = 0 \Longrightarrow tr\left[(-1)^F \left\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\right\}\right] = tr\left[(-1)^F Q_\alpha \bar{Q}_{\dot{\alpha}} + (-1)^F \bar{Q}_{\dot{\alpha}} Q_\alpha\right] =$$

$$tr\left[-Q_\alpha (-1)^F \bar{Q}_{\dot{\alpha}} + (-1)^F \bar{Q}_{\dot{\alpha}} Q_\alpha\right] = 0 \Longrightarrow \sigma^\mu_{\alpha\dot{\alpha}} tr\left[(-1)^F P_\mu\right] |\, p_\mu\,\rangle = \sigma^\mu_{\alpha\dot{\alpha}} p_\mu tr\left[(-1)^F\right] |\, p_\mu\,\rangle = \mathbf{0} \Longrightarrow$$

$$\Longrightarrow tr\left[(-1)^F\right] = n_F - n_B = 0$$

This means that the number of bosons in a non interacting SUSY must always equalize that of the fermions. The trace of the $(-1)^F$ operator is called *Witten index.*

This simplified derivation will be formally demonstrated in the next section.

3.2.2 SUSY representation on particle states

In this section we want to achieve similiar results to those of subsection 1.1.2, where we used the eigenvalues of the Casimirs of $ISO(1,3)$ to label particle states, both in the massive and the maseless cases.

Since the super-Poincarè group is, after all, an extension of a Lie group, one may naively think that they share the same Casimirs:

$$\begin{cases} C_1 = P_\mu P^\mu \\ C_2 = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma} \epsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma} \equiv W_\mu W^\mu \end{cases} \tag{3.11}$$

which are bosonic, and thus, to verify that they do not mutate under SUSY transformations, we have to compute the commutation relations with the generators, and verify if they nullify. We've already proven that C_1 and C_2 do commute with the generators of the Poincarè algebra in subsection 1.1.2, so what remains is to check whether this holds with supercharges, as well.

$$(i) \quad [P_\mu P^\mu, Q_\alpha] = P_\mu [P^\mu, Q_\alpha] + [P^\mu, Q_\alpha] P^\mu \stackrel{3.6}{=} 0$$

$$(ii) \quad [W^\mu W_\mu, Q_\alpha] = \frac{1}{4} [\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma} \epsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma}, Q_\alpha] \propto \epsilon^{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} [P_\nu M_{\rho\sigma} P^\nu M^{\rho\sigma}, Q_\alpha] =$$

$$= \epsilon^{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} \left(P^\nu M^{\rho\sigma} [P_\nu M_{\rho\sigma}, Q_\alpha] + [P_\nu M_{\rho\sigma}, Q_\alpha] P^\nu M^{\rho\sigma}\right) =$$

$$= \epsilon^{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} \left(P^\nu M^{\rho\sigma} (P_\nu [M_{\rho\sigma}, Q_\alpha] + [M_{\rho\sigma}, Q_\alpha] P_\nu) + (P_\nu [M_{\rho\sigma}, Q_\alpha] + [M_{\rho\sigma}, Q_\alpha] P_\nu) P^\nu M^{\rho\sigma}\right) \neq 0$$

We've proven that only the 4-momentum is still a Casimir of the superalgebra, since supercharges are indeed invariant under translation from 3.6. After all, this is exactly what we wanted to achieve: all the particles in a *supermultiplet* share the same mass, but have different spin, breaking the Coleman-Mandula restrictions.

Thus, to build the multiplets, we will start from those of the Poincarè groups, and act with the supercharges.

Maseless multiplets

From the results of subsection 1.1.2, we know that a massless particle can be represented on an Hilbert space as a ket of the type $|\, p_\mu, h\, \rangle$, where h is the helicity.

We derived this by boosting to a frame such that $P^\mu = (E, 0, 0, E)$:

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} \stackrel{3.8}{=} 2\,(\sigma^\mu)_{\alpha\dot\alpha} P_\mu = 2E\left(\mathbf{1} + \sigma^3\right)_{\alpha\dot\alpha} = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Longrightarrow$$

$$\Longrightarrow \langle p_\mu, h|\, \{Q_\alpha, \bar{Q}_{\dot\alpha}\}\, |\, p_\mu, h\,\rangle = 4E\, \langle p_\mu, h| \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} |\, p_\mu, h\,\rangle \Longrightarrow$$

$$\langle p_\mu, h|\, \{Q_2, \bar{Q}_{\dot 2}\}\, |\, p_\mu, h\,\rangle = \langle p_\mu, h|\, \left(Q_2 \bar{Q}_2 + \bar{Q}_2 Q_2\right) |\, p_\mu, h\,\rangle = 0 \Longleftrightarrow$$

$$\Longleftrightarrow Q_2 |\, p_\mu, h\,\rangle = \bar{Q}_2 |\, p_\mu, h\,\rangle = 0$$

where we used the fact that the scalar product is a positive definite form.

Since Q_2 and its conjugate annihilate this state, we may simply cansider the action of the first components of the spinor couple to construct the multiplet, for, after a suitable rescaling, they implement fermionic creation and annihilation operators:

$$\begin{cases} a \equiv \frac{Q_1}{\sqrt{4E}} \\ a^\dagger \equiv \frac{\bar{Q}_1}{\sqrt{4E}} \end{cases} \Longrightarrow \begin{cases} \{a, a^\dagger\} = 1 \\ \{a, a\} = \{a^\dagger, a^\dagger\} = 0 \end{cases} \tag{3.12}$$

As we already know, from the quantization of the Dirac field in subsection 2.2.2, the representation of this algebra consists of:

$$\{\,|\,p_\mu, h\,\rangle, a^\dagger|\,p_\mu, h\,\rangle\,\} \equiv \{|\,0\,\rangle, |\,1\,\rangle\} : \begin{cases} a^\dagger|\,0\,\rangle = |\,1\,\rangle \\ a|\,1\,\rangle = |\,0\,\rangle \end{cases} \tag{3.13}$$

It is now straightforward to evaluate the helicity of these particles, since it represents the eigenvalue of a rotation around the, say, z axis:

$$\begin{cases} \left[M^{12}, Q_\alpha\right] = \left(\sigma^{12}\right)_\alpha^{\ \beta} Q_\beta \\ \left[M^{12}, \bar{Q}^{\dot\alpha}\right] = \left(\sigma^{12}\right)^{\dot\alpha}_{\ \dot\beta} \bar{Q}_{\dot\beta} \end{cases} \iff \begin{cases} \left[M^{12}, Q_1\right] = \left(\sigma^{12}\right)_1^{\ \beta} Q_\beta = \frac{1}{2}Q_1 \\ \left[M^{12}, \bar{Q}^{\dot 1}\right] = \left(\sigma^{12}\right)^{\dot 1}_{\ \dot\beta} \bar{Q}_{\dot\beta} = \frac{1}{2}\bar{Q}^{\dot 1} = -\frac{1}{2}\bar{Q}_1 \end{cases}$$

meaning that Q_1 and $\bar{Q}_1$ respectively raise and lower h by $\frac{1}{2}$, as qualitatively predicted in subsection 3.2.1.

Following the same cosiderations of subsection 1.1.2, we have to complete the multiplet, by add the opposite helicity states, in order to preserve CPT conservation:

$$\left\{|\,p_\mu, h\,\rangle\ ,\ |\,p_\mu, h - \frac{1}{2}\,\rangle = \frac{\bar{Q}_1}{\sqrt{4E}}|\,p_\mu, h\,\rangle\ ,\ |\,p_\mu, -h\,\rangle\ ,\ |\,p_\mu, -h + \frac{1}{2}\,\rangle = \frac{\bar{Q}_1}{\sqrt{4E}}|\,p_\mu, -h\,\rangle\right\} \tag{3.14}$$

To better grasp what we've just built, let's analyze some examples:

1. If $h = \frac{1}{2}$ the multiplet is composed of 4 maseless particles:

$$\left\{\,|\,p_\mu, -\frac{1}{2}\,\rangle\ ,\ |\,p_\mu, 0\,\rangle\ ,\ |\,p_\mu, \frac{1}{2}\,\rangle\,\right\}$$

   which represents a Weyl spinors and a complex scalar particle (double multiplicity).

2. If $h = 1$, the multiplet is given by:

$$\left\{\,|\,p_\mu, -1\,\rangle\ ,\ |\,p_\mu, -\frac{1}{2}\,\rangle\ ,\ |\,p_\mu, \frac{1}{2}\,\rangle\ ,\ |\,p_\mu, 1\,\rangle\,\right\}$$

   which represets a Weyl spinor and a $s = 1$ maseless boson (such as the photon γ) couple.

3. The $h = 2$ case happens to be crucial for our purposes:

$$\left\{\,|\,p_\mu, -2\,\rangle\ ,\ |\,p_\mu, -\frac{3}{2}\,\rangle\ ,\ |\,p_\mu, \frac{3}{2}\,\rangle\ ,\ |\,p_\mu, 2\,\rangle\,\right\}$$

   This multiplets is composed by a $s = \frac{3}{2}$ fermion and a $s = 2$ boson couple, which are respectively the Rarita-Schwinger *gravitino* and the *graviton*. Much more on this later on.

3.2.3 Massive multiplets

Analogously to the derivation in subsection 1.1.2, we can boost the massive state $|\,p_\mu, j, j_3\,\rangle$ to the rest frame, where Equation 3.8 becomes:

$$\left\{Q_\alpha, \bar{Q}_{\dot\alpha}\right\} = 2\left(\sigma^\mu\right)_{\alpha\dot\alpha} P_\mu = 2m\mathbf{1}$$

This time we cannot ignore the second components of the spinors couples: as before, we can define the creation and annihilation operators, by rescaling the supercharges:

$$\begin{cases} a_\alpha \equiv \frac{Q_\alpha}{\sqrt{2m}} \\ a^\dagger_{\dot\alpha} \equiv \frac{Q_{\dot\alpha}}{\sqrt{2m}} \end{cases} \Longrightarrow \begin{cases} \left\{a_\alpha, a^\dagger_{\dot\alpha}\right\} = \sigma_{\alpha\dot\alpha} \\ \{a_\alpha, a_\alpha\} = \left\{a^\dagger_{\dot\alpha}, a_{\dot\alpha}\right\} = 0 \end{cases} \tag{3.15}$$

Considering $|\, p_\mu, j, j_3 \,\rangle$ to be annihilated by a_α, this algebra is fundamentally represented by the following multiplet:

$$\Big\{ \,|\, p_\mu, j, j_3 \,\rangle \;,\; a^\dagger_1 |\, p_\mu, j, j_3 \,\rangle \;,\; a^\dagger_2 |\, p_\mu, j, j_3 \,\rangle \;,\; a^\dagger_1 a^\dagger_2 |\, p_\mu, j, j_3 \,\rangle \,\Big\} \tag{3.16}$$

As in the maseless case, we study the helicity, but this time, by simply considering the $Spin(1, 3)$ representation the states belong to:

$$|\, p_\mu, j, j_3 \,\rangle \in \left(\frac{1}{2}\right) \Longrightarrow \begin{cases} a^\dagger_\alpha |\, p_\mu, j, j_3 \,\rangle \in (j) \otimes (\frac{1}{2}) = (j + \frac{1}{2}) \oplus (j - \frac{1}{2}) \\ a^\dagger_1 a^\dagger_2 |\, p_\mu, j, j_3 \,\rangle \in (j) \end{cases}$$

where the last statement holds since a bosonic operator can't mutate the spin of a particle. Finally the N=1 SUSY massive multiplet is given by:

$$\Big\{ \,|\, p_\mu, j, j_3 \,\rangle \;,\; |\, p_\mu, j - \frac{1}{2}, j_3 \,\rangle \;,\; |\, p_\mu, j + \frac{1}{2}, j_3 \,\rangle \,\Big\} \tag{3.17}$$

where the first state has multiplicity 2, according to the null Witten index requirement.

For example, if $j = \frac{1}{2}$ the multiplet is given by:

$$\Big\{ |\, p_\mu, 0 \,\rangle \;,\; |\, p_\mu, \frac{1}{2} \,\rangle \;,\; |\, p_\mu, 1 \,\rangle \Big\} \tag{3.18}$$

Now that we derived the multiplets, it is time to switch viewpoint, viewing the Poincarè supergroup as a differentiable supermanifold, and introduce a, to say the least, peculiar formalism.

