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Stabilisation of the Angular Directions in Brane-Antibrane Inflation

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Abstract

Brane-antibrane inflation is an exciting framework in string cosmology that aims at connecting fundamental physics with the accelerated expansion of the early universe. However, achieving a successful and viable inflationary phase requires stabilization of all moduli fields, including the angular directions arising in extra-dimensional compactifications. This thesis explores how the angular directions can be stabilized in the context of Type IIB string theory, focusing on warped compactifications and the effects of perturbative and non-perturbative corrections. We examine how flux compactifications and Kähler moduli stabilization contribute to a controlled inflationary scenario and analyze their impact on slow-roll inflation. Our findings contribute to the understanding of moduli stabilization in string cosmology, ensuring a consistent and phenomenologically viable realization of brane-antibrane inflation.

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Lastly, I am grateful for the chance to contribute to the field of string cosmology and for the work of researchers before me, whose studies have inspired and guided this thesis. This journey has been challenging but rewarding, and I deeply appreciate everyone who has been a part of it.

Contents

Chapter 1

Introduction

1.1 Background

Inflation is one of the most important concepts in modern cosmology, explaining how the universe expanded exponentially in its earliest moments. It helps solve key problems in the standard Big Bang model, such as why the universe looks so uniform (horizon problem) and why space appears nearly flat (flatness problem).

Although many models explain inflation using quantum field theory, embedding it within string theory introduces additional challenges. String theory suggests that our universe has extra spatial dimensions, which give rise to many new fields called moduli. These moduli affect how inflation happens, and if they are not properly stabilized, they can cause the inflationary scenario to fail.

In these models, inflation is driven by the motion of branes (higher-dimensional objects) within a compactified space. However, for inflation to work smoothly, the angular directions need to be stabilized properly. This thesis explores how to stabilize these directions within Type IIB string theory, focusing on warped compactifications and the effects of quantum corrections.

1.2 Objectives

This thesis aims to:

- Understand the stabilization of the angular directions in brane-antibrane inflation, so removing unnecessary flat directions in the potential, and understanding how to prevent slow-roll issues and to ensure a controlled evolution driven by non-perturbative effects.

- Explore different ways to stabilize the moduli using techniques from Type IIB string theory.
- Analyze the interplay between perturbative and non-perturbative effects for moduli stabilization.
- Identify conditions that allow for stable slow-roll inflation, making the model consistent with observations.
- Discuss the broader implications of these findings for string cosmology and future research.

1.3 Outline of the Thesis

This thesis is divided into several chapters, each focusing on different aspects:
Chapter 2: Standard Inflationary Cosmology – Introduces basic problems in cosmology and how inflation solves them. It also covers different inflation models and their connection to fundamental physics. **Chapter 3: String Inflation** – Explains why inflation needs to be embedded in string theory. It covers various models, including brane inflation, and discusses the challenges of moduli stabilization. **Chapter 4: Moduli and their Stabilization** – Introduces moduli fields, explaining why they need to be stabilized, and presents different stabilization mechanisms used in string theory. **Chapter 5: The Brane World** – Discusses the physics of branes, their interactions, and how they lead to brane-antibrane inflation models. **Chapter 6: Angular Moduli Stabilization and Inflation** – The core of the thesis, this chapter explores different ways to stabilize angular directions in brane inflation models and examines their effects on inflation.
Finally, we discuss and summarize of our findings, highlight the main conclusions and possible directions for further research.

Chapter 2

Standard Inflationary Cosmology

2.1 An Expanding Universe

On larger (cosmological) scales, the distribution of matter and radiation in the Universe exhibits homogeneity, isotropy, and a dynamical state of expansion. This fundamental assumption dictates that the metric to adopt is of the well-known Friedmann-Robertson-Walker(FRW) form, which serves as the mathematical foundation for modeling the universe large-scale structure and dynamics:

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \right] \tag{2.1}$$

where $a(t)$ is the scale factor which measures the evolution of the Universe in cosmological time, and $k = -1, 0, +1$ for open, flat and closed universes that determines the curvature of the spatial sections at fixed t.
The evolution of the geometry of space-time is controlled by the distribution of energy-momentum within it, as stated by the Einstein equation of General Relativity:

$$G_{\mu\nu} = \frac{1}{M_p^2} T_{\mu\nu} \tag{2.2}$$

where $G_{\mu\nu}$ and $T_{\mu\nu}$ are the Einstein and energy-momentum tensors and $M_p \simeq 2.44 \times 10^{18}$ GeV is the reduced Planck mass.
From injecting the FLRW metric in Eq. (2.1) in the Einstein equation in Eq. (2.2), we find two Friedman equations:

$$H^2 + \frac{k}{a^2} = \frac{\rho}{3M_p^2}, \qquad H = \frac{\dot{a}}{a} \tag{2.3}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_p^2} (\rho + 3\rho) \tag{2.4}$$

Here, the reduced Planck mass M_p is related to the Newton gravitation constant G_N through $M_p^2 \equiv \frac{1}{8\pi G_N}$ and H is the Hubble constant. Eq. (2.4) can be traded for the derived energy conservation equation:

$$\dot{\rho} = -3H(\rho + p) = -3(1+w)H\rho \tag{2.5}$$

With a simple calculation we can conclude the following:

- For a matter dominated universe, we have $\rho_{matter} \propto a^{-3}$
- For a radiation dominated universe, we have $\rho_{radition} \propto a^{-4}$
- For a dark energy dominated universe, $\rho_{vacuum} \propto a^0$

It encodes how each fluid evolves in an expanding universe:

$$\rho_i \propto a^{-3(w_i+1)} \quad , \quad w_i = \frac{p_i}{\rho_i} \tag{2.6}$$

where w_i is the equation of state parameter. For example: $w_i = -1, \frac{1}{3}, 0, -\frac{1}{3}, -1$ for kination, radiation, matter, curvature and vacuum energy, respectively.
The value of k can be determined experimentally by measuring the parameter Ω, defined as the ratio of the energy density of our Universe and a critical density:

$$\Omega \equiv \frac{\rho}{\rho_c} \quad \text{with} \quad \rho_c \equiv \frac{3H^2}{8\pi G_N} \tag{2.7}$$

With this definition the Friedmann Equation is rewritten as:

$$\Omega = 1 + \frac{k}{H^2 a^2} \tag{2.8}$$

with a clear connection between the curvature of the spatial sections and the departure from critical density. Then a flat Universe ($k = 0$) corresponds to $\Omega = 1$, whereas an open ($k = -1$) and closed ($k = 1$) one corresponds to $\Omega < 1$ and $\Omega > 1$, respectively. In the case of multiple contributions to the energy density of the universe, we will have $\Omega = \Sigma_i \Omega_i$.
The Cosmic Microwave Background (CMB) is an extraordinary powerful tool for testing and refining our theoretical models of the early universe. Through CMB observations, we have learned that the universe was in its early stages, remarkably homogeneous, with only small, scale-invariant, Gaussian, and adiabatic primordial temperature fluctuations. These findings align perfectly with the simplest inflationary model.
Before exploring the concept of cosmic inflation, we will first address some of the key issues in Big Bang cosmology. These include the flatness problem, the horizon problem, the baryon asymmetry problem, the issue of spacetime singularity, and the problem of topological defects. However, for the purpose of explaining the inflationary model, we will solely focus on the Flatness problem and the Horizontal problem.

2.1.1 Flatness Problem

From Eq. (2.8) and using $\rho \propto a^{-3(w+1)}$, we can rewrite:

$$\frac{\Omega(t)-1}{\Omega} \propto \frac{1}{\rho a^2} \propto a^{1+3w} . \tag{2.9}$$

Given that in the standard Big Bang picture the evolution of the universe is characterized by either matter ($w = 0$) or radiation ($w = 1/3$) dominance, $(\Omega(t) - 1)/\Omega$ increases from the initial Big Bang singularity to today. Hence the present observed value, $\Omega_{\text{today}} \simeq 1$, can be obtained only by tuning the initial conditions extremely close to $\Omega = 1$. This is the Flatness problem of standard Big Bang cosmology.

2.1.2 Horizon Problem

In order to compute how much of the universe is in causal contact, we define the comoving particle horizon as:

$$\tau = \int_0^t \frac{dt'}{a(t')} = \int_0^a \frac{da}{ah^2} = \int_0^a d\ln\left(\frac{1}{aH}\right) \tag{2.10}$$

The parameter τ represents the maximum possible distance that light could have traveled between an initial time (set to $t = 0$) and a later time t. If the distance between two regions exceeds τ, these regions could not have communicated with each other. Additionally, the comoving Hubble radius, defined as $(aH)^{-1}$, serves as the measure of the maximum distance over which particles can interact, accounting for the expansion of the universe. It is important to note that if the separation between two particles is greater that $(aH)^{-1}$, it means that they are not in causal contact today. However, they could have been in causal contact in the past, particularly during the era of re-ionization. If we now parametrize the evolution of the universe by an equation of state,

$$(aH)^{-1} = H_0^{-1} a^{\frac{1}{2}(1+3w)} \tag{2.11}$$

and substituting (2.11) in (2.10), we find,

$$\tau \propto a^{\frac{1}{2}(1+3w)} \tag{2.12}$$

Eq (2.12) indicates that the comoving horizon increases monotonically over time. As a result, regions that are coming into causal contact today were not in causal contact in the past. The CMB reveals that the temperature of the universe is remarkably uniform across vast regions of the sky, with only tiny fluctuation.

However, this creates a puzzling problem. If the fluctuations in the CMB are so strongly correlated, then the regions responsible for these fluctuations must have been in causal contact in the past. However, if we trace back the history of two distant points, there was not enough time for them to communicate causally. Despite this, these regions exhibit the same temperature which would not be possible without causal contact. This discrepancy is called the Horizon Problem.

2.2 Inflation

The concept of an inflationary universe has been introduced (see [1] for a review from the point of view of string/brane cosmology) as a potential solution for several longstanding issues in cosmology, including the flatness and horizon problems. Subsequently, it was later understood that inflation could also explain the origin of CMB anisotropies and, consequently, the formation of the large-scale structures in the universe. The core idea of the inflation is that, during the early universe, there existed a brief period of accelerated expansion, often exponential in nature. If the inflationary phase lasted long enough, it would:

- Quickly flatten the universe, addressing the flatness problem.
- Allow regions that are currently causally disconnected to have been in causal contact in the past, resolving the horizon problem.

The simplest inflationary model is realized within an effective field theory below M_p. It consists of a scalar field ϕ with a potential $V(\phi)$, whose value provides an effective cosmological constant corresponding to the case $w \simeq -1$, which causes the scale factor $a(t)$ to increase exponentially.

2.2.1 Slow-roll Inflation

Starobinsky [2], and Mukhanov and Chibisov [3] showed that quantum fluctuations produced during slow-roll generate a spectrum of inhomogeneities possibly accounting for the large scale structures in our universe. The simplest possible inflationary model is that of a single scalar field, whose action is given by (setting $M_p = 1$):

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] \tag{2.13}$$

We find the stress-energy tensor:

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_\phi}{\delta g^{\mu\nu}} \tag{2.14}$$

In a Friedmann-Robertson-Walker metric, the energy density and pressure are given by:

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{2.15}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{2.16}$$

From energy conservation we get:

$$\dot{\rho} = -3H(\rho + p) \tag{2.17}$$

$$\dot{\phi}\ddot{\phi} + V'(\phi)\dot{\phi} = -3H\dot{\phi}^2 \tag{2.18}$$

$$\ddot{\phi} + V'(\phi) + 3H\dot{\phi} = 0 \tag{2.19}$$

During inflation the Hubble constant reads (reinstating factors of G_N):

$$H = \sqrt{\frac{8\pi G_N \rho}{3}} = \sqrt{\frac{8\pi G_N}{3}\Big(\frac{1}{2}\dot{\phi}^2 + V(\phi)\Big)} \tag{2.20}$$

Hence, the time derivative of H^2 leads to:

$$2H\dot{H} = \frac{8\pi G_N}{3}\Big(\dot{\phi}\ddot{\phi} + V'(\phi)\dot{\phi}\Big) = -8\pi G_N H\dot{\phi}^2 \tag{2.21}$$

The condition for exponential expansion is $\Big|\dot{H}\Big| \ll H^2$ so,

$$\Big|\dot{H}\Big| \ll H^2 = \frac{2}{3M_p^2}\Big(\frac{1}{2}\dot{\phi}^2 + V(\phi)\Big) \tag{2.22}$$

where $V(\phi)$ is the scalar field potential and $V' \equiv \frac{dV}{d\phi}$. The right side of (2.20) represents the energy density associated with the scalar field ϕ. When the potential V dominates over the kinetic energy, and $V \sim \Lambda > 0$, this corresponds to the equation of state parameter $w = -1$, leading to an exponential expansion of the scale factor, expressed as $a \sim e^{Ht} \sim \exp\sqrt{\frac{\Lambda}{3}}$ in Planck mass units with $k = 0$, with similar expressions for the other values for k. The key-point here is that this exponential growth of the scale factor resolves the flatness and horizon problems that we have mentioned before. From Eq. (2.19) and Eq. (2.20), we can find the evolution of the Hubble parameter,

$$\dot{H} = -\frac{1}{2}\frac{\dot{\phi}^2}{M_p^2} \tag{2.23}$$

Then, from Eq. (2.20) and (2.23), we get the slow-roll ϵ parameter:

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{1}{2}\frac{\dot{\phi}^2}{M_p^2 H^2} = \frac{\frac{3}{2}\dot{\phi}^2}{\frac{1}{2}\dot{\phi}^2 + V} \tag{2.24}$$

Inflation occurs as long as the potential energy dominates over the kinetic energy, so that $\epsilon < 1$. Inflation will continue as long as the acceleration of the field is small. This is parametrized through a second slow-roll parameter:

$$\eta \equiv \frac{\ddot{\phi}}{H\dot{\phi}} \tag{2.25}$$

When ϵ and η are much smaller than 1, we can apply the slow-roll approximation. The condition $\epsilon \ll 1$ indicates that the kinetic energy is negligible, allowing us to simplify Eq. (2.20) accordingly:

$$H^2 \approx \frac{V}{3M_p^2} \tag{2.26}$$

So, during the slow-roll phase, the Hubble rate can be approximated by the nearly constant potential energy. The second parameter simplifies the Klein-Gordon equation in Eq. (2.19) to:

$$3H\dot{\phi} \approx -V' \tag{2.27}$$

Thus, there is a direct relationship between the slope of the potential and the velocity of the inflation, with the Hubble parameter remaining approximately constant. Consequently, the slow-roll parameters can be expressed in an approximate form that depends exclusively on the functional form of the potential,

$$\epsilon_V = \frac{M_p^2}{2}\left(\frac{V'}{V}\right)^2 \tag{2.28}$$

$$\eta_V = M_p^2 \frac{V''}{V} \tag{2.29}$$

Inflation will end when $\epsilon_V = 1$ or $\eta_V = 1$.
If the potential is constant, the universe will undergo a de Sitter expansion, and the amount of this expansion would be given by the size of H (since $a(t) \sim e^{Ht}$). More generally, the total number of e-foldings is expressed as:

$$N(t) = \int_{t_{in}}^{t_{end}} H(t')\,dt' = \int_{\phi_{end}}^{\phi_{in}} \frac{H}{\dot{\phi}}\,d\phi = \frac{1}{M_p^2}\int_{\phi_{end}}^{\phi_{in}} \frac{V}{V'}d\phi \tag{2.30}$$

For inflation to successfully resolve the horizon problem, it requires at least $N \geq 60$ e-foldings. A successful inflation model needs a scalar field potential V that meets the slow-roll condition and produces more than about 60 e-foldings (slightly smaller values are sometimes allowed depending on the scale at which inflation occurs).

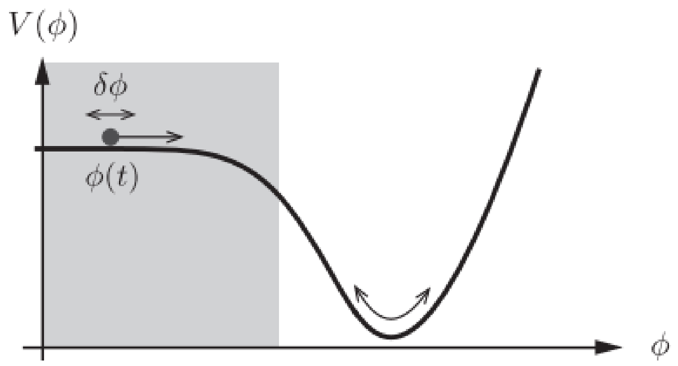

Figure 2.1: Example of a slow-roll potential. Inflation occurs in the shaded part of the potential. In addition to the homogeneous evolution $\phi(t)$, the inflaton experiences spatially varying quantum fluctuations $\delta\phi(t, x)$ [4].

2.3 Effective Theories of Inflation

In case where the UV theory is unknown, it is possible to explicitly derive the Effective Field Theory (EFT) by integrating out the heavy modes. Instead, our lack of knowledge about the UV physics is addressed by postulating the symmetries of the UV theory and formulating the most general effective action consistent with those symmetries:

$$\mathcal{L}_{eff}[\phi] = \mathcal{L}_l[\phi] + \sum_i c_i \frac{\mathcal{O}_i[\phi]}{\Lambda^{\delta_i - 4}} \tag{2.31}$$

The sum includes all operators $\mathcal{O}_i[\phi]$, with dimensions δ_i, that are consistent with the symmetries of the UV theory. The contribution of the higher dimensional operators are estimated in terms of the cutoff scale Λ, while the coefficient c_ia re the dimensionless Wilson coefficients. Eq (2.31) serves as the basis for analyzing inflation within the framework of EFT.

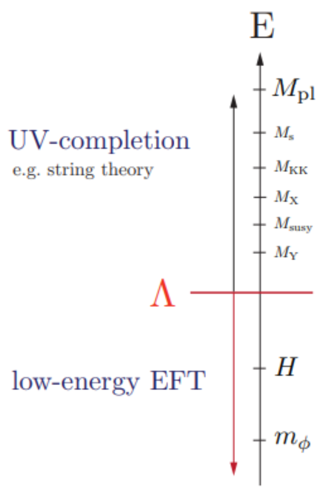

Figure 2.2: The EFT of Inflation. The cut-off Λ of the EFT is defined by the mass of the lightest particle that is not included in the spectrum of the low-energy theory. Particles with masses above the cut-off are integrated out, correcting the Lagrangian for the light fields such as the inflaton [4].

2.3.1 Ultraviolet Completion

The sizes of Wilson coefficients in the effective Lagrangian are influenced by the symmetries of the UV theory. If the symmetry is weakly broken or the light fields couple weakly to symmetry-breaking terms, the EFT exhibits an approximate symmetry, leading to small Wilson coefficients for symmetry-breaking parameter g, and their smallness reflects the approximate UV symmetry. Since the symmetry is restored in the limit $g \to 0$, the Wilson coefficients of all symmetry breaking operators must satisfy:

$$\lim_{g\to 0} c_i(g) = 0.$$

The naturalness of the low-energy theory depends on the symmetries assumed in the UV completion, with guidance from UV-complete theories like string theory offering additional insights.

2.3.2 Inflation in Effective Field Theory

Inflation, as a phenomenon arising from Quantum Field Theory coupled with General Relativity, is not inherently natural. The Lagrangians suitable for inflation represent only a tiny fraction of all possible Lagrangians, and in many models, inflation requires special initial conditions such as low kinetic energy in small-field scenarios. Here we will focus on the naturalness of Lagrangians suitable for inflation.

The starting point is the EFT Lagrangian (2.31) minimally coupled to gravity:

$$S_{eff}[\phi] = \int d^4x \sqrt{-g} \left[\frac{M_p^2}{2} R + \mathcal{L}_l[\phi] + \sum_i c_i \frac{\mathcal{O}_i[\phi]}{\Lambda^{\delta_i - 4}} \right] \tag{2.32}$$

where $\mathcal{L}_l[\phi]$ includes the canonical kinetic term $-\frac{1}{2}(\partial\phi)^2$ as well as any renormalizable interactions. As we explained above, the sum over non-renormalizable terms parameterizes the effects of massive fields on the EFT of the light fields. As usual, the effects of high-scale physics above some cutoff Λ are efficiently described by the coefficients of operators in the low-energy effective theory (see Fig. 2.2). Integrating out particles of mass $M \geq \Lambda$ gives rise to operators of the form:

$$\boxed{\frac{\mathcal{O}_\delta}{M^{\delta-4}}}$$

where δ denotes the mass dimension of the operator. In inflation, however, the flatness of the potential in Planck units introduces sensitivity to $\delta \leq 6$ Planck-suppressed operators, such as:

$$\frac{\mathcal{O}_6}{M_p^2}$$

An understanding of such operators is required to address the smallness of the η-parameter, i.e. to ensure that the theory supports at least about 60 e-foldings of inflationary expansion. This sensitivity to dimension-six Planck-suppressed operators is therefore common to all models of inflation.

2.3.3 The Eta Problem

We have seen that quantum corrections tend to drive scalar masses to the cutoff scale, unless the fields are protected by symmetries. In the case of inflation, this implies the following quantum correction to the inflaton mass:

$$\Delta m^2 \sim \Lambda^2.$$

Since consistency of the EFT treatment requires that $\Lambda > H$, we find a large renormalization of the inflationary η-parameter:

$$\Delta\eta \sim \frac{\Lambda^2}{H^2} \geq 1$$

and sustained slow-roll inflation appears to be unnatural. This difficulty is known as the η-problem.
The η-problem in inflation can be addressed using two strategies: fine tuning the potential or invoking symmetries. However the issue remains challenging because symmetry-based approaches, such as supersymmetry or global internal symmetries, have significant limitations and have only achieved partial success. Supersymmetry only mitigates the problem but cannot resolve it fully, while global symmetries require precise control over Planck-suppressed symmetry-breaking operators, necessitating a quantum gravity framework.

In the context of String Inflation, string theory provides tools to manage Planck-scale corrections and mechanisms like extra-dimensional dynamics or brane configurations to construct UV-complete inflationary models, addressing the η-problem by incorporating natural symmetries and constraints.

Chapter 3

String Inflation

3.1 Motivations for String Inflation

Ensuring a consistent embedding of inflationary models within string theory is essential for several reasons, some of which we outline below following [5].

String inflation is motivated by the need to fit inflationary models within string theory in a way that makes them stable and consistent. One major challenge is that inflation is highly sensitive to ultra-violet (UV) physics, meaning that without a proper UV framework, fine-tuning becomes a problem. Some inflation models, especially those with large tensor-to-scalar ratios, require field values that go beyond the Planck scale, raising deep questions about their validity — questions that string theory is well-equipped to address. Additionally, string theory places strong constraints on the shape of the inflationary potential, especially in setups like Calabi-Yau compactifications with stabilized moduli. Beyond theoretical concerns, string inflation provides a way to test high-scale physics, as the inflationary energy scale is close to the Planck scale, making it possible to extract observational clues about fundamental physics. Finally, understanding how inflation starts and how it ends (through reheating) is crucial, and string theory helps to ensure that the energy from inflation is correctly transferred to known particles without excessive loss to hidden sectors. Together, these factors make string theory an important framework for inflationary cosmology.

