Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE

Corso di Laurea in Matematica

Combinatorial Garside Structures for Braid Groups

Combinatoria

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Anno Accademico 2023-2024

Contents

Introduction

The study of symmetries is widely regarded as one of the most elegant topics in mathematics. In modern times, the interest for symmetries has been responsible for the development of many branches such as group theory and combinatorics, from Galois theory to many areas of algebraic topology. Inevitably, the core of this trending interest can be found in the study of reflectional symmetries. Their comprehension has been possible only through the development of Weyl and Coxeter groups theory. In particular, by studying a discrete group of reflections, one can take advantage of a wide variety of both geometric and algebraic tools to better understand how it behaves, such as chambers, walls, root systems and specific group presentations. Since the works of Hermann Weyl, the interest for some of these tools and techniques grew to give birth to new and independent areas of interest, in the works of Harold Coxeter, Jaques Tits and many others. As an example, Coxeter groups theory itself and the study of hyperplane arrangements are both carried on and still studied nowadays. Coxeter groups theory in particular made it possible to better understand many algebraic aspects of reflection groups, and offered a way to address the problem of their classification, which is still not totally solved. At the same time, the new tools developed made it possible for new questions and constructions to be formulated. In general, from a finite Coxeter group one can construct a new group, called its braid group. The presentation of the braid group can be shown to be very closely related to the one of its Coxeter group, and its geometrical meanings mime many of its properties. From the generalization of braid groups a new important class of groups was defined: Artin groups, destined to take place by right among the most misterious objects in modern mathematics.

Taking a closer look to the theory, one recognizes that many known groups admit a Coxeter group presentation; as an example, symmetric and dihedral groups are cases of spherical (i.e., finite) Coxeter groups. The braid group of the symmetric group is known as the braid group on n-strands, or coharsely *the* braid group, that can be tought as the group of the possible crossing moves on n strands generating a braid. Many classic problems in group theory become particularly intriguing when contextualized to the braid group, mainly because of this realization, such as the word problem and the obvious questions about its torsion freeness and its center. The works that allowed to find a solution for the word problem in the braid group are mostly the ones by Frank Garside. His insights were particularly profund: it turned out that the theory he developed could be extended to a wide class of groups, called Garside groups. The features of Garside theory, as it was later carried on in the works of many scholars such as Luis Paris and Patrick Dehornoy, are the presence of a group with a given generating set, and the choice of an element called Garside element, for which the factorization poset with respect to the given set of generators has some given properties. The main result of the whole theory can be summed up in what follows: the Hasse diagram of the poset embeds in the Cayley graph of the monoid, which embeds in the Cayley graph of the group; this allows to solve the word problem in the group, and to construct a finite $K(G,1)$, with many important consequences. Recently, a new approach to Garside theory has been proposed, considering posets called combinatorial Garside structures. The aim is to make some requiremets to a labelled poset so that it can be found as a factorization poset for a Garside element in some Garside group. That group can be recovered from the poset, so that in the poset the information of a whole Garside structure (namely, the triple formed by a group, a generating set for the group and a Garside element in the group) is already cointained. This modern approach is particularly useful when it comes to apply the results from Garside theory to braid groups. In fact, it can be shown that in every spherical Coxeter group the factorization posets of some elements with respect to opportune sets of generators are combinatorial Garside structures. The Garside group arising from those structures can be shown to be always the braid group of the spherical Coxeter group. This makes it possible to solve the word problem in braid groups, and construct finite $K(\pi,1)$s.

The results mentioned are more historically interesting than it seems at first sight. This approach has actually made it possible to prove the famous $K(\pi,1)$-conjecture in the spherical case, stating that the braid group of every spherical Coxeter group admits a finite $K(\pi,1)$. In very recent times, it has been possible to prove the conjecture also in the case of Artin groups (the analogs of braid groups for an arbitrary Coxeter group) arising from Coxeter groups of the so-called affine type, in the work of Mario Salvetti and Giovanni Paolini. The conjecture is still open for the remaining two classes of Coxeter groups: hyperbolic and higher-rank Coxeter group.

In this dissertation, we present Garside theory with particular interest to the modern approach through combinatorial Garside structures, and then introduce the techniques and the tools that make it possible to recover the two combinatorial structures inside the symmetric group. To do so, we will consider two different presentations for the symmetric group, and for each find an element that we will show to be a Garside element. Finally, we will define and give a brief presentation of the braid group, and show that the groups arising in the two cases of combinatorial Garside structures found are bot isomorphic to the braid group, thus exemplifying what mentioned in the special and most suggestive case of the braid group.

Foundational material

We start by outlining very briefly some material that will be useful in the following. Material in this section includes some notions regarding poset theory, in particular the definition of labeling on a given poset and the lattice property; the section closes with a useful lemma about lattices. In the second section, our interest will shift to the construction of Cayley graphs and the length function.

Poset combinatorics

We call a *poset* a partially ordered set, i.e a set endowed by a reflexive, antisymmetric, transitive relation, called *partial order relation* and often noted $\leqslant$.

Let $(P, \leqslant)$ be a poset. It is said to be *bounded* if exists an element $\hat{1} \in P$ such that $\forall x \in P : x \leqslant \hat{1}$ and another element $\hat{0} \in P$ such that $\forall x \in P : \hat{0} \leqslant x$. It is said to be *graded* if every maximal chain has the same length. It is said to be *of finite height* if it has no infinite chain.

Example 0.0.1 (The product poset.). Let $\mathcal{P}_1 = (P_1, \leqslant_1)$, $\mathcal{P}_2 = (P_2, \leqslant_2)$ two posets. We call the *product poset* of $\mathcal{P}_1$ and $\mathcal{P}_2$ the poset

$$\mathcal{P}_1 \times \mathcal{P}_2 = (P_1 \times P_2, \leqslant)$$

where the relation $\leqslant$ holds between two elements (p_1, p_2) and (q_1, q_2) if and only if $p_1 \leqslant_1 q_1$ and $p_2 \leqslant_2 q_2$.

Particular interest has the following class of relations.

Definition 0.0.2 (Covering relations)**.** A relation $x \leqslant y$ is called a *covering relation* if whenever $x \leqslant z \leqslant y$, it holds that $z = x$ or $z = y$. We say that x *covers* y.

In fact, one can observe that these relations are the ones represented in the Hasse diagram of a poset, since the knowledge of such relations is sufficient to determine the whole poset structure. In fact, every relation in P that is not a covering relation breaks in a sequence of covering relations. The sequence is not unique in general. A sequence of covering relations $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_k$ is called a *chain*. The set of chains starting with x_1 and ending with x_k will be noted $[x_1, x_k]$.

In general, we will note a poset $(P, \leqslant)$ with P and consider it as endowed by the partial order relation $\leqslant$, without any further comments. We consider a poset P, and the set formed by all the relations in P, i.e. the set $R(P) = \{(x, y) \in P^2 \,|\, x \leqslant y\}$, an arbitrary set L that we will call *set of labels* and a set-theoretic map $\Lambda : R \to L$, called *labelling with labels in* L.

Definition 0.0.3. A poset P is a *labelled poset* if it endowed by a set L of labels and a labelling Λ with labels in L.

Remark 0.0.4. Having an arbitrary set L and a map Λ' from the set of covering relations $CR(P)$ to L, there is a way to induce a labelling Λ with labels in $\mathrm{Lang}(L)$ the language of words with letters in L, i.e. the class of sets of words with letters in L, so that the restricted map

$$\Lambda : CR(P) \to L$$

coincides with Λ'. To see this, first consider the chain $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_k$ with $x_i \leqslant x_{i+1}$ for every i. This chain corresponds to a sequence of covering relations $x_i \leqslant x_{i+1}$ for each of which we have $\Lambda'(x_1, x_2)$, so that by reading the labels $\Lambda'(x_1, x_2), \ldots \Lambda'(x_{k-1}, x_k)$ we get a word in $\mathrm{Lang}(L)$. Whenever we have $x \leqslant y$, one can set $\Lambda(x, y)$ to be the class of the words read on chains in $[x, y]$, thus getting a labelling of the whole poset.

We now leave labellings to consider a special property that a poset can have, and that will be crucial in the following.

Definition 0.0.5. We pose for every couple of elements $x, y \in P$ the element $x \wedge y$ as, when it exists, the greatest among the elements that are smaller than both x and y, and call it their *meet*. If such an element exists for every couple of elements in P, the poset is called a *meet semilattice*. Conversely, we pose $x \vee y$ as, if it exists, the lowest among the elements that are greater than both x and y, and call it their *join*. If such an element exists for every couple of elements in P then it will be called a *join semilattice*. If a poset is both a join and meet semilattice, it is called a *lattice*.

Example 0.0.6. We consider the class $\mathcal{P}(X)$ of the subsets of a given set X, with the partial order relation given by inclusion. There exists a lowest subset containing both Y_1 and Y_2, which is clearly the union $Y_1 \cup Y_2$. There exists also a greatest subset contained in both, which is the intersection $Y_1 \cap Y_2$. Thus, $\mathcal{P}(X)$ under the partial order relation given by inclusion is a lattice.

Example 0.0.7. The set of all partitions of a given set P has a natural order of inclusion. We say that a partition A is included in another partition B if every class of the first partition is included in some class of the second, and we set in that case $B \leqslant A$. We now consider the partition whose classes are obtained by intersecting some two classes of A, and observe that setting $A \wedge B$ to be this partition, $(P, \leqslant, \wedge)$ is a meet semilattice. On the other hand, consider the following relation; for any $a \in A$, $b \in B$ respectively in two classes $\mathcal{A}$ and $\mathcal{B}$, we set $a \sim b$ if and only if $\mathcal{A} \cap \mathcal{B} \neq \varnothing$. This is an equivalence relation, and induces a partition. We set $A \vee B$ to be this partition, and $(P, \leqslant, \vee)$ is a join semilattice. Thus $(P, \leqslant, \wedge, \vee)$ is a lattice.

Example 0.0.8. Let P_1, P_2 two posets. We consider their product poset as in example 0.0.1. If P_1, P_2 are lattices, then $P_1 \times P_2$ is a lattice. In fact, if $p_1 \wedge q_1 = r_1$ in P_1 and $p_2 \wedge q_2 = r_2$ in P_2 then $(p_1, q_1) \wedge (p_2, q_2) = (r_1, r_2)$ in $P_1 \times P_2$. The same holds for joins.

The following lemma outlines a sufficient condition for a bounded, finite height poset to be a lattice;

Lemma 0.0.9. Let P be a finite join-semilattice for which there exist a unique minimal element. Then P is a lattice.

Cayley graphs and length function

We start by recalling some basic facts about group combinatorics. The fist definition we give is of a useful tool in group combinatorics that we will use consistently.

Definition 0.0.10 (Length function[1])**.** Let G be a finitely generated group with presentation $\langle S \mid R \rangle$. Among the words $s_1 \cdots s_k$, with $s_i \in S$, representing the element $w \in G$, we define the *length* of w to be the lowest k:

$$l(\sigma) = \min\{k \mid s_1 \cdots s_k = \sigma, s_i \in S\}$$

Words representing the element $g \in G$ that realize this length are called *reduced words* for g.

Length functions play a fundamental role in understanding the way factorizations for a given element in a group behave. There is a nice geometric interpretation for length functions. To see it, we should make the following construction, that will be useful in the following.

[1]This term usually refers to a much wider class of objects, and the object of this definition is often referred to as *word metric*. Since this is the only case we will be interested in, though, we choose to use this terminology.

Definition 0.0.11 (Cayley graph)**.** Let G a finitely generated group, with S set of generators. We construct an oriented, labelled graph as follows; we take the 0-skeleton to be the set of the elements in G, and we take for each $g \in G$ and each $s \in S$ an oriented edge labelled s from g to $g \cdot s$ (respectively, $s \cdot g$). The resulting graph is called *right* (respectively, *left*) *Cayley graph* for (G, S) and will be noted $\mathrm{Cayley}(G, S)$.

Unless otherwise specified, Cayley graphs are usually considered to be right Cayley graphs.

As mentioned, we can interpret the length of an element g as the number of edges crossed by the shortest path among the ones starting in the vertex corresponding to the identity element and ending in the vertex corresponding to the element g, and crossing only edges in the orientation-wise direction.

In general, we will deal with a special case of Cayley graphs, for which in the considered group G every generator is an involution (i.e. every generators g satisfies $g^2 = e$). In that case we will use a sort of *involution convention*, meaning that oriented edges of the Cayley graphs will be crossed orientation-wise when multiplying a firts time by the corresponding generator, and a second time counterorientation-wise when multiplying again, gettin back to the starting point. We will apply this convention without any additional comment.

Chapter 1

Combinatorial Garside structures

Garside structures firstly appeared in Garside's works on the word problem in braid groups [10], and arise naturally from the combinatorics not only of symmetric groups (and thus braid groups), but of a wider class of objects[1]. This section is intended as a brief presentation rather that a proper introduction to Garside theory; in subsection 1.1 notions of Garside theory will be introduced from a combinatorial point of view following the ideas of [5] and [14], by outlining the combinatorial properties of the group-theoretical objects from Garside theory. The link with classical Garside structures will then be sketched, and a solution for the word problem will be given for the arbitrary Garside structure in subsection 1.2. We will then in 1.3 leave this presentation to outline another application of the theory, regarding the construction of finite $K(\pi, 1)$s for groups arising from combinatorial Garside structures; this will be done in full generality for arbitrary Garside groups, relying on the notions introduced.

1.1 Garside structures and Garside groups

From now on, let P be a bounded, graded poset of finite height with a labelling given by a function Λ on the relations, with L set of labels on covering relations. In practical situations, we will define the labels only on covering relations and consider the labelling to be induced as in remark 0.0.4.

We now will make some assumptions on P and its labelling. Let $\mathrm{Pre}(P)$ be the union on $x \in P$ of all sets of the form $[\hat{0}, x]$, corresponding to the

[1] Actually, combinatorial Garside structures arise in general from the combinatorics of Coxeter groups in both the spherical and affine case (see e.g. [14]) and have proved to be effective tools to deal with the combinatorial structure of such groups, ultimately leading to answers to many questions regarding both Coxeter and Artin groups (see [13] for a survey on the whole topic).

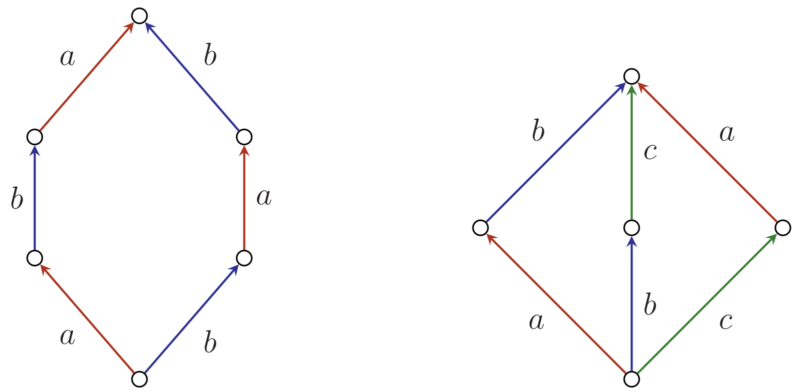


Figure 1.1: Two examples of combinatorial Garside structures

set of all prefixes appearing in some word in $[\hat{0}, \hat{1}]$, and similarly $\mathrm{Suf}(P)$ the union of sets of the form $[x, \hat{1}]$, corresponding to the set of all suffixes.

Definition 1.1.1 (Combinatorial Garside structures)**.** Let P be a bounded, labelled, graded poset of finite height.

- P is said to be *balanced* if $\mathrm{Pre}(P) = \mathrm{Suf}(P)$;
- P is said to be *group-like* if for every $x \leqslant y \leqslant z$, $x' \leqslant y' \leqslant z'$ in P, if labels on two corresponding relations coincide, then labels on the third coincide (e.g. if P is group-like, $\Lambda(x, y) = \Lambda(x', y')$ and $\Lambda(y, z) = \Lambda(y', z')$ imply $\Lambda(x, z) = \Lambda(x', z')$);
- P is said to be *Garside-like* if it is both balanced and group-like;
- P is said to be a *combinatorial Garside structure* if is Garside-like and a lattice;

Example 1.1.2. The Hasse diagram in figure 1.1 on the left has edges labelled a and b. The represented poset P is graded, bounded and of finite height. It holds that $\mathrm{Pre}(P) = \{a, b, ab, ba, aba, bab\} = \mathrm{Suf}(P)$, so that P is balanced. Also, triangles that form are labelled with just one word, so that the poset is group-like as a consequence of the construction in remark 0.0.4. We have therefore the Garside-like property. On the other hand, the lattice property holds since if x divides y then $x \wedge y = x$ and $x \vee y = y$, while if x and y are not ordered then $x \wedge y = \hat{0}$ and $x \vee y = \hat{1}$. Then, P is a combinatorial Garside structure. We leave the poset on the right for the considerations of the reader.

In general, it is possible to define a group from a combinatorial Garside structure, presented using L as set of generators, and relations equating any two words in $[\hat{0}, \hat{1}]$:

$$G(P) := \langle\, x \in L \mid w_1 \cdot w_2^{-1} \quad \forall w_1, w_2 \in [\hat{0}, \hat{1}]\,\rangle.$$

Example 1.1.3. Conside the poset on the left in figure 1.1. Associated to this poset we find the group

$$G(P) = \langle\, a, b \mid\ aba = bab \rangle.$$

We again leave the group obtained from the poset on the right for the considerations of the reader.

While a feature of this approach is to define a group $G(P)$ starting from a poset, classical Garside theory proceeds in the opposite direction, starting with a group G and constructing a poset P in the following sense. Given a finitely generated group G with S set of generators, S also naturally generates a monoid M in G. Assume that G was such that for every element $x \in M$ there is a bound on the length of factorizations of x in G. The choice of an element $\delta \in M$ gives place to a labelled, bounded poset of finite height obtained from its left or right divisors in G under the relation given by divisibility: $\mathrm{Fact}(G, S, \delta)$. Assume that the choice of $\delta \in G$ was such that $\mathrm{Fact}(G, S, \delta)$ is balanced and a lattice.

Definition 1.1.4. A group G for which the latter construction is possible is called a *Garside group*; (G, S, δ) as outlined is said to be a *Garside structure*; δ is said to be a *Garside element* for G.