3.3 Superfields formalism

By the end of this section, we will be able to give a meaning to the *Wess-Zumino Model* Lagrangian:

$$S_{WZ} = \int d^4x\, d^4\theta\, \Phi^\dagger \Phi = \int d^4x \left(\partial_\mu \phi^\dagger \partial^\mu \phi - i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi + F^\dagger F\right) \tag{3.19}$$

representic the most basic non interacting supersymmetric theory, and being a crucial step towards Supergravity.

For such purpose, we need to introduce the concept of *superfield*, as a field defined on a (for our purposes) 6-dimensional manifold, extending $\mathbb{R}^{1,3}$.

3.3.1 Superspace

Conceptually, superspace $\mathbb{R}^{1,3|4}$ is a manifold extending the Minkowski space-time to include both commuting and anticommuting dimensions, meaning that:

$$\left(x^\mu, \theta_\alpha, \bar{\theta}^{\dot{\alpha}}\right) \in \mathbb{R}^{1,3|4} \tag{3.20}$$

where $x^\mu \in \mathbb{R}^{1,3}$, and θ_α and $\bar{\theta}^{\dot{\alpha}}$ are Grassmann-valued spinors.

A Group theoretical point of view

Given a Lie group G, we can define the fundamental representation as its action on the vector space which most naturally accomodates it. For instance, the Poincaré group naturally acts on the Minkowski space.

A less immediate choice is the *coset space*, which is a manifold defined as:

$$\mathcal{M} = G/H \quad : \iff \quad g \cdot h \in \mathcal{M} \quad \forall\, h \in H \ ,\ g \in G \tag{3.21}$$

For example, $SU(2)$ is, as a manifold S^3. Considering now the subgroup $H = U(1) \subset SU(2)$, we get the coset space $S^3/U(1) \sim S^2$.

From this point of view, $\mathbb{R}^{1,3}$ is the coset of $ISO(3,1)$ over $SO(3,1)$:

$$\mathbb{R}^{1,3} = ISO(3,1)/SO(3,1) \tag{3.22}$$

Indeed, the Poincaré group is generated by both $M_{\mu\nu}$ and P_μ, while the Lorentz group is generated only by $M_{\mu\nu}$, implying that the coset can be parametrised by the coordinates with respect to the translation generators P_μ (x^μ), defining $x^\mu \in \mathbb{R}^{1,3}$.

We can proceed in a similiar fashion for the SUSY case: a general transformation of the super-Poincaré group can be expressed as:

$$g(\omega, a, \theta, \bar{\theta}) = exp\left(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} + ia_\mu P^\mu + i\theta^\alpha Q_\alpha + i\bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}\right) \tag{3.23}$$

Thus, if we consider the coset of the coset of the super-Poincaré group over the Lorentz group:

$$\mathbb{R}^{1,3|4} = \frac{ISO(1,3|4)}{SO(1,3)} \tag{3.24}$$

we get a parametrization of the manifold in terms of the coordinates x^μ and the Grassmann-valued spinor coordinates θ_α and $\bar{\theta}^{\dot{\alpha}}$.

SUSY transformations on Superspace

Now we can demonstrate how SUSY transformations act on $R^{1,3|4}$. We will work in the coset representation, and make use of the *Baker-Campbell-Hausdorff* formula:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\dots} \quad \forall\ A,B \in \mathfrak{g} \tag{3.25}$$

A generic element $\{g\}$ of the super-Poincaré group in the coset representation can be factorised as:

$$g(\omega,x,\theta,\bar{\theta}) = \tilde{g}(x,\theta,\bar{\theta})h(\omega) \quad , \quad h(\omega) \in SO(1,3) \quad , \quad \tilde{g}(x,\theta,\bar{\theta}) \in \mathbb{R}^{1,3|4} \tag{3.26}$$

and thus a vector of the coset in this representation can be expressed in the form of $\tilde{g}(x,\theta,\bar{\theta})$, up to a bijection. This is why the BCH formula will be particularly useful in this case.

We thus study at the action of each generator in Equation 3.4 on $\tilde{g}$:

1. ***Translations***:

$$U(a^\mu) = \exp\{ia_\mu P^\mu\}$$

$$U(a)\tilde{g}(x,\theta,\bar{\theta}) \overset{3.25}{=} \exp\left\{i\left(ia_\mu P^\mu + x_\mu P^\mu + \theta_\alpha Q^\alpha + \bar{\theta}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}} + \frac{1}{2}\left[a_\mu P^\mu, x_\mu P^\mu + \theta_\alpha Q^\alpha + \bar{\theta}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}}\right]\right)\right\} =$$

$$\overset{3.6}{=} \exp\left\{i\left(ia_\mu P^\mu + x_\mu P^\mu + \theta_\alpha Q^\alpha + \bar{\theta}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}}\right)\right\} = \tilde{g}(x+a,\theta,\bar{\theta})$$

which is the familiar action of the translation operator on $\mathbb{R}^{1,3}$.

2. ***Lorentz transformations***:

$$K\left(\Omega_{\mu\nu}\right) = \exp\left\{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right\}$$

$$K\left(\Omega_{\mu\nu}\right) g\left(x,\theta,\bar{\theta}\right) = \exp\left\{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right\} exp\left\{ix_\sigma P^\sigma + i\theta^\alpha Q_\alpha + i\bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}\right\} = *$$

$$\left[\omega_{\mu\nu}M^{\mu\nu}, x_\sigma P^\sigma\right] = \omega_{\mu\nu}\left[M^{\mu\nu}, x_\sigma P^\sigma\right] = \omega_{\mu\nu}\left(x_\sigma\left[M^{\mu\nu}, P^\sigma\right] + \left[M^{\mu\nu}, x_\sigma\right]P^\sigma\right) =$$

$$= i\omega_{\mu\nu}\left[x_\sigma\left(\eta^{\sigma\nu}P^\mu - \eta^{\sigma\mu}P^\nu\right) + \left(x^\mu\eta^\nu{}_\sigma - x^\nu\eta^\mu{}_\sigma\right)P^\sigma\right] =$$

$$= i\omega_{\mu\nu}\left[x^\nu P^\mu - x^\mu P^\nu + P^\nu x^\mu - x^\nu P^\mu\right] = 0$$

$$\left[\omega_{\mu\nu}M^{\mu\nu}, \theta^\alpha Q_\alpha\right] = \omega_{\mu\nu}\left[M^{\mu\nu}, \theta^\alpha Q_\alpha\right] = \omega_{\mu\nu}\left(\theta^\alpha\left[M^{\mu\nu}, Q_\alpha\right] + \left[M^{\mu\nu}, \theta^\alpha\right]Q_\alpha\right) =$$

$$= \omega_{\mu\nu}\left[\theta^\alpha(\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta - (\sigma^{\mu\nu})^\alpha{}_\beta\theta^\beta Q_\alpha\right] = 0$$

$$* = \exp\left\{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} + ix_\sigma P^\sigma + i\theta^\alpha Q_\alpha + i\bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}\right\} = \tilde{g}\left(\Lambda x, S\left[\Lambda\right]\theta, S\left[\Lambda\right]^\dagger\bar{\theta}\right)$$

3. ***Supercharges***:

$$V(\xi,\bar{\xi}) = \exp\left\{i\xi_\alpha Q^\alpha + i\bar{\xi}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}}\right\}$$

$$V(\xi,\bar{\xi})\tilde{g}(x,\theta,\bar{\theta}) = \exp\left\{i\xi^\alpha Q_\alpha + i\bar{\xi}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}\right\}\exp\left\{ix_\mu P^\mu + i\theta_\alpha Q^\alpha + i\bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}\right\} = *$$

$$\left[\bar{\theta}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}}, \xi_\alpha Q^\alpha\right] = \bar{\theta}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}}\xi_\alpha Q^\alpha - \xi_\alpha Q^\alpha \bar{\theta}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}} = \xi_\alpha Q^\alpha \bar{Q}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} + \xi_\alpha Q^\alpha \bar{Q}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} =$$

$$\xi^\alpha\left\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\right\}\bar{\theta}^{\dot{\alpha}} \overset{3.8}{=} 2\left(\xi^\alpha \sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\right)P_\mu$$

$$* = \exp\left\{i\xi^\alpha Q_\alpha + i\bar{\xi}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}} + ix_\mu P^\mu + i\theta_\alpha Q^\alpha + i\bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}} + i\left(\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}} - \xi^\alpha \sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\right)P_\mu\right\} =$$

$$= \tilde{g}\left(x + i\theta\sigma\bar{\xi} - i\xi\sigma\bar{\theta}, \theta + \xi, \bar{\theta} + \bar{\xi}\right)$$

Thanks to the coset representation, we were able to derived the SUSY transformations of superspacial coordinates:

$$\begin{cases} \delta x^\mu = a^\mu + i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}} - i\xi^\alpha \sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} \\ \delta\theta^\alpha = \xi^\alpha \\ \delta\bar{\theta}_{\dot{\alpha}} = \bar{\xi}_{\dot{\alpha}} \end{cases} \tag{3.27}$$

Now that we formalised the concept of supermanifold, we can begin exploring some interesting algebraic structures defined on it.

3.3.2 Superfields

For our purposes, a superfield $Y(x,\theta,\bar{\theta})$ is a complex-valued scalar function on $\mathbb{R}^{1,3|4}$.

Since $\theta, \bar{\theta}$ are Grassmann variables, if we Taylor expand Y with respect to them, this truncates at quadratic terms, allowing to obtain an explicit form of the superfield in terms of its components:

$$\begin{aligned}
Y(x,\theta,\bar{\theta}) &= \sum_{n,m=0}^{\infty} \frac{1}{n!\,m!}\frac{\partial^{n+m}Y}{\partial\theta^n\partial\bar{\theta}^m}(\theta^\alpha)^n\left(\bar{\theta}_{\dot{\alpha}}\right)^m\Big|_{\theta=\bar{\theta}=0} = \sum_{n,m=0}^{2} \frac{1}{n!\,m!}\frac{\partial^{n+m}Y}{\partial\theta^n\partial\bar{\theta}^m}(\theta^\alpha)^n\left(\bar{\theta}_{\dot{\alpha}}\right)^m\Big|_{\theta=\bar{\theta}=0} = \\
&= Y(x)\Big|_{(0,0)} + \frac{\partial Y}{\partial\theta^\alpha}\Big|_{(0,0)}\theta^\alpha + \frac{\partial Y}{\partial\bar{\theta}_{\dot{\alpha}}}\Big|_{(0,0)}\bar{\theta}_{\dot{\alpha}} + \frac{\partial^2 Y}{\partial\theta^\alpha\partial\bar{\theta}_{\dot{\alpha}}}\Bigg|_{(0,0)}\theta^\alpha\bar{\theta}_{\dot{\alpha}} + \frac{1}{2}\frac{\partial^2 Y}{\partial\theta^\alpha\partial\theta^\beta}\Big|_{(0,0)}\theta^\alpha\theta^\beta + \\
&+ \frac{1}{2}\frac{\partial^2 Y}{\partial\bar{\theta}_{\dot{\alpha}}\partial\bar{\theta}_{\dot{\beta}}}\Big|_{(0,0)}\bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} + \frac{1}{2}\frac{\partial^2 Y}{\partial\theta^\alpha\partial\theta^\beta\partial\bar{\theta}_{\dot{\alpha}}}\Big|_{(0,0)}\theta^\alpha\theta^\beta\bar{\theta}_{\dot{\alpha}} + \frac{1}{2}\frac{\partial^2 Y}{\partial\theta^\alpha\partial\bar{\theta}_{\dot{\alpha}}\partial\bar{\theta}_{\dot{\beta}}}\Big|_{(0,0)}\theta^\alpha\bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} + \\
&+ \frac{1}{4}\frac{\partial^2 Y}{\partial\theta^\alpha\partial\theta^\beta\partial\bar{\theta}_{\dot{\alpha}}\partial\bar{\theta}_{\dot{\beta}}}\Big|_{(0,0)}\theta^\alpha\theta^\beta\bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = \\
&= Y\Big|_{(0,0)} + \partial_\alpha Y\Big|_{(0,0)}\theta^\alpha + \bar{\partial}^{\dot{\alpha}}Y\Big|_{(0,0)}\bar{\theta}_{\dot{\alpha}} + \partial_\alpha\bar{\partial}^{\dot{\alpha}}Y\Big|_{(0,0)}\theta^\alpha\bar{\theta}_{\dot{\alpha}} + \frac{1}{2}\partial_\alpha\partial_\beta Y\Big|_{(0,0)}\theta^\alpha\theta^\beta + \frac{1}{2}\bar{\partial}^{\dot{\alpha}}\bar{\partial}^{\dot{\beta}}Y\Big|_{(0,0)}\bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} + \\
&+ \frac{1}{2}\partial_\alpha\partial_\beta\bar{\partial}^{\dot{\alpha}}Y\Big|_{(0,0)}\theta^\alpha\theta^\beta\bar{\theta}_{\dot{\alpha}} + \frac{1}{2}\partial_\alpha\bar{\partial}^{\dot{\alpha}}\bar{\partial}^{\dot{\beta}}Y\Big|_{(0,0)}\theta^\alpha\bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} + \frac{1}{4}\partial_\alpha\partial_\beta\bar{\partial}^{\dot{\alpha}}\bar{\partial}^{\dot{\beta}}Y\Big|_{(0,0)}\theta^\alpha\theta^\beta\bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} \equiv \\
&\equiv \phi(x) + \theta^\alpha\psi_\alpha(x) + \bar{\theta}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}(x) + \theta^2 M(x) + \bar{\theta}^2 N(x) + \theta^\alpha\bar{\theta}^{\dot{\alpha}}V_{\alpha\dot{\alpha}} + \theta^2\bar{\theta}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} + \bar{\theta}^2\theta^\alpha\rho_\alpha(x) + \theta^2\bar{\theta}^2 D(x)
\end{aligned} \tag{3.28}$$

Although this is a significant step forward, looking back at the SUSY representation theory, there are way more fields than we would've expected; but this shouldn't be surprising at all: after all, we saw the Dirac spinor to be built from a left hand and a right hand Weyl spinor.

