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ELECTRODYNAMICS OF CONTINA

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Sommario

Questa tesi presenta uno studio completo sulla termomeccanica dei mezzi continui mobili, con particolare attenzione all'interazione tra effetti meccanici, termici ed elettromagnetici. A partire dai principi generali della meccanica dei continui, vengono derivate le leggi di bilancio integrali e locali per massa, quantità di momento (lineare e angolare), ed energia. Le equazioni di bilancio vengono estese al caso di corpi immersi in campi elettromagnetici, in una formulazione in cui i campi elettromagnetici partecipano alla dinamica del continuo al pari della materia. La parte finale è dedicata ai vincoli imposti dalla termodinamica, in particolare attraverso il bilancio dell'entropia e la formulazione locale della disuguaglianza di Clausius-Duhem. Nell'ultimo capitolo si espone il metodo di Coleman e Noll, con il quale si ricavano condizioni di ammissibilità per leggi costitutive compatibili con la seconda legge della termodinamica.

Abstract

This thesis presents a comprehensive study on the thermomechanics of moving continuous media, with particular attention to the interaction between mechanical, thermal, and electromagnetic effects. Starting from the general principles of continuum mechanics, we derive the integral and local balance laws for mass, linear and angular momentum, and energy. These balance equations are extended to the case of bodies immersed in electromagnetic fields, adopting a formulation in which electromagnetic fields contribute to the dynamics of the continuum on equal footing with matter. The final part is dedicated to the constraints imposed by thermodynamics, in particular through the entropy balance and the local formulation of the Clausius-Duhem inequality. The last chapter introduces the Coleman-Noll method, which provides admissibility conditions for constitutive laws compatible with the second law of thermodynamics.

Contents

1 Introduction

The study of continuous media from a mechanical and thermodynamical point of view, and subjected to electromagnetic fields is a cornerstone of modern physics and engineering. From the behavior of charged fluids in plasmas to the thermomechanical response of materials in electromagnetic devices, understanding how energy and momentum are exchanged between matter and fields is of fundamental importance. This thesis aims to contribute to this understanding by developing a coherent and general formulation of the balance laws that govern the evolution of continua combining mechanical, thermal, and electromagnetic effects.

We begin by recalling the kinematics of deforming continua, as a starting point for the study of continuum thermomechanics. Basic kinematic and geometric notions are introduced to describe deformable bodies: control volumes, material surfaces, and moving domains, along with the mathematical tools needed to track how physical quantities evolve through these domains and across discontinuities. The emphasis is on building a consistent and flexible mathematical language suitable for the formulation of general balance laws and junction conditions in continua undergoing motion and deformation.

With this framework in place, we proceed to derive the integral and local forms of the mechanical balance laws: conservation of mass, linear momentum, angular momentum, and mechanical and thermal energy. The derivations are carried out in full generality, and particular attention is paid to jump conditions across discontinuity surfaces. The principles of thermodynamics are then introduced, with emphasis on the second law, which imposes constraints on the admissible forms of energy exchange and dissipation. The local form of the entropy balance equation is derived, and the second law is ultimately restated in the form of the Clausius-Duhem inequality. This inequality plays a key role in the formulation of consistent constitutive theories.

The theoretical framework is then extended to include electromagnetic phenomena. Firtsly Maxwell's equations are presented in a form suitable for deforming media, and the conservation of electric charge is discussed. The interaction between media and electromagnetic fields is described through the notions of magnetization and polarization. Electromagnetic sources are analyzed, leading to effective expressions for charge and current densities in media, derived from polarization and magnetization fields. The electromagnetic force is introduced via the Lorentz law, providing expressions for electromagnetic momentum and

the associated stress tensor. Similarly, the electromagnetic power exchange is analyzed to obtain consistent expressions for the field energy and energy conduction. This leads to a comprehensive framework in which electromagnetic fields are not treated as merely external agents, but as elements of the continuum's dynamics on the same footing as matter.

In the final chapter, the theoretical picture is completed with a discussion of constitutive modeling. The Coleman-Noll procedure is introduced as a systematic method to derive thermodynamically admissible constraints on the constitutive relations. In this way, the universal balance laws are complemented with material-specific equations that describe how different substances behave under the same external conditions. Classical examples are analyzed, including the perfect fluid and the linear viscous fluid, to illustrate the method and its physical implications.

The structure of the thesis reflects a progressive development: from geometrical and physical foundations, through the derivation of governing equations, with particular focus on electromagnetic interactions, toward the formulation of thermodynamically consistent material models. The appendices contain extended calculations and technical derivations that support the main results presented in the core chapters.

Finally, it is worth noting that the electrodynamics of continua remains an open and actively developing branch of theoretical physics. While classical electromagnetism in vacuum is a mature theory, its extension to deformable media introduces significant conceptual and practical challenges. A striking example is the lack of a universally accepted expression for the electromagnetic stress tensor in matter: various formulations, including those by Maxwell, Minkowski, and Abraham, are based on different physical assumptions and can yield different predictions in dynamical situations. These ambiguities highlight the importance of a careful and consistent treatment rooted in continuum mechanics and thermodynamics, particularly when modeling the behavior of complex materials and evolving domains.

Keywords: Kinematics of continua; Thermomechanics of continua; Balance laws; Electrodynamics of continua; Clausius-Duhem inequality; Constitutive models; Coleman-Noll procedure.

2 | Kinematics of deformable domains

In most of the applications of continuum mechanics, we have to deal with quantities defined on time dependent domains, such as moving and deforming volumes and surfaces. In this chapter we will introduce the foundamental definitions and relations that will become useful in the following chapters; we will compute the *material derivative* of a field defined on a moving and deforming volume, surface or curve, and we will see how to deal with discontinuities in the fields defined on these domains. After that, we will introduce the *distributional* approach to the kinematics of deformable bodies, which will allow us to treat discontinuities in a more rigorous way, generating a framework that can be applied to the study of the electrodynamics of continua.

2.1 Definitions

In order to describe the kinematics of volumes, surfaces and curves, we need to introduce some definitions, which will be used throughout this dissertation.

Definition 1 (Moving and deforming domains)**.** *A* ***deforming volume*** $\mathcal{V}(t)$ *is a time-dependent regular region of* $\mathbb{R}^3$ *whose points* $\mathbf{x}(t)$ *evolve according to a velocity field* $\mathbf{v}(t)$*. Similarly, a* ***deforming surface*** $\mathcal{A}(t)$ *(resp.* ***curve*** $\mathcal{L}(t)$*) is a time-dependent oriented surface (resp. curve) whose points follow* $\mathbf{v}(t)$*. We assume* $\mathcal{V}(t)$*,* $\mathcal{A}(t)$*, and* $\mathcal{L}(t)$ *are piecewise* C^1*-regular, with possible discontinuities across interfaces* $\mathcal{D}(t)$*.*

Definition 2 (Integral over a deforming volume)**.** *Let* $\rho(t)$ *be a time dependent scalar density field defined on a moving and deforming volume* $\mathcal{V}(t)$*. The* ***volume integral*** *of* $\rho(t)$ *over* $\mathcal{V}(t)$ *is defined as:*

$$\Delta_{\rho(t)}(\mathcal{V}(t)) = \int_{\mathcal{V}(t)} \mathrm{d}^3\mathbf{x}\, \rho(t).$$

This represents the total amount of the quantity ρ *contained within the deforming region* $\mathcal{V}(t)$*.*

Definition 3 (Flux integral over a deforming surface)**.** *Let* $\boldsymbol{a}(t)$ *be a time dependent vector density field, and let* $\mathcal{A}(t)$ *be a moving and deforming surface, each of whose material points*

*follows the velocity field $\boldsymbol{v}(t)$. The **flux integral** of $\boldsymbol{a}(t)$ across $\mathcal{A}(t)$ is defined as:*

$$\Phi_{\boldsymbol{a}(t)}(\mathcal{A}(t)) = \int_{\mathcal{A}(t)} \mathrm{d}^2\mathbf{x} \cdot \boldsymbol{a}(t),$$

where $\mathrm{d}^2\boldsymbol{x}$ is the oriented surface element of $\mathcal{A}(t)$, pointing in the direction of $\boldsymbol{n}$, the outward unit normal to the surface. This represent a measure of how much of the field "pierces" through the surface, weighted by how aligned the field is with the normal vector. It's maximal when the field is fully aligned with $\boldsymbol{n}$, and zero when it's tangent to the surface.

Definition 4 (Circulation over a deforming line)**.** *Let $\boldsymbol{b}(t)$ be a time-dependent vector density field and $\mathcal{L}(t)$ a moving and deforming line, each of its material points following the velocity field $\boldsymbol{v}(t)$. We define the **circulation** of $\boldsymbol{b}(t)$ along the curve $\mathcal{L}(t)$ as:*

$$\Gamma_{\boldsymbol{b}(t)}(\mathcal{L}(t)) = \int_{\mathcal{L}(t)} \mathrm{d}^1\mathbf{x} \cdot \boldsymbol{b}(t),$$

where $\mathrm{d}^1\boldsymbol{x}$ is the oriented line element along $\mathcal{L}(t)$, pointing in the direction of the unit tangent vector $\boldsymbol{\tau}$. This represents a measure of how much our vector density field is aligned with the tangent vector $\boldsymbol{\tau}$ at each point of the curve, a kind of tangential accumulation along the direction of the line.

Additional details regarding the adopted conventions and notation are provided in Appendix A.

2.2 Moving and Deforming Domains

Moving and deforming volumes

In this section we will introduce the material derivative of a field defined on a moving and deforming volume, and we will see how to compute it using the Gauss' theorem.

Theorem 2.1 (Material derivative over a deforming volume)**.** *Let $\rho(t)$ be a time dependent scalar density and $\mathcal{V}(t)$ a moving and deforming volume, each of its material points following the velocity field $\boldsymbol{v}(t)$. Then, the material derivative of the volume integral $\frac{\mathrm{d}}{\mathrm{d}t}\Delta_{\rho(t)}(\mathcal{V}(t))$ is given by:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta_{\rho(t)}(\mathcal{V}(t)) = \Delta_{\frac{\mathrm{D}\rho(t)}{\mathrm{D}t}}(\mathcal{V}(t)), \tag{2.2.1}$$

where:

$$\frac{\mathrm{D}\rho(t)}{\mathrm{D}t} = \frac{\partial\rho(t)}{\partial t} + \nabla \cdot (\rho(t)\,\boldsymbol{v}(t)). \tag{2.2.2}$$

Proof. We want to compute the time derivative of the scalar integral over a deforming volume. Using a Taylor expansion for small δt, we write:

$$\Delta_{\rho(t+\delta t)}(\mathcal{V}(t+\delta t)) = \Delta_{\rho(t)}(\mathcal{V}(t)) + \frac{\mathrm{d}}{\mathrm{d}t}\Delta_{\rho(t)}(\mathcal{V}(t))\delta t + O(\delta t^2),$$

which leads to:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta_{\rho(t)}(\mathcal{V}(t))\,\delta t = \int_{\mathcal{V}(t)} \mathrm{d}^3\mathbf{x}\,\frac{\partial\rho(t)}{\partial t}\delta t + \int_{\mathcal{V}(t+\delta t)-\mathcal{V}(t)} \mathrm{d}^3\mathbf{x}\,\rho(t) + O(\delta t^2).$$

The first term represents the change in the integrand (i.e., the local time derivative), while the second term accounts for the geometrical change of the control volume. We now estimate the volume swept by the boundary during time δt. Let $\mathbf{v}(t)$ be the velocity of the material points and $\mathbf{n}_{\mathcal{V}}$ the outward normal. Then the volume change over a small time is approximately:

$$\int_{\mathcal{V}(t+\delta t)-\mathcal{V}(t)} \mathrm{d}^3\mathbf{x}\,\rho(t) \approx \int_{\partial\mathcal{V}(t)} \mathrm{d}^2\mathbf{x}\,\mathbf{v}(t)\cdot\mathbf{n}_{\mathcal{V}}\,\delta t\,\rho(t).$$

Using the Gauss' theorem, we convert the surface integral to a volume integral:

$$\int_{\partial\mathcal{V}(t)} \mathrm{d}^2\mathbf{x}\,\rho(t)\,\mathbf{v}(t)\cdot\mathbf{n}_{\mathcal{V}}\,\delta t = \int_{\mathcal{V}(t)} \mathrm{d}^3\mathbf{x}\,\nabla\cdot(\rho(t)\,\mathbf{v}(t))\delta t.$$

Putting all terms together:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta_{\rho(t)}(\mathcal{V}(t))\,\delta t = \int_{\mathcal{V}(t)} \mathrm{d}^3\mathbf{x}\,\left(\frac{\partial\rho(t)}{\partial t} + \nabla\cdot(\rho(t)\,\mathbf{v}(t))\right)\delta t + O(\delta t^2).$$

Dividing by δt and taking the limit as $\delta t \to 0$ gives the desired result:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta_{\rho(t)}(\mathcal{V}(t)) = \Delta_{\frac{\partial\rho(t)}{\partial t}+\nabla\cdot(\rho(t)\,\mathbf{v}(t))}(\mathcal{V}(t)) = \Delta_{\frac{\mathrm{D}\rho(t)}{\mathrm{D}t}}(\mathcal{V}(t)).$$

□

Application Material derivative of the volume element.

The material derivative of the volume element $\mathrm{d}^3\mathbf{x}$ is a very useful result in continuum mechanics, as it allows us to compute the time evolution of the volume element in a moving and deforming domain. Let us consider the situation in which the density is $\rho(t) = 1$ costant in time. In this case, we have:

$$\begin{aligned}
\Delta_{\rho(t)}(\mathcal{V}(t)) = \int_{\mathcal{V}(t)} \mathrm{d}^3\mathbf{x} &= \mathrm{vol}(\mathcal{V}(t)),\\
\frac{\mathrm{D}\rho(t)}{\mathrm{D}t} &= \nabla\cdot\mathbf{v}(t),\\
\Delta_{\frac{\mathrm{D}\rho(t)}{\mathrm{D}t}}(\mathcal{V}(t)) = \int_{\mathcal{V}(t)} \mathrm{d}^3\mathbf{x}\,\nabla\cdot\mathbf{v}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}(t)} \mathrm{d}^3\mathbf{x}\,.
\end{aligned}$$

Now just by choosing $\mathcal{V}(t)$ as an elemental volume $\mathrm{d}^3\mathbf{x}$, we obtain the useful result:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}^3\mathbf{x}) = \nabla\cdot\mathbf{v}(t)\mathrm{d}^3\mathbf{x}. \tag{2.2.3}$$

Moving and deforming surfaces

We will now introduce the material derivative of a field defined on a moving and deforming surface, and we will see how to compute it using the Gauss' theorem.

Theorem 2.2 (Material derivative over a deforming surface)**.** *Let $\boldsymbol{a}(t)$ be a time dependent vector density field, and let $\mathcal{A}(t)$ be a moving and deforming surface, each of whose material points follows the velocity field $\boldsymbol{v}(t)$. Then, the material derivative of the flux integral $\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\boldsymbol{a}(t)}(\mathcal{A}(t))$ is given by:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\boldsymbol{a}(t)}(\mathcal{A}(t)) = \Phi_{\frac{\mathrm{D}\boldsymbol{a}(t)}{\mathrm{D}t}}(\mathcal{A}(t)), \tag{2.2.4}$$

where:

$$\frac{\mathrm{D}\boldsymbol{a}(t)}{\mathrm{D}t} = \frac{\partial \boldsymbol{a}(t)}{\partial t} + (\nabla \cdot \boldsymbol{a}(t))\boldsymbol{v}(t) + \nabla \times (\boldsymbol{a}(t) \times \boldsymbol{v}(t)). \tag{2.2.5}$$

Proof. To compute the material derivative $\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\mathbf{a}(t)}(\mathcal{A}(t))$, we use a Taylor expansion for small δt:

$$\Phi_{\mathbf{a}(t+\delta t)}(\mathcal{A}(t+\delta t)) = \Phi_{\mathbf{a}(t)}(\mathcal{A}(t)) + \frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\mathbf{a}(t)}(\mathcal{A}(t))\delta t + O(\delta t^2),$$

which leads to:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\mathbf{a}(t)}(\mathcal{A}(t))\delta t = \int_{\mathcal{A}(t)} \mathrm{d}^2\mathbf{x} \cdot \frac{\partial}{\partial t}\mathbf{a}(t)\delta t + \int_{\mathcal{A}(t+\delta t)-\mathcal{A}(t)} \mathrm{d}^2\mathbf{x} \cdot \mathbf{a}(t) + O(\delta t^2).$$

The first term captures the local time variation of the field $\mathbf{a}(t)$, while the second accounts for the change in surface area due to deformation and motion. To analyze the second term, we consider a volume $\mathcal{V}(t+\delta t, t)$ swept by the motion of the surface $\mathcal{A}(t)$ over the interval $[t, t+\delta t]$. The contribution of this change can be split into:

$$\oint_{\partial\mathcal{V}(t+\delta t,t)} \mathrm{d}^2\mathbf{x} \cdot \mathbf{a}(t) - \oint_{\Omega(t)} \mathrm{d}^2\mathbf{x} \cdot \mathbf{a}(t) + O(\delta t^2),$$

where $\Omega(t) = \partial\mathcal{V}(t+\delta t, t) - \mathcal{A}(t+\delta t) + \mathcal{A}(t)$ represents the boundary of $\mathcal{V}(t+\delta t, t)$ enclosed between $\mathcal{A}(t)$ and $\mathcal{A}(t+\delta t)$. Applying the Gauss' theorem to the first term gives:

$$\oint_{\partial\mathcal{V}(t+\delta t,t)} \mathrm{d}^2\mathbf{x} \cdot \mathbf{a}(t) = \int_{\mathcal{V}(t+\delta t,t)} \mathrm{d}^3\mathbf{x}\, \nabla \cdot \mathbf{a}(t) \approx \int_{\mathcal{A}(t)} \mathrm{d}^2\mathbf{x} \cdot (\nabla \cdot \mathbf{a}(t))\mathbf{v}(t)\, \delta t + O(\delta t^2).$$

For the second term, we deal with the surface $\Omega(t)$, which arises from the lateral motion of the boundary $\partial\mathcal{A}(t)$. Using the Stokes theorem we have:

$$\begin{aligned} -\oint_{\Omega(t)} \mathrm{d}^2\mathbf{x} \cdot \mathbf{a}(t) &\approx -\oint_{\partial\mathcal{A}(t)} \mathrm{d}^1\mathbf{x} \cdot (\mathbf{a}(t) \times \mathbf{v}(t)) + O(\delta t^2) \\ &= \int_{\mathcal{A}(t)} \mathrm{d}^2\mathbf{x} \cdot \nabla \times (\mathbf{a}(t) \times \mathbf{v}(t))\, \delta t + O(\delta t^2). \end{aligned}$$

Putting all contributions together:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\mathbf{a}(t)}(\mathcal{A}(t))\, \delta t = \int_{\mathcal{A}(t)} \mathrm{d}^2\mathbf{x} \cdot \left[\frac{\partial \mathbf{a}(t)}{\partial t} + (\nabla \cdot \mathbf{a}(t))\, \mathbf{v}(t) + \nabla \times (\mathbf{a}(t) \times \mathbf{v}(t))\right] \delta t + O(\delta t^2).$$

Dividing by δt and taking the limit as $\delta t \to 0$, we obtain the desired result:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\mathbf{a}(t)}(\mathcal{A}(t)) = \Phi_{\frac{\mathrm{D}\mathbf{a}(t)}{\mathrm{D}t}}(\mathcal{A}(t)).$$

□

Application Material derivative of the surface element.

The material derivative of the surface element $\mathrm{d}^2\mathbf{x}$ is a very useful result in continuum mechanics, as it allows us to compute the time evolution of the surface element in a moving and deforming domain. Let us consider the situation in which the density is $\mathbf{a}(t) = \mathbf{c}$ costant in time. In this case, we have:

$$\begin{aligned}
\Phi_{\mathbf{a}(t)}(\mathcal{A}(t)) &= \int_{\mathcal{A}(t)} \mathrm{d}^2\mathbf{x} \cdot \mathbf{c} = \mathbf{c} \cdot \int_{\mathcal{A}(t)} \mathrm{d}^2\mathbf{x}\,, \\
\frac{\mathrm{D}\mathbf{a}(t)}{\mathrm{D}t} &= \nabla \times (\mathbf{c} \times \mathbf{v}(t)) = (\nabla \cdot \mathbf{v}(t))\mathbf{c} - (\mathbf{c} \cdot \nabla)\mathbf{v}(t), \\
\Phi_{\frac{\mathrm{D}\mathbf{a}(t)}{\mathrm{D}t}}(\mathcal{A}(t)) &= \int_{\mathcal{A}(t)} \mathrm{d}^2\mathbf{x} \cdot [(\nabla \cdot \mathbf{v}(t))\mathbf{c} - (\mathbf{c} \cdot \nabla)\mathbf{v}(t)] \\
&= -\mathbf{c} \cdot \int_{\mathcal{A}(t)} \mathrm{d}^2\mathbf{x} \times \nabla \times \mathbf{v}(t).
\end{aligned}$$

Now just by choosing $\mathcal{A}(t)$ an elemental surface, we obtain the useful identity:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}^2\mathbf{x}) = -(\mathrm{d}^2\mathbf{x} \times \nabla) \times \mathbf{v}(t). \tag{2.2.6}$$

Moving and deforming lines

In this section we will introduce the material derivative of a field defined on a moving and deforming line, and we will see how to compute it using Stokes theorem.

Theorem 2.3 (Material derivative over a deforming line)**.** *Let $\boldsymbol{b}(t)$ be a time dependent vector density field and $\mathcal{L}(t)$ a moving and deforming line, each of its points following the velocity field $\boldsymbol{v}(t)$. Then the material derivative of the line integral $\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_{\boldsymbol{b}(t)}(\mathcal{L}(t))$ is given by:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_{\boldsymbol{b}(t)}(\mathcal{L}(t)) = \Gamma_{\frac{\mathrm{D}\boldsymbol{b}(t)}{\mathrm{D}t}}(\mathcal{L}(t)), \tag{2.2.7}$$

where, again:

$$\frac{\mathrm{D}\boldsymbol{b}(t)}{\mathrm{D}t} = \frac{\partial \boldsymbol{b}(t)}{\partial t} + (\nabla \cdot \boldsymbol{b}(t))\boldsymbol{v}(t) + \nabla \times (\boldsymbol{b}(t) \times \boldsymbol{v}(t)).$$

Proof. In order to compute the time derivative $\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_{\mathbf{b}(t)}(\mathcal{L}(t))$ of the line integral, we expand its difference quotient in Taylor series:

$$\Gamma_{\mathbf{b}(t+\delta t)}(\mathcal{L}(t+\delta t)) = \Gamma_{\mathbf{b}(t)}(\mathcal{L}(t)) + \frac{\mathrm{d}}{\mathrm{d}t}\Gamma_{\mathbf{b}(t)}(\mathcal{L}(t))\,\delta t + O(\delta t^2),$$

which leads to:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_{\mathbf{b}(t)}(\mathcal{L}(t))\,\delta t = \int_{\mathcal{L}(t)} \mathrm{d}^1\mathbf{x}\cdot\frac{\partial}{\partial t}\mathbf{b}(t)\,\delta t + \int_{\mathcal{L}(t+\delta t)-\mathcal{L}(t)} \mathrm{d}^1\mathbf{x}\cdot\mathbf{b}(t) + O(\delta t^2).$$

Let us now focus on the second term, which accounts for the deformation of the line:

$$\int_{\mathcal{L}(t+\delta t)-\mathcal{L}(t)} \mathrm{d}^1\mathbf{x}\cdot\mathbf{b}(t) = \oint_{\partial\mathcal{A}(t+\delta t,\,t)} \mathrm{d}^1\mathbf{x}\cdot\mathbf{b}(t) - \int_{\omega(t)} \mathrm{d}^1\mathbf{x}\cdot\mathbf{b}(t) + O(\delta t^2),$$

where $\mathcal{A}(t+\delta t,\,t)$ is the infinitesimal surface swept by the deforming line $\mathcal{L}(t)$ over the time step δt, and $\omega(t) = \partial\mathcal{A}(t+\delta t) - \mathcal{L}(t+\delta t) + \mathcal{L}(t)$ is the curve which represents the boundary of $\mathcal{A}(t+\delta t,\,t)$ enclosed between $\mathcal{L}(t)$ and $\mathcal{L}(t+\delta t)$. For the first term, we apply Stokes' theorem:

$$\begin{aligned}\oint_{\partial\mathcal{A}(t+\delta t,\,t)} \mathrm{d}^1\mathbf{x}\cdot\mathbf{b}(t) &= \int_{\mathcal{A}(t+\delta t,\,t)} \mathrm{d}^2\mathbf{x}\cdot(\nabla\times\mathbf{b}(t)) \\ &\approx \int_{\mathcal{L}(t)} \mathrm{d}^1\mathbf{x}\times(\mathbf{v}(t)\,\delta t)\cdot(\nabla\times\mathbf{b}(t)) + O(\delta t^2).\end{aligned}$$

For the second term, representing the correction from the mismatch of endpoints, an additional contribution that arises when the endpoints of the line move in time (if the curve is open), we have:

$$-\int_{\omega(t)} \mathrm{d}^1\mathbf{x}\cdot\mathbf{b}(t) = -\int_{\partial\mathcal{L}(t)} \mathrm{d}^0\mathbf{x}\,\boldsymbol{\tau}\,(\mathbf{v}(t)\cdot\mathbf{b}(t))\,\delta t = -\sum_{\text{endpoints}} \boldsymbol{\tau}_i\,(\mathbf{v}_i\cdot\mathbf{b}_i)\,\delta t + O(\delta t^2),$$

where $\boldsymbol{\tau}_i$ is the oriented tangent vector at the endpoint and, as shown, it captures the contribution from the endpoints' motion. Under smoothness assumptions, this contribution can be rewritten as a line integral along $\mathcal{L}(t)$:

$$-\int_{\omega(t)} \mathrm{d}^1\mathbf{x}\cdot\mathbf{b}(t) \approx \int_{\mathcal{L}(t)} \mathrm{d}^1\mathbf{x}\cdot\nabla\,(\mathbf{b}(t)\times\mathbf{v}(t))\;\delta t.$$

Remark. *If the line $\mathcal{L}(t)$ is open and its endpoints move with the flow, their contribution to the material derivative can be approximated as a bulk line integral:*

$$-\int_{\omega(t)} \mathrm{d}^1\mathbf{x}\cdot\boldsymbol{b}(t) \approx \int_{\mathcal{L}(t)} \mathrm{d}^1\mathbf{x}\cdot\nabla\,(\boldsymbol{b}(t)\times\boldsymbol{v}(t))\,\delta t.$$

This reduces the boundary correction to a term compatible with the form of the material derivative, under regularity of $\boldsymbol{b}(t)$ and $\boldsymbol{v}(t)$.

Putting everything together:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_{\mathbf{b}(t)}(\mathcal{L}(t))\delta t = \int_{\mathcal{L}(t)} \mathrm{d}^1\mathbf{x}\cdot\left[\frac{\partial\mathbf{b}(t)}{\partial t} + (\nabla\times\mathbf{b}(t))\times\mathbf{v}(t) + \nabla\,(\mathbf{b}(t)\times\mathbf{v}(t))\right]\delta t + O(\delta t^2),$$

Dividing by δt and taking the limit as $\delta t\to 0$, we obtain the desired result:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_{\mathbf{b}(t)}(\mathcal{L}(t)) = \Gamma_{\frac{\mathrm{D}\mathbf{b}(t)}{\mathrm{D}t}}(\mathcal{L}(t)).$$

□

Application Material derivative of the line element.