Remark 1.1.5. This construction does not produce a combinatorial Garside structure, in general. In fact, not always $\mathrm{Fact}(G, S, \delta)$ as a poset is graded; an example is shown in figure 1.2. When it is, (G, S, δ) is said to be a graded Garside structure, and in that case $\mathrm{Fact}(G, S, \delta)$ is a combinatorial Garside structure in the sense of definition 1.1.1[2].

This makes posets defined in 1.1.1 special cases of factorization posets obtained from Garside structures; a justification for this will be complete in remark 1.2.6.

[2]A characterization of those Garside structures (G, S, δ) such that $\mathrm{Fact}(G, S, \delta)$ is a combinatorial Garside structure exists. One can observe that $\mathrm{Fact}(G, S, \delta)$ as a poset is graded if and only if a function sending every element of S in 1 extends to a group homomorphism from G in $\mathbb{Z}$. A group G is called weakly-graded if a weaker condition holds: there exists some function from S to the positive integers which extends to a group homomorphism from G to $\mathbb{Z}$. There exists a bijective correspondence between combinatorial Garside structures as in 1.1.1 and weakly-graded Garside groups. A detailed proof can be found in [5].

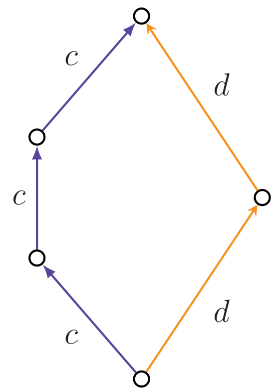


Figure 1.2: Bounded, finite-height, balanced, group-like poset that is not graded.

1.2 Garside monoids and word problem in Garside groups

1.2.1 Garside monoids

Let (G, S, δ) a Garside structure. As mentioned, the set S of generators also generates a monoid M inside G. Following from the properties of Garside structures, the monoid:

1. is atomic (i.e. is generated by its indivisible elements and there is a bound in the length of factorizations on these elements);
2. left and right cancellation laws hold (since it was obtained inside a group);
3. any two elements of M admit a least common multiple and a greatest common divisor on both the left and the right;
4. there exists an element δ such that the left and right divisors of δ form the same finite set of generators for M (because of the choice of δ);

Definition 1.2.1 (Garside monoid)**.** A monoid M satisfying the properties listed is called a *Garside monoid*. When obtained inside a Garside group G, it is called its *positive monoid* and is noted G^+.

The condition for a monoid to be left and right cancellative, and to admit right common multiples, is often referred to as “right Ore condition" (respectively, “left Ore condition"). Right Ore condition implies that the monoid embeds in its group of right fractions, isomorphic to the group of left fractions (a complete study along with a justification for this assertion is found in [6, p.351 etc.]). A Garside group G is thus also isomorphic to the group of fractions of its Garside monoid M (since the group of fractions for M will be a subgroup in G containing the set S of generators). The following theorem regarding groups of this form will be used in section 1.3. A complete proof is the object of [7] and [8].

Theorem 1.2.2. Let G be the group of right fractions of a right- and left-cancellative monoid M for which the partial order relation given by right divisibility has no infinite descending chain, and every couple of elements admits a lowest common right multiple[3]. Then G is torsion free.

In the special case of Garside groups obtained from combinatorial Garside structures, the positive monoid can be recovered from poset P by the monoid presentation

$$M(P) := \langle\, x \in L \mid w_1 \cdot w_2^{-1} \quad \forall w_1, w_2 \in [\hat{0}, \hat{1}]\,\rangle.$$

The following theorem justifies the name "combinatorial Garside structures"; a proof is discussed in [5].

Theorem 1.2.3. If P is a combinatorial Garside structure, then $M(P)$ is a Garside monoid, and its group of fractions is isomorphic to $G(P)$.

Example 1.2.4. For combinatorial Garside structure on the left in figure 1.1, this construction yields the monoid

$$M(P) = \langle\, a, b \mid aba = bab\,\rangle.$$

This monoid is a special case for what is called the *positive braid monoid*. The study of the word problem in braid groups was the main reason for which positive braid monoids were studied, leading ultimately to Garside theory.

Example 1.2.5. The lattice of the subsets of a given set presented in example 0.0.6 for a finite set is an example of what is called a *Boolean lattice*. Finite Boolean lattices such as this are combinatorial Garside structures. For a set of cardinality k, the monoid obtained from this combinatorial Garside structure is $\mathbb{N}^k$, while the group is $\mathbb{Z}^k$. In this example the fact that the group G obtained from the combinatorial Garside structure is the group of fractions for M is particularly evident.

[3] A monoid with those properties is sometimes called a right cancellative right Gaussian monoid. Groups that are groups of fractions of Gaussian monoids are referred to as Gaussian groups.

Remark 1.2.6. As an immediate consequence we have the following. Consider a poset P with covering relations labelled with labels in L, and $G(P)$ the group constructed as above from P. Then $(G(P), L, \hat{1})$ is a Garside structure, where $\hat{1}$ is the element in $G(P)$ represented by any word in $[\hat{0}, \hat{1}]$. This also means that the Hasse diagram $\mathrm{Hasse}(P)$ embeds in the Cayley graph $\mathrm{Cayley}(M(P), S)$, that embeds in $\mathrm{Cayley}(G(P), S)$. As a consequence, for a combinatorial Garside structure (G, S, δ), the factorization poset $\mathrm{Fact}(G, S, \delta)$ is found inside the Cayley graph of G with respect to the generating set S. All the facts listed are studied in detail and generalized in [5].

1.2.2 A solution for the word problem in Garside groups

The presence of a monoid with the properties listed has proved over time to be an effective tool when dealing with word and conjugacy problems in groups, and this was in fact the reason for which the deriving structure was originally studied in [10] in the special case of braid groups (cf. section 3). We will present the solution for the word problem associated to an arbitrary Garside group, as obtained in [9]. Proofs of the results presented in this subsection are the object of the cited paper. *From now on, we will adopt the notation G^+ for Garside monoids, as they will be supposedly have been obtained from a Garside group G whose word problem we are concerned with.*

Let (G, S, δ) be a Garside structure and G^+ the positive monoid associated to G. We will refer to divisors of δ in G^+ as *simple divisors*. The set of simple divisors will be noted D. Because of the choice of δ, they form a set of generators for G^+. We need firstly to solve the word problem in G^+.

Definition 1.2.7 (Greedy normal form)**.** For $k \in \mathbb{N}$, $d_1, \dots, d_k \in D$ such that $d_i = (d_i \cdots d_k) \wedge \delta$ for every $i = 1, \dots, k$, the expression $d_1 \cdots d_k$ is called a *left greedy normal form.* Conversely, if $d_i = \delta \wedge (d_1 \cdots d_i)$ for every $i = 1, \dots, k$, the expression $d_1 \cdots d_k$ is called a *right greedy normal form.*

Note that the definition relies on the properties of Garside monoids when supposing right and left greatest common divisors exist. The form provides an algorithmic solution to the word problem in the monoid G^+, as it satisfies the following property:

Proposition 1.2.8. Let G^+ be a positive monoid. Every $g \in G$ can be expressed in a unique left greedy normal form. The same holds for right greedy normal form.

Example 1.2.9. For the group in example 1.1.3, which we recall was presented as

$$B_3 = \langle\, a, b \mid aba = bab \rangle$$

element $\delta = aba = bab$ is a Garside element, so $(B_3, \{a, b\}, aba = bab)$ is a Garside structure. Consider the element a^2b^2ab in the monoid B_3^+. Its left greedy normal form is $a(ab)\delta$, since $ab \wedge \delta = ab$, $a^2b \wedge \delta = a$ in B_3^+, and $a(ab)\delta = a^2b^2ab$ in B_3^+.

Since M an arbitrary Garside monoid is cancellative, the partial order given by divisibility in M can be extended to a partial order in its group of fractions G, which coincides with the partial order given by divisibility in G. This can be done by posing $g \leqslant h$ if and only if $g^{-1}h \in M$, where M is seen as set-theoretically included in G (this clearly is the case for positive monoids).

The correspondence between the partial order relation induced by divisibility in both M and G makes it possible to extend the solution for the word problem in M to the one in G. In fact, the multiplication of $g \in G$ by a suitable integer power of δ is such that the product lies in G^+, so that the result can be expressed in (left) greedy form. This gives a normal form for the elements in the group G.

Definition 1.2.10 (Deligne normal form)**.** For $k \in N$, $n \in \mathbb{Z}$, $d_1, \ldots, d_k \in D$ such that $d_1 \cdots d_k$ is in G^+ and in left greedy normal form and $d_1 \neq \delta$, the expression $d_1 \cdots d_k \cdot \delta^n$ is called a *Deligne normal form.*

Theorem 1.2.11. Let G be a Garside group. Every $g \in G$ can be expressed uniquely in Deligne normal form.

From now on, the Deligne normal form of an element $g \in G$ will be noted $\mathrm{DNF}(g)$. Deligne normal forms clearly provides a criterion to solve the word problem in G.

Example 1.2.12. Let $\mathcal{G}_1 = (G_1, S_1, \delta_1)$ and $\mathcal{G}_2 = (G_2, S_2, \delta_2)$ two Garside structures. Then

$$\mathcal{G}_1 \times \mathcal{G}_2 = (G_1 \times G_2, S_1 \times S_2, (\delta_1, \delta_2))$$

is in fact a Garside structure; the poset $\mathrm{Fact}(G_1 \times G_2, S_1 \times G_2, (\delta_1, \delta_2))$ is actually the product poset as in example 0.0.1 obtained from the two posets *Fact*(G_1, S_1, δ_1) and *Fact*(G_2, G_2, δ_2). In particular we could define from this observation in the obvious way the *product combinatorial Garside structure.* We incidentally observe that the lattice property holds from the observations in example 0.0.8. It is natural to refer to $\mathcal{G}_1 \times \mathcal{G}_2$ as to the *product Garside structure.* Let g_1 an element in G_1 with Deligne normal form $d_1^1 \ldots d_s^1 \delta_1^n$ and another element g_2 in G_2 with form $d_1^2 \ldots d_t^2 \delta_2^m$. We can obtain the Deligne normal form for (g_1, g_2) as follows. First observe

$$(g_1, g_2) = (d_1^1 \ldots d_s^1 \delta_1^n, d_1^2 \ldots d_t^2 \delta_2^m)$$

. Assume $n \leqslant m$ and $s \leqslant t+m-n$. One can adjust the length of the normal forms to form a factorization with elements of $S_1 \times S_2$ or (δ_1, δ_2). In fact (g_1, g_2) can be written as the product of: n factors of the form (δ_1, δ_2); $m-n$ factors of the form (d^1_{m+i}, δ_2) for $i = 0, \dots, m-n-1$; $s+n-m$ factors of the form $(d^1_i, d^2_{t-s-m+n+i})$ for $i = 1, \dots, s+m-n-1$; $t-s-m+n$ factors of the form (e_1, d^2_i) for $i = 1, \dots, t-s-m+n-1$. We get therefore

$$(e_1, d^2_1) \cdots (d^1_1, d^2_{t-s-m+n}) \cdots (d^1_m, \delta_2) \cdots (\delta_1, \delta_2).$$

This is actually the Deligne normal form for (g_1, g_2) in the product Garside structure. We leave to the considerations of the reader the structure

$$\left(\frac{G_1 * G_2}{\langle \delta_1 \delta_2 \delta_1^{-1} \delta_2^{-1} \rangle}, S_1 \cup S_2, \delta_1 \delta_2 = \delta_2 \delta_1 \right).$$

We now dive a little more in the combinatorics of Garside groups, by observing some symmetries of the set of simple divisors D.

Definition 1.2.13 (Complements)**.** For $d \in D$, an element $d^* \in D$ such that $dd^* = \delta$ is called *right complement* for d. An element ${}^*d \in D$ such that ${}^*dd = \delta$ is called *left complement* for d.

Obviously d^* is a left divisor for δ, while ${}^*d$ is a right divisor. By the choice of δ, this means that D is closed under the operation of taking complements, and the application from D into itself induced by taking complements is clearly injective (since a Garside monoid is cancellative), so that these operations define a bijection of D in itself. Another bijection of D in itself arises from the following property;

Proposition 1.2.14. For every $d \in D$ there exist a permutation σ of the elements of D such that for every $d \in D$

$$\sigma(d) \cdot \delta = \delta \cdot d.$$

Consequently,

- there exists a natural m such that δ^m is central
- if d is a product of irreducible elements, then $\sigma(d)$ is also a product of irreducible elements.

The link between the two bijections is given by observing that from the definition of complements it follows that

$${}^*d\delta = \delta d^*$$

so that $\sigma(d^*)$ and ${}^*d$ coincide. They express therefore the same symmetry, somehow. The following result unveils a useful property of DNFs, and will be useful in the following.

Lemma 1.2.15. Let $d_1 \cdots d_k$ be a word in G^+ in left greedy normal form, where $d_1 \neq \delta$, $\eta \in D$. The left greedy normal form for $d_1 \cdots d_k \cdot \eta$ begins with at most one Δ.

Proof. It is clear that right multiplication by $\eta \in D$ can produce no right factor δ or one righ factor δ (if it coincides with some right complement). We observe that if $\delta^2 \leqslant d_1 \cdots d_k \cdot \eta$ with the right order, then $g \cdot \delta^2 = d_1 \cdots d_k$ for some $g \in G^+$. By using permutation σ as in 1.2.14 one gets

$$\delta \cdot \sigma^{-1}(g) \cdot \delta = d_1 \cdots d_k$$

and then

$$\delta \cdot \sigma^{-1}(g) \cdot^* \eta = d_1 \cdots d_k$$

which is absurd, since it would mean that δ left divides $d_1 \cdots d_k$ in left greedy normal form, where $d_1 \neq \delta$. $\square$

1.3 The construction of finite $K(\pi, 1)$s

We now leave the ongoing brief presentation of Garside theory and its applications to word problems to take a much closer look to another application: the construction of finite $K(\pi, 1)s$ for the abitrary Garside group. The construction will be very explicit, and will rely heavily on the solution presented for the word problem in the arbitrary Garside group. We start with the construction of a $K(\pi, 1)$ in the special case of combinatorial Garside structures.

Definition 1.3.1 (Order complex and interval complex)**.** We associate to P graded poset the simplicial complex obtained by taking as vertices the elements of P as a set, and as simplices the descending chains in P, with attachment maps given by recursively identifying a facet in a simplex (corresponding to a maximal subchain) to the simplex representing said chain. This complex will be called *order complex* (or *geometric realization*) of P and noted $\Delta(P)$. Note that edges (1-simplices) are naturally oriented by the poset structure. If P is labelled, then the extended labelling on descending chains determines a labelling for every simplex in $\Delta(P)$. We quotient the order complex by identifying every vertex, and then inductively identifying simplices of the same dimension with the same label. The complex obtained is called *interval complex* of P and is noted $K(P)$.

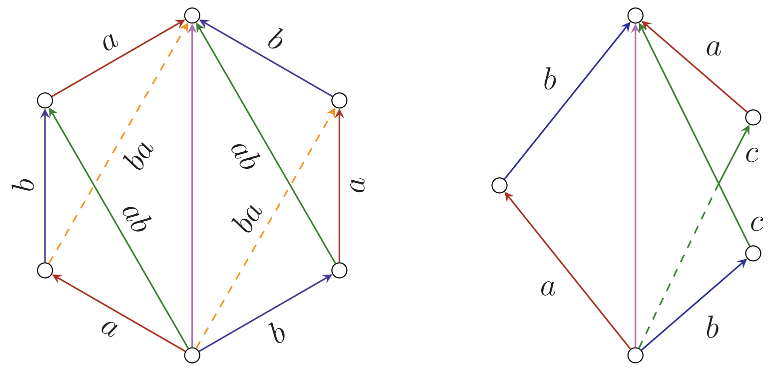


Figure 1.3: Order complexes for posets in figure 1.1. The interval complexes are obtained identifying edges with same labels, facets with edges pairwise identified and tetrahedra with facets pairwise identified.

If P has a minimal element $\hat{0}$, then $\Delta(P)$ is contractible (being a cone on $K(P - \{\hat{0}\})$). This is the case for P combinatorial Garside structure. Let (G, S, δ) be a Garside structure such that $\mathrm{Fact}(G, S, \delta)$ is a combinatorial Garside structure (see Remark 1.1.5). The action of $G(P)$ on vertices by left (right) multiplication extends linearly over simplices to an action on $\Delta(P)$. The quotient of $\Delta(P)$ under the action is $K(P)$. In fact, the action of G is transitive on vertices, and since P is balanced and group-like, the same holds for the action on 1-simplices with same labels; therefore, by induction on the dimension of simplices and by group-like-ness, the same holds for the action on simplices of higher dimension with same labels). The group $G(P)$ is a Garside group by remark 1.2.6, and is therefore torsion-free by theorem 1.2.2. The action of $G(P)$ on $\Delta(P)$ is therefore a covering action (since it is the extension of a free action of a torsion-free group on vertices of a simplicial complex). We have therefore that $K(P)$ is a $K(G(P), 1)$.

This idea, valid in the special case of combinatorial Garside structures, generalizes to arbitrary Garside structures. We start with the following general construction.

Definition 1.3.2 (Flag complex)**.** A *flag complex* is a simplicial complex where every complete sub-graph on n-vertices in the 1-skeleton of an $(n-1)$-simplex.

Let (G, S, δ) a Garside structure in G, with D set of simple divisors. Recalling that D is a generating set for G, and observing that a flag complex is identified by its 1-skeleton, we set $E(G, D)$ the (unique) flag complex having as 1-skeleton the (right) Cayley graph $\mathrm{Cayley}(G, D)$. The argument used in the previous case for the action of G extends identically to this setting, so that we find a covering action of G on $E(G, D)$. It is known that the quotient space will have fundamental group isomorphic to G.

