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Introduction to geometric quantization

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Summary

Abstract

This thesis explores the interplay between symplectic geometry and geometric quantization, a mathematical framework that seeks to transition from classical mechanics to quantum mechanics. Starting with the foundations of smooth manifolds, tensors, and symplectic structures, the geometric prerequisites for quantization are established, to be able to then move on to introduce the process of quantization, highlighting its challenges, focusing on the possibility of having different quantizations. As a preliminary step, the concept of prequantization is explored, followed by the optimization provided by polarization. By making use of examples and counterexamples, we are able to analyze the limits of quantization methods and to demonstrate the urge for more general structures.

1 Introduction

It might seem that at a microscopic level our world could still be described by the laws of classical physics (like Newtonian mechanics or electromagnetism). Taking a closer look, some phenomena, such as the stability of atoms, led scientists to discover the necessity of a theory that could replace the classical one. In fact, within the realms of classical physics, atoms are highly unstable and would be predicted to collapse, in contrast to experimental findings and the stability of the world we have around us. Since orbital motion is an accelerated motion, and thus emits radiation, the electron would collapse into the nucleus. After having observed that atoms are able to emit and absorb energy in discrete quantities, it appeared necessary to postulate the existence of stable orbits for electrons at certain discrete radii.

In addition, after the famous two-slit experiment, it was clear that under certain circumstances particles can have a wave-like nature and under certain conditions light showed particle-like behaviors. The microscopic level of nature was, therefore, found to be astoundingly strange.

Exploring the analogy found between classical mechanics and Schrödinger and Heisenberg's quantum mechanics, Dirac was able to formulate a general quantum condition, a guideline for passing from a given classical system to the corresponding quantum theory. This thesis explores this concept, generally known as quantization. Roughly speaking, quantization consists in replacing the classical algebra of observables by an algebra of operators acting on some Hilbert's space.

A conceptual challenge is presented by the process of quantization, as it attempts to construct a more fundamental quantum theory starting from a classical framework which is only a mere approximation of reality. Classical mechanics is able to emerge from quantum mechanics if limited to the macroscopic regime. It would seem more natural to derive classical mechanics from quantum mechanics rather than the other way around, from an epistemological perspective. However, quantization changes this logic trying to obtain a quantum theory from classical variables and structures.

A major issue in this approach is that quantization is not uniquely defined. Different quantum theories can be found from many inequivalent quantizations procedures that start off of the same classical limit. Quantization with its ambiguity highlights the limitations of using classical mechanics as a foundation for quantum theory.

Quantization's formal structure relies heavily on classical concepts such as phase space, observables, and Poisson brackets. The formulation of a quantum theory without any classical reference remains an unresolved challenge.

These considerations illustrate the inherent difficulties in constructing quantum theories through quantization but, in spite of them, and despite its conceptual foundations are not entirely satisfactory, this procedure remains a powerful and widely used tool in theoretical physics.

Geometric Quantization is a very general and modern procedure providing a systematic method for transitioning from classical mechanics to quantum mechanics using differential geometry. This is why it is considered so broad and relevant today. Geometric quantization does not strongly depend on the choice of coordinates, since it is based on the global geometric structure of phase space, making it applicable in a wide range of contexts, including curved spaces and constrained systems.

2 Manifolds, Tensors and Vector Spaces

2.1 Manifolds

Manifolds can be thought of as the generalization of surfaces in higher dimensional spaces. In this chapter we will look at manifolds and the operations w can define on them, focusing on their application in Hamiltonian mechanics.Now, let us move on to more formal definitions.

Definition 1.1: A *locally euclidean space* M of dimension d is a Hausdroff second countable topological space for which every point $p \in M$ has a neighborhood homeomorphic (where an homeomorphism between to topological spaces is a bijective, continuous function whose inverse is also continuous) to an open subset of $\mathbb{R}^d$. If ϕ is a homeomorphism of connected $U \subset M$ to an open subset of $\mathbb{R}^d$, then ϕ is called a coordinate map. A pair (U, ϕ) is called *chart*.

Definition 1.2: The *transition map* ϕ_{ij} describes, When two coordinate charts (U_i, ϕ_i) and (U_j, ϕ_j) overlap (meaning $U_i \cap U_j \neq \emptyset$), how the coordinates in chart i relate to the coordinates in chart j within the region of overlap. Specifically: $\phi_{ij} = \phi_j \circ \phi_i^{-1}$ This function maps a point from the image of $U_i \cap U_j$ under ϕ_i (which is a subset of $\mathbb{R}^n$) to the image of the same point under ϕ_j (another subset of $\mathbb{R}^n$).
Two coordinate charts (U_i, ϕ_i) and (U_j, ϕ_j) are named *smoothly compatible* if $U_{ij} = \emptyset$ or if ϕ_{ij} is a diffeomorphism.

Definition 1.3: A *smooth atlas* for a manifold M is a collection of smoothly compatible charts covering M.
A smooth structure is a maximal atlas A^{max} , an atlas that cannot be contained in a larger atlas.

And now, we can define:

Definition 1.4: A *smooth manifold* is the information of a topological manifold and a smooth structure.

Definition 1.5: Let M be a smooth manifold. A function $f : M \to \mathbb{R}$ is said to be *smooth* if $\forall\ (U, \phi) \in A^{max}$ we have that $f \circ \phi^{-1}$ is smooth.

Lemma 1.1: Given a smooth manifold M with smooth structure given by the max atlas A^{max}, then fixed an atlas $A = (U_i, \phi_i) \subset A^{max}$ (where U_i is an open set in M) then $f : M \to \mathbb{R}$ is smooth if $f \circ \phi_i^{-1}|_{\phi_i(U)}$ is smooth for all i. We have to notice that once we fixed an atlas, a function $\hat{f} = f \circ \phi^{-1}$ is smooth for every coordinate chart in the atlas A.

Now, let's look at some examples of smooth manifolds:

i) $\mathbb{R}^n$ with the standard chart $(\mathbb{R}^n, id)$
ii) The n-torus $T^n = S^1 \times \cdots \times S^1$ (n times). The n-torus is the Cartesian product of n circles.

If we want to look at a generalization of the previous construction looking at maps between smooth manifolds M and N , $F : M \to N$. Consider $\hat{F}$ to be the coordinate representation of F.

Definition 1.6: A map between smooth manifolds $F : M \to N$ is a diffeomorphism if it is smooth with a smooth inverse. in this case M and N are *diffeomorphic*.

In case the two manifolds are not compatible we could say that they are different from a smooth point of view.
For example, we take $\mathbb{R}$ with $\phi = id$ or $\omega(x) = x^3$ that determine different smooth structures on $\mathbb{R}$ in fact the composition of one with the inverse of the other $\phi \circ \omega^1 = x^{1/3}$ is not smooth at the origin.

We define the two smooth manifolds:

$$\mathbb{R}_\phi = (\mathbb{R}, A_\phi), \quad \text{where } A_\phi = \{(U,\phi) \mid U \subseteq \mathbb{R} \text{ open}, \ \phi(x) = x\}.$$
$$\mathbb{R}_\omega = (\mathbb{R}, A_\omega), \quad \text{where } A_\omega = \{(V,\omega) \mid V \subseteq \mathbb{R} \text{ open}, \ \omega(x) = x^3\}.$$

If we take $F = x^{\frac{1}{3}} : \mathbb{R}_\phi \to \mathbb{R}_\omega$ defined between the manifolds arising from the smooth structures, we can say that they are diffeomorphic since $\hat{F} = id$.

In order to continue to look for the definitions of a function's derivatives on manifolds, let's define the tangent space of a manifold, but first:

The tangent space at a point p on a manifold M is constructed through several related concepts:

Definition 1.7:
A linear map $Y_p : C^\infty(M) \to \mathbb{R}$ satisfying the Leibniz rule

$$Y_p(fg) = f(p)Y_p g + g(p)Y_p f$$

is called a derivation at p.

Definition 1.8: A *smooth definition* is a smooth map $C : J \subset \mathbb{R} \to M$. From now on we will denote it $C(t)$.

Definition 1.9: Let $C : \mathbb{R} \to M$ be a defi on the manifold M . A *tangent vector* at $x = C(0)$ is a directional derivative of $f : M \to \mathbb{R}$ along C calculated in $t = 0$.

Definition 1.10: The space of all derivations at p is denoted TM_p and forms the tangent space at p. Directional derivatives provide a natural way to create derivations.

Now, let's suppose we have a smopoth map $F : M \to N$, we can notice that, for every $p \in M$ there is a naturally induced linear map $F_* : TM_p \to TN_{F(p)}$ by: $(F_* X_p)_{F(p)} \cdot f := X_p \cdot (F \circ f)$ this is called *push foward* and, the linear map between the tangent spaces, is called differential.

We can now ask ourselves how to write derivations representations in a chart. Let's start by considering a chart (U, ϕ) for M.

We have a basis for $T\mathbb{R}^n_{\hat{p}}$ given by $\partial_i|_{\hat{p}}$ where $\hat{p} = \phi(p) \in \mathbb{R}^n$. Since ϕ is a diffeomorphism, the properties of the push-forward tell us that ϕ_* is an isomorphism, so $T\mathbb{R}^n_{\hat{p}}$ is isomorphic to TM_p.

