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Generalized Shioda-Inose structures

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Se non dovessi tornare,
sappiate che non sono mai
partito.
Il mio viaggiare
È stato tutto un restare
qua, dove non fui mai.
- Giorgio Caproni

Abstract

Shioda-Inose structures establish a correspondence between the only two classes of projective compact complex surfaces with trivial canonical bundle, namely $K3$ *surfaces* and *Abelian surfaces*, allowing the study of certain $K3$ surfaces to be reduced to the study of their associated Abelian surface. In this thesis, we extend this classical theory by introducing and analyzing generalized Shioda-Inose structures. In particular, we present Morrison's work on Shioda-Inose structures of order two, the generalization by Garbagnati and Prieto-Montañez to Generalized Shioda-Inose structures of order three and the work of Piroddi on Generalized Shioda-Inose structures of order four. Finally, we summarize the state of the art on Generalized Shioda-Inose structures of order six, and discuss a possible generalization of the very definition of this geometric construction.

Contents

Introduction

Shioda-Inose structures represent a powerful relation between $K3$ surfaces and Abelian surfaces, which are the only two distinct types of projective compact complex surface with trivial canonical bundle. The definition of Shioda-Inose structure was first proposed by D. Morrison, who generalized the work of Shioda-Inose by including all Abelian surfaces. Morrison described a way to associate a $K3$ surface S_A to any Abelian surface A, in such a way that there exists a Hodge isometry between their transcendental lattices, i.e. $T(A) \simeq T(S_A)$. In particular, the unique symplectic involution $-\mathrm{id}_A$ on A gives rise to the Kummer surface X, which, in turn, admits a double cover by a $K3$ surface. Morrison characterized those $K3$ surfaces that admit a Shioda-Inose structure, giving a description in terms of the structure of the transcendental lattice.
After Morrison, many authors provided examples of Shioda-Inose structures and showed some applications. More recently, A. Garbagnati and Y. Prieto-Montañez extended the entire framework by examining Abelian surfaces that admit a symplectic automorphism of order 3, and constructing a covering $K3$ surface S of the generalized Kummer surface $Km_3(A)$ such that the relation between the transcendental lattices of A and S still holds. Subsequently, B. Piroddi proposed a definition of Generalized Shioda-Inose structures of order 4, and showed that the Hodge isometry between the trascendental lattices doesn't hold anymore in general. In this thesis we review the aforementioned works, and we present a possible definition for Generalized Shioda-Inose structures of order 6. In addition, we provide the necessary background for understanding and approaching the topic. We assume the reader has familiarity with basic concepts of Algebraic Geometry and Complex Geometry.

The first chapter focuses on integral even lattices, which are fundamental for understanding the geometry of compact complex surfaces with trivial canonical bundle. Here introduce key tools and important theorems related to even lattices and their applications to the above mentioned surfaces.

In chapter two we introduce $K3$ surfaces, and briefly study their classical invariants. Furthermore, we begin the study of the $K3$ lattice $\Lambda_{K3} \simeq U^3 \oplus E_8(-1)^2$, which leads to the statement of both the *Global Torelli Theorem for K3 surfaces* and the *surjectivity of the period map*. Following this, we present Nikulin's work on *symplectic automorphisms of finite order* of $K3$ surfaces. These automorphisms are of particular interest, because their action yields *isolated singular points of type ADE*, and their minimal resolution yields again a $K3$ surface. We classify cyclic groups acting symplectically on these surfaces and characterize those $K3$ surfaces that admit symplectic automorphisms of finite order.

In chapter three we shift our focus to *Abelian surfaces*. We provide the general definition of a complex torus, together with the proof of its equivalence to a *compact connected complex Lie group*. Furthermore, we characterize projective complex tori using two different approaches. We then review Shioda's work on the *surjectivity of the period map* of Abelian surfaces, which leads to the work of Fujiki on finite order automorphisms of these varieties. Here we reframe the treatment adopting a lattice-theoretic perspective. Finally, we characterize Abelian surfaces that admit a symplectic automorphism of finite order.

In the end, in chapter four, we bring together the concepts introduced in the previous chapters to present *Shioda-Inose structures*. We begin with Morrison's work, introducing a revised notation that ensures consistency with our subsequent treatment of Generalized Shioda-Inose structures. We then follow Garbagnati and Prieto-Montañez's study on Generalized Shioda-Inose structures of order 3. Moreover, we propose a definition for Generalized Shioda-Inose structures which works also for the order 4. After reviewing the results of Piroddi on Generalized Shioda-Inose structures of order 4 we conclude with some considerations on a possible approach to the case of order 6.

Chapter 1

Lattices

1.1 Definitions and examples

We begin with a comprehensive list of definitions and conventions.

Definition 1.1. An *even lattice* L is a finitely generated free $\mathbb{Z}$-module L, equipped with a non-degenerate symmetric even bilinear form $b_L(\cdot,\cdot) : L \times L \to \mathbb{Z}$. Equivalently, an even lattice L is a finitely generated free $\mathbb{Z}$-module L together with a non-degenerate quadratic form.

We may refer to an even lattice L as (L, b_L) or even just as L if its bilinear form is clear by the context.
Now we present a series of useful and common constructions and defitions for even lattices.

Definition 1.2. If L_1 and L_2 are two lattices, then $L_1 \oplus L_2$ is their *orthogonal direct sum*; L^n denotes the orthogonal direct sum $L \oplus L \oplus \cdots \oplus L$.
Given an integer $m \in \mathbb{Z}$, we write $L(m)$ and refer to the lattice $(L, m\, b_L)$ where the bilinear form of L has been multiplied by m.

Definition 1.3. Fixing a basis of vectors of an even lattice L, we denote the matrix associated to its bilinear form b_L by B_L.
The *determinant* of L is the determinant of B_L, and it is independent of the choice of the basis. This is justified by the fact that any base change is given by a matrix in $\mathrm{GL}_n(\mathbb{Z})$, which has determinant ± 1.
An even lattice is *non-degenerate* if its determinant is non-zero. If L is a non-degenerate lattice, then its signature is a pair $(s_{(+)}, s_{(-)})$, where $s_{(\pm)}$ stands for the multiplicity of the eigenvalue ± 1 of the associated real-valued bilinear form on $L \oplus \mathbb{R}$.

Furthermore, L is said to be *positive definite*, *negative definite* or *indefinite* if its corresponding real-valued bilinear form has the same property.

Example 1.4. (i) **Hyperbolic Plane:** It is the even unimodular lattice U of rank 2 with bilinear form given by the matrix

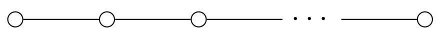
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that U is indefinite and its signature is $(1, -1)$. Moreover for every $m \in \mathbb{Z}$ the lattice $U(m)$ is isometric to $U(-m)$. Indeed, taking $\{e_1, e_2\}$ as a set of generators of U such that $e_1^2 = e_2^2 = 0$ and $b(e_1, e_2) = 1$ we can construct the isomorphism

$$\begin{aligned} \phi : U(m) &\to U(-m) \\ e_1 &\mapsto -e_1 \\ e_2 &\mapsto e_2 \end{aligned}$$

Here it is easy to verify that $\phi(e_1)^2 = \phi(e_2)^2 = e_1^2 = e_2^2 = 0$ and $b(\phi(e_1), \phi(e_2)) = b(e_1, e_2) = m$

(ii) A_n **lattice:** Given the diagram

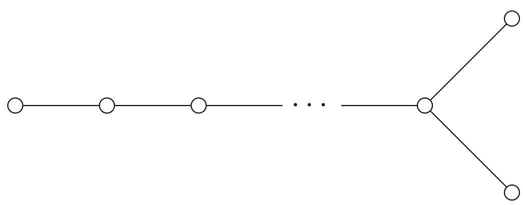


The lattice A_n comes from the associated root lattice, and it corresponds to the free $\mathbb{Z}$-module of rank n together with the bilinear form given by the matrix

$$A_n \text{ matrix:} \quad \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}.$$

Its determinant is equal to $n + 1$, therefore A_n is not unimodular and has signature $(n, 0)$. We will often use the lattice $A_n(-1)$ of signature $(0, n)$.

(iii) D_n **lattice:** Given the diagram

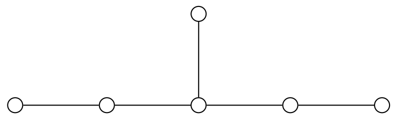


The lattice D_n again comes from the associated root lattice, and it is the free $\mathbb{Z}$-module of rank n defined by the matrix

$$D_n \text{ matrix:} \qquad \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & & & \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 2 \end{pmatrix}.$$

Its matrix is similar to that of A_n, but differs in the last 3 columns, where the relation between the last 3 vertices is made explicit. Its determinant is equal to 4 and its signature is $(n, 0)$.

(iv) E_6 **lattice:** Given the diagram

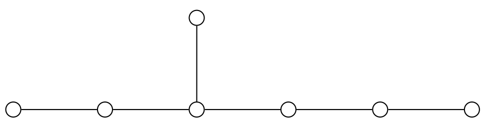


The lattice E_6 is of rank 6 and its bilinear form is given by the matrix:

$$E_6 \text{ matrix:} \qquad \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Its determinant is 3 and it has signature $(6, 0)$. This lattice (or, more properly, the lattice $E_6(-1)$) will be useful in the study of Generalized Shioda-Inose structures of order 3.

(v) E_7 **lattice:** Given the diagram

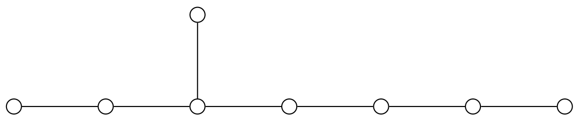


The lattice E_7 is a rank 7 lattice with bilinear form given by the matrix:

$$E_7 \text{ matrix:} \quad \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Its determinant is 2 and it has signature $(7, 0)$.

(vi) E_8 **lattice:** Given by the diagram



The lattice E_8 is given by a free $\mathbb{Z}$-module of rank 8 together with a bilinear form defined by the Matrix:

$$E_8 \text{ matrix:} \quad \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Its determinant is 1, i.e. it is unimodular, and its signature is $(8, 0)$. This is probably the most important Dynkin diagram in the context of K3 surfaces. As we will see in chapter 2, the $K3$ lattice can be described in terms of $E_8(-1)$ and U, the hyperbolic plane. Moreover, this lattice will be crucial when considering Shioda-Inose structures of order two.

1.2 Overlattices

Studying morphisms between objects is in line with customs and practices of mathematics, and we will not shy away from this tradition.

Definition 1.5. Given two even lattices L and M, a *lattice morphism* from L to M is a homomorphism of $\mathbb{Z}$-modules $\phi : L \to M$

Definition 1.6. The *dual lattice* of the lattice L is $L^* := Hom(L, \mathbb{Z})$. Equivalently, it is the subset of $L \otimes \mathbb{Q}$ defined as follows:

$$\{m \in L \otimes \mathbb{Q} \,|\, \tilde{b}_L(m,l) \in \mathbb{Z} \text{ for every } l \in L\},$$

where $\tilde{b}_L$ is the $\mathbb{Q}$-linear extension of b_L to $L \otimes \mathbb{Q}$.
The *discriminant group* of an even lattice L is $A_L := L^*/L$. It can be zero or a product of cyclic groups. In the first case, we call L *unimodular*.

Definition 1.7. An *embedding* of lattices is an injective morphism of even lattices $L \hookrightarrow M$ that preserves the bilinear form, i.e. $b_L(l_1, l_2) = b_M(\phi(l_1), \phi(l_2))$. The embedding is called *primitive* if M/L is free. If M/L is finite we say that M is an overlattice of L.

Definition 1.8. The natural embedding $L \hookrightarrow L^*$ given by $l \mapsto b_L(l, \cdot)$ induces a quadratic form on A_L, called *discriminant form* and denoted $q_L : A_L \to \mathbb{Q}/2\mathbb{Z}$.

Remark 1.9. It is straightforward to verify that $q_{L(-1)} \simeq -q_L$, and that $q_{L\oplus M} \simeq q_L \oplus q_M$.

Remark 1.10. The absolute value of the determinant of the matrix B_L is equal to the order of the group A_L. If the lattice has a positive definite bilinear form, this follows from the fact that for $x \in L \otimes \mathbb{Q}$ we have $x \in L^* \iff \tilde{b}_L(x,y) \in \mathbb{Z}$ for every $y \in L$, and this is equivalent to condition $x^T B y \in \mathbb{Z}$. In other words we are saying that $x^T B = (B^T x)^T$ must be integer valued, implying that the dual basis of L^* is given by the columns of $(B^T)^{-1}$ and thus the Gram matrix of L^* with respect to this basis is given by $B^{-1}B(B^T)^{-1} = (B^T)^{-1}$.
Now, since the determinant of L is equal to the index of L in L^*, we get

$$|\det(L)| = [L^* : L] = |\det(B)| = |\det(B^T)| = |\det(((B^T)^{-1})^{-1}| = |\det(L^*)|^{-1}.$$

This is an instance of a more general fact about sublattices, valid for any symmetric non-degenerate bilinear form: if $L' \subset L$ is a sublattice, then

$$[L : L']^2 = \frac{|\det(L')|}{|\det(L)|}. \tag{1.11}$$

This follows easily by writing $L' = A\,L$ where A is an integer valued matrix. Then $[L : L']^2 = |\det(A)|$ and 1.11 is proved by noticing that $\det(L') = \det(A)\det(L)$.

Consider the following example:

Example 1.12 (Two noteworthy overlattices). Given the even lattice $L = U \oplus A_2 \oplus E_6(-1)^{\oplus 3}$, its determinant is 3^4, since $\det(U) = 1$ and $\det(A_2) = \det(E_6(-1)) = 3$. An overlattice of L can be constructed, following the approach in [5], by considering a basis of each copy of $E_6(-1)$ as follows: let $e_i^{(j)}$ denote the basis vectors for each copy, where $i \in \{1,2,3,4,5,6\}$ and j indexes the corresponding copy of $E_6(-1)$, with the intersection properties given by the matrix written in 1.4 (iv). We define $(E_6(-1)^{\oplus 3})'$ as the overlattice of index 3 of $E_6(-1)^{\oplus 3}$, obtained by adding the vector

$$x = \frac{1}{3}\sum_{j=1}^{3}\left(e_1^{(j)} + 2e_2^{(j)} + e_4^{(j)} + 2e_5^{(j)}\right)$$

In order to verify that its index is indeed equal to 3, we can use 1.11 and choose a basis for the new overlattice simply by swapping an element basis e_i with x. It follows that $\det((E_6(-1)^{\oplus 3})') = 3^2$, so its index is equal to 3.

Now define $v^{(j)} := (e_1^{(j)} + 2e_2^{(j)} + e_4^{(j)} + 2e_5^{(j)})/3$ and consider a basis a_i of A_2 for $i = 1,2$ such that $a_i^2 = 2$ and $a_1a_2 = a_2a_1 = -1$. Using this basis, we construct the vector

$$y := \frac{1}{3}(a_1 + 2a_2) + v^{(1)} + v^{(2)}.$$

The lattice $(U \oplus A_2 \oplus E_6(-1)^{\oplus 3})''$ is defined as the overlattice of L obtained by adding the vectors x and y to L. This results in a unimodular even lattice, as the index satisfies $[(U \oplus A_2 \oplus E_6(-1)^{\oplus 3})'' : L] = 3^2$. Indeed, we've added two vectors of self-intersection 3. We'll discuss this lattice more in depth in the chapter 4.

Theorem 1.13 ([20, Milnor]). *Every indefinite unimodular even lattice L is of the form*

$$L = U^n \oplus E_8(-1)^m \quad \textit{for some positive } m, n \in \mathbb{N}.$$

Theorem 1.14 ([26, Corollary 1.13.3]). *Let L be an indefinite even lattice that verifies $l(A_L) \leq \mathrm{rk}(L) - 2$ where $l(A_L)$ is the minimum number of generators of the group, whereas $\mathrm{rk}(L)$ stands for the rank of L as a free group. Then, up to isometry, L is uniquely determined by its signature and discriminant form q_L.*

Corollary 1.15. *A unimodular indefinite even lattice is uniquely determined by its signature and discriminant form.*

Lemma 1.16 ([22, Lemma 2.3]). *Let M,N be two even lattices with the same signature and discriminant form, and let L be an even lattice uniquely determined by its signature and discriminant form. Then there exists an embedding $M \hookrightarrow L$ if and only if there exists an embedding $N \hookrightarrow L$, too.*

Proof. Suppose $M \hookrightarrow L$ is primitive, and let K be the orthogonal complement of M in L. We have the inclusions:

$$M \oplus K \subset L \subset L^* \subset (M \oplus K)^*$$

because taking the duals inverts the inclusion.
Now, from $A_{M\oplus K} = A_M \oplus A_K$ and the fact that $A_M \simeq A_N$ we get the group isomorphism

$$\phi : A_{N\oplus K} \xrightarrow{\sim} A_{M\oplus K}$$

Define the sublattice

$$L' := \{n \in (N \oplus K)^* \mid \phi([n]) \in L/(M \oplus K)\} \subset (N \oplus K)^*$$

Of course $\phi([n]) = \phi([0]) = [0] \in A_{M\oplus K}$ for every $n \in N$, so we have the embedding $N \hookrightarrow L'$. In order to see that this embedding is primitive, we prove that $N^* \cap L' = N$. In fact, taking $n \in N^* \cap L'$, we have $\phi([n]) \in (M^* \cap L)/(M \oplus K)$, but, since we assumed that $M \hookrightarrow L$ is primitive, then $\phi([n]) \in M$ and $n \in N$. We conclude using the uniqueness of L. In fact, $q_{L'} \simeq q_L$ because ϕ preserves the discriminant form, and from the inclusions $M \oplus K \subset L$ and $N \oplus K \subset L'$ we deduce that L and L' share the same signature. □

Proposition 1.17 ([26, Proposition 1.6.1]). *An embedding of an even lattice M into an even unimodular lattice L verifies:*

$$q_M \simeq -q_M^{\perp}$$

Moreover, given any two even lattices M, N such that $q_M \simeq -q_N$, there exist an even unimodular lattice L and a primitive embedding $M \hookrightarrow L$ with $M^\perp \simeq N$.