Remark 1.3.3. In the special case of a Garside structure for which $\mathrm{Fact}(G, S, \delta)$ is a combinatorial Garside structure, by remark 1.2.6 the construction of $E(G, S)$ is linked to the construction of the order complex as in 1.3.1. In fact, for P combinatorial Garside structure, $E(G(P), P)$ is tiled with copies of $\Delta(\mathrm{Fact}(G, S, \delta))$. Clearly, the copies will be identified by the action of G, so that the complex $K(G, D)$ coincides with the interval complex $K(\mathrm{Fact}(G, S, \delta))$. Nonetheless, while for combinatorial Garside structures the contractibility of $\Delta(P)$ (and thus the $K(G(P), 1)$ property for $K(P)$) is easily granted, contractibility for the complex $E(G, D)$ is less obvious[4].

We now will show that $E(G, D)$ is contractible, thus proving for the general Garside structure (G, S, δ) the existence of a finite $K(G, 1)$. We consider the subcomplex of $E(G, D)$ constructed as follows. By construction, every vertex of $E(G, D)$ is associated to an element in G. This means that one can consider Deligne's normal forms of the vertices as in definition 1.2.10. In particular, we consider those vertices whose DNF contains no factor δ, i.e. those vertices for which n in definition 1.2.10 is 0. We restrict the Cayley graph (as a graph) to these vertices, by considering the subgraph that contains the vertices listed and only the edges that connect two such vertices in the original graph. The flag complex on this graph will be a subcomplex of $E(G, D)$, and will be noted $\tilde{E}(G, D)$.

Remark 1.3.4. The complex $\tilde{E}(G, D)$ can be tough as constructed as follows. We take a vertex for every coset of the cyclic subgroup generated by the Garside element δ. For every coset, taking a representatative and considering its DNF up to the power of δ is a well posed operation. Therefore, we can consider for each vertex (i.e. for each coset) a left greedy normal form. Two vertices are connected by an oriented edge whenever their left greedy normal forms differ by a multiplication for some $d \in D - \{\delta\}$. Edges can be labelled with d. Now, $\tilde{E}(G, D)$ is obtained by taking the flag complex from this graph.

This remark in fact makes one guess a "product" structure for $E(G, D)$, induced by projecting a DNF on its greedy normal form part and along the exponent n as in 1.2.10. Next proposition formalizes this intuition.

Proposition 1.3.5. The complex $E(G, D)$ is homeomorphic to the product $\tilde{E}(G, D) \times \mathbb{R}$.

[4]Althought it is possible to show contractibility for the complex $E(G, D)$ in an easier way in the special case of combinatorial Garside structures, by using the fact that complex $E(G(P), P)$ is tiled with copies of $\Delta(P)$ which we know to be contractible. Some ideas are found in [5] and [14]

Proof. We will use the coharse notations E and $\tilde{E}$, considering P and D fixed. On $\mathbb{R}$ we consider the simplicial complex structure given by taking $\mathbb{Z}$ as 0-skeleton and edges connecting each integer to its following. This induces a simplicial complex structure on the product $\tilde{E} \times \mathbb{R}$ in the following way. This product complex is tiled by objects of the form $\sigma \times I$ where σ is a simplex and I is either $\{n\}$ or $[n, n+1]$ for some n. In the former case we already have the cell structure. In the latter case, we can operate a subdivision of $\sigma \times I$ to give it a cell structure. Vertices of $\Delta \times \{0\}$ will have the form $a_0 < \cdots < a_k$ with $a_i \leqslant \delta$ for each i, and $a_k \leqslant a_0\delta$. Then,

$$\{a_0\delta^n < \cdots < a_k\delta^n < a_0\delta^{n+1}\}, \ldots, \{a_k\delta^n < a_0\delta n + 1 < \cdots < a_k\delta^{n+1}\}$$

induces a simplicial cell structure on the "prism" $\Delta \times [n, n+1]$. As observed, we have natural projections from the 0-skeleton of E onto the 0-skeleta of $\tilde{E}$ and $\mathbb{R}$ defined via the Deligne normal forms. Let $g \in G$ with Deligne normal form $\mathrm{DNF}(g) = d_0 \cdots d_k\delta^n$. We define π_d to be the map sending g to $d_0 \cdots d_k$ and π_δ the map sending g to n. The map π_d sends vertices of E onto vertices of $\tilde{E}$, while π_δ can be viewed as sending the vertices of E onto elements of the infinite cyclic group generated by δ, which can be identified with $\mathbb{Z}$ inside $\mathbb{R}$, so that π_δ sends vertices in vertices of the complex structure we gave to $\mathbb{R}$. We now extend π_d and π_δ linearly over simplices, forming two continuous maps from E to $\tilde{E}$ and $\mathbb{R}$ respectively. Since E, $\tilde{E}$ and $\mathbb{R}$ are flag complexes, they are uniquely determined by their 1-skeleta. Then, to check that those maps are well-defined, it suffices to show that π_d and π_δ take edges of E to edges or vertices of their target spaces, so that they take the 1-skeleton of E onto the 1-skeleta of $\tilde{E}$ and $\mathbb{R}$ respectively. For the rest of this discussion, fix an edge e in E and let $a \in G^+$, δ not smaller that a, and $\mu \in D$ such that the bounding vertices of e correspond to the group elements $a\delta^n$ and $a\delta^n \cdot \mu$. By definition of σ this second element is $a\sigma^n(\mu)\delta^n$.

First we prove our claim for π_δ. Element $a\sigma^n(\mu)$ is divisible by δ at most one time by lemma 1.2.15, so that only two cases can take place when we take the Deligne normal form for $a\delta^n\mu$. If δ does not divide a we have that

$$\mathrm{DNF}(a\delta^n \cdot \mu) = \mathrm{DNF}(a\sigma^n(\mu))\delta^n.$$

If δ divides a, then $a = b({}^*[\sigma^n(\mu)])$ for some b, for which

$$\mathrm{DNF}(a\delta^n \cdot \mu) = \mathrm{DNF}(b)\delta^{n+1}$$

. We have immediately that e is sent to the vertex n by the linear extension of π_δ in one case, and in the edge $[n, n+1]$ in the other. We now turn to π_d.

Clearly $\pi_d(a\delta^n) = a$. Turning to $\pi_d(a\sigma^n(\mu)\delta^n)$, we have that if a is not right divisible by $^*(\sigma(\mu))$ then no δ factor forms, so that

$$\pi_d(a\delta^n\mu) = a\sigma^n(\mu)$$

in this case, while if $a = b \cdot^* \sigma^n(\mu)$ then

$$\pi_d(a\delta^n\mu) = a \cdot^* (\sigma^n(\mu).$$

In each case, unless $\mu = \delta$, the elements corresponding to the bounding vertices of e differ by the multiplication for some element in D, so that e is sent to an edge by the linear extension of π_d. If $\mu = \delta$, however, e is sent to a vertex.

We have obtained two maps, that we can regard as a continuous map $h : E \to \tilde{E} \times \mathbb{R}$, defined as

$$h(g) = (\pi_d(g), \pi_\delta(g)).$$

This map is obviously a bijection (it is injective by uniqueness of Deligne normal forms, and obviously surjective). On the other hand there is also an obvious map $\hbar : E \to \tilde{E} \times \mathbb{R}$ such that on the vertices

$$\hbar((a, n)) = a \cdot \delta^n.$$

Again, we have to check that this map sends the 1-skeleton of the product complex into the 1-skeleton of E. We consider ad edge e in $\tilde{E} \times \mathbb{R}$. We call its bounding vertices (a, n) and (b, m). Up to switching them, it must be $m = n$ or $m = n + 1$, and $a\delta^n < b\delta^m < a\delta^{n+1}$. We check the two cases.

If $m = n$ then the inequality becomes $a < b < a\delta$, so that $b = a\mu$ with $\mu < \delta$ (so that $\mu \in D$). This implies that

$$\hbar((b, m)) = a\mu\delta^n$$

which is connected to $a\delta^n$ by an edge labelled $\sigma^{-n}\mu$ in E. On the other hand, if $m = n + 1$ then the inequality becomes $a < b\delta < a\delta$, so that $b < a$, that means $b = a\mu^{-1}$ for some $\mu \in D$. This implies $b\delta = a\mu^*$ so that

$$\hbar((b, m)) = a\delta^n\sigma^{-n-1}(\mu^*)$$

so that there is again an edge labelled $\sigma^{-n-1}(\mu^*)$ connecting the two images. We conclude the desired homeomorphism. □

In order to prove that $E(G, D)$ is contractible, it is now sufficient to prove contractibility for $\tilde{E}(G, D)$. To do this, we need to introduce a useful tool: the descending link of a vertex.

Definition 1.3.6. Let X a simplicial complex endowed by a map $F : X^0 \to \mathbb{R}$. Then the descending link of a vertex v $Lk_{\downarrow}(v)$ is defined as the link of v inside the subcomplex spanned by the vertices w such that $F(w) \leqslant F(v)$.

The following result will be used to show contractibility for $\tilde{E}(G, D)$; as a reference, one can consult [1].

Lemma 1.3.7. Let a simplicial complex X for which there exists a map $F : X^0 \to \mathbb{R}$ that is linear over each simplex, has discrete image and is noncostant on edges. If every vertex in X has contractible descending link with respect to F, then X is contractible.

Proposition 1.3.8. The complex $\tilde{E}(G, D)$ is contractible.

Proof. We again will write $\tilde{E}$ in place of $\tilde{E}(G, D)$. First we need to construct the map F as in lemma 1.3.7. We consider $\|a\|$ for an element $a \in G$ to be the maximal length of a word for a in the indivisible elements of D. We define the map $\nu : \tilde{E} \to \mathbb{R}$ on the 0-skeleton of $\tilde{E}$ as follows: for each vertex we know there is a corresponding element $a \in G$; we set therefore $\nu(v) = \|a\|$, and then extend the map by linearity over simplices. Note that the linearity of ν over every simplex is granted by construction.

We now want to show that F is non-constant on edges. We recall what was observed in remark 1.3.4 elements corresponding to vertices of $\tilde{E}$ are coset representatives for the infinite cyclic subgroup generated by δ. We consider the two possibilities: for $a \in G^+$ with $a \geqslant \delta$ and $d \in D, d \neq \delta$, the left greedy normal form of $a \cdot d$ begins with at most one δ. If it contains no δ, then $a \cdot d$ is a coset representative for the same coset, and $\nu(a \cdot d) = \|a \cdot d\| \geqslant \|a\| + \|d\| > \|a\|$. Otherwise we have that $a = b\sigma(d^*)$ for some coset representative $b \in G^+$, and right multiplication by d corresponds to the presence of an edge from b to a. Again, $\nu(a) = \|b\sigma(d^*)\| \geqslant \|b\| + \|\sigma(d^*)\| > \|b\|$.

We now turn to prove that the descending link of every vertex is contractible. We begin by reading every vertex v of $\tilde{E}$ as the coset representative $a \in G^+$ such that its left greedy normal form does not begin with δ. By unfolding the definition of descending link, we know that we have to consider the subcomplex induced by vertices $\{w \in \tilde{E}^{(0)} \mid \|w\| \leqslant \|v\|, \text{ and } w \cdot d = a \text{ for some } d \in D\}$. Therefore vertices of $Lk_{\downarrow}(v)$ are in correspondance with those $d \in D$ such that $\delta \leqslant a \cdot d$. We consider now the right greedy normal form for a, say $d_1 \cdots d_k$ with $d_k = \delta \wedge a$, then $\delta < a \cdot d \iff \delta = \delta \wedge (a \cdot d) = \delta \wedge d_k d$. This occurs exactly when $d_k^* \leqslant d$. Thus the descending link is the subcomplex spanned by the simple divisors $d \in D \backslash \{\delta\}$ that are greater than the right complement of $\delta \wedge a$, and the vertex corresponding to $(\delta \wedge a)^*$ is a cone point for $Lk_{\downarrow}(v)$, so that is is contractible. $\square$

Chapter 2

Combinatorial Garside structures in symmetric groups

In this chapter, we will consider two combinatorial Garside structures arising from Σ_n symmetric group on n formal objects. The first combinatorial Garside structure mimes the one originally introduced by Garside in [10] while studying braid groups in the classic Artin presentation (see chapter 3). This structure is then regarded widely as “classic". The second one arises from the much more recent publication [2], where an alternative presentation for the braid group is discovered and studied, leading to the so called “dual" structure. We will for each structure define it by recovering it in some factorization poset in the symmetric group, and observe that the definition of a combinatorial Garside structure is satisfied. The two structures we will study are in fact unique: in the symmetric group no other combinatorial Garside structure can arise, as follows from the general studies in [11].

We start by constructing the following partial order relation on Σ_n; this definition adapts [3].

Definition 2.0.1 (Weak Bruhat order in the symmetric group)**.** Let Σ_n be the symmetric group, S generating set for Σ_n. In this setting we define the *right weak Bruhat order* ($\leqslant_R$) and *left weak Bruhat order* ($\leqslant_L$) as follows. For $u, w \in \Sigma_n$

- $u \leqslant_R w$ if $w = u \cdot s_1 \cdots s_k$, where $s_i \in S$, so that $l(u) = l(w) + i$, for $1 \leqslant i \leqslant k$.

- $u \leqslant_L w$ if $w = s_k \cdots s_1 \cdot u$, where $s_i \in S$, so that $l(w) = l(u) + i$, for $1 \leqslant i \leqslant k$.

Remark 2.0.2. Note that $u \leqslant_R v \iff v^{-1} \leqslant_L u^{-1}$. Clearly, this partial order relation depends on the choice of generating sets for Σ_n. We mention that in terms of words, the relation $u \leqslant_R w$ means that u appears as a prefix in some reduced word representing w in Σ_n, while $u \leqslant_L w$ means that u appears as a suffix in one of such words.

Remark 2.0.3. By orienting edges in the right (left) Cayley graph of Σ_n with respect to generators S in the crescent direction for the length function, we recover the Hasse diagram of the weak right (left) Bruhat order with respect to S.

2.1 Classic structure

We start by considering Σ_n as generated by the set $S_n = \{ \sigma_i \mid 1 \leqslant i \leqslant n-1 \}$ where σ_i is the transposition $(i, i+1)$, under the following relations for distinct $\sigma_i, \sigma_j \in S$:

$$\sigma_i^2 = e$$

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{if } |i-j| = 1$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i-j| > 1$$

which we will call respectively *involution*, *commutation* and *braid* relations. This presentation is sometimes called *Coxeter-Moore presentation*, though we will eventually refer to it as classic presentation, to distinguish it from the dual one we will introduce in this chapter. A detailed discussion of the fact that this set of generators and relations presents Σ_n is beyond the pourpose of this dissertation, but we will nonetheless in the following cite and prove the main theorem that justifies this assertion (see theorem 3.2.3), though with different intentions.

Remark 2.1.1 (Notations). We will adopt the notation σ_i as above for specific elements in S_n the set of generators for Σ_n as above. However, when referring to an arbitrary element in S_n we will often note it s, we will also write S in place of S_n when n is obvious from the context or unimportant. Symbols σ, τ will be used to note arbitrary permutations in Σ_n. The set of all transpositions will be noted $T_n := \{(i,j) \mid 1 \leqslant i < j \leqslant n\}$. Again, the arbitrary element in T_n will be noted t, and index n will be dropped when obvious or unimportant.

2.1.1 Permutahedra and permutahedron order

We consider the action of Σ_n on $\mathbb{R}^n$ by permutation of coordinates. Permutation $\sigma \in \Sigma_n$ acts by

$$\sigma \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_{\sigma(1)} \\ x_{\sigma(2)} \\ \vdots \\ x_{\sigma(n)} \end{pmatrix}.$$

Transposition $(i, j) \in T_n$ acts as the reflection through the hyperplane of equation $x_i = x_j$, and in particular generator $\sigma_i \in S_n$ acts as the reflection through the hyperplane of equation $x_i = x_{i+1}$. This action can be obtained as a linear group representation in the following way.

We start by defining the *simple roots* of Σ_n in $\mathbb{R}^n$ as the vectors of the form

$$\alpha_i := e_i - e_{i+1}$$

for any $i = 1, \ldots, n-1$. Though a general introduction of root theory is beyond the purpose of this dissertation, simple roots are a useful combinatorial tool to outline the following construction. We define for each $\sigma_i \in S$

$$\rho_i(v) = v - 2\frac{\langle v, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}\alpha_i$$

for every v in $\mathbb{R}^n$. Note that α_i is orthogonal to hyperplane $H_i : x_i = x_{i+1}$ for any $i = 1, \ldots, n-1$, so that ρ_i is exactly the reflection through H_i. This extends to a unique group homomorphism $\rho : \Sigma_n \hookrightarrow O_n$ toward the group of orthogonal matrices, for which the image of transposition (i, j) is in fact the reflection through the hyperplane of equation $x_i = x_j$ as expected (i.e., the hyperplane orthogonal to vector $e_i - e_j$ which we choose so that $i < j$).

Note that there is a space fixed by the action of every element of Σ_n: $\mathrm{Span}\,(1, \ldots, 1)$. One can in fact consider the situation so far pictured as projected on the orthogonal space, which is the copy of $\mathbb{R}^{n-1}$ given by the equation $x_1 + \cdots + x_n = 0$. One can formally observe that the representation given for the symmetric group is not indecomposable, as it splits as the direct sum of two irreducible representations of Σ_n: the trivial dimension one representation and the dimension $n-1$ representation. The set of simple roots form a basis for this subspace, since they are $n-1$ linearly independent vectors lying in this subspace. We will from now on consider this "reduced action".

Remark 2.1.2 (Notations). In accordance with the construction of the linear group representation as above, we will note in general the reflection associated to transposition t with $\rho(t)$. Corresponding hyperplane will be $H(t)$. An exception will be made only for elements of S, for which $\rho_i := \rho(\sigma_i)$, with corresponding hyperplane H_i.

Remark 2.1.3. It is known that T set of all transpositions is the conjugacy closure of S. We have therefore that the set of the hyperplanes $\{ H(t) \mid t \in T \}$ is closed under the action of reflections through one of such hyperplanes. In other words, $\rho(t)$ for any $t \in T$ acts as a permutation of the hyperplanes. To see that, we observe that reflection through hyperplane $\rho(t)H(t')$ is actually $\rho(t)\rho(t')\rho(t)^{-1}$, but since ρ is a homomorphism this is $\rho(t \cdot t' \cdot t^{-1})$, and since T is closed under conjugation in Σ this is $\rho(t'')$ for some $t'' \in T$, so that $\rho(t)H(t')$ is $H(t'')$.