3.1.1 Requirements for String Inflation

Let us briefly discuss what are the main conditions that a perfect working model of string inflation should satisfy [5].

For string inflation to work properly, several key conditions have to be met. Moduli stabilization is crucial because, in string theory, extra dimensions come with fields that determine their shapes and sizes. If these are not fixed correctly, they can disrupt inflation or cause unwanted instabilities. A good stabilization mechanism also helps to guide how the universe evolves after inflation, leading to a stable vacuum or a slowly changing dark energy phase. Another important factor is getting the mass scales right — the inflaton mass needs to be below the Hubble scale during inflation but still within a certain range to avoid interference from heavier modes like Kaluza-Klein states ($m < H_{\text{inf}} < M_{\text{KK}} < M_s < M_p$).

Computational control is also necessary since string theory involves quantum effects which are hard to compute, and ensuring the validity of the effective field theory means keeping control over the various perturbative and non-perturbative expansions. Additionally, effects like fluxes and instantons need to be handled carefully to avoid unexpected instabilities. Finally, a proper string embedding, often in a Calabi-Yau compactification, is essential for constructing a realistic model that aligns with known particle physics. A well-structured extra-dimensional space should naturally lead to a Standard Model-like setup and allow a smooth transition from inflation to the physics we observe today. In short, for string inflation to be successful, all these elements must work together to ensure stability, consistency, and a clear connection to real-world physics.

3.1.2 Models of String Inflation

String inflation models explore how the accelerated expansion of the early universe can be explained within string theory. Since string theory naturally includes extra dimensions and additional fields (moduli), the challenge is to find the right setup where inflation can happen smoothly, without leading to unwanted instabilities [6, 7]. Depending on where the inflaton field comes from, these models can be divided into **open string inflation** and **closed string inflation** [5].
In open string models, the inflaton is linked to a scalar field originating from open strings attached to D-branes. Essentially, it describes how a D-brane moves through the compact extra dimensions. In M-theory, a similar mechanism occurs with M5-branes instead of D-branes, producing comparable effects. Well-known examples of open string inflation models include D-brane inflation, slow-roll brane/anti-brane inflation, DBI warped D-brane inflation, and M-theory inflation [8, 9, 10, 11, 12].

In closed string models, the inflaton originates from a closed string modulus. These models are promising due to their well-defined background, which allows certain moduli to have nearly flat potentials. Examples of closed string inflation models include Kähler moduli inflation, axion inflation, Wilson line moduli inflation, and volume modulus inflation [13, 14, 15, 16, 17].

Each inflation model predicts specific values for inflationary observables, represented as distinct in the (r, n_s)-plane. As observations become more precise, many models may soon be ruled out, making an important step towards testing string theory experimentally.

3.2 Different String Theories

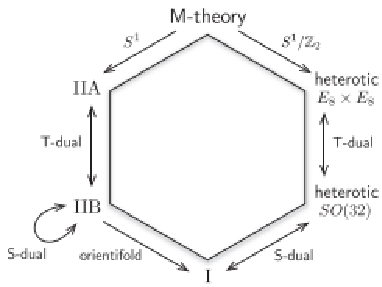

Figure 3.1: Different String Theories and Dualities [18].

There exist five distinct yet consistent string theories: Type I, Type IIA, Type IIB, Heterotic $SO(32)$, and Heterotic $E_8 \times E_8$. Theses theories are interconnected through various dualities, as illustrated in Figure 3.1. At low energy, each string theory, along with M-theory, is described by a supergravity theory, which provides useful insights into quantum gravity. We will explore the spectrum of all five string theories, with a particular focus on Type IIB string theory. This theory is the primary interest of this thesis due to its significant phenomenological implications.

3.2.1 Type IIA

A low energy limit of Type IIA string theory is described by Type IIA supergravity. Here we focus only on its bosonic sector. In superstring theory, fields arise from two sectors based on boundary conditions in the worldsheet. A periodic boundary condition for worldsheet fermions leads to Ramond-Ramond (RR) sector fields, while an anti-periodic boundary condition leads to Neveu-Schwarz (NS-NS) sector fields.

NS-NS Sector: $G_{\mu\nu}, B^{(2)}_{\mu\nu}, \Phi$:
In the NS-NS sector, we have the metric $G_{\mu\nu}$, the anti-symmetric two-form field which is sometimes called the Kalb-Ramond field $B^{(2)}_{\mu\nu}$ and the dilaton Φ.

RR Sector: $C^1_\mu, C^3_{\mu\nu\rho}$:
In the RR sector, we have a one-form field and a three-form field C^1_μ and

$C^3_{\mu\nu\rho}$ respectively. Now, we can write the field strength of these form fields as follows:

$$\begin{aligned} &F^{(p)} = dC^{(p-1)} \\ &\tilde{F}^{(2)} = F^{(2)} \qquad\qquad\qquad ;\text{for } p = 2,4 \quad ; H^{(3)} = dB^{(2)} \\ &\tilde{F}^{(4)} = F^{(4)} - C^{(1)} \wedge H^{(3)} \end{aligned} \tag{3.1}$$

This will help us write the Type IIA action. Let us separate out the NS-NS part and RR part. The NS-NS part is:

$$S_{\text{NS-NS}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x\sqrt{-G}e^{-2\Phi}\left(R_G + 4G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}\frac{1}{3!}G^{\mu\mu'}G^{\nu\nu'}G^{\rho\rho'}H^{(3)}_{\mu\nu\rho}H^{(3)}_{\mu'\nu'\rho'}\right) \tag{3.2}$$

and R-R part is:

$$S_{R-R} = -\frac{1}{2\kappa_{10}^2} \int d^{10}x\sqrt{-G}\left[\frac{1}{4}F^{(2)}_{\mu\nu}F^{(2)}_{\mu'\nu'}G^{\mu\mu'}G^{\nu\nu'} + \frac{1}{2}\frac{1}{4!}\tilde{F}^{(4)}_{\mu'\nu'\rho'\sigma'}\tilde{F}^{(4)}_{\mu\nu\rho\sigma}G^{\mu\mu'}G^{\nu\nu'}G^{\rho\rho'}G^{\sigma\sigma'}\right] \tag{3.3}$$

In addition to the two pieces above, there is a third piece which we call the Chern-Simons term which can be written as:

$$S_{\text{CS}} = -\frac{1}{4\kappa_{10}^2}\int B^{(2)} \wedge F^{(4)} \wedge F^{(4)} \tag{3.4}$$

3.2.2 Type IIB

Type IIB supergravity is the low energy limit of Type IIB string theory. The model discussed and constructed in this thesis is based on the Type IIB framework. Similar to Type IIA, we focus on the bosonic sector and examine the field contents arising from the Ramond-Ramond (RR) and Neveu-Schwarz (NS-NS) sectors.

NS-NS Sector: $G_{\mu\nu}, B^{(2)}_{\mu\nu}, \Phi$:

In the NS-NS sector, we have the metric $G_{\mu\nu}$, the anti-symmetric two-form field $B^{(2)}_{\mu\nu}$ and the dilaton Φ. This is exactly the same as in Type IIA.

RR Sector: $C^0, C^2_{\mu\nu}, C^4_{\mu\nu\rho\sigma}$:

The R-R sector is different in this case. We have a 0-form field C^0, a 2-form field $C^2_{\mu\nu}$ and a 4-form field $C^4_{\mu\nu\rho\sigma}$. We define the field strengths, again, as follows:

$$\begin{aligned} &F^{(1)} = dC^{(0)}, \quad F^{(3)} = dC^{(2)}, \quad F^{(5)} = dC^{(4)}; \quad H^{(3)} = dB^{(2)} \\ &\tilde{F}^{(1)} = F^{(1)}, \quad \tilde{F}^{(3)} = F^{(3)} - C^{(0)} \wedge H^{(3)} \\ &\tilde{F}^{(5)} = F^{(5)} - \frac{1}{2}C^{(2)} \wedge H^{(3)} + \frac{1}{2}B^{(2)} \wedge F^{(3)} \end{aligned} \tag{3.5}$$

The NS-NS part is identical to the one of Type IIA:

$$S_{\text{NS-NS}} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}e^{-2\Phi}\left(R_G + 4G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}\frac{1}{3!}G^{\mu\mu'}G^{\nu\nu'}G^{\rho\rho'}H^{(3)}_{\mu\nu\rho}H^{(3)}_{\mu'\nu'\rho'}\right) \tag{3.6}$$

and the R-R part is:

$$S_{R-R} = -\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left[\frac{1}{2}G^{\mu\nu}\tilde{F}^{(1)}_\mu\tilde{F}^{(1)}_2 + \frac{1}{2}\frac{1}{3!}G^{\mu\mu'}G^{\nu\nu'}G^{\rho\rho'}\tilde{F}^{(3)}_{\mu\nu\rho}\tilde{F}^{(3)}_{\mu'\nu'\rho'} + \frac{1}{2}\frac{1}{5!}G^{\mu\mu'}G^{2\nu'}G^{\rho\rho'}G^{\sigma\sigma'}G^{\tau\tau\tau'}\tilde{F}^{(5)}_{\mu\nu\rho\sigma\tau}\tilde{F}^{(5)}_{\mu'\nu'\rho'\sigma'\tau'}\right] \tag{3.7}$$

In addition to the two pieces above, there is a third piece which we call the Chern-Simons term which takes the following form in Type IIB:

$$S_{\text{CS}} = -\frac{1}{4\kappa_{10}^2}\int C^{(4)}\wedge H^{(3)}\wedge F^{(3)}. \tag{3.8}$$

There are two moduli fields in the action Φ and C^0 (both have no potential term). We can combine these two into a single piece and observe something interesting. Let us introduce the following new definitions:

$$\tau \equiv C^{(0)} + ie^{-\Phi}, \quad G^{(3)} \equiv F^{(3)} - \tau H^{(3)} \tag{3.9}$$

We have written the NS-NS sector 3.6 and 3.7 in string frame, meaning that the Ricci scalar R appears with the dilaton-dependent prefactor $e^{-2\Phi}$. This frame is convenient for comparing to the results of string perturbation theory. However, for many questions involving gravity, it is more practical to work in Einstein frame, in which the dilaton prefactor is absent. The action can be written in Einstein frame by performing the Weyl rescaling:

$$G_{E,MN} \equiv e^{-\Phi/2}G_{MN}\,, \tag{3.10}$$

in terms of which the action, written in Einstein frame, takes the form:

$$S_{\text{IIB}} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G_E}\left[R_E - \frac{|\partial\tau|^2}{2(\text{Im}(\tau))^2} - \frac{|G_3|^2}{2\text{Im}(\tau)} - \frac{|\tilde{F}_5|^2}{4}\right] - \frac{i}{8\kappa^2}\int\frac{C_4\wedge G_3\wedge\bar{G}_3}{\text{Im}(\tau)}. \tag{3.11}$$

The action (3.11) is the starting point for our discussion of Type IIB flux compactifications, which we will see in another chapter.

3.2.3 Type I

Type I string theory is one of the five consistent superstring theories. It includes both open and closed strings. The presence of open strings allows for the existence of gauge symmetries and D-branes, which play a crucial role in string interactions. The field contents in Type I theory are:

- NS-NS Sector: $G_{\mu\nu}, \Phi$
- R-R Sector: $C^{(2)}$
- 32 D9-branes (this gives rise to $SO(32)$ gauge fields which are coming from open strings)

In Type I string theory, the Kalb-Ramond field is absent in the NS-NS sector because it is not invariant under the worldsheet parity operator. Similarly, most R-R fields are removed, leaving only $C^{(2)}$. To cancel the negative R-R charge created by the projection, 32 D9-branes are added, leading to the $SO(32)$ gauge group from the open string sector. Unlike the other string theories, which involve only closed strings, Type I is unique as it includes both the open and closed strings, contributing to its distinct properties. The NS-NS part of the action is:

$$S_{\text{NS-NS}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} e^{-2\Phi} \Big[R_G + 4G^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi \Big] \tag{3.12}$$

and the R-R part is:

$$S_{R-R} = -\frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left[\frac{1}{2} \frac{1}{3!} G^{\mu\mu'} G^{\nu\nu'} G^{\rho\rho'} \tilde{F}^{(3)}_{\mu\nu\rho} \tilde{F}^{(3)}_{\mu'\nu'\rho'} \right] \tag{3.13}$$

and the gauge theory part is:

$$S_{\text{gauge}} = -\frac{1}{2\kappa_{10}^2} \frac{C_I}{2!} \int d^{10}x \sqrt{-G} e^{-\Phi} \Big[G^{\mu\mu'} G^{\nu\nu'} \mathrm{Tr}_V (F_{\mu\nu} F_{\mu'\nu'}) \Big]. \tag{3.14}$$

Here,

$$\tilde{F}^{(3)} = dC^{(2)} - C_I \omega^{(3)}(A)$$

$$\omega^{(3)}(A) = \mathrm{Tr}_V \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

$$C_I = \frac{\alpha'}{4}$$

and $F_{\mu\nu} = F^a_{\mu\nu} T^a = F_{\mu\nu}$ is the Yang-Mills field strength. Also, the trace is over the vector representation of $SO(32)$ with $Tr_V\left(T^a T^b\right) = \delta_{ab}$ Using, $G_{\mu\nu} = e^{\Phi/2}\, g_{\mu\nu}$, we can also write the action in canonical form as follows:

$$\begin{aligned} S_I = \frac{1}{2\kappa_{10}^2} \int d^{10}x\sqrt{-g} \Bigg[& R_g - \frac{1}{2} g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}\frac{1}{3!} e^{\Phi} g^{\mu\mu'} g^{\nu\nu'} g^{\rho\rho'} \tilde{F}^{(3)}_{\mu\nu\rho}\tilde{F}^{(3)}_{\mu'\nu'\rho'} \\ & - \frac{C_I}{2!} e^{\Phi/2} g^{\mu\mu'} g^{\nu\nu'} \mathrm{Tr}_V\left(F_{\mu\nu}F_{\mu'\nu'}\right) \Bigg] \end{aligned} \tag{3.15}$$

3.2.4 Heterotic Theory

There are two heterotic supergravity theories based on two different groups: $SO(32)$ and $E_8 \times E_8$. The number of generators of $SO(32)$ is 496. The field contents are:

- NS-NS Sector: $G_{\mu\nu}, B^{(2)}_{\mu\nu}, \Phi$
- Gauge Fields: $A^a_\mu \quad a = 1, \ldots, 496$

The action can be written as:

$$\begin{aligned} S = \frac{1}{2\kappa_{10}^2} \int d^{10}x\sqrt{-G} e^{-2\Phi} \Bigg[& R_G + 4G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}\frac{1}{3!} G^{\mu\mu'} G^{\nu\nu'} G^{\rho\rho'} \tilde{H}^{(3)}_{\mu\nu\rho}\tilde{H}^{(3)}_{\mu'\nu'\rho'} \\ & - \frac{C_H}{2!} G^{\mu\mu'} G^{\nu\nu'} \mathrm{Tr}_V\left(F_{\mu\nu}F_{\mu'\nu'}\right) \Bigg] \end{aligned} \tag{3.16}$$

where $F_{\mu\nu} = F^a_{\mu\nu} T^a$ is the non-Abelian field strength. We have, $CH = \alpha'/4$ for heterotic supergravity $H^{(3)} = dB^{(2)}$. Thus,

$$\tilde{H}^{(3)} = H^{(3)} - C_H \omega^{(3)}(A)$$

$$\omega^{(3)}(A) = \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

As obvious, the two heterotic theories differ because of the generators T^a satisfying different algebra. In terms of the canonical metric $g_{\mu\nu}$,

$$\begin{aligned} S = \frac{1}{2\kappa_{10}^2} \int d^{10}x\sqrt{-g} \Bigg[& R_g - \frac{1}{2} g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}\frac{1}{3!} e^{-\Phi} g^{\mu\mu'} g^{\nu\nu'} g^{\rho\rho'} \tilde{H}^{(3)}_{\mu\nu\rho}\tilde{H}^{(3)}_{\mu'\nu'\rho'} \\ & - \frac{C_H}{2!} e^{-\Phi/2} g^{\mu\mu'} g^{\nu\nu'} \mathrm{Tr}_V\left(F_{\mu\nu}F_{\mu'\nu'}\right) \Bigg] \end{aligned} \tag{3.17}$$

Type I and $SO(32)$ Heterotic theories are dual to each other, if we identify:

$$\begin{aligned} \Phi_I &= -\Phi_H \\ C_I^{(2)} &= B_H^{(2)} \\ A_I &= A_H \\ g_{\mu\nu}^I &= g_{\mu\nu}^H \end{aligned}$$

The duality can be argued to hold as follows :

$$\begin{aligned} \Phi_I = -\Phi_H \quad &\Rightarrow \quad e^{\Phi_I} = e^{-\Phi_H} \\ &\Rightarrow \quad g_S^I = \frac{1}{g_S^H}. \end{aligned}$$

This implies that when the Type I coupling is small, the Heterotic coupling is strong.

3.3 String Compactification

We mentioned before that superstring theory is a quantum theory of gravity living in 10 spacetime dimensions. So, in string theory we have the existence of the extra spacetime dimensions and therefore, we need to compactify the extra dimensions to four dimensions which describes the world around us. We are doing compactification of the extra dimensions to make a contact with our real world. The 10-dimensional manifold is roughly compactified as follows:

$$\mathbb{R}^{1,3} \times X_6 \tag{3.18}$$

where X_6 is a compact six-manifold. This is referred to as a compactification of string theory on X_6 [19].

3.3.1 Vacuum Compactification

Vacuum compactification is the process of reducing the 10-dimensional spacetime of string theory to an effective 4-dimensional theory by compactifying the extra six dimensions on a compact internal space, such as a Calabi-Yau manifold or an orbifold. The key requirement for a vacuum configuration is that the 10D metric G_{MN} satisfies the vacuum Einstein equations, meaning that both the external and internal Ricci tensors vanish [18]:

$$R_{\mu\nu} = R_{mn} = 0$$

This ensures that the internal manifold, X_6, is Ricci-flat, which is a crucial condition for a stable vacuum solution. The most studied vacuum compactifications use Calabi-Yau threefolds, which satisfy this Ricci-flatness condition and preserve a portion of supersymmetry. We begin by examining vacuum solutions, for which a suitable ansatz is:

$$G_{MN}\, dX^M dX^N = \eta_{\mu\nu}\, dx^\mu dx^\nu + g_{mn}\, dy^m dy^n \tag{3.19}$$

where, $y^m, m = 1, ..., 6$, are coordinates on X_6, and g_{mn} is a metric on X_6.

3.3.2 Warped compactifications

Vacuum configurations solving the vacuum Einstein equations provide a well understood starting point. However, realistic string compactifications often require additional ingredients, such as fluxes, branes, and other stress-energy sources. These non-vacuum solutions modify the Einstein equations by introducing localized energy-momentum sources, leading to a compactification manifold that is not Ricci-flat. Extended objects like D-branes, NS5-branes, and fluxes contribute additional terms to the stress-energy tensor, changing the geometry of the extra dimensions. Instead, the metric 3.19 takes the form of a warped product, where a warp factor modifies the 4D and internal components:

$$G_{MN}\, dX^M dX^N = e^{2A(y)} g_{\mu\nu} dx^\mu dx^\nu + e^{-2A(y)} g_{mn} dy^m dy^n \tag{3.20}$$

where now $g_{\mu\nu}$ is the metric of a maximally symmetric spacetime, the warp factor $A(y)$ is a function on X_6, and the internal metric g_{mn} is not necessarily Ricci-flat [18].

3.3.3 Supersymmetric compactifications

The supergravity actions in ten dimensions possess $\mathcal{N} = 1$ or $\mathcal{N} = 2$ supersymmetry, but their solutions do not necessarily preserve it. However, supersymmetric solutions are the best understood in string theory for several reasons [18]:

1 **Geometric Reason:** Vacuum solutions require Ricci-flatness, which is linked to reduced holonomy. For example, Calabi-Yau threefolds with $SU(3)$ holonomy preserve one-quarter of ten-dimensional supersymmetry, leading to $\mathcal{N} = 2$ supersymmetry in four-dimensional Type II string compactifications.

2 **Theoretical Control:** Supersymmetry ensures stability and imposes constraints on couplings in the effective theory, making calculations more tractable.

3 **Phenomenological Interest:** Finding solutions for $\mathcal{N} = 1$ supersymmetry, broken near the electroweak scale, is crucial for addressing the hierarchy problem in particle physics.