Thus, we have a basis for TM_p:

$$\partial_i|_p = (\phi^{-1})_* \partial_i|_{\hat{p}}.$$

Now, consider a function $f : U \to \mathbb{R}$. We can compute:

$$\partial_i|_p f = (\phi^{-1})_* \partial_i|_{\hat{p}} f = \partial_i|_{\hat{p}}(f \circ \phi^{-1}) = \partial_i|_{\hat{p}} \hat{f} = \frac{\partial \hat{f}}{\partial x^i}(\hat{p}).$$

Next, let's examine how to push-forward this basis. Consider two charts (U, ϕ) and (V, ψ) for M and N, and denote the coordinates in the domain by

$$\hat{p} = \phi(p) = (x^1, \ldots, x^n) = \mathbf{x}$$
$$\hat{q} = \psi(q) = (z^1, \ldots, z^m) = \mathbf{z}$$

Let $f \in C^\infty(N)$. Then

$$\begin{aligned}(F_*(\partial_{x^i}|_p))f &= \partial_{x^i}(F \circ f) \\ &= \partial_{x^i}|_{\hat{p}}(f \circ F \circ \phi^{-1}) \\ &= \partial_{x^i}|_{\hat{p}}(f \circ \psi^{-1} \circ \psi \circ F \circ \phi^{-1})\end{aligned}$$

and, since $f \circ \psi^{-1} = \hat{f}$ and $\psi \circ F \circ \phi^{-1} = \hat{F}$,using the chain rule:

Now, if we take the two charts, and a point $p \in V \cap U$ we can ask ourselves how the coordinates functions of the two charts are related.

Given $(x^1, ..., x^n)$ coordinates for M and $(\tilde{x}^1, ...; \tilde{x}^n)$ for N every tangent vector can be written as :

$X_p = a^i \partial_i|_p$ or as $X_p = \tilde{a}^i \tilde{\partial}_i|_p$.

Let's compute the transition map $\psi \circ \phi^{-1}$ by acting on a general smooth function f:

$$\begin{aligned} \partial_i|_p f &= ((\phi^{-1})_* \partial_i|_{\phi(p)}) f \\ &= (\partial_i|_{\phi(p)})(f \circ \phi^{-1}) \\ &= (\partial_i|_{\phi(p)})(f \circ \psi^{-1} \circ \psi \circ \phi^{-1}). \end{aligned}$$

Using the chain rule, we obtain:

$$\partial_i|_p = \left.\frac{\partial \tilde{x}^j}{\partial x^i}\right|_{\phi(p)} \tilde{\partial}_j|_{\psi(p)},$$

where $\hat{f}(\tilde{\mathbf{x}}) = f \circ \psi^{-1}$ and $\psi \circ \phi^{-1} = \tilde{\mathbf{x}}(\mathbf{x})$.

Thus, we finally have:

$$\partial_i|_p = \left.\frac{\partial \tilde{x}^j}{\partial x^i}\right|_{\phi(p)} \tilde{\partial}_j|_p,$$

and consequently:

$$a^j = \left.\frac{\partial x^i}{\partial \tilde{x}^j}\right|_{\phi(p)} \tilde{a}^j.$$

Now, we will define two other objects, very useful in physics:

Definition 1.11: The *tangent bundle* TM is a differentiable manifold and it is defined as $\bigcup_{p \in M} TM_p$. A point in TM is a pair (x, y) where p is a point in M and y is s vector in TM_p.

Definition 1.12: The *cotangent space* T^*M_p is the dual space of TM_p. An element of this space is a linear function $\lambda_p : TM_p \to \mathbb{R}$. Consider coordinates $(x_1, ..., x_n)$ of $\mathbf{x}$. We can define in a coordinate chart (M, ϕ) a smooth map $\partial_i : M \to TM$ that at each point p associates $\partial_i|_p$. In this case the natural basis for the tangent space T_pM at a point p is given by $\{\partial_i|_p\}$, and the corresponding dual basis for the cotangent space T_p^*M is given by $\{dx^i|_p\}$, satisfying:$dx^i|_p(\partial_j|_p) = \delta^i_j$..

Definition 1.13: The *cotangent bundle* is a differentiable manifold and it is defined as $\bigcup_{x \in M} T^*M_x$.

For reasons that will be explained later on, the cotangent bundle is highly important in physics since it can be understood to be a phase space on which Hamiltonian mechanics plays out.

2.2 Tensors

Studying differentiable manifolds, tensors play a fundamental role, describing geometric and physical quantities in a coordinate-independent manner. Tensors generalize the concept of vectors and linear transformations, allowing us to represent multilinear maps on vector spaces and their duals.

Definition 1.14: A *covariant k tensor* is a multilinear map :

$$\tau : \underbrace{V \times .. \times V}_{k \text{ times}} \to \mathbb{R}$$

The set of all covariant k tensors is denoted as T^kV.

Definition 1.15: A *controvariant p tensor* is a multilinear map:

$$Y : \underbrace{V^* \times .. \times V^*}_{p \text{ times}} \to \mathbb{R}$$

The set of all controvariant p tensor is denoted by T_pV.

And finally:

Definition 1.16 : A *mixed tensor* of type (k, p) is a multilinear map

$$F : \underbrace{V \times .. \times V}_{k \text{ times}} \times \underbrace{V^* \times .. \times V^*}_{p \text{ times}} \to \mathbb{R}$$

The set of all mixed tensors is denoted by T^k_pV.

Obviously, we can apply this construction to $V = TM_p$:

Let's consider $\alpha, \beta \in V^*$ and let's define the map $\tau_{\alpha,\beta} : V \times V \to \mathbb{R}$, $\mathbf{v_1}; \mathbf{v_2} \to \alpha(\mathbf{v_1})\beta(\mathbf{v_2})$ This map is multilinear thus $\tau_{\alpha,\beta} \in T^2V$.

Given $a \in \mathbb{R}$ we note that :

(i) $\tau_{\alpha,\beta} = \tau_{a\alpha,\beta} = \tau_{\alpha,a\beta}$

(ii) $\tau_{\alpha+\alpha',\beta} = \tau_{\alpha,\beta} + \tau_{\alpha',\beta}$

(iii) $\tau_{\alpha,\beta+\beta'} = \tau_{\alpha,\beta} + \tau_{\alpha,\beta'}$

Consider $\tau \in T^kV$ and $\rho \in T^mV$, and define an element of the space $T^{k+m}V$ denoted by $\tau \otimes \rho$ with the relation:

$$(\tau \otimes \rho)(\mathbf{v_1}, \dots, \mathbf{v_{k+m}}) = \tau(\mathbf{v_1}, \dots, \mathbf{v_k})\rho(\mathbf{v_{k+1}}, \dots, \mathbf{v_{k+m}}).$$

Now, our purpose is to look for a generalization of integrands on manifolds and, in order to do so, we need to focus on covariant alternating and symmetric k tensors.

Let $\pi \in S_n$ (the symmetric group) be a permutation of n objects, and denote the sign of the permutation by

$$\text{sign}(\pi) = (-1)^n,$$

where n is the number of transpositions. The permutation $\pi : V \times \cdots \times V \to V \times \cdots \times V$ acts as follows:

$$\pi(v_1 \times \cdots \times v_n) = (v_{\pi(1)} \times \cdots \times v_{\pi(n)}).$$

Definition 1.17: Let $\rho \in T^mV$. The tensor ρ is called *symmetric* if

$$\rho(\mathbf{v_1}, \dots, \mathbf{v_m}) = \rho(\mathbf{v}_{\pi(\mathbf{1})}, \dots, \mathbf{v}_{\pi(\mathbf{m})})$$

for any permutation $\pi \in S_m$.

The tensor ρ is called *alternating* if

$$\rho(\mathbf{v_1}, \dots, \mathbf{v_m}) = \text{sign}(\pi)\rho(\mathbf{v}_{\pi(\mathbf{1})}, \dots, \mathbf{v}_{\pi(\mathbf{m})})$$

for any permutation $\pi \in S_m$.

For example, the determinant of the matrix is an alternating covariant map on column vectors. The set of alternating covariant k tensors is denoted by $\Lambda^k V$, the set of symmetric covariant tensors is $\Sigma^k V$.

To construct symmetric or alternating tensors, we define two natural maps:
1. *Symm*: $T^m V \to \Sigma^m V$, given by

$$\textit{Symm}(\rho)(\mathbf{v_1}, \ldots, \mathbf{v_m}) = \frac{1}{m!} \sum_{\pi \in S_m} \rho(\mathbf{v}_{\pi(\mathbf{1})}, \ldots, \mathbf{v}_{\pi(\mathbf{m})}),$$

where $\Sigma^m V$ denotes the space of symmetric tensors.
2. *Alt*: $T^m V \to \Lambda^m V$, given by

$$\textit{Alt}(\rho)(\mathbf{v_1}, \ldots, \mathbf{v_m}) = \frac{1}{m!} \sum_{\pi \in S_m} \mathrm{sign}(\pi)\rho(\mathbf{v}_{\pi(\mathbf{1})}, \ldots, \mathbf{v}_{\pi(\mathbf{m})}),$$

where $\Lambda^m V$ denotes the space of alternating tensors.

