Alma Mater Studiorum · Università di Bologna

Dipartimento di Fisica e Astronomia "Augusto Righi"
Corso di Laurea in Fisica

Noise and Chaos

Spectral Properties of Stochastically Perturbed Hamiltonian Systems

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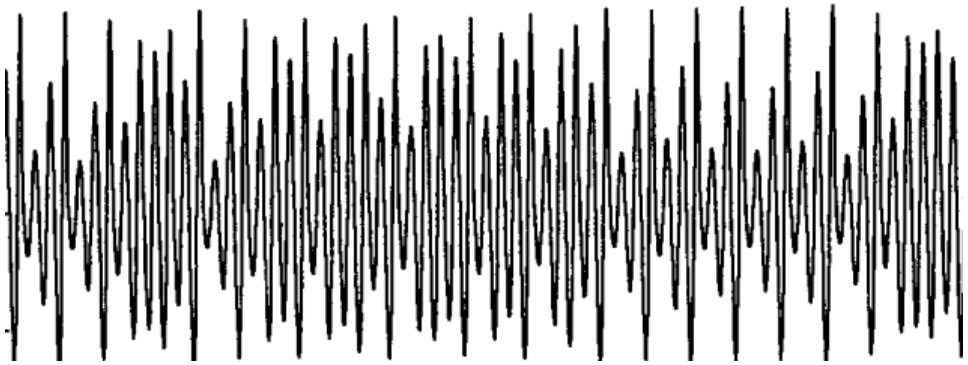
I... I can't hear you!



What!? Who's there?

Mmm... wonder if you hear ***me****.*

Abstract

English

We show that to understand the spectral effects of chaos as an indecomposable object over a periodic backbone, it is useful to develop a quantomechanical description. Noise furnishes such setting in the overdamped approximation of Hamiltonian systems in Thermal Bath. The resulting pure point spectrum statistics exhibits a transition when the underlying classical regularity changes. By applying a perturbative approach, it's showed how non integrability brings couplings that result in energy level repulsion. Both chaotic irregularity and quantum non-perturbative tunnelling connect different otherwise independent phase space regions and this is reflected in spectra as a more correlated statistics. It's then proposed a possible efficient way of computing the relevant spectral statistical quantities by exploiting classical microcanonical averages via single evolution measurements. Under the RMT point of view, computing the power spectra of general Hamiltonians on a reference dynamics, seems allowing a fast discernment of the relevant statistics and by a conjectured link of this latter with the irregular strength, one might try to estimate Lyapunov global divergence from it.

Italiano

Mostriamo che per comprendere gli effetti spettrali di una dinamica caotica, intesa come evoluzione indecomponibile su di uno scheletro periodico, è utile sviluppare una descrizione quantomeccanica. I sistemi hamiltoniani in bagno termico nel limite di alta viscosità risultano essere i migliori candidati a tale scopo. Lo spettro risulata essere puramente discreto ed esibisce una transizione statistica quando la corrispondente dinamica classica passa da integrabile a non. Tramite un approccio perturbativo otteniamo che l'irregolarità classica introduce degli accoppiamenti che inducono una repulsione tra i livelli energetici. La più o meno sviluppata caoticità sulle superfici energetiche e gli effetti tunnel puramente quantistici/stocastici mettono in contatto zone altrimenti isolate ed indipendenti dello spazio delle fasi. Questo si riflette in una statistica spettrale più correlata. Viene poi proposta una possibile tecnica per ottenere in modo computazionalmente efficiente le quantità statistiche rilevanti per il nostro studio. Assumendo come valida l'ipotesi ergodica, si sfruttano delle medie microcanoniche, ottenute da singole evoluzioni su una dinamica di riferimento, per il calcolo di potenze spettrali di altre hamiltoniane, intese come osservabili. Questo è reso possibile dall'isomorfismo tra funzioni e flussi fornito dalla struttura simplettica. In un'ottica di matrici random (il candidato migliore per descrivere spettri caotici), tale tecnica sembrerebbe permettere efficienti constatazioni spettrali della presenza di dinamiche caotiche. Congetturando un diretto collegamento tra la statistica spettrale e gli esponenti di Lyapunov, questi potrebbero essere efficientemente stimati senza alcuna ricostruzione della dinamica tangente.

Contents

A Note for the Reader

Throughout the work you will encounter several contents in coloured boxes. They help to isolate some self-contained considerations, definitions, theorems and mathematical details. Some of them, the gray-blue ones, can be easily skipped without loosing the principal train of thoughts.
Others, the red ones, highlight some small key results and want to capture your attention, they serve as important parallel subsections inserted in the general treatment.
The yellow ones encompass short definitions, statements and remarks.

Supplements

Important Subsections

Definitions and Statements

I haven't used them systematically, only when I felt it was beneficial for the reading to set apart some concepts. For chapters and sections that could be followed all in one breath, I preferred to avoid them as being superfluous.

Introduction

Noise effects on Chaotic dynamics

This work started as an investigation about the effects of stochastic perturbations on chaotic dynamical systems. There's usually a gap between the timescales at which systems' features evolve. The gross phenomenology can be usually modeled by a low dimensional dynamics, whereas the enormous number of fast additional degrees of freedom can be recast as an overall effect on the gross ones. When such division is possible (turbulent flow is an example where no scale gap can be found) we think of the fast "hidden" degrees of freedom as multidimensional highly mixing chaos. When we reduce the dimensionality of the problem to the small set of slow variables and want to include the overall influence of the chaotic part, we are compelled to do so by stochastic means, the partial knowledge of merely the gross part and the timescale gap permits a modeling in terms of noise, we will only deal with delta correlated Gaussian white noise.
When even the low dimensional slow dynamics is chaotic, we end up having extreme sensitivity and a deterministic evolution only at infinite resolution. A statistical description is necessary, only probability densities evolution is meaningful, these are described by suitable PDEs. The operatorial formulation comes with linearity and admits an eigenproblem reformulation, by a time separation ansatz we can reduce our problem to a spectral study, the main object under our investigation is then the spectra, which furnishes the distribution of evolution timescales involved.
In this setting how noise enters the game? At the trajectory level it destroys any possible deterministic hope, the sensitivity of the gross chaotic variables amplifies every little fluctuation and a statistical approach is compulsory. At the density level, it will make trajectories loose their specific character and renders the evolution even more unified, by this I mean that single wandering orbits that eventually visit the entire chaotic region, each of which in its own fashion, will now jump, thanks to fluctuations, to every other possible trajectory and will acquire a general and indistinguishable character from the others: on the long run, every evolution will behave similarly. In this way we suppress eigen-modes that are bond to the fine chaotic structure but we will preserve only modes that behave self-similarly even in the presence of fluctuations.
Our study will focus on the spectral change due to the presence of noise in chaotic dy-

namics.

To do so we have outlined a set of two physically interesting settings where this influence should be studied, chaos has many different manifestations, each of which might have a different spectral response to noise:

- Hamiltonian dynamics in a Thermal Bath.
- Dissipative chaotic systems subject to noise.

The first is the most relevant one but turns out to be hard to tackle. The second comes with geometrical difficulties and it's rather a geometrical manifestation of chaos. Boundness and chaotic divergence of nearby orbits force a dissipative dynamics, the finite room prohibits volume enlargements and the system will eventually collapse on an attractor, the geometry of such lower dimensional set might turn out to be wild (fractal) to permit orbit divergence on a 0-measure space. In these cases, noise will even affect the geometry of the dynamical support.
Hamiltonian conservative dynamics is somewhat a limiting case, volumes are preserved and phase space is entirely foliated with energy invariant surfaces. Chaoticity is restricted on single energy surfaces, no trajectories will wander throughout the entire available space, but it permits to avoid geometrical difficulties and permits a more comprehensible spectral treatment of noise effects (the support of probability distributions remains the same).
We will mainly be interested in conservative evolutions.

We will be treating two limiting cases that admit exact results:

- Deterministic Markov area preserving maps.
- Integrable Hamiltonian system in Thermal Bath.

The first gives an idea of strong mixing chaotic spectra, whereas the second shows regular noisy one.

From these we begin to have some insights about the role played by Lyapunov divergence and noise in evolution spectra, but the aim is to characterize systems in the middle: Hamiltonian non integrability in a Thermal Bath.
The operator involved turns out to be the Kramers one [1], but its non self-adjointness renders the problem too hard to handle for our scopes. We would like to restrict ourselves to real spectra which admits unidimensional statistical studies.
Anyway, we will realize that noise induces a relaxing statistical behaviour that admits a unique equilibrium distribution that turns out to be the Boltzmann one. Additionally,

the spectra becomes discrete, confirming the previously intuited quantization due to the additional noisy requirements to behave self-similarly.
The diffusion process induced by noise renders the density evolution to depend on surrounding points, the involved Laplacian acts as an averaging prescription that smooths out eventual small scale irregularities, and more importantly for conservative systems, it permits to couple different energy shells otherwise independent.

A way to achieve self-adjointness is by taking the overdamped approximation that will turn the spectrum real, the rescaled problem can then be described by a Fokker-Planck operator of the Schroedinger form (after a change of measure).
But the object under investigation changes dramatically, the result is a fixed points gradient flow dissipative dynamics subjected to an external additive white noise. We would loose the primary purpose of studying noise effects on chaos, ad would remain with studying noise on the most trivial dissipative dynamics.

So, are we going off topic?

In the first part of this work we will see how deterministic chaos will almost surely bring to continuous spectra, this makes it little informative. In the deterministic setting the operator that describes densities evolution is often referred as the Frobenius-Perron one, we will see how trace formulas and dynamical zeta functions permit a study of spectra on its own without caring about the corresponding eigenstates [2], [3]. This powerful tool suggests an almost sure continuity, only in uniform hyperbolicity we get a chaotic discrete spectra.
We will see that deterministic chaos is not only about trajectories divergence, dense periodic orbits and flow irreducibility are also required as essential features. Chaos is an indecomposable object built over a periodic backbone (this does even suffice to imply sensitivity), and as I will shortly summarize, it turns out that noise permits a way to study this tight ensamble structure by spectral means. Something that would be impossible at the deterministic level due to the resulting continuity. Especially for Hamiltonian dynamics, Spohn [4] shows how both non integrable and regular systems are characterized by continuous spectra.

Spectral manifestation of Chaos

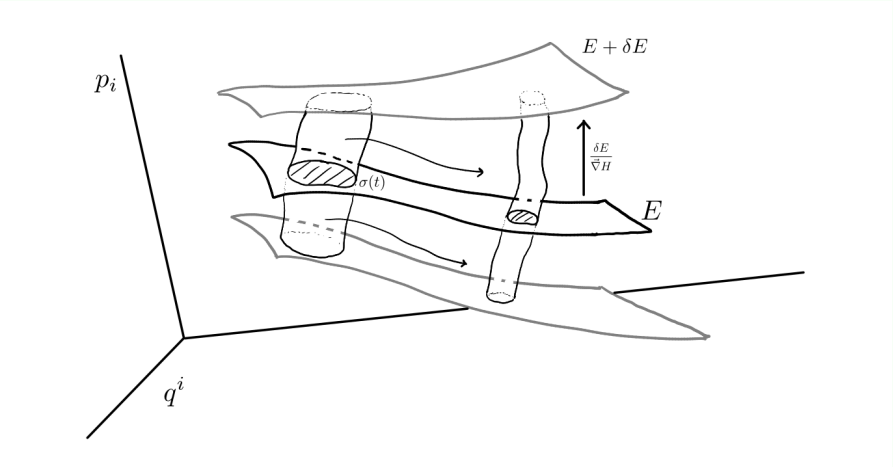
Surprisingly, we are able to have a noise induced conservative chaos characterized by discrete spectra. This peculiar possibility has shifted our attention on the eventual spectral manifestation of the chaotic object. We believe that a density evolution approach that treats the flow as a whole should capture the indecomposable nature of such organic dynamical structure.
We recover Hamiltonian dynamics as a imaginary time weak noise limit, a quantum anal-

ogy is then made available and the non integrability concept resides in this semiclassical recovered Hamiltonian flow.

The main purpose has then become to study the spectral measure properties in the transition from semiclassical Hamilton integrability to irregularity.

Quantum Mechanics as a Surface Chaos Analyser

I've realized in hindsight that the quantization map and correspondence principle turn out to be the only appropriate way for a spectral study of conservative Hamiltonian chaos.
When we consider probability densities evolution in phase space we are forced to use invariant measures, otherwise the flow would not permit conservation of probabilities and a consistent description over time.
The fundamental invariant measure in Hamiltonian phase space is the Liouville volume one, but the energy foliation makes volume densities to evolve on a continuous set of separated invariant dynamics, every volume will evolve on an entire interval of energy hypersurfaces, each of which is of 0-measure. This is ultimately why we always end up with continuous spectra [4].
One then could try to restrict the study on single energy surfaces but it's not obvious that such dynamics will admit an invariant probabilistic measure. What we could use is the microcanonical invariant surface measure that seems a mandatory choice, but it is still rooted on Liouville one, it is indeed a projection of it. The surface density evolution will then only be a projection of the volume one and a continuous spectra will still arise. It seems that a consistent probabilistic description of genuine energy surface chaos cannot be outlined and only a trajectory level description is available.



If we want to find a probabilistic consistent description that can involve single energy

surfaces we have to step back to configuration space.
Here we can write Hamilton-Jacobi equation for the reduced action at a specific energy E.

$$H(q; \nabla S(q; E)) = E$$

The action function solution $S(q; E)$ encapsulates the mechanical evolution on the E-surface. If we then want to attempt a probabilistic description we will have to study the evolution of a configuration space density $\rho(q, t)$ only due to E allowed trajectories. The first requirement should be the probability conservation

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \nabla S(q; E)) = 0$$

We are supposing our spatial density $\rho(q, t)$ to evolve with $p(q) = \nabla S(q; E)$ specified momentum at each point. We see that two functions are needed to specify such density evolution: $S(q; E)$, $\rho(q, t)$, and they have to satisfy the two mentioned equations. It turns out that such dynamics can be reformulated as a geometrical optics limit of the Schroedinger unitary evolution of a complex valued wavefunction $\psi(q) = \sqrt{\rho(q)}\, e^{\frac{i}{\hbar} S(q)}$ that has to satisfy the unique

$$H(\hat{q}, \hat{p})\psi = E\psi$$

with evolution $\psi(q, t) = \psi(q; E)\, e^{-iEt/\hbar}$.
We see that in this picture, single energy probabilistic evolution $\rho(q, t)$ are constant in time. What does it mean? How can chaos appear here?

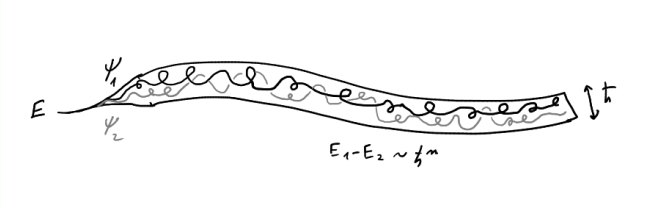
This description is not totally analogous to the classical one, the Hamiltonian operator $\hat{H} = -\hbar^2 \Delta + \hat{V}(q)$ and the Schroedinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$$

reduce to Hamilton-Jacobi and continuity requirements only at the $\hbar \to 0$ limit, but to permit an operatorial description that admits spectral studies we need to keep $\hbar$ finite. This has radical consequences, first of all, an uncertainty principle between positions and momenta arises, no localization in specific phase space points (q, p) will ever be possible, as $p\psi = -i\hbar \nabla \psi$, "some space" is needed for its definition when $\hbar \neq 0$. This prohibits any embedding of ψ as a positive probability distribution in phase space, and every phase space eigenfunction has to occupy at least an $\hbar^N$ volume.
We realize that to overcome the impossibility of a probabilistic description on single energy surfaces we have to introduce these quantum consequences.

The crucial observation is the scale at which these effects appear. It seems as the unitary evolution achieved by the quantomechanical shift freezes the energy surface evolution to a single phase oscillating eigenfunction. But we would have expected a plethora of discretized modes associated to chaotic surface dynamics. The key

fact is that the quantization finest scale of spectral gaps is of $\hbar^N$ order, whereas the phase space thickness of the corresponding eigenfunctions, dictated by the uncertainty principle, is of $\hbar$ order, when $\hbar \ll 1$ this is a huge difference in order, an enormous amount of eigenfunctions share the same phase space $\hbar$-energy shell, but still they evolve independently and orthogonally.



We have effectively lifted the degenerate classical surface chaotic motion to a unitary evolution in a $\hbar$-energy shell, a general ψ in here will be made of a superposition of all the $\hbar^N$-spaced concomitant stationary eigenfunctions. Here is where a spectral study of chaos is recovered: we have a trajectory concept of chaos in terms of periodic density, irreducibility and Lyapunov divergence in the classical analogue restricted on a 0-measure energy surface (the correspondence principle is crucial here), but we gain a discrete spectra amenable of statistical studies thanks to this $\hbar^N$ unitary quantum lifting.

Hamiltonian conservative mechanics permits both regular and irregular motion to be hosted by the same symplectic phase space, without any geometrical support difficulty, quantum mechanics and the correspondence principle permit us to isolate and study single energy surface irreducible chaos by otherwise impossible spectral means.

The main topic of **Part II** will be about studying $\hbar^N$ fine spectral statistics. The key quantity under study is level spacing statistics $P(S)$, it results to be the appropriate property to discern semiclassical integrability from dense irregularity. In [5], a correlation between chaotic levels in a $\hbar$-energy shell is numerically showed, the enhanced regularity that one finds in spectra seems to confirm our intuitive picture of the indecomposable, never repeating, rooted on a periodic backbone flow that defines Chaos as a continuous dynamical organism.
In [5] and [6] is introduced the BGS conjecture that links $P(S)$ Poisson statistics to integrable Hamiltonians and a Wigner-Dyson $P(S)$ profile to chaotic ones.
This conjecture introduces a Random Matrix Theory approach to non integrable semi-classical spectra that models chaotic interrelations with RMT ensambles, the matrix single entries randomness makes the overall ensemble evolution spectrum to be more structured.

Noise permits this quantomechanical discretization and lifting in everyday non-quantum macroscopic descriptions. We have to return to the overdamped approximation of the stochastic Hamiltonian dynamics where we would obtain a self-adjoint (with respect to the equilibrium distribution) Fokker-Planck operator that shares the spectrum with a quantum twin. Noise effectively induces an imaginary time possibly chaotic evolution. Interestingly, this will result in spectral correlations that affect even our stochastic relaxation. Indeed, in the stochastic gradient dynamics we have no difficulties in formulating a probabilistic description, but it will be characterized by a discrete spectrum that is linked, in the weak noise limit, to a concept of Hamiltonian chaoticity. To avoid spectral continuity, we remove the momentum p and consider trivial gradient dissipative dynamics, but we recover some sort of new semiclassical p by adding the $\xi(t)$ white noise external fluctuation. It then becomes a sort of density momentum, this is ultimately the reason why it permits a Hamiltonian classical analogue without spectral continuity, it lacks p at the trajectory level. These fast perturbations permit a quantization that in the weak limit can be characterized by a corresponding classical Hamiltonian governing most probable transition paths and rates. Surprisingly, this weak noise classical limit turns out to be different from the quantum semiclassical one, it has the same potential but flipped.

We will study this duality that turns out to be described by a Wick rotation, we will find out how both pictures will be necessary for a complete study of irregular spectra.

In integrable cases, an exact semiclassical quantization (EBK) can be carried out [7], and by it, it can be rigorously proven to imply a Poisson $P(S)$ in the weak noise/semiclassical limit, a result due to Berry and Tabor [8]. We propose an interpretation of both torus-quantization and Poisson level statistics that lacks mathematical rigour but presumably illustrates better the mechanism that brings to random distributed levels.

Then we consider non integrable irregular spectra, I'll present the theoretical importance of semiclassical trace formulas, already encountered in **Part I** in the deterministic context, their development as spectra inferring tools even in these regimes is principally due to Gutzwiller [9]. The poles of the Green propagator trace will constitute the energy levels. Its resolvent nature deriving from the Fourier time transform of the transition propagator allows a semiclassical interpretation in terms of sums over periodic orbits: a cycle expansion is made available.
In this way we recover some insights about the periodic backbone properties of chaotic dynamical systems: the influence of Lyapunov instability and the coherent resonance that quantizes the levels as a collective property of the entire periodic backbone.
Unfortunately, it will turn out to be computationally impossible and theoretically hard to make some conclusions as for the fine $P(S)$ statistics.

We will prove a short range level repulsion and the initial dip of the Wigner-Dyson statistics by degenerate perturbation theory. It turns out that general two parameters families of systems are required for degeneracies to take place and this entails a diabolical degeneracy structure ([6], [10] appendix X), this will allow us to conclude that $\lim_{S\to 0} P(S) = 0$.

We then exploit the same argument to study tunnelling phenomena. These couplings turn out to be expressible only in a Stochastic Path Integral formulation that explains the flipped sign Hamiltonian dynamics governing non-perturbative effects. I will illustrate a simple system where the transition from Poisson to RMT statistics can be fully analysed, these system prototypes are called quantum graphs and permit a simplification of the coupling dynamics that allows us to hypothesize general mechanisms that bring to RMT statistics. Indeed when we expand an irregular system on a known basis, off diagonal couplings will be weakly correlated due to the energy separation and will apparently behave as random variables.

Chaoticity brings correlation in ergodic regions whereas quantum tunnelling permits to penetrate even geometrical invariant constraints, both entail level repulsion and correlation in the spectrum. Some conjectures about the Lyapunov role in $P(S)$ and the Gaussian randomness of the matrix entries are attempted.
An exact trace formula has to take into account both effects and we mention a possible generalization of it.

I will conclude with an interesting technique for calculating semiclassical matrix entries of general observables [11]. In turns out that the power spectra $\mathcal{P}(\omega; \mathcal{E})$ of classical evolution on single energy surfaces furnishes the distribution of entire bunches of entries. I'll exploit this to attempt a computationally efficient recipe for calculating $P(S)$ statistics. By classical means we cannot deduce the quantization but only the profile of matrix entries distributions, to bridge this gap we suppose to have already found the spectra of a reference $\mathfrak{H}$ system. We will then expand all others H systems on $\mathfrak{H}$ basis and eventually diagonalize it or carry out some theoretical arguments. By considering the H as observables on $\mathfrak{H}$ dynamics we exploit the classical efficient techniques presented in [11] to obtain the distributions and than evaluate them at the $\mathfrak{H}$ quantized levels differences. Since we are interested in $P(S)$ statistics we will try to estimate the RMT character and resulting $P(S)$ from general considerations on power spectra decays, we obtain that under some reasonable hypothesis we can discern $P(S)$ from the FFT computed $\mathcal{P}(\omega)$ decay.

Part I

§§§

Chapter 1

Deterministic Chaos

1.1 Dynamical Systems

The intention of this chapter is to gather the necessary mathematical objects and concepts to define what deterministic chaos is and to explore its various manifestations. It won't be an exhaustive exposition but will focus on its implications and features that will result important for our study. For a more pedagogical and complete introduction to the topic see [12] [3].

We can model only some aspects of the immense complexity that matter around us embodies, in general we will have many possible layers of description but only few of them will highlight some of its non trivial features in a mathematically and computably feasible way, so we suppose to have individuated the relevant macroscopic degrees of freedom that permit a suitable gross description and assign the right dimensionality to the problem.

Dymanical system

For our scope we define *dynamical system* a pair (M, Φ) where M is the state space (assumed to be a Riemannian smooth differential manifold of suitable dimension n) and Φ the dynamics, $\Phi^t : M \to M$ forms a one parameter group of diffeomorphisms of M on itself [a], for many relevant situations we can equip our dynamical system with an invariant measure μ, in terms of push forward we thus have $\Phi^t_*(\mu) = \mu$.

[a] In some cases what is technically formed is a semi-group, we might not have an inverse in situations where the Cauchy-Lipschitz hypothesis are not globally fulfilled.

Invariant measures are crucial for allowing a collective phase flow description that preserves probabilities and that admits a continuity law, moreover the existence of invariant ergodic measures permits equality of time and phase space averages, fundamental for ergodic theory and equilibrium statistical descriptions.
We have even equipped the space with a metric, even though the topology is often enough to define most of chaotic features, the notion of distance permits us to give more precise statements and quantifiers.

The flow is usually thought as induced by the integration along a velocity vectorfield

$$\mathbf{v}(x) = \left.\frac{d\Phi^t x}{dt}\right|_{t=0} \tag{1.1}$$

$\mathbf{v} \in TM$, the tangent bundle.
The *deterministic* [1] evolution given by the solutions $x(t; x_0)$ (x_0 the initial condition) of the differential equation (1.1) will define the flow Φ by $\Phi^t x_0 = x(t; x_0)$.
Given $\mathbf{v}$ we have all the information needed to fully characterize our dynamical system, but compressed in a somewhat physical program that once executed will unroll the trajectory by integrating it along $\mathbf{v}$.

Many systems in nature can be modeled in such a way that we are given $\mathbf{v}$ without watching any evolution, given as a mathematically expressible function of the state; Newtonian Mechanics can tell us what this field is in the kinetic space, it does so via Newton second law that connects the state dependent forces to accelerations hence defining the infinitesimal next-to law $\mathbf{v}$.
We won't focus on the nature of these physically meaningful models and theories but we'll be mainly interested in what these fields comport to the observed phenomenology.

1.2 What is Chaos?

Chaos has an intuitive meaning of sensitivity and unpredictability that has to be defined mathematically. It turns out that it's not easy to fully capture all its features and manifestations in one definition that do not have counterexamples that would have to be considered. A whole hierarchy of it can be outlined, some focus on measure properties, others on stability, see [13] for a wider treatment of this definition difficulty.

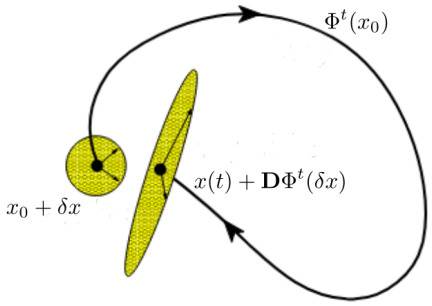
We start by understanding the meaning of sensitivity on initial conditions. It is a property of $\mathbf{v}$'s fine structure; when its Jacobian $D\mathbf{v}(x)$ has positive elements in its

[1] we assume all the necessary hypothesis to be fulfilled by $\mathbf{v}$.

spectra we expect displacements in these principal directions to separate from each other. If this positiveness is roughly preserved in the motion we see by a simple analysis in the linear case that the law governing displacements in the principal directions as time passes will be exponential. $D\mathbf{v}$ gives only local information, to have a global measure of divergence we would have to somewhat compute an average in space. For ergodic regions one may capture this global exponential divergence by the maximal Lyapunov exponent [2]

$$\chi = \max_{\xi} \lim_{t\to\infty} \left\{ \frac{1}{t} \log \frac{||D\Phi_x^t \xi||}{||\xi||} \right\} \tag{1.2}$$

For every ergodic component we find a χ independent of x (given by the μ average over the invariant region). The existence of these limits (regardless of Φ being μ ergodic) is assured by what is called Oseledets Theorem (1969) under mild assumptions.
Here $D\Phi_x^t$ has to be intended as the push forward of the infinitesimal displacements in tangent space.



We are now ready to give a suitable definition of chaos in continuous flows

Chaotic system

Let (M, Φ) be a dynamical system, deterministic chaos arises in an invariant bounded set $K \subseteq M$ when

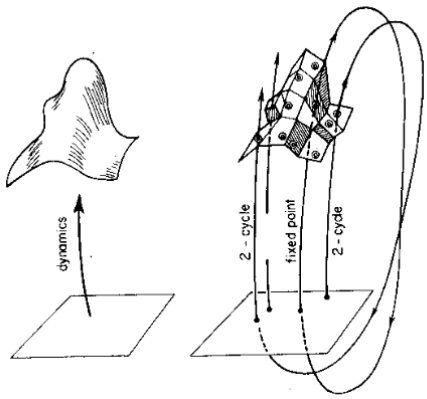
- *Φ_K has a positive global Lyapunov exponent.*
- *Φ_K is topologically transitive.*
  $\forall A, B \subset K, \quad \exists t > 0 \quad$ such that $\quad \Phi^t(A) \cap B \neq \emptyset$.
- *Periodic orbits are dense in K.*

[2] Some insights are given in **Appendix A**.

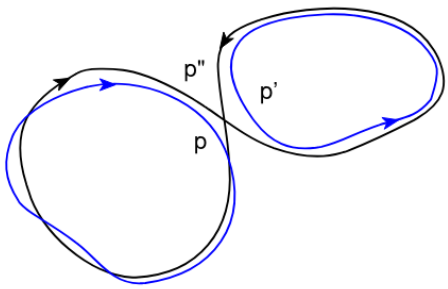
The first point reflects the intuitive idea of chaos as something unpredictable, in the sense that our small error in defining the initial condition (as due to finite resolution of measure devices or numerical error in the digital integration) will be exponentially enlarged by the dynamics, making any prediction at future times very hard.

But another essential ingredient is bounded trajectories, the finite size of the space makes the initial divergence something that cannot merely bring to uncoupled orbits that will no more see each other and wander away as light rays, boundness will make the flow to curl and twist on itself but without any crossing, and initially separating trajectories will eventually get close again. The flow is therefore something that represents an infinite never repeating evolution in finite room.

The irreducibly nature given by topological transitivity tells us that we cannot separate the system in non interacting sub-units, every open set will intersect every other once transported by the flow. The third hypothesis on the other hand gives a backbone regular structure of periodic orbits shadowed by the transitive ones. This robustness of chaos is what we want to eventually put in evidence.



Approximation of a smooth dynamics (left frame) by the skeleton of periodic points, together with their linearized neighborhoods, (right frame).



A longer cycle p'' shadowed by a pair of shorter cycles p and p'.

We immediately see from point one that the mathematics involved is more about topology and other global invariant structures instead of single trajectories that are so hard to compute and that hide its remarkable structure.

Supplement

1.3 Reconstruction from Observations

In one or two dimensions our chaos definition cannot be fulfilled and it's only in three that we start to have this sort of possible phenomenology. Let's start with a dissipative system as the 3D Lorenz attractor [14], the dynamics in this case has a Lyapunov spectrum signature of (+,0,-). We mean that it has a positive global divergence (what previously we called χ that manifests itself on the long run) and a negative one that is bigger in modulus. The null divergence direction is compulsory and represents the average null divergence in the direction of the motion for non single point attractors. We immediately understand why this is the only possibility for chaos: other signatures would bring to a non bounded blow up or to a one dimensional limit cycle without any chaotic feature.
The result is that for this kind of 3D systems we are forced to experience a geometrical manifestation of chaos. What do I mean? The system will admit an attracting limit set due to its overall volume shrinking dynamics, this set will be a 0-measure one and it turns out to be of a fractal nature. To permit the exponential divergence on a 0-measure support dynamics we force the geometry to get crazy; the definition of what a fractal is can be achieved via non integer Hausdorff dimension or other box counting definitions that quantify the never resolving structure. [a]
Indeed, one might ask whether we can extract all the dynamics from studying a single dense orbit evolution, being the flow indecomposable. This is a epistemological question that shifts our attention on the possibility of measuring these things from time series of observable quantities, and by these reconstruct the geometry of the dynamics involved. Even though the computational complexity is the principal effort to tackle, in principle topological transitivity allows to recover the full tangent dynamics if we wait enough. By tangent dynamics information we then extrapolate the global Lyapunov divergence. This is not feasible and only approximate values can be obtained numerically, the difficulty in reconstructing the space from the samples and the neighbourhood topology is a serious problem and cannot be naively done in a less than quadratic algorithm, hence making it explode easily once the number of considered points increases.
Interestingly, we can in principle even reconstruct the diffeomorphic invariant structure (hence all topological properties and even its Hausdorff dimension) of the attractor by sampling only a state observable time series, this is what is known as Takens Embedding Theorem and in [15] this procedure is well exploited. In the article it's even clear how Fourier spectral analysis can come in aid. By projecting our time series on the

Fourier basis we get a representation that globally captures the evolution, they show how the route to chaos experienced by the cymbal due to a slowly varying parameter can be observed via the power spectra that it produces: the Hopf bifurcations involved make the spectrum to be quasi-periodic and when chaos is reached the power spectrum eventually becomes continuous (or possibly a Cantor set) telling us its abandoning of the invariant limiting torus.
But the real takeaway is the importance of these power spectrum profiles and decay rates (the regularity implies a decay). We will see that by these means we can distinguish chaos from noise, i.e. this low-dimensional deterministic chaos from the multi-dimensional often modeled as stochastic chaos (for now quite intuitively).
More details, references and some notes on the small research and simulations that I've done trying to link Lyapunov exponents and these spectrum profiles can be found in **Appendix B**.
This fractal nature of chaos that one might call "geometrical" is not a prerogative of dissipative systems but as it will be clearer even in conservative systems as Hamiltonian ones (where it's not a matter of attractor geometry) one can have some sort of self similar structures of an even subtler and complex relation between stable and unstable regions of phase space (see Homoclinic Tangle [12]). In this latter case we don't have to deal with 0-measure fractal supports of the motion and all differential calculus can be carried out without further mathematical subtleties.

[a] Some conjectures try to link the geometry to the Lyapunov global divergence of orbits, even though we will not be involved with this kind of chaos I just wanted to introduce the topic for the power spectral analysis that it might admit and that we will use at the end of this work.

In this chapter we have introduced what deterministic chaos is from its basic characteristics but this is just its identikit, our aim is to find a way to make chaos manifest its inner core structure as an irreducible-periodic-dense object that resists single trajectory description, we will have to resort to distributions and operators in a way that captures as a whole the system's dynamics and leave back the velocity vectorfield. We will later try to link chaotic behaviour to some properties of these operators hoping to find a such global feature.

Chapter 2

A Statistical Point of View

2.1 Motivation

As we have seen in the previous chapter, chaotic dynamical systems are characterized by local exponential divergence of near trajectories and infinitesimal errors in the measurement of initial conditions will eventually get exponentially amplified and our model prediction would become useless. But even in the usage of numerical methods the discretization breaks the continuous nature of the system bringing to unavoidable errors that will be enlarged.

What we are left with? Some topological arguments can still be outlined and many striking theorems [1] arise in this context and one can get a taste of how they all exploit some global and fundamental structures that result invariant under homeomorphism, i.e. topological properties. They can bring to a global behaviour of the flow but none of these can tell us the exact evolution of the single state.
The question "where will the system be at time t" looses its sense and turns out to be ill posed, it's not expressible by any means, what might be sensate is instead: "What is the probability to be there if I started somewhere here?".