Example 1.18. Consider an even lattice M of signature $(r, n-r)$. Then $M(-1)$ has signature $(n-r, r)$ and, as mentioned earlier, $q_M \simeq -q_{M(-1)}$. The proposition tells us that there exists an even lattice L such that $M^\perp \simeq M(-1)$, but then L has signature (n, n) and is unimodular, which means that $L \simeq U^n$ by Theorem 1.13. The embedding can be made explicit by considering a basis $\{m_1, \dots, m_n\}$ for M and a basis $\{e_1^{(i)}, e_2^{(i)}\}_{i=1,..,n}$ constructing the map this way:

$$m_i \mapsto e_1^{(i)} + \frac{1}{2} m_i^2 e_2^{(i)} + \sum_{j<i} (m_i m_j) e_2^{(j)}.$$

This is summed up by the following proposition:

Proposition 1.19. *An even lattice M of rank n admits a primitive embedding $M \hookrightarrow U^n$.*

We conclude this section with yet another result by Nikulin on lattice embeddings which is related to Theorem 1.14

Theorem 1.20 ([22, Theorem 2.8]). *Let M be an even lattice with signature (s_+, s_-) and discriminant form q_M, and let L be an even unimodular lattice with signature (t_+, t_-) and discriminant form q_L. Suppose that*

1. $s_+ < t_+$;
2. $s_- < t_-$;
3. $l(A_M) \leq \mathrm{rk}(L) - \mathrm{rk}(M) - 2$.

Then there exists a unique embedding $M \hookrightarrow L$.

1.3 Lattices associated to Compact Kähler surfaces

When dealing with a compact Kähler complex surface X, we will focus on studying the second cohomology group, namely $H^2(X, \mathbb{Z})$. If this group is torsion-free, then the intersection pairing gives it an even lattice structure. According to the Hodge Index Theorem, the signature of the resulting lattice is always $(2h^{2,0} + 1, h^{1,1}(X) - 1)$. This lattice proves to be a crucial tool in the study of the geometry of surfaces, particularly when examining the automorphisms of compact Kähler surfaces. In the upcoming chapters, we will explore how the Torelli theorems for both $K3$ surfaces and Abelian varieties offer a fresh algebraic and topological perspective on the geometry of these surfaces. Now, let us introduce the key concepts that will serve as the foundation for this approach. In particular: Hodge theory on complex Kähler manifolds.

Proposition 1.21. *Let X be a compact Kähler manifold. Then, for each integer k, the k-th de Rham cohomology group $H^k(X, \mathbb{C})$ admits a Hodge decomposition:*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

where $H^{p,q}(X)$ denotes the Dolbeault cohomology group of type (p, q). Moreover, the complex conjugation on $H^k(X, \mathbb{C})$ induces an isomorphism:

$$\overline{H^{p,q}(X)} = H^{q,p}(X).$$

Proposition 1.22 ((1,1)-Lefschetz)**.** *Let X be a compact Kähler manifold. Then the image of* $\mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z})$ *is* $H^{1,1}(X, \mathbb{Z}) := H^{1,1}(X) \cap H^2(X, \mathbb{Z})$.

Proof. We begin with a class $\alpha \in H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{C})$. Thanks to the bidegree decomposition, we can write $\alpha = \alpha^{2,0} + \alpha^{1,1} + \alpha^{0,2}$. Since $\alpha \in H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{R})$ it follows that $\alpha = \overline{\alpha}$, therefore $\alpha^{2,0} = \overline{\alpha^{0,2}}$.
By hypothesis $\alpha \in \mathrm{Im}(c_1)$, so it also lies in the kernel of $H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$, and it is easy to verify that the map $H^2(X, \mathbb{C}) \to H^2(X, \mathcal{O}_X)$ given by the sheaf inclusion $\mathbb{C} \subset \mathcal{O}_X$ coincides with the projection map $H^2(X, \mathbb{C}) \to H^{0,2}(X)$; then

$\alpha^{0,2}$ is zero, and thus $\alpha^{2,0} = \overline{\alpha^{0,2}} = 0$.
In conclusion, α reduces to its (1,1)-component: $\alpha = \alpha^{1,1} \in H^{1,1}(X) \cap H^2(X, \mathbb{Z})$. $\square$

In order to give $H^2(X, \mathbb{Z})$ a lattice structure, first we need to define its bilinear form. The next theorem comes into help, giving a thorough description of its signature.

Theorem 1.23 (Hodge Index Theorem)**.** *Let X be a compact complex surface, and consider the intersection pairing*

$$H^2(X, \mathbb{R}) \times H^2(X, \mathbb{R}) \to \mathbb{R}, \quad (\alpha, \beta) \mapsto \int_X \alpha \wedge \beta.$$

Then its signature is equal to $(2h^{2,0}(X) + 1, h^{1,1}(X) - 1)$*, where* $h^{i,j}(X) := \dim H^{i,j}(X)$ *are the Hodge numbers of the surface X. In particular, restricted to* $H^{1,1}(X)$*, the intersection pairing has signature* $(1, h^{1,1}(X) - 1)$*.*

Proof. Here we use the restriction of the bidegree decomposition, $H^2(X, \mathbb{R}) = (H^{2,0}(X) \oplus H^{0,2}(X) \oplus H^{1,1}(X)) \cap H^2(X, \mathbb{R})$, and the fact that $\overline{H^{1,1}} = H^{1,1}$, giving the orthogonal splitting:

$$H^2(X, \mathbb{R}) = \big(H^{2,0}(X) \oplus H^{0,2}(X)\big) \cap H^2(X, \mathbb{R}\big) \oplus H^{1,1}(X, \mathbb{R}).$$

The third component, $H^{1,1}(X) \cap H^2(X, \mathbb{R})$, can be further broken in two orthogonal components: $\langle [\omega] \rangle \oplus H^{1,1}(X, \mathbb{R})_p$, where the last term represents the primitive cohomology. This is due to the Lefschetz Hard Theorem, which states that $H^2(X, \mathbb{R}) = H^2(X, \mathbb{R})_p \oplus LH^0(X, \mathbb{R})_p$ (here the operator L is the wedge product with the Kähler form ω). We've come down to the decomposition

$$H^2(X, \mathbb{R}) = \big(H^{2,0}(X, \mathbb{C}) \oplus H^{0,2}(X, \mathbb{C})\big) \cap H^2(X, \mathbb{R}) \oplus \langle [\omega] \rangle \oplus H^{1,1}(X, \mathbb{R})_p$$

Now for the first summand, take $\alpha = \alpha^{2,0} + \alpha^{0,2} \in (H^{2,0}(X) \oplus H^{0,2}(X)) \cap H^2(X, \mathbb{R})$: it follows that

$$\int_X \alpha \wedge \alpha = 2 \int_X \alpha^{2,0} \wedge \alpha^{0,2} = 2 \int_X \alpha^{2,0} \wedge \bar{\alpha}^{2,0} \geq 0.$$

As for the Kähler class, instead, its square is positive because the Kähler form is a positive form by definition.
Lastly, every form $0 \neq \alpha \in H^{1,1}(X, \mathbb{R})_p$ has strictly negative self-intersection by Hodge-Riemann bilinear relation. $\square$

We need to describe the structure of $H^2(X, \mathbb{Z})$. First of all, let us define what a Hodge structure of weight two is.

Definition 1.24 (Hodge Structure of weight two)**.** Let L be an even lattice. A *Hodge structure of weight two* on L is a decomposition

$$L \otimes \mathbb{C} = L^{2,0} \oplus L^{1,1} \oplus L^{0,2}$$

where, writing b_L for the $\mathbb{C}$-linear extension of the bilinear form of L, the following properties hold:

1. $\overline{L^{2,0}} = L^{0,2}$ and $\overline{L^{1,1}} = L^{1,1}$;
2. $b_L(x, y) = 0$ for $x \in L^{2,0} \oplus L^{0,2}$, $y \in L^{1,1}$;
3. $b_L(x, y) = 0$ for $x, y \in L^{2,0}$;
4. $b_L(x, \bar{x}) > 0$ for $x \in L^{2,0}$.

It is obvious that this definition mimics what we have seen in the proof of Hodge Index Theorem. We need to introduce some more definitions.

Definition 1.25 (Signed Hodge structure of weight two)**.** Given an even lattice L, a *Signed Hodge structure of weight two* on L is given by the following data:

1. a Hodge structure of weight two on L where the quadratic form associated to L, restricted to $L^{1.1} \cap (L \otimes \mathbb{R})$, has signature $(1, n-1)$;
2. a choice of one of the two components of the cone

$$\mathcal{P}(L) := \{x \in L^{1.1} \cap (L \otimes \mathbb{R}) \mid b_L(x, x) > 0\}$$

Definition 1.26 (Polarized Hodge structure of weight two)**.** Given an even lattice L, a *Polarized Hodge structure of weight two* on L is a Hodge structure of weight two on L where the quadratic form associated to L, restricted to $L^{1.1} \cap (L \otimes \mathbb{R})$, is negative definite.

Definition 1.27 ((Signed) Hodge Isometry)**.** Given two even lattices L and M, both with a Hodge structure of weight two, a *Hodge Isometry* is an isometry $\phi : L \xrightarrow{\sim} M$ that preserves the Hodge structure, that is, $\phi(L^{i,j}) = M^{i,j}$ for $i, j \in \{0, 1, 2\}$.
A Signed Hodge Isometry is a Hodge Isometry that preserves the choice of the component of the cone $\mathcal{P}(L)$.

Remark 1.28. Let X be a compact Kähler surface such that $H^2(X, \mathbb{Z})$ is torsion-free. Consider the Stiefel-Whitney class $\omega_2(T_X) \in H^2(X, \mathbb{Z}/2\mathbb{Z})$ of the tangent bundle of X. For any point $v \in H^2(X, \mathbb{Z})$ Wu's formula says that

$$v^2 \equiv \omega_2(T_X) \cdot v \pmod 2.$$

Since $\omega_2(T_X) \equiv c_1(X) \pmod 2$ and $c_1(X) = 0$, we get that $H^2(X, \mathbb{Z})$ is an even lattice. By Poincaré duality, $H^2(X, \mathbb{Z})$ is also unimodular, and its signature is

$(2h^{2,0}+1, h^{1,1}-1)$ due to Hodge Index Theorem. This lattice inherits a Hodge structure of weight two thanks to Hodge Decomposition:

$$H^2(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = H^2(X,\mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X).$$

Furthermore, its Kähler class $[\omega]$ defines a natural choice for the component of the positive cone, namely the one containing all Kähler classes, giving $H^2(X,\mathbb{Z})$ a Signed Hodge structure of weight two.

Additional structure can be defined over this remarkable lattice. To do so recall that, for projective manifolds, the map $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ which associates a divisor D to the line bundle $\mathcal{O}(D)$ is surjective.

Definition 1.29. Let X be an algebraic complex surface; we define an equivalence relation on $\mathrm{Pic}(X)$: if $L_1, L_2 \in \mathrm{Pic}(X)$ are two line bundles and $D_1, D_2 \in Div(X)$ their corresponding divisors then we say that

$$\begin{gathered} L_1 \text{ and } L_2 \text{ are algebraically equivalent} \\ \Updownarrow \\ D_1 - D_2 = D_f, \text{ where } D_f \text{ is a principal divisor.} \end{gathered}$$

The subgroup of line bundles algebraically equivalent to zero is denoted $\mathrm{Pic}^0(X)$, and the associated quotient is called Neron-Severi group:

$$NS(X) := \mathrm{Pic}(X)/\,\mathrm{Pic}^0(X)$$

Its rank as a free $\mathbb{Z}$-module is $\rho(X)$.

Proposition 1.30. *The Kernel of $c_1 : H^1(X, \mathcal{O}_X^*) \to H^2(X,\mathbb{Z})$ coincides with* $\mathrm{Pic}^0(X)$. *In other words $NS(X) \simeq H^{1,1}(X,\mathbb{Z})$.*

Proof. This is due to the fact that for a divisor $D \in Div(X)$, its corresponding line bundle $\mathcal{O}(D)$ is mapped to $[D]$ via the boundary map c_1. It can be easily proven using the language of connections with respect to hermitian structures on line bundles. This is beyond the scope of this thesis, but the curious reader may refer to [12] for a more detailed discussion on this topic. $\square$

Definition 1.31. Let X be a compact Kähler surface. The *transcendental lattice* of X is the orthogonal complement of $NS(X)$ in $H^2(X,\mathbb{Z})$, denoted by $T(X)$.

The following theorem turns out to be very useful when dealing with projective surfaces.

Theorem 1.32 ([16, Kodaira]). *Let X be a compact complex Kähler surface. X is projective if and only if there exists a line bundle F on X of positive self-intersection.*

In the context of projective surfaces, X always admits a line bundle of positive self-intersection: namely, the restriction to X of $\mathcal{O}(1)$. For this reason, we can deduce that projective complex surfaces all have the following properties: $NS(X)$, seen as an even lattice, has signature $(1, \rho(X) - 1)$, and $T(X)$ inherits a Hodge structure which turns out to be a Polarized Hodge structure, since its signature is $(2h^{2,0}(X), h^{1,1}(X) - \rho(X))$.

We conclude with a theorem that justifies the spirit of the following sections.

Theorem 1.33 ([16, Kodaira]). *Let X be a compact complex Kähler surface whose canonical bundle is trivial, i.e. $\omega_X \simeq \mathcal{O}_X$. Then $h^{2,0}(X) = 1$ by Serre Duality and we only have two possibilities:*

- *$X = \mathbb{C}^2/\Lambda$ is a complex torus of dimension 2;*
- *X is a $K3$ surface.*

Chapter 2

K3 surfaces

In this chapter we introduce the foundational definitions and main results about $K3$ surfaces. We will combine the concepts introduced in the first chapter to study closely the lattice structure of the group $H^2(X, \mathbb{Z})$. This structure is capable of fully describing the geometry of a $K3$ surface, as stated in the global Torelli theorem for $K3$ surfaces. In Section 2, we will define the $K3$ lattice, i.e., a particular unimodular lattice to which every $H^2(X, \mathbb{Z})$ is isomorphic, for any $K3$ surface X.
In the treatment of finite-order symplectic automorphisms, thanks to Nikulin's results on the uniqueness of the action of symplectic automorphism groups on the $K3$ lattice, algebraic tools will be provided to determine the existence of such automorphisms on a surface X.

2.1 Core aspects of K3 surfaces

Definition 2.1. A K3 surface is a compact connected complex surface X such that $H^1(X, \mathcal{O}_X) = 0$ and its canonical bundle is trivial, $\omega_X \simeq \mathcal{O}_X$.

Note that there are many equivalent definitions for complex $K3$ surfaces. In a more general context one might allow $K3$ surfaces to be defined on an arbitrary field k, defining them as separated, geometrically integral k-schemes of finite type with the same two triviality conditions: $\omega_{X/k} \simeq \mathcal{O}_X$ and $h^1(\mathcal{O}_X) = 0$. We are interested in the complex case and, thanks to Serre's GAGA[1], we know that these two definitions -one in terms of Complex Manifolds and the other using the language of Schemes- are, indeed, equivalent when considering algebraic $K3$ surfaces over $\mathbb{C}$. Of course, not all complex $K3$ surfaces are projective, which is why we decide to adopt this specific definition.

[1] It stands for 'Géometrie Algébrique et Géométrie Analytique', a theorem by Jean-Pierre Serre.