We now fix n. We call a *chamber* every connected component of the complementary space

$$\mathbb{R}^{n-1} \setminus \bigcup_{t \in T_n} H(t).$$

Hyperplanes touching a chamber are called its *walls*. If chamber C has $H(t_1), \ldots, H(t_i), \ldots, H(t_k)$ as its walls, it follows from the argument in remark 2.1.3 that the chamber on the opposite side of $H(t_i)$ has

$$H(t_i t_1 t_i), \ldots, H(t_i), \ldots, H(t_i t_k t_i)$$

as set of walls. As a quick remark, we observe that since conjugation by a fixed element is a group automorphism, the number of walls for the arbitrary chamber is fixed.

There exists a chamber that has $H_1, \ldots, H_{n-1}$ as walls. To see this, consider the original action of Σ_n on $\mathbb{R}^n$. Each hyperplane $H_1, \ldots, H_{n-1}$ divides the space in two halves. We consider for each H_i the half space $\{x \in \mathbb{R}^n \mid x_i \leqslant x_{i+1}\}$. The region

$$C = \bigcap_{i=1}^{n-1} \{x \in \mathbb{R}^n \mid x_i \leqslant x_{i+1}\}$$

is constituted by points which verify $x_1 \leqslant \cdots \leqslant x_{n-1}$, so that points in its walls verify $x_i = x_{i+1}$ for some i, and therefore lie in some H_i. The region is also uncrossed by any hyperplane of the form $H(t)$ with $t \in T$, because points in the interior of the region cannot satisfy $x_i = x_j$. Region C is therefore a chamber and has $H_1, \ldots, H_{n-1}$ as walls. We will choose one of such chambers and call it *fundamental chamber*.

Definition 2.1.4 (Σ_n-permutahedron)**.** We choose a point in the fundamental chamber that has same distance from every wall of the fundamental chamber,

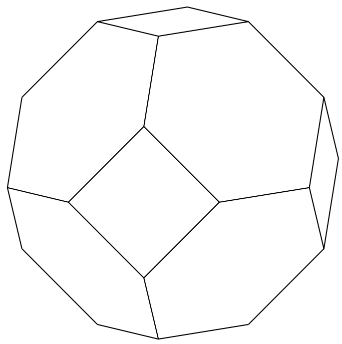


Figure 2.1: Σ_4-permutahedron

and call it P_e. The image of P_e through ρ_i lays on the opposite side of hyperplane H_i. We connect P_e and $\rho_i(P_e)$ with an edge, crossing hyperplane H_i, and label the edge with σ_i. We repeat the process for each wall of the fundamental chamber, each time finding an edge crossing an hyperplane H_j and labelling it σ_j. Now we consider the image $P' := \rho_i P_e$; it was found in a chamber that has (we shift now to a more exaustive notation)

$$\rho(\sigma_i)H(\sigma_1), \ldots, H(\sigma_i), \ldots, \rho(\sigma_i)H(\sigma_{n-1})$$

as walls, and is connected to P_e by an edge that crosses $H(\sigma_i)$. We take the image $\rho(\sigma_i\sigma_j\sigma_i)P'$ by reflecting through hyperplane $\rho(\sigma_i)H(\sigma_j)$, connect P' and this new point with an edge and label it σ_j. We proceed like this inductively, each time crossing hyperplane $\rho(\sigma)H(\sigma_k)$ for some σ with an edge and labelling it with σ_k. Up to rescaling, we find the 1-skeleton of a polyhedron (i.e. the 2-skeleton of a polytope) having each edge of length 1, called the Σ_n*-permutahedron* (or simply *permutahedron* when n is unimportant).

By construction, the permutahedron has a reflection symmetry for each $t \in T$. In the former construction, if the path from vertex P_e to vertex P is labelled $s_1, \ldots, s_k$ for some $s_1, \ldots, s_k \in S$ we can consider vertex P to correspond to the element $s_1 \cdots s_k$ in Σ_n, or to $s_k \cdots s_1$; in the former case we will call the labelled polihedron *right permutahedron* (since we choose to look at successive crossings of labelled paths as right multiplications among the labels), whereas in the latter case we will call it *left permutahedron.* When vertices P_σ and P_τ correspond to permutations σ and τ differing by a multiplication for some $s \in S$, there will be and adge between P_σ and P_τ labelled s, so that in general $\rho_i P_\tau = P_{\tau\cdot\sigma_i}$ in the right permutahedron, $\rho_i P_\tau = P_{\sigma_i\cdot\tau}$ in the left permutahedron. Up to considering multiplication by generator s as crossing an edge labelled s (coherently with the involution relations), we recover the right (left) Cayley graph of Σ_n in the 1-skeleton of the right (left) Σ_n-permutahedron.

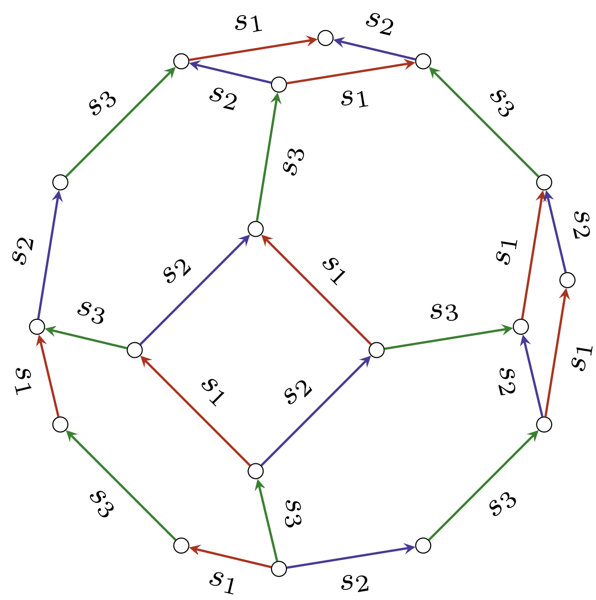


Figure 2.2: The oriented, labelled 1-skeleton of the Σ_4-permutahedron. We choose to draw only the "visible" part of the polytope.

Example 2.1.5. We will constuct as an example the first nontrivial permutahedron, i.e. the one for $n = 3$. We consider thus the action of Σ_3, as generated by $\{\sigma_1, \sigma_2\}$ on $\mathbb{R}^3$ by permutation of coordinates. Reflection hyperplanes are therefore the ones of equations $H(\sigma_1) : x = y$, $H(\sigma_2) : y = z$ and $H((1,3)) : x = z$. The reduced action is the one on the copy of $\mathbb{R}^2$ generated by vectors α_1 and α_2, i.e.,

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

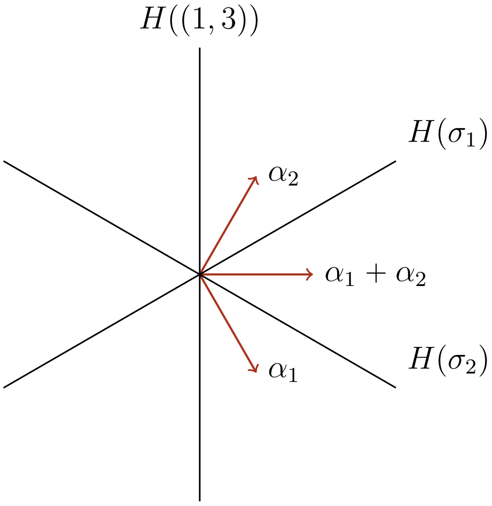
This action is hysomorphic to the action of Σ_3 on the euclidean space $\mathbb{R}^2$ with the geometry induced by the standard scalar product in $\mathbb{R}^3$. That scalar product is associated to the following matrix[1], expressed with respect to the basis $\{\alpha_1, \alpha_2\}$,

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

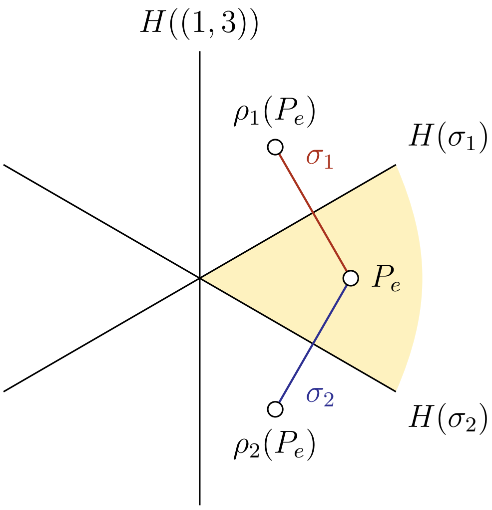
Generators act as reflections through hyperplanes that are the lines of

[1] In general, one gets the Cartan matrix of type A_{n-1}.

equations (expressed in coordinates with respect to the basis above) $H(\sigma_1)$: $y = 0$, $H(\sigma_2) : x = 0$ and $H((1,3)) : x - y = 0$.

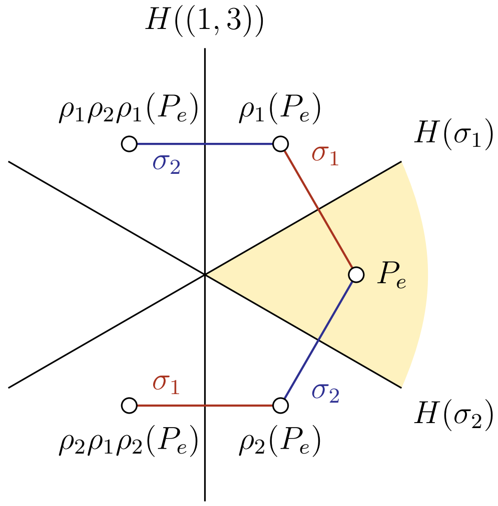


Now we choose as fundamental chamber the one at the far right in the picture, i.e. the one in which lies $\alpha_1 + \alpha_2$. Here we pose the point P_e, and take its images with respect to the reflections $\rho(\sigma_1)$ and $\rho(\sigma_2)$ which we note ρ_1 and ρ_2, respectively through hyperplanes $H(\sigma_1)$ and $H(\sigma_2)$: We connect P_e to each image with an edge, which we label respectively σ_1 and σ_2.

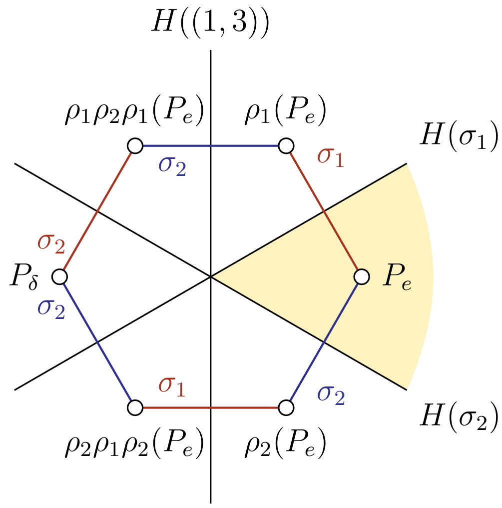


We consider now $\rho_1(P_e)$. It lies in a chamber that has $H((1,3))$ and $H(\sigma_1)$

as walls. Hyperplane $H((1,3))$ here is the image of $H(\sigma_2)$ through reflection ρ_1, so that the image of $\rho_1(P_e)$ on the other side of $H((1,3))$ is $\rho_1\rho_2\rho_1(P_e)$. We draw this point, and connect it to $\rho_1(P_e)$ with an edge labelled σ_2. We do likewise for $\rho_2(P_e)$.



We conclude the permutahedron by taking the images of the last two vertices, which are both $P_\delta := \rho_2\rho_1\rho_2\rho_1\rho_2(P_e) = \rho_1\rho_2\rho_1\rho_2\rho_1(P_e)$.



We now turn back to the general setting. Consider in $\mathbb{R}^n$ a point Q_e in $\mathrm{Span}(P_e)$ external to the permutahedron, and consider on the edges the distance function with respect to Q_e. There is a natural orientation on any edge induced by orienting every edge in the direction of the vertex on which the distance function is maximal (i.e. the furthest vertex from Q_e). Every edge in the 1-skeleton is thus oriented, since no hyperplane contains Q_e, and therefore no edge has both vertices equidistant from Q_e.

This induces a total relation order on each pair of vertices connected by an edge:

$$P \leqslant_{\text{perm}} P' \iff \text{Dist}(Q_e, P) < \text{Dist}(Q_e, P')$$

whenever P and P' are connected by an edge in the permutahedron. In general, this generates a partial order relations for vertices of the permutahedron. There is nonetheless also an order induced on the elements in Σ_n. Since vertices of the permutahedron and permutations are in bijection (though not canonically), the order relation can be described equivalently using both, but specifying if the order refers to the vertices-permutations association on the left or right permutahedron. For two elements $\sigma, \tau \in \Sigma_n$ we say

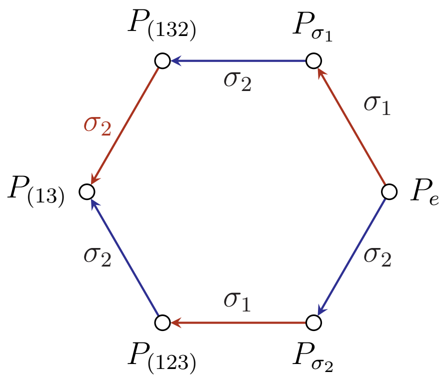
$$\sigma \leqslant_{\text{perm},L} \tau$$

if there is a path from P_σ to P_τ on the edges of the left permutahedron in which every edge is crossed orientation-wise. For two elements $\sigma, \tau \in \Sigma_n$ we say

$$P_\sigma \leqslant_{\text{perm},R} P_\tau$$

if there is a path from P_σ to P_τ on the edges of the right permutahedron in which every edge is crossed orientation-wise.

Example 2.1.6. In example 2.1.5 we have found the Cayley graph of Σ_3 as expected. In fact, we can label each vertex coherently with the path we cross from P_e to get there, and orient each edge as described, obtaining exactly the oriented, labelled Σ_3-permutahedron.



We observe that this is exactly the poset in example 1.1.2.

We will now call *weak order* the weak right Bruhat order (or equivalently the right permutahedron order when considering it on the vertices of the right permutahedron) and note it $\leqslant$, the right permutahedron will be referred to as *permutahedron*; the oriented and labelled 1-skeleton of the Σ_n-permutahedron will be noted M_n or M (when n is obvious or unimportant). The following results are symmetrical in the statement for left weak Bruhat order and left permutahedra.

Remark 2.1.7 (Geometric-combinatorial features). Paths in M_n from P_e to a vertex P_σ correspond to words respresenting σ in Σ_n. In fact, if such a path crosses edges labelled $s_1, \cdots, s_k$ (repetitions of elements in this ordered list are allowed) then $\sigma = s_1 \cdots s_k$. Reduced words correspond to *minimal paths* (which we observe to be edge-paths in which every edge is crossed orientation-wise), so that $l(\sigma)$ is also the length of any of such paths. Facets of any dimension of the permutahedron have also a strong combinatorial meaning. In fact, since we find the Cayley graph for Σ_n in the oriented labelled 1-skeleton of the permutahedron, facets of arbitrary dimension containing P_e as a vertex represent subgroups of Σ_n generated by some subset of S_n. We will call those facets *parabolic facets*, and will refer to such subgroups as *parabolic subgroups* of Σ_n. By construction, for every subset of labels there exist one of these facets such that labels on its edges form exactly that set. We observe that since multiplication in Σ_n induces in the obvious way a transformation from the permutahedron in itself that sends k-facets in k-facets, by the structure of Cayley graphs and this last observations, an arbitrary facet with given labelling on the edges corresponds to a coset of some parabolic subgroup, which is found in correspondance with the parabolic facet with the same labelling. Also, we find the Cayley graph of the parabolic subgroup in the 1-skeleton of the parabolic facet, by inducing on it the same labelling ans orientation that it has inside the permutahedron.

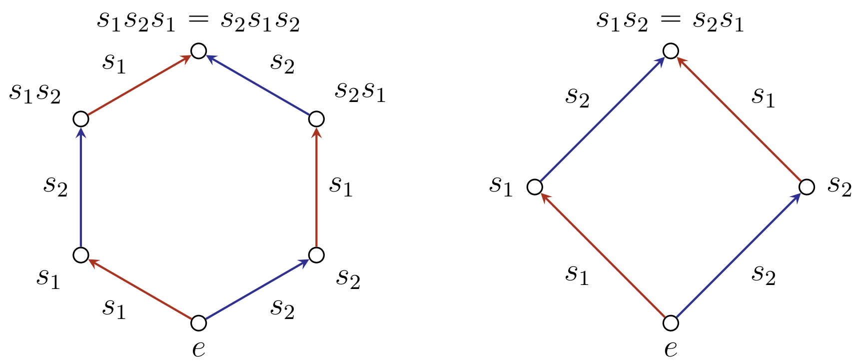


Figure 2.3: The two possibilities for the shape of a 2-facet in a permutahedron are found in correspondance with the two possible labellings of the edges in a parabolic 2-facet. Parabolic 2-facets, in turn, are found in correspondance with the two possible parabolic subgroups with two generators, up to hysomorphisms: the two generators either satisfy the braid relation (left) or they commute (right). Vertices in the image are labelled as if they were found inside the Cayley graph of the parabolic subgroup. Note that the hexagonal facet is also a copy of the Σ_3-permutahedron. This is an example for a more general fact: since some parabolic subgroups of Σ_n are isomorphic to some Σ_m, we find copies of the Σ_m-permutahedron as some facets of the Σ_n-permutahedron.

The following lemma makes it easier to understand the geometric structure of the 2-skeleton of the permutahedron.

Lemma 2.1.8. the action of Σ_n on the 2-facets of the permutahedron with same labels on the edges is transitive.

Proof. As we observed while captioning figure 2.3, there are only two possible shapes for parabolic 2-facets, corresponding to the two possible isomorphism classes of parabolic subgroups with two generators: the hexagonal and the squared facets. The square facet labelled with a, b is stabilized by the action of $\{e, a, b, ab = ba\}$ (note that $ab = ba$ since squared parabolic-facets occur whenever two generators commute). On the other hand, an hexagonal facet labelled with a, b is stabilized by the action of $\{e, a, b, ab, ba, aba = bab\}$. Since Σ_n has $n!$ elements, this means that the orbit of the facet has $n!/4$ elements if it is squared, and $n!/6$ if it is hexagonal. This is also the number of facets with that shape and labelling, since such facets are found in correspondance to cosets of the parabolic subgroup represented by the choosen parabolic facet, as observed in remark 2.1.7. $\square$

This means that the 2-skeleton of the permutahedron is tiled with hexagonal and squared facets, each labelled as we observed, and that patabolic 2-facets form a set of representatives for the classes of labellings on 2-facets of the permutahedron.