Deterministic Density Evolution

We move to a probabilistic view where the fundamental object representing our systems is now a *probability density distribution* $\rho(x,t) : M \to \mathbb{R}$ on state space. This $\rho(x,t)$ initially defined will get stretched and folded by the evolution dynamics

[1] As couple fast examples take the existence of at least one fixed point for a vectorfield on the 2-sphere, known as the Hairy Ball Theorem, or the existence of a limit cycle after a Hopf bifurcation, known as Poincaré-Bendixson Theorem.

in a way that is still hard to be calculated (much more) since at this point we are still resorting to a trajectory picture of evolution

$$\rho_t(x) = (\hat{\mathrm{P}}^t\rho_0)(x) = \int_M \mathrm{d}y\, \delta(x - \Phi^t(y))\rho_0(y) = \frac{\rho_0(\Phi^{-t}x)}{|D\Phi^t(\Phi^{-t}x)|} \tag{2.1}$$

This evolution law for probability densities is often referred as *Frobenius-Perron equation*. In (2.1) Φ^t is the evolution flow acting on state points. As we see, the $\rho_t(x)$ is calculated as a sum over all the y points whose evolution make them end up in x, weighted on the initial distribution density ρ_0.
Chaos makes the needing of a probabilistic approach even for deterministic dynamics.

2.2 Operators

Without being too formal with definitions and assuming that we are quite familiar with these mathematical objects we introduce the concept of *Operator*. In this sense (2.1) can be rewritten as $(\hat{\mathrm{P}}^t\rho_0)(x)$ where $\hat{\mathrm{P}}^t$ is called the Frobenius-Perron operator. Since we have shifted from tracking individual states to studying densities we need a new object that acts on the infinite dimensional space of functions and makes them evolve, as Φ^t does for single state points $\hat{\mathrm{P}}^t$ does on a specific space of functions or distributions that we have to choose and mathematically define, the caveat is that different functional spaces will have different evolution operators even if the underlying dynamics is kept.
The big property of (2.1) is *linearity* in ρ_0 (the integral form makes it explicit), this permits a spectral decomposition and another powerful theoretical approach to the problem. Very important are the mixing properties of transformations, much more stronger than the topological transitivity we have previously required; the measure concept of strong mixing implies that $\lim_{t\to\infty}(T_\mu^t A) \cap B = \mu(A)\mu(B)$, this product shows statistical uncorrelation between state space regions and decay of correlations. Mixing permits a reasonable application of statistical mechanics (more at [16]).

We can now express the infinitesimal dynamical law for $\rho(t, x)$, if

$$\frac{\mathrm{d}x_t}{\mathrm{d}t} = \mathbf{v}(x_t)$$

is the dynamical law for trajectories, than

$$\frac{\partial\rho}{\partial t} = -\operatorname{div}(\mathbf{v}\rho) = \hat{\mathbf{L}}\rho. \tag{2.2}$$

is the continuity equation that express the infinitesimal change in time of ρ in terms of the probability flux given by the divergence of the weighted flow; $\hat{L}$ is called the Liouville operator.
Our equation is now a PDE that given suitable boundary conditions (there are several ways to define them) can in principle, once integrated, give the infinite dimensional evolution of our ensemble of states.
What before was a mere vector $\mathbf{v}(x)$ as mathematical object representing a point displacement has now become an independent operator that maps functions to their infinitesimal displacement in a linear manner. We have thrown all the nonlinearity of $\mathbf{v}(x)$ in higher order spatial derivation operators now available that are remarkably linear.

Now, at the end of the previous chapter we have underlined our ambition to find some properties of these evolution operators to be observed and studied in the transition to chaos.
What is natural to consider when dealing with linear maps is the *Spectrum*.
With a time separation ansatz $\rho_t(x) = e^{\lambda t}\rho(x)$ we obtain the time independent eigenproblem well known in physics, the power of this relying in the possibility of finding a base of eigenfunctions easier to derive (not always this is possible but many times it simplifies the problem) and than project every initial state in the eigen representation and finally recover thanks to linearity its time evolution.

So, a good candidate might be a property of the spectral measure of evolution operators that, by its eigenstates, fully characterizes the dynamical system.
We will try to focus on just the spectrum as a subset of the complex plane or, in case of Hermitian operators, even of the one dimensional real line, without caring to eigenfunctions; we do so as a simplification and reduction to an object that as we will see can be expressed and mathematically treated in its own fashion.
The reduced information contained in it implies that it could be shared by many different systems but as should be clear this might not be a problem as far as all these spectrum-sharing systems (or almost all, the general ones) are characterized by the same global regularity or chaotic behaviour (i.e. have the same global scalar measures of chaos). So at the end, what we are interested in are not specific systems but how some global properties that in some way summarize their behaviour are reflected in the spectra.
If we succeed in finding such theoretical relationship between the trajectory definition of chaos and some aspects of the spectrum we gain a possible different procedure to measure chaos. Is the spectrum something computationally more accessible or convergent than other measures previously mentioned? We will see that at first sight it doesn't seem to be the case but this question needs some further research in the literature and more pondering so I leave it as an open question mark.

So, let's dive into the spectrum in all shapes and forms.

2.3 The Spectrum

There are some beautiful examples of what can be inferred from the spectrum: the way eigenvalue levels occur, their spacing, density, continuity or discreteness; as we will see all of these depend on some properties of the system dynamics represented by its operator PDE. In [17] Kac solves by spectral means the question "Can we hear the shape of a drum?", by modeling the dynamics with a wave equation, the area of the membrane turns out to be linked to the density of modes at high frequencies whereas the boundary conditions (the vanishing of membrane deformation on the borders) dictate the quantization and the discrete spectra character. This theoretically solvable problem necessitates an enormous numerical precision to be carried out but as for the systems we will be concerned, we will instead face not well understood theoretical arguments supporting numerical results.

In our context we are now interested in Frobenius-Perron spectral properties. As it's well illustrated in [2] the starting observation is the integral form of (2.1), $\delta(x - \Phi^t(y))$ is called the kernel of the operator $\hat{P}^t$. The trace is thus defined by

$$\operatorname{tr} \hat{P}^t = \int_M \mathrm{d}\, x \delta(x - \Phi^t(x)). \tag{2.3}$$

with the right hand side of (2.3) to be intended as acting on a space of sufficiently regular test functions, this additional acting-space information is needed for the distributional character of the Dirac δ involved.
The kernel can be interpreted as the infinite equivalent of the matrix representation of a finite dimension operator on a chosen basis, hence by putting $x = y$ in (2.1) we obtain the trace on space representation.
We have introduced the trace quite arbitrarily, for now is just the invariant sum of our spectrum. What is surprising is the physical interpretation of such infinite dimensional formula: it's a sum over all periodic points with t period, hence we are beginning to make out a link between periodic points and spectrum.

Gradient flow

As a first example consider a gradient flow $F(x) = -\nabla\phi(x)$ admitting steady state solutions both stable and unstable, these are the only periodic points that one has in this trivial case. From Dirac δ properties we than obtain a trace expressed as a sum over fixed point solutions $x_{i,s}$ each of which will bring a contribution of

$$\operatorname{tr} \hat{\mathrm{P}}^{\mathrm{t}} = \frac{1}{\left|\det\left[I - \left(\frac{\partial \Phi^t}{\partial x}\right)_{x_s}\right]\right|} \tag{2.4}$$

We see that each contribution depends only on the product $\prod_{k=1}^{d} |1 - \mathrm{e}^{\lambda_k t}|$ where λ_k are the stability exponents, $e^{\lambda_k t}$ being the eigenvalues of the potential Hessian $\frac{\partial^2 \phi}{\partial x_i \partial x_j}$ at the equilibrium points.
In a continuous dynamical system $tr\hat{P}^t$ is related to the trace of the resolvent of the Liouville operator previously introduced in the continuity PDE equation.

$$\operatorname{tr} \frac{1}{s - \widehat{\mathbf{L}}} = \int_0^\infty \mathrm{d}t e^{-st} \operatorname{tr} \hat{\mathbf{P}}^t = \sum_{n_i,\ldots,n_d=0}^{\infty} \frac{1}{s \, - \, n_1\lambda_1 \, - \, \ldots \, \cdot \, - n_d\lambda_d}. \tag{2.5}$$

This comes from the fact that from (2.2) we can formally express $\hat{P}^t = e^{\hat{L}t}$ and by switching the order of integration in (5.7) by applying the Laplace transform to $e^{\hat{L}t}$ before computing the trace, one sees immediately that we have divergences for s eigenvalue of $\hat{L}$.

We arrive to conclude that the spectrum of the Liouville operator is given by the poles of (5.7), which for every stable fixed point brings a spectral contribution of

$$s_{n_1..n_d} = \, n_1\lambda_1 \, + \, \ldots \, \cdot \, n_d\lambda_d, \qquad (n_i = 0,\, 1,\, 2,\, \ldots). \tag{2.6}$$

Calculation and comment

Let's write down the derivation of the above result for a single fixed point:

consider the integral (5.7)

$$I(s) = \int_0^\infty e^{-st} \left[\prod_{\lambda_k > 0} \frac{1}{e^{\lambda_k t} - 1} \prod_{\lambda_k < 0} \frac{1}{1 - e^{\lambda_k t}} \right] dt.$$

For each $Re[\lambda_k] > 0$, note that

$$\frac{1}{e^{\lambda_k t} - 1} = \sum_{m_k=1}^{\infty} e^{-m_k \lambda_k t},$$

and for each $Re[\lambda_k] < 0$

$$\frac{1}{1 - e^{\lambda_k t}} = \sum_{n_k=0}^{\infty} e^{n_k \lambda_k t} = \sum_{n_k=0}^{\infty} e^{-n_k |\lambda_k| t}.$$

We obtain for the Laplace transform

$$I(s) = \sum_{\substack{m_k \geq 1 \\ \lambda_k > 0}} \sum_{\substack{n_k \geq 0 \\ \lambda_k < 0}} \int_0^\infty e^{-t\left[s + \sum_{\lambda_k > 0} m_k \lambda_k + \sum_{\lambda_k < 0} n_k |\lambda_k|\right]} dt.$$

(we have omitted the $Re[\lambda]$ in inequalities)

Now, for $Re[\mu] > 0$ we have

$$\int_0^\infty e^{-\mu t}\, dt = \frac{1}{\mu},$$

we therefore have for $Re[s] > -Re[\sum_{\lambda_k>0} \lambda_k]$

$$I(s) = \sum_{\substack{m_k \geq 1 \\ \lambda_k > 0}} \sum_{\substack{n_k \geq 0 \\ \lambda_k < 0}} \frac{1}{s + \sum_{\lambda_k>0} m_k \lambda_k + \sum_{\lambda_k<0} n_k |\lambda_k|}.$$

we then analytically continue the obtained $I(s)$ for all $s \in \mathbb{C}$.

The poles of this defined $I(s)$ occur when

$$s_{n_k, m_k} = -\left(\sum_{\lambda_k>0} m_k \lambda_k + \sum_{\lambda_k<0} n_k |\lambda_k| \right),$$

We see that relaxation time rates $e^{-s_n t}$ of this dissipative gradient flow are given by all possible combinations of principal curvatures in fixed points. This is something physically sensate: along the principal axes we get an exponential decay given by the eigenvalues but we expect that density distributions, spread in space, will evolve with a superposition of these axial decays, but it's not straightforward to mathematically express the specific profile of such eigenfunctions evolving in a self similar way; even though gradient flows near the minimum will map ellipses to ellipses these get stretched and loose their initial axial ratio, so it must be a more complex eigenfunction. The powerful fact is that by the trace formula we have obtained the spectrum without the need of finding the corresponding expressions for the eigenfunctions and so, we don't need any further work.

One of the essential features of this reasoning is the reduction of our problem to periodic orbits, difficulties arise when the periodicity structure of our system becomes more involved than just fixed points.

Limit Cycles

In this deterministic context a formula for *stable and unstable limit cycles* was proposed by Cvitanovic and Eckhardt [18] which is given in terms of all periodic orbits p of the system

$$\operatorname{tr} \hat{\mathrm{P}}^t = \sum_p T_p \sum_{r=1}^{\infty} \frac{\delta(t - r T_p)}{|\det (\mathrm{I} - \ M_p^r)|} \tag{2.7}$$

where T_p is the period of the primitive orbits and r the number of revolutions.
This is just a non trivial generalization of (2.4) where the M_p^r plays the role of the linearization around the fixed point, here it's the linearization of the Poincarè

mapping in a section plane transverse to the cycle. We will see a similar formula in use in another context later on.

For all the subsequent study it will be important to keep in mind these definitions

Correlation functions and Power Spectra

Given two observables $\varphi, \psi : M \to \mathbb{R}$, we call *measure correlation function*

$$C_{\varphi,\psi}(t) = \int_M \varphi(\Phi^t(x))\psi(x)\mathrm{d}\mu - \int_M \varphi(x)\mathrm{d}\mu \int_M \psi(x)\,\mathrm{d}\mu,$$

and *trajectory correlation function*

$$c_{\varphi,\psi}(t) = \langle \varphi(\Phi^{t+\tau}x)\psi(\Phi^{\tau}x)\rangle_\tau - \langle\varphi\rangle_\tau\langle\psi\rangle_\tau$$

In the *ergodic* case $c(t) = C(t)$.

The *power spectra* is defined as the Fourier transform of $c_{\varphi,\varphi}$

$$S(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}c(t)dt$$

Chaotic Attractor

But more remarkably what I want to show you is the case of a chaotic attractor where all periodic orbits are unstable of the saddle type with stability eigenvalues both bigger and smaller than 1. To be more precise what we assume is the *hyperbolicity assumption* [3] so that the stabilities of all cycles included in the trace sums be exponentially bounded away from unity (the M_p^r eigenvalues). We are still using (2.7) on the dense set of periodic orbits that chaos brings in, hyperbolicity renders the spectrum *discrete* and permits a relatively easy discussion, in turn, if the expansion/contraction is slower than exponential the system might exhibit phase transitions [a], these will entail a *continuous* spectrum.
We can express the poles of (2.7) as the zeros of the expanded (in terms of the stability eigenvalues) Selberg-Smale zeta function [2], which under our assumptions results to be meromorphic and with discrete set of zeros.
These statements about discrete spectrum and the regularity of the zeta function have been outlined for the class of systems called Axiom A, they are due to the research of Ruelle, Pollicott and others [19] [20].

[a] The presence of subexponential stability allows for abrupt changes when a system parameter slightly changes, at the same time this is reflected in the spectrum due to the almost periodicity

of an entire family of metastable trajectories around the periodic one in the neutral directions.

The inverse of Liouville eigenvalues is interpreted as a time scale for relaxation of the corresponding eigenstate, and it has a physical interpretation in terms of measure correlation decay of observable quantities: for strong mixing systems we have mentioned that the memory and correlations will vanish in the ergodic asymptotic limit, but at short times, we still have some strong dependence viewed as peaks in the Fourier transform of the trajectory correlations (that for autocorrelations is called Power Spectrum $\mathcal{P}(\omega)$); it turns out that for Axiom A systems these *resonances* are discrete and localizable as isolated poles of the correlation spectrum [18]. The connection with the periodic expansion given by the trace formula and the zeta function being in the fact that for ergodic systems we can express these resonances as averages over our chaotic set more or less as we have done in the definition of Lyapunov exponents formally introduced with Oseledets theorem. These averages can be carried out by the dense wandering orbit or by averaging over periodic orbits as we have done here.

The reason behind the connection between Ruelle resonances (poles of power spectra in complex plane) and resolvent trace poles, both representing Liouville spectra, can be found in **Appendix C**.

Remark

We are simplifying things a lot and tacitly assuming many hypothesis and hiding technical difficulties. Far from being the author capable within the scope of this work to deeply understand and explain the complete picture I again mention some references where dynamical zeta functions and their regularities are well treated [3]. We have tacitly assumed the (2.1) operator to act on the space of analytic densities that permit a Taylor expansion and a rigorous periodic expansion. Since the kernel in (2.1) is singular the resulting operator is not of the Hilbert-Schmidt type (with L^2 kernel) and this results in a non compact operator. Whereas for compact ones we can assert discreteness it's not true for singular kernel ones as the Frobenius-Perron and the dynamical trace theory has to be carried out carefully, we might encounter essential spectrum. Analytic test function space aid us to essentially make the operator still tractable and the theory easily follows only due to the effective filtering of its singularities, anyway we still need hyperbolicity and axiom A hypothesis; for the Lorenz system and many other continuous dynamics considered this is not fully satisfied and even though the orbits are all unstable of the saddle type we lack the required uniform hyperbolicity.
In [3] is underlined that if we consider spaces of $C^{k+\alpha}$ observables we miss pure discreteness and an essential spectrum in a disc arises. Ruelle proved that in any case, for axiom A systems, we can calculate an upper bound for the radius of this continuous

spectrum and still have meromorphic correlation functions in a strip with resonances that describe the discrete part of the transfer operator.
If than hyperbolicity is not even satisfied we have to rewrite the trace formula and take account of branch cuts that arise in the dynamical zeta function associated, in this case we can still reach a periodic cycle expansion but with special care for marginally stable orbits.
As a last note, averaging over such chaotic fractal sets has to tackle a everywhere singular ergodic measure supported on the fractal attractor, we usually don't do so because zeta functions and periodic cycle expansion permits another way of calculating the spectra and resonances but still, we have to remember this geometrical difficulty.

We will see that noise (to be defined) will permit the kernel of (2.1) to become more regular thanks to a Laplacian entering the Liouville operator. The Laplacian makes the distribution to be dependent on its neighbour and it introduces a smoothening. In [21] it's even displayed how this enables a perturbative expansion since the transfer operator results continuous.
Noise is not magic, it brings regularity only because we are averaging on stochastic realizations, in [22] is pointed out that, since in reality we might be more interest in single noise realizations of the density evolution, other approaches should be taken.

Anyway, we have learned that correlation functions are measurable quantities and obtainable from time series of observables of our system. We have seen that resonances that we might extract from their spectrum can be linked to the eigenvalues of the Liouville operator whose exponentials are the Frobenius-Perron ones. At this point, we then would like to find some connection between these power spectra of correlations and the Lyapunov measure of chaoticity. One such link would permit us to understand how Lyapunov global divergence of trajectories reflects itself in the spectrum of the evolution operator.

This is a good point to read **Appendix B**, some attempts to find this possible connection for continuous chaotic systems are there summarized.

In this mist of partial results and unknown relations there are special kind of systems where many things become mathematically provable and clear, our chaotic dynamical systems are far from being easily reduced to this kind of dynamic but still, as a limiting case, we introduce Markov processes in the following Supplement box.

Supplement

2.4 Markov Approximation

In [2] the authors at a certain point try to map the dynamical system to a stochastic description that can be treated with linear algebra. The shift is radical and has to be treated carefully, we have previously introduced the operator formalism to tackle the difficulty of considering single trajectories and moved to a probability density evolution of ensembles but still, these densities are evolving deterministically with the flow, the uncertainty being only in the knowledge of the initial position that becomes a cloud of possible states but as a whole, the evolution will wander in space with an exact motion fully determined by the field $\mathbf{v}$. Now, when we say stochastic we mean that even the dynamics becomes unpredictable.

A *stochastic variable* Ξ is defined as an object with a set of possible values, the state space S, and a probability distribution over it $P(\xi)$.
A *stochastic process* is a function $X_\Xi(t)$ of time and of the stochastic variable Ξ. It is basically a family of trajectories $f(\xi, t)$ where each element is specified by the realization ξ of Ξ.

Now, a **Markov process** is a stochastic process $X_\Xi(t)$ that satisfies the *Markov property*

$$P_{1|n-1}(x_n, t_n | \, x_1, t_1; \ldots; \, x_{n-1}, t_{n-1}) = P_{1|1}(x_n, t_n | \, x_{n-1}, t_{n-1}). \tag{2.8}$$

The conditional probability density at t_n, given the value x_{n-1} at t_{n-1} is uniquely determined and is not affected by any knowledge of earlier positions.
We have defined

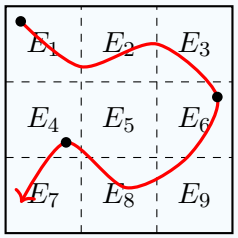
$$P_{1|1}(x_2, t_2 | x_1, t_1) = \frac{\int_S P(\xi)\delta(f(\xi, t_1) - x_1)\delta(f(\xi, t_2) - x_2)d\xi}{\int_S P(\xi)\delta(f(\xi, t_1) - x_1)d\xi}$$

from the knowledge of $P(\xi)$ but in general thanks to (2.8) a Markov process can be fully characterized by the transition probability $P_{1|1}$ and a specified $P_1(x_1, t_1)$ at a certain time. The whore hierarchy $P(x_1, t_1; ...; x_n, t_n)$ (probability of passing through x_i at time t_i) can be constructed from them.

The objective is to reduce the infinite dimensional transfer operator $\hat{P}^t$ to a finite dimensional one (a matrix) while preserving the modeled physics. This would allow us an easier finite dimensional analysis of the spectra. We might try to do so by *partitioning the space*

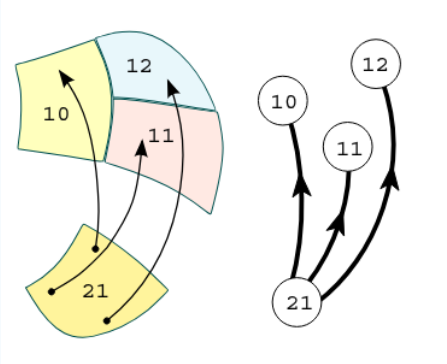
$$\{E_i\}_i$$
$$\bigcup E_i = M$$
$$E_i \cap E_j = \emptyset$$

in small elements $E_i \subset M$ that might represent our limited precision in locating the system at subsequent times. By discretizing time, a trajectory $x(n\tau)$ will be characterized by a parition symbolic trajectory given by all the indices $i_n = x(n\tau) \in E_i$.



$i_n = E_1, E_6, E_4$

We now understand why we have introduced stochastic processes: the trajectories i_n cannot be deterministic, specifying only the partition element E_i is not enough to know the exact $x(t)$, in general we could have all possible transitions $i \to j$ in τ if $\Phi^\tau(E_i) \cap E_j \neq \emptyset$. The symbolic evolution i_n can therefore only be described by a stochastic process. In cases where we start with an initial density **uniformly** distributed in one single E_i, we can conclude that the transition probability will be given by $\Phi^\tau(E_i) \cap E_j$, and for chaotic Φ the $\Phi(E_i)$ will in general intersect many other elements due to the exponential stretching caused by the dynamics.



Representation of the multiple possible transitions from one partition element.

This is captured by what is called *Kolmogorov entropy* which quantifies the additional information needed to specify the deterministic i_n generated by the hidden $x(t)$.
We can think of the partition as our instrumental uncertainty to locate the system and can only give information about the $x(t) \in E_i$, when we perform a measurement, we will furnish the necessary Kolmogorov information to specify the successive E_j and make the stretched E_i to "collapse" in E_j. These successive collapses give the effective stochastic nature to our symbolic dynamics.

We now define the probability vector $\mathbf{p}$ as the ρ equivalent for this partition coarse grained stochastic dynamics by projecting ρ on the partition

$$p_i(t) = \mathbf{\Pi}_i \rho_t(x) = \frac{1}{\mu_i} \int_{E_i} \rho_t(x) \chi_i(x) \, \mathrm{d}\mu(x)$$

where χ_i is the E_i characteristic function and $\mu_i = \mu(E_i)$. $\mathbf{\Pi}_i$ are the projector operators. We are assigning to every p_i the average ρ "content" in E_i, we will define the "leveled" distribution

$$\rho_t^*(x; p_i) = \mathbf{\Pi} \rho_t(x) = \sum_i \chi_i(x) p_i$$

We now proceed this way:

1. start with ρ_0.
2. let it evolve for one time step $\rho(x,t) = \hat{P}^\tau \rho_0$.
3. calculate the p_i vector components.
4. Make the whole ensemble to collapse on $\{E_i\}_i : \rho_\tau^* = \mathbf{\Pi} \rho_\tau$.
5. return to point 1.

The necessity of point 4. is due to the fact that the $\mathbf{p}$ evolution has to be deterministic in the sense that we need to avoid that two ρ_1, ρ_2 that have the same $p_i = \mathbf{\Pi}_i \rho_1 = \mathbf{\Pi}_i \rho_2$ will have different evolution, this would make the $\mathbf{p}(t)$ not a well posed function of t and no possible reduction will ever be possible. So we force the $\hat{P}^\tau \rho$ to collapse in the "representative" ρ^*.
At this point we might hope to find the Frobenius-Perron equivalent operator for this coarse grained dynamics, the only possible candidate is

$$\mathbf{W}_\tau = \mathbf{\Pi} \hat{P}^\tau \mathbf{\Pi}$$

thought as acting on leveled distributions isomorphic to the vectorspace where $\mathbf{p}$ lives.

We have a caveat, to constitute a well defined coarse grained dynamics $\mathbf{W}$ has to form a *semigroup* under composition, this is equivalent to requiring

$$\mathbf{\Pi} \hat{\mathrm{P}}^\tau \, \mathbf{\Pi} = (\mathbf{\Pi} \, \hat{\mathrm{P}} \, \mathbf{\Pi})^\tau \tag{2.9}$$

So that we can write the semigroup as $\mathbf{W}^\tau$ (the τ is now at the exponent).

If (2.9) is satisfied we can express the evolution of the projected vector $\mathbf{p}_t$ with the following master equation

$$p_i(t) = \sum_j W_{ij}^\tau p_j(t - \tau) \tag{2.10}$$

where the transition matrix $\mathbf{W}^\tau$ is fully determined by the dynamics and partition. One might even ask what the nature of W might be, can we effectively model its entries

with random numbers preserving its symmetries (if detailed balance is satisfied we have to keep it symmetric)? We will treat this question much more seriously later on and for now we just underline the word *effectively*.

As for the decay of measure correlations, systems that satisfy (2.10) will show exponential decays, this is a direct consequence of the discretization of space and the finite size matrix representation, we will have a discrete spectrum of finite cardinality and this assures a spectral gap between the leading eigenvalue (1 for the unique equilibrium ergodic measure) and the subleading one (this due to the Frobenius-Perron theorem for mixing systems that allow a stochastic mapping as above); the mixing and decay of well behaved observables will therefore follow a relaxation given by this second timescale smaller than unity.
Even though the vector's evolution has the same structure of a Markov process the single realization represented by a series of symbols given by an underlying trajectory might not satisfy the Markov property (2.8).
So, we have to be careful, in these cases we have some incongruities between measure correlation decay rates of leveled distributions, trajectory correlation decays $\langle x(t)x(t+\tau)\rangle$ and symbolic evolution one, some power law decays might appear even in i_n correlation. As a last remark, the information gained by $\mathbf{W}$ spectrum can capture some Ruelle resonances but might not pointwise converge to the original $\hat{P}$ one.

A limiting case: Markov maps

An important class of processes for which condition (2.8) can be fulfilled with a suitable partition are piecewise linear maps that take the name of Markov maps. In [23] are summarized the results for this particular kind of dynamics that can be well modeled with a finite size transition matrix $\mathbf{W}$. In the article , a possible connection between Lyapunov exponent Λ and correlation decay is carried out for these special systems, and the following upper bound theorem is obtained

Proposition 1. *[23] Let $f : I \to I$ denote a topologically mixing [a] piecewise linear expanding Markov map. If $H = H(D_R)$ denotes a space of piecewise analytic observables (see article for details), then the mixing rate is bounded in terms of the Lyapunov exponent Λ with respect to the piecewise constant invariant density, by*

$$\alpha_H \leq 2\Lambda.$$

If all slopes have the same sign, the sharper estimate

$$\alpha_H \leq \Lambda$$

holds.

We see that there is indeed a connection (here α_H is the typical relaxation exponential rate for functions belonging to the space H, $-\ln|\lambda|$ with λ subleading eigenvalue of the Frobenius-Perron operator associated to f on H. Λ, on the other hand, is linked to properties of the leading eigenvalue).
By exploiting the finite size Markov matrix translation of dynamics we can conclude that Λ reflects itself in the spectrum as an upper bound for the mixing rate (measure decay of correlations). We see that stronger chaos, as for Λ, allows a bigger spectral gap between the first two leading eigenvalues of our Frobenius-Perron operator.
It's just a partial and small result for these very special systems but it gives a first insight of what we will find later as a general trend, the authors even generalize the result for piecewise monotonic smooth expanding interval maps which are mixing with respect to their unique absolutely continuous invariant measure. Then the rate of decay of correlations for functions of bounded variation is bounded by the Lyapunov exponent.

In their conclusion it's pointed out the importance this class of observables as they bring correlation decays more linked to Lyapunov exponents: they are able to transport the microscale sensitivity to the macroscale correlation function thanks to their discontinuity and infinite resolution on it.

[a] Topologically mixing requirement can be proved by finding an n such that $\mathbf{W}^n$ has strictly positive entries entailing the full connectivity of the graph after enough iterations.

We need to return to our physical continuous flows but to avoid the complications involved in the dissipative systems dynamics, characterized by strange attractors, we try to shift our attention to Hamiltonian flows. This shift encloses two objectives: see how chaos manifests itself in one of the most important modeling of classical mechanics and try to avoid continuous spectra. We hope that if we find a way for chaos to manifest itself in discrete spectra it would be clearer what is its effect by some change in disposition of the eigenvalues, if the effect of irregularity and chaotic instability manifest itself only in the continuity of evolution and power spectra, little can be said on the believed organic structure that its indecomposable nature on a periodic skeleton should embody. The impossibility of reducing it to a sum of isolated subsystems should make us realize that the apparent random intertwining of trajectories composes an object that globally cannot be thought as a boring result of the central limit theorem, the apparent randomness is giving a great gluing and relation that build a beautiful structure.

But we are still far from it, for now we introduce Hamilton systems and we will see that their more tractable geometry comes with other new phenomena and intricate be-

haviours.

2.5 Hamiltonian Systems

I start by writing the PDE associated to this class of systems, (2.2) turns out to be expressible as

$$\mathrm{i}\,\frac{\partial}{\partial t}\,\rho(t) = \mathrm{i}\,\{H,\rho\} \tag{2.11}$$

Some remarks are in order to be made.

Recap: Hamilton mechanics

The dynamical field $\mathbf{v}$ is not arbitrary, over the smoothness condition that we have assumed for our general dynamical systems, Hamiltonian flows comes with additional structure on them. These latter are the result of the well known Lagrangian formalism of Newtonian mechanics that synthesizes all mechanical systems in a scalar $\mathcal{L} : J^1(V^{n+1}) \to \mathbb{R}$ (the *Lagrangian*) of the space of kinetic states [a].
This scalar fully embodies all mechanical forces and constraints, we will restrict ourself to the standard systems whose forces are obtainable from a potential and with holonomic constraints (the formalism can go well beyond this but let's stick to the general simple situation).
Once $\mathcal{L}$ is known, we can obtain the equations of motion through a minimum action principle that appears to be the fundamental property shared by all mechanical evolutions

$$S[q_k(t)] = \int_{t_1}^{t_2} \mathcal{L}(q_k, \dot{q}_k, t)\, dt. \tag{2.12}$$

By requiring that the variation of the action vanishes

$$\delta S = 0, \tag{2.13}$$

we obtain the Euler–Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} = 0. \tag{2.14}$$

and these specify the field $\mathbf{v}$.
We are finding the stationary paths in configuration space with fixed boundary conditions (final and initial position states and time passed), we have changed the procedure of how the motion chooses to evolve: before we had a Cauchy problem with initial conditions fully specified and the evolution would simply follow $\mathbf{v}$, now the evolution "chooses" to connect the points through the least action path.
I'm not introducing these concepts and shifts of prospective just by chance and for a sake of completeness, they will play a predominant role in what will follow.

This formalism, by performing a Legendre transform of the generalized velocities, has a counterpart in cotangent bundle T^*V^n, where we have restricted our study to time independent Lagrangians and hence removed the necessity of a jet space. The Legendre transformation maps the Lagrangian to a Hamiltonian function which depends on the conjugate quantities (momentums with respect to velocities). In this setting, Mechanics reaches its maximal brightness, the broader set of transformations of the so called phase space (cotangent bundle) that preserve the structure of the Hamilton equations of motion (the Euler-Lagrange correspondents) permit the most powerful integrability technique, such transformations are called canonical.
It turns out that the structure of these equations is quite peculiar and is called *symplectic*: the flow generated by them can be obtained from a natural differential 2-form embodied in phase space itself, $\omega^2 = \sum dq^k \wedge dp_k$, and a function of it, $H(p,q)$, by the canonical isomorphism between 1-forms and vectors given by $\omega_\xi = \omega^2(\xi, \cdot)$ (we have to invert it to get the dynamical field from the dH form); the flow thus obtained is called Hamiltonian flow of H and preserves the symplectic 2-form ω^2 and all the invariant ones obtainable from it such as the volume one.
At this point we can obtain (2.11) by expanding the divergence in (2.2), we will get two factors once the derivative has been distributed on ρ and $\mathbf{v}$, the first one is the right side of (2.11), the other will be $\rho\left(\frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i}\right)$ and we see that for regular Hamiltonians this has to vanish, and we are left with only the first factor expressible by Poisson brackets; these latter give to the space of observables a Lie algebra structure exploited in the quantization map.

[a] the first order jet bundle, not merely the tangent bundle due to the time fibration of space-time space that we consider in the general theory.

We are interested in the spectrum of such operators. A systematic study of the Liouville spectral properties can be found in [4] where a physical meaning of the results it's given. I will try to summarize some key aspects for our train of thoughts.
The first thing to note is that $i\{H, \cdot\}$ turns out to be selfadjoint in L^2 space(the purpose of the i is to stress an analogy with Schroedinger equation and turns it self-adjoint) and hence has a real spectrum, unlike the Liouville operator for general dissipative systems that usually admits only a Jordan canonical form due to the non vanishing field divergence $\hat{L} + \hat{L}^\dagger = -(\text{div } \mathbf{F})\, \hat{I}$.
Another property of these systems to keep in mind is their conservative nature, the preservation of volumes makes the Frobenius-Perron spectrum to resides on the unit complex circle, it's a unitary operator as quantum mechanical evolution one.
As the author, we are interested in the physical information about a Hamiltonian system that can be obtained by $\hat{L}$ spectrum. It's stressed that knowing merely the spectrum brings much less information than the complete solution of the motion but nevertheless we hope to find something resulting from H chaotic motion.

The zero eigenvalue has the physical meaning of determining the set of bound and scattering states of the systems, mathematically speaking if M_1 is the bounded part of the phase space M, we have that M_1 is the smallest invariant set with the property that the point spectrum of $\hat{L}|L^2(M_1)$ contains the zero and the point spectrum of $\hat{L}|L^2(M \setminus M_1)$ do not.

Apart from the zero eigenvalue we would like to study the properties of the spectrum outside zero as the significative one for the off-equilibrium dynamics and evolution. For this purpose we introduce the following definition of a *weakly relaxing* Hamiltonian system (M, μ, Φ) (μ is the standard $2N$ Lebesgue measure used for scalar products) by requiring that for all $f, g \in L^2$

$$\lim_{t\to\infty} \frac{1}{t} \int_0^\infty \Big| \langle g | f \circ \Phi^t \rangle - \langle g | \mathbf{P}_0 f \rangle \Big| dt = 0 \tag{2.15}$$

and

where $\mathbf{P}_0$ is the projection operator on the zero autospace.
The following theorem gives a reason for the above definition and allows us to characterize the presence of continuous spectra.

Spectra for General Hamiltonians

Theorem [4] *Let (M, μ, Φ^t) be a Hamiltonian system with Liouville operator $\hat{L}$. Then:*
The spectrum of $\hat{L}|(1 - P_0)L^2(M)$ is continuous if and only if (M, μ, Φ^t) is weakly relaxing.

