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GROWTH OF SOBOLEV NORMS OF SOLUTIONS TO NLS ON CLOSED RIEMANNIAN MANIFOLDS

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Abstract

After a brief introduction on the local/global well-posedness theory for the nonlinear Schrödinger equation (NLS) on compact manifolds, we use available Strichartz estimates and suitable "modified energies" to investigate the growth of higher-order Sobolev norms of solutions to the NLS with polynomial nonlinearity (i.e. $|u|^{p-1}u$) on generic 2 and 3 dimensional closed Riemannian manifolds. We provide polynomial in time bounds on the growth of higher-order Sobolev norms of solutions to the $2d$ case with odd nonlinearity $p = 2n + 1$, $n \in \mathbb{N}$, and exponential in time bounds in the cubic $3d$ case. Moreover, we show that the H^2 norm of the solution to the sub-cubic NLS on 3-dimensional manifolds grows at most polynomially in time.

Abstract

Dopo una breve introduzione alla teoria della buona posizione locale/globale dell'equazione di Schrödinger nonlineare (NLS) su varietà compatte, andremo ad utilizzare le stime di Strichartz e le "energie modificate" adeguatamente definite per studiare la crescita delle norme Sobolev delle soluzioni di NLS su varietà Riemanniane chiuse di dimensione 2 o 3, aventi nonlinearità di tipo polinomiale (ovvero della forma $|u|^{p-1}u$). Forniremo stime a priori sulla crescita temporale di tali norme che risulta essere al più polinomiale nel caso 2-dimensionale con nonlinearità dispari $p = 2n+1$, $n \in \mathbb{N}$, e al più esponeziale nel caso 3-dimensionale con nonlinearità cubica. Inoltre, mostreremo che la norma Sobolev H^2 della soluzione di una NLS sub-cubica nel caso 3-dimensionale ha crescita al più polinomiale nel tempo.

Contents

Introduction

In this thesis we will analyze a recent paper by F. Planchon, N. Tzvetkov and N. Visciglia (see [7]) on the growth in time of higher-order Sobolev norms of solutions of certain classes of nonlinear Schrödinger equations (NLS) on closed Riemannian manifolds of dimension 2 and 3.
The NLS is one of the most studied nonlinear dispersive equations. It has a crucial role in the description of many physical phenomena such as, for instance, in Bose-Einstein condensates (BECs), nonlinear optics, solitons behavior, and many others. In recent years, the issue of the growth of higher-order Sobolev norms of solutions to NLS equations has gained more attention due to its implication in the weak wave turbulence theory, i.e. in low-to-high frequency cascade (the energy shifts to increasingly higher frequencies as time goes to infinity).
The aim of the following dissertation is to provide a priori bounds for the H^m Sobolev norms of the solutions to the Cauchy problems

$$\begin{cases} i\partial_t u + \Delta_g u = |u|^{p-1}u, & (t,x) \in \mathbb{R} \times M^d, \\ u(0,\cdot) = u_0 \in H^m(M^d), \end{cases}$$

where (M^d, g) is a generic d-dimensional closed Riemannian manifold, $m \geq 2$, $m \in \mathbb{N}$ and the parameters d, p satisfy one of the following two conditions:

- $(d,p) = (2, 2n+1)$ with $n \in \mathbb{N}$ (2d-case with odd nonlinearity).

- $(d,p) = (3,3)$ (cubic 3d-case).

Moreover we shall investigate the behavior of the H^2 Sobolev norm of the solution to the subcubic Cauchy problem

$$\begin{cases} i\partial_t u + \Delta_g u = |u|^{p-1}u, & (t,x) \in \mathbb{R} \times M^3, \\ u(0,\cdot) = u_0 \in H^2(M^3), \end{cases}$$

where $p \in (2,3)$. From the global Cauchy theory in $H^1(M^2)$ and $H^{1+\epsilon}(M^3)$, these families of Cauchy problems are globally well posed, hence it makes sense to study long-time qualitative properties of their solutions (since no blow-up phenomena occurs). We emphasize that we do not to consider the growth of the H^1 norm as the conservation laws for the NLS immediately imply that the map $t \longmapsto ||u(t,\cdot)||_{H^1(M^d)}$ is bounded by a constant in each of the previous cases. Moreover, we do not deal the growth of higher-order Sobolev norms in the sub-cubic case, since the nonlinearity is not smooth enough to guarantee that the regularity is preserved along the evolution.

To make things clear, we decided to start our dissertation with a brief introduction on the Schrödinger equation in the Euclidean setting. We state homogeneous and nonhomogeneous Strichartz estimates for the linear problem, with the aim to compare these classical results with the ones available on compact manifolds, and introduce local/global well-posedness results in H^1 for the nonlinear problem.

Later on, we shall observe that the dispersion estimate does not hold for the linear Schrödinger equation on compact manifolds, therefore, by using microlocal analysis and a semiclassical version of the aforementioned estimate (that holds for a time interval depending on the spectral localization), we are able to show "classical" and "endpoint" Strichartz estimates with loss of derivatives. These type of estimates are used to implement a contraction argument which provide local/global well posedness results for the NLS on compact manifolds and represent a key element in the proof of qualitative properties of solutions.

Subsequently we focus on the main topic of our dissertation. We shall study the cases $d = 2, 3$ with odd integer polynomial nonlinearity, that is

$(d,p) = (2, 2n+1)$ and $(d,p) = (3,3)$ by studying the even $(m = 2k)$ and the odd $(m = 2k+1)$ cases independently. For each of them we introduce a suitable "modified energy" and use "modified Strichartz estimates" (in the $2d$-case) and "modified endpoint Strichartz estimates" (in the cubic $3d$-case) to find a priori bounds to a small time increment of the H^m Sobolev norms. We shall consider different modified energies for the even and the odd cases in order to use the identification of $||\partial_t^j u||_{H^s}$ with the Sobolev norm $||u||_{H^{2j+s}}$, up to lower order terms. Then, by using a classical iteration argument and the local Cauchy theory, we prove that:

- For every $T > 0$, if u solves the $2d$ problem, we have

  $$\sup_{t\in(0,T)} ||u(t,\cdot)||_{H^m(M^2)} \leq C\left(\max\{1,T\}\right)^{\frac{m-1}{1-s_0}+\epsilon},$$

  where $C = C(||u_0||_{H^m}) > 0$ and $s_0 \in [0, \frac{1}{4}]$.

- For every $T > 0$, if u solves the cubic $3d$ problem, we have

  $$\sup_{t\in(0,T)} ||u(t,\cdot)||_{H^m(M^3)} \leq C_1 \exp(C_2 T),$$

  where $C_{1,2} = C_{1,2}(||u_0||_{H^m}) > 0$.

As a result, these bounds show the polynomial and the exponential growth in time of the H^m norms of the solution to the $2d$ and the cubic $3d$ problem, respectively, for any $m \geq 2$.

Finally, in a similar fashion we prove the polynomial growth in time of the H^2 Sobolev norm of the solution to the sub-cubic $3d$ problem showing that

$$\sup_{t\in(0,T)} ||u(t,\cdot)||_{H^2} \leq C\left(\max\{1,T\}\right)^{\frac{4}{3-p}}.$$

Chapter 1

Preliminaries

1.1 Basics in Riemannian Geometry

In this section we recall some basic notions in Riemannian geometry with the aim to introduce the nonlinear Schrödinger equation (NLS) on compact manifolds.

Definition 1.1.1. *(Smooth Manifold) Let M^d be a d-dimensional topological manifold, a local chart for M^d is a pair (U, κ) where $U \subseteq M^d$ is an open subset and*

$$\kappa : U \longrightarrow \kappa(U) \subseteq \mathbb{R}^d$$

is a homeomorphism of U. A collection of local charts $(U_i, \kappa_i)_{i \in I}$ such that $M^d = \bigcup_{i \in I} U_i$ is an atlas for M^d. We say that an atlas $(U_i, \kappa_i)_{i \in I}$ is smooth if any transition map

$$\kappa_j \circ \kappa_i^{-1} : \kappa_i(U_i \cap U_j) \longrightarrow \kappa_j(U_i \cap U_j)$$

is of class C^∞. A d-dimensional smooth manifold is a connected topological manifold M^d of dimension d together with a maximal smooth atlas.

Notation 1.1.2. *Given a smooth manifol M^d, we denote by $T_x(M^d)$ the tangent space to M^d at $x \in M^d$, and by $T(M^d)$ the disjoint union of the spaces $T_x(M^d)$.*

Notation 1.1.3. *Given a smooth manifold M^d, we denote by $\Gamma(T(M^d))$ the space of smooth vector fields on M^d.*

Remark 1.1.4. *Let M^d be a smooth manifold. For some local chart (U, κ) at x with associated coordinates x^i, we define*

$$\left(\frac{\partial}{\partial x^i}\right)_x (u) := \left(\frac{\partial}{\partial x^i}\right)_{\kappa(x)} (u \circ \kappa_i^{-1}) \quad u \in C^\infty(M^d)$$

We observe that the $\left(\frac{\partial}{\partial x^i}\right)_x$'s form a basis of $T_x(M^d)$.

Definition 1.1.5. *(Riemannian Manifold) Let M^d be a d-dimensional smooth manifold. Riemannian metric g on M^d is given by an inner product on each tangent space $T_x(M^d)$ which depends smoothly on the base point x.*
A Riemannian manifold is a smooth manifold M^d together with a Riemannian metric g.

Notation 1.1.6. *Below we shall use the Einstein summation convention.*

Remark 1.1.7. *Let (M^d, g) be a d-dimensional Riemannian manifold. In any smooth local chart (U, κ) with associated coordinates x^i, the Riemannian metric g can be written as*

$$g = g_{ij} \; dx^i \otimes dx^j$$

where $(g_{ij})_{ij}$ is a positive definite hermitian matrix of smooth functions.

Remark 1.1.8. *From now on we shall assume any Riemannian manifold to be equipped with the Levi-Civita connection.*

Definition 1.1.9. *Let (M^d, g) be a d-dimensional Riemannian manifold and $u \in C^\infty(M^d)$. For a local chart (U, κ) with associated coordinates x^i, we define*

$$\nabla_g u := g^{ij} \frac{\partial u}{\partial x^i} \frac{\partial}{\partial x^j} \in \Gamma(T(M^d)) \qquad (Gradient)$$
$$\Delta_g u := g^{ij} \left(\frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial u}{\partial x^k} \right) \in C^\infty(M^d) \qquad (Laplacian)$$

where

$$\Gamma^k_{ij} := \frac{1}{2} g^{lk} \left(\frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

are the Christoffel symbols (related to the Levi-Civita connection) and g^{ij} *is the inverse of* g_{ij} *i.e.*

$$g^{ij} g_{jk} = g_{kj} g^{ji} = \delta^i_k.$$

Definition 1.1.10. *Let* (M^d, g) *be a compact d-dimensional Riemannian manifold and* $u \in C^\infty(M^d)$. *Let* $(U_i, \kappa_i)_{i\in I}$ *be a smooth atlas and* $(\psi_i)_{i\in I}$ *a partition of the unity subordinated to the atlas, we define*

$$\int_{M^d} u \; dvol_g := \sum_{i\in I} \int_{\kappa_i(U_i)} \left(\psi_i \sqrt{|g|} u \right) \circ \kappa_i^{-1} \; dx,$$

where, in any smooth local coordinates x^i, $|g|$ *is the determinant of the matrix* $(g_{ij})_{ij}$ *and*

$$dvol_g := \sqrt{|g|} \; dx^i \wedge \cdots \wedge dx^d$$

is the volume form related to the Riemannian metric g.

Definition 1.1.11. *(Closed Riemannian Manifold) A closed d-dimensional Riemannian manifold* (M^d, g) *is a compact Riemannian manifold of dimension d such that* $\partial M^d = \emptyset$.

Theorem 1.1.12. *(Integration by Parts) Let* (M^d, g) *be a closed Riemannian manifold of dimension d and* $u \in C^\infty(M^d)$, *then*

$$\int_{M^d} u \; \Delta \overline{u} \; dvol_g = - \int_{M^d} |\nabla_g u|_g^2 \; dvol_g,$$

where $|\cdot|_g$ *is the norm associated to the Riemannian metric* g.

Definition 1.1.13. *(L^p Spaces) Let* (M^d, g) *be a d-dimensional compact Riemannian manifold and* $u \in C^\infty(M^d)$. *We define the norms*

$$||u||_{L^p(M^d)} := \left(\int_{M^d} |u|^p \; dvol_g \right)^{1/p} \quad for \quad p \geq 1,$$

$$||u||_{L^\infty(M^d)} := \sup_{M^d} |u|.$$

Moreover, for any $1 \leq p \leq \infty$*, we define the vector space* $L^p(M^d)$ *as the completion of* $C^\infty(M^d)$ *with respect to the norm* $||\cdot||_{L^p(M^d)}$.

Definition 1.1.14. *Let* (M^d, g) *be a d-dimensional compact Riemannian manifold and* $X \in \Gamma(T(M^d))$*. We define the norm*

$$||X||_{L^p(M^d)} := \left(\int_{M^d} |X|_g^p \, dvol_g \right)^{1/p} \quad for \quad p \geq 1.$$

1.2 Sobolev Spaces on Compact Manifolds

In this section we introduce equivalent notions of Sobolev spaces on compact manifolds. Moreover, we shall state interpolation inequalities which we will use repeadetly in the following dissertation.

Definition 1.2.1. *(Sobolev Space)(1) Let* (M^d, g) *be a d-dimensional compact Riemannian manifold and* $u \in C^\infty(M^d)$*. For any* $k \in \mathbb{N} \cup \{0\}$*, we define the norm*

$$||u||_{W^{k,p}(M^d)} := \sum_{j=0}^{k} \left(\int_{M^d} |\nabla^j u|^p \, dvol_g \right)^{1/p} \quad for \quad p \geq 1,$$

where $\nabla^j u$ *is the* j^{th} *covariant derivative of* u *and the norm* $|\nabla^j u|$ *is defined in a local chart by*

$$|\nabla^j u| := g^{i_1 l_1} \cdots g^{i_j l_j} (\nabla^j u)_{i_1 \cdots i_j} (\nabla^j u)_{l_1 \cdots l_j}.$$

Then, for any $p \geq 1$*, we define the vector space* $W^{k,p}(M^d)$ *as the completion of* $C^\infty(M^d)$ *with respect to the norm* $||\cdot||_{W^{k,p}(M^d)}$.

Remark 1.2.2. *If* $k \geq k'$*, we have*

$$W^{k,p}(M^d) \hookrightarrow W^{k',p}(M^d).$$

Remark 1.2.3. *For any* $u \in C^\infty(M^d)$*, we have*

$$||u||^p_{W^{1,p}(M^d)} = ||u||^p_{L^p(M^d)} + ||\nabla_g u||^p_{L^p(M^d)}.$$

Theorem 1.2.4. *(Sobolev Embedding I) Let (M^d, g) be a d-dimensional compact Riemannian manifold. For any $k, s \in \mathbb{N} \cup \{0\}$ and $p, q \geq 1$ such that $k - d/p \geq s - d/q$, we have*

$$W^{k,p}(M^d) \hookrightarrow W^{s,q}(M^d).$$

Proof. See [4]. □

Theorem 1.2.5. *(Sobolev Embedding II) Let (M^d, g) be a d-dimensional compact Riemannian manifold. For any $k \in \mathbb{N} \cup \{0\}$ and $p \geq 1$ such that $k \geq d/p$, we have*

$$W^{k,p}(M^d) \hookrightarrow L^\infty(M^d).$$

Proof. See [4]. □

In order to define the fractional Sobolev space $H^s(M^d)$ with $s \geq 0$ we recall the definition of the standard Sobolev space $H^s(\mathbb{R}^d)$.

Definition 1.2.6. *(Sobolev Space $H^s(\mathbb{R}^d)$) Let $s \geq 0$. We define the Sobolev space $H^s(\mathbb{R}^d)$ as the set of all the tempered distributions $u \in \mathcal{S}'(\mathbb{R}^d)$ such that*

$$(\mathrm{I} - \Delta)^{s/2} u := \mathcal{F}^{-1}((1 + |\cdot|^2)^{s/2} \mathcal{F}(u)) \in L^2(\mathbb{R}^d).$$

Moreover, we define the norm

$$||u||_{H^s(\mathbb{R}^d)} := ||(\mathrm{I} - \Delta)^{s/2} u||_{L^2(\mathbb{R}^d)}.$$

Proposition 1.2.7. *Let $s_1 \leq s \leq s_2$ be such that $s = \theta s_1 + (1 - \theta) s_2$ for some $0 \leq \theta \leq 1$. Then*

$$||u||_{H^s(\mathbb{R}^d)} \leq ||u||^\theta_{H^{s_1}(\mathbb{R}^d)} ||u||^{1-\theta}_{H^{s_2}(\mathbb{R}^d)}.$$

Proposition 1.2.8. *Let $s \geq 0$, then*

$$||uv||_{H^s(\mathbb{R}^d)} \lesssim ||u||_{H^s(\mathbb{R}^d)} ||v||_{L^\infty(\mathbb{R}^d)} + ||u||_{L^\infty(\mathbb{R}^d)} ||v||_{H^s(\mathbb{R}^d)}.$$

Proof. See [9]. □

Definition 1.2.9. *(Sobolev Space)(2) Let* (M^d, g) *be a d-dimensional compact Riemannian manifold and* $u \in C^\infty(M^d)$. *Let* $(U_i, \kappa_i)_{i \in I}$ *be a smooth atlas and* $(\psi_i)_{i \in I}$ *a partition of unity subordinate to the atlas. For any* $s \geq 0$, *we define the norm*

$$||u||_{H^s(M^d)} := \sum_{i \in I} ||(\psi_i u) \circ \kappa_i^{-1}||_{H^s(\mathbb{R}^d)}.$$

Remark 1.2.10. *Let* (M^d, g) *be a d-dimensional compact Riemannian manifold and* $k \in \mathbb{N} \cup \{0\}$, *then*

$$|| \cdot ||_{H^k(M^d)} \sim || \cdot ||_{W^{k,2}(M^d)}.$$

Now, we take advantage of Definition 1.2.9 and of Propositions 1.2.7 and 1.2.8 to deduce the following results.

Proposition 1.2.11. *Let* (M^d, g) *be a d-dimensional compact Riemannian manifold and* $s_1 \leq s \leq s_2$ *such that* $s = \theta s_1 + (1-\theta)s_2$ *for some* $0 \leq \theta \leq 1$. *Then*

$$||u||_{H^s(M^d)} \lesssim ||u||^{\theta}_{H^{s_1}(M^d)} ||u||^{(1-\theta)}_{H^{s_2}(M^d)}. \tag{1.1}$$

Corollary 1.2.12. *Let* (M^2, g) *be a 2-dimensional compact Riemannian manifold. For all* $\epsilon > 0$

$$||u||_{L^\infty(M^2)} \lesssim_\epsilon ||u||^{1-\epsilon}_{H^1(M^2)} ||u||^{\epsilon}_{H^2(M^2)}. \tag{1.2}$$

Proof. By using the Sobolev Embedding $H^{1+\epsilon}(M^2) \hookrightarrow L^\infty(M^2)$, we have

$$||u||_{L^\infty} \lesssim_\epsilon ||u||_{H^{1+\epsilon}} \lesssim ||u||^{1-\epsilon}_{H^1} ||u||^{\epsilon}_{H^2},$$

where the last inequality is due to Proposition 1.2.11 □

Corollary 1.2.13. *Let* (M^3, g) *a 3-dimensional closed Riemannian manifold. For all* $\epsilon > 0$

$$||u||_{L^\infty(M^3)} \lesssim_\epsilon ||u||^{(1-\epsilon)/2}_{H^1(M^3)} ||u||^{(1+\epsilon)/2}_{H^2(M^3)}. \tag{1.3}$$

Proof. By using the Sobolev Embedding $H^{(3+\epsilon)/2}(M^3) \hookrightarrow L^\infty(M^3)$, we have

$$||u||_{L^\infty} \lesssim_\epsilon ||u||_{H^{\frac{3}{2}+\frac{\epsilon}{2}}} \lesssim ||u||_{H^1}^{\frac{1-\epsilon}{2}} ||u||_{H^2}^{\frac{1+\epsilon}{2}},$$

where we used Proposition 1.2.11 at the last step. ☐

Proposition 1.2.14. *Let (M^d, g) be a d-dimensional compact Riemannian manifold. Then*

$$||uv||_{H^s(M^d)} \lesssim ||u||_{H^s(M^d)} ||v||_{L^\infty(M^d)} + ||v||_{H^s(M^d)} ||u||_{L^\infty(M^d)}. \tag{1.4}$$

Thanks to the functional calculus of the Laplace-Beltrami operator Δ_g, it is possible to define a new norm which is equivalent to $||\cdot||_{H^s(M^d)}$.
We observe that the Laplace-Beltrami operator

$$\Delta_g : H^2(M^d) \subset L^2(M^d) \longrightarrow L^2(M^d)$$

is essentially self-adjoint on $L^2(M^d)$. By an abuse of notation, keep denoting by Δ_g the unique self-adjoint extension of the Laplace-Beltrami operator on $L^2(M^d)$. Since Δ_g is a compact self-adjoint operator acting on $L^2(M^d)$, the Spectral Theorem gives

$$\Delta_g = \sum_{j \in \mathbb{N}} -\lambda_j E_j, \tag{1.5}$$

where $\{-\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^-$ are the eigenvalues of Δ_g and $E_j : L^2(M^d) \to L^2(M^d)$ are the projection operators that project onto the eigenspace $\mathcal{E}_j$ relative to the eigenvalue $-\lambda_j$.

Definition 1.2.15. *Let (M^d, g) be a d-dimensional compact Riemannian manifold and Δ_g the unique self-adjoint extension of the Laplace-Beltrami operator on $L^2(M^d)$. For every $\varphi \in C^\infty(\mathbb{R})$, we define the operator*

$$\varphi(\Delta_g) : L^2(M^d) \longrightarrow L^2(M^d),$$
$$\varphi(\Delta_g) := \sum_{j\in\mathbb{N}} \varphi(-\lambda_j) E_j.$$

The proof of the following equivalences is based on the spectral decomposition of the Laplace-Beltrami operator (1.5).

Proposition 1.2.16. *Let (M^d, g) be a d-dimensional compact Riemannian manifold. For any $s \geq 0$, we have*

$$||\cdot||_{H^s(M^d)} \sim ||(1-\Delta_g)^{s/2}\cdot||_{L^2(M^d)}.$$

Proof. See [8]. □

Proposition 1.2.17. *Let (M^d, g) be a d-dimensional compact Riemannian manifold. For any $s \geq 0$ and $p > 1$, we have*

$$||\cdot||_{W^{s,p}(M^d)} \sim ||(1-\Delta_g)^{s/2}\cdot||_{L^p(M^d)}.$$

Proof. See [8]. □

Chapter 2

The Schrödinger Equation in the Euclidean Space

This chapter is dedicated to a brief introduction to the Schrödinger equation in the Euclidean space $\mathbb{R}^d$. For a complete dissertation see [6].

2.1 The Linear Schrödinger Equation

In this section we introduce some classical results on the linear Schrödinger equation in $\mathbb{R}^d$. In particular we shall state homogeneous and nonhomogeneous Strichartz estimates with the aim of comparing these results with the ones given in Chapter 3.
Let us consider the Cauchy problem

$$\begin{cases} i\partial_t u + \Delta u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,\cdot) = u_0 \in \mathcal{S}'(\mathbb{R}^d). \end{cases} \tag{2.1}$$

It is a well known fact that the unique solution to (2.1) is

$$e^{it\Delta}u_0 := \frac{e^{-|x|^2/4it}}{(4\pi it)^{d/2}} * u_0 = \mathcal{F}^{-1}(e^{-4\pi^2 it|\xi|^2}\mathcal{F}(u_0)).$$

In the following we state a few properties for the class of operators $\{e^{it\Delta}\}_{t\in\mathbb{R}}$.

Proposition 2.1.1. *The family* $\{e^{it\Delta}\}_{t\in\mathbb{R}}$ *is a unitary group of operators acting on* $L^2(\mathbb{R}^d)$.

Proof. See [6]. □

Proposition 2.1.2. *(Dispersive Estimate)*

$$||e^{it\Delta}u||_{L^\infty(\mathbb{R}^d)} \lesssim \frac{1}{|t|^{d/2}}||u||_{L^1(\mathbb{R}^d)}.$$

Proof. See [6]. □

The proof of homogeneous and nonhomogeneous Strichartz estimates for the linear Schrödinger equation in $\mathbb{R}^d$ is mainly based on the dispersive estimate and on the mass conservation. The following functional analytic result by Markus Keel and Terence Tao is crucial to prove the estimates under discussion.

Definition 2.1.3. *Let* $\sigma > 0$, *the pair* (q, r) *is Strichartz* σ*-admissible if*

$$q, r \geq 2, \quad (q, r, \sigma) \neq (2, \infty, 1) \quad \text{and} \quad \frac{2}{q} + \frac{2\sigma}{r} = \sigma.$$

Theorem 2.1.4. *(Keel-Tao) Let* X *be a measure space and* H *an Hilbert space. Suppose that for every* $t \in \mathbb{R}$ *there exist an operator*

$$U(t) : H \longrightarrow L^2(X)$$

such that:

1. *For all* $t \in \mathbb{R}$ *and* $f \in H$

$$||U(t)f||_{L^2(X)} \lesssim ||f||_H.$$

2. *For all* $t, s \in \mathbb{R}$ *such that* $t \neq s$ *and* $g \in L^1(X)$

$$||U(t)(U(s))^*g||_{L^\infty(X)} \lesssim \frac{1}{|t-s|^\sigma}||g||_{L^1(X)},$$

for some $\sigma > 0$.

Then, for every (q,r), $(\tilde{q},\tilde{r})$ Strichartz σ-admissible pairs, the following estimates hold:

$$||U(t)u||_{L^q(\mathbb{R},L^r(X))} \lesssim ||u||_H, \tag{2.2}$$
$$\Big|\Big| \int_{\mathbb{R}} (U(s))^* F(s,\cdot)\ ds \Big|\Big|_H \lesssim ||F||_{L^{q'}(\mathbb{R},L^{r'}(\mathbb{R}^d))},$$
$$\Big|\Big| \int_{-\infty}^{t} U(t)(U(s))^* F(s,\cdot)\ ds \Big|\Big|_{L^q(\mathbb{R},L^r(X))} \lesssim ||F||_{L^{\tilde{q}'}(\mathbb{R},L^{\tilde{r}'}(X))},$$

where q' denotes the Hölder conjugate of q.

Proof. See [5]. □

Theorem 2.1.5. *(Homogeneous Strichartz Estimates) Let (q,r) be Strichartz $d/2$-admissible, then*

$$||e^{it\Delta}u||_{L^q(\mathbb{R},L^r(\mathbb{R}^d))} \lesssim ||u||_{L^2(\mathbb{R}^d)}.$$

Theorem 2.1.6. *(Nonhomogeneous Strichartz Estimates) Let (q,r), $(\tilde{q},\tilde{p})$ be Strichartz $d/2$-admissible, then*

$$\Big|\Big| \int_0^t e^{i(t-s)\Delta} F(s,\cdot)\ ds \Big|\Big|_{L^q(\mathbb{R},L^r(\mathbb{R}^d))} \lesssim ||F||_{L^{\tilde{q}'}(\mathbb{R},L^{\tilde{r}'}(\mathbb{R}^d))}.$$

Theorem 2.1.4 will also be used in Chapter 3 to prove Strichartz estimates on closed Riemannian manifolds.

2.2 The Nonlinear Schrödinger Equation

The aim of this section is to give a description of the nonlinear Schrödinger equation (NLS) in $\mathbb{R}^d$ with polynomial nonlinearity. We shall introduce the notions of classical and strong solution to the NLS and state local and global well-posedness results in the energy space $H^1(\mathbb{R}^d)$ for the Cauchy problem

$$\begin{cases} i\partial_t u + \Delta u = |u|^{p-1}u, & (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ u(0,\cdot) = u_0 \in H^s(\mathbb{R}^d), & s \geq 0. \end{cases} \tag{2.3}$$

Assuming the initial datum u_0 to be smooth enough, broadly speaking we refer to a "classical solution" to the Cauchy problem (2.3) as a function u that solves (at least locally) the equation

$$i\partial_t u + \Delta u = |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d,$$

in a classical sense (i.e. without requiring the theory of weak derivatives), have enough decay and, of course, satisfies the initial condition. The proof of the uniqueness for classical solutions to (2.3) (whenever they exist) is standard (see [9]). In order to study the Cauchy problem (2.3) with "rough" initial data we introduce the notion of strong solution to the NLS.

Definition 2.2.1. *(Strong Solution to the NLS) Let $T > 0$, a function $u \in C([-T,T], H^s(\mathbb{R}^d))$ that solves the integral equation*

$$u(t,x) = e^{it\Delta}u_0(x) - i\int_0^t e^{i(t-s)\Delta}|u(s,x)|^{p-1}u(s,x)\; ds, \quad \forall t \in [-T,T]$$

is said to be a strong solution to (2.3).

Remark 2.2.2. *Whenever a classical solution to (2.3) exists, it coincides with the strong solution to the Cauchy problem.*

Definition 2.2.3. *(Local and Global Well-Posedness) We say that the Cauchy problem (2.3) is locally well-posed in $H^s(\mathbb{R}^d)$ if, given any $u_0 \in H^s(\mathbb{R}^d)$, there exist $T = T(u_0) > 0$, a Banach space $X_T \subset C([-T,T], H^s(\mathbb{R}^d))$ and a unique strong solution $u \in X_T$ to (2.3). Moreover, the map data solution $\tilde{u_0} \longmapsto u$ is locally defined and continuous.*
In case the time of existence T can be taken arbitrarily large, we say that (2.3) is globally well-posed in $H^s(\mathbb{R}^d)$.

Remark 2.2.4. *Let us assume that the Cauchy problem (2.3) is well-posed in $H^s(\mathbb{R}^d)$. By performing a standard regularization of the nonlinearity $|u|^{p-1}u$, every strong solution to the regularized version of (2.3) can be seen as the strong limit (in the H^s topology) of smooth classical solutions generated by any sequence of smooth data converging to the "rough" initial datum u_0.*

As a consequence to Remark 2.2.4, from now on we assume that any strong solution to the Cauchy problem (2.3) has enough regularity to justify the following computations.

Remark 2.2.5. *The notions of solution and local/global well-posedness can be easily adapted to the case of NLS on closed Riemannian manifolds (see Chapter 3).*

Theorem 2.2.6. *(Local Well-Posedness in H^1) Let us assume*

$$\begin{cases} 1 < p \leq \dfrac{d+2}{d-2}, & if\ d > 2, \\ 1 < p < \infty, & if\ d = 1, 2. \end{cases}$$

The Cauchy problem (2.3) is locally well-posed in $H^1(\mathbb{R}^d)$ and the unique strong solution $u \in C([-T, T], H^1(\mathbb{R}^d))$ satisfies the following conservation laws:

1. *Mass Conservation Law*

$$||u(t, \cdot)||_{L^2(\mathbb{R}^d)} = ||u_0||_{L^2(\mathbb{R}^d)}, \quad \forall t \in [-T, T].$$

2. *Energy Conservation Law*

$$\begin{aligned} &\frac{1}{2}||\nabla u(t, \cdot)||^2_{L^2(\mathbb{R}^d)} + \frac{1}{p+1}||u(t, \cdot)||^{p+1}_{L^{p+1}(\mathbb{R}^d)} \\ = &\frac{1}{2}||\nabla u_0||^2_{L^2(\mathbb{R}^d)} + \frac{1}{p+1}||u_0||^{p+1}_{L^{p+1}(\mathbb{R}^d)}, \quad \forall t \in [-T, T]. \end{aligned}$$

In particular, if the polynomial nonlinearity satisfies the condition

$$\begin{cases} 1 < p < \dfrac{d+2}{d-2}, & if\ d > 2, \\ 1 < p < \infty, & if\ d = 1, 2, \end{cases} \tag{2.4}$$

the time of existence T depends only on $||u_0||_{H^1}$ rather than on u_0.

Remark 2.2.7. *By using a standard iterative method based and the conservation laws, it is possible to prove that the Cauchy problem (2.3) is globally well posed whenever condition (2.4) holds.*

Chapter 3

The Schrödinger Equation on Closed Riemannian Manifolds

This chapter is devoted to the description of the Schrödinger equation on an arbitrary d-dimensional compact Riemannian Manifold M^d with $\partial M = \emptyset$.

3.1 The Linear Schrödinger Equation

In this section we introduce some classical results on the linear Schrödinger equation (LS) on a closed Riemannian manifold M^d. In particular, we shall prove homogeneous and nonhomogeneous Strichartz estimates with loss of derivatives for the LS (that is, with a Sobolev norm on the r.h.s.) which are a key tool in the subsequent dissertation on the nonlinear problem.
Let us consider the Cauchy problem

$$\begin{cases} i\partial_t u + \Delta_g u = 0 \quad (t,x) \in \mathbb{R} \times M^d \\ u(0,\cdot) = u_0 \in L^2(M^d). \end{cases} \tag{3.1}$$

The unique solution to (3.1) is $e^{it\Delta_g}u_0$ where, for every $t \in \mathbb{R}$, the linear map

$$e^{it\Delta_g} : L^2(M^d) \longrightarrow L^2(M^d)$$

is defined through the functional calculus of the Laplace-Beltrami operator (see Chapter 1).

Proposition 3.1.1. *The family* $\{e^{it\Delta_g}\}_{t\in\mathbb{R}}$ *is a unitary group of operators acting on* $L^2(M^d)$*.*

Remark 3.1.2. *We recall that the proof of Strichartz estimates in the Euclidean setting is a combination of functional analytic arguments and the dispersive estimate*

$$||e^{it\Delta}u||_{L^\infty(\mathbb{R}^d)} \lesssim \frac{1}{|t|^{d/2}}||u||_{L^1(\mathbb{R}^d)}, \tag{3.2}$$

which fails in the case of compact Riemannian manifolds. In fact, any constant function belongs to $L^2(M^d)$ *(since* M^d *is compact) and solves* (3.1) *but fails to satisfy* (3.2) *with* M^d *in place of* $\mathbb{R}^d$

Since a standard dispersive estimate is not available in M^d (see Remark 3.1.2), we shall need a semiclassical version of the latter (which holds on time intervals that shrink as the frequency of the data is growing) in order to prove Strichartz estimates with loss. To state this result we need some tools from microlocal analysis.

Definition 3.1.3. *A Littlewood-Paley partition of the identity is a pair* $(\tilde{\varphi}, \varphi)$ *such that*

$$Id = \tilde{\varphi}(-\Delta_g) + \sum_{h^{-1}\in 2^{\mathbb{N}}} \varphi(-h^2\Delta_g), \qquad \tilde{\varphi} \in C_0^\infty(\mathbb{R}),\ \varphi \in C_0^\infty(\mathbb{R}^*).$$

Proposition 3.1.4. *Let* $(\tilde{\varphi}, \varphi)$ *be a Littlewood-Paley partition of the identity. For all* $r \geq 2$*, we have*

$$||u||_{L^r(M^d)} \lesssim ||\tilde{\varphi}(-\Delta_g)u||_{L^r(M^d)} + \left(\sum_{h^{-1}\in 2^{\mathbb{N}}} ||\varphi(-h^2\Delta_g)u||^2_{L^r(M^d)}\right)^{1/2}. \tag{3.3}$$

Proof. See Corollary 2.3. in [3]. □

Proposition 3.1.5. *Let* $\varphi \in C_0^\infty(\mathbb{R}^*)$*. For all* $r \geq 1$*, we have*

$$\left|\left| \left(\sum_{h^{-1}\in 2^{\mathbb{N}}} |\varphi(-h^2\Delta_g)u|^2\right)^{1/2} \right|\right|_{L^r(M^d)} \lesssim ||u||_{L^r(M^d)}. \tag{3.4}$$

Proof. See Theorem 0.2.10. in [8]. □

Proposition 3.1.6. *Let* $\varphi \in C_0^\infty(\mathbb{R}^*)$*, we have*

$$\left(\sum_{h^{-1} \in 2^{\mathbb{N}}} h^{-2s} ||\varphi(-h^2 \Delta_g) u||^2_{L^2(M^d)} \right)^{1/2} \lesssim ||u||_{H^s(M^d)} \tag{3.5}$$

Proof. By using the Spectral Theorem and the functional calculus of Δ_g, we get

$$\sum_{h^{-1} \in 2^{\mathbb{N}}} h^{-2s} ||\varphi(-h^2 \Delta_g) u||^2_{L^2(M^d)} \sim \sum_{j \in \mathbb{N}} \left(\sum_{h^{-1} \in 2^{\mathbb{N}}} h^{-2s} \varphi(h^2 \lambda_j)^2 \right) ||E_j u||^2_{L^2}.$$

For any λ_j fixed, since φ is compactly supported, we have

$$\sum_{h^{-1} \in 2^{\mathbb{N}}} h^{-2s} \varphi(h^2 \lambda_j)^2 \lesssim (1 + \lambda_j)^s$$

therefore, by using the Spectral Theorem and the functional calculus of Δ_g, we have

$$\sum_{h^{-1} \in 2^{\mathbb{N}}} h^{-2s} ||\varphi(-h^2 \Delta_g) u||^2_{L^2(M^d)} \lesssim \sum_{j \in \mathbb{N}} (1 + \lambda_j)^s ||E_j u||^2_{L^2} \sim ||u||^2_{H^s}$$

which concludes the proof of the proposition. □

Proposition 3.1.7. *(Semiclassical Dispersive Estimate) Let* $\varphi \in C_0^\infty(\mathbb{R})$*. There exist* $\alpha > 0$ *such that, for all* $h \in]0, 1]$*, we have*

$$||e^{it\Delta_g} \varphi(-h^2 \Delta_g) u||_{L^\infty(M^d)} \lesssim \frac{1}{|t|^{d/2}} ||u||_{L^1(M^d)},$$

for all $t \in [-\alpha h, \alpha h]$*.*

Proof. See [3]. □

Proposition 3.1.8. *Let* $\varphi \in C_0^\infty(\mathbb{R})$ *and* (q, r) *Strichartz* $d/2$*-admissible. There exist* $\alpha > 0$ *such that, for every interval* $J \subset \mathbb{R}$ *of size* $\leq \alpha h$*,* $h \in]0, 1]$*, we have*

$$\left(\int_J ||e^{it\Delta_g} \varphi(-h^2 \Delta_g) u||^q_{L^r(M^d)} dt \right)^{1/q} \lesssim ||\varphi(-h^2 \Delta_g) u||_{L^2(M^d)},$$

Proof. Let $\bar{\varphi} \in C_0^\infty(\mathbb{R})$, $\varphi\bar{\varphi} = \varphi$. By Proposition 3.1.7 there exist $\alpha > 0$ such that, for every $h \in]0,1]$, we have

$$||e^{it\Delta_g}\bar{\varphi}(-h^2\Delta_g)u||_{L^\infty} \lesssim \frac{1}{|t|^{d/2}}||u||_{L^1}, \tag{3.6}$$

for every $t \in [-\alpha h, \alpha h]$. Let $J \subset \mathbb{R}$ be an interval of size $\leq \alpha h$, $h \in]0,1]$, for every $t \in \mathbb{R}$ we define the operator

$$U(t) := 1_J(t)e^{it\Delta_g}\bar{\varphi}(-h^2\Delta).$$

Due to (3.6), for every $t \in \mathbb{R}$ the operator $U(t)$ satisfies the hypotheses of Theorem 2.1.4 with $X \equiv M^d$, $H \equiv L^2(M^d)$ and $\sigma = d/2$, so that, from (2.2), we get

$$\begin{aligned}
&\left(\int_J ||e^{it\Delta_g}\varphi(-h^2\Delta_g)u||_{L^r}^q dt\right)^{1/q} \\
&= \left(\int_{\mathbb{R}} ||1_J(t)e^{it\Delta_g}\bar{\varphi}(-h^2\Delta_g)\varphi(-h^2\Delta_g)u||_{L^r}^q dt\right)^{1/q} \\
&\lesssim ||\varphi(-h^2\Delta_g)u||_{L^2}.
\end{aligned}$$

□

With the previous results at our disposal we can now state the Strichartz estimates with loss on closed Riemannian manifolds.