Proposition 2.1.9 (Permutahedron order and weak Bruhat order coincide)**.** The partial order relation geometrically induced on vertices of the right (left) M_n coincides with weak right (left) Bruhat order on Σ_n, meaning that for every σ,$\tau \in \Sigma_n$ and P_σ,P_τ corresponding vertices on the permutahedron,

$$\sigma \leqslant_{\text{perm},L} \tau \iff \sigma \leqslant_L \tau$$

$$\sigma \leqslant_{\text{perm},R} \tau \iff \sigma \leqslant_R \tau.$$

Proof. Whenever $\sigma \leqslant_{\text{perm},R} \tau$, there is a minimal path from P_σ to P_τ on the oriented labelled 1-skeleton of the right permutahedron. By running on a minimal path from P_e to P_σ and then on that minimal path from from P_σ to P_τ one finds a minimal path from P_e to P_τ. By reading labels on the edges crossed by the paths, we have thus completed a reduced word for σ to a reduced word for τ, so that a reduced word for σ appears as a prefix for a reduced word for τ. By 2.0.2, this means $\sigma \leqslant_L \tau$. On the other hand, this argument can be used on the opposite direction, so that the proof is concluded. □

2.1.2 The factorization poset of the longest element

We now turn to prove that there exists a maximal element for the weak order, and its factorization poset is a combinatorial Garside structure.

Lemma 2.1.10 (Inversions lemma)**.** Let $\sigma \in \Sigma_n$. We note with $i(\sigma)$ the number of inversions of σ, i.e. the number of couples $0 \leqslant i < j \leqslant n$ such that $\sigma(i) > \sigma(j)$. Then $i(\sigma)$ coincides with the length of σ with respect to the set of generators S_n. In particular, for each n there exists a unique element ω_0 in Σ_n such that its length is maximal.

Proof. Multiplication by $\sigma_i \in S_n$ only swaps $\sigma(i)$ and $\sigma(i+1)$, so $i(\sigma)$ changes by at most one. Then $l(\sigma)$ is bounded below by $i(\sigma)$. On the other hand, if $\sigma \neq e$ then $\sigma(i+1) < \sigma(i)$ for some i, so that left multiplication by σ_i decreases $i(\sigma)$ by one. This means that one reaches e by exactly $i(\sigma)$ left multiplications by generators in S_n. The maximal length is therefore

$$\frac{n(n-1)}{2}$$

and is reached only by the “order swapping" permutation, i.e. the one for which $\omega_0(i) = n + 1 - i$. □

Definition 2.1.11. For n fixed, we will call ω_0 as in 2.1.10 the *longest element.*

We mention that the structure of ω_0 as a product of generators is

$$\sigma_{n-1} \cdots \sigma_1 \cdot \sigma_{n-1} \cdots \sigma_2 \cdots \sigma_{n-1} \cdot \sigma_{n-2} \cdot \sigma_{n-1}.$$

This fact follows immediately from its definition.

Note that by following any length-maximizing minimal chain on the permutahedron we have to find the longest element (since it is the unique element with such length). This means that maximal chains in the permutahedron correspond to reduced factorizations for the longest element with respect to S_n set of generators. By maximality of the longest element with respect to the weak order, every reduced word can be completed to a reduced word for ω_0, meaning that every minimal path from e to a vertex of the permutahedron can be completed to a maximal chain. This proves that M coincides with the factorization poset for ω_0 with respect to S.

For every n the poset P given by the elements of S_n with the weak order is bounded, graded and has finite height. In fact, we have that P is bounded by e and ω_0, has a natural gradation induced by the length function, and has finite height since Σ_n is finite. The proof for the lattice structure follows from the general results in [4], where it is proven that in general the poset constructed with the technique we used for the permutahedron, starting from any hyperplane arrangement that induces a simplicial tiling on the unitary sphere is a lattice. The result, although, was originally proven by Deligne. The previous justification for the holding of the lattice property was included to suggest a more geometric approach to the problem, but we will also give for sake of completeness a proof of algebraic flavour using inversions (see lemma 2.1.10).

Lemma 2.1.12. The lattice structure holds for P.

Proof. We already know from lemma 2.1.10 that the inversions of a permutation coincide with its length with respect to S. There is more: by definition of weak Bruhat order, for any $\sigma, \tau \in \Sigma$ the relation $\sigma \leqslant \tau$ holds if and only if $I_\sigma \subseteq I_\tau$ where I is the set of inversions of the permutation. A known characterization for inversion sets is the following: a set of couples of indices is an inversions set for some permutation if and only if both the set and its complementary are transitive. By using this characterization one can observe that for any I_σ and I_τ, by posing $\sigma \vee \tau$ the permutation identified by

$$I_{\sigma \vee \tau} = \overline{I_\sigma \cup I_\tau}$$

where the overline denotes the transitive closure, one gets a join-semilattice. Thus since Σ is finite and has a unique minimal element, it is a lattice by lemma 0.0.9. □

As anticipated, we now consider the (right) factorization poset $P_n := \mathrm{Fact}(\Sigma_n, S_n, \delta)$, or simply $P := \mathrm{Fact}(\Sigma, S, \delta)$ when n is obvious or unimportant. For fixed n, since M coincides with the right Cayley graph for Σ we have that M also coincides with the Hasse diagram for P, so that the group-like property is automatically granted by the properties of Cayley diagrams. By previous results in this subsection we already have that P is a bounded, graded poset of finite height which is group-like and a lattice.

We now consider the sets $Pre(P)$ and $Suf(P)$ as in section 1.1. By remark 2.0.2, whenever $\sigma \leqslant \tau$ in the weak order, it means that there is a minimal path from P_e to P_τ that starts with a minimal path from P_e to P_σ.

Lemma 2.1.13. Poset P is balanced.

Proof. We first observe that if $s_1 \cdots s_k$ is a word representing an element $\sigma \in \Sigma$, then $s_k \cdots s_1$ represents its inverse. There follows that an element and its inverse have same length. We consider now an element $s_k \cdots s_1$ in $Pre(P)$. Its inverse $s_1 \cdots s_k$ corresponds to a minimal length path from P_e to $P_{s_1 \cdots s_k}$ on M. We can complete it to form a length maximizing minimal path on which we read the word $s_1 \ldots s_k \cdot s_{k+1} \cdots s_m$ (otherwise, the poset would not be graded, which it is). This word will represent the longest element by its uniqueness. The inverse $s_m \cdots s_{k+1} \cdot s_k \cdots s_1$ has same length, thus still represents ω_0, again by uniqueness. This means that there is a minimal path of maximal length on M labelled with those labels, so that $s_k \cdots s_1$ is also in $Suf(P)$. The inverse inclusion can be proved with the same technique. $\square$

We have thus proved P to be a combinatorial Garside structure, so that results from chapter 1 hold.

2.2 Dual structure

We now turn to consider another set of generators for Σ_n: the set of all transpositions T_n. Factoring elements with respect to this set gives place to another partial order relation on Σ. This “dual" order has many interesting combinatorial properties, both on its own and when compared with the classical structure. It is known that conjugacy classes in Σ correspond to classes of cyclic structures, in particular T is closed under conjugacy, and we already referred to the fact that it is the closure of S. This has important combinatorial consequences.

Lemma 2.2.1. Let $\sigma \in \Sigma_n$ a permutation with cyclic structure $c_1 \cdots c_k$. Then its length with respect to the factorization by elements in T_n is

$$l_T(\sigma) = \sum_{i=1}^{k}(|c_i| - 1)$$

where $|c_i|$ is the number of indexes that c_i permutes.

Proof. We consider the set of formal indices $1, \ldots, n$ realized by n distinct labelled formal points. We also consider a factorization of $c_1 \cdots c_k$ as a reduced product of transpositions, say $t_1 \cdots t_l$, where l is by definition $l_T(\sigma)$. If t_i permutes indices x_i and y_i, we draw an edge between x_i and y_i as formal points. We repeat this operation for each i, obtaining a graph. Remove from the graph the isolated points (since they represent indices fixed by the permutation). Let C be the number of connected components of the resulting graph. For every c_i, points corresponding to indices permuted ciclically by c_i must lay in the same connected component of the graph. Also, it is known that for an arbitrary graph with n connected components, drawing l edges causes the number of connected components to be at least $n - l$. We have thus that

$$n - l \leqslant C \leqslant k$$

where n is the number of points in the graph (i.e., by construction, the number of indices that appear in some cycle $c_i, \ldots, c_k$), so that $l \geqslant n - k$. If we show that there exists a word for σ with length $n - k$ then the lemma will be proved ($n - k$ is actually the expected length). This word exists, in fact for every cycle

$$(i_1 \ldots i_h) = (i_{h-1} i_h)(i_{h-2} i_{h-1}) \ldots (i_1 i_2)$$

that is a word of length $h - 1$, so that σ can be written as a product of $n - k$ transpositions by writing each cycle in this form and taking their product. $\square$

It is an immediate consequence that length is constant over conjugacy classes; in particular, every n-cycle in Σ_n has the same length: $n - 1$. We now observe another important consequence. Let a reduced factorization $t_1 \cdots t_k$ for an element σ. The conjugate $\sigma' = \tau\sigma\tau^{-1}$ can be factored in $\tau t_1 \tau^{-1} \cdots \tau t_k \tau^{-1}$, corresponding to $t_1' \cdots t_k'$. This is a word for σ', and is reduced since if any two terms $t_i' \cdot t_{i+1}'$ cancel, then $\tau t_i \cdot t_{i+1} \tau^{-1}$ would also cancel, so that the original reduced word for σ was not reduced in the first place. This argument shows that conjugacy sends divisors for a permutation σ in divisors for the conjugate σ' giving place to isomorphic factorization posets. Thus, it is not restrictive to choose an n-cycle arbitrarily. *For the rest of this chapter, we will fix n and an n-cycle c in Σ_n.*

2.2.1 Non-Crossing Partitions

We will now introduce an important tool to deal with the combinatorics of factorizations of c. Definitions and results in this subsection partially follow or adapt [12].

Definition 2.2.2. For all pairs (i,j) of distinct formal indices, we define $\delta(i, j)$ and call it *distance between i and j* the least k such that

$$c^k(i) = j.$$

The definition is well posed, since for every choice of c the orbit of every formal index under the repeated action of c is the whole set of n formal indexes, i.e. $\{i, c(i), c^2(i), \ldots, c^{k-1}(i)\} = \{1, \ldots, n\}$. It also holds for every choice of two distinct formal indices i, j that

$$\delta(i, j) + \delta(j, i) = n$$

since n is the order of c in Σ_n.

Definition 2.2.3. Given two disjoint pairs of formal indices i, j and k, l, we say that the pairs are *crossed* if the integer $\delta(i, j)$ is between the lesser and greater of the the two integers $\delta(i, k)$ and $\delta(i, l)$, else the two are *uncrossed.* Two disjoint subsets X and Y of the set of formal indices $1, \ldots, n$ are said to be *noncrossing* if there does not exist $i, j \in X$ and $k, l \in Y$ all distinct such that i, j and k, l are crossed pairs. Otherwise they are *crossing.* Finally, we call *noncrossing partition* a partition of the set of formal indices $\{1, \ldots, n\}$ in which any two distinct classes are noncrossing.

Noncrossing partitions can be visualized geometrically on the n-gone with vertices labelled clockwise (or counterclockwise) with labels

$$1, c(1), c^2(1), \ldots, c^{n-1}(1).$$

We will note the counterclockwise labelled n-gone as N. Note that we identify the vertices with their labels. Unfolding the definition, pairs i, j and k, l are crossed if the repeated application of c to i finds one among k and l, but not both, before getting to j. Thus, if we draw an edge between vertices i and j, and between k and l on the n-gone, we will find two crossing edges if the pairs were crossing, or uncrossing edges if the pairs were uncrossing.

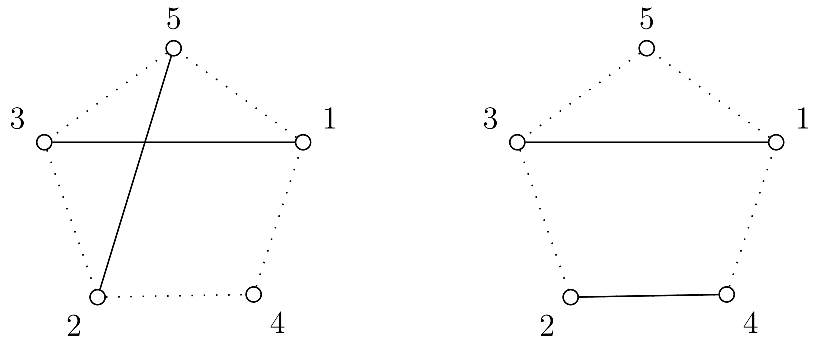


Figure 2.4: Pairs $2, 5$ and $1, 3$ are crossing with respect to $c = (15324)$, while pairs $1, 3$ and $2, 4$ are noncrossing.

Remark 2.2.4. This construction is already useful: it is now intuitively easy to see (and prove) that if i, j and k, l are crossed/uncrossed, then j, i and k, l are crossed/uncrossed, and likewise i, j and l, k.

Two sets of indices X and Y are noncrossing if no pair of distinct indices in X crosses some pair of distinct indices in Y. Since we can visualize crossing/noncrossing pairs as crossing/noncrossing edges, we have that X and Y are noncrossing if and only if the convex hull of the set of vertices on N corresponding to elements of X does not cross the convex hull of the vertices on N corresponding to elements of Y.

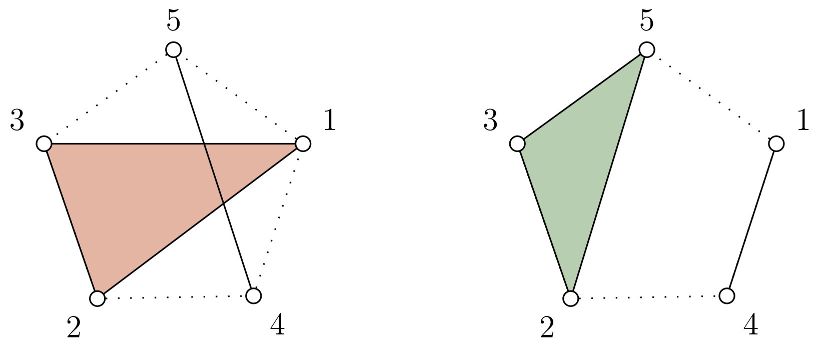


Figure 2.5: Classes $\{1, 3, 2\}$ and $\{4, 5\}$ are crossing with respect to $c = (15324)$, while classes $\{2, 3, 5\}$ and $\{1, 4\}$ are noncrossing.

Finally, a noncrossing partition can be viewed as a partition of the vertices of N such that no two classes give place to intersecting convex hulls.

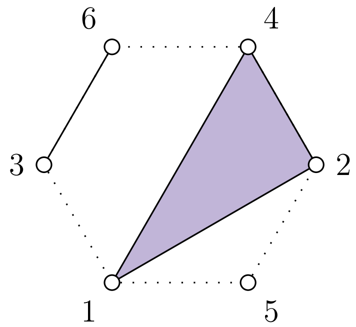


Figure 2.6: The partition $\{\{1, 2, 4\}, \{3, 6\}, \{5\}\}$ is noncrossing with respect to the cycle $c = (152463)$.

Remark 2.2.5. Let $\beth$ be a partition of the set of indices $\{1, \ldots, n\}$. A *singleton* is a class of $\beth$ containing just one index. Note that a singleton and any other class in $\beth$ will always be noncrossing. Moreover, let a class and a singleton $\{j\}$ in the noncrossing partition $\beth$ such that that there is some index i in the class for which i and j find themselves to be the two vertices of an edge in the labelled n-gone (i.e., $c(i) = j$ or vice versa). Then the noncrossing partition obtained from $\beth$ by joining that class and j is still noncrossing. There is another important operation on a noncrossing partition that produces a noncrossing partition. Imagine to draw a straight line that cuts the convex hull representing a class in a noncrossing partition in two, and touches no vertex of N. This operation divides the indices in that class in two sets. The partition of indices obtained by replacing the original class of the noncrossing partition with the two new classes is again noncrossing.

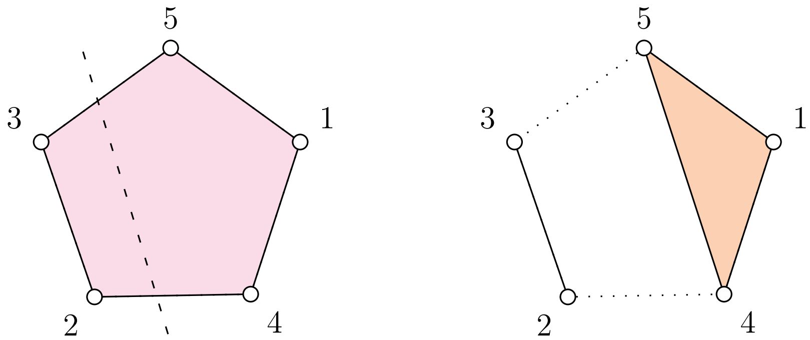


Figure 2.7: A cut in a noncrossing partition as described gives place to another noncrossing partition.

The set of noncrossing partitions has a natural partial order relation given by inclusions of classes, i.e. for any two noncrossing partitions $\aleph_1$ and $\aleph_2$, we set $\aleph_1 \leqslant_{\mathrm{NCP}} \aleph_2$ if and only if any class in $\aleph_1$ is contained in some class of $\aleph_2$.

We need just another construction to prove the main result of this subsection.

Definition 2.2.6. We call *noncrossing closure* of partition $\aleph$, and note it with $\overline{\aleph}$, the noncrossing partition obtained as follows. For every two crossing classes of the partition, we merge them together to form a unique class. We proceed like this, until there are no more crossing couples of classes left.