3.3.4 Kaluza-Klein compactification

In a general **four-dimensional $\mathcal{N} = 1$ supergravity theory**, the bosonic fields include: $g_{\mu\nu}$ (the metric, describing spacetime curvature), A^{α}_{μ} (gauge potentials, associated with gauge interactions), Φ^i (complex scalar fields, which determine the dynamics of the theory). On the other hand, the low-energies interactions are controlled by the **superpotential** $W(\Phi^i)$, which is a holomorphic functions that dictates scalar field interactions and potential energy, and by the **Kähler potential** $K(\Phi^i, \bar{\Phi}^{\bar{i}})$, a real function that defines the kinetic terms and the field space geometry. In the absence of gauge interactions, the Lagrangian for the scalar fields is determined solely by these potentials as:

$$\mathcal{L}_{\Phi} = K_{i\bar{j}}\,\partial^{\mu}\phi^{i}\partial_{\mu}\bar{\phi}^{\bar{j}} - V_F \tag{3.21}$$

where $K_{i\bar{j}} \equiv \partial_i\partial_{\bar{j}}K$ is the Kähler metric. The F-term potential V_F appearing in (3.21) is:

$$V_F(\phi^i, \bar{\phi}^{\bar{i}}) = e^{K/M_p^2}\left[K^{i\bar{j}}D_iW\overline{D_jW} - \frac{3}{M_p^2}|W|^2\right] \tag{3.22}$$

where $K_{i\bar{j}}$ is the inverse Kähler metric and $D_iW \equiv \partial_iW + M_p^{-2}(\partial_iK)W$. A primary task in studying a string compactification with $N = 1$ supersymmetry is to compute the superpotential and Kähler potential in terms of geometric data. Through (3.21) and (3.22) these data determine the four-dimensional effective theory, to leading order in the low-energy (derivative) expansion.
To compute the four-dimensional effective action of a string compactification, one begins with the appropriate ten-dimensional action and performs a Kaluza–Klein reduction. Consider the ten-dimensional geometry:

$$G_{MN}\,dX^M dX^N = e^{-6u(x)}g_{\mu\nu}dx^{\mu}dx^{\nu} + e^{2u(x)}\hat{g}_{mn}dy^m dy^n \tag{3.23}$$

where $\hat{g}_{mn}$ is a reference metric with fixed volume,

$$\int_{X_6} d^6y\sqrt{\hat{g}} \equiv \mathcal{V} \tag{3.24}$$

while $e^{-6u(x)}$ is a "breathing mode" that represents the variations in size of the internal space X_6 as a function of the four-dimensional coordinate x^{μ}. The factor of $e^{-6u(x)}$ in the first term is a convenient choice for which the gravitational action in four dimensions will appear in Einstein frame. We now examine the dimensional reduction of the Einstein–Hilbert term,

$$S^{(10)}_{\text{EH}} = \frac{1}{2\kappa^2}\int d^{10}X\sqrt{-G}\,e^{-2\Phi}R_{10} \tag{3.25}$$

where R_{10} is the Ricci scalar constructed from G_{MN} . We would like to express R_{10} in terms of R_4 and $\hat{R}_6$, the Ricci scalars constructed from $g_{\mu\nu}$ and $\hat{g}_{mn}$, respectively. For this purpose we note that if two D-dimensional metrics g_{MN} and $\bar{g}_{MN}$ are related by the conformal rescaling

$$\bar{g}_{MN} = e^{2\omega(x)} g_{MN} \tag{3.26}$$

then the corresponding Ricci scalars are related by:

$$e^{2\omega}\bar{R} = R - 2(D-1)\nabla^2\omega - (D-2)(D-1)g^{MN}\nabla_M\omega\nabla_N\omega \tag{3.27}$$

Similarly, the Laplacians constructed from g^{MN} and $\hat{g}^{MN}$ are related by

$$e^{2\omega}\bar{\nabla}^2 = \nabla^2 + (D-2)g^{MN}\nabla_M\omega\nabla_N \tag{3.28}$$

Using these results, we find:

$$S_{\text{EH}}^{(10)} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\int_{X_6} d^6y\sqrt{\hat{g}}e^{-2\Phi}\left(R_4 + e^{-8u}\hat{R}_6 + 12\partial_\mu u\partial^\mu u\right) \tag{3.29}$$

If the string coupling $g_s \equiv e^{\Phi}$ is constant over the internal space, then the four-dimensional Einstein–Hilbert term can be written:

$$S_{\text{EH}}^{(4)} = \frac{M_p^2}{2}\int d^4x\sqrt{-g}\,R_4 \tag{3.30}$$

with the four-dimensional Planck mass defined as:

$$M_p^2 \equiv \frac{\mathcal{V}}{g_s^2\kappa^2} \tag{3.31}$$

We recognize the combination of derivatives of $u(x)$ appearing in (3.29) as the kinetic term for a four-dimensional scalar field $u(x)$. This field is a modulus corresponding to a spacetime-dependent deformation of the ten-dimensional solution. As we will see below, in Calabi–Yau compactifications the breathing mode u corresponds to one of the Kähler moduli; the kinetic term for u in (3.29) follows from the Kähler potential:

$$K = -3\ln(T + \bar{T}) \tag{3.32}$$

where we have set $M_p = 1$, and T is a complex scalar field with $\text{Re}(T) = e^{4u}$.
The Ricci scalar $\hat{R}_6$ in six-dimensional compactification generates a potential for the scalar field u in four-dimensions. A positive internal curvature ($\hat{R}_6 > 0$) results in a negative potential ($V \propto -e^{-8u}$), driving the compactification toward smaller volume. Conversely, a negative internal curvature leads to a positive potential($V \propto +e^{-8u}$), causing a decompactification instability. In Ricci-flat compactifications, the internal curvature term is absent, leaving u with a vanishing potential in the classical theory. More general Kaluza-Klein reductions involve complex ten-dimensional actions, including p-form fields and geometric deformations beyond the simple breathing mode. However, the fundamental principles governing these reductions remain the same as in the basic example discussed.

Chapter 4

Moduli and their Stabilization

4.1 Calabi-Yau Moduli

In string theory, compactification on a Calabi-Yau manifold leads to the emergence of moduli fields, which describe the deformations of the internal geometry. These fields remains massless at tree-level, posing phenomenological challenges such as unobserved long-range interactions. Therefore, understanding and stabilizing moduli is essential for constructing realistic four-dimensional models. This chapter introduces different types of moduli in Calabi-Yau compactifications, explains why they are massless at tree level, and explores mechanisms for their stabilization. A more detailed discussion of the geometrical foundations - such as cohomology, holonomy, and the mathematical structure of Calabi-Yau manifolds - is provided in the Appendix.

4.1.1 Kähler and complex structure moduli

A Calabi-Yau threefold X_6 is a complex Kähler manifold with $SU(3)$ holonomy, satisfying the Ricci-flatness condition $R_{mn} = 0$. This ensures the existence of a nowhere-vanishing holomorphic $(3,0)$-form Ω, and the structure of the moduli space is determined by the deformations of the Kähler form J, and the complex structure. The moduli fields in four-dimensional effective theories correspond to these deformations.
To define the fields arising from the compactification, we begin by introducing the moduli fields. These fields will be expressed in terms of a basis of cohomology groups, specifically using harmonic forms. The moduli can be characterized in two different classes:

Kähler Moduli: The Kähler moduli T^i parameterize the deformations of the

Kähler form J, which controls the volumes of two- and four-cycles inside X_6:

$$J = t^i \omega_i, ,$$

here, ω_i are harmonic $(1,1)$-forms, and D_i are the four-cycles of X_6. The internal volume is given by:

$$V = \frac{1}{6}\kappa_{ijk} t^i t^j t^k$$

where κ_{ijk} are the triple intersection numbers of X_6. The Kähler moduli fields are:

$$T_i = \tau_i + i b_i$$

where τ_i is the volume of the corresponding four-cycle, b_i is an axion partner from the C_4 Ramond-Ramond (RR) potential. At tree-level the Kähler moduli do not appear in the flux-induced superpotential, leaving them unstabilized.

Complex Structure Moduli: The complex scalar moduli z^a describe deformations of the complex structure of X_6, modifying its shape but not its volume. These are counted by the number of harmonic $(2,1)$-forms forming a basis of the Dolbeault cohomology group $H^{2,1}(X)$. The metric in the effective theory is given by the Weil-Petersson metric:

$$K_{z^a \bar{z}^b} = -\frac{\int_X \chi_a \wedge \bar{\chi}_b}{\int_X \Omega \wedge \bar{\Omega}}$$

Unlike Kähler moduli, complex structure moduli can be stabilized by fluxes.

So the total geometric moduli space, as stated in the previous subsection, can be separated into the two independent parts correspondent to Kähler moduli (Kähler deformations) and Complex Structure Moduli (Complex Structure Deformations) $\mathcal{M}_{Moduli} = \mathcal{M}_K \times \mathcal{M}_{c.s.}$. In addition to this geometric moduli we have to include another modulus built up from the dilaton and the 0-form C_0: the axio-dilaton $\tau = C_0 + ie^{-\phi}$, where $e^{-\Phi}$ is the dilaton, controlling the string coupling g_s, and C_0 is the RR scalar. Fluxes generate a potential for τ, stabilizing it at tree-level.

4.1.2 Kähler Potential in 4D Supergravity Models

With all the necessary components in place, we can now formulate a general model in four-dimensional supergravity. To achieve this, we must construct the appropriate Lagrangian, which takes the form:

$$\mathcal{L} = K_{i\bar{j}}(\partial_\mu X^i)(\partial^\mu \bar{X}^{\bar{j}}) + \text{gauge, fermion, and other fields,} \tag{4.1}$$

where $K_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$ is the Kähler metric coming from the Kähler potential K and X_i are the moduli, Kähler, complex structure and axio-dilaton ones. To explicitly obtain the Lagrangian, we need to define the Kähler potential, incorporating a term for each type of moduli that we described, while working in Planck units ($M_p = 1$) [20].

Kähler potential for Kähler moduli:

$$K_K = -2 \ln \mathcal{V} \tag{4.2}$$

Kähler potential for complex structure moduli:

$$K_{\text{c.s.}} = -\ln\left(i \int_{X_6} \Omega \wedge \bar{\Omega}\right) = -\ln\left(-i\Pi^{\mathrm{T}}\Sigma\Pi\right) = -\ln\left(-iz^a \mathcal{G}_a(z) + iz^a \overline{\mathcal{G}_a(z)}\right) \tag{4.3}$$

Kähler potential for the axio-dilaton:

$$K_{\text{dil}} = -\ln(-i(\tau - \bar{\tau})) \tag{4.4}$$

With all these 3 parts we have finally a full Type IIB **Kähler Potential** which is quite general and so, for such a 4D Supergravity model:

$$K = K_K(T^i, \bar{T}^{\bar{j}}) + K_{\text{c.s.}}(z^a, \bar{z}^{\bar{a}}) - \ln(-i(\tau - \bar{\tau})) \tag{4.5}$$

$$K = -2 \ln \mathcal{V} - \ln\left(i \int_{X_6} \Omega_3 \wedge \bar{\Omega}_3\right) - \ln\left(-i(\tau - \bar{\tau})\right) \tag{4.6}$$

As the Kähler potential is defined, the next crucial step is **moduli stabilization**, which ensures that moduli fields acquire a mass and do not lead to undesired physical effects, such as long-range forces or cosmological instabilities. Now we move to discuss the details about **moduli stabilization**.

4.2 Moduli Stabilization in Type IIB

In Calabi-Yau compactifications, moduli fields naturally arise as parameters that describe the shape and size of the extra-dimensional space. These fields play a crucial role in both string theory and cosmology, where they can act as inflaton candidates driving cosmic inflation. However, a significant challenge arises: at tree level, the Kähler potential does not generate a potential for the moduli, leaving them as flat directions in field space. This absence of a potential leads to massless moduli, which in turn can cause long-range fifth forces that contradict observational constraints.

To construct a phenomenologically viable model, it is essential to generate a potential that ensures all moduli acquire a positive mass squared, preventing instability and runaway behavior. This process is known as moduli stabilization. Various mechanisms, such as flux compactifications, non-perturbative effects (e.g., instantons, gaugino condensation), and quantum corrections, contribute to moduli stabilization. Popular frameworks include the KKLT scenario, which uses non-perturbative potentials to stabilize the Kähler moduli, and the Large Volume Scenario (LVS), where quantum corrections generate a stable vacuum with large extra-dimensional volumes.
Ultimately, moduli stabilization is a crucial step in string model building, as it not only ensures a consistent low-energy four-dimensional theory but also affects supersymmetry breaking, cosmological evolution, and potential connections to observable physics.

4.2.1 Superpotential

Type IIB compactifications on Calabi-Tau threefolds result in an $N = 2$ supergravity in 4 spacetime dimensions. We then use orientifolds to reduce the amount of supersymmetry to $N = 1$. The number of Kähler moduli is controlled by the Hodge number $h^{1,1}$. However introducing orientifolds projects out half of their amount, and so the Kähler moduli are τ_i where $i = 1, 2, 3...., h^{1,1}_+$. Also, the complex structure moduli z_a are counted by $h^{2,1}_-$.
We consider compactifications where the RR and NS gauge field strengths, $F_3 = dC_2$ and $H_3 = dB_2$, are non-zero, with a warped metric. These compactifications involve a combined 3-form flux $G_3 = F_3 - SH_3$ that is **imaginary self-dual(ISD)**, satisfying $\star_6 G_3 = iG_3$, and are therefore called ISD compactifications. Non-zero field strengths introduce fluxes, characterized by integers $n, m \in \mathbb{Z}$, representing flux quantization [20].

$$\frac{1}{2\pi\alpha'}\int F_3 = 2\pi n \quad , \quad \frac{1}{2\pi\alpha'}\int H_3 = 2\pi m \tag{4.7}$$

Fluxes play a crucial role in stabilizing the geometry of the compactification manifold. They prevent cycles from shrinking and, when distributed over different cycles, help stabilize the shape of the manifold. Since complex structure moduli determine the ratio of 3-cycle volumes, 3-form fluxes stabilize the complex structure moduli, giving them mass. These fluxes introduce a non-zero **superpotential** W_0, which depends on the moduli fields in supergravity models. This superpotential is known as the **Gukov-Vafa-Witten (GVW) superpotential** [21]. In the Type IIB case, it has been rigorously derived from both four-dimensional $\mathcal{N} = 1$ supergravity and ten-dimensional string theory. This superpotential takes the form:

$$W_{\rm GVW} = W_0 = \int_{X_6} G_3 \wedge \Omega_3 \tag{4.8}$$

The usual formula for the scalar potential is:

$$V = e^K \left(K^{i\bar{j}} (D_i W)(D_{\bar{j}} \bar{W}) + K^{a\bar{b}} (D_a W)(D_{\bar{b}} \bar{W}) - 3|W|^2 \right) \tag{4.9}$$

where the covariant derivative is defined as $D_i = \partial_i + K_i$, where the index i runs over all Kähler moduli, and the index a runs over complex structure moduli and the axio-dilaton. The moduli fields are collectively denoted as $z^a = [\tau, z^1, ..., z^{h^{2,1}}]$. The axio-dilaton contribution to the Kähler potential is absorbed into the complex structure term, simplifying it as $K_{c.s.} = K_{c.s.} + K_{\rm dil}$.
The superpotential W_0 depends on these moduli and using the F-term conditions for supersymmetry, stabilization is achieved by solving:

$$D_a W = 0, \quad \text{for} \quad a = 1, \dots, h^{2,1}_{-} + 1 \tag{4.10}$$

This ensures the stabilization of both the complex structure moduli and the axio-dilaton, allowing them to be integrated out.

4.2.2 Kähler Potential

At the sphere level, the moduli space of closed string fields in an $O3/O7$ orientifold of a Calabi-Yau threefold X_6 factorizes into three independent sectors: complex structure moduli, Kähler moduli and axio-dilaton moduli.

$$\mathcal{M} = \mathcal{M}_{\rm c.s.}(X_6) \times \mathcal{M}_{\rm K}(X_6) \times \mathcal{M}_{\rm dil} \tag{4.11}$$

Therefore the metric for the moduli is block diagonal, and the Kähler potential splits into dilaton, complex structure and Kähler moduli terms :

$$K_{\rm tree} = -\ln\left(-i(\tau - \bar{\tau})\right) - \ln\left(-i \int_{X_6} \Omega(z_i) \wedge \bar{\Omega}(\bar{z}_i) \right) - 2\ln\left(\mathcal{V}(T_a, \bar{T}_a)\right) \tag{4.12}$$

In superpotential stabilization, the Kähler moduli remain unstabilized. As in (4.9), only the second term of the potential can be integrated out. Consequently, the scalar potential is given by:

$$V = e^K \left(K^{i\bar{j}} D_i W \bar{D}_{\bar{j}} \bar{W} - 3|W|^2 \right) \tag{4.13}$$

where the Kähler moduli remain unfixed, influencing the vacuum structure and requiring additional mechanisms. This is due to the no-scale structure [22] that arises from the Kähler potential $K_K = -2\ln\mathcal{V}$, where $\mathcal{V}$ is a homogeneous function of degree 3/2 in τ_i. This implies that the Kähler moduli have a no-scale potential, i.e. $V(T,\bar{T}) \equiv 0$. The term "no-scale" reflects that supersymmetry is broken at an undetermined scale, making it a key feature in string compactifications.

$$D^{\bar{T}}W \sim -\frac{W}{(T+\bar{T})} \neq 0 \tag{4.14}$$

Since the Kähler potential depends on T, which remains unstabilized, the supersymmetry breaking scale $\Lambda_{SUSY} = m_{3/2} = e^{K/2}\,W_0$ is also not fixed.
So far we have stabilized all moduli except the Kähler moduli, of which we consider there to be only one, the volume modulus. The other moduli were stabilized by fixing the flux configuration, as they appear in the superpotential. With all moduli stabilized except the volume modulus, the only relevant contribution from the Kähler potential is:

$$K = -3\ln\left(T+\bar{T}\right) \tag{4.15}$$

As we have seen, the stabilization of the T-moduli is possible only once quantum corrections to the scalar potential — through one or more of W_{np}, K_{pert} and K_{np} — impact the vacuum structure and break the no-scale structure. These effects are of 2 kinds: perturbative (pert) and non-perturbative (np). The perturbative ones cannot affect the superpotential due to the non-renormalisation theorem. Hence, calling the previously written Kähler potential as $K \to K_0$:

$$\begin{cases} K = K_0 + K_p + K_{np} \\ W = W_0 + W_{np} \end{cases} \tag{4.16}$$

4.2.3 Perturbative Corrections

In string compactifications, perturbative corrections play a crucial role in stabilizing moduli fields, particularly the Kähler moduli, which remain unstabilized at tree level due to the no-scale structure of the leading-order Kähler potential. These corrections arise from both α' (stringy) and loop effects ($K_p = \delta K_{\alpha'} + \delta K_{g_s}$) and help break the flatness of the moduli potential, leading to stabilized configurations.
The tree-level Kähler potential reads:

$$K_{tree} = -2\ln\mathcal{V} \tag{4.17}$$

Here, $\mathcal{V}$ is the Calabi-Yau volume. $\mathcal{V}$ is related to τ, which is the real part of the Kähler modulus T, defined as $\tau = (T+\bar{T})$, which serves as the appropriate chiral coordinate in the $\mathcal{N}=1$ effective field theory (EFT). At tree level, $\tau = \mathcal{V}^{2/3}$. So, now the Kähler potential takes the form:

$$K_{tree} = -3\ln\tau \tag{4.18}$$

α'-Corrections and Loop Corrections:
α' and g_s-corrections which are needed for our further discussions have been recently classified in [23]. In the effective supergravity description α' corrections correspond to higher derivative terms. In the 4D theory the leading order α' correction in the Kähler potential reads [24]:

$$\frac{K}{M_p^2} = -2\ln\left(\mathcal{V} + \frac{\xi}{2g_s^{3/2}}\right) = -2\ln\mathcal{V} - \frac{\xi}{g_s^{3/2}\mathcal{V}} + \mathcal{O}(1/\mathcal{V}^2),$$

with the constant ξ given by:

$$\xi = -\frac{\chi(X_6)\zeta(3)}{2(2\pi)^3}.$$

Here $\chi(X_6) = 2(h_{1,1} - h_{2,1})$ is the Euler number of the Calabi-Yau X_6, and the relevant value for the Riemann zeta function is $\zeta(3) \equiv \sum_{k=1}^{\infty} 1/k^3 \simeq 1.2$.
For Type IIB compactifications, string loop corrections take the form:

$$\delta K_{\rm gs} = \delta K_{\rm gs}^{\rm KK} + \delta K_{\rm gs}^{\rm W}$$

where the two kinds of corrections come from different sources. In fact, $\delta K_{\rm gs}^{\rm KK}$ comes from the exchange of Kaluza-Klein modes and $\delta K_{\rm gs}^{\rm W}$ originates by the exchange of winding strings. Explicit $N = 2$ $\delta K_{\rm gs}^{\rm KK}$ corrections computed for toroidal orientifolds extend to Calabi-Yau backgrounds, where they take the form [25, 26]:

$$K_{\mathcal{O}(g_s^2\alpha'^2)} \simeq \frac{g_s c_1}{\tau} \tag{4.19}$$

where c_1 is a function of the complex structure moduli. On the other hand, $\delta K_{\rm gs}^{\rm W}$ corrections look like [25, 26]:

$$K_{\mathcal{O}(g_s^2\alpha'^4)} \simeq \frac{c_4}{\tau^2} \tag{4.20}$$

where c_4 depends on the complex structure moduli. Ref. [27] identified the origin of these corrections as arising from loops of Kaluza-Klein modes of open strings stretched between intersecting branes. Additionally, similar corrections are expected to emerge from closed string loop effects [27, 28]. Moduli redefinitions in string theory arise from higher-order effects at $\mathcal{O}(\alpha'^2)$ and $\mathcal{O}(g_s\alpha'^2)$, which modify the Kähler potential primarily through moduli redefinitions rather than direct corrections [29]. These effects introduce logarithmic modifications, transforming the Kähler potential as $K = -3\ln(\tau - \alpha\ln\tau)$ [30, 31, 32]. At $\mathcal{O}(\alpha'^3)$, further corrections appear as [24, 33, 34, 35]:

$$K_{\mathcal{O}(g_s^2\alpha'^3)} = -2\ln\left\{1 + \frac{\xi}{2(g_s\tau)^{3/2}}\left[1 + g_s^2\left(c_2\left(1 - \frac{3T_7}{2}\ln\tau\right) + c_3\right)\right]\right\} \tag{4.21}$$

The term proportional to $c_2 = 2\zeta(2)/\zeta(3)$ represents an $\mathcal{N} = 2$ $\mathcal{O}(g_s^2\alpha'^3)$ correction [36, 37], with logarithmic contributions arising in regions of high localized curvature [38, 39]. Additionally, the small parameter $c_3 \sim 10^{-4}$ governs $\mathcal{N} = 1$ effects at the same order [40, 41]. These contributions stem from localized curvature effects and influence the scalar potential, scaling as [42, 43]:

$$V_F = c_3\sqrt{g_s}\frac{W_0^4}{\tau^{11/2}} \tag{4.22}$$

4.2.4 Non-Perturbative Corrections

Non-perturbative corrections play a crucial role in moduli stabilization and the determination of the low-energy effective action in string theory compactifications. While perturbative corrections modify the Kähler potential and gauge kinetic functions, non-perturbative effects contribute exponentially suppressed terms to the superpotential, which are essential for stabilizing Kähler moduli. The non-renormalization theorem ensures that the superpotential does not receive either α' or g_s corrections [44]. However, it can receive non-perturbative corrections, and we focus specifically on those affecting the superpotential, expressed as (4.16). These non-perturbative contributions typically originate from two key mechanisms: **gaugino condensation [45]** and **Euclidean D3-brane ED3-brane instantons [46]**. We are now discuss about these two types:

- **Gaugino condensation:** In the context of string compactifications, particularly in type IIB string theory, gaugino condensation plays a crucial role in generating a non-perturbative superpotential that stabilizes moduli fields. This mechanism arises when a stack of N $D7$-branes wraps a rigid four-cycle Σ_4 within a compactification manifold. The rigidity of Σ_4 implies that it has no deformations and does not support charged matter fields, ensuring the emergence of a pure $N = 1$ super Yang-Mills theory in the four-dimensional effective field theory (4D EFT) after dimensional reduction. Since the D7-brane action includes a Yang-Mills term with a gauge field A_μ in $4D$ the low-energy effective dynamics of the theory are significantly influenced by the condensation of gauginos. At sufficiently low energies, the strongly coupled gauge theory undergoes gaugino condensation, generating a non-perturbative superpotential of the form:

  $$W_{np} = A\, e^{-aT} \tag{4.23}$$

  where $a = 2\pi/N, A = A(z^a, \rho^\alpha) \sim M_p^3$ with ρ^α brane position moduli and T the Kähler modulus whose real part measures the volume of Σ_4. In general, If multiple D7-brane stacks are present or branes wrap different four-cycles, the superpotential generalizes to a sum over contributions from different wrapped cycles:

$$W_{\rm np} = W_{\rm inst} = \sum_i A_i\, e^{-a_i T_i} \tag{4.24}$$

with i runs over all wrapped cycles, allowing for multiple non-perturbative effects to contribute to the effective theory.