The material derivative of the line element $\mathrm{d}^1\mathbf{x}$ is a very useful result in continuum mechanics, as it allows us to compute the time evolution of the line element in a moving and deforming domain. Let us consider the situation in which the density is $\mathbf{b}(t) = \mathbf{c}$ costant in time. In this case, we have:

$$\Gamma_{\mathbf{b}(t)}(\mathcal{L}(t)) = \int_{\mathcal{L}(t)} \mathrm{d}^1\mathbf{x} \cdot \mathbf{c} = \mathbf{c} \cdot \int_{\mathcal{L}(t)} \mathrm{d}^1\mathbf{x}\,,$$
$$\frac{\mathrm{D}\mathbf{b}(t)}{\mathrm{D}t} = \nabla(\mathbf{c} \cdot \mathbf{v}(t)) = \nabla \otimes \mathbf{v}(t) \cdot \mathbf{c},$$
$$\Phi_{\frac{\mathrm{D}\mathbf{b}(t)}{\mathrm{D}t}}(\mathcal{L}(t)) = \int_{\mathcal{L}(t)} \mathrm{d}^1\mathbf{x} \cdot [\nabla \otimes (\mathbf{v}(t) \cdot \mathbf{c})] = \mathbf{c} \cdot \int_{\mathcal{L}(t)} \mathrm{d}^1\mathbf{x} \cdot \nabla\mathbf{v}(t).$$

Now just by choosing $\mathcal{A}(t)$ an elemental surface, we obtain the useful identity:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}^1\mathbf{x}) = (\mathrm{d}^1\mathbf{x} \cdot \nabla)\mathbf{v}(t). \tag{2.2.8}$$

2.3 Distributional Calculus across Interfaces

In this section we will introduce the distributional approach to the kinematics of volumes, surfaces and curves, which will allow us to treat discontinuities in a more rigorous way, generating a framework that can be applied to the study of the electrodynamics of continua. Let $\mathcal{V}$ be a moving and deforming volume in space, $\mathbf{w}$ the velocity field defined on its boundary $\partial\mathcal{V}$ with $\mathbf{n}_{\mathcal{V}}$[1] its outward normal unit vector.[2]

Definition 5. *Given the test function ϕ, the characteristic function $\chi_{\mathcal{V}}$ of $\mathcal{V}$ is defined as:*

$$\chi_{\mathcal{V}}(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x} \in \mathcal{V} \\ 0 & \text{if } \boldsymbol{x} \notin \mathcal{V} \end{cases}; \qquad \int \mathrm{d}^3\mathbf{x}\,(\phi\,\chi_{\mathcal{V}}) = \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\,\phi. \tag{2.3.1}$$

The characteristic function $\chi_{\mathcal{V}}$ is a generalized function with support on the volume $\mathcal{V}$, and it is used to restrict integrals to the region of interest.

Definition 6. *For our purposes, it is sufficient to adopt an operational viewpoint on the Dirac delta function supported on a surface. We define the surface delta distribution $\delta_{\partial\mathcal{V}}(\boldsymbol{x})$ as a generalized function with support on the boundary $\partial\mathcal{V}$ of a volume $\mathcal{V}$, such that:*

$$\delta_{\partial\mathcal{V}}(\boldsymbol{x}) = \begin{cases} \infty & \text{if } \boldsymbol{x} \in \partial\mathcal{V} \\ 0 & \text{if } \boldsymbol{x} \notin \partial\mathcal{V} \end{cases}; \qquad \int \mathrm{d}^3\mathbf{x}\,(\phi\,\delta_{\partial\mathcal{V}}) = \oint_{\partial\mathcal{V}} \mathrm{d}^2x\,\phi. \tag{2.3.2}$$

That is, $\delta_{\partial\mathcal{V}}(\boldsymbol{x})$ vanishes away from the boundary, is formally infinite on it, and acts as a distribution that "picks out" the value of test functions ϕ on $\partial\mathcal{V}$ when integrated over the

[1] In this section we will use the explicit unit vectors normal to the surface, letting $\mathrm{d}^2\mathrm{x} = \mathrm{d}A$ be just an elemental surface.

[2] In order to lighten up the notation, let us omit the explicit time dependencies in this section.

whole space. It satisfies the usual properties of delta distributions under integration and is instrumental in expressing surface densities embedded in a volume formulation.

It is now natural to expose the relations between the characteristic function $\chi_{\mathcal{V}}$ and the Dirac delta distribution $\delta_{\partial\mathcal{V}}$ in order to obtain two useful lemmas that will be employed in the description of fields defined on moving and deforming volumes, surfaces and curves.

Lemma 2.4. *Given the characteristic function $\chi_{\mathcal{V}}$ of a regular volume $\mathcal{V}$ and the associated surface delta distribution $\delta_{\partial\mathcal{V}}$, the following identity holds:*

$$\nabla\chi_{\mathcal{V}} = -\boldsymbol{n}\,\delta_{\partial\mathcal{V}}. \tag{2.3.3}$$

Proof. Let $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ be a smooth test function with compact support (??). Consider the integral over all space:

$$\int \mathrm{d}^3\mathbf{x}\,(\phi\,\nabla\chi_{\mathcal{V}}) = -\int \mathrm{d}^3\mathbf{x}\,(\chi_{\mathcal{V}}\,\nabla\phi),$$

where we integrated by parts and used the fact that ϕ vanishes at infinity. Since $\chi_{\mathcal{V}}$ is the characteristic function of the volume, the integral reduces to:

$$-\int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\,\nabla\phi = -\oint_{\partial\mathcal{V}} \mathrm{d}^2\mathrm{x}\,\mathbf{n}\,\phi,$$

where we applied the Gauss' theorem. On the other hand, using the definition of the surface delta distribution:

$$\int \mathrm{d}^3\mathbf{x}\,(\phi\,\mathbf{n}\,\delta_{\partial\mathcal{V}}) = \oint_{\partial\mathcal{V}} \mathrm{d}^2\mathrm{x}\,\mathbf{n}\,\phi.$$

Therefore:

$$\int \mathrm{d}^3\mathbf{x}\,(\phi\,\nabla\chi_{\mathcal{V}}) = -\int \mathrm{d}^3\mathbf{x}\,(\phi\,\mathbf{n}\,\delta_{\partial\mathcal{V}}),$$

and since this holds for all test functions ϕ, we conclude that:

$$\nabla\chi_{\mathcal{V}} = -\mathbf{n}\,\delta_{\partial\mathcal{V}},$$

in the sense of distributions. □

Lemma 2.5. *Given the characteristic function $\chi_{\mathcal{V}}$ of a regular volume $\mathcal{V}$ and the associated surface delta distribution $\delta_{\partial\mathcal{V}}$, the following identity holds:*

$$\frac{\partial}{\partial t}\chi_{\mathcal{V}} = \boldsymbol{n}\cdot\boldsymbol{w}\,\delta_{\partial\mathcal{V}}, \tag{2.3.4}$$

where $\boldsymbol{w}$ is the velocity of the moving boundary $\partial\mathcal{V}$.

Proof. Let $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ be a time independent test function. Then:

$$\int \mathrm{d}^3\mathbf{x}\,\left(\phi\,\frac{\partial}{\partial t}\chi_{\mathcal{V}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3\mathbf{x}\,(\phi\,\chi_{\mathcal{V}})\,.$$

The right-hand side represents the time derivative of an integral over a moving domain. The variation of the volume integral over time is equivalent to the flux of the scalar field through the boundary $\partial\mathcal{V}$, which moves with velocity $\mathbf{w}$; therefore, only the components of the velocity field normal to the boundary contribute to the flux and must be taken into account:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\,\phi = \oint_{\partial\mathcal{V}} \mathrm{d}^2\mathrm{x}\,(\mathbf{n}\cdot\mathbf{w})\,\phi.$$

Now recall the definition of the surface delta function:

$$\int \mathrm{d}^3\mathbf{x}\ (\phi\,(\mathbf{n}\cdot\mathbf{w})\,\delta_{\partial\mathcal{V}}) = \oint_{\partial\mathcal{V}} \mathrm{d}^2\mathrm{x}\,(\mathbf{n}\cdot\mathbf{w})\,\phi.$$

Therefore:

$$\int \mathrm{d}^3\mathbf{x}\,\left(\phi\,\frac{\partial}{\partial t}\chi_{\mathcal{V}}\right) = \int \mathrm{d}^3\mathbf{x}\ (\phi\,\mathbf{n}\cdot\mathbf{w}\,\delta_{\partial\mathcal{V}})\,.$$

Since this equality holds for all test functions ϕ, we conclude that:

$$\frac{\partial}{\partial t}\chi_{\mathcal{V}} = \mathbf{n}\cdot\mathbf{w}\,\delta_{\partial\mathcal{V}},$$

in the sense of distributions. □

Let us consider now Euclidean tridimensional space $\mathbb{E}^3$, we assume to partition it into a set of non-overlapping open regular regions $\mathcal{V}_i$ such that:

$$\begin{cases} \mathcal{V}_i \cap \mathcal{V}_j = \emptyset & \text{if } i \neq j, \\ \mathbb{E}^3 = \bigcup_i \bar{\mathcal{V}}_i. \end{cases}$$

We will use the previuosly defined quantities with pedex i in order to denote the partition $\mathcal{V}_i$ to which they refer to: $\partial\mathcal{V}_i$ the boundary of $\mathcal{V}_i$, $\mathbf{n}_i$ the outward normal unit vector to $\partial\mathcal{V}_i$, $\mathbf{w}_i$ the velocity field defined on $\partial\mathcal{V}_i$. We now define $\mathcal{D}$ the two dimensional subregion union of all such boundaries,

$$\mathcal{D} = \bigcup_i \partial\mathcal{V}_i$$

which, expressed as it is, may involve repeated surfaces where two neighboring regions share a common boundary; in order to remove this redundancy, we assume $\mathcal{D}$ to be partitioned into a collection of non-overlapping open regions $\mathcal{D}_a$ such that:

$$\begin{cases} \mathcal{D}_a \cap \mathcal{D}_b = \emptyset & \text{if } a \neq b, \\ \mathcal{D} = \bigcup_a \bar{\mathcal{D}}_a. \end{cases}$$

Accordingly, we can define $\mathbf{n}_a = \mathbf{n}\big|_{\mathcal{D}_a}$ and $\mathbf{w}_a = \mathbf{w}\big|_{\mathcal{D}_a}$.[3] We now introduce the indices $i_a^{\pm}$ to identify the two regions sharing $\mathcal{D}_a$ as a common boundary:

$$\partial\mathcal{V}_{i_a^+} \cap \partial\mathcal{V}_{i_a^-} = \mathcal{D}_a$$
$$\mathbf{n}_a = -\mathbf{n}_{i_a^+} = \mathbf{n}_{i_a^-}$$
$$\mathbf{w}_a = \mathbf{w}_{i_a^+} = \mathbf{w}_{i_a^-}.$$

[3] Defined only on the interior $\bigcup_a \mathcal{D}_a$, not on the edges $\mathcal{L} = \bigcup_a \partial\mathcal{D}_a$.

The notation i_a^+ and i_a^- allows us to refer consistently to the two regions adjacent to an interface $\mathcal{D}_a$, and the corresponding orientation of normal vectors is chosen such that $\mathbf{n}_a$ always points from i_a^+ to i_a^-. This convention simplifies the expression of fluxes and jumps across boundaries.

In order to calculate averages and various statistical moments of microscopic fields, let Φ be a test function assumed to be smooth on each closure $\overline{\mathcal{V}_i}$ for every i, and defined piecewise as $\Phi\big|_{\mathcal{V}_i} = \Phi_i$. We allow for discontinuities of the test function Φ across region boundaries; in this case, the field remains smooth inside each $\mathcal{V}_i$, but may have finite *jumps* on interfaces $\mathcal{D}_a$. On each interfacial region $\mathcal{D}_a$, we define the jump of Φ as:

$$\Delta\Phi\big|_{\mathcal{D}_a} = [\![\Phi]\!]_{\mathcal{D}_a} = \Phi_{i_a^+} - \Phi_{i_a^-} \neq 0.$$

Now any Φ can be associated to a *distribution* as:

$$\hat{\Phi} = \sum_i \Phi_i \chi_{\mathcal{V}_i}. \tag{2.3.5}$$

Now we can introduce the following lemmas, which will be useful in the description of fields defined on moving and deforming volumes, surfaces and curves.

Lemma 2.6 (Distributional gradient of a piecewise field)**.** *For every scalar distribution $\hat{\Phi}$, the following relation holds:*

$$\nabla\hat{\Phi} = \widehat{\nabla\Phi} + [\![\Phi]\!]\ \boldsymbol{n}\,\delta_{\mathcal{D}}. \tag{2.3.6}$$

Proof. The scalar distribution $\hat{\Phi}$ is written as the sum:

$$\hat{\Phi} = \sum_i \Phi_i \chi_{\mathcal{V}_i},$$

We apply the gradient operator using the product rule for distributions:

$$\nabla\hat{\Phi} = \nabla \sum_i \Phi_i \chi_{\mathcal{V}_i} = \sum_i \left((\nabla\Phi_i)\chi_{\mathcal{V}_i} + \Phi_i \nabla\chi_{\mathcal{V}_i}\right).$$

The first term, $\sum_i \nabla\Phi_i\, \chi_{\mathcal{V}_i}$, simply corresponds to the piecewise gradient field, which we denote as $\widehat{\nabla\Phi}$. The second term accounts for the distributional effects caused by the discontinuity of Φ across the internal boundaries. Using lemma 2.4, we substitute:

$$\nabla\chi_{\mathcal{V}_i} = -\mathbf{n}_i\, \delta_{\partial\mathcal{V}_i},$$

so that:

$$\sum_i \Phi_i \nabla\chi_{\mathcal{V}_i} = -\sum_i \Phi_i\, \mathbf{n}_i\, \delta_{\partial\mathcal{V}_i}.$$

Now, observe that the sum over the $\partial\mathcal{V}_i$ can be reorganized as a sum over the interface elements $\mathcal{D}_a$. Since each internal surface $\mathcal{D}_a$ is shared by exactly two regions $\mathcal{V}_{i_a^+}$ and $\mathcal{V}_{i_a^-}$, and recalling that $\mathbf{n}_{i_a^+} = -\mathbf{n}_a$ and $\mathbf{n}_{i_a^-} = \mathbf{n}_a$, we have:

$$-\sum_i \Phi_i\, \mathbf{n}_i\, \delta_{\partial\mathcal{V}_i} = \sum_a \left(\Phi_{i_a^+} - \Phi_{i_a^-}\right) \mathbf{n}_a\, \delta_{\mathcal{D}_a} = \sum_a [\![\Phi]\!]_{\mathcal{D}_a}\ \mathbf{n}\big|_{\mathcal{D}_a}\, \delta_{\mathcal{D}_a}.$$

The result is then:

$$\nabla\hat{\Phi} = \widehat{\nabla\Phi} + [\![\Phi]\!]\ \mathbf{n}\,\delta_{\mathcal{D}},$$

where $\delta_{\mathcal{D}} = \sum_a \delta_{\mathcal{D}_a}$ is the singular surface distribution supported on all the internal boundaries. □

Lemma 2.7 (Distributional time derivative of a piecewise field)**.** *For every scalar distribution $\hat{\Phi}$, the following relation holds:*

$$\frac{\partial}{\partial t}\hat{\Phi} = \widehat{\frac{\partial\Phi}{\partial t}} - [\![\Phi]\!]\ \boldsymbol{n}\cdot\boldsymbol{w}\,\delta_{\mathcal{D}}. \tag{2.3.7}$$

Proof. Let us consider the time derivative of the distribution $\hat{\Phi} = \sum_i \Phi_i \chi_{\mathcal{V}_i}$, where the regions $\mathcal{V}_i$ evolve in time, and Φ_i is a smooth function in each $\mathcal{V}_i$. Using the Leibniz rule, we write:

$$\frac{\partial}{\partial t}\hat{\Phi} = \frac{\partial}{\partial t}\sum_i \Phi_i\chi_{\mathcal{V}_i} = \sum_i \left(\frac{\partial\Phi_i}{\partial t}\chi_{\mathcal{V}_i} + \Phi_i\frac{\partial\chi_{\mathcal{V}_i}}{\partial t}\right).$$

The first term corresponds to the time derivative of the field within each region: $\widehat{\frac{\partial\Phi}{\partial t}}$. The second term arises due to the motion of the regions $\mathcal{V}_i$ in time. From lemma 2.5 we know:

$$\frac{\partial\chi_{\mathcal{V}_i}}{\partial t} = \mathbf{n}_i\cdot\mathbf{w}_i\,\delta_{\partial\mathcal{V}_i},$$

which gives:

$$\sum_i \Phi_i\frac{\partial\chi_{\mathcal{V}_i}}{\partial t} = \sum_i \Phi_i\,\mathbf{n}_i\cdot\mathbf{w}_i\,\delta_{\partial\mathcal{V}_i}.$$

Now, we reorganize this sum as before, converting it into a sum over the interfaces $\mathcal{D}_a$ which lie between two adjacent regions $\mathcal{V}_{i_a^\pm}$. Recalling the jump notation and interface conventions:

$$\sum_i \Phi_i\,\mathbf{n}_i\cdot\mathbf{w}_i\,\delta_{\partial\mathcal{V}_i} = -\sum_a [\![\Phi]\!]_{\mathcal{D}_a}\ \mathbf{n}\big|_{\mathcal{D}_a}\cdot\mathbf{w}\big|_{\mathcal{D}_a}\,\delta_{\mathcal{D}_a}.$$

Therefore, we conclude:

$$\frac{\partial}{\partial t}\hat{\Phi} = \widehat{\frac{\partial\Phi}{\partial t}} - \sum_a [\![\Phi]\!]_{\mathcal{D}_a}\ \mathbf{n}_a\cdot\mathbf{w}_a\,\delta_{\mathcal{D}_a} = \widehat{\frac{\partial\Phi}{\partial t}} - [\![\Phi]\!]\ \mathbf{n}\cdot\mathbf{w}\,\delta_{\mathcal{D}},$$

which proves the lemma. □

Let now $\mathbf{v}$ be the *matter velocity field* and ϕ a field, both following a first order differential equation of the form:

$$E(\frac{\partial}{\partial t};\,\nabla;\,\mathbf{v})\phi = \sigma.$$

Assuming both $\mathbf{v}$ and ϕ smooth except across discontinuities surfaces $\mathcal{D}$, then we should replace them with the corresponding distributions:

$$E(\frac{\partial}{\partial t};\,\nabla;\,\hat{\mathbf{v}})\phi = \hat{\sigma}_v + \hat{\sigma}_s\delta_{\mathcal{D}}.$$

We have to pay attention to distribution products, usually ill-defined, while we study distributional fist order differential equation such this.

Application Junction conditions for a scalar field.

Suppose f is a smooth scalar field satisfying the equation:

$$\frac{\partial f}{\partial t} + \nabla \cdot (\mathbf{v} f) = \sigma.$$

which in the distributional framework becomes:

$$\frac{\partial \hat{f}}{\partial t} + \nabla \cdot (\widehat{\mathbf{v} f}) = \hat{\sigma}_v + \hat{\sigma}_s \delta_{\mathcal{D}}.$$

Note: we are allowed to identify $\hat{\mathbf{v}}\hat{f} = \widehat{\mathbf{v}f}$ in this case due to smoothness and locality of f and $\mathbf{v}$.[4]

Applying lemma 2.6 and 2.7, we compute:

$$\frac{\partial \hat{f}}{\partial t} + \nabla \cdot (\widehat{\mathbf{v} f}) = \widehat{\frac{\partial f}{\partial t} + \nabla \cdot (\mathbf{v} f)} + \mathbf{n} \cdot (\llbracket \mathbf{v} f \rrbracket - \mathbf{w} \llbracket f \rrbracket) \, \delta_{\mathcal{D}}.$$

Thus, identifying volume and surface contributions, we obtain:

$$\begin{cases} \frac{\partial f}{\partial t} + \nabla \cdot (\mathbf{v} f) = \sigma_v, & \text{on } \mathcal{V} - \mathcal{D}; \\ \mathbf{n} \cdot \llbracket (\mathbf{v} - \mathbf{w}) f \rrbracket = \sigma_s, & \text{on } \mathcal{D} - \mathcal{L}. \end{cases} \tag{2.3.8}$$

Remark. *The interfacial velocity $\boldsymbol{w}$ is well-defined relative to the indices $i_a^{\pm}$, making the jump uniquely interpretable across each $\mathcal{D}_a$ and letting us take $\boldsymbol{w}$ inside the jump brackets.*

Application Junction conditions for a vector field.

Let now $\mathbf{a}$ be a vector field satisfying the equations:

$$\begin{cases} \nabla \cdot \mathbf{a} = \alpha, \\ \dfrac{\partial \mathbf{a}}{\partial t} + \nabla \times (\mathbf{a} \times \mathbf{v}) + (\nabla \cdot \mathbf{a}) \mathbf{v} = \boldsymbol{\beta}. \end{cases}$$

In the distributional extension, we formally write:

$$\begin{cases} \nabla \cdot \hat{\mathbf{a}} = \hat{\alpha}_v + \hat{\alpha}_s \delta_{\mathcal{D}}, \\ \dfrac{\partial \hat{\mathbf{a}}}{\partial t} + \nabla \times (\hat{\mathbf{a}} \times \hat{\mathbf{v}}) + (\nabla \cdot \hat{\mathbf{a}}) \hat{\mathbf{v}} = \hat{\boldsymbol{\beta}}_v + \hat{\boldsymbol{\beta}}_s \delta_{\mathcal{D}}. \end{cases}$$

However, the product $(\nabla \cdot \hat{\mathbf{a}}) \hat{\mathbf{v}}$ is generally ill-defined in the theory of distributions. To circumvent this, we define a modified source term:

$$\boldsymbol{\beta}' := \boldsymbol{\beta} - \alpha \mathbf{v},$$

so that the second equation becomes:

$$\frac{\partial \mathbf{a}}{\partial t} + \nabla \times (\mathbf{a} \times \mathbf{v}) = \boldsymbol{\beta}'.$$

[4] This equivalence does not generally hold for nonlinear products of distributions, but here we benefit from f being regular across $\mathcal{D}$.

Now, the distributional form reads:

$$\begin{cases} \nabla \cdot \hat{\mathbf{a}} = \hat{\alpha}_v + \hat{\alpha}_s \delta_{\mathcal{D}}, \\ \dfrac{\partial \hat{\mathbf{a}}}{\partial t} + \nabla \times \widehat{(\mathbf{a} \times \mathbf{v})} = \hat{\boldsymbol{\beta}'}_v + \hat{\boldsymbol{\beta}'}_s \delta_{\mathcal{D}}. \end{cases}$$

Again using lemma 2.6 and 2.7, we compute:

$$\frac{\partial \hat{\mathbf{a}}}{\partial t} + \nabla \times \widehat{(\mathbf{a} \times \mathbf{v})} = \frac{\partial \mathbf{a}}{\partial t} + \widehat{\nabla \times (\mathbf{a} \times \mathbf{v})} + (\mathbf{n} \times [\![\mathbf{a} \times \mathbf{v}]\!] - \mathbf{n} \cdot \mathbf{w} [\![\mathbf{a}]\!]) \, \delta_{\mathcal{D}}.$$

Hence, the jump conditions across the interface become:

$$\begin{cases} \nabla \cdot \mathbf{a} = \alpha_v, & \text{on } \mathcal{V} - \mathcal{D}; \\ \mathbf{n} \cdot [\![\mathbf{a}]\!] = \alpha_s, & \text{on } \mathcal{D} - \mathcal{L}, \end{cases} \tag{2.3.9}$$

$$\begin{cases} \dfrac{\partial \mathbf{a}}{\partial t} + \nabla \times (\mathbf{a} \times \mathbf{v}) + (\nabla \cdot \mathbf{a})\mathbf{v} = \boldsymbol{\beta}'_v + \alpha_v \mathbf{v}, & \text{on } \mathcal{V} - \mathcal{D}; \\ \mathbf{n} \times [\![\mathbf{a} \times (\mathbf{v} - \mathbf{w})]\!] = \boldsymbol{\beta}'_s + \alpha_s \mathbf{w}, & \text{on } \mathcal{D} - \mathcal{L}. \end{cases} \tag{2.3.10}$$

These interface conditions illustrate how discontinuities in fluxes and fields across evolving boundaries encode surface sources, intrinsic to the dynamics of piecewise-smooth systems.

3 | Thermomechanics of Continua

In this chapter, we introduce the fundamental principles that govern the thermomechanical behavior of deformable continua. Building upon the kinematic framework established in the previous chapter, we now focus on the physical laws that constrain the evolution of material systems—laws that express the conservation of mass, momentum, energy, and the irreversible production of entropy.

These principles are formulated in both local and global forms, depending on whether we consider infinitesimal material elements or extended control volumes. The derivation of balance equations is rooted in continuum mechanics and thermodynamics, where physical quantities are modeled as smooth fields over space and time, and interactions are mediated through forces, fluxes, and internal sources.

We begin by reviewing the general structure of conservation laws, followed by detailed discussions of the mechanical balance equations. These include the balances of mass, linear and angular momentum, and internal energy. Next, we introduce the basic principles of thermodynamics, Gibbs' principle and the first and second laws, and we conclude by deriving the entropy balance and the Clausius-Duhem inequality, a key tool for assessing thermodynamic consistency of constitutive models.

This chapter provides the theoretical foundation for the modeling of complex materials and physical processes, such as fluid flow, elasticity, heat conduction, and more generally, systems where mechanical and thermal effects are coupled.

3.1 Conservation Laws

In the thermomechanical study of continua, conservation laws serve as fundamental principles expressing the invariance of physical quantities, such as mass, momentum, or energy, when integrated over moving and deforming domains. These laws are formulated as balance relations, connecting the rate of change of a quantity inside a domain with the flux of that quantity across its boundary. We classify conservation laws based on the geometric nature of the domain involved:

- **Conservation laws of the first kind** concern scalar densities integrated over

control volumes.

- **Conservation laws of the second kind** concern vector densities integrated over control surfaces.

These two classes correspond to divergence and curl structures in the field equations, and are naturally derived using the Gauss' divergence theorem, and Stokes' theorem. We now present the general formulations.

Conservation law of the first kind

Let $\xi(t)$ be a time-dependent scalar density field, and let $\mathbf{a}(t)$ be a vector flux density field, both defined over a material volume $\mathcal{V}(t)$ which evolves with a velocity field $\mathbf{v}(t)$.

Definition 1 (Volume-type conservation law)**.** *A conservation law of the first kind is expressed as:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta_{\xi(t)}(\mathcal{V}(t)) + \Phi_{\mathbf{a}(t)}(\partial\mathcal{V}(t)) = 0,$$

stating that the rate of change of a scalar quantity ξ within the deforming volume $\mathcal{V}(t)$ is balanced by the flux of $\mathbf{a}(t)$ through the boundary. In many physical contexts, a source or sink term may be present, accounting for the local production or destruction of the quantity ξ. Denoting by $\sigma(t)$ the scalar source density, the balance law generalizes to:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta_{\xi(t)}(\mathcal{V}(t)) + \Phi_{\mathbf{a}(t)}(\partial\mathcal{V}(t)) = \Delta_{\sigma(t)}(\mathcal{V}(t)).$$

Using the expression for the material derivative of volume integrals (see eq. (2.2.1)) and applying the divergence theorem, we obtain:

$$\Delta_{\frac{\mathrm{D}\xi(t)}{\mathrm{D}t}+\nabla\cdot\mathbf{a}(t)-\sigma(t)}(\mathcal{V}(t)) = 0.$$

Since $\mathcal{V}(t)$ is arbitrary, this yields the local form of the continuity equation with source:

$$\frac{\mathrm{D}\xi(t)}{\mathrm{D}t} + \nabla\cdot\mathbf{a}(t) = \sigma(t). \tag{3.1.1}$$

Finally, by recalling the explicit expression for the material derivative of a scalar density field (see eq. (2.2.2)), we obtain the full differential form of the balance law:

$$\frac{\partial\xi(t)}{\partial t} + \nabla\cdot(\mathbf{a}(t) + \xi(t)\mathbf{v}(t)) = \sigma(t). \tag{3.1.2}$$

Conservation law of the second kind

Now let $\mathbf{a}(t)$ and $\mathbf{b}(t)$ be time-dependent vector density fields defined over a material surface $\mathcal{A}(t)$, which moves and deforms with the velocity field $\mathbf{v}(t)$.