In the visual construction, we can represent as above the partition $\aleph$ on N by drawing the convex hulls of the sets of vertices corresponding to classes of $\aleph$. For every pair of intersecting convex hulls, we take the convex hull of the union of the two classes of vertices, until there is no intersecting couple of convex hulls left. Classes of vertices in resulting convex hulls individuate a partition of formal indices, which we recognize to be $\overline{\aleph}$.

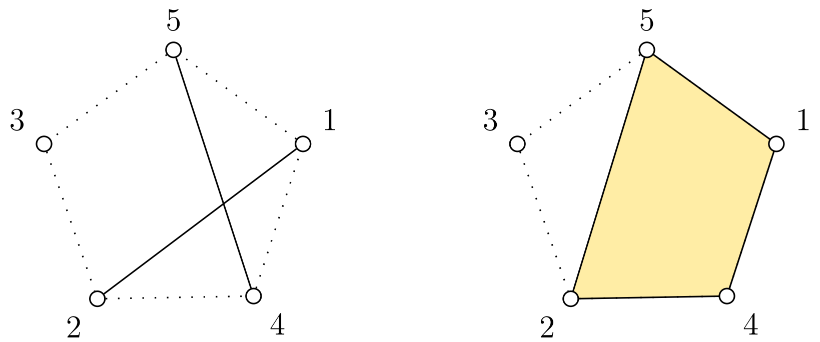


Figure 2.8: The noncrossing closure of partition $\{\{1,2\},\{3\},\{4,5\}\}$ with respect to $c = (15324)$ is $\{\{1,2,4,5\},\{3\}\}$.

We recall (see example 0.0.7) that the set of all partitions of a given set form a lattice, with the two operations $\beth_1 \cup \beth_2$ and $\beth_1 \cap \beth_2$, respectively representing the least fine of the partitions more fine than $\beth_1$ and $\beth_2$ and the most fine of the partitions less fine than both $\beth_1$ and $\beth_2$.

Proposition 2.2.7. The set of noncrossing partitions with the partial order $\leqslant_{\mathrm{NCP}}$ as above and the operations

$$\aleph_1 \wedge \aleph_2 = \aleph_1 \cap \aleph_2$$

$$\aleph_1 \vee \aleph_2 = \overline{\aleph_1 \cup \aleph_2}$$

forms a lattice.

Proof. We start by proving a useful lemma;

Lemma 2.2.8. For every partition $\beth$ of $\{1, \dots, n\}$, every noncrossing partition less fine than $\beth$ is also less fine than $\overline{\beth}$.

Proof. If $\gimel$ is a partition less fine than $\beth$, every class B of $\beth$ is contained in a class G of $\gimel$. We consider two classes G_1, G_2 of $\gimel$ and two classes B_1, B_2 of $\beth$ such that $B_1 \subseteq G_1$ and $B_2 \subseteq G_2$. If $\gimel$ is a noncrossing partition, then G_1, G_2 are noncrossing, so that also B_1, B_2 are. There follows that each time two classes in $\beth$ are crossing, they are contained in the same class of $\gimel$. Therefore, every class of the noncrossing closure $\overline{\beth}$ is contained in some class of $\gimel$, proving $\gimel \leqslant_{\mathrm{NCP}} \overline{\beth}$ □

We now observe that if $\aleph_1$ and $\aleph_2$ are noncrossing partitions, then $\aleph_1 \cap \aleph_2$ is also a noncrossing partition. In fact, every class of $\aleph_1 \cap \aleph_2$ is an intersection of a class of $\aleph_1$ and a class of $\aleph_2$. Since two distinct classes of $\aleph_1$ are noncrossing, the same will be true for their intersections with two distinct classes of $\aleph_2$.

Moreover, if $\aleph_1$ and $\aleph_2$ are two noncrossing partitions, for every noncrossing partition $\aleph$ for which $\aleph \leqslant_{\mathrm{NCP}} \aleph_1, \aleph_2$ it holds that $\aleph \leqslant_{\mathrm{NCP}} \overline{\aleph_1 \cup \aleph_2}$. In fact, every partition less fine than $\aleph_1$ and $\aleph_2$ is less fine than $\aleph_1 \cup \aleph_2$, and if one such partition is also noncrossing, it is less fine than $\overline{\aleph_1 \cup \aleph_2}$ by the lemma above. □

We conclude this subsection by thew following useful remark.

Remark 2.2.9. Though the property for a partition to be uncrossing depends on the choice of the cycle c, the structure of the poset of partitions for n fixed does not. In fact, we could have considered the poset of noncrossing paritions on the unlabelled n-gone, thinking of a noncrossing partition as a partition for which the geometric interpretation of the property of being noncrossing holds. The resulting poset must be the same for every choice of the numeration of the vertices, which corresponds to the choice of a cycle c. This means that the elements in the poset found from noncrossing partitions of a cycle c depend on the choice of c, but the isomorphism class of posets does not.

2.2.2 The factorization poset of an n-cycle

In this subsection, we will use noncrossing partitions from previous subsection to describe the factorization poset of an n-cycle, by finding that the poset formed by noncrossing partitions and the one formed by factorizations of the n-cycle are actually isomorphic. Using this fact, we will be able to show that this poset is also a combinatorial Garside structure. *We keep c an arbitrary n-cycle fixed in Σ_n with n also fixed, unless otherwise specified.*

Definition 2.2.10. Let $\sigma \in \Sigma$ a permutation, and $c_1 \dots c_k$ its cyclic structure. We note with

$$\urcorner(\sigma)$$

the obvious partition of indices induced by $c_1, \dots, c_k$, where fixed points are considered as singletons.

Example 2.2.11. Permutation $(147)(25)$ in Σ_7 induces the partition of indices $\{\{1,4,7\},\{2,5\},\{3\},\{6\}\}$. This partition is crossing with respect to $c = (1234567)$.

To get back from a partition of indices to a permutation, we use the following convention. Once c is fixed, we represent the partition on the n-gone as usual, and write down the cycles c_i by reading the indices in every convex hull in counterclockwise order.

Example 2.2.12. Consider the partition of indices $\{\{1,4,7\},\{2,5\},\{3\},\{6\}\}$ from the previous example. While there is no ambiguity when computing cycles corresponding to classes $\{2,5\}$, $\{3\}$ and $\{6\}$, class $\{1,4,7\}$ could correspond to (147) or (174). If we choose to set $c = (1246375)$ then we find the former, while by setting $c = (1276345)$ we get the latter.

Lemma 2.2.13. *Let σ be a permutation in Σ. Then σ divides c if and only if $\urcorner(\sigma)$ is a noncrossing partition for c.*

Proof. We prove the first implication by induction on the number m of orbits for σ. If $\sigma = c_1 \cdots c_k$ this number is k plus the number of fixed points. If $m = 1$ then $\sigma = c$ and the assert is proved. Otherwise, we write $\sigma = \sigma'(ij)$ for some σ', i and j such that $l(\sigma) = l(\sigma') + 1$. Since l increases, indices i and j must lie in the same class of $\urcorner(\sigma')$ so that multiplication by (ij) splits the class in two distinct classes, as pictured in figure 2.9, this can be deduces from lemma 2.2.1. The number m for σ' is thus less that the one for σ, therefore by inductive hypothesis $\urcorner(\sigma')$ is a noncrossing partition for c. Thus the same holds for $\urcorner(\sigma)$, since it was obtained by splitting in two a class in a noncrossing partition (see remark 2.2.5).

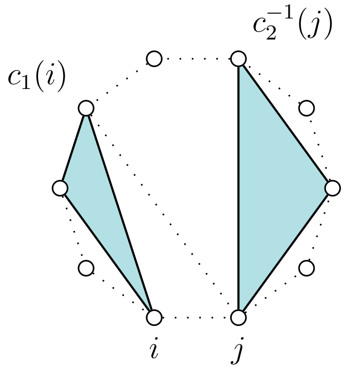


Figure 2.9: Multiplying a cycle by an opportune transposition splits the corresponding class, while multiplying by an opportune transposition we merge two classes.

We now turn to prove the inverse implication: if $\urcorner(\sigma)$ is a noncrossing partition then σ divides c. We compute the cyclical structure of σ as outlined before, with respect to c which is fixed since the very beginning of this section. We call m the number of classes in $\urcorner(\sigma)$, and proceed by induction on this number. Note that m is again equal to k number of disjoint cycles in the cyclic structure of σ plus the number of indices fixed by σ. If $m = 1$ then $\sigma = c$. For $m > 1$, we consider two indices i, j such that they lie in two adjacent vertices of the labelled n-gone, but are contained in two different classes of $\urcorner(\sigma)$. Up to relabelling the cycles, we suppose $i \in c_1$ and $j \in c_2$. The next index we read in c_1 following our convention will be then $c_1(i)$. The situation so far is pictured in 2.9. Note that transposition $(c_1(i)j)$ commutes with every c_i as long as $i \neq 1, 2$, and

$$c_1 \cdot c_2 \cdot (c_1(i)j)$$

is just the cycle

$$(c_1(i) \cdots ij \cdots c_2^{-1}(j)),$$

so that $\sigma \cdot (c_1(i)j)$ has one orbit less that σ. If $\urcorner(\sigma)$ was a noncrossing partition, that the same holds for $\urcorner(\sigma \cdot (c_1(i)j))$ by the geometry of N. By inductive hypothesis, we have that $\sigma \cdot (c_1(i)j)$ divides c, and there follows immediately that σ divides c. $\square$

Lemma 2.2.14. Let σ_1, σ_2 such that both divide c. Then σ_1 divides σ_2 if and only if $\urcorner(\sigma_1)$ is finer than $\urcorner(\sigma_2)$, i.e.,

$$\sigma_1 \leqslant \sigma_2 \iff \urcorner(\sigma_1) \leqslant_{\mathrm{NCP}} \urcorner(\sigma_2).$$

Proof. It is sufficient to show the assert in the case $\sigma_2 = \sigma_1(ij)$ with $l(\sigma_2) = l(\sigma_1) + 1$. Let $\sigma_1 = c_1 \cdots c_k$ cyclic structure. In order for the length to increase, i and j must lie in two different classes of $\urcorner(\sigma_1)$, as follows from lemma 2.2.1. Then, $\urcorner(\sigma_2)$ is obtained from $\urcorner(\sigma_1)$ by breaking a class in two classes (the opposite operation of the merging of two classes pictured in figure 2.9), and clearly this operation gives place to a noncrossing partition.

On the other hand, if $\urcorner(\sigma_1) \leqslant_{\mathrm{NCP}} \urcorner(\sigma_2)$ then $\urcorner(\sigma_1)$ is obtained from $\urcorner(\sigma_2)$ by merging some classes together. It is sufficient to deal with the case in which only two classes are merged. We note c_1 and c_2 the two corresponding cycles. Merging the two classes corresponds to the multiplication by an opportune transposition (again, the concept is pictured in figure 2.9). There follows immediately $\sigma_1 | \sigma_2$. It also holds $l(\sigma_2) = l(\sigma_1) + 1$ because we have merged two cycles together, and again by lemma 2.2.1. $\square$

We conclude the poset isomorphism, and again call the poset found P. We now turn to prove that, with labels induced by the natural labelling of $\mathrm{Fact}(\Sigma_n, T_n, c)$ with labels in T_n, poset P is a combinatorial Garside structure. It is of course bounded and of finite height since it was obtained as a factorization poset in a finite group, and is also graded. By the same reason it is group-like, and by proposition 2.2.7 it is a lattice. What is left to prove is that P is balanced. This will be done only in chapter 3, after the introduction of a useful piece of technology: Hurwitz action of the braid group.

Chapter 3

Braid groups

In this final chapter, we will firstly define braid groups as arising from the algebraic topological context, and then show that groups arising from the combinatorial Garside structures we found within the symmetric group Σ_n are both isomorphic to the braid group on n strands B_n. This will be done by computing a presentation for the group arising from the combinatorial Garside structure in each case; We recall from chapter 1 that, when a combinatorial Garside structure P forms, we can obtain a group by using the labels in the structure as generators and relations equating any two words read on maximal chains. We will also show that the presentation from the “classic" combinatorial Garside structure is equivalent to Artin's presentation for the braid group on n strands, and that the presentations arising from both approaches give place to isomorphic groups. By results in chapter 1, this will lead to the fact that the braid group is a Garside group, so that we have an explicit solution for the word problem and the construction of a finite $K(B_n, 1)$ for any n.

3.1 Braid groups

We start by considering the space constructed starting from the action of Σ by permutation of coordinates on $\mathbb{C}^n$. As in section 2.1, where the action was on $\mathbb{R}^n$, this action can be viewed as an action by reflections through the hyperplanes that have the same equation as the ones in $\mathbb{R}^n$, but should be intended in the complex variable. We again consider the complementary space of the union of the hyperplanes. While in $\mathbb{R}^n$ this complementary space would have been a disjoint union of contractible cones, each of which we called chamber, in the complex setting the complementary space has nontrivial topological structure. We are in particular interested in its fundamental group. Note that we are not yet considering the space obtained by quotienting by the action, but just the complementary space of the union of the reflecting hyperplanes. We can picture an element in this space by viewing its n distinct (since otherwise it would have laid on some removed hyperplane) as n distinct ordered points on the complex plane.

Definition 3.1.1. The space formed by the possible configurations of n distinct ordered points on the complex plane is called the *space of ordered configurations* of n distinct points. It will be noted $\mathrm{ConfOrd}_n$ or ConfOrd when n is obvious or unimportant.

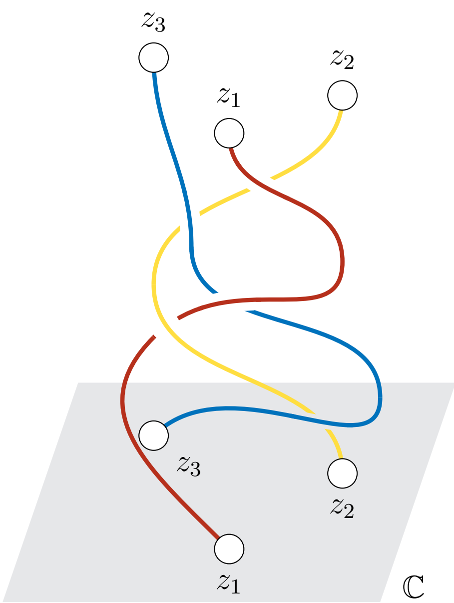
To compute its fundamental group, one can start by considering a loop based at some x_0 in

$$\mathbb{C}^n \backslash \bigcup_{t \in T_n} H^{\mathbb{C}}(t)$$

where we have noted with $H^{\mathbb{C}}(t)$ the hyperplane in $\mathbb{C}^n$ that has the same equation as $H(t)$ from section 2.1.1. When the path variable t varies, in the n-points model one would see each point move along a distinct loop in $\mathbb{C}$. In fact, we can view the path

$$\gamma(r) = \begin{pmatrix} \gamma_1(r) \\ \vdots \\ \gamma_n(r) \end{pmatrix}$$

in ConfOrd based at x_0 as n distinct paths $\gamma_1, \ldots, \gamma_n$ in $\mathbb{C}$, each based at its corresponding $\gamma_i(0)$ which is the i-th coordinate of x_0. Those new n loops in $\mathbb{C}$ will be covered simultaneusly as t varies in $[0,1]$. Moreover, since γ was a path in the complementary of the hyperplanes, at any given time $\gamma_i(t) \neq \gamma_j(t)$ for any i, j, since otherwise γ would cross an hyperplane at the time t. By mapping the value of each loop as t varies, one gets a braid for which the initial and final position of the labels (as read on the start and on the end of the strands) coincide.



Definition 3.1.2. The fundamental group

$$\pi_1(\mathrm{ConfOrd}_n)$$

is called the *pure braid group* on n strands and will be noted PB_n, or PB when n is obvious or unimportant.

We now let Σ act as a group of reflections on ConfOrd. Points in the quotient space correspond to classes of elements in the known complementary space for which an opportune permutation of coordinates takes the one in the other. To visualize one of such classes on the complex plane as for elements of ConfOrd, we will consider n distinct points on the complex plane, but without ordering them.

Definition 3.1.3. The space formed by the possible configurations of n distinct unordered points on the complex plane is called the *space of configurations* of n distinct points. It will be noted Conf_n, or Conf when n is obvious or unimportant.

Note that Conf can be obtained as a quotient of ConfOrd by letting Σ permute the order of the n distinct ordered points in an element of ConfOrd. This operation is the same as the one performed on the complementary space of the hyperplanes, but viewed in the complex plane model. The two quotients actually give place to a covering map from ConfOrd to Conf with n

sheets. When trying to compute its fundamental group, we can use the same technique as for ConfOrd. The main difference is that now a loop based at $\tilde{x_0} \in \text{Conf}$ is induced by any path in the complementary space of the hyperplanes from one and another. In particular, we get many more braids, since a strand of the braid is not bounded to take its complex value in itself anymore, but can reach any other complex value in the initial configuration. We get therefore any braid.

Definition 3.1.4. The fundamental group

$$\pi_1(\text{Conf}_n)$$

is called the *braid group* on n strands and will be noted B_n, or B when n is obvious or unimportant.

Remark 3.1.5 (From braids to permutations). In general a braid induces a permutation of the labels on the strands. To determine the permutation we use the covering structure ConfOrd $\to$ Conf. Fix a point $\tilde{x_0}$ in the fiber of the basepoint x_0. A braid $b \in \mathrm{B}$ lifts to a unique path starting at $\tilde{x_0}$ and ending in some other $\tilde{x_0}'$ also in the fiber. Both x_0 and x_0' are in the same class since they lay in the fiber of a commune point, so that they differ by a permutation of coordinates. We take that permutation to be the permutation associated to b. Geometrically, we can see the permutation by reading the indices on the strands as they appear before and after having crossed the strands. One can observe that different choices of x_0 correspond to different choices of a labelling on the n unlabelled points of $\tilde{x_0}$. This construction thus depends on the choice of $\tilde{x_0}$, but the permutation does not.

The braid pictured induces the trivial permutation. In general the consideration in remark 3.1.5 yields a surjective homomorphism $\pi : \mathrm{B} \to \Sigma$ for which the kernel is PB since pure braids induce the trivial permutation (as they are loops in ConfOrd, so that they connect the choosen $\tilde{x_0}$ to itself). We have therefore that the quotient of B with respect to the normal subgroup PB is isomorphic to Σ. This can be viewed also as a result concerning the covering structure. In fact, from well-known facts about coverings we get that the group of deck transformations is hysomorphic to Σ (corresponding to the fact that the deck transformations correspond to re-labellings of the n points) and the quotient of B with respect to PB is exactly Σ.