- **Euclidean D3-brane instantons:** Another key source of non-perturbative corrections to the superpotential comes from Euclidean D3-brane (ED3) instantons, also known as E3-instantons. These are Euclidean D3-branes that wrap four-cycles in the compactified space and generate quantum corrections in the low-energy effective action.
If Σ_4 instead is wrapped by a Euclidean D3-brane, then we are dealing with ED3-brane instantons. They generate non-perturbative contributions to the path integral.An ED3-brane wrapping a divisor $\sum$ in the Calabi–Yau compactification contributes a non-perturbative term to the superpotential:

$$W_{\rm np} = Ae^{-S_{\rm inst}}$$

Their action includes: **real part:** $\mathrm{Re}(S_{inst}) \propto V_{\Sigma_{p+1}}$(cycle volume) and **imaginary part:** $\mathrm{Im}(S_{inst}) \propto S_{C.S.}$ (Chern-Simons term).
For multiple wrapped cycles, the superpotential is:

$$W_{\rm np} = W_{\rm E3} = \sum_i A_i\, e^{-a_i T_i} \tag{4.25}$$

where $a_i = 2\pi$ and, again, $A = A(z^a, \rho^\alpha) \sim M_p^3$ with ρ^α brane position moduli. In this case, a rigid cycle ensures a non-zero contribution to the superpotential. Additionally, the presence of fluxes allows for a slight relaxation of this rigidity condition while still maintaining a non-vanishing contribution. where A depends on fluxes and additional instanton effects. If multiple instantons contribute, the superpotential may take the form:

$$W = W_{\rm flux} + \sum_i A_i e^{-a_i T}$$

There are two major scenarios for fixing the Kähler moduli. These are the KKLT construction and the Large Volume Scenario (LVS), which we now describe in detail.

- The **KKLT construction [47]** utilizes the ability to tune the vacuum expectation value of the flux superpotential to small values. This acts as a small parameter, enabling different contributions from W_0 and W_{np} to balance each other. As a result, a supersymmetric AdS minimum is achieved.

- The **LVS construction [48]** relies on the perturbative no-scale breaking effect from an α'^3-correction which depends on the volume $\mathcal{V}$. This correction competes with non-perturbative effect on a small blow-up-4-cycle, leading to a non-supersymmetric AdS minimum. At this minimum, the volume scales as $\mathcal{V} \sim e^{1/g_s} \gg 1$ in string units, making it exponentially large. Supersymmetry is broken due to the F-terms of the Kähler moduli.

4.2.5 KKLT Construction

We already saw that in Type-IIB compactification, turning on fluxes generates a potential for the dilaton, and complex structure moduli, while the Kähler moduli remain unstabilized. The first step in the KKLT construction [47] is to integrate out the dilaton and complex structure moduli, reducing the system to a low-energy effective action for the Kähler moduli. While a realistic model includes multiple Kähler moduli, we focus on a single modulus to illustrate the key features of this construction.
The KKLT (Kachru, Kallosh, Linde and Trivedi) proposal is one of the widely accepted solution of dS construction. We know the non-perturbative correction to the super-potential from the previous discussion. So:

$$W = W_0 + A\, e^{-aT} \tag{4.26}$$

For a supersymmetric minimum, $D_T W = 0$.

$$\begin{aligned} 0 &= D_T\left(W_0 + Ae^{-aT}\right) \\ &= \partial_T\left(W_0 + Ae^{-aT}\right) + (\partial_T K)\left(W_0 + Ae^{-aT}\right) \\ &= -aAe^{-aT} - \frac{3}{2}\frac{1}{\tau}\left(W_0 + Ae^{-aT}\right) \end{aligned}$$

Thus, for $W_0 > 0$, $A > 0$ and the axion b fixed at $e^{-iab} = -1$, we have

$$W_0 = A\, e^{-a\tau}\left(\frac{2}{3}a\tau + 1\right) \tag{4.27}$$

Since we are looking at a supersymmetric vacuum, the potential has an AdS minimum at:

$$V_{\text{AdS}} = -3e^K W^2 = -\frac{a^2 A^2 e^{-2a\tau}}{6\tau} \tag{4.28}$$

Since the obtained vacuum is AdS, an uplift to a dS vacuum is required. In the famous KKLT paper, the uplifting of the vacuum is done via adding **anti-branes**, which explicitly breaks supersymmetry and adds a positive contribution to the scalar potential.

Initially, the uplift term was proportional to $1/\tau^3$, but more recent studies have also considered terms proportional to $1/\tau^2$. The final scalar potential takes the form: $V = V_{AdS} + V_{up}$, where the uplift term ensures a stable de Sitter (dS) vacuum. The potential is of the following form:

$$V = \frac{aAe^{-a\tau}}{3\tau}\left(aA\tau e^{-a\tau} - 2W_0 + Ae^{-a\tau}\right) + \frac{D}{\tau^3} \tag{4.29}$$

Here the uplift term D/τ^3 ensures a de Sitter(dS) vacuum. We present in Figure 4.1 a plot illustrating the transition from an AdS vacuum to an uplifted dS vacuum, where the real part of T corresponds to τ.

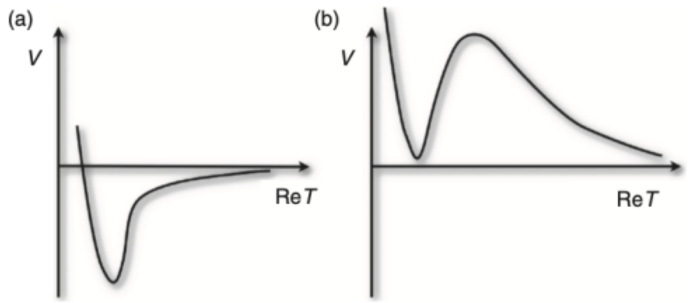

Figure 4.1: (a) The supersymmetric AdS vacuum, and (b) the uplifted dS vacuum from [49].

In GKP flux compcatifictions, the no-scale structure remains intact, whereas in KKLT, it is broken during moduli stabilization, leading to a supersymmetric vacuum where the Kähler modulus acquires a mass exceeding the gravitino mass. This represents a key distinction between KKLT and LVS, as the no-scale structure is preserved at leading LVS order, which we now examine.

4.2.6 Large Volume Scenario

The Large Volume Scenario (LVS) [48] begins similarly to KKLT, using the low-energy effective field theory after integrating out the complex structure moduli and axio-dilaton. However, LVS requires at least two Kähler moduli, with the Calabi-Yau manifold exhibiting a "Swiss-cheese" structure where the overall volume is controlled by one modulus, while additional moduli represent blow-up modes corresponding to geometric holes. The simplest case involves two Kähler moduli, where the Calabi-Yau volume is expressed as:

$$\mathcal{V} = \tau_b^{3/2} - \tau_s^{3/2} \tag{4.30}$$

Here, τ_b represents the volume of a large 4-cycle in the Einstein frame, while τ_s measures the volume of a blow-up cycle, specifically controlling the volume of an exceptional del Pezzo divisor that resolves a point-like singularity. Additionally, the leading α'^3 correction to the kähler potential is:

$$K = -2\ln\left(\mathcal{V} + \frac{\xi}{2g_s^{3/2}}\right) \tag{4.31}$$

with $\xi \equiv -\frac{\chi(X_6)\zeta(3)}{2(2\pi)^3}$ where $\chi(X_6)$ is the Euler number of the Calabi-Yau and ζ is the Riemann zeta function. LVS also requires a non-perturbative effect supported on the small cycle from (4.26).

Working in the limit $\tau_b \gg \tau_s$, after fixing the axionic partner of τ_s at its minimum, the scalar potential takes the form:

$$V = \frac{4}{3}\frac{a_s^2 A_s^2 \sqrt{\tau_s} e^{-2a_s\tau_s}}{s\mathcal{V}} - \frac{2a_s A_s |W_0| \tau_s e^{-a_s\tau_s}}{s\mathcal{V}^2} + \frac{3\sqrt{s}\,\xi |W_0|^2}{8\mathcal{V}^3} \tag{4.32}$$

Minimizing the potential, one finds a minimum at:

$$\langle\mathcal{V}\rangle \simeq \frac{3\sqrt{\langle\tau_s\rangle}|W_0|}{4a_s A_s} e^{a_s\langle\tau_s\rangle} \quad \text{and} \quad \langle\tau_s\rangle \simeq \frac{1}{g_s}\left(\frac{\xi}{2}\right)^{2/3} \tag{4.33}$$

LVS achieves stabilization by balancing the α'^3 corrections and non-perturbative effects, leading to a large overall volume. A high dilaton value ensures that the effective field theory remains under control, and the scenario is viable for natural values of W_0 in the range $\mathcal{O}(1-10)$. The LVS vacuum is Anti-de Sitter (AdS), with a potential at the minimum: $V_{LVS} \sim -m_{3/2}^3 M_p$. It is non-supersymmetric, and the dominant F-term contribution scales as $F^{T_b} \sim \tau_b m_{3/2}$, inherited from the no-scale structure. As a result, the Goldstino associated with T_b is absorbed by the gravitino, giving it a non-zero mass. Overall, the LVS provides a robust framework for moduli stabilization and plays a crucial role in connecting string theory to cosmology.

Chapter 5

The Brane World

5.1 D-brane effective action

A key element in Calabi-Yau flux compactification is the presence of Dp-branes, which must extend across four-dimensional spacetime to preserve Poincaré. Each spacetime-filling Dp-brane supports a $U(1)$ gauge theory on its worldvolume. More generally, a stack of $\mathcal{N}$ D_p-branes gives rise to a non-Abelian $U(N)$ gauge theory. Dp-branes are dynamical objects in string theory, emerging naturally from the boundary conditions of open strings. Just as the Polyakov action describes the string dynamics, Dp branes also have a corresponding action governing their behavior. A relevant action named **Born-Infeld action**, which is a non-linear alternative to Maxwell theory is written as:

$$S = -T_p \int d^{p+1}x\, \xi \sqrt{-\det(\eta_{ab} + 2\pi\alpha' F_{ab})} \tag{5.1}$$

Here, ξ represents the worldvolume coordinates of the Dp-brane, and T_p denotes the brane tension. Since T_p acts as an overall factor in the action, it does not influence the equation of motion. The gauge potential A_a is a function of the worldvolume coordinates, expressed as $A_a = A_a(\xi)$.
For small field strengths, $F_{ab} \ll 1/\alpha'$, the action (5.1) coincides with Maxwell's action. To see this, we need simply expand to get:

$$S = -T_p \int d^{p+1}x\, \xi \left(1 + \frac{(2\pi\alpha')^2}{4} F_{ab}F^{ab} + \dots\right) \tag{5.2}$$

The leading order term, quadratic in field strengths, is the Maxwell action. Terms with higher powers of F_{ab} are suppressed by powers of α'.
This Born-Infeld action originates from the one-loop beta function and provides an exact result for constant field strengths. However, to analyze the dynamics of

gauge fields with large gradients (∂F), one must account for higher-loop corrections to the beta function.

5.2 The DBI Action

We have established that the dynamics of gauge fields on the brane is governed by the Born-Infeld action. However, to understand the functions of the brane itself, we need to examine its embedding in spacetime [50]. From this we can see that the brane action should take the form of the Dirac action. A direct approach to verifying this involves computing the equations of the beta function for the scalar fields Φ^I on the brane. Activating these scalars corresponds to bending the brane and modifying its boundary conditions. By analyzing these equations, one can explicitly show that brane fluctuations are indeed governed by the Dirac action.
More generally, one could consider both the dynamics of the gauge field and the fluctuation of the brane. This is governed by a mixture of the Dirac action and the Born-Infeld action, which is usually referred to as the DBI action:

$$S_{DBI} = -T_p \int d^{p+1}x\, \xi \, \sqrt{-\det(\gamma_{ab} + 2\pi\alpha' F_{ab})} \tag{5.3}$$

Here, γ_{ab} is the pull-back of the the spacetime metric onto the worldvolume,

$$\gamma_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b}\eta_{\mu\nu} \tag{5.4}$$

Coupling to Closed String Fields: The DBI action governs the low-energy dynamics of a Dp-brane in flat space. A natural question arises: How does the motion of a D-brane change when it moves in a background influenced by closed string modes, such as the metric $G_{\mu\nu}$, the B-field $B_{\mu\nu}$, and the dilaton Φ?
Instead of deriving it explicitly, we will state the result and then justify each term in the action accordingly. The DBI action in a general curved background takes the form:

$$S_{\mathrm{DBI}} = -T_p \int d^{p+1}\xi \, e^{-\hat{\Phi}} \sqrt{-\det(\gamma_{ab} + 2\pi\alpha' F_{ab} + B_{ab})} \tag{5.5}$$

The background Metric $G_{\mu\nu}$: Let us start with the coupling to the background metric $G_{\mu\nu}$. It us actually hidden in the notation in this expression: it appears in the pull-back metric γ_{ab} which is now given by:

$$\gamma_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b}G_{\mu\nu} \tag{5.6}$$

The Dilaton field: It us decomposed into a constant part and a varying part, $\Phi = \Phi_0 + \tilde{\Phi}$, where $g_s = e^{\Phi_0}$ represents the asymptotic string coupling. The

D-brane tension, scaling as $T_p \sim 1/g_s$, depends on the local value of the dilaton rather than its asymptotic value. This explains the $e^{-\tilde{\Phi}}$ factor in the action, which accounts for local dilaton variations. The effective string coupling at a spacetime point X is given by $g_s^{eff} = e^{\Phi(X)} = g_s\, e^{\tilde{\Phi}(X)}$. Consequently, the D-brane tension varies with the dilation decreasing in regions where g_s^{eff} is larger.
The $B_{\mu\nu}$ field: This is a 2-form in spacetime. The function B_{ab} appearing in the DBI action is the pull-back onto the worldvolume:

$$B_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} B_{\mu\nu}$$

Its appearance in the DBI action is actually required on grounds of gauge invariance alone. This can be seen by considering an open string, moving in the presence of both a background $B_{\mu\nu}(X)$ in spacetime and a background $A_a(X)$ on the worldvolume of a brane. The relevant terms on the string worldsheet are:

$$\frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d^2\sigma\, \epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu} + \int_{\partial\mathcal{M}} d\tau\, A_a \dot{X}^a$$

5.3 The Chern-Simons action

Since we are interested in R-R charged object, let us move on to them. Potentials in the R-R sector are:

$$\begin{aligned} \text{type IIA} &\quad : \quad C_1, C_3, C_5, C_7 \\ \text{type IIB} &\quad : \quad C_0, C_2, C_4, C_6, C_8 \end{aligned}$$

Dp branes are *p*-dimensional extended objects which couple to all these via an electric coupling of the form (thus R-R charged):

$$\mu_p \int_{M_{p+1}} C_{p+1}$$

In Type IIA string theory there are stable *Dp*-branes with *p* even and in Type IIB string theory there are stable *Dp*-branes with *p* odd. The above action is also called Chern-Simons action and it is the higher dimensional generalization of a charged particle coupled to a gauge potential. Lastly:

$$S_{Dp} = S_{DBI} + S_{CS} \tag{5.7}$$

5.3.1 Brane Cosmology

The discovery of D-branes and the Horava-Witten scenario has led to the development of the brane-world scenario in string theory. In Type IIA, IIB and Type I string theories multiple D-branes can support gauge and matter fields, with at least one accommodating the Standard Model. **Open strings** are confined on D-branes, restricting their endpoints, and **closed strings**, including gravity and the dilaton, propagate through the extra dimensions. For space-times with a product geometry, the four dimensional Planck mass is given by: $M_p^2 \sim M_s^8 R^6$, with R the size of the extra dimension and M_s the string scale. This is the source to claim that large extra dimensions allow the possibility of low M_s as long as the Planck mass is fixed to the experimentally known value.
The Randall-Sundrum model [51] extends the idea of large extra dimensions by introducing a warp factor in a five-dimensional metric:

$$ds^2 = W(y) g_{\mu\nu} dx^\mu dx^\nu + dy^2$$

Here, y represents the extra dimension, and $W(y)$ is the warp factor, which weights variations in the metric at different location along y. **Branes** fixed at different positions in y experience different metric scales due to $W(y)$. **Randall and Sundrum** found an exponential dependence in $W(y)$, enabling small fundamental scales even with moderately large extra dimensions. They also showed that, with 5d anti-de Sitter space and fine-tuning of the cosmological constant, gravity can be localized on the brane, even with infinitely large extra dimensions. These findings sparked significant interest in the physical implications of extra dimensions, extending beyond the string theory community into cosmology.

5.4 Brane-Antibrane Inflation

String theory as a leading candidate for quantum gravity, has inspired various inflationary models, particularly those involving Dp-branes. In this framework, D-branes, as dynamical objects moving through extra dimensions, provide a natural setting for inflation. Among these models, brane-antibrane inflation is one of the most promising, where the interbrane distance serves as the **inflaton field**. It offers a UV-complete realization of supersymmetric hybrid inflation, but faces challenges such as the η-problem, where achieving sufficient inflation in flat space requires an interbrane distance larger than the compactification scale.

Brane-antibrane inflation is a well-established model in string theory, where Dp-branes interact through Ramond-Ramond (RR) fields [23, 52]. In a system of two parallel branes with the same charge, the gravitational attraction is exactly canceled by the repulsive RR interaction, maintaining a static, supersymmetric

configuration. However, when a brane-antibrane pair is introduced, supersymmetry is completely broken, leading to a finite energy density that can drive inflation. In this setup, the RR-mediated interaction becomes attractive, resulting in a net force between the brane and antibrane, which plays a crucial role in inflationary dynamics.
The simplest example of brane-antibrane inflation in Type IIB string theory involves a stack of N_c (the number of branes in the stack) D3-branes and $\tilde{N}_c$ anti-D3-branes spanning the non-compact directions 0123 and separated by a distance r in a six-dimensional flat torus with volume $V_6 = L^6$.

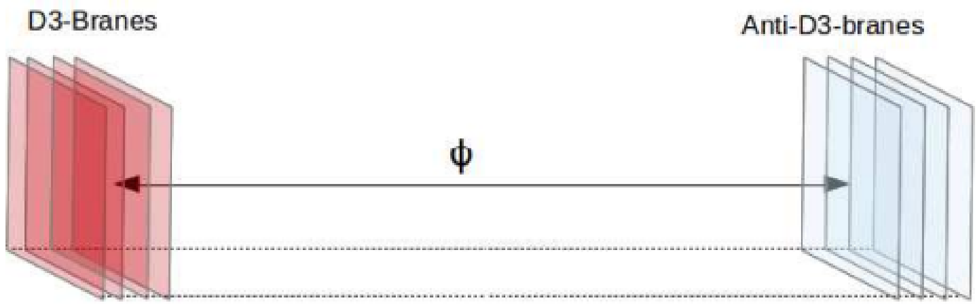

Figure 5.1: Brane-antibrane inflation is implemented with a stack of branes and a stack of antibranes separated within a compact space.

Inflation will occur as the brane and antibrane stacks move towards each other through the higher-dimensional bulk space, with the relative brane-antibrane position parameterizing the inflaton field ϕ and the inflaton potential $V(\phi)$ arising from their interactions. Mainly, the inflaton field corresponds to the interbrane separation, and its effective potential determines whether the model can produce the required inflation. We assume inflation starts with interbrane distance $r \gg \ell_s$, where ℓ_s is the fundamental string length. At large separations, the interaction force between the brane and antibrane, mediated by gravity and Ramond-Ramond (RR) fields, can be well approximated by a Coulomb-like potential. In this regime, the effective potential governing the interaction is given by [52]:

$$V(r) = 2T_3 \left(1 - \frac{1}{2\pi^3 m_{10,p}^8 r^4} T_3\right) \tag{5.8}$$

Here, T_3 represents the D3-brane tension, and the 10D Planck mass is defined as $m_{10,p}^{-8} = 8\pi G_{10}$. The 4D Planck mass M_p is related to the 10D Planck mass through the relation: $M_p^2 = m_{10,p}^{-8} L^6$, where L^6 is the volume of the compact manifold M. This expression can be reformulated in terms of a canonically normalized scalar field ϕ, yielding a potential that, for large field values, can serve as a driving mechanism for inflation [9]:

$$V(\phi) = 2T_3 \left(1 - \frac{1}{2\pi^3 m_{10,p}^8 \phi^4} T_3^3\right) \tag{5.9}$$

We therefore have a potential that can be responsible for driving inflation.
η-problem: Note that the string scale M_s can be expressed in terms of the 4D Planck scale M_p as:

$$M_s = \frac{g_s M_p}{\sqrt{4\pi \mathcal{V}_s}} \tag{5.10}$$

where $\mathcal{V}_s$ denotes the Calabi-Yau volume in string frame measured in units of the string length ℓ_s. The potential (5.8) however would become flat enough to drive inflation only at distances larger than the size of the extra dimensions [52]. This can be easily seen by computing the second slow-roll parameter η:

$$\eta = M_p^2 \frac{V_{\phi\phi}}{V} \simeq -\frac{10}{\pi^3} \frac{\mathcal{V}_s}{(rM_s)^6} \tag{5.11}$$

Here, the derivatives are taken with respect to the canonically normalized inflaton $\phi = \sqrt{T_3} r$. If the Calabi-Yau volume is isotropic, the maximum possible value for r is given by $r_{max} \approx V_s^{1/6} M_s^{-1}$. This leads to:

$$|\eta| \gtrsim \frac{10}{\pi^3} \approx 0.3 \tag{5.12}$$

which is too large to allow for sufficient e-foldings of inflation. For solving this, we focusing on Calabi-Yau spaces with a warped throat, which can help by modifying the potential structure.

5.4.1 Brane-antibrane inflation in warped compactifications

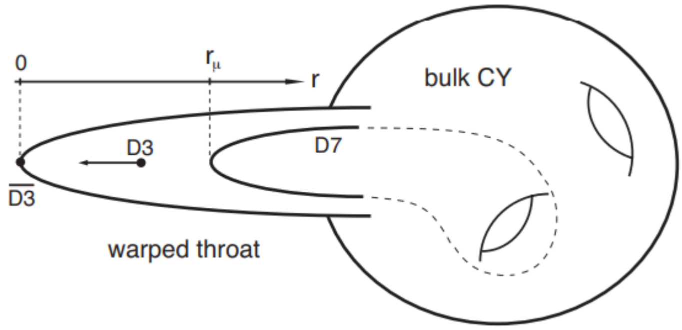

Figure 5.2: Cartoon of an embedded stack of D7-branes wrapping a four-cycle Σ_4, and a mobile D3-brane, in a warped throat region of a compact Calabi-Yau. [53].

String theory offers a promising foundation for the inflationary paradigm, yet constructing explicit and well-controlled inflationary models remains a challenge. This study explores the feasibility of working models within the framework of slow-roll warped D-brane inflation, where the inflaton field corresponds to the position of a mobile D3-brane moving within a warped throat region of the compactification manifold.