The fact that continuity arises as a result of (2.15) is not that strange, the existence of a point spectrum besides zero would imply the existance of proper eigenstates with an evolution given by $e^{i\lambda t}$ which never settles down neither in the weak sense, is does not affect the norm in the integrand of (2.15) that results independent of t.

A comment on Hamilton Spectra

As previously stressed, we are given only gross properties from the spectrum, the question of whether the limiting stationary state depends only on the energy (which is the only explicit constraint) cannot be answered from spectrum alone (ergodic problem). One might even like to know something about the relaxation time of typical observables but again we don't have enough information.
In neat contrast with what has been found for some peculiar dissipative chaotic systems with discrete spectra (on the super restrict and somewhat special case of hyperbolicity

and analytic function space of observables) or even for Markov maps where the spectral gap between the first two leading eigenvalues had a direct meaning for the measure correlation decays.
In this latter case the finite dimensional possible representation and the verified mixing property allows a direct application of Frobenius-Perron theorem for matrices which proves the existence of a leading eigenvalue with a *unique* equilibrium distribution.
This result can than be further generalized for compact operators [a] whose thermodynamic equilibrium turns out to be unique with well specified decay modes given by the lesser eigenvalues. In both these cases an arrow of time clearly stands out and an irreversibility is manifested; as we will mention later on, we are approaching quite deep differences between the unitary conservative evolution that we encounter even in quantum mechanics and the dissipative dynamics that (we anticipate) can be regularized with noise (we have already mentioned that the Laplacian that stochasticity brings in allow operator compactness and as I'm underlying here, irreversibility). Taking noise apart and keeping things deterministic, both, conservative Hamiltonian and not, in the trajectory level description appear reversible due to determinism and uniqueness of the solution (after suitably assumptions of compactness and differentiability of the field) even if sensitive dependence makes these trajectories very unpredictable from a coarse grained view.
When we pass to the statistical point of view and describe distributions evolution given by the same flow, we end up with clear irreversibility and decay rates given by the applicability of Frobenius-Perron theorem or its generalization Krein–Rutman on this side and not clear behaviour on the conservative one. Indeed as the the Theorem above states, Hamiltonian systems can exhibit relaxation only if equipped with a continuous spectrum that renders decay modes and equilibrium properties out of sight from a spectral only analysis. We have indeed to prove ergodicity and as known, proving such property turns out to be very difficult.

We have reached a point to see that there is something more subtle in Hamiltonian dynamics and that resists irreversibility. We have to remember that Hamilton equations of motion have a stronger symmetry which is called time reversibility, i.e. that if $x(t)$ is a solution $x(-t)$ will too be, something which is stronger than the mere group property of the flow and that reflects a symmetry in the equations of motion.
The unitary character of the transfer operator permits a symmetry between initial and final states that we will discover later on, symmetry that we encounter even in the quantum world due to the same unitary property. [b] I've barely touched the surface of this and I cite [24] for more details but I hope it will become clearer later on.

[a] We have already seen that compact operators with non singular kernel, Hilbert Schmidt class, admit a pure discrete spectrum and a much way tractable dynamical zeta function, their regularity admits even a perturbative study.

[b] This can be broken by a Wick rotation making the associated Lagrangian time irreversible,

still in both situation the Euler-Lagrange equations will have time reversal symmetry whereas the quantum evolution will have a full reversibility of the wave equation (Schroedinger one). As we will see, the Wick rotated dynamics can be interpreted as describing a stochastic process.

Returning to [4] I want to quote another interesting result.

One may be tempted in considering the very special class of **Integrable Hamiltonian Systems** foliated with compact invariant sets, these submanifolds result to be invariant tori (see Poisson manifold theory and Arnold-Liouville theorem [see Arnold [10]] for the Action-Angle canonical representation), in this case we have n integrals of motion in involution ($\{f, g\} = 0$) and the equations of motion can be fully integrated giving an evolution that can be brought to a constant drift on the invariant torus specified by the action integrals of motion, being the Hamiltonian dependent only on the action set of n variables $H(I)$.
In this situation it's not clear whether the system will ever admit a relaxation since the flow is completely separated on invariant and non interacting tori. Spohn in his work proves that every integrable system which is not particularly degenerate will exhibit an absolutely continuous spectrum, it then results to be *relaxing*, in the sense that the integrand of (2.15) vanish, not only its average.
The fact is stated as (it is a corollary of a theorem in the article)

Integrable Spectra

Corollary [4] *If for all* $n \in \mathbb{Z}^N \setminus \{0\}$, $\nabla(n \cdot \nabla_I H) \neq 0 \quad \mu_N - a.e.$, *then the spectrum of* $L|(1 - P_0)L^2(M)$ *is absolutely continuous and* (M, μ, Φ^t) *is relaxing.*[a]

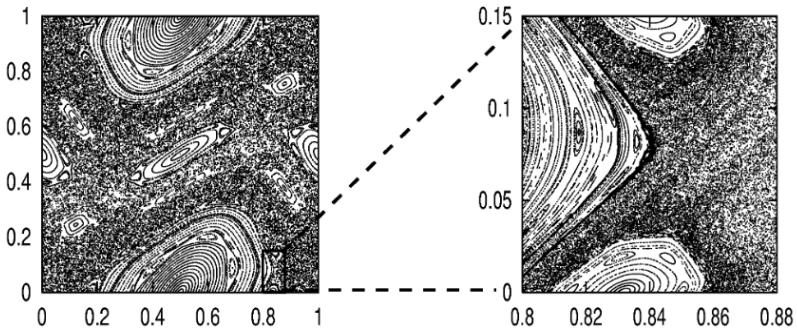
[a] The argument relies on a stationary phase argument for oscillatory integrals.

This tells us that probability distributions will relax even on bounded regions of integrable systems, but here the reason is paradoxically the complete *independence* of the drift evolution between arbitrarily close tori making every initial distribution to evolve on a continuous spectrum of angular drifts eventually bringing to a relaxation in a suitable topology (a single torus is of zero measure and cannot be taken as an open set).

For *weakly relaxing* Hamiltonians and most integrable ones we conclude that we have continuous spectrum, but one might argue that general mechanical systems are neither integrable nor completely chaotic, we nevertheless have that discrete spectra are rather an exception, I will just give an idea, some details can be found in [3].

When we have used dynamical zeta functions poles to express discrete spectra of the

evolution operator, we had to assume exponential instability everywhere but since general Hamiltonian flows exhibit both stable and chaotic components this cannot be taken as true. Discreteness is broken by problems that arise at the borderline between these components where marginally stable orbits and manifolds present serious difficulties.



Typical phase space organization for an area preserving map with mixed phase-space dynamics. Here the standard map for $k = 1.2$.

Dynamical zeta function will exhibit branch cuts that entail sub-exponential decay of correlations without well defined Ruelle resonances in Fourier decomposition (we are assuming ergodicity to compare C and c); this as a consequence of orbits staying near stable islands for arbitrarily long times producing a continuous spectrum for the evolution operator.

We have to conclude that real systems of physical interest modeled by Hamilton mechanics tend to display long correlations in the form of power laws, way far from being explainable with Markov processes or uniform hyperbolic chaos.
As we have already said, our objective is to make discreteness survive while studying general and significative chaotic behaviours present in real physical systems. We want discreteness because of its understandable and relatively easy discussion hoping to reach better investigation tools with respect to the difficulties encountered in analysing correlation decays of power spectra. Dynamical zeta function with branch cuts and all the complications at the stable-unstable boundaries in Hamiltonian systems are still not well understood.

We move to the next section where a fundamental step will be taken and we will finally find a suitable class of systems to be studied.

2.6 All you need is Noise

So what we do?
I start by citing couple of works [25][26] where a procedure for **distinguishing deterministic chaotic time series from systems subject to noise** (we will define it later, for now I want to give the reasons for introducing and studying it). To this end they examine the asymptotic falloff of power spectra $\mathcal{P}(\omega)$ at high frequencies. They indeed want to avoid the computational difficulties of calculating Lyapunov exponents and fractional Hausdorff dimension from time series of observables which we have already pointed out (see **Appendix B**) when trying to reconstruct attractor geometry with Takens theorem or by direct approximation of tangent dynamics, whereas $\mathcal{P}\omega$) decays seemed to give a possible workaround of faster convergence [2].
As in [27] the presence of $\mathcal{P}(\omega) \sim e^{-\lambda\omega}$ is associated to deterministic low dimensional chaos, however it seems still an open question whether a rigorous mathematical proof can be carried out. On the contrary, in [27] they prove that for a general system subject to noise it always corresponds a power-law decay.

To us it's not so surprising, we have already seen that stochastic Markov processes and systems that can be reduced to them with a suitable Markov partition, exhibit a discrete spectrum of evolution (whereas systems such Lorenz cannot be reduced to a Markov symbolic process and have longer correlation tails). This latter, once Fourier transformed, will have a Lorenzian power-law falloff.
More interestingly, here we are dealing with general continuous flows subject to additive white noise and even these are proven ([26]) to show the same ω^{-2} power tail and hence an exponential correlation one. We have learned to relate exponential decays with discrete spectrum and here we see that general stochastic system do have such decay.
On the contrary, we have seen that Lorenz system has an exponential tail for $\mathcal{P}(\omega)$, this confirms that it lacks a discrete spectrum.

[2] By power spectram means in [15] we clearly recognize the bifurcations as a route to chaotic spectra, in [27] the possible connection between decay and Lyapunov, whereas here they want to use power spectra for discerning chaos and stochastic noise

Heuristic motivation

In the article another reasoning is given: if $x(t)$ is differentiable with respect to time than the Fourier transform of $\dot{x}$ is $i\omega\hat{x}(\omega)$. Thus if $\dot{x}$ exists, $\hat{x}(\omega)$ must fall of asymptotically faster than ω^{-1}. Now, for the Lorenz system (and many smooth flows) the variables of a dynamical system are C^∞ functions of time, then one will expect their power spectra to fall off faster than any power of ω^{-1}. This is not satisfied by stochastic realizations that are not even once differentiable, here we expect a falloff as some power of ω^{-1}.

Indeed I've already cited [21] where the regularity that noise brings to otherwise singular transfer kernels is displayed; there, the noise induces a discretization and a continuity in the operator that allows a perturbative expansion (as in quantum mechanics, where for perturbations smaller than the spectral gap, it is always possible to perform such expansions, continuity in the spectrum on the other hand does not permit such stability).

If we want to pursuit a discrete spectrum for continuous flows we are left with these choices:

- *deterministic* gradient flows.
- *deterministic* uniform hyperbolic systems (Axiom A).
- *stochastic* perturbations of a generic dynamical system.

The first one cannot exhibit any chaotic behaviour, the second lacks physical generality whereas in the third case it depends on what we perturb. [3]
If we try to study dissipative chaotic systems subjected to noise we end up with serious complications due to the fractal attractor geometry, that needs further theoretical work to be studied under stochastic perturbation (SRB measures).
On the other hand, we would like to keep Hamiltonian dynamics as a reference modeling of our physical systems. A good candidate is therefore trying to study Hamilton systems in a Thermal Bath that we will define.
It will turn out to be very complicated to treat the associated operator and we will be forced to make some approximations and land to an unexpected choice that will reveal itself as the appropriate one: *stochastic perturbation of gradient flows.* But it will not be

[3] On the distinction between the discreteness that the second point has with respect to the stochastic one I want to mention [28], they outline a random matrix technique to distinguish chaotic signals from stochastic ones. By doing this, they show how the deterministic low dimensional case spectrum statistics and density is much more packed than the noisy ones, which are better modeled by RMT. This as a further reason to choose the third point class of systems whose statistics seems to have a possible RMT modeling that would simplify things a lot and larger spectral gaps to be studied.

a trivial to link it to chaotic continuous flows, discreteness makes us loose the long term tails that these latter usually manifest.

But before, we need to define what noise is and give it a mathematical setting.

Chapter 3

Stochastic Dynamical Systems

3.1 Mathematical construction

A physical Motivation

In physics we are aware that our models are describing only the gross features of real systems. By this I don't mean that they are inopportune or incomplete, they are just fine for capturing the essential phenomenology, very large models with hundreds of parameters to be tuned might risk to tempt more complete views but it's easy to overfit the measures and we end up modeling the data instead of physics, that's what AI is all about, but surprisingly we are usually able to perform a reduction of our real systems to few meaningful variables that can be effectively studied by suitable equations. This is the case of thermodynamics, mechanics, fluid and solid physics and complex systems modeling in general. In these latter cases we focus on some topological properties that can inevitably bring to some qualitative phenomenology in a very solid theory.
Anyway, we might still miss some important features that the microscopic and hidden degrees of freedom we have filtered out can imply. Indeed it's not all about mean field theory, higher moments turn out to be important and that's where noise can bring to visible discrepancies.

So, we remodel what our model lacks, and suppose the hidden degrees of freedom as effectively behave as a multidimensional highly mixing chaos. This means that correlation decays are exponentials and very fast ones. As a picture, one might think of a volume preserving map [a] that stretches itself so fast that the intersection of it with a successive ball around the single trajectory will decay as $e^{-\lambda t}$, this can be easily seen to imply fast correlation decay.
We will than try to craft a suitable noise signal to mimic these high mixing chaotic degrees of freedom by using only their exponential decorrelation feature. We hope that at least the statistical properties of the whole system can be be computed by the gross

drift dynamics plus the crafted noise.
We will bring this assumption to the limit and suppose that these contributions will have delta-like correlations, what is usually called *white noise* approximation. This will simplify the math involved but we see that if we suppose a hidden continuous dynamics even for noise, some memory has to be recovered for infinitesimal times, what *white noise* entails is a total unorrelation even at the infinitesimal level. Than, the question of how an infinitesimally uncorrelated perturbation can give rise to finite contribution to the macroscopic system (which would filter these infinite high frequency disturbs) induces the requirement of infinite amplitude fluctuations, hence we arrive at the self-consistent delta-like *white noise*.
In general we have to prove that such noise modeling will keep the original phenomenology, as for the math it turns out that even under nonlinear deterministic drifts the addition of white noise will keep the stochastic equation mathematically consistent and the classical existence and uniqueness theorems for stochastic differential equations guarantee under suitable conditions that the equation has a unique strong (i.e. pathwise) solution.

[a] Dissipation is a result of hidden degrees of freedom, but we believe these latter to be conservative

White Noise

We can summarize *white noise* properties in an object $\xi_i(t)$

$$\begin{aligned} &\langle \xi_i(t) \rangle = 0 \\ &c_{ij}(\tau) = \langle \xi_i(t)\xi_j(t+\tau) \rangle_t = \delta(\tau)\delta_{ij}. \end{aligned} \tag{3.1}$$

As for the physics, the justification of such ξ is plausible only if the whole evolution can be neatly separated in a fast and a slow part, the former, given by fluctuations, can be therefore modeled with ξ as a good approximation.
We stress its *stationary* character, both (3.1) do not depend on time t itself.

So, starting from our dynamical system differential equation $\dot{x} = \mathbf{v}(x)$ the simplest form of what we might call a *stochastic differential equation* can be written as

$$\dot{x}_i = v_i(x) + \xi_i(t) \tag{3.2}$$

where $\xi(t)$ is what we call the noise term.
We have chosen to use an additive noise to simplify the applicability of CLT and avoid further complications, we have than added it to the dynamical differential law to model the fluctuations of the trajectories around it and assumed its coefficient, 1, to be independent of the state x.

In (3.1) we have specified only its first two moments of distribution but to fully characterize the fluctuation we would need the whole hierarchy. We will do a further assumption: **Gaussianity**, all higher moments will follow the ones of a Gaussian distribution.
Linear drifts preserve Gaussianity[1] and the convolution term that now arises from the integration

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}\xi(s)ds$$

(where A is a matrix) can be fully characterized by its first two moments.
When nonlinearity in the $v_i(x)$ comes in we have to be careful, we cannot write the noise effect as a convolution that transports fluctuations now to the future in an independent way, one fluctuation now might drastically change the evolution map, space points are no more equal under the dynamics.
We now want to rewrite (3.2) in terms of a well defined mathematical object: the Wiener differential.

A Wiener process W_t is defined as the integration over all contributions of a white noise $\xi(t)$

$$W_t = \int_0^t \xi(t)dt$$

This object has the property to be a Gaussian variable $\mathcal{N}(0,t)$ with increasing variance in time. It can be thought as the limit of a random walk where we discretize the integral and as such, with an increasing variance proportional to t.
Its differential can be defined as its small increment $\xi(t)\Delta t$ where we have chosen to use the *Ito* prescription that evaluates ξ at the beginning of the interval $[t, t+\Delta t]$, we understand that its order in t cannot be easily determined.

Wiener differential

More formally we call Wiener differential $d\omega_t$

$$\begin{aligned} d\omega_t &= \lim_{\Delta t\to 0}\int_t^{t+\Delta t}\xi(\tau)d\tau \\ \langle dw_t\rangle &= \lim_{\Delta t\to 0}\int_t^{t+\Delta t}\langle\xi(\tau)\rangle d\tau = 0 \\ \langle dw_t, dw_t\rangle &= \lim_{\substack{\Delta t\to 0\\ \Delta t'\to 0}}\int_t^{t+\Delta t}\int_t^{t+\Delta t'}\langle\xi(\tau),\xi(\tau')\rangle d\tau d\tau' = \int\int\delta(\tau-\tau') = dt \end{aligned} \tag{3.3}$$

[1]This can be understood from the linearity of the displacement $\dot{\delta x} = A\delta x$ evolution that permits to write the convolution, this in turn can be thought as an infinite sum of finite variance independent variables making CLT applicable.

and we consider it to be gaussian even after the limiting process, so these first two moments fully characterize it.
We can finally rewrite (3.2) in differential terms and with a better mathematical understanding using these Wiener differentials [2]

$$dx_i = v_i(x)dt + d\omega_t \tag{3.4}$$

The different order in time of $d\omega\sqrt{dt}$ have serious implications, it comes from the non differentiability of white noise and its infinite amplitude, but even though $d\omega$ turns out to be much more tractable it still reflects this big jumps in the small time scale variance of the stochastic process. Due to this, the integration cannot be done in the standard Riemann way (see Ito calculus) and in general, we loose equation covariance with respect to variables transformation. The physics described by such laws turns out to be model dependent and a change of variables $y = y(x)$ that have $y'' \neq 0$ will respond differently to the introduced noise, we have to resemble that our modeling came from a crafted noise term that would only mimic the statistical properties of our real systems, if we change the gross variables it's quite natural that even if the whole physics is the same we will have to recraft the noise.

Remark

From the previous chapters it might seem that we have arrived to talk about noise only as a tool to make our discussion about chaotic spectra easier and somewhat informative. If that would be the only reason we could have simply introduced $\xi(t)$ and dw_t, or even directly the Laplacian term, as mathematical artefacts to achieve the needed regularity in the equation to make the spectrum discrete. But than, how can we interpret eventual results? The question is even more important for the gradient flow case: we are just adding some smoothening to something that it's already well behaved with a well defined discrete spectrum (2.6), so where one can see chaos here?
That's why we have to give a meaning to these new terms, we will see that it's only with a precise idea of the physical processes that these objects are representing that we can perform some weak noise limit and know how to interpret the chaoticity that is found in the described conditioned evolution, and eventually interpret the spectra properties as physically meaningful reflections of chaotic dynamics.

[2] Here we have done a jump, to construct $d\omega$ we have carried out an integration of ξ as a separate contribution to the infinitesimal displacement dx, as we have seen this is perfectly fine for linear systems due to the applicability of CLT to the convolution term above, for nonlinear systems we can still approximate the evolution operator $U(x)$ with its linearization $DU(x)$ but we will nevertheless end up with an infinitesimal contribution to the displacement that will have a space dependent coefficient. We will skip this difficulty and assume the Wiener Gaussian differential to have a constant coefficient that will have the meaning of temperature; we have just redefined our physics in terms of differentials and used the old (3.2) as a physical starting point but not a mathematical one.

That's why here I've tried to outline its origin and possible modeling, the applicability of the white noise approximation is of crucial importance, we have said that it is founded on a clear and neat separation between two timescales, the gross variables one and the multidimensional hidden dynamics that we interpret as a perturbation of the former. This well distinction, which is widespread in real natural systems, tells us that the discreteness induced by the noise modeling is *physical*, it is a consequence of the huge gap between such dynamics that does exist and it's not an artefact. The fast hidden degrees of freedom make the ensemble evolution (that can be thought physical under certain limitations on the weak interactions that these many body can have between them) to decorrelate faster and forces a quantization on the possible modes whose eigenfunctions have to behave self similarly under the stochastic evolution.
Least but not last, we are interested in real dynamics, and physical sensate systems are always subjected to some sort of fluctuations so our choice to add these stochastic terms has to be viewed as a physical prescription, in the cases where the system under study can be approximated as a physical realization of the ensemble our spectral studies do apply directly. On the fundamental side, we even have that quantum mechanics is deeply rooted on random walks and on a complex kind of stochasticity, the Hamiltonian operators associated to quantum systems share many features with the ones obtained for SDE.

We now want to obtain the ensemble view and distribution evolution for these kind of systems and return to our main problem of studying the spectral properties of the operators involved in the PDEs.

3.2 Fokker-Planck and Kramers equations

We want to find the differential operator that makes evolve our probability distributions. We are no more deterministic and the probability distribution has to embody all possible subsequent realizations of the stochastic noise. Before, we have introduced the ensemble evolution of densities because single trajectories were difficult to integrate but now we are forced to do so, even though our real system will experience a single realization we can compute only probabilities. The shift can be visualized with the following fact: if before points were mapped to points by the trajectory, now points (as initial delta spikes) will be mapped to spreading distributions.

Fokker-Planck Equation

We start by studying how do observables evolve under the flow, this will be enough to define the density evolution. so let $I(x) : M \to \mathbb{R}$ be our state function, by following trajectories we define $I(x_0, t)$: the expectation value of measuring I at time t when the system started at x_0

$$I(x_0, t) = E(\Phi^t(x_0; W(t)))$$

Now, we can use the infinitesimal evolution (3.4) to express the infinitesimal change after dt of $I(x_0, t)$ starting from $(x_0, t = 0)$, we have chosen to use Ito prescription

$$dI = \frac{\partial I}{\partial x_i}(v_i(x_0)dt + d\omega_t) + \frac{1}{2}\frac{\partial^2 I}{\partial x_i \partial x_i}dt \tag{3.5}$$

We have exploited the (3.3) properties and truncated the expansion at the first order in dt. We than average over the realizations to obtain the differential of $I(x_0, t)$.
In this way we can deduce the differential operator governing this initial time variation

$$\frac{\partial I}{\partial t}\Big|_{t=0}(x) = DI(x)$$

$$D = v_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_i}$$

If we than think to start with an initial distribution $\rho_0(x)$ instead of x_0 we can express the mean

$$\bar{I}(t) = \int I(x, t)\rho_0(x)dx = \int I(x')\rho(x', t)dx'$$

where we realize that starting at ρ_0 and observe the system's I mean evolution is the same of making the probability density evolve and compute the average of the time independent original $I(x)$. By using x and x' we are stressing that the first integration is done on the initial measure space (M, ρ), the second on the evolved (M, ρ_t) (these two different specifications are usally called Lagrangian and Eulerian respectively).
Its initial time derivative will be given by

$$\frac{\mathrm{d}\bar{I}}{\mathrm{d}t} = \int DI(x)\rho_0(x)dx$$

We therefore see that its adjoint will, in turn, determine the probability density initial time derivation, dual with respect to space integration scalar product. This

adjoint operator will be valid for all subsequent times since in $\rho(x,t)$ the x is the actual position to evaluate $I(x)$ and not the initial one.
We obtain the Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = D^{\dagger}\rho(x,t) = -\frac{\partial (v_i(x)\rho)}{\partial x_i} + \frac{1}{2}\frac{\partial^2 \rho}{\partial x_i \partial x_i} \tag{3.6}$$

Before continuing I want to comment on the second order nature of (3.6) and the new Laplacian term.
The crucial assumption for this relatively simple equation [3] is the Gaussianity of noise. We have been able to express the whole dynamics without knowing the entire transition kernel $K(x'|x)$ [29] (as in general one should know to write the integro-differential master equation for a general stochastic process) but merely by specifying $v_i(x)$ (we have than assumed to have a state independent noise).[4]

Gaussianity is a universal law, the Central Limit Theorem guarantees that quantities that involve infinite (or very big) summations of random variables will eventually converge to a Gaussian profile, no matter the distribution of single elements. But some hypothesis have to be fulfilled: we need *independent and identically distributed* random elements with *finite variance*. Even though the first two requirements can be weakened, the third is crucial. Finite variance mean local fluctuations, these can be fully characterized by the first two moments thanks to CLT, and Fokker-Planck equation (3.6) does represent a complete description. On the contrary, when we loose locality we are compelled to resort to the general integral form of the master equation.

Internal and External noise

But why am I stressing these difficulties of general Markov processes? We have a Gaussian white noise that allows us to obtain a rigorous equation (3.6), so why should we worry?
The problem resides in the possibility of determining the $v_i(x)$. Only for linear systems we have some phenomenological identification for $v_i(x)$, in these cases we can in fact

[3] With respect to the general Master equation $\frac{\partial P(x,t)}{\partial t} = \int \mathrm{d}x' \left[K(x \mid x')\, P(x',t) - K(x' \mid x)\, P(x,t)\right]$ where the whole kernel $K(x \mid x')$ has to be known.

[4] Indeed the general Kramers-Moyal expansion equivalent to the master equation embodies all subsequent differential orders and in general we have no right to truncate it at the second term. The general treatment of a systematic expansion on a scale parameter can be found in [29], indeed only expanding by powers of something that we know it's small can bring to a motivated truncation. The resulting Fokker-Planck equation will always be linear with this procedure and the author argues that the nonlinear (3.6) cannot be given an easy physical meaning.

write something like $\partial_t \langle x_i \rangle = \langle v_i(x) \rangle = v_i(\langle x \rangle)$ and identify v_i as the macroscopic field. In nonlinear cases, even though we have (3.6), the problem of identifying v remains, it might have some deviations of noise order from the deterministic law.
We will consider only *external noise* sources [29] due to an application of a random force on an otherwise deterministic system, in these cases we can use as v_i the macroscopic laws of the system in isolation. The problem arises when we have internal sources of fluctuations, in these cases the macroscopic law does not correspond to the v_i entering (3.6).
So we have to interpret the results that will follow in the sense of macroscopic dynamics subjected to noise as a result of interactions with noisy environment, otherwise we would lack the meaning of v_i, especially in the case of a physical realization of an ensemble of particles, their internal noise will not be captured here.
In cases when we have to deal with physical systems where noise is introduced naturally (internal), Stratonovich interpretation of stochastic calculus has to be chosen, which uses the midpoint value of the integrand to define the stochastic integral.

The new Laplacian term in (3.6) introduces a relaxation as in the Heat equation, indeed it acts as a low pass filter for irregularities present in ρ, spikes will have a very intense second variation and hence a fast reduction due to the new term. We can understand this behaviour from a random walk point of view: if many realizations are concentrated in a small volume it will be much more probable for them to escape it rather than the fewer option to remain inside.

Kramers Equation

Before considering the spectral properties of such operator, let's introduce a particular case of it coming from our reference physical modeling: Hamilton mechanics.
In this case we take (3.4) but we add the Wiener differential only to the momentum set of coordinates. We believe the external environment to act on our system by some kind of force which is how systems interact in Newtonian mechanics, forces act on momentum and only later this effect will be regularly transported to the positions.
Momentum brings in memory effects and the dynamics complicates a lot. But we have to note that in this way we allow arbitrarily large deviations and no equilibrium will ever be reached. This can be achieved by an additional friction term $-\gamma p$ that will limit the deviations.
At this point we have constructed what is called a Thermal Bath.[a] All of these chit chats can be summarized in the following equation where we have used the

well known Einstein fluctuation-dissipation relation that permits an equilibrium compatible with the Boltzmann distribution.

$$\begin{aligned} dq_i &= \frac{\partial H}{\partial p_i}\, dt \\ dp_i &= -\frac{\partial H}{\partial q_i}\, dt - \gamma p_i\, dt + \sqrt{2\gamma T}\, d\omega_i \end{aligned} \tag{3.7}$$

where T is intended as the temperature of the bath that we suppose state independent as before.
From (3.7) one can obtain the PDE associated, the Kramers equation

$$\frac{\partial \rho}{\partial t} = -\frac{p}{m}\frac{\partial \rho}{\partial q} + V'(q)\frac{\partial \rho}{\partial p} + \gamma\, \frac{\partial}{\partial p}\left(p\rho + \frac{T}{m}\frac{\partial \rho}{\partial p}\right). \tag{3.8}$$

where we have assumed a $H(q,p) = \frac{p^2}{2m} + V(q)$ of the standard form.
Its asymmetric dependence on both q and p makes it not an easy operator to deal with but since we have built the Thermal Bath to make our system relax to Boltzmann distribution we have that for confining potentials with a strong enough divergence to ∞ [b] we can conclude a non degenerate and isolated eigenvalue at zero which entails an exponential relaxation to equilibrium.
Since the operator in (3.8) is neither elliptic nor self-adjoint its treatment is quite involved.

I refer the reader to [1] for further details on the properties of (3.8).

[a] The nature of the fluctuation-dissipation has to be viewed in the environmental dust that keeps on kicking our system, it will experience accelerating fluctuations in the small but if it has an overall macroscopic momentum p, the dust will on average act as a break that kicks it more often in the front.

[b] In [30] some compactness criteria for the resolvent of such Kramers operator are outlined, which suffices for the discreteness of the spectra, in particular there are some results for $|\partial_q V| > C|q|^\alpha$ with $\alpha > 1$

Harmonic Oscillator in a Thermal Bath

As showed in [1], the spectrum of Kramers operator for the particular case of harmonic oscillators can be fully expressed. I will not write here the procedure that involves the creation and destruction operators that one finds even in the quanto mechanical treatment (even if here we have to deal with further complications) but report the final result. In this special case two well distinct scales are in order, the γ friction and the

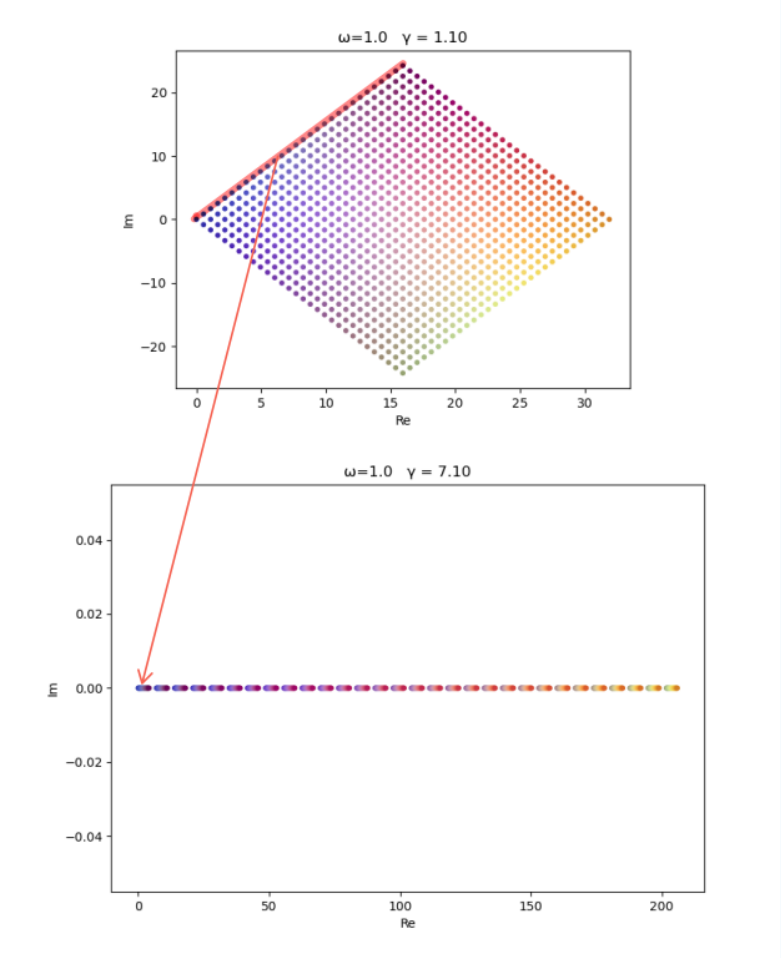
natural frequency ω_0, the spectrum is given by ([1])

$$\lambda_{n_1,n_2} = \tfrac{1}{2}\,\gamma(n_1+n_2) + \tfrac{1}{2}\,\delta(n_1-n_2)\ .$$

with $\delta = \mathrm{i}\sqrt{4\,\omega_0^2-\gamma^2}$.
In the underdamped regime these are complex and form a lattice with imaginary ω_0 and real γ scales.

This extreme case of Hamiltonian intergrability in a thermal bath is a physical starting point but I've opted to limit the subsequent study to the **overdamped** approximation that will allow a special treatment.
In [1] the inverse friction expansion is discussed and from it we might better understand this limiting process.

Harmonic Oscillator: Visual Overdaped Spectrum



Harmonic Oscillator spectrum in complex plane (real horizontal axis and imaginary vertical one), the first plot is for the underdamped case $\omega_0 = 1.0$ $\gamma = 1.10$, whereas the second is the overdamped regime $\omega_0 = 1$ $\gamma = 7.10$. The arrow shows what part of the initial spectrum can be eventually studied at the limit of high friction.

The first image represents λ_{n_1,n_2} for $n_i = 0, ..., 100$ in the underdamped regime. When γ increases and exceeds $2\omega_0$ the spectra becomes completely real and for $\gamma >> 1$ we see that the eigenvalues are organized in packets whose only the first ($n_2 = 0$)

$$\lambda^{\text{ov}}_{n_1,n_2} = \gamma n_2 + \frac{\omega_0^2}{\gamma}(n_1 - n_2)$$

do not diverge as $\gamma \to \infty$. Only these latter will then be studied by our rescaled system.

For general systems the eigenvalue disposition would have been much more involved and not a simple lattice, it's not so clear then what eigenvalues will remain near 0 but still,

when we will pass to the rescaled problem, their physical significance remains as the slow modes of Hamiltonian systems in a high friction thermal bath.

Other studies [31] try to tackle Hamiltonian stochastic evolution by means of the averaging principle (see [10]), they consider only integrable systems with a symplectic-preserving regular noise, and if a division of timescales between the "slow" actions and the "fast" angles can be taken into account, they reduce the dynamics to the former by performing suitable averages on the angles and finally estimate the effective diffusion coefficient of the resulting Fokker-Planck equation in action space by Nekhoroshev estimates.

Since we are interested in the spectral properties of non integrability subjected to noise we have to follow another path. Even though the over-damped approximation will make us land in a stochastic gradient flow dynamics, we will see that some sort of chaoticity can be recovered, but we gain new tools to infer the spectrum, we hope to find the sought after class of systems that allow a clear manifestation of the interrelated structure of chaos.