It is beyond the purpose of this dissertation to calculate a presentation for B, but it was already known to Artin that the geometric braid group has the following presentation, and trough some technology (see [15]) it is possible to show that it is also a presentation for the space Conf_n, as expected. We will call this presentation *Artin presentation*. It consists in the set of generators

S_n, subject to braid relation and commutation relation as in 2.1, but not involutions. I.e.,

$$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \quad \text{if } |i-j| = 1$$
$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{if } |i-j| > 1.$$

This way of presenting the symmetric group has a strong geometrical meaning. In fact, it reflects the realization of a braid by crossings of n strands whose upper ends are disposed in a line. Generator σ_i corresponds to the geometrical act of crossing the i-th and $(i+1)$-th strand. Relations express the equivalences pictured in figure 3.1.

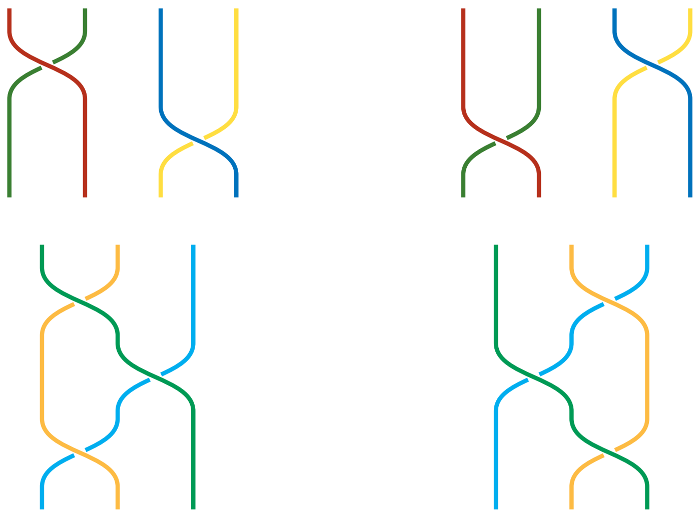


Figure 3.1: Generators s_i and s_j with $|i-j| > 1$ commute (first row), while for $|i-j| = 1$ they do not, but $\sigma_i\sigma_{i+1}s_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ hold (second row).

3.2 Matsumoto's property and classic structure

We fix n. Our task will be now to obtain a presentation for the group found from the classic structure in Σ. We recall that this group is the one presented using S as set of generators and the relations equating every two words read on maximal chains in the poset from classic structure. To find an explicit presentation, we will recall a property of symmetric groups, the exchange property, and then prove that in the symmetric group the so-called Matsumoto's property holds. This will lead to a presentation that we will recognize to coincide with Artin's presentation.

First, we recall what the exchange property is, adapting [3]. On the same text the topic is discussed in detail and a proof is given.

Lemma 3.2.1 (Exchange property for Σ)**.** Let $r \geqslant 1$ fixed and $s_1 \cdots s_r$, $s_1', \cdots s_r'$ be two reduced words for the same permutation μ, where $s_i, s_i' \in S$ for every $i = 1, \ldots, r$. Then for some $q \in${1,. . . r} there exists a reduced word for μ of the form $s_1' s_1 \cdots \hat{s_q} \cdots s_r$.

The following theorem states a crucial property in the study of the combinatorics of the symmetric group. As mentioned in chapter 2, this theorem implies that the relations we used to present the symmetric group as generated by S are sufficient to construct this group. Since the study of the Coxeter-Moore presentation for the symmetric group is beyond the pourpose of this dissertation, we choose nonetheless to present this result only now, since using it we will manage to show our claims. We start with some definitions.

Definition 3.2.2. Let $w = s_1 \cdots s_r$ a reduced word with $s_i \in S$ for every $i = 1, \ldots, r$. Whenever two elements σ_i, σ_j with $|i - j| > 1$ find themselves to be in two consecutive positions in w, we call *commutation move* the substitution $\sigma_i \sigma_j \mapsto \sigma_j \sigma_i$ in the word. Whenever two elements σ_i, σ_j with $|i - j| = 1$ find themselves to be in two consecutive positions in w, we call *braid move* the substitution $\sigma_i \sigma_j \sigma_i \mapsto \sigma_j \sigma_i \sigma_j$ in the word. We say two words on the set of generators S are *classic equivalent* if one can be turned in the other by a finite number of application of commutation and braid moves.

It is a trivial check that the property of being classic equivalent gives place to an equivalence relation.

Theorem 3.2.3 (Matsumoto's property)**.** Any two reduced factorizations with elements in S for the same permutation are classic equivalent.

Proof. We consider again a permutation μ and two reduced words representing μ in Σ_n, say $w = s_1 \cdots s_r$ and $w' = s_1', \cdots s_r'$. We proceed by induction on the length of μ, which we observe to be r. If $r = 1$ the assertion clearly holds.

If $s_1 = s_1'$ or $s_r = s_r'$ then the assertion holds by applying the inductive hypothesis to the reduced words resulting for $s_1\mu = s_1'\mu$ or $s_r\mu = s_r'\mu$ respectively, from our original reduced words. Assume then that this case does not occur.

By the exchange property there exist a reduced word for μ of the form $w'' = s_1' s_1 \cdots \hat{s_x} \cdots s_r$ for some x. By our previous observation, w and w'' are classic equivalent since their first letter coincide. We observe that if w' is equivalent to w'' then it is also equivalent to w. If $x < r$ then w' and w'' coincide again by the observation, since their last letter coincide. We have only one case left: the one for which $x = r$ in the word obtained by applying the exchange property. We are assuming therefore $w'' = s'_1 s_1 \cdots s_{r-1}$. Again applying the exchange property to w and w'', μ has a reduced word of the form $w''' = s_1 s'_1 \cdots \hat{s_y} \cdots s_{r-1}$ Moreover, the deletion in that word must take place strictly after s'_1, since otherwise we would get an unreduced word. Again by the first observation, w and w''' are equivalent. If we showed w'' and w''' are equivalent, we would have concluded. We now therefore claim that $w = s'_1 s_1 \cdots \hat{s_x} \cdots s_r$ and $w' = s_1 s'_1 \cdots \hat{s_y} \cdots s_{r-1}$ (where $1 \leqslant y < r-1$) are classic equivalent.

If $r = 2$ the result is true. In fact, in that case we have that $s_1 s'_1 = s'_1 s_1$ and they are distinct (since the length is 2), so that a commutation move is possible. We assume therefore $r > 3$. If $y < r-1$ again the assertion holds by the initial observation, so that our actual claim is that $w' = s_1 s'_1 s_1 \cdots s_{r-2}$ is classic equivalent to w. Since w' is reduced, we have that $s_1 \neq s'_1$ and $s_1 s'_1 \neq s'_1 s_1$. Therefore s_1 and s'_1 are σ_i and σ_j where $|i-j| = 1$. A braid move is thus sufficient to conclude the proof. □

Remark 3.2.4 (Geometric interpretation). By what we exposed through section 2.1.1, minimal paths in the labelled, oriented 1-skeleton of the permutahedron connecting P_e to P_σ correspond to reduced factorizations for σ. We also know, in particular by remark 2.1.7 and the consequences of lemma 2.1.8, which the shapes of the 2-facets are and how they are labelled and their edges oriented. Let $f = s_1 \cdots s_k$ a reduced factorization for σ. If a commutation relation can be applied turning f into the new reduced factorization f', then two adjacent generators s_i, s_{i+1} in the factorization commute. On the permutahedron, this means that the path corresponding to f crosses the two edges of a squared 2-facet labelled with $\{s_i, s_{i+1}\}$. The path corresponding to f' on M thus coincides with the path for f everywhere except for the edges of that squared facet: if the path for f crossed two edges, then the path for f' crosses the other two. An analogous consideration can be done for the braid relations, corresponding to changing the path on the border of an hexagonal facet. We will call those moves on minimal paths on the permutahedron *squared* and *hexagonal moves*. Both moves are depicted in figure 3.2.

Matsumoto's property can be formulated purely in terms of minimal paths and squared or hexagonal moves, as follows.

Proposition 3.2.5 (Matsumoto's property on the permutahedron). Any two minimal paths on M connecting P_e and a vertex P_σ can be obtained the one from the other by finite applications of squared and hexagonal moves.

The poset we are interested in is again the one found in section 2.1, realized by M and $\mathrm{Fact}(\Sigma, S, \omega_0)$. Generators are the elements of S by construc-

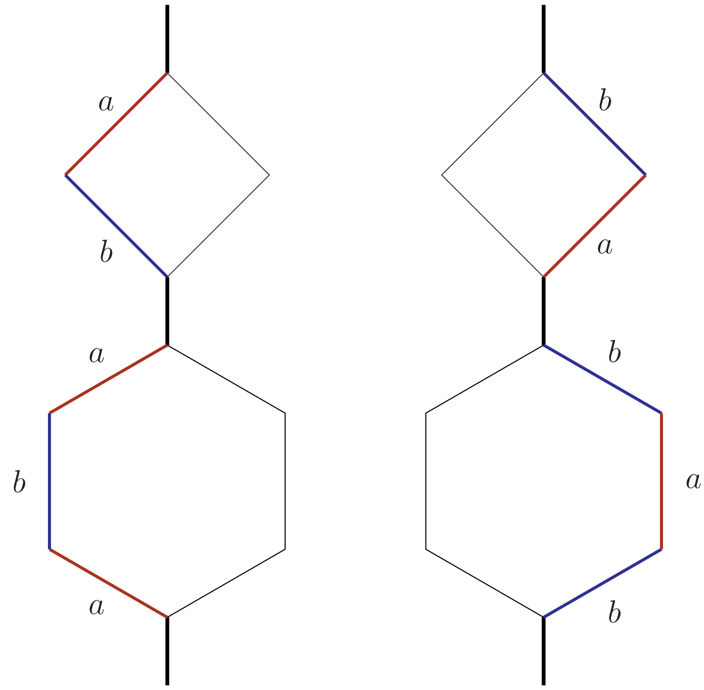


Figure 3.2: The effect of the application of commutation and braid moves to reduced words on corresponding minimal paths on the 1-skeleton of the permutahedron. Switching from the path on the left to the one on the right, a squared and an hexagonal move are being applied.

tion of the labelling on M, and relations are the ones equating words read on any two minimal paths connecting P_e to P_{ω_0}. In particular, words read on two minimal paths differing by a squared or hexagonal move are equated, so that (since they differ on a squared or hexagonal 2-facet of M and coincide everywhere else) applying right and left cancellativity we get braid and commutation relations. On the other hand, relation of such form are enough to present the group, since by Matsumoto's property the equivalence between any two words read on any two minimal paths connecting P_e to P_{ω_0} can be obtained by finite applications of braid and commutation relations. Thus we get that the group arising from classic combinatorial Garside structure in Σ_n has exactly Artin's presentation for the braid group on n strands.

Example 3.2.6. It was already observed that poset on the left in figure 1.1 was a combinatorial Garside structure. It was also noted that it is the classic combinatorial Garside structure taking place in Σ_3. Its Garside group is the braid group on 3 strands, B_3. Thus we have that the braid group on 3 strands is a combinatorial Garside structure, we have a solution for the word problem in this group and we can construct a finite $K(\pi, 1)$, which is depicted on the left in figure 1.3.

3.3 Hurwitz transitivity and dual structure

In this section we will be concerned with the computation of a presentation for the group arising from dual structure, and with a proof of the isomorphism with the classic braid group. The first task will be accomplished through a construction indirectly due to Hurwitz, consisting in an action of braid group B itself on the reduced factorization sets of the n-cycles with transpositions. As a byproduct of the transitivity of this action, we will obtain a complete set of relations to present the group arising from dual structure. Moreover, this presentation will be used to show that the group it individuates and the braid group B_n are isomorphic. We still keep n fixed.

Definition 3.3.1 (Hurwitz action)**.** Let $\mathrm{Red}_n(c)$ be the set of the reduced factorizations of the n-cycle c with elements in T. We consider the action of B_{n-1} on it defined for the generators of B_{n-1} as follows: the image of factorization $(t_1, \ldots, t_{n-1})$ through σ_i has each component unchanged except for components t_i and t_{i+1}, which are changed respectively in $t_i t_{i+1} t_i$ and t_i. The resulting action H will be called *Hurwitz action*.

We observe that this action is indeed an action of the braid group B_n. In fact, whenever $|i - j| = 1$, a quick calculation shows the coincidence

$$H(\sigma_i \sigma_j \sigma_i) = H(\sigma_j \sigma_i \sigma_j)$$

and whenever $|i - j| > 1$ in a similar fashion we check

$$H(\sigma_i \sigma_j) = H(\sigma_i \sigma_j).$$

The morphism H is therefore a homomorphism, and thus we have an action of B expressed with respect to Artin's presentation.

Proposition 3.3.2. Hurwitz action is transitive on reduced factorizations of an n-cycle with elements in T. I.e., for any c and for any two reduced factorizations $w_1, w_2 \in \mathrm{Red}_n(c)$ there exists a braid $b \in B_{n-1}$ for which

$$H(b)(w_1) = w_2.$$

Proof. We proceed by induction on $n - 1$, which we observe to be the length of every n-cycle in Σ_n as we know from lemma 2.2.1. Let

$$(ij) \cdot t_1 \cdots t_{n-2}$$

a reduced factorization for c, so that

$$(ij) \cdot c = t_1 \cdots t_{n-2}.$$

We can assume $i = 1$ and $c = (123 \ldots n)$ since this change can be operated through conjugations and conjugations do not affect the combinatorics we are interested in, as we know from section 2.2. A straightforward calculation shows that

$$(1j) \cdot c = (123 \ldots j-1) \cdot (j \ldots n).$$

We can rearrange $t_1 \cdots t_{n-2}$ using commutations (that can be interpreted as Hurwitz moves whenever they are possible) to form the product of two factorizations, one for $(12 \ldots j-1)$ and the other for $(j \ldots n)$. Up to relabelling, we can assume that $t_1 \cdots t_{j-2}$ was a factorization for the first cycle, and $t_{j-1} \cdots t_{n-2}$ was a factorization for the second cycle.

By inductive hypothesis, the action of B_{j-1} on the reduced factorizations of $(123 \ldots (j-1))$ is transitive, and so is the action of B_{n-j} on the reduced factorizations of $(j \ldots n)$. Up by identifying B_{j-1} with the subgroup of B_{n-1} generated by the first $j-2$ generators, and B_{n-j} with the subgroup of B_{n-1} generated by the last $n-j-1$ generators, we have that by acting with the generators listed, we can change $t_1 \cdots t_{n-1}$ in every desired factorization. In particular through the application of Hurwitz moves of B_n we can get the factorization $(12)(23) \cdots (j-2, j-1)$ for the first cycle as the first fragment of the factorization and the factorization $(j, j+1) \cdots (n-1, n)$ as the second fragment. Through Hurwitz moves thus we get

$$c = (1j)(12)(23) \cdots (j-2, j-1) \cdots (j, j+1) \cdots (n-1, n).$$

But then, by applying $H(\sigma_1^{-1} \cdots \sigma_{j-2}^{-1})$ we get the factorization

$$(12)(23) \cdots (j-2, j-1)(j-1, j)(j, j+1) \cdots (n-1, n).$$

Since we proved that any given factorization for the n-cycle can be taken in the one exhibited, we have the desired transitivity. $\square$

Remark 3.3.3. We start by observing what happens when we repeatedly apply the same Hurwitz move $H(\sigma_i)$ to a reduced word for c. We consider therefore the two distinct transpositions t_1 and t_2, in position i and $i+1$ respectively in the reduced factorization considered for c.

we assume firstly that the two indices switched by t_1 and the ones switched by t_2 have no index in common, so that they commute and therefore $t_2 t_1 t_2 = t_1$. Then $H(\sigma_i)$ acts as

$$t_1, t_2 \mapsto t_2, t_1 \mapsto t_1 t_2.$$

Note that since t_1 and t_2 are found as successive factors in a reduced factorization of the n-cycle, they switch two pairs of indices that are noncrossing with respect to that $n - cycle$. To see this, consider $t_1 \cdots t_k, t_1 \cdots t_{k+1}|c$; in order for the length to increase by one by multiplying by (t_{k+1}), this permutation needs to merge together two classes in $\urcorner(t_1 \cdots t_k)$ to form another noncrossing partition, as can be seen from the explicit computation of the length function with respect to the cyclic structure of a permutation, in lemma 2.2.1. We draw the edge corresponding to t_k in the geometric representation of $\urcorner(t_1 \cdots t_k)$ merging together two classes as in figure 2.9. At this point, the edge corresponding to transposition t_{k+1} needs to merge together two new classes; in particular it cannot cross any convex hull corresponding to a class, in particular the indices it swaps and the indices swapped by t_k form two noncrossing couples.

On the other hand, assume the two pairs of indices have an index in common, so that $t_1 = (ij)$ and $t_2 = (jk)$ for some $i < j < k$. Then the Hurwitz move acts as

$$t_1, t_2 \mapsto t_2, (ik) \mapsto (ik), t_1 \mapsto t_1, t_2.$$

Hurwitz moves therefore act as either the commutation move or the move

$$(ij)(jk) \mapsto (jk)(ki) \mapsto (ki)(ij) \mapsto (ij)(jk)$$

which we will refer to as *dual braid move* for reasons that will be clear in the following. This last observation has an interpretation in terms of the shape of poset P. In fact, from the transitivity of Hurwitz action we get that for any two maximal chains in P we can transform the one in the other by finite applications of either commutation or dual braid moves. While we already observed what effect a commutation move has on the chains in section 3.2, we observe that a dual braid move acts as depicted in figure 3.3.

First we have to pay the debt we contracted in chapter 2: proving the balancedness for the dual poset.

Proposition 3.3.4. Poset P found as the dual structure in Σ is balanced.