Remark 2.2. The well known exponential sequence for a complex manifold X

$$0 \to 2\pi i\mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1 \tag{2.3}$$

yields the long exact sequence in cohomology:

$$\begin{aligned} 0 \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X^*) \xrightarrow{c_1} \\ \to H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X) \to H^2(X,\mathcal{O}_X^*) \to \cdots . \end{aligned}$$

In the case of $K3$ surfaces, using that $H^1(X,\mathcal{O}_X) = 0$, we get $H^1(X,\mathbb{Z}) = 0$ and, by Poincarè duality, $H^3(X,\mathbb{Z}) = 0$ over its free part. Moreover, $H^4(X,\mathbb{Z}) \simeq H^0(X,\mathbb{Z}) \simeq \mathbb{Z}$ and $H^2(X,\mathbb{Z})$ can be computed two ways: the first relies on the fact that all $K3$ surfaces are Kähler, which is somewhat excessive for the complexity of the proof of the result due to Siu[31]. The second one requires only to know that, for X compact complex surface, the Hodge–Frölicher spectral sequence degenerates at the first page:

$$H^1(X,\mathbb{C}) \simeq H^{1,0}(X) \oplus H^{0,1}(X). \tag{2.4}$$

Thus, using Noether's formula for surfaces

$$\mathcal{X}(X,\mathcal{O}_X) = \frac{(\omega_X.\omega_X) + \mathcal{X}(X)}{12}, \tag{2.5}$$

and recalling that $\omega_X \simeq \mathcal{O}_X$, we obtain $\mathcal{X}(X) = 12\mathcal{X}(X,\mathcal{O}_X)$.
Next, applying Serre's duality (which holds for compact complex surfaces in general), we find:

$$h^2(\mathcal{O}_X) = h^{0,2}(X) = h^{2,0}(X) = \dim H^0(X,\wedge^2\Omega_X) = \dim H^0(X,\mathcal{O}_X) = 1.$$

Therefore, $\mathcal{X}(X,\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X) = 1 - 0 + 1 = 2$. From equation (2.5), we get:

$$24 = \mathcal{X}(X) = \sum_{i=0}^{4}(-1)^i h^i(X) = h^0(X) - h^1(X) + h^2(X) - h^3(X) + h^4(X).$$

Now, since $h^1(X) = h^3(X) = 0$ and $h^0(X) = h^4(X) = 1$, we deduce that $h^2(X) = 22$. At this point we can draw the Hodge diamond of every (Kähler) K3 surface:

$$\textit{The K3 Hodge Diamond:} \qquad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} . \tag{2.6}$$

From now on, we assume all our surfaces to be algebraic (meaning they are projective complex surfaces).

Proposition 2.7. *If X is a $K3$ surface* $\rho(X) = \mathrm{rk}(\mathrm{Pic}(X)) \leq 20$.

Proof. Recall that $H^1(X, \mathcal{O}_X^*) \simeq Pic(X)$. The Proof now follows from $(1,1)$-Lefschetz and from the fact that $h^{1,1}(X) = 20$ for $K3$ surfaces. □

Remark 2.8. If X is a $K3$ surface it is clear that, since $H^1(X, \mathcal{O}_X) = 0$, we have the isomorphism $Pic(X) \simeq NS(X)$. This will not be the case for complex tori, as we will prove in the next chapter.

2.2 The K3 lattice

In this section we state the two main results about $K3$ surfaces, namely the *Global Torelli Theorem* and the *Surjectivity of the period map*. The whole theory of $K3$ surfaces is based on the fact that, since $H^2(X, \mathbb{Z})$ is torsion-free, every $K3$ surface X inherits a lattice structure via the intersection pairing. This symmetric bilinear form coincides with the cup product in cohomology, and the resulting lattice is unimodular thanks to Poincaré duality.

Theorem 2.9. *For every $K3$ surface X there exists an isometry*

$$\phi : H^2(X, \mathbb{Z}) \xrightarrow{\sim} U^3 \oplus E_8(-1)^2.$$

Proof. This follows from Theorem 1.13. Precisely, we already proved that $H^2(X, \mathbb{Z})$ is an even unimodular lattice and, since $h^{1,1}(X) = 20 \geq 2$, it is indeed indefinite. Now from the fact that $h^2 = 22$ we have $H^2(X, \mathbb{Z}) \simeq U^n \oplus E_8(-1)^m$ with the following possible values for (n, m): $\{(11, 0), (7, 1), (3, 2)\}$. Comparing the signatures of these even lattices with that of $H^2(X, \mathbb{Z})$ given by Hodge Index Theorem (1.23), namely $(2h^{2,0} + 1, h^{1,1} - 1) = (3, 19)$, we conclude $H^2(X, \mathbb{Z}) \simeq U^3 \oplus E_8(-1)^2$. □

This lattice plays a central role in the theory of $K3$ surface, so we need a proper definition.

Definition 2.10 (The K3 lattice)**.** We define the $K3$ lattice Λ_{K3} as the direct sum $U^3 \oplus E_8(-1)^2$. In contexts where it is obvious, we will denote it simply by Λ.

Remark 2.11. To summarize what has been discussed so far, we can list some properties of the even lattice Λ_{K3}.

- Its signature is $(3, 19)$, hence it is an indefinite lattice.

- It is a unimodular lattice, and consequently, the primitive embedding of $NS(X)$ into $H^2(X,\mathbb{Z})$ is unique up to isomorphism thanks to Theorem 1.20. This also holds for its orthogonal complement, namely $T(X)$.
- Every $K3$ surface admits an isomorphism $\phi : H^2(X,\mathbb{Z}) \xrightarrow{\sim} \Lambda_{K3}$ thanks to Theorem 2.9.

Example 2.12. An interesting even lattice isometric to the $K3$ lattice Λ_{K3} has already been mentioned in Example 1.12. Indeed, the lattice $L := \left(U \oplus A_2 \oplus E_6(-1)^3\right)''$ is unimodular and has signature $(3,19)$. By Corollary 1.15, we conclude that $L \simeq \Lambda_{K3}$. This description of the second integral cohomology group of $K3$ surfaces will be crucial when proving the existence of order three symplectic automorphisms for certain $K3$ surfaces (see 4.2.1).

In order to properly understand the relation between lattices and automorphisms of projective complex surfaces, we need to invoke Global Torelli Theorem for $K3$ surfaces.

Definition 2.13 (Kähler Cone)**.** The Kähler cone of a complex Kähler surface X is the open convex cone of Kähler classes and we denote it by $\mathcal{K}_X$.

Theorem 2.14 (Global Torelli for K3 surfaces)**.** *Let X, Y be two complex $K3$ surfaces. There exist an isomorphism $f : X \xrightarrow{\sim} Y$ of $K3$ surfaces if and only if there exist a Hodge isometry of even lattices $\phi : H^2(Y,\mathbb{Z}) \xrightarrow{\sim} H^2(X,\mathbb{Z})$. Moreover, for every Hodge isometry $\phi : H^2(Y,\mathbb{Z}) \xrightarrow{\sim} H^2(X,\mathbb{Z})$ such that $\phi(\mathcal{K}_Y) \cap \mathcal{K}_X \neq \emptyset$, there exist an isomorphism $f : X \xrightarrow{\sim} Y$ which verifies $f^* = \phi$.*

This theorem allows one to shift focus from the geometric to the algebraic setting and it also provides a powerful new method for constructing such isomorphisms. One might wonder if any lattice isomorphic to Λ comes from the second integral cohomology group of a $K3$ surface. The answer is affirmative when the lattice comes with a signed Hodge structure of weight two, and it follows from the so called *Surjectivity of the Period Map*. We state it without further discussion, as its proof goes somewhat beyond the scope of this thesis. We direct the reader to [11] for a comprehensive treatment of the topic. By the term *period of a K3 surface X* we refer to the associated lattice $H^2(X,\mathbb{Z})$, together with a signed Hodge structure of weight two.

Theorem 2.15 ([13, Theorem 7.4.1])**.** *For every signed Hodge structure of weight two on the lattice Λ_{K3} there exist a $K3$ surface X and a signed Hodge structure isomorphism*

$$\phi : \Lambda_{K3} \xrightarrow{\sim} H^2(X,\mathbb{Z}).$$

2.3 Symplectic Automorphisms of K3 surfaces

In this section we present an overview of the basic results about finite order symplectic automorphisms. The reason for studying these kind of automorphisms is the nature of the singular points of the quotient space: these are DuVal singularities admitting a minimal resolution which yields again a $K3$ surface.

Definition 2.16. Let X be a $K3$ surface. An automorphism $f : X \xrightarrow{\sim} X$ is called *symplectic* if its induced action on the second cohomology group $H^2(X, \mathbb{Z})$ preserves the symplectic form, i.e. $f^*|_{H^{2,0}(X)} \equiv \mathrm{id}\,|_{H^{2,0}(X)}$. Otherwise, we say that f is non-symplectic.

Example 2.17. Let X be the quartic defined by the polynomial $f(z_0, z_1, z_2, z_3) = z_0^4 + z_1^4 + z_2^4 + z_3^4$ in $\mathbb{P}^3$. First note that X is a $K3$ surface, in fact from the adjunction formula $\omega_X \simeq \omega_{\mathbb{P}^3}|_X \otimes \mathcal{O}(4)|_X \simeq \mathcal{O}_X$ and we also have the exact sequence

$$0 \to \mathcal{O}(-4) \to \mathcal{O} \to \mathcal{O}_X \to 0,$$

which yields $H^1(X, \mathcal{O}_X) = 0$. Here we are using many tools, namely Kodaira Vanishing, the fact that $H^i(X, \mathcal{O}_X) = H^i(\mathbb{P}^3, i^*\mathcal{O}_X)$ (under the inclusion $i : X \hookrightarrow \mathbb{P}^3$) and also some knowledge about the cohomology of the projective space. This surface is naturally endowed with automorphisms given by permutations of coordinates in $\mathbb{P}^3$, i.e. S_4 is a group of finite automorphisms. These are obviously automorphisms of the surface, since they come from projective transformations. Using theory of residues, one can check that the action of every permutation is also symplectic. This class of automorphisms on X can be expanded by also multiplying each coordinate by a 4-th root of unity.

Remark 2.18. The condition $f^*|_{H^{2,0}(X)} \equiv \mathrm{id}\,|_{H^{2,0}(X)}$ is equivalent to $f^*|_{T(X)} \equiv \mathrm{id}\,|_{T(X)}$. If f acts as the identity on $H^{2,0}(X)$ then it is obvious that f acts trivially also on $H^{0,2}(X)$, so any integral form α for which $f^*(\alpha) \neq \alpha$ actually belongs to the *Neron-Severi group*. This means that f^* is the identity map on $T(X) = NS(X)^\perp$. Conversely suppose $f^*|_{T(X)} \equiv id|_{T(X)}$, then $f^*|_{T(X)\otimes\mathbb{C}} = \mathrm{id}\,|_{T(X)\otimes\mathbb{C}}$ too, and we are done, since $T(X) \otimes \mathbb{C} = H^{2,0}(X) \oplus H^{0,2}(X)$.

We are interested in symplectic automorphisms of finite order. The hypothesis of finiteness is essential and the following lemma provides a compelling justification for this assumption.

Lemma 2.19. *Let $f : X \xrightarrow{\sim} X$ be a symplectic automorphism of order n of a $K3$ surface X. For every fixed point x there exist local holomorphic coordinates (z_1, z_2) around x and a n-th primitive root of unity λ such that $f(z_1, z_2) = (\lambda z_1, \lambda^{-1} z_2)$.*

Proof. On a proper neighbourhood U of x we have $x = 0 \in U \subset \mathbb{C}^2$. Here there exist local holomorphic coordinates given by $g(y) := \frac{1}{n}\sum_{i=1}^{n}(d_0 f)^{-i} f^i(y)$ where $d_0 f$ is the differential of f in $x = 0$. Indeed, $d_0 g = \frac{1}{n}\sum_{i=1}^{n}(d_0 f)^{-i}(df)^i = \mathrm{id}$. With this choice one has $g(f(y)) = \frac{1}{n}\sum_{i=1}^{n}(d_0 f)^{-i} f^{i+1}(y) = (d_0 f)\frac{1}{n}\sum_{i=1}^{n}(d_0 f)^{-i-1} f^{i+1}(y) = (d_0 f)(g(y))$, where we used the fact that $f^{n+1}(y) = f(y)$ and $(d_0 f)^{-n-1} = (d_0 f)^{-1}$ given by the finite order nature of the automorphism f. As a consequence, using coordinates $(z_1, z_2) = g(y)$, we can rewrite f as a linear application $d_0 f$, which, under the right choice of basis, can be written as a diagonal matrix

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Notice that we excluded the non-diagonal Jordan-block case because $f^n = (d_0 f)^n = \mathrm{id}$. Now f is symplectic, and therefore its determinant must equal 1 (this can be easily seen by writing the non-zero holomorphic symplectic form around x as $dz_1 \wedge dz_2$ and noticing that the action of f preserves it). This yields $\lambda = \lambda_1 = \lambda_2^{-1}$. The finite order condition also ensures that λ is an n-th root of unity. Moreover, we can conclude that it is also a primitive root, since any $k < n$ such that λ is a k-th root of unity would yield $f^k = \mathrm{id}$ in a neighbourhood of $x = 0$, which extends to the whole surface X giving a contradiction. $\square$

In other words, a symplectic automorphism f acts locally on a $K3$ surface as an element of $\mathrm{SL}(2, \mathbb{C})$. As a corollary we have the following:

Corollary 2.20. *Let G be a finite abelian group of automorphisms acting symplectically on a complex $K3$ surface X. For any fixed point $x \in X$, the group $Stab_G(x)$ is a subgroup of* $\mathrm{SL}(2, \mathbb{C})$.

Note that the structure of finite abelian subgroups of $\mathrm{SL}(2, \mathbb{C})$ is thoroughly described via the McKay classification.

Theorem 2.21 ([18, McKay])**.** *Every finite abelian subgroup of* $\mathrm{SL}(2, \mathbb{C})$ *is cyclic.*

Remark 2.22. One could go through the proof of Lemma 2.19 without the assumption that f is symplectic, obtaining that there exist holomorphic local coordinates (z_1, z_2) around a fixed point $x \in X$, in which the automorphism f can be written as a linear application.

The singularities arising from the quotient space of a $K3$ surface X by a symplectic automorphism of finite order f are ADE-type singularities, thanks to Corollary 2.20, and thus they admit a minimal resolution which is crepant (meaning the resolution does not affect the canonical bundle). Moreover, ω_X is preserved by definition of symplectic automorphism. Since any one-form on the resolution of the quotient would yield a one-form on the quotient (and thus on X), one can use Theorem 1.32 to prove that the resulting surface is again a $K3$ surface. This is summarized by the following theorem:

Theorem 2.23. *Let $f : X \xrightarrow{\sim} X$ be a symplectic automorphism of finite order on a $K3$ surface and denote by Y the quotient X/f. Then Y admits a minimal resolution $\tilde{Y} \to Y$ which is crepant and such that $\tilde{Y}$ is a $K3$ surface.*

Corollary 2.24. *Let $f : X \to X$ be a non-trivial symplectic automorphism on a $K3$ surface. Then*

$$|\mathrm{Fix}(f)| = \frac{24}{n} \prod_{p|n} \left(1 + \frac{1}{p}\right)^{-1}.$$

In particular $1 \leq |\mathrm{Fix}(f)| \leq 8$.

Proof. We use the *Lefschetz fixed point formula for biholomorphic automorphisms*:

$$\sum_i (-1)^i \operatorname{Tr}(f^*|_{H^i(X,\mathcal{O}_X)}) = \sum_{x \in Fix(f)} \det(\mathrm{id} - d_x f)^{-1}.$$

One should remember that $h^{0,0} = h^{0,2} = 1$, $h^{0,i} = 0$ for $i \neq 0, 2$ and, since f is symplectic, it acts trivially on $h^{0,2}$. This gives $\sum_i (-1)^i \operatorname{Tr}(f^*|_{H^i(X,\mathcal{O}_X)}) = 2$ and proves $|\mathrm{Fix}(f)| > 0$. For the right hand side of the formula, we know that for any $k \in \mathbb{N}$ such that $(k, n) = 1$ the points fixed by f and those fixed by f^k coincide, and in general the expression can be written as

$$\begin{aligned}
2 &= \sum_{x \in \mathrm{Fix}(f)} \frac{1}{(1 - \lambda_x)(1 - \lambda_x^{-1})} \\
&= \sum_{x \in \mathrm{Fix}(f)} \frac{1}{(1 - \lambda_x^k)(1 - \lambda_x^{-k})} \\
&= \sum_{(k,n)=1} \frac{1}{\phi(n)} \sum_{x \in \mathrm{Fix}(f)} \frac{1}{(1 - \lambda_x^k)(1 - \lambda_x^{-k})}.
\end{aligned}$$

Here by λ_x we mean the primitive n-th root of unity discussed in Lemma 2.19, which depends on the fixed point x. Please note that the inequality $|\mathrm{Fix}(f)| \leq 8$ comes from the fact that $|1 - \lambda_x^{\pm k}| \leq 2$. To conclude, the last term of the above chain of equalities can be simplified using a Lemma by S. Mukai [23, Lemma 1.3] yielding

$$\begin{aligned}
2 &= \sum_{(k,n)=1} \frac{1}{\phi(n)} \sum_{x \in \mathrm{Fix}(f)} \frac{1}{(1 - \lambda_x^k)(1 - \lambda_x^{-k})} \\
&= \sum_{x \in \mathrm{Fix}(f)} \frac{1}{\phi(n)} \frac{n^2}{12} \prod_{p|n} \left(1 - \frac{1}{p^2}\right)^{-1}.
\end{aligned}$$

The corollary is proved using $\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$. $\square$

This corollary is of huge importance in the study of $K3$ surfaces. As an application of it, we provide an example of non-symplectic automorphism.