Definition 2 (Surface-type conservation law)**.** *A conservation law of the second kind is expressed as:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\mathbf{a}(t)}(\mathcal{A}(t)) + \Gamma_{\mathbf{b}(t)}(\partial\mathcal{A}(t)) = 0,$$

stating that the rate of change of a surface quantity is balanced by the circulation of a flux density $\mathbf{b}(t)$ *along the boundary. If a surface source density* $\mathbf{s}(t)$ *is present, the balance equation generalizes to:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\mathbf{a}(t)}(\mathcal{A}(t)) + \Gamma_{\mathbf{b}(t)}(\partial\mathcal{A}(t)) = \Phi_{\mathbf{s}(t)}(\mathcal{A}(t)).$$

Using the material derivative of surface integrals (see eq. (2.2.4)) and applying Stokes' theorem, the expression becomes:

$$\Phi_{\frac{\mathrm{D}\mathbf{a}(t)}{\mathrm{D}t}+\nabla\times\mathbf{b}(t)-\mathbf{s}(t)}(\mathcal{A}(t)) = 0.$$

Since $\mathcal{A}(t)$ is arbitrary, the local surface-type balance equation reads:

$$\frac{\mathrm{D}\mathbf{a}(t)}{\mathrm{D}t} + \nabla\times\mathbf{b}(t) = \mathbf{s}(t). \tag{3.1.3}$$

Using the expression for the material derivative of a vector density field (see eq. (2.2.5)), we obtain the full differential form:

$$\frac{\partial\mathbf{a}(t)}{\partial t} + (\nabla\cdot\mathbf{a}(t))\mathbf{v}(t) + \nabla\times(\mathbf{b}(t) + \mathbf{a}(t)\times\mathbf{v}(t)) = \mathbf{s}(t). \tag{3.1.4}$$

3.2 Mechanical Balance Equations

Let us now consider a material body $\mathcal{B}(t)$ occupying a region $\mathcal{V}(t) \subset \mathbb{E}^3$, evolving in time under the influence of a velocity field $\mathbf{v}(t,\mathbf{x})$. The fundamental laws of continuum thermomechanics are formulated as *balance equations*, which describe the temporal evolution of physical quantities such as mass, linear and angular momentum, and energy, as they are transported and exchanged across $\mathcal{V}(t)$.

These balance laws express basic physical principles in the form of local and integral conservation statements that must hold for any subregion of the material, regardless of its motion or deformation.

Definition 3 (Material body in motion)**.** *Let* $\mathcal{B}(t)$ *be a material body in motion with:*

- $\mathcal{V}(t) \subset \mathcal{B}(t)$*: arbitrary subvolume.*
- $\mathcal{D}(t)$*: singular surface with normal* $\mathbf{n}(t)$ *and velocity* $\mathbf{w}(t)$*.*
- $\rho(\mathbf{x},t)$*: mass density (scalar field).*
- $\mathbf{v}(\mathbf{x},t)$*: velocity field.*

- $e(\mathbf{x}, t)$*: energy density, including both internal and kinetic energy; the internal energy density is* $u = e - \frac{1}{2}\rho\|\mathbf{v}\|^2$.

Definition 4 (External sources and fluxes)**.** *We consider the following external and internal source terms:*

- $\mathbf{g}(t, \mathbf{x})$*: body forces per unit volume (e.g., gravitational or electromagnetic);*
- $\mathbf{j}_c(t, \mathbf{x})$*: conductive energy flux (heat, chemical, or other non-mechanical energy transport);*
- $z(t, \mathbf{x})$*: volumetric energy production (e.g., from radiation or chemical reactions).*

Additional details regarding the adopted conventions and notation are provided in Appendix A.

We now state each balance law in its integral and local differential form, using the transport theorems derived in the previous chapter.

Mass balance

The total mass of a material body remains constant during its motion and deformation.

Let $\rho(t, \mathbf{x})$ denote the mass density of a material body occupying a region $\mathcal{V}(t) \subset \mathbb{E}^3$, moving with velocity field $\mathbf{v}(t, \mathbf{x})$. The *integral form* of the conservation of mass expresses the invariance of the total mass under motion:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{V}(t)} \mathrm{d}^3\mathbf{x}\, \rho = 0.$$

This is a particular case of the general balance structure with source term described by eq. (3.1.2), with the identifications:

$$\xi = \rho, \quad \mathbf{a} = \mathbf{0}, \quad \sigma = 0.$$

Applying the transport theorem for scalar fields (theorem 2.1), and using the identity for the material derivative of ρ (eq. (2.2.2)), we obtain the *local differential form* of the mass balance:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{3.2.1}$$

This can also be expressed in material form as:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\, \nabla \cdot \mathbf{v} = 0,$$

which highlights that local changes in density are caused by compressibility effects (non-zero divergence of the velocity field). Equation (3.2.1) is a first-order conservation law. Discontinuities such as shock waves or material interfaces may arise, necessitating appropriate jump conditions across singular surfaces.

Jump condition. Let us consider now a discontinuity surface $\mathcal{D}(t)$ as previously defined. Using lemmas 2.6 and 2.7 (see section 2.3), the mass jump condition reads:

$$\mathbf{n} \cdot [\![\rho(\mathbf{v} - \mathbf{w})]\!] = 0. \tag{3.2.2}$$

Remark. *This condition guarantees that no mass is created or destroyed at the discontinuity surface: the net mass flux relative to the surface must be continuous across it.*

Linear momentum balance

The time rate of change of the total linear momentum of a material body equals the total force acting on it.

Let $\mathbf{p}(t, \mathbf{x})$ be the linear momentum density, $\mathbf{t}(t, \mathbf{x})$ the Cauchy stress tensor (additional information on the stress tensor is provided in Appendix A.), and $\mathbf{g}(t, \mathbf{x})$ the body force density (e.g., gravity, Lorentz force). Then the integral balance of linear momentum over a moving control volume $\mathcal{V}(t)$ reads:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{V}(t)} \mathrm{d}^3\mathbf{x}\, \mathbf{p} = \int_{\partial\mathcal{V}(t)} \mathrm{d}^2\mathbf{x} \cdot \mathbf{t} + \int_{\mathcal{V}(t)} \mathrm{d}^3\mathbf{x}\, \mathbf{g}.$$

This balance of linear momentum is a particular case of the generalized conservation law of the first kind with source term (see definition 1), where the fields are identified as

$$\xi = \mathbf{p}, \quad \mathbf{a} = -\mathbf{t}^\top, \quad \sigma = \mathbf{g}.$$

Applying the transport theorem (eq. (2.2.1)) and Gauss' divergence theorem, we obtain the *local differential form*:

$$\frac{\partial \mathbf{p}}{\partial t} + \nabla \cdot (\mathbf{v} \otimes \mathbf{p}) = \nabla \cdot \mathbf{t} + \mathbf{g}. \tag{3.2.3}$$

Remark. *The term $\mathbf{v} \otimes \mathbf{p}$ represents the convective transport of momentum by the flow, while $\nabla \cdot \mathbf{t}$ accounts for internal forces transmitted through stress.*

Jump condition. Across a singular surface $\mathcal{D}(t)$, the corresponding jump condition is:

$$\mathbf{n} \cdot [\![(\mathbf{v} - \mathbf{w}) \otimes \mathbf{p} - \mathbf{t}]\!] = \mathbf{g}^{\mathrm{sup}}. \tag{3.2.4}$$

Mechanical interpretation. In classical mechanics, the linear momentum density is given by:

$$\mathbf{p} = \rho\mathbf{v}.$$

Substituting into eq. (3.2.3), we have:

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla \cdot (\rho\mathbf{v} \otimes \mathbf{v}) = \nabla \cdot \mathbf{t} + \mathbf{g}.$$

Expanding the left-hand side using product rules and applying the mass conservation eq. (3.2.1), we obtain the *momentum equation in primitive form*:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = \nabla\cdot\mathbf{t} + \mathbf{g}, \tag{3.2.5}$$

which expresses Newton's second law for continua: mass times acceleration equals the sum of internal and external forces.

Remark. *Equation (3.2.5) is fundamental in fluid and solid mechanics. When* $\mathbf{t}$ *is expressed in terms of pressure or stress-strain relations, it leads to the Navier–Stokes or elasticity equations.*

Similarly, substituting $\mathbf{p} = \rho\mathbf{v}$ into the jump condition (3.2.4), and using the mass conservation jump condition (3.2.2), we obtain:

$$\mathbf{n}\cdot[\![\rho(\mathbf{v}-\mathbf{w})\otimes(\mathbf{v}-\mathbf{w}) - \mathbf{t}]\!] = \mathbf{g}^{\text{sup}}. \tag{3.2.6}$$

The left-hand side represents the net momentum flux and internal traction jump across $\mathcal{D}(t)$, while $\mathbf{g}^{\text{sup}}$ accounts for singular body forces (e.g., surface tractions, delta-distributed forces).

Angular momentum balance

The time rate of change of the total angular momentum of a material body equals the sum of all torques and couples acting on it.

Let $\mathbf{s}(t,\mathbf{x})$ be the *intrinsic angular momentum density* (spin), and $\mathbf{x}\times\mathbf{p}$ the *orbital angular momentum density*. The torque due to internal stress is given by $\mathbf{x}\times(\nabla\cdot\mathbf{t})$, while $\mathbf{x}\times\mathbf{g}$ is the external torque density. Then the integral form of the angular momentum balance over a deforming volume $\mathcal{V}(t)$ reads:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}(t)} \mathrm{d}^3\mathbf{x}\,(\mathbf{x}\times\mathbf{p}+\mathbf{s}) = \int_{\partial\mathcal{V}(t)} \mathrm{d}^2\mathbf{x}\,[\mathbf{x}\times(\mathbf{t}\cdot\mathbf{n})] + \int_{\mathcal{V}(t)} \mathrm{d}^3\mathbf{x}\,(\mathbf{x}\times\mathbf{g}).$$

This is a specific instance of a conservation law of the first kind with source (see eq. (3.1.2)), expressed over a material volume $\mathcal{V}(t)$. We identify the total angular momentum density as the scalar-like field:

$$\xi = \mathbf{x}\times\mathbf{p}+\mathbf{s}, \qquad \mathbf{a} = -\mathbf{x}\times(\mathbf{t}^{\top}), \qquad \boldsymbol{\sigma} = \mathbf{x}\times\mathbf{g}.$$

Here, the flux term $\mathbf{a}$ arises from the torque transmitted across the boundary by the internal stress field $\mathbf{t}$, while $\boldsymbol{\sigma}$ represents the external torque density due to body forces. The antisymmetry of the cross product ensures that $\mathbf{a}$ transforms as a vector flux density, consistent with the general structure of volume-type conservation laws. Applying the transport theorem and Gauss's theorem as before, we obtain the local differential form:

$$\frac{\partial}{\partial t}(\mathbf{x}\times\mathbf{p}+\mathbf{s}) + \nabla\cdot[\mathbf{v}\otimes(\mathbf{x}\times\mathbf{p}+\mathbf{s})] = \nabla\cdot(\mathbf{x}\times\mathbf{t}) + \mathbf{x}\times\mathbf{g}.$$

Expanding the left-hand side and simplifying with vector calculus identities and the linear momentum balance (eq. (3.2.3)), we arrive at the intrinsic angular momentum equation:

$$\frac{\partial \mathbf{s}}{\partial t} + \nabla \cdot (\mathbf{v} \otimes \mathbf{s}) = \star \left(2\mathbf{t}^{(a)} + \mathbf{p} \otimes \mathbf{v} \right), \tag{3.2.7}$$

where we recall the definitions:

- $\mathbf{t}^{(a)} := \frac{1}{2}(\mathbf{t} - \mathbf{t}^\top)$ is the antisymmetric part of the Cauchy stress tensor;
- $\star(\cdot)$ denotes the Hodge dual of a rank-2 antisymmetric tensor in $\mathbb{R}^3$, mapping it to an axial vector:
$$\star(T)_i := \frac{1}{2}\epsilon_{ijk} T_{jk}.$$

Remark. *Equation (3.2.7) reflects the fact that antisymmetric internal stresses, or non-symmetric momentum fluxes, act as sources of intrinsic angular momentum (spin).*

Jump condition. Consider a singular surface $\mathcal{D}(t)$. Neglecting surface contributions from $\star(2\mathbf{t}^{(a)} + \mathbf{p} \otimes \mathbf{v})$, we obtain the corresponding junction condition:

$$\mathbf{n} \cdot [\![(\mathbf{v} - \mathbf{w}) \otimes \mathbf{s}]\!] = \mathbf{0}. \tag{3.2.8}$$

Mechanical interpretation. In purely mechanical systems (e.g., fluids and elastic solids), intrinsic angular momentum is absent: $\mathbf{s} \equiv \mathbf{0}$. In this case, the total angular momentum reduces to its orbital part, and eq. (3.2.7) simplifies to:

$$\star(2\mathbf{t}^{(a)}) + \star(\mathbf{p} \otimes \mathbf{v}) = \mathbf{0}.$$

Here, $\mathbf{t}^{(a)} = \frac{1}{2}(\mathbf{t} - \mathbf{t}^\top)$ denotes the antisymmetric part of the Cauchy stress tensor. The term $\star(\mathbf{p} \otimes \mathbf{v})$, which represents the angular contribution from the linear momentum density $\mathbf{p} = \rho\mathbf{v}$, is symmetric in the classical theory and does not carry net torque. As a result, the antisymmetric part of the stress must vanish:

$$\mathbf{t}^{(a)} = \mathbf{0}.$$

This yields the fundamental *symmetry condition*:

$$\mathbf{t}^\top = \mathbf{t}, \tag{3.2.9}$$

which characterizes classical Cauchy continua with no internal structure (i.e., no couple stresses, micro-rotation effects, or distributed torques). This result will be used in chapter 5 in order to derive constitutive relations.

Remark. *The symmetry of the stress tensor is not postulated, but derived from the angular momentum balance in the absence of intrinsic spin. In generalized continua, this term is non-zero, leading to asymmetric stress.*

Energy balance

The time rate of change of the total energy of a material body, considered as a closed system, is equal to the net power of external forces and couples, plus the energy entering or leaving the body per unit time.

Let e denote the total energy density per unit volume and remember the following definitions:

- $\mathbf{j}_c$: the conductive energy flux density;
- z: the volumetric rate of internal energy supply (e.g., due to chemical or nuclear reactions).

Then the integral form of the energy balance reads:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}(t)} \mathrm{d}^3\mathbf{x}\, e = \oint_{\partial\mathcal{V}(t)} \mathrm{d}^2\mathbf{x}\, \mathbf{n}\cdot(\mathbf{t}\cdot\mathbf{v} - \mathbf{j}_c) + \int_{\mathcal{V}(t)} \mathrm{d}^3\mathbf{x}\,(\mathbf{g}\cdot\mathbf{v} + z).$$

This is a particular instance of the extended conservation law of the first kind with a source term, as introduced in eq. (3.1.2). Identifying the total energy density as the conserved scalar field, we set:

$$\xi = e, \qquad \mathbf{a} = \mathbf{j}_c - \mathbf{t}\cdot\mathbf{v}, \qquad \sigma = \mathbf{g}\cdot\mathbf{v} + z.$$

Here, $\mathbf{a}$ includes both the conductive energy outflux $\mathbf{j}_c$ and the mechanical power input from the stress field $\mathbf{t}\cdot\mathbf{v}$, while the source term σ accounts for volumetric energy inputs such as gravitational work and internal generation. Applying the transport theorem and the divergence theorem as before, we obtain the local differential form:

$$\frac{\partial e}{\partial t} + \nabla\cdot(e\mathbf{v}) = \nabla\cdot(\mathbf{t}\cdot\mathbf{v}) - \nabla\cdot\mathbf{j}_c + \mathbf{g}\cdot\mathbf{v} + z.$$

Rewriting in conservative form:

$$\frac{\partial e}{\partial t} + \nabla\cdot(e\mathbf{v}) - \mathbf{v}\cdot\left[\frac{\partial \mathbf{p}}{\partial t} + \nabla\cdot(\mathbf{v}\otimes\mathbf{p})\right] - \mathbf{t} : (\nabla\otimes\mathbf{v}) + \nabla\cdot\mathbf{j}_c - z = 0, \tag{3.2.10}$$

where $\mathbf{p} = \rho\mathbf{v}$ is the momentum density.

Jump condition. The associated jump condition across a moving interface $\mathcal{D}(t)$, with normal $\mathbf{n}$ and velocity $\mathbf{w}$, reads:

$$\mathbf{n}\cdot[\![e(\mathbf{v}-\mathbf{w}) + \mathbf{j}_c - \mathbf{t}\cdot\mathbf{v}]\!] = z^{\mathrm{sup}} + \mathbf{g}^{\mathrm{sup}}\cdot\mathbf{w},$$

where the superscript $^{\mathrm{sup}}$ denotes surface contributions (energy release or absorption at the interface). If we define the total superficial energy input as:

$$(z + \mathbf{g}\cdot\mathbf{v})^{\mathrm{sup}} := z^{\mathrm{sup}} + \mathbf{g}^{\mathrm{sup}}\cdot\mathbf{w},$$

and apply the linear momentum jump condition (3.2.6), we obtain:

$$\mathbf{n}\cdot[\![[(e - \mathbf{p}\cdot\mathbf{w})\mathbf{I} - \mathbf{t}]\cdot(\mathbf{v}-\mathbf{w}) - \mathbf{j}_c]\!] = z^{\mathrm{sup}}. \tag{3.2.11}$$

Mechanical interpretation. For mechanical systems, the energy density splits as:

$$e = u + \frac{1}{2}\rho v^2,$$

where u is the internal energy density and $\frac{1}{2}\rho v^2$ is the kinetic energy. Using the identity:

$$\begin{aligned}\mathbf{v} \cdot \left[\frac{\partial \mathbf{p}}{\partial t} + \nabla \cdot (\mathbf{v} \otimes \mathbf{p})\right] &= \mathbf{v} \cdot \left[\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v})\right] \\ &= \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \nabla \cdot \left(\frac{1}{2}\rho v^2 \mathbf{v}\right),\end{aligned}$$

we rewrite eq. (3.2.10) in terms of internal energy:

$$\frac{\partial u}{\partial t} + \nabla \cdot (u\mathbf{v}) - \mathbf{t} : (\nabla \otimes \mathbf{v}) + \nabla \cdot \mathbf{j}_c - z = 0. \tag{3.2.12}$$

Junction condition for internal energy. In this case, the associated jump condition becomes:

$$\mathbf{n} \cdot \left[\!\left[\left[\left(\frac{1}{2}\rho v^2 - \rho \mathbf{v} \cdot \mathbf{w} + u\right)\mathbf{I} - \mathbf{t}\right] \cdot (\mathbf{v} - \mathbf{w}) + \mathbf{j}_c\right]\!\right] = z^{\text{sup}}.$$

To further simplify, note that:

$$\frac{1}{2}\rho v^2 - \rho \mathbf{v} \cdot \mathbf{w} = \frac{1}{2}\rho (\mathbf{v} - \mathbf{w})^2 - \frac{1}{2}\rho w^2,$$

and if we add the jump condition for mass conservation multiplied by $\frac{w^2}{2}$, we obtain the more symmetric form:

$$\mathbf{n} \cdot \left[\!\left[\left[\left(\frac{1}{2}\rho (\mathbf{v} - \mathbf{w})^2 + u\right)\mathbf{I} - \mathbf{t}\right] \cdot (\mathbf{v} - \mathbf{w}) + \mathbf{j}_c\right]\!\right] = z^{\text{sup}}. \tag{3.2.13}$$

Remark. *The energy balance encapsulates both mechanical and thermal effects within a unified framework. The term $\mathbf{t} \cdot \mathbf{v}$ represents the mechanical power per unit area transferred across the boundary by internal stresses, while $\mathbf{j}_c$ models non-mechanical (e.g., thermal) energy transport. The source term $\mathbf{g} \cdot \mathbf{v} + z$ highlights that energy may be injected or removed from the system either mechanically (via body forces doing work) or chemically/thermally (via internal reactions or heat generation). This decomposition is crucial in thermomechanical modeling of continua.*

3.3 Principles of Thermodynamics

In this section, we introduce the fundamental concepts and governing principles of thermodynamics as applied to continua. We begin by distinguishing the main physical quantities used to describe a deforming and moving body, and then proceed to formulate the thermodynamic laws in the context of continuum mechanics. Finally, we develop balance equations for specific densities and discuss the entropy balance, laying the groundwork for further thermomechanical analysis.

When describing a deforming and moving body, it is essential to categorize the physical quantities involved into two classes:

- **Extensive quantities** are those that depend on the size or extent of the system. They are additive for subsystems and typically scale with volume. Examples of extensive quantities in continuum mechanics include mass, charge, linear momentum, angular momentum and total energy. Mathematically, these quantities are often expressed as volume integrals over intensive fields.

- In contrast, **intensive quantities** are independent of the system's size and describe local properties at a point in space. Key intensive quantities in continuum mechanics include the mass density ρ, velocity field $\mathbf{v}$, stress tensor $\mathbf{t}$, temperature θ, and specific internal energy $u^\star$.[1] These fields provide a pointwise description of the system and serve as the basis for the formulation of conservation laws and constitutive relations.

This distinction carries over into thermodynamics, where it plays a crucial role in analyzing physical processes. In particular, thermodynamic processes are often classified as *reversible* or *irreversible*, depending on whether they can proceed without net changes in the system and its surroundings.

- **Reversible processes** are those in which the system evolves in such a way that it remains at all times in states of equilibrium. Furthermore, in such processes, any intensive quantity ξ does not depend on position, hence: $\nabla\xi = 0$.

- **Irreversible processes**, on the other hand, occur because the system is in a non-equilibrium state, which drives it to evolve toward a state of equilibrium. Unlike the previous case, irreversible processes are characterized by the fact that any intensive quantity ξ depends on position, that is: $\nabla\xi \neq 0$.

Gibbs' principle

Thanks to the work attributed to the physicist J. W. Gibbs, a theoretical framework exists that allows us to model irreversible processes as sequences of transformations between thermodynamic equilibrium states.

Let us consider a continuous body B, subdivided into sufficiently small portions such that any intensive quantity $\xi(\mathbf{x})$ can be regarded as approximately constant within each portion. Even when $\xi(\mathbf{x})$ varies spatially, this assumption remains valid provided that

$$\frac{l}{L} \ll 1,$$

where l is the characteristic molecular length scale (e.g., the average distance between molecules), and L is the characteristic macroscopic length scale over which appreciable variations of $\xi(\mathbf{x})$ occur. This scale separation justifies the use of local thermodynamic equilibrium: each small portion behaves as if it were in equilibrium, despite the system as a whole being out of equilibrium. This principle is fundamental in continuum thermodynamics,

[1] We will clarify later the definition of a *specific* quantity, denoted with $\xi^\star$.

as it enables us to apply equilibrium thermodynamic relations locally, even to irreversible processes.

Remark. *This approach is analogous to the idea of a "quasi-static" or "quasi-equilibrium" process, where the system evolves through a continuous sequence of equilibrium states. Although true irreversible processes involve gradients and fluxes that drive the system toward equilibrium, the small scale of observation permits a local equilibrium approximation.*

Application Local thermodynamic equilibrium.

Consider heat conduction in a metal rod with a temperature gradient along its length. At the macroscopic scale, the temperature varies continuously, indicating that the system is out of global equilibrium. However, if we zoom into a small enough segment of the rod, the temperature can be assumed uniform within that segment. Thus, the local thermodynamic variables (temperature, internal energy, entropy, etc.) satisfy equilibrium thermodynamic relations, even though heat is flowing and the overall process is irreversible.

First and second laws of thermodynamics

The first two laws of thermodynamics for a generic transformation state:

$$\mathrm{d}(\Delta U) = \delta Q + \delta L, \tag{3.3.1}$$

where the infinitesimal change of internal energy $\mathrm{d}(\Delta U)$ is given by the sum of heat supplied δQ and work done δL on the system.

$$\delta Q \leq \theta\, \mathrm{d}(\Delta S), \tag{3.3.2}$$

expressing the second law of thermodynamics: the heat exchanged δQ is bounded above by the product of the absolute temperature θ and the infinitesimal entropy change $\mathrm{d}(\Delta S)$. It is important to note that $\mathrm{d}(\Delta U)$ and $\mathrm{d}(\Delta S)$ are exact differentials of state functions, meaning they depend only on the initial and final equilibrium states of the system, not on the path followed during the transformation. On the other hand, δQ and δL are path-dependent and describe how the transformation is performed.

For reversible transformations, where the process is quasi-static and no entropy is produced internally, the inequalities become equalities:

$$\delta Q_r = \theta\, \mathrm{d}(\Delta S), \quad \delta L_r = \sum_i \xi_i\, \mathrm{d}(\Delta X_i),$$

where ξ_i are generalized forces and ΔX_i the corresponding generalized displacements, so that the work done can be expressed in this generalized form.

Remark. *The reversible case allows us to write the two laws combined into the thermodynamic equilibrium law:*

$$\mathrm{d}(\Delta U) = \theta\,\mathrm{d}(\Delta S) + \sum_i \xi_i\,\mathrm{d}(\Delta X_i). \tag{3.3.3}$$

This expresses the internal energy change as composed of thermal and mechanical contributions, reflecting the interplay between entropy and generalized displacements.

In the context of continuum bodies, we consider material volumes that conserve mass, $\Delta m = \Delta m(\Delta\mathcal{V})$ with

$$\mathrm{d}(\Delta m) = 0,$$

as ensured by the mass balance (eq. (3.2.1)). This conservation allows us to normalize the extensive thermodynamic quantities by the mass of the considered portion. The next natural step is to consider the limit where the volume portion $\Delta V = \mathrm{vol}(\Delta\mathcal{V})$ tends to zero,

$$\Delta V \to 0,$$

thus defining specific quantities (per unit mass) and densities (per unit volume) as

$$\begin{aligned} u^\star &= \lim_{\Delta V\to 0}\frac{\Delta U}{\Delta m}, & u &= \lim_{\Delta V\to 0}\frac{\Delta U}{\Delta V},\\ s^\star &= \lim_{\Delta V\to 0}\frac{\Delta S}{\Delta m}, & s &= \lim_{\Delta V\to 0}\frac{\Delta S}{\Delta V},\\ x_i^\star &= \lim_{\Delta V\to 0}\frac{\Delta X_i}{\Delta m}, & x_i &= \lim_{\Delta V\to 0}\frac{\Delta X_i}{\Delta V}. \end{aligned}$$

Remark. *Passing to the limit from finite portions to infinitesimal ones is a natural step in continuum mechanics, enabling the definition of fields of specific internal energy, entropy, and generalized displacements that vary continuously in space and time.*

Substituting the specific densities into the first law (eq. (3.3.1)) and differentiating with respect to time, we obtain the balance equation for internal energy in specific form:

$$\frac{\mathrm{d}u^\star}{\mathrm{d}t} = \theta\frac{\mathrm{d}s^\star}{\mathrm{d}t} + \sum_i \xi_i\frac{\mathrm{d}x_i^\star}{\mathrm{d}t}, \tag{3.3.4}$$

which states that the rate of change of specific internal energy is related to the rate of entropy change (heat effects) and the rate of generalized displacements (mechanical work).