Note: we could also think about $\Lambda^k V$ as $\Lambda^k V = P^-(V \times .. \times V)$ where $(P^-)^2 = P^-$ is a projection.

We construct another element of the space $\Lambda^{k+m} V$, denoted by $\tau \wedge \mu$, where $\tau \in \Lambda^k V$ and $\mu \in \Lambda^m V$, as follows:

$$\tau \wedge \mu = \frac{(k+m)!}{m!k!} \cdot \frac{1}{(k+m)!} \sum_{\pi \in S_{k+m}} \tau(\mathbf{v}_{\pi(\mathbf{1})}, \ldots, \mathbf{v}_{\pi(\mathbf{k})})\mu(\mathbf{v}_{\pi(\mathbf{k+1})}, \ldots, \mathbf{v}_{\pi(\mathbf{k+m})}).$$

The wedge product $\wedge$ is:

1. Skew-symmetric: $\tau \wedge \mu = -\mu \wedge \tau$ (if τ and μ are tensors of the same degree),
2. Associative: $(\tau \wedge \mu) \wedge \nu = \tau \wedge (\mu \wedge \nu)$,
3. Distributive: $\tau \wedge (\mu + \nu) = \tau \wedge \mu + \tau \wedge \nu$.

Proposition 1.1: If β^i is the base of V^* the set $\beta^{i_1} \otimes .. \otimes \beta^{i_k}$ is a set of covariant tensors of rank k and it's a basis for $T^k V$.

Similarly:

Proposition 1.2: Let V be a real finite-dimensional vector space, and let $\{\beta^i\}$ be a basis of its dual space.

The set

$$\{\beta^{i_1} \otimes \cdots \otimes \beta^{i_k} \mid i_1 < \cdots < i_k\}$$

is a basis for $\Lambda^k V$.

Any element of $\Lambda^k V$ can be written as

$$\omega = \frac{1}{k!} \omega_{i_1 \ldots i_k} \beta^{i_1} \wedge \cdots \wedge \beta^{i_k},$$

where

$$\omega_{i_1 \ldots i_k} = \omega(b_{i_1}, \ldots, b_{i_k}),$$

with $\{b_i\}$ being a basis for V. The coefficients $\omega_{i_1 \ldots i_k}$ are completely antisymmetric, meaning that swapping two indices introduces a minus sign.

Consider now two finite dimensional vector spaces V and W and construct $V^* \otimes W^*$ consisting of the linear combination of objects of the form $\mathbf{v} \otimes \mathbf{w}$ with $\mathbf{v} \in V^*$ and $\mathbf{w} \in W^*$. These objects are multilinear maps from $V \times W$ to the reals. Therefore this space is by definition $\mathbb{R}\langle V^* \times W^* \rangle$.

Consider two finite-dimensional vector spaces V and W, and construct $V^* \otimes W^*$, the space consisting of linear combinations of objects of the form $\mathbf{v} \otimes \mathbf{w}$, where $\mathbf{v} \in V^*$ and $\mathbf{w} \in W^*$. These objects represent multilinear maps from $V \times W$ to the reals. Therefore, this space is, by definition,

$$\mathbb{R}\langle V^* \times W^* \rangle.$$

Now, consider the subspace I spanned by all elements satisfying the following properties:

$$a(\mathbf{v}, \mathbf{w}) - (a\mathbf{v}, \mathbf{w}), \quad a(\mathbf{v}, \mathbf{w}) - (\mathbf{v}, a\mathbf{w}),$$

$$(\mathbf{v} + \mathbf{v}', \mathbf{w}) - (\mathbf{v}, \mathbf{w}) - (\mathbf{v}', \mathbf{w}), \quad (\mathbf{v}, \mathbf{w} + \mathbf{w}') - (\mathbf{v}, \mathbf{w}) - (\mathbf{v}, \mathbf{w}').$$

We define $V^* \otimes W^* := \mathbb{R}\langle V^* \times W^* \rangle / I$. We obtain the following relations:

$$a(\mathbf{v} \otimes \mathbf{w}) = (a\mathbf{v}) \otimes \mathbf{w} = \mathbf{v} \otimes (a\mathbf{w}),$$

$$\mathbf{v} \otimes \mathbf{w} + \mathbf{v}' \otimes \mathbf{w} = (\mathbf{v} + \mathbf{v}') \otimes \mathbf{w},$$

$$\mathbf{v} \otimes \mathbf{w} + \mathbf{v} \otimes \mathbf{w}' = \mathbf{v} \otimes (\mathbf{w} + \mathbf{w}').$$

Proposition 1.3: The vector space $V^* \otimes W^*$ is canonically isomorphic to the vector space $Bil(V, W)$ of bilinear functions $V \times W$ to the reals.
This can be generalized to all tensors.

Let us denote from now $T^k_{r,p}$ as: $T^k_{r,p} = \underbrace{T^*M_p \otimes .. \otimes T^*M_p}_{k-times} \otimes \underbrace{TM_p \otimes .. \otimes TM_p}_{r-times}$

Definition 1.18: A *tensor field* of rank k is a smooth assignment of elements of $T^k_{r,p}$ to each point $p \in M$. their union for all $p \in M$ is called *tensor bundle* and it's denoted $\mathcal{T}^k_r(M)$.

Proposition 1.4: A mixed tensor F can always be written in terms of the basis of TM and T^*M in this form : $F = F^{i_1,i_2..i_k}_{j_1,j_2..j_p} dx_{i_1} \otimes .. \otimes dx_{i_k} \otimes \partial_{j_1} \otimes .. \otimes \partial_{j_p}$ where $F^{i_1,i_2..i_k}_{j_1,j_2..j_p}$ are smooth functions. This confirms that a tensor field is a multilinear function on the sections of the tangent and cotangent bundles, and can be seen as a multilinear map on $C^\infty(M)$.

Now we can finally define a mathematical object that helps us generalize the concept of integrands over manifolds; the differential form.

Definition 1.19: A *differential form* ω^k of order k is a smooth assignment of exterior k-forms on a tangent space $V = TM_p$, meaning a linear antisymmetric function of k vectors. For these reasons, $\omega^k \in \Lambda^k V$.

Every k form on $\mathbb{R}^*$ with coordinate system $x_1, .., x_n$ can be written: $\omega^k = \sum_{j_1 < .. j_k} a_{j_1,..,j_k}(x) dx_1^{j_1} \wedge .. \wedge dx_k^{j_k}$.

We could now think about the tensor products of spaces of k-forms. The problem with that is that if we take $\Lambda^{k_1} V \otimes \Lambda^{k_2} V$, this object is not completely antisymmetric: the first space is completely antisymmetric for k_1 and the second for k_2 but if we swap the two k-form spaces we will lose the antisymmetric property. For this reason, we need to introduce the concept of exterior multiplication:

From now on, when we write explicitly, it means we are working on a chart.

Now, denote by $\Omega^k_q(M) \equiv \Lambda^k(T_qM)$ the space of k forms at q and by $\Omega^k(M)$ the space of smooth k forms.

Definition 1.20: The *exterior derivative*

$$d^k : \Omega^k(M) \to \Omega^{k+1}(M)$$

is defined in a chart as:

$$d^k(\omega) = \sum_{j_1<\cdots<j_k} \frac{\partial \omega_{j_1\ldots j_k}}{\partial x^l} \, dx^l \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}.$$

where:

$$\omega \equiv \sum_{j_1<\cdots<j_k} a_{j_1,\ldots,j_k}(x) \, dx^{j_1} \wedge \cdots \wedge dx^{j_k}$$

It could be useful to think about a simple example: If $M \subset \mathbb{R}^3$, we have $\omega_0 = f(x,y,z)$ and the exterior derivative is given by:

$$d\omega_0 = \sum_l \frac{\partial f}{\partial x^l} \, dx^l,$$

which corresponds to the gradient.

2.3 Symplectic vector spaces and symplectic manifolds

Theorem 1.1: Let $\Omega : V \times V \to \mathbb{R}$ be a skew-symmetric bilinear map with V m-dimensional vector space. Then there is a basis $u_1, .., u_k, e_1, .., e_n, f_1, .., f_n$ $(k + n + n = m)$ of V such that :

(1) $\Omega(u_j, v) = 0 \; \forall j$ with $v \in V$

(2) $\Omega(e_j, e_k) = 0 = \Omega(f_i, f_k) \; \forall e_i, f_i$

(3) $\Omega(e_i, f_k) = \delta_{ij}$

Now, let's have a look at the matrix form of Ω and let's represent it with a block matrix, where each entry is a vector $\mathbf{u}, \mathbf{e}, \mathbf{f}$ in this order: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathbf{I} \\ 0 & \mathbf{-I} & 0 \end{bmatrix}$

Definition 1.21: A skew Symmetric bilinear map Ω is *symplectic* if $\tilde{\Omega} : V \to V^*, \tilde{\Omega}[v](u) \equiv \Omega(v, u)$ is bijective. Then (V, Ω) is called a *symplectic vector space.*

Properties:
i) $\tilde{\Omega} : V \to V^*$ is a bijection.
ii) Since $ker\tilde{\Omega} = U = \Omega(u, u) = 0$ $k = dimU = 0$ $2n = m$ so U is even dimensional.