Example 2.25. Let X be the Fermat quartic surface in $\mathbb{P}^3$. Consider $\hat{\imath} \in PGL(3, \mathbb{C})$ defined as $\hat{\imath}([z_0 : z_1 : z_2 : z_3]) = [z_0 : iz_1 : -z_2 : -iz_3]$. It is obvious that $\hat{\imath}|_X : X \to X$ is an automorphism on the surface X, and since no points are fixed, f must be *non-symplectic*, due to Corollary 2.24. Its quotient $Y = X/\hat{\imath}$ has a non-trivial canonical bundle, since the symplectic form $\alpha \in \omega_X$ gets killed by the quotient (it is not preserved under the non-symplectic involution), and therefore $h^{2,0}(Y) = 0$. Moreover, the double covering $\pi : X \to Y$ yields the relation $\pi_*\pi^*(\omega_Y) = \omega_Y^{\otimes 2}$, but $\pi^*\omega_Y = \omega_X = \mathcal{O}_X$ and $\pi_*(\mathcal{O}_X) = \mathcal{O}_Y$, therefore $\omega_Y^{\otimes 2} = \mathcal{O}_Y$. This is an example of an *Enriques surface*, i.e. a compact complex surface such that $h^{2,0} = 0$ and $\omega_X^{\otimes 2} = \mathcal{O}_X$. In fact, every quotient of a $K3$ surface by a fixed point free non-symplectic automorphism is an Enriques surface.

Remark 2.26. Non-symplectic automorphisms, by definition, do not act as the identity on $H^{2,0}(X)$; instead, it is easy to verify that they multiply the nowhere vanishing holomorphic two-form by a n-th root of unity, where n is a divisor of the order of the automorphism. Note that this root of unity need not to be primitive, as one can see in the case of some non-symplectic automorphisms of order 4.

Our primary focus will be on the study of symplectic automorphisms. Indeed, in the context of Shioda-Inose structures, both the automorphism on the $K3$ surface and the automorphism on the Abelian surface are symplectic.
Currently, we already have many tools that allow us to enhance our understanding of finite order symplectic automorphisms on $K3$ surface. For instance, Corollary 2.24 states that the number of fixed points depends only on the order of the automorphism, while Lemma 2.19 explains how these automorphisms act locally on the surface. Furthermore, only certain finite orders are allowed for symplectic automorphisms, as stated by the following Lemma:

Lemma 2.27. *For any symplectic automorphism $f : X \xrightarrow{\sim} X$ of finite order n, it holds that $n \leq 8$.*

Proof. First of all we show that the invariant part $H^*(X, \mathbb{C})^f \subset H^*(X, \mathbb{C})$ has complex dimension at least five. This is due to the fact that $H^0(X) \oplus H^{2,0}(X) \oplus H^{0,2}(X) \oplus H^4(X)$ obviously is contained in $H^*(X, \mathbb{C})$, which gives $\dim H^*(X, \mathbb{C})^f \geq 4$, together with the fact that for any ample class $\alpha \in NS(X)$ (we can take the restriction of $\mathcal{O}_{\mathbb{P}^n}(1)$) the sum $\sum_{i=1}^n (f^i)^*\alpha$ is preserved under the action of f and it is again ample (remember that f acts as an isometry on $H^2(X, \mathbb{Z})$). This yields $\dim H^*(X, \mathbb{C})^f \geq 5$. Now $\frac{1}{n}\sum_{i=1}^n (f^i)^*$ can be seen as the projector on the subspace

$H^*(X,\mathbb{C})^f$, since it is the identity on this subspace and it is idempotent. This gives

$$\sum_{i=1}^{n} \mathrm{Tr}((f^i)^*) = n \dim H^*(X,\mathbb{C})^f,$$

which, using Lefschetz formula and the fact that $f^n = id$, can be rewritten as

$$24 + \sum_{i=1}^{n-1} |\mathrm{Fix}(f^i)| = n \dim H^*(X,\mathbb{C})^f.$$

Now, from Corollary 2.24 and the fact that $\dim H^*(X,\mathbb{C})$, we get the upper limit for n. □

Now we turn our attention to lattices and, using results from the previous section, we give necessary and sufficient conditions for the existence of symplectic automorphisms of a given order on a $K3$ surface X. This field has been deeply studied by Nikulin in his PhD thesis (see [25]), and we will make great use of these results in the upcoming chapters.

Definition 2.28. Let $f : X \xrightarrow{\sim} X$ be a symplectic automorphism of finite order n on a $K3$ surface X. We define

$$T_{X,f} := H^*(X,\mathbb{C})^f \text{ and } N_{X,f} := L_{X,f}^{\perp}.$$

Lemma 2.29 ([25, Lemma 4.2])**.** *Let $f : X \xrightarrow{\sim} X$ be a symplectic automorphism of finite order n on a $K3$ surface X. Then*

1) $N_{X,f}$ is negative definite;

2) $N_{X,f}$ contains no element with square -2;

3) $N_{X,f} \subset NS(X)$ and $T(X) \subset T_{X,f}$.

Proof. Assertion 3) follows from the very definition of symplectic automorphism. Assertion 1) is true because, as before, for an ample class $\alpha \in H^{1,1}(X)$ one has that $\hat{\alpha} := \sum_{i=1}^{n}(f^i)^*\alpha$ is again ample and invariant under the action of f. Therefore $\hat{\alpha} \in NS(X)^f$, which means that $N_{X,f} \subset \alpha^{\perp}$. Now we can conclude using Hodge index theorem, which guarantees that $N_{X,f}$ is negative definite. Lastly, thanks to the Lefschetz theorem on $(1,1)$-classes and Riemann-Roch formula, any element with square -2 corresponds to an effective Divisor on the surface X. Effective divisors, though, intersect the ample divisor positively, giving a contradiction which proves assertion 2). □

Nikulin utilized the Global Torelli theorem for $K3$ surfaces (Theorem 2.14), proved by Piatetski-Shapiro and Šafarevič, to provide a pseudo-version of the converse of the statement:

Theorem 2.30. *Let $f \in O(\Lambda_{K3})$ be an isometry of the K3 lattice. Then f acts as a symplectic automorphism on a K3 surface X if the following conditions hold:*

1) $N_f := (\Lambda_{K3}^f)^\perp$ is negative definite;

2) N_f contains no element with square -2;

Nikulin's main result is the following:

Theorem 2.31 ([25, Theorem 4.5])**.** *Let G be a finite abelian group acting symplectically on a K3 surface. Then G has a unique action on Λ_{K3}, hence the lattice $\Omega_G := (\Lambda_{K3}^G)^\perp$ is uniquely determined by G, up to isometry. In particular, the action of any symplectic automorphism on a K3 surface is uniquely determined by its order.*

This allows us to talk about *"the lattice fixed by a symplectic automorphism of order n* and its orthogonal which is denoted as Ω_n". This is possible because Theorem 2.31 tells us that the said lattice is unique up to isometry. Equivalently, we talk about Ω_G as *the lattice orthogonal to the lattice fixed by the symplectic automorphisms group G*. The following answers the question of the existence of a group of symplectic automorphisms on a $K3$ surface.

Theorem 2.32 ([25, Theorem 4.15])**.** *Let X be a K3 surface and let G be a finite abelian group. Then G acts as a group of symplectic automorphisms of X if and only if there is a primitive embedding $\Omega_G \hookrightarrow NS(X)$. In particular, a K3 surface admits an automorphism of order n if and only if there is a primitive embedding $\Omega_n \hookrightarrow NS(X)$.*

Remark 2.33. Note that in order for Ω_G to be well-defined, there must exist a $K3$ surface X admitting an action of G as a group of symplectic automorphisms.

Remark 2.34. For non-abelian groups the same arguments as Theorem 2.31 follow, except for five groups. See [10] for a detailed discussion on the topic.

Remark 2.35. Theorem 2.32 not only represents the core of Nikulin's work, but also gives birth to the study of finite groups acting as symplectic automorphisms on $K3$ surfaces, which was later carried on by S. Mukai in [23], S. Kondo in [17], D. R. Morrison in [22], K. Hashimoto in [10], A. Sarti and A. Garbagnati in [6], [7] and A. Garbagnati in [4] and many others.

Nikulin's work yielded the description of finite abelian groups acting as symplectic automorphisms groups on $K3$ surfaces.

Theorem 2.36. *Let G be an abelian group acting symplectically on a K3 surface. Then G is one of the followings:*

$$\mathbb{Z}/n\mathbb{Z}, \quad 2 \le n \le 8, \quad (\mathbb{Z}/m\mathbb{Z})^2, \quad m = 2, 3, 4,$$

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}, \quad (\mathbb{Z}/2\mathbb{Z})^i, \quad i = 3, 4.$$

We conclude with a useful result by A. Garbagnati and A. Sarti (see [7] for the complete classification, which contains all the abelian groups of Theorem 2.36).

Theorem 2.37. *Let $f : X \xrightarrow{\sim} X$ be a symplectic automorphism of finite order on a K3 surface X. Then f has order $n \in \{2, 3, 4, 5, 6, 7, 8\}$ and the following table lists the invariant and co-invariant lattice for each case:*

n	$rk(\Omega_G)$	$discr(\Omega_G)$	$\Omega_G^\vee/\Omega_G$	$rk(\Omega_G^{\perp_{K3}})$	$\Omega_G^{\perp_{K3}}$
2	8	2^8	$(\mathbb{Z}/2\mathbb{Z})^8$	14	$E_8(-2) \oplus U^{\oplus 3}$
3	12	3^6	$(\mathbb{Z}/3\mathbb{Z})^6$	10	$U \oplus U(3)^{\oplus 2} \oplus A_2^{\oplus 2}$
4	14	2^{10}	$(\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/4\mathbb{Z})^4$	8	Q_4
5	16	5^4	$(\mathbb{Z}/5\mathbb{Z})^4$	6	$U \oplus U(5)^{\oplus 2}$
6	16	6^4	$(\mathbb{Z}/6\mathbb{Z})^4$	6	$U \oplus U(6)^{\oplus 2}$
7	18	7^3	$(\mathbb{Z}/7\mathbb{Z})^3$	4	$U(7) \oplus \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$
8	18	8^3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus (\mathbb{Z}/8\mathbb{Z})^2$	4	$U(8) \oplus \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

where

$$Q_4 := \begin{bmatrix} 0 & 4 & 0 & 2 & 0 & -1 & 0 & 0 \\ 4 & 0 & 4 & 4 & -4 & 0 & 0 & -4 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -4 & 0 & 0 & -2 & -1 & 0 & -2 \\ -1 & 0 & 0 & -1 & -1 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & -4 & 0 & 0 & -2 & 1 & 0 & -2 \end{bmatrix}.$$

Chapter 3

Abelian Surfaces

We now shift our focus to complex tori, beginning with a discussion of their main invariants and properties. Furthermore, we will characterize projective complex tori using two different approaches. In Section 2 we restrict our attention to projective complex tori of dimension two, also known as Abelian varieties. In this context, we will make extensive use of the lattice theory concepts introduced in Chapter 1. We will briefly review the surjectivity of the period map for Abelian surfaces, due to Shioda, before moving on to the work of Fujiki on symplectic automorphisms of finite order on Abelian surfaces. Here, we provide a classification of Abelian surfaces admitting an automorphism of order n, for all possible integer values of n. Finally, we conclude with some examples.

3.1 Core aspects of Complex tori

Definition 3.1. A *(full) lattice in* $\mathbb{C}^n$, denoted Γ, is a lattice obtained as the free group over a set of elements $T = \{z_1, \ldots, z_{2n}\} \subset \mathbb{R}^{2n}$ such that T generates $\mathbb{C}^n$ as a real vector space.

Definition 3.2 (Complex Torus)**.** For any complex vector space $V \simeq \mathbb{C}^n$ we define a *lattice in* V as a lattice $\Gamma \subset \mathbb{R}^{2n} \simeq V$ such that Γ is a set of generating vectors.
In this setting, a *complex torus of dimension* n is a quotient V/Γ.

We will often implicitly choose a basis for V, and refer to a complex torus just as the quotient $\mathbb{C}^n/\Gamma$.

Definition 3.3 (Complex Lie Group)**.** A *complex Lie group of dimension* n is a n-dimensional complex manifold X which is also a group such that the map

$$\mu : X \times X \to X \text{ such that } \mu(x, y) = xy^{-1}$$

is a holomorphic map.

This definition is justified by the following:

Lemma 3.4. *Every connected compact complex Lie group X of dimension n is in a natural way a complex torus.*

Proof. We first show that X is abelian. Consider the holomorphic map $\phi : X \times X \to X$ $\phi(x,y) = xyx^{-1}y^{-1}$ and let U be a neighbourhood of $1 \in X$. For every $x \in X$ $\phi(x,1) = 1$, there exist V_x a neighbourhood of x and W_x a neighbourhood of 1 such that $\phi(V_x, W_x) \subset U$. By compactness of X we deduce that there exist $V_{x_1}, \dots, V_{x_n}$ such that $\bigcup_{i=1}^n V_{x_i} = X$, and denote $W := \bigcap_{i=1}^n W_{x_i}$. It follows that $\phi(X, W) \subset U$, and since for any $y \in W$ the map $\phi(x,y)$ is a holomorphic map on the compact manifold X, $\phi|_{X\times\{y\}}$ is constant. Moreover $\phi(1,y) = \phi(y,1) = 1$ for every $y \in W$, which implies that $\phi|_{X\times W}$ is constant. Now by analytic continuation ϕ is constant and therefore X is abelian.
In order to conclude that X is a complex torus we need a little bit of theory of Lie groups (see [19]), indeed the exponential map $\exp : \mathbb{C}^n \to X$ is a homomorphism which is a local diffeomorphism. Its image is an open subgroup of X, which is connected. Therefore the exponential map is surjective and its kernel is a discrete closed subgroup $\Gamma \subset \mathbb{C}^n$. Since X is compact, Γ must be a full lattice. □

Definition 3.5. A complex torus that is also projective is referred to as an *Abelian variety*. When its complex dimension is one, it is known as an *elliptic curve*, while in dimension two, it is called an *abelian surface*.

Remark 3.6. Let $A = V/\Gamma$ be a complex torus of dimension n. Then $V \to A$ is the universal covering of A, and therefore $\Gamma = \pi_1(A)$. Since this group is abelian, we also have $H_1(A,\mathbb{Z}) = \Gamma$, and by the Universal Coefficient Theorem it follows that $H^1(A,\mathbb{Z}) = \Gamma^*$, as $H^0(A,\mathbb{Z})$ is torsion-free.

Proposition 3.7. *Let $A = V/\Gamma$ be a complex torus of dimension n. Then its integral cohomology groups are all computed as $H^k(A,\mathbb{Z}) = \bigwedge^k H^1(A,\mathbb{Z})$.*

Proof. In order to compute integral cohomology groups, we apply the Künneth formula to $(S^1)^n \simeq A$, which reads as

$$\begin{aligned} H^k\left((S^1)^n\right) = {} & H^k\left((S^1) \times (S^1)^{n-1}, \mathbb{Z}\right) \\ \simeq {} & \bigoplus_{i+j=k} H^i\left(S^1, \mathbb{Z}\right) \otimes H^j\left((S^1)^{n-1}, \mathbb{Z}\right). \end{aligned}$$

Now we proceed by induction and suppose that it holds $H^j(S^{n-1},\mathbb{Z}) = \bigwedge^j H^1(S^{n-1},\mathbb{Z})$.