Theorem 3.1.9. *(Homogeneous Strichartz Estimates) Let (q, r) Strichartz $d/2$-admissible, for any finite interval $I \subset \mathbb{R}$, we have*

$$||e^{it\Delta_g}u||_{L^q(I,L^r(M^d))} \lesssim ||u||_{H^{1/q}(M^d)}.$$

Proof. Let $(\tilde{\varphi}, \varphi)$ be a Littlewood-Paley partition of the identity, by using (3.3) we get

$$||e^{it\Delta_g}u||_{L^r} \lesssim ||e^{it\Delta_g}\tilde{\varphi}(-\Delta_g)u||_{L^r} + \left(\sum_{h^{-1}\in 2^{\mathbb{N}}} ||e^{it\Delta_g}\varphi(-h^2\Delta_g)u||_{L^r}^2\right)^{1/2}.$$

Taking the L^q norm for $t \in I$, by Minkowski inequality we get

$$\begin{aligned} ||e^{it\Delta_g}u||_{L^q(I,L^r)} \lesssim \; & ||e^{it\Delta_g}\tilde{\varphi}(-\Delta_g)u||_{L^q(I,L^r)} + \\ & + \left(\sum_{h^{-1}\in 2^{\mathbb{N}}} ||e^{it\Delta_g}\varphi(-h^2\Delta_g)u||^2_{L^q(I,L^r)} \right)^{1/2}. \end{aligned}$$

Now, we split the interval I into N sub-intervals J_k of size $\leq \alpha h$ (α given by Proposition 3.1.8) with $N \lesssim h^{-1}$ and obtain

$$||e^{it\Delta_g}\tilde{\varphi}(-\Delta_g)u||^q_{L^q(I,L^r)} = \sum_{k=1}^{N} \int_{J_k} ||e^{it\Delta_g}\tilde{\varphi}(-\Delta_g)u||^q_{L^r}$$

$$||e^{it\Delta_g}\varphi(-h^2\Delta_g)u||^q_{L^q(I,L^r)} = \sum_{k=1}^{N} \int_{J_k} ||e^{it\Delta_g}\varphi(-h^2\Delta_g)u||^q_{L^r}$$

so that, applying Proposition 3.1.8 to the right hand side of the previous identities, we get

$$||e^{it\Delta_g}\tilde{\varphi}(-\Delta_g)u||^q_{L^q(I,L^r)} \lesssim N||\tilde{\varphi}(-\Delta_g)u||^q_{L^2} \lesssim ||\tilde{\varphi}(-\Delta_g)u||^q_{L^2},$$

$$||e^{it\Delta_g}\varphi(-h^2\Delta_g)u||^q_{L^q(I,L^r)} \lesssim N||\varphi(-h^2\Delta_g)u||^q_{L^2} \lesssim h^{-1}||\varphi(-h^2\Delta_g)u||^q_{L^2}.$$

We observe that

$$\begin{aligned} ||\tilde{\varphi}(-\Delta_g)u||_{L^2} &= \left(\int_{M^d} |\tilde{\varphi}(-\Delta_g)u|^2 \, dvol_g \right)^{1/2} \\ &= \left(\int_{M^d} \left| \sum_{j\in\mathbb{N}} \tilde{\varphi}(\lambda_j)E_i u \right|^2 \, dvol_g \right)^{1/2}. \end{aligned}$$

Since $\tilde{\varphi}$ is compactly supported, there exist finitely many λ_js such that $\tilde{\varphi}(\lambda_j) \neq 0$, then

$$\left(\int_{M^d} \left| \sum_{j\in\mathbb{N}} \tilde{\varphi}(\lambda_j)E_i u \right|^2 \, dvol_g \right)^{1/2} \lesssim \left(\int_{M^d} \left| \sum_{j\in\mathbb{N}} E_i u \right|^2 \, dvol_g \right)^{1/2} = ||u||_{L^2}$$

therefore

$$||e^{it\Delta_g}\tilde{\varphi}(-\Delta_g)u||_{L^q(I,L^r)} \lesssim ||u||_{L^2},$$

$$||e^{it\Delta_g}\varphi(-h^2\Delta_g)u||_{L^q(I,L^r)} \lesssim h^{-1/q}||\varphi(-h^2\Delta_g)u||_{L^2}.$$

Finally, putting all together, we obtain

$$||e^{it\Delta_g}u||_{L^q(I,L^r)} \lesssim ||u||_{L^2} + \left(\sum_{h^{-1}\in 2^{\mathbb{N}}} h^{-2/q}||\varphi(-h^2\Delta_g)u||_{L^2}^2 \right)^{1/2},$$

hence, due to (3.5), we get

$$||e^{it\Delta_g}u||_{L^q(I,L^r)} \lesssim ||u||_{L^2} + ||u||_{H^{1/q}} \lesssim ||u||_{H^{1/q}},$$

which concludes the proof. □

Notation 3.1.10. *Let X be a Banach space and $T > 0$, we shall use $L^p_T X$ to denote the space $L^p((0,T),X)$ and*

$$|| \cdot ||_{L^p_T X} \equiv \left(\int_0^T || \cdot ||^p_X \, ds \right)^{1/p}.$$

Corollary 3.1.11. *(Nonhomogeneous Strichartz Estimates) For any $T > 0$, we have*

$$\Big|\Big| \int_0^t e^{i(t-s)\Delta_g} F(s,\cdot) \, ds \Big|\Big|_{L^q_T L^r(M^d)} \lesssim ||F||_{L^1_T H^{1/q}(M^d)}.$$

Proof. Defining $\tilde{F}_s(t) := 1_{\leq t}(s)e^{i(t-s)\Delta_g}F(s,\cdot)$, due to Minkowski's inequality, we get

$$\Big|\Big| \int_0^T \tilde{F}_s \, ds \Big|\Big|_{L^q_T L^r} \lesssim \int_0^T ||\tilde{F}_s||_{L^q_T L^r} \, ds$$

which is equivalent to

$$\Big|\Big| \int_0^t e^{i(t-s)\Delta_g} F(s,\cdot) \, ds \Big|\Big|_{L^q_T L^r} \lesssim \int_0^T ||e^{i(t-s)\Delta_g} F(s,\cdot)||_{L^q_T L^r} \, ds.$$

Applying Theorem 3.1.9 to the RHS of the inequality above we obtain

$$\begin{aligned} \Big|\Big| \int_0^t e^{i(t-s)\Delta_g} F(s,\cdot) \, ds \Big|\Big|_{L^q_T L^r} &\lesssim \int_0^T ||e^{-is\Delta_g} F(s,\cdot)||_{H^{1/q}} \, ds \\ &= \int_0^T ||F(s,\cdot)||_{H^{1/q}} \, ds = ||F||_{L^1_T H^{1/q}}. \end{aligned}$$

□

Remark 3.1.12. *We remark that the previous estimates are not optimal, in the sense that it might exist* $s_0 = s_0(M^d) \in [0, \frac{1}{q}]$ *such that*

$$||e^{it\Delta_g}u||_{L^q(I,L^r(M^d))} \lesssim ||u||_{H^{s_0}(M^d)}, \tag{3.7}$$

$$\Big|\Big|\int_0^t e^{i(t-s)\Delta_g}F(s,\cdot)\, ds\Big|\Big|_{L^q_T L^r(M^d)} \lesssim ||F||_{L^1_T H^{s_0}(M^d)}. \tag{3.8}$$

A classical example to that "non-optimality" is given by Bourgain's estimate on $\mathbb{T}^2$ *for which the two inequalities above hold for every* $s_0 > 0$ *(see [2]).*

3.2 The Nonlinear Schrödinger Equation

The aim of this section is to study the nonlinear Schrödinger equation on closed Riemannian manifolds with polynomial nonlinearity; we shall state local and global well-posedness results for the Cauchy problem

$$\begin{cases} i\partial_t u + \Delta_g u = |u|^{p-1}u, & (t,x) \in \mathbb{R} \times M^d, \\ u(0,\cdot) = u_0 \in H^s(M^d), & s \geq 0, \end{cases} \tag{3.9}$$

and show qualitative properties for its solutions (whenever they exist).
By using a contraction argument built on the Strichartz estimates we get the following local well-posedness result.

Theorem 3.2.1. *(Local Well-Posedness in* $H^s(M^d)$*) Let* (M^d, g) *be a* d*-dimensional closed Riemannian manifold and* $p > 1$*, the Cauchy problem* (3.9) *is locally well-posed in* $H^s(M^d)$*, provided*

$$s > \max\Big\{\frac{d}{2} - \frac{1}{p-1}, \ \frac{d}{2} - \frac{1}{2}\Big\}.$$

In particular, assuming that

$$s > \max\Big\{\frac{d}{2} - \frac{1}{p-1}, \ \frac{d}{2} - \frac{1}{2}\Big\}, \quad s \geq 1,$$

the unique strong solution $u \in C([-T,T], H^s(M^d))$ *satisfies the following conservation laws*

1. *Mass Conservation Law*

$$||u(t,\cdot)||_{L^2(M^d)} = ||u_0||_{L^2(M^d)}, \quad \forall t \in [-T,T].$$

2. *Energy Conservation Law*

$$\frac{1}{2}||\nabla_g u(t,\cdot)||^2_{L^2(M^d)} + \frac{1}{p+1}||u(t,\cdot)||^{p+1}_{L^{p+1}(M^d)}$$
$$= \frac{1}{2}||\nabla_g u_0||^2_{L^2(M^d)} + \frac{1}{p+1}||u_0||^{p+1}_{L^{p+1}(M^d)}, \quad \forall t \in [-T,T].$$

Proof. See [3]. □

In the following we take advantage of the previous local well-posedness result in order to state a global well-posedness result in the 2-dimensional case.

Theorem 3.2.2. *(Global Well-Posedness in $H^1(M^2)$) Let (M^2, g) be a 2-dimensional closed Riemannian manifold and $p > 1$, the Cauchy problem*

$$\begin{cases} i\partial_t u + \Delta_g u = |u|^{p-1}u, & (t,x) \in \mathbb{R} \times M^2, \\ u(0,\cdot) = u_0 \in H^1(M^2), \end{cases} \tag{3.10}$$

is globally well posed in $H^1(M^2)$. In particular, the unique strong solution $u \in C(\mathbb{R}, H^1(M^2))$ satisfies the mass and the energy conservation laws.

Proof. See [3]. □

Proposition 3.2.3. *Let $u \in C(\mathbb{R}, H^1(M^2))$ be the unique global solution to (3.10). There exist $s_0 \in [0, \frac{1}{4}]$ such that, for all $\tau \in (0,1)$ and $j \in \mathbb{N} \cup \{0\}$, we have*

$$||\partial_t^j u||_{L^4_\tau L^4(M^2)} \lesssim ||\partial_t^j u(0,\cdot)||_{H^{s_0}(M^2)} + \tau||\partial_t^j(|u|^{p-1}u)||_{L^\infty_\tau H^{s_0}(M^2)}.$$

Proof. At first we observe that if u solves (3.10), then $\partial_t^j u$ is a solution to the Cauchy problem

$$\begin{cases} i\partial_t v + \Delta_g v = \partial_t^j(|u|^{p-1}u), & (t,x) \in \mathbb{R} \times M^2 \\ v(0,\cdot) = \partial_t^j u(0,\cdot). \end{cases}$$

From Duhamel's formula we get

$$\partial_t^j u = e^{it\Delta_g}\partial_t^j u(0,\cdot) - i\int_0^t e^{i(t-s)\Delta_g}\partial_t^j(|u(s,\cdot)|^{p-1}u(s,\cdot))ds,$$

therefore

$$\begin{aligned} ||\partial_t^j u||_{L^4_\tau L^4} \lesssim\ & ||e^{it\Delta_g}\partial_t^j u(0,\cdot)||_{L^4_\tau L^4}+ \\ +\Big|\Big|\int_0^t & e^{i(t-s)\Delta_g}\partial_t^j(|u(s,\cdot)|^{p-1}u(s,\cdot))ds\Big|\Big|_{L^4_\tau L^4}. \end{aligned}$$

By applying (3.7) to the first term on the right hand side of the inequality above and (3.8) to the second term, we get

$$\begin{aligned} ||\partial_t^j u||_{L^4_\tau L^4} &\lesssim ||\partial_t^j u(0,\cdot)||_{H^{s_0}} + ||\partial_t^j(|u|^{p-1}u)||_{L^1_\tau H^{s_0}} \\ &\lesssim ||\partial_t^j u(0,\cdot)||_{H^{s_0}} + \tau||\partial_t^j(|u|^{p-1}u)||_{L^\infty_\tau H^{s_0}} \end{aligned}$$

which concludes the proof of the proposition. □

To establish a global well-posedness result for the cubic NLS on a closed 3-dimensional Riemannian manifolds, it is essential to use an available endpoint Strichartz estimate for spectrally localized data (see Lemma 3.4. [3]) in order to take advantage of the conservation laws.

Theorem 3.2.4. *(Global Well-Posedness in $H^s(M^3)$) Let (M^3, g) be a 3-dimensional closed Riemannian manifold. Then, the Cauchy problem*

$$\begin{cases} i\partial_t u + \Delta_g u = |u|^2 u, & (t,x)\in\mathbb{R}\times M^3, \\ u(0,\cdot) = u_0 \in H^{1+\epsilon}(M^3). \end{cases} \tag{3.11}$$

is globally well-posed in $H^s(M^3)$ for any $s > 1$. In particular the unique strong solution $u \in C(\mathbb{R}, H^s(M^3))$ satisfies the mass and the energy conservation laws.

Proof. See [3]. □

Remark 3.2.5. *A similar global well-posedness result holds for subcubic non-linearities (i.e $|u|^{p-1}u$ with $2 < p < 3$).*

Lemma 3.2.6. *Let $\varphi \in C_0^\infty(\mathbb{R})$, $h \in]0,1]$ and assume that u is the unique global solution to the Cauchy problem*

$$\begin{cases} i\partial_t v + \Delta_g v = \varphi(-h^2\Delta_g)F, & (t,x) \in \mathbb{R} \times M^3, \\ v(0,\cdot) = \varphi(-h^2\Delta_g)v_0. \end{cases}$$

Then, for every $T > 0$, we have

$$||v||_{L^2_T L^6(M^3)} \lesssim ||v||_{L^\infty_T L^2(M^3)} + h^{-1/2}||v||_{L^2_T L^2(M^3)} + ||\varphi(-h^2\Delta_g)F||_{L^2_T L^{6/5}(M^3)}.$$

Proof. See [1]. □

Proposition 3.2.7. *Let $u \in C(\mathbb{R}, H^s(M^3))$ be the unique strong solution to the Cauchy problem (3.11). For all $\tau, \epsilon \in (0,1)$ and $j \in \mathbb{N} \cup \{0\}$, we have*

$$||\partial_t^j u||_{L^2_\tau L^6(M^3)} \lesssim_\epsilon ||\partial_t^j u||_{L^\infty_\tau H^\epsilon(M^3)} + ||\partial_t^j u||_{L^2_\tau H^{1/2}(M^3)} + ||\partial_t^j(|u|^{p-1}u)||_{L^2_\tau L^{6/5}(M^3)}.$$

Proof. We consider a Littlewood-Paley partition of the identity $(\tilde{\varphi}, \varphi)$ so that, by Lemma (3.3) and Minkowski inequality, we get

$$||\partial_t^j u||_{L^2_\tau L^6} \lesssim ||\tilde{\varphi}(-\Delta_g)\partial_t^j u||_{L^2_\tau L^6} + \left(\sum_{h^{-1} \in 2^{\mathbb{N}}} ||\varphi(-h^2\Delta_g)\partial_t^j u||^2_{L^2_\tau L^6} \right)^{1/2}.$$

Since u solves (3.11), then $\tilde{\varphi}(-\Delta_g)\partial_t^j u$ is a solution to the Cauchy problem

$$\begin{cases} i\partial_t v + \Delta_g v = \tilde{\varphi}(-\Delta_g)\partial_t^j(|u|^{p-1}u), & (t,x) \in \mathbb{R} \times M^3, \\ v(0,\cdot) = \tilde{\varphi}(-\Delta_g)\partial_t^j u(0,\cdot), \end{cases}$$

therefore, due to Lemma 3.2.6, we have

$$\begin{aligned} ||\tilde{\varphi}(-\Delta_g)\partial_t^j u||_{L^2_\tau L^6} \lesssim\ & ||\tilde{\varphi}(-\Delta_g)\partial_t^j u||_{L^\infty_\tau L^2} + ||\tilde{\varphi}(-\Delta_g)\partial_t^j u||_{L^2_\tau L^2} \\ & + ||\tilde{\varphi}(-\Delta_g)\partial_t^j(|u|^{p-1}u)||_{L^2_\tau L^{6/5}}. \end{aligned}$$

Arguing as in the proof of Theorem 3.1.9, we get

$$\begin{aligned} ||\tilde{\varphi}(-\Delta_g)\partial_t^j u||_{L^\infty_\tau L^2} &\lesssim ||\partial_t^j u||_{L^\infty_\tau L^2}, \\ ||\tilde{\varphi}(-\Delta_g)\partial_t^j u||_{L^2_\tau L^2} &\lesssim ||\partial_t^j u||_{L^2_\tau L^2}, \\ ||\tilde{\varphi}(-\Delta_g)\partial_t^j(|u|^{p-1}u)||_{L^2_\tau L^{6/5}} &\lesssim ||\partial_t^j(|u|^{p-1}u)||_{L^2_\tau L^{6/5}}, \end{aligned}$$

therefore, putting all together, we get

$$||\tilde{\varphi}(-\Delta_g)\partial_t^j u||_{L^2_\tau L^6} \lesssim ||\partial_t^j u||_{L^\infty_\tau L^2} + ||\partial_t^j u||_{L^2_\tau L^2} + ||\partial_t^j(|u|^{p-1}u)||_{L^2_\tau L^{6/5}}.$$

Since u solves (3.11), then $\varphi(-h^2\Delta_g)\partial_t^j u$ is a solution to the Cauchy problem

$$\begin{cases} i\partial_t v + \Delta_g v = \varphi(-h^2\Delta_g)\partial_t^j(|u|^{p-1}u), \quad (t,x) \in \mathbb{R} \times M^3, \\ v(0,\cdot) = \varphi(-h^2\Delta_g)\partial_t^j u(0,\cdot), \end{cases}$$

therefore, due to Lemma 3.2.6, we have

$$\begin{aligned} ||\varphi(-h^2\Delta_g)\partial_t^j u||^2_{L^2_\tau L^6} \lesssim ||\varphi(-h^2\Delta_g)\partial_t^j u||^2_{L^\infty_\tau L^2} + h^{-1}||\varphi(-h^2\Delta_g)\partial_t^j u||^2_{L^2_\tau L^2} \\ +||\varphi(-h^2\Delta_g)\partial_t^j(|u|^{p-1}u)||_{L^2_\tau L^{6/5}}. \end{aligned}$$

By using Proposition 3.1.6, for every $\epsilon \in (0,1)$ we have

$$\begin{aligned} \sum_{h^{-1}\in 2^{\mathbb{N}}} ||\varphi(-h^2\Delta_g)\partial_t^j u||^2_{L^2} &= \sum_{h^{-1}\in 2^{\mathbb{N}}} h^{2\epsilon}h^{-2\epsilon}||\varphi(-h^2\Delta_g)\partial_t^j u||^2_{L^2} \\ &\lesssim \sum_{m^{-1}\in 2^{\mathbb{N}}} m^{2\epsilon} \sum_{h^{-1}\in 2^{\mathbb{N}}} h^{-2\epsilon}||\varphi(-h^2\Delta_g)\partial_t^j u||^2_{L^2} \\ &\lesssim \sum_{m^{-1}\in 2^{\mathbb{N}}} m^{2\epsilon}||\partial_t^j u||^2_{H^\epsilon} \sim_\epsilon ||\partial_t^j u||^2_{H^\epsilon}, \end{aligned}$$

and

$$\sum_{h^{-1}\in 2^{\mathbb{N}}} h^{-1}||\varphi(-h^2\Delta_g)\partial_t^j u||^2_{L^2} \lesssim ||\partial_t^j u||^2_{H^{1/2}},$$

therefore

$$\sum_{h^{-1}\in 2^{\mathbb{N}}} ||\varphi(-h^2\Delta_g)\partial_t^j u||^2_{L^\infty_\tau L^2} \lesssim_\epsilon ||\partial_t^j u||^2_{L^\infty_\tau H^\epsilon} \tag{3.12}$$

$$\sum_{h^{-1}\in 2^{\mathbb{N}}} h^{-1}||\varphi(-h^2\Delta_g)\partial_t^j u||^2_{L^2_\tau L^2} \lesssim ||\partial_t^j u||^2_{L^2_\tau L^2}. \tag{3.13}$$

Moreover we observe that

$$\begin{aligned} &\sum_{h^{-1}\in 2^{\mathbb{N}}} ||\varphi(-h^2\Delta_g)\partial_t^j(|u|^{p-1}u)||^2_{L^{6/5}} \\ &\lesssim \left(\int_{M^3} \sum_{h^{-1}\in 2^{\mathbb{N}}} |\varphi(-h^2\Delta_g)\partial_t^j(|u|^{p-1}u)|^{6/5}\, dvol_g\right)^{5/3} \end{aligned}$$

$$\lesssim \left(\int_{M^3} \left(\sum_{h^{-1} \in 2^{\mathbb{N}}} |\varphi(-h^2 \Delta_g) \partial_t^j (|u|^{p-1} u)|^2 \right)^{3/5} dvol_g \right)^{5/3},$$

hence, by using (3.4), we deduce that

$$\sum_{h^{-1} \in 2^{\mathbb{N}}} ||\varphi(-h^2 \Delta_g) \partial_t^j (|u|^{p-1} u)||^2_{L^2_\tau L^{6/5}} \lesssim ||\partial_t^j (|u|^{p-1} u)||^2_{L^2_\tau L^{6/5}}. \tag{3.14}$$

Finally, from (3.12), (3.13), (3.14), we obtain

$$\sum_{h^{-1} \in 2^{\mathbb{N}}} ||\varphi(-h^2 \Delta_g) u||^2_{L^2_\tau L^6} \lesssim_\epsilon ||\partial_t^j u||^2_{L^\infty_\tau H^\epsilon} + ||\partial_t^j u||^2_{L^2_\tau H^{1/2}} + ||\partial_t^j (|u|^{p-1} u)||^2_{L^2_\tau L^{6/5}}$$

and we easily deduce the statement of the proposition. □

Chapter 4

Growth of H^{2k} Sobolev Norms of Solutions to NLS on Closed Riemannian Manifolds

In the previous chapter we saw that, given a generic d-dimensional closed Riemannian manifold (M^d, g), the nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta_g u = |u|^{p-1}u, & (t,x) \in \mathbb{R} \times M^d \\ u(0,x) = u_0 \in H^m(M^d) \end{cases}$$

is globally well-posed under certain conditions on d (dimension of the manifold), p (nonlinearity) and m (regularity of the initial datum). In particular the following families of Cauchy problems admit a unique global solution which preserves the regularity along the evolution:

- 2-dimensional manifold and odd integer nonlinearity

$$\begin{cases} i\partial_t u + \Delta_g u = |u|^{p-1}u, & (t,x) \in \mathbb{R} \times M^2, \\ u(0,\cdot) = u_0 \in H^m(M^2), \end{cases} \tag{4.1}$$

  where $m \geq 2$, $m \in \mathbb{N}$ and $p = 2n+1$, $n \in \mathbb{N}$.

- 3-dimensional manifold and cubic nonlinearity

$$\begin{cases} i\partial_t u + \Delta_g u = |u|^2 u, & (t,x) \in \mathbb{R} \times M^3, \\ u(0,\cdot) = u_0 \in H^m(M^3), \end{cases} \tag{4.2}$$

where $m \geq 2$, $m \in \mathbb{N}$.

- 3-dimensional manifold and subcubic nonlinearity

$$\begin{cases} i\partial_t u + \Delta_g u = |u|^{p-1} u, & (t,x) \in \mathbb{R} \times M^3, \\ u(0,\cdot) = u_0 \in H^2(M^3), \end{cases} \tag{4.3}$$

where $p \in (2,3)$.

In the following exposition we want to provide a priori bounds on the growth in time of the Sobolev norms of the solutions to these families of NLS. We shall focus on the Cauchy problems (4.1) and (4.2) by studying independently the two different cases "m odd" (Chapter 4) and "m even" (Chapter 5), then we shall examine the subcubic problem (4.3) (Chapter 6). Let us say right away that the growth in time of these Sobolev norms is only of interest for $m \geq 2$. In fact, given $u \in C(\mathbb{R}, H^m(M^d))$ the unique global solution to either (4.1), (4.2) or (4.3), for every $t \in \mathbb{R}$, the conservation laws give

$$||\nabla_g u(t,\cdot)||^2_{L^2(M^d)} \leq ||\nabla_g u_0||^2_{L^2(M^d)} + \frac{2}{p+1}||u_0||^{p+1}_{L^{p+1}(M^d)},$$
$$||u(t,\cdot)||^2_{L^2(M^d)} = ||u_0||^2_{L^2(M^d)},$$

therefore, by using the Sobolev embedding $H^1(M^d) \hookrightarrow L^{p+1}(M^d)$, we get

$$\sup_{t\in\mathbb{R}} ||u(t,\cdot)||_{H^1(M^d)} \lesssim_{||u_0||_{H^1}} 1 \tag{4.4}$$

proving that the map $\mathbb{R}^+ \ni t \longmapsto ||u(t,\cdot)||_{H^1(M^d)}$ is bounded from above by a constant. Since we deal with positive integers, the meaningful cases are $m \geq 2$.

Remark 4.0.1. *Let $u \in C(\mathbb{R}, H^m(M^d))$ be the unique global solution to either (4.1), (4.2) or (4.3) and $\tau \in (0,1)$. For any $s \geq 1$ and $\gamma \leq \delta$, we have*

$$||u||^{\gamma}_{L^{\infty}_{\tau}H^s(M^d)} \lesssim_{||u_0||_{H^1}} ||u||^{\delta}_{L^{\infty}_{\tau}H^s(M^d)}. \tag{4.5}$$

Indeed, due to the Sobolev embedding $H^s(M^d) \hookrightarrow H^1(M^d)$, we have

$$||u||^{\gamma}_{L^{\infty}_{\tau}H^s}||u_0||^{\delta-\gamma}_{H^1} \lesssim ||u||^{\gamma}_{L^{\infty}_{\tau}H^s}||u_0||^{\delta-\gamma}_{H^s} \lesssim ||u||^{\gamma}_{L^{\infty}_{\tau}H^s}||u||^{\delta-\gamma}_{L^{\infty}_{\tau}H^s} = ||u||^{\delta}_{L^{\infty}_{\tau}H^s}.$$

We shall use (4.5) in the following computations without further notice.

4.1 Comparable Norms

In this section we prove fundamental comparisons between Sobolev norms that we will repeatedly use both in this chapter and in the following one.

Lemma 4.1.1. *Let $u \in C(\mathbb{R}, H^m(M^d))$ be the unique global solution to either (4.1) or (4.2). For every $j \in \mathbb{N}$, we have*

$$\partial_t^j u = i^j \Delta_g^j u + \sum_{h=0}^{j-1} c_h \; \partial_t^h \Delta_g^{j-h-1}(|u|^{p-1}u), \tag{4.6}$$

where c_h are suitable complex coefficients.

Proof. We argue by induction on j:
(Base Case) Trivial.
(Induction Step) By assuming (4.6) holds for j, we have that

$$\partial_t^{j+1} u = \partial_t(\partial_t^j u) = i^j \partial_t \Delta_g^j u + \sum_{h=0}^{j-1} c_h \; \partial_t^{h+1} \Delta_g^{j-h-1}(|u|^{p-1}u)$$

Since u solves (4.1) or (4.2) and ∂_t^j and Δ_g^j commute, we get

$$i^j \Delta_g^j(\partial_t u) = i^{j+1}\Delta_g^{j+1} u - i^{j+1}\Delta_g^j(|u|^{p-1}u),$$

therefore

$$\begin{aligned}\partial_t^{j+1}u &= i^{j+1}\Delta_g^{j+1}u - i^{j+1}\Delta_g^{j}(|u|^{p-1}u) + \sum_{h=0}^{j-1} c_h\ \partial_t^{h+1}\Delta_g^{j-h-1}(|u|^{p-1}u)\\ &= i^{j+1}\Delta_g^{j+1}u - i^{j+1}\Delta_g^{j}(|u|^{p-1}u) + \sum_{h=1}^{j} c_h\ \partial_t^{h}\Delta_g^{j-h}(|u|^{p-1}u)\\ &= i^{j+1}\Delta_g^{j+1}u + \sum_{h=0}^{j} c_h\ \partial_t^{h}\Delta_g^{j-h}(|u|^{p-1}u),\end{aligned}$$

where $c_0 = -i^{j+1}$. □

Proposition 4.1.2. *Let $u \in C(\mathbb{R}, H^m(M^2))$ be the unique global solution to (4.1). For every $j \in \mathbb{N}$ and $s \in \mathbb{N} \cup \{0\}$, we have*

$$||\partial_t^j u - i^j\Delta_g^j u||_{H^s(M^2)} \lesssim_{||u_0||_{H^1}} ||u||_{H^{s+2j-1}(M^2)} \tag{4.7}$$

Proof. We argue by induction on $j \in \mathbb{N}$.
(Base Case) Since u solves (4.1), due to Remark 1.2.10 and the Hölder inequality, we have

$$||\partial_t u - i\Delta_g u||_{H^s} \sim ||(|u|^{p-1}u)||_{W^{s,2}} \lesssim \sum_{s_1+\ldots+s_p=s}\left(\prod_{i=1}^{p}||u||_{W^{s_i,2p}}\right). \tag{4.8}$$

By using the Sobolev embedding $H^{s_i+1}(M^2) \hookrightarrow W^{s_i,2p}(M^2)$ and (1.1), we get

$$(4.8) \lesssim \sum_{s_1+\ldots+s_p=s}\left(\prod_{i=1}^{p}||u||_{H^{s_i+1}}\right) \lesssim \sum_{s_1+\ldots+s_p=s}\left(\prod_{i=1}^{p}||u||_{H^{s+1}}^{\frac{s_i}{s}}||u||_{H^1}^{1-\frac{s_i}{s}}\right).$$

Finally, due to (4.4), we have

$$||\partial_t u - i\Delta_g u||_{H^s} \lesssim_{||u_0||_{H^1}} \sum_{s_1+\ldots+s_p=s}\left(\prod_{i=1}^{p}||u||_{H^{s+1}}^{\frac{s_i}{s}}\right) \lesssim ||u||_{H^{s+1}}.$$

(Induction Step) From Lemma 4.1.1, we have

$$\begin{aligned}||\partial_t^{j+1}u - i^{j+1}\Delta_g^{j+1}u||_{H^s} &\lesssim \sum_{h=0}^{j}||\partial_t^h\Delta_g^{j-h}(|u|^{p-1}u)||_{H^s}\\ &\lesssim \sum_{h=0}^{j}||\partial_t^h(|u|^{p-1}u)||_{H^{2j-2h+s}}\end{aligned} \tag{4.9}$$

By expanding the time derivative, due to Remark 1.2.10, we get

$$(4.9) \lesssim \sum_{h=0}^{j} \sum_{h_1+\ldots+h_p=h} \left|\left| \left(\prod_{i=1,\cdots,\frac{p-1}{2}} \partial_t^{h_i}\overline{u} \right) \left(\prod_{i=\frac{p-1}{2}+1,\cdots,p} \partial_t^{h_i} u \right) \right|\right|_{W^{2j-2h+s,2}},$$

therefore, by using the Hölder inequality, we obtain

$$||\partial_t^{j+1}u - i^{j+1}\Delta_g^{j+1}u||_{H^s} \lesssim \sum_{h=0}^{j} \sum_{\substack{h_1+\cdots+h_p=h \\ s_1+\cdots+s_p=2j-2h+s}} \left(\prod_{i=1}^{p} ||\partial_t^{h_i}u||_{W^{s_i,2p}} \right). \quad (4.10)$$

Further to this, the Sobolev embedding $H^{s_i+1}(M^2) \hookrightarrow W^{s_i,2p}(M^2)$ gives

$$(4.10) \lesssim \sum_{h=0}^{j} \sum_{\substack{h_1+\cdots+h_p=h \\ s_i+\cdots+s_p=2j-2h+s}} \left(\prod_{i=1}^{p} ||\partial_t^{h_i}u||_{H^{s_i+1}} \right). \quad (4.11)$$

Note that, for every $i \in \{1, \cdots, p\}$, we have that $h_i \leq j$ then, by using the inductive hypothesis, we obtain

$$||\partial_t^{h_i}u - i^{h_i}\Delta_g^{h_i}u||_{H^{s_i+1}} \lesssim_{||u_0||_{H^1}} ||u||_{H^{s_i+2h_i}},$$

therefore

$$\begin{aligned} ||\partial_t^{h_i}u||_{H^{s_i+1}} &\lesssim ||\partial_t^{h_i}u - i^{h_i}\Delta_g^{h_i}u||_{H^{s_i+1}} + ||i^{h_i}\Delta_g^{h_i}u||_{H^{s_i+1}} \\ &\lesssim_{||u_0||_{H^1}} ||u||_{H^{s_i+2h_i}} + ||u||_{H^{s_i+2h_i+1}} \lesssim ||u||_{H^{s_i+2h_i+1}}. \quad (4.12) \end{aligned}$$

By using (1.1) and (4.4), we get

$$(4.12) \lesssim ||u||_{H^{s+2j+1}}^{\frac{2h_i+s_i}{2j+s}} ||u||_{H^1}^{1-\frac{2h_i+s_i}{2j+s}} \lesssim_{||u_0||_{H^1}} ||u||_{H^{s+2j+1}}^{\frac{2h_i+s_i}{2j+s}},$$

hence

$$(4.11) \lesssim_{||u_0||_{H^1}} \sum_{h=0}^{j} \sum_{\substack{h_1+\cdots+h_p=h \\ s_1+\cdots+s_p=2j-2h+s}} \left(\prod_{i=1}^{p} ||u||_{H^{s+2j+1}}^{\frac{2h_i+s_i}{2j+s}} \right) \lesssim ||u||_{H^{s+2j+1}},$$

which concludes the induction step. □

Corollary 4.1.3. *Let $u \in C(\mathbb{R}, H^m(M^2))$ be the unique global solution to (4.1). For every $j \in \mathbb{N}$ and $s \in \mathbb{N} \cup \{0\}$, we have*

$$||\partial_t^j u||_{H^s(M^2)} \lesssim_{||u_0||_{H^1}} ||u||_{H^{s+2j}(M^2)}. \tag{4.13}$$

Proof. By using (4.7), we get

$$\begin{aligned} ||\partial_t^j u||_{H^s} &\lesssim ||\partial_t^j u - i^j \Delta_g^j u||_{H^s} + ||i^j \Delta_g^j u||_{H^s} \\ &\lesssim_{||u_0||} ||u||_{H^{s+2j-1}} + ||i^j \Delta_g^j u||_{H^s} \lesssim ||u||_{H^{s+2j}}, \end{aligned}$$

proving the statement. □

Proposition 4.1.4. *Let $u \in C(\mathbb{R}, H^m(M^3))$ be the unique global solution to (4.2). For every $j \in \mathbb{N}$ and $s \in \mathbb{N} \cup \{0\}$, we have*

$$||\partial_t^j u - i^j \Delta_g^j u||_{H^s(M^3)} \lesssim_{||u_0||_{H^1}} ||u||_{H^{s+2j-1}(M^3)} \tag{4.14}$$

Proof. See Proposition 4.1.2. □

Corollary 4.1.5. *Let $u \in C(\mathbb{R}, H^m(M^3))$ be the unique global solution to (4.2). For every $j \in \mathbb{N}$ and $s \in \mathbb{N} \cup \{0\}$, we have*

$$||\partial_t^j u||_{H^s(M^3)} \lesssim_{||u_0||_{H^1}} ||u||_{H^{s+2j}(M^3)} \tag{4.15}$$

Proof. See Corollary 4.1.3. □

4.2 Modified Energies

In this section we introduce suitable "modified energy" for the Cauchy problems (4.1) and (4.2) with $m = 2k$, $k \in \mathbb{N}$ (even-integer Sobolev regularity of the initial datum). These energies will be crucial to establish a priori bounds on the growth of higher order Sobolev norms $||\cdot||_{H^{2k}}$.

Definition 4.2.1. *Let $u \in C(\mathbb{R}, H^{2k}(M^d))$ be the unique global solution to either (4.1) or (4.2) with $m = 2k$, $k \in \mathbb{N}$. We define the modified energy*

$$\mathcal{E}_{2k}(u) := ||\partial_t^k u||^2_{L^2(M^d)} + \mathcal{R}_{2k}(u)$$

where

$$\mathcal{R}_{2k}(u) := -\frac{p-1}{4}\int_{M^d} |\partial_t^{k-1}\nabla_g(|u|^2)|_g^2 |u|^{p-3} dvol_g - \int_{M^d} |\partial_t^{k-1}(|u|^{p-1}u)|^2 dvol_g.$$

Proposition 4.2.2. *Let $u \in C(\mathbb{R}, H^{2k}(M^d))$ be the unique global solution to either (4.1) or (4.2) with $m = 2k$, $k \in \mathbb{N}$. We have*

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{2k}(u) = &-\frac{p-1}{4}\int_{M^d} |\partial_t^{k-1}\nabla_g(|u|^2)|_g^2\ \partial_t(|u|^{p-3})\ dvol_g + \\ &+ 2\int_{M^d} \partial_t^k(|u|^{p-1})\ \partial_t^{k-1}(|\nabla_g u|_g^2)\ dvol_g + \\ &+ \sum_{j=0}^{k=1} c_j \int_{M^d} (\partial_t^j\nabla_g(|u|^2), \partial_t^{k-1}\nabla_g(|u|^2))_g\ \partial_t^{k-j}(|u|^{p-3})\ dvol_g + \\ &+ \mathrm{Re}\sum_{j=0}^{k-1} c_j \int_{M^d} \partial_t^j(|u|^{p-1})\ \partial_t^{k-j}u\ \partial_t^{k-1}(|u|^{p-1}\overline{u})\ dvol_g + \\ &+ \mathrm{Re}\sum_{j=0}^{k-2} c_j \int_{M^d} \partial_t^k(|u|^{p-1})\ \partial_t^j(\Delta_g\overline{u})\ \partial_t^{k-1-j}u\ dvol_g + \\ &+ \mathrm{Im}\sum_{j=1}^{k-1} c_j \int_{M^d} \partial_t^j(|u|^{p-1})\ \partial_t^{k-j}u\ \partial_t^k\overline{u}\ dvol_g \end{aligned} \tag{4.16}$$

where c_j denote real constants that may change from line to line.