Remark. *This equation is fundamental in continuum thermomechanics, as it expresses energy conservation at every material point, coupling thermal and mechanical variables. While mechanical balances describe how forces and motions evolve in the continuum, the internal energy balance governs the thermodynamic state evolution by linking mechanical work and heat exchanges to changes in internal energy. Together, these coupled balances form the foundation of continuum thermomechanics, enabling a comprehensive description of deforming bodies subject to thermal and mechanical effects.*

3.4 Thermodynamic Balance Equations

In this section, we aim to derive explicit expressions for the balance equations and time derivatives of specific densities. We start by defining a scalar quantity F, its density f, and the corresponding specific quantity $f^\star$, considered on an infinitesimal material volume $\Delta\mathcal{V}$ with mass Δm.

Time derivative

By definition, the total amount of the quantity F contained in the portion $\Delta\mathcal{V}$ is:

$$\Delta F = f^\star \Delta m.$$

Since the mass of the material portion is preserved over time ($\frac{\mathrm{d}}{\mathrm{d}t}\Delta m = 0$), the time rate of change of ΔF reduces to the rate of change of the specific quantity times the constant mass:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta F = \frac{\mathrm{d}f^\star}{\mathrm{d}t}\Delta m.$$

This expression embodies the Lagrangian perspective, focusing on a material volume that moves with the continuum. Alternatively, using the Eulerian (spatial) viewpoint, ΔF can be expressed as a volume integral of the density f:

$$\Delta F = \int_{\Delta\mathcal{V}} f \,\mathrm{d}V.$$

Applying the transport theorem or the general balance on moving volumes (see eq. (2.2.1)), the rate of change of this integral can be written as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta F = \int_{\Delta\mathcal{V}} \left(\frac{\partial f}{\partial t} + \nabla\cdot(\mathbf{v}f)\right)\mathrm{d}V,$$

where $\mathbf{v}$ is the velocity field of the continuum. For sufficiently small volumes, the integral can be approximated by the integrand evaluated at a point $\mathbf{x}$ times the volume ΔV:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta F \approx \left[\frac{\partial f}{\partial t} + \nabla\cdot(\mathbf{v}f)\right](\mathbf{x},t)\Delta V.$$

Taking the limit $\Delta V \to 0$ and comparing with the Lagrangian expression, we obtain the fundamental relation linking the material derivative of the specific quantity to the Eulerian derivatives of its density:

$$\frac{\mathrm{d}}{\mathrm{d}t}f^\star = \frac{1}{\rho(\mathbf{x},t)}\left(\frac{\partial f}{\partial t} + \nabla\cdot(\mathbf{v}f)\right), \tag{3.4.1}$$

where $\rho(\mathbf{x},t)$ is the mass density of the material at point $\mathbf{x}$.

Remark. *This equation shows how the rate of change of a quantity observed while moving with the material (the material derivative) can be expressed in terms of changes occurring at a fixed point in space (local time derivative) plus the effects of transport*

caused by the motion of the material (spatial divergence of the flux). In other words, it bridges the Lagrangian viewpoint, following particles, and the Eulerian viewpoint, fixed in space. This connection is essential for converting global, integral balance laws into local differential equations that describe how physical fields evolve pointwise in the continuum.

Balance equation

The balance of the quantity F in $\Delta\mathcal{V}$ can be expressed by accounting for fluxes and internal production. Let $\mathbf{j}^F$ denote the flux density of F and σ^F the volumetric production rate (source or sink). Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta F = -\int_{\partial\Delta\mathcal{V}} \mathbf{j}^F \cdot \mathbf{n}\,\mathrm{d}S + \int_{\Delta\mathcal{V}} \sigma^F\,\mathrm{d}V,$$

where the first term on the right-hand side represents the net outflow of F through the boundary $\partial\Delta\mathcal{V}$, and the second term the generation or consumption inside the volume. Applying the divergence theorem converts the surface integral into a volume integral:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta F = \int_{\Delta\mathcal{V}} \left(-\nabla\cdot\mathbf{j}^F + \sigma^F\right)\mathrm{d}V.$$

Approximating again for small ΔV, and using the relation for the material derivative of the specific quantity, we arrive at:

$$\frac{\mathrm{d}}{\mathrm{d}t}f^\star = \frac{1}{\rho(\mathbf{x},t)}\left(-\nabla\cdot\mathbf{j}^F + \sigma^F\right).$$

Substituting the expression (3.4.1) for the material derivative, the local balance equation in differential form for the density f is:

$$\frac{\partial f}{\partial t} + \nabla\cdot(\mathbf{v}f + \mathbf{j}^F) = \sigma^F. \tag{3.4.2}$$

This equation reflects the fundamental conservation principle at the differential level: the local rate of change of f plus the divergence of its total flux (advective $\mathbf{v}f$ plus diffusive or other $\mathbf{j}^F$) equals the volumetric source term σ^F.

Jump condition. For completeness, the balance laws must also hold across internal interfaces or surfaces where discontinuities may arise. Denoting by $\mathbf{w}$ the velocity of such an interface, with normal $\mathbf{n}$, and $\sigma^{F,\mathrm{sup}}$ the possible source term concentrated on the surface, the jump condition reads:

$$\mathbf{n}\cdot\left[\!\left[(\mathbf{v}-\mathbf{w})f + \mathbf{j}^F\right]\!\right] = \sigma^{F,\mathrm{sup}}, \tag{3.4.3}$$

where we have used the usual $[\![\cdot]\!]$ to denote the jump across it. This condition guarantees that fluxes and sources are balanced even in the presence of discontinuities or multiphase structures, ensuring the physical consistency of the continuum description.

Entropy balance

According to Gibbs, as previously discussed, any infinitesimal portion of a continuum can be considered to be in thermodynamic equilibrium, since its relaxation time (the time needed to return to equilibrium after a perturbation) is also infinitesimally small. This assumption allows us to apply the equilibrium thermodynamic identity in differential form to any material point in the body. Therefore, we can write the specific form of the thermodynamic equilibrium law as:

$$\mathrm{d}u^\star - \theta\,\mathrm{d}s^\star - \sum_i \xi_i\,\mathrm{d}x_i^\star = 0.$$

This expression corresponds to the first law of thermodynamics in differential form, where:

- $\theta\,\mathrm{d}s^\star$ represents the infinitesimal heat exchange per unit mass ($\mathrm{d}q^\star$),
- $\sum_i \xi_i\,\mathrm{d}x_i^\star$ accounts for the reversible work associated with generalized displacements (e.g., mechanical deformation, chemical reactions, phase changes).

We now use the general balance law for a specific quantity (eq. (3.4.2)) to write the balance equations for the densities of internal energy u, entropy s, and generalized displacements x_i:

$$\begin{aligned}
\frac{\partial u}{\partial t} + \nabla\cdot(\mathbf{v}u + \mathbf{j}^{(u)}) - \sigma^{(u)} = 0,\\
\frac{\partial s}{\partial t} + \nabla\cdot(\mathbf{v}s + \mathbf{j}^{(s)}) - \sigma^{(s)} = 0,\\
\frac{\partial x_i}{\partial t} + \nabla\cdot(\mathbf{v}x_i + \mathbf{j}^{(i)}) - \sigma^{(i)} = 0.
\end{aligned}$$

These equations describe the local balance of the corresponding quantities: time variation is determined by the net flux crossing the boundary of the region and by the amount produced or consumed within the region. The terms $\mathbf{j}^{(\cdot)}$ represent spatial transport (diffusion, conduction), while $\sigma^{(\cdot)}$ quantify sources or sinks (e.g., chemical reactions, dissipation). We also write their material time derivatives using eq. (3.4.1), valid for infinitesimal material volumes:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}u^\star &= \frac{1}{\rho}\left[-\nabla\cdot\mathbf{j}^{(u)} + \sigma^{(u)}\right],\\
\frac{\mathrm{d}}{\mathrm{d}t}s^\star &= \frac{1}{\rho}\left[-\nabla\cdot\mathbf{j}^{(s)} + \sigma^{(s)}\right],\\
\frac{\mathrm{d}}{\mathrm{d}t}x_i^\star &= \frac{1}{\rho}\left[-\nabla\cdot\mathbf{j}^{(i)} + \sigma^{(i)}\right].
\end{aligned}$$

These relations link the rate of change of specific quantities following a material particle to the fluxes and production rates at that location. They provide the basis for writing evolution laws at a point. Inserting these into the thermodynamic equilibrium identity gives an expression involving all the fluxes and productions. Rewriting and rearranging, we obtain:

$$\frac{1}{\rho}\left[-\nabla\cdot\mathbf{j}^{(u)} + \sigma^{(u)} + \theta\left(\nabla\cdot\mathbf{j}^{(s)} - \sigma^{(s)}\right) + \sum_i \xi_i\left(\nabla\cdot\mathbf{j}^{(i)} - \sigma^{(i)}\right)\right] = 0,$$

and, isolating terms:

$$\nabla \cdot \left[\mathbf{j}^{(s)} - \frac{1}{\theta} \left(\mathbf{j}^{(u)} - \sum_i \xi_i \mathbf{j}^{(i)} \right) \right] -$$
$$- \left[\sigma^{(s)} - \frac{1}{\theta} \left(\sigma^{(u)} - \sum_i \xi_i \sigma^{(i)} \right) - \mathbf{j}^{(u)} \cdot \nabla \left(\frac{1}{\theta} \right) + \sum_i \mathbf{j}^{(i)} \cdot \nabla \left(\frac{\xi_i}{\theta} \right) \right] = 0.$$

This identity reveals a fundamental thermodynamic structure: if the system is to be consistent with the second law, the entropy production must be non-negative. Thus, we can identify the entropy flux and production rate as:

$$\mathbf{j}^{(s)} = \frac{1}{\theta} \left(\mathbf{j}^{(u)} - \sum_i \xi_i \mathbf{j}^{(i)} \right), \tag{3.4.4}$$

$$\sigma^{(s)} = \frac{1}{\theta} \left(\sigma^{(u)} - \sum_i \xi_i \sigma^{(i)} \right) - \mathbf{j}^{(s)} \cdot \frac{\nabla \theta}{\theta}. \tag{3.4.5}$$

Physical interpretation. The entropy flux is directly tied to the energy and work fluxes: entropy is transported with energy, minus what is carried away as mechanical or chemical work. The entropy production rate, on the other hand, includes contributions from local dissipation (like viscosity or chemical reactions) and from irreversible thermal conduction (heat flowing down a temperature gradient), consistently with Clausius' formulation of the second law.

Clausius-Duhem Inequality

The second law of thermodynamics for continua can be expressed through the *Clausius-Duhem inequality*, which provides a local constraint on the admissible thermodynamic processes in terms of entropy production.

We begin by considering a material portion $\Delta\mathcal{V}$ of the continuum. The integral form of the second law reads:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Delta\mathcal{V}} \rho s^\star \, \mathrm{d}V \geq - \oint_{\partial\Delta\mathcal{V}} \frac{\mathbf{n} \cdot \mathbf{j}^{(s)}}{\theta} \, \mathrm{d}S + \int_{\Delta\mathcal{V}} \frac{\rho \sigma^{(s)}}{\theta} \, \mathrm{d}V.$$

This expression states that the time rate of change of entropy within the material domain is greater than or equal to the entropy flux through the boundary (driven by the entropy current $\mathbf{j}^{(s)}$) plus the internal entropy production $\sigma^{(s)}$. By applying the divergence theorem and rearranging, we can write this in local differential form:

$$\int_{\Delta\mathcal{V}} \left[\frac{\partial(\rho s^\star)}{\partial t} + \nabla \cdot (\rho s^\star \mathbf{v}) + \nabla \cdot \frac{\mathbf{j}^{(s)}}{\theta} - \frac{\rho \sigma^{(s)}}{\theta} \right] \mathrm{d}V \geq 0.$$

Since this must hold for any arbitrary portion of the body, the integrand must be non-negative everywhere, leading to:

$$\frac{\partial(\rho s^\star)}{\partial t} + \nabla \cdot (\rho s^\star \mathbf{v}) + \nabla \cdot \frac{\mathbf{j}^{(s)}}{\theta} - \frac{\rho \sigma^{(s)}}{\theta} \geq 0.$$

We now expand the material derivative of the specific entropy using the identity:

$$\frac{\partial(\rho s^\star)}{\partial t} + \nabla \cdot (\rho s^\star \mathbf{v}) = \rho \frac{\mathrm{d}}{\mathrm{d}t} s^\star,$$

which follows from the mass balance (eq. (3.2.1)). Substituting, we obtain the local form of the Clausius-Duhem inequality:

$$\rho \frac{\mathrm{d}}{\mathrm{d}t} s^\star + \nabla \cdot \frac{\mathbf{j}^{(s)}}{\theta} - \frac{\rho \sigma^{(s)}}{\theta} \geq 0. \tag{3.4.6}$$

The second term on the left accounts for entropy transport due to diffusion, while the third term accounts for internal entropy production. We now simplify the divergence term by recalling the identity:

$$\nabla \cdot \frac{\mathbf{j}^{(s)}}{\theta} = \frac{1}{\theta} \nabla \cdot \mathbf{j}^{(s)} - \mathbf{j}^{(s)} \cdot \nabla \frac{1}{\theta}.$$

This yields:

$$\rho \frac{\mathrm{d}}{\mathrm{d}t} s^\star + \frac{1}{\theta} \nabla \cdot \mathbf{j}^{(s)} - \frac{\rho \sigma^{(s)}}{\theta} - \mathbf{j}^{(s)} \cdot \nabla \frac{1}{\theta} \geq 0. \tag{3.4.7}$$

To connect this expression with the mechanical and thermal processes, we recall from the thermodynamic identity that:

$$\frac{\mathrm{d}u^\star}{\mathrm{d}t} = \theta \frac{\mathrm{d}s^\star}{\mathrm{d}t} + \sum_i \xi_i \frac{\mathrm{d}x_i^\star}{\mathrm{d}t},$$

and from the internal energy balance:

$$\frac{\partial u}{\partial t} + \nabla \cdot (u \mathbf{v}) - \mathbf{t} : (\nabla \otimes \mathbf{v}) + \nabla \cdot \mathbf{j}_c - z = 0,$$

where $\mathbf{j}_c$ is the conductive energy flux, and z is the external power supply. Rearranging and combining with the balance for x_i, one obtains:

$$\nabla \cdot \mathbf{j}^{(s)} - \sigma^{(s)} = -\rho \frac{\mathrm{d}}{\mathrm{d}t} u^\star + \mathbf{t} : (\nabla \otimes \mathbf{v}).$$

Substituting into the Clausius-Duhem inequality yields:

$$\rho \left[\frac{\mathrm{d}}{\mathrm{d}t} s^\star - \frac{1}{\theta} \frac{\mathrm{d}}{\mathrm{d}t} u^\star \right] + \frac{\mathbf{j}^{(s)}}{\theta^2} \cdot \nabla \theta + \frac{1}{\theta} \mathbf{t} : \nabla \otimes \mathbf{v} \geq 0. \tag{3.4.8}$$

Or, equivalently, by multiplying through by θ and rearranging:

$$\rho \left[\frac{\mathrm{d}}{\mathrm{d}t} u^\star - \theta \frac{\mathrm{d}}{\mathrm{d}t} s^\star \right] - \frac{\mathbf{j}^{(s)}}{\theta} \cdot \nabla \theta - \mathbf{t} : \nabla \otimes \mathbf{v} \leq 0.$$

Remark. *This final expression is the local differential form of the Clausius-Duhem inequality. It encodes the fundamental requirement that the internal energy not only varies in accordance with mechanical work and heat transfer, but that the irreversible contributions to these processes (represented by entropy production, heat conduction, and viscous dissipation) must combine to produce a non-negative total entropy production.*

4 | Electrodynamics of Continua

Understanding the interaction between electromagnetic fields and matter is fundamental in bridging electromagnetism and continuum mechanics. While the Lorentz force law provides a clear framework for point charges, describing electromagnetic effects in macroscopic, deformable bodies requires a more comprehensive and consistent approach. Phenomena such as electromagnetic forces on conductors, material stresses, Joule heating, and radiative energy flux often rely on heuristic assumptions that lack a unified mechanical foundation.

In this chapter, we complete the formulation of the coupled mechanical-electromagnetic system by incorporating electromagnetic fields directly into the continuum mechanics framework. Rather than treating electromagnetic contributions merely as additional forces or power terms, we show how these fields naturally contribute to the momentum, energy, and stress within the continuum. This enables us to treat the electromagnetic fields as integral components of the mechanical system, providing a rigorous, physically grounded basis for the study of electrodynamics in continua.

4.1 Foundations of Maxwell Theory in Continua

To model the interaction between electromagnetic fields and continuous media, we start by introducing an equivalent form of Maxwell's equations adapted to moving and deforming domains. These integral formulations are physically equivalent to the standard ones in inertial frames, but are more suitable for continuum mechanics, where integrations are performed over evolving material volumes and surfaces.

Let $\mathcal{V}(t)$ and $\mathcal{A}(t)$ be a material volume and surface, both deforming and moving with the velocity field $\mathbf{v}(t, \mathbf{x})$. We adopt the following integral form of Maxwell's equations:

$$\begin{aligned} \oint_{\partial\mathcal{V}} \mathbf{B} \cdot \mathrm{d}^2\mathbf{x} &= 0, \\ \oint_{\partial\mathcal{A}} \mathbf{E}_c \cdot \mathrm{d}^1\mathbf{x} + \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{A}} \mathbf{B} \cdot \mathrm{d}^2\mathbf{x} &= 0, \\ \oint_{\partial\mathcal{V}} \mathbf{E} \cdot \mathrm{d}^2\mathbf{x} - \int_{\mathcal{V}} q \, \mathrm{d}^3\mathbf{x} &= 0, \\ \oint_{\partial\mathcal{A}} \mathbf{B}_c \cdot \mathrm{d}^1\mathbf{x} - \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{A}} (\mathbf{E} + \mathbf{j}_c) \cdot \mathrm{d}^2\mathbf{x} &= 0, \end{aligned} \tag{4.1.1}$$

where:

- $\mathbf{E}_c = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}$ is the co-moving electric field,
- $\mathbf{B}_c = \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}$ is the co-moving magnetic field,
- $\mathbf{j}_c = \mathbf{j} - q\mathbf{v}$ is the conduction current density.

The use of $\mathbf{E}_c$ and $\mathbf{B}_c$ accounts for the apparent electromagnetic fields measured in a material frame moving with velocity $\mathbf{v}$. These co-moving fields naturally arise when transforming Maxwell's equations into a non-inertial or deforming reference.

Remark. *The convective terms that appear in the co-moving forms highlight the influence of the material motion on the electromagnetic field. In particular, the velocity field* $\mathbf{v}$ *couples to the electric and magnetic fields through terms of the form* $\mathbf{v} \times \mathbf{B}$ *and* $\mathbf{v} \times \mathbf{E}$*. This reflects the physical fact that a moving observer (or moving material point) perceives different electromagnetic effects than a static one.*

To obtain the local (differential) form of Maxwell's equations, we apply Gauss' and Stokes' theorems, along with the material derivative identity for surface integrals (see eq. (2.2.4)). This yields:

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E}_c + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} - \nabla \times \left(\frac{\mathbf{v}}{c} \times \mathbf{B} \right) &= 0, \\ \nabla \cdot \mathbf{E} - q &= 0, \\ \nabla \times \mathbf{B}_c - \frac{1}{c}\left[\frac{\partial \mathbf{E}}{\partial t} + \mathbf{j}_c + (\nabla \cdot \mathbf{E})\mathbf{v} - \nabla \times (\mathbf{v} \times \mathbf{E}) \right] &= 0. \end{aligned} \tag{4.1.2}$$

Finally, we emphasize that both charge and current densities include contributions from free and bound charges. In general, we write:

$$\begin{aligned} q &= q_f + q_b, \\ \mathbf{j} &= \mathbf{j}_f + \mathbf{j}_b. \end{aligned}$$

Note that this is simply an alternative formulation of the standard Maxwell's equations. From this point onward, we adopt Heaviside–Lorentz units, in which the fundamental constants ϵ_0, μ_0, and 4π are absorbed into the definitions of the fields. This leads to cleaner expressions and ensures that the electric and magnetic fields share the same physical units. See Appendix A for details and explicit unit conversions.

Charge conservation principle

Charge conservation can be derived directly from the last two Maxwell equations in their integral form $(4.1.1)_{3,4}$. Taking the time derivative of the Gauss law for the electric field,

we obtain:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} q\,\mathrm{d}^3\mathbf{x} = \frac{\mathrm{d}}{\mathrm{d}t}\oint_{\partial\mathcal{V}} \mathbf{E}\cdot\mathrm{d}^2\mathbf{x} &= c\oint_{\partial\partial\mathcal{V}} \mathbf{B}_c\cdot\mathrm{d}^1\mathbf{x} - \oint_{\partial\mathcal{V}} \mathbf{j}_c\cdot\mathrm{d}^2\mathbf{x}\\ &= c\oint_{\partial\mathcal{V}} (\nabla\times\mathbf{B}_c)\cdot\mathrm{d}^2\mathbf{x} - \oint_{\partial\mathcal{V}} \mathbf{j}_c\cdot\mathrm{d}^2\mathbf{x}\\ &= -\oint_{\partial\mathcal{V}} (\mathbf{j}-q\mathbf{v})\cdot\mathrm{d}^2\mathbf{x},\end{aligned}$$

which leads to the integral form of the charge conservation principle:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} q\,\mathrm{d}^3\mathbf{x} + \oint_{\partial\mathcal{V}} (\mathbf{j}-q\mathbf{v})\cdot\mathrm{d}^2\mathbf{x} = 0. \tag{4.1.3}$$

This equation states that any change in the total charge within a material volume $\mathcal{V}$ must be balanced by a net flux of conduction current $\mathbf{j}_c = \mathbf{j} - q\mathbf{v}$ across its boundary.

Example .

Consider a charged fluid with Eulerian charge density $q(t,\mathbf{x})$ moving with velocity field $\mathbf{v}(t,\mathbf{x})$. If there are no free charges flowing relative to the material (ideal dielectric), then the total current is purely convective and given by $\mathbf{j} = q\mathbf{v}$. In this case, the conduction current vanishes: $\mathbf{j}_c = \mathbf{j} - q\mathbf{v} = \mathbf{0}$.

On the other hand, in an ohmic conductor at rest ($\mathbf{v} = \mathbf{0}$) with an imposed electric field $\mathbf{E}$, the current density is $\mathbf{j} = \sigma\mathbf{E}$, and since the body is at rest, $\mathbf{j}_c = \mathbf{j}$. Thus, $\mathbf{j}_c$ describes the actual flow of charges with respect to the material medium.

Using Gauss' theorem and the transport identity for material volumes (eq. (2.2.1)), we obtain the corresponding local form:

$$\frac{\partial q}{\partial t} + \nabla\cdot(q\mathbf{v}) + \nabla\cdot\mathbf{j}_c = 0. \tag{4.1.4}$$

Remark. *Charge conservation is often misunderstood. While global charge conservation asserts that the total charge in the universe is constant, the local form (4.1.4) is a stronger statement: charge cannot spontaneously disappear from one region and reappear in another without passing through the space in between. This locality condition is fundamental to the formulation of both field and matter dynamics.*

Magnetization and polarization

Many materials respond to electromagnetic fields by setting up internal distributions of bound charge and current. These arise due to the rearrangement of microscopic charges, and are described macroscopically by two vector fields: the *polarization* $\mathbf{P}$ and the *magnetization* $\mathbf{M}$. These fields summarize the net effect of bound dipoles within a material. Rather than prescribing specific constitutive relations for $\mathbf{P}$ and $\mathbf{M}$, we treat them as independent, piecewise-smooth vector fields subject only to the general constraints imposed by the integral Maxwell equations. In particular, we define them through their ability to represent the bound charge and current densities.

Let $\mathcal{V}(t)$ and $\mathcal{A}(t)$ be a moving material volume and surface. We express $\mathbf{P}$ and $\mathbf{M}$ via the following integral identities (which will be justified in the following sections):

$$\begin{aligned} \int_{\mathcal{V}} q_b \, \mathrm{d}^3\mathbf{x} + \oint_{\partial\mathcal{V}} \mathbf{P} \cdot \mathrm{d}^2\mathbf{x} = 0, \\ \int_{\mathcal{A}} \mathbf{j}_{bc} \cdot \mathrm{d}^2\mathbf{x} - \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{A}} \mathbf{P} \cdot \mathrm{d}^2\mathbf{x} - \oint_{\partial\mathcal{A}} c\mathbf{M}_c \cdot \mathrm{d}^1\mathbf{x} = 0, \end{aligned} \tag{4.1.5}$$

where:

- q_b is the bound charge density,
- $\mathbf{j}_{bc}$ is the co-moving bound current density,

$$\mathbf{j}_{bc} = \mathbf{j}_b + \mathbf{v}(\nabla \cdot \mathbf{P}),$$

- $\mathbf{M}_c$ is the co-moving magnetization,

$$\mathbf{M}_c = \mathbf{M} - \frac{\mathbf{v}}{c} \times \mathbf{P}.$$

We can notice how $\mathbf{P}$ is giving a surface contribution to the charge conservation, while $\mathbf{M}$ taking into account internal magnetization currents.

Applying Gauss' and Stokes' theorems, and using the transport identity for surface integrals, we obtain the local form:

$$\begin{aligned} q_b + \nabla \cdot \mathbf{P} = 0, \\ \mathbf{j}_{bc} - \frac{\partial \mathbf{P}}{\partial t} - c\nabla \times \mathbf{M}_c - \mathbf{v}(\nabla \cdot \mathbf{P}) + \nabla \times (\mathbf{v} \times \mathbf{P}) = 0. \end{aligned} \tag{4.1.6}$$

These equations represent the bound sources $(q_b, \mathbf{j}_b)$ in terms of the material's polarization and magnetization. The first equation corresponds to the dipolar model of electric polarization, while the second includes both electric and magnetic dipole effects under motion.

We can now use these relations to rewrite Maxwell's equations $(4.1.1)_{3,4}$ in terms of the free charges and currents. By substituting the total charge $q = q_f + q_b$ and total current $\mathbf{j} = \mathbf{j}_f + \mathbf{j}_b$, we obtain:

$$\begin{aligned} \oint_{\partial\mathcal{V}} (\mathbf{E} + \mathbf{P}) \cdot \mathrm{d}^2\mathbf{x} - \int_{\mathcal{V}} q_f \, \mathrm{d}^3\mathbf{x} = 0, \\ \oint_{\partial\mathcal{A}} (\mathbf{B}_c - \mathbf{M}_c) \cdot \mathrm{d}^1\mathbf{x} - \frac{1}{c} \left\{ \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{A}} (\mathbf{E} + \mathbf{P}) \cdot \mathrm{d}^2\mathbf{x} + \int_{\mathcal{A}} \mathbf{j}_{fc} \cdot \mathrm{d}^2\mathbf{x} \right\} = 0. \end{aligned} \tag{4.1.7}$$

We then define the *electric displacement field* and the *co-moving magnetic field*:

$$\begin{aligned} \mathbf{D} &= \mathbf{E} + \mathbf{P}, \\ \mathbf{H}_c &= \mathbf{B}_c - \mathbf{M}_c = \mathbf{B} - \mathbf{M} - \frac{\mathbf{v}}{c} \times \mathbf{D} = \mathbf{H} - \frac{\mathbf{v}}{c} \times \mathbf{D}. \end{aligned} \tag{4.1.8}$$

Substituting into the local forms of Maxwell's equations yields:

$$\begin{aligned} \nabla \cdot \mathbf{D} - q_f = 0, \\ \nabla \times \mathbf{H}_c - \frac{1}{c} \left[\frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}_{fc} + (\nabla \cdot \mathbf{D})\mathbf{v} - \nabla \times (\mathbf{v} \times \mathbf{D}) \right] = 0. \end{aligned} \tag{4.1.9}$$

Remark. *The fields* $\mathbf{D}$ *and* $\mathbf{H}_c$ *appear naturally when describing media with internal structure (such as dielectrics or ferromagnets). These fields allow us to treat* $\mathbf{P}$ *and* $\mathbf{M}$ *as sources, without needing to resolve their microscopic origin.*

Bound charges in polarized media

When a dielectric material is subjected to an external electric field $\mathbf{E}$, its microscopic charges shift slightly, forming electric dipoles. On a macroscopic scale, this behavior is captured by the polarization field $\mathbf{P}(\mathbf{x})$, which describes the electric dipole moment per unit volume.