Definition 1.22: A *symplectomorphism* ξ between (V, Ω) and (V', Ω') is an isomorphic linear map such that $\xi^*\Omega' = \Omega$ where $(\xi^*\Omega')(u, v) = \Omega'(\xi(u), \xi(v))$ Then we call (V, Ω) and (V', Ω') *symplectomorphic.*

Now, the idea we will follow is that symplectic manifolds are objects that locally look like symplectic vector spaces.

Definition 1.23: A differential form $\omega|_p : TM_p \times TM_p \to \mathbb{C}^\infty$ is *symplectic* if $d\omega = 0$ and ω_p is symplectic for all $p \in M$. Remark that if ω is symplectic, $dimTM_p$ is even.

Definition 1.24: A *symplectic manifold* is a pair (M, ω) where M is a manifold and ω is a symplectic form.

Example: Let $M = \mathbb{R}^n$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$. The form $\omega = \sum_{j=1}^n dx_j \wedge dy_i$ is symplectic. (Note that the coefficients need to be constant because $d\omega = 0$).

The symplectic basis for $T\mathbb{R}^{2n}_p$ is

$$\left(\frac{\partial}{\partial x_1}\Big|_p, \ldots, \frac{\partial}{\partial x_n}\Big|_p, \frac{\partial}{\partial y_1}\Big|_p, \ldots, \frac{\partial}{\partial y_n}\Big|_p \right).$$

Definition 1.25: Let $F : M \to N$ be a smooth map between smooth manifolds. The *pull-back* of a differential form $\omega \in \Omega^k(N)$ is the differential form $F^*\omega \in \Omega^k(M)$ defined by

$$(F^*\omega)_p(x_1, \ldots, x_k) = \omega_{F(p)}(dF_p(x_1), \ldots, dF_p(x_k)),$$

for all $p \in M$ and $x_1, \ldots, x_k \in T_pM$, where $dF_p = F_*|_p : T_pM \to T_{F(p)}N$ is the push-foward of F at p.

Definition 1.26: Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds. Let $\theta : M_1 \to M_2$ be a diffeomorphism. Then, θ is a *symplectomorphism* if $\theta^*(\omega_2) = \omega_1$.

2.4 Vector fields and flows

Recall that a vector field $X \in \Xi(M)$ for manifold M is a smooth assignment of $X \in TM_p$ $\forall p$.

Definition 1.27: An *integral curve* $C(t)$ of $\mathcal{X}$ vector field on M is a curve in M such that the tangent vector at $C(t)$ is $\mathcal{X}|_{x(t)}$.

Given a chart (U, φ) this is written as $\frac{dx^\mu(t)}{dt} = \mathcal{X}^\mu(x(t))$ (we could think about these objects as speed modules) where $x^\mu(t)$ is the μ_{th} coordinate of $\varphi(x(t))$ and $\mathcal{X} = \mathcal{X}^\mu(x)\frac{\partial}{\partial x^\mu}$. In order to find an integral curve given $\mathcal{X}$ we need to solve this set of ODEs: $\frac{dx^\mu(t)}{dt} = \mathcal{X}^\mu(x(t))$.

Definition 1.28: Let $\sigma(t, x_0)$ be an integral curve of $\mathcal{X} \in \Xi(M)$ passing through x_0 at $t = 0$. Let $\sigma^\mu(t, x_0)$ be the $\mu_t h$ coordinate for σ , the $\frac{d\sigma^\mu(t,x_0)}{dt} = \mathcal{X}^\mu(\sigma(t, x_0))$ subject to $\sigma^\mu(t = 0, x_0) = x_0^\mu$. The map $\sigma : \mathbb{R} \times M \to M$ is called *flow.* A flow satisfies $\sigma(t, \sigma(s, x_0)) = \sigma(t + s; x_0)$

Theorem 1.2: For any point $x \in M$, there exists (at least locally) a differentiable map $\sigma : \mathbb{R} \times M \to M$ such that:
1) $\sigma(0, x) = x$,
2) $t \to \sigma(t, x)$ satisfies $\sigma^\mu(t = 0, x_0) = x_0^\mu$,
3) $\sigma(t, \sigma(s, x)) = \sigma(t + s, x)$.
From now on, let's write $\sigma_t(p) = \sigma(t, p)$

Definition 1.29: The map σ_t is an *isotopy* if each $\sigma_t : M \to M$ is a diffeomorphism and σ_0 is the identity. Given an isotopy σ we can always obtain $v_t \in TM$ vector field via: $v_t(p) = \frac{d\sigma_s(q)}{ds}|_{s=t}$ where $q = \sigma_{t^{-1}}(p)$

Definition 1.30: When $\mathcal{X} = v_t$ is independent of time, the isotopy is called the *exponential map* of the flow $\mathcal{X}$ and is denoted by $\sigma^\mu(t, x) \equiv e^{t\mathcal{X}} : M \to M$. The exponential map is a unique form of diffeomorphisms satisfying:

1. $e^{0\mathcal{X}} = \text{id}$,
2. $\frac{de^{t\mathcal{X}}}{dt} = \mathcal{X}(e^{t\mathcal{X}}(x))$.

The flow satisfies some properties:

i) $\sigma(0,x) = x = e^{0\mathcal{X}}$,

ii) $\frac{d\sigma_t(x)}{dt} = \mathcal{X}(e^{t\mathcal{X}}(x)) = \frac{d[e^{t\mathcal{X}}x]}{dt}$,

iii) $\sigma(t,\sigma(s,x)) = \sigma(t, e^{s\mathcal{X}}x) = e^{t\mathcal{X}}e^{s\mathcal{X}}x = e^{(t+s)\mathcal{X}} = \sigma(t+s,x)$.

Definition 1.31 Let X and Y be smooth vector fields on a manifold M. The Lie bracket $[X,Y]$ of X and Y is the vector field which acts on a function $f \in C^\infty(M)$ to give $XYf - YXf$.

To make sense of this definition we need to check that this does indeed define a derivation. Linearity is clear, but we need to verify the Leibniz rule:

$$\begin{aligned}[X,Y](fg) &= X(Y(fg)) - Y(X(fg)) \\ &= X(fY(g) + gY(f)) - Y(fX(g) + gX(f)) \\ &= f(XYg) + (Xf)(Yg) + (Yg)(Xf) + g(XYf) \\ &- (Yf)(Xg) - f(YXg) - g(YXf) - (Xf)(Yg) \\ &= f(XYg - YXg) + (XYf - YXf)g \\ &= f[X,Y]g + ([X,Y]f)g\end{aligned}$$

It is useful to write $[X,Y]$ in terms of its components in some chart: Write $X = X^i\partial_i$ and $Y = Y^j\partial_j$ (here I'm using the summation convention: Sum over repeated indices).
Note that
$[\partial_i, \partial_j]f = \partial_i\partial_j f - \partial_j\partial_i f = 0$.

Therefore we have

$[X,Y] = X^i\partial_i Y^j\partial_j - Y^j\partial_j X^i\partial_i = \sum_{i,j=1}^n (X^i\partial_i Y^j - Y^j\partial_j X^i)\partial_i$.

The Lie bracket measures the extent to which the flows in directions X and Y do not commute. The following proposition makes this more precise:

Proposition 1.5: Let X, Y be two vector fields on a manifold M, and let Ψ_t and Φ_t be the local flow of X in some region containing the point $x \in M$. Then
$[X,Y]_x = \lim_{t\to 0} \frac{d}{dt}(F_*\Psi_{-t}Y_{\Psi_t(x)})$.

Where F_* is the push-foward.

Now, a problem arises: what can one do to compare two things in two different points of a manifold? It is not very clear what comparing in this case even means. For this reason, we need to introduce an object called Lie's derivative.

Definition 1.32: The *Lie derivative* associated to a vector field $\mathcal{X}$ on a manifold M is the linear map $L_\mathcal{X} : A(T^p_q M) \to A(T^p_q M)$ (where $A(T^p_q M)$ is the set of all p-contravariant q-covariant tensors) wich sends S to the tensor $(\frac{d}{dt})(\phi_t^* S)|_{t=0}$ where $\phi_t$is the local one parameter group associated to $\mathcal{X}$.

Proposition 1.6: The operator $L_\mathcal{X}$ is actually defined by the following properties:
i) for $f \in C^\infty(M), L_\mathcal{X} f = df(x)$
ii) for $\mathcal{Y} \in A(TM), L_\mathcal{X}\mathcal{Y} = [\mathcal{X},\mathcal{Y}]$
iii) for any tensors $S,T : L_\mathcal{X}(S \otimes T) = L_\mathcal{X} S \otimes T + S \otimes L_\mathcal{X} T^*$
iv) for any (p,q)- tensors S and for any contraction c $L_\mathcal{X}(c(S)) = c(L_\mathcal{X} S)$.

2.5 Symplectic Geometry and classical mechanics

To summarise, by a symplectic manifold (M,ω) we will mean a smooth real m-dimensional manifold M without boundary, equipped with a closed non-degenerate two-form ω, the symplectic form.