Analogously, Y transforms in a reducible representation of the SUSY algebra, and we need some criteria to constraint it, and evaluate the multiplets.

SUSY transformations on Superfields

While in section 3.3.1 we considered U as belonging to the coset representation of $ISO(1,3|4)$, we are now studying the algebra representation on particle states, and thus we have to switch the point of view, viewing both Y and U as operators acting on an Hilbert space.

Hence, we leverage the transformations properties of operators on the Hilbert space to derive the superfields' ones:

1. ***Translations***:

$$U = \exp\{ia^\mu P_\mu\}$$

$$Y(x+a,\theta,\bar\theta) = UY(x,\theta,\bar\theta)U^\dagger \Longrightarrow Y(x) + a\partial Y(x) + \mathcal{O}(a^2) = \left[1 + iaP + \mathcal{O}(a^2)\right] Y \left[1 - iaP + \mathcal{O}(a^2)\right]^\dagger =$$

$$= Y - iYaP + iaPY + \mathcal{O}(a^2) \Longrightarrow \partial Y = iPY - iYP \Longleftrightarrow [P_\mu, Y] = -i\partial_\mu Y$$

2. ***Lorentz transformations***:

$$K(\Omega_{\mu\nu}) = \exp\left\{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right\}$$

$$KY(x,\theta,\bar\theta)K^\dagger = Y(\Lambda x, S[\Lambda]\theta, S[\Lambda]^\dagger\bar\theta) \Longrightarrow$$

$$\left(1 - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right) Y(x,\theta,\bar\theta)\left(1 + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right) = Y + \frac{\partial Y}{\partial x}\delta x + \frac{\partial Y}{\partial \theta}\epsilon + \frac{\partial Y}{\partial\bar\theta}\bar\epsilon \Longrightarrow$$

$$\Longrightarrow Y + \frac{i}{2}Y\omega_{\mu\nu}M^{\mu\nu} - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}Y = Y - \frac{i}{2}\partial_\mu Y\left(\omega_{\mu\nu}M^{\mu\nu}\right)_\rho^{\ \sigma} x_\sigma +$$

$$+ \partial_\theta Y\left(-\frac{i}{2}\left(\omega_{\mu\nu}\sigma^{\mu\nu}\right)_\alpha^{\ \beta}\theta_\beta\right) + \bar\partial_{\bar\theta} Y\left(-\frac{i}{2}\left(\omega_{\mu\nu}\bar\sigma^{\mu\nu}\right)_{\dot\alpha}^{\ \dot\beta}\bar\theta_{\dot\beta}\right) \Longrightarrow$$

$$\Longrightarrow YM^{\mu\nu} - M^{\mu\nu}Y = \partial_\mu Y\left(M^{\mu\nu}\right)_\rho^{\ \sigma} x_\sigma + \partial_\theta Y\left(-\left(\sigma^{\mu\nu}\right)_\alpha^{\ \beta}\theta_\beta\right) + \bar\partial_{\bar\theta} Y\left(-\left(\bar\sigma^{\mu\nu}\right)_{\dot\alpha}^{\ \dot\beta}\bar\theta_{\dot\beta}\right) \Longrightarrow$$

$$\Longrightarrow [Y, M^{\mu\nu}] = \partial_\mu Y\left(M^{\mu\nu}\right)_\rho^{\ \sigma} x_\sigma + \partial_\theta Y\left(-\left(\sigma^{\mu\nu}\right)_\alpha^{\ \beta}\theta_\beta\right) + \bar\partial_{\bar\theta} Y\left(-\left(\bar\sigma^{\mu\nu}\right)_{\dot\alpha}^{\ \dot\beta}\bar\theta_{\dot\beta}\right)$$

3. ***Supercharges***:

$$V(\epsilon,\bar{\epsilon}) = \exp\left\{i\epsilon^{\alpha}Q_{\alpha} + i\bar{\epsilon}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}\right\}$$

$$VY(x,\theta,\bar{\theta})V^{\dagger} = Y(x + i\theta\sigma^{\mu}\bar{\epsilon} - i\epsilon\sigma^{\mu}\bar{\theta}, \theta+\epsilon, \bar{\theta}+\bar{\epsilon}) \Longrightarrow$$

$$\left(1 + i\epsilon^{\alpha}Q_{\alpha} + i\bar{\epsilon}_{\dot{\alpha}}\bar{Q}^{\alpha}\right) Y \left(1 - i\epsilon^{\alpha}Q_{\alpha} - i\bar{\epsilon}_{\dot{\alpha}}\bar{Q}^{\alpha}\right) = Y + \frac{\partial Y}{\partial x}\delta x + \frac{\partial Y}{\partial \theta}\epsilon + \frac{\partial Y}{\partial \bar{\theta}}\bar{\epsilon} + \mathcal{O}(\epsilon^2) + \mathcal{O}(\bar{\epsilon}^2) \Longrightarrow$$

$$\Longrightarrow Y - iY\epsilon Q - iY\bar{\epsilon}\bar{Q} + i\epsilon QY + i\bar{\epsilon}\bar{Q}Y = Y + \partial_{\mu}Y\left(i\theta\sigma^{\mu}\bar{\epsilon} - i\epsilon\sigma^{\mu}\bar{\theta}\right) + \epsilon\partial_{\theta}Y + \bar{\epsilon}\bar{\partial}_{\bar{\theta}}Y \Longleftrightarrow$$

$$\Longrightarrow [\epsilon Q, Y] + \left[\bar{\epsilon}\bar{Q}, Y\right] = -\theta\sigma^{\mu}\bar{\epsilon}\partial_{\mu}Y + \epsilon\sigma^{\mu}\bar{\theta}\partial_{\mu}Y + i\epsilon\partial_{\theta}Y + i\bar{\epsilon}\bar{\partial}_{\bar{\theta}}Y \Longrightarrow$$

$$\Longrightarrow \begin{cases} [Q,Y] = \left(\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{\mu} + i\frac{\partial}{\partial\theta^{\alpha}}\right)Y \\ \left[\bar{Q},Y\right] = \left(\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu} + i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\right)Y \end{cases}$$

Now we can define the following set of operators:

$$\begin{cases} \mathcal{P}_{\mu} = -i\partial_{\mu} \\ \mathcal{Q}_{\alpha} = -i\partial_{\alpha} - \sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{\mu} \\ \bar{\mathcal{Q}}_{\dot{\alpha}} = i\bar{\partial}_{\dot{\alpha}} + \theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu} \end{cases} \tag{3.29}$$

which we can check to form another representation of the SUSY algebra, now acting on superfields:

$$\left\{\mathcal{Q}_{\alpha},\bar{\mathcal{Q}}_{\dot{\alpha}}\right\} = \left(-i\partial_{\alpha} - \sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{\mu}\right)\left(i\bar{\partial}_{\dot{\alpha}} + \theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu}\right) + \left(i\bar{\partial}_{\dot{\alpha}} + \theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu}\right)\left(-i\partial_{\alpha} - \sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{\mu}\right) =$$

$$= \partial_{\alpha}\bar{\partial}_{\dot{\alpha}} - i\partial_{\alpha}\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu} - i\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{\mu}\bar{\partial}_{\dot{\alpha}} - \sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{\mu}\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu} +$$

$$+ \bar{\partial}_{\dot{\alpha}}\partial_{\alpha} - i\bar{\partial}_{\dot{\alpha}}\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{\mu} - i\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu}\partial_{\alpha} - \theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu}\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{\mu} =$$

$$= \partial_{\alpha}\bar{\partial}_{\dot{\alpha}} - i\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu} + i\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu} - \sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{\mu}\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu} -$$

$$- \partial_{\alpha}\bar{\partial}_{\dot{\alpha}} + i\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu} + i\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu} + \sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{\mu}\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu} =$$

$$= 2i\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu} = 2\sigma^{\mu}_{\alpha\dot{\alpha}}P_{\mu}$$

$$\{Q_{\alpha},Q_{\beta}\} = \left(-i\partial_{\alpha} - \sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{\mu}\right)\left(-i\partial_{\beta} - \sigma^{\mu}_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_{\mu}\right) + \left(-i\partial_{\beta} - \sigma^{\mu}_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_{\mu}\right)\left(-i\partial_{\alpha} - \sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{\mu}\right) =$$

$$= -\partial_{\alpha}\partial_{\beta} + i\partial_{\alpha}\sigma^{\mu}_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_{\mu} + i\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{\mu}\partial_{\beta} + \sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{\mu}\sigma^{\mu}_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_{\mu} +$$

$$- \partial_{\beta}\partial_{\alpha} + i\partial_{\beta}\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{\mu} + i\sigma^{\mu}_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_{\mu}\partial_{\alpha} + \sigma^{\mu}_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_{\mu}\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{\mu} =$$

$$= -\partial_{\alpha}\partial_{\beta} + i\partial_{\alpha}\sigma^{\mu}_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_{\mu} + i\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{\mu}\partial_{\beta} + \sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\sigma^{\mu}_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_{\mu}\partial_{\mu} +$$

$$+ \partial_{\alpha}\partial_{\beta} - i\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{\mu}\partial_{\beta} - i\partial_{\alpha}\sigma^{\mu}_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_{\mu} - \sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\sigma^{\mu}_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_{\mu}\partial_{\mu} = 0 = \left\{\bar{Q}_{\dot{\alpha}},\bar{Q}_{\dot{\beta}}\right\}$$

The infinitesimal variation of Y is defined to be:

$$\delta Y = i\left[\epsilon Q + \bar{\epsilon}\bar{Q}, Y\right] = i\left(\epsilon \mathcal{Q} + \bar{\epsilon}\bar{\mathcal{Q}}\right) Y \tag{3.30}$$

By expanding this variation, one can get the full set of variations under global SUSY, for all the fields. Since this is as huge as straightforward, we will omit the full calculation. In the following, we will not need to know the variations of all the fields appearing in a superfield: sure enough, looking back at 3.19, those who matter for our purposes are the scalar and the Weyl spinor fields' ones:

$$\begin{cases} \delta\phi = \epsilon\psi + \bar{\epsilon}\bar{\chi} \\ \delta\psi = 2\epsilon M + (\sigma^\mu \bar{\epsilon})\left(i\partial_\mu \phi + V_\mu\right) \\ \delta\bar{\chi} = 2\bar{\epsilon}N - (\epsilon\sigma^\mu)\left(i\partial_\mu\phi - V_\mu\right) \end{cases} \tag{3.31}$$

3.3.3 Chiral Superfields

As we've seen by Taylor expanding Y with respect to the Grassman variables, since it transforms in a reducible representation of the super-Poincarè group, there appear to be way too many fields, compared to those forming a multiplet.