Overdamped Limit

Let's write this approximation, we consider the momentum as a fast relaxing variable due to high friction, and therefore, the evolution will not experience any acceleration, hence we can set to 0 the second differential equation in (3.7). By performing a rescale of time $\tau = \frac{t}{m\gamma}$ and substitute $p/m = dq$ we obtain a differential equation in q and t decoupled from p

$$dq_i = -\frac{\partial V}{\partial q_i} d\tau + \sqrt{2T} d\omega_\tau \tag{3.9}$$

which corresponds to the following Fokker-Planck PDE

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial q_i}\frac{\partial V}{\partial q_i}\rho + T\frac{\partial^2 \rho}{\partial q_i \partial q_i} \tag{3.10}$$

called Smoluchowski equation.
This new system's spectra will reflect the slow part of the original Kramers one that do not diverge in the limit of high friction. Further work would have to be done to understand wether its statistical properties can have significative meaning to moderate friction regime[a].

[a] Once the Kramers spectra has become almost real, the eigenfunctions can only "choose" to slow down and move towards zero or have a faster relaxation and move towards $-\infty$, one might conjecture that the ones that slow down will preserve their distance ratios since their relaxation

is only spatial and governed by barrier crossing dynamics and little inertia contribution. At sufficiently high friction this is true, the γ is just a multiplicative factor. This means that the spectra statistics found in (3.10) will reflect the ones between the future slow modes even in moderate regime. One can bring the conjecture further and ask wether the diverging ones do organize as momentum modes packets as for the harmonic oscillator, and if the overall statistics can be reconstructed by the superposition of these packets. We have to remember that for general systems there might not be any packet organization.

We are now in front of our searched class of systems. We are going to understand why in the next section.

3.3 A Quantum Twin

(3.10) is describing a continuous random walk with a potential gradient deterministic drift. We have seen in **Chapter 2** that the Liouville spectrum of such systems were completely discrete and obtainable as the positive linear integer combination of the stability eigenvalues the fixed points (see (2.6)). The dynamics were not showing any clue of possible irregularity [5].
By adding the noise, and hence a multidimensional layer of chaos acting at a much faster time scale, what we obtain is a system that has much more variability and importantly it has a **non degenerate zero eigenvalue**. What before might have been a dynamics allowing many attracting fixed point minima with a well defined basin of attraction is no more true, fluctuations will permit some realizations to escape the potential barriers and put in touch the before uncorrelated minima. It is possible to prove that in such situation the unique equilibrium distribution is again the Boltzmann one $e^{-\frac{V}{2T}}$.[6]
The Fokker-Planck operator

$$L_{FP} = T\Delta\rho + \nabla(\rho\nabla V)$$

turns out to be self-adjoint with respect to the equilibrium metric scalar product, or equivalently, we obtain a self-adjoint operator after the change of basis $\rho = e^{-\frac{V}{2T}}\psi$, to stress the quantum analogy that will become clear we even express the noise amplitude

[5] In some cases we might have some fractal structures on the interfaces between two basins of attraction but for non degenerate smooth potentials they remain regular, anyway it would be only a difficulty related to initial conditions and not to dynamical chaos.

[6] This is a result due to the existence of a Lyapunov function for (3.10) dynamics, the free energy; the vanishing of its time derivative in an isolated system uniquely defines the equilibrium distribution to be the Boltzmann one (a detailed balance equilibrium where all the currents vanish and every subsystem is in equilibrium with the others). As a further confirmation we should make an illuminating analogy with Quantum Mechanics that will unlock to us a plethora of new tools.

with $\hbar$

$$2T \leftrightarrow \hbar \tag{3.11}$$

$$\hat{H} = -\hbar e^{\frac{V}{\hbar}} \mathcal{L}_{FP} e^{-\frac{V}{\hbar}} = -\frac{\hbar^2}{2}\Delta + \hat{\Phi}(q) \tag{3.12}$$

where $\hat{\Phi} = \frac{1}{2}(\nabla V)^2 - \frac{\hbar}{2}\nabla^2 V$.
The obtained $\hat{H}$ is a Hermitian operator of the Schroedinger type. We will study this isomorphism between the potential random walk systems and the quantum ones, but for now I will illustrate just couple essential features.
As such, $\hat{H}$ has a real spectrum and in confining regular potentials the spectrum is purely discrete. This is a milestone for our work since it tells us that for all sort of regular $V(q) \to \infty$ for $|q| \to \infty$ we are sure that noise do not break discreteness.[7]

At this point we can start to make out one possible source of chaos, as in Quantum Mechanics (QM) the $\hat{H}$ in its standard form $-\Delta + V$ can even be given a classical meaning thanks to the correspondence principle [8] and the classical Hamiltonian system can of course exhibit chaotic and irregular features. We see in this analogy that the Laplacian term is the counterpart of kinetic energy and is this noise originated term that allows Hamiltonian mechanics to come back in, but in way to be understood.
We have seen in **Chapter 2** that Hamiltonian Liouville $i\{H, \cdot\}$ [9] spectrum were almost always continuous and many difficulties arose in the study of dynamical zeta functions as a consequence of highly non trivial structures at the borderline between regular islands and irregular motion. We had therefore abandoned Hamiltonian systems, but here we had been able to preserve discreteness while allowing some sort of semiclssical limiting notion of chaos that can be understood only in a path integral formulation of both QM and stochastic processes.

Quantum Integrability

We arrive to the conclusion that if we start by a Hamiltonian system in a thermal bath and study it in the overdamped approximation, we end up with a stochastic system described by a quantum analogous Hamiltonian operator with a transformed potential.

[7] I quote [32] for the rigorous results about $-\Delta + V$ Schroedinger like operators by studying the compactness of its resolvent $(H - \lambda I)^{-1}$ and the normalizability of the eigenfunctions, the article shows that the divergence hypothesis in every direction that I've made can even be relaxed.

[8] the applicability and the true nature of this link is very subtle and we will treat it partly in the subsequent chapters, for quantizations that are not rooted on the configuration space we cannot do any such correspondence.

[9] As a note, we see that even in the deterministic case we had self-adjointness for Hamiltonian dynamics, here we recover it by our modeling of noise fluctuations.

For bonding potentials, the confinement property remains true even after the change of measure and the spectrum is made of a countable set of points. Surprisingly this entails quantum integrability [33]. In the article is proved that an n dimensional $\hat{H}$ with a pure point spectrum admits an entire set of commuting operators $[H, O_i] = 0, i = 1, .., n-1$ that define quantum numbers to uniquely specify the eigenfunctions that make up the complete basis. All our systems are therefore quantum integrable but this notion of integrability has not to be confused with the classical one, the inaccuracy of the quantum map

$$Q : f \to \hat{f}$$

that preserves the Lie algebra structure only up to $O(\hbar^2)$

$$[Q(f), Q(g)] = i\hbar\{f, g\} + O(\hbar^2)$$

allows both integrability and irregularity (provided that it is confining, i.e. with only bound states) to be mapped to QM integrability. This is what we will exploit in the next part of this work.

Before continuing to the second part, I want to point out clearly that the subsequent study is necessary and that very little can be said about the spectra statistical properties that these new systems will have from the deterministic gradient flow one. In [2] it is believed that the introduced stochasticity will produce only some movements of T order in the spectral lines, where T quantifies the intensity of fluctuations, and in a continuous way, so that one recovers the deterministic dynamics at the limit of vanishing T. Even if we suppose this conjecture of continuity to be applicable, we still have that for incommensurate stability eigenvalues (even a pair) the ordering of the Liouville eigenvalues for high positive integer linear combinations will be eventually destroyed by any small noise intensity T due to the spacing between successive levels becoming smaller and smaller as some power law $x^{-\alpha}$ with α the number of incommensurate values [10], and as we will see, it's right there that our studies will focus.

This limit has to be taken carefully and cannot be treated so easily to avoid the naive dissipative dynamics, in this context two possible limiting processes seem to be applicable, one of which (the one mentioned by the authors) seems to be singular.
The semiclassical one considers the quantum mechanical systems in the limit of big quantum numbers (or big actions I with respect to $\hbar$, i.e. $I/\hbar \to \infty$) but $\hbar$ never reaches 0, it preserves some quantomechanical features that eventually converge to the geometric (or ray) optics regime. The dynamics turns out to be well described by a *conservative* classical Hamiltonian flow with $H = p^2/2m + V$ whose action evolution is well described by the Hamilton-Jacobi theory in analogy with ray optics, we loose the tunneling effect

[10] I've been been estimating this by taking the density of multidimensional lattices projected to a single axis, something like the Fermi level density estimation for quantum mechanics.

but we preserve the unitarity of the process.
In the other case we end up with a dissipative dynamics at $\hbar = 0$, so we even loose the symplectic structure, furthermore we have an impossibility of a perturbative expansion in $\hbar$ of the level splitting experienced due to the **non** perturbative effect of tunneling effects of $e^{-1/\hbar}$ order, in this sense the spectrum perturbatively jumps to zero without seeing possible weak noise phenomenology that is where we will mainly work.

The issue is even subtler and one has to understand that the semiclassical limit does not merely converge to the classical Hamiltonian one: as it will become clearer later, we always have to consider small but existing tunneling effects due to the general path integral formulation that takes only its major contributions from classical trajectories.

Part II

§§§

3.4 Introduction II: A Spectral Transition

I will follow [5] to introduce the main topic of this part: an observed statistical spectral transition when the regularity of the quantum twin's classical analogue changes.

Starting from the Hermitian transformed operator in (3.12) we want to study its spectra that we know to be discrete and real.
We proceed with a time separation ansatz $\psi(q,t) = e^{-\lambda t/\hbar}\psi(q)$ and we reduce ourselves to the time independent eigen-problem

$$\hat{H}\psi_\lambda(q) = \lambda\psi_\lambda(q) \tag{3.13}$$

We see that the problem is formally identical to the time independent Schroedinger one. Even though Quantum Mechanics (QM) is rooted on a complex field we have that its eigen-problems are completely real, observables are modeled by Hermitian operators with a real spectrum which represents the possible measurement outcomes (and those must be real) and morover, in a confining potential, we can even choose the basis of energy eigenfunctions, in general complex valued wavefunctions, to be completely real. This does not mean that QM has turned real, to build up general states we still need complex linear combinations but our stochastic process and the quantum system described by $\hat{H}$ share the same Hilbert base and the same spectra, but they are built on different fields.

We will exploit quantum semiclassical results that will serve our stochastic overdamped Hamiltonian treatment. As for the spectrum, it doesn't matter what picture we choose to use (QM or Stochastic), but in what follows one has to remember $\hbar$ to represent our original temperature T of the Thermal Bath, which become the diffusion coefficient of the simpler overdamped Fokker-Planck equation and eventually the fundamental scale of this quantum analogy. We will keep to denote it $\hbar$ to stress the quantum prescription. The semiclassical limit $\hbar \to 0$ has therefore to be intended as the weak noise one $T \to 0$, but as we will see the same limit has different classical analogues depending on the picture we choose (i.e. whhether the time is real or imaginary), but nevertheless the spectra remains unaffected.

Problem (3.13) remains too hard for general Hamiltonians, we have to resort to the old semiclassical quantization prescription of early QM. The old approximation in the weak noise limit permits the classical dynamics of $H = p^2/2 + \phi$ to solve (3.13) through the WKB solutions

$$\Psi_\lambda(q) = \sum_a c_a |\nabla S_a|^{-1/2} \exp\left[\frac{i}{\hbar} S_a(q,\lambda)\right] ,$$

where $S_a(q,\lambda)$ are the action solutions of the Hamilton-Jacobi equation with λ energy.

They can be determined as the action integrals[11] $S_a(q,\lambda) = \int p_a dq$ along the classical trajectories of H. The momentum subscript a already tells us its multivalued nature, this will translate in interesting topological problems and is the main difficulty in the irregular case.

Interesting enough, we know that only the small class of integrable systems can be given an exact quantization semiclassical condition, whereas in the non-integrable case we still have to face a huge difficulty. In both cases it's not straightforward to understand how the semiclassical procedure effectively quantizes the possible energies: classically, they are all allowed.

At this point, we begin to understand in what sense we have been able to accomplish a discrete spectra that might reflect some sort of chaoticity. The introduction of Nnoise allows a semiclassical description in terms of classical trajectories of $H = p^2/2 + \phi$, and here we have notions of what chaos is (divergence, periodicity, etc.): momentum, introduced by stochasticity, has turned a gradient flow into a conservative dynamics where chaos (on energy surfaces) and non integrability can be defined.
A natural question now arises: how do the spectral properties change in the transition from regular integrability to maximum irregularity in H?
We still have to choose what quantity to observe in spectra. It has to characterize it in a somewhat global way, as the Lyapunov exponents did for the overall divergence. We will be concerned with regions $I/\hbar \gg 1$ (weak noise/semiclassical limit), all the spectral information we can obtain from the old quantization based on the correspondence principle belongs to very high quantum numbers. There, we have very packed levels and for confining potentials they will accumulate at infinity, making the level density something divergent and not uniform. The density cannot be therefore used as a proper observable since it has little to do with the regularity of the system but with the volume of energy levels, as in the drum problem [17].

We understand that the sought after quantity has to be a statistical feature, the disposition of single levels cannot be a global property shared by all systems with similar irregular strength. So, since the smoothed density $\varrho(E)$ is the zero moment mean value of the level distribution (given by Thomas-Fermi volume formula), it's natural to consider its oscillations around it.

[11] The action given by (4.5) can be rewritten in cotangent bundle T^*M by using $p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$ and the Legandre transform that connects $H(p,q)$ to $\mathcal{L}$. One has that the action 1-form $\mathcal{L}dt$ of kinetic jet bundle, along the solutions of Euler-Langrange equations, can be recast as $\frac{\partial L}{\partial \dot{q}}\dot{q}dt - Hdt = pdq - Hdt$ which is the Poincarè-Cartan form (we have exploited that along the solutions we have $\dot{q} = dq/dt$), since for autonomous systems we have that H is constant along evolutions, the Hdt term doesn't enter any variational treatment and we can simply neglect it.

Level Spacing Statistics

We thus consider the scaled spacing statistics as the probability distribution of spacings between successive energy levels in units of the mean spacing $\bar{S} = \varrho^{-1}$

$$P(S) = P((\lambda_{i+1} - \lambda_i)/\bar{S}).$$

It's far from being obvious that this distribution will ever converge in semiclassical limit: as the mean density, it might suffer system specific features.

Before trying to outline some theoretical arguments it's good to see some numerical tests. The numerical scheme exploited in [5] is based on expanding $\hat{H}$ in a harmonic oscillator basis and truncating the matrix at a suitably large size. One should in general use a basis which is approximately an eigenbasis, this allows us to neglect eigenstates corresponding to energies way far from the region considered for the statistics, but this can only be done in cases where we have a slight perturbation of an easy solvable system, otherwise one cannot even guess the approximate eigenbasis and is left with choosing a standard one. In these situations, knowing when to stop can be tricky, but one can do a convergence check and see whether further size increase modifies significately or not the energies in the considered region.

$$\hat{H} = \sum_{n,n'=0}^{\infty} |n\rangle\langle n|\hat{H}|n'\rangle\langle n'| = \sum_{n,n'=0}^{\infty} H_{nn'}|n\rangle\langle n'|,$$

and diagonalize the truncated Hamiltonian

$$\hat{H}^{(N)} = \sum_{n,n'=0}^{N} H_{nn'}|n\rangle\langle n'|.$$

In the paper they have chosen to use a potential $V_\epsilon = 2x^4 + \frac{3}{5}\,y^4 + \epsilon xy\,(x-y)^2$, two dimensional to permit chaotic behaviour, from $\epsilon = 0$ completely integrable to various non integrable regimes when $\epsilon > 0$.
To be noted that it's not V_ϵ that enters $\hat{H}$ but its transformed $\phi_\epsilon(x, y)$.
The results $P(S)$ for various ϵ are reported in Figure (3.2).

The authors have checked the energy region considered to calculate $P(S)$ was characterized by approximately the same degree of chaos.

We clearly observe a transition in $P(S)$ once the invariant tori of integrable case (a) are destroyed by the non integrable perturbation (some isolated stable islands will anyway be created even in the non integrable regime), the statistics shifts from a **Poisson** distribution $P(S) = e^{-S}$ to what takes the name of **Wigner-Dyson** $P(S) = (\pi S/2)e^{-\pi S^2/4}$,

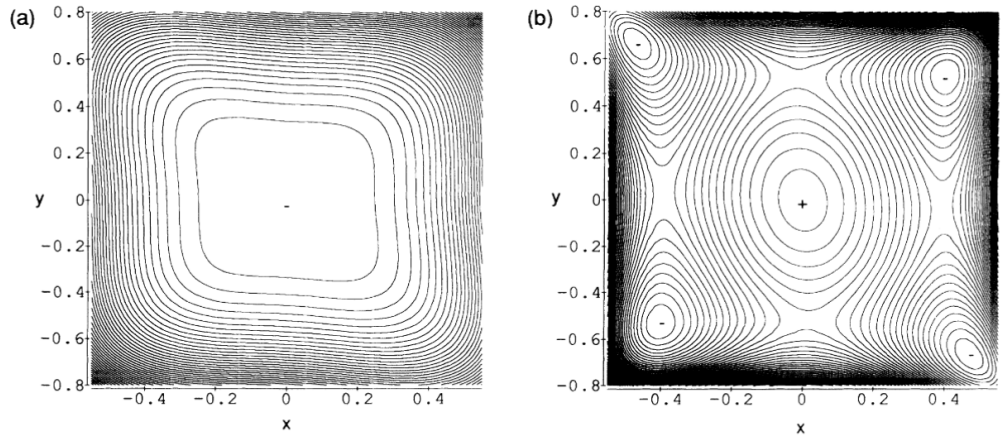


Figure 3.1: Contour plots of the basins of (a) the physical potential V_ϵ, and (b) the noise-dependent transformed potential ϕ_ϵ, for $\epsilon = 0.10$ and the Temperature/Diffusion coefficient $\hbar = 0.2$, which corresponds to case (d) in Figure(3.2). The + and − signs mark the location of the local maxima and minima, respectively. (source: [5]).

which can be related to the spectral properties of random matrix ensembles [34].
For now, these are just qualitative fittings of the numerical data but we begin to realize that irregularity and dense wandering of orbits produces a repulsion between energy levels, indeed irregular $P(S) \to 0$ for $S \to 0$ and an overall shift towards larger gaps is evident.

From these and other similar numerical observations a general trend is observed: *for systems exhibiting chaotic and irregular indecomposable motion (in the sense of our definition) a correlation between levels is observed.*
They do not occur as random variables, on the contrary, in the integrable case, the Poissonian distribution reflects an uncorrelated character of disposition.

Poisson and independence: A Darts Analogy

Think of throwing randomly and uniformly N darts on an interval of the real line $[0, l]$. Independence an uniformity tell us that the probability to have a dart on the infinitesimal interval $[x, x + dx]$ will be independent of x, and will be simply ϱdx, where ϱ is the mean dart density N/l.
Pick a dart, let its position be x, we now calculate the probability to have a s gap in x, to do so, we have to subdivide the interval $[x, x + s]$ in n small pieces and exploit the uniform infinitesimal probability to require that no darts occurred in $[x, x + s]$ and one

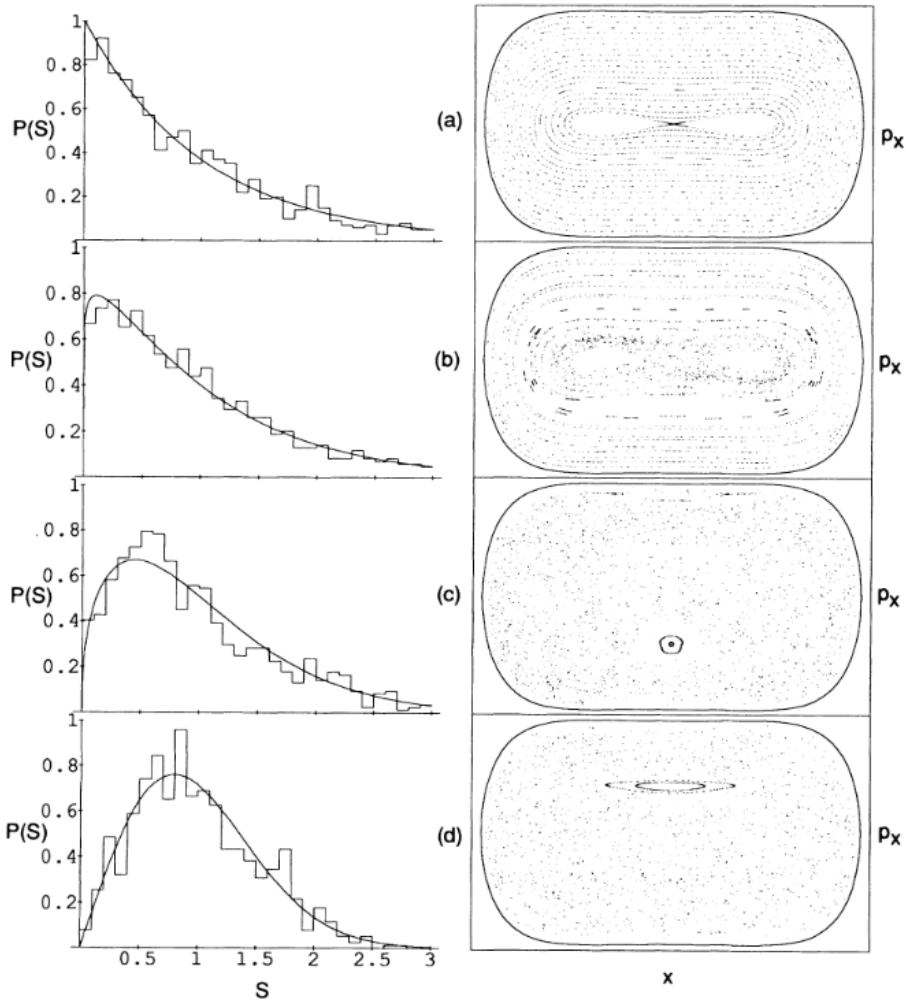


Figure 3.2: Level spacing distributions $P(S)$ and the corresponding Poincare cross sections of the model system with Temperature/Diffusion $\hbar = 0.2$ for (a) $\epsilon = 0.0$ (b)$\epsilon = 0.005$ (c) $\epsilon = 0.04$, (d) $\epsilon = 0.10$. The cross sections are taken for $(y = 0, p_y > 0)$, where $\lambda = 25$. The solid outlines represent the outermost boundary of the $\lambda = 25$ energy surface as projected onto the (x, p_x) plane. (source: [5]).

is found in $[x, x + s + ds]$, by performing the limit $n \to \infty$ we obtain

$$\mathcal{P}(\text{no darts in } [x, x+s]) = \lim_{n\to\infty} \left(1 - \frac{\varrho s}{n}\right)^n = e^{-\varrho s}$$

and by requesting an occurrence in the remaining ds

$$P(s)ds = e^{-\varrho s}\varrho ds$$

and by normalizing in mean spacing units we obtain the Poisson exponential law

$$P(S)dS = e^{-S}dS$$

where $S = s\varrho$.

Wigner and Correlation

On the other hand, we try to consider some sort of correlation between levels to mimic the observed level repulsion in irregular semiclassical dynamics. Start by picking a random dart, let its position be x.
Since our $P(s)ds$ is given as the joint probability of having a dart in $[x+s, x+s+ds]$ (event E_2) and nothing in $[x, x+s]$ (event E_1) we express it as

$$P(s)ds = \mathcal{P}(E_1 \cup E_2) = \mathcal{P}(E_2|E_1)\mathcal{P}(E_1)$$

where we have introduced the *conditioned* probability $\mathcal{P}(E_2|E_1)$, in the uncorrelated case, this is independent of s and is simply ϱds but now we introduce a linear repulsion by

$$\mathcal{P}(E_2|E_1) = \alpha s ds$$

which makes the probability of finding a dart immediately after another to vanish, in this sense the presence of a dart influences the others.
With this specific linear repulsion we have exactly the normalized Wigner-Dyson pdf that seems to fit the globally chaotic data [a] in (d) Fig.(3.2).

Far from being a coincidence, I want to point out that it's exactly the distribution obtained from the GOE (Orthogonal Gaussian Ensemble) eigenvalue spacing statistics in Random Matrix Theory: we might conjecture that the nature of correlation between energy levels of $\hat{H}$ (or decay rates of the associated Fokker-Planck operator) is an independence of matrix entries $H_{ij} = \langle \psi_i | H | \psi_j \rangle$ on a given basis. The choice of the basis is not important since GOE distribution is preserved under orthogonal transformations, using the harmonic oscillator one as a reference should do the job if our chaotic system has no special relation with it[b].

The infinitesimal evolution of a general base ψ_i in ψ_j direction, under the irregular H, seems to have a Gaussian random character

$$(\delta\psi_j)_i = \langle \psi_i | H | \psi_j \rangle \, dt = \mathcal{N}(0, \sigma^2)$$

The crucial thing is independence of the H_{ij}, some results [34] weaken the requirement for H_{ij} to be exactly Gaussian but the joint independence is needed to have GOE spectral statistics. So, essentially we have a dynamics that maps general basis in Gaussian random neighbours, only the exceptional eigen-basis will behave singularly.

From numerical observations we can even deduce something about the eigenfunctions. If we assume GOE conjecture, the following matrix element has to be Gaussian

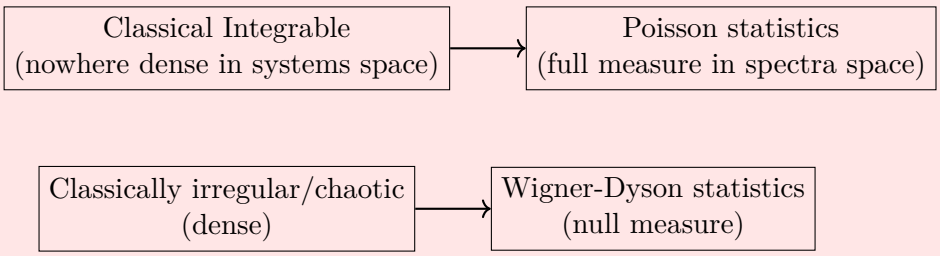
$$\hat{H}\varphi_i = \lambda_i \varphi_i \quad \langle \varphi_i | \varphi_j \rangle = \delta_{ij} \tag{3.14}$$

$$\psi^{(a)} = \xi_i^{(a)} \varphi_i \quad \psi^{(b)} = \xi_i^{(b)} \varphi_i \tag{3.15}$$

$$H_{ab} = \xi_i^{(a)} \xi_i^{(b)} \lambda_i \tag{3.16}$$

and since we have observed a linear repulsion correlation between successive λ_i we need the ξ_i to have an attracting correlation to compensate and preserve the Gaussianity of H_{ab}. From this simple argument we might conclude that for chaotic H we have somewhat coupled energy eigenfunctions φ for successive eigenvalues. This is indeed the case, as we will see, successive levels of integrable systems can have phase-space Wigner distributions (a phase-space generalization) with supports widely separated in phase space, whereas in chaotic case large ergodic regions arise and we will have closer supports and coupling effects.

As a final note I draw the following diagram to display how these characterizations of integrability and irregularity reflect themselves in what we intend as a general system



We see that what is general in H is peculiar in $\hat{H}$, of course the spaces are very different and it's not a deep observation, but highlights the presence of a structured spectra for chaotic systems that do not look random, we have found a space where non integrability plays a distinct role.

[a] We are in an Hamiltonian setting, we still have invariant energy surfaces, chaos has to be intended on these latter.

[b] In the sense that the obtained matrix looks as a representative one, of course the spectral properties are preserved under similarity transforms but a diagonal matrix, which is a possible realization of GOE, does not look generic in the sense of some suitable notion of entropy. If we assume GOE distribution conjecture and take the limit of infinite matrices we eventually reach a spectra with the same asymptotic statistics avoiding possible artefacts present for some finite expansions of chaotic H on a reference basis.

The above observations can be summarized in the following conjectures (the first as

we will see turns out to be a theorem):

Integrable Statistics

The short-range correlations in energy spectra of semi-classical quantum systems which are integrable [a] *in a classical limit obey universal Poisson spacing statistics. Degeneracies are possible.*

[a] Apart from pathological cases where the argument that we will outline do not apply.

Chaotic Statistics

Bohigas, Giannoni, and Schmit (1984) conjecture:
The short-range correlations in energy spectra of semi-classical quantum systems which are strongly chaotic in a classical limit obey universal fluctuation laws based on ensembles of random matrices (GOE [a] *) without free parameters.*
Level crossings (degeneracies) are highly improbable.

[a] In the weaker formulation of the conjecture the class of chaotic systems is technically the Kolmogorov (K) one.

We will have to understand better the influence of classical H dynamics on stochastic quantized evolution of $\hat{H}$; what is the relation between thermal fluctuations and quantomechanical ones? We know they share the same spectra but what is the true link? As for the chaotic regime of H we might wonder if there's a relationship between Lyapunov exponents and Fokker-Planck spectral statistics.

Chapter 4

Integrable Classical Limit

Let's first understand how old quantization techniques permit us to give rigorous support to Poisson statistics conjecture for systems with classical integrable counterpart. Even though our starting focus was Fokker-Planck spectrum and stochastic dynamics, we have seen that the evolution operator is shared by a quantum "twin" described by the same $\hat{H}$. We are therefore allowed to use quatization prescriptions designed for quantum world to obtain such spectral properties for both systems, the difference being in the interpretation of such λ levels: in stochastic systems they represent *decay rates* $e^{-\lambda t}$ whereas in quantum ones they give the frequency of phase oscillations $e^{i\lambda t}$. We will dwell in this difference later on, as we see, the imaginary unit plays a crucial role in the physical effect.

4.1 EBK Quantization

We have already mentioned that high quantal bound states admit an approximate expression based on geometrical optic limit of Schroedinger equation. The following box tries to outline how these approximate solutions arise.

WKB Solutions 1D

(Wentzel–Kramers–Brillouin) one dimensional approximation

We start with the time-independent Schroedinger equation for a particle of mass m in a potential $V(q)$:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(q)}{dq^2} + V(q)\psi(q) = E\psi(q). \tag{4.1}$$

We assume a solution of the form

$$\psi(q) = A(q)\exp\left(\frac{i}{\hbar}S(q)\right), \tag{4.2}$$

where $S(q)$ is a rapidly varying phase and $A(q)$ is a slowly varying amplitude in the sense of the following geometrical optics regime

$$|A(q)'| \ll \left| \frac{S(q)'}{\hbar} A(q) \right|.$$

By substituting expression (4.2) in (4.1) we obtain

$$-\frac{\hbar^2}{2m}\left[\frac{A''(q)}{A(q)} + \frac{2i}{\hbar}\frac{A'(q)}{A(q)}S'(q) - \frac{1}{\hbar^2}(S'(q))^2 + \frac{i}{\hbar}S''(q)\right]\psi(q) + V(q)\psi(q) = E\psi(q). \tag{4.3}$$

We than divide by $\psi(q) \neq 0$ and collect the terms in powers of $\hbar$ and deduce the equivalent system up to $O(\hbar)$. As for the leading Order ($\hbar^0$) we have:

$$\frac{1}{2m}(S'(q))^2 + V(q) = E. \tag{4.4}$$

This is the *Hamilton-Jacobi equation* (H-J) for classical action $S(q)$. $S(q)'$ has therefore to be intended as the conjugated momentum coordinate p and for autonomous systems one has the action integral expression

$$S(q, E) = \int_\gamma p(q')\, dq'. \tag{4.5}$$

Where γ is the classical solution starting at a fixed origin q_0 and ending at q with energy E.
In general, to do so one has to find a complete solution to H-J, i.e. $S(q^k, \alpha_k)$ that depends on N constants α_k, where N is the number of degrees of freedom, here we have only one: energy E.

As for the next order ($\hbar^1$) a calculation leads to the result:

$$A(q) \propto \frac{1}{\sqrt{p(q)}}. \tag{4.6}$$

It confirms the intuitive idea that larger $p(q)$ values will have less contribution due to the little time that the particle passes near q.

In this limit we therefore have an approximate solution given by

$$\psi_E(q) \approx \frac{C}{\sqrt{p(q)}} \exp\left[\frac{i}{\hbar}S(q, E)\right] \tag{4.7}$$

where C is a normalization constant.

We now arrive to the essential point that makes expressions as (4.7) usable only for regular systems. Indeed, to compute the action integrals one needs p_k to be a function of q^k, but we know that in general the system will visit q with many different velocities, p_k will then in general be a multivalued function of the q^k and the integral has to be taken on a topological object that has to take care of this polidromy.
The general WKB solution is a superposition of the multivalued contributions of form (4.7).

$$\Psi_E(\mathbf{q}) = \sum_a c_a |\nabla S_a|^{-1/2} \exp\left[\frac{i}{\hbar} S_a(q^k, E)\right] \tag{4.8}$$

We encounter a problem when motion gets topologically transitive admitting dense orbits on entire energy surfaces or in smaller regions. In these cases the number of possible p values becomes infinite and it's not clear how to take care of infinite summations in (4.8) and regularity requirements that Ψ has to satisfy.

In this chapter we restrict to regular cases which are the classical integrable ones, so that phase space M (in regions foliated by compact invariant sets of dimension N, the numebr of degrees of freedom) can be brought through a canonical transformation f to action-angle coordinates that decomposes the considered region U in $f : U \subset M \to D \times \mathbb{T}^N$, i.e. U is foliated by invariant sets homeomorphic to tori $\mathbb{T}^N$.

Torus Quantization

A somewhat more complete treatment of the following can be found in [7] but here I opted for a personal exposition and derivation.

First of all, where does the quantization happen? In the above semiclassical solution we didn't find any condition to define the allowed discrete energies, it seemed that we could find such solutions for every E.
In general QM, these conditions emerge due to boundary and regularity conditions (think of the infinite potential well where the modes are selected by the vanishing requirement at the boundary) that occur at classical turning points (where $E = V(q)$ that delimits the classically allowed region), but especially there, WKB applicability vacillates being the geometrical optics assumption obviously violated ($p \sim 0$).

We have to introduce something more clever than standard WKB to avoid these turning points difficulties and achieve a suitable quantization condition.

We start by considering the possibility of defining a phase space complex single-valued function $\Psi(q, p; E)$. To do so we start by specifying a torus $\mathbb{T}^N$ by its actions coordinates I_n, it will then have an energy given by $H(I_n) = E$; we than want to associate to $\mathbb{T}^N(I_n)$ a semiclassical solution by finding the multivalued $\psi_a(q)$ on it. We suppose to have a general torus whose frequencies $\nabla_I H = \omega(I)$ do not satisfy any resonance condition $\mathbf{n} \cdot \boldsymbol{\omega} = 0$, we remark that resonances are dense in phase space if $|\partial_I \omega_j| \neq 0$. These toruses have dense orbits on them, so that we cannot reduce them to smaller dimensional invariant sets

Now, we start by defining [a]

$$\Psi(\mathbf{z}=(q^k,p_k);E)=A(\mathbf{z})\exp\left(\frac{i}{\hbar}\int_{\mathbf{z}_0}^{\mathbf{z}}p_k dq^k\right)$$

on a dense $\mathcal{T}\subset\mathbb{T}^N$ given by the *unique* (thanks to non intersection theorem in phase space) phase space trajectory $\Gamma(t)$, solution of the equation of motion starting at $\mathbf{z_0}$ and ending at $\mathbf{z}\in\mathcal{T}$.
We are clearly away from the topological transitive problem, here the density is restricted on the zero measure torus and $p(q)$ can take only few possible values depending on how the torus is projected on q-space.
We now ask whether this $\Psi:\mathbb{T}^N\to\mathbb{C}$ actually exists, indeed we now impose what will result as a quantization condition:

$\Psi:\mathcal{T}\to\mathbb{C}$ *has to be extensible with continuity to the entire* $\mathbb{T}^N$.