Therefore

$$\bigoplus_{i+j=k} H^i\left(S^1,\mathbb{Z}\right)\otimes H^j\left((S^1)^{n-1},\mathbb{Z}\right)\simeq\bigoplus_{i+j=k}\bigwedge^i H^1\left(S^1,\mathbb{Z}\right)\otimes\bigwedge^j H^1\left((S^1)^{n-1},\mathbb{Z}\right).$$

Furthermore, since $H^1((S^1)^n,\mathbb{Z})\simeq H^1(S^1,\mathbb{Z})\oplus H^1((S^1)^{n-1},\mathbb{Z})$ there is an isomorphism $\bigwedge^k H^1((S^1)^n,\mathbb{Z})\simeq\bigoplus_{i+j=k}\left(\bigwedge^i H^1(S^1,\mathbb{Z})\otimes\bigwedge^j H^1((S^1)^{n-1},\mathbb{Z})\right)$. This is true by Vandermonde's Identity, but one can also check this by writing down bases. Now we've shown that $H^k((S^1)^n)\simeq\bigwedge^k H^1((S^1)^n,\mathbb{Z})$, and since the cohomology of S^1 is well known we conclude the proof. □

Remark 3.8. Let A be a complex torus of dimension n. We want to calculate hodge numbers $h^{p,q}$ for A. For simplicity let $A=\mathbb{C}^n/\Gamma$. Since the Kähler form associated to the flat metric on $\mathbb{C}^n$ is preserved under the action of Γ, A is again a Kähler manifold and its hodge numbers can be calculated by counting harmonic forms. Indeed $H^q(A,\Omega_A^{\otimes p})$ is spanned by $\{\alpha_{i_1,\dots,i_p,j_1,\dots,j_q}dz_1\wedge\cdots\wedge dz_p\wedge d\bar{z}_1\wedge\cdots\wedge d\bar{z}_q\mid\Delta\alpha_{i_1,\dots,i_p,j_1,\dots,j_q}=0\}$. All these $\alpha_{i_1,\dots,i_p,j_1,\dots,j_q}$ are global holomorphic form, and from the compactness of A we deduce they must be constant. This yields $h^{p,q}(X)=\binom{n}{p}\binom{n}{q}$ (if $\max\{p,q\}>n$ we just use symmetries of the Hodge diamond). Inputting $n=2$ we obtain the Hodge diamond of complex tori of dimension two:

$$\textit{Complex Tori of dimension two Hodge Diamond:}\qquad \begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array}. \tag{3.9}$$

Proposition 3.10. *For every complex torus A of dimension two there exists an isomorphism*

$$\phi: H^2(X,\mathbb{Z})\xrightarrow{\sim}U^3.$$

Proof. From Remark 3.8 and Hodge Index Theorem (1.23) we deduce that $H^2(A,\mathbb{Z})$ has signature $(3,3)$. Now by Theorem 1.13, there exists an isomorphism $H^2(X,\mathbb{Z})\simeq U^3$ which concludes the proof. □

Lemma 3.11. *A compact Kähler manifold X is projective if and only if the Kähler cone contains some integral classes, i.e. $\mathcal{K}_X\cap H^2(X,\mathbb{Z})\neq\emptyset$.*

Proof. If $\mathcal{K}_X\cap H^2(X,\mathbb{Z})\neq\emptyset$, then consider $\omega\in H^2(X,\mathbb{Z})\cap\mathcal{K}_X$, by definition of Kähler class, one has that $\omega^2>0$. Kodaira embedding ensures that the corresponding line bundle L_ω is ample, and therefore X is projective.

Conversely, if X is projective, then the restriction of the Fubini-Study Kähler form is positive and integral (it is the class corresponding to the restriction of $\mathcal{O}(1)$). $\square$

Proposition 3.12. *[12] Let $A = V/\Gamma$ be a complex torus of dimension n. Then X is projective, if and only if there exists an alternating bilinear form $\omega : V \times V \to \mathbb{R}$ such that:*

1) $\omega(iu, iv) = \omega(u, v)$

2) $\omega(\,\cdot\,, i\,\cdot)$ is positive definite

3) $\omega(u, v) \in \mathbb{Z}$ for $u, v \in \Gamma$

Proof. Thanks to the alternating nature of ω and condition 1), we can interpret ω as a two-form $\omega \in \bigwedge^2 V^* \simeq H^2(A, \mathbb{R}) \subset H^2(A, \mathbb{C})$. Moreover condition 2) ensures that ω is a Kähler form. Since condition 3) is equivalent to $\omega \in H^2(A, \mathbb{Z})$, Lemma 3.11 translates exactly into the statement of this proposition. $\square$

What we've just discussed can be expressed using the language of Hodge structures, which will prove to be very useful in the theory of Abelian surfaces. In this context we will introduce a definition very similar to Definition 1.24.

Definition 3.13. A *integral Hodge structure of weight k* is a free finitely-generated $\mathbb{Z}$-module L, together with a direct sum decomposition

$$L \otimes \mathbb{C} = \bigoplus_{p+q=k} L^{p,q}.$$

No requirement on the associated bilinear form has been done. This is necessary to prove the following:

Proposition 3.14. *There is a natural bijection between the set of isomorphisms classes of complex tori and the set of isomorphisms of integral Hodge structures of weight one.*

Proof. This is canonical. For an integral Hodge structure L we consider the lattice in $\mathbb{C}$ given by $L \cap L^{1,0}$. The quotient $L^{1,0}/L$ is a complex torus, and any isomorphism of two integral Hodge structures of weight one $L' \simeq L$ clearly gives an isomorphism of complex tori $L'^{1,0}/L' \simeq L^{1,0}/L$. Conversely, consider a complex torus $A := \mathbb{C}^n/\Gamma$, as we've already proved $H_1(A, \mathbb{Z}) = \Gamma$, and the natural complex structure on $\mathbb{C}^n$ endows the real vector space $\Gamma \otimes \mathbb{R}$ with a complex structure. Therefore, we have a decomposition

$$(\Gamma \otimes \mathbb{R}) \otimes \mathbb{C} = \Gamma^{1,0} \oplus \Gamma^{0,1},$$

which is a Hodge structure of weight one. These two constructions are inverse to each other, since there exist isomorphisms $\mathbb{C}^n \simeq \Gamma \otimes \mathbb{R} \simeq \Gamma^{1,0}$, of real vector spaces.
Moreover, given any other complex torus $A' := \mathbb{C}^n/\Gamma$ and a complex tori isomorphism $A \simeq A'$, the latter is induced by a unique (up to translation) $\mathbb{C}$-linear isomorphism $\phi : \mathbb{C}^n \xrightarrow{\sim} \mathbb{C}^n$ such that $\phi(\Gamma) = \Gamma'$. □

Now we need to fix a bilinear form, and we come back to lattices as we did in the first chapter.

Definition 3.15. Let L be an integral Hodge structure of weight k. Define $h : \mathbb{C}^* \to \mathrm{GL}(L \otimes \mathbb{C})$ as $h(z)\alpha = (z^p\bar{z}^q) \cdot \alpha$ for every $\alpha \in L^{p,q}$. A polarization on L is a bilinear form

$$(\ , \ ) : L \times L \to \mathbb{Z},$$

such that:

1) $(h(z)\alpha, h(z)\beta)_{\mathbb{C}} = (z\bar{z})^k(\alpha, \beta)_{\mathbb{C}}$ for every $z \in \mathbb{C}$

2) $(\cdot\ , h(i)\cdot)_{\mathbb{C}}$ is symmetric positive definite over $(L \otimes \mathbb{R}) \times (L \otimes \mathbb{R})$.

Where we have used the $\mathbb{C}$-linear extension of the bilinear form.

Please note that this definition is very similar to the definition of a polarized Hodge Structure of weight two. The requirement that $(\cdot\ , h(i)\cdot)$ is symmetric positive definite creates a link between polarized Hodge structures and integral Kähler forms. Now Proposition 3.12 reads as follows

Corollary 3.16. *Let L be an integral Hodge structure of weight one endowed with a polarization. Then its corresponding complex torus is projective.*

Proof. It sufficies to observe that conditions 1) and 2) of Definition 3.15 give a Kähler form which is also integral due to its very definition. □

As a corollary, we have a nice result for elliptic curves:

Corollary 3.17. *All complex tori of dimension one are projective.*

Proof. The corresponding Hodge structure of weight one is $L = \langle\Gamma\rangle = \langle z_1, z_2\rangle$, together with the decomposition $\langle\Gamma\rangle_{\mathbb{C}} = \Gamma^{1,0} \oplus \Gamma^{0,1}$ into two one-dimensional subspaces. This structure admits a natural polarization given by the intersection product. This is true by direct application of the Hodge-Riemann bilinear relations. □

Remark 3.18. We would like to stress the equivalence between the different languages we used so far. For compact Kähler manifolds, being projective is a matter of having a positive class in $H^{1,1}(X,\mathbb{Z}) = NS(X)$. For a complex torus $A = V/\Gamma$ of dimension n, this translates into the existence of an alternating bilinear form on $V^* \simeq \mathbb{R}^{2n}$, by $H^2(X,\mathbb{R}) \simeq \bigwedge^2 V^*$, which is positive definite and compatible with the complex structure of V^* (condition 2 of Proposition 3.12). Moreover, this bilinear form is integer-valued when restricted to the lattice Γ since it belongs to $H^{1,1}(X,\mathbb{Z})$. Using Hodge structures, we are only moving the focus from bilinear forms on V^* to bilinear forms on the lattice Γ.

3.2 Automorphisms of Abelian surfaces

We will now restrict our focus to the case of Abelian surfaces. We are interested in automorphisms of these surfaces; more precisely, we want to study those finite-order automorphisms that preserve the nonzero holomorphic two-form. We will presently prove that the minimal resolution of the quotient by any such automorphism action is a $K3$ surface. The entire discussion is based on the work of A. Fujiki [3], who gave a complete classification of the pairs (A, G) where A is a complex torus of dimension two and G a finite group automorphisms of A. We will discuss many of these results, and we'll provide some examples of finite order symplectic automorphisms on complex 2-tori. Our focus will be the induced action on the lattice $H^2(A,\mathbb{Z})$, the same way we did when treating automorphisms of $K3$ surfaces. This lattice comes as well with a natural signed Hodge structure of weight two, as we have already discussed in Remark 1.28.

While for $K3$ surfaces the lattice $H^2(X,\mathbb{Z})$ encodes all the informations about the manifold, for Abelian surfaces the situation is slightly different. We start by clarifying this situation using a classical result due to T. Shioda.

Theorem 3.19 ([29, Theorem I]). *Let A and B be two Abelian surfaces. If there exists a Hodge isometry $\varphi : H^2(A,\mathbb{Z}) \xrightarrow{\sim} H^2(B,\mathbb{Z})$, then at least one of the followings holds:*

1) *A is isomorphic to B.*

2) *A is isomorphic to the dual torus of B, i.e. $B^* := H^1(B,\mathcal{O}_B)/H^1(B,\mathbb{Z}) \simeq V/\Gamma^*$.*

The period of an Abelian Surface alone can't distinguish between a complex torus and its dual. Indeed, even if the period map is surjective, it is generically a 2:1 map.
Before proving the theorem, we will need some more tools.

Lemma 3.20 ([29, Lemma 1]). *Let $\varphi : H^2(A,\mathbb{Z}) \xrightarrow{\sim} H^2(B,\mathbb{Z})$ be an even lattice isometry such that* $\det\varphi = 1$, *then there exists an isomorphism of $\psi : H^1(A,\mathbb{Z}) \xrightarrow{\sim} H^1(B,\mathbb{Z})$ such that* $\det\psi = 1$ *and $\bigwedge^2 \psi = \varphi$ or $-\varphi$.*

Proof. First, suppose $A = B$. We need to find an isomorphism $\psi \in \mathrm{SL}(4, \mathbb{Z})$ such that $\bigwedge^2 \psi = \varphi$ or $-\varphi$. An isometry on $H^2(A, \mathbb{R})$ corresponds to an element in $O_U := \{M \in \mathrm{GL}(6, \mathbb{R}) \mid M^T U M = U\}$ where

$$U := \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix}.$$

We consider the map $\lambda_{\mathbb{R}} : \mathrm{SL}(4, \mathbb{R}) \to O_U(\mathbb{R})$ given by $\psi \mapsto \bigwedge^2 \psi$. Since $\mathrm{SL}(4, \mathbb{R})$ is connected, the image of $\lambda_{\mathbb{R}}$ is connected, moreover it is a subset of $O_U(\mathbb{R}) \cap \mathrm{SL}(6, \mathbb{R}) \simeq \mathrm{SO}(3, 3)$, and since $I \in \mathrm{Im}(\lambda_{\mathbb{R}})$, it corresponds to $SO_U^+(\mathbb{R}) \simeq \mathrm{SO}^+(3, 3)$. This gives $SO_U(\mathbb{R}) = (\mathrm{Im}\,\lambda_{\mathbb{R}}) \cup (-\mathrm{Im}\,\lambda_{\mathbb{R}})$. In particular, the image consists of isometries that preserve the positive cone.
Now if $A \neq B$, consider any isomorphism $\psi_0 : H^1(A, \mathbb{Z}) \xrightarrow{\sim} H^1(B, \mathbb{Z})$ such that $\det \psi_0 = 1$, and define $\varphi_0 := \bigwedge^2 \psi$. Then $\varphi_0^{-1} \circ \varphi$ is an isometry of $H^2(A, \mathbb{Z})$, and as such admits an automorphism ψ of $H^1(A, \mathbb{Z})$ such that $\bigwedge^2 \psi = \varphi_0^{-1} \circ \varphi$ or $-\varphi_0^{-1} \circ \varphi$. It follows that $\bigwedge^2 \psi_0 \circ \psi = \varphi$ or $-\varphi$ as required. $\square$

This yields another, more familiar, and restricted version of Theorem 3.19:

Theorem 3.21 ([21, Theorem 2.1])**.** *Let A be an Abelian surface. If $\varphi : H^2(A, \mathbb{Z}) \xrightarrow{\sim} H^2(A, \mathbb{Z})$ is a signed Hodge isometry of determinant* 1*, then there exists an automorphism $f : A \xrightarrow{\sim} A$ such that $f^* = \varphi$.*

Proof. This is a direct application of Lemma 3.20. Consider $\psi \in \mathrm{Aut}(H^1(A, \mathbb{Z}))$ as in the Lemma. Since φ is a signed Hodge isometry, it belongs to the connected component of $SO_U(\mathbb{R}) \simeq SO(3, 3)$ containing the Identity. Then, $\bigwedge^2 \psi = \varphi$, and to conclude our proof, we need to show that

$$\psi_{\mathbb{C}}(H^{1,0}(A)) = \psi_{\mathbb{C}}(H^{1,0}(A)),$$

which implies

$$\psi_{\mathbb{C}}(H^{0,1}(A)) = \psi_{\mathbb{C}}(H^{0,1}(A)),$$

thus completing the proof thanks to Proposition 3.14. Now, consider two generators $\omega_1, \omega_2 \in H^{1,0}(A)$ and write

$$\psi_{\mathbb{C}}(\omega_1) = \alpha_1 + \bar{\beta}_1 \text{ and } \psi_{\mathbb{C}}(\omega_2) = \alpha_2 + \bar{\beta}_2,$$

for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in H^{1,0}(A)$. Since

$$\varphi_{\mathbb{C}}(\omega_1 \wedge \omega_2) = \psi_{\mathbb{C}}(\omega_1) \wedge \psi_{\mathbb{C}}(\omega_2),$$

we expand: $\varphi_{\mathbb{C}}(\omega_1 \wedge \omega_2) = \alpha_1 \wedge \alpha_2 + \alpha_1 \wedge \bar{\beta}_2 + \bar{\beta}_1 \wedge \alpha_2 + \bar{\beta}_1 \wedge \bar{\beta}_2$. Since φ is a Hodge isometry, we know that

$$\varphi(\omega_1 \wedge \omega_2) \in H^{2,0}(A).$$

This forces the conditions:

$$\alpha_1 \wedge \bar{\beta}_2 = 0, \quad \bar{\beta}_1 \wedge \alpha_2 = 0, \quad \bar{\beta}_1 \wedge \bar{\beta}_2 = 0.$$

From these, we deduce that $\beta_1 = 0$ and $\beta_2 = 0$, concluding the proof.

□

Conversely, any automorphism f of an Abelian surfaces induces an automorphism $\psi \in \mathrm{Aut}(A)$ such that the isometry $\bigwedge^2 \psi =: \varphi \in O(H^2(A, \mathbb{Z}))$ has determinant 1 and preserves both the Hodge structure and the choice of the positive cone.
Shioda proved that any Abelian surface A admits a Hodge isometry

$$\varphi : H^2(A, \mathbb{Z}) \xrightarrow{\sim} H^2(A, \mathbb{Z})$$

of determinant -1. We will avoid the treatment of the period map, and we refer the curious reader to the work of Shioda [29], where he proves that the period map for Abelian surfaces is surjective, and generically 2:1. Furthermore, the moduli space of complex tori of dimension two has two connected components which map injectively onto the period domain.
We now come back to finite order symplectic automorphisms of Abelian varieties, following the work of A. Fujiki [3], and referring to the modern translation by G. Mongardi, K. Tari and M. Wandel [21].

Definition 3.22. An automorphism $g : A \xrightarrow{\sim} A$ is said to be symplectic if $g^*(\omega) = \omega$ for $0 \neq \omega \in H^{2,0}(A)$. Otherwise we call g non-symplectic. We define the invariant lattice and the coinvariant lattice respectively as

$$T_{A,f} := \{x \in H^2(X, \mathbb{Z}) \mid g^* x = x \text{ for every } g \in G\},$$

$$N_{A,f} := T_{A,f}^{\perp}.$$

Remark 3.23. Let A be an Abelian surface an denote by $Aut_0(A)$ the group of automorphisms of A that fix the identity. Lemma 3.20 shows that there exists a map

$$\nu : Aut_0(A) \to O(H^2(A, \mathbb{Z}))$$

which is surjective and 2:1. Indeed, $\ker(\nu) = \{\pm \mathrm{id}_A\}$.

Definition 3.24. Let $G \subset \mathrm{Aut}_0(A)$ be a group of automorphisms of A. We call G symplectic if every $g \in G$ is a symplectic automorphism. We define the invariant lattice and the coinvariant lattice respectively as

$$T(G) := \{x \in H^2(X, \mathbb{Z}) \mid g^*x = x \text{ for every } g \in G\},$$

$$N(G) := T(G)^{\perp}.$$

Where we used the same notation as in Chapter 2.
It is obvious from the definition that $N_{A,G} \subset NS(X)$. and $T(X) \subset T_{A,G}$.