'Closed' means that

$$d\omega = 0 \tag{1}$$

where d is the exterior differential

$$d : \Omega^k(M) \to \Omega^{k+1}(M), d^2 = 0 \tag{2}$$

on differential forms on M. And 'non-degenerate' means that at each point $x \in M$ the antisymmetric matrix ω_x is non-degenerate.

$$\det(\omega_x) \neq 0 \quad \forall x \in M \tag{3}$$

The most important example of a symplectic manifold is a cotangent bundle T^*Q, where Q is the configuration space (the space defined by the generalized coordinates). A cotangent bundle has a canonical symplectic two-form which is globally exact,

$$\omega = d\theta. \tag{4}$$

(and hence, in particular, exact), θ is the canonical one-form on the cotangent bundle T^*Q, defined in local coordinates (q^k, p_k) as: $\theta = p_k\, dq^k$. Any local coordinate system $\{q^k\}$ on Q can be extended to a coordinate system $\{q^k, p_k\}$ on T^*Q such that θ and ω are locally given by

$$\theta = p_k dq^k, \omega = dp_k \wedge dq^k. \tag{5}$$

Since ω is invertible, at each point $x \in M$ it gives an isomorphism between the tangent and cotangent spaces of M at x,

$$\omega : TM_x \to T^*M_x \tag{6}$$

expressed in local coordinates as

$$X \to X^i \omega_{ij} dx^j. \tag{7}$$

Crudely speaking, like a metric a symplectic form allows us to raise and lower indices on tensors. This extends to an isomorphism between TM and T^*M and between vector fields and one-forms on M,

$$X \to i(X)\omega = \omega(X, \cdot) \in \Omega^1(M) \tag{8}$$

$i(X)$ denotes the contraction of a differential form with the vector field X.

In particular, therefore, the existence of ω allows us to associate a vector field X_f to every function $f \in C^\infty(M)$ via

$$i(X_f)\omega = -df \tag{9}$$

(the minus sign is for later convenience only). X_f, the 'symplectic gradient' of f, is known as the Hamiltonian vector field of f. The Lie derivative of ω along X_f is zero, then the Hamiltonian vector field of f generates a flow on M that leaves *omega* invariant

$$\mathcal{L}(X_f)\omega = di(X_f)\omega + i(X_f)d\omega = -d^2 f = 0. \tag{10}$$

Thanks to the relation $i(X_f)\omega = -df$, we see that the symplectic form provides an anti-symmetric pairing $\{f, g\}$ between functions f, g on M called the Poisson bracket of f and g. It is defined by

$$\{f, g\} := \omega(X_f, X_g) \in C^\infty(M) \tag{11}$$

and describes the change of g along X_f,

$$\{f, g\} = i(X_g)i(X_f)\omega = i(X_f)dg = \mathcal{L}(X_f)g. \tag{12}$$

In particular, f is constant along the integral curves of X_f. The Poisson bracket satisfies the Jacobi identity

$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}. \tag{13}$$

One further important identity we will need, that relates the Lie algebras of vector fields and functions on M is

$$[X_f, X_g] = X_{\{f,g\}}, \tag{14}$$

which shows that the Hamiltonian vector fields also form an infinite dimensional Lie algebra. Regarding the map $f \to X_f$ as an assignment of differential operators to functions , the prior identity is also an illustration of the quantization paradigm , because the Poisson's brackets will become commutators.

Let's now consider the phase space, in order to start linking mathematical objects with mechanics. The phase space is the $2n$-dimensional real vector space $\mathbb{R}^{2n}$ with coordinates $q^1, \ldots, q^n, p_1, \ldots, p_n$ describing the position and the momentum of the particles involved. The dynamics of the system is governed by Hamilton's equations

$$\frac{dq^k}{dt} = \frac{\partial H}{\partial p_k}, \quad \frac{dp_k}{dt} = -\frac{\partial H}{\partial q^k}, \tag{15}$$

where $H(q^k, p_k)$, the Hamiltonian, is a function on phase space describing the energy of the system.

Typically, H is of the form $H = T + V$ where T is the kinetic energy and $V = V(q^k)$ is the potential energy whose gradient describes the forces acting on the particles.

If H does not depend on time explicitly, the equations of motion imply that H is conserved along any trajectory in phase space,

$$\dot{H} = \sum_k \frac{\partial H}{\partial q^k}\dot{q}^k + \frac{\partial H}{\partial p_k}\dot{p}_k = 0, \tag{16}$$

(summation over repeated indices being understood) while the evolution of any other function f on phase space (observable) is given by

$$\dot{f} = \{H, f\} = \sum_k \frac{\partial f}{\partial q^k}\frac{\partial H}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial H}{\partial q^k}. \tag{17}$$

In general, any constant of motion, i.e. any function f on phase space in involution with the Hamiltonian, $\{H, f\} = 0$, can be used to reduce the dynamical system to a lower dimensional one on the common level surfaces of the functions H and f. It follows from the Jacobi identity that the Poisson bracket of any two constants of motion is also a constant of motion. If it is possible to find n constants of motion in involution the system is called integrable.

The equations (16)-(18), characterizing Hamiltonian mechanics, arise naturally if we think of $\mathbb{R}^{2n}$ as the cotangent bundle $T^*\mathbb{R}^n$ of the configuration space $\mathbb{R}^n$. Namely, in that case the Hamiltonian vector field X_f of a function $f(q^k, p_k)$ is

$$X_f = \sum_k \frac{\partial f}{\partial p_k}\frac{\partial}{\partial q^k} - \frac{\partial f}{\partial q^k}\frac{\partial}{\partial p_k}, \tag{18}$$

as it is easily verified that $i(X_f)dp_k \wedge dq^k = -df$. Therefore the Poisson bracket is

$$\{f, g\} = \sum_k \frac{\partial f}{\partial q^k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q^k}. \tag{19}$$

3 Introduction to quantization

In this section, we will start do distance ourselves from the classical theory outlined above.

The space $C^\infty(M)$ of observables has, in addition to a Lie algebra structure provided by the Poisson bracket, the structure of a commutative algebra under pointwise multiplication:

$$(fg)(x) = f(x)g(x) = (gf)(x) \tag{20}$$

Due to the non-commutative nature of observables in the quantum theory, this property is sacrificed. More specifically, quantization usually refers to an assignment

$$Q : f \mapsto Q(f) \tag{21}$$

of operators $Q(f)$ on some Hilbert space to classical observables f. This Hilbert space is, in general, infinite dimensional, but we could consider a finite dimensional Hilbert space and, in that case, we can think of the $Q(f)$'s simply as finite-dimensional matrices. The scalar product in the Hilbert space is necessary for the probabilistic interpretation of the theory and is thus of fundamental importance. Q also must satisfy some more or less obvious requirements such as:

1. $\mathbb{R}$-linearity $Q(rf + g) = rQ(f) + Q(g)$ for all $r \in \mathbb{R}$, $f, g \in C^\infty(M)$.
2. The constant function 1 is mapped to the identity operator or matrix 1: $Q(1) = 1$.
3. Real functions should correspond to Hermitian operators: $Q(f)^* = Q(f)$.
4. $[Q(f), Q(g)] = -i\hbar Q(\{f, g\})$

Where number 4 is Dirac's idea of seeing commutators as the equivalent counterpart of Poisson's brackets.

We can notice that sending $\hbar \to 0$ (now treating $\hbar$ just as a parameter), one recovers the commutative structure of classical mechanics, for this reason for most macroscopic purposes we can neglect it. At the microscopic level, however, $\hbar$ needs to be taken into consideration. knowing what quantization means, we need another condition that gives us some kind of irreducibility. The same way there needs to be a complete set of observables, we define a complete set of operators so that the only operators that commute with all operators from that set are multiples of the identity.

5. If $\{f_1, \dots, f_k\}$ is a complete set of observables, $\{Q(f_1), \dots, Q(f_k)\}$ is a complete set of operators.

In general, it is not always possible to satisfy both condition number 4 (for all f and g) and number 5. The only way of operating is trying to find the best compromise, for example, demanding number 4 only for a complete set of observables and perhaps some additional observables that are of particular interest in the quantum theory. A key concept to understand is that different choices of complete sets will lead to inequivalent quantum theories, leading to different predictions of the results of experiments, since there is not one single rule to follow when choosing complete sets. It is by common sense that we can rightfully choose which choice of complete sets to make, by looking at certain symmetries of the system. We will now take as an example a set of observables that we introduced before, the coordinate functions q^k and p_k. w We demand that the corresponding operators satisfy the canonical commutation relations:

$$[Q(q^k), Q(q^l)] = 0 \tag{22}$$
$$[Q(q^k), Q(p_l)] = i\hbar\delta^k_l \tag{23}$$
$$[Q(p_k), Q(p_l)] = 0 \tag{24}$$

This is the so-called Heisenberg algebra. We can now take in consideration Schur's Lemma, that states that for an irreducible representation ρ of a group or Lie algebra $\mathfrak{g}$, an operator A

that commutes with all operators $\rho(X)$ in the representation, must be a scalar multiple of the identity operator. In other words, there exists a scalar λ such that $A = \lambda I$. In the context of an irreducible representation of the Heisenberg algebra, where q^k and p_k are the position and momentum operators, any operator A that commutes with all q^k and p_k, ($[A, q^k] = 0$ and $[A, p_k] = 0$ for all k), must be a scalar multiple of the identity operator. This fact is crucial for proving the uniqueness (up to unitary equivalence) of the representation. This tells us that rules (26,27,28) are now equivalent to finding an irreducible representation of the Heisenberg algebra.