In analogy with the Dirac spinor, we can imagine to construct some sort of chiral representation, to reveal more fundamental structures.

The canonical way to proceed is to introduce the *covariant derivatives* of the SUSY algebra:

$$\begin{cases} \mathcal{D}_\alpha \equiv \partial_\alpha + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \\ \bar{\mathcal{D}}_{\dot{\alpha}} \equiv -\bar{\partial}_{\dot{\alpha}} - i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \end{cases} \tag{3.32}$$

If we think about the basics of differential geometry on a manifold $\mathcal{M}$ in General Relativity, a covariant derivative is constructed in order to parallelly transport geometrical structures with respect to the diffeomorphisms group $Diff\left(\mathcal{M}\right)$ along a given curve.

In a similiar fashion, we hereby want to define differentiation operations which can parallelly transport a superfield by preserving its supersymmetric transformation properties.

In fact, we can easily chack these operators to anticommute with the supercharges:

$$\{\mathcal{D}_\alpha, \mathcal{Q}_\beta\} = \left\{\mathcal{D}_\alpha, \bar{\mathcal{Q}}_{\dot{\beta}}\right\} = \{\bar{\mathcal{D}}_{\dot{\alpha}}, \mathcal{Q}_\beta\} = \left\{\bar{\mathcal{D}}_{\dot{\alpha}}, \bar{\mathcal{Q}}_{\dot{\beta}}\right\} = 0 \tag{3.33}$$

and thus with an infinitesimal SUSY transformation:

$$\begin{cases} \delta Y = i\left(\epsilon\mathcal{Q} + \bar{\epsilon}\mathcal{Q}\right) Y \\ \left[\epsilon\mathcal{Q} + \bar{\epsilon}\bar{Q}, \mathcal{D}_\alpha\right] = \left[\epsilon\mathcal{Q} + \bar{\epsilon}\bar{Q}, \bar{\mathcal{D}}_{\dot{\alpha}}\right] = 0 \end{cases} \tag{3.34}$$

Importantly, this implies that both $\mathcal{D}_\alpha Y$ and $\bar{\mathcal{D}}_{\dot{\alpha}} Y$ are also superfields, and we can therefore consider to set some conditions on Y thanks to this property.

Specifically, we can impose the following *chirality constraints*:

$$\begin{cases} \bar{\mathcal{D}}_{\dot{\alpha}}\Phi = 0 \quad \text{chiral} \\ D_\alpha \Psi = 0 \quad \text{anti-chiral} \end{cases} \tag{3.35}$$

The multiplet structure can finally be mirrored on a superfield.

To simplify the calculation, we will switch to an appropriate coordinate system:

$$\bar{\mathcal{D}}_{\dot{\alpha}}\Phi\left(x^\mu + i\theta\sigma^\mu\bar{\theta},\theta,\bar{\theta}\right) \equiv \bar{\mathcal{D}}_{\dot{\alpha}}\Phi\left(y^\mu,\theta,\bar{\theta}\right) = 0$$

$$\bar{D}_{\bar{\alpha}}y^\mu = \left(-\bar{\partial}_{\dot{\alpha}} - i\theta^\alpha\sigma^\nu_{\alpha\dot{\alpha}}\partial_\nu\right)\left(x^\mu + i\theta^\beta\sigma^\mu_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}\right) =$$

$$= -\bar{\partial}_{\dot{\alpha}}x^\mu - i\bar{\partial}_{\dot{\alpha}}\left(\theta^\beta\sigma^\mu_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}\right) - i\theta^\alpha\sigma^\nu_{\alpha\dot{\alpha}}\partial_\nu x^\mu + \theta^\alpha\sigma^\nu_{\alpha\dot{\alpha}}\partial_\nu\left(\theta^\beta\sigma^\mu_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}\right) =$$

$$= -i\bar{\partial}_{\dot{\alpha}}\left(\theta^\beta\sigma^\mu_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}\right) - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}} = i\theta^\beta\sigma^\mu_{\beta\dot{\beta}}\delta^{\dot{\alpha}}_{\dot{\beta}}\bar{\theta}^{\dot{\beta}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}} = 0$$

$$\bar{D}_{\dot{\alpha}}\theta_\beta = \left(-\bar{\partial}_{\dot{\alpha}} - i\theta^\alpha\sigma^\nu_{\alpha\dot{\alpha}}\partial_\nu\right)\theta_\beta = -\partial_{\dot{\alpha}}\theta_\beta = 0$$

$$\bar{D}_{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \left(-\bar{\partial}_{\dot{\alpha}} - i\theta^\alpha\sigma^\nu_{\alpha\dot{\alpha}}\partial_\nu\right)\bar{\theta}^{\dot{\beta}} = -\bar{\partial}_{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \delta_{\dot{\alpha}}^{\ \dot{\beta}}$$

Thus the chirality constraint reduces to requesting the superfield to depend only on y^μ and θ, meaning that in this frame, we can literally drop all terms in the expansion 3.28 which do not obey such condition, obtaining:

$$\bar{\mathcal{D}}_{\dot{\alpha}}\Phi(x,\theta,\bar{\theta}) = 0 \Longleftrightarrow \Phi = \Phi(y^\mu,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y) \tag{3.36}$$

where $\sqrt{2}$ is a convention.

By then expanding Y in the originary frame, we finally get:

$$\Phi(x,\theta,\bar{\theta}) = \phi(x) + \sqrt{2}\theta\psi(x) + \theta^2 F(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi(x)\sigma^\mu\bar{\theta} - \frac{1}{4}\theta^2\bar{\theta}^2\Box\phi(x) \tag{3.37}$$

This looks much more similiar to what we've found studying the multiplets, except for an additional field, often referred to as an *auxiliary field*, and we will deal with it very soon.

Of course, a symmetrical analysis can be carried out for the anti-chirality constraint.

3.3.4 Actions over superspace

If we have a superfield $K(x,\theta,\bar{\theta})$ which is functions of other superfields, we can construct an action of the form:

$$S = \int d^4x\, d^4\theta\, K(x,\theta,\bar{\theta}) \tag{3.38}$$

which is real if K is real, meaning $K = K^\dagger$.

This must be invariant under SUSY transformations, and we want to check wheter it is or not:

$$\delta S = \int d^4x\, d^4\theta\, \delta K \overset{3.30}{=} \int d^4x\, d^4\,\theta \left[\xi^\alpha\left(\partial_\alpha K - i\sigma\bar{\theta}\partial_\mu K\right) + \left(-\bar{\partial}_{\dot{\alpha}}K + i\theta^\alpha\sigma^\mu\partial_\mu K\right)\bar{\xi}^{\dot{\alpha}}\right]$$

$$= \int d^4x \int d^2\theta \int d^2\bar{\theta}\, \left[\xi^\alpha\left(\partial_\alpha K - i\sigma\bar{\theta}\partial_\mu K\right) + \left(-\bar{\partial}_{\dot{\alpha}}K + i\theta^\alpha\sigma^\mu\partial_\mu K\right)\bar{\xi}^{\dot{\alpha}}\right]$$

But, for the Taylor expansion of a superfield in 3.28, K is at most second order in $\theta,\bar{\theta}$, so, since an integration over superspace with respect to a Grassmann variable is equivalent to a differentiation, the terms containing

$\partial_\alpha K$ or $\bar{\partial}_{\dot{\alpha}} K$ vanish, while the terms differentiated with respect to x^μ give at most boundary terms, which are assumed to vanish, as usual.

This derivation demonstrates that any action of the form 3.38 is necessarily SUSY invariant.

3.3.5 The free Wess-Zumino model

We reached a point where we are more than able to derive the action in Equation 3.19: since we necessarily have to obtain a real lagrangians, and it must involve both chiral and antichiral superfields, as proposed in the ansatz 3.38.

The simplest possible will be suitable for us:

$$S_{chiral} = \int d^4x\, d^4\theta\, \Phi^\dagger \Phi =$$

$$\int d^4x d^4\theta \left[\phi^\dagger(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) + \bar{\theta}^2 F^\dagger(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi^\dagger(x) + \frac{i}{\sqrt{2}}\bar{\theta}^2\theta\partial_\mu\bar{\psi}(x)\sigma^\mu\theta - \frac{1}{4}\theta^2\bar{\theta}^2\Box\phi^\dagger(x) \right] \cdot$$

$$\cdot \left[\phi(x) + \sqrt{2}\theta\psi(x) + \theta^2 F(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi(x)\sigma^\mu\bar{\theta} - \frac{1}{4}\theta^2\bar{\theta}^2\Box\phi(x) \right]$$

This calculation can be straightforwardly carried out thanks to the mathematical tools that we've introduced in the previous sections. Due to its remarkable length, we will directly discuss the important result:

$$S_{WZ} = \int d^4x\, d^4\theta\, \Phi^\dagger \Phi = \int d^4x \left(\partial_\mu\phi^\dagger\partial^\mu\phi - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + F^\dagger F \right) \tag{3.39}$$

The Lagrangian includes the standard kinetic terms for a real scalar ϕ and a Weyl spinor ψ fields, plus an additional term in the auxiliary field F, which is not kinetic, meaning that, when quantised, it doesn't give rise to any particle state.

This is the action of the *Wess-Zumino model*, and represents the achievement we promised at the beginning of this section. For now, we can shelve it in our backpack, and come back to it in the last chapter, where we'll exhibit the necessity of gravity when gauging SUSY.

Chapter 4

General Relativity as a Gauge Theory

After a long and intense digression on Supersymmetry, it is time to start thinking about overtaking an important step that we've disregarded up to now: locality.

It is well known that General Relativity is built upon the tensorial formalism of differential geometry, and describes the effect of gravity as a matter of geometry of a curved Lorentzian manifold.

So the first question to ask ourselves is wheter the formalism we've developed is compatible with General Relativity, and, if not, if we shall we dismantle the entire apparatus to build a suitable one.

If there is something that dealing with other people taught me, beside ..., is that most of the times a compromise is the way to go: locality in Quantum Field Theory is achieved thanks to gauge transformations, essentially consisting in promoting symmetry parameters to function of space time, in the framework of Yang Mills theories, while in General Relativity with the tensiorial formalism of differential geometry.

Thus..., why not reinterpreting gravity as a gauge theory? Simple as that...

Allow me to add other ... to remark the disconcerting immediacy of this solution.

Reinterpreting gravity as a gauge theory will keep us busy during the first portion of this chapter.

If we proceed along this way, we will most likely bump upon the existance of a *gauge field*, transforming in some kind of tensorial representation of the Lorentz group: this way we will discover the *graviton*.

But Supersymmetry taught us that the graviton multiplet is complemented by the *gravitino*, which is a spinor. Thus, we need a formalism apt to describe spinors subjected to gravity: this is the *tetrads formalism* of General Relativity, and it will have us engaged throughout the whole second part of the chapter.

4.1 Gravity as a gauge theory

Gauging gravity might seem a considerable endeavor, so we want to start from the very basis, by recalling the General Relativity postulates:

1. **Principle of General Covariance**: Laws of physics are valid in any reference frame.
2. **Equivalence Principle**: In any point of the space time, it is possible to define an inertial frame, which is that of a free falling observer.

In the canonical formalism of GR, these can be redrafted under a more geometrical viewpoint:

1. **Principle of General Covariance**: Physical equations are covariant with respect to the diffeomorphism group on the space-time manifold.
2. **Equivalence Principle**: There exists a gaussian reference frame of the tangent space at any point of the manifold.

But we can push even further and, rather bravely, unify them under a unique postulate:

Gravity is the gauge theory of the group:

$$SO(1,3) \ltimes Diff(3,1) \tag{4.1}$$

on the fiber bundle $T_P\mathcal{M} \otimes \mathcal{M}\ \forall P \in \mathcal{M}$.

Let's see what this implies by studying the small perturbations of the metric around the minkowskian one $\eta^{\mu\nu}$.