Proof. Let $t_1 \cdots t_k \in \mathrm{Pre}(P)$. Then there is some reduced factorization $t_1 \cdots t_k \cdot t_{k+1} \cdots t_{n-1}$ for the fixed c. Then by applying Hurwitz move

$$H(\sigma_k \cdots \sigma_{n-2})$$

we have the reduced factorization $t_1 \cdots t_{k-1} t'_{k+1} \cdots t'_{n-1} t_k$. By repeadetly applying Hurwitz moves of the same type we have the reduced word $t''_{k+1} \cdots t''_{n-1} t_1 \cdots t_k$, so that we have proved $\mathrm{Pre}(P) \subseteq \mathrm{Suf}(P)$. The same argument but with the use of the action of the inverses of elements in S shows the inverse inclusion, so that the poset is balanced. □

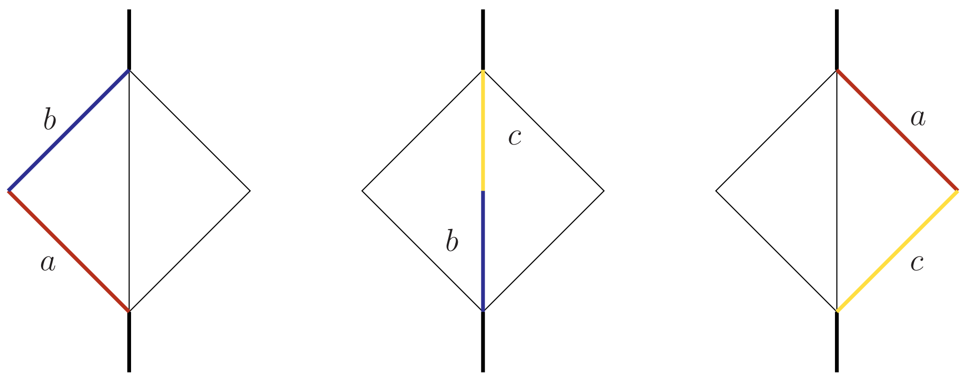


Figure 3.3: The effect of the application of dual braid moves to reduced words on corresponding maximal chains of the dual structure.

We consider the poset found in section 2.2 inside Σ. We again have to form a group by using the set of labels as generators and relations equating words on any two maximal chains. The set of labels is T by construction of the poset as $\mathrm{Fact}(\Sigma, T, c)$, so that we have T as set of generators. Relations equating any two maximal chains in particular equate two chains differing by a commutation or dual braid move, so that for any four distinct indices i, j, k, l,

$$(ij)(jk) = (jk)(ki) = (ki)(ij)$$

and, whenever i, j and k, l are noncrossing pairs,

$$(ij)(kl) = (kl)(ij)$$

also are relations in the resulting group (see remark 3.3.3). By Hurwitz transitivity, those relations are enough to present the group, since they express exactly the moves corresponding to Hurwitz action, which we know to be transitive. This new group will be noted Γ_n (or simply Γ as always when n is obvious or unimportant) and is presented by T as set of generators,

$$(ij)(jk) = (ik)(ij) = (jk)(ik)$$

for any three indices $i < j < k$,

$$(ij)(kl) = (kl)(ij)$$

for any two noncrossing couple i, j and k, l.

The dual presentation arises geometrically by considering the braid group as realized intertwining strands disposed in a circle. Consider in fact n strings labelled with indices $1, \ldots, n$. One end of each strands is fixed so that the ends of the strands form a circle, and the strands are free to dangle. The generator (ij) of the dual presentation corresponds to the act of switching the free end of strands i and j, by making them move the one around the other counterclockwise (obviously the counterclockwise sense was chosen arbitrarily).

This interpretation makes it intuitively easy to see why two generators (ij) and (kl) commute only if the four indices form two noncrossing couples.

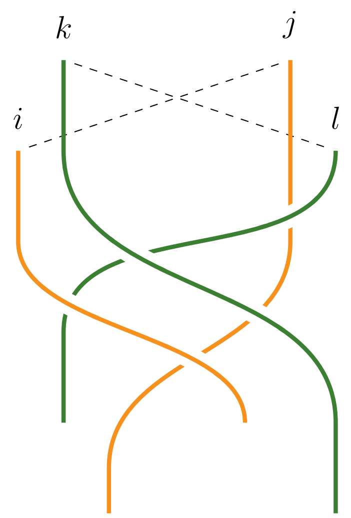


Figure 3.4: When i, j and k, l are crossing pairs, the crossing of strands i and j and the crossing of strands k and l do not commute, since the crossed strands block any attempt to move the crossing between the latter above the crossing between the former.

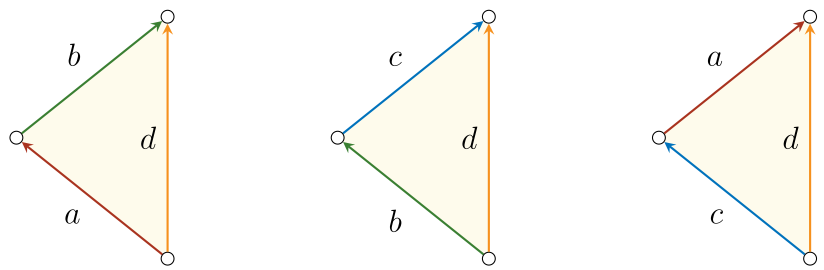
Example 3.3.5. On the right in figure 1.1 we find the dual combinatorial Garside structure from Σ_3. The resulting group is

$$\langle a, b, c \mid ab = bc = ca \rangle$$

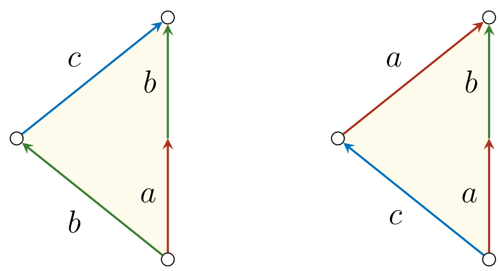
for which the quotient of the object depicted on the right in figure 1.3 is a finite $K(\pi, 1)$.

Lemma 3.3.6. The interval complexes arising from the two combinatorial Garside structures in Σ_3 are homotopically equivalent. In particular the dual interval complex can be deformation retracted on a copy of the classic interval complex.

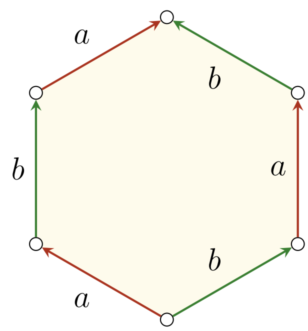
Proof. We consider the order complex arising from the dual approach - as pictured on the right in figure 1.3. It is equivalently obtained by quotienting along the labelled arrows the following spaces.



We start by deformation retracting the first triangle onto the two edges labelled a and b. Thus in the second triangle the edge labelled d is now attached along the edges a and b, both crossed orientation-wise, and similarly happens in the third triangle. We watch now the second triangle, which has now the depicted attachment maps.



It is already evident how this object is the desired space. We make it evident by deformation retracting c in the triangle on the left on the edges labelled b and a. Thus in the third triangle the attachment maps have become the ones of the following complex;



This last object then appears to be a deformation retract of the initial object. □

In the following this special case for the interval complex will play a fundamental role. We will note coharsely the dual interval complex of Σ_3 labelled a, b, c with $D(a, b, c)$ and the classic interval complex labelled a, b with $C(a, b)$.

Proposition 3.3.7. The group Γ is isomorphic to B.

Proof. The general idea for this proof is to construct a cell complex of dimension 2 such that its fundamental group is presented as Γ. Through an algorithm, it will be deformed to become another 2-complex with fundamental group the braid group in Artin presentation. We will show that the final and initial space for wvwery step of the algorithm are homotopic equivalent, and that the algorithm ends in a finite number of steps. Also, some calculations will be necessary to make sure that no 2-cell in the original complex induces some relation in the final complex other that Artin's relations.

First, we construct the initial complex. To do this, we consider the complete graph on n vertices, with the vertices labelled with integers $1, \ldots, n$. We consider in particular the model obtained from the regular n-gone by labelling the vertices counterclockwise. Each edge of the graph has as bounding vertices two vertices with indices $i < j$. We label the edge (ij) and orient the edge from i toward j. This object will be noted F. We now identify the vertices of the graph with a single vertex, getting a wedge sum of oriented circles, each labelled with a transposition. We now will attach some 2-complexes to induce the opportune relations. Whenever a triangle forms in F with vertices labelled $i < j < k$, we attach a copy of $D((ij), (jk), (ik))$ along the oriented circles with same labels, respecting orientations. Whenever two edges labelled $(ij), (kl)$ are noncrossing in F, we attach a square on the circles with those labels inducing a commutation in the fundamental group. We call the resulting 2-complex G.

We now consider the following operation on G. Take a triangle in F labelled $i < j < k$. Attached on the edges $(ij), (jk), (ik)$ in G we find a copy of $D((ij), (jk), (ik))$. By lemma 3.3.6, we can deformation retract it on a copy of $C((ij), (jk))$. There is more: the edge in D that was attached to (ik) now is attached along $(ij)^{-1}(jk)(ij)$, meaning that we have eliminated a generator in the fundamental group of G. Moreover, we have obtained a homotopically equivalent 2-complex. We will refer to this operation as to the *killing of* (ik) *in the triangle* i, j, k (we will assume by using this notation that $i < j < k$).

We now consider the following algorithm. For every edge (ik) from i to k in F we call $k - i$ the *covering number* of (ik), and note it λ. The edges $(j, j + 1)$ with $i \leqslant j \leqslant k - 1$ are called *covered edges* of (ik).

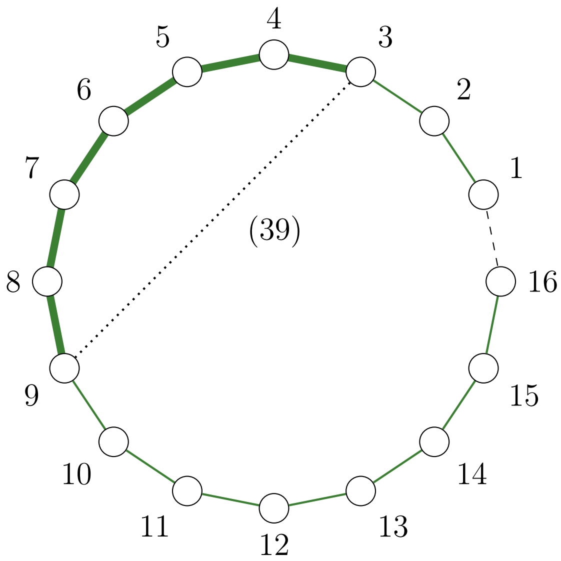


Figure 3.5: The edge (39) here has $\lambda = 6$. Thick edges are the covered edges.

For each $\lambda = n-1, \ldots, 2$ we consider the values of i for which $i + \lambda < n$, so that for those values of i it holds that $(i, i+\lambda)$ is an edge in F. For each of those values of i, we kill $(i, i+\lambda)$ in the triangle $i, i+1, i+\lambda$.

The triangle on which a given step of the algorithm operates has not been touched at some earlier step of the algorithm; in fact for fixed λ and $i \neq j$ the triangles $i, i+1, i+\lambda$ and $j, j+1, j+\lambda$ have no edge in common, and for each λ only edges with covering number λ are killed. The algorithm is thus well-posed. It also ends in a finite number of steps since the edges are finite.

We consider the space obtained at the end of the algorithm. Each circle labelled (ik) has been deformation retract onto some path on the circles labelled as the covering edges of (ik). In particular at each step of the algorithm we have written the generator (ik) of the fundamental group as $\sigma_i^{-1}(i+1,k)\sigma_i$ (where σ_i is usual notation). Prosecuting the algorithm, we have killed every edge that crossed the interior of F, leaving only edges corresponding to elements in S. In the last step of the algorithm (for $\lambda = 2$) we have attached an hexagon along each couple of circles labelled σ_i, σ_{i+1}, and a square is present for each pair of circles labelled σ_i, σ_j with $|i-j| > 1$ from the very beginning of the algorithm since they were noncrossing. Thus, the fundamental group of the resulting space is generated by Artin generators, and Artin relations hold. The hexagonal and squared 2-cells attached on the original complex are still there. All what is left to check is that those 2-cells do not induce relations other that the ones already expressed by Artin relations.

The hexagon attached at the algorithm step operating on the triangle $i, i+1, i+\lambda$ induces the relation

$$\sigma_i(i+1, i+\lambda)\sigma_i = (i+1, i+\lambda)\sigma_i(i+1, i+\lambda).$$

Later steps of the algorithm induce substitutions of the form

$$(i+1, i+\lambda) = \sigma_{i+1}^{-1}(i+2, i+\lambda)\sigma_{i+1},$$

"retracting" circles corresponding to edge (ik) on the circles corresponding to the edges covered by (ik), so that the relation induced in the fundamental group of the resulting space is of the form

$$\begin{gathered} s_1 s_2^{-1} \cdots s_{k-1}^{-1} s_k s_{k-1} \cdots s_2 s_1 = \\ s_2^{-1} \cdots s_{k-1}^{-1} s_k s_{k-1} \cdots s_2 s_1 s_2^{-1} \cdots s_{k-1}^{-1} s_k s_{k-1} \cdots s_2 \end{gathered} \tag{3.1}$$

while by the same reasoning squares induce relations of the form

$$\begin{gathered} s_1^{-1} \cdots s_{k-1}^{-1} s_k s_{k-1} \cdots s_1 \cdot s_i^{-1} \cdots s_{l-1}^{-1} s_l s_{l-1} \cdots s_i = \\ s_i^{-1} \cdots s_{l-1}^{-1} s_l s_{l-1} \cdots s_i \cdot s_2^{-1} \cdots s_{k-1}^{-1} s_k s_{k-1} \cdots s_2. \end{gathered} \tag{3.2}$$

Note that since squares were attached along pairs of circles corresponding to noncrossing pairs of edges in F, $\{s_1, \ldots, s_k\}$ and $\{s_i, \ldots, s_l\}$ are disjoint. Note also that, although in general $s_i \neq \sigma_i$, it is true that if $s_1 = \sigma_i$ then $s_j = \sigma_{i+j}$. Our claim is that those two equations can be realized through the application of Artin relations. We will prove this claim in the following two lemmas.

Lemma 3.3.8. A finite sequence of applications of Artin relations realizes equation 3.1.

Proof. We proceed by induction on k. For $k = 1$ we get an Artin relation.

We consider now the general case. First we multiply each side by $s_2^{-1} \cdots s_{k-1}^{-1}$ getting

$$\begin{gathered} s_1 s_2^{-1} \cdots s_{k-1}^{-1} s_k s_{k-1} \cdots s_2 s_1 s_2^{-1} \cdots s_{k-1}^{-1} = \\ s_2^{-1} \cdots s_{k-1}^{-1} s_k s_{k-1} \cdots s_2 s_1 s_2^{-1} \cdots s_{k-1}^{-1} s_k, \end{gathered}$$

then we apply on both sides the substitution $s_{k-1}^{-1} s_k s_{k-1} = s_k s_{k-1} s_k^{-1}$ which is a trivial consequence of Artin relations.

$$s_1 s_2^{-1} \cdots s_k s_{k-1} s_k^{-1} \cdots s_2 s_1 s_2^{-1} \cdots s_{k-1}^{-1} =$$
$$s_2^{-1} \cdots s_k s_{k-1} s_k^{-1} \cdots s_2 s_1 s_2^{-1} \cdots s_{k-1}^{-1} s_k$$

Now, s_k commutes with each s_j with $j < k$.

$$s_k s_1 s_2^{-1} \cdots s_{k-2}^{-1} s_{k-1} s_{k-2} \cdots s_2 s_1 s_2^{-1} \cdots s_{k-2}^{-1} s_k^{-1} s_{k-1}^{-1} =$$
$$s_k s_2^{-1} \cdots s_{k-2}^{-1} s_{k-1} s_{k-2} \cdots s_2 s_1 s_2^{-1} \cdots s_{k-2}^{-1} s_k^{-1} s_{k-1}^{-1} s_k$$

After the application of $s_k^{-1} s_{k-1}^{-1} s_k = s_k s_{k-1}^{-1} s_k^{-1}$, we have that s_k (the head of each side) and $s_k^{-1} s_{k-1}^{-1}$ (the tail of each side) cancel on both sides;

$$s_1 s_2^{-1} \cdots s_{k-2}^{-1} s_{k-1} s_{k-2} \cdots s_2 s_1 s_2^{-1} \cdots s_{k-2}^{-1} =$$
$$s_2^{-1} \cdots s_{k-2}^{-1} s_{k-1} s_{k-2} \cdots s_2 s_1 s_2^{-1} \cdots s_{k-2}^{-1} s_{k-1}$$

so that we conclude by inductive hypothesis once we multiply each side by $s_{k-2} \cdots s_2$. □

Lemma 3.3.9. A finite sequence of applications of Artin relations realizes equation 3.4.

Proof. We again proceed by induction, but twice. We start by observing that for $k = 1, l = i$ we have an Artin relation.

For the arbitrary k and $l = i$ we have

$$\begin{aligned} &s_1 \cdots s_{k-1} s_k s_{k-1}^{-1} \cdots s_1^{-1} \cdot s_i = \\ &s_i \cdot s_2 \cdots s_{k-1} s_k s_{k-1}^{-1} \cdots s_2^{-1}. \end{aligned} \tag{3.3}$$

By substituting $s_{k-1} s_k s_{k-1}^{-1} = s_k^{-1} s_{k-1} s_k$ and commutations (we observed that s_i is not in $\{s_1, \ldots, s_k\}$ since the original edges were noncrossing),

$$\begin{aligned} &s_k^{-1} s_1 \cdots s_{k-2} s_{k-1} s_{k-2}^{-1} \cdots s_1^{-1} \cdot s_i \cdot s_k = \\ &s_k^{-1} s_i \cdot s_2 \cdots s_{k-1} s_k s_{k-1}^{-1} \cdots s_2^{-1} s_k. \end{aligned} \tag{3.4}$$

By cancelling out s_k and s_k^{-1} we conclude by inductive hypothesis on k. By an analogue induction on $l - i$ one concludes the proof. □

The proof is concluded, since the homotopical equivalence between the starting complex and the final complex yields a group isomorphism between Γ and B as required. □

In conclusion we have a second finite $K(\mathrm{B}_n, 1)$ for each n, and results in chapter 1 apply.

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