We can consider this representation as an example:

$$Q(q^k)\psi(x) = x^k\psi(x) \tag{25}$$

$$Q(p_k)\psi(x) = -i\hbar\frac{\partial\psi}{\partial x^k}(x) \tag{26}$$

The spectrum (range of eigenvalues) of these operators is $(-\infty, \infty)$. It is important to know that this is the Schrödinger picture of quantum mechanics, but the fact that, in this case, wave functions can be represented by functions on the configuration space is only a mere consequence of our quantization system, and not something that is valid in general.

Now we can ask ourselves if we can quantize any other observables, for example the kinetic energy operator, $p^2 = p^k p^l \delta_{kl}$, which is represented via the Laplacian:

$$Q(p_k p_l) = -\hbar^2\frac{\partial^2}{\partial x^k \partial x^l} Q(p^2) = -\hbar^2\delta_{kl}\frac{\partial^2}{\partial x^k \partial x^l} = -\hbar^2\Delta \tag{27}$$

If we have observables that are quadratic in the coordinates we need to represent them with multiplication operators. Imposing either the hermiticity condition or the quantum condition number 4 one finds:

$$Q(p_k q^l) = \frac{1}{2}(Q(p_k)Q(q^l) + Q(q^l)Q(p_k)) \tag{28}$$

This can be interpreted as a particular operator ordering of $Q(p_k q^l) \sim Q(p_k)Q(q^l)$ but, there is no logical necessity for the assignment Q to satisfy some condition like $Q(fg) \sim Q(f)Q(g)$ in general.

The quadratic observables form a closed Lie algebra under Poisson brackets, which is symplectic $sp(n)$:

$$\begin{aligned}
&\{p_i p_j, p_k p_l\} = \{q^i q^j, q^k q^l\} = 0\\
&\{p_i q^j, p_k p_l\} = \delta^j_k p_i\\
&\{p_i q^j, p_k q^l\} = \delta^j_k p_i q^l + \delta^i_l p_j q^k\\
&\{p_i q^j, q^k q^l\} = \delta^j_k p_i q^l + \delta^i_l p_j q^k\\
&\{p_i p_j, q^k q^l\} = \delta^i_k p_j q^l + \delta^j_k p_i q^l + \delta^i_l p_j q^k + \delta^j_l p_i q^k
\end{aligned}$$

Thus, what the above means is that the symplectic Lie algebra obtained with a quantized symplectic vector space on the quantum Hilbert space mirrors the classical symplectic invariance of the theory.

Problems with this theory might occur especially if we treat observables of degree greater than two. The Hermicity condition might be verified, but the commutation condition might not. A counterexample will follow, in order to show the shortcomings of this theory.

3.1 Testing the theory's limits: a counterexample

Consider the classical functions:

$$f(q, p) = q^2 p, \tag{29}$$

$$g(q, p) = q^3. \tag{30}$$

The objective is to quantize these functions and verify whether the commutator condition is satisfied.

We use the canonical representation of the quantized operators:

$$Q(q) = q, \tag{31}$$

$$Q(p) = -i\hbar\frac{\partial}{\partial q}. \tag{32}$$

To obtain a Hermitian operator, we choose the symmetric quantization for $f(q,p)$:

$$Q(f) = \frac{1}{3}(q^2Q(p) + Q(p)q^2 + qQ(p)q). \tag{33}$$

Expanding each term separately:

$$q^2Q(p) = q^2(-i\hbar\frac{\partial}{\partial q}) = -i\hbar q^2\frac{\partial}{\partial q}, \tag{34}$$

$$Q(p)q^2 = (-i\hbar\frac{\partial}{\partial q})q^2 = -i\hbar(2q + q^2\frac{\partial}{\partial q}), \tag{35}$$

$$qQ(p)q = q(-i\hbar\frac{\partial}{\partial q})q = -i\hbar q\frac{\partial}{\partial q}q = -i\hbar q. \tag{36}$$

Substituting these expressions into the definition of $Q(f)$:

$$Q(f) = \frac{1}{3}\left(-i\hbar q^2\frac{\partial}{\partial q} - i\hbar(2q + q^2\frac{\partial}{\partial q}) - i\hbar q\right). \tag{37}$$

Simplifying:

$$Q(f) = -\frac{2}{3}i\hbar q^2\frac{\partial}{\partial q} - i\hbar q. \tag{38}$$

Thus, the quantized operator for $f(q,p)$ is:

$$Q(f) = -\frac{2}{3}i\hbar q^2\frac{\partial}{\partial q} - i\hbar q. \tag{39}$$

For g, the quantized operator is simply:

$$Q(g) = q^3. \tag{40}$$

To verify if $Q(f)$ is Hermitian, we compute its conjugate:

$$Q(f)^\dagger = \frac{1}{3}(q^2Q(p)^\dagger + Q(p)^\dagger q^2 + qQ(p)^\dagger q). \tag{41}$$

For the momentum operator:

$$Q(p) = -i\hbar\frac{\partial}{\partial q}, \tag{42}$$

we compute its adjoint using integration by parts:

$$\int_{-\infty}^{\infty}\psi^*(q)(-i\hbar\frac{\partial}{\partial q}\phi(q))dq = -i\hbar\left[\psi^*(q)\phi(q)\right]_{-\infty}^{\infty} + i\hbar\int_{-\infty}^{\infty}\left(\frac{\partial}{\partial q}\psi^*(q)\right)\phi(q)dq. \tag{43}$$

Assuming that wavefunctions vanish at infinity, the boundary term disappears, leaving:

$$\int_{-\infty}^{\infty}(i\hbar\frac{\partial}{\partial q}\psi^*(q))\phi(q)dq. \tag{44}$$

Comparing with the inner product definition, we conclude: $Q(p)^\dagger = -i\hbar\frac{\partial}{\partial q}$, so we obtain:

$$Q(f)^\dagger = \frac{1}{3}(q^2(-i\hbar\frac{\partial}{\partial q}) + (-i\hbar\frac{\partial}{\partial q})q^2 + q(-i\hbar\frac{\partial}{\partial q})q). \tag{45}$$

Using the product rule for differentiation:

$$\frac{\partial}{\partial q}q^2 = 2q, \quad \frac{\partial}{\partial q}q = 1, \tag{46}$$

we rewrite each term separately:

$$q^2(-i\hbar\frac{\partial}{\partial q}) = -i\hbar q^2\frac{\partial}{\partial q}, \tag{47}$$

$$(-i\hbar\frac{\partial}{\partial q})q^2 = -i\hbar(2q + q^2\frac{\partial}{\partial q}), \tag{48}$$

$$q(-i\hbar\frac{\partial}{\partial q})q = -i\hbar q\frac{\partial}{\partial q}q = -i\hbar q. \tag{49}$$

Substituting these results:

$$Q(f)^\dagger = \frac{1}{3}\left(-i\hbar q^2\frac{\partial}{\partial q} - i\hbar(2q + q^2\frac{\partial}{\partial q}) - i\hbar q\right). \tag{50}$$

Simplifying:

$$Q(f)^\dagger = -\frac{2}{3}i\hbar q^2\frac{\partial}{\partial q} - i\hbar q = Q(f). \tag{51}$$

Thus, $Q(f)$ is Hermitian.
Now, let's verify the commutation condition that tells us that the commutator must satisfy:

$$[Q(f), Q(g)] = -i\hbar Q(\{f, g\}). \tag{52}$$

We compute:

$$[Q(f), Q(g)] = Q(f)Q(g) - Q(g)Q(f). \tag{53}$$

Expanding each term separately:

$$Q(f)Q(g) = \frac{1}{3}(q^2Q(p)q^3 + Q(p)q^5 + qQ(p)q^4), \tag{54}$$

$$= \frac{1}{3}(q^2(-i\hbar\frac{\partial}{\partial q})q^3 + (-i\hbar\frac{\partial}{\partial q})q^5 + q(-i\hbar\frac{\partial}{\partial q})q^4). \tag{55}$$

Using the product rule:

$$\frac{\partial}{\partial q}q^3 = 3q^2, \quad \frac{\partial}{\partial q}q^5 = 5q^4, \quad \frac{\partial}{\partial q}q^4 = 4q^3, \tag{56}$$

we obtain:

$$Q(f)Q(g) = -4i\hbar q^4. \tag{57}$$

Similarly, we compute:

$$Q(g)Q(f) = -\frac{8}{3}i\hbar q^4. \tag{58}$$

Thus:

$$[Q(f), Q(g)] = -\frac{4}{3}i\hbar q^4. \tag{59}$$

Then the classical Poisson bracket is given by:

$$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}. \tag{60}$$

Computing each term:

$$\frac{\partial f}{\partial q} = 2qp, \quad \frac{\partial f}{\partial p} = q^2, \tag{61}$$

$$\frac{\partial g}{\partial q} = 3q^2, \quad \frac{\partial g}{\partial p} = 0. \tag{62}$$

Thus, the Poisson bracket simplifies to:

$$\{f, g\} = 2qp \cdot 0 - q^2 \cdot 3q^2 = -3q^4. \tag{63}$$

Applying the quantization rule:

$$Q(\{f, g\}) = Q(-3q^4) = -3q^4. \tag{64}$$

Multiplying by $-i\hbar$:

$$-i\hbar Q(\{f, g\}) = i\hbar 3q^4. \tag{65}$$

Since:

$$[Q(f), Q(g)] \neq -i\hbar Q(\{f, g\}), \tag{66}$$

The commutator's condition is violated.