4.1.1 Linearised theory

The $SO(1,3)$ symmetry of $T_P(\mathcal{M})$ at any point allows us to continuously perturb the metric $g^{\mu\nu}$ around a gaussian frame:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad , \quad |h_{\mu\nu}| \ll 1 \tag{4.2}$$

From this, we can expand the other fundamental metrical structures to linear order:

- ***Metric connection***:

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} g^{\gamma\lambda} \left(\partial_\alpha g_{\lambda\beta} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta}\right) =$$

$$= \frac{1}{2}\left(\eta^{\gamma\lambda} + h^{\gamma\lambda}\right)\left(\partial_\alpha\left(\eta_{\lambda\beta} + h_{\lambda\beta}\right) + \partial_\beta\left(\eta_{\alpha\lambda} + h_{\alpha\lambda}\right) - \partial_\lambda\left(\eta_{\alpha\beta} + h_{\alpha\beta}\right)\right) \simeq$$

$$\simeq \frac{1}{2}\eta^{\gamma\lambda}\left(\partial_\alpha h_{\lambda\beta} + \partial_\beta h_{\alpha\lambda} - \partial_\lambda h_{\alpha\beta}\right)$$

- ***Riemann tensor***:

$$R_{\sigma\rho\mu\nu} = \frac{1}{2}\left(\partial_\rho\partial_\mu g_{\sigma\nu} + \partial_\nu\partial_\sigma g_{\rho\mu} - \partial_\nu\partial_\mu g_{\sigma\rho} - \partial_\rho\partial_\sigma g_{\mu\nu}\right) \simeq$$

$$\simeq \frac{1}{2}\left(\partial_\rho\partial_\mu h_{\sigma\nu} + \partial_\nu\partial_\sigma h_{\rho\mu} - \partial_\nu\partial_\mu h_{\sigma\rho} - \partial_\rho\partial_\sigma h_{\mu\nu}\right)$$

- ***Ricci tensor***:

$$R^{\rho}_{\rho\mu\nu} \equiv R_{\mu\nu} = \frac{1}{2}\left(\partial^\rho\partial_\mu h_{\rho\nu} + \partial_\nu\partial^\rho h_{\rho\mu} - \partial_\nu\partial_\mu h^{\rho}{}_{\rho} - \partial_\rho\partial^\rho h_{\mu\nu}\right) =$$

$$= \frac{1}{2}\left(\partial^\rho\partial_\mu h_{\rho\nu} + \partial_\nu\partial^\rho h_{\rho\mu} - \partial_\nu\partial_\mu h^{\rho}{}_{\rho} - \Box h_{\mu\nu}\right)$$

- ***Ricci scalar***:

$$R^{\mu}{}_{\mu} \equiv R = \frac{1}{2}\left(\partial^\rho\partial^\mu h_{\rho\mu} + \partial_\mu\partial^\rho h^{\mu}{}_{\rho} - \partial_\mu\partial^\mu h - \Box h\right) = \partial^\rho\partial^\mu h_{\rho\mu} - \Box h$$

- ***Einstein tensor***:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{2}\left(\partial^\rho\partial_\mu h_{\rho\nu} + \partial_\nu\partial^\rho h_{\rho\mu} - \partial_\nu\partial_\mu h^{\rho}{}_{\rho} - \Box h_{\mu\nu} - \partial^\rho\partial^\sigma h_{\rho\sigma}\eta_{\mu\nu} + \eta_{\mu\nu}\Box h\right)$$

Being $T_{\mu\nu}$ the energy-momentum tensor, from the Field Equations in the presence of matter:

$$G_{\mu\nu} = 16\pi G_N T_{\mu\nu} \tag{4.3}$$

we can insert the linearised form of the right-hand side to obtain:

$$\partial^\rho \partial_\mu h_{\nu\rho} + \partial^\rho \partial_\nu h_{\mu\rho} - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h - \eta_{\mu\nu} \left(\partial^\rho \partial^\sigma h_{\rho\sigma} - \Box h\right) = 16\pi G_N T_{\mu\nu} \tag{4.4}$$

They are way too hefty to be actually fully evaluating the symmetry, and the reason is that we still haven't fixed the gauge.

As always, to exhibit a symmetry, it's appropriate to study the invariance of the action: in this case, the action is given by the *Fiertz-Pauli* one, which is the Einstein-Hilbert one, S_{EH}, to quadrtic order in h.

Let's derive it:

$$S_{EH} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} R \propto *$$

$$\sqrt{-g} = \sqrt{-det(\eta_{\mu\nu} + h_{\mu\nu})} = \sqrt{-det(\eta)} \cdot \exp\left\{\frac{1}{2} Tr\left\{\ln(1 + \eta^{-1} h)\right\}\right\} =$$

$$= \exp\left\{\frac{1}{2} Tr\left\{h^\mu_\mu - \frac{1}{2} h^\mu{}_\nu h^\nu{}_\mu + O(h^3)\right\}\right\} = 1 + \frac{1}{2} h + \frac{1}{8} h^2 - \frac{1}{4} h^{\mu\nu} h_{\mu\nu} + O(h^3)$$

$$* \propto \int \left(1 + \frac{1}{2} h + \frac{1}{8} h^2 - \frac{1}{4} h^{\mu\nu} h_{\mu\nu}\right) \left(\partial^\rho \partial^\mu h_{\rho\mu} - \Box h\right) = \int \left(\partial^\rho \partial^\mu h_{\rho\mu} - \Box h + \frac{1}{2} h \partial^\rho \partial^\mu h_{\rho\mu} - \frac{1}{2} h \Box h + \right.$$

$$\left. + \frac{1}{8} h^2 \partial^\rho \partial^\mu h_{\rho\mu} - \frac{1}{8} h^2 \Box h - \frac{1}{4} h^{\mu\nu} h_{\mu\nu} \partial^\rho \partial^\mu h_{\rho\mu} + \frac{1}{4} h^{\mu\nu} h_{\mu\nu} \Box h + O(h^3)\right) =$$

$$= \int \left(-\frac{1}{2} (\partial^\rho h) \partial^\mu h_{\rho\mu} + \frac{1}{2} \partial_\mu h \partial^\mu h - \frac{1}{4} h \partial^\rho h \partial^\mu h_{\mu\rho} + \frac{1}{4} h \partial^\mu h \partial_\mu h + \frac{1}{2} h^{\mu\nu} \partial^\rho h_{\mu\nu} \partial^\mu h_{\rho\mu} - \frac{1}{2} h^{\mu\nu} \partial^\rho h_{\mu\nu} \partial_\rho h\right) =$$

$$= \int \left(-\frac{1}{2} \partial^\rho h \partial^\mu h_{\rho\mu} + \frac{1}{2} \partial_\mu h \partial^\mu h + \frac{1}{4} \partial^\rho h \partial^\mu h_{\mu\rho} - \frac{1}{4} \partial^\mu h \partial_\mu h - \frac{1}{2} \partial^\rho h_{\mu\nu} \partial^\mu h_{\rho\mu} + \frac{1}{2} \partial^\rho h_{\mu\nu} \partial_\rho h^{\mu\nu}\right) =$$

$$= \int \left(-\frac{1}{4} \partial^\rho h \partial^\mu h_{\rho\mu} + \frac{1}{4} \partial_\rho h \partial^\rho h - \frac{1}{2} \partial^\rho h_{\mu\nu} \partial^\mu h_{\rho\mu} + \frac{1}{2} \partial^\rho h_{\mu\nu} \partial_\rho h^{\mu\nu}\right)$$

Integrating by parts and rearranging the indeces, we finally obtain the Fierz-Pauli action:

$$S_{FP} = \frac{1}{8\pi G_N} \int d^4x \left[-\frac{1}{4} \partial_\rho h_{\mu\nu} \partial^\rho h^{\mu\nu} + \frac{1}{2} \partial_\rho h_{\mu\nu} \partial^\nu h^{\rho\mu} + \frac{1}{4} \partial_\mu h \partial^\mu h - \frac{1}{2} \partial_\nu h^{\mu\nu} \partial_\mu h\right] \tag{4.5}$$

Varying it with respect to $h_{\mu\nu}$, it's straightforward to obtain the linearised Einstein equations in the vacuum:

$$\begin{aligned} \delta S_{FP} &= \frac{1}{8\pi G_N} \int d^4x \left(\frac{1}{2} \partial_\rho \partial^\rho h_{\mu\nu} - \partial^\rho \partial_\nu h_{\rho\mu} - \frac{1}{2} \partial^\rho \partial_\rho h \eta^{\mu\nu} + \frac{1}{2} \partial_\nu \partial_\mu h + \frac{1}{2} \partial_\rho \partial_\sigma h^{\rho\sigma} \eta_{\mu\nu}\right) \delta h^{\mu\nu} = \\ &= \frac{1}{8\pi G_N} \int d^4x \left(-G_{\mu\nu} \delta h^{\mu\nu}\right) = 0 \Longrightarrow G_{\mu\nu} = 0 \end{aligned} \tag{4.6}$$

To couple the theory to matter, we can simply add a matter term to the Fierz-Pauli action, and extrapolate the energy-momentum tensor directly from its definition, but we do not really need this for our aims.

We can now exhibit the gauge symmetry of the theory with respect to $Diff(1,3)$ as well: given a local diffeomorphism $x^\mu \to x^\mu - \xi^\mu(x)$, we know the linearised metric to vary as

$$\delta g_{\mu\nu} = (\mathfrak{L}_\xi g)_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \Longrightarrow h_{\mu\nu} \to h_{\mu\nu} + (\mathfrak{L}_\xi h)_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{4.7}$$

and S_{FP} is invariant under such transformation:

$$\delta S_{FP} \propto \int d^4x G_{\mu\nu} \left(\partial^\mu \xi^\nu + \partial^\nu \xi^\mu\right) = \int d^4x 2 G_{\mu\nu} \partial^\mu \xi^\nu = \int d^4x 2 \left(\partial^\mu G_{\mu\nu}\right) \xi^\nu = 0 \tag{4.8}$$

where in the second equality we've neglected the total derivative and in the third one we used the linearised Bianchi identity:

$$\partial_\mu G^{\mu\nu} = 0 \tag{4.9}$$

proving the statement.

Happily, we proved the existance of a very interesting gauge symmetry, which is kindly suggesting to fix the gauge, by pitilessly fobbing us with the horribly looking Field Equations in Equation 4.4.

4.1.2 De Donder Gauge fixing

Fixing the gauge not only will provide us better looking field equation, but, most importantly, will facilitate the count of the physical degrees of freedom of the gauge field associated to the symmetry group.

The most commonly used condition in this framework is the **traceless de Donder** gauge, which is the analogous of the Lorentz gauge for a boson tensor field:

$$\underbrace{\partial_\mu A^\mu = 0}_{\text{Lorentz}} \longrightarrow \underbrace{\partial_\mu \bar{h}^{\mu\nu} \equiv \partial_\mu \left(h^{\mu\nu} - \frac{1}{2} h \eta^{\mu\nu}\right) = 0}_{\text{de Donder}} \tag{4.10}$$

which is always legitimate since, for the covariance:

$$\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = f_\nu \xrightarrow{Diff(1,3)} \partial^\mu \left(h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu\right) - \frac{1}{2}\partial_\nu \left(h + 2\partial^\mu \xi_\mu\right) =$$

$$= \partial^\mu h_{\mu\nu} + \Box \xi_\nu + \partial^\mu \partial_\nu \xi_\mu - \frac{1}{2}\partial_\nu h - \partial_\nu \partial^\mu \xi_\mu = f_\nu + \Box \xi_\nu$$

Meaning that choosing the de Donder Gauge is equivalent to apply a diffeomorphism such that $f_\nu = \Box \xi_\nu$.

We can now simplify the linearised Field Equations in 4.4:

$$\partial_\mu \partial^\rho h_{\rho\nu} + \partial_\nu \partial^\rho h_{\rho\mu} - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h - \eta_{\mu\nu} \left(\partial^\rho \partial^\sigma h_{\sigma\rho} - \Box h\right) = 16\pi G_N T_{\mu\nu} =$$

$$= \partial_\mu \left(\frac{1}{2}\partial_\nu h\right) + \partial_\nu \left(\frac{1}{2}\partial_\mu h\right) - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h - \eta_{\mu\nu} \left(\partial^\rho \left(\frac{1}{2}\partial_\rho h\right) - \Box h\right) =$$

$$= \partial_\mu \partial_\nu h - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h - \frac{1}{2}\eta_{\mu\nu} \Box h - \eta_{\mu\nu} \Box h = -2\Box h_{\mu\nu} - \frac{3}{2}\eta_{\mu\nu} \Box h$$

Upon inserting the $\bar{h}$ definition, the equations become:

$$\Box \bar{h}_{\mu\nu} = -16\pi G_N T_{\mu\nu} \tag{4.11}$$

4.1.3 The Graviton Field

Looking at the linearised theory under the gauge symmetry perspective interestingly suggests us to interpret the first order perturbative field $h_{\mu\nu}$ of the minkowskian metric $\eta_{\mu\nu}$ as a maseless tensorial particle field: the **graviton**.