Lemma 3.25 ([3, Lemma 3.1])**.** *For a group of automorphisms G on an Abelian surface A the following are equivalent:*

1) *G is symplectic*

2) *the minimal resolution of A/G is a K3 surface*

Proof. Consider the following diagram:

$$\begin{tikzcd} & A \arrow[d, "\pi"] \\ X \arrow[r, "\rho"] & A/G \end{tikzcd},$$

where π is the quotient map, and X is the minimal resolution of A/G. Since G is symplectic, there exists $0 \neq \omega \in H^{2,0}(A)$ such that G fixes ω, and therefore this nowhere vanishing holomorphic form passes down to X as a holomorphic two-form whose zeroes lie in $\rho^{-1}(p_i)$ for any fixed point p_i. On the other hand, any for any fixed point, the subgroup $G_{p_i} \subset G$ that fixes p_i only has rational double points, as the group action can be written locally as the action of a subgroup of $\mathrm{SL}(2, \mathbb{C})$. This ensures that the holomorphic form on X is nowhere vanishing, too. Therefore it sufficies to prove that $H^{1,0}(X) = 0$, but this is necessarily true since any non-trivial element of G acts locally as a matrix whose eigenvalues are never 1. This concludes 2) $\rightarrow$ 1).
Conversely, if X is a $K3$ surface, then $\pi^{-1}\rho(\omega)$ gives a G-invariant holomorphic two-form on A. □

Lemma 3.26 ([3, Lemma 3.3])**.** *Let A be an Abelian surface and let $g : A \xrightarrow{\sim} A$ be a finite order symplectic automorphism of A. Then g has order $1, 2, 3, 4$ or 6.*

Remark 3.27. Every complex torus of dimension two $A = V/\Gamma$ comes with a natural automorphism of order two $-id_A$ induced by $-\,\mathrm{id} \in \mathrm{GL}(2, \mathbb{C})$. Equivalently, using the torus complex lie group structure we could say that the operation $v \mapsto -v$ is an automorphism of order two of the torus.

In fact, this is the only symplectic automorphism of order two, and any automorphism of order 6 is the composition of an order 3 automorphism with $-\operatorname{id}_A$. This reduces the study of symplectic automorphisms of Abelian surfaces to the cases of $-\operatorname{id}_A$, order 3 and order 4 automorphisms.

Example 3.28 (Kummer surface). Let us study the case of $g := -\operatorname{id}$. Let $A := \mathbb{C}^2/\Gamma$ be a complex torus of dimension two. The quotient A/g has 16 fixed points, we can calculate it directly:

$$(-a - ib, -c - id) + (\delta_1\gamma_1 + i\delta_2\gamma_2, \delta_3\gamma_3 + \delta_4\gamma_4) = (a + ib, c + id),$$

where δ_i can be either 1 or 0, and $< \gamma_1, \gamma_2, \gamma_3, \gamma_4 >= \Gamma$. Therefore

$$(\delta_1\gamma_1 + i\delta_2\gamma_2, \delta_3\gamma_3 + \delta_4\gamma_4) = 2(a + ib, c + id),$$

which yields 16 solutions, one for each choice of $(\delta_1, \delta_2, \delta_3, \delta_4) \in (\mathbb{Z}/2\mathbb{Z})^4$.
As we have already discussed, these singularities are all rational double points. The minimal resolution X of A/g therefore contains 16 smooth rational curves K_i of self-intersection -2 and such that $K_iK_j = 0$ for $i \neq j$. Moreover, Lemma 3.25 proves that X is a $K3$ surface.

This concept can be generalized to any symplectic automorphism of finite order:

Definition 3.29. Let A be an Abelian surface, and let $g : A \xrightarrow{\sim} A$ be a symplectic automorphism of A of finite order. We call the minimal resolution of A/g a *generalized Kummer surface*, and we denote it by $Km_n(A)$.

Finally, we get this important theorem by Fujiki:

Theorem 3.30 ([3, Theorem 6.9]). *Let A be an Abelian surface. Then A admits an automorphism g of order $n \in \{2, 3, 4, 6\}$ if and only if the two equivalent conditions hold:*

1) There is a primitive embedding of $T(A) \hookrightarrow T_{A,f} = T_f$;

2) There is a primitive embedding of $N_{A,f} = N_f \hookrightarrow NS(A)$.

where T_f and N_f are uniquely determined by the order of the automorphism g, and they do not depend on the torus A.
The following table summarizes the conditions for Abelian surfaces to admit finite-order symplectic automorphisms, all in terms of lattices.

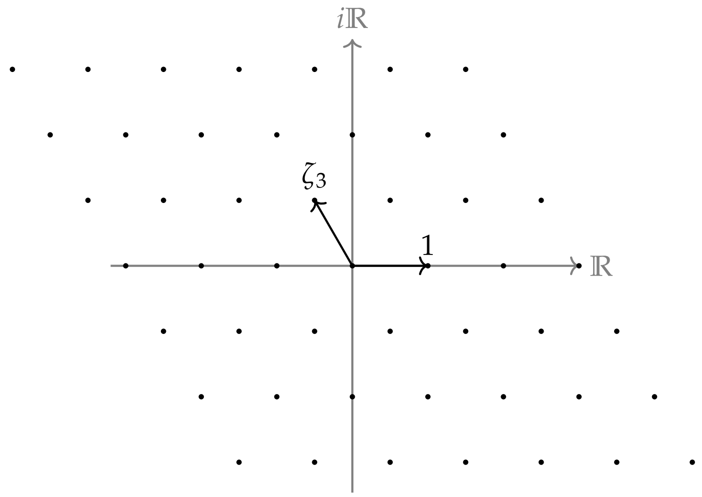
Group	T_f	N_f
$\mathbb{Z}/2\mathbb{Z}$	$H^2(A,\mathbb{Z})$	$\{0\}$
$\mathbb{Z}/3\mathbb{Z}$	$A_2 \oplus U$	$A_2(-1)$
$\mathbb{Z}/4\mathbb{Z}$	$A_1^2 \oplus U$	$A_1(-1)^2$
$\mathbb{Z}/6\mathbb{Z}$	$A_2 \oplus U$	$A_2(-1)$

Remark 3.31. As we already pointed out, having an order 3 symplectic automorphisms is equivalent to having a symplectic automorphism of order 6. In terms of lattices, this is due to the fact that the action of $-\operatorname{id}_A$ on $H^2(A, \mathbb{Z})$ is trivial.

Let's give some straightforward examples of symplectic automorphisms of finite order of an Abelian surface.

Remark 3.32. We've already discussed the case of the symplectic involution $-\operatorname{id}_A$ on an Abelian surface A. Now, let us explore the remaining finite orders for symplectic automorphisms.

Order 3 Consider the lattice in $\mathbb{C}$ defined as $\Gamma := \mathbb{Z} \oplus \zeta_3\mathbb{Z}$ where ζ_3 is a primitive third root of unity and define the complex torus $E := \mathbb{C}/\Gamma$. The following is a representation of Γ:



Since the $\mathbb{C}$-linear morphism given by the multiplication by ζ_3 preserves Γ (i.e. $\zeta_3 \cdot \Gamma = \Gamma$), the curve E inherits an automorphism given by $z \mapsto \zeta_3 z$. On the abelian surface $A := E \times E \simeq \mathbb{C}/(\Gamma \times \Gamma) = \mathbb{C}/\bar{\Gamma}$ we define the automorphism

$$\sigma_A : A \to A, \text{ such that } (z_1, z_2) \mapsto (\zeta_3 z_1, \zeta_3^{-1} z_2).$$

The eigenvalues of σ_A^* are ζ_3 and ζ_3^{-1}, therefore $\bigwedge^2 \sigma_A^*$ acts as the identity on $H^{2,0}(A) \simeq \bigwedge^2 H^{1,0}(A)$.

Order 4 Similarly to the construction of the order 3 automorphism, we could consider an elliptic curve E and the Abelian surface $A := E \times E$. We define the automorphism

$$\tau_A : A \to A, \text{ such that } (z_1, z_2) \mapsto (z_2, -z_1).$$

This automorphism is symplectic, since the action on $H^{2,0}(A)$, by the same considerations as above, is trivial. Further considerations about this automorphism will be discussed in the next chapter.

Order 6 An easy example of automorphism of order 6, proceding as above, is given by considering again the elliptic curve $E := \mathbb{C}/(\mathbb{Z} \oplus \zeta_3\mathbb{Z})$ and constructing, over the abelian surface $A := E \times E$ the automorphism

$$\varphi_A : A \to A, \text{ such that } (z_1, z_2) \mapsto (-\zeta_3 z_1, -\zeta_3^{-1} z_2).$$

This is again symplectic, indeed $\bigwedge^2 \varphi_A(dz_1 \wedge dz_2) = d(-\zeta_3 z_1) \wedge d(-\zeta_3^{-1} z_2) = \zeta_3 \zeta_3^{-1} dz_1 \wedge dz_2 = dz_1 \wedge dz_2$.

Chapter 4

Shioda-Inose structures

As we discussed in Chapter 1, there are only two classes of compact Kähler complex surfaces with trivial canonical bundle: K3 surfaces and Abelian surfaces. The Kummer construction, which we presented in the last Chapter, establishes a relation between these two: from every abelian surface A we can obtain a $K3$ surface X, called Kummer surface, which contains 16 smooth rational curves with self-intersection -2. As we will see, this setup can be further extended by considering a $K3$ surface S with a symplectic involution i_S such that, denoting as X the minimal resolution of S/i_S, $X \simeq Km_2(A)$ and the rational quotient map $\pi : S \dashrightarrow X$ induces a Hodge isometry between the transcendental lattice of the covering $K3$ surface and the abelian surface. This purely geometric construction is called a *Shioda-Inose* structure. It was first defined by D.R. Morrison [22], who expanded the work by Shioda T. and Inose H. [30] to all Abelian surfaces and first set the definition of a Shioda-Inose structure. Subsequently, many authors provided examples and applications of Shioda-Inose structures.
More recently, A. Garbagnati and Y. Prieto-Montañez [5] generalized the entire setup by considering Abelian surfaces with a symplectic automorphism of order 3 and the related Generalized Kummer surface, followed by B. Piroddi [28], who later discussed the order 4 case. In this last chapter we will summarize the literature on Shioda-Inose structures in their most general definition, and we will make some considerations about the order 6 case.

4.1 The order 2 case

We present the definition as it was set by Morrison, slightly rephrased, and we will examine how this definition has evolved as research on the topic progressed over time.

Definition 4.1 (Shioda-Inose structure of order 2)**.** A *Shioda-Inose structure* is a quadruple (S, i_S, A) where S is a $K3$ surface, i_S is a symplectic automorphism of order 2 on S and A is an Abelian surface such that:

1) $Km_2(A)$ is isomorphic to the minimal resolution of S/i_S;

2) $T(S)(2) \simeq T(Km_2(A))$ as even lattices.

Remark 4.2. For any compact complex surface S equipped with an involution i, write $P_1, \dots, P_k$ for the loci fixed by the involution, which we suppose to be isolated points. By repeatedly blowing up these points in S, we obtain a new surface Z, which inherits an involution from S acting trivially on the k exceptional curves E_i. The quotient map $\pi : Z \to X$ is a double cover branched over these curves, therefore we get $\frac{1}{2}\sum_{i=1}^{k} C_i \in NS(X)$, where the C_i are the divisors corresponding to the curves. Note that the viceversa is always true: more generally, for any set $\{C_i\}_{i=1,\dots,k}$ of disjoint smooth irreducible rational curves on a surface X such that $\frac{1}{n}\sum_{i=1}^{k} C_i \in NS(X)$, there exists a covering of degree n branched over $\sum_{i=1}^{k} C_i$. Since X is isomorphic to the minimal resolution of S/i, we obtain the following diagram:

$$\begin{array}{ccc} Z & \xrightarrow{\varphi} & S \\ \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\tilde{\varphi}} & S/i \end{array} .$$

Definition 4.3. We denote $M_{\mathbb{Z}/2\mathbb{Z}}$ the minimal primitive sublattice of $NS(X)$ containing the curves arising from the desingularization of fixed points.

Morrison, using the work of Shioda and Inose, proved the following:

Lemma 4.4 ([22, Lemma 3.1] and [30, Sect. 3])**.** *Using the notation of Remark 4.2, and denoting by H_X the orthogonal complement of the exceptional curves in $H^2(X, \mathbb{Z})$, there exist two natural maps*

$$\pi_* : H^2(S, \mathbb{Z}) \to H_X \ \ \textit{and} \ \ \pi^* : H_X \to H^2(S, \mathbb{Z}),$$

such that

$$\pi_*\pi^*(x) = 2x; \ \ \pi^*\pi_*(s) = s + i^*(s); \ \ x_1x_2 = \frac{1}{2}\pi^*(x_1)\pi^*(x_2),$$

and

$$\pi^*(K_X) = K_S.$$

Furthermore, if there exists a sublattice $T_X \subset L \subset H^2(X, \mathbb{Z})^G$, with $L \simeq U^n$ such that $\pi_*(L)^\perp$ has determinant 2^{2n} it holds that

$$\pi_*|_{T(X)} : T(X) \simeq T(A)(2)$$

is a Hodge isometry.

An Abelian surface A and its related Kummer surface $Km_2(A)$ manifest a peculiar relation between their transcendental lattices. This was proved by Nikulin:

Proposition 4.5 ([24, Remark 2])**.** *The rational quotient map $\pi : A \dashrightarrow Km_2(A)$ induces a Hodge isometry $\pi_* : T(A)(2) \simeq T(Km_2(A))$.*

Before proving the proposition, it is useful to give a characterization of those $K3$ surfaces that arise from the quotient of an Abelian surface by its natural involution. In the next section, we will provide a generalization of this theorem by J. Bertin.

Definition 4.6. The minimal primitive sublattice in $NS(Km_2(A))$ containing the classes of the 16 exceptional curves coming from the desingularization of A/i is called *Kummer lattice of order 2*, and it is denoted $K_{\mathbb{Z}/2\mathbb{Z}}$.

Theorem 4.7. *[24] The following statements about the Kummer lattice hold:*

1) $\det(K_{\mathbb{Z}/2\mathbb{Z}}) = 2^6$*;*

2) A $K3$ surface X is a Kummer surface if and only if there is a primitive embedding $K_{\mathbb{Z}/2\mathbb{Z}} \hookrightarrow NS(X)$;

3) The embedding $K_{\mathbb{Z}/2\mathbb{Z}} \hookrightarrow \Lambda_{K3}$ is unique.

Proof of Proposition 4.5. Since the action of the involution $-\operatorname{id}_A$ is trivial on $H^2(A,\mathbb{Z})$, and $K_{\mathbb{Z}/2\mathbb{Z}}$ is by definition the smallest sublattice of $H^2(Km_2(A),\mathbb{Z})$ containing all the exceptional curves, necessarily $\pi_*(H^2(A,\mathbb{Z}))^\perp \simeq K_{\mathbb{Z}/2\mathbb{Z}}$, which has determinant 2^6. The statement is therefore true thanks to Lemmma 4.4. □

Proposition 4.8. *Let $K_{\mathbb{Z}/2\mathbb{Z}}$ be the Kummer lattice of order 2. Then $q_{K_{\mathbb{Z}/2\mathbb{Z}}} \simeq (q_{U(2)})^3$.*

Proof. As in the proof of proposition 4.5 we get

$$K^\perp_{\mathbb{Z}/2\mathbb{Z}} \simeq \pi_*(H^2(A,\mathbb{Z})) \simeq (U(2))^3,$$

this gives

$$q_{K_{\mathbb{Z}/2\mathbb{Z}}} \simeq -q^\perp_{K_{\mathbb{Z}/2\mathbb{Z}}} \simeq (q_{U(2)})^3.$$

□

Now that the situation regarding the natural involution of Abelian surfaces has been clarified, we turn our focus to involutions of $K3$ surfaces. We will construct a symplectic involution on certain $K3$ surfaces and prove that the resulting resolution of the quotient is indeed a Kummer surface. Moreover, we will establish the necessary and sufficient conditions for this to occur. This follows the work of Morrison [22].

Any involution on a $K3$ surface fixes exactly 8 points which are double rational points (See Nikulin [25, Section 5]). Moreover the lattice $M_{\mathbb{Z}/2\mathbb{Z}}$ is a rank 8 even lattice, constructed as an overlattice of $A_1^8(-1)$. The following theorem sheds light on the importance of this lattice.