We have constructed a quantization operator that is Hermitian ($Q(f)$ is self-adjoint), but does not satisfy the commutator's condition: we have found a counterexample to the quantization relation. We have thus shown the difficulty of choosing the proper quantization. We consider the classical functions on phase space:

$$f(q, p) = q^2 p, \quad g(q, p) = q^3.$$

4 Prequantization

Given the problem regarding the fourth condition of quantization that arose in the previous chapter, we need to construct a representation of the Poisson algebra of functions on the classical phase space M by linear operators on a Hilbert space. This is known as *prequantization* and satisfies every condition of quantization with the exception of the fifth one.

4.1 The Prequantum Hilbert Space

In order to geometrize the notion of quantization, it might be natural to attempt to construct the quantum phase space (the Hilbert space) from the space of functions on the classical phase space M.

We need to state the identity:

$$[X_f, X_g] = X_{\{f,g\}} \tag{67}$$

which shows that Hamiltonian vector fields provide a representation of the Poisson bracket algebra by first-order differential operators on M.

We assign coherently:

$$f \mapsto -i\hbar X_f \tag{68}$$

satisfies the conditions 1 (obviously), 2 (since X_f leaves ω invariant, with respect to the Liouville's measure), and 3 (by the identity above).

However, since the zero vector field is assigned to any constant function, this does not satistfy the fourth condition.

A little further experimenting can lead us to the assignment:

$$P(f) = -i\hbar X_f + \theta(X_f) + f$$

(where θ denotes the canonical one-form, wich only exists on T^*Q and not in general). This thus gives a faithful representation of the Poisson algebra by first-order differential operators on M.

For $M = T^*Q$, where the symplectic form is given by the canonical two-form $\omega = d\theta$, the prequantization operator is given by:

$$P(f) = -i\hbar X_f + \theta(X_f) + f.$$

For $Q = \mathbb{R}^n$, the prequantization operators satisfy:

$$P(q^k) = i\hbar\frac{\partial}{\partial p_k} + q^k,$$

$$P(p_k) = -i\hbar\frac{\partial}{\partial q^k}.$$

This shows that prequantization fails to satisfy the irreducibility condition, as the operator $\frac{\partial}{\partial p_k}$ commutes with all coordinate functions.

4.2 Refining the counterexample

Given the classical functions previously studied, our aim is now to show that introducing prequantization, the problems previously encountered with the counterexample are solved:

$$f(q,p) = q^2 p, \quad g(q,p) = q^3, \tag{69}$$

we compute the Hamiltonian vector fields using:

$$X_f = \frac{\partial f}{\partial p}\frac{\partial}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial}{\partial p}, \tag{70}$$

$$X_g = \frac{\partial g}{\partial p}\frac{\partial}{\partial q} - \frac{\partial g}{\partial q}\frac{\partial}{\partial p}. \tag{71}$$

Computing derivatives:

$$X_f = q^2\frac{\partial}{\partial q} - 2qp\frac{\partial}{\partial p}, \tag{72}$$

$$X_g = 3q^2\frac{\partial}{\partial p}. \tag{73}$$

Let us now compute the canonical one-form correction. In cotangent space $M = T^*Q$, the canonical one-form is given by:

$$\theta = p\,dq. \tag{74}$$

Thus, we compute:

$$\theta(X_f) = pX_f^q = p(q^2) = q^2p, \tag{75}$$

$$\theta(X_g) = pX_g^q = p(0) = 0. \tag{76}$$

Let us now construct the corrected prequantization operators using the prequantization formula:

$$P(f) = -i\hbar X_f + \theta(X_f) + f, \tag{77}$$

$$P(g) = -i\hbar X_g + \theta(X_g) + g. \tag{78}$$

Substituting values:

$$P(f) = -i\hbar\left(q^2\frac{\partial}{\partial q} - 2qp\frac{\partial}{\partial p}\right) + 2q^2p, \tag{79}$$

$$P(g) = -i\hbar\left(3q^2\frac{\partial}{\partial p}\right) + q^3. \tag{80}$$

The commutator is then given by:

$$\begin{aligned}
[P(f), P(g)] &= P(f)P(g) - P(g)P(f) && (81)\\
&= 6(i\hbar)^2q^3\frac{\partial}{\partial p} - 3i\hbar q^4 + 6(i\hbar)^2q^3\frac{\partial}{\partial p} + 6i\hbar q^4 && (82)\\
&= (i\hbar)^2(12q^3\frac{\partial}{\partial p}) + i\hbar 3q^4. && (83)
\end{aligned}$$

After simplifying all terms, we obtain:

$$[P(f), P(g)] = -i\hbar P(\{f, g\}). \tag{84}$$

From our previous calculation, the Poisson bracket is:

$$\{f, g\} = -3q^4. \tag{85}$$

Thus, applying prequantization:

$$-i\hbar P(\{f,g\}) = (i\hbar)^2(12q^3\frac{\partial}{\partial p}) + i\hbar 3q^4. \tag{86}$$

With the introduction of prequantization, the commutator now correctly satisfies the condition:

$$[P(f), P(g)] = -i\hbar P(\{f,g\}). \tag{87}$$

This confirms that prequantization successfully fixes the issue with the condition regarding the Poisson's brackets.

4.3 Explaining prequantization

There is still one minor difficulty with the construction we discussed about above. Instead of θ, we could have chosen a symplectic potential of the form $\theta + df$ for some function f on M. This can be compensated for by multiplying the functions by the phase factor $\exp(if/\hbar)$ (the resulting prequantizations are unitarily equivalent). However, f is, only determined by df up to a constant, resulting in a phase ambiguity of the prequantum wave functions.

In order to proceed we need to define a couple of mathematical concepts:

Definition 4.1: A *complex line bundle* over a smooth manifold M is a surjective map between two smooth manifolds $\pi : L \to M$. There exists an open cover $\{U_i\}$ of M locally trivialized as

$$\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C},$$

with transition functions $\phi_{ij} : U_i \cap U_j \to \mathbb{C}^*$ satisfying the cocycle condition ($\phi_{ij}\phi_{jk} = \phi_{ik}$ on $U_i \cap U_j \cap U_k$)

Definition 4.2: A *section* of a fiber bundle $\pi : L \to M$ is a smooth function $s : M \to L$ such that $\pi(s(x)) = x, \quad \forall x \in M$.

Definition 4.3: Let $L \to M$ be a complex line bundle over a smooth manifold M. A connection D on L is an operator that assigns to each smooth section s of L a differential form Ds with values in L, such that for any smooth function f on M and any section s,

$$D(fs) = df \otimes s + fDs. \tag{88}$$

where $D_s = ds + As$ and A is the connection one-form associated with D.

Definition 4.4: The *curvature* Ω of a connection D is a two-form that measures the failure of the connection to be locally trivial. It is defined as the exterior derivative of the connection one-form,

$$\Omega = dA. \tag{89}$$

where A is the connection one-form associated with D.
Alternatively, in terms of the connection operator, the curvature is given by

$$\Omega(X,Y) = i\left([D_X, D_Y] - D[X,Y]\right), \tag{90}$$

for any pair of vector fields X, Y on M. It is shown that the curvature describes the noncommutativity of the covariant derivatives.

It could thus be more convenient to regard the operators $P(f)$ as acting on the space of sections of a trivial complex line bundle L over M (globally isomorphic to the product space $M \times \mathbb{C}$) equipped with a connection D, which can take the form (in a particular trivialization):

$$D = d - \frac{i}{\hbar}\theta. \tag{91}$$

Let us now look at another way of seeing $\mathcal{P}(f)$ that might be helpful to better understand this concept. Via its Hamiltonian vector field X_f, the function f generates a flow

$$\Phi_t^f : m \mapsto \Phi_t^f(m) \tag{92}$$

of canonical transformations of M. Up to a phase, there is a unique way of lifting this flow to an automorphism of L preserving the Hermitian structure and the compatible connection. This induces a 'pull-back' action

$$\hat{\Phi}_t^f : \psi \mapsto \hat{\Phi}_t^f \psi \tag{93}$$

on sections of L and their local representatives ψ. Introducing the quantity

$$\mathcal{L}_f = \theta(X_f) - f \tag{94}$$

$$= p_k \frac{\partial f}{\partial p_k} - f \tag{95}$$

the *Lagrangian* of f, one finds that (106) is given explicitly by

$$\left(\hat{\Phi}_t^f \psi\right)(m) = \psi(\Phi_t^f(m)) \exp\left(-\frac{i}{\hbar}\int_0^t \mathcal{L}_f(\Phi_{t'}^f(m))dt'\right). \tag{96}$$

Thus, the evolution in time is given by the exponential of the classical action. Since $\mathcal{P}(f)$ can be expressed in terms of $\mathcal{L}_f$ as

$$\mathcal{P}(f) = -i\hbar X_f - \mathcal{L}_f\,, \tag{97}$$

it follows that $\mathcal{P}(f)\psi$ is nothing but the derivative of (110) at $t = 0$,

$$\mathcal{P}(f)\psi = -i\hbar \frac{d}{dt}\left(\hat{\Phi}_t^f \psi\right)\Big|_{t=0}. \tag{98}$$

We can thus interpret $\mathcal{P}(f)$ as the generator of a connection that preserves automorphisms of L lifting the action of the Hamiltonian vector field X_f on M.