Our starting point is a compactification of Type IIB string theory on a Calabi-Yau orientifold in the presence of fluxes. The metric of the 10D space factorises and is given by:

$$ds^2 = h^{-\frac{1}{2}} \left(-dt^2 + d\vec{x}^2\right) + h^{\frac{1}{2}} \left(dr^2 + \frac{r^2}{R^2} \tilde{g}_{ab} dy^a dy^b\right) \tag{5.13}$$

where $\tilde{g}_{ab} dy^a dy^b$ is the line element on X_5, and $h(r)$ is given by

$$h(r) = \frac{R^4}{r^4} \tag{5.14}$$

It is easy to check that $h(r)$ is a harmonic function in a six-dimensional space spanned by r and the directions along X_5, with metric

$$ds_6^2 = dr^2 + \frac{r^2}{R^2} \tilde{g}_{ab} dy^a dy^b \tag{5.15}$$

If the warped factor can be written in the form [54]:

$$\frac{e^{-A(r)}}{\mathcal{V}^{1/6}} \approx \frac{R}{r} \tag{5.16}$$

the dynamics of the brane/antibrane in a warped geometry can be written as [47]:

$$S_{DBI} = T_3 \int d^4x \sqrt{-\det\left(\tilde{\gamma}_{ab} + \tilde{g}_{\alpha\bar{\alpha}} \partial_a z^\alpha \partial_b \bar{z}^{\bar{\alpha}}\right)}$$
$$= \int d^4x \left(T_3 \left(1 + \frac{e^{-4A(r)}}{\mathcal{V}^{2/3}}\right)^{-1} + \frac{1}{2} g_{\alpha\bar{\alpha}} \partial_\mu \left(\sqrt{T_3} z^\alpha\right) \partial^\mu \left(\sqrt{T_3} \bar{z}^{\bar{\alpha}}\right) + \mathcal{O}(\partial z)^4 \right). \tag{5.17}$$

The DBI action for a D3-brane consists of a vacuum energy term and kinetic terms for the brane position moduli. For D3-branes, the vacuum energy cancels due to the Chern-Simons action, while for anti-D3-branes, it doubles, creating a potential. This potential is minimized at the bottom of the warped throat, where the warp factor is maximized, stabilizing the antibrane in the most strongly warped region. This mechanism is crucial for brane-antibrane inflation, ensuring natural stabilization in the extra dimensions. The metric $g_{\alpha\bar{\alpha}}$ appearing in (5.17) is therefore given by:

$$g_{\alpha\bar{\alpha}} = \frac{\partial^2 k(z_\alpha, \bar{z}_{\bar{\alpha}})}{\partial z_\alpha \partial \bar{z}_{\bar{\alpha}}}$$

The inflaton in brane-antibrane inflaton is the radial distance r between the D3-brane and the $\bar{D}$3-brane at the tip of the throat. It correspondence to the complex coordinate $z \sim \bar{z} \sim r^{3/2}$, leading to the metric component $g_{\alpha\bar{\alpha}} \sim r^{-1}$. Substituting this result into the kinetic Lagrangian, the canonically normalized inflaton is found as $\phi = \sqrt{T_3} r$. A proper supergravity embedding requires incorporating the volume mode into the three-level Kähler potential, given by:

$$K = -3 \ln[T + \bar{T} - \gamma k(z_\alpha, \bar{z}_{\bar{\alpha}})] \tag{5.18}$$

The kinetic terms for r can be computed, yielding $\gamma \sim \langle (T + \bar{T}) \rangle T_3$, which aligns with the kinetic terms obtained from the DBI action. This result plays a key role in embedding brane-antibrane inflation within supergravity frameworks.
The Coulomb potential in D3-brane inflation arises from the backreaction of the D3-brane, which modifies the background geometry. When a D3-brane is placed at position r in the warped compactification, it perturbs the warp factor, leading to a correction described by a harmonic function. This backreaction generates an attractive Coulomb-like potential, influencing the motion of the D3-brane and playing a crucial role in inflationary dynamics.

$$h(r) = \frac{R^4}{r^4} + \delta h(r) \qquad or \qquad 1 + \frac{e^{-4A}}{\mathcal{V}^{2/3}} \to 1 + \frac{e^{-4A}}{\mathcal{V}^{2/3}} + \delta h(r) \tag{5.19}$$

In the conifold case the zero mode of the Laplacian in the angular directions gives the profile $\delta h(r) = \beta/(T_3 r^4)$, with $\beta = 27/(32\pi^2)$, while other modes in the multipole decomposition give higher orders in $1/r$. Including this leading effect in the DBI action for the antibrane in the perturbed background, one finds:

$$S_{\rm DBI} = \int d^4x T_3 \big(1 + \frac{e^{-4A(r)}}{\mathcal{V}^{2/3}} + \frac{\beta}{T_3 r^4}\big)^{-1} \tag{5.20}$$

The resulting inflationary potential takes the form:

$$V(r) = C_0 \big(1 - \frac{D_0}{r^4}\big) \tag{5.21}$$

where in terms of the D3-brane tension T_3, the string coupling g_s and the string length $\ell_s = 2\pi\sqrt{\alpha'} = M_s^{-1}$ [23]:

$$C_0 \equiv 2T_3 \mathcal{V}^{2/3} e^{-8\pi K/(3g_s M)} \quad \text{and} \quad D_0 \equiv \frac{27}{32\pi^2 T_3} \mathcal{V}^{2/3} e^{-8\pi K/(3g_s M)} \tag{5.22}$$

Note that the string scale M_s can be expressed in terms of the 4D Planck scale M_p as:

$$M_s = \frac{g_s M_p}{\sqrt{4\pi \mathcal{V}_s}}$$

Here, M represents the quantized flux of F_3 (the R-R 3-form field strength) on the S^3 at the tip of the throat, while K is the quantized flux of H_3 (the NS-NS 3-form field strength) on its Poincaré dual 3-cycle. The Calabi-Yau volume in the Einstein frame is denoted by $\mathcal{V}$ and is related to the volume in the string frame as $\mathcal{V}_s = g_s^{3/2} \mathcal{V}$. The new η-parameter can now be easily very small due to the warping suppression factor:

$$|\eta| \simeq \frac{135}{8\pi^3} \frac{\mathcal{V}_s}{(rM_s)^6} \mathcal{V}^{2/3} e^{-8\pi K/(3g_s M)} \ll 1 \tag{5.23}$$

From the above discussion we can see that, by incorporating the warp-factor, we can mitigate the η-problem, making slow-roll more viable. However, this analysis does not account for the fact that inflation occurs in a compactified setting, where closed string moduli must be properly stabilized throughout the inflationary process.

Chapter 6

Angular Moduli Stabilization and Inflation

6.1 Non-perturbative Corrections and Fine-Tuning

In the previous discussion, we looked at how brane-antibrane inflation works. However, to get a complete picture, we also need to consider moduli stabilization, which is essential for a consistent setup. In this chapter, we will explore brane-antibrane inflation while taking into account all the key parameters needed to properly define the inflation potential. Also in this section we can see the reflection of the famous work "KKLMMT" [52].
A particularly important field in this context is the overall volume mode $\mathcal{V}$, which influences the dynamics of the system. The string scale M_s can be expressed in terms of this modulus as:

$$M_s = \frac{g_s^{1/4} M_p}{\sqrt{4\pi\mathcal{V}}} \tag{6.1}$$

so the inflationary potential is a function of both r and $\mathcal{V}$:

$$V_{\text{inf}}(r, \mathcal{V}) = \frac{C_0}{\mathcal{V}^{4/3}} \left[1 - \frac{\mathcal{D}_0}{(rM_{KK})^4}\right] \tag{6.2}$$

where

$$C_0 = \frac{M_p^4}{4\pi\mathcal{V}^{4/3}} e^{-8\pi K/(3g_s M)} \equiv \frac{C_0}{\mathcal{V}^{4/3}}$$

and

$$\mathcal{D}_0 \equiv \left(\frac{3}{4\pi}\right)^3 e^{-8\pi K/(3g_s M)}$$

M_{KK} is the Kaluza-Klein (KK) scale given in terms of the stabilised volume $\langle\mathcal{V}\rangle$ when the D3-brane is near the tip of the throat:

$$M_{KK} = \frac{M_s}{\langle\mathcal{V}_s\rangle^{1/6}} = \frac{M_p}{\sqrt{4\pi}\langle\mathcal{V}\rangle^{2/3}}$$

Written in terms of the canonically normalized inflaton ϕ, the inflationary potential (6.2) takes the form:

$$V_{\rm inf}(\phi, \mathcal{V}) = \frac{C_0}{\mathcal{V}^{4/3}}\left(1 - \frac{C_1}{\varphi^4}\right) \quad \text{with} \quad C_1 \equiv \frac{\mathcal{D}_0 T_3^2}{M_{KK}^4} \tag{6.3}$$

During inflation, the volume mode $\mathcal{V}$ must be stabilized to prevent unwanted runaway behavior in directions orthogonal to the inflationary trajectory. Previous attempts, such as those in [52], used non-perturbative corrections to the superpotential to fix $\mathcal{V}$. However, these corrections introduce a large mass contribution to the inflaton, disrupting the required flatness of the potential. This issue is a direct consequence of the η-problem, which commonly affects inflationary models in supergravity and string theory.
The core of this problem lies in the distinction between the holomorphic superfield T, which appears in the superpotential W, and the physical Calabi-Yau volume $\mathcal{V}$. The proper relation between these variables is given by:

$$T + \bar{T} = \mathcal{V}^{2/3} + \gamma r^2 \tag{6.4}$$

where the γ is proportional to $T_3\langle T+\bar{T}\rangle$. Consequently, T-dependent non-perturbative corrections generate a potential V_{np} that depends on both $\mathcal{V}$ and r, after using (6.4). In fact, after fixing $(T - \bar{T})$, V_{np} looks like:

$$V_{\rm np}(r, \mathcal{V}) = \frac{1}{\mathcal{V}^{4/3}} U_{\rm np}(T + \bar{T}) \tag{6.5}$$

Using (6.4) and writing (6.2) as $V_{inf} = \mathcal{V}^{-4/3} U_{inf}(r)$, and for $\gamma r^2 \ll (T + \bar{T})$, the total scalar potential hence becomes:

$$V_{\rm tot} = \frac{1}{(T+\bar{T})^2}\left[U_{\rm np}(T+\bar{T}) + U_{\rm inf}(r)\right]\left(1 + \frac{2\gamma r^2}{(T+\bar{T})}\right) \tag{6.6}$$

At the end of inflation, the D3-brane annihilates with the anti-D3-brane, causing the inflationary potential to vanish. What remains is a non-perturbative potential that naturally leads to an AdS vacuum, requiring an additional uplifting term C_{up} to achieve a stable vacuum. The late-time potential is minimized at $\langle T + \bar{T}\rangle$, ensuring $U_{np} + C_{up} \approx 0$. During inflation, the Kähler modulus shift is negligible, and the inflationary potential is expressed in terms of the radial distance r. By rewriting it using the canonically normalized inflaton ϕ, the potential takes the form:

$$V_{inf} = V_0(\phi)\left(1 + \frac{1}{3}\frac{\phi^2}{M_p^2}\right) \tag{6.7}$$

This illustrates how the brane-antibrane interaction governs inflation and post-inflationary vacuum stabilization.
The inflationary potential is modified by the Planck-suppressed 6D operator, which introduces a large correction to the slow-roll parameter η, potentially ruining inflation with a shift of $\Delta\eta = 2/3$. However, [52] suggested that non-perturbative corrections to the superpotential depend on ϕ, requiring an additional subleading term in the potential. This leads to a revised form of the inflationary potential with an extra term $P(\phi)$, defined as the ratio of the subleading correction U_{sub} to U_{inf}. If $P(\phi)$ introduces a correction to η of order $-2/3$, it cancels the dangerous Planck-suppressed contribution, allowing for inflection point inflation around a specific value ϕ_0. This tuning enables inflation to occur only in a localized region, resulting in a finely controlled inflationary scenario.
The fine-tuned microscopic parameters effectively cancel the dangerous contribution to the inflaton mass over a range of ϕ. This relies on the assumption that the non-perturbative superpotential prefactor $A(\phi)$ [53] which contributes to the inflaton potential, contains a quadratic term in ϕ that stabilizes η. However, it is observed that the functional form of $A(\phi)$ does not actually permit a purely quadratic correction, since A is a holomorphic function of the coordinates z_α, which scales as $z_\alpha \propto \phi^{3/2}$. Consequently, the presence of $A(\phi)$ does not lead to the required quadratic terms, undermining the possibility of a fine-tuned cancellation of the inflaton mass across an extended range of ϕ.

6.2 Perturbative Corrections and Slow-Roll

Recent work [23] demonstrates that incorporating perturbative corrections in moduli stabilization for brane-antibrane inflation can help resolve the η-problem. Perturbative corrections in moduli stabilization provide a potential resolution to this issue. The stabilization of moduli fields, particularly the Kähler modulus τ, plays a fundamental role in controlling the dynamics of the extra-dimensional volume. Instead of directly working with τ, it is useful to redefine the physical Calabi–Yau volume σ, which incorporates the D3-brane position modulus r as:

$$\sigma = \tau - \frac{1}{6}(M_{KK}r)^2 \tag{6.8}$$

The stabilization of σ, rather than τ, during inflation ensures the resolution of the η-problem. As inflation progresses, $r \to 0$, leading to $\sigma \to \tau$. In a 4D supergravity effective field theory (EFT), the expansion in powers of a nilpotent superfield X allows for a systematic treatment of non-linear supersymmetry. The Kähler potential and superpotential take the following forms [55, 56, 57]:

$$K = -3\ln[f(\sigma)] + (X+\bar{X})g(\sigma) - X\bar{X}h(\sigma) \tag{6.9}$$

$$W = W_0 + XW_X(r) \tag{6.10}$$

Due to the axionic shift symmetry, K is independent of $(T+\bar{T})$, and perturbative corrections ensure the superpotential remains independent of T. The functions $f(\sigma), g(\sigma), h(\sigma)$ can be expanded in powers of $1/\sigma$, with loop corrections providing logarithmic enhancements. The F-term scalar potential is derived by taking derivatives with respect to τ, r and X, then setting $X = \bar{X} = 0$ and assuming $r \ll M_K^{-1}$, leading to the equation:

$$e^{-K}V = \left[K^{X\bar{X}}W_X^2 + W_0 W_X\left(K^{X\bar{A}}K_A + K^{\bar{X}A}K_{\bar{A}}\right) + W_0^2\left(K^{AB}K_{\bar{A}}K_B - 3\right)\right]\Big|_{X=0} \tag{6.11}$$

where the index A runs just over T and X since $K_r \simeq 0$ for $r \ll M_{KK}^{-1}$, and we assumed without loss of generality that $W_X \in \mathbb{R}$. Using the notation $dy/d\sigma \equiv y'$, the F-term potential becomes:

$$V = \frac{1}{U}\left[(f'W_X - 3g'W_0)^2 - f''\left(fW_X^2 - 6gW_XW_0 - 9hW_0^2\right)\right] \tag{6.12}$$

where

$$U \equiv 3f^2\left(2gf'g' - fg^2 + f'^2h - f''(g^2+fh)\right) \tag{6.13}$$

At tree-level we have $f = \sigma$, $g = 0$ and $h = 1$. In this case one has therefore $f' = 1$ and $g' = f'' = 0$, which implies that (6.12) simply reduces to:

$$V = \frac{W_X(r)^2}{3\sigma^2} \tag{6.14}$$

As expected, this is the standard D3-brane uplift contribution if we identify W_X with the warp factor. Introducing also the dependence on r as:

$$W_X(r) = e^{-2\rho}\sqrt{\frac{3}{4\pi}\left(1 - \frac{\mathcal{D}_0}{(rM_{KK})^4}\right)} \quad \text{with} \quad \rho \equiv \frac{2\pi K}{3g_sM} \tag{6.15}$$

the potential (6.14) reproduces the inflationary potential (6.2). However σ would be an unstable runaway direction at tree-level. We need therefore to add quantum corrections to fix σ.

At the tree level, the potential derived above predicts an unstable trajectory for σ, which would result in a runaway direction during inflation. To stabilize σ, quantum corrections must be incorporated into the function $f(\sigma)$, leading to a modified potential of the form:

$$V \simeq \frac{W_X^2}{3\sigma^2} + 3W_0^2 \left[\frac{\alpha}{\sigma^4} - \frac{\xi\sqrt{g_s}}{4c\sigma^{9/2}} \left(\ln\sigma - \frac{c}{g_s^2} \right) \right] \tag{6.16}$$

Here, the additional terms dependent on W_0 ensure that the potential remains stable throughout inflation. These quantum corrections are of similar magnitude to the late-time minimum τ_{min}, preventing excessive shifts in the minimum during inflation.

Since the potential is primarily controlled by W_X, maintaining a stable minimum requires W_X to remain sufficiently small, such that inflationary and late-time minima remain close to each other. If W_X satisfies

$$W_X \approx e^{-2\rho} \ll 1$$

, then the first term in the above potential contributes only a minor correction to the late-time minimum, resulting in an inflationary vacuum energy given by:

$$V(\sigma_{\rm min}) \simeq \frac{W_X^2}{3\sigma_{\rm min}^2} \simeq \frac{e^{-\frac{2c}{g_s^2}}}{3\lambda_0^2} W_X^2 \tag{6.17}$$

Substituting W_X as a function of r reproduces the brane-antibrane potential (6.2), ensuring a well-stabilized volume modulus through perturbative corrections, thereby solving the η-problem.

6.3 Angular Directions Stabilisation

As mentioned above, in brane-antibrane inflation, the role of the inflaton is played by the radial distance between the D3-brane and the anti-D3-brane at the tip of the throat. To ensure stability of this inflationary trajectory, we now investigate where the angular directions can be appropriately stabilized by exploiting the fact that the prefactor of non-perturbative corrections introduces a dependence of the scalar potential on these angular modes. This whole discussion and the mathematical rigorousness is influenced by this two amazing work [58] and [53].

6.3.1 Warped Volume and Non-perturbative Superpotential

The non-perturbative effects discussed earlier depend exponentially on the warped volume of the associated four-cycle. This warped volume plays a crucial role in two contexts: it governs the instanton action for Euclidean D3-branes and determines

the gauge coupling for strong gauge dynamics on D7-branes.
A warped background metric is given by:

$$\mathrm{d}s^2 = h^{-1/2}(Y)g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + h^{1/2}(Y)g_{ij}\mathrm{d}Y^i \mathrm{d}Y^j \tag{6.18}$$

where $h(Y)$ is the warp factor. Additionally, the Yang-Mills coupling g_7 for the $7+1$ dimensional gauge theory on a stack of D7-branes is given by [59]:

$$g_7^2 = 2(2\pi)^5 g_s(\alpha')^2 \tag{6.19}$$

This expression shows how the gauge coupling is influenced by the string coupling g_s and the string length scale α' emphasizing the impact of warping on non-perturbative effects in string theory. The action for gauge fields on D7-branes that wrap a four-cycle Σ_4 is [58]:

$$S = \frac{1}{2g_7^2}\int_{\Sigma_4} d^4\xi\sqrt{g^{ind}}h(Y)\cdot\int d^4x\sqrt{g}\, g^{\mu\alpha}g^{\nu\beta}\mathrm{Tr}F_{\mu\nu}F_{\alpha\beta} \tag{6.20}$$

Defining the warped volume of Σ_4 [60]:

$$V^w_{\Sigma_4} \equiv \int_{\Sigma_4} d^4\xi\sqrt{g^{ind}}h(Y) \tag{6.21}$$

we read off the gauge coupling of the four-dimensional theory from (6.20) [61]:

$$\frac{1}{g^2} = \frac{V^w_{\Sigma_4}}{g_7^2} = \frac{T_3 V^w_{\Sigma_4}}{8\pi^2} \tag{6.22}$$

In $N=1$ super-Yang-Mills theory, the Wilsonian gauge coupling is given by the real part of a holomorphic function, which receives one-loop corrections but no higher-order perturbative corrections. In the case of $SU(N_{D7})$ super-Yang-Mills, the modulus of the gaugino condensate superpotential depends on the ultraviolet cutoff M_{UV} and plays a crucial role in non-perturbative effects within the theory. So, we get [62, 63, 64]:

$$|W_{\mathrm{np}}| = 16\pi^2 M^3_{\mathrm{UV}}\exp\left(-\frac{1}{N_{D7}}\frac{8\pi^2}{g^2}\right) \propto \exp\left(-\frac{T_3 V^w_{\Sigma_4}}{N_{D7}}\right) \tag{6.23}$$

In the case that the non-perturbative effect comes from a Euclidean D3-brane, the instanton action is:

$$S = T_3\int_{\Sigma_4}\mathrm{d}^4\xi\sqrt{G^{\mathrm{ind}}} = T_3\int_{\Sigma_4}\mathrm{d}^4\xi\sqrt{g^{\mathrm{ind}}}h(Y) \equiv T_3 V^w_{\Sigma_4} \tag{6.24}$$

The modulus of the non-perturbative superpotential is then:

$$|W_{\mathrm{np}}| \propto \exp\left(-T_3 V^w_{\Sigma_4}\right) \tag{6.25}$$

The displacement of a D3-brane in a warped compactification leads to a small distortion δh in the background, affecting the warped volume of four-cycles. This correction is expressed as an integral over the four-cycle Σ_4, allowing the extraction of its dependence on the D3-brane position. The change in volume is related to a holomorphic function $\zeta(X)$, whose imaginary part comes from the integral of the Ramond-Ramond four-form perturbation.
The displacement of a D3-brane in the compactification creates a slight distortion δh of the warped background, and hence affects the warped volumes of four-cycles. The correction takes the form:

$$\delta V^w_{\Sigma_4} \equiv \int_{\Sigma_4} d^4Y \sqrt{g^{ind}(X;Y)}\, \delta h(X;Y) \tag{6.26}$$

This correction modifies the non-perturbative superpotential, which, in the case of gaugino condensation on D7-branes, takes the form:

$$A(X) = A_0 \exp\left(-\frac{T_3 \zeta(X)}{N_{D7}}\right) \tag{6.27}$$

For Euclidean D3-branes, the change in the instanton action is directly proportional to the warped volume [54], leading to a similar expression with $N_{D7} = 1$. A unified expression for both cases is given by:

$$A(X) = A_0 \exp\left(-\frac{T_3 \zeta(X)}{n}\right) \tag{6.28}$$

where $n = N_{D7}$ for D7-brane gaugino condensation and $n = 1$ for Euclidean D3-brane instantons. This result demonstrates how the position of the D3-brane influences non-perturbative effects, affecting moduli stabilization and the inflationary dynamics.
We now recall some relevant geometry. The singular conifold is a non-compact Calabi-Yau threefold defined as the locus in $\mathbb{C}^4$:

$$\sum_{i=1}^{4} z_i^2 = 0 \tag{6.29}$$

After a linear change of variables ($w_1 = z_1 + iz_2, w_2 = z_1 - iz_2$, etc.), the constraint (6.29) becomes:

$$w_1 w_2 - w_3 w_4 = 0 \tag{6.30}$$

The Calabi-Yau metric on the conifold is:

$$ds_6^2 = dr^2 + r^2 ds^2_{T^{1,1}} \tag{6.31}$$

The base of the cone is the $T^{1,1}$ coset space $(SU(2)_A \times SU(2)_B)/U(1)_R$ whose metric in angular coordinates $\theta_i \in [0,\pi], \Phi_i \in [0,2\pi], \psi \in [0,4\pi]$ is:

$$ds^2_{T^{1,1}} = \frac{1}{9}\left(d\psi + \sum_{i=1}^{2} \cos\theta_i d\phi_i\right)^2 + \frac{1}{6}\sum_{i=1}^{2}\left(d\theta_i^2 + \sin^2\theta_i d\phi_i^2\right) \tag{6.32}$$