Proof. We observe that

$$\frac{d}{dt}||\partial_t^k u||_{L^2}^2 = 2\mathrm{Re}\left(\int_{M^d} \partial_t^{k+1}u\ \partial_t^k\overline{u}\ dvol_g\right), \tag{4.17}$$

and that, since u solves (4.1) or (4.2), we have

$$\begin{aligned}(4.17) &= 2\mathrm{Re}\left(\int_{M^d} i\partial_t^k(\Delta_g u)\ \partial_t^k\overline{u}\ dvol_g\right) + \\ &\quad + 2\mathrm{Re}\left(\int_{M^d} -i\partial_t^k(|u|^{p-1}u)\ \partial_t^k\overline{u}\ dvol_g\right) \\ &= 2\mathrm{Re}\left(\int_{M^d} -i\partial_t^k(|u|^{p-1}u)\ \partial_t^k\overline{u}\ dvol_g\right).\end{aligned}$$

Indeed, integration by parts gives

$$2\mathrm{Re}\left(\int_{M^d} i\partial_t^k(\Delta_g u)\ \partial_t^k\overline{u}\ dvol_g\right) = -2\mathrm{Re}\left(i\int_{M^d} |\nabla_g\partial_t^k u|_g^2\ dvol_g\right) = 0.$$

Since

$$\partial_t^k(|u|^{p-1}u) = \partial_t^k(|u|^{p-1})u + |u|^{p-1}\partial_t^k u + \sum_{j=1}^{k-1} c_j\ \partial_t^j(|u|^{p-1})\ \partial_t^{k-j}u, \tag{4.18}$$

where c_j denote suitable real constants that may change in the following computations, we get

$$\begin{aligned}\frac{d}{dt}||\partial_t^k u||_{L^2}^2 =& 2\mathrm{Re}\left(\int_{M^d} -iu\ \partial_t^k(|u|^{p-1})\ \partial_t^k\overline{u}\ dvol_g\right) + \\ &+ \mathrm{Im}\sum_{j=1}^{k-1} c_j\int_{M^d} \partial_t^j(|u|^{p-1})\ \partial_t^{k-j}u\ \partial_t^k\overline{u}\ dvol_g.\end{aligned}$$

By using that u solves (4.1) or (4.2), we have

$$\begin{aligned}&2\mathrm{Re}\left(\int_{M^d} -iu\ \partial_t^k(|u|^{p-1})\ \partial_t^k\overline{u}\ dvol_g\right)\\ &= -2\mathrm{Re}\left(\int_{M^d} u\ \partial_t^k(|u|^{p-1})\ \Delta_g(\partial_t^{k-1}\overline{u})\ dvol_g\right) + \\ &\quad + 2\mathrm{Re}\left(\int_{M^d} u\ \partial_t^k(|u|^{p-1})\ \partial_t^{k-1}(|u|^{p-1}\overline{u})\ dvol_g\right).\end{aligned}$$

Now, using (4.18) again, we obtain

$$\begin{aligned}&2\mathrm{Re}\left(\int_{M^d} -iu\ \partial_t^k(|u|^{p-1})\ \partial_t^k\overline{u}\ dvol_g\right)\\ &= -2\mathrm{Re}\left(\int_{M^d} u\ \partial_t^k(|u|^{p-1})\ \Delta_g(\partial_t^{k-1}\overline{u})\ dvol_g\right) + \\ &\quad + 2\mathrm{Re}\left(\int_{M^d} \partial_t^k(|u|^{p-1}u)\ \partial_t^{k-1}(|u|^{p-1}\overline{u})\ dvol_g\right) + \\ &\quad + \mathrm{Re}\sum_{j=0}^{k-1} c_j\int_{M^d} \partial_t^j(|u|^{p-1})\ \partial_t^{k-j}u\ \partial_t^{k-1}(|u|^{p-1}\overline{u})\ dvol_g.\end{aligned}$$

Putting all together, we get

$$\begin{aligned}\frac{d}{dt}||\partial_t^k u||_{L^2}^2 =& -2\mathrm{Re}\left(\int_{M^d} u\ \partial_t^k(|u|^{p-1})\ \Delta_g(\partial_t^{k-1}\overline{u})\ dvol_g\right)+\\ &+2\mathrm{Re}\left(\int_{M^d}\ \partial_t^k(|u|^{p-1}u)\ \partial_t^{k-1}(|u|^{p-1}\overline{u})\ dvol_g\right)+\\ &+\mathrm{Im}\sum_{j=1}^{k-1} c_j\int_{M^d}\partial_t^j(|u|^{p-1})\ \partial_t^{k-j}u\ \partial_t^k\overline{u}\ dvol_g+\\ &+\mathrm{Re}\sum_{j=0}^{k-1} c_j\int_{M^d}\partial_t^j(|u|^{p-1})\ \partial_t^{k-j}u\ \partial_t^{k-1}(|u|^{p-1}\overline{u})\ dvol_g.\end{aligned}$$

We observe that

$$\frac{d}{dt}|\partial_t^{k-1}(|u|^{p-1}u)|^2 = 2\mathrm{Re}\left(\partial_t^k(|u|^{p-1}u)\ \partial_t^{k-1}(|u|^{p-1}\overline{u})\right),$$

therefore

$$\begin{aligned}\frac{d}{dt}||\partial_t^k u||_{L^2}^2 =& -2\mathrm{Re}\left(\int_{M^d} u\ \partial_t^k(|u|^{p-1})\ \Delta_g(\partial_t^{k-1}\overline{u})\ dvol_g\right)+\\ &+\frac{d}{dt}\int_{M^d}|\partial_t^{k-1}(|u|^{p-1}u)|^2\ dvol_g+\\ &+\mathrm{Im}\sum_{j=1}^{k-1} c_j\int_{M^d}\partial_t^j(|u|^{p-1})\ \partial_t^{k-j}u\ \partial_t^k\overline{u}\ dvol_g+\\ &+\mathrm{Re}\sum_{j=0}^{k-1} c_j\int_{M^d}\partial_t^j(|u|^{p-1})\ \partial_t^{k-j}u\ \partial_t^{k-1}(|u|^{p-1}\overline{u})\ dvol_g.\end{aligned}$$

To prove the result we just have to focus on the term

$$(\mathrm{I}) = -2\mathrm{Re}\left(\int_{M^d} u\ \partial_t^k(|u|^{p-1})\ \Delta_g(\partial_t^{k-1}\overline{u})\ dvol_g\right).$$

Note that

$$\Delta_g(|u|^2) = 2\mathrm{Re}(u\Delta_g\overline{u}) + 2|\nabla_g u|_g^2,$$

hence, by expanding the time derivative, we have

$$\begin{aligned}\partial_t^{k-1}\Delta_g(|u|^2) = 2\mathrm{Re}\left(u\Delta_g(\partial_t^{k-1}\overline{u})\right) + \mathrm{Re}\sum_{j=0}^{k-2} c_j\ \partial_t^j(\Delta_g\overline{u})\ \partial_t^{k-j-1}u+\\ +2\ \partial_t^{k-1}(|\nabla_g u|_g^2)\end{aligned} \tag{4.19}$$

where c_j denote suitable real constants that may change in the following computations. Using (4.19) and the integration by parts Theorem, we get

$$\begin{aligned}(\mathrm{I}) = &\int_{M^d} (\partial_t^k(\nabla_g(|u|^{p-1})), \partial_t^{k-1}(\nabla_g(|u|^2)))_g \, dvol_g + \\ &+ 2\int_{M^d} \partial_t^k(|u|^{p-1}) \, \partial_t^{k-1}(|\nabla_g u|_g^2) \, dvol_g + \\ &+ \mathrm{Re}\sum_{j=0}^{k-2} c_j \int_{M^d} \partial_t^k(|u|^{p-1}) \, \partial_t^j(\Delta_g \overline{u}) \, \partial_t^{k-j-1} u \, dvol_g \end{aligned} \tag{4.20}$$

therefore, the identity

$$\nabla_g(|u|^{p-1}) = \frac{p-1}{2}|u|^{p-3}\nabla_g(|u|^2)$$

gives

$$\begin{aligned}(4.20) = &\frac{p-1}{2}\int_{M^d} (\partial_t^k(|u|^{p-3}\nabla_g(|u|^2)), \partial_t^{k-1}(\nabla_g(|u|^2)))_g \, dvol_g + \\ &+ 2\int_{M^d} \partial_t^k(|u|^{p-1}) \, \partial_t^{k-1}(|\nabla_g u|_g^2) \, dvol_g + \\ &+ \mathrm{Re}\sum_{j=0}^{k-2} c_j \int_{M^d} \partial_t^k(|u|^{p-1}) \, \partial_t^j(\Delta_g \overline{u}) \, \partial_t^{k-1-j} u \, dvol_g. \end{aligned}$$

By expanding the time derivative we obtain

$$\partial_t^k(|u|^{p-3}\nabla_g(|u|^2)) = |u|^{p-3}\partial_t^k(\nabla_g(|u|^2)) + \sum_{j=0}^{k-1} c_j \, \partial_t^j(\nabla_g(|u|^2)) \, \partial_t^{k-j}(|u|^{p-3})$$

therefore

$$\begin{aligned}(\mathrm{I}) = &\frac{p-1}{2}\int_{M^d} (\partial_t^k(\nabla_g(|u|^2)), \partial_t^{k-1}(\nabla_g(|u|^2)))_g \, |u|^{p-3} \, dvol_g + \\ &+ \sum_{j=0}^{k-1} c_j \int_{M^d} (\partial_t^j(\nabla_g(|u|^2)), \partial_t^{k-1}(\nabla_g(|u|^2)))_g \, \partial_t^{k-j}(|u|^{p-3}) \, dvol_g + \\ &+ 2\int_{M^d} \partial_t^k(|u|^{p-1}) \, \partial_t^{k-1}(|\nabla_g u|_g^2) \, dvol_g + \\ &+ \mathrm{Re}\sum_{j=0}^{k-2} c_j \int_{M^d} \partial_t^k(|u|^{p-1}) \, \partial_t^j(\Delta_g \overline{u}) \, \partial_t^{k-1-j} u \, dvol_g. \end{aligned} \tag{4.21}$$

Finally

$$
\begin{aligned}
(4.21) = \frac{p-1}{4}\int_{M^d} & \partial_t|\partial_t^{k-1}(\nabla_g(|u|^2))|_g^2\ |u|^{p-3}\ dvol_g+\\
&+\sum_{j=0}^{k-1} c_j\int_{M^d}(\partial_t^j(\nabla_g(|u|^2)),\partial_t^{k-1}(\nabla_g(|u|^2)))_g\ \partial_t^{k-j}(|u|^{p-3})\ dvol_g+\\
&+2\int_{M^d}\partial_t^k(|u|^{p-1})\ \partial_t^{k-1}(|\nabla_g u|_g^2)\ dvol_g+\\
&+\mathrm{Re}\sum_{j=0}^{k-2} c_j\int_{M^d}\partial_t^k(|u|^{p-1})\ \partial_t^j(\Delta_g\overline{u})\ \partial_t^{k-1-j}u\ dvol_g
\end{aligned}
$$

since

$$
2(\partial_t^k(\nabla_g(|u|^2)),\partial_t^{k-1}(\nabla_g(|u|^2)))_g=\partial_t|\partial_t^{k-1}(\nabla_g(|u|^2))|_g^2.
$$

Putting all together we obtain

$$
\begin{aligned}
\frac{d}{dt}||\partial_t^k u||_{L^2}^2 = 2\int_{M^d} & \partial_t^k(|u|^{p-1})\ \partial_t^{k-1}(|\nabla_g u|_g^2)\ dvol_g+\\
&+\frac{p-1}{4}\int_{M^d}\partial_t|\partial_t^{k-1}(\nabla_g(|u|^2))|_g^2\ |u|^{p-3}\ dvol_g+\\
&+\sum_{j=0}^{k-1} c_j\int_{M^d}(\partial_t^j(\nabla_g(|u|^2)),\partial_t^{k-1}(\nabla_g(|u|^2)))_g\ \partial_t^{k-j}(|u|^{p-3})\ dvol_g+\\
&+\mathrm{Re}\sum_{j=0}^{k-2} c_j\int_{M^d}\partial_t^k(|u|^{p-1})\ \partial_t^j(\Delta_g\overline{u})\ \partial_t^{k-1-j}u\ dvol_g+\\
&+\frac{d}{dt}\int_{M^d}|\partial_t^{k-1}(|u|^{p-1}u)|^2\ dvol_g+\\
&+\mathrm{Im}\sum_{j=1}^{k-1} c_j\int_{M^d}\partial_t^j(|u|^{p-1})\ \partial_t^{k-j}u\ \partial_t^k\overline{u}\ dvol_g\\
&+\mathrm{Re}\sum_{j=0}^{k-1} c_j\int_{M^d}\partial_t^j(|u|^{p-1})\ \partial_t^{k-j}u\ \partial_t^{k-1}(|u|^{p-1}\overline{u})\ dvol_g. \qquad (4.22)
\end{aligned}
$$

By taking the time derivative of the modified energy $\mathcal{E}_{2k}(u)$ and replacing (4.22) in it we have the statement. □

Remark 4.2.3. *Let u be as in Proposition 4.2.2. If $p=3$, we have*

$$
\mathcal{R}_{2k}(u) = -\frac{1}{2}\int_{M^d}|\partial_t^{k-1}\nabla_g(|u|^2)|_g^2\ dvol_g-\int_{M^d}|\partial_t^{k-1}(|u|^2u)|^2\ dvol_g,
$$

$$\frac{d}{dt}\mathcal{E}_{2k}(u) = 2\int_{M^d} \partial_t^k(|u|^2)\ \partial_t^{k-1}(|\nabla_g u|_g^2)\ dvol_g +$$
$$+ \operatorname{Re}\sum_{j=0}^{k-2} c_j \int_{M^d} \partial_t^k(|u|^2)\ \partial_t^j(\Delta_g \overline{u})\ \partial_t^{k-1-j} u\ dvol_g +$$
$$+ \operatorname{Re}\sum_{j=0}^{k-1} c_j \int_{M^d} \partial_t^j(|u|^2)\ \partial_t^{k-j} u\ \partial_t^{k-1}(|u|^2\overline{u})\ dvol_g +$$
$$+ \operatorname{Im}\sum_{j=1}^{k-1} c_j \int_{M^d} \partial_t^j(|u|^2)\ \partial_t^{k-j} u\ \partial_t^k \overline{u}\ dvol_g. \tag{4.23}$$

4.3 The 2-Dimensional Case $(m = 2k)$

The aim of this section is to provide a bound on the growth of the continuous function

$$\mathbb{R}^+ \ni t \longmapsto ||u(t,\cdot)||_{H^{2k}(M^2)} \tag{4.24}$$

where $k \in \mathbb{N}$ and u is the unique global solution to the Cauchy problem (4.1) with $m = 2k$. As a first step we shall prove "modified" Strichartz estimates that we use to establish a priori bounds to small-time increments for (4.24). Finally, we take advantage of an iterative argument to show the polynomial growth of (4.24).

Notation 4.3.1. *In the following computations we call ϵ a positive quantity that may change from line to line. We shall assume ϵ to be small enough at each step.*

Proposition 4.3.2. *Let $u \in C(\mathbb{R}, H^{2k}(M^2))$ be the unique global solution to (4.1) with $m = 2k$, $k \in \mathbb{N}$. For every $\tau \in (0,1)$ and $j \le k$, we have*

$$||\partial_t^j u||_{L^4_\tau W^{s,4}(M^2)} \lesssim_{\epsilon,\ ||u_0||_{H^1}} ||u||^{1-s_0}_{L^\infty_\tau H^{2j+s}(M^2)} ||u||^{s_0}_{L^\infty_\tau H^{2j+s+1}(M^2)} ||u||^{\epsilon}_{L^\infty_\tau H^{2j+2}(M^2)}, \tag{4.25}$$

where $s_0 \in [0, \frac{1}{4}]$ is given by Proposition 3.2.3.

Proof. From Proposition 3.2.3 we have

$$||\partial_t^j u||_{L^4_\tau W^{s,4}} \lesssim ||\partial_t^j u(0,\cdot)||_{H^{s+s_0}} + \tau ||\partial_t^j(|u|^{p-1}u)||_{L^\infty_\tau H^{s+s_0}}. \tag{4.26}$$

By using (1.1) and (4.13), we get

$$\begin{aligned}||\partial_t^j u(0,\cdot)||_{H^{s+s_0}} &\lesssim ||\partial_t^j u(0,\cdot)||_{H^s}^{1-s_0} ||\partial_t^j u(0,\cdot)||_{H^{s+1}}^{s_0} \\ &\lesssim_{||u_0||_{H^1}} ||u||_{L^\infty_\tau H^{2j+s}}^{1-s_0} ||u||_{L^\infty_\tau H^{2j+s+1}}^{s_0}.\end{aligned}$$

We now focus on the nonlinear term of (4.26). Due to (1.1), we have

$$||\partial_t^j (|u|^{p-1}u)||_{L^\infty_\tau H^{s+s_0}} \lesssim ||\partial_t^j (|u|^{p-1}u)||_{L^\infty_\tau H^s}^{1-s_0} ||\partial_t^j (|u|^{p-1}u)||_{L^\infty_\tau H^{s+1}}^{s_0}.$$

By expanding the time derivative, due to (1.4), we deduce that

$$||\partial_t^j (|u|^{p-1}u)||_{H^s} \lesssim \sum_{j_1+\ldots+j_p=j} \left(\prod_{i=2}^p ||\partial_t^{j_i} u||_{L^\infty} \right) ||\partial_t^{j_1} u||_{H^s}, \tag{4.27}$$

therefore, by using (1.2) and (4.13), we get

$$\begin{aligned}(4.27) &\lesssim_\epsilon \sum_{j_1+\ldots+j_p=j} \left(\prod_{i=2}^p ||\partial_t^{j_i} u||_{H^1}^{1-\epsilon} ||\partial_t^{j_i} u||_{H^2}^{\epsilon} \right) ||\partial_t^{j_1} u||_{H^s} \\ &\lesssim_{||u_0||_{H^1}} \sum_{j_1+\ldots+j_p=j} \left(\prod_{i=2}^p ||u||_{H^{2j_i+1}}^{1-\epsilon} ||u||_{H^{2j_i+2}}^{\epsilon} \right) ||u||_{H^{2j_1+s}}. \end{aligned}\tag{4.28}$$

Due to (1.1), we have

$$\begin{aligned}(4.28) \lesssim \sum_{j_1+\ldots+j_p=j} &\left(\prod_{i=2}^p ||u||_{H^{2j+s}}^{\frac{2j_i}{2j+s-1}(1-\epsilon)} ||u||_{H^1}^{\left(1-\frac{2j_1}{2j+s-1}\right)(1-\epsilon)} ||u||_{H^{2j_i+2}}^{\epsilon} \right) \times \\ &\times ||u||_{H^{2j+s}}^{\frac{2j_1+s-1}{2j+s-1}} ||u||_{H^1}^{\frac{2j-2j_1}{2j+s-1}},\end{aligned}\tag{4.29}$$

therefore, (4.4) gives

$$(4.29) \lesssim_{||u_0||_{H^1}} \sum_{j_1+\ldots+j_p=j} \left(\prod_{i=2}^p ||u||_{H^{2j+s}}^{\frac{2j_i}{2j+s-1}(1-\epsilon)} \right) ||u||_{H^{2j+s}}^{\frac{2j_1+s-1}{2j+s-1}} ||u||_{H^{2j+2}}^{\epsilon}.$$

By taking the supremum over $t \in (0,\tau)$ we obtain

$$\begin{aligned}||\partial_t^j (|u|^{p-1}u)||_{L^\infty_\tau H^s} \lesssim_{\epsilon,\ ||u_0||_{H^1}} \sum_{j_1+\ldots+j_p=j} &\left(\prod_{i=2}^p ||u||_{L^\infty_\tau H^{2j+s}}^{\frac{2j_i}{2j+s-1}(1-\epsilon)} \right) \times \\ &\times ||u||_{L^\infty_\tau H^{2j+s}}^{\frac{2j_1+s-1}{2j+s-1}} ||u||_{L^\infty_\tau H^{2j+2}}^{\epsilon},\end{aligned}\tag{4.30}$$

hence

$$(4.30) \lesssim \sum_{j_1+\ldots+j_p=j} \left(\prod_{i=2}^{p} ||u||_{L_\tau^\infty H^{2j+s}}^{\frac{2j_i}{2j+s-1}} \right) ||u||_{L_\tau^\infty H^{2j+s}}^{\frac{2j_1+s-1}{2j+s-1}} ||u||_{L_\tau^\infty H^{2j+2}}^{\epsilon}$$
$$\lesssim ||u||_{L_\tau^\infty H^{2j+s+1}} ||u||_{L_\tau^\infty H^{2j+2}}^{\epsilon}.$$

Similarly, we get

$$||\partial_t^j (|u|^{p-1}u)||_{L_\tau^\infty H^{s+1}} \lesssim_{\epsilon,\ ||u_0||_{H^1}} ||u||_{L_\tau^\infty H^{2j+s+1}} ||u||_{L_\tau^\infty H^{2j+2}}^{\epsilon},$$

therefore

$$||\partial_t^j (|u|^{p-1}u)||_{L_\tau^\infty H^{s+s_0}} \lesssim_{\epsilon,\ ||u_0||_{H^1}} ||u||_{L_\tau^\infty H^{2j+s}}^{1-s_0} ||u||_{L_\tau^\infty H^{2j+s+1}}^{s_0} ||u||_{L_\tau^\infty H^{2j+2}}^{\epsilon}.$$

Finally, putting all together

$$||\partial_t^j u||_{L_\tau^4 W^{s,4}} \lesssim_{\epsilon,\ ||u_0||} ||u||_{L_\tau^\infty H^{2j+s}}^{1-s_0} ||u||_{L_\tau^\infty H^{2j+s+1}}^{s_0} ||u||_{L_\tau^\infty H^{2j+2}}^{\epsilon},$$

which concludes the proof of the proposition. □

Lemma 4.3.3. *Let $u \in C(\mathbb{R}, H^{2k}(M^d))$ be the unique global solution to either (4.1) or (4.2) with $m = 2k$, $k \in \mathbb{N}$. For every $s \in [1, 2k]$, we have*

$$||u||_{L_\tau^\infty H^s(M^d)} \lesssim_{||u_0||_{H^1}} ||u||_{L_\tau^\infty H^{2k}(M^d)}^{\frac{s-1}{2k-1}}. \tag{4.31}$$

Proof. Due to (1.1) and (4.4), for every $s \in [1, 2k]$, we have

$$||u||_{H^s} \lesssim ||u||_{H^{2k}}^{\frac{s-1}{2k-1}} ||u||_{H^1}^{\frac{2k-s}{2k-1}} \lesssim_{||u_0||_{H^1}} ||u||_{H^{2k}}^{\frac{s-1}{2k-1}}.$$

We conclude by taking the supremum over $t \in (0, \tau)$. □

Proposition 4.3.4. *Let $u \in C(\mathbb{R}, H^{2k}(M^2))$ be the unique global solution to (4.1) with $m = 2k$, $k \in \mathbb{N}$. For every $\tau \in (0, 1)$, we have*

$$\int_0^\tau |r.h.s.\ of\ (4.16)|\, ds \lesssim_{\epsilon,\ ||u_0||_{H^1}} \sqrt{\tau} ||u||_{L_\tau^\infty H^{2k}(M^2)}^{\frac{4k-3+2s_0}{2k-1}+\epsilon} + ||u||_{L_\tau^\infty H^{2k}(M^2)}^{\frac{4k-4}{2k-1}+\epsilon}.$$

Proof. We shall estimate each term of (4.16) by assuming $p > 3$; for $p = 3$ the estimate of the first and the third term is trivial.
(First Term) By expanding time and space derivative, we get

$$|\partial_t^{k-1}\nabla_g(|u|^2)|_g^2 \lesssim \sum_{k_1+k_2=k-1} |\partial_t^{k_1}u|^2|\partial_t^{k_2}\nabla_g u|_g^2.$$

Moreover, since $|\partial_t|u|| \leq |\partial_t u|$, we have

$$|\partial_t(|u|^{p-3})| \lesssim |u|^{p-4}|\partial_t u|,$$

therefore

$$\int_0^\tau \left|\int_{M^2} |\partial_t^{k-1}\nabla_g(|u|^2)|_g^2 \; \partial_t(|u|^{p-3}) \; dvol_g\right| \; ds$$
$$\lesssim \sum_{k_1+k_2=k-1} \int_0^\tau \int_{M^2} |\partial_t^{k_1}u|^2 \; |\partial_t^{k_2}\nabla_g u|_g^2 \; |u|^{p-4} \; |\partial_t u| \; dvol_g \; ds.$$

By using Hölder's inequality, we have

$$\begin{aligned}
&\int_{M^2} |\partial_t^{k_1}u|^2 \; |\partial_t^{k_2}\nabla_g u|_g^2 \; |u|^{p-4} \; |\partial_t u| \; dvol_g \\
&\leq \left(\int_{M^2} |\partial_t^{k_1}u|^4|\partial_t^{k_2}\nabla_g u|_g^4 \; dvol_g\right)^{1/2} \left(\int_{M^2} |u|^{2(p-4)} \; |\partial_t u|^2 \; dvol_g\right)^{1/2} \\
&\leq ||\partial_t^{k_1}u||_{L^\infty}^2||u||_{L^\infty}^{p-4} \left(\int_{M^2} |\partial_t^{k_2}\nabla_g u|_g^4 \; dvol_g\right)^{1/2} \left(\int_{M^2} |\partial_t u|^2 \; dvol_g\right)^{1/2} \\
&\lesssim ||\partial_t^{k_1}u||_{L^\infty}^2||u||_{L^\infty}^{p-4}||\partial_t^{k_2}u||_{W^{1,4}}^2||\partial_t u||_{L^2}. \qquad (4.32)
\end{aligned}$$

Moreover, due to (1.2), (4.13) and (4.4), we get

$$\begin{aligned}
(4.32) &\lesssim_\epsilon ||\partial_t^{k_1}u||_{H^1}^{2(1-\epsilon)}||\partial_t^{k_1}u||_{H^2}^{2\epsilon}||u||_{H^1}^{(1-\epsilon)(p-4)}||u||_{H^2}^{\epsilon(p-4)}||\partial_t^{k_2}u||_{W^{1,4}}^2||\partial_t u||_{L^2} \\
&\lesssim_{||u_0||_{H^1}} ||u||_{H^{2k_1+1}}^{2(1-\epsilon)}||u||_{H^{2k_1+2}}^{2\epsilon}||u||_{H^2}^{\epsilon(p-4)}||\partial_t^{k_2}u||_{W^{1,4}}^2||u||_{H^2},
\end{aligned}$$

therefore

$$\int_0^\tau \left|\int_{M^2} |\partial_t^{k-1}\nabla_g(|u|^2)|_g^2 \; \partial_t(|u|^{p-3}) \; dvol_g\right| ds$$
$$\lesssim_{\epsilon,\, ||u_0||_{H^1}} \sum_{k_1+k_2=k-1} \int_0^\tau ||u||_{H^{2k_1+1}}^{2(1-\epsilon)}||u||_{H^{2k_1+2}}^{2\epsilon}||u||_{H^2}^{\epsilon(p-4)}||\partial_t^{k_2}u||_{W^{1,4}}^2||u||_{H^2} \; ds.$$

Due to Hölder's inequality, we have

$$\begin{aligned}
&\sum_{k_1+k_2=k-1} \int_0^\tau ||u||_{H^{2k_1+1}}^{2(1-\epsilon)} ||u||_{H^{2k_1+2}}^{2\epsilon} ||u||_{H^2}^{\epsilon(p-4)} ||\partial_t^{k_2} u||_{W^{1,4}}^2 ||u||_{H^2} \, ds \\
&\leq \sum_{k_1+k_2=k-1} \left(\int_0^\tau ||u||_{H^{2k_1+1}}^{4(1-\epsilon)} ||u||_{H^{2k_1+2}}^{4\epsilon} ||u||_{H^2}^{2\epsilon(p-4)} ||u||_{H^2}^2 \, ds \right)^{1/2} \times \\
&\qquad \times \left(\int_0^\tau ||\partial_t^{k_2} u||_{W^{1,4}}^4 \, ds \right)^{1/2} \\
&\leq \sum_{k_1+k_2=k-1} \sqrt{\tau} ||\partial_t^{k_2} u||_{L_\tau^4 W^{1,4}}^2 ||u||_{L_\tau^\infty H^{2k_1+1}}^{2(1-\epsilon)} ||u||_{L_\tau^\infty H^{2k_1+2}}^{2\epsilon} ||u||_{L_\tau^\infty H^2}^{\epsilon(p-4)} ||u||_{L_\tau^\infty H^2}.
\end{aligned} \tag{4.33}$$

Due to (4.25), we get

$$\begin{aligned}
(4.33) \lesssim_{\epsilon,\ ||u_0||_{H^1}} &\sum_{k_1+k_2=k-1} \sqrt{\tau} ||u||_{L_\tau^\infty H^{2k_2+1}}^{2(1-s_0)} ||u||_{L_\tau^\infty H^{2k_2+2}}^{2s_0} ||u||_{L_\tau^\infty H^{2k_2+2}}^{2\epsilon} \times \\
&\times ||u||_{L_\tau^\infty H^{2k_1+1}}^{2(1-\epsilon)} ||u||_{L_\tau^\infty H^{2k_1+2}}^{2\epsilon} ||u||_{L_\tau^\infty H^2}^{\epsilon(p-4)} ||u||_{L_\tau^\infty H^2}.
\end{aligned} \tag{4.34}$$

By applying (4.31) to each factor we obtain

$$(4.34) \lesssim_{||u_0||_{H^1}} \sum_{k_1+k_2=k-1} \sqrt{\tau} ||u||_{L_\tau^\infty H^{2k}}^{\frac{4k-3+2s_0}{2k-1}+\epsilon} \lesssim \sqrt{\tau} ||u||_{L_\tau^\infty H^{2k}}^{\frac{4k-3+2s_0}{2k-1}+\epsilon},$$

therefore

$$\int_0^\tau \left| \int_{M^2} |\partial_t^{k-1} \nabla_g (|u|^2)|_g^2 \ \partial_t (|u|^{p-3}) \ dvol_g \right| ds \lesssim_{\epsilon,\ ||u_0||_{H^1}} \sqrt{\tau} ||u||_{L_\tau^\infty H^{2k}}^{\frac{4k-3+2s_0}{2k-1}+\epsilon}.$$

(Second Term) By expanding the time derivative, we get

$$|\partial_t^k (|u|^{p-1})| \lesssim \sum_{\substack{j_1+\ldots+j_{p-1}=k \\ j_1=\max\{j_1,\ldots,j_{p-1}\}}} \left(\prod_{l=1}^{p-1} |\partial_t^{j_l} u| \right).$$

Moreover, due to the Cauchy-Schwarz inequality, we have

$$|\partial_t^{k-1} (|\nabla_g u|_g^2)| \lesssim \sum_{k_1+k_2=k-1} |\partial_t^{k_1} \nabla_g u|_g |\partial_t^{k_2} \nabla_g u|_g,$$

then

$$\int_0^\tau \left| \int_{M^2} \partial_t^k (|u|^{p-1}) \ \partial_t^{k-1} (|\nabla_g u|_g^2) \ dvol_g \right| \ ds$$

$$\lesssim \sum_{\substack{k_1+k_2=k-1\\ j_1+\ldots+j_{p-1}=k\\ j_1=\max\{j_1,\ldots,j_{p-1}\}}} \int_0^\tau \int_{M^2} \left(\prod_{l=1}^{p-1} |\partial_t^{j_l} u| \right) |\partial_t^{k_1} \nabla_g u|_g |\partial_t^{k_2} \nabla_g u|_g \; dvol_g \; ds.$$

By using Hölder's inequality, we get

$$\begin{aligned} &\int_{M^2} \left(\prod_{l=1}^{p-1} |\partial_t^{j_l} u| \right) |\partial_t^{k_1} \nabla_g u|_g |\partial_t^{k_2} \nabla_g u|_g \; dvol_g \\ &\leq \left(\int_{M^2} \left(\prod_{l=1}^{p-1} |\partial_t^{j_l} u|^2 \right) dvol_g \right)^{1/2} \left(\int_{M^2} |\partial_t^{k_1} \nabla_g u|_g^4 \; dvol_g \right)^{1/4} \times \\ &\qquad \times \left(\int_{M^2} |\partial_t^{k_2} \nabla_g u|_g^4 \; dvol_g \right)^{1/4} \\ &\leq \left(\prod_{l=2}^{p-1} ||\partial_t^{j_l} u||_{L^\infty} \right) ||\partial_t^{j_1} u||_{L^2} ||\partial_t^{k_1} u||_{W^{1,4}} ||\partial_t^{k_2} u||_{W^{1,4}}. \end{aligned} \tag{4.35}$$

Moreover, due to (1.2) and (4.13), we get

$$\begin{aligned} (4.35) &\lesssim_\epsilon \left(\prod_{l=2}^{p-1} ||\partial_t^{j_l} u||_{H^1}^{1-\epsilon} ||\partial_t^{j_l} u||_{H^2}^{\epsilon} \right) ||\partial_t^{j_1} u||_{L^2} ||\partial_t^{k_1} u||_{W^{1,4}} ||\partial_t^{k_2} u||_{W^{1,4}} \\ &\lesssim_{||u_0||_{H^1}} \left(\prod_{l=2}^{p-1} ||u||_{H^{2j_l+1}}^{1-\epsilon} ||u||_{H^{2j_l+2}}^{\epsilon} \right) ||u||_{H^{2j_1}} ||\partial_t^{k_1} u||_{W^{1,4}} ||\partial_t^{k_2} u||_{W^{1,4}}, \end{aligned}$$

therefore

$$\begin{aligned} &\int_0^\tau \left| \int_{M^2} \partial_t^k (|u|^{p-1}) \; \partial_t^{k-1} (|\nabla_g u|_g^2) \; dvol_g \right| \; ds \\ &\lesssim_{\epsilon, \; ||u_0||_{H^1}} \sum_{\substack{k_1+k_2=k-1\\ j_1+\ldots+j_{p-1}=k\\ j_1=\max\{j_1,\ldots,j_{p-1}\}}} \int_0^\tau \left(\prod_{l=2}^{p-1} ||u||_{H^{2j_l+1}}^{1-\epsilon} ||u||_{H^{2j_l+2}}^{\epsilon} \right) \times \\ &\qquad \times ||u||_{H^{2j_1}} ||\partial_t^{k_1} u||_{W^{1,4}} ||\partial_t^{k_2} u||_{W^{1,4}} \; ds. \end{aligned} \tag{4.36}$$

Due to Hölder's inequality, we have

$$\begin{aligned} (4.36) \leq &\sum_{\substack{k_1+k_2=k-1\\ j_1+\ldots+j_{p-1}=k\\ j_1=\max\{j_1,\ldots,j_{p-1}\}}} \left(\int_0^\tau \left(\prod_{l=2}^{p-1} ||u||_{H^{2j_l+1}}^{1-\epsilon} ||u||_{H^{2j_l+2}}^{\epsilon} \right)^2 ||u||_{H^{2j_1}}^2 \; ds \right)^{1/2} \times \\ &\times \left(\int_0^\tau ||\partial_t^{k_1} u||_{W^{1,4}}^4 \; ds \right)^{1/4} \left(\int_0^\tau ||\partial_t^{k_2} u||_{W^{1,4}}^4 \; ds \right)^{1/4} \end{aligned}$$

$$\leq \sum_{\substack{k_1+k_2=k-1\\ j_1+...+j_{p-1}=k\\ j_1=\max\{j_1,...,j_{p-1}\}}} \sqrt{\tau}\left(\prod_{l=2}^{p-1}||u||^{1-\epsilon}_{L^\infty_\tau H^{2j_l+1}}||u||^{\epsilon}_{L^\infty_\tau H^{2j_l+2}}\right)\times$$
$$\times||u||_{L^\infty_\tau H^{2j_1}}||\partial_t^{k_1}u||_{L^4_\tau W^{1,4}}||\partial_t^{k_2}u||_{L^4_\tau W^{1,4}}. \tag{4.37}$$

Due to (4.25), we get

$$(4.37) \lesssim_{\epsilon,\ ||u_0||_{H^1}} \sum_{\substack{k_1+k_2=k-1\\ j_1+...+j_{p-1}=k\\ j_1=\max\{j_1,...,j_{p-1}\}}} \sqrt{\tau}\left(\prod_{l=2}^{p-1}||u||^{1-\epsilon}_{L^\infty_\tau H^{2j_l+1}}||u||^{\epsilon}_{L^\infty_\tau H^{2j_l+2}}\right)\times$$
$$\times||u||_{L^\infty_\tau H^{2j_1}}||u||^{1-s_0}_{L^\infty_\tau H^{2k_1+1}}||u||^{s_0}_{L^\infty_\tau H^{2k_1+2}}||u||^{\epsilon}_{L^\infty_\tau H^{2k_1+2}}\times$$
$$\times||u||^{1-s_0}_{L^\infty_\tau H^{2k_2+1}}||u||^{s_0}_{L^\infty_\tau H^{2k_2+2}}||u||^{\epsilon}_{L^\infty_\tau H^{2k_2+2}}. \tag{4.38}$$

By applying (4.31) to each factor we obtain

$$(4.38) \lesssim_{||u_0||_{H^1}} \sum_{\substack{k_1+k_2=k-1\\ j_1+...+j_{p-1}=k\\ j_1=\max\{j_1,...,j_{p-1}\}}} \sqrt{\tau}||u||^{\frac{4k-3+2s_0}{2k-1}+\epsilon}_{L^\infty_\tau H^{2k}} \lesssim \sqrt{\tau}||u||^{\frac{4k-3+2s_0}{2k-1}+\epsilon}_{L^\infty_\tau H^{2k}},$$

therefore

$$\int_0^\tau\left|\int_{M^2}\partial_t^k(|u|^{p-1})\ \partial_t^{k-1}(|\nabla_g u|_g^2)\ dvol_g\right|\ ds \lesssim_{\epsilon,\ ||u_0||_{H^1}} \sqrt{\tau}||u||^{\frac{4k-3+2s_0}{2k-1}+\epsilon}_{L^\infty_\tau H^{2k}}.$$

(Third Term) By expanding the time derivative, we get

$$|\partial_t^{k-j}(|u|^{p-3})| \lesssim \sum_{\substack{m_1+...+m_{p-3}=k-j\\ m_1=\max\{m_1,...,m_{p-3}\}}}\left(\prod_{i=1}^{p-3}|\partial_t^{m_i}u|\right).$$

Moreover, due to the Cauchy-Schwarz inequality, we have

$$\left|(\partial_t^j\nabla_g(|u|^2),\partial_t^{k-1}\nabla_g(|u|^2))_g\right| \lesssim \sum_{\substack{j_1+j_2=j\\ k_1+k_2=k-1}}|\partial_t^{j_1}u||\partial_t^{k_1}u||\partial_t^{j_2}\nabla_g u|_g|\partial_t^{k_2}\nabla_g u|_g,$$

then

$$\int_0^\tau\left|\sum_{j=0}^{k-1}c_j\int_{M^2}(\partial_t^j\nabla_g(|u|^2),\partial_t^{k-1}\nabla_g(|u|^2))_g\ \partial_t^{k-j}(|u|^{p-3})\ dvol_g\right|\ ds$$

$$\lesssim \sum_{j=0}^{k-1} \sum_{\substack{j_1+j_2=j \\ k_1+k_2=k-1 \\ m_1+\ldots+m_{p-3}=k-j \\ m_1=\max\{m_1,\ldots,m_{p-3}\}}} \int_0^\tau \int_{M^2} \left(\prod_{i=1}^{p-3} |\partial_t^{m_i} u| \right) |\partial_t^{j_1} u| |\partial_t^{k_1} u| \times$$
$$\times |\partial_t^{j_2} \nabla_g u|_g |\partial_t^{k_2} \nabla_g u|_g \, dvol_g \, ds.$$

By using Hölder's inequality, we get

$$\begin{aligned}
&\int_{M^2} \left(\prod_{i=1}^{p-3} |\partial_t^{m_i} u| \right) |\partial_t^{j_1} u| |\partial_t^{k_1} u| |\partial_t^{j_2} \nabla_g u|_g |\partial_t^{k_2} \nabla_g u|_g \, dvol_g \\
&\leq \left(\int_{M^2} \left(\prod_{i=1}^{p-3} |\partial_t^{m_i} u|^2 \right) |\partial_t^{j_1} u|^2 |\partial_t^{k_1} u|^2 \, dvol_g \right)^{1/2} \times \\
&\times \left(\int_{M^2} |\partial_t^{j_2} \nabla_g u|_g^4 \, dvol_g \right)^{1/4} \left(\int_{M^2} |\partial_t^{k_2} \nabla_g u|_g^4 \, dvol_g \right)^{1/4} \\
&\leq \left(\prod_{i=2}^{p-3} ||\partial_t^{m_i} u||_{L^\infty} \right) ||\partial_t^{j_1} u||_{L^\infty} ||\partial_t^{k_1} u||_{L^\infty} ||\partial_t^{m_1} u||_{L^2} ||\partial_t^{j_2} u||_{W^{1,4}} ||\partial_t^{k_2} u||_{W^{1,4}}.
\end{aligned} \tag{4.39}$$

Moreover, due to (1.2) and (4.13), we get

$$\begin{aligned}
(4.39) &\lesssim_\epsilon \left(\prod_{i=2}^{p-3} ||\partial_t^{m_i} u||_{H^1}^{1-\epsilon} ||\partial_t^{m_i} u||_{H^2}^{\epsilon} \right) ||\partial_t^{j_1} u||_{H^1}^{1-\epsilon} ||\partial_t^{j_1} u||_{H^2}^{\epsilon} \times \\
&\quad \times ||\partial_t^{k_1} u||_{H^1}^{1-\epsilon} ||\partial_t^{k_1} u||_{H^2}^{\epsilon} ||\partial_t^{m_1} u||_{L^2} ||\partial_t^{j_2} u||_{W^{1,4}} ||\partial_t^{k_2} u||_{W^{1,4}} \\
&\lesssim_{||u_0||_{H^1}} \left(\prod_{i=2}^{p-3} ||u||_{H^{2m_i+1}}^{1-\epsilon} ||u||_{H^{2m_i+2}}^{\epsilon} \right) ||u||_{H^{2j_1+1}}^{1-\epsilon} ||u||_{H^{2j_1+2}}^{\epsilon} ||u||_{H^{2k_1+1}}^{1-\epsilon} \times \\
&\quad \times ||u||_{H^{2k_1+2}}^{\epsilon} ||u||_{H^{2m_1}} ||\partial_t^{j_2} u||_{W^{1,4}} ||\partial_t^{k_2} u||_{W^{1,4}},
\end{aligned}$$

then

$$\int_0^\tau \left| \sum_{j=0}^{k-1} c_j \int_{M^2} (\partial_t^j \nabla_g (|u|^2), \partial_t^{k-1} \nabla_g (|u|^2))_g \; \partial_t^{k-j} (|u|^{p-3}) \, dvol_g \right| ds$$
$$\lesssim_{\epsilon,\, ||u_0||_{H^1}} \sum_{j=0}^{k-1} \sum_{\substack{j_1+j_2=j \\ k_1+k_2=k-1 \\ m_1+\ldots+m_{p-3}=k-j \\ m_1=\max\{m_1,\ldots,m_{p-3}\}}} \int_0^\tau \left(\prod_{i=2}^{p-3} ||u||_{H^{2m_i+1}}^{1-\epsilon} ||u||_{H^{2m_i+2}}^{\epsilon} \right) \times$$

$$\times ||u||^{1-\epsilon}_{H^{2j_1+1}}||u||^{\epsilon}_{H^{2j_1+2}}||u||^{1-\epsilon}_{H^{2k_1+1}}||u||^{\epsilon}_{H^{2k_1+2}}||u||_{H^{2m_1}}||\partial_t^{j_2}u||_{W^{1,4}}||\partial_t^{k_2}u||_{W^{1,4}}\ ds. \tag{4.40}$$

Due to Hölder's inequality, we have

$$\begin{aligned}(4.40) \leq \sum_{j=0}^{k-1} \sum_{\substack{j_1+j_2=j\\ k_1+k_2=k-1\\ m_1+\ldots+m_{p-3}=k-j\\ m_1=\max\{m_1,\ldots,m_{p-3}\}}} \sqrt{\tau}\left(\prod_{i=2}^{p-3}||u||^{1-\epsilon}_{L^\infty_\tau H^{2m_i+1}}||u||^{\epsilon}_{L^\infty_\tau H^{2m_i+2}}\right)\times\\ \times||u||^{1-\epsilon}_{L^\infty_\tau H^{2j_1+1}}||u||^{\epsilon}_{L^\infty_\tau H^{2j_1+2}}||u||^{1-\epsilon}_{L^\infty_\tau H^{2k_1+1}}||u||^{\epsilon}_{L^\infty_\tau H^{2k_1+2}}\times\\ \times||u||_{L^\infty_\tau H^{2m_1}}||\partial_t^{j_2}u||_{L^4_\tau W^{1,4}}||\partial_t^{k_2}u||_{L^4_\tau W^{1,4}}.\end{aligned} \tag{4.41}$$