It is well known in topology that for this to be true we only need $\Psi:\mathcal{T}\to\mathbb{C}$ to be continuous [b].

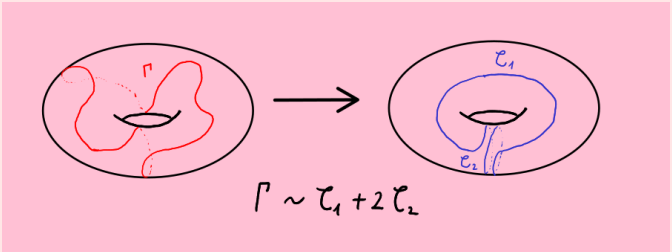
Now, we need a remarkable result: the integration of the canonically invariant form [c] $\omega=\sum_k p_k dq^k$ on a closed loop $\mathcal{C}$ is invariant under continuous deformations of $\mathcal{C}$. This is a consequence of Stokes theorem and the vanishing of ω's exterior derivative restricted to the invariant integrable $\mathbb{T}^N$, indeed one has in action-angle

$$\omega=\sum_k I_k\,d\theta^k$$

$$\mathrm{d}\omega|_{\mathbb{T}^N}=\sum_k \mathrm{d}I_k\wedge\mathrm{d}\theta^k=0$$

since for integrable systems we have no action variation on the foliating invariant tori.

Returning to our continuity extension condition it can only be guaranteed if action integrals along any set of n independent loops $\mathcal{C}_n$ (**no** $\mathcal{C}_i$ can be continuously deformed to any other $\mathcal{C}_j$) is an integer multiple of $2\pi\hbar$, so that the dense orbit Γ that starts at q_1 and eventually passes again near q, say at $q_2\sim q_1$, can be deformed to an approximate sum of various $\mathcal{C}_n$ and yield an overall contribution of approximately $1\sim e^{i(\epsilon+m2\pi)}$ with $\epsilon\to 0$ for $q_2\to q_1$ due to the exact $2\pi\hbar$ multiple phase contributions of every $\mathcal{C}_n$.



We arrive to the conclusion that the sought after condition can be expressed as

$$\oint_{\mathcal{C}_n} \sum_k p_k dq^k = m_n \, 2\pi\hbar \quad m_n \in \mathbb{Z} \tag{4.9}$$

In action-angle coordinates the integrals (scaled by 2π) over these fundamental loops will represent the actions $I_n = m_n\hbar$, the exact condition turns out to be slightly different

$$I_n = (m_n + \alpha_n)\hbar \tag{4.10}$$

$$E_{\mathbf{m}} = H((\mathbf{m} + \boldsymbol{\alpha})\hbar) \quad \mathbf{m} \in \mathbb{Z}^N \tag{4.11}$$

where α_i are not quantum numbers but some constants that take care of the $p_k(q)$ becoming singular at q turning points [d] where one has to pass to momentum representation to smoothly handle these difficulties, the result is a turning contribution represented by α_i (they are called Maslov indexes). With these, condition (4.11) takes the name EKB.

(4.11) is our final result for regular semiclassical energy spectrum.

---

[a] The $A(\mathbf{z})$ are not important for the subsequent reasoning since we are not interested in the exact form of wavefunctions. It doesn't result relevant for the quantization requirement, nevertheless they are given by $|\nabla_{\mathbf{q}} S(\mathbf{z})|$ where $S(\mathbf{z})$ is defined as the action integral.

[b] Indeed all the other requirements are automatically fulfilled: both $M, \mathbb{C}$ are complete metric spaces and being $\mathbb{T}^N$ compact continuity implies uniform continuity.

[c] It was firstly pointed out by Einstein that the quantization prescription should be based on such 1-form to obtain a canonically invariant condition. The earlier Bohr-Sommerfeld theory was coordinate dependent and worked only for separable systems.

[d] In the one dimensional case the invariant tori are just closed loops in phase space, in these cases is evident from the implicit function theorem that at $p = 0$ we have a singular function with $p'(q) \to \infty$.

In the above procedure we are exploiting the periodicity of complex action exponential. By following the trajectory, our solution experiences a phase oscillation $e^{\frac{i}{\hbar}S}$ and when eventually returns near the points previously visited it needs to have almost the same phase to be a continuous function.
This is a condition borrowed from quantum mechanics, it resembles de Broglie explanation for Bohr momentum quantization, only orbits whose wavelength matched the periodicity could exist, the others would immediately break due to a sort of destructive interference with themselves in analogy with geometric optics on a loop. In these cases a single valued reduced action $W_E(q)$ can be defined. When the quantization condition is satisfied one cannot observe any apparent arrow of time, the classical optical evolution will repeat itself again and again as an ideal pendulum. On the contrary if we had a mismatch in the periodicity we would have had an infinite set (if incommensurable ra-

tios) of different $W_E^n(q)$ numbered by the winding number n, so if one imagines to let the wavefronts evolve they will move differently every time allowing us to recognize an evolution and chronologically align the events, this is quantomechanically well captured in the time-energy uncertainty relation where stationary states have no time localization whereas superpositions do.

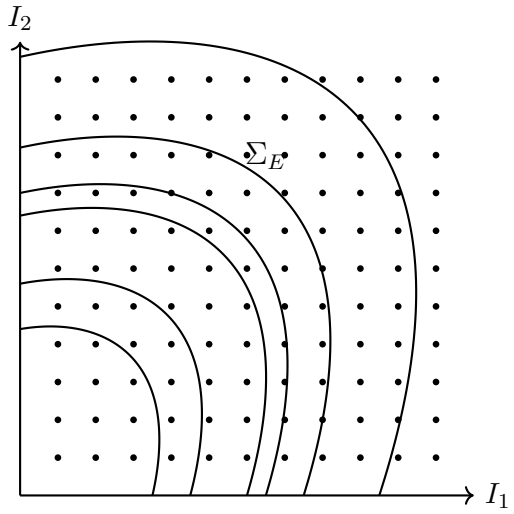
We will see that these considerations taken from both time reversible QM and Hamilton mechanics cannot apply to stochastic systems,however permitted us to find its evolution spectrum but we still have to understand their meaning and whether a genuine stochastic quantization is possible.

But for now our aim is to study $P(S)$ asymptotic spacing statistics of (4.11) and try to prove the Poisson pdf numerically observed.

4.2 A Proof for the Integrable Case

We propose an interpretation and sketch of derivation of the work done by Berry and Tabor [8] who first proved the Poisson semi-classical result for general integrable systems. The authors affirm that the simple-looking e^{-S} result doesn't seem to be obtainable in a more elementary way than the one outlined in the paper, hence the technicalities of some steps cannot be easily bypassed by simpler arguments as the ones I'm going to suggest. In any case they will reflect all the essential features of the rigorous proof.

We start by considering (4.11), we see that (apart from Maslov indexes that do not play any role in the statistics) the allowed energies form an equally spaced lattice in I-space. We wish to study the statistics of the crossing of the energy surface $H(I) = E$ with these points when E is varied.



Now, E is not a good parameter to describe levels, the mean level density $\varrho(E)$ depends on energy. Thomas-Fermi formula tells us that ϱ is equal to the thin shell volume of an energy surface divided by the $\hbar^n$ cube element. Indeed due to the intrinsic quantum uncertainty relations between conjugate coordinates $\Delta p_i \Delta p_j \gtrsim \hbar$ we have that in the whole $2n$ dimensional phase space a quantum state cannot occupy less than approximately $\hbar^n$, so that the number of possible states can be approximated by the volume. [1] As the surface changes by varying E, ϱ cannot be constant and in general will increase due to enlargements of $H(I) = E$.
We have already said that by expressing $P(S)$ fine structure in terms of mean spacing ϱ^{-1} we can obtain a normalization. The authors proceed by crafting a scaled function $U(m)$ as a quantity with uniform average crossing in m-space but still with the original system higher moments.
We instead simply consider some large $E \gg 1$ surface and study the statistics of crossings around it by slightly changing E, we will see that our reasoning will be shape and size independent as the authors conclude after a long calculation.

To perform the semiclassical limit we imagine having $\hbar \ll 1$ and keep of finite size our surface $\Sigma(E) = \{\mathbf{I}/H(I_n) = E\}$ under study. We will consider only **convex** $H(I)$ as a generality requirement but soon we'll understand why. To simplify things further we will consider only a $n = 2$ system that fully captures the meaning of our result. In these cases $\Sigma(E)$ can be parameterized by a single angle θ so that its points are given

[1] This can be viewed as another reason for quantum discrete spectra for Schroedinger like operators, we have previously argued it by citing some technical results about the compactness of operator resolvent thanks to the noise Laplacian, but here it turns out as a consequence of uncertainty relations, ultimately linked to Fourier transform relation between positions and momenta and the wave nature of these latter that need some space to be defined.

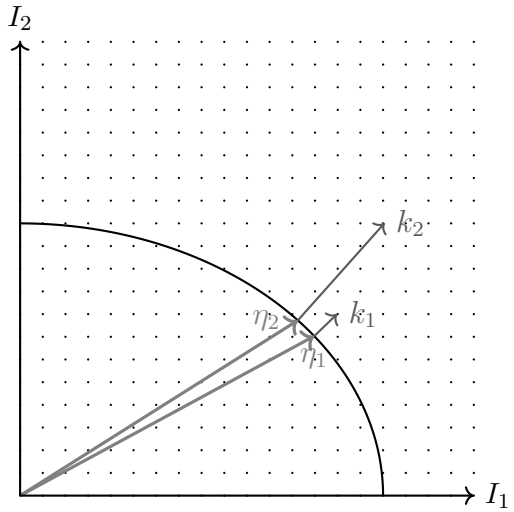
by a vector function $\boldsymbol{\eta}(\theta; E)$.

Now, when we smoothly change E our $\Sigma(E)$ will deform to $\Sigma(E+\delta E)$, we can express the vectorial functional deformation as

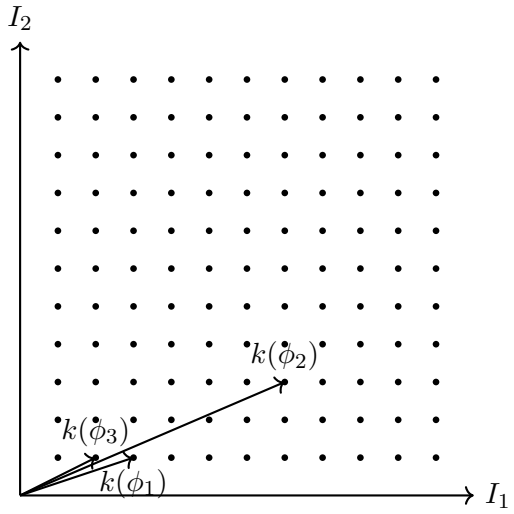
$$\boldsymbol{\delta}\Sigma(\theta) = \frac{\nabla_I H(\boldsymbol{\eta}(\theta))}{||\nabla_I H||^2}\delta E$$

so that we have our velocity of enlargement at $\boldsymbol{\eta}(\theta)$ given by $v(\theta) = \frac{1}{||\nabla H||}$ with direction $\mathbf{n}(\theta) = \frac{\nabla H}{||\nabla H||}$.
We now introduce the length period $k(\theta)$ for crossings of infinitesimal arc elements $\mathrm{d}l$ with lattice points. For $\hbar \ll 1$ even a small energy interval $[E, E + \Delta E]$ will give enough crossings to compute a good statistics, so we assume $v(\theta)$ and $\mathbf{n}(\theta)$ approximately constant in this energy interval.

The essential thing is that $k(\theta)$ is extremely sensitive on θ and it's not even differentiable. We then define $\lambda(\theta) = k(\theta)/v(\theta)$ as the energetic crossing pass at point $\boldsymbol{\eta}(\theta)$.



Thanks to convexity we can even express $\lambda(\theta)$ in terms of angle ϕ that $\mathbf{n}(\theta)$ makes with the horizontal, we than neglect possible initial shifts and define $\lambda(\phi)$ as the length period of lattice crossings of the surface at the unique $\boldsymbol{\eta}(\theta)$ so that $\mathbf{n}(\theta) = (cos(\phi), sin(\phi))$



λ depends on the rationality of $tg(\phi)$ which is a smooth function of a real variable ϕ, we have therefore arbitrarily high irreducible integer fractions $\frac{A}{B}$ in an arbitrarily small $\Delta\phi$ as a consequence of the density of $\mathbb{Q}$ in $\mathbb{R}$. If $tg(\bar{\phi}) = \frac{A}{B}$ we have a periodic energy pass of $\lambda(\bar{\phi}) = \sqrt{A^2 + B^2}$.
If we think about the probability density of rational numbers $\lambda < x$, it should be proportional to x^2 since it scales as the area of a circle [2], so the pdf would be $p(x) \propto x$.
So, on one hand longer passes are favoured but what contribution do they bring to the statistics?
We are now ready for our argument, focus on the probability of finding a crossing in a small ΔE, a point with a specific pass λ will bring a $\Delta E/\lambda$ contribution which is the density of its crossings, so the overall probability to have a level in ΔE is given by

$$\int_{\hbar}^{X} p(x) \cdot \frac{\Delta E}{x} \mathrm{d}x \propto X \Delta E$$

since the probability $p(x)$ is everywhere the same since it's a property of rational numbers.
We have fixed an upper X because we cannot consider passes that exceed our approximation of n, v and ∇H being constant in energy. We have than simply multiplied the probabilities because they are evidently independent, one is a property of the rational lattice and the other is the simple density of λ multiples.
After choosing a suitable constant X, we conclude that the distribution of crossings is therefore uniform. Uniformity implies a Poisson spacing statistics and we have arrived

[2] This is connected to what is called *primitive circle problem*, that is calculating the number of coprime integers (A, B) that satisfy $A^2 + B^2 < x^2$. The probability of coprimality turns out to be a constant $6/\pi^2$ and therefore we have a direct scaling with the area of a disc of radius x, something like $\sim \frac{6}{\pi} x^2$.

to the sought result.

Far from being a rigorous result the heuristics reflects the procedure proposed by [8]. They start by considering the level density of the scaled U

$$\rho(U) = \sum_m \delta(U - U(m))$$

over the integer vectors m and use *Poisson summation formula* to express it as a sum of the Fourier transformed $\tilde{\rho}$ valuated at integer wavevectors $\mathbf{M}$.
At this point every term $\rho_{\mathbf{M}}$ is given by an integral over the $n-1$ dimensional energy surface (the Dirac δ makes the integral to collapse on it). They result to be the integer Fourier components of the shell thickness

$$\rho_{\mathbf{M}}(U) = \int_\Sigma \mathrm{d}^{n-1}\mu \frac{\exp\left(2\pi i \mathbf{M}\cdot\mathbf{m}(\boldsymbol{\mu})\, U^{1/n}\right)}{|\nabla U(\mathbf{m}(\boldsymbol{\mu}))|}.$$

where $\mathbf{M}$ is the integer wavevector and μ the curvilinear coordinates on Σ (the factor U^n is due to a rescaling to the $U(m) = 1$ reference surface, see paper for details).
The crucial step in their reasoning is a *stationary phase approximation* that turns out to be applicable to these integrals, in semiclassical regime $U \gg 1$ the integrands phases oscillate so fast that the overall result gets a non negligible contribution precisely at its stationary points, i.e. at $\mathbf{M}\cdot\nabla_\mu\mathbf{m} = 0$. In our reasoning we had considered the expansion of Σ in its normal directions $\nabla H(\boldsymbol{\eta}(\theta))$, here we find that the stationary point for $\mathbf{M}$ is precisely the $\mathbf{m}(\mu)$ that has $\mathbf{M}$ as surface normal, they reflect $\mathbf{M} \sim \mathbf{n}(\phi)$ and $\mathbf{m}(\mu) \sim \boldsymbol{\eta}(\theta)$ in our considerations.
Later we made probabilistic considerations for a general ϕ, on the other side they have been able, thanks to Poisson formula and stationary approximation, to rigorously reduce all the contributions to *integer* $\mathbf{M}$ normals.

Now, these ρ_M are characterized by a smooth varying amplitude A_M and wavenumber K_M as each component of $\mathbf{M}$ jumps by unity, but their phase ϕ_M jumps many times 2π. For very large U_0 we have that K,A and ϕ are approximately constant for variations $V \ll U_0$, hence for $U = U_0 + V$ they have the form

$$\rho_M(U_0 + V) = A_M(U_0)\exp\left[\mathrm{i}(K_M(U_0)\,\mathrm{V} + \phi_M(U_0))\right],$$

The fast character of ϕ_M with respect to K_M and A_M makes the sum ($\Delta\rho$ because we have subtracted the mean density $\varrho = \rho_0$)

$$\Delta\rho = \sum_{M\neq 0} \rho_M$$

a *random function.*
Think about what if ϕ_M was smoothly dependent on $\mathbf{M}$ integer jumps, we would have obtained that ρ_M itself was so, and the sum $\Delta\rho$ could then be well approximated by an integral $\sum_M \to \int_M$ eventually making $\Delta\rho(V)$ a predictable function. On the contrary, the high variability of successive ϕ_M make the sum essentially a random walk in complex plane (even though it remains a regular function of V but of unknown profile). This ϕ_M unpredictability is the analogous of our discontinuity of λ passes.
But its power spectra can be expressed and from it they can calculate the probability of finding a level in a range $[U_0 + V, U_0 + V + \Delta]$, after performing all the calculations it is obtained the remarkable result that this probability is unity whatever the shape of the contour and the specific $U_0 \gg 1$. We have found the same (non rescaled) constant result as the joint effect of high pass predilection $p(x) \propto x$ and inverse density $d(x) \propto \frac{1}{x}$.
From there it is straightforward to obtain the Poisson level spacing statistics $P(S) = e^{-S}$.

Couple remarks: we have assumed Σ to be convex but didn't emphasize its meaning and importance. The key observation is that we need a surface that "looks" at the lattice from a continuous interval of different angles (as in Euclid's Orchard) so as to give $\lambda(\phi)$

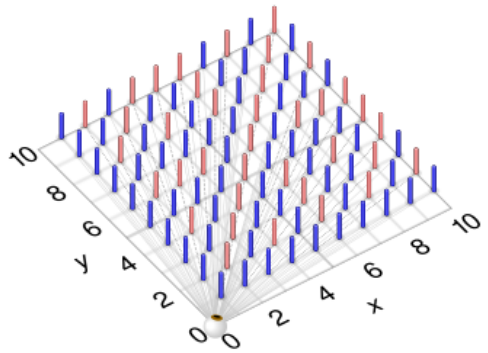


Figure 4.1: Euclid's Orchard, blue lattice points are visible from the origin, they represent our primitive passes where (x, y) are co-primes. (source: Wikipedia)

an extremely variable character.
As for the reasoning carried out by Berry and Tabor it was the condition for the applicability of stationary phase approximations. In the article they show how flat energy surfaces, as the ones of harmonic oscillators, bring to very different statistics, indeed the angle at which their surface elements expand is everywhere the same and the ones that have crossings (many might never cross a lattice point) share the same $\lambda(\phi)$ pass. We see that all the previous arguments are inapplicable (we cannot treat λ as a random function and make probabilistic conclusions), the difference between surface elements being only in initial crossing phases.

We understand that in considering simple systems we are usually comfortable with, has to be done carefully to avoid degenerate cases, even the study previously outlined for harmonic Kramers spectrum cannot be taken as a general result.

This convexity requirement resembles the assumptions of the **Nekhoroshev Theorem** for exponential stability of non integrable perturbations. The argument of Nekhoroshev escape estimates required it (steepness) to avoid fast movements parallel to low resonance surfaces. Hence it seems as convexity brings both perturbation stability and random spectra. By adding a perturbation, in cases of convex energies we are allowed to fulfil the previous study up to exponential times after which tori eigenfunctions cannot be considered as good approximations, but until then, we can think of $P(S)$ as almost Poisson since convexity will be preserved under such small disturbances.
In cases of flat energies, convexity will be dependent on the specific character of perturbations and will change dramatically: the statistics becomes very sensitive.

So, from the lattice surface crossings picture we understand how random nature of integrable spectra comes in: it is due to the independent behaviour of very distant regions in phase space of energy surfaces, the superpositions of all these independent contributions makes up the highly packed Poisson spacing statistics.
As Berry and Tabor observe, this prohibits any possibility of measuring $P(S)$ by exciting a level and observing the emission spectrum produced by its decay since it would fall to some close in phase space eigenstates with $h\omega = \Delta E$ that do not reflect $P(S)$. On the contrary, in chaotic cases, we have already pointed out that every irregular level is expected to be weakly coupled to all other irregular ones with similar energy, making the measurable decay spectra follow $P(S)$ statistics.

Now that we have understood how semi-classical integrability can be associated to Poisson spacing statistics we have to face the main topic: chaotic spectra.

Chapter 5

Chaotic Classical Limit

We now want to study spectra of systems that do not admit torus quantization. In these non integrable cases we end up with dense wandering orbits and no multivalued $S_a(q)$ can be expressed[1]. We need to resort to something else to capture $P(S)$.

When we were dealing with noiseless systems, one of the principal tools to express the spectra without caring about eigenstates were dynamical zeta functions. This technique can still be outlined even quantum mechanics but we have to be careful in what we mean with periodic orbits.

This will only be an overview of these techniques, I've opted to explore other approaches that can show level repulsion in an easier and computationally feasible manner, I will outline them later on. Here we will figure out some difficulties of such expansions that nevertheless remain of important theoretical interest.

5.1 Quantum Trace Formula

One starts by expressing the level density

$$d(E) = \sum_j \delta(E - E_j) = \mathrm{Tr}\delta(E - \hat{H}) \tag{5.1}$$

A semiclassical theory for such $d(E)$ exploiting trace formulas and periodic orbits expansions can be found in one of its main contributors' work, Gutzwiller [9].

[1] In Hamiltonian systems we have to keep in mind that the situation is more complicated, even on single energy surfaces we usually have coexistance of regular and irregular motion, only for fully developed chaotic regimes we should expect most of the surface to be composed by a unique irreducible component.

As we have already done for the integrable case, one separates $d(E) = \varrho(E) + d_{osc}(E)$ in a mean density ϱ and an oscillatory correction.

Now, to expand (5.1) as a trace in space representation we need the quantum propagator. Schroedinger equation can indeed be reformulated as an equation for the propagator function $K(q''t''|q't')$.

$$i\hbar\frac{\partial K}{\partial t''} - \hat{H}K = 0 \tag{5.2}$$

$$\lim_{t''\to t'} K(q''t''|\, q't') = \delta(q'' - q') \tag{5.3}$$

where we have to add the initial condition.
This object is the unitary and complex valued counterpart to Markov stochastic processes propagator $P(q_f t_f|q_0 t_0)$ that we have already seen, indeed it satisfies the equivalent of Chapman-Kologorov equation that allows us to construct a long transition from short ones and eventually allowing a path integral formulation[2].
K can hence be expressed in position representation as

$$K(q''t''|q't') = \langle q'' \mid e^{-iH(t''-t')/\hbar} \mid q'\rangle \tag{5.4}$$

As we have done in **Part I**, we can think of energy levels as the poles of Liouville resolvent trace, in this case the Liouvillian role is played by $\hat{H}$ operator. The resolvent can be expressed by a Fourier transformation of time propagator K in analogy with the real Laplace one for Frobenius-Perron.

$$G(q'', q'; E) = \frac{1}{i\hbar}\int_0^\infty dt K(q''t|q'0)\exp\left[\frac{iEt}{\hbar}\right] \tag{5.5}$$

where we have assumed K to be independent of t but only on its differences due to the autonomy of the systems under study. The integral starts at 0 because we have chosen the *forward* propagator that vanishes for $t < 0$.
Here $G(q'', q'; E)$ is called the Green function which is the spatial representation of $\hat{H}$ resolvent

$$G(q'', q'; E) = \langle q'' \mid \frac{1}{E - \hat{H}} \mid q'\rangle \tag{5.6}$$

G can be interpreted as the solution of the inhomogeneous problem

$$(E - \hat{H})G(q'', q'; E) = \delta(q'' - q') \tag{5.7}$$

that gives sense to the appellative *resolvent*.

[2]In the following chapter we will talk a bit about path integrals but mainly the stochastic ones due to some new phenomena that they can express, for now we will only use some ideas.

In the above (5.5) integral we see that if a stationary solution exists for a given E, K has to describe even the evolution of the corresponding eigenfunction $\psi_E \exp(-iEt/\hbar)$ that satisfies the homogeneous eigen-problem $(E - \hat{H})\psi_E = 0$, in this sense K has to contain an exact resonance at E frequency making (5.5) to diverge.
This can be viewed even as a consequence of (5.6) expression that becomes singular when E matches an $\hat{H}$ eigenvalue, indeed when the homogeneous equation admits solutions the inverse cannot exist and multiple green functions can be defined, one has to choose how to deal with this non uniqueness.

Physical meaning of Green functions

But what is the physical meaning of $G(q'', q'; E)$? We see that it extrapolates from $K(q''t'', q't')$ the energetic E component switching from times to energy representation but to view it better we have to expand (5.5) using (5.4).

Suppose we have an Hamiltonian with a complete set of eigenstates

$$H|n\rangle = E_n|n\rangle$$

$$\sum_n |n\rangle\langle n| = I$$

(5.5) can be defined on the upper complex E plane and analytically continued in the lower part, singularities arise on the real line at energy eigenvalues, to evaluate the integral at divergences we have to shift the poles and choose a prescription to do so, since we have chosen to study the *forward* evolution we are forced to shift them in negative complex plane by adding a small $i\epsilon$ to E.
We then expand (5.5) as

$$G^+(q'', q'; E) = \frac{1}{i\hbar} \sum_n \langle q''|n\rangle \langle n|q'\rangle \int_0^\infty dt\, e^{\frac{i}{\hbar}(E - E_n + i\epsilon)t}. \tag{5.8}$$

where we have called this outgoing Green function G^+.
We are now able to compute the integral and obtain

$$G^+(q'', q'; E) = \sum_n \frac{\langle q''|n\rangle \langle n|q'\rangle}{E - E_n + i\epsilon} \tag{5.9}$$

and by taking its imaginary part [a] we finally have

$$\text{Im } G^+(q'', q'; E) = -\pi \sum_n \delta(E - E_n) \langle q''|n\rangle \langle n|q'\rangle = \langle q''|\delta(E - \hat{H})|q'\rangle. \tag{5.10}$$

We now see clearly that at the $\epsilon \to 0$ limit its imaginary part represents the transition probability from q' to q'' at a given fixed energy E and with no restriction on time.

[a] We use the the Sokhotski–Plemelj formula.

We can now exploit G^+ to expand (5.1) in terms of quantum periodic orbits. From (5.10) we have

$$d(E) = -\frac{1}{\pi}\,\mathrm{Im}\int dq' \lim_{q''\to q'}[G^+(q'',q';E)] \tag{5.11}$$

The semiclassical theory developed in [9] gives an approximation for G^+ as a sum over all *classical* trajectories leading from q' to q'' with energy E. Since we are interested in the asymptotic $P(S)$ statistics we will limit ourselves to this approximation. We have obtained the quantum semiclassical analogue to dynamical zeta functions introduced for noiseless systems. By studying its singularities one might attempt a quantization condition but, as we will see, only some smoothed peaks can be achieved by a finite number of considered orbits.

It turns out that the mean $\varrho(E)$ level density is the result of the dominant direct path $q' \to q''$. This is proportional to energy shell volume divided by $\hbar^N$, the mean spacing is hence of $\hbar^N$ order. The object of interest for our study is on the other side d_{osc} that seems to encapsulate the effect of classical limit regularity or irregularity. We now try to understand the magnitude of these oscillations from ϱ.
To do so, we have to consider all the possible looping back classical paths. The (5.11) formula is expressed in terms of q-space loops but it turns out that only phase space closed ones give non negligible contributions [3]. We now write the *Quantum Trace Formula* (even referred as *Gutzwiller Trace Formula*) in terms of primitive closed orbits p and their repetitions r [6]:

$$d_{osc}(E) = \frac{1}{\hbar}\sum_p \sum_{r=1}^{\infty} \frac{A_{p,r}(E)}{1+l_p/2}\sin\left[\frac{rS_p(E)}{\hbar} + r\alpha_p\right] \tag{5.12}$$

where $S_p(E)$ is the action around the p'th path. If we are not facing a bifurcation we know that the equations of motion depend smoothly on energy and one can use the same indexing for primitive orbits that will only slightly change under small energy shifts and a continuous correspondence can be defined. In this sense $S_p(E)$ is a function of E, when E is varied the p, r contribution will make d_{osc} oscillate with an energy wavelength

$$\Delta E = \frac{h}{r\frac{dS_p}{dE}} = \frac{h}{rT_p}$$

[3] Even though QM cannot be formulated in pq-space, in the semiclassical limit we recover that the periodic backbone resides in phase space; in q-space loops we have that trajectories will not reproduce themselves immediately after the intersection, but will evolve differently, eventually becoming periodic once the initial phase space point is met. This can be thought as the fact that periodic orbit expansion is a Fourier analogy to nonlinear dynamics and as such, takes its contributes only from periodic components that repeat themselves and not to those that get closed by chance and than evolve differently.

where T_p is the period of the p'th primitive orbit [4]. This observation is important in the way that defines the order of these oscillations which results to be $\hbar$, infinitely bigger than the mean scale $\hbar^N$ (in the $\hbar \to 0$ limit). These contributions are therefore visible on much wider scales and one doesn't need $\hbar^N$ resolution to study these looping effects. Even though one wants to try to connect these fluctuations with the chaoticity and stability of underlying classical dynamics we have to remember that it cannot capture $P(S)$ statistics that emerges only at the finest $\hbar^N$ scale where the discretization is visible.

To do so, we have to understand what the terms in (5.12) mean, we have already talked about S_p; α_p is reminiscent of the already cited Maslov indexes that kept track of WKB problems at turning points, where we have a focusing of classical trajectories. We would need to switch to momentum formulation to avoid these singularities at caustics where $p(q)$ cannot be defined but $q(p)$ does, this has an effect of an extra phase that is summarized in the α_p.
More importantly, we have to consider $A_{p,r}$ and l_p. The latter is an integer that measures the degree of isolation: it means that the p'th closed orbit is embedded in a l_p-parameter family of closed orbits. This resembles our considerations for the problem of metastabilities that might arise in Hamiltonian noiseless systems, there we have said that the problem was hard to tackle due to branch cuts arising in dynamical zeta function and continuous spectra arising from them. We were considering coefficients of the form (2.7), here we have something similar

$$\frac{T_p}{\sqrt{|\det(M_p^r - I)|}}$$

the difference being in the square root symptom of complex amplitude interference in QM.
In the cases where the monodromy matrix has a zero eigenvalue we have to integrate out separately the periodic orbit family and this results in an overall effective density measure $A_{p,r}$ for the family.
Anyway, $A_{p,r}$ reflects the better understood behaviour of monodromy matrix: it oscillates with r if the orbit is stable and decays exponentially if the orbit is unstable. In chaotic systems the exponential instability is a direct consequence of global Lyapunov divergence.
We are now starting to see that a global divergence of nearby orbits exponentially fade ($A_{p,r} \propto e^{-\lambda r}$) possible single orbit resonances given by $S_p = 2m\pi\hbar$, when this occur, a level arises only when the system realizes a positive interference between different periodic orbits, and apparently, this is less likely.

[4] The fact that for closed loops $\frac{d}{dE} \oint p(q;E)dq = T$ follows from general classical relations for conservative systems.

One has to take these single orbit arguments with a pinch of salt, as pointed out in [6], taking as energy levels the single orbit singularities is a fallacy.
One can see that by superposing all such levels obtained by every periodic orbit, this would exceed the maximal $\hbar^{-N}$ density! We remember that chaotic systems are characterized by a dense set of primitive periodic orbits. Single orbits contribute as oscillatory clustering with scale ΔE but the determination of individual levels involves the close orbits collectively.

Unfortunately, to spot single levels one has to take into account an exponential number of orbits as $\hbar \to 0$, this makes it computationally infeasible and trace formulas cannot be intended as quantization conditions.

But still, we can affirm that in gross $\hbar$ scale, chaotic and integrable systems show very different clustering. In the latter case families of closed orbits can interfere coherently with longer oscillatory tails so that every torus contributes independently and one can see the Berry and Tabor result illustrated in the previous chapter [8]; in the former case, we need a collective resonance that involves the whole chaotic periodic backbone as a positive interference over different orbits with only the first few repetitions contributing significantly.

Further literature study in this direction may permit to quantify the influence of Lyapunov exponents on this clustering behaviour. We have just mentioned that the $A_{p,r}$ do depend on them and that exponential suppression in r is a consequence of exponential instability, but one may want to understand how Lyapunov divergence influences the overall interference and energy correlations.
Some results [35] seem to confirm that at this $\hbar$ scale one can theoretically calculate some gross quantities (Spectral Form Factors) that behave like the ones obtained from Random Matrix Theory. This does not mean that the same relations hold even at finer scales but still, it's a theoretical argument supporting the connection between RMT and irregular spectra.

We now move to finest scale $\hbar^n$, and reconsider $P(S)$. We present a theoretical argument for the observed initial dip in the statistics, what we will conclude is an initial linearity $P(S) \propto S$ for $S \to 0$ and hence, a repulsion and degeneracy impossibility. However, we will not determine the whole Wigner-Dyson profile.

5.2 Degeneracies

We have seen that in regular cases, where torus quantization is possible, we have a normalized Poisson spacing statistics $P(S) = e^{-S}$, this makes the limit $\lim_{S\to 0} P(S) = 1$ entailing a non zero degeneracy probability. Indeed it's enough to think about how levels appear in the integrable setting: they were due to crossings of the I-space energy surface with torus lattice points, we see that the smooth variation of such surface makes it possible to find multiple crossings at certain energies.
This fact can be rephrased in an equivalent fashion by considering a one parameter family of systems $H(\hat{q}, \hat{p}; A)$, where A is the continuous parameter. If we restrict ourselves on observing just a pair of successive levels, it's highly unlikely that in a specific system $H(\hat{q}, \hat{p}; \bar{A})$ they will result degenerate [5], it's just a consequence of discreteness. But one may wonder whether by varying A it's possible to make the pair under study to cross. In a family of integrable system this is indeed the case and it's just the same argument of before but now, the same energy surface varies smoothly with a continuous change of system.

This evident possibility of arbitrarily close levels is a consequence of the extreme peculiarity of integrable systems and their lattice crossing picture.

But as for general systems with real eigenfunctions (as the bounded ones that we consider in our study), it's not the case: two parameters result to be necessary for decenaracies to take place.
The simple argument can be found in ([10] appendix 10), he uses the inertia ellipsoids as a reference example, here I just reassume the key fact.