Let' count its degrees of freedom: it is a symmetric tensor ($h_{\mu\nu} \in Sym^{+}(1,3)$) and the de Donder gauge is a set of 4 independent constraints:

$$d = \underbrace{\frac{4 \cdot (4-1)}{2}}_{h \in Sym^{+}} - \underbrace{4}_{d.D.g.} = 2 \tag{4.12}$$

Since $h_{\mu\nu} \in Sym^{+}(1,3) \subset GL(1,3)$, it transforms in a tensorial representation of $Spin(1,3)$:

$$(1,1) \equiv (1,0) \otimes (0,1) \text{ of } Spin(1,3) \tag{4.13}$$

and thus has an helicity of 2.

Awesome: we just discovered a boson of a SUSY multiplet: therefore, there must exist a complementary $h = \frac{3}{2}$ fermion: we will discover this missing piece in the next chapter.

Before this, it's essential to to acquire the necessary formalism to deal with spinors on a curved manifold.

4.2 The Vierbein

Since $\mathcal{M}$ is a Lorentzian manifold, it locally "resembles" $\mathbb{R}^{1,3}$, meaning that $\mathbb{T}_P(\mathcal{M}) \simeq \mathbb{R}^{1,3}$, where we know the spinor formalism to hold.

In order to extend the covariance of the Dirac Lagrangian from $Spin(1,3)$ to $Spin(1,3) \ltimes Diff(1,3)$, we now introduce the *vierbein* (or *frame field*) as a linear transformation "bridging" between global frames (intrinsic coordinate systems on $\mathcal{M}$) to local ones (on the tangent space $\mathbb{T}_P(\mathcal{M})$ to a non singular point $P \in \mathcal{M}$):

$$e^a_\alpha \in \mathbb{TM} : g_{\alpha\beta} = e^a_\alpha \, \eta_{ab} \, e^b_\beta \tag{4.14}$$

where $a, b, c...$ denote the *local indices.*

Strictly speaking, a vierbein belongs to the equivalence class of all the possible vierbeins related by a local Lorentz transformation:

$$e'^a_{\ \mu} \, \eta_{ab} \, e'^b_{\ \nu} = \Lambda^{-1a}{}_b e^b_\mu \, \eta_{ac} \Lambda^{-1c}{}_d e^d_\nu = e^a_\mu \eta_{ab} e^b_\nu = g_{\mu\nu} \Longrightarrow e'^a_{\ \mu} \sim \Lambda^{-1a}{}_b e^b_\mu \tag{4.15}$$

since $\Lambda \in ISO(1,3)$, and thus preserves η.

The inverse vielbein is defined as:

$$(e^\alpha_a)^{-1} \quad : \quad \begin{cases} e^a_\alpha e^\alpha_b = \delta^a_b \\ e^\alpha_a e^a_\beta = \delta^\alpha_\beta \end{cases} \tag{4.16}$$

which directly implies:

$$e^\alpha_a g_{\alpha\beta} e^\beta_b = e^\alpha_a e^a_\alpha \eta_{ab} e^b_\beta e^\beta_b = \eta_{ab} \tag{4.17}$$

The role of the vierbein in the tetrads formalism is analogous to that of $g^{\mu\nu}$ in the canonical one.

Therefore, to complete the picture, our next tasks will be:

1. Express the main metrical structures in terms of the vierbein.
2. Write a $Diff(1,3)$ invariant Dirac action.
3. Reinterpret the perturbative graviton field $h_{\mu\nu}$ in this framework, which will be much more apt to perform SUSY transformations on $\mathcal{M}$.

4.2.1 Metrical structures in the tetrads formalism

Volume forms and integration

The first requirement to satisfy to write a $Diff(1,3)$ invariant action is of course a volume form.

In the classical formalism the usual choice is the metric-compatible *canonical volume form* $dV = \sqrt{-g}\, d^4x$, which we also used in the first sections of this chapter to compute the Fiertz-Pauli action.

It is immediate to write it in terms of the vierbein:

$$g = det\left(e^\mu_a \eta^{ab} e^\nu_b\right) = det(e^\mu_a) det(\eta^{ab}) det(e^\nu_b) = -det(e^\mu_a)^2 \equiv -e^2 \Longrightarrow \sqrt{-g}\, d^4x = e\, d^4x \tag{4.18}$$

The Spin Connection

We now need to evaluate the metric compatible parallel transport "translating" the Cristoffel symbol $\Gamma^\mu_{\nu\rho}$ in the vierbein formalism: this way, we will be able to define the covariant derivative with respect to $Diff(1,3)$.

We can hence write a covariant derivative of a local vector, according to the usual ansatz, to then impose the compatibility with that of a global one:

$$D_\alpha V^\beta = e^\beta_a D^{(\omega)}_\alpha V^a \equiv e^\beta_a \left(\partial_\alpha V^a + \omega_\alpha{}^a{}_b V^b\right) = e^\beta_a \left[\partial_\alpha \left(e^a{}_\gamma V^\gamma\right) + \omega_\alpha{}^a{}_b V^b\right] =$$

$$= e^\beta_a \left[e^a_\gamma \partial_\alpha V^\gamma + \left(\partial_\alpha e^a_\gamma\right) V^\gamma + \omega_\alpha{}^a{}_b V^b\right] = \partial_\alpha V^\beta + e^\beta_a \left(\partial_\alpha e^a_\gamma\right) V^\gamma + \omega_\alpha{}^a{}_b e^b_\gamma V^\gamma$$

$$\partial_\alpha V^\beta + e^\beta_a \left(\partial_\alpha e^a_\gamma\right) V^\gamma + \omega_\alpha{}^a{}_b e^b_\gamma V^\gamma = \partial_\alpha V^\beta + \Gamma^\beta{}_{\alpha\gamma} V^\gamma \Longrightarrow$$

$$\Longrightarrow e^\beta_a \left(\partial_\alpha e^a_\gamma\right) V^\gamma + \omega_\alpha{}^a{}_b e^b_\gamma V^\gamma = \Gamma^\beta{}_{\alpha\gamma} V^\gamma \Longrightarrow \Gamma^\beta{}_{\alpha\gamma} = e^\beta_a \left(\partial_\alpha e^a_\gamma\right) + \omega_\alpha{}^a{}_b e^\beta_a e^b_\gamma \Longrightarrow$$

$$\Longrightarrow \omega_\alpha{}^a{}_b = \Gamma^\beta_{\alpha\gamma} e^\gamma_b e^a_\beta - e^\beta_a (\partial_\alpha e^a_\gamma) e^\gamma_b e^a_\beta = \Gamma^\beta_{\alpha\gamma} e^\gamma_b e^a_\beta - e^\beta_a (\partial_\alpha e^a_\beta \delta^\beta_\gamma) e^\gamma_b e^a_\beta =$$

$$= \Gamma^\beta_{\alpha\gamma} e^\gamma_b e^a_\beta - e^\beta_a (\partial_\alpha e^a_\beta) \delta^\beta_\gamma e^\gamma_b e^a_\beta = \Gamma^\beta_{\alpha\gamma} e^\gamma_b e^a_\beta - e^\beta_a (\partial_\alpha e^a_\beta) e^\beta_b e^a_\beta =$$

$$= \Gamma^\beta_{\alpha\gamma} e^\gamma_b e^a_\beta - e^\beta_a \delta^a_b (\partial_\alpha e^a_\beta) = \Gamma^\beta_{\alpha\gamma} e^\gamma_b e^a_\beta - e^\beta_b (\partial_\alpha e^a_\beta)$$

Since in this formalism, the affine connection is tortionless, it is also symmetric, so we can finally define the *spin connection* as:

$$\omega_\alpha{}^a{}_b = e^a_\beta e^\gamma_b \Gamma^\beta_{\gamma\alpha} - e^\beta_b (\partial_\alpha e^a_\beta) \tag{4.19}$$

Thus, if the Cristoffel symbol defines a metric compatible parallel transport under the action of $Diff(1,3)$, $\omega_\alpha{}^a{}_b$ defines the same operation with respect to the local $SO(1,3)$ symmetry.

This way, we can evaluate the *Minimal Coupling* principle with respect to the local Lorentz group, defining the covariant derivative of a spinor as:

$$D_\mu \equiv \left(\partial_\mu - \frac{i}{4}\omega_\mu{}^{bc}\sigma_{bc}\right) \tag{4.20}$$

4.3 The generalised Dirac action

We now have finally covered all the necessary steps to describe spinors on a curved manifold, thanks to the power of the reinterpretation of gravity as a gauge theory.

First, we shall recall the Dirac action on Minkowski space-time:

$$S = \int d^4x\, \bar{\psi}(i\gamma^a \partial_a - m)\psi \tag{4.21}$$

To obtain the invariance under the symmetry group in 4.1, we have to:

1. Replace the measure with the canonical volume form, to integrate over the manifold:

$$d^4x \longrightarrow \sqrt{-g}d^4x = ed^4x \tag{4.22}$$

2. Express the gamma matrices in the local frame:

$$\gamma^a \to \gamma^\mu = e^\mu_a \gamma^a \tag{4.23}$$

Which can be checked to satisfy a generalised Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} = \left\{e^\mu_a \gamma^a, e^\nu_b \gamma^b\right\} = e^\mu_a \gamma^a e^\nu_b \gamma^b + e^\nu_b \gamma^b e^\mu_a \gamma^a = e^\mu_a \left\{\gamma^a, \gamma^b\right\} e^\nu_b = 2g^{\mu\nu} \tag{4.24}$$

3. Evaluate the Minimal Coupling Principle, by substituting the partial derivative with the covariant one 4.20:

$$\partial_a \longrightarrow D_a = \partial_a - \frac{i}{4}\omega_a^{bc}\sigma_{bc} \tag{4.25}$$

where $\sigma_{bc} = \frac{i}{2}[\gamma^b, \gamma^c]$ are the generators of the Lorentz group.

The resulting action is therefore:

$$S = \int d^4x \, e\bar{\psi}(i\gamma^\mu D_\mu - m)\psi \tag{4.26}$$

and the generalised Dirac equation is:

$$\frac{\delta S}{\delta \bar{\psi}} = 0 \Longrightarrow (i\gamma^\mu D_\mu - m)\,\psi = 0 \tag{4.27}$$

Chapter 5

Rarita-Schwinger Theory

In the previous chapter we found the graviton $h_{\mu\nu}$ to emerge as a first order perturbation of the metric of the space-time manifold $\mathcal{M}$, and, by counting its degrees of freedom, we discovered it to belong to the $h = 2$ maseless multiplet of the SUSY algebra representation on particle states, and we anticipated that we would've had to complete it with a $h = \frac{3}{2}$ fermion, namely the *gravitino*.

Although the necessity of such a particle for Supergravity will be fully clearified in the final chapter, we can already begin to foresee the reason: many times throughout this dissertation we've recalled SUGRA to be the gauge theory of SUSY, and this essentially means that the spinorial parameter of a Supersymmetry transformation is promoted to a space-time function.

Both from our backing knowledges of gauge theories, and from the observations of the last chapter, we know a gauge symmetry to require the existence of one or more tensor fields for the action to be invariant under such transformations: if such parameter is a supercharge, then its variation will unavoidably carry both a spinor and a vector index.

Although this will be demonstrated in the following chapter, these considerations are apt to lay the ground to approach the present analysis.

The Rarita-Schwinger theory describes the dynamics of such fermions on a minkoskian manifold.

5.1 Maseless Rarita-Schwinger Field

5.1.1 The Action

As always, in order to show the symmetries of a theory, it's suitable to start from an action. The dynamics of the maseless gravitino is described by the Rarita-Schwinger action:

$$\mathcal{S}_{RS} = -\int d^4x\, \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \Psi_\rho \tag{5.1}$$

where:

$$\gamma^{\mu\nu\rho} = \frac{1}{3!} \sum_{perm} sign(\sigma) \gamma^{\sigma(\mu)} \gamma^{\sigma(\nu)} \gamma^{\sigma(\rho)} \tag{5.2}$$

representing the antisymmetrized product of gamma matrices.