Due to (4.25), we get

$$\begin{aligned}(4.41) \lesssim_{\epsilon,\ ||u_0||_{H^1}} \sum_{j=0}^{k-1} \sum_{\substack{j_1+j_2=j\\ k_1+k_2=k-1\\ m_1+\ldots+m_{p-3}=k-j\\ m_1=\max\{m_1,\ldots,m_{p-3}\}}} \sqrt{\tau}\left(\prod_{i=2}^{p-3}||u||^{1-\epsilon}_{L^\infty_\tau H^{2m_i+1}}||u||^{\epsilon}_{L^\infty_\tau H^{2m_i+2}}\right)\\ \times||u||^{1-\epsilon}_{L^\infty_\tau H^{2j_1+1}}||u||^{\epsilon}_{L^\infty_\tau H^{2j_1+2}}||u||^{1-\epsilon}_{L^\infty_\tau H^{2k_1+1}}||u||^{\epsilon}_{L^\infty_\tau H^{2k_1+2}}||u||_{L^\infty_\tau H^{2m_1}}\times\\ \times||u||^{1-s_0}_{L^\infty_\tau H^{2j_2+1}}||u||^{s_0}_{L^\infty_\tau H^{2j_2+2}}||u||^{\epsilon}_{L^\infty_\tau H^{2j_2+2}}\times\\ \times||u||^{1-s_0}_{L^\infty_\tau H^{2k_2+1}}||u||^{s_0}_{L^\infty_\tau H^{2k_2+2}}||u||^{\epsilon}_{L^\infty_\tau H^{2k_2+2}}.\end{aligned} \tag{4.42}$$

By applying (4.31) to each factor we obtain

$$(4.42) \lesssim_{||u_0||_{H^1}} \sum_{j=0}^{k-1} \sum_{\substack{j_1+j_2=j\\ k_1+k_2=k-1\\ m_1+\ldots+m_{p-3}=k-j\\ m_1=\max\{m_1,\ldots,m_{p-3}\}}} \sqrt{\tau}||u||^{\frac{4k-3+2s_0}{2k-1}+\epsilon}_{L^\infty_\tau H^{2k}} \lesssim \sqrt{\tau}||u||^{\frac{4k-3+2s_0}{2k-1}+\epsilon}_{L^\infty_\tau H^{2k}},$$

therefore

$$\begin{aligned}\int_0^\tau\left|\sum_{j=0}^{k-1}c_j\int_{M^2}(\partial_t^j\nabla_g(|u|^2),\partial_t^{k-1}\nabla_g(|u|^2))_g\ \partial_t^{k-j}(|u|^{p-3})\ dvol_g\right|\ ds\\ \lesssim_{\epsilon,\ ||u_0||_{H^1}}\sqrt{\tau}||u||^{\frac{4k-3+2s_0}{2k-1}+\epsilon}_{L^\infty_\tau H^{2k}}.\end{aligned}$$

(Fourth Term) By expanding the time derivative, we get

$$|\partial_t^j(|u|^{p-1})| \lesssim \sum_{\substack{j_1+...+j_{p-1}=j \\ j_1=\max\{j_1,...,j_{p-1}\}}} \left(\prod_{i=1}^{p-1} |\partial_t^{j_i} u| \right),$$

$$|\partial_t^{k-1}(|u|^{p-1}\overline{u})| \lesssim \sum_{\substack{k_1+...+k_p=k-1 \\ k_1=\max\{k_1,...,k_p\}}} \left(\prod_{i=1}^{p} |\partial_t^{k_i} u| \right),$$

then

$$\int_0^\tau \left| Re \sum_{j=0}^{k-1} c_j \int_{M^2} \partial_t^j(|u|^{p-1})\ \partial_t^{k-j} u\ \partial_t^{k-1}(|u|^{p-1}\overline{u})\ dvol_g \right| ds$$

$$\lesssim \sum_{\substack{k_1+...+k_p=k-1 \\ k_1=\max\{k_1,...,k_p\}}} \int_0^\tau \int_{M^2} \left(\prod_{i=1}^{p} |\partial_t^{k_i} u| \right) |u|^{p-1}\ |\partial_t^k u|\ dvol_g\ ds +$$

$$+ \sum_{j=1}^{k-1} \sum_{\substack{j_1+...+j_{p-1}=j \\ j_1=\max\{j_1,...,j_{p-1}\} \\ k_1+...+k_p=k-1 \\ k_1=\max\{k_1,...,k_p\}}} \int_0^\tau \int_{M^2} \left(\prod_{i=1}^{p-1} |\partial_t^{j_i} u| \right) \left(\prod_{i=1}^{p} |\partial_t^{k_i} u| \right) |\partial_t^{k-j} u|\ dvol_g\ ds.$$

We consider at first the case $j = 0$. By using Hölder's inequality, we get

$$\int_{M^2} \left(\prod_{i=1}^{p} |\partial_t^{k_i} u| \right) |u|^{p-1}\ |\partial_t^k u|\ dvol_g$$

$$\leq \left(\int_{M^2} \left(\prod_{i=1}^{p} |\partial_t^{k_i} u|^2 \right) |u|^{2(p-1)}\ dvol_g \right)^{1/2} \left(\int_{M^2} |\partial_t^k u|^2\ dvol_g \right)^{1/2}$$

$$\leq \left(\prod_{i=2}^{p} ||\partial_t^{k_i} u||_{L^\infty} \right) ||\partial_t^{k_1} u||_{L^2} ||u||_{L^\infty}^{p-1} ||\partial_t^k u||_{L^2}. \tag{4.43}$$

Moreover, due to (1.2), (4.13) and (4.4), we get

$$(4.43) \lesssim_\epsilon \left(\prod_{i=2}^{p} ||\partial_t^{k_i} u||_{H^1}^{(1-\epsilon)} ||\partial_t^{k_i} u||_{H^2}^{\epsilon} \right) ||\partial_t^{k_1} u||_{L^2} ||u||_{H^1}^{(1-\epsilon)(p-1)} ||u||_{H^2}^{\epsilon(p-1)} ||\partial_t^k u||_{L^2}$$

$$\lesssim_{||u_0||_{H^1}} \left(\prod_{i=2}^{p} ||u||_{H^{2k_i+1}}^{(1-\epsilon)} ||u||_{H^{2k_i+2}}^{\epsilon} \right) ||u||_{H^{2k_1}} ||u||_{H^2}^{\epsilon(p-1)} ||u||_{H^{2k}},$$

therefore

$$\sum_{\substack{k_1+...+k_p=k-1\\ k_1=\max\{k_1,...,k_p\}}} \int_0^\tau \int_{M^2} \left(\prod_{i=1}^{p} |\partial_t^{k_i} u| \right) |u|^{p-1} \, |\partial_t^k u| \, dvol_g \, ds$$

$$\lesssim_{\epsilon,\ ||u_0||_{H^1}} \sum_{\substack{k_1+...+k_p=k-1\\ k_1=\max\{k_1,...,k_p\}}} \int_0^\tau \left(\prod_{i=2}^{p} ||u||^{(1-\epsilon)}_{H^{2k_i+1}} ||u||^{\epsilon}_{H^{2k_i+2}} \right) \times$$

$$\times ||u||_{H^{2k_1}} ||u||^{\epsilon(p-1)}_{H^2} ||u||_{H^{2k}} \, ds$$

$$\leq \sum_{\substack{k_1+...+k_p=k-1\\ k_1=\max\{k_1,...,k_p\}}} \tau \left(\prod_{i=2}^{p} ||u||^{(1-\epsilon)}_{L^\infty_\tau H^{2k_i+1}} ||u||^{\epsilon}_{L^\infty_\tau H^{2k_i+2}} \right) \times$$

$$\times ||u||_{L^\infty_\tau H^{2k_1}} ||u||^{\epsilon(p-1)}_{L^\infty_\tau H^2} ||u||_{L^\infty_\tau H^{2k}}. \tag{4.44}$$

By applying (4.31) to each factor, we get

$$(4.44) \lesssim_{||u_0||_{H^1}} \sum_{\substack{k_1+...+k_p=k-1\\ k_1=\max\{k_1,...,k_p\}}} \tau ||u||^{\frac{4k-4}{2k-1}+\epsilon}_{L^\infty_\tau H^{2k}} \lesssim \tau ||u||^{\frac{4k-4}{2k-1}+\epsilon}_{L^\infty_\tau H^{2k}},$$

therefore

$$\sum_{\substack{k_1+...+k_p=k-1\\ k_1=\max\{k_1,...,k_p\}}} \int_0^\tau \int_{M^2} \left(\prod_{i=1}^{p} |\partial_t^{k_i} u| \right) |u|^{p-1} \, |\partial_t^k u| \, dvol_g \, ds$$

$$\lesssim_{\epsilon,\ ||u_0||_{H^1}} ||u||^{\frac{4k-4}{2k-1}+\epsilon}_{L^\infty_\tau H^{2k}}.$$

Let us now consider the case $j \in \{1, ..., k-1\}$. By using Hölder's inequality, we get

$$\int_{M^2} \left(\prod_{i=1}^{p-1} |\partial_t^{j_i} u| \right) \left(\prod_{i=1}^{p} |\partial_t^{k_i} u| \right) |\partial_t^{k-j} u| \, dvol_g$$

$$\leq \left(\int_{M^2} \left(\prod_{i=2}^{p-1} |\partial_t^{j_i} u|^2 \right) |\partial_t^{j_1} u|^2 |\partial_t^{k-j} u|^2 dvol_g \right)^{1/2} \times$$

$$\times \left(\int_{M^2} \left(\prod_{i=2}^{p} |\partial_t^{k_i} u|^2 \right) |\partial_t^{k_1} u|^2 \, dvol_g \right)^{1/2}$$

$$\leq \left(\prod_{i=2}^{p-1} ||\partial_t^{j_i} u||_{L^\infty} \right) \left(\prod_{i=2}^{p} ||\partial_t^{k_i} u||_{L^\infty} \right) ||\partial_t^{k-j} u||_{L^\infty} ||\partial_t^{j_1} u||_{L^2} ||\partial_t^{k_1} u||_{L^2}. \tag{4.45}$$

Moreover, due to (1.2) and (4.13), we get

$$\begin{aligned}
(4.45) \lesssim_\epsilon & \left(\prod_{i=2}^{p-1} ||\partial_t^{j_i} u||_{H^1}^{1-\epsilon} ||\partial_t^{j_i} u||_{H^2}^{\epsilon} \right) \left(\prod_{i=2}^{p} ||\partial_t^{k_i} u||_{H^1}^{1-\epsilon} ||\partial_t^{k_i} u||_{H^2}^{\epsilon} \right) \times \\
& \times ||\partial_t^{k-j} u||_{H^1}^{1-\epsilon} ||\partial_t^{k-j} u||_{H^2}^{\epsilon} ||\partial_t^{j_1} u||_{L^2} ||\partial_t^{k_1} u||_{L^2} \\
\lesssim_{||u_0||_{H^1}} & \left(\prod_{i=2}^{p} ||u||_{H^{2k_i+1}}^{1-\epsilon} ||u||_{H^{2k_i+2}}^{\epsilon} \right) \left(\prod_{i=2}^{p-1} ||u||_{H^{2j_i+1}}^{1-\epsilon} ||u||_{H^{2j_i+2}}^{\epsilon} \right) \times \\
& \times ||u||_{H^{2k-2j+1}}^{1-\epsilon} ||u||_{H^{2k-2j+2}}^{\epsilon} ||u||_{H^{2j_1}} ||u||_{H^{2k_1}},
\end{aligned}$$

therefore

$$\begin{aligned}
& \sum_{j=1}^{k-1} \sum_{\substack{j_1+\ldots+j_{p-1}=j \\ j_1=\max\{j_1,\ldots,j_{p-1}\} \\ k_1+\ldots+k_p=k-1 \\ k_1=\max\{k_1,\ldots,k_p\}}} \int_0^\tau \int_{M^2} \left(\prod_{i=1}^{p-1} |\partial_t^{j_i} u| \right) \left(\prod_{i=1}^{p} |\partial_t^{k_i} u| \right) |\partial_t^{k-j} u| \, dvol_g \, ds \\
& \lesssim_{\epsilon,\ ||u_0||_{H^1}} \sum_{j=1}^{k-1} \sum_{\substack{j_1+\ldots+j_{p-1}=j \\ j_1=\max\{j_1,\ldots,j_{p-1}\} \\ k_1+\ldots+k_p=k-1 \\ k_1=\max\{k_1,\ldots,k_p\}}} \int_0^\tau \left(\prod_{i=2}^{p} ||u||_{H^{2k_i+1}}^{1-\epsilon} ||u||_{H^{2k_i+2}}^{\epsilon} \right) \times \\
& \times \left(\prod_{i=2}^{p-1} ||u||_{H^{2j_i+1}}^{1-\epsilon} ||u||_{H^{2j_i+2}}^{\epsilon} \right) ||u||_{H^{2k-2j+1}}^{1-\epsilon} ||u||_{H^{2k-2j+2}}^{\epsilon} ||u||_{H^{2j_1}} ||u||_{H^{2k_1}} \, ds \\
& \leq \sum_{j=1}^{k-1} \sum_{\substack{j_1+\ldots+j_{p-1}=j \\ j_1=\max\{j_1,\ldots,j_{p-1}\} \\ k_1+\ldots+k_p=k-1 \\ k_1=\max\{k_1,\ldots,k_p\}}} \tau \left(\prod_{i=2}^{p} ||u||_{L_\tau^\infty H^{2k_i+1}}^{1-\epsilon} ||u||_{L_\tau^\infty H^{2k_i+2}}^{\epsilon} \right) ||u||_{L_\tau^\infty H^{2k_1}} \times \\
& \times \left(\prod_{i=2}^{p-1} ||u||_{L_\tau^\infty H^{2j_i+1}}^{1-\epsilon} ||u||_{L_\tau^\infty H^{2j_i+2}}^{\epsilon} \right) ||u||_{L_\tau^\infty H^{2j_1}} ||u||_{L_\tau^\infty H^{2k-2j+1}}^{1-\epsilon} ||u||_{L_\tau^\infty H^{2k-2j+2}}^{\epsilon}.
\end{aligned} \tag{4.46}$$

By applying (4.31) to each factor, we get

$$(4.46) \lesssim_{||u_0||_{H^1}} \sum_{j=1}^{k-1} \sum_{\substack{j_1+...+j_{p-1}=j \\ j_1=\max\{j_1,...,j_{p-1}\} \\ k_1+...+k_p=k-1 \\ k_1=\max\{k_1,...,k_p\}}} \tau ||u||^{\frac{4k-4}{2k-1}+\epsilon}_{L^\infty_\tau H^{2k}} \lesssim \tau ||u||^{\frac{4k-4}{2k-1}+\epsilon}_{L^\infty_\tau H^{2k}},$$

therefore

$$\sum_{j=1}^{k-1} \sum_{\substack{j_1+...+j_{p-1}=j \\ j_1=\max\{j_1,...,j_{p-1}\} \\ k_1+...+k_p=k-1 \\ k_1=\max\{k_1,...,k_p\}}} \int_0^\tau \int_{M^2} \left(\prod_{i=1}^{p-1} |\partial_t^{j_i} u| \right) \left(\prod_{i=1}^{p} |\partial_t^{k_i} u| \right) |\partial_t^{k-j} u| \; dvol_g \; ds$$

$$\lesssim_{\epsilon,\; ||u_0||_{H^1}} ||u||^{\frac{4k-4}{2k-1}+\epsilon}_{L^\infty_\tau H^{2k}}.$$

(Fifth Term) By expanding the time derivative, we get

$$|\partial_t^k (|u|^{p-1})| \lesssim \sum_{\substack{k_1+...+k_{p-1}=k \\ k_1=\max\{k_1,...,k_{p-1}\}}} \left(\prod_{i=1}^{p-1} |\partial_t^{k_i} u| \right),$$

then

$$\int_0^\tau \left| Re \sum_{j=0}^{k-2} c_j \int_{M^2} \partial_t^k (|u|^{p-1}) \; \partial_t^j (\Delta_g \overline{u}) \; \partial_t^{k-1-j} u \; dvol_g \right| \; ds$$

$$\lesssim \sum_{j=0}^{k-2} \sum_{\substack{k_1+...+k_{p-1}=k \\ k_1=\max\{k_1,...,k_{p-1}\}}} \int_0^\tau \int_{M^2} \left(\prod_{i=1}^{p-1} |\partial_t^{k_i} u| \right) |\partial_t^j (\Delta_g \overline{u})| \; |\partial_t^{k-1-j} u| \; dvol_g \; ds.$$

By using Hölder's inequality, we get

$$\int_{M^2} \left(\prod_{i=1}^{p-1} |\partial_t^{k_i} u| \right) \; |\partial_t^j (\Delta_g \overline{u})| \; |\partial_t^{k-1-j} u| \; dvol_g$$

$$\leq \left(\int_{M^2} \left(\prod_{i=1}^{p-1} |\partial_t^{k_i} u|^2 \right) dvol_g \right)^{1/2} \left(\int_{M^2} |\partial_t^j (\Delta_g \overline{u})|^4 \; dvol_g \right)^{1/4} \times$$

$$\times \left(\int_{M^2} |\partial_t^{k-1-j} u|^4 \; dvol_g \right)^{1/4}$$

$$\leq \left(\prod_{i=2}^{p-1} ||\partial_t^{k_i} u||_{L^\infty} \right) ||\partial_t^{k_1} u||_{L^2} ||\partial_t^j \Delta_g u||_{L^4} ||\partial_t^{k-1-j} u||_{L^4}. \qquad (4.47)$$

Moreover, due to (1.2) and (4.13), we have

$$(4.47) \lesssim_{\epsilon} \left(\prod_{i=2}^{p-1} ||\partial_t^{k_i} u||_{H^1}^{1-\epsilon} ||\partial_t^{k_i} u||_{H^2}^{\epsilon} \right) ||\partial_t^{k_1} u||_{L^2} ||\partial_t^{j} u||_{W^{2,4}} ||\partial_t^{k-1-j} u||_{L^4}$$
$$\lesssim_{||u_0||_{H^1}} \left(\prod_{i=2}^{p-1} ||u||_{H^{2k_i+1}}^{1-\epsilon} ||u||_{H^{2k_i+2}}^{\epsilon} \right) ||u||_{H^{2k_1}} ||\partial_t^{j} u||_{W^{2,4}} ||\partial_t^{k-1-j} u||_{L^4},$$

then

$$\int_0^\tau \left| Re \sum_{j=0}^{k-2} c_j \int_{M^2} \partial_t^k(|u|^{p-1})\ \partial_t^j(\Delta_g \overline{u})\ \partial_t^{k-1-j} u\ dvol_g \right| ds$$
$$\lesssim_{\epsilon,\ ||u_0||_{H^1}} \sum_{j=0}^{k-2} \sum_{\substack{k_1+\ldots+k_{p-1}=k \\ k_1=\max\{k_1,\ldots,k_{p-1}\}}} \int_0^\tau \left(\prod_{i=2}^{p-1} ||u||_{H^{2k_i+1}}^{1-\epsilon} ||u||_{H^{2k_i+2}}^{\epsilon} \right) \times$$
$$\times ||u||_{H^{2k_1}} ||\partial_t^j u||_{W^{2,4}} ||\partial_t^{k-1-j} u||_{L^4}\ ds. \tag{4.48}$$

Due to Hölder's inequality, we have

$$(4.48) \leq \sum_{j=0}^{k-2} \sum_{\substack{k_1+\ldots+k_{p-1}=k \\ k_1=\max\{k_1,\ldots,k_{p-1}\}}} \left(\int_0^\tau \left(\prod_{i=2}^{p-1} ||u||_{H^{2k_i+1}}^{1-\epsilon} ||u||_{H^{2k_i+2}}^{\epsilon} \right)^2 ||u||_{H^{2k_1}}^2\ ds \right)^{1/2}$$
$$\times \left(\int_0^\tau ||\partial_t^j u||_{W^{2,4}}^4\ ds \right)^{1/4} \left(\int_0^\tau ||\partial_t^{k-1-j} u||_{L^4}^4\ ds \right)^{1/4}$$
$$\leq \sum_{j=0}^{k-2} \sum_{\substack{k_1+\ldots+k_{p-1}=k \\ k_1=\max\{k_1,\ldots,k_{p-1}\}}} \sqrt{\tau} \left(\prod_{i=2}^{p-1} ||u||_{L_\tau^\infty H^{2k_i+1}}^{1-\epsilon} ||u||_{L_\tau^\infty H^{2k_i+2}}^{\epsilon} \right) ||u||_{L_\tau^\infty H^{2k_1}} \times$$
$$\times ||\partial_t^j u||_{L_\tau^4 W^{2,4}} ||\partial_t^{k-1-j} u||_{L_\tau^4 L^4}. \tag{4.49}$$

Moreover, due to (4.25), we get

$$(4.49) \lesssim_{\epsilon,\ ||u_0||_{H^1}} \sum_{j=0}^{k-2} \sum_{\substack{k_1+\ldots+k_{p-1}=k \\ k_1=\max\{k_1,\ldots,k_{p-1}\}}} \sqrt{\tau} \left(\prod_{i=2}^{p-1} ||u||_{L_\tau^\infty H^{2k_i+1}}^{1-\epsilon} ||u||_{L_\tau^\infty H^{2k_i+2}}^{\epsilon} \right) \times$$
$$\times ||u||_{L_\tau^\infty H^{2k_1}} ||u||_{L_\tau^\infty H^{2j+2}}^{1-s_0} ||u||_{L_\tau^\infty H^{2j+3}}^{s_0} ||u||_{L_\tau^\infty H^{2j+2}}^{\epsilon} \times$$
$$\times ||u||_{L_\tau^\infty H^{2k-2j-2}}^{1-s_0} ||u||_{L_\tau^\infty H^{2k-2j-1}}^{s_0} ||u||_{L_\tau^\infty H^{2k-2j}}^{\epsilon}. \tag{4.50}$$

By applying (4.31) to each factor, we get

$$(4.50) \lesssim_{||u_0||_{H^1}} \sum_{j=0}^{k-2} \sum_{\substack{k_1+\ldots+k_{p-1}=k \\ k_1=\max\{k_1,\ldots,k_{p-1}\}}} \sqrt{\tau}||u||_{L^\infty_\tau H^{2k}}^{\frac{4k-3+2s_0}{2k-1}+\epsilon} \lesssim \sqrt{\tau}||u||_{L^\infty_\tau H^{2k}}^{\frac{4k-3+2s_0}{2k-1}+\epsilon},$$

therefore

$$\int_0^\tau \left| Re \sum_{j=0}^{k-2} c_j \int_{M^2} \partial_t^k(|u|^{p-1})\ \partial_t^j(\Delta_g \overline{u})\ \partial_t^{k-1-j} u\ dvol_g \right| ds$$
$$\lesssim_{\epsilon,\ ||u_0||_{H^1}} \sqrt{\tau}||u||_{L^\infty_\tau H^{2k}}^{\frac{4k-3+2s_0}{2k-1}+\epsilon}.$$

(Sixth Term) By expanding the time derivative, we get

$$|\partial_t^j(|u|^{p-1})| \lesssim \sum_{\substack{j_1+\ldots+j_{p-1}=j \\ j_1=\max\{j_1,\ldots,j_{p-1}\}}} \left(\prod_{i=1}^{p-1} |\partial_t^{j_i} u| \right),$$

then

$$\int_0^\tau \left| Im \sum_{j=1}^{k-1} c_j \int_{M^2} \partial_t^j(|u|^{p-1})\ \partial_t^{k-j} u\ \partial_t^k \overline{u}\ dvol_g \right| ds$$
$$\lesssim \sum_{j=1}^{k-1} \sum_{\substack{j_1+\ldots+j_{p-1}=j \\ j_1=\max\{j_1,\ldots,j_{p-1}\}}} \int_0^\tau \int_{M^2} \left(\prod_{i=1}^{p-1} |\partial_t^{j_i} u| \right) |\partial_t^{k-j} u|\ |\partial_t^k u|\ dvol_g\ ds.$$

By using Hölder's inequality, we get

$$\int_{M^2} \left(\prod_{i=1}^{p-1} |\partial_t^{j_i} u| \right) |\partial_t^{k-j} u|\ |\partial_t^k u|\ dvol_g\ ds$$
$$\leq \left(\int_{M^2} \left(\prod_{i=1}^{p-1} |\partial_t^{j_i} u|^4 \right) dvol_g \right)^{1/4} \left(\int_{M^2} |\partial_t^{k-j} u|^4\ dvol_g \right)^{1/4} \times$$
$$\times \left(\int_{M^2} |\partial_t^k u|^2\ dvol_g \right)^{1/2}$$
$$\leq \left(\prod_{i=2}^{p-1} ||\partial_t^{j_i} u||_{L^\infty} \right) ||\partial_t^{j_1} u||_{L^4} ||\partial_t^{k-j} u||_{L^4} ||\partial_t^k u||_{L^2}. \qquad (4.51)$$

Moreover, due to (1.2) and (4.13), we get

$$(4.51) \lesssim_{\epsilon} \left(\prod_{i=2}^{p-1} ||\partial_t^{j_i} u||_{H^1}^{1-\epsilon} ||\partial_t^{j_i} u||_{H^2}^{\epsilon} \right) ||\partial_t^{j_1} u||_{L^4} ||\partial_t^{k-j} u||_{L^4} ||\partial_t^{k} u||_{L^2}$$

$$\lesssim_{||u_0||_{H^1}} \left(\prod_{i=2}^{p-1} ||u||_{H^{2j_i+1}}^{1-\epsilon} ||u||_{H^{2j_i+2}}^{\epsilon} \right) ||\partial_t^{j_1} u||_{L^4} ||\partial_t^{k-j} u||_{L^4} ||u||_{H^{2k}},$$

then

$$\int_0^{\tau} \left| Im \sum_{j=1}^{k-1} c_j \int_{M^2} \partial_t^{j} (|u|^{p-1})\ \partial_t^{k-j} u\ \partial_t^{k} \overline{u}\ dvol_g \right| \ ds$$

$$\lesssim_{\epsilon,\ ||u_0||_{H^1}} \sum_{j=1}^{k-1} \sum_{\substack{j_1+\ldots+j_{p-1}=j \\ j_1=\max\{j_1,\ldots,j_{p-1}\}}} \int_0^{\tau} \left(\prod_{i=2}^{p-1} ||u||_{H^{2j_i+1}}^{1-\epsilon} ||u||_{H^{2j_i+2}}^{\epsilon} \right) \times$$

$$\times ||\partial_t^{j_1} u||_{L^4} ||\partial_t^{k-j} u||_{L^4} ||u||_{H^{2k}}\ ds. \tag{4.52}$$

Due to Hölder's inequality, we have

$$(4.52) \leq \sum_{j=1}^{k-1} \sum_{\substack{j_1+\ldots+j_{p-1}=j \\ j_1=\max\{j_1,\ldots,j_{p-1}\}}} \left(\int_0^{\tau} ||\partial_t^{j_1} u||_{L^4}^4\ ds \right)^{1/4} \left(\int_0^{\tau} ||\partial_t^{k-j} u||_{L^4}^4\ ds \right)^{1/4} \times$$

$$\times \left(\int_0^{\tau} \left(\prod_{i=2}^{p-1} ||u||_{H^{2j_i+1}}^{1-\epsilon} ||u||_{H^{2j_i+2}}^{\epsilon} \right)^2 ||u||_{H^{2k}}^2\ ds \right)^{1/2}$$

$$\leq \sum_{j=1}^{k-1} \sum_{\substack{j_1+\ldots+j_{p-1}=j \\ j_1=\max\{j_1,\ldots,j_{p-1}\}}} \sqrt{\tau} \left(\prod_{i=2}^{p-1} ||u||_{L_\tau^\infty H^{2j_i+1}}^{1-\epsilon} ||u||_{L_\tau^\infty H^{2j_i+2}}^{\epsilon} \right) \times$$

$$\times ||u||_{L_\tau^\infty H^{2k}} ||\partial_t^{j_1} u||_{L_\tau^4 L^4} ||\partial_t^{k-j} u||_{L_\tau^4 L^4}. \tag{4.53}$$

Moreover, due to (4.25), we have

$$(4.53) \lesssim_{\epsilon,\ ||u_0||_{H^1}} \sum_{j=1}^{k-1} \sum_{\substack{j_1+\ldots+j_{p-1}=j \\ j_1=\max\{j_1,\ldots,j_{p-1}\}}} \sqrt{\tau} \left(\prod_{i=2}^{p-1} ||u||_{L_\tau^\infty H^{2j_i+1}}^{1-\epsilon} ||u||_{L_\tau^\infty H^{2j_i+2}}^{\epsilon} \right) \times$$

$$\times ||u||_{L_\tau^\infty H^{2k}} ||u||_{L_\tau^\infty H^{2j_1}}^{1-s_0} ||u||_{L_\tau^\infty H^{2j_1+1}}^{s_0} ||u||_{L_\tau^\infty H^{2j_1+2}}^{\epsilon} \times$$

$$\times ||u||_{L_\tau^\infty H^{2k-2j}}^{1-s_0} ||u||_{L_\tau^\infty H^{2k-2j+1}}^{s_0} ||u||_{L_\tau^\infty H^{2k-2j+2}}^{\epsilon}.$$

By applying (4.31) to each factor, we get

$$(4.3) \lesssim_{||u_0||_{H^1}} \sum_{j=1}^{k-1} \sum_{\substack{j_1+\ldots+j_{p-1}=j \\ j_1=\max\{j_1,\ldots,j_{p-1}\}}} \sqrt{\tau}||u||_{L^\infty_\tau H^{2k}(M^2)}^{\frac{4k-3+2s_0}{2k-1}+\epsilon} \lesssim \sqrt{\tau}||u||_{L^\infty_\tau H^{2k}(M^2)}^{\frac{4k-3+2s_0}{2k-1}+\epsilon},$$

therefore

$$\int_0^\tau \left| Im \sum_{j=1}^{k-1} c_j \int_{M^2} \partial_t^j(|u|^{p-1})\ \partial_t^{k-j} u\ \partial_t^k \overline{u}\ dvol_g \right| ds$$
$$\lesssim_{\epsilon,\ ||u_0||_{H^1}} \sqrt{\tau}||u||_{L^\infty_\tau H^{2k}}^{\frac{4k-3+2s_0}{2k-1}+\epsilon}.$$

Putting all together, we finally get

$$\int_0^\tau |r.h.s.\ of\ (4.16)|\, ds \lesssim_{\epsilon,\ ||u_0||_{H^1}} \sqrt{\tau}||u||_{L^\infty_\tau H^{2k}(M^2)}^{\frac{4k-3+2s_0}{2k-1}+\epsilon} + ||u||_{L^\infty_\tau H^{2k}(M^2)}^{\frac{4k-4}{2k-1}+\epsilon}.$$

which concludes the proof of the proposition. □

Proposition 4.3.5. *Let $u \in C(\mathbb{R}, H^{2k}(M^2))$ be the unique global solution to (4.1) with $m = 2k$, $k \in \mathbb{N}$. For every $\tau \in (0,1)$, we have*

$$||u(\tau,\cdot)||_{H^{2k}(M^2)}^2 - ||u(0,\cdot)||_{H^{2k}(M^2)}^2$$
$$\lesssim_{\epsilon,\ ||u_0||_{H^1}} \sqrt{\tau}||u||_{L^\infty_\tau H^{2k}(M^2)}^{\frac{4k-3+2s_0}{2k-1}+\epsilon} + ||u||_{L^\infty_\tau H^{2k}(M^2)}^{\frac{4k-4}{2k-1}+\epsilon}.$$

Proof. Let $\mathcal{R}_{2k}(u)$ be as in Definition 4.2.1. We have already seen that

$$|\partial_t^{k-1}\nabla_g(|u|^2)|_g^2 \lesssim \sum_{k_1+k_2=k-1} |\partial_t^{k_1} u|^2 |\partial_t^{k_2}\nabla_g u|_g^2,$$

therefore

$$\int_{M^2} |\partial_t^{k-1}\nabla_g(|u|^2)|_g^2\ |u|^{p-3} dvol_g$$
$$\lesssim \sum_{k_1+k_2=k-1} \int_{M^2} |\partial_t^{k_1} u|^2\ |\partial_t^{k_2}\nabla_g u|_g^2\ |u|^{p-3}\ dvol_g$$
$$\leq \sum_{k_1+k_2=k-1} ||\partial_t^{k_1} u||_{L^\infty}^2 ||\partial_t^{k_2} u||_{H^1}^2 ||u||_{L^\infty}^{p-3}. \tag{4.54}$$

Due to (1.2), (4.13) and (4.4), we have

$$(4.54) \lesssim_{\epsilon} \sum_{k_1+k_2=k-1} ||\partial_t^{k_1} u||_{H^1}^{2(1-\epsilon)} ||\partial_t^{k_1} u||_{H^2}^{2\epsilon} ||\partial_t^{k_2} u||_{H^1}^2 ||u||_{H^1}^{(1-\epsilon)(p-3)} ||u||_{H^2}^{\epsilon(p-3)}$$

$$\lesssim_{||u_0||_{H^1}} \sum_{k_1+k_2=k-1} ||u||_{H^{2k_1+1}}^{2(1-\epsilon)} ||u||_{H^{2k_1+2}}^{2\epsilon} ||u||_{H^{2k_2+1}}^2 ||u||_{H^2}^{\epsilon(p-3)}, \qquad (4.55)$$

therefore, by applying (4.31) to each factor we, obtain

$$(4.55) \lesssim_{||u_0||_{H^1}} \sum_{k_1+k_2=k-1} ||u||_{H^{2k}}^{\frac{4k-4}{2k-1}+\epsilon} \lesssim ||u||_{H^{2k}}^{\frac{4k-4}{2k-1}+\epsilon}.$$

We have already seen that

$$|\partial_t^{k-1}(|u|^{p-1}u)|^2 \lesssim \sum_{\substack{k_1+\ldots+k_p=k-1 \\ k_1=\max\{k_1,\ldots,k_p\}}} \left(\prod_{i=1}^{p} |\partial_t^{k_i} u|^2 \right),$$

therefore

$$\int_{M^2} |\partial_t^{k-1}(|u|^{p-1}u)|^2 \, dvol_g \lesssim \sum_{\substack{k_1+\ldots+k_p=k-1 \\ k_1=\max\{k_1,\ldots,k_p\}}} \int_{M^2} \left(\prod_{i=1}^{p} |\partial_t^{k_i} u|^2 \right) dvol_g$$

$$\leq \sum_{\substack{k_1+\ldots+k_p=k-1 \\ k_1=\max\{k_1,\ldots,k_p\}}} \left(\prod_{i=2}^{p} ||\partial_t^{k_i} u||_{L^\infty}^2 \right) ||\partial_t^{k_1} u||_{L^2}^2. \qquad (4.56)$$

Due to (1.2) and (4.13), we have

$$(4.56) \lesssim_{\epsilon} \sum_{\substack{k_1+\ldots+k_p=k-1 \\ k_1=\max\{k_1,\ldots,k_p\}}} \left(\prod_{i=2}^{p} ||\partial_t^{k_i} u||_{H^1}^{2(1-\epsilon)} ||\partial_t^{k_i} u||_{H^2}^{2\epsilon} \right) ||\partial_t^{k_1} u||_{L^2}^2$$

$$\lesssim_{||u_0||_{H^1}} \sum_{\substack{k_1+\ldots+k_p=k-1 \\ k_1=\max\{k_1,\ldots,k_p\}}} ||u||_{H^{2k_1}}^2 \left(\prod_{i=2}^{p} ||u||_{H^{2k_i+1}}^{2(1-\epsilon)} ||u||_{H^{2k_i+2}}^{2\epsilon} \right) \qquad (4.57)$$

therefore, by applying (4.31) to each factor, we obtain

$$(4.57) \lesssim_{||u_0|_{H^1}} \sum_{\substack{k_1+\ldots+k_p=k-1 \\ k_1=\max\{k_1,\ldots,k_p\}}} ||u||_{H^{2k}}^{\frac{4k-6}{2k-1}+\epsilon} \lesssim ||u||_{H^{2k}}^{\frac{4k-6}{2k-1}+\epsilon}.$$

Putting all together, we deduce that

$$|\mathcal{R}_{2k}(u)| \lesssim_{\epsilon,\ ||u_0||_{H^1}} ||u||_{H^{2k}}^{\frac{4k-4}{2k-1}+\epsilon} + ||u||_{H^{2k}}^{\frac{4k-6}{2k-1}+\epsilon}. \tag{4.58}$$

From Proposition 4.3.4, we have

$$\left|\int_0^\tau \frac{d}{ds}\mathcal{E}_{2k}(u)\ ds\right| \leq \int_0^\tau |r.h.s.\ of\ (4.16)|\ ds$$
$$\lesssim_{\epsilon,\ ||u_0||_{H^1}} \sqrt{\tau}||u||_{L^\infty_\tau H^{2k}}^{\frac{4k-3+2s_0}{2k-1}+\epsilon} + ||u||_{L^\infty_\tau H^{2k}}^{\frac{4k-4}{2k-1}+\epsilon}. \tag{4.59}$$

By applying the Fundamental Theorem of Calculus on the l.h.s. of (4.59), we get

$$\left|||\partial_t^k u(\tau,\cdot)||_{L^2}^2 - ||\partial_t^k u(0,\cdot)||_{L^2}^2 + \mathcal{R}_{2k}(u)(\tau) - \mathcal{R}_{2k}(u)(0)\right|$$
$$\lesssim_{\epsilon,\ ||u_0||_{H^1}} \sqrt{\tau}||u||_{L^\infty_\tau H^{2k}}^{\frac{4k-3+2s_0}{2k-1}+\epsilon} + ||u||_{L^\infty_\tau H^{2k}}^{\frac{4k-4}{2k-1}+\epsilon},$$

therefore

$$||\partial_t^k u(\tau,\cdot)||_{L^2}^2 - ||\partial_t^k u(0,\cdot)||_{L^2}^2 - 2\sup_{t\in(0,\tau)} |\mathcal{R}_{2k}(u)(t)|$$
$$\lesssim_{\epsilon,\ ||u_0||_{H^1}} \sqrt{\tau}||u||_{L^\infty_\tau H^{2k}}^{\frac{4k-3+2s_0}{2k-1}+\epsilon} + ||u||_{L^\infty_\tau H^{2k}}^{\frac{4k-4}{2k-1}+\epsilon}.$$

By using (4.58), we get

$$||\partial_t^k u(\tau,\cdot)||_{L^2}^2 - ||\partial_t^k u(0,\cdot)||_{L^2}^2 \lesssim \sqrt{\tau}||u||_{L^\infty_\tau H^{2k}}^{\frac{4k-3+2s_0}{2k-1}+\epsilon} + ||u||_{L^\infty_\tau H^{2k}}^{\frac{4k-4}{2k-1}+\epsilon}. \tag{4.60}$$

Finally, due to (4.7), (4.13) and (4.31), we have

$$||u(\tau,\cdot)||_{H^{2k}}^2 - ||\partial_t^k u(\tau,\cdot)||_{L^2}^2$$
$$\leq \left(||\partial_t^k u(\tau,\cdot) - i^k\Delta_g^k u(\tau,\cdot)||_{L^2} + ||\partial_t^k u(\tau,\cdot)||_{L^2}\right)^2 +$$
$$+||u(\tau,\cdot)||_{H^{2k-1}}^2 - ||\partial_t^k u(\tau,\cdot)||_{L^2}^2$$
$$\lesssim_{||u_0||_{H^1}} ||u(\tau,\cdot)||_{H^{2k}}^{\frac{4k-3}{2k-1}} + ||u(\tau,\cdot)||_{H^{2k}}^{\frac{4k-4}{2k-1}}$$

and

$$||\partial_t^k u(0,\cdot)||_{L^2}^2 - ||u(0,\cdot)||_{H^{2k}}^2$$
$$\leq \left(||\partial_t^k u(0,\cdot) - i^k\Delta_g^k u(0,\cdot)||_{L^2} + ||i^k\Delta_g^k u(0,\cdot)||_{L^2}\right)^2 - ||i^k\Delta_g^k u(0,\cdot)||_{L^2}^2$$
$$\lesssim_{||u_0||_{H^1}} ||u(0,\cdot)||_{H^{2k}}^{\frac{4k-3}{2k-1}} + ||u(0,\cdot)||_{H^{2k}}^{\frac{4k-4}{2k-1}},$$

therefore, from (4.60), we get

$$\begin{aligned}
&||u(\tau,\cdot)||_{H^{2k}}^2 - ||u(0,\cdot)||_{H^{2k}}^2 \\
&\lesssim_{\epsilon,\ ||u_0||_{H^1}} \sqrt{\tau}||u||_{L_\tau^\infty H^{2k}}^{\frac{4k-3+2s_0}{2k-1}+\epsilon} + ||u||_{L_\tau^\infty H^{2k}}^{\frac{4k-4}{2k-1}+\epsilon} + ||u(\tau,\cdot)||_{H^{2k}}^{\frac{4k-3}{2k-1}} + \\
&+||u(\tau,\cdot)||_{H^{2k}}^{\frac{4k-4}{2k-1}} + ||u(0,\cdot)||_{H^{2k}}^{\frac{4k-3}{2k-1}} + ||u(0,\cdot)||_{H^{2k}}^{\frac{4k-4}{2k-1}} \\
&\lesssim \sqrt{\tau}||u||_{L_\tau^\infty H^{2k}}^{\frac{4k-3+2s_0}{2k-1}+\epsilon} + ||u||_{L_\tau^\infty H^{2k}}^{\frac{4k-4}{2k-1}+\epsilon},
\end{aligned}$$

which concludes the proof of the proposition. □

Now we take advantage of the previous estimate to show the polynomial growth in time of the Sobolev norm $||\cdot||_{H^{2k}}$

Theorem 4.3.6. *Let $u \in C(\mathbb{R}, H^{2k}(M^2))$ be the unique global solution to (4.1) with $m = 2k$, $k \in \mathbb{N}$. For every $T > 0$, we have the following bound:*

$$\sup_{t\in(0,T)} ||u(t,\cdot)||_{H^m(M^2)} \leq C(\max\{1,T\})^{\frac{2k-1}{1-s_0}+\epsilon},$$

where $C = C(\epsilon, k, ||u_0||_{H^{2k}})$ and $s_0 \in [0, \frac{1}{4}]$ is given by Proposition 3.2.3.