A stack of N D3-branes placed at the singularity $w_i = 0$ back-reacts on the geometry, producing the ten-dimensional metric:

$$ds_6^2 = dr^2 + r^2 ds^2_{T^{1,1}} \tag{6.33}$$

where the warp factor is:

$$h(r) = \frac{27\pi g_s N(\alpha')^2}{4r^4} \tag{6.34}$$

This describes the $AdS_5 \times T^{1,1}$, background in Type IIB string theory, which is dually related to an $N = 1$ supersymmetric conformal gauge theory. The dual field theory consists of an $SU(N)\times SU(N)$ gauge group coupled to bi-fundamental chiral superfields A_1, A_2, B_1, B_2, each with an R-charge of 1/2, and transforming as doublets under the $SU(2)_A \times SU(2)_b$ global symmetry.
The complex coordinates z_i are related to the real coordinates

$$r \in [0,\infty], \qquad \theta_i \in [0,\pi], \qquad \Phi_i \in [0,2\pi], \qquad \psi \in [0,4\pi] \tag{6.35}$$

via:

$$z_1 = \frac{r^{3/2}}{\sqrt{2}} e^{t\psi}\left[\cos\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\phi_1+\phi_2}{2}\right) + i\cos\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\phi_1+\phi_2}{2}\right)\right],$$
$$z_2 = \frac{r^{3/2}}{\sqrt{2}} e^{t\psi}\left[-\cos\left(\frac{\theta_1+\theta_2}{2}\right)\sin\left(\frac{\phi_1+\phi_2}{2}\right) + i\cos\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\phi_1+\phi_2}{2}\right)\right],$$
$$z_3 = \frac{r^{3/2}}{\sqrt{2}} e^{t\psi}\left[-\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\phi_1-\phi_2}{2}\right) + i\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\phi_1-\phi_2}{2}\right)\right],$$
$$z_4 = \frac{r^{3/2}}{\sqrt{2}} e^{t\psi}\left[-\sin\left(\frac{\theta_1+\theta_2}{2}\right)\sin\left(\frac{\phi_1-\phi_2}{2}\right) - i\sin\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\phi_1-\phi_2}{2}\right)\right].$$

The complex coordinates w_i can be written as:

$$w_1 = r^{3/2} e^{\frac{i}{2}(\psi-\phi_1-\phi_2)} \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}, \tag{6.36}$$

$$w_2 = r^{3/2} e^{\frac{i}{2}(\psi+\phi_1+\phi_2)} \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}, \tag{6.37}$$

$$w_3 = r^{3/2} e^{\frac{i}{2}(\psi-\phi_1+\phi_2)} \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}, \tag{6.38}$$

$$w_4 = r^{3/2} e^{\frac{i}{2}(\psi-\phi_1+\phi_2)} \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}. \tag{6.39}$$

It was shown in [65] that the following holomorphic four-cycles admit supersymmetric D7-branes:

$$f(w_i) \equiv \prod_{i=1}^{4} w_i^{p_i} - \mu^P = 0 \tag{6.40}$$

Here $p^i \in \mathbb{Z}, P \equiv \sum_{i=1}^4 P_i$, and $\mu \in \mathbb{C}$ are constants defining the embedding of the D7-branes. Now if we moving the D3-brane to X from some reference point X_0. If we choose the reference point X_0 to be at the tip of the cone, $r = 0$, then:

$$\delta h = \frac{27\pi g_s(\alpha')^2}{4r^4} \left[\sum_i \frac{c_i f_i(\theta_1, \theta_2, \phi_1, \phi_2, \psi)}{r^{\Delta_i}} \right] \tag{6.41}$$

Integrating (6.41) term by term as prescribed in (6.26), we find that the final result for a general embedding (6.40) is:

$$T_3 \delta V^w_{\Sigma_4} = T_3 \operatorname{Re}\left(\zeta(w_i)\right) = -\operatorname{Re}\left(\log \left[\frac{\mu^P - \prod_{i=1}^4 w_i^{p_i}}{\mu^P} \right] \right) \tag{6.42}$$

so that:

$$A = A_0 \left(\frac{\mu^P - \prod_{i=1}^4 w_i^{p_i}}{\mu^P} \right)^{1/n} \tag{6.43}$$

6.3.2 F-term Potential Dependence on the Angular Directions

Before examining how the F-term potential depends on the angular directions, we first need to understand the form of the Kähler potential in terms of the angles. According to [53], the combined Kähler potential for the volume modulus ρ and the three open string moduli (D3-brane positions) z^α follows the form proposed by De Wolfe and Giddings:

$$\kappa^2 \mathcal{K}(T, \bar{T}, z_\alpha, \bar{z}_\alpha) = -3 \ln[T + \bar{T} - \gamma k(z_\alpha, \bar{z}_\alpha)] \equiv -3 \ln U \tag{6.44}$$

where in general $k(z_\alpha, \bar{z_\alpha})$ denotes the Kähler potential of the Calabi-Yau manifold. The normalization constant γ is: (we find out that value in the Appendix)

$$\gamma \equiv \frac{\sigma_0 T_3}{3M_P^2} \tag{6.45}$$

σ_0 is the stabilized value of σ when the D3-brane is at its stabilized configuration. We can represent Eq. (6.43) in the form:

$$A(z^\alpha) = A_0 \left[\frac{f(z^\alpha)}{f(0)} \right]^{1/n} \tag{6.46}$$

where A_0 depends on the stabilized complex structure moduli and has mass dimension 3. The dependence on the position of D3-branes shows up through the embedding function $f(z^\alpha) = 0$ of the four cycle in the Calabi-Yau space, where $f(0)$ represents the value of the embedding function when the D3-brane is stabilized. The total superpotential:

$$W = W_0 + A_0 \left[\frac{f(z^\alpha)}{f(0)}\right]^{1/n} e^{-aT} \tag{6.47}$$

and the Kähler potential (6.44) give rise to the F-term contribution to the scalar potential which depends on the Kähler moduli and the D3-positions:

$$V_F = e^{\kappa^2 \mathcal{K}} \left[\mathcal{K}^{\Sigma\Omega} D_\Sigma W D_\Omega \bar{W} - 3\kappa^2 |W|^2\right] \tag{6.48}$$

The Kähler metric $K_{\Omega\Sigma} \equiv K_{,\Omega\Sigma}$ assumes the form:

$$\mathcal{K}_{\Omega\bar{\Sigma}} = \frac{3}{\kappa^2 U^2} \begin{pmatrix} 1 & -\gamma k_{\bar{\beta}} \\ -\gamma k_\alpha & U\gamma k_{\alpha\bar{\beta}} + \gamma^2 k_\alpha k_{\bar{\beta}} \end{pmatrix} \tag{6.49}$$

Substituting the general superpotential (6.47) as well as the explicit expression for the inverse metric $K^{\Sigma\Omega}$ (see [52, 58]) into (6.48), the explicit form for the non-perturbative F-term scalar potential $V_{np}(\sigma, z^\alpha)$ is given by:

$$\begin{aligned} V_{np}(\sigma, z_\alpha) = \frac{\kappa^2}{3U^2} \Bigg[& \left(T + \bar{T} + \gamma(k_\gamma k^{\bar{\gamma}\beta} k_{\bar{\beta}} - k)\right) |W_T|^2 - 3(\bar{W} W_{,T} + \text{c.c.}) \\ & + \underbrace{\left(k^{\alpha\bar{\delta}} k_{\bar{\delta}} W_T W_\alpha + \text{c.c.}\right) + \frac{1}{\gamma} k^{\alpha\bar{\beta}} W_\alpha \bar{W}_{\bar{\beta}}}_{\Delta V_{np}} \Bigg]. \end{aligned} \tag{6.50}$$

We assume that the mobile D3-brane and the fixed D7-branes are positioned far from the tip of the throat, allowing us to neglect the deformation parameter ϵ. By choosing $z_\alpha = z_1, z_2, z_3$ as the three independent coordinates, the conifold constraint enables us to express z_4 as:

$$z_4 = \pm i \left(\sum_{\alpha=1}^{3} z_\alpha^2\right)^{1/2} . \tag{6.51}$$

Using this basis and the Kähler potential from $k = \frac{3}{2}\left(\sum_{i=1}^4 |z_i|^2\right)^{2/3} = \frac{3}{2} r^2 = \hat{\tau}^2$, we then derive the conifold metric:

$$\begin{aligned} k_{\alpha\bar{\beta}} &= \frac{3}{2} \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \left(\sum_{\gamma=1}^3 |z_\gamma|^2 + \sum_{\gamma=1}^3 z_\gamma^2\right)^{\frac{2}{3}} \\ &= \frac{1}{r}\left[\delta_{\alpha\bar{\beta}} + \frac{z_\alpha \bar{z}_\beta}{|z_4|^2} - \frac{1}{3r^3}\left(z_\alpha \bar{z}_\beta + z_\beta \bar{z}_\alpha - \frac{z_4}{\bar{z}_4} z_\alpha \bar{z}_\beta - \frac{\bar{z}_4}{z_4} z_\alpha z_\beta\right)\right]. \end{aligned} \tag{6.52}$$

Its inverse assumes the simple form:

$$k^{\alpha\bar{\beta}} = r\left[\delta^{\alpha\bar{\beta}} + \frac{1}{2}\frac{z_\alpha \bar{z}_\beta}{r^3} - \frac{z_\beta \bar{z}_\alpha}{r^3}\right] \tag{6.53}$$

This expression, together with $U(\sigma, r) = T + \bar{T} - \frac{3\gamma}{2}r^2$, $\quad k^{\bar{\gamma}\bar{\delta}}k_\delta = \frac{3}{2}\gamma$, $\quad k_\alpha k^{\bar{\alpha}\bar{\beta}}k_\beta = k$, can be inserted into (6.50) to simplify the non-perturbative scalar potential. This potential should be added to the inflationary one generated by perturbative effects, and given by (6.17). Hence the total potential takes the form:

$$V = \frac{W_X^2}{3\sigma_{min}^2} + V_{np}(\sigma, z_\alpha) \tag{6.54}$$

6.3.3 Kuperstein Embedding

The holomorphic embedding proposed by Kuperstein [66] is defined by the equation:

$$f(z_1) = \mu - z_1 = 0 \tag{6.55}$$

This embedding preserves an $SO(3)$ subgroup of the $SO(4)$ global symmetry, which acts on the coordinates of the deformed conifold. It has been shown to be supersymmetric not only for the singular conifold but also within the fully warped deformed conifold background with three-form fluxes.
In a non-compact throat, introducing a mobile D3-brane does not break supersymmetry, ensuring that its interaction with D7-branes vanishes in this limit. However, when the throat is embedded in a compactification, the D3-brane potential can receive contributions from non-perturbative superpotential terms.
The inflaton potential $V(\sigma, r, z_i)$ generally depends on the Kähler modulus as well as the D3-brane radial and angular coordinates. By systematically integrating out all fields except the radial coordinate, an effective single-field potential for the radial inflaton can be obtained.
Equation (6.55) implies:

$$A(z_1) = A_0\left(1 - \frac{z_1}{\mu}\right)^{1/n} \tag{6.56}$$

So the potential ΔV_{np} can be written as:

$$\Delta V_{np} = \frac{\kappa^2 a\,|A(z_1)|^2\, e^{-a(\rho+\bar{\rho})}}{3U(\rho, r)^2}\Big(- 3\,\mathrm{Re}(\alpha_{z_1} z_1) + \frac{r}{a\gamma}\left(1 - \frac{|z_1|^2}{2r^3}\right)|\alpha_{z_1}|^2\Big) \tag{6.57}$$

where

$$\alpha_{z_1} \equiv \frac{A_{z_1}}{A} = -\frac{1}{n(\mu - z_1)} \tag{6.58}$$

and:

$$\text{Re}(\alpha_{z_1} z_1) = -\frac{1}{2n}\frac{\mu(z_1+\bar{z}_1)-2|z_1|^2}{|\mu-z_1|^2} \tag{6.59}$$

We integrated out the imaginary part of the Kähler modulus by setting $T=\tau+ib$, and so the whole non-perturbative F-term potential of (6.50) can be written as:

$$V_{np} = \frac{\kappa^2 a|A|^2 e^{-2a\tau}}{3U^2}\left[(2a\tau+6)+6W_0 e^{a\tau}\,\text{Re}\left(\frac{e^{iab}}{A}\right)-3\,\text{Re}(\alpha_{z_1}z_1)+\frac{r}{a\gamma}\left(1-\frac{|z_1|^2}{2r^3}\right)|\alpha_{z_1}|^2\right] \tag{6.60}$$

Note that this potential only depends on the τ, r and z_1. Therefore, it is invariant under the $SO(3)$ that acts on z_2, z_3, z_4.

6.3.4 Angular Moduli Stabilization

The position of the D3-brane is determined by the radial coordinate r and five angular coordinates Ψ_i on the base of the cone. Since these angles are periodic in a compact space, the potential in Ψ_i is either constant or has discrete minima at specific values Ψ_i^*. To simplify the motion, we focus on trajectories that are stable in the angular directions, ensuring that the brane moves only along the radial direction. This allows us to fix the angular coordinates at their potential minima, effectively reducing the number of degrees of freedom in the system. We will find out the trajectory now by doing some explicit calculation.

Trajectory: As previously mentioned, our goal is to extremize the potential in the angular directions, which requires that $\frac{dV}{d\Psi_i}=0$ for all r. To achieve this, we need to identify specific points in $T^{(1,1)}$ that satisfy this condition, ensuring that the motion of the D3-brane is stable in the angular directions while allowing dynamics along the radial direction. So,

$$\frac{\partial |z_1|^2}{\partial \Psi_i} = 0 = \frac{\partial (z_1+\bar{z}_1)}{\partial \Psi_i} \tag{6.61}$$

The coordinates are introduced in the vicinity of a fiducial point given by $z_0 = (z_1', z_2', z_3', z_4')$. The local coordinates are governed by the five generators of $SO(4)$ acting non-trivially on z_0, given by:

$$z(r,\Psi_i) = \exp(T) z_0 \tag{6.62}$$

The transformation matrix for this stability group is:

$$T = \begin{pmatrix} 0 & \alpha_2 & \alpha_3 & \alpha_4 \\ -\alpha_2 & 0 & \beta_3 & \beta_4 \\ -\alpha_3 & -\beta_3 & 0 & 0 \\ -\alpha_4 & -\beta_4 & 0 & 0 \end{pmatrix} \tag{6.63}$$

where $\Psi_i = (\alpha_i, \beta_i)$ are the local coordinates of the cone.
Equation (6.62) indicates that under a small variation, the change in z_1 (denoted as δz_1) can be written as:

$$\delta z_1 = \sum_{i=2}^{4} \alpha_i z_i' \tag{6.64}$$

This equation means that a small change in the coordinates leads to a shift in z_1, which is a linear combination of z_2', z_3', z_4' weighted by α_i. The squared norm of z_1 is given by :

$$|z_1|^2 = z_1 \bar{z}_1 \tag{6.65}$$

By applying the product rule we found out that:

$$\delta |z_1|^2 = (\delta z_1)\bar{z}_1 + z_1(\delta \bar{z}_1) \tag{6.66}$$

Using the expression for 6.64:

$$\delta |z_1|^2 = \sum_{i=2}^{4} \alpha_i z_i' \bar{z}_1 + \sum_{i=2}^{4} \alpha_i \bar{z}_i' z_1. = \sum_{i=2}^{4} \alpha_i (z_i' \bar{z}_1 + \bar{z}_i' z_1). \tag{6.67}$$

For extremal trajectories, the variation must vanish:

$$\sum_{i=2}^{4} \alpha_i (z_i' \bar{z}_1 + \bar{z}_i' z_1) = 0. \tag{6.68}$$

To satisfy this for arbitrary α_i, we require:

$$z_i' \bar{z}_1 + \bar{z}_i' z_1 = 0, \quad \forall i = 2, 3, 4. \tag{6.69}$$

Rewriting this in terms of real and imaginary parts:

$$z_i' = i \rho_i z_1', \tag{6.70}$$

where $\rho_i \in \mathbb{R}$, we know the conifold equation $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$. By setting $\beta_3 = \beta_4 = 0$, we keep ρ_2 finite. The constraint then simplifies to $z_1'^2 + z_2'^2 = 0$. Since $z_2' = i\rho_2 z_1'$, squaring both sides:

$$z_1'^2 + (i\rho_2 z_1')^2 = 0. \tag{6.71}$$

$$z_1'^2 - \rho_2^2 z_1'^2 = 0. \tag{6.72}$$

$$z_1'^2 (1 - \rho_2^2) = 0. \tag{6.73}$$

This implies $\rho_2 = \pm 1$ so:

$$z_2' = \pm i z_1' \tag{6.74}$$

while the requirement is :

$$\delta(z_1 + \bar{z}_1) = a_2(z_2' + \bar{z}_2) = 0 \tag{6.75}$$

This makes z_2' purely imaginary and z_1' real. This proves that the following is an extremal trajectory of the brane potential for the Kuperstein potential:

$$z_1' = \pm\frac{1}{\sqrt{2}}r^{3/2} \tag{6.76}$$

Stability Now we examine the matrix of second derivatives, $\frac{\partial^2 V}{\partial\Psi_i\partial\Psi_j}$, and find the conditions under which these extrema are stable minima. So, we start with taking small perturbations of (6.62), and we expand $\exp(T)$ to first order:

$$\exp(T) \approx I + T + \frac{1}{2}T^2 + \ldots$$

and using the transformation matrix T from (6.63), the first-order correction to z_1 from (6.64) and from (6.74), and applying it to the expansion, we get:

$$z_1 = z_1'\left[1 - \frac{1}{2}(\alpha_2^2 + \alpha_3^2 + \alpha_4^2) + \frac{i}{2}\rho_2(2\alpha_2 - \alpha_3\beta_3 - \alpha_4\beta_4) + \ldots\right] \tag{6.77}$$

Adding z_1 and $\bar{z}_1$, we have:

$$\begin{aligned} z_1 + \bar{z}_1 &= z_1'\left[1 - \frac{1}{2}(\alpha_2^2 + \alpha_3^2 + \alpha_4^2) + \ldots\right] + z_1'\left[1 - \frac{1}{2}(\alpha_2^2 + \alpha_3^2 + \alpha_4^2) + \ldots\right]. \\ &= 2z_1'\left[1 - \frac{1}{2}(\alpha_2^2 + \alpha_3^2 + \alpha_4^2) + \ldots\right]. \end{aligned} \tag{6.78}$$

Now computing: $|z_1|^2 = z_1\bar{z}_1$ and substituting 6.77 we get:

$$|z_1|^2 = z_1'\left(1 - \frac{1}{2}(\alpha_2^2 + \alpha_3^2 + \alpha_4^2) + \ldots\right) \times z_1'\left(1 - \frac{1}{2}(\alpha_2^2 + \alpha_3^2 + \alpha_4^2) + \ldots\right).$$

Expanding:

$$|z_1|^2 = (z_1')^2\left[1 - (\alpha_3^2 + \alpha_4^2) + \ldots\right] \tag{6.79}$$

Since the Kuperstein potential (6.60) depends only on r, $z_1 + \bar{z}_1$ and $z_1\bar{z}_1$, and because

$$\left.\frac{\partial|z_1|^2}{\partial\Psi_i}\right|_0 = \left.\frac{\partial(z_1 + \bar{z}_1)}{\partial\Psi_i}\right|_0 = 0 \tag{6.80}$$

where $\big(...\big)|_0$ denotes evaluation at z_0, we find:

$$\left.\frac{\partial V}{\partial\Psi_i}\right|_0 = 0 \tag{6.81}$$

and

$$\left.\frac{\partial^2 V}{\partial\Psi_i\partial\Psi_j}\right|_0 = \left.\left[\frac{\partial V}{\partial|z_1|^2}\frac{\partial^2|z_1|^2}{\partial\Psi_i\partial\Psi_j} + \frac{\partial V}{\partial(z_1+\bar{z}_1)}\frac{\partial^2(z_1+\bar{z}_1)}{\partial\Psi_i\partial\Psi_j}\right]\right|_0 \tag{6.82}$$

The first derivative of (6.78) reads:

$$\begin{aligned}\partial_i(z_1+\bar{z}_1) &= 2z_1'\left(-\frac{1}{2}\partial_i(\alpha_2^2+\alpha_3^2+\alpha_4^2)\right)\\ &= -z_1'\partial_i(\alpha_2^2+\alpha_3^2+\alpha_4^2).\end{aligned} \tag{6.83}$$

while the second derivative is:

$$\partial_i\partial_j(z_1+\bar{z}_1) = -z_1'\partial_i\partial_j(\alpha_2^2+\alpha_3^2+\alpha_4^2) \tag{6.84}$$

Since $\alpha_2^2, \alpha_3^2, \alpha_4^2$ are Kronecker deltas:

$$\partial_i\partial_j(\alpha_2^2+\alpha_3^2+\alpha_4^2) = 2(\delta_{i2}\delta_{j2}+\delta_{i3}\delta_{j3}+\delta_{i4}\delta_{j4}) \tag{6.85}$$

we have:

$$\partial_i\partial_j(z_1+\bar{z}_1) = -2z_1'(\delta_{i2}\delta_{j2}+\delta_{i3}\delta_{j3}+\delta_{i4}\delta_{j4}) \tag{6.86}$$

Similarly if we take the first and second derivative of (6.79) we get:

$$\partial_i\partial_j|z_1|^2 = -(z_1')^2(\delta_{i3}\delta_{j3}+\delta_{i4}\delta_{j4}) \tag{6.87}$$

Substituting $z_1' = \pm\frac{r^{3/2}}{\sqrt{2}}$, we obtain:

$$\left.\partial_i\partial_j|z_1|^2\right|_0 = \pm\frac{r^{3/2}}{\sqrt{2}}\left.\partial_i\partial_j(z_1+\bar{z}_1)\right|_0 + r^3\delta_{i2}\delta_{j2} = -r^3\delta_{i3}\delta_{j3} - r^3\delta_{i4}\delta_{j4} \tag{6.88}$$

Substitute the result of (6.86) and (6.87) into (6.82):

$$\left.\frac{\partial^2 V}{\partial\Psi_i\partial\Psi_j}\right|_0 = -r^3\frac{\partial V}{\partial|z_1|^2}(\delta_{i3}\delta_{j3}+\delta_{i4}\delta_{j4}) \pm \frac{2r^{3/2}}{\sqrt{2}}\frac{\partial V}{\partial(z_1+\bar{z}_1)}(\delta_{i2}\delta_{j2}+\delta_{i3}\delta_{j3}+\delta_{i4}\delta_{j4})$$