General Degeneracies

Consider a quantum system $\hat{H}$ whose Hilbert space $\mathcal{H} = \mathcal{H}_K \oplus \mathcal{H}_D$ is subdivided in two orthogonal subspaces D, K where K is 2-dimensional and the matrix representation of $\hat{H}$ is diagonal on D, this means that we have been able to find a basis for $\mathcal{H}$ composed of eigenstates apart from the two spanning K, we refer to these latter as ψ_1, ψ_2. We are therefore left with only a 2×2 real symmetric block which is not diagonal and we will focus on the possibility of degeneracies of its eigenvalues. Since we are considering systems admitting purely real basis we write it as

$$\hat{H}|_K = \begin{pmatrix} x & \xi \\ \xi & y \end{pmatrix}$$

[5] Here we are considering $\hat{H}$ without any symmetry that would imply an entire class of degeneracies, we want to study the occurrence of isolated ones.

where $\xi = \langle\psi_1|\hat{H}|\psi_2\rangle$, $x = \langle\psi_1|\hat{H}|\psi_1\rangle$ and $y = \langle\psi_2|\hat{H}|\psi_2\rangle$ are real values.

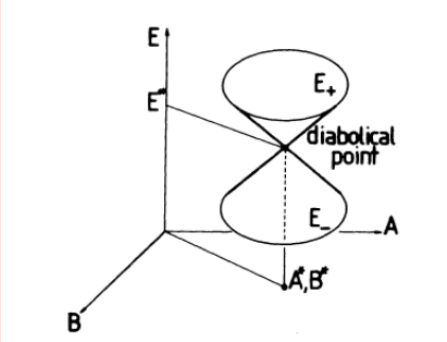
For a degeneracy to take place we need the matrix to be similar to a multiple of the identity, the only way to achieve it is to fulfil these *two* requirements

$$x = y \quad \xi = 0$$

and in general if we consider a one parameter family of such systems $H(\hat{q}, \hat{p}; A)$ it's not possible to satisfy both by varying only A.[a]
In general, we are going to need a 2 parameter family of systems $H(\hat{q}, \hat{p}; A, B)$ to do so. To have a clearer view of what we are saying, think of the two surfaces that the pair of eigenvalues under study will draw by varying A and B. These $E_{\pm}(A, B)$ in (E,A,B) space will form a double cone with a contact point for a specific (A^*, B^*) that realize the degeneracy.

$$E_{\pm}(A, B) = \frac{(x + y) \pm \sqrt{(x - y)^2 + 4\xi^2}}{2}$$

with $x = x(A, B)$, $y = y(A, B)$, $\xi = \xi(A, B)$.



[a]The purpose of keeping a 2×2 non diagonal block was to make evident the necessity of *two* conditions, if one starts with a diagonal block he/she may think that it suffice to have $x = y$, but by varying the parameter the system will couple ψ_1 and ψ_2 making it no more diagonal.

Now, we hypothesize that a one-parameter family of non-symmetric classically chaotic systems will be typical, the crossing point will almost surely be missed and the eigenvalue pair will draw two branches of a hyperbola.

We now want to deduce something for $P(S)$ from these observations. We consider a generic Hamiltonian $\hat{H}$ (irregular), suppose we are given the spectrum and its eigenbasis. Now, take randomly a pair of successive eigenvalues, we almost surely choose a non degenerate pair in both regular and irregular spectra.
We have to find a way to embed our H in suitable 2-parameter family of systems that

preserves the subdivision $\mathcal{H} = \mathcal{H}_D \oplus \mathcal{H}_K$ (with K 2-dimensional), the existence of such family it's not obvious to me.
We have to be careful in what we call generic and how to intend these parametric families.[6] We can always construct such embedding by making only the matrix elements H^K_{kl} , corresponding to $\langle\psi_k|H|\psi_l\rangle$ with $\psi_k, \psi_l \in K$, to be dependent on the introduced parameters. In this way we are crafting the parametric dependence of the system's matrix representation by surgically modify specific state couplings.
Besides being artificial, in this way it's not clear whether we still have a classical counterpart, if we shift to general quantum systems not constructed from classical analogues $H(q,p)$ we loose our chaoticity concept as a semiclassical Hamiltonian limit.
We should therefore look for a family of non integrable classical systems $H(q,p;A,B)$ which have a quantum correspondent $H(\hat{q},\hat{p};A,B)$ that preserves the direct sum scomposition, allowing us to focus on K 2-dimensional subspace.
Asserting the existence of such family does not seem to be straightforward and I limit myself to an encouraging observation: for general ergodic Hamiltonian systems (in some energy surface regions at least) the topological transitivity implies an irreducibility of the classical flow making a possible energy eigenstate [7] to be widespread on the whole region as the wandering classical trajectories bring everywhere non vanishing probability even for quantum measurement localization; we have no geometrical constraints as the ones in integrable cases, where different regions of the same energy surface can have their own torus-eigenfunctions [8]. This entails that they can couple almost only with the next-to energy neighbour, making the existence of such approximate (in the sense that the statistics is not destroyed) family more plausible since one does have to care only about close-to-diagonal elements to preserve the direct sum [9].

Anyway, when we are allowed to do so, our randomly chosen pair of successive eigenvalues will draw a diabolo as the parameters are varied.
Our system will correspond to a specific pair $(\bar{A},\bar{B})$, if we consider the line in (E,A,B) space with $A = \bar{A}$ and $B = \bar{B}$ it will cross the diabolo shape at eigenvalues $E_\pm$ corresponding to our system K subspace.
Now, $P(S)$ can be thought as the probability distribution of spacing between two successive randomly taken levels. If one instead of sampling these successive pairs from the energy line does from (A,B) space, he/she should obtain the same result since we are

[6] Remember the diagram I've drew in the introduction to this **Part II** (at the end of *Wigner and Correlation* box)

[7] The treatment of phase space wavefunctions is delicate and here we are referring to it as an intuitive idea.

[8] This as an additional confirmation of the higher packing of classically integrable spectrum, the same energy surface can embody many more eigenfunctions. We are always considering some intrinsic thickening due to the uncertainty principle.

[9] We have already seen this when we talked about the measurability of such spectra.

averaging over a family of systems with the same chaotic behaviour ([6]).
For very small S, the the probability of S spacing comes from regions near the diabolical point, more specifically it will be given by the intersection of two parallel energy planes separated by S with the diabolo. This length of intersection (probability measure) will be always proportional to S, whatever the ellipticity, angle and orientation of the double cone, in definitive this implies a linear law of the form

$$P(S) \propto S \quad S \to 0$$

We have been able to provide a theoretical argument for the distinction of these two classes of systems based on $\lim_{S\to 0} P(S)$, but to avoid misconceptions, the following remark underlines how this spectral correspondence has to be intended.

Remark: Ambiguous Relation

What we have heuristically proved is that non integrability and developed irregular motion almost surely implies $\lim_{S\to 0} P(S) = 0$*, but the converse is not necessarily true. Level repulsion is not something limited to chaotic behaviour but embodies broader sets of systems that are partially integrable or exhibit* ***complexity*** *rather than chaos as we defined it*[a].

RMT theory does therefore cover a wider class of systems and one cannot use spectra statistics alone to discern underlying classical chaoticity, but nevertheless it highlights possible transitions to more correlated global behaviours.

[a] In any case I expect noise/quantum tunnelling non perturbative effects to provoke some splittings of order $e^{-1/\hbar^2}$ that make the vanishing $P(S) \to 0$ for $S \to 0$ a universal law. Only in the rare cases of *separability* we have an exact quantum correspondence that preserves separability even in the quantum world, this entails quantum independence and a true Poisson character down to vanishing S.

These are all semiclassical considerations and as such, can be true only for perturbative spectral studies. At exponentially small scales $e^{-k/\hbar}$ in $\hbar \to 0$ one faces tunneling phenomena that will make torus quantization just an approximation: small overlaps between eigenfunctions will arise causing a tiny repulsion that will eventually bring $P(S)$ to vanish for $S \ll e^{-k/\hbar}$. Only when the torus quantization can be made exact the semiclassical result will be quanomechanically correct even at the finest scale, these are the cases when the system results *separable* in the coordinates allowing an exact quantization map

$$\{f, H\} = 0 \implies [\hat{f}, \hat{H}] = 0$$

This implies that tunnelling overlaps will be prohibited by the exact commutation vanishing relations that make semiclassical eigenfunctions a basis.

From trace formula (5.11) we have intuited a repulsion realized by the irreducible nature of dense orbits, but this refers only to classically allowed dynamics. We would like to understand the effect of non perturbative couplings and how these can be integrated in periodic orbit expansion.

Level repulsion produced by non integrable perturbations and tunnelling couplings can be illustrated via diabolical means by using one of the two parameters as coupling coefficient and studying how the remaining one-family of systems will draw a more and more separated hyperbola. We will outline a very simple example that will try to capture its essential features.
The observed statistics is not always a result of just classical underlying chaos. We can consider barrier crossing quantum phenomenology as another face of chaotic behaviour that permits transitivity even between classically separated regions, this further suggests that irregular wavefunctions will cover entire energy surfaces and not merely some ergodic components as suggested by the classical limit, in this way even some geometrical constraints (invariant manifolds) can be bypassed by tunnelling means.

Interesting enough, it turns out that to study barrier crossing phenomena the suitable setting are the old stochastic processes we have started with. They will permit us to express quantal tunnelling probabilities and will enlighten the duality between stochastic and quantum twins.

Chapter 6

Tunnelling Couplings

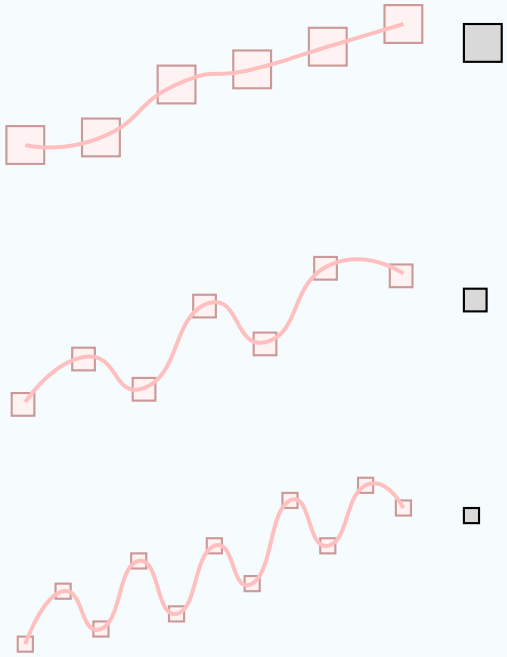
6.1 Stochastic and Quantum Evolution

This chapters tries to address the connection between the two possible physical interpretations that we have given to the same spectrum: the quantum unitary evolution and the stochastic relaxation to equilibrium.
We have seen that the origin of such possible duality arises as a consequence of Hermitian Fokker-Planck operators reality with respect to Boltzmann equilibrium measure. Indeed, the starting gradient flow dynamics, thanks to noise, became governed by a Schroedinger like operator with a transformed potential.

Even though in QM one cannot study single trajectories, in the following supplement we see how a forced localization brings to stochastic phenomenology.

Trajectories and Wavepackets

In QM we have the well known Heisemberg uncertainty relation between conjugate coordinated q^i, p_i, this results in an impossibility of defining a probability distribution in phase space. By this I mean that canonically conjugate coordinates when repetitively measured will not make the other converge, when the position is frequently sampled the path becomes more and more irregular.
In this way the observed trajectory at fine spatial resolution will have an irregular and non differentiable behaviour that resembles a Wiener process. They are very distinct evolutions but under a sequence of measurements QM might acquire a stochastic phenomenology.
We want to better formalize these ideas.



Incompatibility of position and momentum measurements.
Pink squares represent τ sampled grid data.
The small squares on the right hand side indicate the grid space resolution.

Think of a quantum particle that evolves, as time passes we will try to measure its positions through a measuring grid with Δx resolution; the result is something analogous to symbolic dynamics we have introduced as a reduction attempt for dynamical systems. In the present case, we don't have to care about the real hidden deterministic dynamics since we have none.
We proceed by setting a sample frequency $f = 1/\tau$ so that, every τ seconds, our measuring apparatus localizes the particle in one Δx, the new and essential thing is that now the physical state collapses in that portion of space and assumes the profile of its characteristic function or something peaked in it.
Anyway, the process is not Markovian in general, momentum introduces memory, Markovianity will be proportional to $1/\Delta x$: only when we have small enough Δx and consequently large $\Delta p = \hbar/\Delta x$, momentum uncertainty becomes so high that it cannot play any informative role.

We are going to use Gaussian wave-packets as solutions for both quantum and stochastic evolution between successive measurements, their importance resides in the minimization of position-momentum uncertainties (we have $\sigma_x \sigma_p = \frac{\hbar}{2}$) and their complete characterization in terms of their first two moments, which are the physical quantities under interest in what follows.
As for the 1-moment we know thanks to Ehrenfest theorem that the quantum mean

evolution of p and q will follow classical Hamiltonian trajectories.

$$\frac{d}{dt}\langle x\rangle = \frac{\langle p\rangle}{m}, \quad \frac{d}{dt}\langle p\rangle = -\langle V'(x)\rangle,$$

Beyond this classical correspondence, as for the second moment Gaussian packets exhibits quantum spreading due to the nonlinear dispersion relation. After the small τ before the next measurement and collapse, the packet will have a spread given by [a]

$$\sqrt{\tau\sigma_p}$$

The momentum uncertainty governs the spreading. We immediately see an equivalence with the spreading velocity of the stochastic heat Gaussian solution, $\sqrt{2Tdt}$ where T is the temperature/strength of noise.
But we have to remember that they represent quite different objects, the heat solution is a probability distribution whereas the quantum wavefunction cannot be interpreted as such, only $|\psi|^2$ does; additionally, in this latter case the spreading is a consequence of oscillating waves with different velocities whereas in the former we have a real relaxation of different modes superposed without interference.
So it's remarkable that these two processes cause the same spreading rate $\sqrt{t}$

$$\sqrt{\frac{\hbar\tau}{\sigma_x}} \leftrightarrow \sqrt{T\tau}$$

We now consider $|\psi|^2$ which can be given a probability meaning. Its spreading results to be governed by $\frac{\hbar\tau}{\sigma_x}$, we see that the order of initial dispersion is infinitely slower with respect to the stochastic one.
But we have the freedom to choose any spatial resolution Δx for our apparatus to increase the dispersion: by setting the resolution to $\Delta x = \sqrt{\tau}$ we end up with a spread of $\hbar\sqrt{\tau}$.

Suppose we don't have any potential and deterministic average evolution, the result is that at a certain spatial scale, expressible in terms of the sample frequency, we obtain a process with a coarse grained trajectory that can be modeled by a stochastic differential equation since the spreading has the same order of the Wiener differential. [b]

We have been able to force a stochastic evolution of our quantum system by repetitive measurements, these are somewhat perturbing the system and eventually have an effect equivalent to noise, but this is made possible only by the relation existing between spreading rates. We are now going to see that this is ultimately a consequence of the existence of a complex time rotation connecting the two processes.

[a] This simply follows from the Gaussian solution of Schroedinger equation.

[b] In presence of a potential we have to be careful about treating the momentum induced memory, and how to intend the possible deterministic drift in the resulting SDE.

Let's now look at the PDEs.

$$\text{(Fokker-Planck)} \quad -\frac{\partial \psi}{\partial t} = \hat{H}\psi \tag{6.1}$$

$$\text{(Schroedinger)} \quad i\frac{\partial \psi}{\partial t} = \hat{H}\psi \tag{6.2}$$

with $\hat{H} = -\Delta + \hat{\Phi}$. (Fokker-Planck equation is written after the change of measure).
We observe that by switching to imaginary time we can pass from one to the other. The transformation $t = -i\tau$ is called a **Wick Rotation** which represents a $\pi/2$ rotation in complex time plane, here we are directly jumping from the real axis to the imaginary one.
One might think of a general evolution at complex times $z \in \mathbb{C}$, we see that such evolution would embody both (6.2). We suppose [1] the analytic continuation from real times to define the whole $\psi(z, \mathbf{q})$, we therefore see that in this sense, one evolution uniquely defines the other.

We would like to understand what are the implications for the semi-classical limit on one hand and the weak noise one on the other. To do so, we have to study the propagator kernels.

Stochastic Propagator

White noise modeling as a zero memory limit makes the process Markovian. The transition probability $P(q', t'|q, t)$ (propagator) depends only on current state (q, t) and fulfils the *Chapman-Kolmogorov equation* that permits the construction of a transition rate from intermediate ones

$$P(q_f, t_f \mid q_0, t_0) \;=\; \int_M dq_i P(q_f, t_f \mid q_i, t_i) P(q_i, t_i \mid q_0, t_0).$$

Where (q_i, t_i) is an intermediate step at t_i integrated over the whole space.
The propagator fully characterizes the evolution. And the above formula permits a path-dependent representation of the propagator by subdividing the time interval in N steps and performing an integration over all possible intermediate positions.

$$P(q_f, t_f \mid q_0, t_0) = \int_M \cdots \int_M dq_1 dq_2 \ldots dq_{N-1}\; P(q_f, t_f \mid q_{N-1}, t_{N-1}) \ldots$$
$$\ldots\; P(q_2, t_2 \mid q_1, t_1)\; P(q_1, t_1 \mid q_0, t_0)$$

It then takes the meaning of integration over all possible paths.
To integrate over paths instead of points we have to define a suitable probability

[1] In general one has to deal with possible singularities and branch cuts but we suppose to be able to treat them in a physically meaningful way and define a unique $\psi(z, \mathbf{q})$.

measure for these. To do so, one calculates what is called the short time propagator $P(q_0 + dq, t + dt \mid q_0, t)$, this has an easy explicit expression since we suppose that for such small steps one has a unique possible path, which can be thought as the probability to have $\xi = \frac{dq}{dt} + \nabla V$ outcome from the Gaussian white noise that models our stochastic process $\frac{dq}{dt} = -\nabla V + \sqrt{T}\xi(t)$; the probability of the whole trajectory $P[q(t)]$ will be the product of the single steps.
We can therefore express it as

$$P[q(t)] \sim \exp\left[-\frac{1}{2T} \int_{t_0}^{t_f} (\dot{q} + \nabla V)^2 dt \right]$$

The $(\dot{q} + \nabla V)^2$ term at the exponential can be expanded and the cross term turns out to be a total derivative that pulls out an $\exp(-\Delta V/T)$ term; what remains is an action integral over a Lagrangian of the form $\mathcal{L}_F = \dot{q}^2 + (\nabla V)^2$. We called this integral an *action* because by minimizing it, one finds the most probable paths that allow a given transition, they represent the main contributions to the overall propagator.

$$P(q_f, t_f | q_0, t_0) = \exp\left(-\frac{\Delta V}{T}\right) \int \mathcal{D}q \exp\left[-\frac{1}{2T} \int_{t_0}^{t_f} \left(\dot{q}^2 + |\nabla V|^2\right) \mathrm{d}t\right]$$

where we have denoted with $\mathcal{D}q$ the integration over paths. [a]
In this picture we immediately see how the $\exp(-2\Delta V)$ coefficient that do not enter the variational study does give an arrow of time: if we consider the time reversal trajectory, we have that $\mathcal{L}_F$ is invariant under such transformation but the ΔV flips sign so that positive ΔV trajectory will be exponentially less likely than its reverse.

So, in the *weak noise* limit $T \to 0$ we have this path integral transition probability $P(q_f, t_f | q_0, t_0)$ dominated by action-minimizing paths.
We therefore conclude that *most probable stochastic paths in V* are obtained by *Hamilton classical dynamics in $H_S = \dot{q}^2 - |\nabla V|^2$*.

[a] Smooth paths are of zero measure in space of stochastic paths; really we are considering trajectories that lie in a tube around them.

Even quantum mechanics admits a path integral formulation, here we cannot treat it well, but we'll rather comment some of its results.
Even though WKB sometimes permits to find ψ_E eigenfunctions, to do so, we need an entire set of trajectories running on invariant tori. In these cases the geometric optics reduction of Schroedinger equation looks for a $\psi = Ae^{\frac{i}{\hbar}S}$ solutions with action

S satisfying the classical Hamilton-Jacobi equation, but for the general propagator we need a path integral over all possible trajectories and not merely the classical ones, we quote the result without proving it [2]

$$\langle q_f, t_f | q_0, t_0 \rangle = \int \mathcal{D}q \exp \frac{i}{\hbar} \int_{t_0}^{t_f} \left(\dot{q}^2 - |\nabla V|^2 \right) \mathrm{d}t$$

Anyway, the main contribution still arrives from stationary action integrals of classical trajectories, but these cannot capture tunnelling phenomena.
Unfortunately, once one considers the complete quantum path integral he/she encounters convergence problems that do not allow a rigorous mathematical formulation. We are not going to show here how a sigma additive measure cannot exist for oscillating action integrals but we underline that to do so, one needs exponential suppression as in the stochastic case.

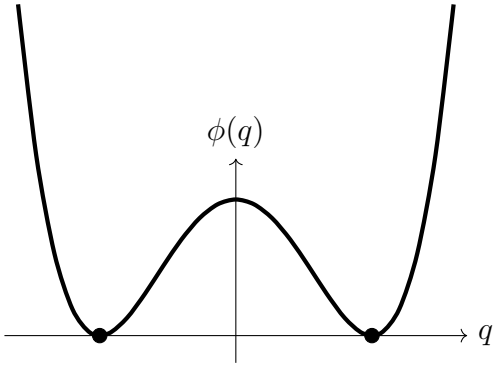
In torus quantization we have exploited never dying oscillation of action phases to force the 2π exact periodicity, in the new path integral quantomechanical picture, every other case would have resulted in a destructive interference.
The difference between how path contributions are added to form the overall propagator is huge: stochastically, one is always adding positive real contributions, exponentially suppressed for long actions, whereas in the QM case, phasors can interfere and no suppression is involved. In weak noise limit we have for both cases that the main contribution arrives from stationary action paths but these result to be described by a *different* Hamiltonian.

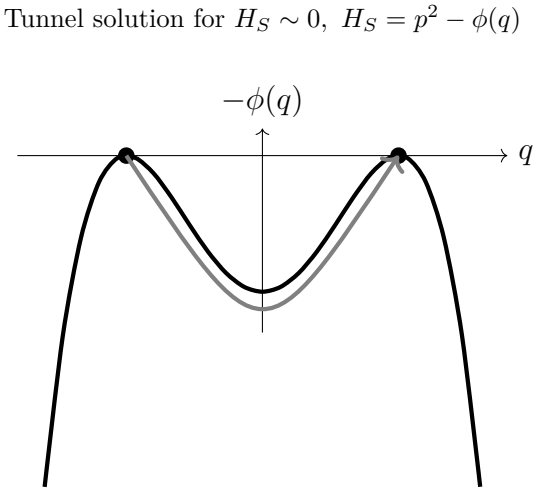
We will now shift to the Wick rotated view where non perturbative phenomena probabilities can be expressed. We have to note that in this switch the Hamiltonian H_S describing the main contributions has a flipped sign potential. *What before was a barrier has now become an accessible hollow*, thanks to this, it can capture classically prohibited transitions.

Remember the contour plots of the basins of noise-dependent transformed potential ϕ_ϵ illustrated in **Part II** introduction, by varying the coupling ϵ parameter we eventually reach chaoticity in some phase space regions with enough energy. At the same time, we now observe that a nontrivial potential landscape is realized, with multiple minima, saddles and maxima. This entails a much more involved tunnelling phenomenology described by stochastic Hamiltonian $H_S = p^2 - \phi_\epsilon$. *One may ask what effect does chaoticity in H_S cause to spectra statistics.* We have always considered the regularity of imaginary time semiclassical trajectories given by $H = p^2 + \phi_\epsilon$ and never considered the low energy non perturbative weak noise limit H_S dynamics.

[2] This is often called the *Feynman Path Integral* .

We now want to understand the importance of these most probable stochastic paths for our study of spectra statistics.





6.2 Tunnelling Level Repulsion

By adding noise to a gradient dissipative dynamics we have understood that it is possible to study its spectra with quantum techniques. The quantum twin system permits barrier tunneling probabilities that cannot be easily captured by its path integral formulation, but after a Wick rotation to our old stochastic process, these noise induced transitions can be studied by a new H_S Hamiltonian dynamics.
We want to show that these tunnelling effects put into contact different previously separated regions of the potential landscape, and by this, they can introduce level repulsion in spectra.

The existence of a unique equilibrium distribution (non degeneracy of the zero eigenvalue) was a result of such transitions. The old $\dot{q} = -\nabla V$ dynamics had a spectrum given by the superposition of all the stationary points linear combinations of stability exponents and we had a zero eigenvalue for each of them whereas now, this degeneracy has been destroyed by noise induced communication between previously separated local systems.
As in integrable cases, separate systems would instead produce a superposition of independent spectra with possibly high clustering.

These tunneling probabilities represent off diagonal contributions to the overall Hamiltonian matrix represented on a noiseless basis. To illustrate this, we start by using some completely solvable quantum systems separated by an infinite well, later relaxed, allowing more and more transitions and consequently an overlap between the new evolutions.

In [36] the authors try to study the transition that one observes in the spectrum when isolated quantum systems become coupled this way. The tunnelling rate should be calculated by the real action integral, but here we are crafting an idealized Dirac δ barrier with an adjustable transition probability that does not follow from physical potentials.

A Simple Tunnelling System

Both ergodic classical overlap and quantum/stochastic barrier penetration bring to couplings between regions of phase space. These couplings can be viewed as interactions between some otherwise isolated systems, if we bring this to an idealization we can start with two completely independent classical and quantum dynamics and introduce some couplings between them in a schematic and oversimplifying manner, that is by a network abstraction. It is analogous as what we were trying to do for the infinite dimensional probabilistic description governed by the Frobenius-Perron operator: when a Markov partition was possible, we were allowed to reduce the dynamics to a finite dimensional symbolic representation described by a finite matrix evolution master equation (Supplement box *Markov Approximation*). Matrix elements were representing transition probabilities for the Markov stochastic process.
Here the abstraction is usually done by connecting the systems (nodes) by bond paths along which a one dimensional Schroedinger equation has to be satisfied (with suitable boundary conditions). They are called *Quantum Graphs*.

We simplify things further and consider as a quantum graph prototype an infinite square well (ISW) with one Dirac delta barrier inside.

$$V_{isw}(x;L) = \begin{cases} 0, & 0 < x < L \\ \infty, & \text{otherwise} \end{cases}$$

$$V_\delta(x;D) = L\beta\delta(x-D)$$

and the overall potential

$$V(x) = V_{isw} + V_\delta$$

where L is the size of the entire well, β is an energy that quantifies the barrier strength and $D \in [0, L]$ its position.

Since we are concerned with $P(S)$, and in general this depends on the energy region considered, we have to apply some sort of normalization (as we have already done before). Here we can overcome this difficulty by employing a very large β so that one has enough eigenvalues with approximately the same transmission probability to compute a good statistics, or we can simply make the barrier response energy dependent by scaling $\beta(E)$.[a]

Now, we calculate the transmission probability that furnishes a more intuitive parameter to consider

$$T = \frac{1}{1 + \left(\frac{L\beta}{k}\right)^2}$$

As we have said, to build our statistics we consider an energy interval over which T is approximately constant and we suppose that by acting on β we are able to vary it from $T = 0$ separate systems to coupled ones.

We will now take our own road and leave the more formal treatment of [36] to exploit what we have seen in the previous chapter about general two-parameters degeneracies and the diabolic shape.

We choose as parameters T and D, so that our two-parameter family will be $\hat{H}(T, D)$. Start off with considering the one-parameter family given by $\hat{H}(0, D)$. When $T = 0$ we have two completely independent ISW potential systems and as such, the spectra for this limiting case is simply given by the union of the two ISW ones, i. e.

$$\left\{\frac{n^2\pi^2}{2D^2}\right\}_n \cup \left\{\frac{m^2\pi^2}{2(L-D)^2}\right\}_m$$

With $T = 0$ our D family is made of completely integrable systems. We then expect that by considering a random pair of successive eigenvalues, we will eventually make them cross by varying D. It's not difficult to show that we can have arbitrarily close eigenvalues for large n, m. We thus focus on a pair of successive levels and call D^* the value that realizes the degeneracy (for the special $D = L/2$ they are all organized in degenerate pairs).

We interpret this situation as $(T^* = 0, D^*)$ being the diabolic coordinate where the two cones intersect.

What happens when T is varied and becomes positive?
The old well eigenfunctions ψ_n^1, ψ_m^2 are no more stationary and begin to spread across the barrier. Now, we are not going to compute the new eigenfunctions but use scattering solutions as initial evolution of the old ψ_n.
We want to expand the perturbed $\hat{H}$ on the old basis

$$H_{mn} = \langle \psi_m^1 | \hat{H} | \psi_n^2 \rangle$$

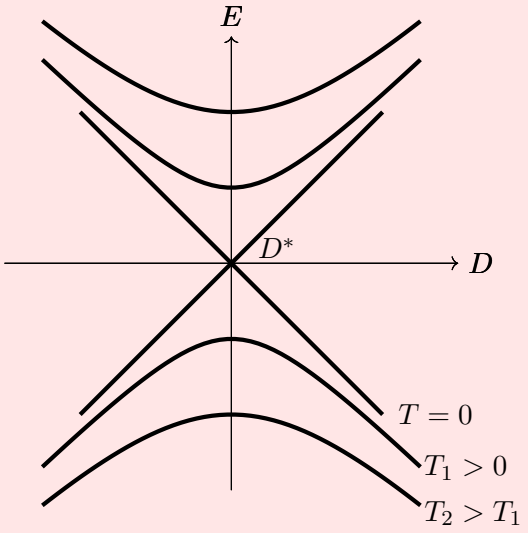
where ψ^1_n and ψ^2_m are eigenfunctions of the left and right wells respectively (the other terms involving ψ of the same well are not relevant, they remain approximately diagonal).
We can imagine this as the time derivative of the evolution of ψ_n which is no more an eigenfunction.

$$H_{mn} = i\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi^1_m|\exp(-it\hat{H})|\psi^2_n\rangle$$

I think it's plausible from scattering theory that the evolution will make ψ_n spread in the right region of the well, I expect with an amplitude proportional to $\sqrt{T}$ and a velocity proportional to $\sqrt{E}$, whereas the wavenumber will be preserved. Since we suppose to restrict ourselves to a small energy region we can take both T and the spreading group velocity almost the same for all the wavefunctions under consideration.
We end up asserting that H_{mn} will be given by the inner product of ψ^2_m and the spread part of ψ^1_n, this will approximately preserve orthogonality between different wavenumbers and the only important off diagonal elements will be the ones with equal k_n, the old degenerate.
This is like asserting that the combinations $\Psi_\pm = \psi^1_m \pm \psi^2_n$ with $k_m \sim k_n$ will represent the approximate eigenfunctions of the whole system.
The off diagonal terms of the resulting 2×2 matrix of these 2-dimensional sub-spaces will be proportional to T and therefore we obtain a spacing $S(T)$ of

$$S(T) \propto T \quad T \ll 1$$

In the diabolo picture we are moving in T direction and the hyperbola that the D-family will draw will then be separated by $S \propto T$, at its minimum D^* value.



Now, suppose we are computing the spacing statistics by averaging over the D-family. We assume that they all share the same $P(S)$ and we compute it by randomly sampling D and compute $\Delta E = S(T, D)$ at a fixed T. What we are effectively calculating is the probability distribution of the hyperbola function $S_T(D)$ of a uniform random variable D.

In the $T = 0$ case we have a linear function and the uniformity is preserved even in S, so that we have a $P(S) \neq 0$ for $S \to 0$ and a constant distribution of spacing for $S \ll 1$. We would expect a Poisson e^{-S} with a $1 - S$ expansion around zero, but here we have considered only two coupled wells. In general, to mimic integrable torus quantization one has to place many Dirac delta barriers eventually superposing their uniform spectra making the distribution to fall of as e^{-S}.

On the other hand, consider the $T > 0$ hyperbola and apply the same reasoning. We end up with a shifted $P(S)$ by the minimum distance $S_T(D^*) \propto T$ and a rounded shape no more linear but as a first approximation quadratic.
The resulting $P(S)$ will therefore present a dip for $S < S_T(D^*)$ and a faster decay for $S > S_T(D^*)$ since the probability distribution $S_T(D)$ is no more uniform but decreases for larger ΔE (the curvature makes the inverse of a $[\Delta E, \Delta E + \delta E]$ smaller for larger ΔE).
This observation would confirm the stronger decay after the peak that one has in the Wigner-Dyson distribution for GOE spacings.

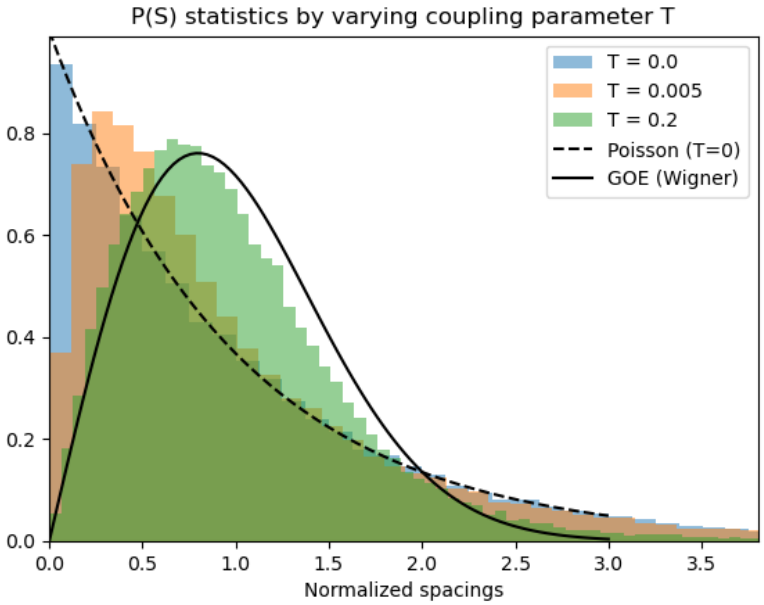
The results in ([36], FIG. 1) do show a shift of $P_T(S)$ peak that is approximately equal to T. The fact that we still have many spacings below the apparent minimum $S_T(D^*) \propto T$ is not captured by our rough reasoning. The approximation of just pair couplings is too simplistic, probably we have many more off diagonal terms that do not permit a diabolic treatment and closer spacings may arise.

[a] In [36] some other technical precautions are taken: they avoid possible resonances between periodic orbits by ensuring an irrational ratio between D and $L - D$. This even helps avoiding accidental degeneracies in the decoupled system.

We have presented a possible physical system that illustrates the coupling effect on spectra. We could even think of bypassing all the system specific technicalities and directly use a network picture with random coupling connections scaled by T strength. We are simply saying: take many isolated systems and suppose that by varying a parameter T we introduce some unknown interactions and couplings of T order between them, what is then the characteristic $P(S)$ of such models?

As a simple simulation I've taken a $N \times N$ matrix with a diagonal given by a uniform distribution with zero mean and with off diagonal elements given by a Gaussian distribution with zero mean and T standard deviation. Then I've calculated the eigenvalues

with a simple numerical routine and obtained $P(S)$ from the successive spacings, the result is the following.