Proof. Let us consider $\tau \in (0,1)$ given by the local Cauchy theory. From Proposition 4.3.5, Remark 4.5 and elementary computations, we get

$$||u(\tau,\cdot)||_{H^{2k}}^2 \leq ||u(0.\cdot)||_{H^{2k}}^2 + C||u||_{L_\tau^\infty H^{2k}}^{2-2\gamma},$$

where $\gamma := \frac{1-2s_0}{4k-2} + \epsilon$ and $C = C(\epsilon, k, ||u_0||_{H^{2k}}) > 0$ may change in the following computations. By iteration, for every $n \in \mathbb{N} \cup \{0\}$, we obtain

$$||u(n\tau+\tau,\cdot)||_{H^{2k}}^2 \leq ||u(n\tau,\cdot)||_{H^{2k}}^2 + C\left(\sup_{t\in(n\tau,n\tau+\tau)} ||u(t,\cdot)||_{H^{2k}}^{2-2\gamma}\right),$$

therefore, since the map data solution is continuous (see 2.2.3), we have

$$||u(n\tau+\tau,\cdot)||_{H^{2k}}^2 \leq ||u(n\tau,\cdot)||_{H^{2k}}^2 + C||u(n\tau,\cdot)||_{H^{2k}}^{2-2\gamma}.$$

Now, for every $n \in \mathbb{N} \cup \{0\}$, we claim that

$$||u(n\tau,\cdot)||_{H^{2k}}^2 \leq C\left(\max\{1,n\}\right)^{1/\gamma}. \tag{4.61}$$

For $n = 0$ the claim is trivially satisfied; we prove it in case $n \geq 1$. Let us define $\alpha_n = ||u(n\tau, \cdot)||^2_{H^{2k}}$, by summing up we obtain

$$\alpha_n - \alpha_0 = \sum_{j=0}^{n-1} (\alpha_{j+1} - \alpha_j) \leq C \sum_{j=0}^{n-1} \alpha_j^{1-\gamma} \leq Cn \left(\sup_{j\in\{0,\cdots,n\}} \alpha_j \right)^{1-\gamma},$$

therefore

$$\left(\sup_{j\in\{0,\cdots,n\}} \alpha_j \right) \leq Cn \left(\sup_{j\in\{0,\cdots,n\}} \alpha_j \right)^{1-\gamma} + \alpha_0 \leq Cn \left(\sup_{j\in\{0,\cdots,n\}} \alpha_j \right)^{1-\gamma}.$$

By elementary computations, we get

$$||u(n\tau, \cdot)||^2_{H^{2k}} \leq \left(\sup_{j\in\{0,\cdots,n\}} \alpha_j \right) \leq Cn^{1/\gamma},$$

which proves the claim.

Now, from the continuity of the map data solution and due to (4.61), we have

$$\sup_{t\in(n\tau,n\tau+\tau)} ||u(t, \cdot)||^2_{H^{2k}} \leq C||u(n\tau, \cdot)||^2_{H^{2k}} \leq C \left(\max\{1, n\}\right)^{1/\gamma}.$$

Finally, given $N \in \mathbb{N}$ such that $N\tau \leq T \leq N\tau + \tau$, we have

$$\begin{aligned}\sup_{t\in(0,T)} ||u(t, \cdot)||^2_{H^{2k}} &\leq \max_{n\in\{0,\cdots,N\}} \left(\sup_{t\in(n\tau,n\tau+\tau)} ||u(t, \cdot)||^2_{H^{2k}} \right) \leq C \max\{1, N\}^{1/\gamma} \\ &\leq C \left(\max\{\tau, T\}\right)^{1/\gamma} \leq C \left(\max\{1, T\}\right)^{1/\gamma}\end{aligned}$$

since $\tau < 1$, therefore

$$\sup_{t\in(0,T)} ||u(t, \cdot)||_{H^{2k}} \leq C \left(\max\{1, T\}\right)^{1/2\gamma} = C \left(\max\{1, T\}\right)^{\frac{1-2s_0}{2k-1}+\epsilon},$$

which concludes the proof of the Theorem. □

4.4 The Cubic 3-Dimensional Case ($m = 2k$)

The aim of this section is to provide a bound on the growth of the continuous function

$$\mathbb{R}^+ \ni t \longmapsto ||u(t, \cdot)||_{H^{2k}(M^3)} \tag{4.62}$$

where $k \in \mathbb{N}$ and u is the unique global solution to the Cauchy problem (4.2) with $m = 2k$. As a first step we shall prove "modified" endpoint Strichartz estimates that we use to establish a priori bounds to small time increments for (4.62). Finally, we take advantage of an iterative argument to show the exponential growth of (4.62).

Notation 4.4.1. *In the following computations we call ϵ a positive quantity that may change from line to line. We shall assume ϵ to be small enough at each step.*

Proposition 4.4.2. *Let $u \in C(\mathbb{R}, H^{2k}(M^3))$ be the unique global solution to (4.2) with $m = 2k$, $k \in \mathbb{N}$. For every $\tau \in (0,1)$ and $j \leq k$, we have*

$$\begin{aligned}||\partial_t^j u||_{L^2_\tau L^6(M^3)} \lesssim_{\epsilon,\ ||u_0||_{H^1}} ||u||^{1-\epsilon}_{L^\infty_\tau H^{2j}(M^3)} ||u||^{\epsilon}_{L^\infty_\tau H^{2j+1}(M^3)} + \\ + \sqrt{\tau} ||u||^{1/2}_{L^\infty_\tau H^{2j}(M^3)} ||u||^{1/2}_{L^\infty_\tau H^{2j+1}(M^3)} + \\ + \sqrt{\tau} \sum_{\substack{j_1+j_2+j_3=j \\ j_1=max\{j_1,j_2,j_3\}}} ||u||_{L^\infty_\tau H^{2j_1}(M^3)} ||u||_{L^\infty_\tau H^{2j_2+1}(M^3)} ||u||_{L^\infty_\tau H^{2j_3+1}(M^3)},\end{aligned} \tag{4.63}$$

and

$$\begin{aligned}||\partial_t^j u||_{L^2_\tau W^{1,6}(M^3)} \lesssim_{\epsilon,\ ||u_0||_{H^1}} ||u||^{1-\epsilon}_{L^\infty_\tau H^{2j+1}(M^3)} ||u||^{\epsilon}_{L^\infty_\tau H^{2j+2}(M^3)} + \\ + \sqrt{\tau} ||u||^{1/2}_{L^\infty_\tau H^{2j+1}(M^3)} ||u||^{1/2}_{L^\infty_\tau H^{2j+2}(M^3)} + \\ + \sqrt{\tau} \sum_{j_1+j_2+j_3=j} ||u||_{L^\infty_\tau H^{2j_1+1}(M^3)} ||u||_{L^\infty_\tau H^{2j_2+1}(M^3)} ||u||_{L^\infty_\tau H^{2j_3+1}(M^3)}.\end{aligned} \tag{4.64}$$

Proof. From Proposition 3.2.7, we have

$$||\partial_t^j u||_{L^2_\tau L^6} \lesssim_\epsilon ||\partial_t^j u||_{L^\infty_\tau H^\epsilon} + ||\partial_t^j u||_{L^2_\tau H^{1/2}} + ||\partial_t^j (|u|^2 u)||_{L^2_\tau L^{6/5}}, \tag{4.65}$$

$$||\partial_t^j u||_{L^2_\tau W^{1,6}} \lesssim_\epsilon ||\partial_t^j u||_{L^\infty_\tau H^{1+\epsilon}} + ||\partial_t^j u||_{L^2_\tau H^{3/2}} + ||\partial_t^j (|u|^2 u)||_{L^2_\tau W^{1,6/5}}. \tag{4.66}$$

We prove (4.63) first. By using (1.1) and (4.15), we get

$$||\partial_t^j u||_{H^\epsilon} \lesssim_\epsilon ||\partial_t^j u||^{1-\epsilon}_{L^2} ||\partial_t^j u||^{\epsilon}_{H^1} \lesssim_{||u_0||_{H^1}} ||u||^{1-\epsilon}_{H^{2j}} ||u||^{\epsilon}_{H^{2j+1}},$$

therefore

$$||\partial_t^j u||_{L^\infty_\tau H^\epsilon} \lesssim_{\epsilon,\ ||u_0||_{H^1}} ||u||^{1-\epsilon}_{L^\infty_\tau H^{2j}} ||u||^{\epsilon}_{L^\infty_\tau H^{2j+1}}.$$

From (1.1) and (4.15), we get

$$||\partial_t^j u||_{H^{1/2}} \lesssim ||\partial_t^j u||_{L^2}^{1/2} ||\partial_t^j u||_{H^1}^{1/2} \lesssim_{||u_0||_{H^1}} ||u||_{H^{2j}}^{1/2} ||u||_{H^{2j+1}}^{1/2},$$

therefore

$$\begin{aligned} ||\partial_t^j u||_{L^2_\tau H^{1/2}} &\lesssim_{||u_0||_{H^1}} \left(\int_0^\tau ||u||_{H^{2j}} ||u||_{H^{2j+1}} \, ds \right)^{1/2} \\ &\leq \sqrt{\tau} ||u||_{L^\infty_\tau H^{2j}}^{1/2} ||u||_{L^\infty_\tau H^{2j+1}}^{1/2}. \end{aligned}$$

Now we focus on the nonlinear term of (4.65). By expanding the time derivative, we get

$$\begin{aligned} ||\partial_t^j (|u|^2 u)||_{L^{6/5}} &= \left(\int_{M^3} |\partial_t^j (|u|^2 u)|^{6/5} \, dvol_g \right)^{5/6} \\ \lesssim \sum_{\substack{j_1+j_2+j_3=j \\ j_1=\max\{j_1,j_2,j_3\}}} &\left(\int_{M^3} |\partial_t^{j_1} u|^{6/5} |\partial_t^{j_2} u|^{6/5} |\partial_t^{j_3} u|^{6/5} \, dvol_g \right)^{5/6}. \end{aligned} \tag{4.67}$$

Moreover, by using the Hölder inequality, we obtain

$$(4.67) \leq \sum_{\substack{j_1+j_2+j_3=j \\ j_1=\max\{j_1,j_2,j_3\}}} ||\partial_t^{j_1} u||_{L^2} ||\partial_t^{j_2} u||_{L^6} ||\partial_t^{j_3} u||_{L^6}, \tag{4.68}$$

so that, by the Sobolev embedding $H^1(M^3) \hookrightarrow L^6(M^3)$ and (4.15), we have

$$\begin{aligned} (4.68) &\lesssim \sum_{\substack{j_1+j_2+j_3=j \\ j_1=\max\{j_1,j_2,j_3\}}} ||\partial_t^{j_1} u||_{L^2} ||\partial_t^{j_2} u||_{H^1} ||\partial_t^{j_3} u||_{H^1} \\ &\lesssim_{||u_0||_{H^1}} \sum_{\substack{j_1+j_2+j_3=j \\ j_1=max\{j_1,j_2,j_3\}}} ||u||_{H^{2j_1}} ||u||_{H^{2j_2+1}} ||u||_{H^{2j_3+1}}. \end{aligned}$$

Hence

$$\begin{aligned} &||\partial_t^j (|u|^2 u)||_{L^2_\tau L^{6/5}} \\ \lesssim_{||u_0||_{H^1}} &\sum_{\substack{j_1+j_2+j_3=j \\ j_1=\max\{j_1,j_2,j_3\}}} \left(\int_0^\tau ||u||_{H^{2j_1}}^2 ||u||_{H^{2j_2+1}}^2 ||u||_{H^{2j_3+1}}^2 \, ds \right)^{1/2} \\ \leq \sqrt{\tau} &\sum_{\substack{j_1+j_2+j_3=j \\ j_1=\max\{j_1,j_2,j_3\}}} ||u||_{L^\infty_\tau H^{2j_1}} ||u||_{L^\infty_\tau H^{2j_2+1}} ||u||_{L^\infty_\tau H^{2j_3+1}} \end{aligned}$$

and putting all together we get (4.63).

Now we prove (4.64). Arguing as before, we get

$$||\partial_t^j u||_{L^\infty_\tau H^{1+\epsilon}} \lesssim_{\epsilon,\ ||u_0||_{H^1}} ||u||^{1-\epsilon}_{L^\infty_\tau H^{2j+1}} ||u||^{\epsilon}_{L^\infty_\tau H^{2j+2}}$$
$$||\partial_t^j u||_{L^2_\tau H^{3/2}} \lesssim_{||u_0||_{H^1}} \sqrt{\tau} ||u||^{1/2}_{L^\infty_\tau H^{2j+1}} ||u||^{1/2}_{L^\infty_\tau H^{2j+2}}.$$

We focus then on the nonlinear term of (4.66). We observe that

$$||\partial_t^j(|u|^2 u)||_{W^{1,6/5}} \sim ||\partial_t^j(|u|^2 u)||_{L^{6/5}} + ||\nabla_g \partial_t^j(|u|^2 u)||_{L^{6/5}}.$$

By expanding the time derivative and using the Hölder inequality, we get

$$\begin{aligned} ||\nabla_g \partial_t^j(|u|^2 u)||_{L^{6/5}} &\lesssim \sum_{j_1+j_2+j_3=j} \left(\int_{M^3} |\nabla_g \partial_t^{j_1} u|_g^{6/5} |\partial_t^{j_2} u|^{6/5} |\partial_t^{j_3} u|^{6/5} \, dvol_g \right)^{5/6} \\ &\leq \sum_{j_1+j_2+j_3=j} ||\nabla_g \partial_t^{j_1} u||_{L^2} ||\partial_t^{j_2} u||_{L^6} ||\partial_t^{j_3} u||_{L^6}. \end{aligned} \tag{4.69}$$

Due to the Sobolev embedding $H^1(M^3) \hookrightarrow L^6(M^3)$ and (4.15), we have

$$\begin{aligned} (4.69) &\lesssim \sum_{j_1+j_2+j_3=j} ||\partial_t^{j_1} u||_{H^1} ||\partial_t^{j_2} u||_{H^1} ||\partial_t^{j_3} u||_{H^1} \\ &\lesssim_{||u_0||_{H^1}} \sum_{j_1+j_2+j_3=j} ||u||_{H^{2j_1+1}} ||u||_{H^{2j_2+1}} ||u||_{H^{2j_3+1}}. \end{aligned}$$

Moreover, by previous computations, we deduce that

$$||\partial_t^j(|u|^2 u)||_{L^{6/5}} \lesssim_{||u_0||_{H^1}} \sum_{j_1+j_2+j_3=j} ||u||_{H^{2j_1+1}} ||u||_{H^{2j_2+1}} ||u||_{H^{2j_3+1}},$$

therefore

$$||\partial_t^j(|u|^2 u)||_{W^{1,6/5}} \lesssim_{||u_0||_{H^1}} \sum_{j_1+j_2+j_3=j} ||u||_{H^{2j_1+1}} ||u||_{H^{2j_2+1}} ||u||_{H^{2j_3+1}},$$

and we conclude as before. □

Proposition 4.4.3. *Let $u \in C(\mathbb{R}, H^{2k}(M^3))$ be the unique global solution to (4.2) with $m = 2k$, $k \in \mathbb{N}$. For every $\tau \in (0,1)$ we have*

$$\int_0^\tau |r.h.s.\ of\ (4.23)| \, ds \lesssim_{||u_0||_{H^1}} \tau ||u||^2_{L^\infty_\tau H^{2k}} + ||u||^\gamma_{L^\infty_\tau H^{2k}}$$

for some $\gamma \in (0,2)$.

Proof. By applying (4.31) to each term of (4.63) and (4.64), we get

$$||\partial_t^j u||_{L^2_\tau L^6} \lesssim_{\epsilon,\ ||u_0||_{H^1}} ||u||_{L^\infty_\tau H^{2k}}^{\frac{2j-1}{2k-1}+\epsilon} + \sqrt{\tau}||u||_{L^\infty_\tau H^{2k}}^{\frac{4j-1}{4k-2}} + \sqrt{\tau}||u||_{L^\infty_\tau H^{2k}}^{\frac{2j-1}{2k-1}}, \tag{4.70}$$

$$||\partial_t^j u||_{L^2_\tau W^{1,6}} \lesssim_{\epsilon,\ ||u_0||_{H^1}} ||u||_{L^\infty_\tau H^{2k}}^{\frac{2j}{2k-1}+\epsilon} + \sqrt{\tau}||u||_{L^\infty_\tau H^{2k}}^{\frac{4j+1}{4k-2}} + \sqrt{\tau}||u||_{L^\infty_\tau H^{2k}}^{\frac{2j}{2k-1}}. \tag{4.71}$$

We shall use (4.70) and (4.71) in order to estimate each term of (4.23).

(First Term) By expanding the time derivative and using the Cauchy-Schwarz inequality we get

$$\begin{aligned}
&\int_0^\tau \left| \int_{M^3} \partial_t^k(|u|^2)\partial_t^{k-1}(|\nabla_g u|_g^2)\ dvol_g \right|\ ds \\
\lesssim \sum_{\substack{m_1+m_2=k-1\\ k_1+k_2=k\\ k_1=\max\{k_1,k_2\}}} &\int_0^\tau \int_{M^3} |\partial_t^{k_1} u||\partial_t^{k_2} u||\partial_t^{m_1}\nabla_g u|_g|\partial_t^{m_2}\nabla_g u|_g\ dvol_g\ ds.
\end{aligned}$$

By using Hölder's inequality we obtain

$$\begin{aligned}
&\int_{M^3} |\partial_t^{k_1} u||\partial_t^{k_2} u||\partial_t^{m_1}\nabla_g u|_g|\partial_t^{m_2}\nabla_g u|_g\ dvol_g \\
&\leq ||\partial_t^{k_1} u||_{L^2}||\partial_t^{k_2} u||_{L^6}||\partial_t^{m_1}\nabla_g u||_{L^6}||\partial_t^{m_2}\nabla_g u||_{L^6} \\
&\lesssim ||\partial_t^{k_1} u||_{L^2}||\partial_t^{k_2} u||_{L^6}||\partial_t^{m_1} u||_{W^{1,6}}||\partial_t^{m_2} u||_{W^{1,6}}.
\end{aligned} \tag{4.72}$$

Moreover, due to the Sobolev embedding $H^1(M^3) \hookrightarrow L^6(M^3)$ and (4.15), we have

$$\begin{aligned}
(4.72) &\lesssim ||\partial_t^{k_1} u||_{L^2}||\partial_t^{k_2} u||_{H^1}||\partial_t^{m_1} u||_{W^{1,6}}||\partial_t^{m_2} u||_{W^{1,6}} \\
&\lesssim_{||u_0||_{H^1}} ||u||_{H^{2k_1}}||u||_{H^{2k_2+1}}||\partial_t^{m_1} u||_{W^{1,6}}||\partial_t^{m_2} u||_{W^{1,6}},
\end{aligned}$$

therefore

$$\begin{aligned}
&\int_0^\tau \left| \int_{M^3} \partial_t^k(|u|^2)\partial_t^{k-1}(|\nabla_g u|_g^2)\ dvol_g \right|\ ds \\
\lesssim_{||u_0||_{H^1}} \sum_{\substack{m_1+m_2=k-1\\ k_1+k_2=k\\ k_1=\max\{k_1,k_2\}}} &\int_0^\tau ||u||_{H^{2k_1}}||u||_{H^{2k_2+1}}||\partial_t^{m_1} u||_{W^{1,6}}||\partial_t^{m_2} u||_{W^{1,6}}\ ds.
\end{aligned} \tag{4.73}$$

By using Hölder's inequality and (4.31), we have

$$(4.73) \leq \sum_{\substack{m_1+m_2=k-1\\ k_1+k_2=k\\ k_1=\max\{k_1,k_2\}}} ||u||_{L^\infty_\tau H^{2k_1}} ||u||_{L^\infty_\tau H^{2k_2+1}} ||\partial_t^{m_1} u||_{L^2_\tau W^{1,6}} ||\partial_t^{m_2} u||_{L^2_\tau W^{1,6}}$$

$$\lesssim_{||u_0||_{H^1}} \sum_{\substack{m_1+m_2=k-1\\ k_1+k_2=k\\ k_1=\max\{k_1,k_2\}}} ||u||_{L^\infty_\tau H^{2k}} ||\partial_t^{m_1} u||_{L^2_\tau W^{1,6}} ||\partial_t^{m_2} u||_{L^2_\tau W^{1,6}}. \qquad (4.74)$$

From (4.71) we have

$$||\partial_t^{m_1} u||_{L^2_\tau W^{1,6}} ||\partial_t^{m_2} u||_{L^2_\tau W^{1,6}}$$

$$\lesssim_{\epsilon,\ ||u_0||_{H^1}} \left(||u||^{\frac{2m_1}{2k-1}+\epsilon}_{L^\infty_\tau H^{2k}} + \sqrt{\tau}||u||^{\frac{4m_1+1}{4k-2}}_{L^\infty_\tau H^{2k}} + \sqrt{\tau}||u||^{\frac{2m_1}{2k-1}}_{L^\infty_\tau H^{2k}} \right) \times$$

$$\times \left(||u||^{\frac{2m_2}{2k-1}+\epsilon}_{L^\infty_\tau H^{2k}} + \sqrt{\tau}||u||^{\frac{4m_2+1}{4k-2}}_{L^\infty_\tau H^{2k}} + \sqrt{\tau}||u||^{\frac{2m_2}{2k-1}}_{L^\infty_\tau H^{2k}} \right)$$

$$\lesssim \tau ||u||^{\frac{2m_1+2m_2+1}{2k-1}}_{L^\infty_\tau H^{2k}} + ||u||^{\alpha}_{L^\infty_\tau H^{2k}},$$

where $\alpha < \frac{2m_1+2m_2+1}{2k-1}$, since

$$\frac{2m_1}{2k-1} + \epsilon, \frac{2m_1}{2k-1} < \frac{4m_1+1}{4k-2},$$
$$\frac{2m_2}{2k-1} + \epsilon, \frac{2m_2}{2k-1} < \frac{4m_2+1}{4k-2}.$$

Therefore

$$(4.74) \lesssim_{||u_0||_{H^1}} \sum_{\substack{m_1+m_2=k-1\\ k_1+k_2=k\\ k_1=\max\{k_1,k_2\}}} \left(\tau ||u||^2_{L^\infty_\tau H^{2k}} + ||u||^\gamma_{L^\infty_\tau H^{2k}} \right),$$

where $\gamma := 1 + \alpha < 2$. Since γ depends only on k, we finally get

$$\int_0^\tau \left| \int_{M^3} \partial_t^k(|u|^2) \partial_t^{k-1}(|\nabla_g u|_g^2)\ dvol_g \right|\ ds \lesssim_{||u_0||_{H^1}} \tau ||u||^2_{L^\infty_\tau H^{2k}} + ||u||^\gamma_{L^\infty_\tau H^{2k}}.$$

(Second Term) By expanding the time derivative we get

$$\int_0^\tau \left| Re \sum_{j=0}^{k-2} c_j \int_{M^3} \partial_t^k(|u|^2)\ \partial_t^j(\Delta_g u)\ \partial_t^{k-1-j}\overline{u}\ dvol_g \right|\ ds$$

$$\lesssim \sum_{j=0}^{k-2} \sum_{\substack{k_1+k_2=k\\ k_2=\max\{k_1,k_2\}}} \int_0^\tau \int_{M^3} |\partial_t^{k_1} u||\partial_t^{k_2} u||\partial_t^j(\Delta_g u)||\partial_t^{k-1-j} u|\ dvol_g\ ds.$$

By using Hölder's inequality, we obtain

$$\begin{aligned}\int_{M^3} &|\partial_t^{k_1} u||\partial_t^{k_2} u||\partial_t^{j}(\Delta_g u)||\partial_t^{k-1-j} u|\ dvol_g \\ &\leq ||\partial_t^{k_1} u||_{L^2}||\partial_t^{k_2} u||_{L^6}||\partial_t^{j}(\Delta_g u)||_{L^6}||\partial_t^{k-1-j} u||_{L^6}. \end{aligned} \tag{4.75}$$

Moreover, due to the Sobolev embedding $H^1(M^3) \hookrightarrow L^6(M^3)$ and (4.15), we have

$$\begin{aligned}(4.75) &\lesssim ||\partial_t^{k_1} u||_{L^2}||\partial_t^{k_2} u||_{L^6}||\partial_t^{j}(\Delta_g u)||_{H^1}||\partial_t^{k-1-j} u||_{L^6} \\ &\lesssim_{||u_0||_{H^1}} ||u||_{H^{2k_1}}||\partial_t^{k_2} u||_{L^6}||u||_{H^{2j+3}}||\partial_t^{k-1-j} u||_{L^6}\end{aligned}$$

therefore

$$\begin{aligned}&\int_0^\tau \left| Re \sum_{j=0}^{k-2} c_j \int_{M^3} \partial_t^k(|u|^2)\ \partial_t^j(\Delta_g u)\ \partial_t^{k-1-j}\overline{u}\ dvol_g \right| ds \\ &\lesssim_{||u_0||_{H^1}} \sum_{j=0}^{k-2} \sum_{\substack{k_1+k_2=k \\ k_2=\max\{k_1,k_2\}}} \int_0^\tau ||u||_{H^{2k_1}}||\partial_t^{k_2} u||_{L^6}||u||_{H^{2j+3}}||\partial_t^{k-1-j} u||_{L^6}\ ds.\end{aligned} \tag{4.76}$$

By using Hölder's inequality and (4.31), we have

$$\begin{aligned}(4.76) &\leq \sum_{j=0}^{k-2} \sum_{\substack{k_1+k_2=k \\ k_2=\max\{k_1,k_2\}}} ||u||_{L^\infty_\tau H^{2k_1}}||u||_{L^\infty_\tau H^{2j+3}}||\partial_t^{k_2} u||_{L^2_\tau L^6}||\partial_t^{k-1-j} u||_{L^2_\tau L^6} \\ &\lesssim_{||u_0||_{H^1}} \sum_{j=0}^{k-2} \sum_{\substack{k_1+k_2=k \\ k_2=\max\{k_1,k_2\}}} ||u||^{\frac{2k_1+2j+1}{2k-1}}_{L^\infty_\tau H^{2k}} ||\partial_t^{k_2} u||_{L^2_\tau L^6}||\partial_t^{k-1-j} u||_{L^2_\tau L^6}.\end{aligned} \tag{4.77}$$

From (4.70) we have

$$\begin{aligned}&||\partial_t^{k_2} u||_{L^2_\tau L^6}||\partial_t^{k-1-j} u||_{L^2_\tau L^6} \\ &\lesssim_{\epsilon,\ ||u_0||_{H^1}} \left(||u||^{\frac{2k_2-1}{2k-1}+\epsilon}_{L^\infty_\tau H^{2k}} + \sqrt{\tau}||u||^{\frac{4k_2-1}{4k-2}}_{L^\infty_\tau H^{2k}} + \sqrt{\tau}||u||^{\frac{2k_2-1}{2k-1}}_{L^\infty_\tau H^{2k}} \right) \times \\ &\times \left(||u||^{\frac{2k-2j-3}{2k-1}+\epsilon}_{L^\infty_\tau H^{2k}} + \sqrt{\tau}||u||^{\frac{4k-4j-5}{4k-2}}_{L^\infty_\tau H^{2k}} + \sqrt{\tau}||u||^{\frac{2k-2j-3}{2k-1}}_{L^\infty_\tau H^{2k}} \right) \\ &\lesssim \tau||u||^{\frac{2k_2+2k-2j-3}{2k-1}}_{L^\infty_\tau H^{2k}} + ||u||^{\alpha}_{L^\infty_\tau H^{2k}},\end{aligned}$$

where $\alpha < \frac{2k_2+2k-2j-3}{2k-1}$, since

$$\frac{2k_2-1}{2k-1}+\epsilon, \frac{2k_2-1}{2k-1} < \frac{4k_2-1}{4k-2},$$
$$\frac{2k-2j-3}{2k-1}+\epsilon, \frac{2k-2j-3}{2k-1} < \frac{4k-4j-5}{4k-2}.$$

Therefore

$$(4.77) \lesssim_{||u_0||_{H^1}} \sum_{j=0}^{k-2} \sum_{\substack{k_1+k_2=k \\ k_2=\max\{k_1,k_2\}}} \left(\tau ||u||^2_{L^\infty_\tau H^{2k}} + ||u||^\gamma_{L^\infty_\tau H^{2k}} \right),$$

where $\gamma := \frac{2k_1+2j+1}{2k-1} + \alpha < 2$. Since γ depends only on k, we finally get

$$\int_0^\tau \left| Re \sum_{j=0}^{k-2} c_j \int_{M^3} \partial_t^k(|u|^2)\ \partial_t^j(\Delta_g u)\ \partial_t^{k-1-j}\overline{u}\ dvol_g \right| ds$$
$$\lesssim_{||u_0||_{H^1}} \tau ||u||^2_{L^\infty_\tau H^{2k}} + ||u||^\gamma_{L^\infty_\tau H^{2k}}.$$

(Third Term) By expanding the time derivative, we get

$$\int_0^\tau \left| Re \sum_{j=0}^{k-1} c_j \int_{M^3} \partial_t^j(|u|^2)\ \partial_t^{k-j} u\ \partial_t^{k-1}(|u|^2\overline{u})\ dvol_g \right| ds$$
$$\lesssim \sum_{j=0}^{k-1} \sum_{\substack{j_1+j_2=j \\ k_1+k_2+k_3=k-1}} \int_0^\tau \int_{M^3} |\partial_t^{j_1} u||\partial_t^{j_2} u||\partial_t^{k-j} u||\partial_t^{k_1} u||\partial_t^{k_2} u||\partial_t^{k_3} u|\ dvol_g\ ds.$$

By using Hölder's inequality we obtain

$$\int_{M^3} |\partial_t^{j_1} u||\partial_t^{j_2} u||\partial_t^{k-j} u||\partial_t^{k_1} u||\partial_t^{k_2} u||\partial_t^{k_3} u|\ dvol_g$$
$$\leq ||\partial_t^{j_1} u||_{L^\infty} ||\partial_t^{j_2} u||_{L^\infty} ||\partial_t^{k-j} u||_{L^2} ||\partial_t^{k_1} u||_{L^6} ||\partial_t^{k_2} u||_{L^6} ||\partial_t^{k_3} u||_{L^6}, \quad (4.78)$$

hence, due to the Sobolev embeddings $H^1(M^3) \hookrightarrow L^6(M^3)$, $H^2(M^3) \hookrightarrow L^\infty(M^3)$ and (4.15), we have

$$(4.78) \lesssim ||\partial_t^{j_1} u||_{H^2} ||\partial_t^{j_2} u||_{H^2} ||\partial_t^{k-j} u||_{L^2} ||\partial_t^{k_1} u||_{L^6} ||\partial_t^{k_2} u||_{L^6} ||\partial_t^{k_3} u||_{H^1}$$
$$\lesssim_{||u_0||_{H^1}} ||u||_{H^{2j_1+2}} ||u||_{H^{2j_2+2}} ||u||_{H^{2k-2j}} ||\partial_t^{k_1} u||_{L^6} ||\partial_t^{k_2} u||_{L^6} ||u||_{H^{2k_3+1}}.$$

Therefore

$$\int_0^\tau \left| Re \sum_{j=0}^{k-1} c_j \int_{M^3} \partial_t^j(|u|^2)\ \partial_t^{k-j} u\ \partial_t^{k-1}(|u|^2\overline{u})\ dvol_g \right|\ ds$$
$$\lesssim \sum_{j=0}^{k-1} \sum_{\substack{j_1+j_2=j \\ k_1+k_2+k_3=k-1}} \int_0^\tau ||u||_{H^{2j_1+2}} ||u||_{H^{2j_2+2}} ||u||_{H^{2k-2j}} \times$$
$$\times ||\partial_t^{k_1} u||_{L^6} ||\partial_t^{k_2} u||_{L^6} ||u||_{H^{2k_3+1}}\ ds. \tag{4.79}$$

By using Hölder's inequality and (4.31), we have

$$(4.79) \leq \sum_{j=0}^{k-1} \sum_{\substack{j_1+j_2=j \\ k_1+k_2+k_3=k-1}} \leq ||u||_{L_\tau^\infty H^{2j_1+2}} ||u||_{L_\tau^\infty H^{2j_2+2}} ||u||_{L_\tau^\infty H^{2k-2j}} \times$$
$$\times\ ||u||_{L_\tau^\infty H^{2k_3+1}} ||\partial_t^{k_1} u||_{L_\tau^2 L^6} ||\partial_t^{k_2} u||_{L_\tau^2 L^6}$$
$$\lesssim_{||u_0||_{H^1}} \sum_{j=0}^{k-1} \sum_{\substack{j_1+j_2=j \\ k_1+k_2+k_3=k-1}} ||u||_{L_\tau^\infty H^{2k}}^{\frac{2k+2k_3-1}{2k-1}} ||\partial_t^{k_1} u||_{L_\tau^2 L^6} ||\partial_t^{k_2} u||_{L_\tau^2 L^6} \tag{4.80}$$

From (4.70) we have

$$||\partial_t^{k_1} u||_{L_\tau^2 L^6} ||\partial_t^{k_2} u||_{L_\tau^2 L^6}$$
$$\lesssim_{\epsilon,\ ||u_0||_{H^1}} \left(||u||_{L_\tau^\infty H^{2k}}^{\frac{2k_1-1}{2k-1}+\epsilon} + \sqrt{\tau} ||u||_{L_\tau^\infty H^{2k}}^{\frac{4k_1-1}{4k-2}} + \sqrt{\tau} ||u||_{L_\tau^\infty H^{2k}}^{\frac{2k_1-1}{2k-1}} \right) \times$$
$$\times \left(||u||_{L_\tau^\infty H^{2k}}^{\frac{2k_2-1}{2k-1}+\epsilon} + \sqrt{\tau} ||u||_{L_\tau^\infty H^{2k}}^{\frac{4k_2-1}{4k-2}} + \sqrt{\tau} ||u||_{L_\tau^\infty H^{2k}}^{\frac{2k_2-1}{2k-1}} \right)$$
$$\lesssim \tau ||u||_{L_\tau^\infty H^{2k}}^{\frac{2k_1+2k_2-1}{2k-1}} + ||u||_{L_\tau^\infty H^{2k}}^{\alpha},$$

where $\alpha < \frac{2k_1+2k_2-1}{2k-1}$, since

$$\frac{2k_1-1}{2k-1} + \epsilon, \frac{2k_1-1}{2k-1} < \frac{4k_1-1}{4k-2},$$
$$\frac{2k_2-1}{2k-1} + \epsilon, \frac{2k_2-1}{2k-1} < \frac{4k_2-1}{4k-2}.$$

Therefore

$$(4.80) \lesssim_{||u_0||_{H^1}} \sum_{j=0}^{k-1} \sum_{\substack{j_1+j_2=j \\ k_1+k_2+k_3=k-1}} \left(\tau ||u_0||_{L_\tau^\infty H^{2k}}^2 + ||u||_{L_\tau^\infty H^{2k}}^{\gamma} \right),$$

where $\gamma := \frac{2k+2k_3-1}{2k-1} + \alpha < 2$. Since γ depends only on k, we finally get

$$\int_0^\tau \left| Re \sum_{j=0}^{k-1} c_j \int_{M^3} \partial_t^j(|u|^2)\ \partial_t^{k-j} u\ \partial_t^{k-1}(|u|^2\overline{u})\ dvol_g \right| ds$$
$$\lesssim_{||u_0||_{H^1}} \tau ||u_0||^2_{L^\infty_\tau H^{2k}} + ||u||^\gamma_{L^\infty_\tau H^{2k}}.$$

(Fourth Term) By expanding the time derivative we get

$$\int_0^\tau \left| Im \sum_{j=1}^{k-1} c_j \int_{M^3} \partial_t^j(|u|)\ \partial_t^{k-j} u\ \partial_t^k \overline{u}\ dvol_g \right| ds$$
$$\lesssim \sum_{j=1}^{k-1} \sum_{j_1+j_2=j} \int_0^\tau \int_{M^3} |\partial_t^{j_1} u||\partial_t^{j_2} u||\partial_t^{k-j} u||\partial_t^k u|\ dvol_g\ ds.$$

By using Hölder's inequality we obtain

$$\int_{M^3} |\partial_t^{j_1} u||\partial_t^{j_2} u||\partial_t^{k-j} u||\partial_t^k u|\ dvol_g$$
$$\leq ||\partial_t^{j_1} u||_{L^6}||\partial_t^{j_2} u||_{L^6}||\partial_t^{k-j} u||_{L^6}||\partial_t^k u||_{L^2}, \tag{4.81}$$

hence, due to the Sobolev embedding $H^1(M^3) \hookrightarrow L^6(M^3)$ and (4.15), we have

$$(4.81) \lesssim ||\partial_t^{j_1} u||_{L^6}||\partial_t^{j_2} u||_{L^6}||\partial_t^{k-j} u||_{H^1}||\partial_t^k u||_{L^2}$$
$$\lesssim_{||u_0||_{H^1}} ||\partial_t^{j_1} u||_{L^6}||\partial_t^{j_2} u||_{L^6}||u||_{H^{2k-2j+1}}||u||_{H^{2k}}.$$

Therefore

$$\int_0^\tau \left| Im \sum_{j=1}^{k-1} c_j \int_{M^3} \partial_t^j(|u|)\ \partial_t^{k-j} u\ \partial_t^k \overline{u}\ dvol_g \right| ds$$
$$\lesssim_{||u_0||_{H^1}} \sum_{j=1}^{k-1} \sum_{j_1+j_2=j} \int_0^\tau ||\partial_t^{j_1} u||_{L^6}||\partial_t^{j_2} u||_{L^6}||u||_{H^{2k-2j+1}}||u||_{H^{2k}}\ ds. \tag{4.82}$$

By using Hölder's inequality and (4.31), we have

$$(4.82) \leq \sum_{j=1}^{k-1} \sum_{j_1+j_2=j} ||u||_{L^\infty_\tau H^{2k-2j+1}}||u||_{L^\infty_\tau H^{2k}}||\partial_t^{j_1} u||_{L^2_\tau L^6}||\partial_t^{j_2} u||_{L^2_\tau L^6}$$
$$\lesssim_{||u_0||_{H^1}} \sum_{j=1}^{k-1} \sum_{j_1+j_2=j} ||u||^{\frac{4k-2j-1}{2k-1}}_{L^\infty_\tau H^{2k}} ||\partial_t^{j_1} u||_{L^2_\tau L^6}||\partial_t^{j_2} u||_{L^2_\tau L^6}. \tag{4.83}$$

From (4.70), we have

$$\begin{aligned} &||\partial_t^{j_1} u||_{L^2_\tau L^6} ||\partial_t^{j_2} u||_{L^2_\tau L^6} \\ &\lesssim_{\epsilon,\ ||u_0||_{H^1}} \left(||u||_{L^\infty_\tau H^{2k}}^{\frac{2j_1-1}{2k-1}+\epsilon} + \sqrt{\tau}||u||_{L^\infty_\tau H^{2k}}^{\frac{4j_1-1}{4k-2}} + \sqrt{\tau}||u||_{L^\infty_\tau H^{2k}}^{\frac{2j_1-1}{2k-1}} \right) \times \\ &\times \left(||u||_{L^\infty_\tau H^{2k}}^{\frac{2j_2-1}{2k-1}+\epsilon} + \sqrt{\tau}||u||_{L^\infty_\tau H^{2k}}^{\frac{4j_2-1}{4k-2}} + \sqrt{\tau}||u||_{L^\infty_\tau H^{2k}}^{\frac{2j_2-1}{2k-1}} \right) \\ &\lesssim \tau ||u||_{L^\infty_\tau H^{2k}}^{\frac{2j-1}{2k-1}} + ||u||^\alpha_{L^\infty_\tau H^{2k}}, \end{aligned}$$

where $\alpha < \frac{2j-1}{2k-1}$, since

$$\begin{aligned} \frac{2j_1-1}{2k-1} + \epsilon, \frac{2j_1-1}{2k-1} < \frac{4j_1-1}{4k-2}, \\ \frac{2j_2-1}{2k-1} + \epsilon, \frac{2j_2-1}{2k-1} < \frac{4j_2-1}{4k-2}. \end{aligned}$$

Therefore

$$(4.83) \lesssim_{||u_0||_{H^1}} \sum_{j=1}^{k-1} \sum_{j_1+j_2=j} \tau ||u||^2_{L^\infty_\tau H^{2k}} + ||u||^\gamma_{L^\infty_\tau H^{2k}},$$

where $\gamma := \frac{4k-2j-1}{2k-1} + \alpha < 2$. Since γ depends only on k, we finally get

$$\begin{aligned} &\int_0^\tau \left| Im \sum_{j=1}^{k-1} c_j \int_{M^3} \partial_t^j(|u|)\ \partial_t^{k-j} u\ \partial_t^k \overline{u}\ dvol_g \right| ds \\ &\lesssim_{||u_0||_{H^1}} \tau ||u||^2_{L^\infty_\tau H^{2k}} + ||u||^\gamma_{L^\infty_\tau H^{2k}}. \end{aligned}$$

Putting all together we get

$$\int_0^\tau |r.h.s.\ of\ (4.23)|\ ds \lesssim_{||u_0||_{H^1}} \tau ||u||^2_{L^\infty_\tau H^{2k}} + ||u||^\gamma_{L^\infty_\tau H^{2k}},$$

for some $\gamma \in (0, 2)$, which concludes the proof of the proposition. □

Proposition 4.4.4. *Let $u \in C(\mathbb{R}, H^{2k}(M^3))$ be the unique global solution to (4.2) with $m = 2k$, $k \in \mathbb{N}$. For every $\tau \in (0, 1)$, we have*

$$||u(\tau,\cdot)||^2_{H^{2k}(M^3)} - ||u(0,\cdot)||^2_{H^{2k}(M^3)} \lesssim_{||u_0||_{H^1}} \tau ||u||^2_{L^\infty_\tau H^{2k}(M^3)} + ||u||^\gamma_{L^\infty_\tau H^{2k}(M^3)}$$

for some $\gamma \in (0, 2)$.