We now want to show and/or demonstrate its main properties:

1. ***First order*** in space-time derivatives.
2. ***Lorentz invariance***: the Lagrangian is manifestly a Lorentz scalar.
3. ***Local SUSY invariance***: if we promote the spinor parameter ϵ to a function on space-time $\epsilon(x)$, the gravitino variation under such transformation becomes:

$$\delta\Psi_{\mu\alpha}(x) = \partial_\mu \epsilon(x) \tag{5.3}$$

and, crucially, we can show it to be a symmetry of the Rarita-Schwinger theory:

$$\begin{aligned}
\mathcal{L}_{RS} &= \bar{\Psi}_\mu \gamma^{\mu\nu\rho}\partial_\nu \Psi_\rho \longrightarrow \left(\bar{\Psi}_{\mu\dot{\alpha}} + \delta\bar{\Psi}_{\mu\dot{\alpha}}\right)\gamma^{\mu\nu\rho}\partial_\nu\left(\Psi_{\rho\alpha} + \delta\Psi_{\rho_\alpha}\right) = \\
&= \left(\bar{\Psi}_{\mu\dot{\alpha}} + \partial_\mu\bar{\epsilon}_{\dot{\alpha}}\right)\gamma^{\mu\nu\rho}\partial_\nu\left(\Psi_{\rho\alpha} + \partial_\rho\epsilon_\alpha(x)\right) = \\
&= \bar{\Psi}_{\mu\dot{\alpha}}\gamma^{\mu\nu\rho}\partial_\nu\Psi_{\rho\alpha} + \bar{\Psi}_{\mu\dot{\alpha}}\gamma^{\mu\nu\rho}\partial_\nu\partial_\rho\epsilon_\alpha + \left(\partial_\mu\bar{\epsilon}_{\dot{\alpha}}\right)\gamma^{\mu\nu\rho}\partial_\nu\Psi_{\rho\alpha} + \left(\partial_\mu\bar{\epsilon}_{\dot{\alpha}}\right)\gamma^{\mu\nu\rho}\partial_\nu\partial_\rho\epsilon_\alpha(x) = \\
&= \mathcal{L}_{RS} + \left(\partial_\mu\bar{\epsilon}_{\dot{\alpha}}\right)\gamma^{\mu\nu\rho}\partial_\nu\Psi_{\rho\alpha} + \left[\bar{\Psi}_{\mu\dot{\alpha}} + \partial_\mu\bar{\epsilon}_{\dot{\alpha}}\right]\gamma^{\mu\nu\rho}\partial_\nu\partial_\rho\epsilon_\alpha(x) = \\
&= \mathcal{L}_{RS} + \left(\partial_\mu\bar{\epsilon}_{\dot{\alpha}}\right)\gamma^{\mu\nu\rho}\partial_\nu\Psi_{\rho\alpha} = \mathcal{L}_{RS} + \partial_\mu\left(\bar{\epsilon}_{\dot{\alpha}}\gamma^{\mu\nu\rho}\partial_\nu\Psi_{\rho\alpha}\right)
\end{aligned}$$

where in the last two lines we used the fact that the contraction of a symmetric with an antisymmetric tensor is always null. Since the Lagrangian varies to a boundary term, local SUSY is an invariance of the action.

4. Hermitian, so that equation of motion for $\bar{\Psi}$ is the Dirac conjugate of that of Ψ.

We can directly derive the equation of motion from the Euler-Lagrangian equations:

$$\partial_\mu\left(\frac{\delta\mathcal{L}_{RS}}{\delta\left(\partial_\mu\bar{\Psi}\right)}\right) - \frac{\delta\mathcal{L}_{RS}}{\delta\bar{\Psi}} = 0 \Longrightarrow \gamma^{\mu\nu\rho}\partial_\nu\Psi_\rho = 0 \tag{5.4}$$

with a conjugate equation for $\bar{\Psi}$.

5.1.2 Degrees of Freedom

Just as we did for the graviton, we want to count the number of degrees of freedom of the gravitino: the presence of the gauge symmetry in Equation 5.3 suggests us to impose some kind of constraint to isolate the physical ones.

A possible constraint is given by an analogous of the Coulomb gauge:

$$\gamma^i\Psi_i = 0 \quad , \quad i = 1,2,3 \tag{5.5}$$

By introducing the gravitino strength, in analogy with Yang-Mills theories:

$$\Psi_{\mu\nu} \equiv \partial_\mu\Psi_\nu - \partial_\nu\Psi_\mu \equiv 2\partial_{[\mu}\Psi_{\nu]} \tag{5.6}$$

the equations of motion can be rewritten by separating in components:

$$\gamma^\mu\Psi_{\mu\sigma} = 0 \Longleftrightarrow \begin{cases}\gamma^i\partial_i\Psi_0 - \partial_0\gamma^i\Psi_i = 0\\ \gamma^0\partial_0\Psi_i - \partial_i\gamma^0\Psi_0 = 0\end{cases} \tag{5.7}$$

From the gauge condition in Equation 5.5, we see that:

$$\gamma^i \partial_i \Psi_0 - \partial_0 \gamma^i \Psi_i = \gamma^i \partial_i \Psi_0 = 0 \Longrightarrow \left(\gamma^i \partial_i\right)^2 \Psi_0 = 0 \Longrightarrow \nabla^2 \Psi_0 = 0 \Longrightarrow \Psi_0 = 0$$

By contracting with γ^i the spacial components of the equation of motions we can derive another constraint:

$$\partial^i \Psi_i = 0 \tag{5.8}$$

Hence, we obtained a set of $(4-1)2^{\frac{4}{2}} = 12$ constraints. The remaining $4 \cdot 2^{\frac{4}{2}} - 12 = 4$ are halved by the equation of motions.

The total number of degrees of freedom before imposing the equations of motion and the gauge is called *off-shell* degrees of freedom, while the resulting number is called *on-shell* degrees of freedom.

5.2 Quantization of the Rarita-Schwinger field

As we mentioned in the introduction, the maseless Rarita-Schwinger field belongs to the

$$\Psi \in \left[\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)\right] \otimes \left(\frac{1}{2}, \frac{1}{2}\right) \tag{5.9}$$

representation of $Spin(1,3)$, meaning that it carries both a vector and a spinor indices.

Since we are dealing with the free limit, the spatial components of the field can be expanded as a superposition of plane waves:

$$\Psi_i(x) = \exp\{ip \cdot x\}\, v_i\left(\vec{p}\right) u(\vec{p}) \tag{5.10}$$

where $u(\vec{p}) \equiv u_{\vec{p}}$ is four component spinor, hence given by the sum of positive and negative energy states, since Ψ_i satisfies the Dirac equation, whereas the vector $v_{i\,\vec{p}}$ can be expanded in the basis of the transverse polarization vectors and the momentum:

$$v_{i\,\vec{p}} = a p_i + b \epsilon_i\left(\vec{p}, +\right) + c \epsilon_i\left(\vec{p}, -\right) \tag{5.11}$$

where $\epsilon\left(\pm\right)$ are the polarization vectors of a quantized vector field, satisfying that $p^i \epsilon_i = 0$. For the last constraint in Equation 5.7, we see that $a = 0$, and thus:

$$\Psi_i(x) = \exp\{ip \cdot x\}\left[b^+ \epsilon^+_{\vec{p}} u^+_{i\,\vec{p}} + c^+ \epsilon^-_{i\,\vec{p}} u^+_{\vec{p}} + b^- \epsilon^+_{\vec{p}} u^-_{\vec{p}} + c^- \epsilon^-_{\vec{p}} u^-_{\vec{p}}\right] \tag{5.12}$$

Moreover, the constraint $\gamma^i \Psi_i = 0$ ultimately determines $c^+ = b^- = 0$, and thus the quantized maseless gravitino field can be written as:

$$\Psi_\mu(x) = \int \frac{d^3p}{\left(2\pi\right)^3 2p^0} \sum_h \left[e^{i\vec{p}\cdot x} \epsilon^h_{\vec{p}\,\mu} u^h_{\vec{p}} c^h_{\vec{p}} + e^{-i\vec{p}\cdot x} \epsilon^{*\,h}_{\vec{p}\,\mu} v^h_{\vec{p}} d^{*\,h}_{\vec{p}}\right] \tag{5.13}$$

where the sum is extended over the $\pm\frac{3}{2}$ helicity states, $c^h_{\vec{p}}$ is the annihilation operator for particles and $d^\dagger_{\vec{p}}$ is that of the antiparticle.

Chapter 6

Supergravity

After an arduous, but extremely rewarding journey, everything is ready to approach Supergravity.

As anticipated in the introduction to chapter 5, we will hereby craft the SUGRA Lagrangian as a necessary additional term to sum to the Wess-Zumino Model to achieve local SUSY invariance for the one chiral multiplet $\{\phi, \psi\}$: we will evaluate the $h = \frac{3}{2}$ multiplet by showing superpartners to naturally arise. Ultimately, this will permit us to isolate the SUGRA term, and identify it as the Lagrangian of the pure theory.

The following and last aim of this analysis will be to covariantize $\mathcal{L}_{SUGRA}$ under the action of $Diff(1,3)$ on a curved space-time manifold $\mathcal{M}$, by exploiting the vierbein formalism.

Hopefully, we will appreciate how all the knowledges we acquired throughout the previous chapters will condense in a harmonious theory.

6.1 Gauging SUSY in the WZ model

To begin, we rewrite the Wess-Zumino Lagrangian, eliminating the ghost field term, and in terms of Majorana spinors (end of section 2.2):

$$\begin{aligned}
&\mathcal{L}_{WZ} = \partial_\mu \phi^\dagger \partial^\mu \phi - i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi = * \\
&\chi = \begin{pmatrix} \psi^\alpha \\ \bar{\psi}_{\dot{\alpha}} \end{pmatrix} \quad , \quad \bar{\chi} = \chi^\dagger \gamma^0 = \begin{pmatrix} \bar{\psi}^{\dot{\alpha}} \\ \psi_\alpha \end{pmatrix}^\dagger \quad , \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \\
&\bar{\chi}_L \gamma^\mu \partial_\mu \chi_R + \bar{\chi}_R \gamma^\mu \partial_\mu \chi_L = \left(\bar{\psi}, 0\right) \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \partial_\mu \begin{pmatrix} \psi \\ 0 \end{pmatrix} + \left(0, \psi\right) \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \bar{\psi} \end{pmatrix} = \\
&= \bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi + \psi \sigma^\mu \partial_\mu \bar{\psi} = \bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi + \left(\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi\right)^\dagger = \partial_\mu \left(\bar{\psi}\bar{\sigma}^\mu \psi\right) - \partial_\mu \bar{\psi}\bar{\sigma}^\mu \psi + \\
&+ \left(\partial_\mu \left(\bar{\psi}\bar{\sigma}^\mu \psi\right) - \partial_\mu \bar{\psi}\bar{\sigma}^\mu \psi\right)^\dagger = -2i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi + t.d. \\
&* = \underbrace{-\partial_\mu \phi \partial^\mu \phi^*}_{\mathcal{L}_{scalar}} \underbrace{-\frac{1}{2}\left(\bar{\chi}_R \gamma^\mu \partial_\mu \chi_L + \bar{\chi}_L \gamma^\mu \partial_\mu \chi_R\right)}_{\mathcal{L}_{spinor}} + t.d.
\end{aligned} \tag{6.1}$$

In section 3.3.4 we've proven it to be invariant under global SUSY transformations up to boundary terms, leveraging the superfields formalism and neglecting an explicit calculation, since an action of the type in Equation 3.38 is necessarily endowed with such property:

$$\delta\mathcal{L}_{WZ} = \partial_\mu K^\mu \tag{6.2}$$

Here we will instead need to develop an explicit calculation, to show how the Noether supercurrent differs in local from the global variations.