This gives two terms:

$$Y = -r^3\frac{\partial V}{\partial|z_1|^2} \tag{6.89}$$

and

$$X = \mp\frac{\sqrt{2}r^{3/2}}{2}\frac{\partial V}{\partial(z_1+\bar{z}_1)} \tag{6.90}$$

Hence, the angular mass matrix at z_0 has the form:

$$\left.\frac{\partial^2 V}{\partial\Psi_i\partial\Psi_j}\right|_0 = \begin{pmatrix} X & 0 & 0 & 0 & 0 & 0 \\ 0 & X+Y & 0 & 0 & 0 & 0 \\ 0 & 0 & X+Y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \beta_3 \\ \beta_4 \end{pmatrix} \tag{6.91}$$

The flat directions in the β_3, β_4 angles parameterize the symmetry group $SO(3)/SO(2)$ that leaves the Kuperstein embedding $z_1 = \mu$ invariant. The angles α_3 and α_4 have degenerate eigenvalues.
To calculate the eigenvalues X and Y we note that the non-perturbative F-term potential (6.60) may be written as:

$$\begin{aligned} V_{np} = \mathcal{C}(r,\tau)\mathcal{G}^{1/n}\Bigg[& (2a\tau+6) - \frac{6\,e^{a\tau}|W_0|}{|A_0|}\mathcal{G}^{-1/2n} \\ & + \frac{3}{2n}\left(\frac{1}{\mu}(z_1+\bar{z}_1) - \frac{2}{\mu^2}|z_1|^2\right)\mathcal{G}^{-1} + \frac{4c}{n}\frac{r}{r_\mu}\left(1 - \frac{|z_1|^2}{2r^3}\mathcal{G}^{-1}\right)\Bigg] \end{aligned} \tag{6.92}$$

Where:

$$\begin{aligned} \mathcal{C}(r,\tau) &\equiv \frac{\kappa^2 a|A_0|^2 e^{-2a\tau}}{3U^2} > 0, \quad \partial_{z_1+\bar{z}_1}\mathcal{C} = \partial_{z_1}\mathcal{C} = 0, \\ \mathcal{G} &\equiv \frac{|A|}{|A_0|} = \left|1 - \frac{z_1}{\mu}\right|^2, \quad \partial_{z_1+\bar{z}_1}\mathcal{G} = \frac{1}{\mu}, \quad \left.\partial_{z_1}\mathcal{G}\right|_{z_1} = \frac{1}{\mu^2}, \\ \mathcal{G}_0 &\equiv \left.\mathcal{G}\right|_0 = \left|1 \pm x^{3/2}\right|^2 = g(x)^2. \end{aligned} \tag{6.93}$$

and the variable:

$$x \equiv \frac{r}{r_\mu} = \frac{\phi}{\phi_\mu} \tag{6.94}$$

After a lengthy calculation of the derivative of (6.89) and (6.90), we obtain:

$$\begin{aligned} X = \pm\frac{2\mathcal{C}}{n}\frac{x^{3/2}}{|1\pm x^{3/2}|^{2(1-1/n)}} & \Bigg[2a\tau + \frac{9}{2} - 3e^{a\tau}\frac{|W_0|}{|A_0|}\frac{1}{|1\pm x^{3/2}|^{1/n}} \\ & \mp 3\left(1-\frac{1}{n}\right)\frac{x^{3/2}(1\pm x^{3/2})}{|1\pm x^{3/2}|^2} - 3c\left(1-\frac{1}{n}\right)\frac{x}{|1\pm x^{3/2}|^2}\Bigg], \end{aligned} \tag{6.95}$$

and

$$\begin{aligned} X+Y = {} & (1\mp x^{3/2})X \\ & + \frac{2\mathcal{C}c}{n}\frac{x}{|1\mp x^{3/2}|^{2(1-1/n)}}\left(1+\frac{3}{2c}x^2\right). \end{aligned} \tag{6.96}$$

where

$$c \equiv \frac{9}{4na\sigma_{min}} \frac{\phi_\mu^2}{M_P^2} \tag{6.97}$$

The stability of the trajectory $z_1 = \pm\frac{r^{3/2}}{\sqrt{2}}$, for both positive and negative real z_1, only requires $X > 0$. From (6.96), this automatically ensures that $X + Y > 0$ within the relevant range $r < r_\mu$. Therefore, using (6.95), a simple numerical check can determine whether a given scenario remains stable in the angular directions. For all potential inflationary trajectories, this stability test has been systematically performed.

So, the the angular mass matrix $\delta^2 V$ shows that the trajectory along $z_1 = +\frac{r^{3/2}}{\sqrt{2}}$ is unstable, whereas the trajectory along $z_1 = -\frac{r^{3/2}}{\sqrt{2}}$ remains stable in all angular directions.

So along the trajectory $z_1 = -\frac{r^{3/2}}{\sqrt{2}}$, we can obtain the final non-perturbative potential as:

$$\begin{aligned} V_{np}(\phi,\tau) = \frac{a|A_0|^2}{3} \frac{e^{-2a\tau}}{U^2(\phi,\tau)} g(\phi)^{2/n} \Bigg[& 2a\tau + 6 \\ & - 6e^{a\tau} \frac{|W_0|}{|A_0|} \frac{1}{g(\phi)^{1/n}} + \frac{3c}{n} \frac{\phi}{\phi_\mu g(\phi)^2} \\ & - \frac{3}{n} \frac{1}{g(\phi)} \frac{\phi^{3/2}}{\phi_\mu^{3/2}} \Bigg]. \end{aligned} \tag{6.98}$$

Due to (6.4), which can be rewritten as $\tau = \sigma + \gamma r^2$, V_{np} depends on the inflaton r, and so the non-perturbative potential (6.98) could reintroduce an η-problem, unless it is subdominant with respect to the inflationary potential (6.17).

Chapter 7

Results and Discussion

By incorporating non-perturbative corrections, we introduced a dependence on the angular directions in the potential and successfully stabilized them, leading to the final potential. However, an important question arises: Will these non-perturbative corrections introduce an η-problem in inflation?
For answering this question, we need to compare the non-perturbative potential (6.98) with (6.17) whose value is $\approx 10^{-13}$ for $W_X = \sqrt{10^{-7}}$ and $\sigma_{min} = 370$, according to [23]. When $\sigma_{min} \gg 1$ and $\gamma r^2 \ll (T + \bar{T})$, the non-perturbative scalar potential (6.98) can very well be approximated as:

$$V_{np}(\phi, \sigma_{min}) \simeq -\frac{2a|A_0||W_0|g(\phi)^{1/n}}{\sigma_{min}^2} e^{-a\sigma_{min}} \tag{7.1}$$

By taking the values as $g(\phi) = 1 + \left(\frac{\phi}{\phi_\mu}\right)^{3/2}$, $\phi_\mu = 0.25$, $A_0 = 1$, $a = \frac{2\pi}{n}, n = 8$, $W_0 = 0.16$ and $\sigma_{min} = 370$ and by substituting the values into the expression, we find that the contribution from the non-perturbative correction is negligibly small compared to the reference value in (6.17) since:

$$V_{np}(\phi, \sigma_{min}) \simeq -\frac{2\pi}{\sigma_{min}^2} \left(1 + 8\phi^{3/2}\right)^{1/8} \times 10^{-132} \tag{7.2}$$

This confirms that adding this term does not significantly impact the inflationary potential and does not disrupt the slow-roll inflation condition. Therefore, the correction remains well within acceptable limits, ensuring the stability of the inflationary dynamics.

Chapter 8

Conclusion and Future Work

In this thesis, alongside stabilizing the moduli fields, we focused on stabilizing the angular directions to ensure a more consistent inflationary model and to avoid the η-problem which can hinder slow-roll inflation. We first examined how perturbative corrections or fine-tuned non-perturbative effects contribute to build a viable inflationary model. Then, we explored ways to avoid fine-tuning, aiming to create a more robust and naturally stabilized framework for brane-antibrane inflation within Type IIB string theory. While this thesis lays the groundwork for understanding angular directions stabilization, there are still open questions and directions for future research:

- **Exploring other Compactifications:** Our study focused on Type IIB warped compactifications, but it would be interesting to see if the same stabilization methods work in F-theory or M-theory setups.
- **Extending to Multi-Field Inflation:** We mainly looked at a single-field dynamics, but brane inflation models often involve multiple evolving fields.

By exploring these areas, future research can strengthen our understanding of angular directions stabilization and its role in string cosmology and observational physics.

Appendix A

Cohomology And Homology

The Appendix contains the summary of the mathematical contents needed as a basis for this thesis. This discussion is based on [18, 67, 68, 69, 70, 71].

A.1 Holonomy

The holonomy group at a point p in a differentiable manifold M describes how tangent vectors are transformed when they undergo parallel transport along closed loops that start and end at p. It is defined for a given connection Γ on the tangent bundle of M, which has dimension k.
Mathematically, the holonomy group consists of all transformations acting on the tangent space at p, forming a subset of the general linear group $GL(k, \mathbb{R})$, which represents all invertible linear transformations in k-dimensional space:

$$\mathrm{Hol}_p(\Gamma) = \{G_c : T_pM \to T_pM\} \subset GL(k, \mathbb{R})$$

If two points p and q in a connected manifold M are linked by a path, their holonomy groups are related by a conjugation transformation:

$$\mathrm{Hol}_p(\Gamma) = g\,\mathrm{Hol}_q(\Gamma)\,g^{-1}$$

When the connection Γ is compatible with a metric, parallel transport preserves lengths, restricting the holonomy group to $SO(n)$ or a subgroup. A key case of interest is when Γ is the Levi-Civita connection for a Riemannian metric on M, leading to the Riemannian holonomy group, denoted as $Hol(M)$.
Berger's classification provides a systematic way to understand holonomy groups in Riemannian manifolds, particularly in physics. It states that if M is a simply connected Riemannian manifold, then it falls into one of three categories: A product of lower-dimensional manifolds, A symmetric space, represented as a coset space G/H or One of a specific set of exceptional holonomy groups:

1. $\text{Hol}(M) = SO(n)$
2. $\text{Hol}(M) = U(k) \subset SO(2k)$ for $n = 2k$
3. $\text{Hol}(m) = Sp(k) \subset SO(4k)$ for $n = 4k$
4. $\text{Hol}(M) = Sp_p(k) \subset S_0(4k)$ for $n = 4K$
5. $\text{Hol}(M) = Sp_p(k)Sp(1) \subset S_0(4k)$ for $n = 4K$
6. $\text{Hol}(M) = G_2 \subset SO(7)$ for $n = 7$
7. $\text{Hol}(M) = \text{Spin}(7) \subset SO(8)$ for $n = 8$

This classification is crucial for understanding the geometry and symmetry properties of manifolds, especially in contexts like string theory and supergravity.
Now we will see how each of these classes is relevent for the string compactification:

- A manifold with holonomy $U(k)$, case (2), is a *Kähler manifold*. In this case the Christoffel symbols have only holomorphic (or only antiholomorphic) indices, and hence parallel transport does not transform holomorphic vectors into antiholomorphic vectors.
- A manifold with holonomy $SU(k)$, case (3), is a *Calabi–Yau manifold*. Such a manifold admits a Ricci-flat Kähler metric, i.e. a Kähler metric for which the Ricci tensor vanishes, $R_{ij} = 0$. Taking $k = 3$, one has the celebrated case of Calabi–Yau threefolds.
- A manifold with holonomy $Sp(k) \subseteq SU(2k)$, case (4), is a *hyper-Kähler manifold*. Such a manifold admits a Ricci-flat Kähler metric, and is a special case of a Calabi–Yau manifold.
- A manifold with holonomy $Sp(k)Sp(1)$, case (5), is a *quaternionic Kähler manifold*. The corresponding metric is neither Kähler nor Ricci-flat.
- Manifolds with holonomy G_2 or Spin(7), cases (6) and (7), are said to have *exceptional holonomy*. The corresponding metrics are Ricci-flat.

A.1.1 Homology

A k-cycle is a k-dimensional subspace C_k of a manifold M that has no boundary, meaning $\partial C_k = 0$. For example, a 1-cycle is a closed curve on a manifold, which may be contractible (can shrink to a point) or non-contractible.
A k-cycle is called exact if it is the boundary of a higher-dimensional subspace B_{k+1}, written as $C_k = \partial B_{k+1}$. Since the boundary of a boundary always vanishes,

two k-cycles that differ only by a boundary are considered equivalent, forming a homology class.
This defines the homology group $H_k(M)$, which classifies topological features of a space. A practical example shows how two distinct boundaries can cancel out, reinforcing the concept of homology equivalence.

A.1.2 Differential forms and Cohomology

String theory contains several differential form fields, such as B_2 from the bosonic string and (C_2, C_4) in Type IIB string theory. Even scalars like ϕ and C_0 can be viewed as 0-forms in differential geometry.
Since these fields play a fundamental role, differential geometry becomes an essential tool for analyzing string theory physics, particularly in compactifications. To facilitate this, we summarize key concepts, focusing on the relationship between harmonic forms, cohomology, and homology groups in a compact space, which are crucial in understanding the topological properties of string compactifications.
A differential r-form w_r on a manifold M is a completely antisymmetric tensor of type $(0, r)$, defined as:

$$\omega_r = \frac{1}{r!}\omega_{\mu_1\mu_2\ldots\mu_r} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_r}$$

The space of such r-forms at a point $p \in \mathcal{M}$ is denoted $\Omega_p^r(\mathcal{M})$. The exterior product between two forms ξ_q and η_r produces a $(q+r)$-form:

$$\xi_q \wedge \eta_r = \frac{1}{q!r!}\xi_{\mu_1\ldots\mu_q}\eta_{\nu_1\ldots\nu_r} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_q} \wedge dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_r}$$

which is nonzero only if $q+r \leq m$, where $m = dim(\mathcal{M})$. A top form $w_m \in \Omega_p^m(\mathcal{M})$ is the only type of form that can be integrated over M.
The exterior derivative operator d acts on a smooth r-form $w_r \in \Omega^r(\mathcal{M})$, mapping it to an $(r+1)$-form:

$$d\omega = \frac{1}{(r+1)!}\partial_\nu \omega_{\mu_1\ldots\mu_r} dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}$$

If a metric g is defined on M, we can introduce the Hodge star operator $*$, which maps an r-form to an $(m-r)$-form:

$$\star\omega = \frac{\sqrt{|g|}}{(m-r)!}\left(\frac{1}{r!}\omega_{\mu_1\ldots\mu_r}\right)\epsilon^{\mu_1\ldots\mu_r}_{\nu_{r+1}\ldots\nu_m} dx^{\nu_{r+1}} \wedge \cdots \wedge dx^{\nu_m}$$

where $\epsilon^{\mu_1\ldots\mu_m}$ is the Levi-Civita symbol satisfying:

$$\epsilon^{\mu_1\ldots\mu_m} = g^{\mu_1\nu_1}\ldots g^{\mu_m\nu_m}\epsilon_{\nu_1\ldots\nu_m}, \quad g^{\mu\nu} = g^{-1}_{\mu\nu}$$

This formulation provides a natural way to define inner products on differential forms, which is essential in cohomology, duality, and field theory applications.
In differential geometry, r-forms $w_r, \eta_r \in \Omega^r(\mathcal{M})$ are antisymmetric tensors on a manifold M, and their inner product is given by:

$$(\omega, \eta) \equiv \int \omega \wedge \star\eta = \frac{1}{r!} \int \omega_{\mu_1 \ldots \mu_r} \eta^{\mu_1 \ldots \mu_r} \sqrt{|g|} dx^1 \ldots dx^m$$

For a Riemannian metric, this inner product is positive definite, ensuring $(w, w) \geq 0$, and equals zero only if $w = 0$.
The adjoint exterior derivative $d^\dagger$ is defined using the inner product:

$$(d\omega, \eta) = (\omega, d^\dagger \eta) \Longrightarrow d^\dagger \equiv (-1)^{1+m(r+1)} \star d \star .$$

The Laplacian operator Δ is defined in terms of d and $d^\dagger$ as:

$$\Delta = dd^\dagger + d^\dagger d$$

For a o-form w, the Laplacian expression in terms of covariant derivatives, Riemann tensor, and Ricci tensor is:

$$\Delta\omega_{m_1 \ldots m_p} = -\nabla^q \nabla_q \omega_{m_1 \ldots m_p} + pR_{q[m_1} \omega^q_{m_2 \ldots m_p]} - \frac{1}{2} p(p-1) R_{qr[m_1 m_2} \omega^{qr}_{m_3 \ldots m_p]}.$$

Differential forms can be classified based on how they behave under d, $d^\dagger$, and Δ:

- **Closed**: $d\omega = 0$
- **Exact**: $\omega = d\eta$
- **Co-closed**: $d^\dagger \omega = 0$
- **Co-exact**: $\omega = d^\dagger \xi$
- **Harmonic**: $\Delta\omega = 0$

A form is harmonic if it is both closed and co-closed: $\Delta w = 0 \Leftrightarrow dw = 0 \ and \ d^\dagger w = 0$ **de Rham Cohomology:** From $d^2 = 0$ and $(d^\dagger) = 0$, any co-exact form is also co-closed. However, the reverse is not always true globally. Some closed forms may not be exact, meaning $dw = 0$ but $w \neq d\eta$.
This leads to the definition of the equivalence class of forms:

$$[\omega_p] = \{\tilde{\omega}_p : \tilde{\omega}_p = \omega_p + d\eta_{p-1}\} \in H^p(\mathcal{M})$$

where $H^p(\mathcal{M})$ is the de Rham cohomology group, classifying forms that differ by exact forms. **Homology and Cohomology:** The homology group is defined similarly to cohomology but through the boundary operator δ, which maps C^p submanifolds to their boundaries in C^{p-1}. The definitions follow:

- Closed (cycles): $\delta C^p = \emptyset$
- Exact (boundaries): $C^p = \delta C^{p+1}$

Since δ is nilpotent ($\delta^2 = 0$), all boundaries are cycles, but not all cycles are boundaries. Homology equivalence classes group cycles that differ by a boundary:

$$[C^p] = \{\tilde{C}^p : C^p + \delta C^{p+1}\} \in H_p$$

A p-form w_p and a p-dimensional submanifold C^p can be paired through integration:

$$(C^p, \omega_p) = \int_{C^p} \omega_p$$

By **Stokes' theorem**, this inner product is well-defined in $H_p(\mathcal{M}) \times H^p(\mathcal{M})$ since elements differing by exact components yield the same integral. This leads to de Rham's theorem.
de Rham's Theorem: For a compact manifold $\mathcal{M}$, the cohomology group $H^p(\mathcal{M})$ and the homology group $H_p(\mathcal{M})$ finite-dimensional and dual vector spaces. They are isomorphic:

$$H^p(\mathcal{M}) \simeq H_{m-p}(\mathcal{M})$$

This means we can study topology via homology to understand cohomology classes $[w_p]$. **Hodge Decomposition Theorem:** For a compact Riemannian manifold $(\mathcal{M}, g)$ without a boundary, any differential r-form w can be uniquely decomposed as:

$$\omega = d\eta + d^\dagger \xi + \gamma, \quad \text{where} \quad \gamma \text{ is a harmonic form}$$

This decomposition is fundamental because it connects the space of harmonic forms on $\mathcal{M}$ with the cohomology group $H^p(\mathcal{M})$, highlighting the structure of differential forms in geometry and physics. **Hodge's Theorem:** Hodge's theorem states that on a compact orientable Riemannian manifold $(\mathcal{M}, g)$, the cohomology group $H^p(\mathcal{M})$ is isomorphic to the space of harmonic p-forms, denoted as:

$$H^p(\mathcal{M}) \cong \text{Harm}^p(\mathcal{M})$$

This result is significant because it relates the solutions of differential equations to the topology of the manifold. Specifically, the number of harmonic forms is equal to the Betti numbers b_p, which count the independent cycles that cannot be continuously deformed into one another.

$$\Delta\omega_p = 0 \Longleftrightarrow \text{dim} H_{d-p}(\mathcal{M}) = b_{d-p}$$

These Betti numbers are topological invariants and contribute to the Euler characteristic:

$$\chi(\mathcal{M}) = \sum_{p=0}^{m} (-1)^p b_p(\mathcal{M})$$

Poincaré Duality: Poincaré duality establishes a relationship between cohomology and homology classes on a compact manifold. It states that for a given cohomology class $H^p(\mathcal{M})$, there exists a corresponding class in $H^{m-p}(\mathcal{M})$, defined via the inner product:

$$\langle \omega, \eta \rangle = \int_{\mathcal{M}} \omega \wedge \eta$$

This implies the isomorphism:

$$H^p(\mathcal{M}) \cong H^{m-p}(\mathcal{M})$$

Furthermore, a $(m-p)$-form w_{m-p} is said to be Poincaré dual to a p-cycle C^p if:

$$\int_{C^p} \eta_p = \int_{\mathcal{M}} \eta_p \wedge \omega_{m-p}.$$

This establishes a deep connection between differential forms and homology cycles, showing how topology and geometry are intertwined in mathematical physics and string theory.

Appendix B

Kähler and Calabi–Yau geometry

B.1 Complex manifolds

Complex manifolds are even-dimensional real manifolds locally modeled on $\mathbb{C}^k$. Similar to how complex analysis refines real analysis, the study of complex manifolds imposes additional structure, making them a crucial tool in mathematical physics. They are important in string theory for two main reasons: (1) they serve as moduli spaces in supersymmetric theories, and (2) they play a key role in Ricci-flat compactifications, such as vacuum solutions. While not all Ricci-flat manifolds are complex, complex structures simplify construction and analysis.
A complex manifold M is a real manifold of even dimension $n = 2k$ where coordinate charts transition holomorphically. The structure group of the tangent bundle is a subgroup of $GL(2k, \mathbb{R})$, distinguishing it from general Riemannian manifolds. A useful definition involves an almost complex structure $\mathcal{J}$ a tensor field of type (1,1) that satisfies $\mathcal{J}^2 = -1$. This structure acts as a map on the tangent bundle:

$$\mathcal{J} : TM \to TM$$

When M is fully complex, the complexified tangent space is defined as:

$$T_pM^{\mathbb{C}} = \{X + iY \mid X, Y \in T_pM\}$$

This allows the splitting of the space into holomorphic and antiholomorphic components. In terms of local holomorphic coordinates $z^\mu = x^\mu + iy^\mu$, a basis for $T_pM^{\mathbb{C}}$ consists of: $\frac{\partial}{\partial z^\mu} \frac{\partial}{\partial \bar{z}^{\bar{\mu}}}$ The complex structure operator $\mathcal{J}$ acts on these basis vectors as:

$$\mathcal{J}\left(\frac{\partial}{\partial z^\mu}\right) = i\frac{\partial}{\partial z^\mu}, \quad \mathcal{J}\left(\frac{\partial}{\partial \bar{z}^{\bar{\mu}}}\right) = -i\frac{\partial}{\partial \bar{z}^{\bar{\mu}}}$$

Thus, the tangent space is naturally divided into a holomorphic eigenspace spanned by $\frac{\partial}{\partial z^\mu}$ and an antiholomorphic eigenspace spanned by $\frac{\partial}{\partial \bar{z}^{\bar{\mu}}}$. These eigenspaces play a crucial role in defining complex structures and are essential in string theory and compactifications.