Theorem 4.9 ([22, Theorem 5.7])**.** *Let S be a K3 surface and suppose that $E_8(-1)^2 \hookrightarrow NS(S)$ is a primitive embedding. Then there exists a symplectic involution i_S of S such that, if X is the minimal resolution of S/i_S and $\pi : S \dashrightarrow X$ is the rational quotient map, the following holds:*

1) *There is a primitive embedding of $M_{\mathbb{Z}/2\mathbb{Z}} \oplus E_8(-1) \hookrightarrow NS(X)$;*
2) *There is a Hodge isometry $\pi_* : T(S)(2) \simeq T(X)$;*
3) *$M_{\mathbb{Z}/2\mathbb{Z}} \oplus E_8(-1)$ is a rank 16 even lattice with discriminant form $(q_{U(2)})^3$.*

Proof. Consider a basis $\{c_i^{(j)}\}$ of $E_8(-1)^2$, where $j \in \{1,2\}$, such that $\{c_i^{(j)}\}$ is a basis for the jt-th copy of $E_8(-1)$ yielding the associated matrix equal to the negative of the matrix in Example 1.4. The isometry of Theorem 2.9 specializes to an embedding $\varphi : E_8(-1)^2 \to H^2(S,\mathbb{Z})$.
Define an isometry g on $H^2(S,\mathbb{Z})$ that switches the two copies of $E_8(-1)$ by sending $c_i^{(1)} \mapsto c_i^{(2)}$ and viceversa, fixing the rest.
The lattice $N_{S,G} := (H^2(S,\mathbb{Z})^G)^\perp$ is generated by

$$\{\varphi(c_i^{(1)}) - \varphi(c_i^{(2)})\},$$

which means that

$$E_8(-2) \simeq N_{S,G} \subset NS(S).$$

This is an even, negative definite lattice which surely contains no element of square -2. Therefore, there exists a symplectic involution i_S on S such that, up to isometry, $i_S^* = g$.
Let $\pi : S \dashrightarrow X$ be the rational quotient map. Thanks to Lemma 4.4, we know that the classes $\pi_*(c_1^{(1)}), \dots, \pi_*(c_8^{(1)})$ are orthogonal to $M_{\mathbb{Z}/2\mathbb{Z}}$, due to the definition of the map π_* (with abuse of notation, by $c_i^{(1)}$ we refer to its image in $H^2(S,\mathbb{Z})$, which is defined by the symplectic action of i_S^*). Moreover, using again Lemma 4.4 we get

$$\begin{aligned}
\left(\pi_*(c_j^{(i)}), \pi_*(c_k^{(1)})\right) &= \frac{1}{2}\left(\pi^*\pi_* c_j^{(1)}, \pi^*\pi_* c_k^{(1)}\right) \\
&= \frac{1}{2}\left(c_j^{(1)} + i^* c_j^{(1)}, c_k^{(1)} + i^* c_k^{(1)}\right) \\
&= \frac{1}{2}\left(c_j^{(1)}, c_k^{(1)}\right) + \frac{1}{2}\left(c_j^{(2)}, c_k^{(2)}\right) \\
&= \left(c_j^{(1)}, c_k^{(1)}\right)
\end{aligned}.$$

This implies that $\{\pi_* c_j^{(1)}\}$ is a basis for a copy of $E_8(-1)$, which is therefore primitively embedded in $NS(X)$ (here we are using the fact that $E_8(-1)$ is unimodular). Combining this with the fact that $M_{\mathbb{Z}/2\mathbb{Z}}$ is primitively embedded in $NS(X)$ by its very definition, we see that there is a primitive embedding $M_{\mathbb{Z}/2\mathbb{Z}} \oplus E_8(-1) \hookrightarrow NS(X)$.
To conclude, we consider the orthogonal complement of the two copies of $E_8(-1)$ switched by i_S^*, and denote it as L. Note that $L \simeq U^3$ contains the transcendental lattice of S and is contained in the invariant lattice. Moreover, $M_{\mathbb{Z}/2\mathbb{Z}} \oplus E_8(-1)$ is a rank 16 lattice because any involution of a $K3$ surface S contains exactly 8 isolated fixed points, and a direct calculation shows that $\det(M_{\mathbb{Z}/2\mathbb{Z}} \oplus E_8(-1)) = 2^6$.
Using Lemma 4.4 we see that π_* induces a Hodge isometry $T(S)(2) \simeq T(X)$, and $\pi_*(L) \simeq U(2)^3$. Finally, the sublattice $\pi_*(L) \subset H^2(X, \mathbb{Z})$ is primitive and therefore

$$q_{M_{\mathbb{Z}/2\mathbb{Z}} \oplus E_8(-1)} = -q_{\pi_*(L)} = (q_{U(2)})^3.$$

□

Remark 4.10. This ensures that whenever a Shioda-Inose structure exists, there are Hodge isometries:

$$\begin{tikzcd} T(S)(2) \arrow[rr, "\rho"] \arrow[dr, "\tilde{\pi}_*"'] & & T(A)(2) \arrow[dl, "\pi_*"] \\ & T(Km_2(A)) & \end{tikzcd} .$$

We will need one more lemma:

Lemma 4.11 ([22, Corollary 2.10]). *Let S be a $K3$ surface such that $\rho(S) \geq 12$. Then there exists only one embedding $T(S) \hookrightarrow \Lambda_{K3}$, namely the restriction of $H^2(S, \mathbb{Z}) \xrightarrow{\sim} \Lambda_{K3}$.*

Proof. Note that $l(A_{T(S)}) \leq 22 - \rho(S) \leq 22 - (22 - \rho(S)) - 2$. By hypothesis, $\rho(S) \geq 12$, therefore the above inequality is satisfied and Theorem 1.20 applies and the lemma is proved. □

We are ready to prove the central Theorem for Shioda-Inose structures of order 2.

Theorem 4.12. *Let S be a projective $K3$ surface. Then the following are equivalent:*

1) *There exists an Abelian surface A and a symplectic involution i_S of S such that (S, i_S, A) is a Shioda-Inose structure;*

2) *There exists an Abelian surface A together with a Hodge isometry $T(A) \simeq T(S)$;*

3) *There exists a primitive embedding $T(S) \hookrightarrow U^3$;*

4) There is an embedding $E_8(-1)^2 \hookrightarrow NS(S)$.

Proof. 1) $\to$ 2): Obvious from the very definition of Shioda-Inose structure.
2) $\to$ 3): Since there exists an isometry $\varphi : H^2(A,\mathbb{Z}) \xrightarrow{\sim} U^3$ (see Proposition 3.10), we get the embedding by composing $T(S) \simeq T(A) \hookrightarrow H^2(A,\mathbb{Z}) \simeq U^3$
3) $\to$ 4): Since there is an embedding $T(S) \hookrightarrow U^3$, we have that $22 - \rho(S) \leq 5$, which gives $\rho(S) \geq 17$. Moreover $U^3 = \{0\} \oplus U^3 \subset E_8(-1)^2 \oplus U^3 = \Lambda_{K3}$. Then the lemma ensures that the embedding

$$T(S) \hookrightarrow U^3 \hookrightarrow \Lambda_{K3}$$

corresponds to $T(S) \hookrightarrow H^2(S,\mathbb{Z}) \xrightarrow{\sim} \Lambda_{K3}$. We conclude by taking the orthogonal complements

$$E_8(-1)^2 = (U^3)^\perp \hookrightarrow T(S)^\perp = NS(S),$$

which is again a primitive embedding.
4) $\to$ 1): This is just a review of what we've seen so far in this Chapter. Indeed by Theorem 4.9 there exists an involution i_S on S such that, denoting by X the minimal resolution of its quotient and by $\pi : S \dashrightarrow X$ the rational quotient map, there is a primitive embedding $M_{\mathbb{Z}/2\mathbb{Z}} \oplus E_8(-1) \hookrightarrow NS(X)$, and it holds that $T(S)(2) \simeq T(X)$.
We show that $NS(X)$ is uniquely determined by its signature and discriminant form. This is due to the fact that $E_8(-1)$ is a unimodular lattice that is primitively embedded in the Neron-Severi group, which means that for sure $A_{NS(X)} \leq \rho(X) - 2$, giving the uniqueness of $NS(X)$ by Theorem 1.14. Furthermore, since the lattices $M_{\mathbb{Z}/2\mathbb{Z}} \oplus E_8(-1)$ and $K_{\mathbb{Z}/2\mathbb{Z}}$ share isomorphic discriminant-forms, we conclude using Lemma 1.16: the primitive embedding $M_{\mathbb{Z}/2\mathbb{Z}} \oplus E_8(-1) \hookrightarrow NS(X)$ determines a primitive embedding $K_{\mathbb{Z}/2\mathbb{Z}} \hookrightarrow NS(X)$. Then X is isomorphic to $Km_2(A)$ for an Abelian surface A, and Proposition 4.5 gives the Hodge isometry $\pi_* : T(A)(2) \xrightarrow{\sim} T(X)$. ☐

Corollary 4.13. *Let S be a projective K3 surface.*

1) If $\rho(S) = 19$ *or* 20*, then S admits a Shioda-Inose structure;*

2) If $\rho(S) = 18$*, then S admits a Shioda-Inose structure if and only if* $T(S) \simeq U \oplus T'$*;*

3) If $\rho(S) = 17$ *then S admits a Shioda-Inose structure if and only if* $T(S) \simeq U^2 \oplus T'$*;*

4) If $\rho(S) < 17$ *then S does not admit a Shioda-Inose structure.*

Proof. Case 1): Since $\operatorname{rk} T(S) = 2$ or 3, by Proposition 1.19 there is a primitive embedding $T(S) \hookrightarrow U^3$.
Case 2: If S admits a Shioda-Inose structure, there is a primitive embedding $T(S) \hookrightarrow$

U^3. Let $T' := T(S)^{\perp_{U^3}}$ and consider the even lattice $U \oplus T'(-1)$; it has the same signature as $T(S)$ and, by Proposition 1.17, they share the same discriminant form. Furthermore,

$$l(A_{T(S)}) = l(A_{T'}) \leq 2 = \mathrm{rk}(A_{T(S)}) - 2$$

and from Theorem 1.14 $T(S) \simeq U \oplus T'(-1)$. Conversely, if $T(S) \simeq U \oplus T'$, then $T' \hookrightarrow U^2$ as in case 1), and then $T \hookrightarrow U^3$.
Case 3: this is almost identical to case 2.
Case 4: If $\rho(S) \leq 16$ the condition $E_8(-1)^2 \hookrightarrow NS(S)$ can never be satisfied. This is obvious for $\rho(S) < 16$, and is still true for $\rho(S) = 16$ because X is projective and, from Kodaira embedding, we know there is always an ample divisor. □

Remark 4.14. Theorem 4.12 creates a powerful link between $K3$ surfaces and Abelian surfaces. Indeed, given any $K3$ surface S with a primitive embedding $E_8(-1)^2 \hookrightarrow NS(S)$, there are exactly two (just one if the torus is a principally polarized abelian surface) complex tori that fit in a Shioda-Inose structure $(S, i_S, \cdot)$, one being the dual of the other. Conversely, for any abelian surface A there exists exactly one $K3$ surface S with a symplectic involution i_S such that (S, i_S, A) is a Shioda-Inose structure. This is due to the fact that $T(S) \simeq T(A)$ is a Hodge isometry, and then the statement follows from the surjectivity of the period map.

4.2 Generalized Shioda-Inose structures

A natural way to generalize the work of Morrison is by substituting the involution i_S with a symplectic automorphism of order n of the $K3$ surface S, and the natural map $-\mathrm{id}_A$ with a symplectic automorphism of order n on the torus. This is the leading path for *Generalized Shioda-Inose structures* which was first suggested by H. Onsiper and S. Sertoz in [27].

Definition 4.15 (Generalized Shioda-Inose structure of order n)**.** A Generalized Shioda-Inose structure of order n is a quadruple $(S, \sigma_S, A, \sigma_A)$ such that:

1) S is a $K3$ surface and σ_S is a symplectic automorphism of order n on S;

2) A is an abelian surface and σ_A is a symplectic automorphism of order n on A;

3) the minimal resolution of S/σ_S is isomorphic to $Km_n(A)$.

We will often refer to the Generalized Shioda-Inose structure of order n omitting the term 'Generalized'.

One should keep this diagram in mind:

$$\begin{tikzcd}
S \arrow[d] \arrow[dr, dashed] & & A \arrow[d] \arrow[dl, dashed] \\
S/\sigma_S & X \simeq Km_n(A) \arrow[l] \arrow[r] & A/\sigma_A
\end{tikzcd}$$

This is a weak definition, because there's no more trace of an isometry between the transcendental lattices of A, S and $Km_n(A)$. We will see that a relation of this type still holds for Generalized Shioda-Inose structures of order 3, but this will be no longer the case for order 4 structures.

Remark 4.16. The careful reader knows that there are only few possible values for n other than $n = 2$, namely $n = 3, n = 4$ and $n = 6$ (see Lemma 3.26).

Proposition 4.17. *for $n \geq 3$, if a K3 surface S admits a Generalized Shioda-Inose structure of order n, then $\rho(S) \geq 19$.*

Proof. As shown in [14], $\rho(S) = \rho(Km_n(A))$. Moreover, since $\rho(Km_n(A)) \geq 19$ for any $n \geq 3$ and any Abelian surface A, the proposition follows. See [15] for the result on $\rho(Km_n(A))$. □

Definition 4.18. We extend the definitions of $M_{\mathbb{Z}/2\mathbb{Z}}$ and $K_{\mathbb{Z}/2\mathbb{Z}}$ to $M_{\mathbb{Z}/n\mathbb{Z}}$ and $K_{\mathbb{Z}/n\mathbb{Z}}$ naturally by considering the exceptional curves arising from the minimal resolutions of the two different quotients.
In particular $M_{\mathbb{Z}/n\mathbb{Z}}$ is the minimal primitive sublattice of $NS(X)$ that contains the exceptional curves corresponding to the points with a non-trivial stabilizer under the action of a symplectic automorphism of order n on a $K3$ surface S, whereas $K_{\mathbb{Z}/n\mathbb{Z}} \subset NS(Km_n(A))$ is the minimal primitive sublattice containing the exceptional curves corresponding to the fixed points of the symplectic automorphism of order n on the Abelian surface A.

We conclude with a generalization of the results on the order 2 about Kummer surfaces and the lattice $K_{\mathbb{Z}/2\mathbb{Z}}$.

Theorem 4.19 ([2, Theorem 2.5])**.** *A K3 surface S is a Generalized Kummer surface of order n if and only if the lattice $K_{\mathbb{Z}/n\mathbb{Z}}$ is primitively embedded in $NS(S)$.*

Proposition 4.20 ([25, Proposition 7.1 and Lemma 10.2])**.** *The lattices $M_{\mathbb{Z}/n\mathbb{Z}}$ and Ω_n are negative definite lattice with the same rank.*

4.2.1 The order 3 case

Recall from example 1.12 the definition of the unimodular even lattice $(E_6(-1)^3)'$. The following is the work of A. Garbagnati and Y. Prieto-Montañez contained in [5].

Remark 4.21 (The lattice $M_{\mathbb{Z}/3\mathbb{Z}}$). the lattice $M_{\mathbb{Z}/3\mathbb{Z}}$ is described by Nikulin in [25, Section 6]. We give an explicit construction for it as an abstract lattice:
Let a_1, a_2 be an usual basis of $A_2(-1)$ as in example 1.12. Consider the vector $b := (a_1 + 2a_2)/3$ and extend this construction to $A_2(-1)^6$ using the notation $b^{(j)}$ for the definition of b on the j-th copy of $A_2(-1)$, and $\{a_i^{(j)}\}$ for its basis. Then $M_{\mathbb{Z}/3\mathbb{Z}}$ is the overlattice of $A_2(-1)^3$ obtained by adding the vector $B := \sum_{j=1}^{6} b^{(j)}$. In addition, we denote as $(M_{\mathbb{Z}/3\mathbb{Z}} \oplus E_6(-1))'$ the overlattice of index 3 of $M_{\mathbb{Z}/3\mathbb{Z}} \oplus E_6(-1)$ obtained by adding the vector $n := (e_1 + 2e_2 + e_4 + 2e_5)/3 - b^{(1)} + b^{(3)} - b^{(4)} + b^{(5)}$ where $\{e_i\}$ stands for the usual basis for $E_6(-1)$.

Remark 4.22. The construction in Example 1.12 shows that $(U \oplus A_2 \oplus E_6(-1)^3)''$ is a unimodular even lattice of rank 22, and its signature is $(3, 19)$. By Theorem 1.13, there exists an isometry $(U \oplus A_2 \oplus E_6(-1)^3)'' \simeq \Lambda_{K3}$. We will often make use of this throughout this chapter.

Definition 4.23. Given the usual sets of generators:

- $\{a_1, a_2\}$ for A_2 that yield a matrix in the form of Example 1.4;
- $\{b_1, b_2\}$ for $A_2(3)$;
- $\{u_1, u_2\}$ for U as in Example 1.4;
- $\{v_1, v_2\}$ for $U(3)$;

we define the map

$$\begin{aligned} \gamma : U \oplus A_2 &\to U(3) \oplus A_2 \subset (U(3) \oplus A_2(3)) \otimes \mathbb{Q} \\ u_i \mapsto v_i, \quad a_1 &\mapsto (b_1 + b_2)/3, \quad a_2 \mapsto b_2 \end{aligned}.$$

Theorem 4.24 ([5, Theorem 1.24]). *Let S be a K3 surface and suppose that $(E_6(-1)^3)' \hookrightarrow NS(S)$ is a primitive embedding. Then there exists a symplectic automorphism σ_S of S of order 3 such that, if X is the minimal resolution of S/σ_S and $\pi : S \dashrightarrow X$ is the rational quotient map, the following holds:*

1) *There is a primitive embedding $(M_{\mathbb{Z}/3\mathbb{Z}} \oplus E_6(-1))' \hookrightarrow NS(X)$;*
2) *The transcendental lattice $T(S)$ is primitively embedded in $U \oplus A_2$, and the map π_* acts on $T(S)$ as the restriction of γ;*

3) $(M_{\mathbb{Z}/3\mathbb{Z}} \oplus E_6(-1))'$ *is a rank* 18 *negative definite even lattice with discriminant form equal to that of* $U(-3) \oplus A_2$.