Proof. Let $\mathcal{R}_{2k}(u)$ be as in 4.2.3. By expanding the time derivative, we get

$$\int_{M^3} |\partial_t^{k-1}\nabla_g(|u|^2)|_g^2 \, dvol_g \lesssim \sum_{k_1+k_2=k-1} \int_{M^3} |\partial_t^{k_1}u|^2 |\partial_t^{k_2}\nabla_g u|_g^2 \, dvol_g$$
$$\lesssim \sum_{k_1+k_2=k-1} ||\partial_t^{k_1}u||_{L^\infty}^2 ||\partial_t^{k_2}u||_{H^1}^2. \tag{4.84}$$

Due to (1.3) and (4.15), we have

$$(4.84) \lesssim_\epsilon \sum_{k_1+k_2=k-1} ||\partial_t^{k_1}u||_{H^1}^{1-\epsilon} ||\partial_t^{k_1}u||_{H^2}^{1+\epsilon} ||\partial_t^{k_2}u||_{H^1}^2$$
$$\lesssim_{||u_0||} \sum_{k_1+k_2=k-1} ||u||_{H^{2k_1+1}}^{1-\epsilon} ||u||_{H^{2k_1+2}}^{1+\epsilon} ||u||_{H^{2k_2+1}}^2, \tag{4.85}$$

hence, by applying (4.31) to each term, we obtain

$$(4.85) \lesssim_{\epsilon,\ ||u_0||_{H^1}} \sum_{k_1+k_2=k-1} ||u||_{H^{2k}}^{\frac{4k-3}{2k-1}+\epsilon} \lesssim ||u||_{H^{2k}}^{\frac{4k-3}{2k-1}+\epsilon}.$$

By expanding the time derivative, we get

$$\int_{M^3} |\partial_t^{k-1}(|u|^2 u)|^2 \, dvol_g \lesssim \sum_{k_1+k_2+k_3=k-1} \int_{M^3} |\partial_t^{k_1}u|^2 |\partial_t^{k_2}u|^2 |\partial_t^{k_3}u|^2$$
$$\lesssim \sum_{k_1+k_2+k_3=k-1} ||\partial_t^{k_1}u||_{L^\infty}^2 ||\partial_t^{k_2}u||_{L^\infty}^2 ||\partial_t^{k_3}u||_{L^2}^2. \tag{4.86}$$

Due to (1.3) and (4.15), we have

$$(4.86) \lesssim_\epsilon \sum_{k_1+k_2+k_3=k-1} ||\partial_t^{k_1}u||_{H^1}^{1-\epsilon} ||\partial_t^{k_1}u||_{H^2}^{1+\epsilon} ||\partial_t^{k_2}u||_{H^1}^{1-\epsilon} ||\partial_t^{k_2}u||_{H^2}^{1+\epsilon} ||\partial_t^{k_3}u||_{L^2}^2$$
$$\lesssim_{||u_0||_{H^1}} \sum_{k_1+k_2+k_3=k-1} ||u||_{H^{2k_1+1}}^{1-\epsilon} ||u||_{H^{2k_1+2}}^{1+\epsilon} ||u||_{H^{2k_2+1}}^{1-\epsilon} ||u||_{H^{2k_2+2}}^{1+\epsilon} ||u||_{H^{2k_3}}^2, \tag{4.87}$$

hence, by applying (4.31) to each term, we obtain

$$(4.87) \lesssim_{\epsilon,\ ||u_0||_{H^1}} \sum_{k_1+k_2+k_3=k-1} ||u||_{H^{2k}}^{\frac{4k-4}{2k-1}+\epsilon} \lesssim ||u||_{H^{2k}}^{\frac{4k-4}{2k-1}+\epsilon},$$

therefore

$$|\mathcal{R}_{2k}(u)| \lesssim_{\epsilon,\ ||u_0||_{H^1}} ||u||_{H^{2k}}^{\frac{4k-3}{2k-1}+\epsilon} + ||u||_{H^{2k}}^{\frac{4k-4}{2k-1}+\epsilon}. \tag{4.88}$$

From Proposition 4.4.3 we have

$$\left|\int_0^\tau \frac{d}{ds}\mathcal{E}_{2k}(u)\ ds\right| \le \int_0^\tau |r.h.s.\ of\ (4.23)|\ ds \lesssim_{||u_0||_{H^1}} \tau||u||^2_{L^\infty_\tau H^{2k}} + ||u||^\gamma_{L^\infty_\tau H^{2k}} \tag{4.89}$$

for some $\gamma \in (0,2)$. By applying the Fundamental Theorem of Calculus on the l.h.s. of (4.89), we get

$$\Big|||\partial_t^k u(\tau,\cdot)||^2_{L^2} - ||\partial_t^k u(0,\cdot)||^2_{L^2} + \mathcal{R}_{2k}(u)(\tau) - \mathcal{R}_{2k}(u)(0)\Big| \lesssim_{||u_0||_{H^1}} \tau||u||^2_{L^\infty_\tau H^{2k}} + ||u||^\gamma_{L^\infty_\tau H^{2k}},$$

therefore

$$||\partial_t^k u(\tau,\cdot)||^2_{L^2} - ||\partial_t^k u(0,\cdot)||^2_{L^2} - 2\sup_{t\in(0,\tau)}|\mathcal{R}_{2k}(u)(t)| \lesssim_{||u_0||_{H^1}} \tau||u||^2_{L^\infty_\tau H^{2k}} + ||u||^\gamma_{L^\infty_\tau H^{2k}}.$$

By using (4.88), we get

$$||\partial_t^k u(\tau,\cdot)||^2_{L^2} - ||\partial_t^k u(0,\cdot)||^2_{L^2} \lesssim_{\epsilon,\ ||u_0||_{H^1}} \tau||u||^2_{L^\infty_\tau H^{2k}} + ||u||^\gamma_{L^\infty_\tau H^{2k}} \tag{4.90}$$

for some $\gamma \in (0,2)$. Finally, due to (4.14), (4.15) and (4.31), we have

$$\begin{aligned}
&||u(\tau,\cdot)||^2_{H^{2k}} - ||\partial_t^k u(\tau,\cdot)||^2_{L^2} \\
&\le \left(||\partial_t^k u(\tau,\cdot) - i^k\Delta_g^k u(\tau,\cdot)||_{L^2} + ||\partial_t^k u(\tau,\cdot)||_{L^2}\right)^2 + \\
&\quad + ||u(\tau,\cdot)||^2_{H^{2k-1}} - ||\partial_t^k u(\tau,\cdot)||^2_{L^2} \\
&\lesssim_{||u_0||_{H^1}} ||u(\tau,\cdot)||^{\frac{4k-3}{2k-1}}_{H^{2k}} + ||u(\tau,\cdot)||^{\frac{4k-4}{2k-1}}_{H^{2k}},
\end{aligned}$$

and

$$\begin{aligned}
&||\partial_t^k u(0,\cdot)||^2_{L^2} - ||u(0,\cdot)||^2_{H^{2k}} \\
&\le \left(||\partial_t^k u(0,\cdot) - i^k\Delta_g^k u(0,\cdot)||_{L^2} + ||i^k\Delta_g^k u(0,\cdot)||_{L^2}\right)^2 - ||i^k\Delta_g^k u(0,\cdot)||^2_{L^2} \\
&\lesssim_{||u_0||_{H^1}} ||u(0,\cdot)||^{\frac{4k-3}{2k-1}}_{H^{2k}} + ||u(0,\cdot)||^{\frac{4k-4}{2k-1}}_{H^{2k}},
\end{aligned}$$

therefore, from (4.90), we get

$$\begin{aligned} & ||u(\tau,\cdot)||^2_{H^{2k}} - ||u(0,\cdot)||^2_{H^{2k}} \\ \lesssim_{\epsilon,\ ||u_0||_{H^1}} & \ \tau||u||^2_{L^\infty_\tau H^{2k}} + ||u||^\gamma_{L^\infty_\tau H^{2k}} + ||u(\tau,\cdot)||^{\frac{4k-3}{2k-1}}_{H^{2k}} + \\ & + ||u(\tau,\cdot)||^{\frac{4k-4}{2k-1}}_{H^{2k}} + ||u(0,\cdot)||^{\frac{4k-3}{2k-1}}_{H^{2k}} + ||u(0,\cdot)||^{\frac{4k-4}{2k-1}}_{H^{2k}} \\ & \lesssim \tau||u||^2_{L^\infty_\tau H^{2k}} + ||u||^\gamma_{L^\infty_\tau H^{2k}}, \end{aligned}$$

for some $\gamma \in (0, 2)$, which concludes the proof of the proposition. □

Theorem 4.4.5. *Let $u \in C(\mathbb{R}, H^{2k}(M^3))$ be the unique global solution to (4.2) with $m = 2k$, $k \in \mathbb{N}$. For every $T > 0$, we have the following bound:*

$$\sup_{t\in(0,T)} ||u(t,\cdot)||_{H^{2k}} \le C_1 \exp(C_2 T)$$

where $C_{1,2} = C_{1,2}(k, ||u_0||_{H^{2k}}) > 0$.

Proof. Let us consider $\tau \in (0, 1)$ given by the local Cauchy theory. From Proposition 4.4.4 and Remark 4.5 we get

$$||u(\tau,\cdot)||^2_{H^{2k}} \le ||u(0,\cdot)||^2_{H^{2k}} + C||u||^2_{L^\infty_\tau H^{2k}},$$

where $C = C(k, ||u_0||_{H^{2k}}) > 0$ may change in the following computations. By iteration, for every $n \in \mathbb{N} \cup \{0\}$, we obtain

$$||u(n\tau+\tau,\cdot)||^2_{H^{2k}} \le ||u(n\tau,\cdot)||^2_{H^{2k}} + C\left(\sup_{t\in(n\tau,n\tau+\tau)} ||u(t,\cdot)||^2_{H^{2k}}\right),$$

therefore, since the map data solution is continuous (see 2.2.3), we have

$$||u(n\tau+\tau,\cdot)||^2_{H^{2k}} \le C||u(n\tau,\cdot)||^2_{H^{2k}}. \tag{4.91}$$

Now, for every $n \in \mathbb{N} \cup \{0\}$, we claim that

$$||u(n\tau,\cdot)||^2_{H^{2k}} \le C_1 \exp(C_2 n),$$

where $C_{1,2} = C_{1,2}(k, ||u_0||_{H^{2k}})$ may change in the following computations. For $n = 0$ the claim is trivially satisfied; we prove it in case $n \ge 1$. By using (4.91) we obtain

$$||u(n\tau,\cdot)||^2_{H^{2k}} \le ||u_0||^2_{H^{2k}}\ C^n = ||u_0||^2_{H^{2k}} \exp(\ln(C)n) = C_1 \exp(C_2 n)$$

which proves the claim.

Now, from the continuity of the map data solution an due to 4.4, we have

$$\sup_{t\in(n\tau,n\tau+\tau)} ||u(t,\cdot)||^2_{H^{2k}} \leq C||u(n\tau,\cdot)||^2_{H^{2k}} \leq C_1\exp(C_2 n).$$

Finally, given $N \in \mathbb{N}$ such that $N\tau \leq T \leq N\tau + \tau$, we have

$$\begin{aligned}\sup_{t\in(0,T)} ||u(t,\cdot)||^2_{H^{2k}} &\leq \max_{n\in\{0,\cdots,N\}} \left(\sup_{t\in(n\tau,n\tau+\tau)} ||u(t,\cdot)||^2_{H^{2k}} \right) \\ &\leq C_1\exp(C_2 N) \leq C_1\exp(C_2 T),\end{aligned}$$

therefore

$$\sup_{t\in(0,T)} ||u(t,\cdot)||_{H^{2k}} \leq C_1\exp(C_2 T),$$

which concludes the proof of the Theorem. □

Chapter 5

Growth of H^{2k+1} Sobolev Norms of Solutions to NLS on Closed Riemannian Manifolds

In this chapter we use an approach similar to the one used in Chapter 4 to provide a priori bounds on the growth in time of higher order Sobolev norms of the solutions to the families of Cauchy problems (4.1) and (4.2) with $m = 2k+1$, $k \in \mathbb{N}$.

5.1 Modified Energies

In this section we introduce new suitable "modified energy" (different from the modified energies introduced in the case $m = 2k$) for the Cauchy problems (4.1) and (4.2) with $m = 2k+1$, $k \in \mathbb{N}$ (odd-integer Sobolev regularity of the initial datum). These energies, as before for $m = 2k$, will be crucial to establish a priori bounds on the growth of the higher-order Sobolev norms $||\cdot||_{H^{2k+1}}$.

Definition 5.1.1. *Let $u \in C(\mathbb{R}, H^{2k+1}(M^d))$, be the unique global solution to either (4.1) or (4.2) where $m = 2k+1$, $k \in \mathbb{N}$. We define the modified* *energy as*

$$\mathcal{E}_{2k+1}(u) := \frac{1}{2}||\partial_t^k \nabla_g u||^2_{L^2(M^d)} + \mathcal{R}_{2k+1}(u)$$

where

$$\begin{aligned}\mathcal{R}_{2k+1}(u) := \frac{1}{2}\int_{M^d} |u|^{p-1}|\partial_t^k u|^2 \, dvol_g + \frac{p-1}{8}\int_{M^d} |u|^{p-3}|\partial_t^k(|u|^2)|^2 \, dvol_g + \\ - \operatorname{Re}\sum_{j=1}^{k-1} c_j \int_{M^d} \partial_t^j u \; \partial_t^{k-j}(|u|^{p-1}) \; \partial_t^k \overline{u} \, dvol_g + \\ - \sum_{j=1}^{k-1} c_j \int_{M^d} \partial_t^{k-j}(|u|^{p-3}) \; \partial_t^j(|u|^2) \; \partial_t^k(|u|^2) \, dvol_g\end{aligned}$$

and c_j, for all $j = 1, \cdots, k-1$, are suitable real constants that may change from line to line (the values of the c_j's are given by the following proposition).

Proposition 5.1.2. *Let $u \in C(\mathbb{R}, H^{2k+1}(M^d))$ be the unique global solution to either (4.1) or (4.2). Then*

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{2k+1}(u) = \frac{1}{2}\int_{M^d} \partial_t(|u|^{p-1})|\partial_t^k u|^2 \, dvol_g + \\ + \frac{p-1}{8}\int_{M^d} \partial_t(|u|^{p-3})|\partial_t^k(|u|^2)|^2 \, dvol_g + \\ - \operatorname{Re}\sum_{j=1}^{k-1} c_j \int_{M^d} \partial_t^{j+1} u \; \partial_t^{k-j}(|u|^{p-1}) \; \partial_t^k \overline{u} \, dvol_g + \\ - \operatorname{Re}\sum_{j=1}^{k-1} c_j \int_{M^d} \partial_t^{j} u \; \partial_t^{k-j+1}(|u|^{p-1}) \; \partial_t^k \overline{u} \, dvol_g + \\ + \sum_{j=1}^{k-1} c_j \int_{M^d} \partial_t^{k-j+1}(|u|^{p-3}) \; \partial_t^j(|u|^2) \; \partial_t^k(|u|^2) \, dvol_g + \\ + \sum_{j=1}^{k-1} c_j \int_{M^d} \partial_t^{k-j}(|u|^{p-3}) \; \partial_t^{j+1}(|u|^2) \; \partial_t^k(|u|^2) \, dvol_g + \\ + \sum_{j=1}^{k} c_j \int_{M^d} \partial_t^{k}(|u|^{p-1}) \; \partial_t^j u \; \partial_t^{k+1-j} \overline{u} \, dvol_g \qquad (5.1)\end{aligned}$$

where the c_j's denote real constants that may change from line to line.

Proof. Since u solves (4.1) or (4.2), using integration by parts, we have

$$\mathrm{Re}\left(i\int_{M^d}\partial_t^{k+1}u\ \partial_t^k\overline{u}\ dvol_g\right)=||\partial_t^k\nabla_g u||_{L^2}^2+$$
$$+\mathrm{Re}\left(\int_{M^d}\partial_t^k(|u|^{p-1}u)\ \partial_t^k\overline{u}\ dvol_g\right),$$

which gives

$$\frac{d}{dt}\mathrm{Re}\left(\int_{M^d}\partial_t^k(|u|^{p-1}u)\ \partial_t^k\overline{u}\ dvol_g\right)+\frac{d}{dt}||\partial_t^k\nabla_g u||_{L^2}^2$$
$$=\frac{d}{dt}\mathrm{Re}\left(i\int_{M^d}\partial_t^{k+1}u\ \partial_t^k\overline{u}\ dvol_g\right)=\mathrm{Re}\left(i\int_{M^d}\partial_t^{k+2}u\ \partial_t^k\overline{u}\ dvol_g\right). \tag{5.2}$$

Moreover, since u solves (4.1) or (4.2), using again integration by parts, we have

$$(5.2)=\mathrm{Re}\left(\int_{M^d}\partial_t^{k+1}(|u|^{p-1}u)\ \partial_t^k\overline{u}\ dvol_g\right)+\frac{1}{2}\frac{d}{dt}||\partial_t^k\nabla_g u||_{L^2}^2.$$

Putting all together we obtain

$$\frac{1}{2}\frac{d}{dt}||\partial_t^k\nabla_g u||_{L^2}^2=-\frac{d}{dt}\mathrm{Re}\left(\int_{M^d}\partial_t^k(|u|^{p-1}u)\ \partial_t^k\overline{u}\ dvol_g\right)+$$
$$+\mathrm{Re}\left(\int_{M^d}\partial_t^{k+1}(|u|^{p-1}u)\ \partial_t^k\overline{u}\ dvol_g\right),$$

and since

$$\frac{d}{dt}\mathrm{Re}\left(\int_{M^d}\partial_t^k(|u|^{p-1}u)\ \partial_t^k\overline{u}\ dvol_g\right)=\mathrm{Re}\left(\int_{M^d}\partial_t^{k+1}(|u|^{p-1}u)\ \partial_t^k\overline{u}\ dvol_g\right)+$$
$$+\mathrm{Re}\left(\int_{M^d}\partial_t^k(|u|^{p-1}u)\ \partial_t^{k+1}\overline{u}\ dvol_g\right)$$

we get

$$\frac{1}{2}\frac{d}{dt}||\partial_t^k\nabla_g u||_{L^2}^2=-\mathrm{Re}\left(\int_{M^d}\partial_t^k(|u|^{p-1}u)\ \partial_t^{k+1}\overline{u}\ dvol_g\right).$$

By using the identity

$$\partial_t^k(|u|^{p-1}u)=\partial_t^k(|u|^{p-1})u+|u|^{p-1}\partial_t^k u+\sum_{j=1}^{k-1}c_j\ \partial_t^j u\ \partial_t^{k-j}(|u|^{p-1}),$$

where c_j denote suitable real constants that may change in the following computations, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\partial_t^k\nabla_g u||_{L^2}^2 =& -\operatorname{Re}\left(\int_{M^d}\partial_t^k(|u|^{p-1})u\ \partial_t^{k+1}\overline{u}\ dvol_g\right)+\\ &-\operatorname{Re}\left(\int_{M^d}|u|^{p-1}\partial_t^k u\ \partial_t^{k+1}\overline{u}\ dvol_g\right)+\\ &+\operatorname{Re}\sum_{j=1}^{k-1}c_j\int_{M^d}\partial_t^j u\ \partial_t^{k-j}(|u|^{p-1})\ \partial_t^{k+1}\overline{u}\ dvol_g.\end{aligned}$$

Elementary computations yield

$$\begin{aligned}\operatorname{Re}\left(\int_{M^d}|u|^{p-1}\partial_t^k u\ \partial_t^{k+1}\overline{u}\ dvol_g\right) =& \frac{1}{2}\frac{d}{dt}\int_{M^d}|u|^{p-1}|\partial_t^k u|^2\ dvol_g+\\ &-\frac{1}{2}\int_{M^d}\partial_t(|u|^{p-1})|\partial_t^k u|^2\ dvol_g,\end{aligned}$$

which lead to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\partial_t^k\nabla_g u||_{L^2}^2 =& -\operatorname{Re}\left(\int_{M^d}\partial_t^k(|u|^{p-1})u\ \partial_t^{k+1}\overline{u}\ dvol_g\right)+\\ &-\frac{1}{2}\frac{d}{dt}\int_{M^d}|u|^{p-1}|\partial_t^k u|^2\ dvol_g+\\ &+\frac{1}{2}\int_{M^d}\partial_t(|u|^{p-1})|\partial_t^k u|^2\ dvol_g+\\ &+\operatorname{Re}\sum_{j=1}^{k-1}c_j\int_{M^d}\partial_t^j u\ \partial_t^{k-j}(|u|^{p-1})\ \partial_t^{k+1}\overline{u}\ dvol_g.\end{aligned}$$

Moreover, by expanding the time derivative, we get

$$\begin{aligned}\frac{d}{dt}\left(\partial_t^j u\ \partial_t^{k-j}(|u|^{p-1})\ \partial_t^k\overline{u}\right) =& \partial_t^{j+1}u\ \partial_t^{k-j}(|u|^{p-1})\ \partial_t^k\overline{u}+\\ &+\partial_t^j u\ \partial_t^{k-j+1}(|u|^{p-1})\ \partial_t^k\overline{u}+\partial_t^j u\ \partial_t^{k-j}(|u|^{p-1})\ \partial_t^{k+1}\overline{u},\end{aligned}$$

therefore

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\partial_t^k\nabla_g u||_{L^2}^2 =& -\operatorname{Re}\left(\int_{M^d}\partial_t^k(|u|^{p-1})u\ \partial_t^{k+1}\overline{u}\ dvol_g\right)+\\ &-\frac{1}{2}\frac{d}{dt}\int_{M^d}|u|^{p-1}|\partial_t^k u|^2\ dvol_g+\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{2}\int_{M^d} \partial_t(|u|^{p-1})|\partial_t^k u|^2 \; dvol_g + \\
&+ \frac{d}{dt}\mathrm{Re}\sum_{j=1}^{k-1} c_j \int_{M^d} \partial_t^j u \; \partial_t^{k-j}(|u|^{p-1}) \; \partial_t^k \overline{u} \; dvol_g + \\
&- \mathrm{Re}\sum_{j=1}^{k-1} c_j \int_{M^d} \partial_t^{j+1} u \; \partial_t^{k-j}(|u|^{p-1}) \; \partial_t^k \overline{u} \; dvol_g + \\
&- \mathrm{Re}\sum_{j=1}^{k-1} c_j \int_{M^d} \partial_t^j u \; \partial_t^{k-j+1}(|u|^{p-1}) \; \partial_t^k \overline{u}.
\end{aligned} \tag{5.3}
$$

Now, we focus on the first term of (5.3)

$$
(\mathrm{I}) := -\mathrm{Re}\left(\int_{M^d} \partial_t^k(|u|^{p-1})u \; \partial_t^{k+1}\overline{u} \; dvol_g\right).
$$

By expanding the time derivative we get

$$
\frac{1}{2}\partial_t^{k+1}(|u|^2) = \mathrm{Re}(u \; \partial_t^{k+1}\overline{u}) + \sum_{j=1}^{k} c_j \; \partial_t^j u \; \partial_t^{k-j+1}\overline{u},
$$

therefore

$$
\begin{aligned}
(\mathrm{I}) = &-\frac{1}{2}\int_{M^d} \partial_t^k(|u|^{p-1}) \; \partial_t^{k+1}(|u|^2) \; dvol_g + \\
&+ \sum_{j=1}^{k} c_j \int_{M^d} \partial_t^k(|u|^{p-1}) \; \partial_t^j u \; \partial_t^{k+1-j}\overline{u} \; dvol_g.
\end{aligned} \tag{5.4}
$$

We notice that

$$
\begin{aligned}
\partial_t^k(|u|^{p-1}) &= \frac{p-1}{2}\partial_t^{k-1}(\partial_t(|u|^2)|u|^{p-3}) \\
&= \frac{p-1}{2}|u|^{p-3}\partial_t^k(|u|^2) + \sum_{j=1}^{k-1} c_j \; \partial_t^j(|u|^2) \; \partial_t^{k-j}(|u|^{p-3}),
\end{aligned}
$$

therefore

$$
\begin{aligned}
(5.4) = &-\frac{p-1}{4}\int_{M^d} |u|^{p-3}\partial_t^k(|u|^2) \; \partial_t^{k+1}(|u|^2) \; dvol_g + \\
&+ \sum_{j=1}^{k-1} c_j \int_{M^d} \partial_t^j(|u|^2) \; \partial_t^{k-j}(|u|^{p-3}) \; \partial_t^{k+1}(|u|^2) \; dvol_g + \\
&+ \sum_{j=1}^{k} c_j \int_{M^d} \partial_t^k(|u|^{p-1}) \; \partial_t^j u \; \partial_t^{k+1-j}\overline{u} \; dvol_g.
\end{aligned}
$$

From elementary computations we get

$$|u|^{p-3}\ \partial_t^k(|u|^2)\ \partial_t^{k+1}(|u|^2) = \frac{1}{2}\frac{d}{dt}\Big(|u|^{p-3}|\partial_t^k(|u|^2)|^2\Big) - \frac{1}{2}\partial_t(|u|^{p-3})|\partial_t^k(|u|^2)|^2,$$

therefore

$$\begin{aligned}(\mathrm{I}) = &-\frac{p-1}{8}\frac{d}{dt}\int_{M^d} |u|^{p-3}|\partial_t^k(|u|^2)|^2\ dvol_g +\\ &+\frac{p-1}{8}\int_{M^d} \partial_t(|u|^{p-3})|\partial_t^k(|u|^2)|^2\ dvol_g +\\ &+\sum_{j=1}^{k-1} c_j \int_{M^d} \partial_t^j(|u|^2)\ \partial_t^{k-j}(|u|^{p-3})\ \partial_t^{k+1}(|u|^2)\ dvol_g +\\ &+\sum_{j=1}^{k} c_j \int_{M^d} \partial_t^k(|u|^{p-1})\ \partial_t^j u\ \partial_t^{k+1-j}\overline{u}\ dvol_g. \end{aligned} \tag{5.5}$$

By expanding the time derivative we get

$$\begin{aligned}\frac{d}{dt}\Big(\partial_t^j(|u|^2)\ \partial_t^{k-j}(|u|^{p-3})\ \partial_t^k(|u|^2)\Big) &= \partial_t^{j+1}(|u|^2)\ \partial_t^{k-j}(|u|^{p-3})\ \partial_t^k(|u|^2) +\\ +\partial_t^j(|u|^2)\ \partial_t^{k-j+1}(|u|^{p-3})\ \partial_t^k(|u|^2) &+ \partial_t^j(|u|^2)\ \partial_t^{k-j}(|u|^{p-3})\ \partial_t^{k+1}(|u|^2)\end{aligned}$$

which yields

$$\begin{aligned}(5.5) = &-\frac{p-1}{8}\frac{d}{dt}\int_{M^d} |u|^{p-3}|\partial_t^k(|u|^2)|^2\ dvol_g +\\ &+\frac{p-1}{8}\int_{M^d} \partial_t(|u|^{p-3})|\partial_t^k(|u|^2)|^2\ dvol_g +\\ &+\frac{d}{dt}\sum_{j=1}^{k-1} c_j \int_{M^d} \partial_t^j(|u|^2)\ \partial_t^{k-j}(|u|^{p-3})\ \partial_t^k(|u|^2)\ dvol_g +\\ &+\sum_{j=1}^{k-1} c_j \int_{M^d} \partial_t^j(|u|^2)\ \partial_t^{k-j+1}(|u|^{p-3})\ \partial_t^k(|u|^2)\ dvol_g +\\ &+\sum_{j=1}^{k-1} c_j \int_{M^d} \partial_t^{j+1}(|u|^2)\ \partial_t^{k-j}(|u|^{p-3})\ \partial_t^k(|u|^2)\ dvol_g +\\ &+\sum_{j=1}^{k} c_j \int_{M^d} \partial_t^k(|u|^{p-1})\ \partial_t^j u\ \partial_t^{k+1-j}\overline{u}\ dvol_g \end{aligned} \tag{5.6}$$

Replacing (5.6) in (5.3) we find an explicit expression for $\frac{1}{2}\frac{d}{dt}||\partial_t^k \nabla_g u||_{L^2}^2$ and we conclude as in the proof of Proposition 4.2.2. □

Remark 5.1.3. *Let u be as in Proposition 5.1.2, if $p = 3$ we have*

$$\begin{aligned}\mathcal{R}_{2k+1}(u) =& \frac{1}{2}\int_{M^d} |u|^2 |\partial_t^k u|^2 \, dvol_g + \frac{1}{4}\int_{M^d} |\partial_t^k (|u|^2)|^2 \, dvol_g + \\ &- \mathrm{Re} \sum_{j=1}^{k-1} c_j \int_{M^d} \partial_t^j u \; \partial_t^{k-j}(|u|^2) \; \partial_t^k \overline{u} \, dvol_g\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{2k+1}(u) = & \frac{1}{2}\int_{M^d} \partial_t(|u|^2) |\partial_t^k u|^2 \, dvol_g + \\ &- \mathrm{Re} \sum_{j=1}^{k-1} c_j \int_{M^d} \partial_t^{j+1} u \; \partial_t^{k-j}(|u|^2) \; \partial_t^k \overline{u} \, dvol_g + \\ &- \mathrm{Re} \sum_{j=1}^{k-1} c_j \int_{M^d} \partial_t^{j} u \; \partial_t^{k-j+1}(|u|^2) \; \partial_t^k \overline{u} \, dvol_g + \\ &+ \sum_{j=1}^{k} c_j \int_{M^d} \partial_t^k(|u|^2) \; \partial_t^j u \; \partial_t^{k+1-j} \overline{u} \, dvol_g. \end{aligned} \tag{5.7}$$

5.2 The 2-Dimensional case $(m = 2k + 1)$

The aim of this section is to provide a bound on the growth of the continuous function

$$\mathbb{R}^+ \ni t \longmapsto ||u(t, \cdot)||_{H^{2k+1}(M^2)} \tag{5.8}$$

where $k \in \mathbb{N}$ and u is the unique global solution to the Cauchy problem (4.1) with $m = 2k + 1$. As a first step we shall prove a finer version of Proposition 4.3.2 that we use to establish a priori bounds to small-time increments for (5.8). Finally we take advantage of an iterative argument to show the polynomial growth of (5.8).

Notation 5.2.1. *In the following computations we call ϵ a positive quantity that may change from line to line. We shall assume ϵ to be small enough at each step.*

Proposition 5.2.2. *Let $u \in C(\mathbb{R}, H^{2k+1}(M^2))$ be the unique global solution to (4.1) with $m = 2k+1$, $k \in \mathbb{N}$. For every $\tau \in (0,1)$, we have*

$$||\partial_t^j u||_{L^4_\tau L^4(M^2)} \lesssim_{\epsilon,\ ||u_0||_{H^1}} ||u||^{(1-s_0)}_{L^\infty_\tau H^{2j}(M^2)} ||u||^{s_0+\epsilon}_{L^\infty_\tau H^{2j+1}(M^2)}.$$

where $s_0 \in [0, \frac{1}{4}]$ is given by Proposition 3.2.3.

Proof. From Proposition 3.2.3, we have

$$||\partial_t^j u||_{L^4_\tau L^4} \lesssim ||\partial_t^j u(0,\cdot)||_{H^{s_0}} + \tau ||\partial_t^j (|u|^{p-1}u)||_{L^\infty_\tau H^{s_0}}. \tag{5.9}$$

By using (1.1) and (4.13), we get

$$\begin{aligned} ||\partial_t^j u(0,\cdot)||_{H^{s_0}} &\lesssim ||\partial_t^j u(0,\cdot)||^{1-s_0}_{L^2} ||\partial_t^j u(0,\cdot)||^{s_0}_{H^1} \\ &\lesssim ||u||^{1-s_0}_{L^\infty_\tau H^{2j}} ||u||^{s_0}_{L^\infty_\tau H^{2j+1}}. \end{aligned}$$

We now focus on the nonlinear term (5.9). Due to (1.1), we have

$$||\partial_t^j (|u|^{p-1}u)||_{L^\infty_\tau H^{s_0}} \lesssim ||\partial_t^j (|u|^{p-1}u)||^{1-s_0}_{L^\infty_\tau L^2} ||\partial_t^j (|u|^{p-1}u)||^{s_0}_{L^\infty_\tau H^1}.$$

By expanding the time derivative, we get

$$||\partial_t^j (|u|^{p-1}u)||_{L^2} \lesssim \sum_{\substack{j_1+\ldots+j_p=j \\ j_1=\max\{j_1,\ldots,j_p\}}} \left(\prod_{i=2}^{p} ||\partial_t^{j_i} u||_{L^\infty} \right) ||\partial_t^{j_1} u||_{L^2}, \tag{5.10}$$

therefore, due to (1.2) and (4.13), we have

$$(5.10) \lesssim_{\epsilon,\ ||u_0||_{H^1}} \sum_{\substack{j_1+\ldots+j_p=j \\ j_1=\max\{j_1,\ldots,j_p\}}} \left(\prod_{i=2}^{p} ||u||^{1-\epsilon}_{H^{2j_i+1}} ||u||^{\epsilon}_{H^{2j_i+2}} \right) ||u||_{H^{2j_1}}. \tag{5.11}$$

By using (1.1) and (4.4), we get

$$\begin{aligned} (5.11) &\lesssim \sum_{\substack{j_1+\ldots+j_p=j \\ j_1=\max\{j_1,\ldots,j_p\}}} \left(\prod_{i=2}^{p} ||u||^{\frac{2j_i}{2j-1}(1-\epsilon)}_{H^{2j}} ||u||^{(1-\frac{2j_i}{2j-1})\epsilon}_{H^1} ||u||^{\epsilon}_{H^{2j_i+2}} \right) \times \\ &\qquad \times ||u||^{\frac{2j_1-1}{2j-1}}_{H^{2j}} ||u||^{1-\frac{2j_1-1}{2j-1}}_{H^1} \\ &\lesssim_{||u_0||_{H^1}} \sum_{\substack{j_1+\ldots+j_p=j \\ j_1=\max\{j_1,\ldots,j_p\}}} \left(\prod_{i=2}^{p} ||u||^{\frac{2j_i}{2j-1}(1-\epsilon)}_{H^{2j}} ||u||^{\epsilon}_{H^{2j_i+2}} \right) ||u||^{\frac{2j_1-1}{2j-1}}_{H^{2j}}, \end{aligned}$$

hence, by taking the supremum over $t \in (0, \tau)$ and using Remark 4.0.1, we obtain

$$\begin{aligned} &||\partial_t^j(|u|^{p-1}u)||_{L^\infty_\tau L^2} \\ \lesssim_{||u_0||_{H^1}} &\sum_{\substack{j_1+\ldots+j_p=j \\ j_1=\max\{j_1,\ldots,j_p\}}} \left(\prod_{i=2}^p ||u||^{\frac{2j_i}{2j-1}}_{L^\infty_\tau H^{2j}} \right) ||u||^{\frac{2j_1-1}{2j-1}}_{L^\infty_\tau H^{2j}} ||u||^{\epsilon}_{L^\infty_\tau H^{2j}} \\ \lesssim &\sum_{\substack{j_1+\ldots+j_p=j \\ j_1=\max\{j_1,\ldots,j_p\}}} ||u||^{1+\epsilon}_{L^\infty_\tau H^{2j}} \lesssim ||u||^{1+\epsilon}_{L^\infty_\tau H^{2j}}. \end{aligned}$$

Now, we observe that

$$\begin{aligned} ||\partial_t^j(|u|^{p-1}u)||_{L^\infty_\tau H^1} &\sim ||\partial_t^j(|u|^{p-1}u)||_{L^\infty_\tau L^2} + ||\nabla_g \partial_t^j(|u|^{p-1}u)||_{L^\infty_\tau L^2} \\ &\lesssim_{\epsilon,\ ||u_0||_{H^1}} ||u||^{1+\epsilon}_{L^\infty_\tau H^{2j}} + ||\nabla_g \partial_t^j(|u|^{p-1}u)||_{L^\infty_\tau L^2}. \end{aligned}$$

By expanding the time derivative, we get

$$\begin{aligned} ||\nabla_g \partial_t^j(|u|^{p-1}u)||_{L^2} \lesssim &\sum_{\substack{j_1+\ldots+j_p=j \\ j_1=\max\{j_1,\ldots,j_p\}}} \left(\prod_{i=2}^p ||\partial_t^{j_i} u||_{L^\infty} \right) ||\nabla_g \partial_t^{j_1} u||_{L^2} + \\ + &\sum_{\substack{j_1+\ldots+j_p=j \\ j_1=\max\{j_1,\ldots,j_p\}}} \left(\prod_{i=3}^p ||\partial_t^{j_i} u||_{L^\infty} \right) ||\partial_t^{j_1} u||_{L^2} ||\nabla_g \partial_t^{j_2} u||_{L^\infty}, \end{aligned} \tag{5.12}$$

and denote by (I) and (II) the first and the second term on the r.h.s. of (5.12). Due to (1.2), (4.13), (1.1) and (4.4), we have

$$\begin{aligned} \text{(I)} \lesssim_{\epsilon,\ ||u_0||_{H^1}} &\sum_{\substack{j_1+\ldots+j_p=j \\ j_1=\max\{j_1,\ldots,j_p\}}} \left(\prod_{i=2}^p ||u||^{1-\epsilon}_{H^{2j_i+1}} ||u||^{\epsilon}_{H^{2j_i+2}} \right) ||u||_{H^{2j_1+1}} \\ \lesssim &\sum_{\substack{j_1+\ldots+j_p=j \\ j_1=\max\{j_1,\ldots,j_p\}}} \left(\prod_{i=2}^p ||u||^{\frac{2j_i}{2j}(1-\epsilon)}_{H^{2j+1}} ||u||^{(1-\frac{2j_i}{2j})(1-\epsilon)}_{H^1} \right) ||u||^{\frac{2j_1}{2j}}_{H^{2j+1}} \times \\ &\times ||u||^{1-\frac{2j_1}{2j}}_{H^1} ||u||^{(p-1)\epsilon}_{H^{2j+1}} \\ \lesssim_{||u_0||_{H^1}} &\sum_{\substack{j_1+\ldots+j_p=j \\ j_1=\max\{j_1,\ldots,j_p\}}} \left(\prod_{i=2}^p ||u||^{\frac{2j_i}{2j}(1-\epsilon)}_{H^{2j+1}} \right) ||u||^{\frac{2j_1}{2j}}_{H^{2j+1}} ||u||^{\epsilon}_{H^{2j+1}}. \end{aligned}$$