Eigenvalue spacing statistics for $N = 100$ $N \times N$ matrix with diagonal given by a uniform distribution $\mathcal{U}([-1, 1])$ and Gaussian off diagonal elements $\mathcal{N}(0, T^2)$. The statistics is calculated for three different values of std T. Only a small connectivity variance is needed to have a transition from Poisson to Wigner RMT, this reflects the higher order of off diagonal entries $\mathcal{O}(N^2)$ with respect to the diagonal ones $\mathcal{O}(N)$.

It is evident how random couplings produce a shift in $P(S)$ towards GOE statistics. Here it's much stronger due to the full connected nature of the matrix graph, whereas in the tunneling system considered in the box, only few off diagonal elements were contributing to the splitting of almost degenerate unperturbed levels.

We can draw the same conclusion even for general non integrable Hamiltonian perturbations.[3] Take a classical perturbation problem $H = H_0 + V_\epsilon$, its quantum correspondent will then be $\hat{H} = \hat{H}_0 + \hat{V}_\epsilon$. Suppose $\hat{V}$ does not alter the diagonal terms of $\hat{H}_0$ (otherwise we would have included them in the original $\hat{H}_0$) and that the initial integrable H_0 is diagonal on the $|\psi_i^{(0)}\rangle$ eigen-basis, perturbation theory tells us that the average energy of the no more eigenstates $|\psi_i^{(0)}\rangle$ increases by the diagonal term $\langle\psi_i^{(0)}|\hat{V}_\epsilon|\psi_i^{(0)}\rangle$ that we

[3]We mention that another possibly fruitful approach might be the variational LCAO method for molecular orbitals. To compute the new energy levels what one does is constructing a wavefunction as a linear superposition of a starting basis often given by the uncoupled systems (in our case the integrable ones) and than calculates both the off diagonal elements and what is called the overlap matrix (here absent), then one proceeds by minimizing the mean energy associated to the constructed trial wavefunction.

may include in the $\hat{H}_0$ definition.
We now compute the unperturbed $E_i^{(0)}$ in terms of the overlaps $\langle\psi_j|\psi_i^{(0)}\rangle$ between the true eigenbasis of the whole H and the unperturbed one H_0

$$E_i^{(0)} = \langle\psi_i^{(0)}|\hat{H}|\psi_i^{(0)}\rangle = \sum_j E_j |\langle\psi_j|\psi_i^{(0)}\rangle|^2$$

where E_j are the true eigenvalues of $\hat{H}$.
The square of overlaps $|\langle\psi_j|\psi_i^{(0)}\rangle|^2$ has a probabilistic interpretation and the unperturbed energy eigenvalues are therefore quantum *averages* over the true E_j of the system.
We intuitively see that the distribution of the $E_i^{(0)}$ must therefore, loosely speaking, be closer to each other than the distribution of true E_j.

6.3 Some Final Remarks and Conjectures

Finally we realize that the essential phenomena that causes level repulsion and RMT spacing statistics is the coupling given by non integrable perturbations.

We divide its possible origins in ***classical chaos*** and ***quantum tunnelling***, both participate in the overall statistics and have to be embodied in an exact trace formula.

Classical Chaos:

The effect of semiclassical trajectories becoming irregular and filling entire phase space ergodic regions of non zero measure brings to decays of correlations [4], and this results in an overlap between the corresponding quantum wavefunctions.
Start with the set of tori phase space eigenfunctions ϕ_i (we should consider the Wigner semi-probability distribution associated but for now forget about the details) and let them evolve. Since we have a semiclassical path integral representation of the evolution kernel we can hypothesize the coupling rate with other tori to be given by the mixing classical rate. Indeed, the small perturbation will form an Arnold possibly ergodic and connected web that permits trajectories to wander from one torus to another with an escape rate that can be estimated by Nekhoroshev means [37] [38]. We have to say that these Arnold diffusion rates are extremely small and they cannot be even observed numerically unless the perturbation is strong enough.

[4]We should require the mixing property that doesn't automatically follow from topological transitivity or even ergodicity, only for uniformly hyperbolic regions one can deduce correlation decay; in turn, positive Lyapunov exponents on invariant energy surfaces should imply it but in presence of non uniformity we might not have exponential laws.

By expanding $\hat{H}$ on the tori unperturbed basis, one therefore obtains off diagonal terms given by

$$\frac{d}{dt}|_{t=0}\langle\phi_i|\Phi^t\phi_j\rangle \sim \langle\psi_i|\hat{H}|\psi_j\rangle$$

where we have approximated the evolution as given by the classical phase flow just for estimating couplings order.
We will consider only close ϕ_i, ϕ_j since far away pairs will have no overlap at all even after a small evolution. They therefore share approximately the same unperturbed energy $E_i \sim E_j$, on the new basis they will be constituted by a superposition of multiple energies but we suppose them to be approximately concentrated in true eigenvalues near the old $E_i \sim E_j$ so that the spreading of the ϕ will be approximately limited on those true energy surfaces within an interval centred at $E_i \sim E_j$.

But now we remember that the mixing rate couldn't be obtained by spectral gaps in case of continuous spectra, usually related to Hamiltonian dynamics. We have to be careful, here we are considering only the small phase space region between the close ϕ_i, ϕ_j. On such small scale we cannot neglect the finite $\hbar$ and the fine spacing statistics $P(S)$. What one might conjecture is that the mixing rate of general ϕ_i, ϕ_j is approximately given by the mean oscillatory spectral amplitude (the normalized first moment of $P(S)$ times the mean gap $\varrho(E)^{-1}$) [5] between true levels at about the unperturbed energy $E_i \sim E_j$.
We are conjecturing an order of these band (in the sense that they are limited to close eigenfunctions) overlaps given by $\bar{P}(S)\varrho(E)^{-1}$ mean spacing, the previous section seems to confirm such hypothesis but it was limited to quantum graphs and schematized tunneling phenomena.
But the diffusion and mixing rates are believed to be linked to Lyapunov divergence of the region under study, we have mentioned it in **Part I** when we tried to study power spectra decays (Fourier transform of mixing rates) and Lyapunov spectra. We had found many difficulties but we remember that a spectral gap entails a power law power spectrum tail, different from the dissipative systems there considered.
Despite this, we had found that for Markov maps such correspondence might be outlined, here we are more similar to Markov maps due to the noise induced discretization of spectra and Markovianity of stochastic processes. We finally conjecture that the Lyapunov exponent of the classical analogue system might quantify the $\bar{P}(S)\varrho(E)^{-1}$ shift.

This is another picture of what is more rigorously captured by the *Quantum Trace Formula* presented in the previous chapter. There, we have been able to express the non

[5] As a fast check think of ψ_i, ψ_j, $\langle\psi_i|\psi_j\rangle = 0$ as a general basis that we expand on the real one Ψ_i. They have approximate E energy and involve only few Ψ_i. By this we can say $\hat{H}\psi_j \sim E\psi_j \pm \Delta E\psi$ so that $\langle\psi_i|\hat{H}|\psi_j\rangle \sim \langle\psi_i|E\psi_j \pm \Delta E\psi\rangle \sim \Delta E$, where ΔE is the mean non integrable spacing amplitude.

integrable spectra by studying the poles of Green functions, obtained by a periodic orbit expansion. The closed orbits that brought non negligible contributions were the phase space solutions of classical Hamiltonian correspondent dynamics.
We have seen how instability and Lyapunov divergence would influence the parameters entering the formula and intuited how chaotic behaviour could bring level repulsion and larger spacings, but even there we have only alluded to it without carrying out any proper proof.
In this picture, off diagonal elements are, in part, a result of the chaotic mixing behaviour, numerical observations suggest that the produced spectra can be modeled by RMT, meaning that choticity brings to randomly distributed mixing rates between general wavefunctions; we are now in a position to attempt a reasoning.
We start off by saying that are noisy signals rather than deterministic chaos ones that exhibit RMT statistics [28]. In turn, stochastic processes spectra in the weak noise limit can be treated by semiclassical means in a Hamiltonian setting. Noise therefore seems to suggest that it is Hamiltonian non integrability that is better described by RMT statistics.
Consider the $H_{ij} = \langle \psi_i | \hat{H} | \psi_j \rangle$ expanded on a generic basis, in this case we don't treat integrability perturbations but fully developed chaos on energy surfaces and a basis that has nothing to do with $\hat{H}$. Then ψ_i will be composed of a large variety of $\hat{H}$ energy eigenfunctions. We think of the ψ_i as energy eigenfunctions of another system $\hat{H}'$; as so, they are too organized on energy surfaces of H' or in special integrable cases, on tori.
This Hamiltonian separation of very different energy eigenfauctions is shared by both ψ_i of $\hat{H}'$ and the true Ψ_i of $\hat{H}$. This means that the resulting matrix will be concentrated around the diagonal and that very distant matrix elements will be given by uncorrelated mixing rates since they belong to very distinct parts of phase space and only weak correlations on nearby elements can be formed. The contribution to mixing rates of ψ_i comes from a superposition of many different Ψ_j and maybe one can invoke the CLT (even though we have seen that we have correlations between $\hat{H}$ energy levels, we may want to estimate if they are enough weak) to conclude Gaussianity of the matrix entries and independence (of course this does not constitute a proof). This discussion is related to the one given in box *Wigner and Correlation* (introduction to **Part II**), where we had instead focused on correlations.

Classical mixing overlap does not incorporate barrier crossing terms and consequently trace formulas calculated only on classical solutions cannot take into account such non perturbative effects.

Quantum Tunnelling:

The weak noise limit dynamics of these classically prohibited transitions is governed

by a Hamiltonian with flipped potential. This is the result of the Wick rotation that connects quantum and stochastic path integrals. One usually thinks of them being the time complexification of the other, and the analytical continuation of one evolution uniquely defines the other propagator. This correspondence is reflected in the shared spreading rate of wave-packets and the possible reduction of QM to a stochastic process through an intermittent grid measuring apparatus (box *Trajectories and Wavepackets*), but more importantly, by the shared spectrum.
Indeed, when we have expressed the spectra as the poles of Green function traces we used the quantum propagator and the Fourier transform to fix energy instead of time, but we could as well proceed by stochastic means, indeed for a Hamiltonian

$$H = p^2 + \phi(q)$$

the quantum propagator is given by

$$K(q_f, q_i; t) = \langle q_f | e^{-i\hat{H}t/\hbar} | q_i \rangle = \int \mathcal{D}[q(t)] e^{\frac{iS[q(t)]}{\hbar}}.$$

We than take the Fourier transform to obtain the Green function at a specified energy

$$G(q_f, q_i; E) = \frac{1}{i\hbar} \int_0^\infty dt\, e^{i(E+i\epsilon)t/\hbar}\, K(q_i, q_f; t).$$

where the small $i\epsilon$ assures convergence.

Under a Wick rotation we obtain the stochastic picture of the propagator and its Fourier transformation that becomes the Laplace one. We could have derived their expression directly from the stochastic PDE and their resolvent, we would have found out that the transformation is exactly the Wick one.

Anyway, by $t = -i\tau$ we obtain

$$K_S(q_f, q_i; \tau) = \langle q_f | e^{-\frac{\hat{H}\tau}{\hbar}} | q_i \rangle = \int \mathcal{D}[q(\tau)] e^{-\frac{\mathcal{S}_S(q)}{\hbar}},$$

with stochastic action $\mathcal{S}_S$

$$\mathcal{S}_S = \int_{\tau_{q_i}}^{\tau_{q_f}} \left(\left(\frac{dq}{d\tau} \right)^2 + \phi(q) \right) d\tau.$$

which mirrors the underlying Hamiltonian limit given by

$$H_S = p^2 - \phi(q)$$

and the stochastic resolvent given by what becomes the Laplace transform

$$G_S(q_f, q_i; \lambda) = \int_0^\infty d\tau\, e^{-E\tau}\, K_S(q_f, q_i; \tau).$$

G and G_S can be viewed as the time integration of the same time complexified function $e^{\frac{izE}{\hbar}}K(q_f, q_i; z)$ on the positive real and complex axis respectively. We can invoke Jordan's lemma and conclude that if no poles are present in the first quadrant of complex times we have $G = G_S$ and the poles of their trace will match.

We see that the periodic orbit expansion is consistent in both settings and yields the same result. But surprisingly their semiclassical/weak noise trajectories and transition probabilities are describes by different Hamiltonian dynamics with flipped potential sign.

In particular if one chooses to use $H_S = p^2 - \phi$ when ϕ is a confining potential, he/she obtains a dynamics that can be described by geodesic flows on appropriate Riemannian manifolds, this comes from the possibility of applying the Maupertius principle with a reduced action without turning points singularities in the metric.[6]

Especially when we deal with potentials with high and thin barriers, one might be concerned with the tunnelling topology between classically allowed regions and study the corresponding quantum graph, when such reduction can be made. In weak noise limit, usually one supposes that the penetrating trajectory is given by the most probable path at a certain energy, given by an arbitrarily long[7] time trajectory of H_S, and hence, generally, one obtains unique connections between different classical region at a certain energy. This might permit a reduction to a quantum graph with different classically allowed regions as nodes and tunnelling probabilities as edges.

If one then considers the tunnelling graph alone (without caring about the dynamics inside every node), he/she can study its spectral character and statistics as a toy model that is typical, not too simplistic, and mathematically feasible. Quantum graphs permit easier periodic orbit expansions of trace formulas and many conjectures we made as for the Lyapunov influence entering the coefficients can be tested using some equivalent definition of divergence on networks, and even its overall effect on $P(S)$ statistics (we are not going to discuss it here). Indeed in [39] they motivate their study on statistical spectral properties of such graphs by saying that it might permit a mathematically friendly setting for studying the mechanisms that induce the spectral transition from integrable Poisson statistics to RMT Wigner one. To this end, they even construct the corresponding underlying classical phase space evolution to enable a notion of chaos and integrability.

[6] I've not studied possible approaches to periodic expansions by Riemannian geodesics means but it could be an interesting topic.

[7] If one wants to keep time information and compute finite time most probable trajectories, he/she can find $H_S > 0$ solutions. One minimizes the crossing probability with respect to the constant noise rate needed (i.e. H_S) to perform the jump in the time given, and could find that higher barrier crossings, with more noise required, permit faster transitions that are the ones observed, if one doesn't wait enough.

With graphs we can additionally study how spectral statistics is influenced by changes in the connectivity. The fast shift observed in the simple simulation plot presented above is a result of full graph connectivity.[8]

We now want to mention the possibility of incorporating into a unique trace formula both perturbative and non perturbative effects to yield an exact weak noise expression, from which it may be possible to extrapolate their overall effect on spectral statistics.

An Exact Trace Formula

A generalization of *Gutzwiller's approximate Trace Formula* would not consider only the perturbative classical cycles contribution but even the non perturbative purely quantum/noise ones and compositions of them.

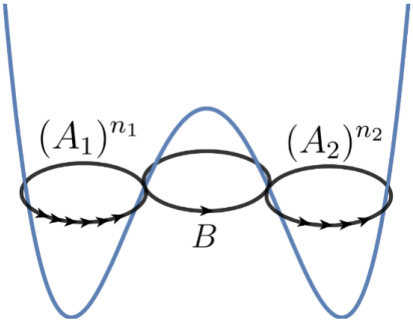
We will then write

$$G(E) = G_{\mathrm{p}}(E) + G_{\mathrm{np}}(E)$$

where the perturbative spectral path integral G_{p} can be obtained by the standard formula (5.12), where single cycles are characterized by weight factors $e^{r\frac{i}{\hbar}S_p(E)}$.
The G_{np} term turns out to be more complicated. First of all, we can exploit the stochastic path integral to quantify weights associated to tunnelling cycles quantum corrections (it is remarkable that the leading quantum correction that the general Feynman path integral embodies can be captured by the stochastic Wick rotated one associated to H_S dynamics), they are exponentially small and real $e^{-\frac{r}{\hbar}\mathcal{S}_S(E)}$.
The difficulties arise when we try to calculate these terms since at each barrier penetration the system can oscillate in perturbative closed orbits as many times as it wants and than switch back choosing another classically admitted periodic orbit, so on and so forth.
G_{np} spectral path contributions are exponentially suppressed so one does not usually care about them as for the spectral spacing statistics $P(S)$. Even if this latter does encapsulates information about the finest discrete scale of spectra, the spacings are normalized on the mean $O(\hbar^N)$ one, so that $e^{-\mathcal{S}_S}$ scale seems to play no role.

This is true in most situations but in one important case they might be relevant. When we consider slightly perturbed integrable systems we know from KAM theorem that phase space will still be filled up with many invariant N-tori (the sufficiently irrational unperturbed ones that persist even if slightly deformed) and this puts a geometrical

[8]This turns out to be intimately related to the semi-classical theory of Anderson localization: one observes a Poissonian randomly distributed spectrum when high localization is possible and a transition to RMT statistics when the connectivity increases and the number of possible localizable states decreases. The localization length turns out to be linked to the Lyapunov exponent.



One of many possible minimal nonperturbative orbits. A_i and B are the imaginary and real exponential weights associated to each illustrated cycle.

barrier and possibly isolates some regions from others.
In these cases, quantum non perturbative corrections can permit dynamical tunnelling[9] of these invariant constraints allowing wavefunctions to wander through the entire energy surface.[10] Since we are almost integrable it could seem that many torus-eigenfunctions can be found, with the annexed one-parameter degeneracies and their Poisson statistics; this means that we would have many torus-eigenfunctions approximately on the same energy surface and with nearly degenerate eigenvalues, this brings to a much more dense spectrum and even a more packed $P(S)$ normalized statistics with nonzero probability for infinitesimal spacings.
In this situation we shouldn't neglect these extra interference terms in trace formulas and spectral studies: can they bring to splittings that are comparable to the mean level spacing?

As a last insight I want to mention *Chaos-Assisted Tunnelling*.
In systems with mixed phase space, certain quantum states can be localized in regular regions (or islands) that are dynamically separated [11]. Although direct tunnelling between these regions might be very weak, the presence of a surrounding chaotic sea provides many additional routes or intermediate steps for the tunnelling process. In effect, these chaotic states *assist* the tunnelling, leading to an enhanced coupling between the localized states that would otherwise be nearly isolated.
This constitutes another motivation for studying non perturbative contributions that can be amplified by such phenomena.

[9] Unlike traditional tunnelling across a potential barrier, dynamical tunnelling occurs between regions in phase space that are separated by invariant structures.

[10] This comes from the fact that these invariant manifolds have zero measure, and phase space wavefunctions must have a minimum thickness given by the uncertainty principle, so that they will always have a small amplitute in a neighbouring region.

[11] We lack global topological transitivity, even on the same energy surface.

Chapter 7

Power Spectra and Random Matrices

Treating trace formulas analytically is hard, the density of periodic orbits and the exponential number needed to resolve single levels down to the spacing statistics scale $\hbar^N$, seems to prevent any precise theoretical argument. Even if cycle expansion gives an interesting physical meaning to the occurrence of irregular energy levels (see box below), computationally speaking is still too hard to use, chaotic flows are notoriously difficult to integrate due to exponential amplification of numerical errors, finding the classical periodic orbits seems to be even harder than the quantum eigen-problem.
Only if we schematize and reduce the dynamics to graphs, one can attempt a finite dimensional theoretical analysis based on trace formulas that might permit an analytical study. One has then to understand whether these systems are too simplistic to infer some phenomenology for the real cases.
As for numerical approaches, noise/quantum discretization suggests to use truncated expansions on known complete basis. In [5] we have seen that $P(S)$ was computed by expanding the irregular Hamiltonian operator on the Harmonic Oscillator basis, truncating it at a suitable size and verifying the stability of the subsequent results with respect to the truncation. We end up with a finite dimensional matrix whose entries are computed by applying the differential Hamiltonian operator on analytically expressible functions, then numerically integrated on state space. This matrix can then be diagonalized by numerical routines. Importantly, all steps involved in this derivation after the truncation have errors that can be controlled and that do not get amplified by divergences. So, this approach is preferable but still it remains a computational challenge, diagonalizing very large matrices is time consuming and one has to repeat it multiple times to verify convergence.
We see that even if the eigen-problem can be tackled as a whole, by calculating matrix entries and diagonalizing them to get bunches of levels, one may wonder whether the information of the exact positions of spectral lines is really necessary to compute their

spacing statistics, after all we are not concerned with system specific spectra but with an asymptotic statistical trend, that we wish to correlate to some gross measures of chaos in the corresponding classical dynamics.

Here I present an interesting old study [11] about the possibility of extrapolating matrix elements distributions from time series. This resembles our trial of computing Lyapunov exponents directly from power spectra decays, by measuring just an observable of a dense evolution, this was to avoid time consuming reconstructions of phase space and tangent dynamics. This has very important physical relevance, we might be interested in deducing chaoticity of unmodeled real systems, for which we have no idea of what the dynamical vectorfield is, but we can still measure some observables of their single time evolution realization.
Takens embedding theorem gave a theoretical basis for the reconstruction of a diffeomorphic attractor from just measured time series, but it involves topology reconstruction of neighbouring points in space but distant in time, a computationally consuming task.
If one were capable of computing such quantities of physical relevance, as Lyapunov exponents, directly from time series, two point correlations and perhaps even their Fourier transform, he/she would succeed in obtaining them *live*.

Here the objective is spacing statistics $P(S)$, we wish to bypass complexity of periodic orbits expansions and the time consuming computation of large matrix eigenvalues.
The authors in [11] are more involved in showing the failing of quantum perturbation theory, but in doing so, they display some time series analysis that permit to estimate energy representation matrix entries by power spectral means.
I will attempt to outline a possible procedure based on these techniques and results to compute *live*, from time series (and FFT transformed ones), the $P(S)$ statistics. We will do this by using a reference $\mathfrak{H}$ system as a litmus test.

If then a connection between $P(S)$ and Lyapunov spectra can be carried out (as we have said multiple times, there are clues of this link in trace formulas coefficients, heuristic arguments, quantum graph reduced studies and numerical evidence), we would eventually build a new tool for inferring Lyapunov exponents directly from time series passing by $P(S)$ statistics. Unfortunately I've had not enough time to carry out some simulations supporting and testing it.

Fourier generalization and Computational Complexity

A Fourier Generalization:

It's interesting to note that trace formulas can be thought as Fourier decomposition

generalization for nonlinear flows.

Consider a signal $s(t)$, if we compute its Fourier transform $\tilde{s}(\omega)$, the $\tilde{s}(\omega)$ singularities represent the oscillatory components $e^{-i\omega t}$ present in $s(t)$. These are the energy eigenvalues of every Schroedinger equation (it always allows time separation for autonomous systems). But the evolution of a wavefunction embodies an infinite number of signals and even in the semiclassical limit we would have a signal for every flow trajectory. We would like to find the time oscillatory components of the overall flow evolution.
To do so one observes that in chaotic regimes the flow is composed by a dense set of periodic orbits, i.e. of signals with a specific frequency. The density permits us to concentrate only on this periodic backbone since every trajectory can be well approximated by a composition of periodic ones. One therefore can guess the sense of finding the singularities of a quantity that involves all periodic orbits compositions and interference between phases gained at each orbit revolution.

A note on computational complexity:

When we have used the word *live*, we were referring to the different order in operations needed among the techniques to compute the relevant quantities.

For topological and tangential dynamics reconstruction we need $\mathcal{O}(N^2)$ operations for N samples considered, we have to measure the relative distance of each new point with the others to recover a space vicinity notion and eventually compute the tangent dynamics.
For a possible direct power spectra analysis the FFT algorithm has $\mathcal{O}(N\log(N))$ complexity for the same number of samples. This effectively reduces by an enormous amount the time required for eventual precise estimates.

7.1 Microcanonical Statistics

As we have already noted, the authors point out that for quantum systems whose classical analogue exhibits chaos, in the semiclassical limit $\hbar \to 0$, the energy eigenfunctions $|E\rangle$ have phase space Wigner distributions that fill the entire energy shell with thickness $\Delta E \sim \mathcal{O}(\hbar)$. Their amplitude fluctuates around the classical microcanonical phase-space density.

Microcanonical Surface Measure

The microcanonical density is not necessarily uniform: even classically one has to associate to states a finite phase-space volume and than eventually take the vanishing limit, quantomechanically one does the same in the semiclassical limit, we cannot take area elements as states. This means that when we define the density of states on a specific energy surface $H(q,p) = E$ in reality what we do is to take a small δE shell and count the volume elements above every surface area element within it. Due to the varying magnitude of the Hamiltonian gradient ∇H we end up having different shell thicknesses at different surface points.
The thickness $h(\eta)$ (where η are the curvilinear coordinates on the $2N-1$ energy surface) is inversely proportional to the gradient at that point

$$h(\eta) = \frac{\delta E}{\nabla H(\eta)}$$

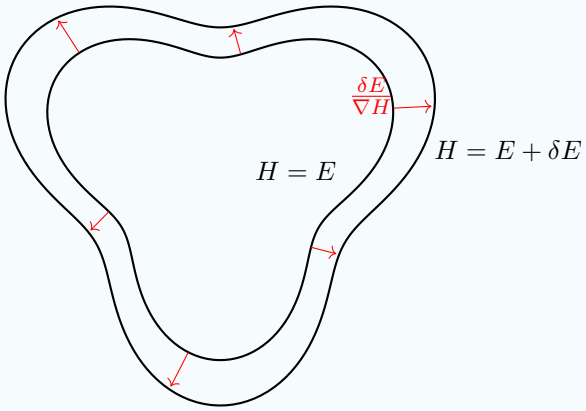
We now define a surface measure to compute averages given by

$$d\mu(\eta) = \frac{h(\eta) d\eta}{\hbar^N}$$

where $d\eta$ is the surface element and $\hbar^N$ the volume associated to single states.
$d\mu(\eta)$ represents the microcanonical ensemble density for a given energy surface up to normalization.
By it, one can compute microcanonical phase space averages on energy surfaces. Its importance resides in being *invariant* under the flow, this is a consequence of its derivation from the standard invariant Liouville phase-space measure, indeed if one performs with care the limits $\delta E, \hbar, d\eta \to 0$ obtains that

$$\int_M \delta(H(p,q) - E) f(p,q) dV = \int_{\Sigma(E)} f(p,q) d\mu(\eta) \tag{7.1}$$

where $f(p,q)$ is a general phase space observable. Since dV, H, E are invariant under the flow one concludes that even $d\mu$ does.
In ergodic systems, $d\mu$ will play the role of an *invariant ergodic* measure that will permit equality of phase space and time averages on single energy surfaces.



When chaotic behaviour makes reasonable the ergodic hypothesis on energy surfaces relative to eigenstates $|E\rangle$, we should have that for quantum observables $\hat{A}$ with a well behaved classical limit, $A(p,q)$ for $\hbar \to 0$, the expectation value $\langle E|\hat{A}|E\rangle$ tends to the microcanonical phase-space average

$$\langle\, E \mid \hat{A} \mid E \,\rangle \simeq \{A\,(E)\} = \frac{\int A\,(p,q)\,\delta(E - H(p,q))dp\,dq}{\int \delta(E - H(p,q))dp\,dq}\ .$$

where we have denoted with $\{\}$ the microcanonical average.

We now want to show that for such observables it is possible to calculate their matrix entries in the energy representation $A_{jk} = \langle E_j|\hat{A}|E_k\rangle$ by time series power spectra.

We start by writing

$$\sum_k \exp[i\,(E_j - E_k)t/\hbar\,]\,|\,A_{jk}\,|^2 = \sum_k \langle\, E_j \,|\, \exp(iE_j t\,/\hbar)|\, E_k\rangle \exp(-iE_k t\,/\hbar)\,\langle\, E_k \,|\, \hat{A} \,|\, E_j\,\rangle \tag{7.2}$$

$$= \langle E_j \mid e^{iHt/\hbar}\hat{A}e^{-iHt/\hbar}\hat{A} \mid E_j\rangle = \langle E_j \mid \hat{A}(t)\hat{A}(0) \mid E_j\rangle\ , \tag{7.3}$$

where $\hat{A}(t), \hat{A}(0)$ are given in the Heisenberg picture. For what we have said before, the last term should be well approximated by its microcanonical average $\{A(t)A(0)\}$ on E_j surface. Since we are considering ergodic regions we can go further and express it as a time average over a single evolution

$$c_{A,A}(t) = \{\, A\,(t)A\,(0)\,\} = \lim_{T\to\infty} \frac{1}{T} \int_0^T A\,(t+\tau)A\,(\tau)d\tau \tag{7.4}$$

where we recognize $c_{A,A}(t)$ as the already encountered *trajectory correlation function*, in this case *autocorrelation* of $A(p,q)$.

By considering (7.3) we can extract the $|A_{jk}|^2$ terms by Fourier transforming $c(t)$ and evaluating it at the right energy differences $(E_j - E_k)/\hbar$. We thus write the power spectra of A as

$$\mathcal{P}(\omega; E) = \int_{-\infty}^{\infty} c(t)e^{-i\omega t} \tag{7.5}$$

Now, since we will be concerned in evaluating $\mathcal{P}(\omega)$ even for very small ω, we have to avoid the singularity at $\omega = 0$ caused by the constant term $A_{jj}^2 \sim \{A\}^2$ in (7.3). It will be enough to subtract the average $\{A\}$ of the considered energy surface from the observable entering (7.3), i.e. we redefine it with $\hat{A}' = \hat{A} - \{A\}$, so that its power spectrum will be well defined even in proximity of zero.

From (7.5) we obtain

$$\mid A'_{jk} \mid^2 \simeq \mathcal{P}((E_j - E_k)/\hbar; E_j)/2\pi\varrho(E_k) \tag{7.6}$$

where $\varrho(E)$ is the coarse-grained mean level density calculated by the Thomas-Fermi volume formula. We need to include this factor because in performing the continuous Fourier transform (7.5) we have to replace the discrete sum in (7.3) with an integral over energy states

$$\sum_k \cdots \to \int \cdots \varrho(E_k) dE_k$$

Even though expression (7.4) involves only a single dense trajectory, it's tough to compute the two point correlation for every t lag. However, (7.5) can be directly approximated with an ensemble average (over a family of trajectories on the same energy surface) of A's Fourier components

$$\mathcal{P}(\omega; E) = \lim_{T\to\infty} \frac{1}{T} \left\langle \left| \int_0^T A'(t) e^{i\omega t} dt \right|^2 \right\rangle. \tag{7.7}$$

Where $\langle\rangle$ denotes the family average.

The authors conclude with some implications for quantum perturbation theory.

Consider a Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$, then it is a standard result that a necessary[1] condition for the eigenvectors of $\hat{H}$ to be near the ones of $\hat{H}_0$ is that

$$|V_{jk}| \ll |E_j - E_k| \tag{7.8}$$

Thanks to the presented microcanonical averages, they have been able to write (7.8) as an inequality of the form

$$\hbar^{2N+2} \gg C.Q.$$

where the $C.Q.$ stands for Classical Quantities that do not depend on $\hbar$. From this, one concludes that in the semiclassical limit $\hbar \to 0$, (7.8) cannot be satisfied and that perturbation theory for chaotic systems is bound to fail: arbitrarily close Hamiltonians have, in general, completely different sets of eigenvectors[2].

[1] It does not guarantee perturbation series convergence.

[2] In regular systems, the presence of selection rules makes most of the V_{jk} to vanish, (7.8) condition is therefore always satisfied. In this sense one can distinguish regular and irregular spectra from its perturbation response.

7.2 A Litmus test for Chaos

From their study, it seems that in semiclassical limit $\hbar \to 0$, one cannot study smooth transitions to non integrability and eventually chaos, arbitrarily close Hamiltonians have very different eigenstates that cannot be identified as analytic variations of the original ones, and energy levels will cross many times destroying any initial ordering after any small perturbation.

But what about spectral spacing statistics $P(S)$?
When $\hbar \to 0$ it's true that we cannot perform any perturbative expansion since levels get closer and closer eventually becoming continuous, but that's why we need a quantity that scales hand in hand. $P(S)$ is defined at the finest scale, $\hbar^N$, and remains a well defined statistical distribution even at the limit.
When we say that eigenstates radically change, we mean that their inner products with the original ones gets far away from unity not analytically. But this as far as the Hilbert space metric is concerned, two distant eigenstates in quantomechanical *ray space* (the projective Hilbert space) might still be close in classical *phase-space*. Indeed even though phases and level orderings might change a lot, one still has small variations in the corresponding phase space energy surfaces [3], and therefore, the couplings triggered by the perturbation will mainly concern close corresponding Wigner distributions.

From these considerations I believe $P(S)$ to remain a meaningful statistical property that permits to observe a transition from overall regularity to widespread chaoticity.[4,5]

We now try to outline a procedure to compute $P(S)$ exploiting the results of this chapter.
The key ingredient of what follows is (7.6), once we know the energy levels of one Hamiltonian system, we can easily compute all the matrix entries at j-th row and column of a generic observable $\hat{A}$ (with well behaved semiclassical limit), we only have to compute its power spectra on the j-th energy surface (after subtracting its average $\{A\}_j$). This,

[3]The perturbation of an integrable system is a singular exception, but nevertheless even in these cases KAM restores some sort of continuity in measure, in the sense that most tori will survive.

[4]One should carry out some numerical tests to test this continuity under small perturbations, in [5] we have already seen that it seems indeed the case.

[5]In many cases, if one takes BGS conjecture as true, an abrupt change in $P(S)$ following a small perturbation would mean that arbitrarily small changes in the dynamical law provoke a transition from regularity to fully developed chaos. We have to remember that BGS should be intended in only one direction: chaos brings to RMT statistics and integrability to Poisson, the opposite implication is not true and some counterexamples exist. Maybe one can even conjecture that by observing abrupt changes in $P(S)$ the underlying dynamical phenomenology might not be traced back to chaos but maybe to some other sorts of complexity. Bifurcations can happen even in chaotic settings but they do not entail such abrupt changes in the measure of overall irregularity.

thanks to (7.7), can be computed *live* (with little computational effort with respect to other techniques).

Of course, to obtain the whole energy representation of $\hat{A}$, one has to repeat the process for all energy levels, but if the power spectral profile $\mathcal{P}_j$ varies slowly with E_j, we can use the same for entire energy intervals.

Now, what if we take as $\hat{A}$ another Hamiltonian system $\hat{H}$?

We call our reference dynamics $\mathfrak{H}$, with which we calculate microcanonical averages and power spectra of all other $\hat{H}$ systems by evaluating the evolution of $H(p(t), q(t))$ on wandering trajectories of $\mathfrak{H}$.

We proceed as follows:

A Take a reference irregular system $\mathfrak{H}$.

B Compute its energy levels by expanding it on a known complete basis.

C Extrapolate from the obtained spectrum its $\mathfrak{P}(S)$ statistics and mean $\varrho(E)$.