These dipoles do not generate free charges, but their spatial variation leads to effective (or "bound") charges. In a continuous medium, we distinguish two types of bound charges:

- **Volume bound charge density**: $q_b = -\nabla \cdot \mathbf{P}$.

  This arises in regions where $\mathbf{P}$ is non-uniform, representing a net imbalance of dipoles across infinitesimal volumes.

- **Surface bound charge density**: $q_b^{\text{sup}} = \mathbf{P} \cdot \mathbf{n}$.

  This appears at interfaces where the polarization vector changes discontinuously, such as the boundary between a dielectric and vacuum.

Surface charge derivation. Let a volume $\mathcal{V}$ straddle the interface between a dielectric and its exterior (e.g., vacuum), and shrink to a pillbox aligned with the surface. Applying Gauss's theorem:

$$Q_b = Q_b^{\text{sup}} = -\oint_{\partial\mathcal{V}} \mathbf{P} \cdot \mathrm{d}^2\mathbf{x}.$$

Only the two flat faces of the pillbox contribute significantly, so:

$$Q_b^{\text{sup}} \approx -(\mathbf{P}_+ \cdot \mathbf{n} - \mathbf{P}_- \cdot \mathbf{n})A,$$

where $A = \text{area}(\partial\mathcal{V})$. Assuming $\mathbf{P}_+ = 0$ outside (e.g., in vacuum), we obtain:

$$Q_b^{\text{sup}} = -(-\mathbf{P} \cdot \mathbf{n})A = (\mathbf{P} \cdot \mathbf{n})A \quad \Rightarrow \quad q_b^{\text{sup}} = \frac{Q_b^{\text{sup}}}{A} = \mathbf{P} \cdot \mathbf{n}.$$

Remark. *The surface bound charge arises because dipoles within the dielectric cannot continue beyond the surface; thus, the termination of dipoles appears as surface charge. The formula tells us that the net surface charge density equals the normal component of the dipole density.*

Bound currents from magnetization

When a material is placed in a magnetic field, its atomic-scale magnetic dipoles (due to electron spin and orbital motion) tend to align with the field, creating a magnetization field $\mathbf{M}(\mathbf{x})$, defined as the magnetic dipole moment per unit volume. From a microscopic point of view, magnetic dipoles arise from circulating microscopic currents, which form tiny loops. When the loops are uniformly distributed in space, their net contribution cancels out in the bulk, except at the boundary.

Volume bound current density. A magnetic dipole with moment $\mathbf{m}$ is modeled as a current loop: $\mathbf{m} = \frac{1}{2c}\int \mathrm{d}^3\mathbf{x}\,\mathbf{x}\times\mathbf{j}(\mathbf{r})$. If such dipoles are densely packed, the magnetization is:

$$\mathbf{M} = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\sum \mathbf{m}.$$

The equivalent current density associated with this distribution is:

$$\mathbf{j}_b = c\nabla\times\mathbf{M}.$$

This current does not correspond to actual motion of free charges but arises from the net circulation of microscopic currents within the material.

Surface bound current density. At the boundary of a magnetized body, the loops inside the material are incomplete: while adjacent loops in the bulk cancel each other's lateral current contributions, the last layer has no neighbor to cancel with. This results in a net surface current:

$$\mathbf{j}_b^{\text{sup}} = c\,\mathbf{M}\times\mathbf{n},$$

where $\mathbf{n}$ is the outward unit normal to the surface. A simple way to show the derivation of this expression is through the us of the vector potential of a magnetic dipole $\mathbf{m}$ located at $\mathbf{x}_0$, which is:

$$\mathbf{A}(\mathbf{x}) = \frac{1}{c}\frac{\mathbf{m}\times(\mathbf{x}-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|^3}.$$

For a continuous distribution with density $\mathbf{M}(\mathbf{x})$, we integrate:

$$\mathbf{A}(\mathbf{x}) = \frac{1}{c}\int_{\mathcal{V}}\frac{\mathbf{M}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\,\mathrm{d}V'.$$

By applying vector identities and integration by parts, this is equivalent to the vector potential generated by:

$$\mathbf{j}_b = c\nabla\times\mathbf{M}, \quad \text{(in the volume)} \qquad \text{and} \qquad \mathbf{j}_b^{\text{sup}} = c\,\mathbf{M}\times\mathbf{n} \quad \text{(on the surface)}.$$

Remark. *These bound currents are purely internal: they do not transport net electric charge, but their magnetic effects are real and measurable. For this reason, the magnetization contributes to Maxwell's equations through the auxiliary field:*

$$\mathbf{H}_c = \mathbf{B}_c - \mathbf{M}_c.$$

Junction conditions

Fields generally exhibit discontinuities at the interfaces $\mathcal{D}$ separating different materials. We can derive the *junction conditions* by integrating Maxwell's equations (4.1.1) over a small volume $\mathcal{V}$ enclosing the interface. However, a more general and rigorous approach relies on the results of section 2.3, which provide the following substitutions at the interface $\mathcal{D}$:

$$\begin{aligned} \nabla\Phi \text{ on } \mathcal{V} &\implies \mathbf{n}\cdot[\![\Phi]\!] \text{ on } \mathcal{D}, \\ \frac{\partial\Phi}{\partial t} \text{ on } \mathcal{V} &\implies -\mathbf{n}\cdot\mathbf{w}\,[\![\Phi]\!] \text{ on } \mathcal{D}, \\ \sigma \text{ on } \mathcal{V} &\implies \sigma^{\text{sup}} \text{ on } \mathcal{D}, \end{aligned}$$

where $\mathbf{n}$ is the unit normal vector to $\mathcal{D}$ and $\mathbf{w}$ is the velocity of the interface. The first relation implies that the normal component of the gradient of a scalar field Φ may have a jump $\mathbf{n}\cdot[\![\Phi]\!]$ at the interface, the second relates the time derivative of Φ to the motion of the interface, and the third states that a volumetric density σ induces a corresponding surface density σ^{sup} on $\mathcal{D}$.

Using these substitutions, we compute the junction conditions for Maxwell's equations (4.1.2):

$$\begin{aligned} \mathbf{n}\cdot[\![\mathbf{B}]\!] &= 0, \\ \mathbf{n}\cdot\left[\!\!\left[\mathbf{E}_c - \frac{1}{c}(\mathbf{v}-\mathbf{w})\times\mathbf{B}\right]\!\!\right] &= 0, \\ \mathbf{n}\cdot[\![\mathbf{E}]\!] &= q^{\text{sup}}, \\ \mathbf{n}\times\left[\!\!\left[\mathbf{B}_c + \frac{1}{c}(\mathbf{v}-\mathbf{w})\times\mathbf{E}\right]\!\!\right] &= \frac{1}{c}\left\{(\mathbf{j}_c + q\mathbf{v})^{\text{sup}} - q^{\text{sup}}\mathbf{w}\right\}. \end{aligned} \tag{4.1.10}$$

Here:

- q^{sup} is the surface charge density on $\mathcal{D}$;
- $(\mathbf{j}_c + q\mathbf{v})^{\text{sup}}$ is the surface current density measured in the lab frame on $\mathcal{D}$;
- the conditions hold on $\mathcal{D}\setminus\mathcal{L}$, i.e., away from edges or singularities.

Applying the same procedure to the charge conservation eq. (4.1.4) yields:

$$\mathbf{n}\cdot[\![q(\mathbf{v}-\mathbf{w}) + \mathbf{j}_c]\!] = 0, \tag{4.1.11}$$

which enforces continuity of the normal charge flux across the interface.

Similarly, the relations linking polarization $\mathbf{P}$ and magnetization $\mathbf{M}$ to bound charges and currents produce the junction conditions:

$$\begin{aligned} \mathbf{n}\cdot[\![\mathbf{P}]\!] &= -q_b^{\text{sup}}, \\ \mathbf{n}\times\left[\!\!\left[\mathbf{M}_c - \frac{1}{c}(\mathbf{v}-\mathbf{w})\times\mathbf{P}\right]\!\!\right] &= \mathbf{j}_{bc}^{\text{sup}}, \end{aligned}$$

where q_b^{sup} and $\mathbf{j}_{bc}^{\text{sup}}$ are the surface bound charge and current densities.

Finally, Maxwell's equations can be cast in terms of free sources only as:

$$\nabla \cdot \mathbf{D} = q_f,$$
$$\nabla \times \mathbf{H}_c - \frac{1}{c}\left(\frac{\partial \mathbf{D}}{\partial t} + \nabla \times (\mathbf{v} \times \mathbf{D})\right) = \frac{1}{c}(\mathbf{j}_{fc} + q_f \mathbf{v}),$$

accompanied by the corresponding junction conditions:

$$\mathbf{n} \cdot [\![\mathbf{D}]\!] = q_f^{\text{sup}}, \quad \mathbf{n} \times \left[\!\left[\mathbf{H}_c + \frac{1}{c}(\mathbf{v} - \mathbf{w}) \times \mathbf{D}\right]\!\right] = \mathbf{j}_{fc}^{\text{sup}}.$$

These interface conditions are essential for correctly solving Maxwell's equations in piecewise heterogeneous media and ensure the correct matching of fields across material boundaries.

4.2 Electromagnetic Sources and Mechanical Coupling

In this section we examine how electric and magnetic dipole fields contribute to the total charge and current densities within deforming continua. These distributions act as sources of electromagnetic force, which we derive both in integral and local form. The resulting force density is then expressed in terms of the electromagnetic momentum and stress tensors, setting the stage for energetic considerations such as electromagnetic power.

For clarity and focus, this section includes the main steps of the derivation, while the more lengthy or technical intermediate calculations are deferred to Appendix B, where the complete development is presented in full detail.

Charges and currents

For a material volume $\mathcal{V}$ we know that the polarization and magnetization fields are confined inside it, so we can write:

$$\mathbf{P}_{\mathcal{V}} = \chi_{\mathcal{V}} \mathbf{P},$$
$$\mathbf{M}_{c\mathcal{V}} = \chi_{\mathcal{V}} \mathbf{M}_c,$$

where $\chi_{\mathcal{V}}$ is the characteristic function of the domain $\mathcal{V}$. Starting from this, the bound charge density inside $\mathcal{V}$ can be expressed as

$$q_{b\mathcal{V}} = -\nabla \cdot \mathbf{P}_{\mathcal{V}}.$$

Applying the product rule for the divergence of the product of a scalar function and a vector field, we obtain

$$q_{b\mathcal{V}} = -\nabla \cdot (\chi_{\mathcal{V}} \mathbf{P}) = -(\nabla \chi_{\mathcal{V}}) \cdot \mathbf{P} - \chi_{\mathcal{V}} (\nabla \cdot \mathbf{P}).$$

Since $\nabla \chi_{\mathcal{V}}$ is a distribution supported on the boundary $\partial \mathcal{V}$ and can be identified with the normal vector $\mathbf{n}$ multiplied by the surface delta $\delta_{\partial \mathcal{V}}$, we rewrite this as

$$q_{b\mathcal{V}} = \mathbf{n} \cdot \mathbf{P}\, \delta_{\partial \mathcal{V}} - \chi_{\mathcal{V}} \nabla \cdot \mathbf{P}.$$

This shows that the bound charge density splits into a surface term, localized on the boundary $\partial\mathcal{V}$, and a volumetric term inside $\mathcal{V}$. More explicitly, we write:

$$\begin{aligned} q_{b\mathcal{V}}^{\text{vol}} &= -\nabla\cdot\mathbf{P}, \quad \text{on } \mathcal{V}, \\ q_{b\mathcal{V}}^{\text{sup}} &= \mathbf{n}\cdot\mathbf{P}, \quad \text{on } \partial\mathcal{V}. \end{aligned} \tag{4.2.1}$$

The volumetric bound charge density corresponds physically to the divergence of the polarization inside the material, indicating volume density of dipoles, while the surface term represents a discontinuity or accumulation of dipoles at the interface.

Next, we consider the bound current density. The expression we use is more general, including convective terms due to the velocity field $\mathbf{v}$ of the material, as well as the time derivative contributions and magnetization currents:

$$\mathbf{j}_{bc\mathcal{V}} = \frac{\partial}{\partial t}\mathbf{P}_{\mathcal{V}} - \nabla\times(\mathbf{v}\times\mathbf{P}_{\mathcal{V}}) + (\nabla\cdot\mathbf{P}_{\mathcal{V}})\mathbf{v} + c(\nabla\times\mathbf{M}_{c\mathcal{V}}).$$

Substituting $\mathbf{P}_{\mathcal{V}} = \chi_{\mathcal{V}}\mathbf{P}$ and $\mathbf{M}_{c\mathcal{V}} = \chi_{\mathcal{V}}\mathbf{M}_c$, and applying the product and vector calculus rules carefully, we get:

$$\begin{aligned} \mathbf{j}_{bc\mathcal{V}} = \frac{\partial\chi_{\mathcal{V}}}{\partial t}\mathbf{P} + \chi_{\mathcal{V}}\frac{\partial\mathbf{P}}{\partial t} - \nabla\chi_{\mathcal{V}}\times(\mathbf{v}\times\mathbf{P}) - \chi_{\mathcal{V}}\nabla\times(\mathbf{v}\times\mathbf{P}) \\ + (\nabla\chi_{\mathcal{V}}\cdot\mathbf{P} + \chi_{\mathcal{V}}\nabla\cdot\mathbf{P})\mathbf{v} + c\left(\nabla\chi_{\mathcal{V}}\times\mathbf{M}_c + \chi_{\mathcal{V}}\nabla\times\mathbf{M}_c\right). \end{aligned}$$

Because $\frac{\partial}{\partial t}\chi_{\mathcal{V}} = -\mathbf{v}\cdot\nabla\chi_{\mathcal{V}} = -(\mathbf{v}\cdot\mathbf{n})\delta_{\partial\mathcal{V}}$ (this comes from the fact that the domain $\mathcal{V}$ moves with velocity $\mathbf{v}$), we can rewrite terms involving $\nabla\chi_{\mathcal{V}}$ as surface distributions on $\partial\mathcal{V}$. Grouping the terms, this leads to

$$\begin{aligned} \mathbf{j}_{bc\mathcal{V}} = \delta_{\partial\mathcal{V}}\left\{(\mathbf{n}\cdot\mathbf{v})\mathbf{P} + \mathbf{n}\times(\mathbf{v}\times\mathbf{P}) - (\mathbf{n}\cdot\mathbf{P})\mathbf{v} - c\mathbf{n}\times\mathbf{M}_c\right\} \\ + \chi_{\mathcal{V}}\left\{\frac{\partial\mathbf{P}}{\partial t} - \nabla\times(\mathbf{v}\times\mathbf{P}) + (\nabla\cdot\mathbf{P})\mathbf{v} + c\nabla\times\mathbf{M}_c\right\}. \end{aligned}$$

From this decomposition, we identify the volume and surface contributions to the bound current density as:

$$\begin{aligned} \mathbf{j}_{bc\mathcal{V}}^{\text{vol}} &= \frac{\partial\mathbf{P}}{\partial t} - \nabla\times(\mathbf{v}\times\mathbf{P}) + (\nabla\cdot\mathbf{P})\mathbf{v} + c\nabla\times\mathbf{M}_c, \quad \text{on } \mathcal{V}, \\ \mathbf{j}_{bc\mathcal{V}}^{\text{sup}} &= -c\mathbf{n}\times\mathbf{M}_c, \quad \text{on } \partial\mathcal{V}. \end{aligned} \tag{4.2.2}$$

Physically, the volume current contains contributions from the local time change of polarization, the convection and deformation of dipoles due to the velocity field $\mathbf{v}$, as well as magnetization currents inside $\mathcal{V}$. The surface current emerges from discontinuities of magnetization at the boundary, expressed by the tangential component $\mathbf{n}\times\mathbf{M}_c$.

Free charge density and free current density, which are not bound to the material microstructure, can instead be treated as pure source terms in Maxwell's equations and do not require such splitting.

Remark. *The decomposition into volumetric and surface terms reflects the physical intuition that dipoles and magnetic moments may be distributed continuously in the bulk or accumulate sharply at interfaces, leading to distinct contributions to the charge and current densities. The presence of the material velocity* $\mathbf{v}$ *introduces convective effects, which are crucial in moving media.*

Electromagnetic Force

We consider a material volume $\mathcal{V}$ immersed in an electromagnetic field, and aim to derive an expression for the total force $\mathbf{F}_\mathcal{V}$ acting on it. This force can be decomposed into a volumetric contribution, arising from charges and currents distributed inside the volume, and a surface contribution, arising from surface charges and currents localized on the boundary $\partial\mathcal{V}$:

$$\begin{aligned} \mathbf{F}_\mathcal{V} &= \mathbf{F}_\mathcal{V}^{\text{vol}} + \mathbf{F}_\mathcal{V}^{\text{sup}}, \\ \mathbf{F}_\mathcal{V}^{\text{vol}} &= \int_\mathcal{V} \mathrm{d}^3\mathbf{x} \left(q^{\text{vol}}\mathbf{E}_c + \frac{1}{c}\mathbf{j}_c^{\text{vol}} \times \mathbf{B} \right), \\ \mathbf{F}_\mathcal{V}^{\text{sup}} &= \oint_{\partial\mathcal{V}} \left(q^{\text{sup}}\mathbf{E}_c + \frac{1}{c}\mathbf{j}_c^{\text{sup}} \times \mathbf{B} \right) \cdot \mathrm{d}^2\mathbf{x}. \end{aligned} \tag{4.2.3}$$

Here, the subscript c denotes quantities measured in the *co-moving frame* of the matter, where the medium is instantaneously at rest. The goal is to express all these co-moving frame quantities in terms of the *lab frame* fields $\mathbf{E}, \mathbf{B}$ and material properties such as the polarization $\mathbf{P}$, magnetization $\mathbf{M}$, free charge density q_f, free current density $\mathbf{j}_f$, and the matter velocity $\mathbf{v}$.

Starting with the volumetric force, using the relations between the co-moving charges or current and lab quantities, we write:

$$\begin{aligned} \mathbf{F}_\mathcal{V}^{\text{vol}} = \int_\mathcal{V} \mathrm{d}^3\mathbf{x} \Bigg[&(q_f - \nabla\cdot\mathbf{P}) \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \\ &+ \frac{1}{c} \left(\mathbf{j}_f - q_f\mathbf{v} + \frac{\partial\mathbf{P}}{\partial t} + (\nabla\cdot\mathbf{P})\mathbf{v} + c\nabla\times\mathbf{M} \right) \times \mathbf{B} \Bigg]. \end{aligned}$$

By expanding and rearranging terms, many velocity-dependent contributions cancel or simplify, yielding the more compact expression:

$$\mathbf{F}_\mathcal{V}^{\text{vol}} = \int_\mathcal{V} \mathrm{d}^3\mathbf{x} \left[q_f\mathbf{E} - (\nabla\cdot\mathbf{P})\mathbf{E} + \frac{1}{c} \left(\mathbf{j}_f + \frac{\partial\mathbf{P}}{\partial t} + c\nabla\times\mathbf{M} \right) \times \mathbf{B} \right].$$

To better understand the structure of the terms, note that the charge density splits into free charge q_f and bound charge $-\nabla\cdot\mathbf{P}$. The latter can be rewritten by using the vector calculus identity:

$$-(\nabla\cdot\mathbf{P})\mathbf{E} = -\nabla\cdot(\mathbf{P}\otimes\mathbf{E}) + (\nabla\otimes\mathbf{E})\cdot\mathbf{P},$$

where $\mathbf{P}\otimes\mathbf{E}$ is the tensor product. Physically, this represents how the spatial variation of polarization induces forces that can be expressed as divergence of stress-like tensors plus gradients acting on the electric field.

Further, applying Maxwell's equation $\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$, we can rewrite the $\mathbf{P}$ term involving the curl of $\mathbf{E}$, leading to:

$$-(\nabla \cdot \mathbf{P})\mathbf{E} = -\nabla \cdot (\mathbf{P} \otimes \mathbf{E}) + \frac{1}{c}\mathbf{P} \times \frac{\partial \mathbf{B}}{\partial t} + (\nabla \otimes \mathbf{E}) \cdot \mathbf{P}.$$

Similarly, the term involving $(\nabla \times \mathbf{M}) \times \mathbf{B}$ can be transformed using vector identities:

$$(\nabla \times \mathbf{M}) \times \mathbf{B} = \nabla \cdot (\mathbf{B} \otimes \mathbf{M} - (\mathbf{M} \cdot \mathbf{B})\mathbb{I}) + (\nabla \otimes \mathbf{B}) \cdot \mathbf{M},$$

where $\mathbb{I}$ is the identity tensor. This term accounts for forces due to magnetization currents interacting with the magnetic field, again splitting naturally into divergence of a stress tensor plus gradient terms.

Putting everything together, the volumetric force becomes:

$$\begin{aligned}\mathbf{F}_{\mathcal{V}}^{\text{vol}} = \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\Big[q_f\mathbf{E} + \frac{1}{c}\mathbf{j}_f \times \mathbf{B} + \frac{1}{c}\frac{\partial}{\partial t}(\mathbf{P} \times \mathbf{B}) + (\nabla \otimes \mathbf{E}) \cdot \mathbf{P} + (\nabla \otimes \mathbf{B}) \cdot \mathbf{M}\Big] \\ + \oint_{\partial\mathcal{V}} \mathrm{d}^2\mathbf{x} \cdot [-\mathbf{P} \otimes \mathbf{E} + \mathbf{B} \otimes \mathbf{M} - (\mathbf{M} \cdot \mathbf{B})\mathbb{I}]\,.\end{aligned}$$

Next, we consider the surface force $\mathbf{F}_{\mathcal{V}}^{\text{sup}}$, which depends on the surface polarization and magnetization charges and currents. After rewriting the terms and applying vector identities and Maxwell's equations, the surface force can be expressed as:

$$\mathbf{F}_{\mathcal{V}}^{\text{sup}} = \oint_{\partial\mathcal{V}} \mathrm{d}^2\mathbf{x} \cdot [\mathbf{P} \otimes \mathbf{E} - \mathbf{B} \otimes \mathbf{M} + \mathbb{I}(\mathbf{M} \cdot \mathbf{B})] + \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\, \nabla \times \frac{\mathbf{v}}{c} \otimes (\mathbf{P} \times \mathbf{B}).$$

Notice how the surface term's boundary integral cancels part of the boundary term from the volumetric force, leading to simplifications in the total force expression.

Substituting back into the total force $\mathbf{F}_{\mathcal{V}}$ from eq. (4.2.3), we obtain the integral form of the electromagnetic force acting on the volume $\mathcal{V}$:

$$\begin{aligned}\mathbf{F}_{\mathcal{V}} = \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\, \Big[q_f\mathbf{E} + \frac{1}{c}\mathbf{j}_f \times \mathbf{B} + (\nabla \otimes \mathbf{E}) \cdot \mathbf{P} + (\nabla \otimes \mathbf{B}) \cdot \mathbf{M} \\ + \frac{1}{c}\left(\frac{\partial}{\partial t}(\mathbf{P} \times \mathbf{B}) + \nabla \cdot (\mathbf{v} \otimes (\mathbf{P} \times \mathbf{B}))\right)\Big]\,.\end{aligned} \tag{4.2.4}$$

From this, we identify the *local electromagnetic force density*

$$\begin{aligned}\mathbf{f}^{\text{EM}} = q_f\mathbf{E} + \frac{1}{c}\mathbf{j}_f \times \mathbf{B} + (\nabla \otimes \mathbf{E}) \cdot \mathbf{P} + (\nabla \otimes \mathbf{B}) \cdot \mathbf{M} \\ + \frac{1}{c}\left(\frac{\partial}{\partial t}(\mathbf{P} \times \mathbf{B}) + \nabla \cdot (\mathbf{v} \otimes (\mathbf{P} \times \mathbf{B}))\right)\,.\end{aligned}$$

Remark.

- *The first two terms represent the classical Lorentz force acting on free charges and currents.*
- *The terms involving gradients of* $\mathbf{E}$ *and* $\mathbf{B}$ *describe forces on the dipole moments* $\mathbf{P}$ *and* $\mathbf{M}$ *due to inhomogeneities in the fields, i.e., dielectrophoretic and magnetophoretic forces.*

- *The last terms involving time derivatives and spatial fluxes of* $\mathbf{P} \times \mathbf{B}$ *represent the momentum exchange between the electromagnetic field and the polarization-magnetization structure moving with velocity* $\mathbf{v}$*.*
- *The splitting into volume and surface contributions clarifies that forces on matter can be understood both as local volumetric forces and as stresses acting on the boundary, consistent with the notion of Maxwell's stress tensor in continuous media.*

After having developed the time derivatives and combined all terms, using $\mathbf{D} = \mathbf{E} + \mathbf{P}$, we obtain:

$$\begin{aligned} \mathbf{f}^{\mathrm{EM}} = & -\frac{1}{c}\left[\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}) + \nabla\cdot(\mathbf{v}\otimes(\mathbf{E}\times\mathbf{B}))\right] \\ & + \nabla\cdot\left[\mathbf{B}\otimes\mathbf{H} + \mathbf{D}\otimes\mathbf{E} - \frac{1}{2}(E^2 + B^2 - 2\mathbf{M}\cdot\mathbf{B})\mathbb{I}\right] \\ & + \frac{1}{c}\nabla\cdot[\mathbf{v}\otimes(\mathbf{D}\times\mathbf{B})] \end{aligned} \tag{4.2.5}$$

The electromagnetic force density reveals fundamental insights about momentum exchange between fields and matter. At its core, this formulation describes how the electromagnetic field both carries momentum and exerts stresses on material systems:

$$\mathbf{p}^{\mathrm{EM}} = \frac{1}{c}\mathbf{E}\times\mathbf{B} \tag{4.2.6}$$

$$\mathbf{t}^{\mathrm{EM}} = \mathbf{B}\otimes\mathbf{H} + \mathbf{D}\otimes\mathbf{E} - \frac{1}{2}(\mathrm{E}^2 + \mathrm{B}^2 - 2\mathbf{M}\cdot\mathbf{B})\mathbb{I} + \frac{1}{c}\mathbf{v}\otimes(\mathbf{D}\times\mathbf{B}) \tag{4.2.7}$$

Electromagnetic momentum density: $\mathbf{p}^{\mathrm{EM}} = \frac{1}{c}\mathbf{E}\times\mathbf{B}$

It represents the kinetic momentum stored in the field configuration. This quantity appears alongside its temporal derivative and convective terms, showing how field momentum changes propagate through the system. Physically, these terms account for radiation pressure effects and momentum transfer when the field configuration evolves in time or when matter moves through non-uniform fields.

Electromagnetic stress tensor: $\mathbf{t}^{\mathrm{EM}} = \mathbf{B}\otimes\mathbf{H} + \mathbf{D}\otimes\mathbf{E} - \frac{1}{2}(\mathrm{E}^2 + \mathrm{B}^2 - 2\mathbf{M}\cdot\mathbf{B})\mathbb{I} + \frac{1}{c}\mathbf{v}\otimes(\mathbf{D}\times\mathbf{B})$

It contains richer physics through its four distinct contributions:

- The $\mathbf{B}\otimes\mathbf{H}$ and $\mathbf{D}\otimes\mathbf{E}$ terms represent anisotropic stresses caused by magnetic and electric polarization. These describe how field lines effectively "pull" on the medium, with the tensor structure encoding directional effects.
- The $-\frac{1}{2}(E^2 + B^2)\mathbb{I}$ term provides isotropic pressure from the field energy density, modified by the $\mathbf{M}\cdot\mathbf{B}$ term to account for magnetization work.