B.1.1 Complex differential forms

In the context of complex manifolds, we extend differential forms to include complex-valued forms. A complex r-form is defined as a sum of real r-forms with complex coefficients:

$$\gamma_r \equiv \alpha_r + i\beta_r$$

Where α_r and β_r are real r-form. The vector space of such forms is denoted as $\Omega^r_{\mathbb{C}}(M)$, and the conjugate of γ_r is:

$$\bar{\gamma}_r \equiv \alpha_r - i\beta_r$$

A complex differential form can be further decomposed into (r, s)-form, where r and s denote the number of holomorphic and anti-holomorphic indices, respectively. A general (r, s)-form is written as:

$$dz_{\mu_1} \wedge \cdots \wedge dz_{\mu_r} \wedge d\bar{z}_{\nu_1} \wedge \cdots \wedge d\bar{z}_{\nu_s} \equiv dz_M \wedge d\bar{z}_{\bar{N}}$$

where the multi-indices are defined as:

$$M = (\mu_1, \dots, \mu_r), \quad N = (\nu_1, \dots, \nu_s)$$

The space of all complex k-forms is then expressed as a direct sum:

$$\Omega^k_{\mathbb{C}} = \bigoplus_{r+s=k} \Omega^{r,s}$$

Dolbeault Operators: To respect the complex structure, we refine the exterior derivative d into the Dolbeault operators:

- **The holomorphic Dolbeault operator** ∂ acts as:

$$\partial\gamma_{r,s} = \left(\frac{\partial}{\partial z^\kappa}\gamma_{MN}\right) dz^\kappa \wedge dz^M \wedge d\bar{z}^{\bar{N}}.$$

- **The anti-holomorphic Dolbeault operator** $\bar{\partial}$ acts as:

$$\bar{\partial}\gamma_{r,s} = \left(\frac{\partial}{\partial \bar{z}^\kappa}\gamma_{MN}\right) dz^M \wedge d\bar{z}^{\bar{N}} \wedge d\bar{z}^\kappa.$$

These satisfy the fundamental relations:

$$d = \partial + \bar{\partial}, \qquad \partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$$

A holomorphic $(r, 0)$-form satisfies:

$$\bar{\partial}\gamma_{r,0} = 0$$

This ensures that a holomorphic zero-form is simply a holomorphic function.

B.1.2 Dolbeault cohomology

The Dolbeault cohomology group is defined similarly to the de Rham cohomology but using the Dolbeault operator $\bar{\partial}$. the space of $\bar{\partial}$-closed (r, s)-forms is denoted as $Z^{r,s}_{\bar{\partial}}(M)$, while the space of $\bar{\partial}$-exact forms is $B^{r,s}_{\bar{\partial}}(M)$. The Dolbeault cohomology group is then:

$$H^{r,s}_{\bar{\partial}}(M, \mathbb{C}) \equiv \frac{Z^{r,s}_{\bar{\partial}}(M)}{B^{r,s}_{\bar{\partial}}(M)}$$

Hodge theory for complex manifolds extends the real case by defining an inner product on the complexified tangent space, using the Hodge star operator. This allows the introduction of adjoint operators $\partial^\dagger$ and $\bar{\partial}^\dagger$. Two Laplacians are defined:

$$\Delta_\partial = \partial\partial^\dagger + \partial^\dagger\partial$$

$$\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}$$

The set of $\bar{\partial}$-harmonic (r, s)-forms, denoted as $\mathcal{H}^{r,s}_{\bar{\partial}}(M)$, consists of forms annihilated by $\Delta_{\bar{\partial}}$. Hodge's theorem states:

$$H^{r,s}_{\bar{\partial}}(M, \mathbb{C}) \cong \mathcal{H}^{r,s}_{\bar{\partial}}(M)$$

The Hodge numbers $h^{r,s}$ represent the dimensions of the Dolbeault cohomology groups:

$$h^{r,s} = dim H^{r,s}_{\bar{\partial}}(M, \mathbb{C})$$

By applying the Hodge star operator, one finds a symmetry relation:

$$h^{r,s} = h^{k-r,k-s}$$

for a complex manifold of dimension k. This plays a crucial role in understanding the topology of Kähler and Calabi-Yau manifolds.

B.2 Kähler and Calabi–Yau geometry

Kähler manifolds are a special class of complex manifolds with significant properties that make them fundamental in string theory. They serve as the foundation for vacuum solutions in string compactifications, particularly Calabi–Yau manifolds. In supersymmetric theories, the moduli space of a theory with $N = 1$ supersymmetry is always a Kähler manifold, and higher supersymmetry theories ($N > 1$) also often exhibit Kähler structure in their moduli spaces. This makes them essential in understanding the geometric aspects of string compactifications and supersymmetric field theories.

B.2.1 Kähler Manifold

A Kähler manifold is a special type of Hermitian manifold where the Kähler form J ia closed $dJ = 0$. Equivalently, a complex manifold M of complex dimension k is Kähler if its holonomy group satisfies $Hol(M) = U(k)$. The Kähler metric can be locally expressed in terms of a Kähler potential k, stisfying:

$$J = i\partial\bar{\partial}k, \qquad g_{\mu\bar{\nu}} = \partial_\mu\partial_{\bar{\nu}}k$$

It remains invariant under Kähler transformations of the form: $k \to k + f(z^i) + \bar{f}(\bar{z}^i)$. On a Kähler manifold, the Ricci tensor is derived from the Christoffel symbols:

$$R_{\mu\bar{\nu}\rho\bar{\sigma}} = -\partial_\rho \Gamma^\sigma_{\mu\bar{\nu}}$$

The Ricci form is then defined as:

$$\mathcal{R} = iR_{\mu\bar{\nu}}dx^\mu \wedge d\bar{z}^{\bar{\nu}}$$

where the Ricci tensor components satisfy:

$$R_{\mu\bar{\nu}} = -\partial_\mu \Gamma_{\bar{\nu}}$$

and the Ricci form follows:

$$\mathcal{R} = \partial\bar{\partial}\ln g$$

The Ricci form defines a de Rham cohomology class, known as the **first Chern class:**

$$c_1 \equiv \frac{1}{2\pi}[\mathcal{R}] \in H^2(M, \mathbb{C})$$

On a Kähler manifold, the Laplacians for $d, \partial, \bar{\partial}$ are relted:

$$\Delta = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$$

The Hodge numbers satisfy:

$$h^{r,s} = h^{s,r}$$

The Betti numbers b_k, which count the dimensions of the cohomology groups $H^k(M, \mathbb{R})$, are related to Hodge numbers as:

$$b_k = \sum_{r+s=k} h^{r,s}$$

An important consequence is that odd Betti numbers on Kähler manifolds are even:

$$b_{2k-1} = 2 \times \sum_{r+s=2k-1, r<s} h^{r,s}$$

Finally, the Euler characteristic is given by:

$$\mathcal{X}(M) = \sum_{r,s} (-1)^{r+s} h^{r,s}$$

B.2.2 Calabi–Yau manifolds

A Calabi-Yau k-fold is a compact Kähler manifold M of complex dimension k satisfies the following equivalent conditions:

1. M has a Kähler metric with holonomy in $SU(k)$.
2. There exists a nowhere-vanishing $(k, 0)$-form Ω on M.
3. M has a Kähler metric with vanishing Ricci curvature.
4. The first Chern class $c_1(M)$ vanishes.

Variations in definitions exist; some do not require compactness, while others require $Hol(M) = SU(k)$, not a subgroup. The existence of a Ricci-flat metric on a Calabi-Yau k-fold implies that compactification of string theory on M (for $k \leq 4$) satisfies the vacuum Einstein equations $R_{MN} = 0$. For Calabi-Yau threefolds ($k = 3$), compactification preserves $\mathcal{N} = 1$ supersymmetry in four dimensions because the holonomy $SU(3) \subset SO(2K)$ is small enough to leave a single invariant spinor. The decomposition of the six-dimensional spinor representation under $SU(3)$ follows:

$$4 = 3 + 1$$

The Hodge numbers of a Calabi-Yau threefold are:

$$h^{0,0} = 1, \quad h^{3,0} = h^{0,3} = 1, \quad h^{2,1} = h^{1,2}$$

The Euler characteristic is given by:

$$\mathcal{X}(CY_3) = 2(h^{1,1} - h^{2,1})$$

This characterization of Calabi-Yau manifolds makes them significant in string compactifications due to their role in preserving supersymmetry.

B.3 Moduli space

The moduli space of a Calabi-Yau three-fold, a Ricci-flat Kähler manifold with $SU(3)$ holonomy, consists of two independent deformations: Kähler and complex structure moduli. To preserve Ricci-flatness, deformations of the metric δg must satisfy the condition: $\nabla(\delta g) = 0$. For Kähler moduli t^i, which correspond to harmonic $(1,1)$-forms, the deformations satisfy:

$$\Delta \delta g_{m\bar{n}} = 0$$

and can be expanded as:

$$\delta g_{m\bar{n}} = it^i(\hat{D}_i)_{m\bar{n}}$$

where $\hat{D}_i$ forms a basis of $H_{1,1}(X)$. To ensure the positivity of the new metric, the conditions:

$$\int_C J > 0, \quad \int_S J \wedge J > 0, \quad \int_X J \wedge J \wedge J > 0$$

must hold.
For complex structure moduli U^α, the deformations correspond to harmonic $(2,0)$-forms, which are related to the holomorphic $(3,0)$-form Ω by:

$$\delta g_{mn} = \frac{i}{|\Omega|^2} \bar{U}^a (\bar{\chi}_a)_{mn\bar{p}} g^{\bar{p}q}$$

where $\bar{\chi}_a$ is a basis of $H_{1,2}(X)$. The complex moduli space is also a special Kähler manifold with a tree-level Kähler potential:

$$K_{\text{cs}} = -\ln\left(-i\int_X \Omega \wedge \bar{\Omega}\right)$$

and metric:

$$g_{ab} = \frac{\partial^2 K_{CS}}{\partial U^a \partial \bar{U}^{\bar{b}}}$$

The total moduli space factorizes as:

$$\mathcal{M} = \mathcal{M}^{CS}_{h_{1,2}} \times \mathcal{M}^{K}_{h_{1,1}}$$

These metric deformations generate massless scalar fields in four dimensions, crucial for understanding the dynamics of moduli stabilization in string compactifications.

B.4 Mirror symmetry

Mirror symmetry is a duality in string theory that relates pairs of Calabi-Yau threefolds, exchanging their Hodge numbers and moduli spaces. It corresponds to a reflection about the diagonal axis in the Hodge diamond, interchanging odd and even cohomologies. Specifically, for a given Calabi-Yau manifold X, there exists a mirror manifold $\tilde{X}$ with reversed Hodge numbers:

$$h_{1,1}(X) = h_{1,2}(\tilde{X}), \quad h_{1,2}(X) = h_{1,1}(\tilde{X}).$$

Mirror symmetry also exchanges the Kähler and complex structure moduli spaces:

$$\mathcal{M}^{cs}_{h_{1,2}}(X) \equiv \mathcal{M}^{K}_{h_{1,1}}(\tilde{X}), \quad \mathcal{M}^{K}_{h_{1,1}}(X) \equiv \mathcal{M}^{cs}_{h_{1,2}}(\tilde{X}).$$

This leads to the equality of their prepotentials:

$$\mathcal{F}(X) = \mathcal{F}(\tilde{X})$$

In type II string theories, mirror symmetry manifests as a T-duality, relating type IIA on X to type IIB on $\tilde{X}$:

$$\text{IIA in background } \mathbb{R}^{3,1} \times X \equiv \text{IIB in background } \mathbb{R}^{3,1} \times \tilde{X}.$$

This symmetry plays a crucial role in understanding string compactifications, allowing insights into type IIB setups by studying their type IIA mirrors.

Appendix C

Type IIB Flux Compactifications

C.1 N=2 type IIB compactifications

This section presents Calabi-Yau compactifications of type IIB string theory, which preserve $N = 2$ supersymmetry (or 8 supercharges) in four dimensions. The low-energy effective action follows $N = 2$ supergravity, including vector, hyper, and tensor multiplets.

C.1.1 The Spectrum

The compactification is performed on a Calabi-Yau three-fold X, leading to the background metric:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + g_{m\bar{n}} dy^m dy^{\bar{n}}$$

where $g_{\mu\nu}$ is the Minkowski metric, and $g_{m\bar{n}}$ is the Calabi-Yau metric.

- The **low-energy spectrum** is obtained by keeping **massless bosonic fields**.
- The ten-dimensional metric leads to a **four-dimensional metric tensor** $g_{\mu\nu}$, a scalar **dilaton** $\hat{\phi}$, and a one-form vector V^0.
- The **NS-NS sector** includes:

$$\hat{\phi} = \phi(x), \quad \hat{B}_2 = B_2(x) + b^i(x)\hat{D}_i$$

- The **R-R sector** includes:

$$\hat{C}_0 = C_0(x), \quad \hat{C}_2 = C_2(x) + c^i(x)\hat{D}_i$$

$$\hat{C}_4 = Q_i^2(x) \wedge D_i + V^a(x) \wedge \alpha_a - V_a(x) \wedge \beta^a + \rho_a(x)\hat{D}^a$$

These fields assemble into $N = 2$ multiplets.

C.1.2 Tree-Level Effective Action

The tree-level four-dimensional low-energy effective action is given by:

$$S_{IIB}^{(4D)} = -\int \frac{1}{2}R * 1 - \frac{1}{4}\text{Re}(M_{ab})F^a \wedge F^b - \frac{1}{4}\text{Im}(M_{ab})F^a \wedge *F^b$$

$$-g_{ab}dU^a \wedge *dU^b + h_{IJ}dq^I \wedge *dq^J$$

- The **gauge kinetic functions** $M_{ab}(U)$ are determined by the **holomorphic prepotential** $\mathcal{F}(U)$.
- The **Kähler metric** g_{ab} and **quaternionic metric** h_{IJ} define the moduli space structure.

The total moduli space is a product of:

$$\mathcal{M} = \mathcal{M}^{cs}_{h^{1,2}} \times \mathcal{M}^{Q}_{2(h^{1,1}+1)}$$

Where,

- $\mathcal{M}^{cs}$ is the **special Kähler manifold** of **complex structure moduli**.
- $\mathcal{M}^{Q}$ is a **quaternionic manifold** of **hypermultiplet scalars**.

Since the $N = 2$ moduli space contains geometrical moduli, it is entirely determined by two prepotentials $\mathcal{F}(U)$ and $\mathcal{V}$, which are mirror symmetric.
This formalism plays a crucial role in type IIB flux compactifications, providing a well-defined low-energy theory in four dimensions.

C.2 $N = 1$ Type IIB Compactifications

This section discusses compactifications of Type IIB string theory that preserve $N = 1$ supersymmetry. These arise from breaking $N = 2$ supersymmetry in ten-dimensional string theory through an orientifold projection.

C.2.1 Orientifold Projection

A four-dimensional $N = 1$ orientifold is derived from an $N = 2$ compactification by gauging a discrete symmetry:

$$(-1)^{F_L}\Omega_p\sigma$$

Where, Ω_p denotes world-sheet parity, F_L left-moving fermion number, σ An involution on the Calabi-Yau space X, which preserves its isometry and holomorphy but leaves the four-dimensional Minkowski space untouched. The projection acts on the fundamental forms of the Calabi-Yau as:

$$\sigma^* J = J, \quad \sigma^* \Omega = (-1)^\epsilon \Omega$$

Depending on the value of ϵ, there are two classes of models to consider:

1. $\epsilon = 0$: theories with $O5/O9$ orientifold planes, in which the fixed point set of σ is either one or three complex dimensional;

2. $\epsilon = 1$: theories with $O3/O7$-planes, with σ leaving invariant zero or two complex dimensional submanifolds of X.

The focus is on the second case ($O3/O7$-planes), leading to a four-dimensional $N = 1$ low-energy spectrum, which can be expressed in terms of geometrical and topological properties of the orientifold.

C.2.2 Spectrum

The $N = 1$ spectrum is derived by truncating the $N = 2$ theory, keeping only fields invariant under the orientifold action. The surviving Kähler deformations satisfy:

$$J = t^{i_+}(x)\hat{D}_{i_+}, \quad i_+ = 1, \ldots, h^+_{1,1}$$

Similarly, the complex structure deformations kept in the spectrum belong to $H^{1,2}_-$:

$$\delta g_{mn} = \frac{i}{|\Omega|^2} \bar{U}^{a_-} (\bar{\chi}_{a_-})_{mp\bar{q}} \, \bar{\Omega}^{\bar{p}\bar{q}}_n, \quad a_- = 1, \ldots, h^{1,2}$$

where $\bar{\chi}_{a_-}$ are defined using a basis of $H^{1,2}_-$.
The orientifold constraints enforce:

$$\sigma^* \hat{g} = \hat{g}, \quad \sigma^* \hat{\phi} = \hat{\phi}, \quad \sigma^* \hat{B}_2 = -\hat{B}_2,$$

$$\sigma^* \hat{C}_0 = \hat{C}_0, \quad \sigma^* \hat{C}_2 = -\hat{C}_2, \quad \sigma^* \hat{C}_4 = \hat{C}_4,$$

The NS-NS sector contains:

$$\hat{\phi} = \phi(x), \quad \hat{B}_2 = b^{i_-}(x)\hat{D}_{i_-}, \quad i_- = 1, \ldots, h^{1,1}_-,$$

The RR-sector contains:

$$\hat{C}_0 = C_0(x), \quad \hat{C}_2 = c^{i_-}(x)\hat{D}_{i_-}, \quad i_- = 1, \ldots, h^{1,1}_-,$$

$$\hat{C}_4 = Q^{i_+}_2(x) \wedge \hat{D}_{i_+} + V^{a_+}(x) \wedge \alpha_{a_+} - \tilde{V}_{a_+}(x) \wedge \beta^{a_+} + \rho_{i_+}(x)\tilde{D}^{i_+}, \quad a_+ = 1, \ldots, h^{1,2}_+,$$

here, $\hat{D}^a$ is a basis of $H^{2,2}$ dual to $\hat{D}_i$.

C.2.3 Effective Action

The low-energy action for orientifold compactifications is derived from the N = 2 action:

$$S_{IIB}^{(4D)} = -\int \frac{1}{2}R\star 1 + K_{I\bar{J}}D\Phi^I \wedge \star D\bar{\Phi}^{\bar{J}} + \frac{1}{2}\mathrm{Re}(f_{ab})F^a \wedge \star F^b + \frac{1}{2}\mathrm{Im}(f_{ab})F^a \wedge F^b + V.$$

where $F^a = dV^a$, and the moduli space factorizes as:

$$\mathcal{M} = \mathcal{M}^{cs}_{h^{1,2}_-} \times \mathcal{M}^{K}_{h^{1,1}_+ +1},$$

indicating a product of the complex structure moduli space and a quaternionic hyper-multiplet space. **Kähler Moduli and Volume**
The Kähler moduli are defined as:

$$T_i = \tau_i + ib_i, \quad i = 1, \ldots, h_{1,1}(X).$$

where:

$$\tau_i = \frac{\partial \mathcal{V}}{\partial t^i} = \frac{1}{2}\int_X \hat{D}_i \wedge J \wedge J = \frac{1}{2}k_{ijk}t^j t^k,$$

Using these moduli, the tree-level Kähler potential takes the form:

$$\frac{K_{\text{tree}}}{M_p^2} = -2\ln\left[\mathcal{V}(T+\bar{T})\right] - \ln(S+\bar{S}) - \ln\left(-i\int_X \Omega(U) \wedge \bar{\Omega}(U)\right)$$

where Ω is the Calabi-Yau holomorphic (3,0)-form.

C.3 Background fluxes

Here we will discuss the stabilization of scalar fields in Type IIB Calabi-Yau orientifold compactifications. It explains that the tree-level superpotential vanishes, but a potential can be generated by turning on background fluxes. This stabilizes the axio-dilaton and complex structure moduli, while the Kähler moduli remain unfixed due to no-scale structure.

C.3.1 Type IIB fluxes

Type IIB flux compactifications, focusing on the role of background fluxes in shaping the low-energy supergravity theory. The flux of a p-form field strength F_p through a p-cycle γ_i^p in the compact space X is given by:

$$\int_{\gamma_i^p \in X} F_p = n_i \neq 0.$$

This generalizes the concept of electromagnetic flux. A more geometrical understanding of fluxes arises from expanding F_p in terms of harmonic forms ω_p^i, such that:

$$F_p = \langle F_p, \omega_p^i \rangle \, \omega_p^i, \quad \omega_p^i \in H_p(X).$$

Due to Poincaré duality, the flux integral is related to the expansion coefficients, ensuring that fluxes are quantized as:

$$\frac{1}{2\pi\alpha'} \int_{\gamma_i^p \in X} F_p = n_i \in \mathbb{Z}.$$

In the context of Type IIB Calabi-Yau compactifications, the three-form field strengths $\mathbf{F_3}$ (from the Ramond-Ramond sector) and $\mathbf{H_3}$ (from the NS-NS sector) are expanded in terms of a symplectic basis (α_a, β^b):

$$F_3 = m_a^{RR} \alpha^a + n_b^{RR} \beta^b, \quad H_3 = m_a^{NS} \alpha^a + n_b^{NS} \beta^b.$$

A key result is that the fluxes define a combined three-form $\mathbf{G_3}$:

$$G_3 = F_3 - iSH_3 = (m_a^{RR} - iSm_a^{NS})\alpha^a + (n_b^{RR} - iSn_b^{NS})\beta^b.$$

These fluxes generate a potential energy term that lifts the vacuum degeneracy and can lead to spontaneous breaking of $\mathcal{N} = 1$ supersymmetry. The tree-level superpotential, derived from the Gukov-Vafa-Witten form, is given by:

$$W_{\text{tree}}(S, U) \sim \int G_3 \wedge \Omega.$$

Furthermore, the presence of background fluxes modifies the internal geometry, introducing a warp factor in the metric:

$$ds^2 = e^{2A(y)} g_{\mu\nu} dx^\mu dx^\nu + e^{-2A(y)} g_{m\bar{n}} dy^m dy^{\bar{n}},$$

This warping is particularly relevant for addressing the hierarchy problem, as it induces red-shifting effects in strongly warped regions, potentially localizing chiral matter at the end of a warped throat. However, in the regime of exponentially large compactification volumes, the effect of warping can often be neglected.

Thus, the presence of background fluxes plays a crucial role in moduli stabilization, vacuum structure, and supersymmetry breaking within Type IIB string compactifications.

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