Then:

1. *Take any other general system $\hat{H}$ with classical analogue $H(p, q)$.*
2. *Pick a level E_j of $\mathfrak{H}$, compute the power spectrum $\mathcal{P}(\omega; E_j)$ of $H - \{H\}_j$ from a bunch of time series on the E_j energy surface of $\mathfrak{H}$ dynamics.*
3. *Compute $|H_{jk}|^2$ by simply evaluating $\mathcal{P}(|(E_j - E_k)|/\hbar; E_j)/(2\pi\varrho(E_k))$ by using the E_k of point B (then add the mean $\{H\}_j$ to H_{jj}).*

3 You might want to use the $\mathfrak{P}(S)$ statistics to generate the E_k.*

4. *Assign randomly the sign to the matrix entry $H_{jk} = \pm\sqrt{|H_{jk}|^2}$.*
5. *Repeat point 1. and 2. for every j-th row/column to be calculated.*

This is the general recipe, points *A,B,C* have to be done only once, so we don't care about the difficulty of finding $\mathfrak{H}$ spectrum since once we obtained it we will use points *1,2,3,4* to compute all other system's spectral statistics.

Lots of comments are needed. First, $\mathfrak{H}$ cannot be taken regular otherwise we wouldn't give sense to time averages and power spectra, additionally this avoids the degenerate possibility to choose H that are in involution with $\mathfrak{H}$, in those cases the trajectories of $\mathfrak{H}$

would remain on a single level set of H and no statistical information about the whole j-th row could be extracted, indeed one does not even have ergodicity. So we assume $[\mathfrak{H}, H]_{PB} \neq 0$ (where $[\cdot]_{PB}$ denotes Poisson Brackets).
Point *4.* sounds bad since forces us to statistical arguments, but in a RMT theoretical setting, this shouldn't look awkward, the power spectra approximation, after all, quantifies the variance in the family of sampled trajectories of their Fourier components. We understand that $\mathcal{P}(\omega, E)$ is more about giving the modulus profile of matrix entries, and again, invoking RMT and BGS conjecture we are supposing their distribution to give equally probable positive and negative couplings (One could even think of taking $\mathcal{P}$ as the variance of a *Gaussian random entry*).[6]
This supports even more the usage of point 3^* as a more statistical and system independent approach.

Apart from the $\mathfrak{H}$ spectrum steps, all the subsequent one are very efficient and permit to calculate entire row/columns at once. The only possible criticality is point *5.*, we have to repeat the calculation of $\mathcal{P}(\omega; E_j)$ for every E_j, we shouldn't generally expect it to be uniform in E.
I will return a bit on this difficulty later, eventually comparing its complexity with standard expansions on known basis (that have nothing to do with the system, but are complete).

But now, the key observation is that $\mathcal{P}_E(\omega) \to 0$ for $\omega \to \infty$, with a rate to be specified. This decay permits us to limit ourselves on a suitable *band* centred on the diagonal.
We don't know the character of this decay, for some surfaces and systems it may be exponential (as in the one considered by [11]), for others a power law decay. For ergodic energy regions we expect little coupling between far eigenfunctions so that $\langle E_j|H|E_k\rangle$ (with $\mathfrak{H}$ $|E_j\rangle$ basis) should be small for far off-diagonal elements. For uniform hyperbolic regions we have clues for $\mathcal{P}$ to have power law tails. When both regular islands and irregular (but connected) sea are present, we might even encounter exponential decays.
We begin to make out that a suitable balance between chaoticity and regularity is recommended in $\mathfrak{H}$: partial regularity allows faster decays but with more packed $P(S)$, whereas fully developed chaos has larger gaps that help the effective decay of matrix elements but the $\mathcal{P}$ will more probably have longer tails.
This trade off between decay and spectral packing is fundamental, otherwise one would conclude that by taking a $\mathfrak{H}$ with both large gaps and fast decay all other systems are regular and Poisson like since they will have a very thin band with almost no couplings. One such system would be super-chaotic, with no regularity in both trajectory and spec-

[6]We mention that even $\mathcal{P}$ itself does not converge to a well defined limit for $T \to \infty$, and may even be a Cantor set, thus we see how the random assignment is not that bad.

tra.

Now, even if one proceeds with the recipe for many E_j up to truncation, will it be more efficient than just do the same with the standard truncated expansion on known basis?
The two methods are after all linked, both use another system as reference, but ours tries to exploit the ergodicity of the flow on energy surfaces to collect by FFT means the information needed for an entire set of entries. We are exploiting the knowledge that chaotic Wigner distributions fill the entire energy surface, and that they are weakly coupled with non close in energy states, whereas when we pick a general known basis as the Harmonic Osillator one, we have to compute all the $N \times N$ entries since it generally have couplings with arbitrarily far away off diaconal elements (due to its integrability and level packing). We usually use it due to the available expression of its eigenstates but in our procedure we don't need any expression of $\mathfrak{H}$ eigenbasis since we will compute inner products $\langle E_j|H|E_k\rangle$ by a classically computed $\mathcal{P}$, but we still need the $\mathfrak{H}$ spectrum or its statistics (that can be even tackled by trace formulas).

To sum up, if we want to compute the spectra of H using standard truncated expansions of $N \times N$ dimension we end up having a computational complexity of $\mathcal{O}(N^3)$ for diagonalization, since we generally have to use routines for dense matrices.
We then have to add $\mathcal{O}(N^2) \times$ (numerical complexity of calculating phase space integrals), this takes account of the matrix construction.
On the contrary, with our recipe, we end up with a band matrix with possible long tails but still when it's possible to truncate it (exponential case or particularly large gaps) we will have a reduced diagonalization complexity to $\mathcal{O}(Nk^2)$ where k is the band width.
Additionally, the calculation of matrix entries takes only N cycles of the recipe, but the exact complexity of each of them depends on the precision needed for the trajectory evolution and the number of points used for the FFT to compute $\mathcal{P}$.
Even at this point, we can see that there are promising clues for its efficiency.

We then have to check whether the overall statistical study is stable with respect to $P(S)$ (in case of point *3**), the randomly assignment of H_{jk} sign and eventually its Gaussian sample $H_{jk} \sim \mathcal{N}(0, \mathcal{P}(\Delta E/\hbar; E_j))$.

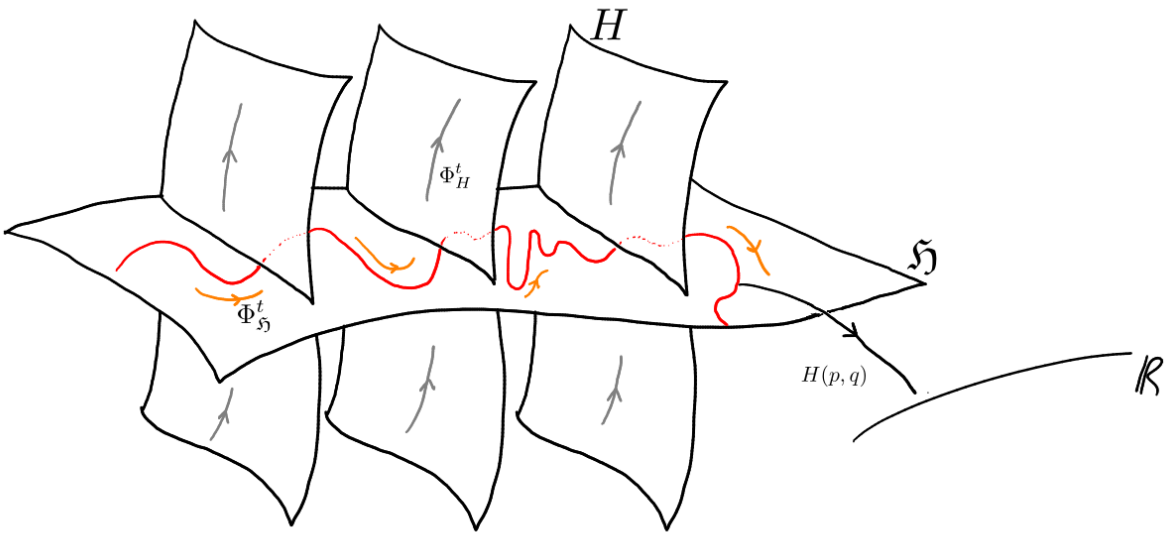
A crucial remark is in order.
We are concerned only about spectral statistical information of $\hat{H}$ and not its precise spectrum, we are even forced to proceed statistically since point *4.* essentially breaks any precise spectral hope, this can even be viewed as a consequence of our classical averaging approximation procedure.
So one is happy even with just the $\mathfrak{H}$'s $\mathfrak{P}(S)$ statistics and the general $\mathcal{P}$ decay. Then one can proceed in different ways to estimate the $\hat{H}$'s $P(S)$.

For effective exponential band decays that are rather uniform throughout energies we essentially have a Random Band Matrix, and one can refer to their general theory to assert that in the infinite size limit we will have a Poissonian character. This is a result of H being much more regular than $\mathfrak{H}$.
Whereas if one deals with Power-law Random Banded Matrices (PLRBM) he/she will find that there are some critical exponents that separate the Poissonian from the Wigner statistical behaviour.

We can therefore estimate the chaotic degree of general Hamiltonians and $P(S)$ quantum/stochastic spectral statistics by these decays of matrix entries. One is therefore left with understanding what is the general decay of $\mathcal{P}(\omega; E)$, sampled with gaps given by $\mathfrak{H}$'s $\mathfrak{P}(S)$.
In some cases, a uniform estimate cannot be found and one is left with the question of whether general statistical studies can be even carried out.
It turns out that one can still capture some local spectral properties as $P(S)$, by focusing on a *bulk* region of $\mathfrak{H}$ energy levels. Treating the applicability of such finite dimensional reductions is out of the scope of this work, but it surely depends on matrix homogeneity and band decay.

We conclude with a curious question: *What is the role played by $H(p, q)$ Lyapunov spectra in this picture?*

We have seen that H's $P(S)$ statistics depends on the decay character of $\mathcal{P}$ which in turn depends on both $\mathfrak{H}$ and H. We expect Lyapunov divergence to be a measure of the statistical shift from Poissonian to RMT, here H is treated as an observable, we see how the decay of correlations is dependent on the observables chosen. Anyway, the specific off-diagonal decays should ultimately depend only on H but it's not clear how the divergence on H energy surfaces and dynamics manifests itself in the power spectra decays of $H(p, q)$ observable on $\mathfrak{H}$ surfaces and dynamics.



The recipe presented here is just a sketch for some possible further work, many points have to be well understood and tested before making any risky conclusions. In any case, I've tried to give you a rough idea of it.

Conclusions

We have defined the concept of chaos in low dimensional dynamical systems by focusing on extreme sensitivity (i.e. positive Lyapunov divergence) and topological transitivity (i.e. irreducibility). We have moved to a statistical description in terms of probability density evolution and tried to understand its global behaviour from the evolution spectra of the Frobenius-Perron operator. Unfortunately, continuous spectra was the general result for deterministic non trivial systems. We tried to avoid geometrical difficulties by considering Hamiltonian conservative dynamics but then we had to consider a limited concept of chaos on single energy surfaces. We then introduced Noise as a physical prescription to have both discrete spectra and a classical limiting concept of regularity and non integrability. We then have realized that the Fokker-Planck operator obtained by a stochastic perturbation of a simple gradient flow dissipative dynamics was of the Schroedinger type. We explited this quantomechanical analogy to infer spectral statistical quantities by semiclassical quantization prescriptions of early QM. We tried to furnish theoretical arguments for a conjecture about the level spacing statistics $P(S)$. A Poisson statistics is usually a result of classical integrability whereas RMT statistics is a consequence of irregularity and chaos.
The key takeaway is that introduced non integrability by a perturbation causes some couplings between otherwise independent regular eigenstates (the torus-eigenstates of EBK quantization) and these off-diagonal elements are ultimately causing the onset of RMT spectral statistics. Additionally, quantum tunnelling effects may play a significant role especially when some invariant constraints cannot be classically overcome. The effect of these two concomitant effects produce, at the matrix level, off diagonal couplings that make the spectrum more correlated and far from random. This collective behaviour not reducible to a superposition of independent spectra finds a counterpart in the third requirement for a genuine chaos: a dense set of periodic orbits. Indeed, it turns out that we can fully characterize probability ensemble evolution only by this periodic backbone, every trajectory will shadow an arbitrarily large number of these closed cycles and can be reconstructed with high precision from them. This point of view results to be well represented by Gutzwiller semiclassical trace formulas that furnish a periodic orbit expansion of the Green propagator. The poles of this latter do represent the energy spectra of our system and only with a collective coherent interference of all possible composi-

tions of primitive periodic orbits one obtains a level, confirming our intuitive idea of a quantization condition that has to involve the dynamics as a whole.
A question about the irregularity of the Wick rotated weak noise Hamiltonian is considered, the flipped sign dynamics is way different but could capture the irregularity of the potential morphology and the overall tunnelling level repulsion effect on spectra.
For integrable Poisson statistics we have a rigorous semiclassical result whereas we haven't been able to carry out any rigorous proof for RMT chaotic spectra, from trace formulas we have been able to make qualitative considerations that seem to confirm it but we are far from having understood it. Only for the vanishing spacing limit we have a quite precise result from degenerate perturbative theory, here, the diabolical structure for the general two-parameter family of crossings imply a linear initial $P(S) \propto S$ that entails repulsion of very close levels. This can be well understood in quantum graphs where the dynamics is simplified to a schematic representation, we mentioned that in this setting many rigorous results can be carried out and some links with the network-equivalent of Lyapunov spectra can be found.
The connection between correlation decays and spectral gaps found for deterministic Markovian maps is conjectured to be generalizable to our Hamiltonian, quantum-lifted (in the sense that noise has acted as a microscope to discretize the continuous spectra and only through the correspondence principle we recover a Hamilton flows), systems. Can the $P(S)$ statistics through some light on energy surface Lyapunov divergence?
We have to underline that the RMT conjecture is only in one direction: we have counterexamples of partially integrable systems whose statistics shows level repulsion, we should consider more carefully the many ways that one can obtain RMT from physical systems.
At the end we have seen that we can exploit classically computed quantities and a quantized reference system to obtain a lot of statistical information about all other spectra. In particular we have used the power spectra obtained by an efficient FFT of correlation functions, by it one can estimate the decay rates of the off diagonal elements and by the irregularity of it he/she can even presume the randomness of the matrix entries. By the RMT hypothesis and these so estimated decays we might conclude the RMT or Poisson character of $P(S)$.

Appendix

A. Lyapunov insight

The evolution of initial points is fully qualified and described by the laws of motion that determine the field f (we assume a well posed Cauchy problem) and we can associate to every point its unique evolution trajectory in space-time, as time passes, close points will diverge and will touch different locations at different times experiencing different drift fields f but eventually they will meet again and again due to the finite space available to the motion.
Imagine mapping $M_t : V \to \mathcal{T}$ initial conditions of the state space to a space of trajectories both embodied with a suitable metric (the trajectory one given by the normalized integral of the relative distance in time), at time $t = 0$ we have that M is an isometry, at later times we will see a global enlargement due to the mean positive λ that prohibits close trajectories to remain close for arbitrarily long time (they must almost always have a tangent component in the positive λ direction), the Jacobian of M is acting as a microscope and enlarges continuous space infinitesimal dynamics f by amplifying its effects on the future states, so as time passes M is telling us this fine structure invisible at a starting resolution. To be clearer think of a grid in V of elementary small edge ϵ that gets enlarged in some directions and contracted in others, the overall result will be an exponential enlargement after χ exponent has manifested himself, once the dynamics has settled down to eventual attracting invariant subspaces and even for volume preserving maps (null sum of divergences) we have that close lattice points will eventually evolve farther apart. At this point I need the hypothesis of weak mixing [7] to affirm that we can use the ergodic measure even on product spaces. If we wait enough we would see this enlargement to come to an end due to the compactness of the space under consideration, and the dense points that evolve densely in space will settle down to a constant distance from one another in the infinite dimensional space of functions $x(t, x_0)$ $\mathcal{T}$ as $t \to \infty$, this is due to the weak mixing hypothesis that allows convergence of time averages of the functions $\phi(x, y) = ||x - y||, \phi : V \times V \to R$ under the evolution of the point pairs $(x(t), y(t))$:

$$\lim_{T\to\infty} \{\frac{1}{T} \int_0^T \phi(x(t), y(t))dt\} = D$$

and D reflects the macroscopic size of the system.
Every trajectory started in the ϵ-grid is now equally distant from almost all the others! This is possible because the space is infinite dimensional, but more importantly it means that the map M has destroyed the metric structure and notion of neighbourhood in the domain V, still with a continuous evolution in the smooth f.

[7] We make this assumption just to analyse in these special cases what can we can say and visualize. We see that chaos is build upon a hierarchy of topological and measure properties that some systems might lack even if we usually consider them to be true. When a specific system has to be analysed we must pay attention to what chaotic features it has and be aware that most of the time these are tough to be mathematically proved

Chaos will even have a dense subset of periodic points whose orbit don't satisfy the convergence of time averages to the space one and in the $\mathcal{T}$ space these will have different distances from D.
Arbitrarily close and far points will be mapped to trajectories that will have a completely distorted metric and a dense set of equally distant neighbours.

B. Dissipative dynamics, a personal attempt

I've learned the hard way how this connection is not trivial and not well understood for continuous flows. The following is quite condensed and might might seem incomprehensible but the interested reader will find the details elsewhere or in the references mentioned. I've first tried to focus on the expansion of the small scale distances of the Takens reconstructed attractor [15] when longer samples are taken. When a point in the embedding is the result of a longer trajectory sampled with the appropriate time lag we have that, on average, labelled close trajectories (we will keep the numer of trajectories considered fixed) will be exponentially less frequent in a certain small volume due to Lyapunov divergence (of coure they are still dense but since we will work with finite resolution and amout of data this is what is seen). By varying the strength of chaos through a parameter we might measure the Kolmogorov complexity by considering how the density of points reduces at small scales in the embedding. This is effectively a known technique even illustrated in [12] but still it requires some sort of neighbour reconstruction that we have already said it's computationally demanding [8].
At this point one may want to study the Fourier Takens embedding by embed the fourier transform of the trajectories sampled. Power spectra of observables can be interpreted even as the variances of the Fourier components of the sampled observable $P(f,\tau,N) = \frac{\tau}{N} < |X(f,N,\tau)|^2 >$ (see [40] for more details on the definition of these spectra) and as the embedding time τN becomes very long we end up having (for weakly mixing systems where the ergodic measure can be used on Cartesian products of the space with itself) a distance between every pair of points in the embedding converges to something connected to the total integral of the power spectrum (all pairs are allowed to have the same distance because we are in a space of dimension N as the samples considered in the Takens reconstruction). This latter intended as the total variance of the embedding for long trajectories and should give an order or even be proportional to this length scale eventually reached.
Now, if we think of Kolmogorov entropy $\mathcal{K}$ as the rate of information needed to specify a chaotic trajectory and we need additional information to dynamics because in a pysically meaningful finite resolution partition of space exponential divergence makes beams of orbits starting at the same point in the discretized space to squeeze in space covering many elements of the partition at the same time, say n after a time Δt, we will then need to furnish $log_2(n)$ bits of information

[8] we might gain some speed by considering only a point in the embedding and don't loose the global view since the embedded points could represent long trajectories, anyway it turns out that to have a sufficient density of points one needs to run the simulation for very long times and the efficiency is not much better than the standard techniques to approximate Lyapunov exponents and Kolmogorov complexity

to specify the evolution in the coarse grained space; we can than conjecture that $\mathcal{K}\tau$ should be equal to the power spectrum integral that gives how all the points in the embedding repel from each other when scaled by N, so even in this high dimensional embedding where pieces of trajectories are represented by points we will need an amount of additional information to specify the position of the trajectory in the embedding since everything got enlarged by something like the total variance. I've found that if we choose the same length scale for the discretization as in the Kolmogorov entropy definition we should have a liner proportionality relation between these two quantities.
I'm not displaying these results with the space that they would require only because it's not the primary aim of this work but even because the numerical tests that I've run are not so convincing and seem to tell that the power spectra integral does not always quantify the true unknown variability between trajectories and indeed we are not considering the fact that dynamics has to reduce some of this unpredictability and the amount of information needed, indeed the total variance as I've called it should give an upper bound for the Kolmogorov complexity that in turn under suitable hypothesis (Pesin's entropy formula) is given by the total sum of positive Lyapunov exponents giving it a meaning of volume stretching measure.
The simulations seem to confirm the conjecture but with different proportionality coefficients for different differential laws, whereas by varying the parameters (constants entering the equations) of the same differential laws we obtain a family of systems with different Lyapunov exponents and chaotic strengths but with a somewhat equal differential structure, in these families it seems that the conjecture holds without any adjustment of the proportionality constant.

Later I've found that some research was carried out to connect the decays of the power spectrum and the Lyapunov exponents, the idea of Sigeti [27] was to find the cause of the exponential decay rates in the singularities of the analytical continuation for complex times of the trajectory function $x(t)$. A sketched derivation even though not mathematically rigorous (to see some rigorous partial results [41]) is outlined and by the residue theorem to compute The Fourier transform of the correlation function gives a reason for linking these poles to the exponential decay, the nearest pole to the real axis giving the major contribution. The connection with the Lyapunov divergence is that even when integrating the tangent dynamics on the real evolution, when we pass under the closest pole we get the major contribution to the Lyapunov exponent.
This, as explained in the paper gives additional support to the conjecture that $\mathcal{K} \sim \sum \lambda_+$.
Much more can and should be said but I want to end this supplement with a possible suggestion that I haven't worked out. The correlation spectra resonance poles studied by Ruelle that compose the transfer operator spectrum can be connected to the poles studied by Sigeti? I mean, what can be said about the spectrum of the Frobenius-Perron operator from the exponential decays of the power spectrum? Sigeti shows as a possible reason for the decay the presence of poles in the time complexified trajectory (for dissipative chaotic systems exhibiting fractal attractors) and since a well behaved observable of position should share the same poles so the correlation function, as a function of this evolution, will share them too; at the end can't we assert that Frobenius-Perron spectrum and exponential decay rates of power spectra are bond

from these singularities of the complex evolution?
This possible relation is far from being trivial: we are comparing how a single orbit behaves when analytically continued in complex times and the spectrum of the evolution operator for densities of states coming from the poles of correlation function spectra (resonances), expressible as the zeros of a zeta function with a cycle expansion obtained by averaging on the periodic structure, something that globally captures the ensamble dynamics. Far from being obvious it is anyway plausible since we expect the dense trajectory to fully represent the entire dynamics, it would be interesting if it does so by sharing some properties between complex poles.

C. Frobenius–Perron Spectra and Power Spectrum Poles

Consider a dynamical system with evolution flowΦ^t associated Frobenius–Perron operator $\hat{P}^t$. The spectral decomposition of $\hat{P}^t$ is

$$\hat{P}^t \psi_j(x) = e^{-s_j t}\, \psi_j(x),$$

with corresponding left eigenfunctions $\phi_j(x)$ such that

$$\langle \phi_j, \psi_k \rangle = \delta_{jk}.$$

For an observable $A(x)$, its time evolution under Φ^t can be written as

$$A(\Phi^t(x)) = \sum_j e^{-s_j t}\, \psi_j(x)\, \langle \phi_j, A \rangle.$$

Define the correlation function between observables A and B by

$$C(t) = \langle A(\Phi^t(x))\, B(x) \rangle - \langle A \rangle\, \langle B \rangle.$$

Omitting the invariant contribution (typically $s_0 = 0$), we have

$$C(t) = \sum_{j \neq 0} c_j\, e^{-s_j t}, \quad \text{with} \quad c_j = \langle \phi_j, A \rangle\, \langle B, \psi_j \rangle.$$

Taking the Fourier transform,

$$\tilde{C}(\omega) = \int_0^\infty e^{i\omega t}\, C(t)\, dt = \sum_{j \neq 0} c_j \int_0^\infty e^{(i\omega - s_j)t}\, dt,$$

which yields

$$\tilde{C}(\omega) = \sum_{j \neq 0} \frac{c_j}{s_j - i\omega}.$$

Thus, the Fourier-transformed correlation function $\tilde{C}(\omega)$ has poles at

$$\omega = -is_j,$$

demonstrating that the resonances ω do represent the exponential rates of Frobenius-Perron spectra. In particular the imaginary part of ω determines the decay rates, so that the existence of a strip of analyticity $|Im(\omega)| < \delta$ would imply a gap from the equilibrium one and an exponential decay of correlations.

Bibliography

[1] Hannes Risken. *The Fokker-Planck Equation: Methods of Solution and Applications*. Springer Berlin Heidelberg, 1996. ISBN 9783642615443. doi: 10.1007/978-3-642-61544-3. URL http://dx.doi.org/10.1007/978-3-642-61544-3.

[2] G. Nicolis and P. Gaspard. Toward a probabilistic approach to complex systems. *Chaos, Solitons and Fractals*, 4(1):41–57, 1994. ISSN 0960-0779. doi: https://doi.org/10.1016/0960-0779(94)90016-7. URL https://www.sciencedirect.com/science/article/pii/0960077994900167. Chaos and Order in Symbolic Sequences.

[3] P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner, and G. Vattay. *Chaos: Classical and Quantum*. Niels Bohr Inst., Copenhagen, 2016. URL http://ChaosBook.org/.

[4] H. Spohn. Spectral properties of liouville operators and their physical interpretation. *Physica A: Statistical Mechanics and its Applications*, 80(4):323–338, January 1975. ISSN 0378-4371. doi: 10.1016/0378-4371(75)90124-7. URL http://dx.doi.org/10.1016/0378-4371(75)90124-7.

[5] Mark M. Millonas and Linda E. Reichl. Stochastic chaos in a class or fokker-planck equations. *Physical Review Letters*, 68(21):3125–3128, May 1992. ISSN 0031-9007. doi: 10.1103/physrevlett.68.3125. URL http://dx.doi.org/10.1103/PhysRevLett.68.3125.

[6] M. V. Berry. *Structures in Semiclassical Spectra: A Question of Scale*, page 231–252. Springer Netherlands, 1984. ISBN 9789400962866. doi: 10.1007/978-94-009-6286-6_12. URL http://dx.doi.org/10.1007/978-94-009-6286-6_12.

[7] Ian C. Percival. Semiclassical theory of bound states, January 1977. ISSN 1934-4791. URL http://dx.doi.org/10.1002/9780470142554.ch1.

[8] *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 356 (1686):375–394, September 1977. ISSN 0080-4630. doi: 10.1098/rspa.1977.0140. URL http://dx.doi.org/10.1098/rspa.1977.0140.

[9] Martin C. Gutzwiller. Periodic orbits and classical quantization conditions. *Journal of Mathematical Physics*, 12(3):343–358, March 1971. ISSN 1089-7658. doi: 10.1063/1.1665596. URL http://dx.doi.org/10.1063/1.1665596.

[10] V. I. Arnold. *Mathematical Methods of Classical Mechanics*. Springer New York, 1989. ISBN 9781475720631. doi: 10.1007/978-1-4757-2063-1. URL http://dx.doi.org/10.1007/978-1-4757-2063-1.

[11] Mario Feingold and Asher Peres. Distribution of matrix elements of chaotic systems. *Physical Review A*, 34(1):591–595, July 1986. ISSN 0556-2791. doi: 10.1103/physreva.34.591. URL http://dx.doi.org/10.1103/PhysRevA.34.591.

[12] Robert Hilborn. *Chaos and Nonlinear Dynamics*. 09 2000. ISBN 9780198507239. doi: 10.1093/acprof:oso/9780198507239.001.0001.

[13] Robert Bishop. Chaos. In Edward N. Zalta and Uri Nodelman, editors, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, Winter 2024 edition, 2024.

[14] Edward Norton Lorenz. Deterministic nonperiodic flow. *Journal of the Atmospheric Sciences*, 20:130–141, 1963. URL https://api.semanticscholar.org/CorpusID:15359559.

[15] Cyril Touzé and Antoine Chaigne. Lyapunov exponents from experimental time series. application to cymbal vibrations. *Acustica*, 86, 05 2000.

[16] Calvin C. Moore. Ergodic theorem, ergodic theory, and statistical mechanics. *Proceedings of the National Academy of Sciences*, 112(7):1907–1911, February 2015. ISSN 1091-6490. doi: 10.1073/pnas.1421798112. URL http://dx.doi.org/10.1073/pnas.1421798112.

[17] M. H. Protter. Can one hear the shape of a drum? revisted. *SIAM Review*, 29(2):185–197, 1987. ISSN 00361445, 10957200. URL http://www.jstor.org/stable/2031658.

[18] P Cvitanovic and B Eckhardt. Periodic orbit expansions for classical smooth flows. *Journal of Physics A: Mathematical and General*, 24(5):L237–L241, March 1991. ISSN 1361-6447. doi: 10.1088/0305-4470/24/5/005. URL http://dx.doi.org/10.1088/0305-4470/24/5/005.

[19] David Ruelle. Resonances of chaotic dynamical systems. *Physical Review Letters*, 56 (5):405–407, February 1986. ISSN 0031-9007. doi: 10.1103/physrevlett.56.405. URL http://dx.doi.org/10.1103/PhysRevLett.56.405.

[20] D. Ruelle. Resonances for axiom a flows. *Journal of Differential Geometry*, 25(1), January 1987. ISSN 0022-040X. doi: 10.4310/jdg/1214440726. URL http://dx.doi.org/10.4310/jdg/1214440726.

[21] Jeffrey M. Heninger, Domenico Lippolis, and Predrag Cvitanović. Perturbation theory for the fokker–planck operator in chaos. *Communications in Nonlinear Science and Numerical Simulation*, 55:16–28, February 2018. ISSN 1007-5704. doi: 10.1016/j.cnsns.2017.06.025. URL http://dx.doi.org/10.1016/j.cnsns.2017.06.025.

[22] Mickaël D. Chekroun, Eric Simonnet, and Michael Ghil. Stochastic climate dynamics: Random attractors and time-dependent invariant measures. *Physica D: Nonlinear Phenomena*, 240(21):1685–1700, October 2011. ISSN 0167-2789. doi: 10.1016/j.physd.2011.06.005. URL http://dx.doi.org/10.1016/j.physd.2011.06.005.

[23] Julia Slipantschuk, Oscar F Bandtlow, and Wolfram Just. On the relation between lyapunov exponents and exponential decay of correlations. *Journal of Physics A: Mathematical and Theoretical*, 46(7):075101, February 2013. ISSN 1751-8121. doi: 10.1088/1751-8113/46/7/075101. URL http://dx.doi.org/10.1088/1751-8113/46/7/075101.

[24] L. Garrido, D. Luri, and M. San Miguel. Stochastic quantization and detailed balance in fokker-planck dynamics. *Journal of Statistical Physics*, 21(3):313–335, September 1979. ISSN 1572-9613. doi: 10.1007/bf01011473. URL http://dx.doi.org/10.1007/BF01011473.

[25] D. Sigeti and W. Horsthemke. High-frequency power spectra for systems subject to noise. *Physical Review A*, 35(5):2276–2282, March 1987. ISSN 0556-2791. doi: 10.1103/physreva.35.2276. URL http://dx.doi.org/10.1103/PhysRevA.35.2276.

[26] B. Caroli, C. Caroli, and B. Roulet. High-frequency power spectra of systems described by non-linear langevin equations. *Physica A: Statistical Mechanics and its Applications*, 112(3):517–522, June 1982. ISSN 0378-4371. doi: 10.1016/0378-4371(82)90193-5. URL http://dx.doi.org/10.1016/0378-4371(82)90193-5.

[27] David E. Sigeti. Exponential decay of power spectra at high frequency and positive lyapunov exponents. *Physica D: Nonlinear Phenomena*, 82(1–2):136–153, April 1995. ISSN 0167-2789. doi: 10.1016/0167-2789(94)00225-f. URL http://dx.doi.org/10.1016/0167-2789(94)00225-F.

[28] Bin Ye, Jianxing Chen, Chen Ju, Huijun Li, and Xuesong Wang. Distinguishing chaotic time series from noise: A random matrix approach. *Communications in Nonlinear Science and Numerical Simulation*, 44:284–291, March 2017. ISSN 1007-5704. doi: 10.1016/j.cnsns.2016.08.018. URL http://dx.doi.org/10.1016/j.cnsns.2016.08.018.

[29] Elsevier, 2007. ISBN 9780444529657. doi: 10.1016/b978-0-444-52965-7.x5000-4. URL http://dx.doi.org/10.1016/B978-0-444-52965-7.X5000-4.

[30] Mona Ben Said. *Decoding the Kramers–Fokker–Planck Operator: An Overview*, page 127–154. Springer Nature Switzerland, 2024. ISBN 9783031622656. doi: 10.1007/978-3-031-62265-6_7. URL http://dx.doi.org/10.1007/978-3-031-62265-6_7.

[31] Carlo Emilio Montanari. Diffusive models and chaos indicators for non-linear betatron motion in circular hadron accelerators. 2023. doi: 10.48676/UNIBO/AMSDOTTORATO/10811. URL http://amsdottorato.unibo.it/id/eprint/10811.

[32] Barry Simon. Schrodinger operators with purely discrete spectrum, 2008. URL https://arxiv.org/abs/0810.3275.

[33] A. Enciso and D. Peralta-Salas. On quantum integrability and hamiltonians with pure point spectrum. 2004. doi: 10.48550/ARXIV.MATH-PH/0406022. URL https://arxiv.org/abs/math-ph/0406022.

[34] Terence Tao. The asymptotic distribution of a single eigenvalue gap of a wigner matrix. *Probability Theory and Related Fields*, 157(1–2):81–106, September 2012. ISSN 1432-2064. doi: 10.1007/s00440-012-0450-3. URL http://dx.doi.org/10.1007/s00440-012-0450-3.

[35] Adel A. Rahman. The gutzwiller trace formula and the quantum-classical correspondence. 2020. URL https://api.semanticscholar.org/CorpusID:236165298.

[36] Todd K. Timberlake and Kya Wiggins. Transition in eigenvalue statistics due to tunneling in a simple quantum system. *Physical Review E*, 108(2), August 2023. ISSN 2470-0053. doi: 10.1103/physreve.108.024205. URL http://dx.doi.org/10.1103/PhysRevE.108.024205.

[37] Vladimir I. Arnold. Instability of dynamical systems with several degrees of freedom. *Hamiltonian Dynamical Systems*, 2020. URL https://api.semanticscholar.org/CorpusID:122498489.

[38] N N Nekhoroshev. An exponential estimate of the time of stability of nearly-integrable hamiltonian systems. *Russian Mathematical Surveys*, 32(6):1–65, December 1977. ISSN 1468-4829. doi: 10.1070/rm1977v032n06abeh003859. URL http://dx.doi.org/10.1070/RM1977v032n06ABEH003859.

[39] Tsampikos Kottos and Uzy Smilansky. Periodic orbit theory and spectral statistics for quantum graphs. *Annals of Physics*, 274(1):76–124, May 1999. ISSN 0003-4916. doi: 10.1006/aphy.1999.5904. URL http://dx.doi.org/10.1006/aphy.1999.5904.

[40] M C Valsakumar, S V M Satyanarayana, and V Sridhar. Signature of chaos in power spectrum. *Pramana*, 48(1):69–85, January 1997. ISSN 0973-7111. doi: 10.1007/bf02845623. URL http://dx.doi.org/10.1007/BF02845623.

[41] A. Cheskidov, M. S. Jolly, and E. S. Van Vleck. On a relation between lyapunov exponents and the radius of analyticity. *Indiana University Mathematics Journal*, 57(6):2663–2680, 2008. ISSN 0022-2518. doi: 10.1512/iumj.2008.57.3826. URL http://dx.doi.org/10.1512/iumj.2008.57.3826.