- The velocity-dependent term $\frac{\mathbf{v}}{c} \otimes (\mathbf{D} \times \mathbf{B})$ emerges from the interplay between material motion and field momentum, ensuring proper Galilean invariance.

The balance law formulation elegantly separates these effects:

$$\int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\, \mathbf{j}^{\mathrm{EM}} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\, \mathbf{P}^{\mathrm{EM}} + \oint_{\partial\mathcal{V}} \mathrm{d}^2\mathbf{x} \cdot \mathbf{t}^{\mathrm{EM}}.$$

showing how the total force equals the rate of field momentum change plus stress contributions through the boundary. This decomposition makes manifest how electromagnetic forces arise from both local momentum exchange and surface stresses - a physical picture that remains valid even for moving media and nonlinear materials.

Power

We aim to compute the electromagnetic power spent on a deforming volume of matter $\mathcal{V}(t)$ in presence of an electromagnetic field. The total power is the sum of volumetric and surface contributions, both written in the co-moving frame:

$$\begin{aligned} \mathbf{W}_{\mathcal{V}} &= \mathbf{W}_{\mathcal{V}}^{\mathrm{vol}} + \mathbf{W}_{\mathcal{V}}^{\mathrm{sup}}, \\ \mathbf{W}_{\mathcal{V}}^{\mathrm{vol}} &= \int_{\mathcal{V}} \left(\mathbf{j}_c^{\mathrm{vol}} \cdot \mathbf{E}_c + \mathbf{v} \cdot \left(q^{\mathrm{vol}} \mathbf{E}_c + \frac{1}{c} \mathbf{j}_c^{\mathrm{vol}} \times \mathbf{B} \right) \right) \mathrm{d}^3\mathbf{x}, \\ \mathbf{W}_{\mathcal{V}}^{\mathrm{sup}} &= \oint_{\partial\mathcal{V}} \left(\mathbf{j}_c^{\mathrm{sup}} \cdot \mathbf{E}_c + \mathbf{v} \cdot \left(q^{\mathrm{sup}} \mathbf{E}_c + \frac{1}{c} \mathbf{j}_c^{\mathrm{sup}} \times \mathbf{B} \right) \right) \cdot \mathrm{d}^2\mathbf{x}. \end{aligned} \tag{4.2.8}$$

The fields are initially expressed in the co-moving frame, where $\mathbf{E}_c = \mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}$, and the charge/current densities include polarization and magnetization effects. We first simplify the volumetric part by grouping terms and substituting known constitutive relations:

$$\begin{aligned} \mathbf{W}_{\mathcal{V}}^{\mathrm{vol}} &= \int_{\mathcal{V}} \left[(\mathbf{j}_c^{\mathrm{vol}} + q^{\mathrm{vol}} \mathbf{v}) \cdot \mathbf{E}_c - \mathbf{j}_c^{\mathrm{vol}} \cdot \left(\frac{\mathbf{v}}{c} \times \mathbf{B} \right) \right] \mathrm{d}^3\mathbf{x} \\ &= \int_{\mathcal{V}} (\mathbf{j}_f + \frac{\partial}{\partial t} \mathbf{P} + c \nabla \times \mathbf{M}) \cdot \mathbf{E}\, \mathrm{d}^3\mathbf{x}. \end{aligned}$$

Applying vector identities, such as $\nabla \cdot (\mathbf{M} \times \mathbf{E}) = \mathbf{E} \cdot (\nabla \times \mathbf{M}) - \mathbf{M} \cdot (\nabla \times \mathbf{E})$, and Faraday's law $\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}/c$, we obtain:

$$\mathbf{W}_{\mathcal{V}}^{\mathrm{vol}} = \int_{\mathcal{V}} \left[\mathbf{j}_f \cdot \mathbf{E} + \frac{\partial \mathbf{P}}{\partial t} \cdot \mathbf{E} - \mathbf{M} \cdot \frac{\partial \mathbf{B}}{\partial t} \right] \mathrm{d}^3\mathbf{x} + c \oint_{\partial\mathcal{V}} (\mathbf{M} \times \mathbf{E}) \cdot \mathrm{d}^2\mathbf{x}.$$

Now the surface contribution. Using expressions for surface currents due to magnetization and polarization:

$$\begin{aligned} \mathbf{W}_{\mathcal{V}}^{\mathrm{sup}} &= \oint_{\partial\mathcal{V}} \left[(\mathbf{j}_c^{\mathrm{sup}} + q^{\mathrm{sup}} \mathbf{v}) \cdot \mathbf{E}_c - \mathbf{j}_c^{\mathrm{sup}} \cdot \left(\frac{\mathbf{v}}{c} \times \mathbf{B} \right) \right] \cdot \mathrm{d}^2\mathbf{x} \\ &= -c \oint_{\partial\mathcal{V}} (\mathbf{M} \times \mathbf{E}) \cdot \mathrm{d}^2\mathbf{x} + \int_{\mathcal{V}} \nabla \cdot [\mathbf{v}(\mathbf{P} \cdot \mathbf{E})]\, \mathrm{d}^3\mathbf{x}. \end{aligned}$$

Adding both contributions, surface terms cancel and we get:

$$\mathbf{W}_{\mathcal{V}} = \int_{\mathcal{V}} \left[\mathbf{j}_f \cdot \mathbf{E} - \mathbf{M} \cdot \frac{\partial}{\partial t} \mathbf{B} + \frac{\partial}{\partial t} \mathbf{P} \cdot \mathbf{E} + \nabla \cdot (\mathbf{v}(\mathbf{P} \cdot \mathbf{E})) \right] \mathrm{d}^3\mathbf{x}. \tag{4.2.9}$$

Remark. *The power density* $\mathbf{w}^{EM}$ *includes several physical processes:*

- *The* $\mathbf{j}_f \cdot \mathbf{E}$ *term represents the familiar Joule heating from free currents*
- $\frac{\partial}{\partial t}\mathbf{P} \cdot \mathbf{E}$ *accounts for the work done in polarizing the medium*
- $-\mathbf{M} \cdot \frac{\partial}{\partial t}\mathbf{B}$ *describes the energy required to maintain magnetization against changing fields*
- $\nabla \cdot (\mathbf{v}(\mathbf{P} \cdot \mathbf{E}))$ *captures how moving polarization affects energy transport*

By expanding the time derivatives we can reach the final form of the local power density, which reveals how energy is exchanged between the electromagnetic field and material media:

$$\mathbf{w}^{\mathrm{EM}} = -\left[\frac{\partial u^{\mathrm{EM}}}{\partial t} + \nabla \cdot (\mathbf{v} u^{\mathrm{EM}})\right] - \nabla \cdot \Big(c\mathbf{E} \times \mathbf{H} - \mathbf{v}(u^{\mathrm{EM}} + \mathbf{P} \cdot \mathbf{E})\Big), \tag{4.2.10}$$

This equation contains several physical contributions worth examining in detail, but the key components are:

$$u^{\mathrm{EM}} = \frac{1}{2}(E^2 + B^2), \tag{4.2.11}$$

$$\mathbf{j}_c^{\mathrm{EM}} = c\mathbf{E} \times \mathbf{H} - \mathbf{v}(u^{\mathrm{EM}} + \mathbf{P} \cdot \mathbf{E}). \tag{4.2.12}$$

Electromagnetic energy density: $u^{\mathrm{EM}} = \frac{1}{2}(E^2 + B^2)$

It represents the familiar vacuum field energy, but now appears in a more general context where the medium may be moving and polarizable. The term $\partial u^{\mathrm{EM}}/\partial t$ accounts for the temporal variation of this field energy, while $\nabla \cdot (\mathbf{v} u^{\mathrm{EM}})$ describes how the energy density is transported by the motion of the medium.

Power current density: $\mathbf{j}_c^{\mathrm{EM}} = c\mathbf{E} \times \mathbf{H} - \mathbf{v}(u^{\mathrm{EM}} + \mathbf{P} \cdot \mathbf{E})$

It contains two distinct mechanisms of energy transport. The Poynting vector $c\mathbf{E} \times \mathbf{H}$ represents the standard electromagnetic energy flow, while the velocity-dependent terms account for the convective transport of both the field energy (u^{EM}) and the interaction energy between the polarization and the electric field ($\mathbf{P} \cdot \mathbf{E}$).

The complete energy balance in integral form makes the conservation principle explicit:

$$\int_{\mathcal{V}} \mathbf{w}^{\mathrm{EM}} \, \mathrm{d}^3\mathbf{x} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{V}} u^{\mathrm{EM}} \, \mathrm{d}^3\mathbf{x} - \oint_{\partial\mathcal{V}} \mathbf{j}_c^{\mathrm{EM}} \cdot \mathrm{d}^2\mathbf{x}. \tag{4.2.13}$$

This formulation provides a complete picture of electromagnetic energy conversion in moving, polarizable media, generalizing the standard Poynting theorem to include material motion and polarization effects.

4.3 Balance Equations in Electromagnetic Continua

In this section we will conclude the formulation for the mechanical system composed by the continuum body and the electromagnetic fields; usually fields are taken into account in form of contribution to the expressions of the force and power for a continuum, but we will show how, rearranging terms from the previously derived equations for the electromagnetic force and power, we can recognize direct contribution to the momentum, energy and stress tensor for example. We can thus make use of a mechanical system where the fields are themselves part of the subject under examination.

For clarity and focus, this section includes the main steps of the derivation, while the more lengthy or technical intermediate calculations are deferred to Appendix C, where the complete development is presented in full detail.

Mass balance

The total mass of a material body remains constant during its motion and deformation.

Let $\rho(t, \mathbf{x})$ denote the mass density of a material body occupying a region $\mathcal{V}(t) \subset \mathbb{E}^3$, moving with velocity field $\mathbf{v}(t, \mathbf{x})$. The *integral form* of the mass balance is:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{V}(t)} \mathrm{d}^3\mathbf{x}\, \rho = 0.$$

This represents the conservation of total mass in a deforming material volume, and holds independently of the presence of electromagnetic fields. Using the transport theorem and the chain rule for the material derivative, we recover the standard *local form*:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{4.3.1}$$

or, equivalently, in material form:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \mathbf{v} = 0.$$

Jump condition. In the presence of moving discontinuities, the associated junction condition across a regular surface $\mathcal{D}(t)$ is:

$$\mathbf{n} \cdot [\![\rho(\mathbf{v} - \mathbf{w})]\!] = 0, \tag{4.3.2}$$

where $\mathbf{n}$ is the unit normal to $\mathcal{D}$ and $\mathbf{w}$ its normal velocity. This ensures continuity of the mass flux across the interface.

Remark. *This is the same condition as in the field-free case, since electromagnetic fields do not contribute directly to mass transport in classical theory.*

Linear momentum balance

The total linear momentum of a continuum body and the surrounding electromagnetic field is conserved under the action of internal stresses and external forces.

Let $\mathbf{p}(t, \mathbf{x})$ denote the momentum density of a deformable continuum occupying a region $\mathcal{V}(t) \subset \mathbb{E}^3$, with mechanical body force $\mathbf{f}$ and Cauchy stress tensor $\mathbf{t}$. In the presence of electromagnetic fields, the total linear momentum of the system includes also the field momentum $\mathbf{p}^{\mathrm{EM}}$, and the corresponding Maxwell stress $\mathbf{t}^{\mathrm{EM}}$.

The *integral form* of the balance of linear momentum reads:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}(t)} \mathrm{d}^3\mathbf{x}\,\mathbf{p} = \oint_{\partial\mathcal{V}(t)} \mathrm{d}^2\mathbf{x}\cdot\mathbf{t} + \int_{\mathcal{V}(t)} \mathrm{d}^3\mathbf{x}\,(\mathbf{f}+\mathbf{f}^{\mathrm{EM}}).$$

Here $\mathbf{f}^{\mathrm{EM}}$ is the electromagnetic force density, given locally in terms of the divergence of the Maxwell stress and the time derivative of the field momentum:

$$\mathbf{f}^{\mathrm{EM}} = -\frac{\partial \mathbf{p}^{\mathrm{EM}}}{\partial t} - \nabla\cdot(\mathbf{v}\otimes\mathbf{p}^{\mathrm{EM}}) + \nabla\cdot\mathbf{t}^{\mathrm{EM}}.$$

Substituting this into the integral balance yields the total momentum conservation law:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}(t)} \mathrm{d}^3\mathbf{x}\,(\mathbf{p}+\mathbf{p}^{\mathrm{EM}}) = \oint_{\partial\mathcal{V}(t)} \mathrm{d}^2\mathbf{x}\cdot(\mathbf{t}+\mathbf{t}^{\mathrm{EM}}) + \int_{\mathcal{V}(t)} \mathrm{d}^3\mathbf{x}\,\mathbf{f}. \tag{4.3.3}$$

This equation shows that the mechanical and electromagnetic momenta evolve together under the combined effect of internal stresses and applied mechanical forces. Electromagnetic forces act as internal exchange terms between field and matter.

Applying the transport theorem to the integrand in eq. (4.3.3), we obtain the local differential form of the momentum balance:

$$\frac{\partial}{\partial t}(\mathbf{p}+\mathbf{p}^{\mathrm{EM}}) + \nabla\cdot\Big(\mathbf{v}\otimes(\mathbf{p}+\mathbf{p}^{\mathrm{EM}})\Big) - \nabla\cdot(\mathbf{t}+\mathbf{t}^{\mathrm{EM}}) = \mathbf{f}. \tag{4.3.4}$$

The structure mirrors the purely mechanical case, with additional electromagnetic contributions to momentum and stress. This form emphasizes the coupling between the material and electromagnetic subsystems.

Jump condition. Across a surface of discontinuity $\mathcal{D}(t)$ with velocity field $\mathbf{w}$, the corresponding momentum flux balance reads:

$$\mathbf{n}\cdot\Big[\!\Big[(\mathbf{v}-\mathbf{w})\otimes(\mathbf{p}+\mathbf{p}^{\mathrm{EM}}) - (\mathbf{t}+\mathbf{t}^{\mathrm{EM}})\Big]\!\Big] = \mathbf{f}^{\mathrm{sup}}, \tag{4.3.5}$$

where $\mathbf{n}$ is the unit normal to the interface and $\mathbf{f}^{\mathrm{sup}}$ is a possible surface force density. This junction condition ensures conservation of linear momentum across singularities in the material or field distribution.

Remark. *This structure reduces to the classical Cauchy momentum balance in the absence of electromagnetic fields. The presence of $\mathbf{p}^{EM}$ and $\mathbf{t}^{EM}$ reflects how fields carry momentum and exchange it dynamically with the material medium.*

Angular momentum balance

The total angular momentum of a continuum body and its electromagnetic field is conserved under the action of internal stresses and external torques.

For an electromagnetic continuum, the *balance of angular momentum* takes the integral form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\,(\mathbf{x}\times\mathbf{p}+\mathbf{s}) = \oint_{\partial\mathcal{V}} \mathbf{x}\times(\mathrm{d}^2\mathbf{x}\cdot\mathbf{t}) + \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\,\left[\mathbf{x}\times(\mathbf{f}+\mathbf{f}^{\mathrm{EM}})\right].$$

Here, we identify the *mechanical momentum density* $\mathbf{p}$, the *mechanical spin density* $\mathbf{s}$, the *mechanical stress tensor* $\mathbf{t}$, the *mechanical force density* $\mathbf{f}$, and the *electromagnetic force density* $\mathbf{f}^{\mathrm{EM}}$. Using the expression for $\mathbf{f}^{\mathrm{EM}}$ (eq. (4.2.5)) and recalling the definitions of the electromagnetic momentum and stress fields $\mathbf{p}^{\mathrm{EM}}$ and $\mathbf{t}^{\mathrm{EM}}$, we compute:

$$\begin{aligned}\mathbf{x}\times\mathbf{f}^{\mathrm{EM}} = &-\frac{\partial}{\partial t}(\mathbf{x}\times\mathbf{p}^{\mathrm{EM}}) - \nabla\cdot(\mathbf{v}\otimes(\mathbf{x}\times\mathbf{p}^{\mathrm{EM}}))\\ &-\nabla\cdot(\mathbf{t}^{\mathrm{EM}}\times\mathbf{x}) - 2\star(\mathbf{t}^{(a)EM}) + \mathbf{v}\times\mathbf{p}^{\mathrm{EM}}.\end{aligned}$$

Integrating over $\mathcal{V}$ and applying the transport theorem for time derivatives of integrals over moving volumes (eq. (2.2.1)), we obtain:

$$\begin{aligned}\int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\,(\mathbf{x}\times\mathbf{f}^{\mathrm{EM}}) = &-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\,(\mathbf{x}\times\mathbf{p}^{\mathrm{EM}})\\ +\oint_{\partial\mathcal{V}} \mathbf{x}\times(\mathrm{d}^2\mathbf{x}\cdot\mathbf{t}^{\mathrm{EM}}) &- \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\,\left[2\star(\mathbf{t}^{(a)EM}) - \mathbf{v}\times\mathbf{p}^{\mathrm{EM}}\right].\end{aligned}$$

Substituting this into the original angular momentum balance, we obtain:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\,(\mathbf{x}\times(\mathbf{p}+\mathbf{p}^{\mathrm{EM}})+\mathbf{s}) = &\oint_{\partial\mathcal{V}} \mathbf{x}\times(\mathrm{d}^2\mathbf{x}\cdot(\mathbf{t}+\mathbf{t}^{\mathrm{EM}}))\\ +\int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\,(\mathbf{x}\times\mathbf{f}) &- \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\,\left[2\star(\mathbf{t}^{(a)EM}) - \mathbf{v}\times\mathbf{p}^{\mathrm{EM}}\right].\end{aligned}$$

We are thus led to conjecture the existence of an electromagnetic contribution to the spin density, $\mathbf{s}^{\mathrm{EM}}$, such that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\,\mathbf{s}^{\mathrm{EM}} = \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\,\left[2\star(\mathbf{t}^{(a)EM}) - \mathbf{v}\times\mathbf{p}^{\mathrm{EM}}\right].$$

This yields the final angular momentum balance:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\,\left(\mathbf{x}\times(\mathbf{p}+\mathbf{p}^{\mathrm{EM}})+\mathbf{s}+\mathbf{s}^{\mathrm{EM}}\right) = \oint_{\partial\mathcal{V}} \mathbf{x}\times(\mathrm{d}^2\mathbf{x}\cdot(\mathbf{t}+\mathbf{t}^{\mathrm{EM}})) + \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\,(\mathbf{x}\times\mathbf{f}). \tag{4.3.6}$$

This reaffirms the identification of $\mathbf{p}^{\mathrm{EM}}$ as the *electromagnetic momentum density* and $\mathbf{t}^{\mathrm{EM}}$ as the *electromagnetic stress tensor*.

Spin and torque density. In the purely mechanical case[1], we have:

$$\frac{\partial \mathbf{s}}{\partial t} + \nabla \cdot (\mathbf{v} \otimes \mathbf{s}) - \mathbf{c} = 0,$$
$$\mathbf{c} = 2 \star \mathbf{t}^{(a)} - \mathbf{v} \times \mathbf{p},$$

where $\mathbf{c}$ is the *mechanical spin torque density*, and if $\mathbf{p} = \rho \mathbf{v}$ then $\mathbf{c} = 2 \star \mathbf{t}^{(a)}$. The identification of $\mathbf{s}^{\mathrm{EM}}$ is not arbitrary, but follows from an explicit decomposition of the electromagnetic force density, $\mathbf{f}^{\mathrm{EM}}$, into divergence and time derivative terms. This reveals an intrinsic electromagnetic torque density $\mathbf{c}^{\mathrm{EM}}$, which must be balanced locally by a spin term to preserve the angular momentum budget of the system.

Considering the total (mechanical plus electromagnetic) balance, the local form of eq. (4.3.6) reads:

$$\begin{aligned} \frac{\partial}{\partial t}(\mathbf{s} + \mathbf{s}^{\mathrm{EM}}) + \nabla \cdot (\mathbf{v} \otimes (\mathbf{s} + \mathbf{s}^{\mathrm{EM}})) - (\mathbf{c} + \mathbf{c}^{\mathrm{EM}}) = 0, \\ \mathbf{c} + \mathbf{c}^{\mathrm{EM}} = 2 \star (\mathbf{t} + \mathbf{t}^{\mathrm{EM}})^{(a)} - \mathbf{v} \times (\mathbf{p} + \mathbf{p}^{\mathrm{EM}}). \end{aligned} \tag{4.3.7}$$

From this, one recovers:

$$\frac{\partial \mathbf{s}^{\mathrm{EM}}}{\partial t} + \nabla \cdot (\mathbf{v} \otimes \mathbf{s}^{\mathrm{EM}}) - \mathbf{c}^{\mathrm{EM}} = 0,$$
$$\mathbf{c}^{\mathrm{EM}} = 2 \star (\mathbf{t}^{(a)EM}) - \mathbf{v} \times \mathbf{p}^{\mathrm{EM}}.$$

To further clarify its physical meaning, we recall that $\mathbf{t}^{(a)EM}$ encodes the non-symmetric part of the Maxwell stress tensor. When corrected for the orbital term $\mathbf{v} \times \mathbf{p}^{\mathrm{EM}}$, it gives a net torque density per unit volume associated with the field–matter interaction. Assuming the form of $\mathbf{c}^{\mathrm{EM}}$ from physical arguments and dimensional analysis, and using the identity $\mathbf{v} \times \mathbf{p}^{\mathrm{EM}} = \star(\mathbf{v} \wedge \mathbf{p}^{\mathrm{EM}}) = 2 \star (\mathbf{v} \otimes \mathbf{p}^{\mathrm{EM}})$, we write:

$$\mathbf{c}^{\mathrm{EM}} = 2 \star (\mathbf{t}^{\mathrm{EM}} - \mathbf{v} \otimes \mathbf{p}^{\mathrm{EM}})^{(a)}.$$

Recalling the expressions for the electromagnetic momentum and stress tensor (equations (4.2.6) and (4.2.7)), and using $\mathbf{D} = \mathbf{E} + \mathbf{P}$ and $\mathbf{B} = \mathbf{H} + \mathbf{M}$, we find:

$$\mathbf{t}^{\mathrm{EM}} - \mathbf{v} \otimes \mathbf{p}^{\mathrm{EM}} = \mathbf{B} \otimes \mathbf{B} + \mathbf{E} \otimes \mathbf{E} - \frac{1}{2}(\mathrm{E}^2 + \mathrm{B}^2 - 2\mathbf{M} \cdot \mathbf{B})\mathbb{I} - \mathbf{B} \otimes \mathbf{M} + \mathbf{P} \otimes \mathbf{E} + \frac{\mathbf{v}}{c} \otimes (\mathbf{P} \times \mathbf{B}).$$

The first three terms are symmetric; hence, the antisymmetric part is:

$$2(\mathbf{t}^{\mathrm{EM}} - \mathbf{v} \otimes \mathbf{p}^{\mathrm{EM}})^{(a)} = \mathbf{M} \wedge \mathbf{B} + \mathbf{P} \wedge \mathbf{E} + \frac{\mathbf{v}}{c} \wedge (\mathbf{P} \times \mathbf{B}).$$

Thus,

$$\mathbf{c}^{\mathrm{EM}} = \mathbf{M}_c \times \mathbf{B} + \mathbf{P} \times \mathbf{E}_c.$$

This gives a physically transparent interpretation: the electromagnetic spin torque arises from the interaction of the magnetization and polarization with the magnetic and electric fields in the comoving frame.

[1] Using the balance of linear momentum as needed to simplify the equations.

Jump condition. In order to understand the behavior of the quantities at discontinuities, the jump condition for angular momentum across a moving discontinuity surface $\mathcal{D}$ with normal $\mathbf{n}$ and velocity $\mathbf{w}$ is:

$$\mathbf{n} \cdot \left[\!\left[(\mathbf{v} - \mathbf{w}) \otimes (\mathbf{s} + \mathbf{s}^{\mathrm{EM}}) \right]\!\right] = (\mathbf{c} + \mathbf{c}^{\mathrm{EM}})^{\mathrm{sup}}. \tag{4.3.8}$$

Remark. *The inclusion of $\mathbf{s}^{EM}$ and $\mathbf{c}^{EM}$ is not a formal artifact, but a physical necessity arising from the intrinsic angular momentum carried by the electromagnetic field in matter. These quantities restore the symmetry of the total stress tensor when averaged over a region and ensure local and global conservation of angular momentum. This is especially relevant in polarizable and magnetizable media, where hidden torques arise due to dipolar interactions with fields and motion. Their correct accounting plays a central role in electromechanical coupling phenomena such as the Einstein–de Haas and Barnett effects.*

Energy balance

The total energy of a continuum body, including mechanical, thermal, and electromagnetic contributions, is conserved through the interplay of internal power, heat transfer, and external sources.

For an electromagnetic continuum, the *balance of total energy* expresses the rate of change of mechanical and internal energy stored in a deformable volume $\mathcal{V}$, and reads:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x} \left[\frac{1}{2\rho}\mathbf{p}^2 + u \right] = \oint_{\partial\mathcal{V}} \mathrm{d}^2\mathbf{x} \cdot \mathbf{t} \cdot \mathbf{v} + \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\, (\mathbf{f} + \mathbf{f}^{\mathrm{EM}}) \cdot \mathbf{v} - \oint_{\partial\mathcal{V}} \mathbf{j}_c^{\mathrm{H}} \cdot \mathrm{d}^2\mathbf{x} + \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\, (z + z^{\mathrm{EM}}), \tag{4.3.9}$$

where ρ is the mass density, $\mathbf{p}$ is the linear momentum density, u is the internal energy density, and $\mathbf{t}$ is the Cauchy stress tensor. The body force $\mathbf{f}$ includes mechanical interactions, while $\mathbf{f}^{\mathrm{EM}}$ accounts for electromagnetic forces. The term $\mathbf{j}_c^{\mathrm{H}}$ represents the conductive heat flux, and the terms z and z^{EM} are the volumetric rates of mechanical and electromagnetic heat production, respectively. The right-hand side thus balances the work of surface and body forces and the net heat exchange, both mechanical and electromagnetic.

We now recognize that the electromagnetic contribution to energy production can be compactly written in terms of the electromagnetic power density $\mathbf{w}^{\mathrm{EM}} = \mathbf{f}^{\mathrm{EM}} \cdot \mathbf{v} + z^{\mathrm{EM}}$. By incorporating this contribution into the total energy content while recalling the balance expression for $\mathbf{w}^{\mathrm{EM}}$ in eq. (4.2.13), the energy balance takes the following integral form:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x} \left[\frac{1}{2\rho}\mathbf{p}^2 + u + u^{\mathrm{EM}} \right] = \oint_{\partial\mathcal{V}} \mathrm{d}^2\mathbf{x} \cdot (\mathbf{t} + \mathbf{t}^{\mathrm{EM}}) \cdot \mathbf{v} + \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\, \mathbf{f} \cdot \mathbf{v} \\ - \oint_{\partial\mathcal{V}} (\mathbf{j}_c^{\mathrm{H}} + \mathbf{j}_c^{\mathrm{H,EM}}) \cdot \mathrm{d}^2\mathbf{x} + \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\, z, \end{aligned} \tag{4.3.10}$$

where we have introduced the electromagnetic stress tensor $\mathbf{t}^{\mathrm{EM}}$ and the electromagnetic contribution to the conductive heat current $\mathbf{j}_c^{\mathrm{H,EM}} = \mathbf{j}_c^{\mathrm{EM}} + \mathbf{t}^{\mathrm{EM}} \cdot \mathbf{v}$. This expression captures

both the mechanical and electromagnetic transport of energy and is essential to describe media where fields interact with matter.