Proof. As in the case of sympectic involutions, we build the automorphisms by working on the abstract lattice Λ_{K3}. Consider the isometry g of the $K3$ lattice that permutes cyclically the three copies of $E_6(-1) \subset (E_6(-1))' \hookrightarrow NS(S)$ and leaves the rest fixed. Let's verify the conditions for Theorem 2.30:

i) $N_g = K_{12}$, where K_{12} is the opposite of the lattice described by Coxeter and Todd. This is shown explicitly in [8, Proposition 3.1].

ii) N_g has no element of square -2. This is true by the very definition of K_{12}. Alternatively, one could use Lemma 4.2 of [25] which proves that the coinvariant lattice never contains such elements.

iii) $N_g \subset NS(S)$: this follows easily from the embedding $(E_6(-1)^3)' \hookrightarrow NS(S)$, since the fixed part obviously contains $T(S)$.

Therefore, there exists a symplectic automorphism $\sigma_S : S \xrightarrow{\sim} S$ of order 3 such that σ_S acts on $H^2(S, \mathbb{Z})$ by permuting cyclically the three copies of $(E_6(-1)^3)'$.
In order to prove assertion 1), we rely again on the work of Garbagnati and Prieto-Montañez in [8, Section 3.5], where it is proved that $H^2(X, \mathbb{Z})$ is an overlattice of finite index of $M_{\mathbb{Z}/3\mathbb{Z}} \oplus \pi_*(H^2(S, \mathbb{Z}))$. The gluing vectors needed to obtain $H^2(X, \mathbb{Z})$ as an overlattice of $M_{\mathbb{Z}/3\mathbb{Z}} \oplus \pi_*(H^2(S, \mathbb{Z}))$ are the same needed to obtain $NS(X)$ as an overlattice of $M_{\mathbb{Z}/3\mathbb{Z}} \oplus \pi_*(NS(S))$, since $M_{\mathbb{Z}/3\mathbb{Z}} \hookrightarrow NS(X)$. Moreover, thanks to the work done in [8, Proposition 3.2] we can interpret the map $\pi_* : H^2(S, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ as the extension of the map

$$\begin{array}{rccccccccc} \pi_* : & U \oplus & A_2 \oplus & E_6(-1) \oplus & E_6(-1) \oplus & E_6(-1) & \to & U(3) \oplus & A_2(3) \oplus & E_6(-1) \\ & (u, & a, & e, & f, & g) & \mapsto & (u, & a, & e+f+g) \end{array}$$

to the overlattice $(U \oplus A_2 \oplus (E_6(-1))^3)''$. In addition, it is proved that $E_6(-1) \simeq \pi_*((E_6(-1)^3)')$, and therefore $E_6(-1)$ is primitively embedded in $NS(X)$. This proves that $NS(X)$ is a finite index overlattice of $M_{\mathbb{Z}/3\mathbb{Z}} \oplus E_6(-1)$. Furthermore, it is also shown that the class n of remark 4.21 is contained in $H^2(X, \mathbb{Z})$, but any primitive sublattice containing $M_{\mathbb{Z}/3\mathbb{Z}} \oplus E_6(-1)$ also contains n, since $n \in A_{M_{\mathbb{Z}/3\mathbb{Z}} \oplus E_6(-1)}$. Therefore $NS(X)$ contains the overlattice $(M_{\mathbb{Z}/3\mathbb{Z}} \oplus E_6(-1))'$, and this embedding is primitive (see [8, Proposition 3.4]) proving assertion 1).
For assertion 3), by hypothesis $(E_6(-1)^3)' \hookrightarrow NS(S)$, which implies $T(S) \hookrightarrow U \oplus A_2$ by the uniqueness of the embedding $(E_6(-1)^3)' \hookrightarrow H^2(S, \mathbb{Z})$.
Finally, the discriminant form of $(M_{\mathbb{Z}/3\mathbb{Z}} \oplus E_6(-1))'$ can be directly calculated from its description in remark 4.21. $\square$

The following is the analogue of theorem 4.12:

Theorem 4.25. *Let S be a projective K3 surface. Then the following are equivalent:*

1) *There exists a symplectic automorphism σ of S of order 3 and an Abelian surface A admitting a symplectic automorphism σ_A of order 3 such that $(S, \sigma_S, A, \sigma_A)$ is a Generalized Shioda-Inose structure of order 3. Moreover there is an isometry $T(A) \simeq T(S)$;*

2) *There exists an Abelian surface A admitting a symplectic automorphism σ_A of order 3 together with an isometry $T(A) \simeq T(S)$;*

3) *There exists a primitive embedding $T(S) \hookrightarrow U \oplus A_2$;*

4) *There is an embedding $(E_6(-1)^3)' \hookrightarrow NS(S)$.*

Proof. We proceed similarly to the proof of Theorem 4.12. 1) $\to$ 2): Obvious from the very definition of Shioda-Inose structure.
2) $\to$ 3): Recall from chapter 3 that an Abelian surface A admits a symplectic automorphism of order 3 if and only if there is a primitive embedding $T(X) \hookrightarrow U \oplus A_2$.
3) $\to$ 4): Since there is an embedding $T(S) \hookrightarrow U \oplus A_2$, $\operatorname{rk} T(S) \leq 3$ and its signature is $(2, t)$ for $t \in \{0, 1\}$, we can apply Theorem 1.20 so that there exists a unique primitive embedding

$$T(S) \hookrightarrow (U \oplus A_2 \oplus E_6(-1)^3)'' \simeq \Lambda_{K3}.$$

We assume it is embedded onto the first two components, then

$$(U \oplus A_2)^{\perp} = (E_6(-1)^3)' \subset NS(S) = T(S)^{\perp}.$$

4) $\to$ 1): Thanks to Theorem 4.24 we know that there exists a symplectic automorphism σ_S on S such that, denoting the minimal resolution of the quotient as X, $T(X) \simeq \gamma(T(S))$ and there is a primitive embedding

$$(M_{\mathbb{Z}/3\mathbb{Z}} \oplus E_6(-1))' \hookrightarrow NS(X).$$

Now, by Theorem 1.14 we know that $NS(X)$ is uniquely determined by its signature and discriminant form. As in the proof of Theorem 4.12, we observe that the lattices $K_{\mathbb{Z}/3\mathbb{Z}}$ and $(M_{\mathbb{Z}/3\mathbb{Z}} \oplus E_6(-1))'$ have the same rank and discriminant form, therefore we conclude using Lemma 1.16: the primitive embedding $(M_{\mathbb{Z}/3\mathbb{Z}} \oplus E_6(-1))' \hookrightarrow NS(X)$ determines a primitive embedding $K_{\mathbb{Z}/3\mathbb{Z}} \hookrightarrow NS(X)$. Then X is isomorphic to $Km_3(A)$ for an Abelian surface A. We need to prove the isometry between transcendental lattices. Note that $T(Km_3(A)) = (\pi_A)_*(T(A))$. We refer the reader to [1] for the description of the map $(\pi_A)_*$, which for our sake is sufficient to sum up as follows: the action of $(\pi_A)_*$ on $U \oplus A_2$ is equal to the action of

γ. This gives $T(Km_3(A)) = \gamma(T(A))$.
We already proved that $T(Km_3(A)) \simeq T(X) \simeq \gamma(T(S))$, so now $\gamma(T(S)) \simeq \gamma(T(A))$. It follows that $T(A)$ is embedded in $U \oplus A_2$ as $T(X)$, which yields $\gamma(T(A)) = \gamma(T(X))$. □

The isometry between $T(S)$ and $T(A)$ still holds, as in the case of Shioda-Inose structures of order 2. One might think that a similar relation exists between $T(S)$ and $T(Km_3(A))$ or between $T(A)$ and $T(Km_3(A))$. However, this is absolutely false, as examples have been provided showing very different behaviours of the lattice $T(Km_3(A))$. See [5]. For this reason, the definition of Generalized Shioda-Inose structure had to be modified by taking into consideration that the isometry holds only between the $K3$ covering surface and the Abelian surface A. Furthermore, in definition 4.15 we also removed this latter condition, and this is due to the work of B. Piroddi [28], who found that any relation of this type between the transcendental lattices would restrict too much the possibilities for admissible Generalized Shioda-Inose structures of order 4.

4.2.2 The order 4 case

In order to generalize Shioda-Inose structures to automorphisms of order 4, one needs to understand the behavior of these automorphisms on the $K3$ lattice. This work was conducted by Benedetta Piroddi in her PhD thesis, and we will present the main results of her research.

Remark 4.26. From Theorem 2.37 we know that Ω_4 is a rank 14 lattice. We will consider Π, the overlattice of $D_4^4 \oplus \langle -4 \rangle^2$ obtained this way: set basis e_i, f_i, g_i, h_i for the four copies of D_4, and denote as $a_1 - a_2$ and σ the generators of the two copies of $\langle -4 \rangle$. Then the overlattice Π is obtained by adding the following gluing vectors:

$$\zeta_1 = (\sigma + e_1 - g_1 + e_2 - f_2 + f_4 - g_4)/2,$$
$$\zeta_2 = (e_1 - g_1 + f_1 - h_1 + e_2 - g_2 + f_4 - h_4)/2,$$
$$\zeta_3 = (\sigma + f_1 - h_1 + e_2 - h_2 + f_4 - e_4)/2,$$
$$\zeta_4 = (e_2 - g_2 + e_4 - g_4 + a_1 - a_2 + \sigma)/2.$$

Theorem 4.27 ([28, Theorem 4.2.4.11]). *Let S be a projective $K3$ surface. Then S admits an order 4 automorphism such that the induced action on $H^2(S, \mathbb{Z})$ permutes cyclically the four copies of D_4 if and only if there is a primitive embedding $\Pi \hookrightarrow NS(S)$.*

In order to describe the lattice $M_{\mathbb{Z}/4\mathbb{Z}}$ we want to understand the action of a symplectic automorphism σ_S of order 4.

Remark 4.28. Let σ_S be a symplectic automorphism of order 4 on a $K3$ surface S. We know from Nikulin's work [25, Chapter 5, case 2] that there are exactly 4 points fixed by σ_S and 4 points fixed by σ_S^2. This means that the quotient S/σ_S will have 6 fixed points, four of which are singularities of type A_3, and two of which are A_1 singular points. The resulting lattice $M_{\mathbb{Z}/4\mathbb{Z}}$ can be described as the overlattice of $A_3^4 \oplus A_1^2$ obtained by adding the vector

$$v := \frac{1}{4}\sum_{i=1}^{4}\left(m_1^i + 2m_2^i + 3m_3^i\right) + \frac{1}{2}\left(\tilde{m}^1 + \tilde{m}^2\right),$$

where $\{m_1^i, m_2^i, m_3^i\}$ stands for a basis of the i-th copy of A_3, and $\tilde{m}^j$ is a generator of the j-th copy of A_1.

Definition 4.29. Using the notation of Remark 4.28, we define $(M_{\mathbb{Z}/4\mathbb{Z}} \oplus D_4)'$ as the overlattice of $M_{\mathbb{Z}/4\mathbb{Z}} \oplus D_4$ obtained by adding the vector

$$w := \frac{m_1^2 + m_3^2 + m_1^3 + m_3^3 + \tilde{m}^1 + \tilde{m}^2 + e_2 + e_4}{2},$$

where $\{e_1, e_2, e_3, e_4\}$ is a standard basis of D_4 which gives the matrix associated to the bilinear form as in example 1.4.

The following is the analogue of Theorems 4.24 and 4.9.

Theorem 4.30 ([28, Theorem 4.2.4.12])**.** *Let S be a $K3$ surface and suppose that $\Pi \hookrightarrow NS(S)$ is a primitive embedding. Then there exists a symplectic automorphism σ_S of S of order 4 such that, if X is the minimal resolution of S/σ_S and $\pi : S \dashrightarrow X$ is the rational quotient map, the following holds:*

1) *There is a primitive embedding $(M_{\mathbb{Z}/4\mathbb{Z}} \oplus D_4)' \hookrightarrow NS(X)$;*
2) *There exists an abelian surface A admitting a symplectic automorphism σ_A of order 4 such that $X \simeq Km_4(A)$.*

The approach followed by B. Piroddi, in this case, is to preserve condition 1) of Theorem 4.12 and give up on condition 2). In fact, the existence of an isometry $T(S) \simeq T(A)$ turns out to be rare in the general case, and we refer the reader to [28, Theorem 4.2.4.14] for an explicit treatment. Piroddi found that there can be up to two different families of $K3$ surfaces to which an abelian surface A, admitting a symplectic automorphism of order 4, can be associated via Generalized Shioda-Inose structures of order 4. In other words, Generalized Shioda-Inose structures of order 4 no longer have the interesting property described in Remark 4.14.

We conclude this subsection with the analogue of 4.12. We omit the proofs of these Theorems, and we refer the reader to the work of B. Piroddi [28, Chapter 4].

Theorem 4.31. *Let S be a projective $K3$ surface. Then the following are equivalent:*

1) *S admits a Generalized Shioda-Inose structure of order 4.*

2) *There exists an Abelian surface A with a symplectic automorphism σ_A of order 4 and a symplectic automorphism σ_S of S of order 4 such that $Km_4(A)$ is isomorphic to the minimal resolution of S/σ_S, and there is a correspondence between the projective families of A and S.*

3) *There is a primitive embedding $\Pi \hookrightarrow NS(S)$.*

4.2.3 The order 6 case

Remark 4.32. If S is a $K3$ surface admitting a Generalized Shioda Inose structure of order 3 $(S, \sigma_S, A, \sigma_A)$, then X also admits a Shioda-Inose structure of order 2 (S, i_S, B). Indeed, $T(S) \simeq T(A)$ for an abelian surface A, therefore $T(S) \hookrightarrow H^2(A, \mathbb{Z}) \simeq U^3$ is a primitive embedding and Theorem 4.12 applies. However, nothing assures that $A \simeq B$.

If S is a $K3$ surface admitting a Generalized Shioda-Inose structure of order 3 $(S, \sigma_S, A, \sigma_A)$, then one could compose σ_A with the natural symplectic involution $i_A = -\,\mathrm{id}_A$ and obtain a symplectic automorphism of order 6. The situation is as depicted in the diagram:

$$\begin{tikzcd}
S \arrow[dr, dashed] & & A \arrow[dl, dashed] \arrow[ddl, dashed] \\
& Km_3(A) \arrow[d, dashed] & \\
& Km_6(A) &
\end{tikzcd}$$

At this point one might ask whether S admits a symplectic automorphism φ_S of order 6 such that the minimal resolution of S/φ_S is isomorphic to $Km_6(A)$, so that $(S, \varphi_S, A, \sigma_A \circ i_A)$ is a Generalized Shioda-Inose structure of order 6. The following proposition gives a negative answer:

Proposition 4.33. *Let A be an Abelian surface such that $T(A) \simeq U \oplus \langle 2 \rangle$. Then there exists $Km_6(A)$ but A is not part of a Generalized Shioda-Inose structure of order 6 $(S, \varphi_S, A, \varphi_A)$ such that $T(A) \simeq T(S)$.*

Proof. This is due to the fact that any primitive embedding $T(A) \simeq U \oplus \langle 2 \rangle \hookrightarrow U \oplus A_2$ provides the existence of a symplectic automorphism $\varphi_A = \sigma_A \circ i_A$ of order 6, by Theorem 3.30. This proves that $Km_6(A)$ is well-defined. Furthermore, any $K3$ surface S with an automorphism of order 6 admits a primitive embedding

$T(S) \hookrightarrow U \oplus U(6)^2$ (see Theorem 2.37). If we suppose that $T(S) \simeq T(A)$ then, denoting as N the orthogonal of $T(S)$ inside $U \oplus U(6)^2$, we get

$$A_{N \oplus T(S)} = A_N \oplus A_{T(S)}$$

and we know that $A_{T(S)} = \mathbb{Z}/2\mathbb{Z}$ and that $A_{U \oplus U(6)^2} = (\mathbb{Z}/6\mathbb{Z})^4$ (see Theorem 3.30). Since N is a rank 3 lattice, this leads to a contradiction. $\square$

This suggests that the existence of an isometry between the transcendental lattices of S and A is something unique to Shioda Inose structures of order 2 and 3. Any Generalization to the order 6 forbids this kind of relation. This is the reason why definition 4.15 seems like the natural choice for the general definition of Shioda Inose structure. Please note that the very existence of Shioda-Inose structures of order 6 is still an open question.

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