Due to (1.2), (4.13), (1.1) and (4.4), we obtain

$$\begin{aligned}
(\mathrm{II}) \lesssim_{\epsilon,\ ||u_0||_{H^1}} & \sum_{\substack{j_1+\ldots+j_p=j\\ j_1=\max\{j_1,\ldots,j_p\}}} \left(\prod_{i=3}^{p} ||u||_{H^{2j_i+1}}^{1-\epsilon} ||u||_{H^{2j_i+2}}^{\epsilon}\right) \times \\
& \times ||u||_{H^{2j_1}} ||u||_{H^{2j_2+2}}^{1-\epsilon} ||u||_{H^{2j_2+3}}^{\epsilon} \\
\lesssim & \sum_{\substack{j_1+\ldots+j_p=j\\ j_1=\max\{j_1,\ldots,j_p\}}} \left(\prod_{i=3}^{p} ||u||_{H^{2j+1}}^{\frac{2j_i}{2j}(1-\epsilon)} ||u||_{H^1}^{(1-\frac{2j_i}{2j})(1-\epsilon)}\right) ||u||_{H^{2j+1}}^{(p-2)\epsilon} ||u||_{H^{2j+1}}^{\frac{2j_1-1}{2j}} \times \\
& \times ||u||_{H^1}^{1-\frac{2j_1-1}{2j}} ||u||_{H^{2j+1}}^{\frac{2j_2+1}{2j}(1-\epsilon)} ||u||_{H^1}^{(1-\frac{2j_2+1}{2j})(1-\epsilon)} ||u||_{H^{2j+1}}^{\frac{2j_2+2}{2j}\epsilon} ||u||_{H^1}^{(1-\frac{2j_2+2}{2j})\epsilon} \\
\lesssim_{||u_0||_{H^1}} & \sum_{\substack{j_1+\ldots+j_p=j\\ j_1=\max\{j_1,\ldots,j_p\}}} \left(\prod_{i=3}^{p} ||u||_{H^{2j+1}}^{\frac{2j_i}{2j}(1-\epsilon)}\right) ||u||_{H^{2j+1}}^{\epsilon} ||u||_{H^{2j+1}}^{\frac{2j_1-1}{2j}} ||u||_{H^{2j+1}}^{\frac{2j_2+1}{2j}(1-\epsilon)},
\end{aligned}$$

therefore, taking the supremum over $t \in (0,\tau)$ and using Remark 4.0.1, we get

$$\begin{aligned}
& ||\nabla_g \partial_t^j (|u|^{p-1}u)||_{L_\tau^\infty L^2} \\
\lesssim_{\epsilon,\ ||u_0||_{H^1}} & \sum_{\substack{j_1+\ldots+j_p=j\\ j_1=\max\{j_1,\ldots,j_p\}}} \left(\prod_{i=2}^{p} ||u||_{L_\tau^\infty H^{2j+1}}^{\frac{2j_i}{2j}}\right) ||u||_{L_\tau^\infty H^{2j+1}}^{\frac{2j_1}{2j}} ||u||_{L_\tau^\infty H^{2j+1}}^{\epsilon} + \\
+ & \sum_{\substack{j_1+\ldots+j_p=j\\ j_1=\max\{j_1,\ldots,j_p\}}} \left(\prod_{i=3}^{p} ||u||_{L_\tau^\infty H^{2j+1}}^{\frac{2j_i}{2j}}\right) ||u||_{L_\tau^\infty H^{2j+1}}^{\epsilon} ||u||_{L_\tau^\infty H^{2j+1}}^{\frac{2j_1-1}{2j}} ||u||_{L_\tau^\infty H^{2j+1}}^{\frac{2j_2+1}{2j}} \\
& \lesssim \sum_{\substack{j_1+\ldots+j_p=j\\ j_1=\max\{j_1,\ldots,j_p\}}} ||u||_{H^{2j+1}}^{1+\epsilon} \lesssim ||u||_{H^{2j+1}}^{1+\epsilon}.
\end{aligned}$$

Finally, putting all together, we get

$$\begin{aligned}
||\partial_t^j (|u|^{p-1}u)||_{L_\tau^\infty H^{s_0}} &\lesssim ||u||_{L_\tau^\infty H^{2j}}^{(1+\epsilon)(1-s_0)} ||u||_{L_\tau^\infty H^{2j+1}}^{(1+\epsilon)s_0} \\
&\lesssim ||u||_{L_\tau^\infty H^{2j}}^{1-s_0} ||u||_{L_\tau^\infty H^{2j+1}}^{s_0+\epsilon},
\end{aligned}$$

hence

$$\begin{aligned}
||\partial_t^j u||_{L_\tau^4 L^4} &\lesssim_{\epsilon,\ ||u_0||_{H^1}} ||u||_{L_\tau^\infty H^{2j}}^{1-s_0} ||u||_{L_\tau^\infty H^{2j+1}}^{s_0} + ||u||_{L_\tau^\infty H^{2j}}^{1-s_0} ||u||_{L_\tau^\infty H^{2j+1}}^{s_0+\epsilon} \\
&\lesssim ||u||_{L_\tau^\infty H^{2j}}^{1-s_0} ||u||_{L_\tau^\infty H^{2j+1}}^{s_0+\epsilon},
\end{aligned}$$

which concludes the proof of the proposition. □

Lemma 5.2.3. *Let $u \in C(\mathbb{R}, H^{2k+1}(M^d))$, be the unique global solution to either (4.1) or (4.2) with $m = 2k+1$, $k \in \mathbb{N}$. For every $s \in [1, 2k+1]$, we have*

$$||u||_{L^\infty_\tau H^s(M^d)} \lesssim_{||u_0||_{H^1}} ||u||^{\frac{s-1}{2k}}_{L^\infty_\tau H^{2k+1}(M^d)}.$$

Proof. See Lemma 4.3.3. □

Proposition 5.2.4. *Let $u \in C(\mathbb{R}, H^{2k+1}(M^2))$ be the unique global solution to (4.1) with $m = 2k+1$, $k \in \mathbb{N}$. For every $\tau \in (0,1)$, we have*

$$\int_0^\tau |r.h.s.\ of\ (5.1)|\, ds \lesssim_{\epsilon,\ ||u_0||_{H^1}} \sqrt{\tau}||u||^{\frac{4k-1+2s_0}{2k}+\epsilon}_{L^\infty_\tau H^{2k+1}(M^2)}.$$

Proof. (Sketch) We shall estimate each term of (5.1) by assuming $p > 3$. For $p = 3$ the proof is similar.
(First Term) By expanding the time derivative, we get

$$\int_0^\tau \left| \int_{M^2} \partial_t(|u|^{p-1})|\partial_t^k u|^2\, dvol_g \right| ds \lesssim \int_0^\tau \int_{M^2} |u|^{p-2}\ |\partial_t u|\ |\partial_t^k u|^2\, dvol_g\, ds,$$

therefore, due to Hölder's inequality, we have

$$\begin{aligned}\int_0^\tau \left| \int_{M^2} \partial_t(|u|^{p-1})|\partial_t^k u|^2\, dvol_g \right| ds &\lesssim \int_0^\tau ||u||^{p-2}_{L^\infty}||\partial_t u||_{L^2}||\partial_t^k u||^2_{L^4}\, ds \\ &\lesssim \sqrt{\tau}||u||^{p-2}_{L^\infty_\tau L^\infty}||\partial_t u||_{L^\infty_\tau L^2}||\partial_t^k u||^2_{L^4_\tau L^4}.\end{aligned}$$

By using Proposition 5.2.2 and Lemma 5.2.3 together with the same techniques used in the proof of Proposition 4.3.4, we get

$$\int_0^\tau \left| \int_{M^2} \partial_t(|u|^{p-1})|\partial_t^k u|^2\, dvol_g \right| ds \lesssim_{\epsilon,\ ||u_0||_{H^1}} \sqrt{\tau}||u||^{\frac{4k-1+2s_0}{2k}+\epsilon}_{L^\infty_\tau H^{2k+1}}.$$

(Second Term) By expanding the time derivative, we get

$$\begin{aligned}&\int_0^\tau \left| \int_{M^2} \partial_t(|u|^{p-3})|\partial_t^k(|u|^2)|^2\, dvol_g \right| ds \\ \lesssim &\sum_{\substack{k_1+k_2=k \\ k_1=\max\{k_1,k_2\}}} \int_0^\tau \int_{M^2} |u|^{p-4}\ |\partial_t u|\ |\partial_t^{k_1} u|^2\ |\partial_t^{k_2} u|^2\, dvol_g\, ds,\end{aligned}$$

therefore, due to Hölder's inequality, we have

$$\begin{aligned}
&\int_0^\tau \left| \int_{M^2} \partial_t(|u|^{p-3}) |\partial_t^k(|u|^2)|^2 \, dvol_g \right| \, ds \\
&\lesssim \sum_{\substack{k_1+k_2=k \\ k_1=\max\{k_1,k_2\}}} \int_0^\tau ||u||_{L^\infty}^{p-4} ||\partial_t u||_{L^2} ||\partial_t^{k_1} u||_{L^4}^2 ||\partial_t^{k_2} u||_{L^\infty}^2 \\
&\lesssim \sum_{\substack{k_1+k_2=k \\ k_1=\max\{k_1,k_2\}}} \sqrt{\tau} ||u||_{L_\tau^\infty L^\infty}^{p-4} ||\partial_t u||_{L_\tau^\infty L^2} ||\partial_t^{k_1} u||_{L_\tau^4 L^4}^2 ||\partial_t^{k_2} u||_{L_\tau^\infty L^\infty}^2 .
\end{aligned}$$

By using Proposition 5.2.2 and Lemma 5.2.3 together with the same techniques used in the proof of Proposition 4.3.4, we get

$$\int_0^\tau \left| \int_{M^2} \partial_t(|u|^{p-3}) |\partial_t^k(|u|^2)|^2 \, dvol_g \right| \, ds \lesssim_{\epsilon,\ ||u_0||_{H^1}} \sqrt{\tau} ||u||_{L_\tau^\infty H^{2k+1}}^{\frac{4k-1+2s_0}{2k}+\epsilon} .$$

(Third Term) By expanding the time derivative, we get

$$\begin{aligned}
&\int_0^\tau \left| \mathrm{Re} \sum_{j=1}^{k-1} c_j \int_{M^2} \partial_t^{j+1} u \ \partial_t^{k-j}(|u|^{p-1}) \ \partial_t^k \overline{u} \ dvol_g \right| \, ds \\
&\lesssim \sum_{j=1}^{k-1} \sum_{\substack{m_1+\ldots+m_{p-1}=k-j \\ m_1=\max\{m_1,\ldots,m_{p-1}\}}} \int_0^\tau \int_{M^2} \left(\prod_{i=1}^{p-1} |\partial_t^{m_i} u| \right) |\partial_t^{j+1} u| \ |\partial_t^k u| \ dvol_g \ ds,
\end{aligned}$$

therefore, due to Hölder's inequality, we have

$$\begin{aligned}
&\int_0^\tau \left| \mathrm{Re} \sum_{j=1}^{k-1} c_j \int_{M^2} \partial_t^{j+1} u \ \partial_t^{k-j}(|u|^{p-1}) \ \partial_t^k \overline{u} \ dvol_g \right| \, ds \\
&\lesssim \sum_{j=1}^{k-1} \sum_{\substack{m_1+\ldots+m_{p-1}=k-j \\ m_1=\max\{m_1,\ldots,m_{p-1}\}}} \int_0^\tau \left(\prod_{i=2}^{p-1} ||\partial_t^{m_i} u||_{L^\infty} \right) ||\partial_t^{j+1} u||_{L^4} \times \\
&\qquad \times ||\partial_t^k u||_{L^4} ||\partial_t^{m_1} u||_{L^2} \ ds \\
&\lesssim \sum_{j=1}^{k-1} \sum_{\substack{m_1+\ldots+m_{p-1}=k-j \\ m_1=\max\{m_1,\ldots,m_{p-1}\}}} \sqrt{\tau} \left(\prod_{i=2}^{p-1} ||\partial_t^{m_i} u||_{L_\tau^\infty L^\infty} \right) ||\partial_t^{j+1} u||_{L_\tau^4 L^4} \times \\
&\qquad \times ||\partial_t^k u||_{L_\tau^4 L^4} ||\partial_t^{m_1} u||_{L_\tau^\infty L^2} .
\end{aligned}$$

By using Proposition 5.2.2 and Lemma 5.2.3 together with the same techniques used in the proof of Proposition 4.3.4 we get

$$\int_0^\tau \left| \mathrm{Re} \sum_{j=1}^{k-1} c_j \int_{M^2} \partial_t^{j+1} u \ \partial_t^{k-j}(|u|^{p-1}) \ \partial_t^k \overline{u} \ dvol_g \right| \ ds$$
$$\lesssim_{\epsilon,\ ||u_0||_{H^1}} \sqrt{\tau} ||u||_{L_\tau^\infty H^{2k+1}}^{\frac{4k-1+2s_0}{2k}+\epsilon}.$$

(Fourth Term) By expanding the time derivative, we get

$$\int_0^\tau \left| \mathrm{Re} \sum_{j=1}^{k-1} c_j \int_{M^2} \partial_t^{j} u \ \partial_t^{k-j+1}(|u|^{p-1}) \ \partial_t^k \overline{u} \ dvol_g \right| \ ds$$
$$\lesssim \sum_{j=1}^{k-1} \sum_{\substack{m_1+\ldots+m_{p-1}=k-j+1 \\ m_1=\max\{m_1,\ldots,m_{p-1}\}}} \int_0^\tau \int_{M^2} \left(\prod_{i=2}^{p-1} |\partial_t^{m_i} u| \right) |\partial_t^j u| \ |\partial_t^k u| \ |\partial_t^{m_1} u| \ dvol_g \ ds,$$

therefore, due to Hölder's inequality, we have

$$\int_0^\tau \left| \mathrm{Re} \sum_{j=1}^{k-1} c_j \int_{M^2} \partial_t^{j} u \ \partial_t^{k-j+1}(|u|^{p-1}) \ \partial_t^k \overline{u} \ dvol_g \right| \ ds$$
$$\lesssim \sum_{j=1}^{k-1} \sum_{\substack{m_1+\ldots+m_{p-1}=k-j+1 \\ m_1=\max\{m_1,\ldots,m_{p-1}\}}} \int_0^\tau \left(\prod_{i=2}^{p-1} ||\partial_t^{m_i} u||_{L^\infty} \right) ||\partial_t^j u||_{L^2} ||\partial_t^k u||_{L^4} ||\partial_t^{m_1} u||_{L^4} \ ds$$

$$\lesssim \sum_{j=1}^{k-1} \sum_{\substack{m_1+\ldots+m_{p-1}=k-j+1 \\ m_1=\max\{m_1,\ldots,m_{p-1}\}}} \sqrt{\tau} \left(\prod_{i=2}^{p-1} ||\partial_t^{m_i} u||_{L_\tau^\infty L^\infty} \right) ||\partial_t^j u||_{L_\tau^\infty L^2} \times$$
$$\times ||\partial_t^k u||_{L_\tau^4 L^4} ||\partial_t^{m_1} u||_{L_\tau^4 L^4}.$$

By using Proposition 5.2.2 and Lemma 5.2.3 together with the same techniques used in the proof of Proposition 4.3.4, we get

$$\int_0^\tau \left| \mathrm{Re} \sum_{j=1}^{k-1} c_j \int_{M^2} \partial_t^{j} u \ \partial_t^{k-j+1}(|u|^{p-1}) \ \partial_t^k \overline{u} \ dvol_g \right| \ ds$$
$$\lesssim_{\epsilon,\ ||u_0||_{H^1}} \sqrt{\tau} ||u||_{L_\tau^\infty H^{2k+1}}^{\frac{4k-1+2s_0}{2k}+\epsilon}.$$

(Fifth Term) By expanding the time derivative, we get

$$\int_0^\tau \left| \sum_{j=1}^{k-1} c_j \int_{M^2} \partial_t^{k-j+1}(|u|^{p-3})\ \partial_t^j(|u|^2)\ \partial_t^k(|u|^2)\ dvol_g \right| ds$$
$$\lesssim \sum_{j=1}^{k-1} \sum_{\substack{m_1+\ldots+m_{p-3}=k-j+1 \\ m_1=\max\{m_1,\ldots,m_{p-3}\} \\ k_1+k_2=k \\ k_1=\max\{k_1,k_2\} \\ j_1+j_2=j}} \int_0^\tau \int_{M^2} \left(\prod_{i=2}^{p-3} |\partial_t^{m_i} u| \right) |\partial_t^{m_1} u|\ |\partial_t^{j_1} u| \times$$
$$\times |\partial_t^{j_2} u|\ |\partial_t^{k_1} u|\ |\partial_t^{k_2} u|\ dvol_g\ ds,$$

therefore, due to Hölder's inequality, we have

$$\int_0^\tau \left| \sum_{j=1}^{k-1} c_j \int_{M^2} \partial_t^{k-j+1}(|u|^{p-3})\ \partial_t^j(|u|^2)\ \partial_t^k(|u|^2)\ dvol_g \right| ds$$
$$\lesssim \sum_{j=1}^{k-1} \sum_{\substack{m_1+\ldots+m_{p-3}=k-j+1 \\ m_1=\max\{m_1,\ldots,m_{p-3}\} \\ k_1+k_2=k \\ k_1=\max\{k_1,k_2\} \\ j_1+j_2=j}} \int_0^\tau \left(\prod_{i=2}^{p-3} ||\partial_t^{m_i} u||_{L^\infty} \right) ||\partial_t^{m_1} u||_{L^4} ||\partial_t^{j_1} u||_{L^2} \times$$
$$\times ||\partial_t^{j_2} u||_{L^\infty} ||\partial_t^{k_1} u||_{L^4} ||\partial_t^{k_2} u||_{L^\infty}\ ds$$
$$\lesssim \sum_{j=1}^{k-1} \sum_{\substack{m_1+\ldots+m_{p-3}=k-j+1 \\ m_1=\max\{m_1,\ldots,m_{p-3}\} \\ k_1+k_2=k \\ k_1=\max\{k_1,k_2\} \\ j_1+j_2=j}} \sqrt{\tau} \left(\prod_{i=2}^{p-3} ||\partial_t^{m_i} u||_{L^\infty_\tau L^\infty} \right) ||\partial_t^{m_1} u||_{L^4_\tau L^4} ||\partial_t^{j_1} u||_{L^\infty_\tau L^2} \times$$
$$\times ||\partial_t^{j_2} u||_{L^\infty_\tau L^\infty} ||\partial_t^{k_1} u||_{L^4_\tau L^4} ||\partial_t^{k_2} u||_{L^\infty_\tau L^\infty}.$$

By using Proposition 5.2.2 and Lemma 5.2.3 together with the same techniques used in the proof of Proposition 4.3.4, we get

$$\int_0^\tau \left| \sum_{j=1}^{k-1} c_j \int_{M^2} \partial_t^{k-j+1}(|u|^{p-3})\ \partial_t^j(|u|^2)\ \partial_t^k(|u|^2)\ dvol_g \right| ds$$
$$\lesssim_{\epsilon,\ ||u_0||_{H^1}} \sqrt{\tau} ||u||_{L^\infty_\tau H^{2k+1}}^{\frac{4k-1+2s_0}{2k}+\epsilon}.$$

(Sixth Term) By expanding the time derivative, we get

$$\int_0^\tau \left| \sum_{j=1}^{k-1} c_j \int_{M^2} \partial_t^{k-j}(|u|^{p-3})\ \partial_t^{j+1}(|u|^2)\ \partial_t^k(|u|^2)\ dvol_g \right|\ ds$$
$$\lesssim \sum_{j=1}^{k-1} \sum_{\substack{m_1+\ldots+m_{p-3}=k-j \\ j_1+j_2=j+1 \\ j_1=\max\{j_1,j_2\} \\ k_1+k_2=k \\ k_1=\max\{k_1,k_2\}}} \int_0^\tau \int_{M^2} \left(\prod_{i=1}^{p-3} |\partial_t^{m_i} u| \right) |\partial_t^{j_1} u| \times$$
$$\times |\partial_t^{j_2} u|\ |\partial_t^{k_1} u|\ |\partial_t^{k_2} u|\ dvol_g\ ds,$$

therefore, due to Hölder's inequality, we have

$$\int_0^\tau \left| \sum_{j=1}^{k-1} c_j \int_{M^2} \partial_t^{k-j}(|u|^{p-3})\ \partial_t^{j+1}(|u|^2)\ \partial_t^k(|u|^2)\ dvol_g \right|\ ds$$
$$\lesssim \sum_{j=1}^{k-1} \sum_{\substack{m_1+\ldots+m_{p-3}=k-j \\ j_1+j_2=j+1 \\ j_1=\max\{j_1,j_2\} \\ k_1+k_2=k \\ k_1=\max\{k_1,k_2\}}} \int_0^\tau \left(\prod_{i=2}^{p-3} ||\partial_t^{m_i} u||_{L^\infty} \right) ||\partial_t^{m_1} u||_{L^2} ||\partial_t^{j_1} u||_{L^4} \times$$
$$\times ||\partial_t^{j_2} u||_{L^\infty} ||\partial_t^{k_1} u||_{L^4} ||\partial_t^{k_2} u||_{L^\infty}\ ds$$
$$\lesssim \sum_{j=1}^{k-1} \sum_{\substack{m_1+\ldots+m_{p-3}=k-j \\ j_1+j_2=j+1 \\ j_1=\max\{j_1,j_2\} \\ k_1+k_2=k \\ k_1=\max\{k_1,k_2\}}} \sqrt{\tau} \left(\prod_{i=2}^{p-3} ||\partial_t^{m_i} u||_{L^\infty_\tau L^\infty} \right) ||\partial_t^{m_1} u||_{L^\infty_\tau L^2} ||\partial_t^{j_1} u||_{L^4_\tau L^4} \times$$
$$\times ||\partial_t^{j_2} u||_{L^\infty_\tau L^\infty} ||\partial_t^{k_1} u||_{L^4_\tau L^4} ||\partial_t^{k_2} u||_{L^\infty_\tau L^\infty}.$$

By using Proposition 5.2.2 and Lemma 5.2.3 together with the same techniques used in the proof of Proposition 4.3.4, we get

$$\int_0^\tau \left| \sum_{j=1}^{k-1} c_j \int_{M^2} \partial_t^{k-j}(|u|^{p-3})\ \partial_t^{j+1}(|u|^2)\ \partial_t^k(|u|^2)\ dvol_g \right|\ ds$$
$$\lesssim_{\epsilon,\ ||u_0||_{H^1}} \sqrt{\tau} ||u||_{L^\infty_\tau H^{2k+1}}^{\frac{4k-1+2s_0}{2k}+\epsilon}.$$

(Seventh Term) By expanding the time derivative, we get

$$\int_0^\tau \left| \sum_{j=1}^{k} c_j \int_{M^2} \partial_t^k (|u|^{p-1})\ \partial_t^j u\ \partial_t^{k+1-j} \overline{u}\ dvol_g \right|\ ds$$
$$\lesssim \sum_{j=1}^{k} \sum_{\substack{k_1+\ldots+k_{p-1}=k \\ k_1=\max\{k_1,\ldots,k_{p-1}\}}} \int_0^\tau \int_{M^2} \left(\prod_{i=2}^{p-2} |\partial_t^{k_i} u| \right) |\partial_t^{k_1} u|\ |\partial_t^j u|\ |\partial_t^{k+1-j} u|\ dvol_g\ ds,$$

therefore, due to Hölder's inequality, we have

$$\int_0^\tau \left| \sum_{j=1}^{k} c_j \int_{M^2} \partial_t^k (|u|^{p-1})\ \partial_t^j u\ \partial_t^{k+1-j} \overline{u}\ dvol_g \right|\ ds$$
$$\lesssim \sum_{j=1}^{k} \sum_{\substack{k_1+\ldots+k_{p-1}=k \\ k_1=\max\{k_1,\ldots,k_{p-1}\}}} \int_0^\tau \left(\prod_{i=2}^{p-2} ||\partial_t^{k_i} u||_{L^\infty} \right) ||\partial_t^{k_1} u||_{L^4} \times$$
$$\times ||\partial_t^j u||_{L^2} ||\partial_t^{k+1-j} u||_{L^4}\ ds$$
$$\lesssim \sum_{j=1}^{k} \sum_{\substack{k_1+\ldots+k_{p-1}=k \\ k_1=\max\{k_1,\ldots,k_{p-1}\}}} \sqrt{\tau} \left(\prod_{i=2}^{p-2} ||\partial_t^{k_i} u||_{L^\infty_\tau L^\infty} \right) ||\partial_t^{k_1} u||_{L^4_\tau L^4} \times$$
$$\times ||\partial_t^j u||_{L^\infty_\tau L^2} ||\partial_t^{k+1-j} u||_{L^4_\tau L^4}.$$

By using Proposition 5.2.2 and Lemma 5.2.3 together with the same techniques used in the proof of Proposition 4.3.4, we get

$$\int_0^\tau \left| \sum_{j=1}^{k} c_j \int_{M^2} \partial_t^k (|u|^{p-1})\ \partial_t^j u\ \partial_t^{k+1-j} \overline{u}\ dvol_g \right|\ ds$$
$$\lesssim_{\epsilon,\ ||u_0||_{H^1}} \sqrt{\tau} ||u||_{L^\infty_\tau H^{2k+1}}^{\frac{4k-1+2s_0}{2k}+\epsilon}.$$

Finally, putting all together, we get

$$\int_0^\tau |r.h.s.\ of\ (5.1)|\ ds \lesssim_{\epsilon,\ ||u_0||_{H^1}} \sqrt{\tau} ||u||_{L^\infty_\tau H^{2k+1}(M^2)}^{\frac{4k-1+2s_0}{2k}+\epsilon}.$$

□

Proposition 5.2.5. *Let* $u \in C(\mathbb{R}, H^{2k+1}(M^2))$ *be the unique global solution to* (4.1) *with* $m = 2k + 1$, $k \in \mathbb{N}$*. For every* $\tau \in (0, 1)$*, we have*

$$||u(\tau,\cdot)||^2_{H^{2k+1}(M^2)} - ||u(0,\cdot)||^2_{H^{2k+1}(M^2)} \lesssim_{\epsilon, ||u_0||_{H^1}} \sqrt{\tau}||u||^{\frac{4k-1+2s_0}{2k}+\epsilon}_{L^\infty_\tau H^{2k+1}(M^2)} + ||u||^{\frac{4k-2}{2k}+\epsilon}_{L^\infty_\tau H^{2k+1}(M^2)}.$$

Proof. (Sketch) We prove the proposition assuming that $p > 3$; if $p = 3$ the proof is similar. Let $\mathcal{R}_{2k+1}$ be defined as in 5.1.1, we claim that

$$|\mathcal{R}_{2k+1}| \lesssim_{\epsilon,\ ||u_0||_{H^1}} ||u||^{\frac{4k-2}{2k}+\epsilon}_{H^{2k+1}}. \tag{5.13}$$

In order to prove (5.13), we shall estimate each term of $\mathcal{R}_{2k+1}$ independently. (First Term) We observe that

$$\left|\int_{M^2} |u|^{p-1}|\partial_t^k u|^2 \ dvol_g\right| \lesssim ||u||^{p-1}_{L^\infty}||\partial_t^k u||^2_{L^2},$$

therefore, by using (1.2), (4.4), (4.13) and Lemma 5.2.3, we get

$$\left|\int_{M^2} |u|^{p-1}|\partial_t^k u|^2 \ dvol_g\right| \lesssim_{\epsilon,\ ||u_0||_{H^1}} ||u||^{\frac{4k-2}{2k}+\epsilon}_{H^{2k+1}}.$$

(Second Term) By expanding the time derivative, we get

$$\begin{aligned}\left|\int_{M^2} |u|^{p-3}|\partial_t^k(|u|^2)|^2 \ dvol_g\right| &\lesssim \sum_{\substack{k_1+k_2=k \\ k_1=\max\{k_1,k_2\}}} \int_{M^2} |u|^{p-3}\ |\partial_t^{k_1}u|^2\ |\partial_t^{k_2}u|^2 \ dvol_g \\ &\lesssim \sum_{\substack{k_1+k_2=k \\ k_1=\max\{k_1,k_2\}}} ||u||^{p-3}_{L^\infty}||\partial_t^{k_1}u||^2_{L^2}||\partial_t^{k_2}u||^2_{L^\infty}\end{aligned}$$

therefore, by using (1.2), (4.4), (4.13) and Lemma 5.2.3, we get

$$\left|\int_{M^2} |u|^{p-3}|\partial_t^k(|u|^2)|^2 \ dvol_g\right| \lesssim_{\epsilon,\ ||u_0||_{H^1}} ||u||^{\frac{4k-2}{2k}+\epsilon}_{H^{2k+1}}.$$

(Third Term) By expanding the time derivative, we get

$$\left|\mathrm{Re}\sum_{j=1}^{k-1} c_j \int_{M^2} \partial_t^j u\ \partial_t^{k-j}(|u|^{p-1})\ \partial_t^k \overline{u} \ dvol_g\right|$$
$$\lesssim \sum_{j=1}^{k-1} \sum_{m_1+\ldots+m_{p-1}=k-j} \int_{M^2} \left(\prod_{i=1}^{p-1} |\partial_t^{m_i}u|\right) |\partial_t^j u|\ |\partial_t^k u| \ dvol_g.$$

Due to Hölder's inequality, we have

$$\left|\mathrm{Re}\sum_{j=1}^{k-1} c_j \int_{M^2} \partial_t^j u\ \partial_t^{k-j}(|u|^{p-1})\ \partial_t^k \overline{u}\ dvol_g\right|$$
$$\lesssim \sum_{j=1}^{k-1} \sum_{m_1+\ldots+m_{p-1}=k-j} \left(\prod_{i=1}^{p-1} ||\partial_t^{m_i} u||_{L^\infty}\right) ||\partial_t^j u||_{L^2} ||\partial_t^k u||_{L^2},$$

therefore, by using (1.2), (4.13) and Lemma 5.2.3, we get

$$\left|\mathrm{Re}\sum_{j=1}^{k-1} c_j \int_{M^2} \partial_t^j u\ \partial_t^{k-j}(|u|^{p-1})\ \partial_t^k \overline{u}\ dvol_g\right| \lesssim_{\epsilon,\ ||u_0||_{H^1}} ||u||_{H^{2k+1}}^{\frac{4k-2}{2k}+\epsilon}.$$

(Fourth Term) By expanding the time derivative, we get

$$\left|\sum_{j=1}^{k-1} c_j \int_{M^2} \partial_t^{k-j}(|u|^{p-3})\ \partial_t^j(|u|^2)\ \partial_t^k(|u|^2)\ dvol_g\right|$$
$$\lesssim \sum_{j=1}^{k-1} \sum_{\substack{m_1+\ldots+m_{p-1}=k-j \\ j_1+j_2=j \\ k_1+k_2=k \\ k_1=\max\{k_1,k_2\}}} \int_{M^2} \left(\prod_{i=1}^{p-3} |\partial_t^{m_i} u|\right) |\partial_t^{j_1} u|\ |\partial_t^{j_2} u|\ |\partial_t^{k_1} u|\ |\partial_t^{k_2} u|.$$

Due to Hölder's inequality, we have

$$\left|\sum_{j=1}^{k-1} c_j \int_{M^2} \partial_t^{k-j}(|u|^{p-3})\ \partial_t^j(|u|^2)\ \partial_t^k(|u|^2)\ dvol_g\right|$$
$$\lesssim \sum_{j=1}^{k-1} \sum_{\substack{m_1+\ldots+m_{p-1}=k-j \\ j_1+j_2=j \\ k_1+k_2=k \\ k_1=\max\{k_1,k_2\}}} \left(\prod_{i=1}^{p-3} ||\partial_t^{m_i} u||_{L^\infty}\right) ||\partial_t^{j_1} u||_{L^2} \times$$
$$\times ||\partial_t^{j_2} u||_{L^\infty} ||\partial_t^{k_1} u||_{L^2} ||\partial_t^{k_2} u||_{L^\infty},$$

therefore, by using (1.2), (4.13) and Lemma 5.2.3, we get

$$\left|\sum_{j=1}^{k-1} c_j \int_{M^2} \partial_t^{k-j}(|u|^{p-3})\ \partial_t^j(|u|^2)\ \partial_t^k(|u|^2)\ dvol_g\right| \lesssim_{\epsilon,\ ||u_0||_{H^1}} ||u||_{H^{2k+1}}^{\frac{4k-2}{2k}+\epsilon}.$$

Now, we observe that Proposition (5.2.4) gives

$$\left|\int_0^\tau \frac{d}{ds}\mathcal{E}_{2k+1}(u)\ ds\right| \lesssim \int_0^\tau |r.h.s.\ of\ (5.1)|\ ds \lesssim_{\epsilon,\ ||u_0||_{H^1}} \sqrt{\tau}||u||_{L^\infty_\tau H^{2k+1}}^{\frac{4k-1+2s_0}{2k}+\epsilon},$$

therefore, due to the Fundamental Theorem of Calculus,

$$\left|\frac{1}{2}||\partial_t^k \nabla_g u(\tau,\cdot)||_{L^2}^2 - \frac{1}{2}||\partial_t^k \nabla_g(0,\cdot)||_{L^2}^2 + \mathcal{R}_{2k+1}(u)(\tau) - \mathcal{R}_{2k+1}(u)(0)\right|$$
$$\lesssim_{\epsilon,\ ||u_0||_{H^1}} \sqrt{\tau}||u||_{L^\infty_\tau H^{2k+1}}^{\frac{4k-1+2s_0}{2k}+\epsilon}.$$

Due to (5.13), we have

$$\begin{aligned}
&||\partial_t^k \nabla_g u(\tau,\cdot)||_{L^2}^2 - ||\partial_t^k \nabla_g(0,\cdot)||_{L^2}^2 \\
&\lesssim_{\epsilon,\ ||u_0||_{H^1}} \sqrt{\tau}||u||_{L^\infty_\tau H^{2k+1}}^{\frac{4k-1+2s_0}{2k}+\epsilon} + 2\sup_{t\in(0,\tau)} |\mathcal{R}_{2k+1}(u)(t)| \\
&\lesssim \sqrt{\tau}||u||_{L^\infty_\tau H^{2k+1}}^{\frac{4k-1+2s_0}{2k}+\epsilon} + ||u||_{L^\infty_\tau H^{2k+1}}^{\frac{4k-2}{2k}+\epsilon}.
\end{aligned}$$

Moreover, since $||\cdot||_{H^1}^2 = ||\cdot||_{L^2}^2 + ||\nabla_g \cdot||_{L^2}^2$, we have

$$\begin{aligned}
&||\partial_t^k u(\tau,\cdot)||_{H^1}^2 - ||\partial_t^k u(0,\cdot)||_{H^1}^2 \\
&\lesssim_{\epsilon,\ ||u_0||_{H^1}} \sqrt{\tau}||u||_{L^\infty_\tau H^{2k+1}}^{\frac{4k-1+2s_0}{2k}+\epsilon} + ||u||_{L^\infty_\tau H^{2k+1}}^{\frac{4k-2}{2k}+\epsilon} + ||\partial_t^k u(\tau,\cdot)||_{L^2}^2 \\
&\lesssim_{||u_0||_{H^1}} ||u||_{L^\infty_\tau H^{2k+1}}^{\frac{4k-1+2s_0}{2k}+\epsilon} + ||u||_{L^\infty_\tau H^{2k+1}}^{\frac{4k-2}{2k}+\epsilon},
\end{aligned}$$

where we used (4.13) and Lemma 5.2.3 at the last step. We conclude by arguing as in the proof of Proposition 4.3.5. □

Theorem 5.2.6. *Let $u \in C(\mathbb{R}, H^{2k+1}(M^2))$ be the unique global solution to (4.1) with $m = 2k+1$, $k \in \mathbb{N}$. For every $T > 0$, we have the following bound:*

$$\sup_{t\in(0,T)} ||u(t,\cdot)||_{H^{2k+1}(M^2)} \leq C(\max\{1,T\})^{\frac{2k}{1-s_0}+\epsilon}$$

where $C = C(\epsilon, k, ||u_0||_{H^{2k+1}})$ and $s_0 \in [0, \frac{1}{4}]$ is given by Proposition 3.2.3.

Proof. See Theorem 4.3.6. □

5.3 The 3-Dimensional Case $(m = 2k+1)$

The aim of this section is to provide a bound on the growth of the continuous function

$$\mathbb{R}^+ \ni t \longmapsto ||u(t,\cdot)||_{H^{2k+1}(M^3)} \tag{5.14}$$

where $k \in \mathbb{N}$ and u is the unique global solution to the Cauchy problem (4.2) with $m = 2k+1$. As a first step we use the same "modified" endpoint Strichartz estimates used in Chapter 4 to provide a priori bounds to small-time increments for (5.14). Then, we take advantage of an iterative argument to show the exponential growth of (5.14).

Notation 5.3.1. *In the following computations we call ϵ a positive quantity that may change from line to line. We shall assume ϵ to be small enough at each step.*

Proposition 5.3.2. *Let $u \in C(\mathbb{R}, H^{2k+1}(M^3))$ be the unique global solution to (4.2) with $m = 2k+1$, $k \in \mathbb{N}$. For every $\tau \in (0,1)$, we have*

$$\int_0^\tau |r.h.s.\ of\ (5.7)|\, ds \lesssim_{||u_0||_{H^1}} \tau ||u||^2_{L^\infty_\tau H^{2k+1}(M^2)} + ||u||^\gamma_{L^\infty_\tau H^{2k+1}(M^2)}$$

for some $\gamma \in (0,2)$.

Proof. (Sketch) By applying Lemma 5.2.3 to each term of (4.63), we have

$$||\partial_t^j u||_{L^2_\tau L^6} \lesssim_{\epsilon,\ ||u_0||_{H^1}} ||u||^{\frac{2j-1}{2k}+\epsilon}_{L^\infty_\tau H^{2k+1}} + \sqrt{\tau}||u||^{\frac{4j-1}{4k}}_{L^\infty_\tau H^{2k+1}} + \sqrt{\tau}||u||^{\frac{2j-1}{2k}}_{L^\infty_\tau H^{2k+1}}. \tag{5.15}$$

In the following computations we use (5.15) to estimate each term of (5.7). (First Term) By expanding the time derivative, we get

$$\int_0^\tau \left| \int_{M^3} \partial_t(|u|^2)\ |\partial_t^k u|^2\ dvol_g \right|\ ds \lesssim \int_0^\tau \int_{M^3} |u|\ |\partial_t u|\ |\partial_t^k u|^2\ dvol_g\ ds.$$

Due to Hölder's inequality, we have

$$\begin{aligned}\int_0^\tau \left| \int_{M^3} \partial_t(|u|^2)\ |\partial_t^k u|^2\ dvol_g \right|\ ds &\lesssim \int_0^\tau ||u||_{L^6}||\partial_t u||_{L^2}||\partial_t^k u||^2_{L^6}\ ds \\ &\lesssim \sqrt{\tau}||u||_{L^\infty_\tau L^6}||\partial_t u||_{L^\infty_\tau L^2}||\partial_t^k u||^2_{L^2_\tau L^6},\end{aligned}$$

therefore, by using the Sobolev embedding $H^1(M^3) \hookrightarrow L^6(M^3)$, (4.4), (4.15), (5.15) and Lemma 5.2.3, we get

$$\int_0^\tau \left| \int_{M^3} \partial_t(|u|^2)\ |\partial_t^k u|^2\ dvol_g \right|\ ds \lesssim_{||u_0||_{H^1}} \tau ||u||^2_{L^\infty_\tau H^{2k+1}} + ||u||^\gamma_{L^\infty_\tau H^{2k+1}}$$

for some $\gamma \in (0, 2)$.

(Second Term) By expanding the time derivative, we get

$$\int_0^\tau \left| \mathrm{Re} \sum_{j=1}^{k-1} c_j \int_{M^3} \partial_t^{j+1} u\ \partial_t^{k-j}(|u|^2)\ \partial_t^k \overline{u}\ dvol_g \right|\ ds$$
$$\lesssim \sum_{j=1}^{k-1} \sum_{m_1+m_2=k-j} \int_0^\tau \int_{M^3} |\partial_t^{j+1} u|\ |\partial_t^{m_1} u|\ |\partial_t^{m_2} u|\ |\partial_t^k u|\ dvol_g\ ds.$$

Due to Hölder's inequality, we have

$$\int_0^\tau \left| \mathrm{Re} \sum_{j=1}^{k-1} c_j \int_{M^3} \partial_t^{j+1} u\ \partial_t^{k-j}(|u|^2)\ \partial_t^k \overline{u}\ dvol_g \right|\ ds$$
$$\lesssim \sum_{j=1}^{k-1} \sum_{m_1+m_2=k-j} \int_0^\tau ||\partial_t^{j+1} u||_{L^6} ||\partial_t^{m_1} u||_{L^2} ||\partial_t^{m_2} u||_{L^6} ||\partial_t^k u||_{L^6}\ ds$$
$$\lesssim \sum_{j=1}^{k-1} \sum_{m_1+m_2=k-j} ||\partial_t^{j+1} u||_{L^2_\tau L^6} ||\partial_t^{m_1} u||_{L^\infty_\tau L^2} ||\partial_t^{m_2} u||_{L^\infty_\tau L^6} ||\partial_t^k u||_{L^2_\tau L^6},$$

therefore, by using the Sobolev embedding $H^1(M^3) \hookrightarrow L^6(M^3)$, (4.15), (5.15) and Lemma 5.2.3, we get

$$\int_0^\tau \left| \mathrm{Re} \sum_{j=1}^{k-1} c_j \int_{M^3} \partial_t^{j+1} u\ \partial_t^{k-j}(|u|^2)\ \partial_t^k \overline{u}\ dvol_g \right|\ ds$$
$$\lesssim_{||u_0||_{H^1}} \tau ||u||^2_{L^\infty_\tau H^{2k+1}} + ||u||^\gamma_{L^\infty_\tau H^{2k+1}}$$

for some $\gamma \in (0, 2)$.