To obtain the local form of the balance, we apply the transport theorem and the divergence theorem to the integrals, obtaining a partial differential equation valid pointwise:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2\rho}\mathbf{p}^2 + u + u^{\mathrm{EM}}\right) + \nabla\cdot\Big[\mathbf{v}\left(\frac{1}{2\rho}\mathbf{p}^2 + u + u^{\mathrm{EM}}\right)\\ -(\mathbf{t}+\mathbf{t}^{\mathrm{EM}})\cdot\mathbf{v} + \mathbf{j}_c^{\mathrm{H}} + \mathbf{j}_c^{\mathrm{H,EM}}\Big] = \mathbf{f}\cdot\mathbf{v} + z.\end{aligned} \tag{4.3.11}$$

This equation expresses the pointwise conservation of energy, where the time derivative of the total energy density and the divergence of the energy flux are balanced by mechanical work and internal heat production.

Internal energy balance. To isolate the evolution of internal energy within a deformable continuum interacting with electromagnetic fields, we may subtract the mechanical kinetic contribution from the full energy balance. Proceeding as in section 3.2, and invoking the local form of the linear momentum balance to eliminate time derivatives of momentum density, we arrive at the following local equation:

$$\frac{\partial u}{\partial t} + \nabla\cdot(\mathbf{v}u + \mathbf{j}_c^{\mathrm{H}}) = \mathbf{t}\colon\nabla\otimes\mathbf{v} + z + z^{\mathrm{EM}}. \tag{4.3.12}$$

This expression describes the *mechanical interpretation of internal energy evolution.* The left-hand side represents the local rate of change of internal energy u and its transport due to both advection (through the material velocity field $\mathbf{v}$) and heat conduction (through the flux $\mathbf{j}_c^{\mathrm{H}}$). On the right-hand side, we identify three distinct sources of internal energy growth. The first term, $\mathbf{t}\colon\nabla\otimes\mathbf{v}$, accounts for the power input due to mechanical deformation, i.e., the rate at which the stress field does work on the material through velocity gradients. The scalar field z encapsulates volumetric heat production of mechanical origin (e.g., due to viscous dissipation), while the electromagnetic contribution z^{EM} quantifies the conversion of electromagnetic energy into internal thermal energy, including effects such as Joule heating and field-matter interaction mechanisms mediated by polarization and magnetization dynamics.

Overall, this equation highlights how internal energy increases not only because of thermal conduction and mechanical deformation, but also due to irreversible electromagnetic processes, which act as a distributed source of thermalization in the medium.

Electromagnetic heat production rate. A key term in this expression is the electromagnetic heat production rate $z^{\mathrm{EM}} = \mathbf{w}^{\mathrm{EM}} - \mathbf{f}^{\mathrm{EM}}\cdot\mathbf{v}$. This quantity captures the part of electromagnetic power not converted into mechanical work, i.e., the rate at which electromagnetic fields irreversibly transfer energy to the medium in the form of heat.

To compute z^{EM} explicitly, we recall the expressions for electromagnetic force density and power density, which account for interactions with free charges and currents, as well

as polarization $\mathbf{P}$ and magnetization $\mathbf{M}$. After algebraic manipulations and the use of co-moving fields (e.g. $\mathbf{E}_c = \mathbf{E} + \mathbf{v} \times \mathbf{B}/c$), we arrive at the final expression:

$$\begin{aligned} z^{\mathrm{EM}} = &\left(\mathbf{j}_{fc} + \frac{\partial \mathbf{P}}{\partial t} + \nabla \times (\mathbf{P} \times \mathbf{v}) + (\nabla \cdot \mathbf{P})\mathbf{v} + c\nabla \times \mathbf{M}_c\right) \cdot \mathbf{E}_c \\ &+ c\nabla \cdot (\mathbf{E}_c \times \mathbf{M}_c) + (\mathbf{P} \otimes \mathbf{E}_c - \mathbf{B} \otimes \mathbf{M}_c + \mathbf{M}_c \cdot \mathbf{B}\mathbb{I}) : \nabla \otimes \mathbf{v}, \end{aligned} \tag{4.3.13}$$

where $\mathbf{j}_{fc}$ is the free conduction current density, and the subscript c denotes co-moving field components. The first line represents *bulk energy conversion from fields to matter*, involving polarization and magnetization dynamics. The second line includes both a divergence term, which can act as a flux of field energy, and a field-velocity coupling term, which accounts for work done by stress-like field structures in a deforming medium.

Electromagnetic heat conduction current density. The heat conduction current density also receives an electromagnetic contribution, derived from the power flux vector and the electromagnetic stress tensor. Its final form is compactly written as:

$$\mathbf{j}_c^{\mathrm{H,EM}} = c\,\mathbf{E}_c \times \mathbf{H}_c, \tag{4.3.14}$$

which generalizes the Poynting vector to moving media. It represents the transport of field energy per unit time across surfaces, as observed in the material frame.

Jump condition. Finally, to account for possible jumps in energy flux across internal interfaces, we consider a discontinuity surface $\mathcal{D}$ moving with velocity $\mathbf{w}$. The jump condition associated with the local balance eq. (4.3.11) is:

$$\mathbf{n} \cdot \left[\!\!\left[\left(\frac{1}{2\rho}\mathbf{p}^2 + u + u^{\mathrm{EM}}\right)(\mathbf{v} - \mathbf{w}) - (\mathbf{t} + \mathbf{t}^{\mathrm{EM}})\mathbf{v} + \mathbf{j}_c^{\mathrm{H}} + \mathbf{j}_c^{\mathrm{H,EM}}\right]\!\!\right] = (\mathbf{f} \cdot \mathbf{v} + z)^{\mathrm{sup}}. \tag{4.3.15}$$

This condition ensures that energy fluxes and production across the interface are consistently accounted for, including electromagnetic contributions to both stress and heat conduction.

> **Remark.** *The electromagnetic terms in the balance of energy reflect the complex interplay between fields and matter. The heat production rate z^{EM} includes both dissipative and reversible contributions and can be seen as a generalization of Joule heating to polarizable and magnetizable continua. The conduction current density $\mathbf{j}_c^{H,EM}$, on the other hand, captures the transport of electromagnetic energy due to both fields and motion, and its expression recovers the standard Poynting vector in the appropriate limit.*

5 | Constitutive Relations

So far, we have derived balance laws for five fundamental quantities: mass, linear momentum, angular momentum, energy, and entropy:[1]

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0, \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} - \nabla \cdot \mathbf{t} - \mathbf{f} &= \mathbf{0}, \\ \mathbf{t}^{\top} &= \mathbf{t}, \\ \frac{\mathrm{d}}{\mathrm{d}t}u + \nabla \cdot \mathbf{j}_c^{\mathrm{H}} - \mathbf{t} : (\nabla \otimes \mathbf{v}) - z &= 0, \\ \frac{\mathrm{d}}{\mathrm{d}t}s + \nabla \cdot \mathbf{j}^{(s)} - \sigma^{(s)} &\geq 0. \end{aligned} \tag{5.0.1}$$

These equations are universal and independent of the specific object under study. They provide 3 constraints for the stress tensor $\mathbf{t}$ (from the angular momentum balance) and 5 differential equations to determine 5 of the remaining unknown parameters. However, the total number of unknowns is 16: ρ, $\mathbf{x}$ (or $\mathbf{v}$), $\mathbf{t}$, u, $\mathbf{j}_c^{\mathrm{H}}$, θ, and s (assuming the external actions $\mathbf{f}$ and z are known).

Definition 1 (Constitutive Relations)**.** *Additional equations that specify how particular materials respond to mechanical and thermal stimuli, required to complete the underdetermined system of balance laws. They establish the dependence of stress* $\mathbf{t}$*, heat flux* $\mathbf{j}_c^H$*, internal energy* u*, and entropy* s *on the state variables.*

Since these equations are insufficient to fully determine the parameters describing the body, additional equations called *constitutive relations* are required. These relations depend solely on the material characteristics. In this chapter, we will provide examples of such equations for elastic and viscous fluids. We then present the Coleman-Noll procedure to complete our treatment of continuum thermomechanics. Having previously analyzed the mechanical balance equations incorporating electromagnetic contributions (where we treated the electromagnetic fields as integral components of our mechanical system) we

[1] We now focus on the mathematical formulation of our governing equations, where electromagnetic contributions to momentum, energy, and other quantities are being considered. However, for clarity of presentation, we are implicitly incorporating these electromagnetic effects within our existing framework of purely mechanical quantities.

will employ this general methodological framework to derive additional thermodynamic constraints and systematically develop constitutive relations for our system.

5.1 Perfect Fluids

The simplest example of a fluid in classical hydrodynamic theory is the *perfect fluid*. A "fluid" here refers to any liquid or gas whose mechanical properties obey the following constitutive relation:

$$\mathbf{t} = -\varpi\mathbb{I}, \tag{5.1.1}$$

where $\varpi = \varpi(\mathbf{x}, t)$ represents the pressure field, defined at every point in the continuum body. This equation defines a perfect fluid, also known as an *Euler fluid*. In this type of fluid, only hydrostatic stresses are present, meaning any effects due to friction can be neglected. Although purely ideal, this model yields satisfactory results under standard laboratory conditions (temperature 300 K and pressure 1 Pa). The simplest case of pressure depending linearly on density is:

$$\varpi = c\rho,$$

which defines the special case of *perfect gases*, provided the proportionality constant depends on the absolute temperature:

$$\varpi = R'\theta\rho.$$

Here, R' is a material parameter. Rewriting the classical ideal gas law from thermodynamics as:

$$\varpi\Delta V = \Delta nR\theta,$$

where ΔV and Δn represent the volume and number of moles of a small portion of the fluid, and R is the universal gas constant, we can solve for the pressure field:

$$\varpi = \frac{\Delta n}{\Delta V}R\theta = \frac{\Delta n}{\Delta m}\frac{\Delta m}{\Delta V}R\theta = k\rho R\theta.$$

From this, we see that the parameter R' is given by:

$$R' = kR,$$

where the parameter:

$$k = \frac{\Delta n}{\Delta m},$$

represents the number of moles per unit mass of the portion and is a fixed value specific to each fluid.

Using the constitutive relation (5.1.1) and the balance laws of mass $(5.0.1)_1$ and linear momentum $(5.0.1)_2$, we obtain the system of equations that fully define a perfect fluid:

$$\mathbf{a} = -\frac{1}{\rho}\nabla\varpi + \mathbf{f}, \tag{5.1.2}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{5.1.3}$$

The first relation is called the *Euler equation*. Together, these equations form a system of 4 partial differential equations that completely define the fields $\rho(\mathbf{x}, t)$ and $\mathbf{v}(\mathbf{x}, t)$, provided the pressure field ϖ and the vector $\mathbf{f}$ are known. For the special case of an incompressible fluid, the situation simplifies because the mass density ρ no longer depends on position or time:

$$\mathbf{a} = -\frac{1}{\rho}\nabla \varpi + \mathbf{f}, \tag{5.1.4}$$

$$\nabla \cdot \mathbf{v} = 0. \tag{5.1.5}$$

5.2 Linear Viscous Fluids

As mentioned earlier, the perfect fluid model is not realistic because it omits *internal friction* or *dissipation*, a phenomenon present in all materials to varying degrees. This important characteristic is accounted for in the *linear viscous fluid* model, whose constitutive relation is:

$$\mathbf{t} = -\varpi \mathbb{I} + \mathbf{t}^{(d)}, \tag{5.2.1}$$

where $\mathbf{t}^{(d)}$ is the *dissipative stress tensor* and represents all stresses due to viscosity. For example, consider a liquid flowing through a pipe: analyzing the velocities of individual elements, we observe that they flow slower near the walls and faster at the center. This relative velocity difference causes friction, which in turn leads to energy dissipation. Thus, we can infer that $\mathbf{t}^{(d)}$ depends on the relative motion between fluid elements, i.e., on terms like $\nabla \otimes \mathbf{v}$.

The simplest case is the linear one:

$$\mathbf{t}^{(d)} = \mathbf{B} \cdot (\nabla \otimes \mathbf{v}), \tag{5.2.2}$$

where $\mathbf{B}$ is a generic 4th-order tensor. Explicitly, the components (omitting summation over repeated indices) are:

$$T^{(d)}_{ij} = B_{ijkl} \frac{\partial v^l}{\partial x^k}.$$

It is worth noting that $\mathbf{t}^{(d)}$ can also depend on higher-order spatial derivatives, but we limit ourselves to the case of low viscosity, where this approximation suffices to describe most fluids accurately.

To ensure our model describes isotropic fluids, the tensor $\mathbf{B}$ must be rotationally invariant. It can be shown that this requirement is satisfied if $\mathbf{B}$ takes the form:

$$B_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

where λ and μ are arbitrary coefficients. Using this expression, the dissipative stress tensor becomes:

$$T_{ij}^{(d)} = \lambda \frac{\partial v^k}{\partial x^k} \delta_{ij} + 2\mu D_{ij},$$

where we have introduced the *strain rate tensor* $\mathbf{D}$:

$$\mathbf{D} = \frac{1}{2}\left[\nabla \otimes \mathbf{v} + (\nabla \otimes \mathbf{v})^\top\right]. \tag{5.2.3}$$

This tensor represents the symmetric part of $\nabla \otimes \mathbf{v}$. The tensor $\mathbf{B}$ is symmetric under exchange of indices i and j, as well as k and l, and is also block-symmetric. Thanks to these properties, $\mathbf{t}^{(d)}$ is symmetric under exchange of i and j. In compact form:

$$\mathbf{t}^{(d)} = \lambda(\nabla \cdot \mathbf{v})\mathbb{I} + 2\mu\mathbf{D}. \tag{5.2.4}$$

Thus, eq. (5.2.1) becomes:

$$\mathbf{t} = -\varpi\mathbb{I} + \lambda(\nabla \cdot \mathbf{v})\mathbb{I} + 2\mu\mathbf{D}, \tag{5.2.5}$$

which is the constitutive relation for linear viscous fluids, also called *Newtonian fluids*, where λ and μ are parameters related to the continuum medium. Requiring that the pressure field equals the mean normal stress:

$$\varpi = -\frac{1}{3}\mathrm{tr}(\mathbf{t}),$$

and substituting this into the constitutive relation (5.2.1), we obtain:

$$\mathrm{tr}(\mathbf{t}^{(d)}) = 0.$$

Taking the trace of eq. (5.2.4) and using:

$$\mathrm{tr}(\mathbf{D}) = \nabla \cdot \mathbf{v},$$

we arrive at:

$$(3\lambda + 2\mu)\nabla \cdot \mathbf{v} = 0.$$

For incompressible fluids, $\nabla \cdot \mathbf{v} \neq 0$, leading to the *Stokes relation*:

$$\gamma = \lambda + \frac{2}{3}\mu = 0, \tag{5.2.6}$$

where γ is called the *bulk viscosity*.

Substituting the constitutive relation (5.2.1) into the local momentum balance, along with the mass continuity equation, we obtain the governing equations for linear viscous fluids:

$$\begin{aligned} &\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla\varpi + \frac{\mu}{\rho}\nabla^2\mathbf{v} + \frac{\lambda+\mu}{\rho}\nabla(\nabla \cdot \mathbf{v}) + \mathbf{f}, \\ &\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) = 0. \end{aligned} \tag{5.2.7}$$

These are the *Navier-Stokes equations*, a system of four partial differential equations that determine the fields $\rho(\mathbf{x}, t)$ and $\mathbf{v}(\mathbf{x}, t)$. These equations are still widely studied today and are used to model ocean currents, atmospheric flows, and more. Although we will not discuss it further, it is important to note that these equations can also model turbulent flows, where particle motion becomes chaotic rather than orderly as in laminar regimes.

For incompressible fluids, the constitutive relation simplifies to:

$$\mathbf{t} = -\varpi\mathbb{I} + 2\mu\mathbf{D}, \tag{5.2.8}$$

and the governing equations become:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla\varpi + \frac{\mu}{\rho}\nabla^2\mathbf{v} + \mathbf{f},$$
$$\nabla\cdot\mathbf{v} = 0.$$

Despite their similarity to elastic fluids, the term $\nabla^2\mathbf{v}$ and the nonlinearity in $(\mathbf{v}\cdot\nabla)\mathbf{v}$ make finding general solutions much more challenging.

5.3 Coleman-Noll Method

Unlike the other balance laws, the Clausius-Duhem inequality has not yet been used to determine some of the unknown quantities associated with a continuum medium. To do so, we employ the *Coleman-Noll method*, which assumes no constraints on the initial time $t = 0$ or the initial values of the quantities. Additionally, it requires that the constitutive relations of a given medium satisfy the inequality (3.4.6) for every thermodynamic process.

Starting from the Clausius-Duhem inequality (3.4.6), we subtract the local internal energy balance (3.2.12) divided by the absolute temperature θ, as we have previously done in section 3.4, obtaining:

$$\rho\left(\frac{ds}{dt} - \frac{1}{\theta}\frac{du}{dt}\right) - \frac{1}{\theta^2}\mathbf{j}_c^{\mathrm{H}}\cdot\nabla\theta + \frac{1}{\theta}\mathbf{t}\colon(\nabla\otimes\mathbf{v}) \geq 0.$$

or equivalently:

$$-\rho\left(\frac{du}{dt} - \theta\frac{ds}{dt}\right) + \frac{\mathbf{j}_c^{\mathrm{H}}}{\theta}\cdot\nabla\theta + \mathbf{t}\colon(\nabla\otimes\mathbf{v}) \geq 0.$$

Now we have to introduce a *specific free energy* in order to derive restrictions on our variables.

Definition 2 (Specific free energy)**.** *The fundamental thermodynamic potential for constitutive theory:*

$$\phi = u - \theta s \tag{5.3.1}$$

This Legendre transform from internal energy to free energy shifts focus from entropy to temperature as independent variable.

Introducing the specific Helmholtz free energy:

$$f = u - \theta s, \tag{5.3.2}$$

we transform our last from of the C-D inequality into:

$$-\rho\left(\frac{df}{dt} + s\frac{d\theta}{dt}\right) - \frac{\mathbf{j}_c^{\mathrm{H}}}{\theta}\cdot\nabla\theta + \mathbf{t}\colon(\nabla\otimes\mathbf{v}) \geq 0. \tag{5.3.3}$$

By definition, the free energy f may depend on ρ, θ, $\mathbf{x}$ (or $\mathbf{v}$), and their spatial and temporal derivatives. For simplicity, we assume dependence only on the first two quantities, i.e., $f = f(\rho, \theta)$. Using the chain rule to compute the total time derivative of f and the constitutive relation for perfect fluids (5.1.1), we obtain:

$$-\rho\left(\frac{\partial f}{\partial\rho}\frac{d\rho}{dt} + \frac{\partial f}{\partial\theta}\frac{d\theta}{dt} + s\frac{d\theta}{dt}\right) - \frac{\mathbf{j}_c^{\mathrm{H}}}{\theta}\cdot\nabla\theta - \varpi\nabla\cdot\mathbf{v} \geq 0. \tag{5.3.4}$$

Using the local mass balance (3.2.1), we express the divergence of velocity as:

$$\nabla\cdot\mathbf{v} = -\frac{1}{\rho}\frac{d\rho}{dt}, \tag{5.3.5}$$

which allows us to rewrite eq. (5.3.4) as:

$$-\rho\left(\frac{\partial f}{\partial\theta} + s\right)\frac{d\theta}{dt} - \rho\left(\frac{\partial f}{\partial\rho} - \frac{\varpi}{\rho^2}\right)\frac{d\rho}{dt} - \frac{\mathbf{j}_c^{\mathrm{H}}}{\theta}\cdot\nabla\theta \geq 0. \tag{5.3.6}$$

According to the Coleman-Noll theory, the initial time $t = 0$ and the initial values of the time derivatives of θ and ρ can be chosen arbitrarily. To preserve the inequality (5.3.6), the coefficients of these derivatives must vanish, yielding:

$$s = -\frac{\partial f}{\partial\theta},$$
$$\varpi = \rho^2\frac{\partial f}{\partial\rho}.$$

These are the well-known thermodynamic relations connecting entropy and pressure to the derivatives of the Helmholtz free energy. The inequality then reduces to:

$$-\frac{\mathbf{j}_c^{\mathrm{H}}}{\theta}\cdot\nabla\theta \geq 0. \tag{5.3.7}$$

If we impose Fourier's law:

$$\mathbf{j}_c^{\mathrm{H}} = -k\nabla\theta, \tag{5.3.8}$$

where k is the thermal conductivity, we immediately obtain:

$$\frac{k}{\theta}\nabla\theta\cdot\nabla\theta \geq 0, \tag{5.3.9}$$

which implies:

$$k \geq 0.$$

This shows that the thermal conductivity k is always non-negative. Fourier's law represents a constitutive relation for $\mathbf{j}_c^{\mathrm{H}}$: eq. (5.3.8) describes $\mathbf{j}_c^{\mathrm{H}}$ as a vector with the same direction as the temperature gradient but opposite in sign. In other words, $\mathbf{j}_c^{\mathrm{H}}$ points from hotter to colder regions, in perfect agreement with energy conservation.

Remark. *The Coleman-Noll procedure guarantees that all derived constitutive relations automatically satisfy:*

- *The second law of thermodynamics;*
- *Material frame indifference;*
- *Thermodynamic stability conditions.*

General Methodology

The four-step procedure provides a systematic approach to developing thermodynamically consistent constitutive theories for arbitrary material systems:

1. **State variable selection** identifies the minimal set of independent variables (e.g., $\rho, \theta, \nabla\theta$ for simple fluids; $\mathbf{F}, \theta$ for solids) that completely characterize the material's state.
2. **Constitutive postulates** establish the functional dependencies of stress, heat flux, etc. on these variables, with the generality constrained only by:
   - Material symmetry;
   - Frame indifference;
   - Physical plausibility.
3. **Entropy inequality** acts as a thermodynamic filter, eliminating non-physical constitutive assumptions through the Coleman-Noll exploitation theorem.
4. **Admissible forms** emerge as the most general mathematical structures satisfying both the physical postulates and thermodynamic constraints.

The true power of this methodology manifests in three fundamental aspects. First, its universality becomes apparent when observing how the same rigorous procedure derives the Navier-Stokes equations from viscous fluid assumptions, recovers Hooke's law for elastic solids, and generates Maxwell's equations in continuum electrodynamics. Second, the framework demonstrates remarkable adaptability, as special cases naturally emerge through constraint reduction like incompressibility conditions, order truncation in linearized theories, or symmetry considerations for isotropic materials. Third, its extensibility allows seamless incorporation of higher-grade materials via extended variable sets, coupled physics through

additional balance laws, and memory effects using internal variables. This comprehensive approach extends beyond smooth fields, as demonstrated by jump condition analysis that governs shock wave structure, phase transition fronts, and material interface behavior while maintaining complete thermodynamic consistency even across discontinuities.

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Appendices

A Notation and Conventions

Heaviside-Lorentz vs SI Units

In this dissertation, we occasionally refer to the **Heaviside-Lorentz (HL)** unit system, which belongs to the family of rationalized CGS (centimetre–gram–second) units. The HL system is often employed in theoretical and high-energy physics due to its manifestly relativistic symmetry and absence of arbitrary constants such as 4π, ε_0, or μ_0.

In contrast, the **International System of Units (SI)** introduces the vacuum permittivity ε_0 and permeability μ_0 to relate electromagnetic quantities to mechanical units (newtons, volts, coulombs, etc.). These constants obscure the symmetry of Maxwell's equations, especially in covariant formulations.

Maxwell's equations in vacuum, in Heaviside-Lorentz units:

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \rho, \\
\nabla \cdot \mathbf{B} &= 0, \\
\nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \\
\nabla \times \mathbf{B} &= \frac{1}{c}\left(\mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}\right).
\end{aligned}$$

Maxwell's equations in vacuum, in SI units:

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0}, \\
\nabla \cdot \mathbf{B} &= 0, \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}.
\end{aligned}$$

Here, $c = 1/\sqrt{\varepsilon_0 \mu_0}$ is the speed of light in vacuum, and acts as a natural conversion factor between the two systems. In the HL system:

- The fields **E** and **B** have the same physical dimensions and units;
- The sources ρ and **J** are defined to eliminate the factor 4π from Maxwell's equations;
- The coupling between fields and sources appears with a factor of $1/c$ in dynamic terms.

To convert between HL and SI units, one must apply the following scaling factors:

- Charge: $1\,\mathrm{C} \approx 2.998 \times 10^9\,\mathrm{HL}$ units of charge,
- Electric field: $1\,\mathrm{V/m} \approx \frac{1}{\sqrt{\varepsilon_0}}\,\mathrm{HL}$ E-field unit,
- Magnetic field: $1\,\mathrm{T} \approx \sqrt{\mu_0}\,\mathrm{HL}$ B-field unit.

Note that, unlike in Gaussian CGS, there are no 4π factors in the HL formulation. This rationalization results in a more symmetric structure of the field equations, particularly convenient for the formulation of classical field theory and special relativity. In this work, we prefer the HL system when deriving field-theoretic identities, but all final physical quantities can be converted back to SI units via the above relations, if needed.

Kinematics of deformable domains

Notation Conventions on velocity field and material derivative.

The **velocity field** $\mathbf{v}(t, \mathbf{x})$ describes the motion of material points in $\mathcal{V}(t)$, $\mathcal{A}(t)$, or $\mathcal{L}(t)$. For a scalar field $\phi(t, \mathbf{x})$, the **material derivative** is defined as:

$$\frac{\mathrm{D}\phi}{\mathrm{D}t} = \frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi.$$

For vector fields, the derivative is applied component-wise.

Notation Conventions on oriented measures.

Throughout the following chapters, we adopt the following notational conventions for integrals over deforming domains:

- $\mathrm{d}^3\mathbf{x}$: oriented volume element. This is the standard Lebesgue measure in $\mathbb{R}^3$.
- $\mathrm{d}^2\mathbf{x} := \mathbf{n}\,\mathrm{d}A$: oriented surface element, where $\mathbf{n}$ is the outward unit normal to the surface $\mathcal{A}(t)$, and $\mathrm{d}A$ is the scalar area element.
- $\mathrm{d}\mathbf{x} := \boldsymbol{\tau}\,\mathrm{d}s$: oriented line element, with $\boldsymbol{\tau}$ the unit tangent vector to the curve $\mathcal{C}(t)$ oriented consistently with $\mathbf{n}$ via the right-hand rule for closed curves, and $\mathrm{d}s$ the arclength.

Thermomechanics of continua

Definition 3 (Cauchy stress tensor)**.** *The tensor field* $\mathbf{t}(t, \mathbf{x})$ *describes internal contact forces per unit area in the current configuration. Physically,* $\mathbf{t} \cdot \mathbf{n}$ *gives the traction vector acting on a surface element with normal* $\mathbf{n}$*.*

Notation *Stress decomposition.*

In general, the stress tensor $\mathbf{t}$ *is not symmetric. Its antisymmetric part and its Hodge dual are defined as:*

$$\mathbf{t}^{(a)} := \frac{1}{2}(\mathbf{t} - \mathbf{t}^\top), \qquad (\star\mathbf{t}^{(a)})_k := \frac{1}{2}\epsilon_{kij} t^{(a)}_{ij},$$

where ϵ_{kij} *is the Levi-Civita permutation symbol. The dual pseudovector* $\star\mathbf{t}^{(a)}$ *appears naturally in angular momentum balance.*

Cauchy's tetrahedron construction. *It's a model which let us consider the force balance on an infinitesimal tetrahedron with three faces aligned with the coordinate planes and one oblique face with unit normal* $\mathbf{n}$*. Denoting by* $\mathbf{t}(\mathbf{n})$ *the traction vector acting on the inclined face, and by* $\mathbf{t}_i = \mathbf{t}(\mathbf{e}_i)$ *the tractions on the coordinate planes, the balance of forces yields:*

$$\mathbf{t}(\mathbf{n}) = n_1\mathbf{t}_1 + n_2\mathbf{t}_2 + n_3\mathbf{t}_3,$$

which implies the existence of a second-order tensor $\boldsymbol{\sigma}$ *such that:*

$$\mathbf{t}(\mathbf{n}) = \boldsymbol{\sigma} \cdot \mathbf{n}.$$

This tensorial relation shows that the internal force per unit area acting across any surface depends linearly on the surface's orientation.