Moving on, the curvature Ω of L, defined by

$$\Omega(X, Y) = i\left([D(X), D(Y)] - D([X, Y])\right)\,, \tag{99}$$

is

$$\Omega = iD^2 = (1/\hbar)d\theta = (1/\hbar)\omega\,. \tag{100}$$

Let us now look at this relation:

$$[\mathcal{P}(f), \mathcal{P}(g)] = -i\hbar\mathcal{P}(\{f, g\}) \tag{101}$$

is satisfied for all f and g provided that L is a line bundle with connection D whose curvature two-form is $(1/\hbar)\omega$.

As $\omega = d\theta$ is real, there always exists a compatible Hermitian structure on L, and we thus can get to the following:

Definition 4.5: A *prequantization* of a symplectic manifold (M, ω) is a pair (L, D) where L is a complex Hermitian line bundle over M and D a compatible connection with curvature $(1/\hbar)\omega$. The prequantum Hilbert space $\mathcal{H}$ is the completion of the space of smooth sections of L, square-integrable with respect to the Liouville measure on M (and the Hermitian structure on the fibers).

5 Polarization

Up until now, Geometric Quantization has been quite straightforward and elegant. However, we need some additional structures to obtain a quantization of a symplectic manifold. One of the structures we need is a polarization, and this leads to rather severe technical complications, this is the reason why it will be briefly treated in this context. Most of the problems are related to the fact that there is no natural measure on the space of quantum states. This leads us to modify the quantization scheme to what is known as half-form or metaplectic quantization. And even if at this stage geometric quantization becomes quite successful, it simultaneously becomes rather complicated.

Later on, it will be shown that the concept of a polarization arises rather naturally when one tries to 'cut down' the prequantum Hilbert space.

5.1 The emergence of Polarization

The possible generalization of Schrödinger quantum mechanics on $T^*Q = \mathbb{R}^{2n}$ has been treated before, which is based not on the concept of a complete set of observables, but on the concept of a maximal commuting set. We also treated the possibility of seeing the Hilbert space $L^2(Q)$ as the space of functions on the phase space constant along the leaves of a polarization.

It will now be shown how the concept of a polarization arises quite naturally if one attempts to construct the quantum Hilbert space from the prequantum Hilbert space $\mathcal{H}$.

The issue with the prequantum Hilbert space $\mathcal{H}$ is that it is too wide, consisting of functions ψ which depend on all $2n$ coordinates of the symplectic manifold (M, ω). If we demand that wave functions are constant along n vector fields on M, we find a way of eliminating 'half' of these coordinates. Since ordinary differentiation has no invariant meaning for sections of a bundle, this must be understood as them being covariantly constant. A way to proceed is to choose some n-dimensional subbundle P of the tangent bundle TM of M and to consider only those wave functions that satisfy

$$D(X)\psi = 0 \quad \forall X \in P \tag{102}$$

(where '$X \in P$' means 'X is a section of P').There may now exist non-trivial integrability conditions for these equations, which could become an obstruction to finding any solutions to (116). From (116), we have that $[D(X), D(Y)]\psi = 0$ for all $X, Y \in P$. If we combine this expression with (104), we are led to the integrability condition

$$D([X, Y])\psi - (i/\hbar)\omega(X, Y)\psi = 0 \quad \forall X, Y \in P. \tag{103}$$

Condition is which is satisfied if

$$X \in P, Y \in P \Rightarrow [X, Y] \in P \tag{104}$$

and

$$X \in P, Y \in P \Rightarrow \omega(X, Y) = 0\,. \tag{105}$$

The first condition tells us that P needs to be integrable, so that locally there exist integral manifolds in M through P. As these manifolds are n-dimensional, the second condition tells us that these integral manifolds are Lagrangian, where integral means a submanifold whose tangent space at every point is completely contained within a given distribution (a smooth assignment of a subspace of the tangent space at each point). If we demand the wave functions to be covariantly constant along the leaves (integral submanifolds corresponding to this polarization, seen as the maximal involutive distribution of the tangent bundle) of a polarization P, we can see that there are no local integrability condition. We are now able to see how this concept, which is a generalization of that based on a maximal commuting set of observables, arises naturally from prequantization.

Definition 5.1: Let (M, ω) be a symplectic manifold. A submanifold $L \subset M$ is called a *Lagrangian submanifold* if:

1. L is isotropic, meaning that the symplectic form ω restricts to zero on L
2. $\dim L = \frac{1}{2} \dim M$.

Definition 5.2: Let (M, ω) be a symplectic manifold. A *polarization* P of (M, ω) is an integrable maximally isotropic (Lagrangian) subbundle of the complexified tangent bundle TM^c of M.

Let us now explore the concept of real polarization, which is characterized by the property $P = \bar{P}$, that implies that $P = D^c$, where D^c is obtained by extending the vector spaces D_p at each point p to the complex numbers. An example of a real polarization can be the *vertical polarization* of a cotangent bundle $M = T^*Q$. It is spanned by the vectors, in local coordinates, $(\partial/\partial p_k)$ tangent to the fibers of T^*Q. Thus D is the vertical tangent bundle, P is its complexification, and the integral manifolds of D are the fibers T_p^*Q, isomorphic to $\mathbb{R}^n$. The space of integral manifolds is just the configuration space Q itself.

This vertical polarization always exists for cotangent bundles, so does the Schrödinger representation of quantum mechanics on M, based on the Hilbert space $L^2(M)$.

There can be real polarizations which are not vertical polarizations of some cotangent bundle, but there are not many more possibilities that satisfy our regularity conditions. To see an example, we can look at the cylinder $M = T^*S^1$. Instead of choosing the vertical polarization, spanned by $(\partial/\partial p)$, we can also choose a *'horizontal'* polarization spanned by $(\partial/\partial q)$. This leads to what is known (for $M = \mathbb{R}^n$) as the momentum representation. In this case the integral manifolds of D are circles S^1.

Now finally, after having accumulated all these bits and pieces of information, we come to the quantization of symplectic manifolds. This involves the determination of the quantum Hilbert space $\mathcal{H}_P$ corresponding to a polarization P, and the construction of operators acting on $\mathcal{H}_P$.

6 Conclusion

In this thesis, the fundamental aspects of symplectic geometry and its role in geometric quantization, the mathematical framework constructed to establish a transition from classical to quantum mechanics, were explored. We began with the foundations in differential geometry, treating manifolds, tensor fields, and symplectic structures, providing the language for Hamiltonian mechanics and the representations of the phase space.

The main aim of this work has been to study quantization, a procedure focused on the assignment of quantum operators to classical observables, preserving the underlying algebraic structures. Later on, the difficulties inherent in this transition were analyzed. We focused in particular on the challenge of finding a consistent way to replace Poisson brackets with commutators. After having explored prequantization in detail, we noticed that while it provides a successful representation of the Poisson algebra, it cannot be the base for a fully developed quantum theory.

The construction of a counterexample that explicitly demonstrates that prequantization fixes the problems with the commutator relations was a crucial step to understand the limitations of this theory, and the resolutions proposed by prequantization. We then highlighted the necessity of additional structures, such as polarization. This process is vital to be able to define the quantum Hilbert space and eliminate extraneous variables. The concept of polarization was only briefly introduced in this thesis, focusing on real polarization, but it represents a fundamental step in completing geometric quantization, leading to a more accurate and physically relevant quantum model and letting us manage the problems found with prequantization.

Beyond its mathematical elegance, geometric quantization serves as a powerful theoretical tool in modern physics. It finds applications not only in quantum mechanics but also in quantum field theory, representation theory, and even in recent developments related to quantum computing and quantum gravity. The study of alternative quantization methods, such as deformation quantization or path integral approaches, remains an active area of research, offering potential solutions to some of the ambiguities present in geometric quantization.

In conclusion, the theory of geometric quantization provides a rich framework for understanding the transition from classical mechanics to quantum mechanics. There obviously are inherent challenges and open problems, but in spite of them, its interplay with symplectic geometry and mathematical physics continues to be an evolving field. Our understanding of quantum systems is deepening, and geometric quantization will likely remain an essential tool in creating a bridge between mathematical formalisms and physical intuition, paving the way for further developments in theoretical physics.

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