(Third Term) By expanding the time derivative, we get

$$\left| \mathrm{Re} \sum_{j=1}^{k-1} c_j \int_{M^3} \partial_t^{j} u\ \partial_t^{k-j+1}(|u|^2)\ \partial_t^k \overline{u}\ dvol_g \right|\ ds$$

$$\lesssim \sum_{j=1}^{k-1} \sum_{m_1+m_2=k-j+1} \int_0^\tau \int_{M^3} |\partial_t^j u| \, |\partial_t^{m_1} u| \, |\partial_t^{m_2} u| \, |\partial_t^k u| \, dvol_g \, ds.$$

Due to Hölder's inequality, we have

$$\left| \mathrm{Re} \sum_{j=1}^{k-1} c_j \int_{M^3} \partial_t^j u \; \partial_t^{k-j+1}(|u|^2) \; \partial_t^k \overline{u} \; dvol_g \right| \; ds$$

$$\lesssim \sum_{j=1}^{k-1} \sum_{m_1+m_2=k-j+1} \int_0^\tau ||\partial_t^j u||_{L^2} ||\partial_t^{m_1} u||_{L^6} ||\partial_t^{m_2} u||_{L^6} ||\partial_t^k u||_{L^6} \; ds$$

$$\lesssim \sum_{j=1}^{k-1} \sum_{m_1+m_2=k-j+1} ||\partial_t^j u||_{L_\tau^\infty L^2} ||\partial_t^{m_1} u||_{L_\tau^2 L^6} ||\partial_t^{m_2} u||_{L_\tau^2 L^6} ||\partial_t^k u||_{L_\tau^\infty L^6},$$

therefore, by using the Sobolev embedding $H^1(M^3) \hookrightarrow L^6(M^3)$, (4.15), (5.15) and Lemma 5.2.3, we get

$$\left| \mathrm{Re} \sum_{j=1}^{k-1} c_j \int_{M^3} \partial_t^j u \; \partial_t^{k-j+1}(|u|^2) \; \partial_t^k \overline{u} \; dvol_g \right| \; ds$$

$$\lesssim_{||u_0||_{H^1}} \tau ||u||^2_{L_\tau^\infty H^{2k+1}} + ||u||^\gamma_{L_\tau^\infty H^{2k+1}}$$

for some $\gamma \in (0, 2)$.

(Fourth Term) By expanding the time derivative, we get

$$\int_0^\tau \left| \sum_{j=1}^{k} c_j \int_{M^3} \partial_t^k(|u|^2) \; \partial_t^j u \; \partial_t^{k+1-j} \overline{u} \; dvol_g \right| \; ds$$

$$\lesssim \sum_{j=1}^{k} \sum_{k_1+k_2=k} \int_0^\tau \int_{M^3} |\partial_t^{k_1} u| \, |\partial_t^{k_2} u| \, |\partial_t^j u| \, |\partial_t^{k+1-j} u| \, dvol_g \, ds.$$

Due to Hölder's inequality, we have

$$\int_0^\tau \left| \sum_{j=1}^{k} c_j \int_{M^3} \partial_t^k(|u|^2) \; \partial_t^j u \; \partial_t^{k+1-j} \overline{u} \; dvol_g \right| \; ds$$

$$\lesssim \sum_{j=1}^{k} \sum_{k_1+k_2=k} \int_0^\tau ||\partial_t^{k_1} u||_{L^6} ||\partial_t^{k_2} u||_{L^6} ||\partial_t^j u||_{L^6} ||\partial_t^{k+1-j} u||_{L^2} \; ds$$

$$\lesssim \sum_{j=1}^{k} \sum_{k_1+k_2=k} \int_0^\tau ||\partial_t^{k_1} u||_{L_\tau^2 L^6} ||\partial_t^{k_2} u||_{L_\tau^2 L^6} ||\partial_t^j u||_{L_\tau^\infty L^6} ||\partial_t^{k+1-j} u||_{L_\tau^\infty L^2},$$

therefore, by using the Sobolev embedding $H^1(M^3) \hookrightarrow L^6(M^3)$, (4.15), (5.15) and Lemma 5.2.3, we get

$$\int_0^\tau \left| \sum_{j=1}^k c_j \int_{M^3} \partial_t^k(|u|^2)\ \partial_t^j u\ \partial_t^{k+1-j}\overline{u}\ dvol_g \right|\ ds$$
$$\lesssim_{||u_0||_{H^1}} \tau ||u||^2_{L^\infty_\tau H^{2k+1}} + ||u||^\gamma_{L^\infty_\tau H^{2k+1}}$$

for some $\gamma \in (0, 2)$. Finally putting all together we obtain

$$\int_0^\tau |r.h.s.\ of\ (5.7)|\, ds \lesssim_{\ ||u_0||_{H^1}} \tau ||u||^2_{L^\infty_\tau H^{2k+1}(M^2)} + ||u||^\gamma_{L^\infty_\tau H^{2k+1}(M^2)}$$

for some $\gamma \in (0, 2)$. □

Proposition 5.3.3. *Let $u \in C(\mathbb{R}, H^{2k+1}(M^3))$ be the unique global solution to (4.2) with $m = 2k + 1$, $k \in \mathbb{N}$. For every $\tau \in (0, 1)$, we have:*

$$||u(\tau,\cdot)||^2_{H^{2k+1}(M^3)} - ||u(0,\cdot)||^2_{H^{2k+1}(M^3)}$$
$$\lesssim_{||u_0||_{H^1}} \tau ||u||^2_{L^\infty_\tau H^{2k+1}(M^3)} + ||u||^\gamma_{L^\infty_\tau H^{2k+1}(M^3)}$$

for some $\gamma \in (0, 2)$.

Proof. (Sketch) Let $\mathcal{R}_{2k+1}$ be defined as in 5.1.3, we claim that

$$|\mathcal{R}_{2k+1}(u)| \lesssim ||u||^\gamma_{H^{2k+1}} \tag{5.16}$$

for some $\gamma \in (0, 2)$. In order to prove (5.16), we shall estimate each term of $\mathcal{R}_{2k+1}$ independently.

(First Term) We observe that

$$\left| \int_{M^3} |u|^2 |\partial_t^k u|^2\ dvol_g \right| \lesssim ||u||^2_{L^\infty} ||\partial_t^k u||^2_{L^2},$$

therefore, by using (1.3), (4.4), (4.15) and Lemma 5.2.3, we have

$$\left| \int_{M^3} |u|^2 |\partial_t^k u|^2\ dvol_g \right| \lesssim_{\epsilon,\ ||u||_{H^1}} ||u||^{\frac{4k-1}{2k}+\epsilon}_{H^{2k+1}}.$$

(Second Term) By expanding the time derivative, we get

$$\left|\int_{M^3} |\partial_t^k(|u|^2)|^2 \ dvol_g\right| \lesssim \sum_{\substack{k_1+k_2=k \\ k_1=\max\{k_1,k_2\}}} \int_{M^3} |\partial_t^{k_1} u|^2 \ |\partial_t^{k_2} u|^2 \ dvol_g$$
$$\lesssim \sum_{\substack{k_1+k_2=k \\ k_1=\max\{k_1,k_2\}}} ||\partial_t^{k_1} u||_{L^2}^2 ||\partial_t^{k_2} u||_{L^\infty}^2,$$

therefore, by using (1.3), (4.15) and Lemma 5.2.3, we have

$$\left|\int_{M^3} |\partial_t^k(|u|^2)|^2 \ dvol_g\right| \lesssim ||u||_{H^{2k+1}}^{\frac{4k-1}{2k}+\epsilon}.$$

(Third Term) By expanding the time derivative, we get

$$\left|\mathrm{Re}\sum_{j=1}^{k-1} c_j \int_{M^3} \partial_t^j u \ \partial_t^{k-j}(|u|^2) \ \partial_t^k \overline{u} \ dvol_g\right|$$
$$\lesssim \sum_{j=1}^{k-1} \sum_{m_1+m_2=k-j} \int_{M^3} |\partial_t^j u| \ |\partial_t^{m_1} u| \ |\partial_t^{m_2} u| \ |\partial_t^k u| \ dvol_g.$$

Due to Hölder's inequality and the Sobolev embedding $H^1(M^3) \hookrightarrow L^6(M^3)$, we have

$$\left|\mathrm{Re}\sum_{j=1}^{k-1} c_j \int_{M^3} \partial_t^j u \ \partial_t^{k-j}(|u|^2) \ \partial_t^k \overline{u} \ dvol_g\right|$$
$$\lesssim \sum_{j=1}^{k-1} \sum_{m_1+m_2=k-j} ||\partial_t^j u||_{L^6} ||\partial_t^{m_1} u||_{L^6} ||\partial_t^{m_2} u||_{L^6} ||\partial_t^k u||_{L^2}$$
$$\lesssim \sum_{j=1}^{k-1} \sum_{m_1+m_2=k-j} ||\partial_t^j u||_{H^1} ||\partial_t^{m_1} u||_{H^1} ||\partial_t^{m_2} u||_{H^1} ||\partial_t^k u||_{L^2},$$

therefore, by using (4.15) and Lemma 5.2.3, we obtain

$$\left|\mathrm{Re}\sum_{j=1}^{k-1} c_j \int_{M^3} \partial_t^j u \ \partial_t^{k-j}(|u|^2) \ \partial_t^k \overline{u} \ dvol_g\right| \lesssim_{||u_0||_{H^1}} ||u||_{H^{2k+1}}^{\frac{4k-1}{2k}}.$$

Putting all together we have

$$|\mathcal{R}_{2k+1}(u)| \lesssim_{\epsilon, \ ||u_0||_{H^1}} ||u||_{H^{2k+1}}^{\frac{4k-1}{2k}+\epsilon} + ||u||_{H^{2k+1}}^{\frac{4k-1}{2k}} \lesssim ||u||_{H^{2k+1}}^{\gamma}$$

for some $\gamma \in (0, 2)$, which proves the claim. Now, Proposition 5.3.2 gives

$$\left| \int_0^\tau \frac{d}{ds} \mathcal{E}_{2k+1}(u)\, ds \right| \lesssim \int_0^\tau |r.h.s.\ of\ (5.7)|\, ds$$
$$\lesssim_{||u_0||_{H^1}} \tau ||u||^2_{L^\infty_\tau H^{2k+1}} + ||u||^\gamma_{L^\infty_\tau H^{2k+1}}.$$

Due to the Fundamental Theorem of Calculus we have

$$\left| \frac{1}{2} ||\partial_t^k \nabla_g u(\tau, \cdot)||^2_{L^2} - \frac{1}{2} ||\partial_t^k \nabla_g (0, \cdot)||^2_{L^2} + \mathcal{R}_{2k+1}(u)(\tau) - \mathcal{R}_{2k+1}(u)(0) \right|$$
$$\lesssim_{||u_0||_{H^1}} \tau ||u||^2_{L^\infty_\tau H^{2k+1}} + ||u||^\gamma_{L^\infty_\tau H^{2k+1}},$$

therefore, (5.16) gives

$$||\partial_t^k \nabla_g u(\tau, \cdot)||^2_{L^2} - ||\partial_t^k \nabla_g (0, \cdot)||^2_{L^2}$$
$$\lesssim_{||u_0||_{H^1}} \tau ||u||^2_{L^\infty_\tau H^{2k+1}} + ||u||^\gamma_{L^\infty_\tau H^{2k+1}} + 2 \sup_{t \in (0,\tau)} |\mathcal{R}_{2k+1}(u)(t)|$$
$$\lesssim \tau ||u||^2_{L^\infty_\tau H^{2k+1}} + ||u||^\gamma_{L^\infty_\tau H^{2k+1}}$$

for some $\gamma \in (0, 2)$. We conclude by arguing as in the proof of Proposition 5.2.5. □

Theorem 5.3.4. *Let $u \in C(\mathbb{R}, H^{2k+1}(M^3))$ be the unique global solution to (4.2) with $m = 2k+1$, $k \in \mathbb{N}$. For every $T > 0$, we have the following bound:*

$$\sup_{t \in (0,T)} ||u(t, \cdot)||_{H^m(M^3)} \leq C_1 \exp(C_2 T)$$

where $C_{1,2} = C_{1,2}(k, ||u_0||_{H^{2k+1}}) > 0$.

Proof. See Theorem 4.4.5. □

Chapter 6

Growth of the H^2 Sobolev Norm of the Solution to the subcubic NLS on Closed Riemannian Manifolds

The aim of this chapter is to provide a priori bounds on the growth in time of the Sobolev norm $||\cdot||_{H^2}$ of the solutions to the family of Cauchy problems (4.3). At first, we prove a fundamental comparison between Sobolev norms that we will use throughout this chapter

Proposition 6.0.1. *Let $u \in C(\mathbb{R}, H^2(M^3))$ be the unique global solution to (4.3). We have*

$$||\partial_t u - i\Delta_g u||_{L^2(M^3)} \lesssim_{||u_0||_{H^1}} ||u||_{H^1(M^3)}. \tag{6.1}$$

Proof. Since u solves (4.3), we have

$$||\partial_t u - i\Delta_g u||_{L^2} = ||(|u|^{p-1}u)||_{L^2} = ||u||^p_{L^{2p}},$$

therefore, due to the Sobolev embedding $H^1(M^3) \hookrightarrow L^{2p}(M^3)$ and (4.4), we have

$$||\partial_t u - i\Delta_g u||_{L^2} \lesssim ||u||^p_{H^1} = ||u||^{p-1}_{H^1}||u||_{H^1} \lesssim_{||u_0||} ||u||_{H^1},$$

which concludes the proof of the proposition. □

Corollary 6.0.2. *Let $u \in C(\mathbb{R}, H^2(M^3))$ be the unique global solution to (4.3). We have*

$$||\partial_t u||_{L^2(M^3)} \lesssim_{||u_0||_{H^1}} ||u||_{H^2(M^3)}. \tag{6.2}$$

Proof. Due to (6.1), we have

$$||\partial_t u||_{L^2} \leq ||\partial_t u - i\Delta_g u||_{L^2} + ||\Delta_g u||_{L^2} \lesssim_{||u_0||_{H^1}} ||u||_{H^1} + ||u||_{H^2} \lesssim ||u||_{H^2},$$

proving the statement of the corollary. □

6.1 Modified Energy

In this section we introduce suitable "modified energy" for the Cauchy problem (4.3). We shall use this modified energy in order to provide a priori bounds on the growth of the Sobolev norm $||\cdot||_{H^2}$.

Definition 6.1.1. *Let $u \in C(\mathbb{R}, H^2(M^3))$ be the unique global solution to (4.3). We define the modified energy as*

$$\mathcal{F}_2(u) := ||\partial_t u||^2_{L^2(M^3)} + \mathcal{G}_2(u),$$

where

$$\mathcal{G}_2(u) := -(p-1)\int_{M^3} |u|^{p-1}|\nabla_g |u||_g^2 \, dvol_g - \frac{p-1}{p}\int_{M^3} |u|^{2p} \, dvol_g.$$

Proposition 6.1.2. *Let $u \in C(\mathbb{R}, H^2(M^3))$ be the unique global solution to (4.3). We have*

$$\frac{d}{dt}\mathcal{F}_2(u) = (p-1)(p-3)\int_{M^3} |u|^{p-2}\partial_t|u|\big|\nabla_g|u|\big|_g^2 \, dvol_g + \\ +2(p-1)\int_{M^3} |u|^{p-2}\partial_t|u|\big|\nabla_g u\big|_g^2 \, dvol_g. \tag{6.3}$$

Proof. We observe that

$$\frac{d}{dt}||\partial_t u||_{L^2}^2 = 2\mathrm{Re}\left(\int_{M^3} \partial_t^2 u \ \partial_t \overline{u} \ dvol_g\right). \tag{6.4}$$

Moreover, since u solves (4.3), we have

$$\begin{aligned}(6.4) &= 2\mathrm{Re}\left(i\int_{M^3} \Delta_g(\partial_t u) \ \partial_t \overline{u} \ dvol_g\right) - 2\mathrm{Re}\left(i\int_{M^3} \partial_t(|u|^{p-1}u) \ \partial_t \overline{u} \ dvol_g\right) \\ &= -2\mathrm{Re}\left(i\int_{M^3} \partial_t(|u|^{p-1}u) \ \partial_t \overline{u} \ dvol_g\right)\end{aligned}$$

since

$$2\mathrm{Re}\left(i\int_{M^3} \Delta_g(\partial_t u) \ \partial_t \overline{u} \ dvol_g\right) = -2\mathrm{Re}\left(i\int_{M^3} |\partial_t \nabla_g u|_g^2 \ dvol_g\right) = 0.$$

By expanding the time derivative we get

$$\partial_t(|u|^{p-1}u) = \partial_t(|u|^{p-1})u + |u|^{p-1}\partial_t u,$$

therefore

$$\frac{d}{dt}||\partial_t u||_{L^2}^2 = -2\mathrm{Re}\left(i\int_{M^3} \partial_t(|u|^{p-1})u \ \partial_t \overline{u} \ dvol_g\right).$$

Now, using that u solves (4.3), we have

$$\begin{aligned}-2\mathrm{Re}\left(i\int_{M^3} \partial_t(|u|^{p-1})u \ \partial_t \overline{u} \ dvol_g\right) = &- 2\mathrm{Re}\left(\int_{M^3} \partial_t(|u|^{p-1})u \ \Delta_g \overline{u} \ dvol_g\right) \\ &+ 2\mathrm{Re}\left(\int_{M^3} \partial_t(|u|^{p-1})|u|^{p+1} \ dvol_g\right).\end{aligned}$$

By elementary computations

$$2\mathrm{Re}\left(\int_{M^3} \partial_t(|u|^{p-1})|u|^{p+1} \ dvol_g\right) = 2(p-1)\int_{M^3} |u|^{2p-1} \ \partial_t|u| \ dvol_g,$$

$$\frac{d}{dt}\int_{M^3} |u|^{2p} \ dvol_g = 2p\int_{M^3} |u|^{2p-1}\partial_t|u| \ dvol_g,$$

therefore

$$2\mathrm{Re}\left(\int_{M^3} \partial_t(|u|^{p-1})|u|^{p+1} \ dvol_g\right) = \frac{p-1}{p}\frac{d}{dt}\int_{M^3} |u|^{2p} \ dvol_g.$$

Moreover, due to the identity

$$\Delta_g(|u|^2) = 2\mathrm{Re}(u\Delta_g\overline{u}) + 2|\nabla_g u|_g^2,$$

we have

$$-2\mathrm{Re}\left(\int_{M^3} \partial_t(|u|^{p-1})u\ \Delta_g\overline{u}\ dvol_g\right) = -\int_{M^3} \partial_t(|u|^{p-1})\ \Delta_g(|u|^2)\ dvol_g + \\ +2\int_{M^3} \partial_t(|u|^{p-1})\ |\nabla_g u|_g^2\ dvol_g.$$

Putting all together and using integration by parts we get

$$\begin{aligned}\frac{d}{dt}||\partial_t u||_{L^2}^2 &= \frac{p-1}{p}\frac{d}{dt}\int_{M^3} |u|^{2p}\ dvol_g + 2\int_{M^3} \partial_t(|u|^{p-1})\ |\nabla_g u|_g^2\ dvol_g + \\ &\quad - \int_{M^3} \partial_t(|u|^{p-1})\ \Delta_g(|u|^2)\ dvol_g \\ &= \frac{p-1}{p}\frac{d}{dt}\int_{M^3} |u|^{2p}\ dvol_g + 2\int_{M^3} \partial_t(|u|^{p-1})\ |\nabla_g u|_g^2\ dvol_g + \\ &\quad + \int_{M^3} \left(\partial_t \nabla_g(|u|^{p-1}), \nabla_g(|u|^2)\right)_g\ dvol_g \\ &= \frac{p-1}{p}\frac{d}{dt}\int_{M^3} |u|^{2p}\ dvol_g + 2\int_{M^3} \partial_t(|u|^{p-1})\ |\nabla_g u|_g^2\ dvol_g + \\ &\quad + 2(p-1)\int_{M^3} \left(\partial_t(|u|^{p-2}\nabla_g|u|), |u|\nabla_g|u|\right)_g\ dvol_g. \end{aligned} \tag{6.5}$$

where in the last line we used the identities

$$\nabla_g|u|^{p-1} = (p-1)|u|^{p-2}\nabla_g|u|, \quad \nabla_g|u|^2 = 2|u|\nabla_g|u|.$$

By expanding the time derivative and from elementary computations we get

$$\frac{d}{dt}\left(|u|^{p-1}|\nabla_g|u||_g^2\right) = \left(\partial_t(|u|^{p-2}\nabla_g|u|), |u|\nabla_g|u|\right)_g + \\ +|u|^{p-2}\partial_t|u||\nabla_g|u||_g^2 + |u|^{p-1}\left(\nabla_g|u|, \nabla_g\ \partial_t|u|\right)_g$$

therefore

$$
\begin{aligned}
(6.5) = \frac{p-1}{p}\frac{d}{dt}\int_{M^3} |u|^{2p}\, dvol_g + 2\int_{M^3} \partial_t(|u|^{p-1})\; |\nabla_g u|_g^2\, dvol_g + \\
+ 2(p-1)\frac{d}{dt}\int_{M^3} |u|^{p-1}\big|\nabla_g|u|\big|_g^2\, dvol_g + \\
- 2(p-1)\int_{M^3} |u|^{p-2}\partial_t|u|\big|\nabla_g|u|\big|_g^2\, dvol_g + \\
- 2(p-1)\int_{M^3} |u|^{p-1}\left(\nabla_g|u|, \nabla_g\partial_t|u|\right)_g\; dvol_g. \qquad (6.6)
\end{aligned}
$$

Finally, the identity

$$
\frac{d}{dt}\left(|u|^{p-1}\big|\nabla_g|u|\big|_g^2\right) = \partial_t(|u|^{p-1})\big|\nabla_g|u|\big|_g^2 + 2|u|^{p-1}\left(\nabla_g|u|, \nabla_g\partial_t|u|\right)_g
$$

gives

$$
\begin{aligned}
(6.6) = & \frac{p-1}{p}\frac{d}{dt}\int_{M^3} |u|^{2p}\, dvol_g + 2\int_{M^3} \partial_t(|u|^{p-1})\; |\nabla_g u|_g^2\, dvol_g + \\
& + (p-1)\frac{d}{dt}\int_{M^3} |u|^{p-1}\big|\nabla_g|u|\big|_g^2\, dvol_g + \\
& - 2(p-1)\int_{M^3} |u|^{p-2}\partial_t|u|\big|\nabla_g|u|\big|_g^2\, dvol_g + \\
& + (p-1)\int_{M^3} \partial_t(|u|^{p-1})\big|\nabla_g|u|\big|_g^2\, dvol_g \\
= & \frac{p-1}{p}\frac{d}{dt}\int_{M^3} |u|^{2p}\, dvol_g + 2(p-1)\int_{M^3} |u|^{p-2}\partial_t|u|\; |\nabla_g u|_g^2\, dvol_g + \\
& + (p-1)\frac{d}{dt}\int_{M^3} |u|^{p-1}\big|\nabla_g|u|\big|_g^2\, dvol_g + \\
& + (p-1)(p-3)\int_{M^3} |u|^{p-2}\partial_t|u|\big|\nabla_g|u|\big|_g^2\, dvol_g. \qquad (6.7)
\end{aligned}
$$

Now, taking the time derivative of the modified energy $\mathcal{F}_2(u)$, and replacing (6.7) in it, we have the statement. □

6.2 The Subcubic Case

The aim of this section is to provide a bound on the growth of the continuous function

$$
\mathbb{R}^+ \ni t \longmapsto ||u(t,\cdot)||_{H^2(M^3)}
$$

where u is the unique global solution to the Cauchy problem (4.3). We argue similarly to the previous cases.

Notation 6.2.1. *In the following computations we call ϵ a positive quantity that may change from line to line. We shall assume ϵ to be small enough at each step.*

Proposition 6.2.2. *Let $u \in C(\mathbb{R}, H^2(M^3))$ be the unique global solution to (4.3). For every $\tau \in (0,1)$, we have*

$$||u||_{L^2_\tau W^{1,6}(M^3)} \lesssim_{\epsilon,\ ||u_0||_{H^1}} \sqrt{\tau}||u||^{1/2}_{L^\infty_\tau H^2(M^3)} + ||u||^{\epsilon}_{L^\infty_\tau H^2(M^3)}. \tag{6.8}$$

Proof. Since u is a solution to (4.3), from Proposition 3.2.7, we deduce that

$$||u||_{L^2_\tau W^{1,6}} \lesssim_\epsilon ||u||_{L^\infty_\tau H^{1+\epsilon}} + ||u||_{L^2_\tau H^{3/2}} + ||(|u|^{p-1}u)||_{L^2_\tau W^{1,6/5}}.$$

Arguing as in the proof of Proposition 4.4.2, we get

$$||u||_{L^\infty_\tau H^{1+\epsilon}} \lesssim_\epsilon ||u||^{1-\epsilon}_{L^\infty_\tau H^1}||u||^{\epsilon}_{L^\infty_\tau H^2},$$
$$||u||_{L^2_\tau H^{3/2}} \lesssim \sqrt{\tau}||u||^{1/2}_{L^\infty_\tau H^1}||u||^{1/2}_{L^\infty_\tau H^2},$$

therefore, by using (4.4), we obtain

$$||u||_{L^\infty_\tau H^{1+\epsilon}} \lesssim_{\epsilon,\ ||u_0||_{H^1}} ||u||^{\epsilon}_{L^\infty_\tau H^2}, \quad ||u||_{L^2_\tau H^{3/2}} \lesssim_{||u_0||_{H^1}} \sqrt{\tau}||u||^{1/2}_{L^\infty_\tau H^2}.$$

By using the Sobolev embedding $H^1(M^3) \hookrightarrow L^{6p/5}(M^3)$ and (4.4), we get

$$||(|u|^{p-1}u)||_{L^{6/5}} = ||u||^p_{L^{6p/5}} \lesssim ||u||^p_{H^1} = ||u||^{p-\epsilon}_{H^1}||u||^{\epsilon}_{H^1} \lesssim_{||u_0||_{H^1}} ||u||^{\epsilon}_{H^2}.$$

Since $|\nabla_g|u|| \leq |\nabla_g u|$, we get

$$\begin{aligned}||\nabla_g(|u|^{p-1}u)||_{L^{6/5}} &= \left(\int_{M^3} |\nabla_g(|u|^{p-1}u)|^{6/5}_g \, dvol_g\right)^{5/6} \\ &\lesssim \left(\int_{M^3} |\nabla_g u|^{6/5}_g |u|^{(6p-6)/5} \, dvol_g\right)^{5/6},\end{aligned}$$

therefore, due to the Hölder inequality, the Sobolev embedding $H^1(M^3) \hookrightarrow L^{3(p-1)}(M^3)$ and (4.4), we have

$$||\nabla_g(|u|^{p-1}u)||_{L^{6/5}} \lesssim ||u||_{H^1}||u||^{p-1}_{L^{3(p-1)}} \lesssim ||u||^p_{H^1} \lesssim_{||u_0||_{H^1}} ||u||^{\epsilon}_{H^2},$$

so that

$$||(|u|^{p-1}u)||_{W^{1,6/5}} \sim ||(|u|^{p-1}u)||_{L^{6/5}} + ||\nabla_g(|u|^{p-1}u)||_{L^{6/5}} \lesssim_{||u_0||_{H^1}} ||u||^{\epsilon}_{H^2}.$$

Putting all together we obtain the statement of the proposition. □

Proposition 6.2.3. *Let $u \in C(\mathbb{R}, H^2(M^2))$ be the unique global solution to (4.3). For every $\tau \in (0,1)$, we have*

$$\int_0^\tau |r.h.s.\ of\ (6.3)|\ ds \lesssim_{||u_0||_{H^1}} \tau ||u||^{\frac{p+5}{4}}_{L^\infty_\tau H^2(M^3)} + ||u||^{\gamma}_{L^\infty_\tau H^2(M^3)}$$

for some $\gamma \in (0, \frac{p+5}{4})$.

Proof. We shall estimate the first term of (6.3), the second one is identical. Since $|\nabla_g|u|| \leq |\nabla_g u|$, we get

$$\int_0^\tau \left| \int_{M^3} |u|^{p-2} \partial_t |u|\, |\nabla_g |u||^2\ dvol_g \right| ds \leq \int_0^\tau \int_{M^3} |u|^{p-2} |\partial_t u|\, |\nabla_g u|^2_g\ dvol_g\ ds. \tag{6.9}$$

By using Hölder's inequality, the Sobolev embedding $H^1(M^3) \hookrightarrow L^6(M^3)$, (6.2) and (4.4) we get

$$\begin{aligned} &\int_{M^3} |u|^{p-2} |\partial_t u|\, |\nabla_g u|^2_g\ dvol_g \\ &\leq \left(\int_{M^3} |u|^6\ dvol_g \right)^{(p-2)/6} \left(\int_{M^3} |\partial_t u|^2\ dvol_g \right)^{1/2} \times \\ &\quad \times \left(\int_{M^3} |\nabla_g u|^{12/(5-p)}\ dvol_g \right)^{(5-p)/6} \\ &\lesssim ||u||^{p-2}_{H^1} ||\partial_t u||_{L^2} ||u||^2_{W^{1,\frac{12}{5-p}}} \lesssim_{||u_0||_{H^1}} ||u||_{H^2} ||u||^2_{W^{1,\frac{12}{5-p}}}, \end{aligned}$$

therefore

$$(6.9) \lesssim_{||u_0||_{H^1}} \int_0^\tau ||u||_{H^2} ||u||^2_{W^{1,\frac{12}{5-p}}}\ ds \leq ||u||_{L^\infty_\tau H^2} \int_0^\tau ||u||^2_{W^{1,\frac{12}{5-p}}}\ ds. \tag{6.10}$$

By using Hölder's inequality, we get

$$\int_0^\tau ||u||^2_{W^{1,\frac{12}{5-p}}}\ ds \leq \tau^{\frac{3-p}{4}} \left(\int_0^\tau ||u||^{\frac{8}{p+1}}_{W^{1,\frac{12}{5-p}}}\ ds \right)^{\frac{p+1}{4}}. \tag{6.11}$$

Now we claim that

$$||u||_{W^{1,\frac{12}{5-p}}} \lesssim_{||u_0||_{H^1}} ||u||_{W^{1,6}}^{\frac{p+1}{4}}. \tag{6.12}$$

Due to the Hölder inequality and (4.4), we have

$$||u||_{L^{\frac{12}{5-p}}} = \left(\int_{M^3} |u|^{12/(5-p)}\, dvol_g\right)^{(5-p)/12}$$
$$= \left(\int_{M^3} |u|^{3(3-p)/(5-p)}|u|^{3(p+1)/(5-p)}\, dvol_g\right)^{(5-p)/12}$$
$$\leq \left(\int_{M^3} |u|^2\, dvol_g\right)^{(3-p)/8} \left(\int_{M^3} |u|^6\, dvol_g\right)^{(p+1)/24}$$
$$\leq ||u||_{L^2}^{\frac{3-p}{4}} ||u||_{L^6}^{\frac{p+1}{4}} \lesssim ||u||_{H^1}^{\frac{3-p}{4}} ||u||_{L^6}^{\frac{p+1}{4}} \lesssim_{||u_0||_{H^1}} ||u||_{L^6}^{\frac{p+1}{4}}.$$

Similarly, we get

$$||\nabla_g u||_{L^{\frac{12}{5-p}}} \leq ||u||_{H^1}^{\frac{3-p}{4}} ||\nabla_g u||_{L^6}^{\frac{p+1}{4}} \lesssim_{||u_0||_{H^1}} ||\nabla_g u||_{L^6}^{\frac{p+1}{4}},$$

therefore

$$||u||_{W^{1,\frac{12}{5-p}}} \sim ||u||_{L^{\frac{12}{5-p}}} + ||\nabla_g u||_{L^{\frac{12}{5-p}}}$$
$$\lesssim_{||u_0||_{H^1}} ||u||_{L^6}^{\frac{p+1}{4}} + ||\nabla_g u||_{L^6}^{\frac{p+1}{4}} \sim ||u||_{W^{1,6}}^{\frac{p+1}{4}},$$

which proves the claim. By using (6.12) and (6.8), since $\tau < 1$ we get

$$(6.11) \lesssim_{||u_0||_{H^1}} \tau^{\frac{3-p}{4}} ||u||_{L^2_\tau W^{1,6}}^{\frac{p+1}{2}} \lesssim_{\epsilon,\ ||u_0||_{H^1}} \tau ||u||_{L^\infty_\tau H^2}^{\frac{p+1}{4}} + ||u||_{L^\infty_\tau H^2}^{\epsilon},$$

therefore

$$(6.10) \lesssim_{\epsilon,\ ||u_0||_{H^1}} \tau ||u||_{L^\infty_\tau H^2}^{\frac{p+5}{4}} + ||u||_{L^\infty_\tau H^2}^{\gamma}$$

for some $\gamma \in (0, \frac{p+5}{4})$, which concludes the proof of the proposition. □

Proposition 6.2.4. *Let $u \in C(\mathbb{R}, H^2(M^3))$ be the unique global solution to (4.3). For every $\tau \in (0,1)$, we have the following bound:*

$$||u(\tau,\cdot)||_{H^2(M^3)}^2 - ||u(0,\cdot)||_{H^2(M^3)}^2 \lesssim_{||u_0||_{H^1}} \tau ||u||_{L^\infty_\tau H^2(M^3)}^{\frac{p+5}{4}} + ||u||_{L^\infty_\tau H^2(M^3)}^{\gamma}$$

for some $\gamma \in (0, \frac{p+5}{4})$.

Proof. Let $\mathcal{G}_2(u)$ be as in Definition 6.1.1. Due to the Sobolev embedding $H^2(M^3) \hookrightarrow L^\infty(M^3)$ and (4.4), since $|\nabla_g |u||_g \leq |\nabla_g u|_g$, we have

$$\int_{M^3} |u|^{p-1} |\nabla_g |u||_g^2 \ dvol_g \leq ||u||_{L^\infty}^{p-1} \int_{M^3} |\nabla_g u|_g^2 \ dvol_g$$
$$\lesssim ||u||_{L^\infty}^{p-1} ||u||_{H^1}^2 \lesssim_{||u_0||_{H^1}} ||u||_{H^2}^{p-1}$$

Moreover, due to the Sobolev embedding $H^1(M^3) \hookrightarrow L^{2p}(M^3)$ and (4.4), we have

$$\int_{M^3} |u|^{2p} \ dvol_g = ||u||_{L^{2p}}^{2p} \lesssim ||u||_{H^1}^{2p} = ||u||_{H^1}^{p+1} ||u||_{H^1}^{p-1} \lesssim_{||u_0||_{H^1}} ||u||_{H^1}^{p-1},$$

so that

$$|\mathcal{G}_2(u)| \lesssim_{||u_0||_{H^1}} ||u||_{H^2}^{p-1} + ||u||_{H^1}^{p-1} \lesssim ||u||_{H^2}^{p-1}. \tag{6.13}$$

From Proposition 6.2.3, we have

$$\left| \int_0^\tau \frac{d}{ds} \mathcal{E}_{2k}(u) \ ds \right| \leq \int_0^\tau |r.h.s. \ of \ (6.3)| \ ds$$
$$\lesssim_{||u_0||_{H^1}} \tau ||u||_{L_\tau^\infty H^2}^{\frac{p+5}{4}} + ||u||_{L_\tau^\infty H^2}^\gamma$$

for some $\gamma \in (0, \frac{p+5}{4})$. By applying the Fundamental Theorem of Calculus, we get

$$\left| ||\partial_t u(\tau, \cdot)||_{L^2}^2 - ||\partial_t u(0, \cdot)||_{L^2}^2 + \mathcal{G}_2(u)(\tau) - \mathcal{G}_2(u)(0) \right|$$
$$\lesssim_{||u_0||_{H^1}} \tau ||u||_{L_\tau^\infty H^2}^{\frac{p+5}{4}} + ||u||_{L_\tau^\infty H^2}^\gamma,$$

hence

$$||\partial_t u(\tau, \cdot)||_{L^2}^2 - ||\partial_t u(0, \cdot)||_{L^2}^2 - 2 \sup_{t \in (0,\tau)} |\mathcal{G}_2(u)(t)|$$
$$\lesssim_{||u_0||_{H^1}} \tau ||u||_{L_\tau^\infty H^2}^{\frac{p+5}{4}} + ||u||_{L_\tau^\infty H^2}^\gamma.$$

Due to (6.13), since $p - 1 < \frac{p+5}{4}$, we have

$$||\partial_t u(\tau, \cdot)||_{L^2}^2 - ||\partial_t u(0, \cdot)||_{L^2}^2 \lesssim_{||u_0||_{H^1}} \tau ||u||_{L_\tau^\infty H^2}^{\frac{p+5}{4}} + ||u||_{L_\tau^\infty H^2}^\gamma$$

for some $\gamma \in (0, \frac{p+5}{4})$. Finally, due to (6.1) and (4.4), we have

$$\begin{aligned} &||u(\tau,\cdot)||_{H^2}^2 - ||\partial_t u(\tau,\cdot)||_{L^2}^2 \\ &\leq (||\partial_t u(\tau,\cdot) - i\Delta_g u(\tau,\cdot)||_{L^2} + ||\partial_t u(\tau,\cdot)||_{L^2})^2 + \\ &+||u(\tau,\cdot)||_{H^1}^2 - ||\partial_t u(\tau,\cdot)||_{L^2}^2 \\ &\lesssim_{||u_0||_{H^1}} ||u(\tau,\cdot)||_{H^2}, \end{aligned}$$

and

$$\begin{aligned} &||\partial_t u(0,\cdot)||_{L^2}^2 - ||u(0,\cdot)||_{H^2}^2 \\ &\leq (||\partial_t u(0,\cdot) - i\Delta_g u(0,\cdot)||_{L^2} + ||\Delta_g u(0,\cdot)||_{L^2})^2 - ||\Delta_g u(0,\cdot)||_{L^2}^2 \\ &\lesssim_{||u_0||_{H^1}} ||u(0,\cdot)||_{H^2}, \end{aligned}$$

therefore

$$\begin{aligned} &||u(\tau,\cdot)||_{H^2}^2 - ||u(0,\cdot)||_{H^2}^2 \\ &\lesssim_{||u_0||_{H^1}} \tau ||u||_{L^\infty_\tau H^2}^{\frac{p+5}{4}} + ||u||_{L^\infty_\tau H^2}^{\gamma} + ||u(\tau,\cdot)||_{H^2} + ||u(0,\cdot)||_{H^2} \\ &\lesssim \tau ||u||_{L^\infty_\tau H^2}^{\frac{p+5}{4}} + ||u||_{L^\infty_\tau H^2}^{\gamma} \end{aligned}$$

for some $\gamma \in (0, \frac{p+5}{4})$, proving the statement of the proposition. □

Theorem 6.2.5. *Let $u \in C(\mathbb{R}, H^2(M^3))$ be the unique global solution to (4.3). For any $T > 0$, we have the following bound:*

$$\sup_{t \in (0,T)} ||u(t,\cdot)||_{H^2(M^3)} \leq C \left(\max\{1,T\}\right)^{\frac{4}{3-p}},$$

where $C = C(||u_0||_{H^2}) > 0$.

Proof. Let us consider $\tau \in (0,1)$ given by the local Cauchy theory. From Proposition 6.2.4, Remark 4.0.1 and elementary computations, we get

$$||u(\tau,\cdot)||_{H^2}^2 \leq ||u(0,\cdot)||_{H^2}^2 + C||u||_{L^\infty_\tau H^2}^{2-2\gamma}$$

where $\gamma := \frac{3-p}{8}$ and $C = C(||u_0||_{H^2}) > 0$ may change in the following computations. By iteration, for every $n \in \mathbb{N} \cup \{0\}$, we obtain

$$||u(n\tau+\tau,\cdot)||_{H^2}^2 \leq ||u(n\tau,\cdot)||_{H^2}^2 + C \left(\sup_{t \in (n\tau, n\tau+\tau)} ||u(t,\cdot)||_{H^2}^{2-2\gamma} \right),$$

hence, since the map data solution is continuous (see 2.2.3), we have

$$||u(n\tau+\tau,\cdot)||_{H^2}^2 \le ||u(n\tau,\cdot)||_{H^2}^2 + C||u(n\tau,\cdot)||_{H^2}^{2-2\gamma}.$$

Now, arguing as in the proof of Theorem 4.3.6, for every $n \in \mathbb{N} \cup \{0\}$, we obtain

$$||u(n\tau,\cdot)||_{H^2}^2 \le C\left(\max\{1,n\}\right)^{1/\gamma},$$

therefore, due to the continuity of the map data solution we have

$$\sup_{t\in(n\tau,n\tau+\tau)} ||u(t,\cdot)||_{H^2}^2 \le C||u(n\tau,\cdot)||_{H^2}^2 \le C\left(\max\{1,n\}\right)^{1/\gamma}.$$

Finally, given $N \in \mathbb{N}$ such that $N\tau \le T \le N\tau+\tau$, we have

$$\begin{aligned}\sup_{t\in(0,T)} ||u(t,\cdot)||_{H^2}^2 \le \max_{n\in\{0,\cdots,N\}} \left(\sup_{t\in(n\tau,n\tau+\tau)} ||u(t,\cdot)||_{H^2}^2\right) &\le C\left(\max\{1,N\}\right)^{1/\gamma} \\ &\le C\left(\max\{\tau,T\}\right)^{1/\gamma} \le C\left(\max\{1,T\}\right)^{1/\gamma}\end{aligned}$$

since $\tau < 1$, therefore

$$\sup_{t\in(0,T)} ||u(t,\cdot)||_{H^2} \le C\left(\max\{1,T\}\right)^{1/2\gamma} = C\left(\max\{1,T\}\right)^{\frac{4}{3-p}},$$

which concludes the proof of the theorem. □

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