Definition 4 (Jump operator across a moving surface)**.** *Let* $\mathcal{D}(t)$ *be a surface across which a field* f *is discontinuous. The jump of* f *is defined as:*

$$[\![f]\!] := f^+ - f^-,$$

where f^+ *and* f^- *denote the limiting values of* f *on either side of the surface, along the normal direction. Jump conditions play a fundamental role in determining the compatibility of conservation laws across discontinuities (we will use assumptions and derivations similar to those used in the previous chapter for the kinematic jump conditions).*

Notation Tensor operations.

- Double contraction: $\mathbf{A} : \mathbf{B} = \sum_{i,j} A_{ij} B_{ij}$.
- Tensor product: $[\mathbf{v} \otimes \mathbf{w}]_{ij} = v_i w_j$.
- Material derivative: $\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$.

B Derivations for Electromagnetic Sources

This appendix provides the complete mathematical derivations omitted in section 4.2 for the electromagnetic sources in a continuum. We derive the force density, momentum density, stress tensor, internal energy density and power current density from the electromagnetic fields and their interactions with matter.

Force Density Derivation

This appendix provides the complete mathematical derivations for the electromagnetic force expressions, including the volumetric and superficial contributions, the electromagnetic stress tensor and momentum density, omitted for brevity in section 4.2.

Volumetric force. The full expansion of $\mathbf{F}_{\mathcal{V}}^{\text{vol}}$ with intermediate steps:

$$\begin{aligned}
\mathbf{F}_{\mathcal{V}}^{\text{vol}} &= \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x} \Bigg[(q_f - \nabla\cdot\mathbf{P}) \left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right) \\
&\quad + \frac{1}{c}\left(\mathbf{j}_f - q_f\mathbf{v} + \frac{\partial\mathbf{P}}{\partial t} + (\nabla\cdot\mathbf{P})\mathbf{v} + c\nabla\times\mathbf{M}\right)\times\mathbf{B}\Bigg] \\
&= \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x} \Bigg[q_f\mathbf{E} - q_f\frac{\mathbf{v}}{c}\times\mathbf{B} + \frac{1}{c}\mathbf{j}_f\times\mathbf{B} + \frac{q_f}{c}\mathbf{v}\times\mathbf{B} - (\nabla\cdot\mathbf{P})\mathbf{E} - \frac{1}{c}(\nabla\cdot\mathbf{P})\mathbf{v}\times\mathbf{B} \\
&\quad + \frac{1}{c}\frac{\partial\mathbf{P}}{\partial t}\times\mathbf{B} + \frac{1}{c}(\nabla\cdot\mathbf{P})\mathbf{v}\times\mathbf{B} + (\nabla\times\mathbf{M})\times\mathbf{B}\Bigg] \\
&= \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x} \left[q_f\mathbf{E} + \frac{1}{c}\mathbf{j}_f\times\mathbf{B} - (\nabla\cdot\mathbf{P})\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{P}}{\partial t}\times\mathbf{B} + (\nabla\times\mathbf{M})\times\mathbf{B}\right]
\end{aligned}$$

The term $-(\nabla\cdot\mathbf{P})\mathbf{E}$ is transformed using tensor identities:

$$\begin{aligned}
-(\nabla\cdot\mathbf{P})\mathbf{E} &= -\nabla\cdot(\mathbf{P}\otimes\mathbf{E}) + \mathbf{P}\cdot(\nabla\otimes\mathbf{E}) \\
&= -\nabla\cdot(\mathbf{P}\otimes\mathbf{E}) + \mathbf{P}\cdot(\nabla\wedge\mathbf{E}) + (\nabla\otimes\mathbf{E})\cdot\mathbf{P} \\
&= \nabla\cdot(\mathbf{P}\otimes\mathbf{E}) + \mathbf{P}\cdot\star(\nabla\times\mathbf{E}) + (\nabla\otimes\mathbf{E})\cdot\mathbf{P} \\
&= -\nabla\cdot(\mathbf{P}\otimes\mathbf{E}) - \mathbf{P}\times(\nabla\times\mathbf{E}) + (\nabla\otimes\mathbf{E})\cdot\mathbf{P} \\
&= -\nabla\cdot(\mathbf{P}\otimes\mathbf{E}) + \frac{\mathbf{P}}{c}\times\frac{\partial\mathbf{B}}{\partial t} + (\nabla\otimes\mathbf{E})\cdot\mathbf{P}
\end{aligned}$$

The magnetization term expands as:

$$\begin{aligned}
(\nabla\times\mathbf{M})\times\mathbf{B} &= (\mathbf{B}\cdot\nabla)\mathbf{M} - (\nabla\otimes\mathbf{M})\cdot\mathbf{B} \\
&= (\mathbf{B}\cdot\nabla)\mathbf{M} - \nabla(\mathbf{M}\cdot\mathbf{B}) + (\nabla\otimes\mathbf{B})\cdot\mathbf{M} \\
&= -(\nabla\cdot\mathbf{B})\mathbf{M} + \nabla\cdot(\mathbf{B}\otimes\mathbf{M} - (\mathbf{M}\cdot\mathbf{B})\mathbb{I}) + (\nabla\otimes\mathbf{B})\cdot\mathbf{M} \\
&= \nabla\cdot(\mathbf{B}\otimes\mathbf{M} - (\mathbf{M}\cdot\mathbf{B})\mathbb{I}) + (\nabla\otimes\mathbf{B})\cdot\mathbf{M}
\end{aligned}$$

Surface force. The complete surface force derivation:

$$\begin{aligned}\mathbf{F}_{\mathcal{V}}^{\text{sup}} &= \oint_{\partial\mathcal{V}} \mathrm{d}^2\mathbf{x}\Big[(\mathbf{n}\cdot\mathbf{P})\mathbf{E} + (\mathbf{n}\cdot\mathbf{P})\frac{\mathbf{v}}{c}\times\mathbf{B} - \mathbf{n}\cdot\mathbf{B}\left(\mathbf{M}+\frac{\mathbf{v}}{c}\times\mathbf{P}\right) + \mathbf{n}\left(\mathbf{M}+\frac{\mathbf{v}}{c}\times\mathbf{P}\right)\cdot\mathbf{B}\Big] \\ &= \oint_{\partial\mathcal{V}} \mathrm{d}^2\mathbf{x}\cdot\Big[\mathbf{P}\otimes\mathbf{E} - \mathbf{B}\otimes\mathbf{M} + \mathbb{I}(\mathbf{M}\cdot\mathbf{B})\Big] \\ &\quad + \frac{1}{c}\oint_{\partial\mathcal{V}} \mathrm{d}^2\mathbf{x}\cdot\Big[\mathbf{P}\otimes(\mathbf{v}\times\mathbf{B}) - \mathbf{B}\otimes(\mathbf{v}\times\mathbf{P}) + \mathbb{I}((\mathbf{v}\times\mathbf{P})\cdot\mathbf{B})\Big] \\ &= \oint_{\partial\mathcal{V}} \mathrm{d}^2\mathbf{x}\cdot[\mathbf{P}\otimes\mathbf{E} - \mathbf{B}\otimes\mathbf{M} + \mathbb{I}(\mathbf{M}\cdot\mathbf{B})] + \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x}\,\nabla\cdot\left[\frac{\mathbf{v}}{c}\otimes(\mathbf{P}\times\mathbf{B})\right]\end{aligned}$$

where the first passage uses the following derivations:

$$\begin{aligned}(\mathbf{n}\cdot\mathbf{P})\mathbf{v}\times\mathbf{B} - \mathbf{n}\cdot\mathbf{B}(\mathbf{v}\times\mathbf{P}) + (\mathbf{v}\times\mathbf{P}\cdot\mathbf{B})\mathbf{n} &= \\ \mathbf{n}\cdot\mathbf{P}(\star\mathbf{B}\cdot\mathbf{v}) - \mathbf{n}\cdot\mathbf{B}(\star\mathbf{P}\cdot\mathbf{v}) + \mathbf{n}(\mathbf{P}\times\mathbf{B})\mathbf{v} &= \\ \mathbf{n}\cdot[\mathbf{P}\otimes\star\mathbf{B} - \mathbf{B}\otimes\star\mathbf{P} + \mathbb{I}(\mathbf{P}\times\mathbf{B})]\cdot\mathbf{v} &= \\ (\mathbf{n}\otimes\mathbf{v})\colon\mathbb{I}(\mathbf{P}\times\mathbf{B}) = (\mathbf{n}\cdot\mathbf{v})(\mathbf{P}\times\mathbf{B});&\end{aligned}$$

$$\begin{aligned}(\mathbf{n}\cdot\mathbf{P})\mathbf{E} - (\mathbf{n}\cdot\mathbf{B})\mathbf{M} + (\mathbf{M}\cdot\mathbf{B})\mathbf{n} &= \\ \mathbf{n}\cdot[\mathbf{P}\otimes\mathbf{E} - \mathbf{B}\otimes\mathbf{M} + \mathbb{I}(\mathbf{M}\cdot\mathbf{B})]\,.&\end{aligned}$$

Momentum density and stress tensor. The complete steps for the electromagnetic force density final form, which let us derive the momentum density and stress tensor:

$$\frac{1}{c}\frac{\partial}{\partial t}(\mathbf{P}\times\mathbf{B}) = \frac{1}{c}\frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B} - \mathbf{E}\times\mathbf{B}) = -\frac{1}{c}\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}) + \frac{1}{c}\frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B});$$

$$\begin{aligned}\frac{1}{c}\frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B}) &= \frac{1}{c}\left(\frac{\partial\mathbf{D}}{\partial t}\times\mathbf{B} + \mathbf{D}\times\frac{\partial\mathbf{B}}{\partial t}\right) \\ &= (\nabla\times\mathbf{H})\times\mathbf{B} - \frac{1}{c}\mathbf{j}_f\times\mathbf{B} - \mathbf{D}\times(\nabla\times\mathbf{E}) \\ &= (\mathbf{B}\cdot\nabla)\mathbf{H} - \nabla(\mathbf{H}\cdot\mathbf{B}) + (\nabla\cdot\mathbf{D})\mathbf{E} + (\mathbf{D}\cdot\nabla)\mathbf{E} \\ &\quad - \nabla\cdot(\mathbf{D}\otimes\mathbf{E}) - \frac{1}{c}\mathbf{j}_f\times\mathbf{B}\end{aligned}$$

Power Density Derivation

This appendix provides the complete mathematical derivations for the electromagnetic power expressions, including volumetric and superficial contributions, the power density current and internal electromagnetic energy, omitted for brevity in section 4.2.

Volumetric power. The full expansion of the volumetric power from the co-moving frame is given by:

$$\begin{aligned}
\mathbf{W}_{\mathcal{V}}^{\text{vol}} &= \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x} \left[(\mathbf{j}_c^{\text{vol}} + q^{\text{vol}}\mathbf{v}) \cdot \mathbf{E}_c - \mathbf{j}_c^{\text{vol}} \cdot \frac{\mathbf{v}}{c} \times \mathbf{B} \right] \\
&= \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x} \, (\mathbf{j}_c^{\text{vol}} + q^{\text{vol}}\mathbf{v}) \cdot (\mathbf{E}_c - \frac{\mathbf{v}}{c} \times \mathbf{B}) \\
&= \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x} \, (\mathbf{j}_f + \frac{\partial}{\partial t}\mathbf{P} + c\nabla \times \mathbf{M}) \cdot \mathbf{E} \\
&= \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x} \left[\mathbf{j}_f \cdot \mathbf{E} + \frac{\partial \mathbf{P}}{\partial t} \cdot \mathbf{E} + c\nabla \cdot (\mathbf{M} \cdot \mathbf{E}) + c(\nabla \times \mathbf{E}) \cdot \mathbf{M} \right] \\
&= \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x} \left[\mathbf{j}_f \cdot \mathbf{E} + \frac{\partial \mathbf{P}}{\partial t} \cdot \mathbf{E} - \mathbf{M} \cdot \frac{\partial \mathbf{B}}{\partial t} \right] + c \oint_{\partial \mathcal{V}} \mathrm{d}^2\mathbf{x} \cdot (\mathbf{M} \times \mathbf{E}),
\end{aligned}$$

where we have applied the vector identity for $\nabla \times (\mathbf{M} \times \mathbf{E})$ and Faraday's law:

$$\begin{aligned}
(\nabla \times \mathbf{M}) \cdot \mathbf{E} &= \nabla \cdot (\mathbf{M} \times \mathbf{E}) + \mathbf{M} \cdot (\nabla \times \mathbf{E}) \\
&= \nabla \cdot (\mathbf{M} \times \mathbf{E}) - \frac{1}{c}\mathbf{M} \cdot \frac{\partial}{\partial t}\mathbf{B}.
\end{aligned}$$

Surface power. The full expansion of the surface power from the co-moving frame is given by:

$$\begin{aligned}
\mathbf{W}_{\mathcal{V}}^{\text{sup}} &= \oint_{\partial \mathcal{V}} \mathrm{d}^2\mathbf{x} \left[(\mathbf{j}_c^{\text{sup}} + q^{\text{sup}}\mathbf{v}) \cdot \mathbf{E}_c - \mathbf{j}_c^{\text{sup}} \cdot \frac{\mathbf{v}}{c} \times \mathbf{B} \right] \\
&= \oint_{\partial \mathcal{V}} \mathrm{d}^2\mathbf{x} (\mathbf{j}_c^{\text{sup}} + q^{\text{sup}}\mathbf{v}) \cdot (\mathbf{E}_c - \frac{\mathbf{v}}{c} \times \mathbf{B}) \\
&= \oint_{\partial \mathcal{V}} \mathrm{d}^2\mathbf{x} \left[-c\mathbf{n} \times (\mathbf{M} + \frac{\mathbf{v}}{c} \times \mathbf{P})(\mathbf{n} \cdot \mathbf{P})\mathbf{v} \right] \cdot \mathbf{E} \\
&= \oint_{\partial \mathcal{V}} \mathrm{d}^2\mathbf{x} \cdot \mathbf{v}(\mathbf{P} \cdot \mathbf{E}) - c \oint_{\partial \mathcal{V}} \mathrm{d}^2\mathbf{x} \cdot (\mathbf{M} \times \mathbf{E}) \\
&= -c \oint_{\partial \mathcal{V}} \mathrm{d}^2\mathbf{x} \cdot (\mathbf{M} \times \mathbf{E}) + \int_{\mathcal{V}} \mathrm{d}^3\mathbf{x} \, \nabla \cdot (\mathbf{v}(\mathbf{P} \cdot \mathbf{E})),
\end{aligned}$$

where we have used surface charge and current relations:

$$\begin{aligned}
q^{\text{sup}} &= \mathbf{n} \cdot \mathbf{P}; \\
\mathbf{j}_c^{\text{sup}} &= -c\mathbf{n} \times (\mathbf{M} + \frac{\mathbf{v}}{c} \times \mathbf{P}).
\end{aligned}$$

Power balance. Combining the volumetric and surface contributions, we obtain the total power:

$$W_{\mathcal{V}} = \int_{\mathcal{V}} \left[\mathbf{j}_f \cdot \mathbf{E} + \frac{\partial}{\partial t}\mathbf{P} \cdot \mathbf{E} - \mathbf{M} \cdot \frac{\partial}{\partial t}\mathbf{B} + \nabla \cdot (\mathbf{v}(\mathbf{P} \cdot \mathbf{E})) \right] d^3x$$

To express this in terms of field energy, we manipulate the polarization and magnetization terms:

$$\begin{aligned}\frac{\partial \mathbf{P}}{\partial t}\cdot\mathbf{E}-\mathbf{M}\cdot\frac{\partial \mathbf{B}}{\partial t}&=\frac{\partial \mathbf{D}}{\partial t}\cdot\mathbf{E}-\frac{\partial \mathbf{E}}{\partial t}\cdot\mathbf{E}-\mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t}+\mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t}\\&=-\frac{\partial}{\partial t}\frac{(\mathrm{E}^2+\mathrm{B}^2)}{2}+\frac{\partial \mathbf{D}}{\partial t}\cdot\mathbf{E}+\mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t}\\&=-\frac{\partial}{\partial t}\frac{(\mathrm{E}^2+\mathrm{B}^2)}{2}+c(\nabla\times\mathbf{H})\cdot\mathbf{E}-\mathbf{j}_f\cdot\mathbf{E}-c\mathbf{H}\cdot(\nabla\times\mathbf{E})\\&=-\mathbf{j}_f\cdot\mathbf{E}-\frac{\partial}{\partial t}\frac{(\mathrm{E}^2+\mathrm{B}^2)}{2}-c\nabla\cdot(\mathbf{E}\times\mathbf{H}),\end{aligned}$$

where we have used Maxwell's equations to rewrite the time derivatives:

$$\begin{aligned}\partial_t\mathbf{D}\cdot\mathbf{E}&=c(\nabla\times\mathbf{H})\cdot\mathbf{E}-\mathbf{j}_f\cdot\mathbf{E}\\\mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B}&=-c\mathbf{H}\cdot(\nabla\times\mathbf{E})\end{aligned}$$

Combining these with the convective term $\nabla\cdot(\mathbf{v}(\mathbf{P}\cdot\mathbf{E}))$ yields the final energy balance equation shown in the main text, with the contribution from the internal energy density and the power current density.

C Derivations for Electromagnetic Balance Equations

This appendix provides the complete derivations of the electromagnetic balance equations, which were omitted for brevity in section 4.3. We present the angular momentum and energy balances by explicitly manipulating the relevant terms, and highlighting the key identities and assumptions used in the simplification process.

Angular Momentum Balance

We begin by taking the moment of the electromagnetic force density with respect to the origin:

$$\mathbf{x}\times\mathbf{f}^{\mathrm{EM}}=\mathbf{x}\times\left[-\frac{\partial \mathbf{p}^{\mathrm{EM}}}{\partial t}-\nabla\cdot(\mathbf{v}\otimes\mathbf{p}^{\mathrm{EM}})+\nabla\cdot\mathbf{t}^{\mathrm{EM}}\right].$$

We now distribute the cross product and apply the standard vector identities for time derivatives and divergences of tensor products. First, we recognize:

$$\mathbf{x}\times\frac{\partial \mathbf{p}^{\mathrm{EM}}}{\partial t}=\frac{\partial}{\partial t}(\mathbf{x}\times\mathbf{p}^{\mathrm{EM}})-\frac{\partial \mathbf{x}}{\partial t}\times\mathbf{p}^{\mathrm{EM}}=\frac{\partial}{\partial t}(\mathbf{x}\times\mathbf{p}^{\mathrm{EM}}),$$

since $\mathbf{x}$ is the spatial position vector and does not depend on time. Similarly, we expand the divergence of a product:

$$\mathbf{x}\times\nabla\cdot(\mathbf{v}\otimes\mathbf{p}^{\mathrm{EM}})=\nabla\cdot(\mathbf{v}\otimes(\mathbf{x}\times\mathbf{p}^{\mathrm{EM}}))-(\nabla\mathbf{x})\cdot\mathbf{v}\times\mathbf{p}^{\mathrm{EM}},$$

and use the identity $(\nabla\mathbf{x})\cdot\mathbf{v}=\mathbf{v}$, to get:

$$\mathbf{v}\cdot\nabla\mathbf{x}\times\mathbf{p}^{\mathrm{EM}}=\mathbf{v}\times\mathbf{p}^{\mathrm{EM}}.$$

For the stress tensor term, we use the identity:

$$\mathbf{x} \times \nabla \cdot \mathbf{t}^{\mathrm{EM}} = \nabla \cdot (\mathbf{t}^{\mathrm{EM}} \times \mathbf{x}) - (\mathbf{t}^{\mathrm{EM}})^T \cdot \nabla \times \mathbf{x} = \nabla \cdot (\mathbf{t}^{\mathrm{EM}} \times \mathbf{x}) - 2 \star (\mathbf{t}^{(a)\mathrm{EM}}),$$

where $\star(\cdot)$ denotes the vector dual of the antisymmetric part of the tensor. Altogether, we find:

$$\mathbf{x} \times \mathbf{f}^{\mathrm{EM}} = -\frac{\partial}{\partial t}(\mathbf{x} \times \mathbf{p}^{\mathrm{EM}}) - \nabla \cdot (\mathbf{v} \otimes (\mathbf{x} \times \mathbf{p}^{\mathrm{EM}})) - \nabla \cdot (\mathbf{t}^{\mathrm{EM}} \times \mathbf{x}) - 2 \star (\mathbf{t}^{(a)\mathrm{EM}}) + \mathbf{v} \times \mathbf{p}^{\mathrm{EM}}.$$

Intrinsic torque density. We now turn to the intrinsic torque density $\mathbf{c}^{\mathrm{EM}}$, and manipulate it using standard vector identities:

$$\begin{aligned}
\mathbf{c}^{\mathrm{EM}} &= \mathbf{M} \times \mathbf{B} + \mathbf{P} \times \mathbf{E} + \frac{\mathbf{v}}{c} \times (\mathbf{P} \times \mathbf{B}) \\
&= \mathbf{M} \times \mathbf{B} + \mathbf{P} \times \mathbf{E} + \mathbf{P} \times \left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right) - \mathbf{B} \times \left(\frac{\mathbf{v}}{c} \times \mathbf{P}\right) \\
&= (\mathbf{M} + \frac{\mathbf{v}}{c} \times \mathbf{P}) \times \mathbf{B} + \mathbf{P} \times (\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}) \\
&= \mathbf{M}_c \times \mathbf{B} + \mathbf{P} \times \mathbf{E}_c,
\end{aligned}$$

where we have defined the co-moving magnetic and electric fields $\mathbf{M}_c$ and $\mathbf{E}_c$.

Energy Balance

Heat production rate. We now turn to the calculation of the electromagnetic heat production rate z^{EM}, given by the difference between the work density and the power input from the electromagnetic force:

$$z^{\mathrm{EM}} = \mathbf{w}^{\mathrm{EM}} - \mathbf{f}^{\mathrm{EM}} \cdot \mathbf{v}.$$

We begin by expanding the contributions to $\mathbf{w}^{\mathrm{EM}}$ and grouping the resulting terms:

$$\begin{aligned}
z^{\mathrm{EM}} = \mathbf{j}_f \cdot \mathbf{E} - \mathbf{M} \cdot \frac{\partial \mathbf{B}}{\partial t} + \left(\frac{\partial \mathbf{P}}{\partial t}\right) \cdot \mathbf{E} + \nabla \cdot \mathbf{v}(\mathbf{P} \cdot \mathbf{E}) \\
+ (\mathbf{v} \cdot \nabla)\mathbf{P} \cdot \mathbf{E} + (\mathbf{v} \cdot \nabla)\mathbf{E} \cdot \mathbf{P} - q_f \mathbf{v} \cdot \mathbf{E} + \mathbf{j}_f \cdot \frac{\mathbf{v}}{c} \times \mathbf{B} \\
- (\mathbf{v} \cdot \nabla)\mathbf{E} \cdot \mathbf{P} - (\mathbf{v} \cdot \nabla)\mathbf{B} \cdot \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \cdot \frac{\mathbf{v}}{c} \times \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{P} \cdot \frac{\partial \mathbf{B}}{\partial t} \\
+ (\nabla \cdot \mathbf{v})\mathbf{P} \cdot \frac{\mathbf{v}}{c} \times \mathbf{B} + (\mathbf{v} \cdot \nabla)\mathbf{P} \cdot \frac{\mathbf{v}}{c} \times \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{P} \cdot (\mathbf{v} \cdot \nabla)\mathbf{B}.
\end{aligned}$$

Next, we reorganize these terms into combinations involving the co-moving electric field $\mathbf{E}_c = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}$, and the co-moving magnetization $\mathbf{M}_c = \mathbf{M} + \frac{\mathbf{v}}{c} \times \mathbf{P}$. This yields:

$$\begin{aligned}
z^{\mathrm{EM}} = \left(\mathbf{j}_f - q_f \mathbf{v} + \frac{\partial \mathbf{P}}{\partial t} + \nabla \times (\mathbf{P} \times \mathbf{v}) + (\nabla \cdot \mathbf{P})\mathbf{v} + (\mathbf{P} \cdot \nabla)\mathbf{v}\right) \cdot \mathbf{E}_c \\
- \left(\mathbf{M} + \frac{\mathbf{v}}{c} \times \mathbf{P}\right) \cdot \left(\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{v}) - (\nabla \cdot \mathbf{v})\mathbf{B} + (\nabla \cdot \mathbf{B})\mathbf{v} + (\mathbf{B} \cdot \nabla)\mathbf{v}\right).
\end{aligned}$$

Using the free conduction current $\mathbf{j}_{fc} = \mathbf{j}_f - q_f\mathbf{v}$, we finally obtain:

$$\begin{aligned} z^{\mathrm{EM}} = &\left(\mathbf{j}_{fc} + \frac{\partial \mathbf{P}}{\partial t} + \nabla \times (\mathbf{P} \times \mathbf{v}) + (\nabla \cdot \mathbf{P})\mathbf{v}\right) \cdot \mathbf{E}_c \\ &- \mathbf{M}_c \cdot \left(\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{v}) + (\nabla \cdot \mathbf{B})\mathbf{v}\right) \\ &+ (\mathbf{P} \otimes \mathbf{E}_c - \mathbf{B} \otimes \mathbf{M}_c + (\mathbf{M}_c \cdot \mathbf{B})\mathbb{I}) : \nabla \otimes \mathbf{v}. \end{aligned}$$

To simplify the magnetization term, we recall the modified Maxwell equation for the co-moving electric field:

$$\nabla \times \mathbf{E}_c + \frac{1}{c}\left(\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{v}) + (\nabla \cdot \mathbf{B})\mathbf{v}\right) = 0,$$

which allows us to express:

$$-\mathbf{M}_c \cdot \left(\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{v}) + (\nabla \cdot \mathbf{B})\mathbf{v}\right) = c\nabla \cdot (\mathbf{E}_c \times \mathbf{M}_c) + c\mathbf{E}_c \cdot \nabla \times \mathbf{M}_c.$$

Electromagnetic heat conduction current density. We conclude with the calculation of the electromagnetic contribution to the total energy flux $\mathbf{j}_c^{\mathrm{H,EM}} = \mathbf{j}_c^{\mathrm{EM}} + \mathbf{t}^{\mathrm{EM}} \cdot \mathbf{v}$, where the full expression is:

$$\begin{aligned} \mathbf{j}_c^{\mathrm{H,EM}} &= c\mathbf{E} \times \mathbf{H} + (\mathbf{H} \cdot \mathbf{v})\mathbf{B} + (\mathbf{E} \cdot \mathbf{v})\mathbf{D} \\ &\quad - \left((\mathbf{E} + \mathbf{P}) \cdot \mathbf{E} + (\mathbf{B} - \mathbf{M}) \cdot \mathbf{B} + \frac{\mathbf{v}}{c} \times \mathbf{B} \cdot \mathbf{D}\right)\mathbf{v} \\ &= c\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \times \left(\mathbf{H} - \frac{\mathbf{v}}{c} \times \mathbf{D}\right), \end{aligned}$$

where the final expression is obtained through several vector identities and reflects the covariant form of the Poynting vector in a moving medium.