From the variations of the required components of a superfield, from in section 3.3.2, it is straightforward to rewrite the spinorial ones in the Majorana basis:

$$\begin{cases} \delta\phi = \epsilon\psi + \bar{\epsilon}\bar{\chi} \\ \delta\psi = i\sigma^\mu\bar{\epsilon}\partial_\mu\phi \\ \delta\bar{\chi} = -i\epsilon\bar{\sigma}^\mu\partial_\mu\phi \end{cases} \Longleftrightarrow \begin{cases} \delta\phi = \bar{\epsilon}_L\chi_L \quad , \quad \delta\phi^* = \bar{\epsilon}_R\chi_R \\ \delta\chi_L = \frac{1}{2}\gamma^\mu\partial_\mu\phi\,\epsilon_R \quad , \quad \delta\chi_R = \frac{1}{2}\gamma^\mu\partial_\mu\phi^*\,\epsilon_L \\ \delta\bar{\chi}_L = -\frac{1}{2}\bar{\epsilon}_R\gamma^\mu\partial_\mu\phi \quad , \quad \delta\bar{\chi}_R = -\frac{1}{2}\bar{\epsilon}_L\gamma^\mu\partial_\mu\phi^* \end{cases} \tag{6.3}$$

With these, we may compute the variation of the action:

$$\delta S = \int d^4x\,(\delta\mathcal{L}) = \int d^4x\,(\delta\mathcal{L}_{scalar} + \delta\mathcal{L}_{spinor}) = *$$

$$\mathcal{L}_{spinor} = -\bar{\chi}_L\gamma^\mu\partial_\mu\chi_R - \bar{\chi}_R\gamma^\mu\partial_\mu\chi_L = -\bar{\chi}_R\gamma^\mu\partial_\mu\chi_L + \partial_\mu\left(\bar{\chi}_R\right)\gamma^\mu\chi_L$$

$$* = \int d^4x\,(-\partial_\mu(\delta\phi)\partial^\mu\phi^* - \delta\bar{\chi}_R\gamma^\mu\partial_\mu\chi_L + \partial_\mu(\delta\bar{\chi}_R)\gamma^\mu\chi_L + h.c.) =$$

$$\overset{I.b.P.}{=} \int d^4x\left(\delta\phi\Box\phi^{+}2\partial_\mu\left(\delta\bar{\chi}_R\right)\gamma^\mu\chi_L + \partial_\mu\left(-\delta\phi\partial^\mu\phi^* - \delta\bar{\chi}_R\gamma^\mu\chi_L\right) + h.c.\right) =$$

$$\int d^4\left(\bar{\epsilon}_L\chi_L\Box\phi^* - \partial_\mu\left(\bar{\epsilon}_L\gamma^\mu\partial_\mu\phi^*\right)\gamma^\mu\chi_L + \partial_\mu\mathcal{K}^\mu + h.c.\right) =$$

$$= \int d^4x\left[\bar{\epsilon}_L\chi_L\Box\phi^* - \partial_\mu\left(\bar{\epsilon}_L\right)\gamma^\mu\partial_\mu\phi^*\gamma^\mu\chi_L - \bar{\epsilon}_L\partial_\mu\partial_\nu\phi^*\gamma^\nu\gamma^\mu\chi_L + \partial_\mu\mathcal{K}^\mu + \text{ h.c }\right] =$$

$$\overset{Cl}{=} \int d^4x\left[-\partial_\mu\left(\bar{\epsilon}_L\right)\gamma^\mu\partial_\mu\phi^*\gamma^\mu\chi_L + \partial_\mu\mathcal{K}_\mu + \text{ h.c }\right] = 0 \Longleftrightarrow \partial_\mu\bar{\epsilon}_{L,R} = 0$$

which shows that, if the spinorial parameter is constant, the action is invariant, as expected, since the Lagrangian varies by a total derivative:

$$\delta\mathcal{L} = \partial_\mu\left(\mathcal{K}^\mu + \mathcal{K}^{\mu*}\right) \equiv \partial_\mu\mathcal{K}^\mu \tag{6.4}$$

where $\mathcal{K}^\mu$ is the Noether supercurrent:

$$\mathcal{K}^\mu = \left[\left(-\delta\phi\partial^\mu\phi^* - \delta\bar{\chi}_R\gamma^\mu\chi_L\right) + h.c.\right] = \left[\left(-\bar{\epsilon}_L\chi_L\partial^\mu\phi^* + \frac{1}{2}\bar{\epsilon}_L\gamma^\mu\partial_\mu\phi^*\gamma^\mu\chi_L\right) + h.c.\right] \tag{6.5}$$

On the other hand, if we were to gauge SUSY, the spinorial parameter would be promoted to a function of space-time, $\epsilon \longrightarrow \epsilon(x)$, restricting to a so called *Kahler manifold* in the superspace, and the Lagrangian would no longer be invariant up to a boundary term:

$$\begin{cases} \delta\mathcal{L} = (\partial_\mu \bar{\epsilon}) j^\mu = (\partial^\mu \bar{\epsilon}_L) j^\mu_L + (\partial_\mu \bar{\epsilon}_R) j^\mu_R \\ j^\mu_L \equiv -\gamma^\mu \partial_\mu \phi^* \gamma^\mu \chi_L \\ j^\mu_R \equiv -\gamma^\mu \partial_\mu \phi \gamma^\mu \chi_R \end{cases} \tag{6.6}$$

where $j^\mu = j^\mu_L + j^\mu_R$ is the super-Noether current, which is conserved, as it can be proven by showing that it has null divergence, by leveraging the equation of motions of the fields.

6.1.1 Adding the SUGRA multiplet

In order to compensate the variations and achieve the invariance of the action, in a similiar fashion to Yang-Mills theories, we can associate a gauge field Ψ to the supercurrent, which, as forseen in chapter 5, necessarily carries both a spinorial and a vectorial index, resulting in an additional term $\mathcal{L}'_{WZ}$ to $\mathcal{L}_{WZ}$:

$$\mathcal{L}'_{WZ} = -\frac{1}{M_P}\left(\bar{\Psi}_{L\,\mu} j^\mu_L + \bar{\Psi}_{R\,\mu} j^\mu_R\right) = -\frac{1}{M_P}\bar{\Psi}_{\mu\dot{\alpha}} j^{\mu\dot{\alpha}} \quad : \quad \begin{cases} \delta\Psi_{L,R\,\mu} = M_P \partial_\mu \epsilon_{L,R} \\ \delta\bar{\Psi}_{L,R\,\mu} = M_P \partial_\mu \bar{\epsilon}_{L,R} \end{cases} \tag{6.7}$$

where is the M_P is the reduced Planck mass, necessary to equate the mass dimensions.

Now we can prove that the variations we intended to compensate actually vanishes, but a new piece, generated by $\delta j^\mu_{R,L}$ generates:

$$\begin{aligned} &\delta\mathcal{L}'_{WZ} = \delta\mathcal{L}_{WZ} + \delta\mathcal{L}'_{WZ} = (\partial_\mu \bar{\epsilon})\, j^\mu + \partial_\mu \mathcal{K}^\mu - \frac{1}{M_P}\left(\delta\bar{\Psi}_{\mu\dot{\alpha}} j^{\mu\dot{\alpha}} + \bar{\Psi}_{\mu\dot{\alpha}} \delta j^{\mu\dot{\alpha}}\right) = \\ &= (\partial_\mu \bar{\epsilon})\, j^\mu + \partial_\mu \mathcal{K}^\mu - \frac{1}{M_P}\left(\delta\bar{\Psi}_{\mu L} j^\mu_L + \delta\bar{\Psi}_{\mu R} j^\mu_R + \bar{\Psi}_{\mu\dot{\alpha}} \delta j^{\mu\dot{\alpha}}\right) = \\ &= (\partial_\mu \bar{\epsilon})\, j^\mu + \partial_\mu \mathcal{K}^\mu - \frac{1}{M_P}\left((M_P \partial_\mu \bar{\epsilon}_L)\, j^\mu_L + (M_P \partial_\mu \bar{\epsilon}_L)\, j^\mu_R + \bar{\Psi}_{\mu\dot{\alpha}} \delta j^{\mu\dot{\alpha}}\right) = \\ &= (\partial_\mu \bar{\epsilon})\, j^\mu + \partial_\mu \mathcal{K}^\mu - (\partial_\mu \bar{\epsilon} j^\mu) + \frac{1}{M_P}\bar{\Psi}_{\mu\dot{\alpha}} \delta j^{\mu\dot{\alpha}} = \partial_\mu \mathcal{K}^\mu + \frac{1}{M_P}\bar{\Psi}_\mu \left(j^\mu_L + j^\mu_R\right) = \\ &= \partial_\mu \mathcal{K}^\mu + \frac{1}{M_P}\left[\bar{\Psi}_{\mu L} \delta j^\mu_L + \bar{\Psi}_{\mu R} \delta j^\mu_R\right] \simeq * \\ &\bar{\Psi}_{\mu L} \delta j^\mu_L = \bar{\Psi}_{\mu L} \delta\left(-\gamma^\mu \partial_\mu \phi^* \gamma^\mu \chi_L\right) \\ &* \simeq \partial_\mu \mathcal{K}^\mu - \frac{1}{M_P}\bar{\epsilon}\gamma_\mu \Psi_\nu T^{\mu\nu} \end{aligned} \tag{6.8}$$

In order to cancel the non boundary term, we introduce a symmetric tensor $g_{\mu\nu}$, obeying the transformation rule:

$$\delta g_{\mu\nu} \sim \frac{1}{2M_P}\left(\bar{\epsilon}\gamma_\mu \psi_\nu + \bar{\epsilon}\gamma_\nu \psi_\mu\right) \equiv \frac{1}{M_P}\bar{\epsilon}\gamma_{(\mu}\psi_{\nu)} \tag{6.9}$$

and insert a respective term in the Lagrangian:

$$\mathcal{L}''_{WZ} \sim -g_{\mu\nu} T^{\mu\nu} \tag{6.10}$$

Since the only bilinear form which can enter the Lagrangian coupled with the energy-momentum tensor is the space-time metric, we demonstrated that local SUSY invariance requires to complete the Wess-Zumino multiplet with the graviton $g_{\mu\nu}$ and the gravitino $\Psi_{\mu\alpha}$, which are indeed superpartners.

The resulting Wess-Zumino Lagrangian is:

$$\mathcal{L} = \mathcal{L}_{WZ} + \underbrace{\mathcal{L}'_{WZ} + \mathcal{L}''_{WZ}}_{\mathcal{L}_{int}(\phi,\chi,g,\Psi)} + \mathcal{L}_{kin}(g_{\mu\nu}) + \mathcal{L}_{kin}(\psi_\mu) \tag{6.11}$$

From the analysis of SUSY algebra representation, we see that the resulting multiplet is reducible to the sum of the $h = \frac{3}{2}$ and $h = \frac{1}{2}$ multiplets, respectively corresponding to $\{\Psi, g\}$ and $\{\phi, \chi\}$.

This suggests to identify the Lagrangian of pure SUGRA on Minkowski spacetime as:

$$\tilde{\mathcal{L}}_{SUGRA} = \mathcal{L}_{kin}(g_{\mu\nu}) + \mathcal{L}_{kin}(\psi_\mu) \tag{6.12}$$

To complete the analysis, we need to covariantize it under $Diff(\mathcal{M})$ on a generalised spacetime manifold.

6.2 Pure SUGRA Lagrangian

In order to achieve the covariance under the diffeomorphisms group, we can leverage our knowledges of the tetrads formalism and the Rarita-Schwinger theory and retrace the same steps we followed in section 4.3, to covariantize the pure SUGRA action:

1. Replace the measure d^4x with the canonical volume form:

$$d^4x \longrightarrow e d^4x \quad : \quad e = \det(e^a_\mu) = \sqrt{-g} \tag{6.13}$$

2. Express the gamma matrices in the vierbein basis:

$$\gamma^a \longrightarrow \gamma^\mu = e^\mu_a \gamma^a \tag{6.14}$$

3. Evaluate the Minimal Coupling Principle, by leveraging Equation 4.20:

$$\partial_\mu \longrightarrow D_\mu = \partial_\mu - \frac{i}{4}\omega^{ab}_\mu \sigma_{ab} \tag{6.15}$$

This way, we find the free SUGRA action in the hypothesis of maseless multiplet:

$$S_{SUGRA} = \int d^4x \; \mathcal{L}_{SUGRA} = \int d^4x \; (\mathcal{L}_{EH} + \mathcal{L}_{RS}) = \int d^4x \, e \left(\frac{M_P^2}{2} R - \frac{1}{2} \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho \right) \tag{6.16}$$

which is invariant under the covariantized version of the variations of the fields in Equation 6.7 and Equation 6.9[1].

[1]For a complete demonstration of the invariance of S_{SUGRA} under SUGRA variations of the fields lies outside the purposes of this thesis, and we refer to excellent textbooks such as [1] and [4] for anyone interested in a complete proof.

The action of pure Supergravity represents the final aim of this dissertation: under its charm, it encapsulates the amazingly broad formalism we explored throughout this work.

No conclusion paragraph would ever manage to offer such a pleasing farewell.

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